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ON STRUCTURE PRESERVING GROUPS OF LATIN SQUARES AND THEIR APPLICATIONS TO STATISTICS

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ABSTRACT

ON STRUCTURE PRESERVING GROUPS OF LATIN SQUARES
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We consider a symmetric property, invariance of probability distribution under a group of transformations of the sample space, of Latin square designs. The group of transformations of the sample space will be called "the structure preserving group for a Latin square design." We show that Latin square designs from the same transformation set have isomorphic structure preserving groups. The commutator algebras of the representation of the structure preserving groups are then studied. The structure preserving group and commutator algebra are computed for one Latin square design from each transformation set of Latin square designs of orders three, four and five. Associated with the symmetric property, random assignment of treatments (Latin square design as a fractional three factor design) to subjects and randomization tests for Latin square designs are then studied.

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INTRODUCTION

Latin square designs are used in agricultural experiments. Suppose we wish to find out by experiments whether there is any significant difference among yields of m different varieties $v_1,\ldots,v_m$. The experimental field is subdivided into m^2 plots laid out in m rows and m columns and each plot is assigned to one of the m varieties. If each variety appears once and only once in each row and each column, we have a Latin square arrangement. Latin square designs are also used in biological experiments to provide a method of controlling individual differences among experimental units. Another important use of Latin square designs is in the area of behavioral sciences to counterbalance order effects in repeated measurements plans. The dual balance, i.e. the varieties or treatments to be compared are equally represented across each row and each column, makes the statistical analysis for Latin square designs more precise than those designs without such balance. Wald [15] and Ehrenfeld [3] studied the problem of testing linear hypotheses for a linear regression model. With respect to the problem, Wald [15] stated an optimality criterion (called D-optimality by Kiefer [10]) for designs in the setting of two-way heterogeneity (m treatments are assigned to a $m \times m$ array of plots in such a way that each plot receives one treatment) and showed that Latin square designs are optimal among them. With respect to the same problem, Ehrenfeld [3] stated another optimality criterion (called E-optimality by Kiefer [10]) for designs in the setting of two-way heterogeneity and showed that Latin square designs are optimal among them. Their results thus enhance the superiority of Latin square designs. Kiefer [10] generalized their ideas and defined several different optimality criteria for non-randomized designs, with respect to the same problem of testing linear hypotheses for a linear regression model. He proved that balanced incomplete block designs are optimal among designs in the setting of one-way heterogeneity (m treatments are assigned to b blocks, each of which contains k plots). He also proved that Youden square designs are optimal among designs in the setting of two-way heterogeneity (m treatments are assigned to a $k_1 \times k_2$ array of plots). So, with respect to the problem of testing linear hypotheses for a linear regression model, all Latin square designs of the same order are equally good.

James [6] studied the algebraic structures of randomized block designs, Latin square designs and balanced incomplete block designs. He introduced the concept of relationship algebra for those designs and showed that the relationship algebras of Latin square designs are always isomorphic to the algebra of all diagonal 5×5 matrices. So, all Latin square designs of the same order are the same with respect to the structure of their relationship algebras. Dubenko, Sysoev and Shaikin ([1],[2],[12],[13]) studied the symmetry properties of the designs studied by James. They introduced the concepts of commutator algebras, associated with the symmetry properties, for those designs. They proved that the relationship algebra of any one of those designs is always a subalgebra of the commutator algebra of the design. For

Latin square designs, they stated that "It can be shown by direct verification that the commutator algebra coincides with the relationship algebra for all 2×2 and 3×3 squares and for 4×4 square of the form $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}$. Latin squares of larger dimensions have not yet been investigated. But it is clear that the dimensions of the commutator algebra, first of all, depend on the type of square, second, increase with the order of the square."

In the first section of this dissertation, we study the symmetry properties of designs in the setting of two-way heterogeneity (it includes Latin square designs in particular). We show that designs from a transformation set (similar to the transformation set defined for Latin square designs in Fisher and Yates [5]) have similar symmetry property and similar commutator algebras associated with them. The symmetry properties introduced by Dubenko, Sysoev and Shaikin thus distinguish designs from different transformation sets. In the second section, we compute the commutator algebras of Latin square designs, from different transformation sets of orders three, four and five. Commutator algebras, associated with designs from different transformation sets, are quite different for Latin square designs of orders four and five. It also gives one counter example to the statement that "the dimensions of the commutator algebra increase with the order of the square." In the third section, we study different schemes to assign treatments (Latin square design as a fractional three factor design) to subjects and the impact of the different randomization schemes on the analysis of Latin square designs. We show that a particular scheme will justify the assumption about the covariance matrix of the observed random vector. In the last section, we study the randomization test (Fisher [4]) for Latin square designs. We show that only those designs, from the same transformation set as the design actually used, should be included in the randomization test.

§1. A SYMMETRIC PROPERTY OF LATIN SQUARE DESIGNS AND SOME OF ITS RELATED CONCEPTS

Many statistical experimental designs exhibit symmetries, which provide natural restrictions to impose on the probability distribution of the observed random variable. In this section, we study symmetric properties of Latin square designs.

A Latin square design of order k can be represented by a $k \times k$ matrix with elements from a finite set of k symbols, say $\{1,2,\dots,k\}$, such that each of the k symbols appears once and only once in each row and each column. Latin square designs are commonly used in small scale pilot experiments to remove the heterogeneity of experimental material in two directions. For example, a 4×4 matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ represents a Latin square design of order 4. A Latin square design can also be described as an incomplete 3-factor design (row factor, column factor and treatment factor) which is balanced with respect to main effects but only partially balanced with respect to two-factor interactions (i.e. each row level occurs in combination with each column level and each treatment level but does not occur in combination with all possible pairs of column level and treatment level). In the following, we discuss experimental designs which can be represented as a $b \times k$ matrix with elements from a finite set, say $\{1,2,\dots,v\}$. The finite set $\{1,2,\dots,v\}$ represents possible treatment levels. Latin square designs are a specific type of designs out of such designs. So all the results in this section are applicable to Latin square designs in particular. The cell in the i^{th} row and j^{th} column is ordered as the $[(i-1)k+j]^{th}$ cell of a $b \times k$ matrix. In a given design, there are three numbers (r_x, c_x, t_x) associated with each cell x, where r_x is the row level of the cell, c_x is the column level of the cell, and t_x is the treatment level. A given design thus can also be represented by $\begin{pmatrix} t_1 t_2 \cdots\cdots\cdots t_k \\ \cdots\cdots\cdots\cdots \\ t_{(b-1)k+1} \cdots t_{bk} \end{pmatrix}$, denoted also by GDM (Given Design Matrix). In the Latin square design $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}$, the following equality holds: $(r_9, c_9, t_9) = (3,1,3)$.

Different kinds of permutations of cells will be defined next.

Definition 1.1: A row preserving permutation g of cells with respect to a given design is a permutation of $\{1,2,\ldots,bk\}$ satisfying the condition that $r_x = r_y$ if and only if $r_{g(x)} = r_{g(y)}$ for $x,y = 1,2,\ldots,bk$.

Definition 1.2: A column preserving permutation g of cells with respect to a given design is a permutation of $\{1,2,\ldots,bk\}$ satisfying the condition that $c_x = c_y$ if and only if $c_{g(x)} = c_{g(y)}$ for $x,y = 1,2,\ldots,bk$.

Definition 1.3: A treatment preserving permutation g of cells with respect to a given design is a permutation of $\{1,2,\ldots,bk\}$ satisfying the condition that $t_x = t_y$ if and only if $t_{g(x)} = t_{g(y)}$ for $x,y = 1,2,\ldots,bk$.

Let

G_r = {g|g is a row preserving permutation of cells}

G_c = {g|g is a column preserving permutation of cells}

G_t = {g|g is a treatment preserving permutation of cells}.

It follows immediately that G_r, G_c and G_t are subgroups of the symmetric group S_{bk}, i.e. the group of all permutations of $\{1,2,\ldots,bk\}$ with its binary operation "composition of mappings." A permutation belonging to the set of intersection of G_r, G_c and G_t (i.e. $G_r \cap G_c \cap G_t$) is called an admissible permutation of cells. In the paper of (Dubenko, Sysoev and Shaikin 1976 [2]), they call the group $G_r \cap G_c \cap G_t$ "the symmetry group for a given design." To avoid the confusion between symmetry group and symmetric group, let us call $G_r \cap G_c \cap G_t$ "The <u>structure preserving group</u> for a given design." It is also denoted by $K\begin{pmatrix} t_1 t_2 \ldots\ldots\ldots t_k \\ \ldots\ldots\ldots\ldots \\ t_{(b-1)k+1} \ldots t_{bk} \end{pmatrix}$, i.e. K(GDM), to emphasize its dependency on the given design GDM. Note that $G_r \cap G_c$ is a group which is independent of the given design. A permutation $g \in G_r \cap G_c$ has the following representation:

$$\begin{pmatrix} g(1) \ldots\ldots\ldots\ldots & g(k) \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ g((b-1)k+1) \ldots\ldots & g(bk) \end{pmatrix} = L_g \cdot \begin{pmatrix} 1 \; 2 \ldots\ldots\ldots & k \\ \ldots\ldots\ldots\ldots\ldots & \\ (b-1)k+1 \ldots\ldots & b_k \end{pmatrix} \cdot R_g$$

where L_g and R_g are two elementary matrices obtained from the identity matrices I_b and I_k, respectively, through permutation of rows or columns of the identity matrices. Let us call

$\begin{pmatrix} 1\ 2\ \ldots\ldots\ldots\ k \\ \ldots\ldots\ldots\ldots\ldots \\ (b-1)k+1 \ldots\ldots\ bk \end{pmatrix}$, the Cell Matrix, abbreviated as CM. Let us call $\begin{pmatrix} g(1)\ \ldots\ldots\ldots\ldots\ g(k) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ g((b-1)k+1) \ldots\ldots\ g(bk) \end{pmatrix}$, the Transformed Cell Matrix determined by g and abbreviated as TCM_g. So the above representation becomes $TCM_g = L_g \cdot CM \cdot R_g$. The left multiplication of CM by L_g amounts to a row permutation of CM. The right multiplication of CM by R_g amounts to a column permutation of CM. Moreover, the representation mentioned above is unique in the sense that if

$$\begin{pmatrix} g(1)\ \ldots\ldots\ldots\ g(k) \\ \ldots\ldots\ldots\ldots\ldots \\ g((b-1)k+1) \ldots\ g(bk) \end{pmatrix} = L_g \cdot \begin{pmatrix} 1\ 2\ \ldots\ldots\ k \\ \ldots\ldots\ldots\ldots \\ (b-1)k+1\ ..\ bk \end{pmatrix} \cdot R_g = L_g' \cdot \begin{pmatrix} 1\ 2\ \ldots\ldots k \\ \ldots\ldots\ldots\ldots \\ (b-1)k+1\ ..bk \end{pmatrix} \cdot R_g'$$

then $L_g = L_g'$ and $R_g = R_g'$, where L_g' and R_g' also are elementary matrices. There are $b!$ different elementary matrices which can be obtained from I_b by permutation of rows or columns. From the above representation, the order of the group $G_r \cap G_c$, denoted by $|G_r \cap G_c|$, is equal to $b!k!$. At this point, it is important to discuss the following lemmas which will be used later.

<u>Lemma 1.1</u>: Let h be a permutation of $\{1,2,\ldots,bk\}$ then

$$L_g \cdot \begin{pmatrix} h(1)\ \ldots\ldots\ldots\ h(k) \\ \ldots\ldots\ldots\ldots\ldots \\ h((b-1)k+1) \ldots\ h(bk) \end{pmatrix} \cdot R_g = \begin{pmatrix} h\circ g(1)\ \ldots\ldots\ldots\ldots\ h\circ g(k) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ h\circ g((b-1)k+1) \ldots\ldots\ h\circ g(bk) \end{pmatrix}$$

i.e. $L_g \cdot TCM_h \cdot R_g = TCM_{h\circ g}$, where $L_g \cdot CM \cdot R_g = TCM_g$ and $h\circ g$ is the composition of h and g with $h\circ g(i) = h(g(i))$.

