This is to certify that the

thesis entitled

ADDITIONAL PROPERTIES OF WEIGHTED SHIFTS

presented by

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has been accepted towards fulfillment
of the requirements for

Ph. D. degree in Mathematics

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Major professor

Date April 27, 1981

O-7639

ADDITIONAL PROPERTIES OF WEIGHTED SHIFTS

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A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1981

ABSTRACT

ADDITIONAL PROPERTIES OF WEIGHTED SHIFTS

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In this paper we examine some properties of weighted shifts which were previously unknown. Some of these properties are stimulated by properties of the unweighted bilateral shift. In particular, we define Toeplitz and Hankel operators for weighted shifts. Some of the properties that hold for Toeplitz and Hankel operators for the unweighted shift carry over for weighted shifts in general. However, there are some striking differences which we point out with several examples. Some properties for unweighted shifts carry over for weighted shifts with only minor modifications. This happens mainly when the weighted shift under consideration has a periodic weight sequence. Thus we also prove some properties for weighted shifts with periodic weight sequences. The material above is found in Chapters I, II, III, and IV.

In Chapters V and VI our work takes a slightly different direction. In Chapter V, we concern ourselves with answering Question 11 in Allen Shields' survey article

on weighted shifts [23]. In particular, we give a general sufficient condition for the "analytic" projection on $L^{\infty}(\beta)$ to be bounded.

In Chapter VI, we concern ourselves with spectral sets. We give a new proof of von Neumann's Theorem which says the closed unit disc is a spectral set for all contractions. We then investigate what happens when we replace the disc with an annulus. In particular, we answer the last half of Question 7 in Shields' article with an example. Finally, we examine what happens when we restrict ourselves to operators on two-dimensional spaces.

[23] Shields, A., Weighted Shift Operators and Analytic Function Theory, Amer. Math. Soc. Surveys 13 (1974), 49-128.

ACKNOWLEDGEMENTS

First and foremost, I would like to thank my thesis advisor Dr. Sheldon Axler for his encouragement and guidance during the course of this research. I greatly appreciate the time and energy he gave to me and his willingness to discuss questions I had. He has given me a sense for what is important mathematically by steering me in the right direction when I was tempted to divert my attention to tangential interests.

I am also greatly indebted to my wife Peggy who has supported and encouraged me constantly during our time at Michigan State University. Her unfailing love kept me going at times when I felt it was not worth it. Words cannot express the importance that her life is to mine.

I would also like to thank Cindy Lou Smith for typing this manuscript. I have great admiration for anyone who is able to read my writing and put up with my careless errors in the original draft. I thank her again for the excellent job she did in taking my sloppy draft and putting it into good form. I also would like to thank other members of the faculty and staff in the Mathematics Department of Michigan State University for their support during my time as a graduate student there.

Finally, I would like to thank my parents Lowell and Doris Hartman for all their love and support. Their help and guidance in my life has been invaluable. I would also like to thank my wife's parents Art and Lydia Kursch, our friends Tom and Doris Schumacher, and Gene and Jewel Ratzlaff. Last, but certainly not least, I would like to thank Pam Gorkin with whom I spent many hours discussing the intricacies of graduate school.

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CHAPTER I

WEIGHTED SHIFT OPERATORS

Let H be a complex separable Hilbert space with orthonormal basis $\{e_n : n \in \mathbb{Z}\}$. We denote the set of bounded linear operators on H by $\mathcal{B}(H)$. A bilateral weighted shift T is a bounded linear operator on H that maps the basis vector e_n into a scalar multiple w_n of e_{n+1}, i.e. $Te_n = w_n e_{n+1}$. Weighted shifts have been used generously through the years to provide examples and counterexamples to questions in operator theory. However, the first specific study of weighted shifts was done only recently by R.L. Kelley in his unpublished dissertation at the University of Michigan in 1966. Since that time many other properties of weighted shifts have been identified and examined. Most of what is known about weighted shifts is compiled in the survey article by Allen Shields, "Weighted Shift Operators and Analytic Function Theory," [23]. This article contains all of the basic facts one needs when working with weighted shifts. It also contains a list of unsolved problems concerning weighted shifts. We will rely heavily on the material found in Shields' article and use the notation developed there. The article contains facts about both bilateral and unilateral weighted shifts. We will restrict our attention here to bilateral weighted shifts which are injective and also whose weight sequences $\{w_n : n \in \mathbb{Z}\}$ consist only of positive terms. That it is sufficient to do this is pointed out by the following proposition. Before that proposition, we should note that when we say weighted shift, we mean an injective, bilateral weighted shift.

<u>Proposition 1.1</u>: (Shields,[23]) If T is a weighted shift with weight sequence $\{w_n\}$, then T is unitarily equivalent to the weighted shift with weight sequence $\{|w_n|\}$.

The assumption that T is injective guarantees that there are no weights which are zero. Noninjective shifts may be studied by considering them as a direct sum of weighted shifts in which some of the summands may operate on finite dimensional spaces. Thus from here on, if T is a weighted shift with weight sequence $\{w_n\}$, it will be understood that $w_n > 0$ for all integers n. Given such a weighted shift, the following definitions are made.

<u>Definition 1.1</u>: Let T be a weighted shift with weight sequence $\{w_n\}$. Then

i) $\beta(n) = \prod_{k=0}^{n-1} w_k$ if $n > 0$

ii) $\beta(n) = 1$ if $n = 0$

iii) $\beta(n) = \left(\prod_{k=n}^{-1} w_k\right)^{-1}$ if $n < 0$

<u>Definition 1.2</u>: Let T and $\beta(n)$ be as in Definition 1.1. Then

$$L^2(\beta) = \{f = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n :$$

$$\hat{f}(n) \in ¢ \text{ for all } n \text{ and } \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2\beta^2(n) < \infty\}$$

We note that the sum $\sum_{n=-\infty}^{\infty} \hat{f}(n)z^n$ is not taken in a literal sense at this point. It is taken in the formal sense that $L^2(\beta)$ is a set of sequences indexed by the integers with the summand $\hat{f}(n)z^n$ indicating that $\hat{f}(n)$ is the nth term of the sequence f. For $f,g \in L^2(\beta)$ we define an inner product

$$(f,g) = \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}\beta^2(n)$$

With this inner product, $L^2(\beta)$ is a Hilbert space with addition and scalar multiplication of vectors being componentwise. The set $\{z^n : n \in \mathbb{Z}\}$ can be thought of as an orthogonal basis of $L^2(\beta)$.

For $f,g \in L^2(\beta)$ we define a formal product $h = fg$ by $h = \sum_{n=-\infty}^{\infty} \hat{h}(n)z^n$ where $\hat{h}(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\hat{g}(n-k)$ if all of the latter sums converge. We note that this mimics the multiplication of analytic functions whose Laurent series would be given as f and g are given. Now let $L^\infty(\beta) = \{\varphi \in L^2(\beta) : \varphi f \in L^2(\beta) \text{ for all } f \in L^2(\beta)\}$.

Then for $\varphi \in L^{\infty}(\beta)$ we can define the linear map $M_{\varphi} : L^2(\beta) \to L^2(\beta)$ by $M_{\varphi} f = \varphi f$. Under these definitions we have the following theorem.

<u>Theorem 1.1</u>: (Shields,[23]) For $\varphi \in L^{\infty}(\beta)$, M_{φ} is a bounded linear operator on $L^2(\beta)$ and M_z is unitarily equivalent to the weighted shift $T \in \mathcal{B}(H)$. Furthermore, under this unitary equivalence, $\{M_{\varphi} : \varphi \in L^{\infty}(\beta)\}$ corresponds to the commutant $\{T\}' = \{S \in \mathcal{B}(H) : ST = TS\}$ of T.

The theorem above says that a weighted shift, which weightedly shifts an orthonormal basis, can also be thought of as an unweighted shift of a weighted basis. This follows from the equalities below.

$$M_z(f) = zf = \sum_{n=-\infty}^{\infty} \hat{f}(n-1)z^n = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^{n+1}$$

The second equality comes from the identity:

$$\widehat{zf}(n) = \sum_{k=-\infty}^{\infty} \hat{z}(k)\hat{f}(n-k) = \hat{f}(n-1) .$$

At times it is convenient to think of T strictly as a weighted shift on H. At other times, though, it is helpful to think of T as M_z on $L^2(\beta)$. During the first four chapters, I will basically think of a weighted shift as M_z on $L^2(\beta)$. However, in the last two chapters, I will sometimes think of them as weighted shifts on unweighted spaces. Some further notations, definitions, and facts are as below:

$$H^2(\beta) = \{f \in L^2(\beta) : \hat{f}(n) = 0 \text{ for all } n < 0\}$$

$$B(\beta) = \{\varphi \in L^\infty(\beta) : M_\varphi^* = M_\psi \text{ for some } \psi \in L^\infty(\beta)\}$$

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$$

$$r(T) = \sup\{|z| : z \in \sigma(T)\}$$

For $\varphi \in L^\infty(\beta)$, we let $\|\varphi\|_\infty = \|M_\varphi\|$. When $L^\infty(\beta)$ is endowed with this norm, it is a commutative Banach algebra.

<u>Definition 1.3</u>: For $f \in L^2(\beta)$ we define $\overline{f} \in L^2(\beta)$ by

$$\overset{\Delta}{f}(n) = \overline{\hat{f}(-n)\beta(-n)}/\beta(n)$$

From the definitions above, the following facts are easy to verify.

1. $H^2(\beta)$ is a closed subspace of $L^2(\beta)$. (The coefficient maps $\Gamma_n : L^2(\beta) \to \mathbb{C}$ given by $\Gamma_n(f) = \hat{f}(n) = (f, z^n)/\beta^2(n)$ are continuous.)
2. $\{\lambda I : \lambda \in \mathbb{C}\} \subset B(\beta) \subset L^\infty(\beta)$
3. For $f \in L^2(\beta)$, $\|f\|_2 = (f,f)^{1/2} = \|\overline{f}\|_2$
4. If T is invertible then $\sigma(T) = \{z \in \mathbb{C} : r(T^{-1})^{-1} \leq |z| \leq r(T)\}$. If T is not invertible then $\sigma(T) = \{z \in \mathbb{C} : |z| \leq r(T)\}$.
5. For $\varphi \in L^\infty(\beta)$, $\|\varphi\|_2 = \|M_\varphi(1)\|_2 \leq \|M_\varphi\| \|1\|_2 \leq \|\varphi\|_\infty$.

Also, we will let $P : L^2(\beta) \to H^2(\beta)$ be the orthogonal projection of $L^2(\beta)$ onto its closed subspace $H^2(\beta)$. This projection is described by the formula:

$$P\left(\sum_{n=-\infty}^{\infty} \hat{f}(n)z^n\right) = \sum_{n=0}^{\infty} \hat{f}(n)z^n \quad \text{for} \quad f \in L^2(\beta) .$$

If we let $\pi_0(T^*)$ be the point spectrum of T^* (i.e. the set of eigenvalues of T^*), then we have the following theorem.

<u>Theorem 1.2</u>: (Shields,[23]) Let T be a weighted shift represented as M_z on $L^2(\beta)$. Then the following properties hold:

a) If T is invertible and $r(T^{-1})^{-1} < |w| < r(T)$, then $\lambda_w : L^\infty(\beta) \to \mathbb{C}$ given by $\lambda_w(\varphi) = \sum_{n=-\infty}^{\infty} \hat{\varphi}(n)w^n = \varphi(w)$ is a multiplicative linear functional on $L^\infty(\beta)$. Thus $|\varphi(w)| \leq \|\varphi\|_\infty$ when $r(T^{-1})^{-1} < |w| < r(T)$.

b) If $w \in \pi_0(T^*)$, then there exists $k_w \in L^2(\beta)$ such that $M_\varphi^* k_w = \overline{\varphi(w)}k_w$ for all $\varphi \in L^\infty(\beta)$.

c) For $w \in \pi_0(T^*)$ and k_w as above, we have $(f,k_w) = \sum_{n=-\infty}^{\infty} \hat{f}(n)w^n = f(w)$ for all $f \in L^2(\beta)$.

We note that this theorem implicitly says that the series $\sum_{n=-\infty}^{\infty} \hat{f}(n)w^n$ converges when w is as given.

Before we go on to Chapter II, we should mention a very important weighted shift. It is called the unweighted shift because all of its weights are 1's. For the unweighted shift, $L^2(\beta)$ is the space of measurable functions on $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ whose absolute values are square integrable with respect to arclength measure. We will denote this by $L^2(\partial\mathbb{D})$. Furthermore, $L^\infty(\beta)$ is the set of essentially bounded measurable functions on $\partial\mathbb{D}$. We will denote this by $L^\infty(\partial\mathbb{D})$. (See Douglas, [6]). We will refer to the unweighted shift and its properties quite frequently. In particular, we will use it as a model for some of our definitions and lines of thought.

CHAPTER II

SHIFTS WITH PERIODIC WEIGHT SEQUENCES

We recall that $\{\lambda I : \lambda \in \mathbb{C}\} \subset B(\beta) \subset L^\infty(\beta)$. One may ask whether equality holds at either end of this chain of inequalities. To answer this question, we present the following lemmas and theorem.

<u>Lemma 2.1</u>: If $\varphi \in B(\beta)$ and $\hat{\varphi}(N) \neq 0$, then $\beta(N+k) = \beta(N)\beta(k)$ for every integer k.

<u>Proof</u>: $$\begin{aligned}\hat{\varphi}(N) &= (\varphi z^k, z^{N+k})/\beta^2(N+k) \\ &= (M_\varphi z^k, z^{N+k})/\beta^2(N+k) \\ &= (z^k, M_\varphi^* z^{N+k})/\beta^2(N+k) \\ &= (z^k, M_\psi z^{N+k})/\beta^2(N+k) \quad \text{where } M_\psi = M_\varphi^* \\ &= \overline{\hat{\psi}(-N)}\, \beta^2(k)/\beta^2(N+k) \quad \text{for each } k.\end{aligned}$$

Thus $\beta^2(N+k)/\beta^2(k) = \overline{\hat{\psi}(-N)}/\hat{\varphi}(N)$ for all k. Letting $k = 0$ we get $\overline{\hat{\psi}(-N)}/\hat{\varphi}(N) = \beta^2(N)$. Hence $\beta^2(N+k)/\beta^2(k) = \beta^2(N)$. Hence $\beta^2(N+k)/\beta^2(k) = \beta^2(N)$ for every integer k. This is the desired result since $\beta(i) > 0$ for every integer i. Q.E.D.

We also note at this point that $\beta(N+k) = \beta(N)\beta(k)$ for all k implies $\beta(-N) = 1/\beta(N)$.

Lemma 2.2: Assume there exists an integer N such that $\beta(N+k) = \beta(N)\beta(k)$ for all k. Then $w_{N+k} = w_k$ for all k (i.e. the weight sequence for the weighted shift is periodic).

Proof: $w_{N+k} = \beta(N+K+1)/\beta(N+k) = \beta(N)\beta(k+1)/\beta(N)\beta(k) = \beta(k+1)/\beta(k) = w_k$. The second equality above holds because of the assumption on the sequence $\{\beta(n) : n \in \mathbb{Z}\}$. Q.E.D.

Theorem 2.1: Let T be a bilateral weighted shift with periodic weight sequence of least period N. Then $B(\beta) = \{\varphi \in L^{\infty}(\beta) : \hat{\varphi}(n) = 0$ for all n which are not integer multiples of $N\}$.

Proof: Suppose $\varphi \in B(\beta)$ and $\hat{\varphi}(n) \neq 0$. Then Lemma 2.1 implies $\beta(n+k) = \beta(n)\beta(k)$ for all k. Lemma 2.2 then implies $w_{n+k} = w_k$ for every k. This implies that $n = mN$ for some integer m since the least period of the weight sequence is N. Thus $B(\beta) \subset E = \{\varphi \in L^{\infty}(\beta) : \hat{\varphi}(n) = 0$ for n not an integer multiple of $N\}$.

Now let $\psi \in E$. We will show that $M_{\psi}^{*} = M_{\bar{\psi}}$. This will be true if and only if $M_{\psi}^{*}z^k = \bar{\psi}z^k = M_z^k(\bar{\psi})$ for all k. The reason for writing this in such a strange way is that it is unknown whether $\psi \in L^{\infty}(\beta)$ implies $\bar{\psi} \in L^{\infty}(\beta)$. Also, we should mention that an injective bilateral shift with a periodic weight sequence is invertible. Hence $M_z^k \in \mathcal{B}(L^2(\beta))$

for every integer k. Continuing with the proof now, we have:

$$(M_\psi^* z^k, z^n)/\beta^2(n) = (z^k, \psi z^n)/\beta^2(n)$$

$$= \overline{\hat{\psi}(k-n)}\ \beta^2(k)/\beta^2(n)$$

$$(M_z^k \bar{\psi}, z^n)/\beta^2(n) = (z^k \bar{\psi}, z^n)/\beta^2(n)$$

$$= \hat{\bar{\psi}}(n-k)$$

$$= \overline{\hat{\psi}(k-n)}\ \beta(k-n)/\beta(n-k)\ .$$

Now if $\hat{\psi}(k-n) \neq 0$ then $k-n = mN$ for some integer m. Hence

$$\beta^2(k)/\beta^2(n) = \beta^2(n+mN)/\beta^2(n) = \beta^2(mN)$$

$$= \beta(mN)/\beta(-mN) = \beta(k-n)/\beta(n-k)\ .$$

So we have $(M_\psi^* z^k, z^n) = (M_z^k \bar{\psi}, z^n)$ for all integers n. This shows that $M_\psi^* z^k = \bar{\psi}\, z^k$ for every integer k. Q.E.D.

<u>Corollary 2.1</u>: Under the involution $\psi \to \bar{\psi}$, $B(\beta)$ is a commutative C^*-subalgebra of $L^\infty(\beta)$.

<u>Corollary 2.2</u>: For shifts with periodic weight sequences with least period N, M_{z^n} is normal if and only if n is an integer multiple of N.

<u>Proof</u>: For $n = kN$, $z^n \in B(\beta)$. Thus $M_{z^n}^* = M_{\bar{z}^n}$ which implies $M_{z^n}^* M_{z^n} = M_{z^n} M_{z^n}^*$ since $L^\infty(\beta)$ is a commutative Banach algebra. For the converse, we assume $M_{z^n}^* M_{z^n} = M_{z^n} M_{z^n}^*$. This implies that

$$
\begin{aligned}
(z^{n+k},z^{n+m}) &= (M_{z^n}z^k, M_{z^n}z^m) \\
&= (M^*_{z^n}\, M_{z^n}z^k, z^m) \\
&= (M_{z^n}\, M^*_{z^n}z^k, z^m) \\
&= (M^*_{z^n}z^k, M^*_{z^n}z^m) \\
&= (z^{k-n}, z^{m-n})\beta^2(k)\beta^2(m)/\beta^2(k-n)\beta^2(m-n) \\
&= \begin{cases} \beta^4(m)/\beta^2(m-n) & \text{if } m = k \\ 0 & \text{if } m \neq k \end{cases} .
\end{aligned}
$$

Also, $(z^{n+k},z^{n+m}) = \begin{cases} \beta^2(m+n) & \text{if } m = k \\ 0 & \text{if } m \neq k \end{cases}$. Thus we must have $\beta(m+n)\beta(m-n) = \beta^2(m)$ for all m. Hence $\beta(m+n)/\beta(m) = \beta(m)/\beta(m-n)$ for all m, which implies $\beta(m+n+1)/\beta(m+1) = \beta(m+1)/\beta(m-n+1)$ by using $m+1$ instead of m. Now dividing the latter equality by the former one gets

$$[\beta(m+n+1)/\beta(m+n)][\beta(m)/\beta(m+1)] = [\beta(m+1)/\beta(m)][\beta(m-n)/\beta(m-n+1)]$$

or equivalently

$$w_{m+n} = w_m^2/w_{m-n} .$$

Now we prove by induction that $w_{kn+m} = (w_{n+m}/w_m)^k w_m$ for all $k \geq 0$. The equality is certainly true for $k = 0$ and $k = 1$. So assume it is true for $k \leq \ell$. Then

$$w_{(\ell+1)n+m} = w_{(\ell n+m)+n}$$

$$= w^2_{\ell n+m}/w_{(\ell-1)n+m}$$

$$= [(w_{n+m}/w_m)^{2\ell}(w_m)^2] \;/\; [(w_{n+m}/w_m)^{\ell-1}w_m]$$

$$= (w_{n+m}/w_m)^{\ell+1}\; w_m \;.$$

Thus the identity is established for $k = \ell+1$. By induction then, we have $w_{kn+m} = (w_{n+m}/w_m)^k w_m$ for all $k \geq 0$.

