# NON-MANIFOLD FACTORS OF EUCLIDEAN SPACES

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K. W. Kwun

Major professor

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## ABSTRACT

## NON-MANIFOLD FACTORS OF EUCLIDEAN SPACES

by Alfred John Boals

This thesis is a study of a class C of decomposition spaces which are shown to be factors of Euclidean space.

Suppose $A$ and $B$ are disjoint compact subsets of $E^n$. Then we know it is possible to find disjoint compact sets $A^*$ and $B^*$ such that $A \subset \text{int } A^*$ and $B \subset \text{int } B^*$. In Chapter I we give sufficient conditions for $A$ and $B$ to insure that $A^*$ and $B^*$ can be picked to be cells.

In Chapter II we define the class C of decomposition spaces and prove that the product of any member of C and a line is topologically $E^n$ for some integer $n$.

In Chapter III we prove that the product of any two members of the class C is topologically $E^{n+m}$ for suitably chosen integers $n$ and $m$.

# NON-MANIFOLD FACTORS OF EUCLIDEAN SPACES

By

Alfred John Boals

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# INTRODUCTION

In 1957 R. H. Bing [3] gave an example of a decomposition of $E^3$ into tame arcs and points such that the associated decomposition (the "dogbone space") is not topologically $E^3$. In fact this space is not even a manifold (i.e. there exist points which do not have Euclidean neighborhoods). The "dogbone space" was constructed in answer to a question of G. T. Whyburn [8, p. 70] which asked: Is it true that if $G$ is an upper semi-continuous decomposition of $E^3$ into point like compact continua, then the decomposition space is homeomorphic to $E^3$?

In [5] Bing gave examples (i.e. "unused example" and "segment space") of two other decompositions of $E^3$ into tame arcs and points. The "unused example" is known to be distinct from $E^3$, however, whether or not the "segment space" is $E^3$ is still unknown. Bing [4] proved that the product of the "dogbone space" and the line is 4-dimensional Euclidean space ($E^4$). It is reported [5] that John Hempel has proved that the product of the "segment space" and $E^1$ is $E^4$. In [2] S. Armentrout asked if the same were true of Bing's "unused example". An affirmative answer is given to this question in Chapter II.

In proving that the product of a dogbone space and a line is $E^4$, Bing showed that $E^4$ has non-manifold factors. J. J. Andrews anu M. L. Curtis [1] gave another factorization of $E^4$ into factors one of which is not Euclidean.

They proved that if $\alpha$ is an arc in $E^n$ then $E^n/\alpha \times E^1$ is $E^{n+1}$. K. W. Kwun proved that if $\alpha \subset E^n$ and $\beta \subset E^m$ where $\alpha$ and $\beta$ are arcs then $E^n/\alpha \times E^m/\beta$ is $E^{n+m}$. Thus Kwun has shown that $E^n$ where $n \geq 6$ can be written as the product of two non-manifolds.

In view of the relationship of these last two results, it is natural to ask the following question. Is the product of dogbone spaces, unused examples, or segment spaces topologically $E^6$? This question is answered in the affirmative in Chapter III.

## Notation and Terminology

Suppose $X$ is a topological space. A collection of subsets $G$ of $X$ will be called a <u>decomposition</u> of $X$ if $\cup\{s \mid s \in G\} = X$ and $s_1 \cap s_2 = \Phi$ for any two distinct elements of G. $G$ will be called an <u>upper semi-continuous</u> decomposition of $X$ if for any element $g \in G$ and any open set $U \subset X$ which contains $g$, there exists an open set $V \subset U$ such that $g \subset V$ and $V$ is the union of elements of $G$.

Suppose $X$ is a topological space and $G$ is a decomposition of $X$. The <u>decomposition space</u> associated with $G$, say $Y$, is defined as follows. The points of $Y$ are the elements of $G$ and a base for the topology of $Y$ is the collection of sets of elements of $G$ whose union in $X$ is open. More information on decomposition spaces can be found in [8] and [6, Chap. 3].

A <u>pseudo-isotopy</u> of a topological space $X$ into a topological space $Y$ is a continuous function $H$ on $X \times I$ into $Y$ such that $H_t(x) = H(x,t)$ is a homeomorphism for each $0 \leq t < 1$.

Any subset of a topological space which is homeomorphic to $I^n$ where $I = [0,1]$ will be called an n-<u>cell</u>. Any paracompact Hausdorff space in which every point has a neighborhood whose closure is an n-cell will be called an n-<u>manifold</u>.

# CHAPTER I

## SEPARATION THEOREMS

In proving that the product of the "dogbone space" and a line is $E^4$ Bing was forced to give a rather lengthy construction of a sequence of 4-cells. Theorem 1.1 asserts the existence of suitable n-cells in a more general context. Theorem 1.1 is applied in proving the main result of Chapter II. Theorem 1.2, in addition to being of interest in itself, plays an important role in the proof of the main result of Chapter III.

<u>Definition 1.1</u>: If $X$ is a topological space and $D \subset X$ then by $\operatorname{int} D$ is meant the set $X - \overline{X - D}$, where $\overline{X - D}$ is the set theoretic closure of $X - D$ in $X$.

<u>Theorem 1.1</u>: Let $C_1, C_2, \ldots, C_p$ be disjoint compact subsets of a Hausdorff space $X$. Let $D_1, D_2, \ldots, D_p$ be (not necessarily disjoint) n-cells such that for each $i = 1,2, \ldots, p$

$$C_i \subset \operatorname{int} D_i .$$

Then for any $[a,b] \subset E^1$ and $\varepsilon > 0$ there exist disjoint $(n + 1)$-cells $E_1, E_2, \ldots, E_p$ contained in $X \times (a-\varepsilon, b+\varepsilon)$ such that for each $i = 1,2, \ldots, p$

(1) $C_i \times [a,b] \subset \operatorname{int} E_i$

(2) $\pi_1 E_i = D_i$ ;

where $\pi_1$ is the projection of $X \times E^1$ onto $X$.

Proof: Let $f:[-\varepsilon, r+\varepsilon] \longrightarrow [a-\varepsilon, b+\varepsilon]$ be the homeomorphism given by

$$f(x) = \begin{cases} a + x & ; \text{ if } x \in [-\varepsilon,0] \\ (\frac{b - a}{r})x + a; & \text{ if } x \in [0,r] \\ b + x - r & ; \text{ if } x \in [r,r+\varepsilon] \end{cases}$$

Let $k:[-\varepsilon,r+\varepsilon] \xrightarrow{\text{onto}} [-\varepsilon,2p-1+\varepsilon]$ be the homeomorphism given by

$$k(t) = \frac{2(p+\varepsilon) - 1}{r+2\varepsilon}(t + \varepsilon) - \varepsilon$$

For each $j = 1,2, \ldots, p$ let $k_j$ be a homeomorphism of $[-\varepsilon,r+\varepsilon]$ onto $[-\varepsilon,2p-1+\varepsilon]$ with the properties:

1. $k_j(-\varepsilon) = -\varepsilon$ and $k_j(r + \varepsilon) = (2p - 1 + \varepsilon)$
2. $k_j(0) = 2j - 2$
3. $k_j(r) = 2j - 1.$

Let $A = \bigcup_i D_i \subset X$ and note that since $A$ is a compact Hausdorff space it is normal. Bdy $A = A - \text{int } A$ and $C_j$ for $j = 1, \ldots, p$ are closed sets. Thus there exist open sets $U_j$ for $j = 1,2, \ldots, p$ satisfying

1. $U_i \cap U_j = \Phi$ if $i \neq j$,
2. $C_i \subset U_i$ for all $i = 1,2,\ldots, p$,
3. $U_i \subset \text{int } A$.

By the Urysohn lemma there exists a continuous function $g$ mapping $A$ onto $I = [0,1]$ such that

1. $g(\bigcup_i C_i) = 1$ and
2. $g(A - \bigcup_i U_i) = 0$.

Construct $h: A \times [-\varepsilon, r+\varepsilon] \longrightarrow A \times [-\varepsilon, 2p-1+\varepsilon]$ as follows:

$$h(x,t) = \begin{cases} (x,\ g(x)k_j(t) + (1-g(x))k(t)); & \text{on } \overline{U}_j \times [-\varepsilon, r+\varepsilon] \\ (x,\ k(t)); & \text{on } (A - \bigcup_i U_i) \times [-\varepsilon, r+\varepsilon]. \end{cases}$$

For each $j = 1, 2, \ldots, p$ $h = id \times k$ on Bdy $\overline{U}_j \times [-\varepsilon, k+\varepsilon]$, hence $h$ is well defined. $h$ is continuous since $g$, $k$ and $k_j$ are all continuous. Suppose

$$h(x_1, t_1) = h(x_2, t_2)$$

then $x_1 = x_2 = x$.

Case 1:

If $x \in U_j$ then

$$g(x)k_j(t_1) + [1-g(x)]k(t_1) = g(x\ )k_j(t_2)+[1-g(x\ )]k(t_2)$$

or

$$g(x)[k_j(t_1)-k_j(t_2)]+[1-g(x)][k(t_1)-k(t_2)] = 0$$

But $g(x)$ and $1-g(x) \geq 0$ and both $k$ and $k_j$ are order preserving homeomorphisms, whence $k_j(t_1) - k_j(t_2)$ and $k(t_1) - k(t_2)$ are both positive, zero, or negative together. Therefore

$$k(t_1) - k(t_2) = 0$$

and $\qquad t_1 = t_2$.

Case 2:

If $x \in A - \bigcup_i U_i$ then $k(t_1) = k(t_2)$ and again $t_1 = t_2$.