<u>Proof</u>: The mapping of $CM \to L_g \cdot CM \cdot R_g$ amounts to moving the element i from i^{th} cell to j^{th} cell where $g(j) = i$, and $i = 1,2,\ldots,bk$. So

the mapping $TCM_h \to L_g \cdot TCM_h \cdot R_g$ amounts to moving the element $h(i)$ from i^{th} cell to j^{th} cell where $h \circ g(j) = h(i)$ and $i = 1,2,\ldots,bk$. It follows that $L_g \cdot TCM_h \cdot R_g = TCM_{h \circ g}$. Q.E.D.

Lemma 1.2: If $g_1, g_2 \in G_r \cap G_c$ then $L_{g_1} \cdot L_{g_2} = L_{g_2 \circ g_1}$, $R_{g_2} \cdot R_{g_1} = R_{g_2 \circ g_1}$, $L_{g_1^{-1}} = L_{g_1}^{-1}$ and $R_{g_1^{-1}} = R_{g_1}^{-1}$.

Proof:

$$L_{g_1} \cdot L_{g_2} \cdot CM \cdot R_{g_2} \cdot R_{g_1}$$

$$= TCM_{g_2 \circ g_1}$$

$$= L_{g_2 \circ g_1} \cdot CM \cdot R_{g_2 \circ g_1}$$

$\therefore$ $L_{g_1} \cdot L_{g_2} = L_{g_2 \circ g_1}$ and $R_{g_2} \cdot R_{g_1} = R_{g_2 \circ g_1}$ follows from the uniqueness of representation.

$\because$ $L_{g_1} \cdot L_{g_1^{-1}} \cdot CM \cdot R_{g_1^{-1}} \cdot R_{g_1}$

$$= L_{g_1^{-1} \circ g_1} \cdot CM \cdot R_{g_1^{-1} \circ g_1}$$

$$= L_{g_1} \cdot L_{g_1}^{-1} \cdot CM \cdot R_{g_1}^{-1} \cdot R_{g_1}$$

$\therefore$ $L_{g_1^{-1}} = L_{g_1}^{-1}$ and $R_{g_1^{-1}} = R_{g_1}^{-1}$ follows from the uniqueness of representation again. Q.E.D.

Let us define the set $D = \{L_g \cdot GDM \cdot R_g | g \in G_r \cap G_c\}$ where GDM is a given design. The number of elements in D (counting the possible repetitions) is $b! \times k!$. In other words, D is the collection of all the possible designs which can be obtained from the given design by a row permutation and a column permutation.

<u>Definition 1.4</u>: Two designs $L_g \cdot GDM \cdot R_g$ and $L_{g'} \cdot GDM \cdot R_{g'}$, are defined to be equivalent, $L_g \cdot GDM \cdot R_g \sim L_{g'} \cdot GDM \cdot R_{g'}$, if and only if one of them can be obtained from the other by a permutation of treatments.

Let us define the set $D_a = \{L_g \cdot GDM \cdot R_g | g \in G_r \cap G_c \cap G_t\}$. We use the notation D_a because an element of $G_r \cap G_c \cap G_t$ is called an admissible permutation. It can also be defined as $D_a = \{A | A \in D$ and $A \sim GDM\}$. The number of elements in D_a (counting the possible repetitions) is equal to the order of the group $G_r \cap G_c \cap G_t$, $|G_r \cap G_c \cap G_t|$. Concerning the invariant property of the given design, up to equivalence with respect to a row permutation and a column permutation of the given design, the cardinality of D_a is an indicator for this invariance property. The equivalence relation defined above partitions the set D into equivalence classes. It will be shown that all the equivalence classes have the same number of elements, and D_a is one of the equivalence classes. If the number of equivalence classes is given, then we know the order of the structure preserving group, and vice versa.

<u>Theorem 1.1</u>: The equivalence relation, which states that two designs in the set D are equivalent to each other if one of them can be obtained from the other by a permutation of treatments, partitions the set D into a number of equivalence classes which have the same number of elements as the structure preserving group does.

<u>Proof</u>: Let

$$L_{g_0} \cdot GDM \cdot R_{g_0} \sim L_{g'} \cdot GDM \cdot R_{g'}$$

then

$$L_{g_o}^{-1}\cdot L_{g_o}\cdot GDM\cdot R_{g_o}\cdot R_{g_o}^{-1}\sim L_{g_o}^{-1}\cdot L_{g'}\cdot GDM\cdot R_{g'}\cdot R_{g_o}^{-1}$$

i.e. $GDM \sim L_{g'\circ g_o^{-1}}\cdot GDM\cdot R_{g'\circ g_o^{-1}}$

$\therefore \quad g'og_o^{-1} \in G_r \cap G_c \cap G_t$

$\therefore \quad L_{g'}\cdot GDM\cdot R_{g'} = L_{g_o}\cdot L_{g_i}\cdot GDM\cdot R_{g_i}\cdot R_{g_o}$ for some $g_i \in G_r \cap G_c \cap G_t$.

The equivalance class determined by $L_{g_o}\cdot GDM\cdot R_{g_o}$, denoted by D_{g_o}, is as follows:

$$D_{g_o} = \{L_{g_o}\cdot L_{g_i}\cdot GDM\cdot R_{g_i}\cdot R_{g_o} | g_i \in G_r \cap G_c \cap G_t\}$$

$$= \{L_g\cdot GDM\cdot R_g | g \in (G_r \cap G_c \cap G_t)\cdot g_o\}$$

where $(G_r \cap G_c \cap G_t)\cdot g_o$ is a coset of $G_r \cap G_c \cap G_t$ and the cardinality of $(G_r \cap G_c \cap G_t)\cdot g_o$, denoted by $|(G_r \cap G_c \cap G_t)\cdot g_o|$, is equal to $|G_r \cap G_c \cap G_t|$. Note that $D = \bigcup_{g \in G_r \cap G_c} D_g$, and we thus have shown that D is partitioned into a number of equivalence classes that have the same number of elements as the structure preserving group. Q.E.D.

From the representation,

$$D = \{L_g\cdot GDM\cdot R_g | g \in G_r \cap G_c\}$$

$$= \{L_g\cdot \begin{pmatrix} t_{g_o(1)} & \cdots\cdots & t_{g_o(k)} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ t_{g_o((b-1)k+1)} & \cdot & t_{g_o(bk)} \end{pmatrix} \cdot R_g | g \in G_r \cap G_c\}.$$

Let us call $\begin{pmatrix} t_{g_0(1)} & \cdots\cdots & t_{g_0(k)} \\ \cdots & \cdots & \cdots \\ t_{g_0((b-1)k+1)} & \cdot & t_{g_0(bk)} \end{pmatrix}$, the Transformed Design Matrix determined by g_0 and abbreviate it as TDM_{g_0} where $g_0 \in G_r \cap G_c$. So if we start with a design TDM_{g_0} from the set D, then the set of all possible designs which can be obtained from this design by a row permutation and a column permutation is still the set D.

Theorem 1.2: Let GDM be a given design and $g \in G_r \cap G_c$. Then the following equality holds:

$$K(TDM_g) = g^{-1} \cdot K(GDM) \cdot g.$$

Proof:

$$h \in K(TDM_g)$$

iff $\quad L_h \cdot L_g \cdot GDM \cdot R_g \cdot R_h \sim L_g \cdot GDM \cdot R_g$

iff $\quad L_g^{-1} \cdot L_h \cdot L_g \cdot GDM \cdot R_g \cdot R_h \cdot R_g^{-1} \sim GDM$

iff $\quad g \circ h \circ g^{-1} \in K \cdot (GDM)$

iff $\quad h \in g^{-1} \cdot K(GDM) \cdot g$ Q.E.D.

Definition 1.5: A transformation set of designs is a set of designs which can be generated from any one of its members by permutation of rows, columns, and treatments.

Theorem 1.3: All designs from a tranformation set of designs have isomorphic structure preserving groups.

Proof: From Theorem 1.2, it is proved that any two designs in a transformation set that may be obtained from one another by permutation of rows and columns have isomorphic structure preserving groups. Also, any two designs which can be obtained from one another by permutation of treatments have the identical structure preserving group, so the theorem follows. Q.E.D.

Theorem 1.4: If we define an equivalence relation for designs in a transformation set as in Definition 1.4, then the number of equivalence classes in a transformation set is equal to $|G_r \cap G_c|/|G_r \cap G_c \cap G_t| = (b!\times k!)/|G_r \cap G_c \cap G_t|$.

Proof: A transformation set of designs can also be represented as a set of all designs which can be obtained by permutation of treatments from those designs in D. Theorem 1.4 follows directly from Theorem 1.1. Q.E.D.

Relating to the concept of the structure preserving group, a commutator algebra to represent the group is to be defined for later discussion.

Definition 1.6: A commutator algebra C associated with a Latin square design is defined as $C = \{\Sigma | \Sigma$ is a $bk \times bk$ matrix over the reals and $M_g \cdot \Sigma \cdot M_g^T = \Sigma$ for all $g \in G_r \cap G_c \cap G_t\}$ where M_g, a $bk \times bk$ matrix, denotes the permutation g, i.e. $(g(1), g(2),\ldots,g(bk))^T = M_g \cdot (1,2,\ldots,bk)^T$ and T denotes the transpose operator on matrices.

If $\Sigma = (c_{ij})$ then $M_g \cdot \Sigma \cdot M_g^T = (c_{g(i)g(j)})$. So, the commutator algebra can also be characterized as $C = \{(c_{ij}) | c_{ij} = c_{i'j'}$ for all

$(i,j),(i',j')$ with $g(i) = i'$ and $g(j) = j'$ for some $g \in G_r \cap G_c \cap G_t\}$.

By using addition, scalar multiplication and multiplication product for matrices, it can be shown that C is an associative algebra over the reals in the usual algebraic sense. Since C is defined according to a specific $G_r \cap G_c \cap G_t$ which depends on a given design, C also depends on this design. So, C is also denoted by $C(\text{GDM})$ to emphasize this dependency. Let z_i denote the observed value in cell i of the given design and g is a permutation of the set $\{1,2,\ldots,bk\}$, $(z_{g(1)},\ldots,z_{g(bk)})^T = M_g\cdot(z_1,\ldots,z_{bk})^T$ with M_g defined as in Definition 1.6. Let Σ be the covariance matrix for $(z_1,z_2,\ldots,z_{bk})$. If we assume that $(z_1,z_2,\ldots,z_{bk})$ and $(z_{g(1)},z_{g(2)},\ldots,z_{g(bk)})$ have the same probability distribution for $g \in G_r \cap G_c \cap G_t$ then $M_g\cdot\Sigma\cdot M_g^T = \Sigma$ for all $g \in G_r \cap G_c \cap G_t$. In other words, if we assume the invariance of probability distribution of $(z_1,z_2,\ldots,z_{bk})$ under the permutations from the structure preserving group then the covariance matrix of $(z_1,z_2,\ldots,z_{bk})$ is an element of the commutator algebra C.

<u>Theorem 1.5</u>: The dimension of the vector space $C(\text{GDM})$ is the same as the dimension of $C(\text{TDM}_g)$, where $g \in G_r \cap G_c$.

<u>Proof</u>: Let us define two equivalence relations for ordered pair of indices as follows:

$$(i,j) \sim (i',j') \quad \text{if} \quad i' = h(i) \quad \text{and} \quad j' = h(j)$$

for some $h \in K(\text{GDM})$

$$(i,j) \underset{g}{\sim} (i',j') \quad \text{if} \quad i' = g^{-1}\circ h\circ g(i) \quad \text{and} \quad j' = g^{-1}\circ h\circ g(j)$$

for some $h \in K(\text{GDM})$.

Then we have that

$C(GDM) = \{(c_{ij}) | c_{ij} = c_{i'j'}$ for all $(i,j),(i',j')$ with $(i,j) \sim (i',j')\}$

and

$C(TDM_g) = \{(c_{ij}) | c_{ij} = c_{i'j'}$ for all $(i,j),(i',j')$ with $(i,j) \underset{g}{\sim} (i',j')\}$.