If we now let m be fixed and let k get large, we see that $w_{n+m} = w_m$ since a periodic weight sequence is bounded above and bounded away from zero. This is true for all m and hence $\{w_m\}$ is periodic with period $|n|$. This implies that $n = kN$ for some integer k. Q.E.D.

<u>Corollary 2.3</u>: For $\varphi \in \mathcal{B}(\beta)$ and $f \in L^2(\beta)$ we have $\bar{\varphi}f = \overline{\varphi\bar{f}}$.

<u>Proof</u>: $$\widehat{(\bar{\varphi}f)}(n) = \sum_{k=-\infty}^{\infty} \hat{\bar{\varphi}}(k)\hat{f}(n-k)$$

$$= \sum_{k=-\infty}^{\infty} \overline{\hat{\varphi}(-k)}\; f(n-k)\beta(-k)/\beta(k)$$

$$= [\beta(-n)/\beta(n)]\Big[\sum_{k=-\infty}^{\infty} \overline{\hat{\varphi}(-k)}\hat{f}(n-k)\; \beta(n)\beta(-k)/\beta(k)\beta(-n)\Big]$$

$$= [\beta(-n)/\beta(n)]\Big[\sum_{k=-\infty}^{\infty} \overline{\hat{\varphi}(-k)}\hat{f}(n-k)\; \beta(n-k)/\beta(k-n)\Big]$$

since $\hat{\varphi}(-k) \neq 0$ implies $\beta(-k)\beta(n) = \beta(n-k)$ and $\beta(k)\beta(-n) = \beta(k-n)$. Thus $(\bar{\varphi}f)(n) = [\beta(-n)/\beta(n)]$ $[\sum_{k=-\infty}^{\infty} \overline{\hat{\varphi}(-k)}\ \overline{\hat{\bar{f}}(k-n)}] = [\beta(-n)/\beta(n)]\overline{\widehat{\varphi\bar{f}}(-n)}$. This last term is just $\widehat{\overline{\varphi\bar{f}}}(n)$, however. Thus $(\widehat{\bar{\varphi}f})(n) = \widehat{\overline{\varphi\bar{f}}}(n)$ for all n implying $\bar{\varphi}f = \overline{\varphi\bar{f}}$. Q.E.D.

<u>Corollary 2.4</u>: The equality $B(\beta) = L^{\infty}(\beta)$ holds if and only if all the weights are equal.

<u>Proof</u>: If $B(\beta) = L^{\infty}(\beta)$, then $z \in B(\beta)$. This implies the weight sequence is periodic with period one. Thus $w_{k+1} = w_k = w_0$ for every integer k. The converse is immediate from the characterization of $B(\beta)$ given in Theorem 2.1. Q.E.D.

The following corollary follows immediately from Theorem 2.1.

<u>Corollary 2.5</u>: The equality $B(\beta) = \{\lambda I : \lambda \in \mathbb{C}\}$ holds if and only if the weight sequence is not periodic.

We now recall that a von Neumann algebra of operators in $\mathcal{B}(H)$ is a selfadjoint algebra of operators in $\mathcal{B}(H)$ which is closed in the weak operator topology. The algebra $B(\beta)$ is an abelian von Neumann algebra. However, the following proposition says that it is not maximal.

<u>Proposition 2.1</u>: If $\{\lambda I : \lambda \in \mathbb{C}\} \subset B(\beta) \subsetneq L^\infty(\beta)$, then $\{M_\varphi : \varphi \in \mathcal{B}(\beta)\}$ is not a maximal abelian von Neumann algebra.

<u>Proof</u>: If $B(\beta) = \{\lambda I : \lambda \in \mathbb{C}\}$, we let S be any nontrivial, selfadjoint operator and consider the von Neumann algebra generated by $B(\beta)$ and S. This will be abelian and properly contain $B(\beta)$. If $B(\beta) \neq \{\lambda I : \lambda \in \mathbb{C}\}$, let $N > 1$ be the least period of the weight sequence. Let $A \in B(L^2(\beta))$ be given by $A(z^{kN}) = z^{(k+1)N}$ for any integer k and $A(z^n) = 0$ otherwise. Then a direct computation shows that $A^*(z^{kN}) = \beta^2(N)z^{(k-1)N}$ and $A^*(z^n) = 0$ for $n \neq kN$. Hence, $AA^*(z^{kn}) = A(\beta^2(N)z^{(k-1)N}) = \beta^2(N)z^{kN} = A^*(Az^{kN})$ and $AA^*(z^n) = 0 = AA^*(z^n)$ for $n \neq kN$. Thus A is normal. Also for $\varphi \in B(\beta)$, $AM_\varphi(z^k) = A(\varphi z^k)$

$$= A\left(\sum_{\ell=-\infty}^{\infty} \hat{\varphi}(\ell N) z^{\ell N+k}\right)$$

$$= \begin{cases} 0 & \text{if } k \neq mN \\ \sum_{\ell=-\infty}^{\infty} \hat{\varphi}(\ell N) z^{\ell N+k+N} & \text{if } k = mN \end{cases}.$$

Now if we look at $M_\varphi A$, we see that $M_\varphi A(z^k) = 0$ if $k \neq mN$. If $k = mN$, then $M_\varphi A(z^k) = M_\varphi(z^{k+N})$

$$= \varphi\, z^{k+N}$$

$$= \sum_{\ell=-\infty}^{\infty} \hat{\varphi}(\ell N) z^{\ell N+k+N}.$$

Hence $AM_\varphi(z^k) = M_\varphi A(z^k)$ for all k, implying $AM_\varphi = M_\varphi A$ for all $\varphi \in B(\beta)$. By Fuglede's theorem (Rudin,[17],Theorem 12.16), A^* also commutes with everything in $\{M_\varphi : \varphi \in B(\beta)\}$. Hence the von Neumann algebra generated by $\{M_\varphi : \varphi \in B(\beta)\}$ and A properly contains $\{M_\varphi : \varphi \in B(\beta)\}$ since $A \notin \{M_\varphi : \varphi \in B(\beta)\}$. Q.E.D.

We now discuss some properties of weighted shifts which have periodic weight sequences.

<u>Proposition 2.2</u>: Let T be an injective bilateral shift having a periodic weight sequence with least period N. Then $\beta(n) = r^n\alpha(n)$ where $r = (\prod_{k=0}^{N-1} w_k)^{1/N}$ and $\alpha(n)$ is periodic.

<u>Proof</u>: Suppose $n \geq 1$ and $n = tN+s$ where $0 \leq s < N$. Then

$$\beta(n) = \prod_{k=0}^{n-1} w_k = (\prod_{k=0}^{N-1} w_k)^t (w_0 w_1 \cdots w_{s-1}) .$$

$$= (\prod_{k=0}^{N-1} w_k)^{(tN+s)/N} (w_0 w_1 \cdots w_{s-1}) / (\prod_{k=0}^{N-1} w_k)^{s/N}$$

$$= r^n[(w_0 w_1 \cdots w_{s-1})/(w_0 \cdots\cdot w_{N-1})^{s/N}] .$$

Since $0 \leq s < N$, the right half of the product above is a bounded sequence $\alpha(n)$ which is periodic and has $\alpha(kN) = 1$ for all $k \geq 0$.

Now suppose $n = tN+s$ where $t < 0$ and $-N < s \leq 0$. Then

$$\beta(n) = (\prod_{k=-N}^{-1} w_k)^{-|t|}(w_{-1}\cdot\cdots\cdot w_s)^{-1}$$

$$= (\prod_{k=0}^{N-1} w_k)^{\frac{tN+s}{N}} (w_{-1}\cdot\cdots\cdot w_s)^{-1}/(w_0\cdot\cdots\cdot w_{N-1})^{s/N}$$

$$= r^n(w_{-1}\cdot\cdots\cdot w_s)^{-1}/(w_0\cdot\cdots\cdot w_{N-1})^{s/N} .$$

Again since $-N < s \leq 0$, the right half of the product is a periodic sequence $\alpha(n)$ with $\alpha(-kN) = 1$ for all $k \geq 0$.

Thus $\beta(n) = r^n\alpha(n)$ where $\alpha(n)$ is bounded. To see that $\alpha(n)$ is periodic overall we note that for all integers k:

$$\alpha(N+k)/\alpha(k) = \beta(N+k)r^k/\beta(k)r^{N+k}$$

$$= \beta(N)\beta(k)/\beta(k)r^N = \beta(N)/r^n = 1. \quad \text{Q.E.D.}$$

<u>Proposition 2.3</u>: For injective bilateral shifts having periodic weight sequences with least period N and $r = \beta(N)^{1/N}$, we have the following:

i) $r(T^{-1})^{-1} = r(T) = r$

ii) $\|T\| = \max\{w_0,\cdots,w_{N-1}\}$ and $\|T^{-1}\|^{-1} = \min\{w_0,\cdots,w_{N-1}\}$

iii) If $N \neq 1$, then $\|T^{-1}\|^{-1} < r(T^{-1})^{-1} = r(T) < \|T\|$

iv) There exists a constant $\ell > 0$ such that $\ell \sup\{\beta(k-n)/\beta(k) : k \geq 0\} \leq \inf\{\beta(k-n)/\beta(k) : k \geq 0\}$ for all n

v) $f \in L^2(\beta)$ if and only if $\sum_{-\infty}^{\infty} \hat{f}(n)r^nz^n \in L^2(\partial\mathbb{D})$

<u>Proof</u>: Parts i) and ii) follow from the corollary to Proposition 7 in Shields [23] and the equation $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$. Part iii) then follows immediately from parts i) and ii).

For part iv) we consider

$$\sup\{\beta(k-n)/\beta(k) : k \geq 0\} = \sup\{r^{k-n}\alpha(k-n)/r^k\alpha(k) : k \geq 0\}$$

$$= r^{-n} \sup\{\alpha(k-n)/\alpha(k) : k \geq 0\}$$

$$= r^{-n} \max\{\alpha(k-n)/\alpha(k) : N \geq k \geq 0\}.$$

Likewise

$$\inf\{\beta(k-n)/\beta(k) : k \geq 0\} = r^{-n} \min\{\alpha(k-n)/\alpha(k) : N \geq k \geq 0\}.$$

Now let $\ell = \min\{\alpha(k-n)/\alpha(k) : N \geq k \geq 0\}/\max\{\alpha(k-n)/\alpha(k) : N \geq k \geq 0\}$. Since $\alpha(n)$ is bounded away from zero, we have $\ell > 0$. This constant ℓ satisfies the desired property.

To prove v), note that $f \in L^2(\beta)$ implies $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2\beta^2(n) < \infty$, so $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 r^{2n}\alpha^2(n) < \infty$. Using the boundedness of $\{\alpha(n) : n \in \mathbb{Z}\}$ again, we see that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 r^{2n} < \infty$. This implies $\sum_{n=-\infty}^{\infty} \hat{f}(n)r^n z^n \in L^2(\partial\mathbb{D})$. The converse is established by tracing this argument backwards since the statements are equivalent at each point. Q.E.D.

<u>Corollary 2.6</u>: Let $R : L^2(\beta) \to L^2(\partial\mathbb{D})$ be given by $R(f) = \sum_{n=-\infty}^{\infty} \hat{f}(n)r^n z^n$. Then R is a similarity between M_z on $L^2(\beta)$ and a scalar multiple of the unweighted shift.

Proof: The fact that R is bounded and invertible follows from the boundedness of $\alpha(n)$; or one may appeal to the closed graph theorem. For $f \in L^2(\partial \mathbb{D})$ we have

$$RM_z R^{-1} f = RM_z\left(\sum_{n=-\infty}^{\infty} \hat{f}(n) r^{-n} z^n\right)$$

$$= R\left(\sum_{n=-\infty}^{\infty} \hat{f}(n) r^{-n} z^{n+1}\right)$$

$$= \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{-n} r^{n+1} z^{n+1}$$

$$= r\left(\sum_{n=-\infty}^{\infty} \hat{f}(n) z^{n+1}\right). \qquad \text{Q.E.D.}$$

Proposition 2.4: If $\varphi \in L^\infty(\beta)$ and $f \in L^2(\beta)$, then $R(\varphi f) = R(\varphi)R(f)$.

Proof: $\widehat{R(\varphi f)}(n) = r^n (\varphi f)(n)$

$$= r^n \sum_{k=-\infty}^{\infty} \hat{\varphi}(k) \hat{f}(n-k)$$

$$= \sum_{k=-\infty}^{\infty} \hat{\varphi}(k) r^k \hat{f}(n-k) r^{n-k}$$

$$= [\widehat{R(\varphi)R(f)}](n).$$

This is true for all n. Hence $R(\varphi f) = R(\varphi)R(f)$. Q.E.D.

Corollary 2.7: The map R is a Banach algebra isomorphism between $L^\infty(\beta)$ and $L^\infty(\partial \mathbb{D})$.

<u>Proof</u>: If $\varphi \in L^\infty(\beta)$, then $\varphi f \in L^2(\beta)$ for all $f \in L^2(\beta)$. Then by Proposition 2.4, $R(\varphi)R(f) \in L^2(\partial\mathbb{D})$ for all $f \in L^2(\beta)$. This says $R(\varphi) \in L^\infty(\partial\mathbb{D})$ since $\{R(f) : f \in L^2(\beta)\} = L^2(\partial\mathbb{D})$. The converse again is achieved by tracing the argument above backwards. Thus $\varphi \in L^\infty(\beta)$ iff $R(\varphi) \in L^\infty(\partial\mathbb{D})$. We also have that

$$\begin{aligned}\|R(\varphi)R(f)\|_{L^2(\partial\mathbb{D})} &\leq \|R(\varphi f)\|_{L^2(\partial\mathbb{D})} \\ &\leq \|R\|\|\varphi f\|_2 \\ &\leq \|R\|\|\varphi\|_\infty\|f\|_2 \\ &\leq \|R\|\|R^{-1}\|\|\varphi\|_\infty\|Rf\|_{L^2(\partial\mathbb{D})}.\end{aligned}$$

Hence $\|R(\varphi)\|_{L^\infty(\partial\mathbb{D})} \leq \|R\|\|R^{-1}\|\|\varphi\|_\infty$. Thus $R : L^\infty(\beta) \to L^\infty(\partial\mathbb{D})$ is continuous. By the open mapping theorem, so is R^{-1}. The only thing left to verify is that $R(\varphi\psi) = R(\varphi)R(\psi)$ for $\varphi, \psi \in L^\infty(\beta)$. This follows from Proposition 2.4. Q.E.D.

<u>Corollary 2.8</u>: Suppose T is a weighted shift with periodic weight sequence. If $\varphi \in L^\infty(\beta)$ and $0 \neq f \in H^2(\beta)$, then $\varphi f = 0$ implies $\varphi = 0$.

<u>Proof</u>: If $\varphi f = 0$ then $R(\varphi f) = R(\varphi)R(f) = 0$. But $0 \neq f \in H^2(\beta)$ implies $0 \neq R(f) \in H^2(\partial\mathbb{D})$. However, $R(\varphi) \in L^\infty(\partial\mathbb{D})$ and $R(\varphi)R(f) = 0$ implies $R(\varphi) = 0$ from the F. and M. Riesz theorem (see Douglas,[6]). Finally, $R(\varphi) = 0$ if and only if $\varphi = 0$. Q.E.D.

From the work above several questions may have arisen. First of all, we know that $\varphi \in B(\beta)$ implies $\bar{\varphi} \in B(\beta)$, and hence $\bar{\varphi} \in L^\infty(\beta)$. At this point we may ask the question; does $\varphi \in L^\infty(\beta)$ imply $\bar{\varphi} \in L^\infty(\beta)$? The answer to this question is unknown. However, in special cases one can answer the question affirmatively. For example, if the weighted shift is rationally strictly cyclic (see Shields, [23], p.101), then the answer is yes. An affirmative answer is also obtained if $\beta(k) = r^k$ for all k or if $\beta(k) = \beta(-k)$ for all k. For bilateral shifts with periodic weight sequence the answer is determined by whether $\alpha(-n)/\alpha(n)$ is a multiplier on $L^\infty(\partial\mathbb{D})$.

The other question involves Corollary 2.8. Can one say that this corollary holds for all bilateral weighted shifts and not just for those with periodic weight sequences? Here again, one can say something about particular weighted shifts. For example, if $\pi_0(T^*)$ contains an open annulus, then one can answer yes since in this case the functions φ and f have Laurent series which converge on the annulus. Thus φ and f are analytic on this annulus. However, it is not true that $\pi_0(T^*)$ contains an open annulus for all weighted shifts T.

CHAPTER III

TOEPLITZ OPERATORS FOR WEIGHTED SHIFTS

Toeplitz operators $T_\varphi : H^2(\partial\mathbb{D}) \to H^2(\partial\mathbb{D})$ for $\varphi \in L^\infty(\partial\mathbb{D})$ have been studied quite widely in recent years. (Douglas, [6]; Halmos, [8], or Sarason, [21]). Some interesting properties have emerged from these studies along with generalizations to $\mathbb{C}^n$ and other H^2 spaces. (Gohberg, [7]; Abrahamse, [1]; or Devinatz, [5]). The idea used here is that M_z on $L^2(\partial\mathbb{D})$ is a weighted shift with all weights equal to 1. Then $\{M_\varphi : \varphi \in L^\infty(\partial\mathbb{D})\}$ is just the commutant $\{M_z\}'$ of M_z on $L^2(\partial\mathbb{D})$. For weighted shifts whose weights are not all 1, we can then generalize the idea of a Toeplitz operator $T_\varphi : H^2(\beta) \to H^2(\beta)$ for $\varphi \in L^\infty(\beta)$. This follows from thinking of a weighted shift as M_z on $L^2(\beta)$. The commutant of T is then identified with $\{M_\varphi : \varphi \in L^\infty(\beta)\}$. Hence we have the following definitions.

<u>Definition 3.1</u>: For $\varphi \in L^\infty(\beta)$, let $T_\varphi \in \mathcal{B}(H^2(\beta))$ be given by $T_\varphi(f) = P(\varphi f)$. Here P is the orthogonal projection of $L^2(\beta)$ onto $H^2(\beta)$ given in Chapter I.

<u>Definition 3.2</u>: $\mathcal{H}^\infty(\beta) = \{\varphi \in L^\infty(\beta) : \hat{\varphi}(n) = 0$ for all $n < 0\}$.

