This is to certify that the

thesis entitled

COEFFICIENTS OF BLOCH FUNCTIONS

presented by

John Joseph Neitzke

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

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Major professor

Date July 30, 1980

COEFFICIENTS OF BLOCH FUNCTIONS

By

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A DISSERTATION

Submitted to

Michigan State University

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1980

ABSTRACT

COEFFICIENTS OF BLOCH FUNCTIONS

By

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A Bloch function is a function analytic in the open unit disc D which satisfies the restrictive growth condition that $(1-|z|^2)|f'(z)|$ is bounded in D. The space of all Bloch functions is denoted by $\mathcal{B}$. The subspace of $\mathcal{B}$ consisting of those functions for which $(1-|z|^2)|f'(z)|$ tends to zero as $|z| \to 1$ is called $\mathcal{B}_0$. The Bloch norm is defined by $\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in D}(1-|z|^2)|f'(z)|$. With this norm, the Bloch functions form a Banach space.

In this dissertation, the coefficients of the power series expansion $\sum_{k=0}^{\infty} a_k z^k$ of functions f in $\mathcal{B}$ and $\mathcal{B}_0$ are examined. Conditions on the coefficients which imply that f is in $\mathcal{B}$ or in $\mathcal{B}_0$ are given, as well as other conditions which are necessary for f to be in $\mathcal{B}$ or in $\mathcal{B}_0$.

Certain facts concerning coefficients of Bloch functions are known. If f is Bloch, the coefficients are bounded but need not tend to zero. If f is in $\mathcal{B}_0$, the coefficients do converge to zero.

It can be shown that for Bloch functions, $\sum_{k=1}^{n} k|a_k|^2 = O(n)$ and $\sum_{k=m}^{2m} |a_k|^2$ is bounded. A natural place to search for conditions which characterize Bloch functions is among expressions similar to these.

Mathews [1] proved that $\sum_{k=1}^{n} k^J |a_k| = O(n^J)$ for some integer $J \geq 1$ is sufficient, and $\sum_{k=1}^{n} k|a_k| = O(n^{3/2})$ is necessary for f to

be a Bloch function, and that the first is also necessary if the coefficients have arguments lying in an interval of length $\pi/2$. Mathews also gave an example of a Bloch function satisfying $\sum_{k=1}^{n} k|a_k| = O(n^P)$, for P between 1 and 3/2.

In this dissertation, we extend one of Mathews' results, that $\sum_{k=1}^{\infty} k|a_k|^2$ being bounded is sufficient for f to be Bloch, to accommodate a wider class of functions. We then state and prove analogues of Mathew's theorems for functions in $\mathcal{B}_0$. We show that if for some $J \geq 1$, $\sum_{k=1}^{n} k^J|a_k| = o(n^J)$, then f is in $\mathcal{B}_0$, and if the arguments of the coefficients lie in an interval of length $\pi/2$, this condition is also necessary. We use this to prove that a Hadamard gap series with coefficients tending to zero is in $\mathcal{B}_0$. We also show that $\sum_{k=1}^{n} k|a_k| = o(n^{3/2})$ is necessary for f to be in $\mathcal{B}_0$.

We then examine conditions involving sums from m to $2m$. If all the a_k are real and nonnegative, then $\sum_{k=m}^{2m} |a_k| \leq M < \infty$ is necessary and sufficient for f to be Bloch, and the convergence of these partial sums to zero is necessary and sufficient for f to be in $\mathcal{B}_0$. Without the real nonnegative restriction, we obtain a series of results which parallel those for sums taken from 1 to n. We prove that f is Bloch if $\sum_{k=m}^{2m} k^{J-1}|a_k| = O(m^{J-1})$ for some integer $J \geq 1$; with the arguments of the coefficients restricted to an interval of length $\pi/2$, this is also necessary. Without this restriction, a necessary condition is that $\sum_{k=m}^{2m} k^{P-1}|a_k| = O(m^{P-1/2})$. For functions in $\mathcal{B}_0$, these results hold with o in place of O.

We give an example of a Bloch function which is bounded with coefficients tending to zero, with the property that if the part of the power series expansion of f from $k = m$ to $k = 2m$ is considered as a function, then for selected m, the sum of the coefficients and the Bloch norm of the function both diverge to infinity as m increases.

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1. J. H. Mathews, Coefficients of uniformly normal-Bloch functions, Yokahama Math. J. 25 (1973), 27-31.

ACKNOWLEDGMENTS

The author wishes to express appreciation to those who have assisted in the writing of this dissertation, especially Dr. Peter Lappan, his major professor. Thanks are also expressed to Dr. Stephen Dragosh and Dr. Thomas McCoy for their careful reading of the manuscript.

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I. PRELIMINARIES

The beginnings of Bloch functions date back to 1925, when André Bloch [6] studied the class G of functions f holomorphic in the unit disc $D = \{z : |z| < 1\}$ with normalization $f'(0) = 1$. A schlicht disc in the Riemann image surface W of f is an open disc Γ in W such that there is an open connected set A in D which f maps one-to-one onto Γ. Let $d_f(z)$ denote the radius of the largest schlicht disc in W, centered at $f(z)$. Set

$$r(f) = \sup_{z \in D} d_f(z)$$

and

$$b = \inf \{r(f) : f \in G\}.$$

Bloch proved that b is positive.

Since then, the work of Bloch has prompted results of various types. Bounds for b were calculated. Other mathematicians generalized Bloch's result to R^n and C^n. Another group considered the function-theoretic implications for the unit disc. More recently, a Banach space of functions related to those examined by Bloch, called the Bloch functions, has been studied [10].

The first activity following Bloch's paper was in the search for bounds for b. Landau first proved that $b > .396$ [15]. Grunsky and Ahlfors proved $b < .472$ [2]. Ahlfors subsequently proved $b > \frac{1}{4}\sqrt{3} > .433$ [1].

Generalizations of Bloch's result to R^n and C^n are found in the papers of Bochner [7], Takahashi [25] and Sakaguchi [23].

Recently, applications have been found for Bloch functions in the study of functions of bounded mean oscillation (BMOA) and vanishing mean oscillation (VMOA).

Since much of the interest in Bloch functions has stemmed from their position as a subset of the space of normal functions, we will state the definition of normal function to show how it relates to the definition of Bloch function.

A function $f(z)$ meromorphic in the unit disc D is called normal if the family $\mathcal{F} = \{f(S(z))\}_{S\in\mathcal{S}}$ is a normal family in the sense of Montel; that is, every sequence of elements of $\mathcal{F}$ has a subsequence which converges uniformly on compact subsets of D, either to a function meromorphic in D or identically to infinity. Here, $\mathcal{S}$ is the set of all conformal self-mappings of D,

$$\mathcal{S} = \left\{S(z) = \mu \frac{a+z}{1+\bar{a}z} : |a| < 1, \ |\mu| = 1\right\}.$$

The following characterization is equivalent and could be used as a definition: A function $f(z)$ meromorphic in D is normal if and only if there is a finite constant M, depending only on $f(z)$, such that

$$\sup\, (1-|z|^2) \frac{|f'(z)|}{1+|f(z)|^2} \leq M \quad [18]. \tag{1}$$

There are several alternative characterizations of Bloch functions. We shall use only the second two below; we include the others for reference.

1. A function f analytic in the unit disc D is a Bloch function if and only if it is finitely normal; that is, if the family $\mathcal{F} = \{f(S(z)) - f(S(0))\}_{S\in\mathcal{S}}$ forms a normal family where the constant infinity is not permitted as a limit [3].

2. Equivalently, a function f analytic in D is a Bloch function if and only if there is a finite constant M such that

(2) $$\sup (1-|z|^2)|f'(z)| \leq M \quad [3].$$

One reason why $(1-|z|^2)|f'(z)|$ is used in (2) rather than, say, $(1-|z|)|f'(z)|$, is that it is an "invariant" form of the derivative: with $S(z) = \frac{a+z}{1+\bar{a}z}$, where $|a| < 1$, the derivative of $f(S(z))$ is

$$f'\left(\frac{a+z}{1+\bar{a}z}\right) \frac{1-\bar{a}a}{(1+\bar{a}z)^2} \quad ;$$

with $z = 0$, this reduces to $(1-|a|^2)|f'(a)|$.