The vector space dimension of $C(GDM)$ is equal to the number of different equivalence classes, corresponding to the equivalence relation "$\sim$", in the Cartesian product $\{1,2,\ldots,bk\} \times \{1,2,\ldots,bk\}$. Similarly, the dimension of $C(TDM_g)$ is equal to the number of different equivalence classes, corresponding to the equivalence relation "$\underset{g}{\sim}$", in $\{1,2,\ldots,bk\} \times \{1,2,\ldots,bk\}$. Let us denote the equivalence class, corresponding to the equivalence relation "$\sim$", containing the pair (i,j) by $[(i,j)]$. So we have

$$[(i,j)] = \{(h(i),h(j)) | h \in K(GDM)\}$$

Similarly, define

$$[(i,j)]_g = \{(h(i),h(j)) | h \in K(TDM_g)\}.$$

From Theorem 1.2, we have that

$$[(i,j)]_g = \{(g^{-1} \circ h \circ g(i), g^{-1} \circ h \circ g(j)) | h \in K(GDM)\}.$$

Given a permutation f of $\{1,2,\ldots,bk\}$, define a function (f,f) on $\{1,2,\ldots,bk\} \times \{1,2,\ldots,bk\}$ as follows:

$$(f,f): (i,j) \to (f(i),f(j)).$$

It is clear that the function (f,f) is a one-to-one function. Moreover, we have the following equality:

(A) $[(i,j)]_g = (g^{-1},g^{-1})([(g(i),g(j))])$, where $(g^{-1},g^{-1})([(g(i),g(j))])$ is the image of $[(g(i),g(j))]$ under (g^{-1},g^{-1}). Let m be the dimension of the vector space $C(GDM)$ and the set of different equivalence classes be $\{[(i_1,j_1)], [(i_2,j_2)],\ldots,[(i_m,j_m)]\}$. From (A), it follows that the set of different equivalence classes, corresponding to the equivalence relation "$\tilde{g}$" is as follows:

$$\{(g^{-1},g^{-1})([(i_1,j_1)]), (g^{-1},g^{-1})([(i_2,j_2)]),\ldots,(g^{-1},g^{-1})([(i_m,j_m)])\}.$$

So the dimension of $C(TDM_g)$ is also equal to m. Q.E.D.

Theorem 1.5 shows that the dimension of the vector space $C(GDM)$ is the same for all the designs from the transformation set generated from GDM. This is expected because the designs from a transformation set have isomorphic structure preserving groups.

§2. COMMUTATOR ALGEBRAS OF LATIN SQUARE DESIGNS

From the discussion in section one, we know that Latin square designs from the same transformation set have isomorphic structure preserving groups and their associated commutator algebras have similar structures. So, it is enough to consider one design from each transformation set. In the following examples, the elements of structure preserving groups are found through computer programming. The generators of structure preserving groups are determined from their elements.

<u>Example 1</u>: Latin Square Designs of Order Three

There is only one transformation set in this case. Let us compute the structure preserving group and commutator algebra for the design $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$. The structure preserving group of $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$ is generated by the following permutations of $\{1,2,\ldots,9\}$.

$$g_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 1 & 5 & 6 & 4 & 8 & 9 & 7 \end{pmatrix}$$

$$g_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 4 & 6 & 2 & 1 & 3 & 8 & 7 & 9 \end{pmatrix}$$

$$g_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 7 & 8 & 3 & 1 & 2 & 6 & 4 & 5 \end{pmatrix}$$

The order of $K\begin{pmatrix}1&2&3\\2&3&1\\3&1&2\end{pmatrix}$ is equal to 18 and there are two equivalence classes for the transformation set (Theorem 1.4). The general form of a matrix in the commutator algebra is as follows:

$$\left[\begin{array}{ccc|ccc|ccc}
a & b & b & c & d & e & c & e & d \\
b & a & b & e & c & d & d & c & e \\
b & b & a & d & e & c & e & d & c \\
\hline
c & e & d & a & b & b & c & d & e \\
d & c & e & b & a & b & e & c & d \\
e & d & c & b & b & a & d & e & c \\
\hline
c & d & e & c & e & d & a & b & b \\
e & c & d & d & c & e & b & a & b \\
d & e & c & e & d & c & b & b & a
\end{array}\right]$$

The commutator algebra is a five dimensional vector space. The five equivalence classes of index pairs are as follows:

$\{[(1,1)], [(1,2)], [(1,4)], [(1,5)], [(1,6)]\}$ where

$[(1,1)] = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8),(9,9)\}$

$[(1,2)] = \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2),(4,5),(4,6),(5,4),(5,6),$
$(6,4),(6,5),(7,8),(7,9),(8,7),(8,9),(9,7),(9,8)\}$

$[(1,5)] = \{(1,5),(1,9),(2,6),(2,7),(3,4),(3,8),(4,3),(4,8),(5,1),(5,9),$
$(6,2),(6,7),(7,2),(7,6),(8,3),(8,4),(9,1),(9,5)\}$

$[(1,4)] = \{(1,4),(1,7),(2,5),(2,8),(3,6),(3,9),(4,1),(4,7),(5,2),(5,8),$
$(6,3),(6,9),(7,1),(7,4),(8,2),(8,5),(9,3),(9,6)\}$

$[(1,6)] = \{(1,6),(1,8),(2,4),(2,9),(3,5),(3,7),(4,2),(4,9),(5,3),(5,7),$
$(6,1),(6,8),(7,3),(7,5),(8,1),(8,6),(9,2),(9,4)\}$

Example 2: Latin Square Designs of Order Four

There are two transformation sets in this case. Let us consider one design from each transformation set.

(I) The structure preserving group of $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ is generated by the following permutations: (Dubenko, Sysoev and Shaikin, 1976 [2])

$$g_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 2 & 3 & 4 & 1 & 14 & 15 & 16 & 13 & 10 & 11 & 12 & 9 & 6 & 7 & 8 & 5 \end{pmatrix},$$

$$g_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 15 & 14 & 13 & 16 & 3 & 2 & 1 & 4 & 7 & 6 & 5 & 8 & 11 & 10 & 9 & 12 \end{pmatrix},$$

$$g_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 2 & 4 & 3 & 5 & 6 & 8 & 7 & 13 & 14 & 16 & 15 & 9 & 10 & 12 & 11 \end{pmatrix}.$$

The order of $K\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ is equal to 96 and there are six equivalence classes for the transformation set (Theorem 1.4). The general form of a matrix in the commutator algebra is as follows.

$$
(A)\qquad \left[\begin{array}{cccc|cccc|cccc|cccc}
a & b & b & b & c & d & e & e & c & e & d & e & c & e & e & d \\
b & a & b & b & d & c & e & e & e & c & e & d & e & c & d & e \\
b & b & a & b & e & e & c & d & d & e & c & e & e & d & c & e \\
b & b & b & a & e & e & d & c & e & d & e & c & d & e & e & c \\
\hline
c & d & e & e & a & b & b & b & c & e & c & d & c & e & d & e \\
d & c & e & e & b & a & b & b & e & c & d & e & e & c & e & d \\
e & e & c & d & b & b & a & b & e & d & c & e & d & e & c & e \\
e & e & d & c & b & b & b & a & d & e & e & c & e & d & e & c \\
\hline
c & e & d & e & c & e & e & d & a & b & b & b & c & d & e & e \\
e & c & e & d & e & c & d & e & b & a & b & b & d & c & e & e \\
d & e & c & e & e & d & c & e & b & b & a & b & e & e & c & d \\
e & d & e & c & d & e & e & c & b & b & b & a & e & e & d & c \\
\hline
c & e & e & d & c & e & d & e & c & d & e & e & a & b & b & b \\
e & c & d & e & e & c & e & d & d & c & e & e & b & a & b & b \\
e & d & c & e & d & e & c & e & e & e & c & d & b & b & a & b \\
d & e & e & c & e & d & e & c & e & e & d & c & b & b & b & a
\end{array}\right]
$$

The commutator algebra is a five dimensional vector space. Since any matrix in the commutator algebra is symmetric, it follows that the algebra is commutative. An algebra of square matrices which can be generated by symmetric matrices is a semi-simple algebra (James, 1957 [6]). According to a theorem of Wedderburn, a semi-simple algebra is isomorphic to a direct sum of complete matrix algebras. (Van Der Waerden, 1950 [14], Chapter XVI). So, the commutator algebra is isomorphic to the algebra of all diagonal 5×5 matrices.

(II) The structure preserving group of $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}$ is generated by the following permutations:

$$g_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 4 & 1 & 2 & 3 & 8 & 5 & 6 & 7 & 12 & 9 & 10 & 11 & 16 & 13 & 14 & 15 \end{pmatrix}$$

$$g_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 11 & 10 & 9 & 12 & 7 & 6 & 5 & 8 & 3 & 2 & 1 & 4 & 15 & 14 & 13 & 16 \end{pmatrix}$$

$$g_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 13 & 14 & 15 & 16 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{pmatrix}$$

The order of $K\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}$ is equal to 32 and there are eighteen equivalence classes for the transformation set (Thoerem 1.4). The general form of a matrix in the commutator algebra is as follows.

$$(B) \quad \left[\begin{array}{cccc|cccc|cccc|cccc}
a & b & c & b & d & e & f & g & h & i & j & i & d & g & f & e \\
b & a & b & c & g & d & e & f & i & h & i & j & e & d & g & f \\
c & b & a & b & f & g & d & e & j & i & h & i & f & e & d & g \\
b & c & b & a & e & f & g & d & i & j & i & h & g & f & e & d \\
\hline
d & g & f & e & a & b & c & b & d & e & f & g & h & i & j & i \\
e & d & g & f & b & a & b & c & g & d & e & f & i & h & i & j \\
f & e & d & g & c & b & a & b & f & g & d & e & j & i & h & i \\
g & f & e & d & b & c & b & a & e & f & g & d & i & j & i & h \\
\hline
h & i & j & i & d & g & f & e & a & b & c & b & d & e & f & g \\
i & h & i & j & e & d & g & f & b & a & b & c & g & d & e & f \\
j & i & h & i & f & e & d & g & c & b & a & b & f & g & d & e \\
i & j & i & h & g & f & e & d & b & c & b & a & e & f & g & d \\
\hline
d & e & f & g & h & i & j & i & d & g & f & e & a & b & c & b \\
g & d & e & f & i & h & i & j & e & d & g & f & b & a & b & c \\
f & g & d & e & j & i & h & i & f & e & d & g & c & b & a & b \\
e & f & g & d & i & j & i & h & g & f & e & d & b & c & b & a
\end{array}\right]$$

The commutator algebra is a ten dimensional vector space. It is also a commutative algebra. The commutator algebra is isomorphic to the algebra of all diagonal 10×10 matrices (follows again from the Wedderburn theorem). Applying the results in section one, the commutator algebra of a design which belongs to the same transformation set as $\begin{pmatrix} 1&2&3&4\\ 2&3&4&1\\ 3&4&1&2\\ 4&1&2&3 \end{pmatrix}$ will be examined.

Let $g = \begin{pmatrix} 1&2&3&4&5&6&7&8&9&10&11&12&13&14&15&16\\ 7&6&8&5&3&2&4&1&11&10&12&9&15&14&16&13 \end{pmatrix}$, which is an element of the group $G_r \cap G_c$.

$$\begin{pmatrix} t_1&t_2&t_3&t_4\\ t_5&t_6&t_7&t_8\\ t_9&t_{10}&t_{11}&t_{12}\\ t_{13}&t_{14}&t_{15}&t_{16} \end{pmatrix} = \begin{pmatrix} 1&2&3&4\\ 2&3&4&1\\ 3&4&1&2\\ 4&1&2&3 \end{pmatrix}$$

$$\begin{pmatrix} t_{g(1)}&t_{g(2)}&t_{g(3)}&t_{g(4)}\\ t_{g(5)}&t_{g(6)}&t_{g(7)}&t_{g(8)}\\ t_{g(9)}&t_{g(10)}&t_{g(11)}&t_{g(12)}\\ t_{g(13)}&t_{g(14)}&t_{g(15)}&t_{g(16)} \end{pmatrix} = \begin{pmatrix} 4&3&1&2\\ 3&2&4&1\\ 1&4&2&3\\ 2&1&3&4 \end{pmatrix},\text{ and}$$

$$g^{-1} = \begin{pmatrix} 1&2&3&4&5&6&7&8&9&10&11&12&13&14&15&16\\ 8&6&5&7&4&2&1&3&12&10&9&11&16&14&13&15 \end{pmatrix}.$$

The ten equivalence classes of ordered pairs of indices, for the design $\begin{pmatrix} 1&2&3&4\\ 2&3&4&1\\ 3&4&1&2\\ 4&1&2&3 \end{pmatrix}$, are as follows.