We first note that $\mathcal{H}^\infty(\beta)$ is a closed subalgebra of $L^\infty(\beta)$. It is closed since $\Gamma_n : L^\infty(\beta) \to \mathbb{C}$ given by $\Gamma_n(\varphi) = \hat{\varphi}(n)$ is continuous for each n. It is an

algebra since if $\varphi, \psi \in \mathcal{N}^{\infty}(\beta)$, then $\widehat{(\varphi\psi)}(n) = \sum_{k=0}^{\infty} \hat{\varphi}(k)\hat{\psi}(n-k) = 0$ if $n < 0$. Thus $\varphi\psi \in \mathcal{N}^{\infty}(\beta)$.
At this point, one may ask which properties of Toeplitz operators on $H^2(\partial\mathbb{D})$ carry over for Toeplitz operators on $H^2(\beta)$. Many of the same properties do hold, some with minor modifications. For example, if $\varphi \in L^{\infty}(\partial\mathbb{D})$ then $M_{\varphi}^{*} = M_{\bar{\varphi}}$. This property does not hold for all $\varphi \in L^{\infty}(\beta)$ in general. It will hold, though, if $\varphi \in B(\beta)$. Hence some of the properties for Toeplitz operators on $H^2(\beta)$ will not hold for all $\varphi \in L^{\infty}(\beta)$, only for some subset of $L^{\infty}(\beta)$. Also, we may be able to show some properties hold for certain classes of weighted shifts (e.g. those with periodic weight sequences). However, there are some striking differences which will be pointed out. We will now examine properties of Toeplitz operators on $L^2(\beta)$.

Proposition 3.1: If $\varphi \in B(\beta)$, then $T_{\varphi}^{*} = T_{\bar{\varphi}}$.

Proof: Suppose $\varphi \in B(\beta)$, then $\bar{\varphi} \in L^{\infty}(\beta)$ and $M_{\varphi}^{*} = M_{\bar{\varphi}}$. It is well known that $A \in \mathcal{B}(L^2(\beta))$ and $S = PA|_{H^2(\beta)}$ implies that $S^{*} = PA^{*}|_{H^2(\beta)}$. Thus

$$T_{\varphi}^{*} = PM_{\varphi}^{*}|_{H^2(\beta)} = PM_{\bar{\varphi}}|_{H^2(\beta)} = T_{\bar{\varphi}} . \qquad \text{Q.E.D.}$$

The "converse" of Proposition 3.1 is not true, however. That is, if $T_\varphi^* = T_\psi$ for some $\psi \in L^\infty(\beta)$, then it is not necessarily true that $\psi = \bar{\varphi}$. To see this, consider the following example.

<u>Example 3.1</u>: Let the weight sequence for the weighted shift be given by i) $w_n = 1$ for $n \neq -1$ and ii) $w_{-1} = \frac{1}{2}$. Then $\beta(n) = 1$ for all $n \geq 0$ and $\beta(n) = 2$ for all $n < 0$.

For $n,k \geq 0$, $(T_z z^k, z^n) = 1$ for $k = n-1$ and $(T_z z^k, z^n) = 0$ for $k \neq n-1$. Thus $T_z^*(z^n) = P(z^{n-1}) = T_{z^{-1}}(z^n)$. However, $\bar{z} = z^{-1}\beta(1)/\beta(-1) = \frac{1}{2} z^{-1}$ implying $\bar{z} \in L^\infty(\beta)$ and $T_{\bar{z}} = \frac{1}{2} T_z^*$.

<u>Proposition 3.2</u>: Let $\Phi : L^\infty(\beta) \to \mathcal{B}(H^2(\beta))$ be defined by $\Phi(\varphi) = T_\varphi$. Then Φ is linear and contractive. Moreover, $\Phi|_{B(\beta)}$ is *-linear and contractive.

<u>Proof</u>: It is easy to see that Φ is linear. Also, by Proposition 3.1 we need only verify that Φ is contractive. This is easy, however, since for $f \in H^2(\beta)$ and $\varphi \in L^\infty(\beta)$ we have:

$$\|T_\varphi f\|_2 = \|P(\varphi f)\|_2 \leq \|\varphi f\|_2 \leq \|\varphi\|_\infty \|f\|_2 .$$

The last inequality holds, of course, by the definition of $\|\varphi\|_\infty$. We thus have $\|T_\varphi\| \leq \|\varphi\|_\infty$. Q.E.D.

We now have the following proposition which was proved by Brown and Halmos [9] in the unweighted case. We note that in the statement of the proposition, we say $\overline{\psi} \in H^2(\beta)$. This just says that $\hat{\psi}(n) = 0$ for $n > 0$. We use this instead of $\overline{\psi} \in \mathcal{H}^{\infty}(\beta)$ since it is unknown whether $\psi \in L^{\infty}(\beta)$ implies $\overline{\psi} \in L^{\infty}(\beta)$ as mentioned before.

<u>Proposition 3.3</u>: Let $\varphi, \psi \in L^{\infty}(\beta)$. Then $T_{\psi}T_{\varphi} = T_{\psi\varphi}$ if and only if $\varphi \in \mathcal{H}^{\infty}(\beta)$ or $\overline{\psi} \in H^2(\beta)$.

<u>Proof</u>: Suppose $\varphi \in \mathcal{H}^{\infty}(\beta)$. Then for $f \in H^2(\beta)$ it is easy to see that $\varphi f \in H^2(\beta)$. Thus $T_{\varphi}f = \varphi f$. Then $T_{\psi}T_{\varphi}f = T_{\psi}(\varphi f) = P((\psi\varphi)(f)) = T_{\psi\varphi}(f)$. Now suppose $\overline{\psi} \in H^2(\beta)$. For $k \geq 0$ let $h = T_{\psi}T_{\varphi}(z^k)$. Then for $n \geq 0$

$$
\begin{aligned}
\beta^2(n)\hat{h}(n) &= (T_{\psi}T_{\varphi}(z^k), z^n) \\
&= (T_{\psi}(\sum_{\ell=0}^{\infty} \hat{\varphi}(\ell-k)z^{\ell}), z^n) \\
&= (\psi(\sum_{\ell=0}^{\infty} \hat{\varphi}(\ell-k)z^{\ell}), z^n) \\
&= \beta^2(n)(\sum_{\ell=0}^{\infty} \hat{\varphi}(\ell-k)\hat{\psi}(n-\ell)) \\
&= \beta^2(n)(\sum_{\ell=n}^{\infty} \hat{\varphi}(\ell-k)\hat{\psi}(n-\ell)) \ .
\end{aligned}
$$

If $f = T_{\psi\varphi}(z^k)$, then for $n \geq 0$

$$\beta^2(n)\hat{f}(n) = ((\psi\varphi)(z^k), z^n)$$

$$= \beta^2(n)(\psi\varphi)(n-k)$$

$$= \beta^2(n) \sum_{\ell=-\infty}^{\infty} \hat{\varphi}(\ell)\hat{\psi}(n-k-\ell) \; .$$

Letting $\ell' = \ell + k$ we get:

$$\beta^2(n)\hat{f}(n) = \beta^2(n) \sum_{\ell'=-\infty}^{\infty} \hat{\varphi}(\ell'-k)\hat{\psi}(n-\ell')$$

$$= \beta^2(n) \sum_{\ell'=n}^{\infty} \hat{\varphi}(\ell'-k)\hat{\psi}(n-\ell')$$

$$= \beta^2(n)\hat{h}(n) \; .$$

Thus $T_\psi T_\varphi(z^k) = T_{\psi\varphi}(z^k)$ for all $k \geq 0$. This implies $T_\psi T_\varphi = T_{\psi\varphi}$ since $\{z^k : k \geq 0\}$ forms an orthogonal basis for $H^2(\beta)$.

Now for the converse, suppose $T_\psi T_\varphi = T_{\psi\varphi}$ and $\varphi \notin \mathcal{N}^\infty(\beta)$. Then for $k,n \geq 0$, we must have $\sum_{\ell=-\infty}^{\infty} \hat{\varphi}(\ell-k)\hat{\psi}(n-\ell) = \sum_{\ell=0}^{\infty} \hat{\varphi}(\ell-k)\hat{\psi}(n-\ell)$ from the calculations above. Thus $\sum_{\ell=-\infty}^{-1} \hat{\varphi}(\ell-k)\hat{\psi}(n-\ell) = 0$ for $k,n \geq 0$ or equivalently $\sum_{\ell=1}^{\infty} \hat{\varphi}(-\ell-k)\hat{\psi}(n+\ell) = 0$ for all $k,n \geq 0$. Taking $k = 0$ and $n = m$ we get

$$(*) \qquad \sum_{\ell=1}^{\infty} \hat{\varphi}(-\ell)\hat{\psi}(m+\ell) = 0 \text{ for all } m \geq 0 \; .$$

Also letting $\ell' = \ell + k$ we get $\sum_{\ell'=k+1}^{\infty} \hat{\varphi}(-\ell')\hat{\psi}(n-k+\ell') = 0$.

Then for $n = k+m$ we have

$$(**) \qquad \sum_{\ell=k+1}^{\infty} \hat{\varphi}(-\ell)\hat{\psi}(m+\ell) = 0 \quad \text{for all } m \geq 0,\ k \geq 0 .$$

Putting (*) and (**) together we get $\sum_{\ell=1}^{k} \hat{\varphi}(-\ell)\hat{\psi}(m+\ell) = 0$ for all $m \geq 0$, $k \geq 1$. Since $\varphi \notin \mathcal{H}^{\infty}(\beta)$ there exists $N > 0$ such that $\hat{\varphi}(-N) \neq 0$. Now by the first part of this proposition $T_{\psi}T_{\varphi}T_{z^{N-1}} = T_{\psi}T_{\varphi z^{N-1}} = T_{\psi\varphi z^{N-1}}$. So we may assume that $\hat{\varphi}(-1) \neq 0$. Letting $k = 1$ in our last sum above, we get $\hat{\psi}(m+1) = 0$ for all $m \geq 0$. Thus $\bar{\psi} \in H^2(\beta)$ as desired. Q.E.D.

In the case of the unweighted shift, it has been shown (Douglas, [6]) that if $\varphi \in L^{\infty}(\partial\mathbb{D})$, then $\sigma(M_{\varphi}) \subset \sigma(T_{\varphi})$. This inclusion is used to show that $\|T_{\varphi}\| = \|M_{\varphi}\|$. The proof is given by the chain of inequalities: $\|M_{\varphi}\| = r(M_{\varphi}) \leq r(T_{\varphi}) \leq \|T_{\varphi}\| \leq \|M_{\varphi}\|$. Thus $\Phi : L^{\infty}(\partial\mathbb{D}) \to \mathcal{B}(H^2(\partial\mathbb{D}))$ is an isometric *-homomorphism between $L^{\infty}(\partial\mathbb{D})$ and a closed subspace of $\mathcal{B}(H^2(\partial\mathbb{D}))$. However, in the case when not all of the weights are 1, it is not necessarily true that $r(M_{\varphi}) = \|M_{\varphi}\|$. This will be seen in Chapter 6. One may still ask, though, whether there is a spectral inclusion theorem $(\sigma(M_{\varphi}) \subset \sigma(T_{\varphi}))$. This property is examined in the next two results and the example following them.

<u>Proposition 3.4</u>: Let T, a weighted shift with periodic weight sequence, be represented as M_z on $L^2(\beta)$. If $\varphi \in L^\infty(\beta)$ and T_φ is invertible then M_φ is invertible.

<u>Proof</u>: Since T_φ is invertible there is a constant $c_1 > 0$ such that $\|\varphi f\|_2 \geq \|T_\varphi f\|_2 \geq c_1\|f\|_2$ for all $f \in H^2(\beta)$. Also, there is a constant $c_2 > 0$ such that $\|T_\varphi^* f\|_2 \geq c_2\|f\|_2$ for all $f \in L^2(\beta)$. Now let $n > 0$ and consider

$$\|z^{-n}f\|_2 = (\sum_{k=0}^{\infty} |\hat{f}(k)|^2\beta^2(k-n))^{1/2}$$

$$= [\sum_{k=0}^{\infty} |\hat{f}(k)|^2\beta^2(k)\beta^2(k-n)/\beta^2(k)]^{1/2}$$

$$\leq (\sum_{k=0}^{\infty} |\hat{f}(k)|^2\beta^2(k))^{1/2}\sup\{\beta(k-n)/\beta(k) : k \geq 0\} .$$

By Proposition 2.3 there exists $\ell > 0$ such that $\sup\{\beta(k-n)/\beta(k) : k \geq 0\} \leq \inf\{\beta(k-n)/\beta(k) : k \geq 0\}/\ell$. Thus

$$\|z^{-n}f\|_2 \leq \|f\|_2 \inf\{\beta(k-n)/\beta(k) : k \geq 0\}/\ell$$

$$\leq \|T_\varphi f\|_2 \inf\{\beta(k-n)/\beta(k) : k \geq 0\}/\ell c_1$$

$$\leq \|z^{-n}T_\varphi f\|_2/\ell c_1$$

$$\leq \|z^{-n}M_\varphi f\|_2/\ell c_1$$

$$\leq \|M_\varphi(z^{-n}f)\|_2/\ell c_1 \text{ since } M_{z^{-n}}M_\varphi = M_\varphi M_{z^{-n}} .$$

Now since $\{z^{-n}f : f \in H^2(\beta), n > 0\}$ is dense in $L^2(\beta)$, we have M_φ is bounded below on $L^2(\beta)$.

We will now show that M_φ^* is bounded below. The two conditions that M_φ is bounded below and M_φ^* is bounded below then imply M_φ is invertible (Douglas, [6], p.84). To prove M_φ^* is bounded below we attempt to imitate the proof that M_φ is bounded below. A new difficulty is encountered here since $M_{z^{-n}}$ does not necessarily commute with M_φ^*. However, we note that it is sufficient to use $n = kN$ where N is the period of the weight sequence and k is a nonnegative integer. It is sufficient since $\{z^{-kN}f : f \in H^2(\beta),\ k \geq 0\}$ is also dense in $L^2(\beta)$. Now we have

$$\|z^{-kN}f\|_2 \leq \|z^{-kN}T_\varphi^* f\|_2/c_2\ell \leq \|z^{-kN}M_\varphi^* f\|_2/\ell c_2 \leq \|M_\varphi^*(z^{-kN}f)\|_2/\ell c_2 \ .$$

Noting that $z^{-kN} \in B(\beta)$, the last inequality is a result of the following equation.

$$M_{z^{-kN}} M_\varphi^* = M^*_{\overline{z^{-kN}}} M_\varphi^* = M_\varphi^* M^*_{\overline{z^{-kN}}} = M_\varphi^* M_{z^{-kN}} \ .$$

Thus both M_φ and M_φ^* are bounded below. Q.E.D.

Thus we do have a spectral inclusion theorem for shifts with periodic weight sequence. We note that a key part of the proof involved the existence of a constant $\ell > 0$ such that $\ell \sup\{\beta(k-n)/\beta(k) : k \geq 0\} \leq \inf\{\beta(k-n)/\beta(k) : k \geq 0\}$. The following theorem shows that this is a sufficient condition for a spectral inclusion theorem.

<u>Theorem 3.1</u>: Let T be an invertible weighted shift for which there exists a constant $\ell > 0$ such that $\ell \sup\{\beta(k-n)/\beta(k) : k \geq 0\} \leq \inf\{\beta(k-n)/\beta(k) : k \geq 0\}$. If $\varphi \in L^\infty(\beta)$ and T_φ is invertible, then M_φ is invertible.

<u>Proof</u>: We again prove that both M_φ and M_φ^* are bounded below. The proof that M_φ is bounded below is exactly the same as in Proposition 3.4. We note, however, that if the weight sequence is not periodic, then $B(\beta) = \{\lambda I : \lambda \in \mathbb{C}\}$. Thus we cannot use the same idea as in the last part of Proposition 3.4. To show that M_φ^* is bounded below we will first show that $U = \{M_{z^n}^* f : f \in H^2(\beta),\ n \geq 0\}$ is dense in $L^2(\beta)$.

If $g = \sum_{k=-n}^{\infty} \hat{g}(k) z^k \in L^2(\beta)$, then $(M_{z^n}^*)^{-1}(g) \in H^2(\beta)$ as shown by the following computation:

$$\begin{aligned}((M_{z^n}^*)^{-1}(g), z^k) &= (g, M_{z^n}^{-1} z^k)\\ &= (g, M_{z^{-n}} z^k)\\ &= (g, z^{k-n})\\ &= \hat{g}(k-n)\beta^2(k-n)\\ &= 0 \quad \text{if} \quad k < 0.\end{aligned}$$

This implies $g = M_{z^n}^*((M_{z^n}^*)^{-1}(g)) \in U$ and shows that U is dense in $L^2(\beta)$.

Now

$$\|M^*_{z^n} f\|_2 = \Big(\sum_{k=-n}^{\infty} |\hat{f}(k+n)|^2 \beta^2(k+n)\beta^2(k+n)/\beta^2(k)\Big)^{1/2}$$

$$\leq \Big(\sum_{k=-n}^{\infty} |f(k+n)|^2 \beta^2(k+n)\Big)^{1/2} \sup\{\beta(k+n)/\beta(k) : k \geq -n\}$$

$$\leq \|f\|_2 \sup\{\beta(k)/\beta(k-n) : k \geq 0\}$$

$$\leq \|f\|_2 (\inf\{\beta(k-n)/\beta(k) : k \geq 0\})^{-1}$$

$$\leq \|f\|_2 (\sup\{\beta(k-n)/\beta(k) : k \geq 0\})^{-1} \ell^{-1}$$

$$\leq \|f\|_2 \inf\{\beta(k)/\beta(k-n) : k \geq 0\}/\ell$$

$$\leq \|T^*_{\varphi} f\|_2 \inf\{\beta(k+n)/\beta(k) : k \geq -n\}/\ell c_2$$

$$\leq \|M^*_{z^n} T^*_{\varphi} f\|_2/\ell c_2$$

$$\leq \|M^*_{z^n} M^*_{\varphi} f\|_2/\ell c_2$$

$$\leq \|M^*_{\varphi}(M^*_{z^n} f)\|_2/\ell c_2 \quad .$$

We note that the first equality comes from a direct computation of $M^*_{z^n} f = \sum_{k=-n}^{\infty} \hat{f}(k+n)(\beta^2(k+n)/\beta^2(k))z^k$. We have hence verified that M^*_{φ} is bounded below on $L^2(\beta)$.

Thus we have both M_{φ} and M^*_{φ} bounded below on $L^2(\beta)$ which again implies that M_{φ} is invertible. Q.E.D.

However, a spectral inclusion theorem does not hold for all invertible weighted shifts. The following example illustrates this point.

Example 3.1: Let T be the weighted shift with weight sequence given as below:

i) $w_n = 1$ if $n \geq -1$

ii) $w_n = \frac{1}{2}$ if $n < -1$

Then for $k \geq 0$, $\|M^k_{z^{-1}}\| = 2^k$. It is also not difficult to verify that $\|M_{z^{-1}}|_{H^2(\beta)}\| = 1$. Hence, $r(M_{z^{-1}}) = 2$ from the first equality and $r(T_{z^{-1}}) \leq \|M_{z^{-1}}|_{H^2(\beta)}\| \leq 1$. Thus it is not possible that $\sigma(M_{z^{-1}}) \subset \sigma(T_{z^{-1}})$.

We now examine other conditions on T_φ which imply something about the invertibility of M_φ. The first result has been proven for the unweighted shift. Its proof can be found in Douglas, [6].

Proposition 3.5: If T is a weighted shift with periodic weight sequence, then either $\operatorname{Ker} T_\varphi = \{0\}$ or $\operatorname{Ker} T^*_\varphi = \{0\}$ for all $\varphi \in L^\infty(\beta)$, $\varphi \not\equiv 0$.