3. The preceding condition (2) may be replaced by

(3) $$|f'(z)| = O((1-|z|)^{-1}) \text{ as } |z| \to 1 \quad [19].$$

4. Defining $d_f(z)$ as previously, $f(z)$ is a Bloch function if and only if

$$\sup d_f(z) < \infty \quad [3].$$

5. $f(z)$ is a Bloch function if and only if its indefinite integral satisfies a certain smoothness condition: Let

$$\phi(z) = z^{-1} \int_0^z f(w)\, dw = \sum_{k=0}^{\infty} a_k (k+1)^{-1} z^k.$$

Then $f(z)$ is a Bloch function if and only if $\phi(z)$ is continuous in $|z| \leq 1$ and

(4) $$\max_{|z| \leq 1} |\phi(e^{it}z) + \phi(e^{-it}z) - 2\phi(z)| = O(t) \text{ as } t \to 0 \quad [3].$$

6. Let G denote the complex linear space of all functions $g(z) = \sum_{n=0}^{\infty} b_n z^n$ analytic in D for which

$$||g||_G = |g(0)| + \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 |g'(re^{i\theta})|\, dr\, d\theta < \infty.$$

Then the space of Bloch functions is the dual of G [3].

Denote the space of all Bloch functions by $\mathcal{B}$. With the Bloch norm, defined by

$$(5) \qquad \|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in D}(1-|z|^2)|f'(z)|,$$

$\mathcal{B}$ is a Banach space. It is clear from a comparison of the definitions that the Bloch functions are contained in the space of normal functions; however, since the sum of two normal functions need not be normal [17], the normal functions are not even a linear space.

In the paper of Pommerenke [21] in which the Bloch functions were defined, the definitions above included the additional restriction $f(0) = 0$. Anderson, Clunie and Pommerenke dropped this assumption; the changes this entails are minor [3].

Let $\mathcal{B}_0$ denote the subspace of $\mathcal{B}$ consisting of all $f(z)$ for which

$$(6) \qquad (1-|z|^2)|f'(z)| \to 0 \quad \text{as} \quad |z| \to 1.$$

As a subspace of $\mathcal{B}$, $\mathcal{B}_0$ is separable, (strongly) closed, nowhere dense, and is the closure of the polynomials in the Bloch norm [3].

We have the following alternative characterizations of $\mathcal{B}_0$: $f(z)$ is in $\mathcal{B}_0$ if and only if $|f'(z)| = o((1-|z|)^{-1})$ as $|z| \to 1$. Equivalently, $f(z)$ is in $\mathcal{B}_0$ if and only if $d_f(z)$ has limit 0 as $|z| \to 1$. Another characterization is that $f(z) \in \mathcal{B}_0$ if and only if the maximum in (4) is equal to $o(t)$ as $t \to 1$ [3].

More recently, Bloch functions have been studied in connection with BMOA and VMOA functions. BMOA, or "bounded mean oscillation" denotes the space of functions $f \in H^2$ for which

$$\|f_w\|_2^2 = \frac{1}{2\pi}\int_{\partial D} |f_w(z)|^2 \frac{1-|w|^2}{|z-w|^2} |dz|$$

is bounded for w in D, where we set

$$f_w(s) = f\left(\frac{s+w}{1+\bar{w}s}\right) - f(w) \qquad (s \in D, w \in D).$$

VMOA, or vanishing mean oscillation, denotes the subspace of functions with $\|f_w\|_2 \to 0$ as $|w| \to 1$. It can be shown that $\mathrm{BMOA} \subset \mathcal{B}$ and $\mathrm{VMOA} \subset \mathcal{B}_0$ [22]. Campbell, Cima and Stephenson [8] gave an example of a Bloch function which is in all H^p classes, but not in BMOA.

As an analytic function in D, a Bloch function $f(z)$ has a power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$. The coefficients $\{a_n\}$ are known to satisfy several conditions. If $f(z) \in \mathcal{B}$, then the coefficients are bounded but need not tend to zero. Hayman [14] raised the question of finding further conditions which imply $a_n \to 0$. This was answered in part by Pommerenke [21]: If the Bloch function $f(z)$ has radial limits almost everywhere, then $a_n \to 0$ as $n \to \infty$.

Pommerenke [21] gave an example of a Bloch function with radial limits almost nowhere. Timoney [26] gave an example of a Bloch function which has radial limits almost everywhere, but which is not in any H^p space for $0 < p \leq \infty$.

Anderson, Clunie and Pommerenke [3] gave two results on power series where the modului a_n of the coefficients are given but the arguments α_n are independent random variables with the uniform distribution on $[0, 2\pi]$. Set

$$f(z,\alpha) = \sum_{n=0}^{\infty} a_n e^{i(\alpha_n)} z^n.$$

As a first result, suppose that

$$\left\{\sum_{k=1}^{n} k^2(a_k)^2\right\}^{1/2} = O(n(\log(n))^{-1/2}).$$

Then $f(z,\alpha)$ is a Bloch function with probability 1. In particular, if $\sum_{n=1}^{\infty} a_n \log(n) < \infty$, then $f(z,\alpha)$ is in $\mathcal{B}_0$ with probability 1.

In the other direction, given a decreasing sequence $\{p_n\}$ of positive numbers, where $p_n \to 0$ as $n \to \infty$, there exists a sequence $\{a_n\}$ with $a_n > 0$ and $\sum_{n=1}^{\infty} a_n{}^2 p_n \log(n) < \infty$ such that with probability 1, $f(z,\alpha)$ is not Bloch.

There is also the following characterization of the Bloch functions by a quadratic form: Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be a Bloch function. Then

$$\left|\sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \frac{a_{u+v+1}}{u+v+1} w_u w_v\right| \le K \sum_{v=0}^{\infty} \frac{|w_v|^2}{2v+1}, \tag{6}$$

where the w_v are complex numbers for $v = 1, 2, \ldots$, and $K = 2\|f\|_{\mathcal{B}}$. The double sum converges if the right-hand sum does. Conversely, (6) implies f is a Bloch function and $\|f\|_{\mathcal{B}} \le 2K$ [3].

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a Bloch function. By (2), there is a number $M < \infty$ such that $|f'(z)| \le M/(1 - |z|^2)$ for all z in D. Then [3]

$$\int_0^p \int_0^{2\pi} |f'(re^{i\theta})|^2 \, d\theta \, r \, dr \le \int_0^p \int_0^{2\pi} \frac{M^2}{(1 - r^2)^2} \, d\theta \, r \, dr. \tag{7}$$

Examining first the right side of (7), we find that it is equal to

$$2\pi \int_0^p \frac{M^2}{(1 - r^2)^2} \, r \, dr =$$

$$= \pi M^2/(1 - r^2) \Big|_0^p$$

$$= \pi M^2 p^2/(1 - p^2).$$

For the left side of (7), Parseval's formula gives

$$2\pi \sum_{k=1}^{\infty} k^2 |a_k|^2 r^{2k-2} = \int_0^{2\pi} |f'(re^{i\theta})|^2 \, d\theta.$$

Thus, the left side of (7) is

$$\int_0^p 2\pi \sum_{k=1}^{\infty} k^2 |a_k|^2 r^{2k-2} r \, dr$$

$$= 2\pi \sum_{k=1}^{\infty} k^2 |a_k|^2 \int_0^p r^{2k-1} \, dr$$

$$= 2\pi \sum_{k=1}^{\infty} k^2 |a_k|^2 p^{2k}/(2k)$$

$$= \pi \sum_{k=1}^{\infty} k |a_k|^2 p^{2k} \quad [3].$$

Thus, (7) reduces to

$$\sum_{k=1}^{\infty} k |a_k|^2 p^{2k} \le M^2 p^2/(1-p^2)$$

or equivalently

$$\sum_{k=1}^{\infty} k |a_k|^2 p^{2(k-1)} (1-p^2) \le M^2.$$

Setting $p^2 = 1 - 1/n$, we obtain

$$\sum_{k=1}^{\infty} k |a_k|^2 (1 - 1/n)^{k-1} n^{-1} \le M^2$$

Since the partial sums of this series are increasing for fixed n, and $(1 - 1/n)^{n-1} \le (1 - 1/n)^{k-1}$ for $k = 1, 2, \ldots , n$,

$$(1 - 1/n)^{n-1}\, n^{-1} \sum_{k=1}^{n} k|a_k|^2 \le \sum_{k=1}^{n} k|a_k|^2 (1 - 1/n)^{k-1}\, n^{-1}.$$

We have

$$(1 - 1/n)^{n-1}\, n^{-1} \sum_{k=1}^{n} k|a_k|^2 \le M^2. \tag{8}$$

Since $(1 - 1/n)^{n-1}$ approaches e^{-1} from above as $n \to \infty$, (8) yields

$$e^{-1}\, n^{-1} \sum_{k=1}^{n} k|a_k|^2 \le M^2$$

or

$$\sum_{k=1}^{n} k|a_k|^2 \le neM^2. \tag{9}$$

Since it is also true that

$$\sum_{k=m}^{2m} k|a_k|^2 \le 2m \sum_{k=m}^{2m} |a_k|^2,$$

and all of the terms in the sum in (9) are positive, we also have

$$\sum_{k=m}^{2m} |a_k|^2 \le 2eM^2. \tag{10}$$

We note that the last inequality is independent of m.