$[(1,1)] = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8),(9,9),$
$(10,10),(11,11),(12,12),(13,13),(14,14),(15,15),(16,16)\}$

$[(1,2)] = \{(1,2),(1,4),(2,1),(2,3),(3,2),(3,4),(4,1),(4,3),(5,6),$
$(5,8),(6,5),(6,7),(7,6),(7,8),(8,5),(8,7),(9,10),(9,12),$
$(10,9),(10,11),(11,10),(11,12),(12,9),(12,11),(13,14),$
$(13,16),(14,13),(14,15),(15,14),(15,16),(16,13),(16,15)\}$

$[(1,3)] = \{(1,3),(2,4),(3,1),(4,2),(5,7),(6,8),(7,5),(8,6),(9,11),$
$(10,12),(11,9),(12,10),(13,15),(14,16),(15,13),(16,14)\}$

$[(1,5)] = \{(1,5),(1,13),(2,6),(2,14),(3,7),(3,15),(4,8),(4,16),(5,1),$
$(5,9),(6,2),(6,10),(7,3),(7,11),(8,4),(8,12),(9,5),(9,13),$
$(10,6),(10,14),(11,7),(11,15),(12,8),(12,16),(13,1),$
$(13,9),(14,2),(14,10),(15,3),(15,11),(16,4),(16,12)\}$

$[(1,6)] = \{(1,6),(1,16),(2,7),(2,13),(3,8),(3,14),(4,5),(4,15),(5,4),$
$(5,10),(6,1),(6,11),(7,2),(7,12),(8,3),(8,9),(9,8),(9,14),$
$(10,5),(10,15),(11,6),(11,16),(12,7),(12,13),(13,2),$
$(13,12),(14,3),(14,9),(15,4),(15,10),(16,1),(16,11)\}$

$[(1,7)] = \{(1,7),(1,15),(2,8),(2,16),(3,5),(3,13),(4,6),(4,14),(5,3),$
$(5,11),(6,4),(6,12),(7,1),(7,9),(8,2),(8,10),(9,7),(9,15),$
$(10,8),(10,16),(11,5),(11,13),(12,6),(12,14),(13,3),$
$(13,11),(14,4),(14,12),(15,1),(15,9),(16,2),(16,10)\}$

$[(1,8)] = \{(1,8),(1,14),(2,5),(2,15),(3,6),(3,16),(4,7),(4,13),(5,2),$
$(5,12),(6,3),(6,9),(7,4),(7,10),(8,1),(8,11),(9,6),(9,16),$
$(10,7),(10,13),(11,8),(11,14),(12,5),(12,15),(13,4),$
$(13,10),(14,1),(14,11),(15,2),(15,12),(16,3),(16,9)\}$

$[(1,9)] = \{(1,9),(2,10),(3,11),(4,12),(5,13),(6,14),(7,15),(8,16),$
$(9,1),(10,2),(11,3),(12,4),(13,5),(14,6),(15,7),(16,8)\}$

$[(1,10)] = \{(1,10),(1,12),(2,9),(2,11),(3,10),(3,12),(4,9),(4,11),$
$(5,14),(5,16),(6,13),(6,15),(7,14),(7,16),(8,13),(8,15),$
$(9,2),(9,4),(10,1),(10,3),(11,2),(11,4),(12,1),(12,3),$
$(13,6),(13,8),(14,5),(14,7),(15,6),(15,8),(16,5),(16,7)\}$

$[(1,11)] = \{(1,11),(2,12),(3,9),(4,10),(5,15),(6,16),(7,13),(8,14),$
$(9,3),(10,4),(11,1),(12,2),(13,7),(14,8),(15,5),(16,6)\}$

The images of the ten equivalence classes under the mapping (g^{-1},g^{-1}) are as follows:

$(g^{-1},g^{-1})([(1,1)]) = [(1,1)]$

$(g^{-1},g^{-1})([(1,2)]) = \{(8,6),(8,7),(6,8),(6,5),(5,6),(5,7),(7,8),(7,5),$
$(4,2),(4,3),(2,4),(2,1),(1,2),(1,3),(3,4),(3,1),$
$(12,10),(12,11),(10,12),(10,9),(9,10),(9,11),$
$(11,12),(11,9),(16,14),(16,15),(14,16),(14,13),$
$(13,14),(13,15),(15,16),(15,13)\}$

$(g^{-1},g^{-1})([(1,3)]) = \{(8,5),(6,7),(5,8),(7,6),(4,1),(2,3),(1,4),(3,2),$
$(12,9),(10,11),(9,12),(11,10),(16,13),(14,15),$
$(13,16),(15,14)\}$

$(g^{-1},g^{-1})([(1,5)]) = \{(8,4),(8,16),(6,2),(6,14),(5,1),(5,13),(7,3),$
$(7,15),(4,8),(4,12),(2,6),(2,10),(1,5),(1,9),$
$(3,7),(3,11),(12,4),(12,16),(10,2),(10,14),(9,1),$
$(9,13),(11,3),(11,15),(16,8),(16,12),(14,6),$
$(14,10),(13,5),(13,9),(15,7),(15,11)\}$

$(g^{-1},g^{-1})([(1,6)])$ = {(8,2),(8,15),(6,1),(6,16),(5,3),(5,14),(7,4),
(7,13),(4,7),(4,10),(2,8),(2,9),(1,6),(1,11),
(3,5),(3,12),(12,3),(12,14),(10,4),(10,13),
(9,2),(9,15),(11,1),(11,16),(16,6),(16,11),
(14,5),(14,12),(13,7),(13,10),(15,8),(15,9)}

$(g^{-1},g^{-1})([(1,7)])$ = {(8,1),(8,13),(6,3),(6,15),(5,4),(5,16),(7,2),
(7,14),(4,5),(4,9),(2,7),(2,11),(1,8),(1,12),
(3,6),(3,10),(12,1),(12,13),(10,3),(10,15),
(9,4),(9,16),(11,2),(11,14),(16,5),(16,9),
(14,7),(14,11),(13,8),(13,12),(15,6),(15,10)}

$(g^{-1},g^{-1})([(1,8)])$ = {(8,3),(8,14),(6,4),(6,13),(5,2),(5,15),(7,1),
(7,16),(4,6),(4,11),(2,5),(2,12),(1,7),(1,10),
(3,8),(3,9),(12,2),(12,15),(10,1),(10,16),
(9,3),(9,14),(11,4),(11,13),(16,7),(16,10),(14,8),
(14,9),(13,6),(13,11),(15,5),(15,12)}

$(g^{-1},g^{-1})([(1,9)])$ = {(8,12),(6,10),(5,9),(7,11),(4,16),(2,14),(1,13),
(3,15),(12,8),(10,6),(9,5),(11,7),(16,4),(14,2),
(13,1),(15,3)}

$(g^{-1},g^{-1})([(1,10)])$ = {(8,10),(8,11),(6,12),(6,9),(5,10),(5,11),(7,12),
(7,9),(4,14),(4,15),(2,16),(2,13),(1,14),(1,15),
(3,16),(3,13),(12,6),(12,7),(10,8),(10,5),(9,6),
(9,7),(11,8),(11,5),(16,2),(16,3),(14,4),(14,1),
(13,2),(13,3),(15,4),(15,1)}

$(g^{-1},g^{-1})([(1,11)])$ = {(8,9),(6,11),(5,12),(7,10),(4,13),(2,15),(1,16),
(3,14),(12,5),(10,7),(9,8),(11,6),(16,1),(14,3),
(13,4),(15,2)}

The general form of a matrix in the commutator algebra $C\begin{pmatrix} 4 & 3 & 1 & 2 \\ 3 & 2 & 4 & 1 \\ 1 & 4 & 2 & 3 \\ 2 & 1 & 3 & 4 \end{pmatrix}$ is as follows:

$$
(C) \quad \left[\begin{array}{cccc|cccc|cccc|cccc}
a & b & b & c & d & j & f & e & d & f & j & e & g & h & h & i \\
b & a & c & b & f & d & e & j & j & d & e & f & h & g & i & h \\
b & c & a & b & j & e & d & f & f & e & d & j & h & i & g & h \\
c & b & b & a & e & f & j & d & e & j & f & d & i & h & h & g \\
\hline
d & f & j & e & a & b & b & c & g & h & h & i & d & j & f & e \\
j & d & e & f & b & a & c & b & h & g & i & h & f & d & e & j \\
f & e & d & j & b & c & a & b & h & i & g & h & j & e & d & f \\
e & j & f & d & c & b & b & a & i & h & h & g & e & f & j & d \\
\hline
d & j & f & e & g & h & h & i & a & b & b & c & d & f & j & e \\
f & d & e & j & h & g & i & h & b & a & c & b & j & d & e & f \\
j & e & d & f & h & i & g & h & b & c & a & b & f & e & d & j \\
e & f & j & d & i & h & h & g & c & b & b & a & e & j & f & d \\
\hline
g & h & h & i & d & f & j & e & d & j & f & e & a & b & b & c \\
h & g & i & h & j & d & e & f & f & d & e & j & b & a & c & b \\
h & i & g & h & f & e & d & j & j & e & d & f & b & c & a & b \\
i & h & h & g & e & j & f & d & e & f & j & d & c & b & b & a
\end{array}\right]
$$

The commutator algebra $C\begin{pmatrix} 4 & 3 & 1 & 2 \\ 3 & 2 & 4 & 1 \\ 1 & 4 & 2 & 3 \\ 2 & 1 & 3 & 4 \end{pmatrix}$ is still isomorphic to the algebra of all diagonal 10×10 matrices.

Let us do more comparisons for Latin square designs from different transformation sets. Assume the observed values from the

Latin square designs $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}$ are arranged

correspondingly in a matrix $\begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ z_5 & z_6 & z_7 & z_8 \\ z_9 & z_{10} & z_{11} & z_{12} \\ z_{13} & z_{14} & z_{15} & z_{16} \end{pmatrix}$.

The following fixed effect model is assumed for the analysis of the design $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}$.

$$\begin{aligned}
z_1 &= \mu + \alpha_1 + \beta_1 + \gamma_1 + \varepsilon_1 \\
z_2 &= \mu + \alpha_1 + \beta_2 + \gamma_2 + \varepsilon_2 \\
z_3 &= \mu + \alpha_1 + \beta_3 + \gamma_3 + \varepsilon_3 \\
z_4 &= \mu + \alpha_1 + \beta_4 + \gamma_4 + \varepsilon_4 \\
z_5 &= \mu + \alpha_2 + \beta_1 + \gamma_2 + \varepsilon_5 \\
z_6 &= \mu + \alpha_2 + \beta_2 + \gamma_1 + \varepsilon_6 \\
z_7 &= \mu + \alpha_2 + \beta_3 + \gamma_4 + \varepsilon_7 \\
z_8 &= \mu + \alpha_2 + \beta_4 + \gamma_3 + \varepsilon_8 \\
z_9 &= \mu + \alpha_3 + \beta_1 + \gamma_3 + \varepsilon_9 \\
z_{10} &= \mu + \alpha_3 + \beta_2 + \gamma_4 + \varepsilon_{10} \\
z_{11} &= \mu + \alpha_3 + \beta_3 + \gamma_1 + \varepsilon_{11} \\
z_{12} &= \mu + \alpha_3 + \beta_4 + \gamma_2 + \varepsilon_{12} \\
z_{13} &= \mu + \alpha_4 + \beta_1 + \gamma_4 + \varepsilon_{13} \\
z_{14} &= \mu + \alpha_4 + \beta_2 + \gamma_3 + \varepsilon_{14} \\
z_{15} &= \mu + \alpha_4 + \beta_3 + \gamma_2 + \varepsilon_{15} \\
z_{16} &= \mu + \alpha_4 + \beta_4 + \gamma_1 + \varepsilon_{16}
\end{aligned}$$

where μ is the overall effect, α, β, γ are main effects and

$\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_{16})$ is assumed to have (A) as its covariance matrix.