Thus $h$ is one-to-one and continuous hence a homeomorphism. $h$ can be extended to a homeomorphism of $X \times [-\varepsilon, r+\varepsilon] \longrightarrow X \times [-\varepsilon, 2p-1+\varepsilon]$ by defining

$h(x, t) = (x, k(t))$ on $X - A$.

For each $j = 1, \ldots, p$ let $E_j'$ be the $(n+1)$-cells defined by

$$E_j' = D_j \times [2j - 2 - \tfrac{1}{4}, 2j - 1 + \tfrac{1}{4}].$$

Now define

$$E_j = (id \times f)[h^{-1}(E_j')]$$

Clearly $E_j \cap E_i = \phi$ if $i \neq j$ and $\pi_1 E_j = D_j$. Moreover $E_i \subset \operatorname{int} A \times (a - \varepsilon, b + \varepsilon)$.

If $x \in C_i \times [a,b]$ then

$(id \times f)^{-1}(x) \in C_i \times [0, 2r-1]$. and

$h \cdot (id \times f)^{-1}(x) \in C_i \times [2i - 2, 2i - 1]$.

But

$$C_i \times [2i - 2, 2i - 1] \subset \operatorname{int} (D_i \times [2i-2-\tfrac{1}{4}, 2i-1+\tfrac{1}{4}])$$

whence

$$(id \times f) \cdot h^{-1} \cdot h \cdot (id \times f)^{-1}(x) \in (id \times f)h^{-1}(D_i \times [2i-2-\tfrac{1}{4}, 2i-1+\tfrac{1}{4}])$$

and $x \in E_i$. Thus the $E_i$ $i = 1,2, \ldots, p$ satisfy all the claims of the theorem.

The above theorem will be applied in Chapter II in the following form.

<u>Corollary 1.2</u>: Let $A$ be a compact n-manifold in $E^n$ and $C_1, C_2, \ldots, C_p$ be disjoint compact sets in int $A$ such that there exist (not necessarily disjoint) n-cells $D_1, D_2, \ldots, D_p$ with the property that $C_i \subset \operatorname{int} D_i \subset \operatorname{int} A$. Then for any $[a,b] \subset E^1$ and $\varepsilon > 0$ there exist disjoint $(n + 1)$-cells $E_1, E_2, \ldots, E_p$ contained in $\operatorname{int} A \times (a - \varepsilon, b + \varepsilon)$ such that $C_i \times [a,b] \subset \operatorname{int} E_i$ and $\pi_n E_i = D_i$, where $\pi_n$ is the projection of $E^n \times E^1$ onto $E^n$.

<u>Theorem 1.3</u>: Suppose $B$ is a compact subset of $\operatorname{int} I^n$ and $C$ is a compact subset of $I^n$ disjoint from $B$. Similarly suppose $D$ is a compact subset of $\operatorname{int} I^m$ and $E$ is a compact subset of $I^m$ disjoint from $D$. Then there exists an $(n + m)$-cell $G$ with the following properties:

(1) $B \times D \subset \operatorname{int} G \subset G \subset \operatorname{int} I^n \times \operatorname{int} I^m$

(2) $G \cap [(B \times E) \cup (C \times D) \cup (C \times E)] = \Phi$.

<u>Proof</u>: Let $T \subset \operatorname{int} I^n$ be an n-cell such that $T \cap (B \cup C) = \Phi$ and $T$ is the product of its projections. Such an n-cell exists since $C \cap B = \Phi$ and they are each closed. Similarly let $R \subset \operatorname{int} I^m$ be an m-cell such that $R \cap (D \cup E) = \Phi$ and $R$ is the product of its projections. Let $\pi_i T = [t_i, t_i']$ for each $i = 1,2, \ldots, n$ and $\pi_j R = [r_j, r_j']$ for each $j = 1,2, \ldots, m$, where $\pi_\alpha$ is the projection onto the $\alpha$-th coordinate.

Let $\delta_1 = \min[\text{distance from } B \text{ to } (\operatorname{Bdy} I^n \cup C), \text{distance from } T \text{ to } (B \cup C \cup \operatorname{Bdy} I^n)]$. Let $\delta_2 = \min[\text{distance from } D \text{ to } (\operatorname{Bdy} I^m \cup E), \text{distance from } R \text{ to } (D \cup E \cup \operatorname{Bdy} I^m)]$. Set $\delta = \min(\delta_1, \delta_2)$. Let $k, k_1, k_2, \ldots, k_m$ be homeomorphisms defined as follows,

(1) $k : [0,1] \longrightarrow [0,1]$ such that
$k(0) = 0, \; k(1) = 1$ and
$k[\delta/2, 1-\delta/2] = [1/4, 3/4]$.

(2) For each $i = 1,2, \ldots, m$ let
$k_i : [0,1] \longrightarrow [0,1]$ such that
$k_i(0) = 0, \; k_i(1) = 1$ and
$k_i[r_i, r_i'] = [1/4, 3/4]$.

Let $U_n$ be an open subset of $(\delta/2,\ 1 - \delta/2)^n$ such that $B \subset U_n$ and $U_n \cap C = \phi$. Set $W_n = I^n - [\delta/3,\ 1 - \delta/3]^n$. By the Urysohn Lemma there exists a continuous function $g : I^n \longrightarrow [0,1]$ such that

$$g(B \cup \operatorname{Bdy} I^n) = 1 \qquad \text{and}$$
$$g(I^n - (U_n \cup W_n)) = 0.$$

Consider the following collection of maps

$$h_i : I^n \times I^m \longrightarrow I^n \times I^m,\ i = 1,2,\ \dots,\ m.$$

For $x \in I^n$ and $(y_1,\ \dots,\ y_m) \in I^m$

$$h_i(x,(y_1,\dots,y_m)) = \begin{cases} (x,(y_1,\dots,y_{i-1},g(x)k(y_i) + [1-g(x)]k_i(y_i),y_{i+1},\dots,y_m) \text{ for } (x,(y_1,\dots,y_m)) \in \overline{U}_n \times I^m. \\ (x,(y_1,\dots,y_{i-1},g(x)y_i + [1-g(x)]k_i(y_i),y_{i+1},\dots,y_m)) \text{ for } (x,(y_1,\dots,y_m)) \in \overline{W}_n \times I^m. \\ (x,(y_1,\dots,y_{i-1},k_i(y_i),y_{i+1},\dots,y_m)) \text{ for } (x,(y_1,\dots,y_m)) \in [I^n-(W_n \cup U_n)] \times I^m. \end{cases}$$

Each $h_i$ is well defined since $\overline{U}_n \cap \overline{W}_n = \phi$,

$$\pi_j h_i/\operatorname{Bdy} U_n \times I^m = \pi_j \quad \text{for all } j \neq i$$

$$\pi_i h_i/\operatorname{Bdy} U_n \times I^m = k_i \pi_i,$$ where again $\pi_i$ is the projection onto the i-th coordinate axis. And

$$\pi_i h_i/\operatorname{Bdy} W_n \times I^m = \pi_j \quad \text{for all } j \neq i$$

$$\pi_i h_i/\operatorname{Bdy} W_n \times I^m = k_i \pi_i.$$

Clearly each $h_i$ is continuous and onto $I^n \times I^m$. Suppose for $x, x' \in I$ and $(y_1, \ldots, y_m), (z_1, \ldots, z_m) \in I^m$

$$h_i[x,(y_1, \ldots, y_m)] = h_i[x',(z_1, \ldots, z_m)]$$

then $x = x'$ and $y_j = z_j$ for $j \neq i$.

Consider the three cases:

(1) $x \in U_n$ ,

(2) $x \in W_n$ , or

(3) $x \in I^n - (U_n \cup W_n)$.

Case 1: $g(x)k(y_i)+[1-g(x)]k_i(y_i)=g(x)k(z_i)+[1-g(x)]k_i(z_i)$ and $g(x)[k(y_i)-k(z_i)]+[1-g(x)][k_i(y_i)-k_i(z_i)] = 0.$

Note that $g(x)$ and $1 - g(x) \geq 0$ also

$[k(y_i) - k(z_i)][k_i(y_i) - k_i(z_i)] \geq 0$ since $k$ and $k_i$ preserve order. Thus $y_i = z_i$.

Similar arguments show that for cases 2 and 3 $y_i = z_i$. Thus for each $i$, $h_i$ is an injection consequently a homeomorphism.

Define $H : I^n \times I^m \longrightarrow I^n \times I^m$ to be the homeomorphism $h_1 \cdot h_2 \cdot \ \ldots \ \cdot h_m$.

Set $J = [\delta/2, 1- \delta/2]^n \times [1/4, 3/4]^m \subset I^n \times I^m$. If $(x,y) \in B \times D$ then $x \in U_n$ and $H(x,y) \in J$. Thus $H(B \times D) \subset J$. Let $(x,y) \in [C \times (D \cup E)]$ then $x \in I - U_n$ and there exists a $j$ such that $\pi_j(y) \in I-[r_j,r_j']$. If $x \in W_n$ then $H(x,y) \notin J$. If $x \in I^n - (W_n \times U_n)$ then $\pi_j h_j(x,y) \in I - [1/4,3/4]$ and $H(x,y) \notin J$. Thus $H[C \times (D \cup E)] \cap J = \Phi$.

Note that $H/B \times I^m = id \times k^*$, where $k^*$ is the m-fold product of $k$; (i.e. $k^* = (k \times k \times \ldots \times k)$ with $m$ factors).