The two inequalities (9) and (10) are conditions on the coefficients of a function which are necessary for that function to be a Bloch function. A natural place to search for sufficient conditions is among inequalities which are similar to but weaker than these two. It is also of interest to see how far these inequalities can be weakened and still remain necessary.

Mathews [19] gave conditions for which, as in (9), the summation is from 1 to n. He proved that

(11) $$\sum_{k=1}^{n} k|a_k| = O(n) \quad \text{as} \quad n \to \infty$$

is a sufficient condition, and

(12) $$\sum_{k=1}^{n} k|a_k| = O(n^{3/2}) \quad \text{as} \quad n \to \infty$$

is a necessary condition for $f(z)$ to be a Bloch function, and if the coefficients are restricted to lie in one sector of the plane with central angle $\pi/2$, (11) is also necessary.

Theorem 1 of Chapter II is a consequence of Mathews result (11). Using the Hölder inequality, we establish that for $1 < q < \infty$,

$$\sum_{k=1}^{\infty} k^{q-1}|a_k|^q < \infty$$

is a sufficient condition for $f(z)$ to be a Bloch function. Two examples follow, illustrating the use of the theorem.

In Chapter III, we prove a series of theorems which are the analogues of Mathews' results for functions in $\mathcal{B}_0$. The first theorem of the chapter shows that if for some integer $J \geq 1$,

(13) $$\sum_{k=1}^{n} k^J|a_k| = o(n^J),$$

then $f(z)$ is in $\mathcal{B}_0$. Two consequences of this are proved. One is the analogue of the theorem of Chapter II, the other is that a Hadamard gap series with coefficients approaching zero as $n \to \infty$ is in $\mathcal{B}_0$. Theorem 5 then shows that if the coefficients of $f(z)$ are constrained to lie in one sector of the plane with central angle $\pi/2$, then (13) is also necessary. An example demonstrates that there are functions in $\mathcal{B}_0$ for which

$$\sum_{k=1}^{n} k|a_k| = o(n^p)$$

for $1 < p < 3/2$. The last theorem of the chapter shows

$$\sum_{k=1}^{n} k|a_k| = o(n^{3/2})$$

is a necessary condition, where the arguments of the a_k's are not restricted.

In Chapter IV, motivated by (10), we examine conditions on "blocks" of coefficients, looking at sums taken from m to $2m$. First we prove that if all of the coefficients are real and nonnegative, then $\sum_{k=m}^{2m} a_k \leq K < \infty$ is a necessary and sufficient condition for $f(z)$ to be Bloch, and $\sum_{k=m}^{2m} a_k \to 0$ is a necessary and sufficient condition for $f(z)$ to be in $\mathcal{B}_0$.

Without the real nonnegative restriction on the coefficients, we obtain a series of theorems which parallel those of Mathews and of Chapter III. The change made from those theorems is, in general, a substitution of the limits m to $2m$ for the limits 1 to n on the finite sums, and removal of a factor of n from the bounds on those sums.

In Chapter V, we give a single example of a bounded Bloch function, with coefficients summing to 1, for which the behavior of sections of the power series from m to $2m$ is bad for selected values of m: Considering this section of the power series as determining a function, as m increases, the sum of the coefficients and the Bloch norm both diverge to infinity.

The last chapter contains a short list of open problems.

II. AN EXTENSION OF A THEOREM OF MATHEWS

Mathews [19] proved that a sufficient condition for $f(z) = \sum_{k=0}^{\infty} a_k z^k$ to be a Bloch function is: If there exists a fixed constant C, $C < \infty$, such that

$$\sum_{k=1}^{\infty} k|a_k|^2 \leq C.$$

The method of proof in Mathews' paper relies on the Cauchy - Schwarz inequality. By using the Hölder inequality, we can establish the following:

<u>Theorem 1</u>: If $\sum_{k=1}^{\infty} k^{q-1}|a_k|^q \leq C$, where $1 < q < \infty$ and C is a positive constant, then $f(z)$ is a Bloch function.

<u>Proof:</u> Set $p = q/(q - 1)$. Then $1 < p < \infty$ and $p + q = pq$. By using the Hölder inequality, we can show that

$$\sum_{k=1}^{n} k|a_k| = \sum_{k=1}^{n} k^{(p-1)/p}\, k^{1/p}\, |a_k|$$

$$\leq \left\{\sum_{k=1}^{n} (k^{(p-1)/p})^p\right\}^{1/p} \cdot \left\{\sum_{k=1}^{n} (k^{1/p})^q\, |a_k|^q\right\}^{1/q}$$

$$= \left\{\sum_{k=1}^{n} k^{p-1}\right\}^{1/p} \cdot \left\{\sum_{k=1}^{n} k^{q-1}\, |a_k|^q\right\}^{1/q}$$

$$= (O(n^p))^{1/p} \cdot \left\{\sum_{k=1}^{n} k^{q-1}\, |a_k|^q\right\}^{1/q}$$

$$= O(n) \cdot C$$

$$\leq BCn$$

for a suitable constant B. By [19, Theorem 2(i)], $f(z)$ is a Bloch function.

We remark that if $q = 2$, the preceding theorem is exactly the theorem of Mathews quoted earlier.

Next, we will give two examples and use the preceding theorem to show that each is indeed a Bloch function. The first uses a value of q near 1, the second employs a large q.

Example 1: Let $\varepsilon > 0$. Then

$$\sum_{k=1}^{\infty} k^{-\varepsilon/(1+\varepsilon)}(k^{1/k} - 1)^{k/(1+\varepsilon)} z^k$$

is a Bloch function.

Proof: Set $q = 1 + \varepsilon$ and $a_k = k^{(1/q)-1}(k^{1/k} - 1)^{k/q}$. Then

$$\sum_{k=1}^{\infty} k^{q-1}|a_k|^q$$

$$= \sum_{k=1}^{\infty} k^{q-1}(k^{(1/q)-1}(k^{1/k} - 1)^{k/q})^q$$

$$= \sum_{k=1}^{\infty} k^{q-1} k^{1-q} (k^{1/k} - 1)^k$$

$$= \sum_{k=1}^{\infty} (k^{1/k} - 1)^k.$$

The last sum is bounded [20, p.219], so by the preceding theorem,

$$\sum_{k=1}^{\infty} k^{(1/q)-1}(k^{1/k} - 1)^{k/q} z^k$$

is a Bloch function. Since $1/q - 1 = -\varepsilon/(1 + \varepsilon)$, this is exactly what we wished to prove.

Example 2: For any $\varepsilon > 0$,

$$\sum_{k=1}^{\infty} (k(\log(k))^{\varepsilon})^{-1} z^k$$

is a Bloch function.

Proof: We have

$$\sum_{k=1}^{\infty} k^{q-1} ((k(\log(k))^{\varepsilon})^{-1})^q$$

$$= \sum_{k=1}^{\infty} (k(\log(k))^{q\varepsilon})^{-1}.$$

This converges for $q > 1/\varepsilon$ [20, p.210], and by Theorem 1, we are done.

III. CONDITIONS ON COEFFICIENTS FOR THE CLASS $\mathcal{B}_0$

In this chapter, we will adapt several results from Mathews' paper, stated for Bloch functions, to functions in $\mathcal{B}_0$. In the theorem following, we will give a proof, using Cauchy's formula, of the known fact [3] that the coefficients of a function in $\mathcal{B}_0$ tend to zero.

<u>Theorem 2</u>: Let $f(z)$ be a function in $\mathcal{B}_0$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Then $a_k \to 0$.

<u>Proof</u>: From Cauchy's formula,

$$
\begin{aligned}
|a_k| &= \left| \frac{1}{2\pi i k} \int_{|z|=r} \frac{f'(z)}{z^k}\, dz \right| \\
&\le \frac{1}{2\pi k} \int_0^{2\pi} \left| \frac{f'(re^{i\theta})}{r^k e^{ik\theta}}\, rie^{i\theta} \right| d\theta \\
&= \frac{1}{2\pi k}\, o((1-r)^{-1}) r^{1-k} \int_0^{2\pi} \left| e^{-i(k-1)\theta} \right| d\theta \\
&= k^{-1}\, o((1-r)^{-1})\, r^{1-k} \\
&= o(k^{-1}\, r^{1-k}\, (1-r)^{-1}).
\end{aligned}
$$

The minimum value of the last term occurs for $r = 1 - k^{-1}$. Evaluating for this r, we obtain

$$\frac{1}{k} \left(1 - \frac{1}{k}\right)^{1-k} \left(\frac{1}{k}\right)^{-1}$$

which tends to e^{-1} as $k \to \infty$. Thus $a_k \to 0$.