The estimators of elementary contrasts of treatment effects are as follows.

$$\hat{\gamma}_1 - \hat{\gamma}_2 = (z_1 + z_6 + z_{11} + z_{16})/4 - (z_2 + z_5 + z_{12} + z_{15})/4$$

$$\hat{\gamma}_1 - \hat{\gamma}_3 = (z_1 + z_6 + z_{11} + z_{16})/4 - (z_3 + z_8 + z_9 + z_{14})/4$$

$$\hat{\gamma}_1 - \hat{\gamma}_4 = (z_1 + z_6 + z_{11} + z_{16})/4 - (z_4 + z_7 + z_{10} + z_{13})/4$$

The variances and covariances of the estimators are as follows.

(2.1)

$$\mathrm{Var}(\hat{\gamma}_i - \hat{\gamma}_j) = (8a + 24d - 8(b + 2e + c))/16 \qquad \text{for } i \neq j$$

$$\mathrm{Cov}(\hat{\gamma}_{j_1} - \hat{\gamma}_{j_2}, \hat{\gamma}_{j_2} - \hat{\gamma}_{j_3}) = (-4a - 12d + 4b + 8e + 4c)/16$$

for j_1, j_2, j_3 not equal

$$\mathrm{Cov}(\hat{\gamma}_{j_1} - \hat{\gamma}_{j_2}, \hat{\gamma}_{j_3} - \hat{\gamma}_{j_4}) = 0 \qquad \text{for } j_1, j_2, j_3, j_4 \text{ not equal}$$

A similar analysis of the design $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}$ assuming $\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_{16})$ to have (B) as its covariance matrix has the following results.

(2.2)

$$\mathrm{Var}(\hat{\gamma}_1 - \hat{\gamma}_2) = \mathrm{Var}(\hat{\gamma}_1 - \hat{\gamma}_4) = \mathrm{Var}(\hat{\gamma}_2 - \hat{\gamma}_3) = \mathrm{Var}(\hat{\gamma}_3 - \hat{\gamma}_4)$$
$$= (8(a + 2g + j) - 8(b + d + i + f))/16$$

$$\mathrm{Var}(\hat{\gamma}_1 - \hat{\gamma}_3) = \mathrm{Var}(\hat{\gamma}_2 - \hat{\gamma}_4) = (8(a + 2g + j) - 8(c + 2e + h))/16$$

$$\mathrm{Cov}(\hat{\gamma}_1 - \hat{\gamma}_2, \hat{\gamma}_2 - \hat{\gamma}_3) = (8(b + f + i + d) - 4(a + 2g + j) - 4(c + 2e + h))/16$$

$$\mathrm{Cov}(\hat{\gamma}_1 - \hat{\gamma}_2, \hat{\gamma}_2 - \hat{\gamma}_4) = (4(c + 2e + h) - 4(a + 2g + j))/16$$

and so forth.

From (2.1) and (2.2), the two designs are shown to have different statistical properties.

Example 3: Latin Square Designs of Order Five

There are two transformation sets in this case. Let us consider one design from each transformation set.

(I) The structure preserving group of $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$ is generated by the following permutations.

$$g_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 9 & 10 & 6 & 12 & 13 & 14 & 15 & 11 & 17 & 18 & 19 & 20 & 16 & 22 & 23 & 24 & 25 & 21 \end{pmatrix}$$

$$g_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 \\ 12 & 15 & 13 & 11 & 14 & 2 & 5 & 3 & 1 & 4 & 17 & 20 & 18 & 16 & 19 & 7 & 10 & 8 & 6 & 9 & 22 & 25 & 23 & 21 & 24 \end{pmatrix}$$

$$g_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 \\ 23 & 24 & 25 & 21 & 22 & 3 & 4 & 5 & 1 & 2 & 8 & 9 & 10 & 6 & 7 & 13 & 14 & 15 & 11 & 12 & 18 & 19 & 20 & 16 & 17 \end{pmatrix}$$

The order of the structure preserving group is equal to 100 and there are 144 equivalence classes for the transformation set. The general form of a matrix in the commutator algebra is as follows.

$$
\left[\begin{array}{ccccc|ccccc|ccccc|ccccc|ccccc}
a & b & b & b & b & c & d & e & f & g & c & f & d & g & e & c & e & g & d & f & c & g & f & e & d \\
b & a & b & b & b & g & c & d & e & f & e & c & f & d & g & f & c & e & g & d & d & c & g & f & e \\
b & b & a & b & b & f & g & c & d & e & g & e & c & f & d & d & f & c & e & g & e & d & c & g & f \\
b & b & b & a & b & e & f & g & c & d & d & g & e & c & f & g & d & f & c & e & f & e & d & c & g \\
b & b & b & b & a & d & e & f & g & c & f & d & g & e & c & e & g & d & f & c & g & f & e & d & c \\
\hline
c & g & f & e & d & a & b & b & b & b & c & d & e & f & g & c & f & d & g & e & c & e & g & d & f \\
d & c & g & f & e & b & a & b & b & b & g & c & d & e & f & e & c & f & d & g & f & c & e & g & d \\
e & d & c & g & f & b & b & a & b & b & f & g & c & d & e & g & e & c & f & d & d & f & c & e & g \\
f & e & d & c & g & b & b & b & a & b & e & f & g & c & d & d & g & e & c & f & g & d & f & c & e \\
g & f & e & d & c & b & b & b & b & a & d & e & f & g & c & f & d & g & e & c & e & g & d & f & c \\
\hline
c & e & g & d & f & c & g & f & e & d & a & b & b & b & b & c & d & e & f & g & c & f & d & g & e \\
f & c & e & g & d & d & c & g & f & e & b & a & b & b & b & g & c & d & e & f & e & c & f & d & g \\
d & f & c & e & g & e & d & c & g & f & b & b & a & b & b & f & g & c & d & e & g & e & c & f & d \\
g & d & f & c & e & f & e & d & c & g & b & b & b & a & b & e & f & g & c & d & d & g & e & c & f \\
e & g & d & f & c & g & f & e & d & c & b & b & b & b & a & d & e & f & g & c & f & d & g & e & c \\
\hline
c & f & d & g & e & c & e & g & d & f & c & g & f & e & d & a & b & b & b & b & c & d & e & f & g \\
e & c & f & d & g & f & c & e & g & d & d & c & g & f & e & b & a & b & b & b & g & c & d & e & f \\
g & e & c & f & d & d & f & c & e & g & e & d & c & g & f & b & b & a & b & b & f & g & c & d & e \\
d & g & e & c & f & g & d & f & c & e & f & e & d & c & g & b & b & b & a & b & e & f & g & c & d \\
f & d & g & e & c & e & g & d & f & c & g & f & e & d & c & b & b & b & b & a & d & e & f & g & c \\
\hline
c & d & e & f & g & c & f & d & g & e & c & e & g & d & f & c & g & f & e & d & a & b & b & b & b \\
g & c & d & e & f & e & c & f & d & g & f & c & e & g & d & d & c & g & f & e & b & a & b & b & b \\
f & g & c & d & e & g & e & c & f & d & d & f & c & e & g & e & d & c & g & f & b & b & a & b & b \\
e & f & g & c & d & d & g & e & c & f & g & d & f & c & e & f & e & d & c & g & b & b & b & a & b \\
d & e & f & g & c & f & d & g & e & c & e & g & d & f & c & g & f & e & d & c & b & b & b & b & a
\end{array}\right]
$$

The commutator algebra is a 7-dimensional vector space. It is also a commutative algebra. The commutator algebra is thus isomorphic to the algebra of all diagonal 7×7 matrices.

(II) The structure preserving group of $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 3 & 4 \\ 3 & 4 & 1 & 5 & 2 \\ 4 & 5 & 2 & 1 & 3 \\ 5 & 3 & 4 & 2 & 1 \end{pmatrix}$ is generated by the following permutations.

$g_1 =$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 \\ 1 & 4 & 2 & 3 & 5 & 16 & 19 & 17 & 18 & 20 & 6 & 9 & 7 & 8 & 10 & 11 & 14 & 12 & 13 & 15 & 21 & 24 & 22 & 23 & 25 \end{pmatrix}$$

$g_2 =$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 \\ 1 & 5 & 4 & 3 & 2 & 21 & 25 & 24 & 23 & 22 & 16 & 20 & 19 & 18 & 17 & 11 & 15 & 14 & 13 & 12 & 6 & 10 & 9 & 8 & 7 \end{pmatrix}$$

$g_3 =$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 \\ 1 & 4 & 5 & 2 & 3 & 16 & 19 & 20 & 17 & 18 & 21 & 24 & 25 & 22 & 23 & 6 & 9 & 10 & 7 & 8 & 11 & 14 & 15 & 12 & 13 \end{pmatrix}$$

The order of the structure preserving group is equal to 12 and there are 1200 equivalence classes for the transformation set. The general form of a matrix in the commutator algebra is as follows.

1	6	6	6	6	7	8	9	9	9	7	9	8	9	9	7	9	9	8	9	7	9	9	9	8
10	2	14	14	14	15	16	17	17	17	18	19	20	21	22	18	19	22	20	21	18	19	21	22	20
10	14	2	14	14	18	20	10	22	21	15	17	16	17	17	18	21	19	20	22	18	22	19	21	20
10	14	14	2	14	18	20	21	19	22	18	22	20	19	21	15	17	17	16	17	18	21	22	19	20
10	14	14	14	2	18	20	22	21	19	18	21	20	22	19	18	22	21	20	19	15	17	17	17	16
11	23	26	26	26	3	31	33	33	33	35	36	37	38	39	35	36	39	37	38	35	36	38	39	37
12	24	28	28	28	32	4	40	40	40	41	42	43	44	45	41	42	45	43	44	41	42	44	45	43
13	25	27	29	30	34	46	5	53	54	47	55	50	56	57	48	58	59	51	60	49	61	62	63	52
13	25	30	27	29	34	46	54	5	53	49	61	52	62	63	47	55	57	50	56	48	58	60	59	51
13	25	29	30	27	34	46	53	54	5	48	58	51	60	59	49	61	63	52	62	47	55	56	57	50
11	26	23	26	26	35	57	56	59	58	3	33	31	33	33	35	38	36	37	39	35	39	36	38	37
13	27	25	30	29	47	50	55	57	56	34	5	46	54	53	49	62	61	52	63	48	59	58	60	51
12	28	24	28	28	41	43	42	45	44	32	40	4	40	40	41	44	42	43	45	41	45	42	44	43
13	29	25	27	30	48	51	58	59	60	34	53	46	5	54	47	56	55	50	57	49	63	61	62	52
13	30	25	29	27	49	52	61	63	62	34	54	46	53	5	48	60	58	51	59	47	57	55	56	50
11	26	26	23	26	35	37	38	36	39	35	39	37	36	38	3	33	33	31	33	35	38	39	36	37
13	27	29	25	30	47	50	56	55	57	48	59	51	58	60	34	5	53	46	54	49	62	63	61	52
13	30	27	25	29	49	52	62	61	63	47	57	50	55	56	34	54	5	46	53	48	60	59	58	51
12	28	28	24	28	41	43	44	42	45	41	45	43	42	44	32	40	40	4	40	41	44	45	42	43
13	29	30	25	27	48	51	60	58	59	49	63	52	61	62	34	53	54	46	5	47	56	57	55	50
11	26	26	26	23	35	37	39	38	36	35	38	37	39	36	35	39	38	37	36	3	33	33	33	31
13	27	30	29	25	47	50	57	56	55	49	62	52	63	61	48	59	60	51	58	34	5	54	53	46
13	29	27	30	25	48	51	59	60	58	47	56	50	57	55	49	63	62	52	61	34	53	5	54	46
13	30	29	27	25	49	52	63	62	61	48	60	51	59	58	47	57	56	50	55	34	54	53	5	46
12	28	28	28	24	41	43	45	44	42	41	44	43	45	42	41	45	44	43	42	32	40	40	40	4

The elements corresponding to the same number are equal to each other. The structure of this commutator algebra is completely different from the one in (I). It is a 63 dimensional vector space and a noncommutative algebra.