Proof: Suppose both $\operatorname{Ker} T_\varphi \neq \{0\}$ and $\operatorname{Ker} T^*_\varphi \neq \{0\}$. Then there exist nonzero elements $f, g \in H^2(\beta)$ such that $T_\varphi g = 0 = T^*_\varphi f$. Since $T_\varphi g = 0$ we have $\overline{\varphi g} \in H^2_0(\beta)$

where $H_0^2(\beta) = \{h \in H^2(\beta) : \hat{h}(0) = 0\}$. Also $T_\varphi^* f = 0$ implies $(T_\varphi^* f, z^n) = (M_\varphi^* f, z^n) = (f, \varphi z^n) = 0$ for all $n \geq 0$. This last equation says $\sum_{k=0}^{\infty} \hat{f}(k)\overline{\hat{\varphi}(k-n)}\beta^2(k) = 0$ for all $n \geq 0$.

Now let $h \in H^2(\beta)$ be given by $\hat{h}(k) = \hat{f}(k)\beta(-k)\beta(k)$. (We note that since the shift is periodic $\beta(k)\beta(-k)$ is bounded.) Now

$$\begin{aligned}
\widehat{(\varphi\overline{h})}(n) &= \sum_{k=-\infty}^{\infty} \hat{\overline{h}}(-k)\hat{\varphi}(n+k) \\
&= \sum_{k=-\infty}^{\infty} \overline{\hat{h}(k)}(\beta(k)/\beta(-k))\hat{\varphi}(n+k) \\
&= \sum_{k=0}^{\infty} \overline{\hat{f}(k)}\hat{\varphi}(n+k)\beta^2(k) \\
&= 0 \quad \text{for all} \quad n \leq 0 \quad \text{from the last line}
\end{aligned}$$

of the previous paragraph. Thus $\varphi\overline{h} \in H_0^2(\beta)$.

So $R(\varphi\overline{h}) = R(\varphi)R(\overline{h}) \in H_0^2(\partial\mathbb{D})$ and $\overline{R(\varphi g)} = \overline{R(\varphi)}\ \overline{R(g)} \in H_0^2(\partial\mathbb{D})$. This says $R(\varphi)R(\overline{h})R(g) \in H_0^1(\partial\mathbb{D})$ and $\overline{R(\varphi)R(\overline{h})R(g)} \in H_0^1(\partial\mathbb{D})$ since $\overline{R(\overline{h})} \in H^2(\partial\mathbb{D})$ and $R(g) \in H^2(\partial\mathbb{D})$. Now by Douglas, ([6], Corollary 6.7) we have $R(\varphi)R(\overline{h})R(g) = 0$ which implies $R(\varphi g)R(\overline{h}) = 0$. By the F. and M. Riesz Theorem if $R(\overline{h}) \not\equiv 0$, then $R(\varphi g) = 0$. Corollary 2.6 of this paper then implies that $\varphi = 0$. This is a contradiction, and so we are done. Q.E.D.

The next two corollaries have also been proven in the unweighted case. Their proofs are also found in Douglas [6].

<u>Corollary 3.1</u>: If $0 \neq \varphi \in B(\beta)$ and T_φ has closed range, then M_φ is invertible.

<u>Proof</u>: We may assume without loss of generality that the weighted shift has periodic weight sequence. If not, $B(\beta) = \{\lambda I : \lambda \in \mathbb{C}\}$ in which case the result is trivial.

Now if $\varphi \in B(\beta)$ then $T_\varphi^* = T_{\overline{\varphi}}$. Also by Proposition 3.5, we may assume $\text{Ker } T_\varphi = \{0\}$. Then since T_φ has closed range and $\text{Ker } T_\varphi = \{0\}$, we have T_φ is bounded below on $H^2(\beta)$. One can then show, as before, that M_φ is bounded below on $L^2(\beta)$. That is, there exists a constant $c > 0$ such that $\|\varphi f\|_2 \geq c\|f\|_2$ for all $f \in L^2(\beta)$. Then for $f \in L^2(\beta)$, we have:

$$\|\overline{\varphi}f\|_2 = \|\overline{\varphi\overline{f}}\|_2 = \|\varphi\overline{f}\|_2 \geq c\|\overline{f}\|_2 = c\|f\|_2 \text{ .}$$

The second equality holds by Corollary 2.3. The above chain shows that $M_\varphi^* = M_{\overline{\varphi}}$ is bounded below on $L^2(\beta)$. As before, we conclude that M_φ is invertible. Q.E.D.

We recall now that $S \in \mathcal{B}(H)$ is said to be Fredholm if the range of S is closed and if both the kernel of S and S^* are finite dimensional. If S

is Fredholm, we define the index $i(S)$ of S by $i(S) = \dim(\operatorname{Ker} S) - \dim(\operatorname{Ker} S^*)$.

Corollary 3.2: If T is a weighted shift with periodic weight sequence and $\varphi \in L^\infty(\beta)$, then T_φ is invertible if and only if T_φ is Fredholm and $i(T_\varphi) = 0$.

Proof: It is easy to verify that if T_φ is invertible then T_φ is Fredholm and $i(T_\varphi) = 0$. So if T_φ is Fredholm and $i(T_\varphi) = 0$, then $\operatorname{Ker} T_\varphi = \{0\}$ and $\operatorname{Ker} T_\varphi^* = \{0\}$ by Proposition 3.5. This implies both T_φ and T_φ^* are bounded below on $H^2(\beta)$ since they both have closed range. Thus T_φ is invertible. Q.E.D.

We now present the last result concerning $B(\beta)$.

Proposition 3.6: $\Phi : B(\beta) \to \mathcal{B}(H^2(\beta))$ is *-linear and isometric.

Proof: By Proposition 3.2, we need only show that Φ is isometric on $B(\beta)$. The C*-algebra $B(\beta)$ is commutative and hence $r(M_\varphi) = \|M_\varphi\|$ for all $\varphi \in B(\beta)$. Thus $\|M_\varphi\| = r(M_\varphi) \le r(T_\varphi) \le \|T_\varphi\| \le \|M_\varphi\|$ for $\varphi \in B(\beta)$. The first inequality holds from the spectral inclusion theorem. The result is achieved since equality must hold throughout. Q.E.D.

We now discuss some miscellaneous problems and results concerning multiplication operators, Toeplitz operators, and $\mathcal{H}^{\infty}(\beta)$. First of all we note that for $\varphi \in L^{\infty}(\beta)$, $(M_{\varphi}z^i, z^j)/\beta^2(j) = \hat{\varphi}(j-i) = (M_{\varphi}z^{i+1}, z^{j+1})/\beta^2(j+1)$. So suppose $L \in \mathcal{B}(L^2(\beta))$ is such that $(Lz^j, z^j)/\beta^2(j) = (Lz^{i+1}, z^{j+1})/\beta^2(j+1)$. Is it then true that there is $\varphi \in L^{\infty}(\beta)$ such that $L = M_{\varphi}$? The answer to this question is given below.

<u>Proposition 3.6</u>: Let $L \in \mathcal{B}(L^2(\beta))$ satisfy the equation $(Lz^n, z^m)/\beta^2(m) = (Lz^{n+1}, z^{m+1})/\beta^2(m+1)$. Then there exists $\varphi \in L^{\infty}(\beta)$ such that $L = M_{\varphi}$.

<u>Proof</u>: Since $\{M_{\varphi} : \varphi \in L^{\infty}(\beta)\} = \{M_z\}'$, the commutant of M_z on $L^2(\beta)$, we only need to show that $LM_z = M_zL$. This will be true if and only if $(LM_zz^i, z^j) = (M_zLz^i, z^j)$ for all integers i and j.

Now $(LM_zz^i, z^j) = (Lz^{i+1}, z^j)$ and

$$
\begin{aligned}
(M_zLz^i, z^j) &= (Lz^i, M_z^*z^j) \\
&= (Lz^i, z^{j-1})\beta^2(j)/\beta^2(j-1) \\
&= (Lz^{i+1}, z^j) \quad \text{by the property of } L \text{ given.}
\end{aligned}
$$

Thus $(M_zLz^j, z^j) = (LM_zz^i, z^j)$ and we are done. Q.E.D.

Now note that for Toeplitz operators T_φ with $\varphi \in L^\infty(\beta)$, we have the same type of property. The only difference is that $(T_\varphi z^i, z^j)/\beta^2(j) = (T_\varphi z^{i+1}, z^{j+1})/\beta^2(j+1)$ holds only for $i,j \geq 0$. The question here is like the question above. Suppose $S \in \mathcal{B}(H^2(\beta))$ satisfies the property above; is it true that there exists $\varphi \in L^\infty(\beta)$ such that $S = T_\varphi$? The answer to this question is no and the solution follows. We begin with the following proposition.

<u>Proposition 3.7</u>: Let T be an invertible weighted shift with $\beta(n) = 1$ for $n \geq 0$ and $\sup\{\beta(n) : n < 0\} = \infty$. Then there exists $f \in C(\partial\mathbb{D})$ such that $\sum_{n=-\infty}^{\infty} \hat{f}(n) z^n \notin L^\infty(\beta)$ where $\hat{f}(n) = \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi}$.

<u>Proof</u>: Suppose $f \in C(\partial\mathbb{D})$ implies $\sum_{n=-\infty}^{\infty} \hat{f}(n) z^n \in L^\infty(\beta)$. Let $I : C(\partial\mathbb{D}) \to L^\infty(\beta)$ be the map that sends f to $\sum_{n=-\infty}^{\infty} \hat{f}(n) z^n$. Then I is continuous as a result of the closed graph theorem. The graph $\mathcal{G} = \{(f, If) : f \in C(\partial\mathbb{D})\}$ is closed since the coefficient functional Γ_n which sends f to $\hat{f}(n)$ is continuous on both $C(\partial\mathbb{D})$ and $L^\infty(\beta)$. Now since I is continuous, we have:

$$\beta(n) = \|z^n\|_2 \leq \|z^n\|_\infty \leq \|I\| \|z^n\|_{\partial\mathbb{D}} \leq \|I\| \quad \text{for all } n.$$

But this contradicts the assumption that $\sup\{\beta(n) : n < 0\} = \infty$. Thus there must exist $f \in C(\partial \mathbb{D})$ such that $\sum_{n=-\infty}^{\infty} \hat{f}(n)z^n \notin L^\infty(\beta)$. Q.E.D.

Corollary 3.3: Let T be an invertible weighted shift with $\sup\{\beta(n) : n < 0\} = \infty$ and $\beta(n) = 1$ for $n \geq 0$. Then there exists $S \in \mathcal{B}(H^2(\beta))$ such that $(Sz^i, z^j)/\beta^2(j) = (Sz^{i+1}, z^{j+1})/\beta^2(j+1)$ for all $i,j \geq 0$, but $S \neq T_\varphi$ for any $\varphi \in L^\infty(\beta)$.

Proof: We note that $H^2(\beta) = H^2(\partial \mathbb{D})$. Let S on $H^2(\beta)$ be defined by $(Sz^i, z^j) = \hat{f}(j-i)$ where $f \in C(\partial \mathbb{D})$ but $\sum_{n=-\infty}^{\infty} \hat{f}(n)z^n \notin L^\infty(\beta)$. Then $S \in \mathcal{B}(H^2(\beta))$ by Halmos, [8], Problem 194. But if $S = M_\varphi$ for some $\varphi \in L^\infty(\beta)$, then $\hat{\varphi}(k) = \hat{f}(k)$ for all k by picking $i,j \geq 0$ such that $j - i = k$. This equality implies $\sum_{n=-\infty}^{\infty} \hat{f}(n)z^n \in L^\infty(\beta)$, a contradiction. Q.E.D.

Corollary 3.4: If T is an invertible weighted shift with $\sup\{\beta(n) : n < 0\} = \infty$ and $\beta(n) = 1$ for all $n \geq 0$, then $\{T_\varphi : \varphi \in L^\infty(\beta)\}$ is not closed in $\mathcal{B}(H^2(\beta))$.

Proof: Let $f \in C(\partial \mathbb{D})$ be such that $\sum_{n=-\infty}^{\infty} \hat{f}(n)z^n \notin L^\infty(\beta)$. Then $\sigma_n(f) = \sum_{k=-n}^{n} (1 - \frac{|k|}{n+1})\hat{f}(k)z^k \in L^\infty(\beta)$.

Also, $\sigma_n(f) \in C(\partial \mathbb{D})$ and $\|\sigma_n(f)-f\|_{\partial \mathbb{D}} \to 0$ as $n \to \infty$ (Katznelson, [13], Thm. 2.11). Let S be as in Corollary 3.3. Then

$$\|T_{\sigma_{n(f)}}-S\| \leq \|\sigma_n(f)-f\|_{\partial \mathbb{D}} \quad \text{(Douglas, [6], Prop. 7.4).}$$

Hence $S \notin \{T_\varphi : \varphi \in L^\infty(\beta)\}$ implies this set is not closed in $\mathcal{B}(H^2(\beta))$. Q.E.D.

We note that these last two corollaries show a remarkable difference from the unweighted case, $\beta(n) = 1$ for all n. In this case it is true that if $S \in \mathcal{B}(H^2(\beta))$ satisfies $(Sz^i,z^j) = (Sz^{i+1},z^{j+1})$ then $S = T_\varphi$ for some $\varphi \in L^\infty(\beta)$. This also says that $\{T_\varphi : \varphi \in L^\infty(\beta)\}$ is closed in $\mathcal{B}(H^2(\beta))$.

In the last part of this chapter, we consider the following. Let $H^\infty(\beta) = \{\psi = \sum_{n=0}^{\infty} \hat{\psi}(n)z^n : \psi f \in H^2(\beta)$ all $f \in H^2(\beta)\}$ where the multiplication ψf is defined as before. Then it can be shown (Shields, [23], Thm. 3) that $\{T_z\}' = \{M_\psi : \psi \in H^\infty(\beta)\}$. The problem we want to consider here is: what is the relationship between $\mathcal{H}^\infty(\beta)$ and $H^\infty(\beta)$ when considered as sets of sequences? It is easy to verify that $\mathcal{H}^\infty(\beta) \subset H^\infty(\beta)$ since $\mathcal{H}^\infty(\beta) \subset L^\infty(\beta)$. For the three cases: $\beta(n) = 1$ for all n; $\beta(n) = r^n$ for all n; and the case where the shift is rationally strictly cyclic, we have $\mathcal{H}^\infty(\beta) = H^\infty(\beta)$. However, in general, it is not true that $\mathcal{H}^\infty(\beta) = H^\infty(\beta)$ as the following example indicates.

<u>Example 3.2</u>: Let $\beta(k) = 2^{-|k|}$ for all k.

Then it can be shown that $\psi \in H^{\infty}(\beta)$ if and only if $\psi(z) = \sum_{n=0}^{\infty} \hat{\psi}(n) z^n$ is a bounded analytic function on $\{z \in ¢ : |z| < 1/2\}$. (Shields, [23], Theorem 10′).

Thus $\psi = \sum_{k=0}^{\infty} z^k \in H^{\infty}(\beta)$. Also, $g = \sum_{k=1}^{\infty} \frac{2^k}{k} z^{-k} \in L^2(\beta)$.

If it were true that $\psi \in \mathcal{K}^{\infty}(\beta)$ then $(\widehat{\psi g})(0)$ would be defined (and finite). However

$$(\widehat{\psi g})(0) = \sum_{k=0}^{\infty} \hat{\psi}(k)\hat{g}(-k) = \sum_{k=1}^{\infty} 1 \cdot \frac{2^k}{k} = \infty .$$

Thus $\psi \in H^{\infty}(\beta)$ but $\psi \notin \mathcal{K}^{\infty}(\beta)$.

CHAPTER IV

HANKEL OPERATORS FOR WEIGHTED SHIFTS

The study of Hankel operators is a natural outgrowth of the study of Toeplitz operators. Again, as in the case of Toeplitz operators, Hankel operators for the unweighted shift have been studied quite thoroughly. One area of study has been to determine which Hankel operators are compact. We will be considering this question for weighted shifts. We will also point out during the course of the investigation differences between the unweighted case and other cases. We start with the following definitions.

<u>Definition 4.1</u>: Let T be an invertible weighted shift represented as M_z on $L^2(\beta)$. For $\varphi \in L^\infty(\beta)$ we define the Hankel operator $H_\varphi : H^2(\beta) \to L^2(\beta) \ominus H^2(\beta)$ with symbol φ by

$$H_\varphi(f) = (1-P)(\varphi f) \quad \text{for all} \quad f \in H^2(\beta) .$$

We recall that P is the orthogonal projection of $L^2(\beta)$ onto $H^2(\beta)$ given in Chapter I. Thus $1-P$ is the orthogonal projection of $L^2(\beta)$ onto the orthogonal complement $L^2(\beta) \ominus H^2(\beta)$ of $H^2(\beta)$ in $L^2(\beta)$. Also, we note there is no loss in assuming T is invertible. If T is not invertible then $L^\infty(\beta) = H^\infty(\beta)$ (Shields, [23], p.68). We will show later that $H_\psi = 0$ for all

$\psi \in \mathcal{N}^\infty(\beta)$. For the rest of this paper, the weighted shift T will be invertible unless otherwise stated.

Definition 4.2: $C(\beta)$ is the closed linear span in $L^\infty(\beta)$ of the (Laurent) polynomials.

Equivalently, $f \in L^\infty(\beta)$ is in $C(\beta)$ if and only if for every $\varepsilon > 0$ there exists $q = \sum_{k=-N}^{N} \hat{q}(k)z^k$ such that $\|f - q\|_\infty < \varepsilon$. This definition is motivated by the unweighted case. In that case, $C(\beta) = C(\partial\mathbb{D})$; and $C(\partial\mathbb{D})$ is the closed linear span of the (Laurent) polynomials in $L^\infty(\partial\mathbb{D})$.

We now begin our study with some easy results concerning Hankel operators.

Proposition 4.1: The map $\varphi \to H_\varphi$ is a contractive linear map from $L^\infty(\beta)$ into $\mathcal{B}(H^2(\beta), L^2(\beta) \ominus H^2(\beta))$.

Proof: It is clear that the map is linear. The proof that it is contractive is just like the proof for Toeplitz operators.

$$\|H_\varphi f\|_2 = \|(1-P)(\varphi f)\|_2 \leq \|\varphi f\|_2 \leq \|\varphi\|_\infty \|f\|_2 \quad \text{for all } f \in H^2(\beta).$$

Thus $\|H_\varphi\| \leq \|\varphi\|_\infty$. Q.E.D.

Proposition 4.2: If $\varphi \in \mathcal{N}^\infty(\beta)$, then $H_\varphi = 0$.

<u>Proof</u>: For $\varphi \in \mathscr{H}^\infty(\beta)$ and $f \in H^2(\beta)$ we have shown $\varphi f \in H^2(\beta)$. Thus $H_\varphi(f) = (1-P)(\varphi f) = 0$ for all $f \in H^2(\beta)$. Q.E.D.

<u>Definition 4.3</u>: For $\varphi \in L^\infty(\beta)$ we define the distance $d(\varphi, \mathscr{H}^\infty(\beta))$ between φ and $\mathscr{H}^\infty(\beta)$ by

$$d(\varphi, \mathscr{H}^\infty(\beta)) = \inf\{\|\varphi - \psi\|_\infty : \psi \in \mathscr{H}^\infty(\beta)\} .$$

<u>Proposition 4.3</u>: If $\varphi \in L^\infty(\beta)$, then $\|H_\varphi\| \le d(\varphi, \mathscr{H}^\infty(\beta))$.