In Theorem 3, we will establish conditions sufficient for a holomorphic function to be in the class $\mathcal{B}_0$. We will then investigate necessary conditions.

Theorem 3: We have the following sufficient conditions:

(i). Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$. If there exists a fixed integer J, $J \geq 1$, such that

$$\sum_{k=1}^{n} k^J |a_k| = o(n^J),$$

then $f(z)$ is in $\mathcal{B}_0$.

(ii). Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$. If there exists a p, $1 < p < \infty$, and a w, $w < p - 1$, such that

$$(14) \qquad \sum_{k=1}^{n} k^{(p-w)/(p-1)} |a_k|^{p/(p-1)} = o(n^{(p-w-1)/(p-1)}),$$

then $f(z)$ is in $\mathcal{B}_0$.

(iii). Let $\alpha > 1$, and let $\{k_j\}$ be a sequence for which $k_j > \alpha k_{j-1}$. If $a_{k_j} \to 0$, then

$$\sum_{j=0}^{\infty} a_{k_j} (z)^{k_j} \in \mathcal{B}_0$$

We shall need the following lemmas, which are extensions of a theorem of Titchmarsh [28, p.224].

Lemma 1: Let $h(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$, where $a_k \geq 0$, $b_k \geq 0$, and both series converge for $|x| < 1$ and diverge for $|x| \geq 1$. If $a_k = o(b_k)$ as $k \to \infty$, then $h(x) = o(g(x))$ as $x \to 1$.

Proof: Given $\varepsilon > 0$, there exists an integer N such that $a_n < \frac{\varepsilon}{2} b_n$ for $n \geq N$. Then

$$h(x) = \sum_{k=0}^{\infty} a_k x^k =$$

$$= \sum_{k=0}^{N} a_k x^k + \sum_{k=N+1}^{\infty} a_k x^k$$

$$\leq \sum_{k=0}^{N} a_k x^k + \frac{\varepsilon}{2} \sum_{k=N+1}^{\infty} b_k x^k$$

$$\leq \sum_{k=0}^{N} a_k x^k + \frac{\varepsilon}{2}\, g(x).$$

Let N be fixed. Since $g(x) \to \infty$ as $x \to 1$, we may choose $\delta > 0$ so that $\sum_{k=0}^{N} a_k x^k < \frac{\varepsilon}{2} g(x)$ for $x > 1 - \delta$. Thus $h(x) < \varepsilon g(x)$ for $x > 1 - \delta$, or $h(x)/g(x) < \varepsilon$ for $x > 1 - \delta$. Since ε is arbitrary, $h(x) = o(g(x))$ as $x \to 1$.

Lemma 2: Let $h(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$ both converge for $|x| < 1$ and diverge for $|x| \geq 1$, and set $s_n = \sum_{k=0}^{n} a_k$ and $t_n = \sum_{k=0}^{n} b_k$. Suppose that s_n and t_n are positive, $\sum s_n$ and $\sum t_n$ are divergent, and $s_n = o(t_n)$. Then $h(x) = o(g(x))$.

Proof: We have $h(x) = (1 - x) \sum_{k=0}^{n} s_k x^k$ and $g(x) = (1 - x) \sum_{k=0}^{n} t_k x^k$. By the preceding lemma,

$$\sum_{n=0}^{\infty} s_n x^n = o\left(\sum_{n=0}^{\infty} t_n x^n\right),$$

and the result is immediate.

We remark that if, in particular, $s_n = o(n)$, then $h(x) = o((1 - x)^{-1})$.

We will now prove Theorem 3.

Proof of (i): Set $s_n = \sum_{k=1}^{n} k^J a_k$ and $t_n = n^J$. For $J = 1$, $s_n = o(n)$, so by the remark above, with $h(x) = \sum_{k=1}^{\infty} k|a_k|x^k$ and $x = |z|$, $|f'(z)| = o((1 - |z|)^{-1})$. For $J > 1$, we can show

$$f^{(J)}(z) = o((1 - |z|^J)^{-1}).$$

Successive integration then completes the proof of part (i)

Proof of (ii): Suppose $1 < p < \infty$, $p + q = pq$, and $w < p - 1$. Applying the Hölder inequality,

$$\sum_{k=1}^{n} k|a_k|$$

$$= \sum_{k=1}^{n} k^{w/p}\, k^{(p-w)/p}\, |a_k|$$

$$\leq \left(\sum_{k=1}^{n} (k^{w/p})^p\right)^{1/p} \cdot \left(\sum_{k=1}^{n} (k^{(p-w)/p})^q\, |a_k|^q\right)^{1/q}$$

$$= (O(n^{w+1}))^{1/p} \cdot \left(\sum_{k=1}^{n} k^{((p-w)/p)\cdot(p/(p-1))} |a_k|^{p/(p-1)}\right)^{(p-1)/p}$$

$$= O(n^{(w+1)/p}) \cdot \left(\sum_{k=1}^{n} k^{(p-w)/(p-1)} |a_k|^{p/(p-1)}\right)^{(p-1)/p}$$

$$= O(n^{(w+1)/p}) \cdot (o(n^{(p-w-1)/(p-1)}))^{(p-1)/p}$$

$$= O(n^{(w+1)/p}) \cdot o(n^{(p-w-1)/p})$$

$$= o(n^{(w+1)/p + (p-w-1)/p})$$

$$= o(n)$$

An application of the first part of this theorem now completes the proof of part (ii).

We note that if $p = 2$ and $w = 0$, then $f(z) \in \mathcal{B}_0$ if

$$\sum_{k=1}^{n} k^2\, |a_k|^2 = o(n).$$

Proof of (iii): Since $\alpha k_{j-1} < k_j$, we have $k_{j-1} < \frac{1}{\alpha} k_j$, so $k_j - k_{j-1} > k_j - \frac{1}{\alpha} k_j = (1 - \frac{1}{\alpha})k_j$. We also note that, writing $k(j)$ instead of k_j to make the formulas more readable,

$$(1 - \frac{1}{\alpha})\ k(j)|z|^{k(j)-1}$$

$$< (k(j) - k(j-1))|z|^{k(j)-1}$$

$$< |z|^{k(j-1)} + |z|^{k(j-1)+1} + \dots + |z|^{k(j)-1}.$$

Let $\varepsilon > 0$, and let J be such that for $j \geq J$, $a_{k(j)} < \frac{\varepsilon}{2}(1 - \frac{1}{\alpha})$. Then

$$|f'(z)| = \left|\sum_{j=1}^{\infty} k(j)\ a_{k(j)}\ z^{k(j)}\right|$$

$$\leq \left|\sum_{j=1}^{J-1} k(j)\ a_{k(j)}\ z^{k(j)-1}\right| + \sum_{j=J}^{\infty} k(j)\ |a_{k(j)}||z|^{k(j)-1}.$$

The second sum is bounded by

$$\frac{\varepsilon}{2}(1 - \frac{1}{\alpha}) \sum_{j=J}^{\infty} k(j)\ |z|^{k(j)-1}$$

$$\leq \frac{\varepsilon}{2}(1 - \frac{1}{\alpha}) \sum_{t=k(J-1)}^{\infty} \frac{1}{1 - \frac{1}{\alpha}}\ |z|^{t}$$

$$= \frac{\varepsilon}{2} \sum_{t=k(J-1)}^{\infty} |z|^{t}$$

$$< \frac{\varepsilon}{2}\ \frac{1}{1 - |z|}.$$

Since the first sum is a polynomial of degree $k(J-1)-1$, it is bounded, and hence is less than $\frac{\varepsilon}{2}\ \frac{1}{1 - |z|}$ if $|z|$ is sufficiently near 1. Thus,

$$|f'(z)| < \varepsilon \frac{1}{1 - |z|}$$

for $|z|$ near 1. Since ε is arbitrary, $|f'(z)| = o\left(\frac{1}{1 - |z|}\right)$ as $|z| \to 1$.

We note that the functions in (iii) are called Hadamard gap series. If the coefficients converge to zero, the series is in $\mathcal{B}_0$.

Next we consider necessary conditions on the coefficients. The first theorem in this direction shows when the sufficient condition of Theorem 3, with an additional condition on the arguments of the coefficients, is also necessary.