Remark: For Latin square designs of order k, the number of equivalence classes in a transformation set is equal to the product of the number of standard Latin squares in the transformation set and the number $(k-1)!$. From the table in Fisher and Yates, 1938 [5], the number of standard squares are listed for Latin square designs of order up to 6. There are 22 transformation sets of Latin squares of order 6. Some of them have equal number of standard squares, so they have the same order for their structure preserving groups. Are the vector space dimensions of the commutator algebras (representing structure preserving groups of larger order) smaller than the dimensions of the commutator algebras (representing structure preserving groups of smaller order)? We need a computer to study these problems. A computer program was written to compute the structure preserving groups for 4×4 designs, however, it is quite expensive to run and further work is needed to improve its efficiency. It is hoped that further results for Latin square designs of order 6 will be obtained in the future.

§3. DIFFERENT SCHEMES FOR ASSIGNING TREATMENTS TO SUBJECTS

Concerning the problem of estimation of covariance matrices of Latin square designs, it is desirable to assume that the covariance matrix of the observed values belongs to the commutator algebra associated with the Latin square design (Dubenko, Sysoev and Shaikin, 1976 [2]). A special randomization scheme, depending on a group of permutations of the cells, to assign treatments to subjects is to be discussed for Latin square designs.

As mentioned in section one, a Latin square design can be considered as a 3-factor design. Let L be a given Latin square design of order m, with m^2 subjects corresponding to m^2 cells in this given design. The level of each of the three factors (i.e. row factor, column factor and treatment factor) for a specific x is denoted by (r_x, c_x, t_x) according to section one. Let G be a subgroup of the symmetric group S_{m^2}. The randomization scheme is described as follows. First, one element (g) will be randomly chosen from G. Then the condition $(r_x, c_x, t_{g(x)})$ will be assigned to the x^{th} experimental subject (r_x is the row level associated with cell x, c_x is the column level associated with cell x and $t_{g(x)}$ is the treatment level associated with cell $g(x)$ in the given design L). For example, if the given design is $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ and

$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 6 & 5 & 7 & 8 & 2 & 1 & 3 & 4 & 10 & 9 & 11 & 12 & 14 & 13 & 15 & 16 \end{pmatrix}$ is randomly chosen from G, then the condition $(r_1, c_1, t_{g(x)})$ will be assigned to the first subject, where $(r_1, c_1, t_{g(1)}) = (1,1,1)$. Similarly, the condition $(r_2, c_2, t_{g(2)})$ will be assigned to the second subject, where $(r_2, c_2, t_{g(2)}) = (1,2,2)$, and so forth. With this randomization scheme, the observed value for the x^{th} subject can be written as follows (fixed effect model):

$$z_x = \mu + \alpha_{r_x} + \beta_{c_x} + \gamma_{t_{g(x)}} + (\alpha\beta)_{r_x c_x} + \varepsilon_x$$

where μ is the overall effect; α, β, r are the main effects; $(\alpha\beta)$ is the row-column interaction and ε_x is the random error which is independent of the randomization scheme. The row-treatment interaction, column-treatment interaction and three-way interaction are assumed to be negligible. Let us rewrite z_x as follows: $z_x = u_x + \eta_x$, where $u_x = \mu + \alpha_{r_x} + \beta_{c_x} + (\alpha\beta)_{r_x c_x} + \varepsilon_x$ and $\eta_x = \gamma_{t_{g(x)}}$. u_x is not affected by the randomization scheme, but η_x is affected by the randomization scheme. It is assumed that $\sum_{i=1}^{m} \alpha_i = \sum_{i=1}^{m} \beta_i = \sum_{i=1}^{m} \gamma_i = \sum_{i=1}^{m} (\alpha\beta)_{ij} = \sum_{j=1}^{m} (\alpha\beta)_{ij} = 0$. Under the above model, we have that

(3.1) $$Ez_x = Eu_x + E\eta_x$$

and

(3.2) $$\mathrm{Cov}(z_x, z_y) = \mathrm{Cov}(u_x, u_y) + \mathrm{Cov}(\eta_x, \eta_y).$$

It is interesting to study what the impact of the randomization scheme is upon the η_x's. Let us introduce the following notations. $\pi^{(x)}(k) = P(t_{g(x)} = k)$, $\pi^{(x)}(k)$ is the probability that the condition

(r_x, c_x, k) will be assigned to the x^{th} subject under the randomization scheme. $\Pi^{(x,y)}(k,k') = P(t_{g(x)} = k$ and $t_{g(y)} = k')$, $\Pi^{(x,y)}(k,k')$ is the joint probability that the condition (r_x, c_x, k) will be assigned to the x^{th} subject and the condition (r_y, c_y, k') will be assigned to the y^{th} subject. Note that the probability comes from the random choice of g from the group G and the following equalities hold.

$$(3.3) \qquad E\eta_x = \sum_k (\Pi^{(x)}(k) \times \gamma_k)$$

$$(3.4) \qquad Var(\eta_x) = \sum_k (\Pi^{(x)}(k) \times \gamma_k^2) - (E\eta_x)^2$$

$$(3.5) \qquad Cov(\eta_x, \eta_y) = \sum_{k,k'} (\Pi^{(x,y)}(k,k') \times \gamma_k \times \gamma_{k'}) - (E\eta_x)(E\eta_y)$$

where summations are over all possible k and k'. Some theorems concerning $E\eta_x$, $Var(\eta_x)$ and $Cov(\eta_x, \eta_y)$ will be shown next.

<u>Lemma 3.1</u>:

a) $\Pi^{(x)}(k) = \Pi^{(y)}(k)$ for all k if $x = g(y)$ for some $g \in G$

b) $\Pi^{(x,y)}(k,k') = \Pi^{(x',y')}(k,k')$ for all k,k', if $(x,y) = (g(x'), g(y'))$ for some $g \in G$.

<u>Proof</u>:

a) The probability distribution of $\Pi^{(x)}(k)$ is completely dependent on $G_x = \{g(x) | g \in G\}$, the orbit of x under G. We also know that $x = g(Y)$ with $g \in G$ implies that $G_x = G_y$.
$\therefore$ $\Pi^{(x)}(k) = \Pi^{(y)}(k)$ for all k, if $x = g(y)$ for some $g \in G$.

b) Similarly, the probability distribution of $\Pi^{(x,y)}(k,k')$ is completely determined by the set $\{(g(x), g(y)) | g \in G\}$. Also, if $(x,y) = (g(x'), g(y'))$ for some $g \in G$, then

$\{(g(x),g(y))|g \in G\} = \{(g(x'),g(y'))|g \in G\}$.

$\therefore \quad \Pi^{(x,y)}(k,k') = \Pi^{(x',y')}(k,k')$ for all k,k' if

$(x,y) = (g(x'),g(y'))$ for some $g \in G$. Q.E.D.

Theorem 3.1:

a) If $x = g(y)$ for some $g \in G$ then $E\eta_x = E\eta_y$ and $Var(\eta_x) = Var(\eta_y)$.

b) If $(x,y) = g(x'),g(y'))$ for some $g \in G$ then $Cov(\eta_x,\eta_y) = Cov(\eta_{x'},\eta_{y'})$.

Proof: Theorem 3.1 follows directly from lemma 3.1. Q.E.D.

Theorem 3.2: If G is a subgroup of the symmetric group S_{m^2} which induces one and only one orbit on $\{1,2,\ldots,m^2\}$ then $E\eta_x = 0$ for $x = 1,2,\ldots,m^2$. (A group of transformations which induces one and only one orbit on its domain is called a transitive group.)

Proof: From the assumption that G is transitive, we have that $\{1,2,\ldots,m^2\} = \{g(1)|g \in G\}$. Let $H_1 = \{g|g(1) = 1\}$, $H_2 = \{g|g(1) = 2\},\ldots,H_{m^2} = \{g|g(1) = m^2\}$ and then H_1 is a subgroup G. Next, we prove the following equalities:

(*) $\qquad H_i = g \cdot H_1$ for some $g \in G$, $i = 2,3,\ldots,m^2$.

Proof of (*):

$$
\begin{aligned}
g \in H_i \quad & \text{iff} \quad g(1) = i \\
& \text{iff} \quad g(1) = g_0(1) \text{ for some } g_0 \in G \\
& \text{iff} \quad g_0^{-1} \circ g(1) = 1 \text{ for some } g_0 \in G \\
& \text{iff} \quad g \in g_0 \cdot H_1 \text{ for some } g_0 \in G
\end{aligned}
$$

(*) is thus proved.

From (*), we have that $|H_1| = |H_2| = \ldots = |H_{m^2}|$, thus

$$\pi^{(1)}(1) = \pi^{(1)}(2) = \ldots = \pi^{(1)}(m) = 1/m$$

$$\therefore \quad E\eta_1 = \sum_k r_k/m = 0$$

$$\therefore \quad E\eta_1 = E\eta_2 = \ldots\ldots = E\eta_{m^2} = 0$$

follows from theorem 3.1. Q.E.D.

<u>Theorem 3.3</u>: If we choose $G = G_r \cap G_c \cap G_t$, the structure preserving group of the given Latin square design, and let $\vec{\eta} = (\eta_1,\ldots,\eta_{m^2})$, then $\mathrm{Cov}(\vec{\eta}) \in C$ (where C is the commutator algebra of Definition 1.6).

<u>Proof</u>: From Theorem 3.1, $\mathrm{Cov}(\eta_i,\eta_j) = \mathrm{Cov}(\eta_{i'},\eta_{j'})$ if $(i,j) = (g(i'),g(j'))$ for some $g \in G_r \cap G_c \cap G_t$ and $i,j,i',j' \in \{1,2,\ldots,m^2\}$. $\therefore$ $\mathrm{Cov}(\vec{\eta}) \in C$ follows from the definition of C. Q.E.D.

Let $g \in G_r \cap G_c$ and (g,g) be a function defined on $\{1,2,\ldots,m^2\} \times \{1,2,\ldots,m^2\}$ as in the proof of Theorem 1.5. Let $[(i,j)] = \{(g,g)(i,j) | g \in G_r \cap G_c\}$, it is the orbit of (i,j) under the set of transformations $\{(g,g) | g \in G_r \cap G_c\}$. We have that $\{1,2,\ldots,m^2\} \times \{1,2,\ldots,m^2\} = [(1,1)] \cup [(1,2)] \cup [(1,m+1)] \cup [(1,m+2)]$. From Lemma 3.1, the next theorem completely specifies the value $\pi^{(x,y)}(k',k')$ for all $x,y = 1,2,\ldots,m^2$ and $k,k' = 1,2,\ldots,m$.