<u>Proof</u>: Let $\varphi \in L^\infty(\beta)$ and $\psi \in \mathscr{H}^\infty(\beta)$. Then $H_{\varphi-\psi} = H_\varphi$ by Proposition 4.2. Now by Proposition 4.1, $\|H_\varphi\| = \|H_{\varphi-\psi}\| \le \|\varphi-\psi\|_\infty$. Since this is true for all $\psi \in \mathscr{H}^\infty(\beta)$, we have the result. Q.E.D.

The first three propositions and their proofs are identical with ones for the unweighted case. However, in that case it can be shown that $\|H_\varphi\| = d(\varphi, \mathscr{H}^\infty(\beta))$ for all $\varphi \in L^\infty(\beta)$ (Nehari, [15]). We will see that this is not true for all weighted shifts when we study compact Hankel operators. We will begin this study after one more definition and a theorem following it.

<u>Definition 4.4</u>: $\mathscr{H}^\infty(\beta) + C(\beta) = \{\psi + \varphi : \psi \in \mathscr{H}^\infty(\beta)$ and $\varphi \in C(\beta)\}$.

For the unweighted case $\mathcal{X}^\infty(\beta)+C(\beta)$ has been shown to be a closed subalgebra of $L^\infty(\beta)$ (Sarason, [22]). We prove the same result for an arbitrary weighted shift by using a theorem of Rudin, [19].

Theorem 4.1: $\mathcal{X}^\infty(\beta)+C(\beta)$ is closed in $L^\infty(\beta)$.

Proof: We use the same notation as in Rudin, [19]. For $\varphi \in L^\infty(\beta)$ let $\sigma_n(\varphi) = \sum_{k=-n}^{n} (1-\frac{|k|}{n+1})\hat{\varphi}(k)z^k$ be the n'th Cesaro mean of φ. Let $\Phi = \{\sigma_n : n \in \mathbb{N}\}$, $y = C(\beta)$, and $Z = \mathcal{X}^\infty(\beta)$. The latter two sets are then closed subspaces of the Banach space $L^\infty(\beta)$. Now for $\sigma \in \Phi$, $f \in y$, $g \in Z$, and $h \in L^\infty(\beta)$ we have the following results:

i) $\sigma(h) \in y$ (i.e. $\sigma(L^\infty(\beta)) \subset C(\beta)$)

ii) $\sigma(g) \in Z$ (i.e. $\sigma(\mathcal{X}^\infty(\beta)) \subset \mathcal{X}^\infty(\beta)$)

iii) $\|\sigma\| \leq 1$ (i.e. $\sup\{\|\sigma\| : \sigma \in \Phi\} < \infty$).

Parts i) and ii) are easy to verify using the definition of σ_n. For part iii) see Shields ([23], p.89). Rudin's theorem says we must only verify one more thing to conclude that $\mathcal{X}^\infty(\beta)+C(\beta)$ is closed. We must show that for each $f \in y$ and $\varepsilon > 0$ there exists $\sigma \in \Phi$ such that $\|\sigma(f)-f\|_\infty < \varepsilon$. To see this let $f \in C(\beta)$ and $\varepsilon > 0$ be given. Then there exists a (Laurent) polynomial $p(z) = \sum_{k=-N}^{N} \hat{p}(k)z^k$ such that $\|f-p\|_\infty < \varepsilon/3$. Therefore,

$$\|\sigma_n(f) - f\|_\infty \le \|\sigma_n(f - p)\|_\infty + \|\sigma_n(p) - p\|_\infty + \|p - f\|_\infty \quad \text{or}$$

$$\|\sigma_n(f) - f\|_\infty \le 2\|f - p\|_\infty + \|\sigma_n(p) - p\|_\infty \quad \text{(by part iii))}$$

$$\le 2\varepsilon/3 + \|\sigma_n(p) - p\|_\infty \; .$$

Now for $n > N$, $\sigma_n(p) - p = \sum_{k=-N}^{N} \frac{|k|}{n+1} \hat{p}(k)z^k$.

Using the results of Shields ([23], Prop.29), it can be shown that

$$\|\sigma_n(p) - p\|_\infty \le \|p\|_\infty \left(\int_{\partial \mathbb{D}} |q_n(\bar{w})| \frac{|dw|}{2\pi}\right) \quad \text{where}$$

$q_n(w) = \sum_{k=-N}^{N} |k|w^k/(n+1)$. Thus

$$\|\sigma_n(p)-p\|_\infty \le \|p\|_\infty \left(\int_{\partial \mathbb{D}} |q_n(w)|^2 \frac{|dw|}{2\pi}\right)^{1/2}$$

$$\le \|p\|_\infty \left(2 \sum_{k=0}^{N} k^2\right)^{1/2} /(n+1) \; .$$

So there exists $N_0 > 0$ such that $\|\sigma_n(p) - p\|_\infty < \varepsilon/3$ if $n > N_0$. Finally if $n > N_0$, then $\|\sigma_n(f) - f\|_\infty < \varepsilon$. Q.E.D.

<u>Proposition 4.4</u>: If $\varphi \in \mathcal{N}^\infty(\beta)$, then $z^n\varphi \in \mathcal{N}^\infty(\beta) + C(\beta)$ for all n.

<u>Proof</u>: For $n \ge 0$, $z^n\varphi \in \mathcal{N}^\infty(\beta)$. Thus we need only consider the case when $n < 0$. In this case, $z^n\varphi = \sum_{k=n}^{-1} \hat{\varphi}(k-n)z^k + \sum_{k=0}^{\infty} \hat{\varphi}(k-n)z^k$. Since $g = \sum_{k=n}^{-1} \hat{\varphi}(k-n)z^k$

is in $C(\beta)$, we see that $\sum_{k=0}^{\infty} \hat{\varphi}(k-n)z^k = z^n\varphi - g$ is in $\mathcal{H}^\infty(\beta)$. Thus $z^n\varphi \in \mathcal{H}^\infty(\beta) + C(\beta)$. Q.E.D.

<u>Corollary 4.1</u>: $\mathcal{H}^\infty(\beta) + C(\beta)$ is a closed subalgebra of $L^\infty(\beta)$.

<u>Proof</u>: By Theorem 4.1, $\mathcal{H}^\infty(\beta) + C(\beta)$ is closed. Thus we need only show that it is a subalgebra of $L^\infty(\beta)$. Because it is closed, it is sufficient to show that $p\varphi \in \mathcal{H}^\infty(\beta) + C(\beta)$ for all $\varphi \in \mathcal{H}^\infty(\beta)$ and (Laurent) polynomials $p = \sum_{k=-N}^{N} \hat{p}(k)z^k$. However, $p\varphi = \sum_{k=-N}^{N} \hat{p}(k)(z^k\varphi)$ and by Proposition 4.4 we know that $z^k\varphi \in \mathcal{H}^\infty(\beta) + C(\beta)$ for all k. Hence $p\varphi \in \mathcal{H}^\infty(\beta) + C(\beta)$. Q.E.D.

We will now discuss the set $\{\varphi \in L^\infty(\beta) : H_\varphi$ is compact$\}$. In the unweighted case, it is known that the set above is exactly $\mathcal{H}^\infty(\beta) + C(\beta)$ (See P. Hartman, [12]). For weighted shifts in general this is not true as we will see later. However, we will show that H_φ is compact for all φ in $\mathcal{H}^\infty(\beta) + C(\beta)$. We begin with the following proposition.

<u>Proposition 4.5</u>: If $n > 0$ and $\varphi \in \mathcal{H}^\infty(\beta)$, then $H_{z^{-n}\varphi}$ is a finite rank operator and hence compact.

<u>Proof</u>: If $q = \sum_{k=-n}^{-1} \hat{\varphi}(n+k)z^k$ and $\psi = \sum_{k=0}^{\infty} \hat{\varphi}(n+k)z^k$, then $q \in C(\beta)$, $\psi \in \mathscr{H}^\infty(\beta)$, and $z^{-n}\varphi = q + \psi$. Thus by Proposition 4.2, $H_{z^{-n}\varphi} = H_q$.

Now let $f \in H^2(\beta)$; then $\widehat{(qf)}(m) = \sum_{k=0}^{\infty} \hat{f}(k)\hat{q}(m-k) = 0$ for $m < -n$ since $\hat{q}(\ell) = 0$ for $\ell < -n$. Thus $qf = \sum_{k=-n}^{\infty} \widehat{(qf)}(n)z^k$, implying $H_{z^{-n}\varphi}(f) = (1-P)(qf) = \sum_{k=-n}^{-1} \widehat{(qf)}(k)z^k$. Hence $\{z^{-n}, z^{-n+1}, \cdots, z^{-1}\}$ spans the range of $H_{z^{-n}\varphi}$. Q.E.D.

<u>Corollary 4.2</u>: If $\varphi \in \mathscr{H}^\infty(\beta) + C(\beta)$ then H_φ is compact.

<u>Proof</u>: Let $\varphi = q + \psi$ where $\psi \in \mathscr{H}^\infty(\beta)$ and $q \in C(\beta)$. Let $q_k \in C(\beta)$ be (Laurent) polynomials such that $\|q_k - q\|_\infty \to 0$ as $k \to \infty$. Then

$$\|H_\varphi - H_{q_k}\| = \|H_q - H_{q_k}\| \leq \|q - q_k\|_\infty \to 0 \quad \text{as} \quad k \to \infty .$$

By Proposition 4.5, H_{q_k} is compact. Also, it is known that the collection of compact operators is closed in the norm topology on $\mathscr{B}(H^2(\beta), L^2(\beta) \ominus H^2(\beta))$ (See Rudin, [17], Theorem 4.18). Thus H_φ is compact since $\|H_\varphi - H_{q_k}\| \to 0$ as $k \to \infty$ and H_{q_k} is compact for all k. Q.E.D.

Before we continue with our discussion of compact Hankel operators, we need to take a short digression. There are some general properties of operators on Hilbert space which we will need in our discussion. The results we will give in this digression have been known for many years, but for the sake of completeness, we will include their proofs. The following result is due to I. Schur. Its proof is found in Hardy, [10].

Theorem 4.2: (Schur) Let H and K be separable Hilbert spaces with orthonormal bases $\{e_j : j = 0,1,2,\cdots\}$ and $\{e_i : i = -1,-2,\cdots\}$ respectively. Let $A \in \mathcal{B}(H,K)$ have matrix $[a_{ij}]$ where $a_{ij} = (Ae_j,e_i)$. If $b_{ij} = (f_j,g_i)$ for f_j, $g_i \in H$ with $\sup\{\|f_j\| : j = 0,1,\cdots\} = M < \infty$ and $\sup\{\|g_i\| : i = -1,-2,\cdots\} = N < \infty$, then the operator D with matrix $[a_{ij}\, b_{ij}]$ is in $\mathcal{B}(H,K)$. Furthermore, $\|D\| \leq \|A\|MN$.

Proof: Let $f_j = \sum_{\ell=0}^{\infty} \hat{f}_j(\ell)e_\ell$ and $g_i = \sum_{\ell=0}^{\infty} \hat{g}_i(\ell)e_\ell$. If $x = \sum_{k=0}^{\infty} \lambda_k e_k \in H$ and $y = \sum_{j=-\infty}^{-1} \beta_j e_j \in K$ then

$$(Dx,y) = \sum_{k=0}^{\infty} \sum_{j=-\infty}^{-1} \lambda_k \overline{\beta}_j\, a_{jk}\, b_{jk}$$

$$= \sum_{k=0}^{\infty} \sum_{j=-\infty}^{-1} \lambda_k \overline{\beta}_j\, a_{jk}\Big(\sum_{e=0}^{\infty} \hat{f}_k(\ell)\overline{\hat{g}_j(\ell)}\Big)$$

$$= \sum_{\ell=0}^{\infty} \Big(\sum_{k=0}^{\infty} \sum_{j=-\infty}^{-1} \lambda_k\, \hat{f}_k(\ell)\overline{\beta_j \hat{g}_j(\ell) a_{jk}}\Big)$$

Now if $h_\ell = \sum_{k=0}^{\infty} \lambda_k \hat{f}_k(\ell) e_k$ and $m_\ell = \sum_{j=-\infty}^{-1} \beta_j \hat{g}_j(\ell) e_j$ then $(Dx,y) = \sum_{\ell=0}^{\infty} (Ah_\ell, m_\ell)$. So

$$
\begin{aligned}
|(Dx,y)| &\le \sum_{\ell=0}^{\infty} |(Ah_\ell, m_\ell)| \\
&\le \sum_{\ell=0}^{\infty} \|A\| \|h_\ell\| \|m_\ell\| \\
&\le \|A\| \Big(\sum_{\ell=0}^{\infty} \|h_\ell\|^2 \Big)^{1/2} \Big(\sum_{\ell=0}^{\infty} \|m_\ell\|^2 \Big)^{1/2} \quad \text{by Cauchy-Schwartz.}
\end{aligned}
$$

However,

$$
\begin{aligned}
\sum_{\ell=0}^{\infty} \|h_\ell\|^2 &= \sum_{\ell=0}^{\infty} \Big(\sum_{k=0}^{\infty} |\lambda_k|^2 |\hat{f}_k(\ell)|^2 \Big) \\
&= \sum_{k=0}^{\infty} |\lambda_k|^2 \Big(\sum_{\ell=0}^{\infty} |\hat{f}_k(\ell)|^2 \Big) \\
&\le \sum_{k=0}^{\infty} |\lambda_k|^2 M^2 \\
&\le M^2 \|x\|^2 .
\end{aligned}
$$

Thus $\Big(\sum_{\ell=0}^{\infty} \|h_\ell\|^2 \Big)^{1/2} \le M\|x\|$ and similarly $\Big(\sum_{\ell=0}^{\infty} \|m_\ell\|^2 \Big)^{1/2} \le N\|y\|$. Therefore $|(Dx,y)| \le \|A\| MN \|x\| \|y\|$ implying $\|D\| \le \|A\| MN$.

We note that we should be a little more careful in switching the order of summation. This can be done by letting x and y be only finite linear combinations of the basis vectors. We would then get the norm inequality on a dense set of vectors and be able to extend by continuity. Q.E.D.

We now have the following two corollaries which we will need in discussing compact Hankel operators. The operator $A_n(A)$ defined below is like the $(n-1)$th Cesaro mean of a function. If we let $B_k(A)$ be the operator whose matrix entries are those of A above the kth cross diagonal and zeros on or below it, then $A_n(A) = \sum_{k=1}^{n} B_k(A)/n$.

Corollary 4.3: (O'Donovan, [16]) Let $A \in \mathcal{B}(H,K)$ where H and K are as in Theorem 4.2. Let $a_{ij} = (Ae_j, e_i)$ and let

$$(A_n(A)e_j, e_i) = \begin{cases} 0 & \text{if} \quad |i|+j \geq n \\ a_{ij}(n-|i|-j)/n & \text{if} \quad |i|+j < n \end{cases} .$$

Then $A_n(A) \in B(H,K)$, $\|A_n(A)\| \leq \|A\|$, and $A_n(A) \to A$ in the strong operator topology (SOT) as $n \to \infty$.

Proof: For $j = 0,1,2,\cdots,n-1$, let $f_j = (\sum_{\ell=1}^{n-j} e_\ell)/\sqrt{n}$ and for $j \geq n$ let $f_j = 0$. Also for $i = -1,-2,\cdots,-n+1$, let $g_i = (\sum_{\ell=|i|+1}^{n} e_\ell)/\sqrt{n}$ and for $i \leq -n$ let $g_i = 0$.

Then $\|f_j\| \leq 1$, $\|g_i\| \leq 1$, and it is easy to verify that

$$(f_j, g_i) = \begin{cases} 0 & \text{if} \quad |i|+j \geq n \\ (n-|i|-j)/n & \text{if} \quad |i|+j < n \end{cases} .$$

Hence by Theorem 4.2, $A_n(A) \in \mathcal{B}(H,K)$ and $\|A_n(A)\| \leq \|A\|$. We note that in fact $A_n(A)$ is a finite rank operator.

To show convergence in the SOT we start with a basis vector e_j for $j \geq 0$. If $\varepsilon > 0$ is given then there exists an integer $k < 0$ such that

$$\sum_{\ell=-\infty}^{k} |(Ae_j,e_\ell)|^2 < \varepsilon^2/2 \text{ since } \|Ae_j\|^2 = \sum_{\ell=-\infty}^{-1} |(Ae_j,e_\ell)|^2 .$$

Thus if $n > |k| + j + 1$ we have $k > j - n + 1$ and

$$\|A_n(A)e_j - Ae_j\|^2$$

$$= \sum_{\ell=-\infty}^{j-n} |(Ae_j,e_\ell)|^2 + \sum_{\ell=j-n+1}^{-1} ((|\ell|+j)/n)^2 |(Ae_j,e_\ell)|^2$$

$$\leq \sum_{\ell=-\infty}^{k-1} |(Ae_j,e_\ell)|^2 + \sum_{\ell=k}^{-1} ((|\ell|+j)/n)^2 |(Ae_j,e_\ell)|^2$$

$$\leq \varepsilon^2/2 + (|k|+j)^2 |k| \|A\|/n^2 .$$

Thus there exists $N_0 > 0$ such that $(|k|+j)^2|k|\|A\|/n^2 < \varepsilon^2/2$ if $n > N_0$. Therefore if $n > N_0$, then $\|A_n(A)e_j - Ae_j\| < \varepsilon$. Or in other words $\|A_n(A)e_j - Ae_j\| \to 0$ as $n \to \infty$.

To show convergence in the strong operator topology we must show $\|A_n(A)x - Ax\| \to 0$ as $n \to \infty$ for each $x \in H$. We have shown this for $x = e_j$; $j = 0,1,2,\cdots$. So let $x = \sum_{j=0}^{\infty} \lambda_j e_j \in H$ and let $\varepsilon > 0$.

Then there exists $M > 0$ such that

$\| \sum_{j=M+1}^{\infty} \lambda_j e_j \| < \varepsilon/4\|A\|$. Now let $y = \sum_{j=M+1}^{\infty} \lambda_j e_j$. Then

$$\|A_n(A)x - Ax\| \leq \sum_{j=0}^{M} |\lambda_j| \|A_n(A)e_j - Ae_j\| + \|A_n(A)y\| + \|Ay\|$$

$$\leq (M+1)\|x\| \max\{\|A_n(A)e_j - Ae_j\| : 0 \leq j \leq M\} + \varepsilon/2 .$$

By the part directly above we can make $\max\{\|A_n(A)e_j - Ae_j\| : 0 \leq j \leq M\}$ "small" by taking n large. Thus for each $x \in H$ and $\varepsilon > 0$ there exists $N > 0$ such that $\|A_n(A)x - Ax\| < \varepsilon$ if $n > N$. Q.E.D.

<u>Corollary 4.4</u>: If A is compact then $A_n(A)$ converges to A in the norm topology on $\mathcal{B}(H,K)$ as $n \to \infty$.