Thus it follows that $\pi_m^* H(B \times D)$ and $\pi_m^* H(B \times E)$ are disjoint compact subsets of $[1/4, 3/4]^m \subset I^m$, where $\pi_m^*$ is the projection of $I^n \times I^m$ onto $I^m$. Also $\pi_m^* H(B \times D) \subset (1/4, 3/4)^m$. Let $\gamma = \min(\text{distance from } \pi_m^* H(B \times D)$ to $\text{Bdy}[1/4, 3/4]^m, \delta/2)$. Let $U_m$ be an open set in $(1/4 + \gamma/2, 3/4 - \gamma/2)^m$ such that $\pi_m^* H(B \times D) \subset U_m$ and and $U_m \cap \pi_m^* H(B \times E) = \Phi$. Let $W_m = [1/4, 3/4]^m - [1/4 + \gamma/3, 3/4 - \gamma/3]^m$. There exists a continuous function

$$f : [1/4, 3/4]^m \longrightarrow [0,1]$$

such that

(1) $f/\pi_m^*[H(B \times D) \cup \text{Bdy}[1/4, 3/4]^m] = 1$

(2) $f/[1/4,3/4]^m - (U_m \cup W_m) = 0$

Let $\Psi, \Psi_1, \Psi_2, \ldots, \Psi_n$ be homeomorphisms defined as follows

(1) $\Psi : [\delta/2, 1-\delta/2] \longrightarrow [\delta/2, 1-\delta/2]$ such that

$\Psi(\delta/2) = \delta/2$ , $\Psi(1-\delta/2) = 1-\delta/2$

and

$$\Psi[\delta/2 + \gamma/2, 1-\delta/2-\gamma/2] = [1/4, 3/4]$$

(2) For each $i = 1,2, \ldots, n$ let

$\Psi_i : [\delta/2, 1-\delta/2] \longrightarrow [\delta/2, 1-\delta/2]$ such that

$\Psi_i(\delta/2) = \delta/2$ , $\Psi_i(1-\delta/2) = 1-\delta/2$

and

$$\Psi_i[t_i, t_i'] = [1/4, 3/4].$$

Consider the following collection of maps

$$\theta_i : I^n \times I^m \longrightarrow I^n \times I^m , \quad i = 1,2, \ldots, n.$$

For $(x_1, x_2, \ldots, x_n) \in I^n$ and $y \in I^m$

$$\theta_i[(x_1,\ldots,x_n),y] = \begin{cases} \text{id} \quad \text{for } ((x_1,\ldots,x_n),y) \in (I^n \times I^m)-\text{int } J \\[1ex] ((x_1,\ldots,x_{i-1},f(y)\Psi(x_i) + \\ {[1-f(y)]}\Psi(x_i),x_{i+1},\ldots,x_n),y) \quad \text{for} \\ ((x_1,x_2,\ldots,x_n),y) \in [\delta/2,1-\delta/2]^n \times U_m \\[1ex] ((x_1,\ldots,x_{i-1},f(y)x_i + \\ {[1-f(y)]}\Psi_i(x_i),x_{i+1},\ldots,x_n),y) \quad \text{for} \\ ((x_1,x_2,\ldots,x_n),y) \in [\delta/2,1-\delta/2]^n \times W_m \\[1ex] ((x_1,\ldots,x_{i-1},\Psi_i(x_i),x_{i+1},\ldots,x_n),y) \\ \text{on } [\delta/2,1-\delta/2]^n \times ([1/3,3/4]^m-(U_m \cup W_m)) \end{cases}$$

Each $\theta_i$ is well defined since $\overline{W}_m \cap \overline{U}_m = \phi$,

$\pi_j\theta_i/[\delta/2,1-\delta/2]^n \times \text{Bdy } U_m = \pi_j$ for $i \neq j$

$\pi_j\theta_i\ [\delta/2,1-\delta/2] \times \text{Bdy } U_m = \Psi_i\pi_i$

and

$\pi_j\theta_i/[\delta/2,1-\delta/2]^n \times ([\delta/2,1-\delta/2]^m-(U_m \cup W_m))$ is $\pi_j$ if $j \neq i$ and $\Psi_i\pi_i$ if $j = i$. By an argument exactly like the one given above for $h_i$, each $\theta_i$ is a homeomorphism. Define $\theta = \theta_1 \cdot \theta_2 \cdot \ldots \cdot \theta_n$. Set $J' = [1/4,3/4]^n \times [1/4 + \gamma/2,\ 3/4 - \gamma/2]^m \subset J$. If $(x,y) \in B \times D$ and $x = (x_1,\ \ldots,\ x_n)$, $y = (y_1,y_2,\ \ldots,\ y_m)$.

Then $\theta[H(x,y)] = \theta(x,[k(y_1),\ \ldots,\ k(y_m)])$

$$= [(\Psi(x_1),\ldots,\Psi(x_n)),(k(y_1)\cdot k(y_2),\ldots k(y_m))].$$

Thus $\theta \cdot H(B \times D) \subset J'$. If $(x,y) \in [(C \times D) \cup (C \times E)]$ then $H(x,y) \notin J$ hence $\theta \cdot H(x,y) \notin J'$. Suppose $(x,y) \in B \times E$ then $\pi_m^* H(x,y) \in [1/4,\ 3/4]^m - U_m$ and there exists a $j$ such that $\pi_j(x) \notin [t_j, t_j']$. If $\pi_m^* H(x,y) \in W_m$ then $\theta \cdot H(x,y) \notin J'$ since $\pi_m^* \theta \cdot H(x,y) = \pi_m^* H(x,y) \in W_m$ and $W_m \cap \pi_m^* J' = \Phi$. If $\pi_m^* H(x,y) \in \pi_m^*(J) - W_m - U_m$ then

$$\pi_j \pi_m^* H(x,y) \notin [1/4, 3/4] \text{ and } \theta \cdot H(x,y) \notin J'.$$

Therefore $\theta \cdot H[(B \times E) \cup (C \times D) \cup (C \times E)] \subset I^n \times I^m - J'$.

Define $G = H^{-1} \cdot \theta^{-1}(J')$. $G$ is the $(n + m)$-cell contained in $\operatorname{int} I^n \times \operatorname{int} I^m$ satisfying properties 1 and 2 of the theorem.

Note that $J'$ defined in the above proof is a product of cells. Thus a proof similar to that of Theorem 1.1 would prove the following corollary.

Corollary 1.4: Suppose $B_i$, $i = 1,2, \ldots, p$, are disjoint compact subsets of $\operatorname{int} I^n$ and $C$ is a compact subset of $I^n$ disjoint from $\bigcup_i B_i = B$. Similarly suppose $D_j$, $j = 1,2, \ldots, q$, are disjoint compact subsets of $\operatorname{int} I^m$ and $E$ is a compact subset of $I^m$ disjoint from $\bigcup_i D_i = D$. Then there exist $(n + m)$-cells $G_{ij}$, $i = 1,2, \ldots, p$; $j = 1,2, \ldots, q$, such that

(1) $G_{ij} \cap G_{rs} = \Phi$ if $i \neq r$ or $j \neq s$,

(2) $B_i \times D_j \subset \operatorname{int} G_{ij} \subset G_{ij} \subset \operatorname{int}(I^n \times I^m)$,

(3) $G_{ij} \cap [(C \times D) \cup (B \times E) \cup (C \times E)]$.

# CHAPTER II

## A CLASS OF DECOMPOSITION SPACES

In this chapter we define a class of upper semi-continuous decompositions of $E^n$ and prove that the associated decomposition spaces are factors of $E^{n+1}$. This class contains the decompositions for each of the spaces (a) "dogbone space", (b) "unused example" and (c) the "segment space" [see 5].

<u>Definition 2.1</u>: Suppose $\alpha$ is an arc in $E^n$ (i.e. $\alpha = h[0,1]$ for some homeomorphism $h : I \longrightarrow E^n$) such that $P = \pi_1/\alpha$ is an injection, where $\pi_1$ is the projection of $E^n$ onto the 1st coordinate. In this case $\alpha$ will be said to have property QS.

Let $\alpha$ be an arc with property QS and assume that $\pi_1\, h(1) = b$ and $\pi_1\, h(0) = a$ with $a < b$. Define the continuous function $f : E^1 \longrightarrow E^n$ by

$$f(t) = \begin{cases} P^{-1}(a) & \text{for } t \leq a \\ P^{-1}(t) & \text{for } a \leq t \leq b \\ P^{-1}(b) & \text{for } b \leq t\ . \end{cases}$$

Define the homeomorphism $k : E^1 \times E^{n-1} \longrightarrow E^1 \times E^{n-1}$ by $k(t,x) = (t,\ x - f(t))$. For any $\varepsilon > 0$ let

$C_1 = \{z \mid z \in E^n,\ \|z - a\| \leq \varepsilon\}$

$C_2 = \{z \mid z \in E^n,\ \|z - b\| \leq \varepsilon\}$

$C_3 = \{z \mid z \in E^n,\ a \leq \pi_1 z \leq b \text{ and } \|z - \pi_1 z\| \leq \varepsilon\}$

then $Q_\varepsilon = C_1 \cup C_2 \cup C_3$ is an n-cell containing $\pi_1(\alpha)$. The n-cell $k^{-1}(Q_\varepsilon)$ shall be called an $\varepsilon$-radial neighborhood of $\alpha$.

Remark 2.1: Note that if $\alpha$ is an arc with property QS then for any $\varepsilon > 0$ the $\varepsilon$-radial neighborhood of $\alpha$ intersects the planes $\pi_1^{-1}(t) = R_t = \{(t,y) \mid (t,y) \in t \times E^{n-1}\}$ in the void set, a point, or an (n-1)-cell.