Theorem 4: Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be in $\mathcal{B}_0$. If there is some α such that for each k, $\alpha \leq \arg(a_k) \leq \alpha + \pi/2$, then

$$\sum_{k=1}^{n} k|a_k| = o(n).$$

Proof: There is no loss of generality in assuming that $\alpha = 0$. Since $f(z)$ is in $\mathcal{B}_0$, we have

$$\left|\sum_{k=0}^{\infty} k\ a_k\ z^{k-1}\right| = |f'(z)| = o((1 - |z|)^{-1})$$

and

$$\left|\sum_{k=0}^{\infty} k\ \overline{a_k}\ z^{k-1}\right| = |\overline{f'(\overline{z})}| = o((1 - |z|)^{-1}).$$

Therefore,

$$\sum_{k=1}^{\infty} k\ \mathrm{Re}(a_k)\ z^{k-1} = o((1 - |z|)^{-1}),$$

where $0 \leq \mathrm{Re}(a_k)$. Similarly,

$$\sum_{k=1}^{\infty} k\ \mathrm{Im}(a_k)\ z^{k-1} = o((1 - |z|)^{-1}),$$

where $0 \leq \mathrm{Im}(a_k)$. Thus,

$$\left|\sum_{k=1}^{\infty} k|a_k|z^{k-1}\right| = 2\ o((1 - |z|)^{-1}) = o((1 - |z|)^{-1}).$$

By [12, Theorem 96], if $c_k \geq 0$, $g(x) = \sum_{k=0}^{\infty} c_k x^k$ and $g(x) = o((1 - x)^{-1})$ as $x \to 1$, then $\sum_{k=0}^{\infty} c_k = o(n)$. Here, we set

$c_{k-1} = k|a_k|$ and note that

$$|f'(z)| \le \left|\sum_{k=1}^{\infty} k\, a_k\, z^{k-1}\right| \le \sum_{k=1}^{\infty} k|a_k|\ |z|^k.$$

Hence $\sum_{k=0}^{n} k|a_k| = o(n)$.

Corollary: If $F(z) = \sum_{k=0}^{\infty} |a_k|z^k$ is in $\mathcal{B}_0$, then so is $f(z) = \sum_{k=0}^{\infty} a_k z^k$.

Example 3: Let $1 < p < 3/2$. There exists a function $f(z)$ in $\mathcal{B}_0$ for which $\sum_{k=1}^{n} k\, a_k = o(n^p)$. Let $\delta > 0$. Then

$$f(z) = \sum_{k=1}^{\infty} k^{p-2-\delta}\, e^{ik^{2p-2}}\, z^k$$

is the required function.

Proof: Hardy [13] has shown that if $F(z) = \sum_{k=1}^{\infty} k^{-b}\, e^{ik^{\alpha}}\, z^k$ where $0 < \alpha < 1$, then F is unbounded if $1 - b - \frac{1}{2}\alpha > 0$, and

$$|F(z)| = 0\left(\frac{1}{(1-|z|)^{1-b-\alpha/2}}\right).$$

Set $\alpha = 2p - 2$, and note that $1 < p < 3/2$, so $0 < \alpha < 1$. Then with $b = 1 - p + \delta$,

$$F(z) = \sum_{k=1}^{\infty} k^{p-1-\delta}\, e^{ik^{2p-2}}\, z^k$$

and

$$|F(z)| = 0\left(\frac{1}{(1-|z|)^{1-(1-p+\delta)-(2p-2)/2}}\right)$$

$$= 0\left(\frac{1}{(1-|z|)^{1-\delta}}\right) =$$

$$= o((1 - |z|)^{-1}).$$

Setting $F(z) = zf'(z)$, it follows that $f(z)$ is in $\mathcal{B}_0$. The coefficients of this function are

$$a_k = k^{p-2-\delta} e^{ik^{2p-2}}$$

so

$$\sum_{k=1}^{n} k|a_k| = \sum_{k=1}^{n} k^{p-1-\delta} = O(n^{p-\delta}) = o(n^p).$$

In the proofs of the following lemma and of Theorem 5, we will use the following fact: It follows from an exercise in Titchmarsh [28, p.242] that if $\alpha > 1$, and the a_k are real, $a_k \geq 0$, then for

$$f(z) = \sum_{k=1}^{\infty} a_k z^k = (1 - z)^{-\alpha},$$

a_k is asymptotic to $k^{\alpha-1} \Gamma(\alpha)$ and $f(z)$ is asymptotic to

$$\Gamma(\alpha) \sum_{k=0}^{\infty} k^{\alpha-1} z^k$$

as $|z| \to 1$.

For the proof of the next theorem, we will need the following lemma, which is adapted from [11, p.45].

<u>Lemma 3</u>: If $g(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in D and $g(z) = o((1 - |z|)^{-\alpha})$, where $\alpha > 0$, then for $-\infty < \gamma < \alpha + 1/2$, setting $r = |z|$, we have that

$$G(r) = \sum_{n=1}^{\infty} n^{-\gamma} |a_n| r^n$$

satisfies

$$G(r) = o\left(\frac{1}{(1 - r)^{\alpha-\gamma+1/2}}\right).$$

<u>Proof</u>: Assume that $\gamma < 1/2$. Set $s_N = \sum_{k=1}^{N} k^{-\gamma}|a_k|r^k$. By the Cauchy - Schwarz inequality,

$$s_N \le \left\{\sum_{k=1}^{N} (k^{-\gamma})^2\right\}^{1/2} \cdot \left\{\sum_{k=1}^{N} (|a_k|r^k)^2\right\}^{1/2}$$

$$\le (O(N^{-2\gamma+1}))^{1/2} \cdot \left\{\frac{1}{2\pi}\int_0^{2\pi} |g(re^{i\theta})|^2\, d\theta\right\}^{1/2}$$

$$\le O(N^{-\gamma+1/2}) \cdot \left\{\frac{1}{2\pi}\int_0^{2\pi} |o((1-r)^{-\alpha})|^2\, d\theta\right\}^{1/2}$$

$$= O(N^{-\gamma+1/2}) \cdot o((1-r)^{-\alpha}).$$

Thus

$$G(r) = (1-r)\sum_{k=1}^{\infty} s_k\, r^k$$

$$= (1-r)\, o((1-r)^{-\alpha}) \cdot \sum_{k=1}^{\infty} O(k^{-\gamma+1/2})\, r^k$$

$$= o((1-r)^{1-\alpha}) \cdot \sum_{k=1}^{\infty} k^{-\gamma+1/2}\, r^k$$

$$= o((1-r)^{1-\alpha}) \cdot \sum_{k=1}^{\infty} k^{3/2-\gamma-1}\, r^k$$

$$= o((1-r)^{1-\alpha})\;\; O((1-r)^{-(3/2-\gamma)})$$

$$= o((1-r)^{1-\alpha})\;\; O((1-r)^{\gamma-3/2})$$

$$= o((1-r)^{\gamma-\alpha-1/2}).$$

For $\gamma \geq 1/2$, the result follows from the case when $\gamma < 1/2$ by successive integration.

We now consider conditions on the coefficients which are necessary for a function $f(z)$ to be in $\mathcal{B}_0$.

Theorem 5: Suppose that $f(z) = \sum_{k=1}^{\infty} a_k z^k$ is in $\mathcal{B}_0$. Then for each fixed integer $P \geq 1$, we have

$$\sum_{k=1}^{n} k^P |a_k| = o(n^{P+1/2}).$$

Proof: Applying Lemma 3 with $zf'(z) = g(z)$, $\alpha = 1$, $\gamma = -P + 1$, we obtain

$$\sum_{k=1}^{\infty} k^P |a_k| \; |z|^k = o((1 - |z|)^{-(P+1/2)}).$$

Since the coefficients are all real and nonnegative,

$$|z|^n \sum_{k=1}^{n} k^P |a_k| \leq \sum_{k=1}^{n} |a_k| \; |z|^k \leq \sum_{k=1}^{n} k|a_k||z|^k$$

for all $|z|$ and n. Taking $|z| = e^{-1/n}$, and noting that $1 - e^{-1/n}$ is asymptotic to $1/n$ as n increases without bound,

$$e^{-1} \sum_{k=1}^{n} k^P |a_k| = o((1 - e^{-1/n})^{-(P+1/2)}) = o(n^{P+1/2}).$$

IV. CONDITIONS ON BLOCKS OF COEFFICIENTS

Let $f(z) = \sum_{k=1}^{\infty} a_k z^k$. In this chapter we shall give several results concerning "blocks" of coefficients for Bloch and $\mathcal{B}_0$ functions. We will consider the portion of the power series for $f(z)$ between $k = m$ and $k = 2m$. The integer "2" of the upper bound is not critical to the following analysis; simple revisions will accommodate any number exceeding 1.