<u>Proof</u>: We first assume A is Hilbert-Schmidt and let $\varepsilon > 0$. Then there exists $N > 0$ such that $\sum_{n=N+1}^{\infty} \|Ae_n\|^2 < \varepsilon^2/9$. We also note that for $j \geq 0$

$$\|A_n(A)e_j\| \leq \|Ae_j\| \quad \text{since} \quad |(A_n(A)e_j, e_i)| \leq |(Ae_j, e_i)| .$$

Thus if $x = \sum_{k=0}^{\infty} \lambda_k e_k \in H$, we let $x_1 = \sum_{k=0}^{N} \lambda_k e_k$ and $x_2 = \sum_{k=N+1}^{\infty} \lambda_k e_k$. Then

$$\|A_n(A)x - Ax\| \leq \|A_n(A)x_1 - Ax_1\| + \|A_n(A)x_2\| + \|Ax_2\|$$

$$\leq \sum_{k=0}^{N} |\lambda_k| \|A_n(A)e_k - Ae_k\| + \|A_n(A)x_2\| + \|Ax_2\| .$$

But $$\|A_n(A)x_2\| \le \sum_{k=N+1}^{\infty} |\lambda_k| \, \|A_n(A)e_k\|$$

$$\le \left(\sum_{k=N+1}^{\infty} |\lambda_k|^2\right)^{1/2} \left(\sum_{k=N+1}^{\infty} \|A_n(A)e_k\|^2\right)^{1/2}$$

$$\le \|x_2\| \left(\sum_{k=N+1}^{\infty} \|Ae_k\|^2\right)^{1/2}$$

$$\le \|x\|\varepsilon/3 \quad .$$

The above inequalities also show that $\|Ax_2\| \le \|x\|\varepsilon/3$. Therefore

$$\|A_n(A)x - Ax\| \le [(N+1)\max\{\|A_n(A)e_k - Ae_k\| : 0 \le k \le N\} + \tfrac{2\varepsilon}{3}]\,\|x\| .$$

However, by Corollary 4.3 there exists $N_0 > 0$ such that $\max\{\|A_n(A)e_k - Ae_k\| : 0 \le k \le N\} < \varepsilon/3(N+1)$ for all $n > N_0$. Thus if $n > N_0$, $\|A_n(A)x - Ax\| < \varepsilon\|x\|$ for all $x \in H$. We note that N_0 does not depend on the vector x, only on the operator A and the ε given. Thus $\|A_n(A) - A\| < \varepsilon$ if $n > N_0$. This says $A_n(A) \to A$ in the norm topology if A is Hilbert-Schmidt.

However, the set of Hilbert-Schmidt operators is norm dense in the set of compact operators. Thus if A is compact and $\varepsilon > 0$ is given, there exists a Hilbert-Schmidt operator B such that $\|A - B\| < \varepsilon/3$. Therefore

$$\|A_n(A) - A\| \le \|A_n(A) - A_n(B)\| + \|A_n(B) - B\| + \|B - A\|$$

$$\le \|A_n(A-B)\| + \varepsilon/3 + \|A_n(B) - B\|$$

$$\le 2\varepsilon/3 + \|A_n(B) - B\| \quad \text{by Corollary 4.3} \quad .$$

Since B is Hilbert-Schmidt there exists $N_0 > 0$ such that $\|A_n(B) - B\| < \varepsilon/3$ if $n > N_0$ by the first part of this corollary. Hence $\|A_n(A) - A\| < \varepsilon$ if $n > N_0$. Q.E.D.

We now relate the material above to the consideration of compact Hankel operators.

<u>Proposition 4.6</u>: If $\varphi \in L^\infty(\beta)$ then $H_{\sigma_n(\varphi)} = A_{n+1}(H_\varphi)$ where $\sigma_n(\varphi)$ and $A_{n+1}(H_\varphi)$ are defined as before.

<u>Proof</u>: From before $\sigma_n(\varphi) = \sum_{k=-n}^{n} (1 - \frac{|k|}{n+1})\hat{\varphi}(k)z^k$.
An orthonormal basis for $H^2(\beta)$ is $\{z^j/\beta(j) : j = 0,1,2,\cdots\}$ and an orthonormal basis for $L^2(\beta) \ominus H^2(\beta)$ is $\{z^i/\beta(i) : i = -1,-2,\cdots\}$. Thus if $|i-j| = |i| + j < n+1$, then

$$\begin{aligned}
(H_{\sigma_n(\varphi)}z^j,z^i)/\beta(j)\beta(i) &= (\sigma_n(\varphi)z^j,z^i)/\beta(j)\beta(i) \\
&= \sigma_n(\varphi)(i-j)\beta(i)/\beta(j) \\
&= \hat{\varphi}(i-j)[(n+1-|i-j|)/(n+1)]\beta(i)/\beta(j) \\
&= \hat{\varphi}(i-j)[(n+1-|i|-j)/(n+1)]\beta(i)/\beta(j).
\end{aligned}$$

Similarly

$$\begin{aligned}
&(A_{n+1}(H_\varphi)z^j,z^i)/\beta(i)\beta(j) \\
&\quad = (H_\varphi z^j,z^i)[(n+1-|i|-j)/(n+1)]/\beta(i)\beta(j) \\
&\quad = \hat{\varphi}(i-j)[(n+1-|i|-j)/(n+1)]\beta(i)/\beta(j)
\end{aligned}$$

$$= (H_{\sigma_n(\varphi)} z^j, z^i)/\beta(i)\beta(j) .$$

If $|i-j| \geq n+1$ then $(H_{\sigma_n(\varphi)} z^j, z^i) = 0 = (A_{n+1}(H_\varphi) z^j, z^i)$, Hence we must have $H_{\sigma_n(\varphi)} = A_{n+1}(H_\varphi)$. Q.E.D.

<u>Theorem 4.3</u>: Suppose there exists a constant $c > 0$ such that $\|H_\varphi\| \geq c\,d(\varphi, \mathcal{H}^\infty(\beta))$ for all $\varphi \in L^\infty(\beta)$. Then H_φ is compact if and only if $\varphi \in \mathcal{H}^\infty(\beta) + C(\beta)$.

<u>Proof</u>: We have already shown that if $\varphi \in \mathcal{H}^\infty(\beta) + C(\beta)$ then H_φ is compact. So now we assume H_φ is compact. Then by Corollary 4.4 and Proposition 4.6 $\|H_{\sigma_n(\varphi)} - H_\varphi\| \to 0$ as $n \to \infty$. Now by our hypothesis $\|H_{\sigma_n(\varphi)} - H_\varphi\| = \|H_{\sigma_n(\varphi)-\varphi}\| \geq c\,d(\sigma_n(\varphi)-\varphi, \mathcal{H}^\infty(\beta))$. Thus there exists $\{\psi_n : n = 1,2,\cdots\} \subset \mathcal{H}^\infty(\beta)$ such that $\|\sigma_n(\varphi) + \psi_n - \varphi\| \to 0$ as $n \to \infty$. However, $\sigma_n(\varphi) + \psi_n \in \mathcal{H}^\infty(\beta) + C(\beta)$ since $\sigma_n(\varphi) \in C(\beta)$. Now by Theorem 4.1 we conclude that $\varphi \in \mathcal{H}^\infty(\beta) + C(\beta)$ since this space is closed. Q.E.D.

We note that this also gives us a proof that H_φ is compact if and only if $\varphi \in \mathcal{H}^\infty(\beta) + C(\beta)$ in the unweighted case. This follows from the equation: $\|H_\varphi\| = d(\varphi, \mathcal{H}^\infty(\beta))$. We now consider the problem of knowing whether such a constant c (in Theorem 4.3) exists for every weighted shift. The answer is no. We illustrate this by using the following example.

<u>Example 4.1</u>: Let T be the weighted shift with weight sequence as below:

i) $w_n = 1$ if $n \geq 0$

ii) $w_n = 1$ if $-[(k+1)^2+1] < n \leq -[k(k+1)+1]$ for $k = 0,1,2,\cdots$

iii) $w_n = \frac{1}{2}$ otherwise

Then for $m > 0$, I claim $\|z^{-m}\|_\infty = d(z^{-m}, \mathcal{N}^\infty(\beta)) = 2^m$. To see this pick an integer $k > m$. Then for $\varphi \in \mathcal{N}^\infty(\beta)$ and $n = -[(k+1)^2+1]$,

$$\|(z^{-m}+\varphi)z^n\|_2^2 = \|z^{-m+n}\|_2^2 + \|\varphi_{z^n}\|_2^2$$

$$\geq \|z^{-m+n}\|_2^2 \quad .$$

Thus $\|z^{-m}+\varphi\|_\infty \geq \|z^{-m+n}\|_2/\|z^n\|_2 = \beta(-m+n)/\beta(n) = \left(\prod_{k=n-m}^{n-1} w_k\right)^{-1} = 2^m$. Since this is true for all $\varphi \in \mathcal{N}^\infty(\beta)$, we have $d(z^{-m}, \mathcal{N}^\infty(\beta)) \geq 2^m$. But also $\|z^{-m}\|_\infty \leq \|z^{-1}\|_\infty^m \leq 2^m$ implying the equality desired.

Now however,

$$\|H_{z^{-m}}\| \leq \|M_{z^{-m}}|H^2(\beta)\|$$

$$\leq \|z^{-m}\|_2 \quad \text{since} \quad \|z^{-m}z^k\|_2 \leq \|z^{-m}\|_2 \quad \text{for} \quad k \geq 0$$

$$\leq \beta(-m)$$

$$\leq 2^{m/2} \; .$$

The last inequality holds by a simple calculation. Thus $\|H_{z^{-m}}\|/d(z^{-m},\mathcal{H}^{\infty}(\beta)) \leq (\sqrt{2})^{-m} \to 0$ as $m \to \infty$. Therefore no such constant as in Theorem 4.3 can exist.

One may ask at this point what is the set $\{\varphi \in L^{\infty}(\beta) : H_{\varphi}$ is compact$\}$ for Example 4.1? The answer is given below.

For the weighted shift given in Example 4.1, $r(T) = \|T\| = 1$ and $r(T^{-1})^{-1} = \|T^{-1}\|^{-1} = 1/2$. Now by the remarks following the proof of Theorem 10′ (Shields, [23]) and Theorem 1.2 of this paper, $f \in L^{\infty}(\beta)$ if and only if $f = \sum_{k=-\infty}^{\infty} \hat{f}(k)z^k$ is a bounded analytic function on the annulus $A = \{z \in \mathbb{C} : \frac{1}{2} < |z| < 1\}$. Also, $\|f\|_A = \sup\{|f(z)| : z \in A\} \leq \|f\|_{\infty}$ for all $f \in L^{\infty}(\beta)$. It is also easy to show that if $f \in C(\beta)$ then f is continuous on $\{z \in \mathbb{C} : |z| = \frac{1}{2}\} \cup \{z \in \mathbb{C} : |z| = 1\}$. This is done using (Laurent) polynomials and the norm inequality above. Now let $f(z) = \sum_{k=0}^{\infty} \hat{f}(k)z^k$ be a bounded analytic function on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For $z \in A$, let $g(z) = f(\frac{1}{2z})$. Then $g(z) = \sum_{k=0}^{\infty} 2^{-k}\hat{f}(k)z^{-k}$ is a bounded analytic function on A. We recall that $\{z^n/\beta(n) : n = 0,1,2,\cdots\}$ and $\{z^m/\beta(m) : m = -1,-2,\cdots\}$ are orthonormal bases for $H^2(\beta)$ and $L^2(\beta) \ominus H^2(\beta)$ respectively. We then have the following:

$(H_g z^n, z^m)/\beta(n)\beta(m) = \hat{g}(m-n)\beta(m)$ since $\beta(n) = 1$ for $n \geq 0$.

Thus
$$\sum_{n=0}^{\infty} \|H_g(z^n/\beta(n))\|^2 = \sum_{n=0}^{\infty} \Big(\sum_{m=-\infty}^{-1} |\hat{g}(m-n)|^2 \beta^2(m) \Big)$$
$$= \sum_{n=0}^{\infty} \Big(\sum_{m=1}^{\infty} |\hat{g}(-m-n)|^2 \beta^2(-m) \Big)$$
$$= \sum_{n=0}^{\infty} \Big(\sum_{m=1}^{\infty} |\hat{f}(n+m)|^2 2^{-2n-2m} \beta^2(m) \Big)$$
$$\leq \sum_{n=0}^{\infty} \Big(\sum_{m=1}^{\infty} |\hat{f}(n+m)|^2 4^{-n} \Big) \quad \text{since}$$
$$\beta(-m) \leq 2^{m/2}$$
$$\leq \sum_{n=0}^{\infty} 4^{-n} \Big(\sum_{m=1}^{\infty} |\hat{f}(n+m)|^2 \Big)$$
$$\leq \sum_{n=0}^{\infty} 4^{-n} \|f\|_A^2$$
$$\leq 2\|f\|_A^2 .$$

This condition says H_g is Hilbert-Schmidt and hence compact. In fact, the proof shows that if g is a bounded analytic function for $|z| > 1/2$, then H_g is compact.

We now note that if $\varphi = \sum_{k=-\infty}^{\infty} \hat{\varphi}(k) z^k \in L^\infty(\beta)$, then $\psi_1 = \sum_{k=0}^{\infty} \hat{\varphi}(k) z^k$ and $\psi_2 = \sum_{-\infty}^{-1} \hat{\varphi}(k) z^k$ are bounded analytic functions for $|z| < 1$ and $|z| > 1/2$ respectively; and hence are in $L^\infty(\beta)$. Thus $H_\varphi = H_{\psi_1+\psi_2} = H_{\psi_2}$ is compact (in fact Hilbert-Schmidt) by the remark above. Thus H_φ is compact for all $\varphi \in L^\infty(\beta)$.

However, for some weighted shifts $L^{\infty}(\beta) = \mathcal{H}^{\infty}(\beta) + C(\beta)$. This is not true here. If $g \in C(\beta)$, then g is continuous for $|z| = \frac{1}{2}$. There are, however, bounded analytic functions on A which are not continuous for $|z| = \frac{1}{2}$. Such a function would be in $L^{\infty}(\beta)$ but not in $\mathcal{H}^{\infty}(\beta) + C(\beta)$.

CHAPTER V

ANALYTIC PROJECTIONS FOR WEIGHTED SHIFTS

Let T be an invertible weighted shift represented as M_z on $L^2(\beta)$. Then, as before, $L^\infty(\beta) = \{\varphi = \sum_{k=-\infty}^{\infty} \hat{\varphi}(k)z^k : \varphi f \in L^2(\beta) \text{ for all } f \in L^2(\beta)\}$.
If we consider $\varphi = \sum_{k=-\infty}^{\infty} \hat{\varphi}(k)z^k$ as a formal Laurent series, then we can define the "analytic" projection $\mathcal{P} : L^\infty(\beta) \to L^2(\beta)$ as below.

<u>Definition 5.1</u>: For $\varphi = \sum_{k=-\infty}^{\infty} \hat{\varphi}(k)z^k \in L^\infty(\beta)$ we define the "analytic" projection $\mathcal{P} : L^\infty(\beta) \to L^2(\beta)$ by

$$\mathcal{P}(\varphi) = \sum_{k=0}^{\infty} \hat{\varphi}(k)z^k$$

It is easy to see that $\mathcal{P} : L^\infty(\beta) \to L^2(\beta)$ is a bounded linear map since $\|\mathcal{P}(\varphi)\|_2 \le \|\varphi\|_2 \le \|\varphi\|_\infty$. The problem we want to consider concerns the range of $\mathcal{P}$. For which weighted shifts is the range of $\mathcal{P}$ contained in $L^\infty(\beta)$? By appealing to the closed graph theorem and the continuity of the coefficient functionals Γ_n on $L^\infty(\beta)$, one can ask equivalently: for which weighted shifts is $\mathcal{P}$ a bounded linear map from the Banach space $L^\infty(\beta)$ to $L^\infty(\beta)$? So $\mathcal{P} \in \mathcal{B}(L^\infty(\beta))$ if and only if $\mathcal{P}(\varphi) \in L^\infty(\beta)$ for all $\varphi \in L^\infty(\beta)$. In what follows, we will give a general sufficient condition for $\mathcal{P}$

to be bounded. We will then prove several corollaries of this theorem. Finally, we will provide several examples illustrating various aspects of this problem. Before we give the theorem, however, we need the following definition.

<u>Definition 5.2</u>: Let $K \subset \mathbb{C}$ be compact and let f be a function analytic on an open set containing K. We then define the norm of f on K by

$$\|f\|_K = \sup\{|f(z)| : z \in K\}$$

We are now able to state the theorem.

<u>Theorem 5.1</u>: Let T be an invertible weighted shift with $r(T^{-1})^{-1} < r(T)$. If there exists a constant $c > 0$ such that $\|p\|_\infty \le c\|p\|_{\sigma(T)}$ for all polynomials $p(z) = \sum_{k=0}^{N} \hat{p}(k)z^k$ in z or if $\|q\|_\infty \le c\|q\|_{\sigma(T)}$ for all polynomials $q(z) = \sum_{k=1}^{N} \hat{q}(-k)z^{-k}$ in z^{-1}, then $\mathcal{P}$ is a bounded linear map from $L^\infty(\beta)$ into $L^\infty(\beta)$.

<u>Proof</u>: We first assume that $\|p\|_\infty \le c\|p\|_{\sigma(T)}$ for all polynomials in z. Let $f = \sum_{-N}^{N} f(k)z^k$ and consider $\|\mathcal{P}(f)\|_\infty \le c\|\mathcal{P}(f)\|_{\sigma(T)}$. At this point, we note that $\sigma(T) = \{z \in \mathbb{C} : r(T^{-1})^{-1} \le |z| \le r(T)\}$.

Since $r(T^{-1})^{-1} < r(T)$, there exists a constant $d > 0$ such that $\|\mathcal{G}(f)\|_{\sigma(T)} \leq d\|f\|_{\sigma(T)}$ (See Shields, [23], p.81). Thus

$$\|\mathcal{G}(f)\|_\infty \leq c\|\mathcal{G}(f)\|_{\sigma(T)} \leq cd\|f\|_{\sigma(T)} \leq cd\|f\|_\infty$$

The last inequality follows from the spectral mapping theorem. For f, as above, $f(\sigma(T)) = \sigma(M_f)$. Thus $\|f\|_{\sigma(T)} = r(M_f) \leq \|M_f\| = \|f\|_\infty$.

Now let $f = \sum_{k=-\infty}^{\infty} \hat{f}(k)z^k$ be any element of $L^\infty(\beta)$. Then $\sigma_n(f) = \sum_{k=-n}^{n} [(n+1-|k|)/(n+1)]\hat{f}(k)z^k$ and $\sigma_n(\mathcal{G}(f)) = \mathcal{G}(\sigma_n(f))$. Hence

$$\|\sigma_n(\mathcal{G}(f))\|_\infty = \|\mathcal{G}(\sigma_n(f))\|_\infty \leq cd\|\sigma_n(f)\|_\infty \leq cd\|f\|_\infty .$$

The first inequality comes from the first part of this proof and the second follows from the inequality $\|\sigma_n(f)\|_\infty \leq \|f\|_\infty$ for all $f \in L^\infty(\beta)$. Thus $\{M_{\sigma_n(\mathcal{G}f)} : n = 1,2,\cdots)$ is a norm bounded sequence in $\mathcal{B}(L^2(\beta))$ and hence must have a convergent subsequence in the weak operator topology (WOT). Assume without loss of generality that $M_{\sigma_n(\mathcal{G}f)} \to S$ in the WOT as $n \to \infty$. Then $S = M_\psi$ for some $\psi \in L^\infty(\beta)$ since $L^\infty(\beta)$ is closed in the WOT. Thus

$$\hat{\psi}(\ell) = (\psi, z^{\ell})/\beta^2(\ell)$$

$$= \lim_{n\to\infty} (\sigma_n(\mathcal{P}f), z^{\ell})/\beta^2(\ell)$$

$$= \lim_{n\to\infty} (\sigma_n(\mathcal{P}f))(\ell)$$

$$= \begin{cases} 0 & \text{if } \ell < 0 \\ \lim\limits_{n\to\infty} [(n+1-|\ell|)/(n+1)]\hat{f}(\ell) & \text{if } \ell \geq 0 \end{cases}$$

$$= \begin{cases} 0 & \text{if } \ell < 0 \\ \hat{f}(\ell) & \text{if } \ell \geq 0 \end{cases} .$$

Thus $\psi = \mathcal{P}(f) \in L^{\infty}(\beta)$ and $\|\mathcal{P}(f)\|_{\infty} \leq cd\|f\|_{\infty}$ for all $f \in L^{\infty}(\beta)$.