Remark 2.2: Suppose $\alpha$ is an arc which has property QS. Since the homeomorphism used to define radial neighborhood is uniformly continuous, it follows that for any $\varepsilon > 0$ there exists a $\delta > 0$ and a collection of planes $R_i = \pi^{-1}(t_i)$ with $t_1 = a < t_2 < \ldots < t_p = b$ such that the $R_i$ cut the $\delta$-radial neighborhood of $\alpha$ into (p + 1) n-cells $C_i$ ; $i = 0,1,\ldots, p$ and diam $C_i \leq \varepsilon$.

Let $A_1, A_2, \ldots$ be a sequence of compact n-manifolds (not necessarily connected) in $E^n$ satisfying

P1. $A_{i+1} \subset \operatorname{int} A_i$ for all $i = 1,2,3, \ldots$

P2. Each component of $A_\infty = \bigcap_i A_i$ is an arc with property QS.

Lemma 2.1: Suppose $\varepsilon > 0$ and $A_i$ and $A_\infty$ are as defined above, then there exists a finite collection of n-cells $U_i$ satisfying

1. For each $U_i$ there exists an arc $\alpha_i \subset A_\infty \cap \operatorname{int} U_i$ such that the distance from $x$ to the boundary of $U_i$ is less than $\varepsilon$ for all $x \in \alpha_i$.

2. There exists an integer $m$ such that if $A$ is a component of $A_m$ then $A \subset \text{int } U_j$ for some $j$.

<u>Proof</u>: For each arc $\alpha \in A_\infty$ let $N_\alpha$ be the $\frac{\varepsilon}{2}$-radial neighborhood of $\alpha$. For each $N_\alpha$ there exists a neighborhood $V_\alpha \subset N_\alpha$ with the property that if an arc $\beta \subset A_\infty$ intersects $V_\alpha$ non-trivially then $\beta \subset N_\alpha$. The existence of such $V_\alpha$'s follows from the fact that the decomposition of $E^n$ into the arcs of $A_\infty$ and the points of $E^n - A_\infty$ is an upper semi-continuous decomposition. The collection of sets $\{V_\alpha | \alpha \subset A_\infty\}$ is an open cover of the compact set $A_\infty$. Thus there is a finite subcollection $V_1, V_2, \ldots, V_p$ which cover $A_\infty$. Let $N_1, N_2, \ldots, N_p$ be the corresponding $N_\alpha$'s. Note that by the choice of the $V_\alpha$'s we have each arc $\alpha \subset A_\infty$ contained in the interior of at least one $N_i$. For each arc $\alpha \subset A_\infty$ there exists an integer $m(\alpha)$ such that

1. $\alpha \subset A^\alpha_{m(\alpha)} \subset A_{m(\alpha)}$, where $A^\alpha_{m(\alpha)}$ is the component of $A_{m(\alpha)}$ containing $\alpha$;
2. $A^\alpha_{m(\alpha)} \subset \text{int } N_i$, for some $i = 1,2, \ldots, p$.

The collection $\{\text{int } A^\alpha_{m(\alpha)} | \alpha \subset A_\infty\}$ is an open cover of $A_\infty$. Therefore there is a finite subcover. From this collection of $A^\alpha_{m(\alpha)}$'s there is one with largest subscript $m(\alpha)$. $m = m(\alpha)$ is the desired integer. Each $N_i$ is the $\frac{\varepsilon}{2}$-radial neighborhood of some $\alpha \subset A_\infty$. Therefore the collection $U_i = N_i$ satisfies the claims of the Lemma.

<u>Lemma 2.2</u>: Suppose $A_i$, $i = 1,2, \ldots,$ are defined as above and $A$ is a component of $A_r$ for some $r$. Given $\varepsilon > 0$ then there exist integers $\gamma(1), \gamma(2), \ldots, \gamma(m+1)$ and sets $K_{ij} \subset A \times E^1$ $i = 1,2, \ldots, s;$ $j = 1,2, \ldots, m,$ which satisfy the following conditions.

1. For each $i$, $K_{io}$ is an $(n+1)$-cell and $K_{ij}$ is the disjoint union of $(n+1)$-cells $K_{ijk}$, $k = 1,2, \ldots, \mu(i,j)$.
2. $K_{io} \cap K_{eo} = \phi$ if $i \neq e$.
3. $\bigcup_i K_{ij} \subset (A_{\gamma(j)} \cap A) \times [j, 2m+1-j]$
   $(\text{int } A_{\gamma(j)} \cap A) \times (j, 2m+1-j) \supset \bigcup_i K_{ij+1}$
   for each $j$
4. For each $i$ $K_{io}$ can be written as the union of $(n+1)$-cells $D_{ie}$, $e = 0, 1, \ldots, m,$ such that $D_{ie} \cap D_{iv} = \text{Bdy } D_{ie} \cap \text{Bdy } D_{iv}$ is an $n$-cell if $|e - v| = 1$ and is void if $|e - v| > 1$.
5. Diameter of $\pi_n^*(D_{ie}) < \varepsilon$ for all $i,e$, where $\pi_n^*$ is the projection $E^n \times E^1 \longrightarrow E^n$.
6. $D_{ie} \cap D_{iv} \cap K_{ijk}$ is either void or an $n$-cell.

<u>Proof</u>: Let the $\varepsilon$ of Lemma 2.1 be the $\min(\varepsilon,$ distance from $A_\infty \cap A$ to Bdy $A)$ hence there exists a finite set of $n$-cells $K_{io}'$, $i = 1,2, \ldots, s$ and an integer $\gamma(1)$ satisfying;

a. $K_{io}' \subset \text{int } A$ for all $i$.
b. If $A'$ is a component of $A_{\gamma(1)} \cap A$ then $A' \subset \text{int } K_{io}'$ for some $i$.

Note that the $K'_{io}$ may not be disjoint. By Remark 2.2 each n-cell $K'_{io}$ can be chosen so that there is a finite set of planes $R_{ij}$, $j = 1,2, \ldots, m_i$ which cut $K'_{io}$ into $(m_i + 1)$ n-cells $D'_{ij}$ such that

$$D'_{ij} \cap D'_{iv} = \text{Bdy } D'_{ij} \cap \text{Bdy } D'_{iv}$$

is an $(n - 1)$-cell if $|j - v| = 1$ and is void if $|j - v| > 1$. Without loss of generality assume $m_i = m$ for all $i$.

Similarly apply Lemma 2.1 to each component of $A_{\gamma(1)} \cap A$ to obtain an integer $\gamma(2)$ and sets $K'_{i1}$, where $K'_{i1}$ is the union of n-cells $K'_{i1k}$, $k = 1,2,\ldots \mu(i,1)$, satisfying;

i. If $A^*$ is a component of $A_{\gamma(2)} \cap A$ then

$$A^* \subset \text{int } K'_{i1k} \subset K'_{i1k} \subset \text{int } A' \subset A' \subset \text{int } K'_{io}$$

for some $k$ and some component $A'$ of $A_{\gamma(1)} \cap A$.

ii. $K'_{i1k} \cap R_{ij}$ is either void or an $(n - 1)$-cell.

Condition ii actually follows from the proof of Lemma 2.1. Continue this procedure to obtain the integers $\gamma(3)$, $\gamma(4)$. $\ldots$, $\gamma(m + 1)$ and sets $K'_{ij}$ and n-cells $K'_{ijk}$ satisfying conditions analogous to i and ii above.

For each $i$ and $j$ define $W_{ij\ell}$ to be the union of the components of $A_{\gamma(j+1)} \cap A$ which are contained in $K'_{ij\ell}$ but not in $K'_{ijp}$ for any $p < \ell$. Note that $W_{ij\ell}$ are compact and $W_{ijk} \cap W_{ij\ell} = \phi$ if $k \neq \ell$. Let $\{W_{io1}\}$ and $\{K'_{io}\}$ be respectively $\{C_i\}$ and $\{D_i\}$ of Theorem 1.1 and let $a - \varepsilon = 0$ and $b + \varepsilon = 2m + 1$. Then define $K_{io} = E_i$ of Theorem 1.1. By the proof of Theorem 1.1 we see that $K_{io}$ can be written as the union of $(n + 1)$-cells $D_{i\ell}$ such that $\pi_n^* D_{i\ell} = D'_{i\ell}$. Further the $D_{i\ell}$ satisfy condition 4.

In general let $\{W_{ijk}\}$ and $\{K'_{ijk}\}$ be respectively $\{C_{ik}\}$ and $\{D_{ik}\}$ of Theorem 1.1 and let $\varepsilon = \frac{1}{2}$, $a = j$ and $b = 2m + 1 - j$. If $K_{ijk} = E_{ik}$ of Theorem 1.1 and $K_{ij} = \bigcup_k E_{ik}$ then conditions 1 through 5 are clearly satisfied and condition 6 follows from ii above.

Remark 2.3: Note that if $i \neq r$ and $A'$ is a component of $A_{\gamma(j+1)} \cap A$ contained in $K'_{rj}$ then $K_{i,j+1} \cap A' \times E^1 = \Phi$ since $K'_{ij} \cap K'_{rj} \subset A - A_{\gamma(j+1)}$. Also $K'_{ijp} \cap K'_{ijq} = \Phi$ if they are not contained in the same n-cell of $K'_{ij-1}$.

The proof of the next lemma is based on the following known result.

Theorem: Suppose that $A$ is an n-cell which is the union of two n-cells $A_1$ and $A_2$ with the properties that $A_1 \cap A_2$ and Bdy $A_1 \cap$ Bdy $A_2$ are (n-1)-cells and $A_1 \cap A_2 \subset$ Bdy $A_1 \cap$ Bdy $A_2$. If $B \subset A$, $B$ is compact and $B \cap \text{Bdy } A \subset A_2$ then there exists a homeomorphism $h$ of $A$ onto $A$ which is fixed on the Bdy $A$ and such that $h(B) \subset A_2$.