Define $\beta_m = \sum_{k=m}^{2m} |a_k|$. We will first show that for functions whose power series have real nonnegative coefficients, we can characterize $\mathcal{B}$ and $\mathcal{B}_0$ by conditions on β_m.

<u>Theorem 6</u>: Suppose that $f(z) = \sum_{k=1}^{\infty} a_k z^k$, where a_k is real and nonnegative for all k. Then $f(z)$ is a Bloch function if and only if there is a constant $M < \infty$ such that $\beta_m \leq M$ for all m.

<u>Proof:</u> Suppose that $\beta_m \leq M < \infty$ for all M. By [19, Theorem 2(i)], it is sufficient to show that there is a constant C such that $\sum_{k=1}^{n} k|a_k| \leq Cn$ for all n. Fix n, and suppose that p is an integer such that $2^p \leq n < 2^{p+1}$. Then

$$\sum_{k=1}^{n} k|a_k| = \sum_{k=1}^{n} ka_k$$

$$= \sum_{k=1}^{1} ka_k + \sum_{k=2}^{3} ka_k + \dots + \sum_{k=2^p}^{n} ka_k \leq$$

$$\leq \sum_{k=2^0}^{2^1} ka_k + \sum_{k=2^1}^{2^2} ka_k + \ldots + \sum_{k=2^p}^{2^{p+1}} ka_k$$

$$\leq 2 \sum_{k=2^0}^{2^1} a_k + 2^2 \sum_{k=2^1}^{2^2} a_k + \ldots + 2^{p-1} \sum_{k=2^p}^{2^{p+1}} a_k$$

$$\leq (2 + 2^2 + \ldots + 2^{p+1})\ M$$

$$\leq 2^{p+2}\ M$$

$$\leq 4Mn.$$

Taking $C = 4M$, we see that $f(z)$ is a Bloch function.

Conversely, suppose that $f(z)$ is a Bloch function. Since all of the a_k's are real nonnegative, all of the arguments are zero. By a result of Mathews [19, Theorem 3], there is a constant C such that

$$\sum_{k=1}^{n} k\ a_k \leq Cn$$

for all n. Then

$$m \sum_{k=m}^{2m} a_k \leq \sum_{k=m}^{2m} k\ a_k \leq 2mC$$

so

$$\sum_{k=m}^{2m} a_k \leq 2C.$$

<u>Theorem 7</u>: Suppose $f(z) = \sum_{k=1}^{\infty} a_k z^k$ with all a_k real and nonnegative. Then $\beta_m \to 0$ as $m \to \infty$ if and only if $f(z)$ is in $\mathcal{B}_0$.

<u>Proof</u>: Suppose $\beta_m \to 0$ as $m \to \infty$. To show that $f(z)$ is in $\mathcal{B}_0$, it will suffice to show that $\sum_{k=1}^{n} k\ a_k = o(n)$, or that

$n^{-1} \sum_{k=1}^{n} k\ a_k \to 0$ as $n \to \infty$.

Let $\varepsilon > 0$, and let N be large enough so that for $p \geq N$,

$$\sum_{k=2^p}^{2^{p+1}} a_k < \varepsilon/8.$$

We have

$$n^{-1}\sum_{k=1}^{n} ka_k = n^{-1}\sum_{k=1}^{2^N} ka_k + n^{-1}\sum_{k=2^N+1}^{n} ka_k.$$

If n is large enough, say, $n \geq Q \geq 2^N$, then $n^{-1}\sum_{k=1}^{2^N} ka_k < \varepsilon/2.$

Let us now concentrate on the second sum. If $2^r \leq n < 2^{r+1}$, then

$$\begin{aligned}
& n^{-1}\sum_{k=2^N}^{n} ka_k \\
& = n^{-1}\left[\sum_{k=2^N}^{2^{N+1}-1} ka_k + \sum_{k=2^{N+1}}^{2^{N+2}-1} ka_k + \dots + \sum_{k=2^r}^{n} ka_k\right] \\
& \leq n^{-1}\left[\sum_{k=2^N}^{2^{N+1}} ka_k + \dots + \sum_{k=2^r}^{2^{r+1}} ka_k\right] \\
& \leq n^{-1}\left[2^{N+1}\sum_{k=2^N}^{2^{N+1}} a_k + \dots + 2^{r+1}\sum_{k=2^r}^{2^{r+1}} a_k\right] \\
& \leq \frac{\varepsilon}{8}\left[\frac{2^{N+1}}{2^r} + \frac{2^{N+2}}{2^r} + \dots + \frac{2^{r+1}}{2^r}\right] \\
& \leq \frac{\varepsilon}{8}\left(2^{1+N-r} + \dots + 2^{-2} + 2^{-1} + 1 + 2\right) \\
& < \frac{\varepsilon}{2}.
\end{aligned}$$

Therefore, for $n \geq Q$, $n^{-1}\sum_{k=1}^{n} ka_k < \varepsilon$. By Theorem 3(i), $f(z)$ is in B_0.

Conversely, if f is in B_0, then by Theorem 4,

$$\frac{1}{2}\beta_m = \frac{m}{2m}\sum_{k=m}^{2m} a_k \le \frac{1}{2m}\sum_{k=m}^{2m} ka_k \le \frac{1}{2m}\sum_{k=1}^{2m} ka_k \to 0,$$

so $\beta_m \to 0$.

If we now consider functions with coefficients not restricted to be real nonnegative, we obtain several theorems which parallel the results of Mathews [19], with conditions on the coefficients between a_m and a_{2m}.

Theorem 8: The following conditions are sufficient for $f(z) = \sum_{k=0}^{\infty} a_k z^k$ to be a Bloch function:

(i). If there is a constant M^* such that

$$\sum_{k=m}^{2m} k^{J-1}|a_k| \le M^* m^{J-1}, \tag{15}$$

where J is an integer greater than or equal to 1.

(ii). If there is a constant C such that for some q satisfying $1 < q < \infty$, $\sum_{k=1}^{\infty} k^{q-1}|a_k|^q \le C$.

Proof of (i): Suppose that equation (15) is satisfied and $2^r \le n < 2^{r+1}$. Then

$$\begin{aligned}
\sum_{k=1}^{n} k^J|a_k| &\le \sum_{k=1}^{2} k^J|a_k| + \sum_{k=2}^{2^2} k^J|a_k| + \dots + \sum_{k=2^r}^{2^{r+1}} k^J|a_k| \\
&\le 2^1 \sum_{k=2^0}^{2^1} k^{J-1}|a_k| + \dots + 2^{r+1} \sum_{k=2^r}^{2^{r+1}} k^{J-1}|a_k| \\
&\le 2\, M^* (1 + 2^J + 2^{2J} + \dots + 2^{rJ}) \\
&< 2\, M^*\, 2^{(r+1)J} \\
&\le 2^{J+1} M^* n^J.
\end{aligned}$$

By [19, Theorem 2(i)], $f(z)$ is a Bloch function.

<u>Proof of (ii)</u>: By part (i), with $J = 2$, it suffices to show that

$$\sum_{k=m}^{2m} k|a_k| \leq M^*m$$

for some finite constant M^*. The details of the proof, using the Hölder inequality, are similar to those of Theorem 1; we omit them.

The next theorem gives additional restrictions on the location of the coefficients, under which the sufficient condition of Theorem 8 for $J = 1$, that the β_m are bounded, is also necessary.

<u>Theorem 9</u>: Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be a Bloch function. If there exists an α such that $\alpha \leq \arg(a_k) \leq \alpha + \pi/2$ for all k, then there is a constant C^* such that $\beta_m \leq C^*$ for all m.

<u>Proof</u>: By a theorem of Mathews [19], there exists a constant C such that $\sum_{k=1}^{n} k|a_k| \leq Cn$ for all n. Then

$$m\beta_m \leq \sum_{k=m}^{2m} k|a_k| \leq 2Cm.$$

Hence $\beta_m \leq 2C$. Set $C^* = 2C$.

We remark that an example used by Mathews [19] to illustrate that a Bloch function $f(z)$ may satisfy $\sum_{k=1}^{n} k|a_k| = O(n^p)$ for $1 < p < 3/2$ also satisfies $\beta_m = O(m^{p-1})$. For,

$$m \sum_{k=m}^{2m} |a_k| \leq \sum_{k=1}^{2m} k|a_k| = O((2m)^p) = O(m^p),$$

so

$$\sum_{k=m}^{2m} |a_k| = O(m^{p-1}).$$

Next, we consider necessary conditions.