The proof in the case where $\|q\|_{\infty} \leq c\|q\|_{\sigma(T)}$ is very similar. Here, one shows that $\mathcal{P}_1 = 1 - \mathcal{P}$ is a bounded linear operator on $L^{\infty}(\beta)$, and hence so is $\mathcal{P} = 1 - \mathcal{P}_1$. Q.E.D.

<u>Corollary 5.1</u>: If $r(T^{-1})^{-1} < r(T)$ and $r(T) = \|T\|$ or $r(T^{-1}) = \|T^{-1}\|$, then $\mathcal{P}$ is a bounded linear operator on $L^{\infty}(\beta)$.

<u>Proof</u>: Assume $r(T) = \|T\|$. Then for $p(z) = \sum_{k=0}^{N} \hat{p}(k)z^k$ we have

$$\|p\|_{\infty} = \|M_p\| = \|p(M_z)\| \leq \|p\|_K \quad \text{where} \quad K = \{z \in \mathbb{C} : |z| = \|T\|\}.$$

The inequality is a result of von Neumann's Theorem (von Neumann, [26]) and the maximum modulus theorem. Now, since $r(T) = \|T\|$, we have $K \subset \sigma(T)$. Thus $\|p\|_K \leq \|p\|_{\sigma(T)}$ implying that $\|p\|_\infty \leq \|p\|_{\sigma(T)}$ for all polynomials p in z. The result then follows from Theorem 5.1.

On the other hand, if $r(T^{-1}) = \|T^{-1}\|$ and $q(z) = \sum_{k=1}^{N} \hat{q}(-k)z^{-k}$ then

$$\|q\|_\infty = \|\sum_{k=1}^{N} \hat{q}(-k)(M_z^{-1})^k\|$$

$$\leq \|\sum_{k=1}^{N} \hat{q}(-k)z^k\|_{K'}, \quad \text{where } K' = \{z \in ₵ : |z| = \|T^{-1}\|\}.$$

Letting $K'' = \{z \in ₵ : |z| = \|T^{-1}\|^{-1} = r(T^{-1})^{-1}\}$ and replacing z with z^{-1}, we get $\|q\|_\infty \leq \|q\|_{K''} \leq \|q\|_{\sigma(T)}$ since $K'' \subset \sigma(T)$. Again Theorem 5.1 applies and we are done. Q.E.D.

<u>Corollary 5.2</u>: If $r(T^{-1})^{-1} < r(T)$ and M_z is similar to an operator S such that $r(S) = \|S\|$ or $r(S^{-1}) = \|S^{-1}\|$, then $\mathcal{P}$ is bounded.

<u>Proof</u>: Let $M_z = RSR^{-1}$ where $R \in \mathcal{B}(L^2(\beta))$. Assume $r(S) = \|S\|$ and let $p(z) = \sum_{k=0}^{N} \hat{p}(k)z^k$. Then

$$\|p\|_\infty = \|p(M_z)\| = \|Rp(S)R^{-1}\| \leq \|R\|\|R^{-1}\|\|p(S)\| .$$

But $\|p(S)\| \leq \|p\|_{\sigma(T)}$ since $\sigma(T) = \sigma(S)$ and $\{z \in ¢ : |z| = r(S) = \|S\|\} \subset \sigma(T)$. Thus $\|p\|_{\infty} \leq \|R\|\|R^{-1}\|\|p\|_{\sigma(T)}$ and by Theorem 5.1 the result follows.

The proof in the case $r(S^{-1}) = \|S^{-1}\|$ is similar to the case above, just as in Corollary 5.1. Q.E.D.

Before we present the third corollary, we need the following definition.

Definition 5.3: The numerical radius $w(A)$ for $A \in \mathcal{B}(L^2(\beta))$ is given by

$$w(A) = \sup\{|(Af,f)| : f \in L^2(\beta) \text{ and } \|f\|_2 = 1\} .$$

Corollary 5.3: If $r(T^{-1})^{-1} < r(T)$ and $r(T) = w(T)$ or $r(T^{-1}) = w(T^{-1})$, then $\mathcal{P}$ is a bounded linear operator on $L^{\infty}(\beta)$.

Proof: Assume $w(T) = r(T)$ and $p(z) = \sum_{k=0}^{N} \hat{p}(k)z^k$. Then p is a bounded analytic function on $\{z \in ¢ : |z| \leq w(T)\}$. Also, it is well known that $\|A\| \leq 2w(A)$ for all $A \in \mathcal{B}(L^2(\beta))$. If $g(z) = p(z) - \hat{p}(0)$, then

$$\|p\|_\infty = \|g + \hat{p}(0)\|_\infty$$

$$\leq \|g\|_\infty + |\hat{p}(0)|$$

$$\leq \|g(M_z)\| + \|p\|_{\sigma(T)} \text{ since } \hat{p}(0) = \int_0^{2\pi} p(r(T)e^{i\theta})\frac{d\theta}{2\pi r(T)} .$$

Thus $\|p\|_\infty \leq 2w(g(M_z)) + \|p\|_{\sigma(T)}$.

We now apply Theorem 4 of Berger and Stampfli, [3]. This is a mapping theorem for the numerical range and says:

$$w(g(M_z)) \leq \|g\|_B \quad \text{where} \quad B = \{z \in \mathbb{C} : |z| \leq w(T)\} .$$

Thus $\|p\|_\infty \leq 2\|g\|_B + \|p\|_{\sigma(T)}$

$$\leq 2\|g\|_{\sigma(T)} + \|p\|_{\sigma(T)}$$

since $r(T) = w(T)$ and $\|g\|_B = \|g\|_{B'} \leq \|g\|_{\sigma(T)}$, where $B' = \{z \in \mathbb{C} : |z| = w(T)\} \subset \sigma(T)$. Finally,

$$\|p\|_\infty \leq 2\|p - \hat{p}(0)\|_{\sigma(T)} + \|p\|_{\sigma(T)} \leq 5\|p\|_{\sigma(T)}$$

and Theorem 5.1 applies again.

Also, as before, the case where $w(T^{-1}) = r(T^{-1})$ is similar using $q(z) = \sum_{k=1}^{N} \hat{q}(-k)z^{-k}$ instead of $p(z)$. Q.E.D.

<u>Corollary 5.4</u>: Let $r = r(T^{-1})^{-1} < r(T) = t$ and $A(r,t) = \{z \in \mathbb{C} : r \leq |z| \leq t\}$. If there exists a constant $c > 0$ such that $\|f\|_\infty \leq C\|f\|_{A(r,t)}$ for all $f \in L^\infty(\beta)$ then $\mathcal{P}$ is a bounded linear operator on $L^\infty(\beta)$.

<u>Proof</u>: The proof is obvious. If the norm inequality holds for all $f \in L^\infty(\beta)$, then it certainly holds for all polynomials p is z. Q.E.D.

This ends the set of corollaries to Theorem 5.1. We will finish this chapter with four examples related to Theorem 5.1. The first example is used to answer the following question. Do the hypotheses of Theorem 5.1 imply the hypotheses for Corollary 5.4? That is, does $\|p\|_\infty \leq C\|p\|_{\sigma(T)}$ for all polynomials p in z imply $\|f\|_\infty \leq C\|f\|_{A(r,t)}$ for all $f \in L^\infty(\beta)$? The answer to this question is no, as illustrated below.

<u>Example 5.1</u>: Let T be the weighted shift with weight sequence as below:

i) $w_n = \frac{1}{2}$ if $n \geq 0$

ii) $w_n = 1/4$ if $n = -k(k+1)/2$ $k = 1,2,3,\cdots$

iii) $w_n = 1/3$ if $-(k+1)(k+2)/2 < n < -k(k+1)/2$ $k = 1,2,\cdots$

Then it is easy to show that $r(T) = \|T\| = 1/2$ and $1/4 = \|T^{-1}\|^{-1} < r(T^{-1})^{-1} = 1/3$. Since $r(T) = \|T\|$ we have $\|p\|_\infty \leq \|p\|_{\sigma(T)}$ for all polynomials p in z by von Neumann's theorem.

Now for $k < 0$, let $\ell(k)$ be the number of times $1/4$ appears in $\{w_{-1}, w_{-2}, \cdots, w_k\}$. Then $\ell(k) \to \infty$ as $k \to -\infty$. Also, let $q_k(z) = z^k$ for $k < 0$. Then $\|q_k\|_{\sigma(T)} = 3^{-k}$, but $\|q_k\|_\infty \geq \|q_k\|_2 = \beta(k)$ and $\beta(k) = 3^{-k-\ell(k)} 4^{\ell(k)}$. Thus

$$\|q_k\|_\infty / \|q_k\|_{\sigma(T)} \geq (4/3)^{\ell(k)} \to \infty \quad \text{as} \quad k \to -\infty .$$

Hence there can be no constant $c > 0$ such that $\|f\|_\infty \leq c\|f\|_{\sigma(T)}$ for all $f \in L^\infty(\beta)$.

In all of the previous situations we have taken $r(T^{-1})^{-1} < r(T)$. We have used this condition so that $\sigma(T)$ contains an open annulus. We were then able to use the boundedness of the "analytic" projection on the space of bounded analytic functions on this annulus. The condition $r(T^{-1})^{-1} < r(T)$ is not necessary, however, as the following example shows.

<u>Example 5.2</u>: Let T be the weighted shift represented as M_z on $L^2(\beta)$ where $\beta(n) = |n| + 1$. The weight sequence $\{w_n\}$ for T is then given by

$$w_n = \beta(n+1)/\beta(n) = (|n+1| + 1)/(|n|+1) .$$

For this weighted shift $r(T) = r(T^{-1})^{-1} = 1$. This weighted shift is known to be rationally strictly cyclic (See Shields, [23], p.101). For rationally strictly cyclic weighted shifts, $L^2(\beta) = L^\infty(\beta)$ as formal Laurent series and the norms on these spaces are equivalent. Hence, the projection $\mathcal{P}$ is bounded since $\mathcal{P}: L^{\infty}(\beta) \to L^{2}(\beta)$ is bounded.

Again, we note that we have usually taken $r(T^{-1})^{-1} < r(T)$. This, of course implies that $\|T^{-1}\|^{-1} < \|T\|$. One might ask whether the condition $\|T^{-1}\|^{-1} < \|T\|$ is sufficient for the boundedness of $\mathcal{P}$. The answer to this question is no and is illustrated by the following example.

Example 5.3: Let T be the weighted shift with weight sequence as below:

i) $w_n = 1$ if $n \neq -1$

ii) $w_n = 1/2$ if $n = -1$.

For this shift $\|T\| = 1$ and $\|T^{-1}\|^{-1} = 1/2$. This shift, though, is similar to the unweighted shift. Thus the "analytic" projection on $L^{\infty}(\beta)$ will be bounded if and only if it is bounded for the unweighted shift. However, it is known that the "analytic" projection for the unweighted shift is unbounded (See Rudin, [18], Prob. 9, Chapter 14). Thus the analytic projection on $L^{\infty}(\beta)$ is unbounded.

Our last example concerns the hypotheses of Theorem 5.1 again. If $r(T^{-1})^{-1} < r(T)$, is the condition that $\|p\|_{\infty} \leq c\|p\|_{\sigma(T)}$ for all polynomials

p in z or $\|q\|_\infty \le c\|q\|_{\sigma(T)}$ for all polynomials q in z^{-1} necessary for $\mathcal{P}$ to be bounded on $L^\infty(\beta)$? The answer here is again no. We use the following example to show this.

Example 5.4: Let T be the weighted shift with weight sequence given below. For $k = 1,2,3,\cdots$ let

i) $w_n = 1$ if $n = (k(k+1)/2) - 1$

ii) $w_n = 1/2$ if $n > 0$ and $n \neq (k(k+1)/2) - 1$

iii) $w_n = 1/3$ if $n < 0$ and $n \neq -[(k(k+1)/2) - 1$

iv) $w_n = 1/4$ if $n = -[(k(k+1)/2) - 1]$.

Then $\|T^{-1}\|^{-1} = 1/4 < r(T^{-1})^{-1} = 1/3 < \frac{1}{2} = r(T) < \|T\| = 1$. For $k > 0$, let $\ell(k)$ be the number of 1's appearing in $\{w_0,\cdots,w_{k-1}\}$. For $k < 0$, let $\ell(k)$ be the number of 1/4's appearing in $\{w_{-1},w_{-2},\cdots,w_k\}$; and let $\ell(0) = 0$. Then

a) $\beta(k) = 2^{\ell(k)-k}$ if $k \ge 0$

b) $\beta(k) = 3^{-k-\ell(k)}4^{\ell(k)}$ if $k < 0$.

We note that $\ell(k) \to \infty$ as either $k \to \infty$ or $k \to -\infty$.

For $k \ge 0$, $\|z^k\|_\infty \ge \|z^k\|_2 = \beta(k) = 2^{\ell(k)-k}$ and $\|z^k\|_{\sigma(T)} = 2^{-k}$. Hence, $\|z^k\|_\infty/\|z^k\|_{\sigma(T)} \ge 2^{\ell(k)}$ for $k > 0$. Similarly for $k < 0$, $\|z^k\|_\infty \ge \|z^k\|_2 = 3^{-\ell(k)-k}4^{\ell(k)}$ and $\|z^k\|_{\sigma(T)} = 3^{-k}$. Hence $\|z^k\|_\infty/\|z^k\|_{\sigma(T)} \ge (4/3)^{\ell(k)}$ for $k < 0$. Thus there can be no constant $C > 0$ such that either $\|p\|_\infty \le C\|p\|_{\sigma(T)}$ for all polynomials p in z or $\|q\|_\infty \le C\|q\|_{\sigma(T)}$ for all polynomials q in z^{-1}. For this weighted shift, however, the "analytic" projection $\mathcal{P}$ is bounded on $L^\infty(\beta)$. To prove this, we will show that T is a rationally strictly cyclic weighted shift. We do this by verifying the condition in Proposition 32 of Shields, [23]. (Note that this can be extended to bilateral shifts according to the remarks after Proposition 36 of Shields, [23].)

To simplify notation let $\beta(n,k) = \beta(n)/\beta(k)\beta(n-k)$. Thus we want to show that $\sup\{\sum_{k=-\infty}^{\infty} \beta^2(n,k) : n \in \mathbb{Z}\}$ is finite. For $n \ge 0$,

$$\sum_{k=-\infty}^{\infty} \beta^2(n,k) = \sum_{k=-\infty}^{-1} \beta^2(n,k) + \sum_{k=0}^{n} \beta^2(n,k) + \sum_{k=n+1}^{\infty} \beta^2(n,k)$$

$$\sum_{k=-\infty}^{-1} \beta^2(n,k) = \sum_{k=-\infty}^{-1} 4^{-n}\, 4^{\ell(n)}\, 9^k\, 9^{\ell(k)}\, 16^{-\ell(k)}\, 4^{n-k}\, 4^{-\ell(n-k)}$$

$$= \sum_{k=-\infty}^{-1} (4/9)^{-k} (9/16)^{\ell(k)} 4^{\ell(n)-\ell(n-k)}$$

$$\le \sum_{k=-\infty}^{-1} (4/9)^{-k} \quad \text{since for } k < 0,$$

$$\ell(k) \ge 0 \quad\text{and}\quad \ell(n-k) \ge \ell(n).$$

So $\sum_{k=-\infty}^{-1} \beta^2(n,k) \le \sum_{k=1}^{\infty} (4/9)^k = 4/5$. Also,

$$\sum_{k=n+1}^{\infty} \beta^2(n,k)$$

$$= \sum_{k=n+1}^{\infty} 4^{-n}\, 4^{\ell(n)}\, 4^{k}\, 4^{-\ell(k)}\, 9^{n-k}\, 9^{\ell(n-k)}\, 16^{-\ell(n-k)}$$

$$= \sum_{k=n+1}^{\infty} (4/9)^{k-n}\, 4^{\ell(n)-\ell(k)} (9/16)^{\ell(n-k)}$$

$$\leq \sum_{k=n+1}^{\infty} (4/9)^{k-n} \quad \text{since for } k \geq n, \ \ell(k) \geq \ell(n)$$

$$\text{and } \ \ell(n-k) \geq 0 \ .$$

So $\sum_{k=n+1}^{\infty} \beta^2(n,k) \leq \sum_{k=1}^{\infty} (4/9)^k \leq 4/5$. The other term for $n > 0$ is $\sum_{k=0}^{n} \beta^2(n,k) = \sum_{k=0}^{n} 4^{\ell(n)-\ell(k)-\ell(n-k)}$, which we will deal with a little later. First we will see what happens if $n < 0$. For the case when $n < 0$,

$$\sum_{k=-\infty}^{\infty} \beta^2(n,k) = \sum_{k=-\infty}^{n-1} \beta^2(n,k) + \sum_{k=n}^{0} \beta^2(n,k) + \sum_{k=1}^{\infty} \beta^2(n,k).$$

$$\sum_{k=-\infty}^{n-1} \beta^2(n,k)$$

$$= \sum_{k=-\infty}^{n-1} 9^{-n}\, 9^{-\ell(n)}\, 16^{\ell(n)}\, 9^{k}\, 9^{\ell(k)}\, 16^{-\ell(k)}\, 4^{n-k}\, 4^{-\ell(n-k)}$$

$$= \sum_{k=-\infty}^{n-1} (4/9)^{-k+n} (9/16)^{\ell(k)-\ell(n)}\, 4^{-\ell(n-k)}$$

$$\leq \sum_{k=-\infty}^{n-1} (4/9)^{-k+n} \quad \text{since for } k < n, \ \ell(k) \geq \ell(n)$$

$$\text{and } \ \ell(n-k) \geq 0$$

$$\leq 4/5 \ .$$

Similarly $\sum_{k=1}^{\infty} \beta^2(n,k) \leq 4/5$, but again

$\sum_{k=n}^{0} \beta^2(n,k) = \sum_{k=n}^{0} (16/9)^{\ell(n)-\ell(k)-\ell(n-k)}$.

One can show that for $k \geq 0$, $\ell(k) = [\,-\frac{1}{2}+\sqrt{\frac{1}{4}+2k}\,]$ where $[\,\cdot\,]$ is the greatest integer function. Thus $\sqrt{2}\sqrt{k}-1 \leq \ell(k) \leq \sqrt{2}\sqrt{k}+1$ for $k \geq 0$. So

$$\sum_{k=0}^{4} 4^{\ell(n)-\ell(k)-\ell(n-k)} \leq 4^3 \sum_{k=0}^{4} (4^{\sqrt{2}})^{\sqrt{n}-\sqrt{k}-\sqrt{n-k}}$$

Let $0 < r < 1$ and $\alpha(n,k) = \sqrt{k}+\sqrt{n-k}-\sqrt{n}$ for $0 \leq k \leq [\frac{n}{2}]$. Then by symmetry

$$\sum_{k=0}^{n} r^{\sqrt{k}+\sqrt{n-k}-\sqrt{n}} \leq 2 \sum_{k=0}^{[n/2]} r^{\alpha(n,k)} .$$

One can now show that $\alpha(n,k) \leq \alpha(n+1,k)$ by a direct computation. Also if n is even, then $\alpha(n,n/2) = (\sqrt{2}-1)\sqrt{n}$ which tends to ∞ as $n \to \infty$. If

$$S_n = 2 \sum_{k=0}^{[n/2]} r^{\alpha(n,k)}, \text{ then } S_{n+1} \leq S_n + 2r^{\alpha(n+1,[(n+1)/2])}$$

Thus $S_{n+1} \leq S_n + 2r^{(\sqrt{2}-1)\sqrt{n}}$ and $\sum_{n=0}^{\infty} r^{(\sqrt{2}-1)\sqrt{n}} < \infty$.