Lemma 2.3: For $\varepsilon > 0$ and $A$ a component of $A_r$ (where $A_i$ $i = 1,2,\ldots$ are defined as above) let $\gamma(f)$, $D_{i\ell}$, $K_{ij}$, and $K_{ijk}$ be defined as in Lemma 2.2. Then there exists a homeomorphism $h : E^n \times E^1 \longrightarrow E^n \times E^1$ such that the following hold.

1. $h = id$ on the complement of $\bigcup_i K_{i1}$.

2. $h = id$ on the complement of

$$\bigcup_i([K_{i1} \cap (D_{io} \cup D_{i1})] \cup [K_{i2} \cap (D_{i1} \cap D_{i2})] \cup \ldots \cup [K_{im} \cap (D_{im-1} \cup D_{im})])$$

3. If $A' \subset A_{\gamma(j+1)} \cap A$ and $K_{ijk} \supset A' \times [j, 2m+1-j]$ then

$$h((D_{io} \cup \ldots \cup D_{ij}) \cap A' \times ([j,j+1] \cup [2m-j, 2m-j+1])) \subset D_{i\Psi} \cup D_{i\Psi+1}$$

where $\Psi = \min(j, \max\{e | K_{iek} \cap D_{ie} \neq \Phi, K_{iek} \supset A'\})$.

Before reading the proof of Lemma 2.3 it may be helpful to look at Figures 1 and 2. The homeomorphism $h$ will be obtained as the composition of homeomorphisms $h_{m-1} \cdot h_{m-2} \cdot \ldots \cdot h_1$. Figure 1 illustrates how the $h_j$ will be constructed. The shaded region of Figure 2 is that part of $A \times [0, 2m+1]$ which is not moved by $h$.

__Proof__: Let $h_1 : E^n \times E^1 \longrightarrow E^n \times E^1$ be a homeomorphism defined as follows

$$h_1 = id \text{ on } E^n \times E^1 - \bigcup_i (K_{i1} \cap (D_{io} \cup D_{i1}))$$

For each $i$ and $A'$ a component of $A_{\gamma(2)}$ with $A' \times [1,2m] \subset K_{i1k}$, then

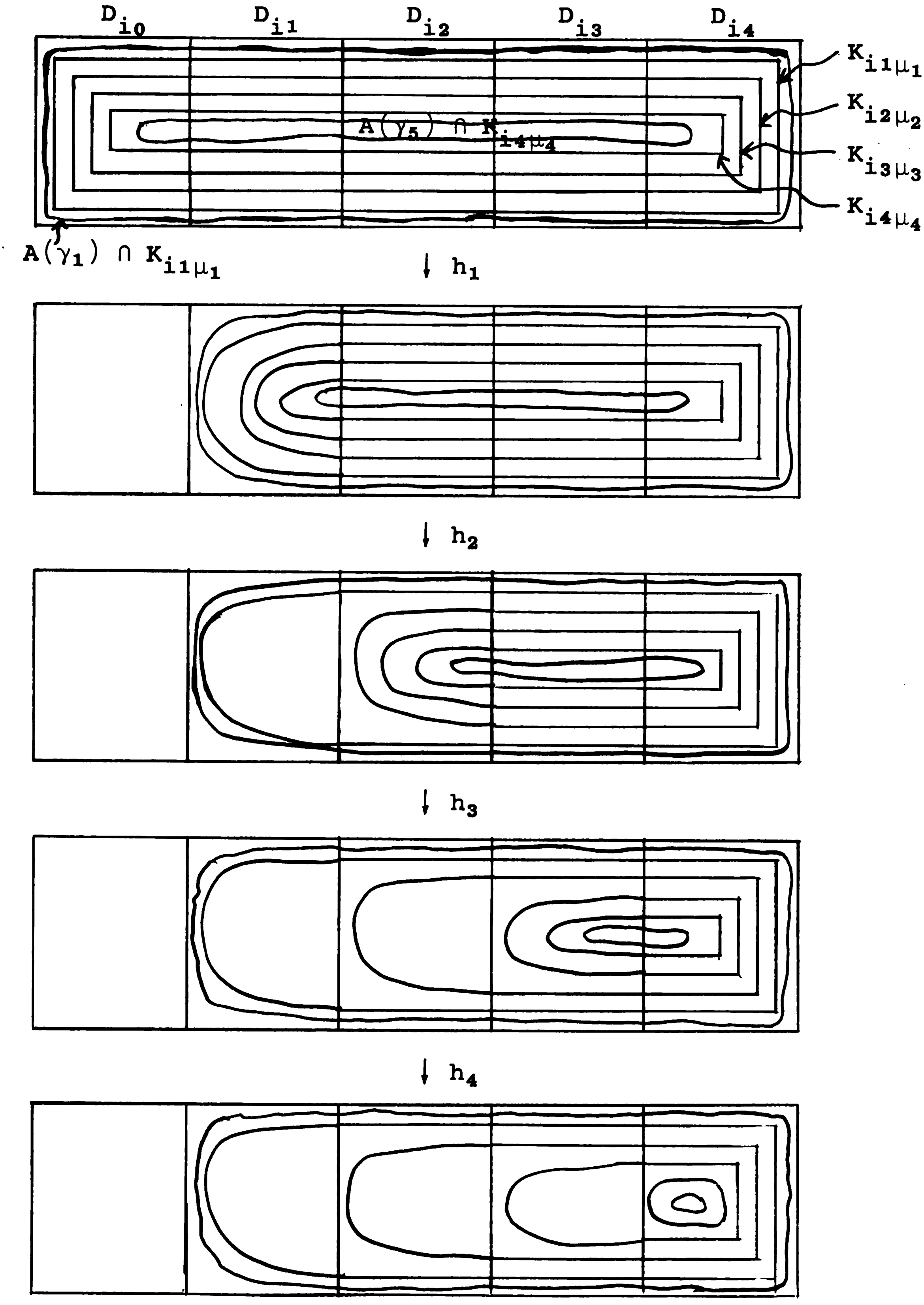
a. If $K_{i1k} \cap D_{io} = \Phi$ or $K_{i1k} \cap D_{i1} = \Phi$ then $h_1 = id$ on $K_{i1k}$

b. If $K_{i1k} \cap K_{io} \cap D_{i1} \neq \Phi$ then

$$h_1 = id \text{ on } \operatorname{Bdy} K_{i1} \cap (D_{io} \cup D_{i1})$$

$$h_1(A' \times [1,2m] \cap (D_{io} \cup D_{i1})) \subset D_{i1}$$

$h_1$ as defined exists since $A' \times [1,2m]$ is compact,



$D_{i0}$
$D_{i1}$
$D_{i2}$
$D_{i3}$
$D_{i4}$
$K_{i1\mu_1}$
$K_{i2\mu_2}$
$K_{i3\mu_3}$
$K_{i4\mu_4}$
$A(\gamma_5) \cap K_{i4\mu_4}$
$A(\gamma_1) \cap K_{i1\mu_1}$
$h_1$
$h_2$
$h_3$
$h_4$


Figure 1.

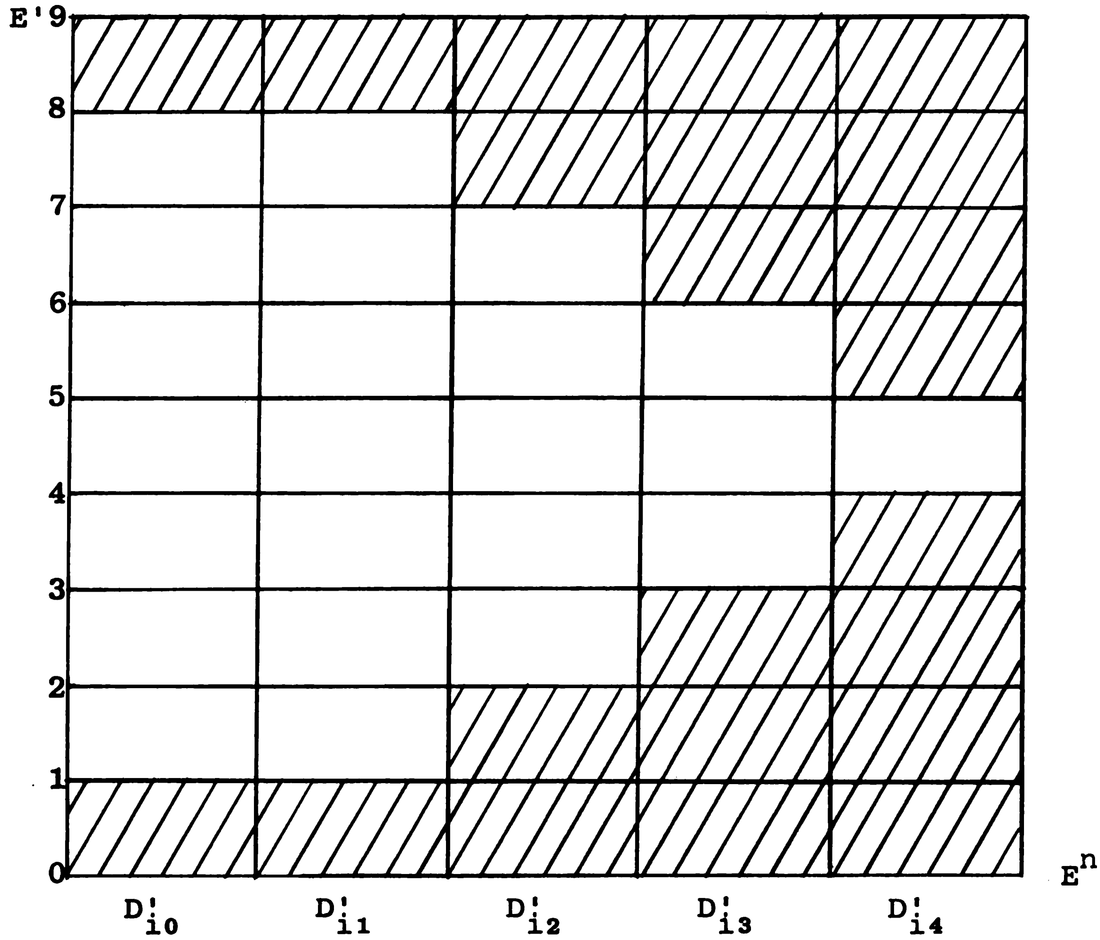



Figure 2.