<u>Theorem 10</u>: Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be a Bloch function. Then for each fixed integer $P \geq 1$ there is a constant $C^{\#}$ such that

$$\sum_{k=m}^{2m} k^{P-1}|a_k| \leq C^{\#} m^{P-1/2}.$$

<u>Proof</u>: By a theorem of Mathews [19], there is a constant C such that

$$\sum_{k=1}^{n} k^{P}|a_k| \leq Cn^{P+1/2}.$$

Then

$$m \sum_{k=m}^{2m} k^{P-1}|a_k| \leq \sum_{k=m}^{2m} k^{P}|a_k| \leq \sum_{k=1}^{2m} k^{P}|a_k| \leq C(2m)^{P+1/2}.$$

Thus

$$\sum_{k=m}^{2m} k^{P-1}|a_k| \leq 2^{P+1/2}Cm^{P-1/2}.$$

Setting $C^{\#} = 2^{P+1/2}C$, we are done.

We next turn our attention to functions in $\mathcal{B}_0$. These results are similar to the preceding Theorems 8, 9 and 10. The proofs of all but part of one of the following theorems are similar to those of earlier theorems and will be omitted.

<u>Theorem 11</u>: The following are sufficient conditions for the function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ to be in $\mathcal{B}_0$:

(i). If there is an integer $J \geq 1$ such that

$$\sum_{k=m}^{2m} k^{J-1}|a_k| = o(m^{J-1}).$$

(ii). If there is a number p, $1 < p < \infty$, and a number $w < p - 1$ such that

$$\sum_{k=m}^{2m} k^{(p-w)/(p-1)}|a_k|^{p/(p-1)} = o(n^{(p-w-1)/(p-1)}).$$

<u>Proof of (i)</u>: Let $\varepsilon > 0$ be given, and set $u(m) = \sum_{k=m}^{2m} k^{J-1}|a_k|$. Since $\sum_{k=m}^{2m} k^{J-1}|a_k| = o(m^{J-1})$, there is an increasing function $s(j)$ on the positive integers such that for $m > 2^{s(j)}$,

$$\sum_{k=m}^{2m} k^{J-1}|a_k| < \varepsilon 2^{-(j+3)} m^{J-1}.$$

If n is sufficiently large, say, $n \geq N_1$, then

$$n^{-J} \sum_{k=1}^{2^{s(1)}} k^{J-1}|a_k| < \varepsilon/2.$$

Let n be a fixed integer greater that N_1, and p an integer such that $2^{s(p)} \leq n < 2^{s(p+1)}$. Then

$$\sum_{k=2^{s(1)}}^{n} k^J|a_k|$$

$$\leq \sum_{k=2^{s(1)}}^{2^{s(2)}} k^J|a_k| + \sum_{k=2^{s(2)}}^{2^{s(3)}} k^J|a_k| + \dots + \sum_{k=2^{s(p)}}^{n} k^J|a_k|.$$

For $1 \leq j < p$,

$$\sum_{k=2^{s(j)}}^{2^{s(j+1)}} k^J|a_k|$$

$$\leq \sum_{k=2^{s(j)}}^{2^{s(j)+1}} k^J|a_k| + \sum_{k=2^{s(j)+1}}^{2^{s(j)+2}} k^J|a_k| + \dots + \sum_{k=2^{s(j+1)-1}}^{2^{s(j+1)}} k^J|a_k|$$

$$\leq 2^{s(j)+1}u(2^{s(j)}) + 2^{s(j)+2}u(2^{s(j)+1}) + \dots + 2^{s(j+1)}u(2^{s(j+1)-1})$$

$$< \frac{2}{2^{j+3}}((2^{s(j)})^J + (2^{s(j)+1})^J + \dots + (2^{s(j+1)-1})^J).$$

Dividing this by n^J gives a result less than that obtained by dividing by $(2^{s(j+1)-1})^J$, which is

$$\frac{\varepsilon}{2^{j+1}}\left[\left[\frac{2^{s(j)}}{2^{s(j+1)-1}}\right]^J + \ldots + \left[\frac{2^{s(j+1)-1}}{2^{s(j+1)-1}}\right]^J\right]$$

$$= \frac{\varepsilon}{2^{J+1}}\left((2^{s(j)-s(j+1)+1})^J + \ldots + 2^{-J} + 1^{-J}\right)$$

$$< \frac{\varepsilon}{2^{J+1}}.$$

Thus,

$$n^{-1}\sum_{k=2^{s(j)}}^{2^{s(j+1)}} k^J|a_k| < \frac{\varepsilon}{2^{j+1}}.$$

A similar procedure shows that

$$n^{-1}\sum_{k=2^{s(p)}}^{n} k^J|a_k| < \frac{\varepsilon}{2^{p+1}}.$$

It then follows that

$$n^{-1}\sum_{k=2^{s(1)}}^{n} k^J|a_k| < \frac{\varepsilon}{2^2} + \frac{\varepsilon}{2^3} + \ldots + \frac{\varepsilon}{2^{p+1}} < \frac{\varepsilon}{2}.$$

This shows that for $n \geq N_1$, $\sum_{k=1}^{n} k^J|a_k| = o(n^J)$, and so by Theorem 3(i), $f(z)$ is a Bloch function.

The proof of part (ii) of this theorem is similar to that of Theorem 3(ii); we will omit it.

<u>Theorem 12</u>: Let $f(z)$ be a function in $\mathcal{B}_0$. If there is an α such that $\alpha \leq \arg(a_k) \leq \alpha + \pi/2$ for all k, then $\beta_m \to 0$ as $m \to \infty$.

<u>Theorem 13</u>: Let $f(z)$ be a function in $\mathcal{B}_0$, and p an integer, $p \geq 1$. Then

$$\sum_{k=m}^{2m} k^{p-1}|a_k| = o(m^{p-1/2}).$$

V. A BOUNDED BLOCH FUNCTION OF WHICH SECTIONS ARE ARBITRARILY LARGE

In this chapter we will give an example of a Bloch function with real coefficients for which the sum of the coefficients is finite, β_m is large for selected values of m, and if the the same portion of the power series from which the β_m are taken is considered as a function $g_m(z)$, the Bloch norm of $g_m(z)$ increases without bound as $m \to \infty$.

Landau [15] gave a theorem which shows that a function $f(z)$, analytic and bounded by 1 in D, satisfies $s_n \le G_n$ for all n, where $s_n(z) = \sum_{k=0}^{\infty} a_k z^k$, $s_n = s_n(1)$, and

$$G_n = \sum_{k=0}^{n} \left(\frac{1\cdot 3\cdot \ \ldots \ \cdot(2k-1)}{2\cdot 4\cdot \ \ldots \ \cdot 2k}\right)^2 .$$

It is noted that G_n is asymptotic to $\frac{1}{\pi}\log(n)$.

Landau also gave the following example of a function for which $s_n = G_n$: Define

$$K_n(z) = \sum_{k=0}^{n} \binom{-1/2}{k} (-z)^k$$

$$= 1 + \frac{1}{2} z + \frac{1\cdot 3}{2\cdot 4} z^2 + \ldots + \frac{1\cdot 3\cdot \ \ldots \ \cdot(2n-1)}{2\cdot 4\cdot \ \ldots \ \cdot 2n} z^n$$

and set

$$f_n(z) = \sum_{k=0}^{\infty} a_{n,k} z^k$$

$$= \frac{z^n K_n(z^{-1})}{K_n(z)} =$$

$$= \frac{\frac{1\cdot 3\cdot \ \dots \ \cdot(2n-1)}{2\cdot 4\cdot \ \dots \ \cdot 2n} + \dots + \frac{1}{2} z^{n-1} + z^n}{1 + \frac{1}{2} z + \dots + \frac{1\cdot 3\cdot \ \dots \ \cdot(2n-1)}{2\cdot 4\cdot \ \dots \ \cdot 2n} z^n} .$$

For this function, all the coefficients are real, $a_{n,k} > 0$ for $0 \le k \le n$, $\sum_{k=0}^{n} a_{n,k} = G_n$, $|f_n(z)| = 1$ for $|z| = 1$, $f_n(1) = 1$, and $|f_n(z)| \le 1$ for $|z| \le 1$. Since $|f_n(z)|$ is bounded by 1 on the closed unit disc, $f_n(z)$ is a Bloch function with Bloch norm $\|f_n\|_{\mathcal{B}} \le 2$ [3]. As a bounded Bloch function, the coefficients of $f_n(z)$ satisfy $a_{n,k} \to 0$ as $k \to \infty$ [21].