<u>Theorem 3.4</u>: If we choose $G = G_r \cap G_c$ for our randomization scheme, then

a) $\Pi^{(x)}(1) = \Pi^{(x)}(2) = \ldots = \Pi^{(x)}(m) = 1/m$ for $x = 1,2,\ldots,m^2$

b) $\Pi^{(1,2)}(i,j) = 1/m(m-1)$ for $i \neq j$ and $i,j = 1,2,\ldots,m$

$= 0$ for $i = j$ and $i = 1,2,\ldots,m$

c) $\Pi^{(1,m+1)}(i,j) = 1/m(m-1)$ for $i \neq j$ and $i,j = 1,2,\ldots,m$

$= 0$ for $i = j$ and $i = 1,2,\ldots,m$

d) $\Pi^{(1,m+2)}(i,j) = (m-2)/m(m-1)^2$ for $i \neq j$ and $i,j = 1,2,\ldots,m$

$= 1/m(m-1)$ for $i = j$ and $i = 1,2,\ldots,m$

Proof:

a) It follows from the proof in Theorem 3.2 and Lemma 3.1 ($\because G_r \cap G_c$ is a transitive group)

b), c), d) Let

$M_1 = \{(i,j) \mid i \neq j$ and $i,j = 1,2,\ldots,m\}$

$M_2 = \{(k.j) \mid i \neq j$ and $i,j = m+1,m+2,\ldots,2m\}$

....

$M_m = \{(i,j) \mid i \neq j$ and $i,j = (m-1)m+1,\ldots,m^2\}$

$N_1 = \{(i,j) \mid i \neq j$ and $i,j = 1,m+1,2m+1,\ldots,(m-1)m+1\}$

$N_2 = \{(i,j) \mid i \neq j$ and $i,j = 2,m+2,2m+2,\ldots,(m-1)m+2\}$

....

$N_m = \{(i,j) \mid i \neq j$ and $i,j = m,2m,\ldots,m^2\}$

$R_1 = \{(1,j) \mid j \neq 1, j \neq 2,\ldots, j \neq m, j \neq 1, j \neq m+1, j \neq 2m+1,\ldots,$
$j \neq (m-1)m+1\}$

$R_2 = \{(2,j) \mid j \neq 1, j \neq 2,\ldots, j \neq m, j \neq 2, j \neq m+2, j \neq 2m+2,\ldots,$
$j \neq (m-1)m+2\}$

....

$R_m = \{(m,j) | j \neq 1, j \neq 2, \ldots, j \neq m, j \neq 2m, j \neq 3m, \ldots, j \neq m^2\}$

$R_{m+1} = \{(m+1,j) | j \neq m+1, j \neq m+2, \ldots, j \neq 2m, j \neq 1, j \neq m+1, \ldots,$
$j \neq (m-1)m+1\}$

....

$R_{2m} = \{(2m,j) | j \neq m+1, j \neq m+2, \ldots, j \neq 2m, j \neq m, j \neq 2m, \ldots, j \neq m^2\}$

....

$R_{(m-1)m+1} = \{((m-1)m+1,j) | j \neq (m-1)m+1, j \neq (m-1)m+2, \ldots, j \neq m^2,$
$j \neq 1, j \neq m+1, \ldots, j \neq (m-1)m+1\}$

....

$R_{m^2} = \{(m^2,j) | j \neq (m-1)m+1, j \neq (m-1)m+2, \ldots, j \neq m^2, j \neq m, j \neq 2m,$
$\ldots, j \neq m^2\}$

where $M_1, \ldots, M_m, N_1, \ldots, N_m, R_1, R_2, \ldots, R_{m^2}$ are disjoint sets.

Note that

$$[(1,2)] = M_1 \cup M_2 \cup \ldots \cup M_m$$

$$[(1,m+1)] = N_1 \cup N_2 \cup \ldots \cup N_m$$

$$[(1,m+2)] = R_1 \cup R_2 \cup \ldots \cup R_{m^2}$$

and

$$|M_1| = |M_2| = \ldots = |M_m| = m(m-1)$$

$$|N_1| = |N_2| = \ldots = |N_m| = m(m-1)$$

$$|R_1| = |R_2| = \ldots = |R_{m^2}| = (m-1)^2$$

$\therefore \quad |[(1,2)]| = m^2(m-1), \ |[(1,m+1)]| = m^2(m-1)$ and $|[(1,m+2)]| = m^2(m-1)^2$

Let $[(1,2)] = \{(1,2),(i_2,j_2),(i_3,j_3),\ldots,(i_{m^2(m-1)},j_{m^2(m-1)})\}$

$H_1 = \{g \mid g \in G_r \cap G_c \text{ and } (g(1),g(2)) = (1,2)\}$

$H_2 = \{g \mid g \in G_r \cap G_c \text{ and } (g(1),g(2)) = (i_2,j_2)\}$

....

$H_{m^2(m-1)} = \{g \mid g \in G_r \cap G_c \text{ and } (g(1),g(2)) = (i_{m^2(m-1)},j_{m^2(m-1)})\}$

then $H_1,H_2,\ldots,H_{m^2(m-1)}$ are disjoint sets and

$$(1) \qquad G_r \cap G_c = \bigcup_{i=1}^{m^2(m-1)} H_i.$$

Moreover, H_1 is a subgroup of $G_r \cap G_c$. Next, we prove the following equalities:

$$(2) \qquad H_k = g\,H_1 \text{ for some } g \in G_r \cap G_c,\ k = 2,3,\ldots,m^2(m-1).$$

<u>Proof of (2)</u>:

$$g \in H_k \quad \begin{aligned} &\text{iff } (g(1),g(2)) = (i_k,j_k) \\ &\text{iff } (g(1),g(2)) = (g_0(1),g_0(2)) \text{ for some } g_0 \in G_r \cap G_c \\ &\text{iff } (g_0^{-1}\circ g(1),g_0^{-1}\circ g(2)) = (1,2) \text{ for some } g_0 \in G_r \cap G_c \\ &\text{iff } g \in g_0\cdot H_1 \text{ for some } g_0 \in G \end{aligned}$$

(2) is thus proved.

$\therefore\ |H_1| = |H_2| = \ldots = |H_{m^2(m-1)}|$ follows from (2).

Note that

(3) $\{(t_i,t_j)|(i,j) \in M_1\} = \{(t_i,t_j)|(i,j) \in M_2\} = \ldots =$

$\{(t_i,t_j)|(i,j) \in M_m\}$.

($\because$ In a Latin square design each treatment appears once in each row.)

Let $(i_0,j_0) \in M_1$ then

$|\{(i,j)|(i,j) \in [(1,2)]$ and $(t_i,t_j) = (i_0,j_0)\}| = m$ ($\because$ (3))

i.e. $|\{H_i|(t_{g(1)},t_{g(2)}) = (i_0,j_0), i = 1,2,\ldots,m^2(m-1)\}| = m$

$$\therefore \pi^{(1,2)}(i_0,j_0) = \frac{m\cdot|H_1|}{|G_r \cap G_c|} = \frac{m\cdot|H_1|}{m^2(m-1)|H_1|} = \frac{1}{m(m-1)}$$

for $(i_0,j_0) \in M_1$ ($\because$ (1)) b) is thus proved.

The proof of c) is completely similar to b), so we omit it.

For d): Let

$[(1,m+2)] = \{(1,m+2),(i_2',j_2'),\ldots,(i'_{m^2(m-1)^2}, j'_{m^2(m-1)^2})\}$

$Q_1 = \{g|(g(1),g(m+2)) = (1,m+2)$ and $g \in G_r \cap G_c\}$,

....

$Q_{m^2(m-1)^2} = \{g|(g(1),g(m+2)) = (i'_{m^2(m-1)^2}, j'_{m^2(m-1)^2})\}$

Similar to the proof of (2) for $\{H_1,\ldots,H_{m^2(m-1)}\}$, we have

$Q_k = g\cdot Q_1$ for some $g \in G_r \cap G_c$, $k = 2,3,\ldots,m^2(m-1)^2$

$\therefore$ $|Q_1| = |Q_2| = \ldots = |Q_{m^2(m-1)^2}|$.

Note that

$$(4) \qquad G_r \cap G_c = \bigcup_{i=1}^{m^2(m-1)^2} Q_i$$

$$(5) \qquad |\{(t_i,t_j)|(t_i,t_j) = (1,1) \text{ and } (i,j) \in R_1\}| = m - 1$$

and

$$|\{(t_i,t_j)\ (t_i,t_j) = (1,k) \text{ and } (i,j) \in R_1\}| = m - 2$$
$$\text{for } k = 2,3,\ldots,m.$$

($\because$ structure of Latin square design.) Moreover,

$$(6) \qquad |\{i|R_i \text{ has the same property as that of } R_1 \text{ in } (5)\}| = m.$$

Similarly,

$$(7) \qquad |\{(t_i,t_j)|(t_i,t_j) = (2,2) \text{ and } (i,j) \in R_2\}| = m - 1$$

and

$$|\{(t_i,t_j)|(t_i,t_j) = (2,k) \text{ and } (i,j) \in R_2\}| = m - 2$$
$$\text{for } k \in \{1,2,\ldots,m\}\backslash\{2\}$$

and

$$(8) \qquad |\{i|R_i \text{ has the same property as that of } R_2 \text{ in } (7)\}| = m$$

and so forth.

$$\therefore \ \Pi^{(1,m+2)}(i,j) = \frac{m(m-2)|Q_1|}{|G_r \cap G_c|} = \frac{m-2}{m(m-1)^2} \quad \text{for } i \neq j$$

and $i,j = 1,2,\ldots,m$ then follows from (4), (5), (6), (7), (8), and

$$\Pi^{(1,m+2)}(i,j) = \frac{m(m-1)|Q_1|}{G_r \cap G_c} = \frac{1}{m(m-1)} \quad \text{for } i = j$$

and $i,j = 1,2,\ldots,m$. d) is thus proved. Q.E.D.

<u>Discussion</u>: In practice, $E\eta_x = 0$ for all x are required for (3.1) so that no bias is introduced through the randomization scheme. Theorem 3.2 tells us that a transitive group G will fulfill the requirement. The structure preserving groups for Latin square designs of order 3, order 4 and the Latin square designs from the transformation set containing $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$ are transitive groups.

S_{m^2} and $G_r \cap G_c$ are two other transitive groups. In case we would like to assume that all the observed values z_x's have equal variances, the randomization scheme with a transitive group G will justify the assumption. The reasoning is as follows:

$\text{Var}(\eta_1) = \text{Var}(\eta_2) = \ldots = \text{Var}(\eta_{m^2})$ if G is transitive ($\because$ Theorem 3.1). Moreover, $\text{Var}(z_x) = \text{Var}(u_x) + \text{Var}(\eta_x)$ for $x = 1,2,\ldots,m^2$ ($\because$ (3.2)). If the original $\text{Var}(u_x)$'s are not equal to each other, then $\text{Var}(\eta_x)$'s have the stabilization effect to make the assumption "$\text{Var}(z_x)$'s are all equal" justified. Similarly, if we would like to assume that $\text{Cov}(\vec{z})$ belongs to the commutator algebra associated with the Latin square design. From (3.2) we have that $\text{Cov}(\vec{z}) = \text{Cov}(\vec{u}) + \text{Cov}(\vec{\eta})$, Theorem 3.3 tells us that if G is the structure preserving group then $\text{Cov}(\vec{\eta})$ belongs to the commutator algebra. So when $\text{Cov}(\vec{u})$ is an element of the commutator algebra or very close to an element of the commutator algebra, the randomization scheme using the structure preserving group will have a similar stabilization effect. The randomization scheme using $G_r \cap G_c$ corresponds to the usual Fisher and Yates randomization for Latin square designs. Under the Fisher and Yates randomization scheme, $\mathrm{Cov}(\eta_1,\eta_2)$ is not equal to $\mathrm{Cov}(\eta_1,\eta_{m+2})$ in general. In fact,

$$\mathrm{Cov}(\eta_1,\eta_2) = \sum_{k,k'}(\Pi^{(1,2)}(K,K') \times r_k \times r_{k'}) = \sum_{i \neq j}(r_i r_j)/m(m-1)$$

$$(\because \text{ Theorem 3.4c}))$$

$$\mathrm{Cov}(\eta_1,\eta_{m+1}) = \frac{m-2}{m(m-1)^2} \sum_{i \neq j}(r_i r_j) + \frac{1}{m(m-1)}\sum_{i=1}^{m} r_i^2 \quad (\because \text{ Theorem 3.4d}))$$

$$= \frac{m-2}{m(m-1)^2} \sum_{i \neq j}(r_i r_j) - \frac{1}{m(m-1)} \sum_{i \neq j}(r_i r_j) \quad (\because \sum_{i=1}^{m} r_i = 0)$$

$$= \frac{-1}{m(m-1)} \sum_{i \neq j}(r_i r_j).$$

We thus introduce some bias into $\mathrm{Cov}(\vec{z})$ under the Fisher and Yates randomization scheme. In this section, I attempt to understand more about the impact of the randomization scheme on the statistical analysis that follows it. I think that randomization changes the probability distributions of the observed values. However, I doubt that it will make the normal theory analysis more appropriate as is claimed by some people.