$K_{i1k} \cap (D_{io} \cup D_{i1})$ is the union of two $(n + 1)$-cells which intersect in an n-cell in their common boundary and $A' \times [1,2m] \cap \mathrm{Bdy}\, [K_{ijk} \cap (D_{io} \cup D_{i1})]$ is contained in $D_{i1}$ and $K_{ijk} \cap K_{ij\ell} = \Phi$ if $k \neq \ell$.

Now proceed inductively to define $h_j$ for $j = 2, 3, \ldots, m-1$. As a notational aid define $L_{ij} = (D_{io} \cup D_{i1} \cup \ldots \cup D_{ij}) \cap K_{ij}$. Define

$h_j : E^n \times E^1 \longrightarrow E^n \times E^1$ as follows

$h_j = \mathrm{id}$ on $E^n \times E^1 - (h_{j-1} \cdot h_{j-2} \cdot \ldots \cdot h_1 (\bigcup_i L_{ij}))$

For each $i$ and $A'$ a component of $A_{\gamma(j+1)} \cap A$ with $A' \times [j, 2m+1-j] \subset K_{ijk}$ then

a. If $H \cap D_{ij-1} = \Phi$ or $H \cap D_{ij} = \Phi$ then $h_j = \mathrm{id}$ on $H$ where $H = (h_{j-1} \cdot h_{j-2} \cdot \ldots \cdot h_1 (K_{ijk}))$.

b. If $H \cap D_{ij-1} \cap D_{ij} \neq \Phi$ then

$$h_j (h_{j-1} \cdot h_{j-2} \cdot \ldots \cdot h_1 (A' \times [j, 2m+1-j] \cap D_{ij-1} \cup D_{ij}))) \subset D_{ij}.$$

$h_j$ exists since $A' \times [j, 2m\ 1-j]$ is compact, $h_{j-1} \cdot \ldots \cdot h_1 (A' \times [j, 2m+1-j] \cap (D_{ij-1} \cup D_{ij}))$ is the union of two $(n + 1)$-cells which intersect in an n-cell in their common boundary and

$$h_{j-1} \cdot \ldots \cdot h_1 (A' \times [j, 2m+1-j]) \cap \mathrm{Bdy}(h_{j-1} \cdot \ldots \cdot h_1 [k_{ijk} \cap (D_{ij-1} \cup D_{ij})])$$

is contained in $D_{ij}$.

Define $h : E^n \times E^1 \to E^n \times E^1$ as $h = h_{m-1} \cdot h_{m-2} \cdot \ldots \cdot h_1$. Clearly conditions 1 and 2 are satisfied by $h$. To see that condition 3 is satisfied let

$x \in (A_{\gamma(j+1)} \cap A) \times ([j,j+1] \cup [2m-j,2m+1-j])$.

There exists some component $A' \subset A_{\gamma(j+1)} \cap A$ such that $x \in A' \times ([j,j+1] \cup [2m-j,2m+1-j])$ and a unique $k_{ijk}$ containing $x$. Let $\Psi = \min(j, \max\{e \mid K_{iek} \cap D_{ie} \neq \Phi, K_{iek} \supset A'\})$.

Case 1. If $\Psi < j$ then

$h(x) = h_{m-1} \ldots h_{\Psi} \ldots h_1(x) = h_{\Psi} \ldots h_1(x) \subset D_{i\Psi} \cup D_{i\Psi+1}$ .

Case 2. If $\Psi = j$ then

$$h(x) = h_{m-1} \ldots h_{j+1} \cdot h_j \ldots h_1(x) = h_{j+1}\, h_j \ldots h_1(x) \subset D_{ij} \cup D_{ij+1}$$

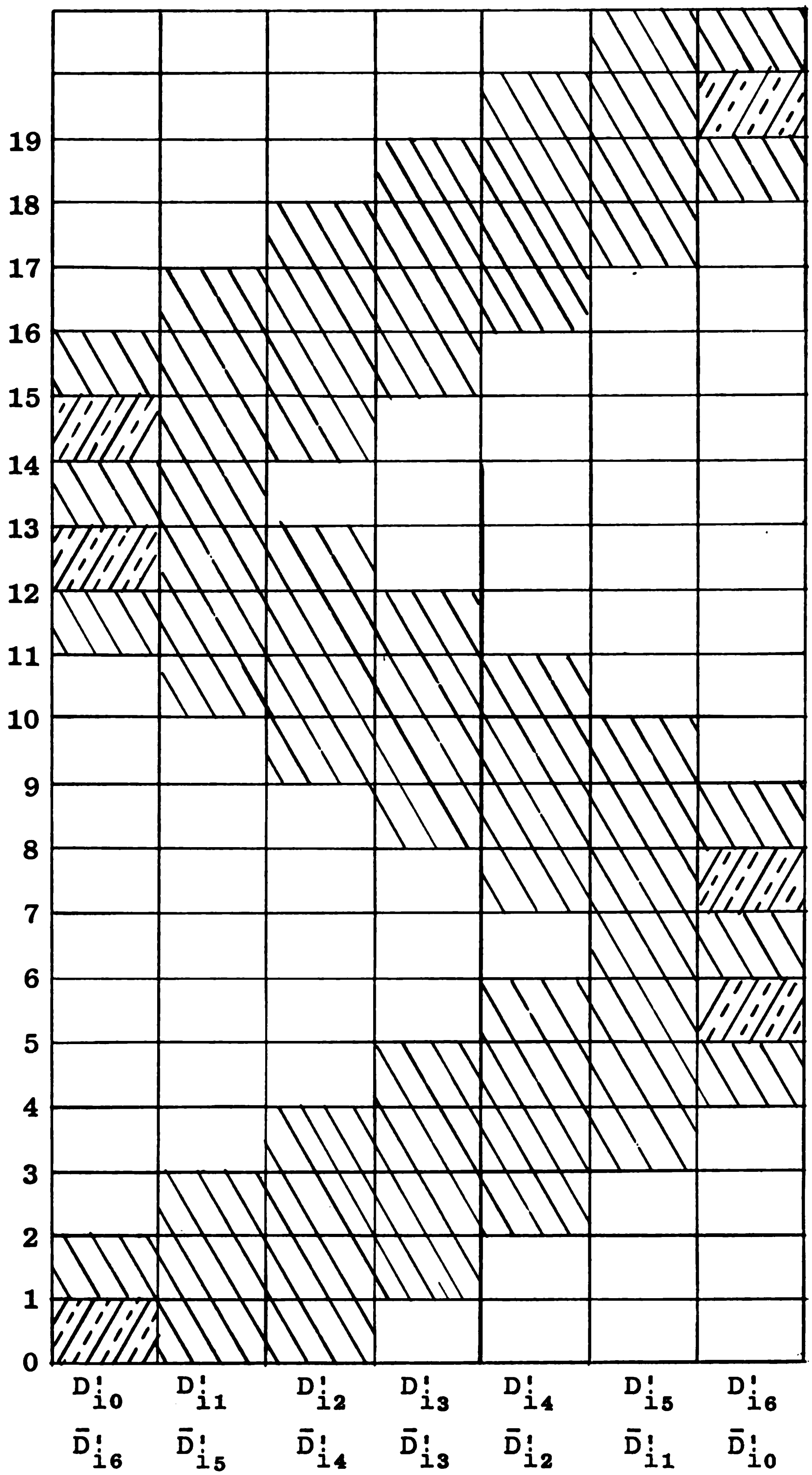
<u>Lemma 2.4:</u> Suppose $\varepsilon > 0$ and $A$ is a component of $A_r$ (where $A_i$ $i = 1,2, \ldots.$ are defined as above). Then there exists an integer $N$ and a uniformly continuous homeomorphism $h : E^n \times E^1 \longrightarrow E^n \times E^1$ which is the identity on $E^{n+1} - (A \times E^1)$ and such that for each $w \in E'$

(1) $\pi_{n+1}(h(A \times w)) \subset [w - 2m - 1, w + 2m + 1]$.

(2) $\operatorname{diam} (\pi_n^* (A' \times w)) < 4\varepsilon$,

where $A'$ is a component of $A_n \cap A$, $\pi_{n+1}$ is the projection of $E^n \times E^1$ onto $E^1$, and $\pi_n^*$ is the projection onto $E^n$.

Figure 3 shows how to apply Lemma 2.3 to prove Lemma 2.4. In Figure 3 only one sequence $K_{i1k}, K_{i2k}, \ldots, K_{im+1\,k}$, containing a component of $A_N \cap A$, is shown. The $(n + 1)$-cells in the figure are shown as if they intersect each of the $(n + 1)$-cells $D_{io}, D_{i1}, \ldots, D_{im}$. This may not be the case, however, an analogous figure is obvious.