We inductively define a function $n(i)$ on the positive integers as follows: Set $n(1) = 2$. If $n(j)$ has been defined for $j < i$, let $n(i)$ be an integer sufficiently large so that for $k > n(i)$, we have $n(i) > 2n(i-1)$, $G_{n(i)} \ge 2^{2i}$, and

$$a_{n(j),k-n(j)} \le (2n(i)(n(i)+1))^{-1}$$

for $j = 1, 2, \dots , i-1$.

Now, we define the function $f(z) = \sum_{k=2}^{\infty} d_k z^k$ by

$$f(z) = \sum_{i=1}^{\infty} 2^{-i} z^{n(i)} f_{n(i)}(z).$$

If $|z| = 1$, then $|f(z)| \le \sum_{i=1}^{\infty} 2^{-i} |f_{n(i)}(z)| = 1$. For $|z| \le 1$, $|f(z)| \le 1$. Thus $f(z)$ is a Bloch function, with Bloch norm $\|f\|_{\mathcal{B}} \le 2$ [21], [3]. Summing the coefficients of f,

$$\sum_{k=2}^{\infty} d_k = f(1) = \sum_{i=1}^{\infty} 2^{-1} = 1.$$

Let us now consider β_m for selected values of m, in particular for $m = n(i)$. For $1 \le j < i$,

$$\sum_{k=n(i)}^{2n(i)} 2^{-j}|a_{n(j),k-n(j)}| \leq (n(i)+1)(2n(i)(n(i)+1)))^{-1}2^{-j}$$

$$< 2^{-j}/n(i)$$

so

$$\sum_{j=1}^{i-1} \sum_{k=n(i)}^{2n(i)} 2^{-j}|a_{n(j),k-n(j)}| < 1/n(i).$$

We also have that

$$\sum_{k=n(i)}^{2n(i)} 2^{-i}|a_{n(i),k-n(i)}| = 2^{-i} \sum_{k=0}^{n(i)} a_{n(i),k}$$

$$= 2^{-i}\, G_{n(i)}$$

$$> 2^{i}.$$

Thus, $\beta_{n(i)} > 2^{i} - 1/n(i)$, which is unbounded as $i \to \infty$.

Consider next $\sum_{k=n(i)}^{2n(i)} d_k z^k$ as a function $g_i(z)$ in its own right. We will show that the Bloch norm of $g_i(z)$ is unbounded as $i \to \infty$.

We may write

$$g_i(z) = \sum_{j=1}^{i-1} 2^{-j} \sum_{k=n(i)}^{2n(i)} a_{n(j),k-n(j)}\, z^k + \tag{16}$$

$$2^{-i} \sum_{k=n(i)}^{2n(i)} a_{n(i),k-n(i)}\, z^k.$$

Differentiating

$$2^{-j} \sum_{k=n(i)}^{2n(i)} a_{n(j),k-n(j)}\, z^k$$

for $j < i$, we get

$$2^{-j} \sum_{k=n(i)}^{2n(i)} k\, a_{n(j),k-n(j)}\, z^{k-1}$$

which in absolute value is less than

$$2^{-j}\, 2\, n(i) \sum_{k=n(i)}^{2n(i)} |a_{n(j),k-n(j)}|$$

$$\leq 2^{1-j}\, n(i)\, (n(i)+1)\, (2n(i)(n(i)+1))^{-1}$$

$$= 2^{-j}.$$

Thus the value of the derivative of the double sum in (16) is less than 1 in absolute value.

The second, single sum in (16) may be rewritten as

$$(17) \qquad 2^{-i}\, z^{n(i)} \sum_{k=0}^{n(i)} a_{n(i),k}\, z^k;$$

we will denote the sum in (17) by $t_i(z)$. This function is a polynomial of degree $n(i)$ with positive real coefficients, and hence the maximum absolute value of (17) occurs for $z = 1$. Differentiating $2^{-i}\, z^{n(i)}\, t_i(z)$, we obtain

$$(18) \qquad 2^{-i}\, z^{n(i)-1}\, n(i)\, t_i(z) + 2^{-i}\, z^{n(i)}\, t_i'(z).$$

All of the coefficients on $t_i'(z)$ are real nonnegative as well, so $|t_i'(z)|$ has a maximum at 1. We have

$$t_i'(1) = \sum_{k=0}^{n(i)} k\, a_{n(i),k}$$

$$\leq n(i)\, (t_i(1) - a_{n(i),0})$$

$$\leq n(i)\, t_i(1).$$

To find the Bloch norm of $g_i(z)$, we must find

$$\sup_{z \in D} (1 - |z|^2) |g_i'(z)|.$$

Because of the bounds found earlier on the double sum in (16),

$$\|g_i(z)\|_{\mathcal{B}} \geq \|2^{-i} z^{n(i)} t_i(z)\|_{\mathcal{B}} - 2.$$

Since the second term in (18) is positive, it will suffice for us to show that

$$(1 - |z|^2)\, 2^{-i} z^{n(i)-1}\, n(i)\, t_i(z) \tag{19}$$

is bounded below on $(0, 1)$ by a term which increases without bound as $i \to \infty$. The expression in (19) is greater than

$$(1 - |z|)\, 2^{-i} z^{n(i)-1}\, n(i)\, t_i(z).$$

Set $z = 1 - 1/(4n(i))$. Then

$$\begin{aligned}(1 - |z|)\, z^{n(i)-1}\, n(i) &= 1/(4n(i))(1 - 1/(4n(i)))^{n(i)-1}\, n(i)\\ &= 1/4\, (1 - 1/(4n(i)))^{n(i)-1},\end{aligned}$$

which approaches the limit $1/4\, e^{-1/4}$ as i, and hence $n(i)$, tends to infinity.

We also have

$$\begin{aligned} t_i(1 - 1/(4n(i))) &\geq t_i(1) - (1/(4n(i)))\sup_{z \in D} |t_i'(z)|\\ &\geq t_i(1) - (1/(4n(i)))\, n(i)\, t_i(1)\\ &= 3/4\, G_{n(i)}.\end{aligned}$$

Thus, the expression in (19) is greater than the product of 2^{-i}, a term which tends to $4^{-1} e^{-1/4}$ as $i \to \infty$, and a term which is asymptotically greater than $3/4\, G_{n(i)} \geq 3/4(2^{2i})$ as $i \to \infty$. Hence the Bloch norm of $g_i(z)$ is asymptotically greater than $2^{i-4} e^{-1/4}$, which increases without bound as $i \to \infty$.

VI. OPEN PROBLEMS

We list some open problems concerning coefficients of Bloch functions.

1. Mathews [19] gave an example of a holomorphic function with bounded coefficients which is not normal. Does there exist a holomorphic function with coefficients tending to zero which is not normal?

Let Γ be the family of univalent functions $g(z) = z + \ldots$ on D. Let $\mathcal{B}_S$ denote the family of functions

$$\mathcal{B}_S = \{f(z) = \log(g'(z))\}_{g\epsilon\Gamma}$$

It is known that

$$\{f \epsilon \mathcal{B} : \|f\|_{\mathcal{B}} \leq 1\} \subset \mathcal{B}_S \subset \{f \epsilon \mathcal{B} : \|f\|_{\mathcal{B}} \leq 3\}$$

Let $\mathcal{B}_Q$ be the subclass of $\mathcal{B}_S$ coming from functions in Γ having quasiconformal extensions to C. It is known that

$$\{f \epsilon \mathcal{B} : \|f\|_{\mathcal{B}} \leq 1\} \subset \mathcal{B}_Q \quad [3].$$

2. Give conditions on the coefficients which characterize $\mathcal{B}_S$ and $\mathcal{B}_Q$.

3. Evaluate, or estimate,

$$\sup_{f\epsilon\mathcal{B}_S} \sup_n |a_n| \quad [3].$$

4. Let $f \epsilon \mathcal{B}$ and $(1 - |z_n|^2)|f'(z)| \to 0$ as $n \to \infty$ for every sequence $\{z_n\}$ with $|f(z_n)| \to \infty$ as $n \to \infty$. Is it true that f is in $\mathcal{B}_0$? [3], [14].

5. In Theorem 9, we proved that if $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is a Bloch function, and there is an α such that $\alpha \leq \arg(a_k) \leq \alpha + \pi/2$ for all k, then there is a constant C^* such that $\beta_m \leq C^*$ for all m. Is this still true if the arguments of the coefficients within the separate blocks satisfy a similar restriction? Specifically, suppose that there is a sequence $\{\alpha_j\}$ such that for $2^j \leq k \leq 2^{j+1}-1$, we have $\alpha_j \leq \arg(a_k) \leq \alpha_j + \pi/2$. Is

$$\sum_{k=2^j}^{2^{j+1}-1} |a_k|$$

uniformly bounded?

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