§4. RANDOMIZATION TEST FOR THE LATIN SQUARE DESIGN ANALYSIS

From the computation in section two, it is seen that the intrinsic structures for Latin square designs from different transformation sets are quite different. However, Latin square designs from the same transformation set share some common properties. Let us compare different Latin square designs from the point of view of estimation of the covariance matrix of the observation values. Among Latin square designs of the same order, those with small vector space dimension for their associated commutator algebras should be preferred (since less parameters need to be estimated). So, for Latin square designs of order 4, designs from the transformation set containing $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ are preferred to designs from the transformation set containing $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}$. For Latin square designs of order 5, designs the transformation set containing $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$ are preferred to designs from the transformation set containing $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 3 & 4 \\ 3 & 4 & 1 & 5 & 2 \\ 4 & 5 & 2 & 1 & 3 \\ 5 & 3 & 4 & 2 & 1 \end{pmatrix}$. In this section, we discuss the comparison for different Latin square designs from the point of view of randomization.

The idea of randomization tests was originated by R.A. Fisher (Fisher, 1935 [4]). Let us quote and reexamine an example of a randomization test (Kempthorne, 1952 [7]).

Example 4.1: Suppose we have 8 experimental objects, a,b,c,d,e,f,g,h, of which four a,b,c,d have received treatment 1 and the other four treatment 2, and let the experimental results be:

Treatment 1:	a,	b,	c,	d
	18	13	3	17
Treatment 2:	e	f	g	h
	9	16	17	17

There are 70 possible ways of assigning treatment 1 to four objects and treatment 2 to the other four objects. Using all the seventy possible ways of assigning the eight observed values, the absolute value of the difference between treatment averages was computed. The frequency distribution of these seventy values was as follows.

(A)

Value	0	.5	1	2	2.5	3	4	4.5	5	6	6.5	7
Frequency	6	8	6	6	8	6	6	8	6	2	6	2

The computed value for the assignment actually used is 2. The significance probability is 50/70 (i.e. approximately 71 percent). Suppose the way of assigning treatments to objects was obtained through a randomization scheme, each object receiving either treatment 1 or treatment 2 according to the outcome of flipping a coin. Then the randomization test should allow for those other ways of assigning treatments to objects. For example, three objects receive one of the two treatments and five objects receive the other treatment (call it

{3,5} ways of assigning treatments to objects) and so forth. Totally there are 254 ways excluding the two ways which assign one of the two treatments to all objects. The absolute value of the difference between treatment averages is computed for each of the 254 ways. The frequency distribution for such computed values are tabled as follows:

The frequency distribution of the computed values for {3,5} ways of assigning treatments to objects:

(B)

Value	.4	$.\dot{6}$	$.9\dot{3}$	1.20	$1.4\dot{6}$	$1.7\dot{3}$	$2.2\dot{6}$	$2.5\dot{3}$	2.80	$3.0\dot{6}$
Frequency	6	2	8	6	6	8	8	6	6	8

Value	3.6	$3.8\dot{6}$	4.40	$4.\dot{6}$	$4.9\dot{3}$	5.2	$5.7\dot{3}$	6	$6.5\dot{3}$	$7.0\dot{6}$	$8.\dot{6}$
Frequency	6	2	6	6	2	8	6	2	6	2	2

The frequency distritution of the computed values for {2,6} ways of assigning treatment to objects:

(C)

Value	$.\dot{3}$	1	$1.\dot{6}$	$2.\dot{3}$	$3.\dot{6}$	$4.\dot{3}$	5.0	$5.\dot{6}$	$7.\dot{6}$	$10.\dot{3}$
Frequency	2	8	8	2	8	10	12	2	2	2

The frequency distribution of the computed values for {1,7} ways of assigning treatments to objects:

(D)

Value	.8571	2.57	3.714	4.857	5.428	12.28
Frequency	2	2	6	2	2	2

The mean values and standard deviations for the four distributions are as follows:

(A) Mean Value: 3.07 Standard Deviation: 2.11

(B) Mean Value: 3.29 Standard Deviation: 2.00

(C) Mean Value: 3.69 Standard Deviation: 2.21

(D) Mean Value: 4.64 Standard Deviation: 3.27

A comparison between the four cases from another point of view is as follows: Let $z_1, z_2, \ldots, z_8$ denote the observed values for the experiment and assume they are independently distributed with the same variance σ^2. The computed values in (A),(B),(C),(D) can be represented in the following forms:

(A) $(z_{i_1} + z_{i_2} + z_{i_3} + z_{i_4})/4 - (z_{i_5} + z_{i_6} + z_{i_7} + z_{i_8})/4$

(B) $(z_{i_1} + z_{i_2} + z_{i_3})/3 - (z_{i_3} + \ldots + z_{i_8})/5$

(C) $(z_{i_1} + z_{i_2})/2 - (z_{i_3} + \ldots + z_{i_8})/6$

(D) $(z_{i_1} - (z_{i_2} + \ldots + z_{i_8})/7$

The variances of the computed values are as follows: (A) $\frac{\sigma^2}{2}$, (B) $\frac{8}{15}\sigma^2$, (C) $\frac{2}{3}\sigma^2$, (D) $\frac{8}{7}\sigma^2$. This suggests that the randomization scheme according to the outcome of flipping a coin is inappropriate. In practice, the number of objects to receive treatment 1 and the number of objects to receive treatment 2 should be predetermined before doing the randomization procedure. In other words, randomization should be performed among those designs with the same structure or symmetric property. So in practice, a particular transformation set of Latin square designs should be chosen according to some criterion and a Latin square then randomly chosen from the transformation set. A criterion for the choice between different transformation sets is discussed below in connection with the randomization test.

A Latin square design of order k is randomly chosen from a transformation set. The observed values from the Latin square design experiment are arranged in a matrix $\begin{pmatrix} z_1 \cdots\cdots\cdots z_k \\ \cdots\cdots\cdots\cdots \\ z_{(k-1)k+1} \cdots z_{k^2} \end{pmatrix}$ as before.

Under the null hypothesis that k treatment effects are all equal, the observed values would have been the same had any other Latin square design been used from the transformation set. If we superimpose the observed values on all the possible Latin square designs in the transformation set, then we can compute the mean treatment sum of squares, mean error sum of squares and the F^* statistics as in the usual analysis of variance $\left(F^* = \frac{\text{mean trt SS}}{\text{mean error SS}}\right)$. The computed F^* value, using the actual design used, is the test statistic for the randomization test. The computed F^* values, using designs from the same equivalence class of a transformation set, have the same value.

Let me explain more clearly, from the point of view of analysis of variance, why we should include only those designs in the same transformation set as the design actually used. We know that two Latin square designs from different transformation sets can not be obtained from each other by permutations of rows and columns. In other words, if we superimpose the observed values on designs from the other transformation set, then the computed values for the sum of squares due to rows and columns are vitiated in this case. The computed F^* value is thus inappropriate for comparison. For Latin square designs of order 3, there is only one transformation set and it has two equivalence classes. The significance probability for testing the null hypothesis with one replicate is either .5 or 1.0. For Latin square designs of order 4, the attainable significance probabilities are multiples of 1/6 for $\begin{pmatrix}1&2&3&4\\2&1&4&3\\3&4&1&2\\4&3&2&1\end{pmatrix}$ and the attainable significance probabilities are multiples of 1/18 for $\begin{pmatrix}1&2&3&4\\2&3&4&1\\3&4&1&2\\4&1&2&3\end{pmatrix}$.

We say that $\begin{pmatrix}1&2&3&4\\2&3&4&1\\3&4&1&2\\4&1&2&3\end{pmatrix}$ is more precise than $\begin{pmatrix}1&2&3&4\\2&1&4&3\\3&4&1&2\\4&3&2&1\end{pmatrix}$ (or more sensitive) with respect to the randomization test. The transformation set containing $\begin{pmatrix}1&2&3&4\\2&3&4&1\\3&4&1&2\\4&1&2&3\end{pmatrix}$ is preferred to the transformation set containing $\begin{pmatrix}1&2&3&4\\2&1&4&3\\3&4&1&2\\4&3&2&1\end{pmatrix}$ in terms of the randomization test. Similarly, applying the results of Example 3, Section two, the transformation set containing $\begin{pmatrix}1&2&3&4&5\\2&1&5&3&4\\3&4&1&5&2\\4&5&2&1&3\\5&3&4&2&1\end{pmatrix}$ is preferred to the transformation set containing $\begin{pmatrix}1&2&3&4&5\\2&3&4&5&1\\3&4&5&1&2\\4&5&1&2&3\\5&1&2&3&4\end{pmatrix}$ in terms of the randomization test. These are contrary to the results in terms of estimation of covariance matrix. It shows that the main interest of statistical study should be determined before an optimal design for an experiment can be found. Let us look at two examples of randomization tests for a Latin square design of order 4.

Example 4.2: The observed values from a randomly chosen Latin square design $\begin{pmatrix}2&1&4&3\\3&2&1&6\\4&3&2&1\\1&4&3&2\end{pmatrix}$ are arranged correspondingly in a matrix

$$\begin{pmatrix} 21 & 8 & 17 & 9 \\ 5 & 10 & 3 & 12 \\ 20 & 10 & 15 & 21 \\ 4 & 15 & 3 & 9 \end{pmatrix} .$$

The analysis of variance table is as follows:

Source	Sum of Squares	d.f.	Mean Square	F ratio
Row	240.25	3	80.08	
Column	28.25	3	9.42	
Treatment	216.25	3	72.08	4.55
Error	95.00	6	15.83	
Total	579.75	15		

The significance probability $\approx$ .07

The computed F^* values, using designs from 18 equivalence classes of the transformation set containing the design $\begin{pmatrix} 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & 4 & 3 & 2 \end{pmatrix}$ are as follows:

(4.1) F^*values: .096, .207, .231, .284, .318, .390, .422, .485, .638, 1.051, 1.242, 2.07, 2.61, 2.75, 4.55, 4.62, 4.92

Mean value of F^* = 1.53

Standard Deviation of F^* values = 1.67

The computed F^* values using designs from six equivalence classes of the other transformation set are as follows:

(4.2) F^*values: .128, .318, .441, .594, 4.29, 4.92

Mean Value of F^* = 1.78, Standard Deviation of F^* = 2.20

Example 4.3: The observed values from a randomly chosen Latin square design are arranged in a matrix $\begin{pmatrix} 1.1 & 1.5 & 1.0 & 1.7 \\ 1.4 & 1.9 & 1.6 & 1.5 \\ 2.8 & 2.2 & 2.7 & 2.1 \\ 3.4 & 2.5 & 2.9 & 2.7 \end{pmatrix}$.

The computed F^* values, using designs from eighteen equivalence classes of the transformation set containing the design $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}$ are as follows:

(4.3) F^* values: .0905, .127, .155, .155, .205, .205, .3095, .332, .378, 1.03, 1.19, 1.44, 1.67, 1.85, 2.85, 7.33, 10.76, 11.1

Mean Value of F^* value = 2.29

Standard Deviation of F^* value = 3.58

The computed F^* values, using designs from six equivalence classes of the other transformation set are as follows:

(4.4) F^* values: .0905, .155, .205, .377, 5.95, 15.34

Mean Value of F^* value = 3.67

Standard Deviation of F^* value = 6.15

The computed F^* values, using designs from different transformation sets in the above two examples suggest that distributions of computed F^* values for different transformation sets might be quite different for Latin square designs of order greater than 4. It indicates again, that the randomization test should be performed using only those designs in the same transformation set as the design actually used.

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