19
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4
3
2
1
0
D'i0
D'i1
D'i2
D'i3
D'i4
D'i5
D'i6
D̄'i6
D̄'i5
D̄'i4
D̄'i3
D̄'i2
D̄'i1
D̄'i0

Figure 3

Proof: Apply Lemma 2.3 to $A \times E^1$ and integers $m$ and $\gamma(m+1)$ and sets $D_{i\ell}$, $K_{ij}$ and $K_{ijk}$. Set $N = \gamma(m+1)$ for $g = 0, \pm 1, \pm 2, \ldots$ let

$$x_g = g(2m+2) \qquad x'_g = x_g + m + 1$$
$$y_g = g(2m+2) + 2m + 1 \qquad y'_g = y_g + m + 1.$$

Note that $D_{i\ell}, K_{ij}, K_{ijk} \subset A \times [x_0, y_0]$, by suitable translations of $E^1$ we get sets analogous to $D_{i\ell}$, $K_{i\ell}$, and $K_{ijk}$ in $A \times [x_g, y_g]$ for each $g$. Apply Lemma 2.3 to $A \times [x_g, y_g]$ for each $g$. Define $\overline{D}_{i\ell} = D_{im-\ell}$ and apply Lemma 2.3 to $A \times [x'_\ell, y'_\ell]$ using $\overline{D}_{i\ell}$ in place of $D_{i\ell}$. Thus there exists a homeomorphism $h: E^n \times E^1 \to E^n \times E^1$ which is uniformly continuous. By the choice of $x_g$, $y_g$, $x'_g$ and $y'_g$ and Remark 2.3 there exist integers $i$ and $k$ such that

$$\pi_n^* h(A' \times w) \subset \pi_n^*(D_{ik} \cup D_{ik+1} \cup \overline{D}_{im-k+2} \cup \overline{D}_{im-k+3})$$

for each component $A' \subset A \cap A_N$ and $w \in E^1$. Note that $i$ and $k$ depend on $A'$ and $w$. Diam $\pi_n^*(D_{i\ell}) < \varepsilon$ for all $i$ and $\ell$. Thus condition 2 is satisfied. For $w \in E^1$ there exist $x_g$, $y_g$, $x'_{g+\delta}$ and $y'_{g+\delta}$, where $\delta = 0, -1$, such that $w \in [x_g, y_g] \cap [x'_{g+\delta}, y'_{g+\delta}]$. Thus

$$\pi_{n+1}(A' \times w) \subset [x_g, y_g] \cup [x'_{g+\delta}, y'_{g+\delta}]$$

ard condition 1 is satisfied.

Theorem 2.5: For each component $A \subset A_r$ (where $A_i$, $i = 1,2,\ldots$ are as defined above) and each $\varepsilon > 0$ there exists an integer $N$ and an uniformly continuous homeomorphism $h : E^n \times E^1 \to E^n \times E^1$ such that

1. $h = id$ on $E^{n+1} - A \times E^1$

2. $|\pi_{n+1}[h(x)] - \pi_{n+1}(x)| < \varepsilon$

3. For each $w \in E^1$ diameter of each component of $A_N \times w$ is less than $\varepsilon$.

Proof: Let $\varepsilon' = \frac{\varepsilon}{8}$ then by Lemma 2.4 there exists a uniformly continuous homeomorphism $h_1$ and an integer $N$ satisfying

a. $h_1 = id$ on $E^{n+1} - A \times E^1$

b. $|\pi_{n+1}h_1(x) - \pi_{n+1}(x)| < 4m + 2$ for some positive integer $m$.

c. diam $\pi_n^*(A' \times w) < 4\varepsilon'$

for all $w \in E^1$ and components $A' \subset A_N \cap A$.

Let $h_2$ be the homeomorphism $h_2 : E^N \times E^1 \to E^n \times E^1$ given by

$$h_2(x,t) = (x, \frac{4m + 2}{\varepsilon'} t)$$

The homeomorphism $h = h_2^{-1}h_1 h_2$ is the desired homeomorphism.

The homeomorphism $h$ is isotopic to the identity since the homeomorphisms of Lemmas 2.3 and 2.4 were.

Suppose $A_i$ $i = 1,2, \ldots,$ be a sequence of manifolds as defined above (see page 15). Let $G_n$ be the upper semi-continuous decomposition of $E^n$ into the arcs of $A_\infty$ and the points of $E^n - A_\infty$. Denote the decomposition space of $G_n$ by $X_n$.

<u>Theorem 2.6:</u> $X_n \times E^1 = E^{n+1}$

This theorem follows from Theorem 2.1 and the following theorem which is due to R. H. Bing [4].

<u>Theorem:</u> Let $A_i$ $i = 1,2,\ldots,$ and $X_n$ be defined as above. Further suppose that for each $i$ and $\varepsilon > 0$ there is an integer $N$ and an isotopy $\mu$ of $E^{n+1}$ onto $E^{n+1}$ such that $\mu_0$ is the identity $\mu_1$ is uniformly continuous and

1. $\mu_t = \text{id}$ on $E^{n+1} - (A_j \times E^1)$
2. $|\pi_{n+1}\,\mu_t(x) - \pi_{n+1}(x)| < \varepsilon$ where $\pi_{n+1}$ is the projection of $E^{n+1}$ onto the $(n+1)$-st coordinate.
3. For each $w \in E^1$ the diameter of each component of $\mu_1(A_N \times w)$ is less than $\varepsilon$.

Then $X_n \times E^1 = E^{n+1}$.

<u>Remark 2.4:</u> Note that there exists a countable collection of compact sets $R_i$ such that

1. $A \times E^1 = \bigcup_i R_i$.
2. $h(R_i) \subset R_i$ for all $i = 1,2,\ldots$
3. $h/\text{bdy } R_i = \text{id}$ for all $i = 1,2,\ldots$
4. $\text{diam } [\pi_{n+1}(R_i)] < \varepsilon/8$.
5. $\text{diam } h[R_i \cap (A_n \times E^1)] < \varepsilon/2$.

Where $h$ is the homeomorphism of Theorem 2.5.

# CHAPTER III

## THE "DOGBONE SPACE" SQUARED IS $E^6$

In [7] Kwun showed that there exists two non-manifolds whose product is $E^n$ for $n \geq 6$. In this chapter we give another factorization of $E^6$ into non-manifold factors.

Let $B_i$ be a collection of m-manifolds in $E^m$ which are analogous to the $A_i$ defined in Chapter II. That is $B_i$ $(i = 1,2,\cdots)$ is a collection of compact manifolds in $E^m$ satisfying P1 and P2 (see page 15) and $B_\infty = \bigcap_i B_i$.

Throughout this chapter let $A_i$ $(i = 1,2,\cdots)$ be as defined in Chapter II.

<u>Lemma 3.1</u>: Given $A$ and $B$ components of $A_r$ and $B_s$ respectively and $\varepsilon > 0$ then there exists an integer $N > \max(r,s)$ and a homeomorphism $h: E^n \times E^m \longrightarrow E^n \times E^m$ such that;

1. $h = \text{id}$ on $E^{n+m} - (A \times B)$
2. $\text{Diam}\, h(A' \times B') < \varepsilon$ for each component $A' \subset A_N \cap A$ and $B' \subset B_N \cap B$.

<u>Proof</u>: By Lemma 2.1 there exists integers $J$ and $K$, a set of n-cells $E_1, E_2, \cdots, E_p$, and a set of m-cells $F_1, F_2, \cdots, F_q$ such that

1. $E_i \subset \text{int } A$ for each $i = 1,2,\cdots, p$.
2. $F_j \subset \text{int } B$ for each $j = 1,2,\cdots, q$.

3. For each component $A' \subset A_J \cap A$ there is at least one $i$ such that $A' \subset \text{int } E_i$.

4. For each component $B' \subset B_K \cap B$ there is at least one $j$ such that $B' \subset \text{int } F_j$.

Let $N = \max(J,K)$ and note that for each component $A' \times B' \subset (A_N \times B_N) \cap (A \times B)$ there exist integers $i$ and $j$ such that $A' \times B' \subset \text{int } E_i \times \text{int } F_j$.

By Theorem 1.2 there exists a collection of $(n+m)$-cells $G_1, G_2, \ldots, G_\ell$ such that;

1. For each component $A' \times B'$ of $(A_N \times B_N) \cap (A \times B)$ there exists a unique $k$ such that $A' \times B' \subset \text{int } G_k$ and $A' \times B' \cap G_j = \Phi$ for all $j \neq k$.

2. $G_k \subset \text{int } E_i \times \text{int } F_j \subset A \times B$ for some $i$ and $j$.

Note that even though $i \neq j$ it may be the case that $G_i \cap G_j \neq \Phi$. Since $(\bigcup_{j \neq i} G_j) \cap G_i$ is compact for each $i = 1,2, \ldots, \ell$, there exists an $(n+m)$-cell $Q_i =$

$\{z \in E^{m+n} / \|z - z_i\| \leq \delta$, for some $z_i \in \text{int } G_i$ and $\delta \leq \varepsilon/2\}$ $\subset G_i$ such that $Q_i \cap G_j = \Phi$ for $i \neq j$. For each component $A' \times B' \subset [(A_N \times B_N) \cap (A \times B)]$ there exists an integer $i$ and a homeomorphism $h_i : E^{n+m} \longrightarrow E^{n+m}$ such that

1. $A' \times B' \subset G_i$

2. $h_i = \text{id}$ on $E^{n+m} - G_i$

3. $h_i(A' \times B') \subset Q_i$.

Define $h = h_1 \cdot h_2 \cdot \ldots \cdot h_\ell$. Even though the $G_i$'s are not disjoint, $h_i$ is the identity on $G_j \cap (A_N \times B_N)$

for $j \neq i$. Thus $h$ satisfies conditions 1 and 2 of the theorem.

Remark 3.1: Since the homeomorphism $h$ of Lemma 3.1 is the identity outside a compact set $h$ is uniformly continuous and isotopic to the identity.

Theorem 3.2: Let $A_i$, $i = 1,2, \ldots$; $B_j$, $j = 1,2, \ldots$ be defined as above then there exists a pseudo-isotopy $H : E^{n+m} \times I \longrightarrow E^{n+m}$ such that ;

a. $H(x,0) = x$

b. If $H_t(x) = H(x,t)$ then for all $t < 1$ $H_t$ is a homeomorphism of $E^{n+m}$ onto itself which is the identity outside a compact set.

c. $H_1$ maps $E^{n+m}$ onto itself and maps each component of $A_\infty \times B_\infty$ onto a distinct point.

d. If $x \in E^{m+n} - (A_\infty \times B_\infty)$ then $H_1^{-1}(H_1(x)) = x$.

Proof: Let $\varepsilon_0 = \operatorname{diam}(A_1 \times B_1)$ and $\varepsilon_i = 1/2^i$ for $i = 1,2, \ldots$ . A sequence of integers $1 = N(1), N(2), \ldots$ and isotopies.

$$H^i : E^{n+m} \times [\tfrac{i-1}{i}, \tfrac{i}{i+1}] \longrightarrow E^{n+m}$$

for $i = 1,2, \ldots$ which satisfy

1. $H^1(x,0) = x$

2. $H^{i-1}(x,\frac{i-1}{i}) = H^i(x,\frac{i-1}{i})$ for $i = 2,3, \ldots$

3. diam $H^i(A' \times B', \frac{i}{i+1}) < \varepsilon_i$ for each component $A' \times B' \subset A_{N(i+1)} \times B_{N(i+1)}$

4. $H^i(x,t) = H^{i-1}(x,\frac{i-1}{i})$ for $x \in E^{n+m} - (A_{N(i)} \times B_{N(i)})$ and $i = 2,3, \ldots$ .

5. $||H^i(x,t) - H^i(x,t')|| < \varepsilon_{i-1}$ for all $x \in E^{n+m}$ and $t,t' \in [\frac{i-1}{i},\frac{i}{i+1}]$,

are defined inductively as follows. Let $A_r$ and $B_s$ of Lemma 3.1 be $A_1$ and $B_1$ respectively and let $\varepsilon$ of Lemma 3.1 be $\varepsilon_1$. Then there exists a uniformly continuous isotopy

$$h_1 : E^{n+m} \times I \longrightarrow E^{n+m}$$

and an integer $N(2)$ such that

$$h_1(x,0) = 0,$$

$$\text{diam } h_1(A' \times B',1) < \varepsilon_1 \quad \text{for each component}$$

$$A' \times B' \subset A_{N(2)} \times B_{N(2)},$$

$$h_1(x,t) = x \quad \text{on} \quad E^{n+m} - (A_1 \times B_1).$$

Define $H^1(x,t) = h_1(x,2t)$ , $0 \leq t \leq \frac{1}{2}$ .

Suppose $H^k$ and $N_{k+1}$ are defined. Since $H^k_w$ is uniformly continuous for $w = \frac{k}{k+1}$ there exists a $\delta > 0$ such that if the diameter of $V \subset E^{n+m}$ is less than $\delta$ then the diameter of $H^k_w(V)$ is less than $\varepsilon_{k+1}$. Lemma 3.1 implies the existence of an integer $N_{k+2}$ and an isotopy such that

$$h_{k+1}(x,0) = x \quad \text{on} \quad E^{n+m},$$

$$h_{k+1}(x,t) = x \quad \text{on} \quad E^{n+m} - [A_{N(k+1)} \times B_{N(k+1)}],$$

$$\operatorname{diam}(A^* \times B^*, 1) < \delta \quad \text{for each component}$$

$$A^* \times B^* \subset A_{N(k+2)} \times B_{N(k+2)},$$

and $h_{k+1}$ is uniformly continuous.

Define

$$H^{k+1}(x,t) = H_w^k\, h_{k+1}[x,(k+1)(k+2)(t - \tfrac{k}{k+1})]$$

for $\frac{k}{k+1} \leq t \leq \frac{k+1}{k+2}$. Clearly 1 and 2 are satisfied.

Now

$$H^{k+1}(x,\tfrac{k+1}{k+2}) = H_w^k\, h_{k+1}(x,1)$$

thus by choice of $\delta$ condition 3 is satisfied. $h_{k+1}(x,t) = x$ for $x \in E^{n+m} - [A_{N(k+1)} \times A_{N(k+1)}]$ hence condition 4 is satisfied. $h_{k+1}(A'' \times B'',t) \subset A'' \times B''$ for each component $A'' \times B'' \subset A_{N(k+1)} \times B_{N(k+1)}$. $\operatorname{Diam}[H_w^k(A'' \times B'')] < \varepsilon_k$ by condition 3 thus condition 5 is satisfied.

Define

$$H(x,t) = H^i(x,t) \quad \text{on} \quad E^{n+m} \times [\tfrac{i+1}{i}, \tfrac{i}{i+1}] \quad \text{for}$$

$$i = 1,2, \ldots .$$

Define

$$H_1(x) = \lim_{t \to 1} H(x,t).$$

$H_1(x)$ is continuous map of $E^{n+m}$ onto $E^{n+m}$ by condition 5. Clearly 1 implies that a. is satisfied by H. Condition 4 along with the definition of $H^1$ implies b. is satisfied by H. Suppose $\varepsilon > 0$ and $\alpha \times \beta$ is a component of $A_\infty \times B_\infty$ then there exists an integer p such that $\frac{1}{2^p} = \varepsilon_p < \varepsilon$. For all $t > \frac{p}{p+1}$, $\operatorname{diam} H(A^* \times B^*, t) < \varepsilon_p$

where $A^* \times B^*$ is the component of $A_{N(p)} \times B_{N(p)}$ containing $\alpha \times \beta$. Thus $H(\alpha \times \beta, 1)$ is a point. Let $x \in E^{n-m} - A_\infty \times B_\infty$ then there exists an integer $N(q)$ such that $x \in E^{n+m} - [A_{N(q)} \times B_{N(q)}]$ thus 4 implies that $H(x,t) = H(x, \frac{q-1}{q})$ for all $t > \frac{q-1}{q}$. $H/E^{n+m} \times [0, \frac{q-1}{q}]$ is an isotopy thus $H_1^{-1}[H_1(x)] = x$ and d. is satisfied by H. Let $\alpha_1 \times \beta_1$ and $\alpha_2 \times \beta_2$ be distinct components of $A_\infty \times B_\infty$ then there exists an integer $N(j)$ such that $\alpha_1 \times \beta_1 \subset A' \times B'$ and $\alpha_2 \times \beta_2 \subset A'' \times B''$, where $A' \times B'$ and $A'' \times B''$ are distinct components of $A_{N(j)} \times B_{N(j)}$. Thus $H_1(\alpha_1 \times \beta_1) \neq H_1(\alpha_2 \times \beta_2)$ and c. is satisfied. Therefore H is the desired pseudo-isotopy.

<u>Corollary 3.2</u>: Suppose F is an upper semi-continuous decomposition of $E^{n+m}$ consisting of the 2-cells $\alpha \times \beta$, where $\alpha \subset A_\infty$ and $\beta \subset B_\infty$, and the points of $E^{n+m} - (A_\infty \times B_\infty)$. If Z is the decomposition space associated with F then Z is topologically $E^{n+m}$. Moreover, there exists a uniformly continuous homeomorphism carrying Z onto $E^{n+m}$.

Let $G_n$ be the decomposition of $E^n$ into the arcs of $A_\infty$ and points of $E^n - A_\infty$. Similarly let $G_m$ be the decomposition of $E^m$ into the arcs of $B_\infty$ and points of $E^m - B_\infty$. Suppose that $X_i$ $(i = 1,2)$ is the decomposition space associated with $G_i$ $(i = 1,2)$.

<u>Theorem 3.3</u>: $X_n \times X_m$ is topologically $E^{n+m}$.

Proof: By Corollary 3.2 there exists a pseudo-isotopy $H$ of $E^{n+m}$ onto itself which shrinks each of the 2-cells $\alpha \times \beta$ for $\alpha \subset A_\infty$ and $\beta \subset B_\infty$. Let $f = H_1$. The proof will be completed by constructing a pseudo-isotopy $K$ of $f(E^{n+m})$ onto itself which shrinks each of the arcs $f(\alpha \times y)$, $f(z \times \beta)$ where $\alpha$ is an arc of $A_\infty$, $\beta$ is an arc of $B_\infty$, $z \in E^n$ and $y \in E^m$.

Let

$$U_1 = \bigcup_i f(\text{int } A_i \times [E^m - B_i])$$

and

$$U_2 = \bigcup_i f([E^n - A_i] \times \text{int } B_i).$$

Note that each arc $f(\alpha \times y) \subset U_1$ and $f(z \times \beta) \subset U_2$. Also $U_1 \cap U_2 = \Phi$.

The pseudo-isotopy $K$ can be constructed by amending the construction of the pseudo-isotopy in [7] as follows.

(1) Replace the compact neighborhoods $T_i$ and $T_i'$ with $A_i$ and $B_i$ respectively.

(2) In the proof of the Lemma replace Theorem 1 of [1] with Theorem 2.6 of this thesis. And further replace the $R_i'$ by $R_i$ of Remark 2.4.

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