Hence $\sup\{S_n : n = 0,1,\cdots\} < \infty$. This says

$\sup\{\sum_{k=0}^{n} 4^{\ell(n)-\ell(k)-\ell(n-k)} : n = 0,1,2,\cdots\} < \infty$. Similarly

one can show $\sup\{\sum_{k=n}^{0} (16/9)^{\ell(n)-\ell(k)-\ell(n-k)} : n = -1,-2,\cdots\} < \infty$.

Now putting all the parts together, we have $\sup\{\sum_{k=-\infty}^{\infty} \beta^2(n,k) : n \in \mathbb{Z}\} < \infty$. Thus T is a rationally strictly cyclic weighted shift. Hence $L^\infty(\beta) = L^2(\beta)$ and the norms are equivalent. Thus $\mathcal{P}$ is bounded on $L^\infty(\beta)$.

We end this chapter with a conjecture. We have seen that the condition $r(T^{-1})^{-1} < r(T)$ is not necessary for the boundedness of $\mathcal{P}$. However, we have used it extensively in most of the results in this chapter. Is $r(T^{-1})^{-1} < r(T)$ a sufficient condition for $\mathcal{P}$ to be bounded? Another condition, stronger than $r(T^{-1})^{-1} < r(T)$, which might be sufficient is to require that $\pi_0(T^*)$ contain an open annulus.

CHAPTER VI

SPECTRAL SETS

Let T be a bounded linear operator on a Hilbert space H. If $p(z) = \sum_{k=0}^{N} \hat{p}(k)z^k$, then we can define $p(T) \in \mathcal{B}(H)$ by $p(T) = \sum_{k=0}^{N} \hat{p}(k)T^k$ where $T^0 = 1$, the identity operator on H. Also, if $h(z) = (z-\lambda)^{-1}$ and $\lambda \notin \sigma(T)$ then we can define $h(T) \in \mathcal{B}(H)$ by $h(T) = (T-\lambda)^{-1}$. Putting these two things together, if $q(z) = p_1(z)/p_2(z)$ is any rational function with poles (the zeroes of $p_2(z)$) off $\sigma(T)$, then $q(T) = p_1(T)p_2(T)^{-1} \in \mathcal{B}(H)$. One can investigate certain relationships between the function q (which is analytic on a neighborhood of $\sigma(T)$) and the operator $q(T) \in \mathcal{B}(H)$. In this chapter, we want to consider the relationship between $\|q(T)\|$ and $\|q\|_K$ where K is a compact set containing $\sigma(T)$. Thus we have the following definitions.

<u>Definition 6.1</u>: Let $K \subset \mathbb{C}$ be compact. Then $\mathrm{Rat}(K)$ is the set of rational functions with poles off K.

<u>Definition 6.2</u>: Let $T \in \mathcal{B}(H)$. A compact set K containing $\sigma(T)$ is said to be a spectral set for T if and only if $\|f(T)\| \leq \|f\|_K$ for all $f \in \mathrm{Rat}(K)$.

<u>Definition 6.3</u>: Let $T \in \mathcal{B}(H)$ and let $c > 0$. A compact set K containing $\sigma(T)$ is said to be a c-spectral set for T if and only if $\|f(T)\| \le c\|f\|_K$ for all $f \in \mathrm{Rat}(K)$.

We note at this point that it is not possible for $\|f\|_{\sigma(T)}$ to be greater than $\|f(T)\|$. From the spectral mapping theorem, we have $\|f\|_{\sigma(T)} = r(f(T)) \le \|f(T)\|$. Thus in most cases we will have $c \ge 1$ since we will consider sets K not much "larger" than $\sigma(T)$. Two questions are raised by Definition 6.2.

i) If $T \in \mathcal{B}(H)$ what kind of sets K containing the $\sigma(T)$ can be chosen so that K is a spectral set for T?

ii) What conditions can be placed upon the operator $T \in \mathcal{B}(H)$ so that $\sigma(T)$ is a spectral set for T?

In this chapter we will concentrate upon the first question. However, in some cases we may limit our study to weighted shifts. In order to provide some type of answer to the second question, we note that if $T \in \mathcal{B}(H)$ is normal then $\sigma(T)$ is a spectral set for T (Rudin, [17], p.309). In particular, $\sigma(T)$ is a spectral set for T if T is hermitian.

Returning to the first question, we note that if $T \in \mathcal{B}(H)$ then $\sigma(T) \subset \{z \in \mathbb{C} : |z| \leq \|T\|\}$. This containment may not be strict as there are some operators with $\sigma(T) = \{z \in \mathbb{C} : |z| \leq \|T\|\}$. The question of whether $\{z \in \mathbb{C} : |z| \leq \|T\|\}$ is a spectral set for T was answered in 1951 by von Neumann [26]. It is sufficient, by scaling, to consider only operators T with $\|T\| = 1$. In this case, we let $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$, $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{D} = \overline{\mathbb{D}} \setminus \partial\mathbb{D}$. We will give a new proof of von Neumann's theorem using some of his ideas. In place of one of his arguments, we will use the solution of the Caratheodory-Schur problem suggested in the section on Hankel operators in Sarason's VPI notes [21]. The result needed is stated and proved below.

<u>Lemma 6.1</u>: Let $p(z) = \sum_{k=0}^{N} \hat{p}(k) z^k$ be a polynomial. Then for each $n > 0$, there exists a set $\{\alpha_1, \cdots \alpha_{\ell(n)}\} \subset \mathbb{D}$, a constant $\lambda \in \mathbb{C}$ and a function h analytic on a neighborhood of $\overline{\mathbb{D}}$ such that:

i) $|\lambda| \leq \|p\|_{\overline{\mathbb{D}}}$

ii) $\|h\|_{\overline{\mathbb{D}}} \leq 2\|p\|_{\overline{\mathbb{D}}}$

iii) $p(z) = \lambda \sum_{k=1}^{\ell(n)} (z - \alpha_i)(1 - \overline{\alpha}_i z)^{-1} + z^{n+1} h(z).$

<u>Proof</u>: Let T, the unweighted shift ($w_n = 1$ for all n), be represented as M_z on $L^2(\partial\mathbb{D})$. Let $f(z) = z^{-(n+1)} p(z)$ and consider the Hankel operator H_f.

Since $f \in C(\partial\mathbb{D})$, H_f is compact. Thus if B is the closed unit ball of $H^2(\partial\mathbb{D})$, then $H_f(B)$ is compact. Therefore there is $g_1 \in B$ such that $\|H_f\, g_1\|_2 = \|H_f\| = \lambda$. Also there is a function $h \in H^\infty(\partial\mathbb{D})$ such that $\|f - h\|_\infty = \|H_f\| = \lambda$. Now $H_f(g_1) = H_f(g)$ where $g = \sum_{k=0}^{n+1} \hat{g}_1(k)z^k$ since for $m < 0$ $(H_f(g_1),z^m) =$

$$(fg_1,z^m) = \sum_{k=0}^{\infty} \hat{g}_1(k)\hat{f}(m-k) = \sum_{k=0}^{n+1} \hat{g}_1(k)\hat{f}(m-k) = (H_f g,z^m).$$

Thus $\lambda = \|H_f\| = \|H_f g\|_2 = \|(1-P)[(f-h)g]\|_2 \leq \|(f-h)g\|_2 \leq \|f-h\|_\infty \|g\|_2 \leq \lambda$. Hence equality must hold throughout. This implies that $H_f g = (f-h)g = \sum_{k=-(n+1)}^{-1} b_k z^k$.

From the string of inequalities above, we must have $|f-h| = \lambda$ almost everywhere on $\partial\mathbb{D}$. Also, $z^{n+1}H_f g = z^{n+1}(f-h)g$ is a polynomial and hence in $H^\infty(\partial\mathbb{D})$. We note that $z^{n+1}(f-h)g = [\prod_{i=1}^{N_1}(z-\alpha_i)][\prod_{i=1}^{N_2}(z-\lambda_i)]$ where $\{\alpha_i : i = 1,\cdots,N_1\} \subset \mathbb{D}$ and $\{\lambda_i : i = 1,\cdots,N_2\} \subset \mathbb{C}\backslash\mathbb{D}$. It is known that $z^{n+1}(f-h)g$ has a unique inner-outer factorization. Since $z^{n+1}(f-h)g$ has only a finite number of zeroes in $\mathbb{D}$, the right hand factor is an outer function, and the left hand factor is bounded away from zero on $\partial\mathbb{D}$, the inner factor of $z^{n+1}(f-h)g$ must be a finite Blaschke product (see Rudin, [18], Th. 17.17). However, $z^{n+1}(f-h)/\lambda$ is an inner function and hence must be a finite Blaschke product b. Thus $p(z) = z^{n+1}f = \lambda b(z) + z^{n+1}h(z)$. Also,

$\lambda = \|H_f\| \leq \|f\|_{\partial\mathbb{D}} = \|p\|_{\partial\mathbb{D}}$. Consequently, $|h(z)| = |f(z) - \lambda z^{-(n+1)}b(z)| \leq \|f\|_{\partial\mathbb{D}} + |\lambda| \leq 2\|p\|_{\partial\mathbb{D}}$. Thus parts i), ii), and iii) are satisfied. Q.E.D.

We now state and prove von Neumann's theorem. We use a complex function theory argument. A purely operator theoretic proof can be given using unitary power dilations (see Halmos, [8]).

<u>Theorem 6.1</u>: If $T \in \mathcal{B}(H)$ and $\|T\| = 1$, then $\overline{\mathbb{D}}$ is a spectral set for T.

<u>Proof</u>: First let $\alpha \in \mathbb{D}$ and let $0 < r < 1$. We now want to consider the operator $A = (rT - \alpha)(1 - \bar{\alpha}rT)^{-1}$. If $f \in H$ and $g = (1 - \bar{\alpha}rT)^{-1}f$, then

$$\begin{aligned}\|Af\|^2 &= (Af, Af)\\ &= ((rT - \alpha)g, (rT - \alpha)g)\\ &= r^2\|Tg\|^2 - 2\,\mathrm{Re}\,(\alpha g, rTg) + |\alpha|^2\|g\|^2 .\end{aligned}$$

Also

$$\begin{aligned}\|f\|^2 &= ((1 - \bar{\alpha}rT)g, (1 - \bar{\alpha}rT)g)\\ &= \|g\|^2 - 2\,\mathrm{Re}\,(\alpha g, rTg) + |\alpha|^2 r^2\|Tg\|^2 .\end{aligned}$$

Hence
$$\begin{aligned}\|f\|^2 - \|Af\|^2 &= \|g\|^2(1 - |\alpha|^2) - r^2\|Tg\|^2(1 - |\alpha|^2)\\ &= (1 - |\alpha|^2)[\|g\|^2 - r^2\|Tg\|^2]\\ &\geq 0 \quad \text{since} \quad |\alpha| < 1 \quad \text{and} \quad r\|Tg\| \leq \|g\| .\end{aligned}$$

Thus $\|Af\| \leq \|f\|$ for all $f \in H$ implying $\|A\| \leq 1$.

Now let $p(z) = \sum_{k=0}^{N} \hat{p}(k) z^k$ and let $\{\alpha_1, \cdots, \alpha_{\ell(n)}\}$, λ and h be as in Lemma 6.1. Then for $0 < r < 1$

$$p(rT) = \lambda \prod_{k=1}^{\ell(n)} (rT - \alpha_k)(1 - \bar{\alpha}_k rT)^{-1} + r^{n+1} T^{n+1} h(rT) .$$

Thus $\|p(rT)\| \leq |\lambda| + r^{n+1} \|h(rT)\|$ by the triangle inequality and the first part of this proof.

Now since $h \in H^\infty(\mathbb{D})$, $h(rz) = \sum_{h=0}^{\infty} \hat{h}(n) r^n z^n$. Hence $h(rT) = \sum_{h=0}^{\infty} \hat{h}(n) r^n T^n$. Thus $\|h(rT)\| \leq \sum_{n=0}^{\infty} |\hat{h}(n)| r^n \leq \|h\|_\infty \sum_{n=0}^{\infty} r^n \leq \|h\|_\infty / (1-r)$. We now have

$$\|p(rT)\| \leq |\lambda| + \|h\|_{\partial \mathbb{D}} r^{n+1}/(1-r) \leq \|p\|_{\overline{\mathbb{D}}} + 2\|p\|_{\overline{\mathbb{D}}} r^{n+1}/(1-r)$$

by Lemma 6.1 and the maximum modulus theorem. This is true, however, for all $n > 0$. Hence

$$\|p(rT)\| \leq \|p\|_{\overline{\mathbb{D}}} \text{ for all } r \text{ between } 0 \text{ and } 1 .$$

Letting r increase to 1 and using the continuity of the $\|p(rT)\|$ as a function of r, we get $\|p(T)\| \leq \|p\|_{\overline{\mathbb{D}}}$ for all polynomials p. The polynomials are dense in $\mathrm{Rat}(\overline{\mathbb{D}})$, though. Thus we are done. Q.E.D.

Now let $T \in \mathcal{B}(H)$ be invertible. Then we know that $\sigma(T)$ is contained in the annulus $\{z \in \mathbb{C} : \|T^{-1}\|^{-1} \leq |z| \leq \|T\|\}$. It will be sufficient (as in the case of the disc) to concern ourselves only with operators T for which $\|T\| = 1$ and $0 < \|T^{-1}\|^{-1} < 1$. (We note that if $\|T\| = 1 = \|T^{-1}\|^{-1}$, then T is unitary and hence $\|f(T)\| \leq \|f\|_{\partial \mathbb{D}}$ for all $f \in \mathrm{Rat}(\partial\mathbb{D})$ (Rudin, [17],Th. 12:13). In order to simplify our writing, we introduce the following notation: $A(r) = \{z \in \mathbb{C} : r \leq |z| \leq 1\}$. We now state a result which appears in Shields ([23], Prop.23).

<u>Theorem 6.2</u>: Let $T \in \mathcal{B}(H)$ be such that $\|T\| = 1$ and $\|T^{-1}\|^{-1} = r$ where $0 < r < 1$. Then there exists a constant c_r such that $\|f(T)\| \leq c_r\|f\|_{A(r)}$ for all $f \in \mathrm{Rat}(A(r))$.

This theorem says the annulus $A(r)$ is a c_r-spectral set for all $T \in \mathcal{B}(H)$ satisfying the norm requirements above. However, this theorem (as stated) is unsatisfying in the sense that it does not tell us anything about c_r. Can we take $c_r = 1$? If not, how large must c_r be? In 1967, J. Williams, [28], showed that we cannot take $c_r = 1$ for all $T \in \mathcal{B}(H)$ satisfying the norm requirements. In the result by

Shields, cited above, it is shown that the constant $c_r = 2+\sqrt{(1+r)/(1-r)}$ will suffice. However, we will show that a slightly smaller constant works. The question of determining the best constant for c_r is still open.

<u>Proposition 6.1</u>: Let $T \in \mathcal{B}(H)$ be such that $\|T\| = 1$, $\|T^{-1}\|^{-1} = r$. Then $\|f(T)\| \leq k_r\|f\|_{A(r)}$ for all $f \in \mathrm{Rat}(A(r))$ where $k_r = \min[3+4\pi^{-1}\ln(16(1-r^2))^{-1}, 2+\sqrt{(1+r)/(1-r)}]$.

<u>Proof</u>: Let $f \in \mathrm{Rat}(A(r))$ be uniquely decomposed as follows:

$$f(z) = \varphi_1(z)+\varphi_2(z);\ \varphi_1 \text{ is analytic on } \mathbb{D},$$

$$\varphi_2 \text{ is analytic on } \{z \in \mathbb{C} : |z| > r\}$$

and $\varphi_2(z) \to 0$ as $|z| \to \infty$.

Then $f(T) = \varphi_1(T)+\varphi_2(T)$. Also $\|\varphi_1(T)\| \leq \|\varphi_1\|_{\partial\mathbb{D}}$ by von Neumann's theorem and $\|\varphi_2(T)\| \leq \|\varphi_2\|_{r\partial\mathbb{D}}$ where $r\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = r\}$. The second part follows by using von Neumann's theorem for T^{-1} and the maximum modulus theorem. Now $\|\varphi_2\|_{r\partial\mathbb{D}} \leq \|f\|_{r\partial\mathbb{D}} + \|\varphi_1\|_{r\partial\mathbb{D}} \leq \|f\|_{A(r)} + \|\varphi_1\|_{r\partial\mathbb{D}}$.

Using the Cauchy Integral Formula, we get

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$$|\varphi_1(rw)| = \frac{1}{2\pi}\left|\int_{\partial\mathbb{D}} f(z)(z-rw)^{-1}dz\right| \quad \text{for all} \quad w \in \partial\mathbb{D}.$$

$$\le \|f\|_{\partial\mathbb{D}}\int_{\partial\mathbb{D}} |z-rw|^{-1}\frac{|dz|}{2\pi}$$

$$\le \|f\|_{A(r)}\int_{\partial\mathbb{D}} |1-rz|^{-1}\frac{|dz|}{2\pi} \quad \text{since} \quad |z| = 1$$

$$\text{for} \quad z \in \partial\mathbb{D}$$

$$\le \|f\|_{A(r)}\int_0^{2\pi} |1-re^{i\theta}|^{-1}\frac{d\theta}{2\pi}$$

$$\le \|f\|_{A(r)}\int_0^{2\pi} (1-2r\cos\theta + r^2)^{-1/2}\frac{d\theta}{2\pi}.$$

Now $\int_0^{2\pi}(1-2r\cos\theta+r^2)^{-1/2}\frac{d\theta}{2\pi} = 2\pi^{-1}k(r)$ where $k(r) = \int_0^{\pi/2}(1-r^2\sin^2\theta)^{-1/2}d\theta$ from Ryshik, [20]. However, $\lim_{r\to 1^-}(k(r)+\frac{1}{2}\ln 16(1-r^2)) = 0$ by Byrd [4]. Thus $k(r) \le \ln(16(1-r^2))^{-1}$ if $1 > r > r_0$ for some r_0 between 0 and 1. Thus

$$\|\varphi_1\|_{r\partial\mathbb{D}} \le \|f\|_{A(r)}(2\pi^{-1}\ln(16(1-r^2))^{-1} \quad \text{if} \quad r_0 < r < 1,$$

implying

$$\|\varphi_2(T)\| \le \|f\|_{A(r)}(1+2\pi^{-1}\ln(16(1-r^2))^{-1} \quad \text{if} \quad r_0 < r < 1.$$

Similarly

$$\|\varphi_1\|_{\partial\mathbb{D}} \le \|f\|_{\partial\mathbb{D}} + \|\varphi_2\|_{\partial\mathbb{D}} \le \|f\|_{A(r)} + \|\varphi_2\|_{r\partial\mathbb{D}} \quad \text{by the}$$

maximum modulus theorem. Putting both of these together we get:

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$$\|f(T)\| \le \|\varphi_1(T)\| + \|\varphi_2(T)\|$$

$$\le \|\varphi_1\|_{\partial\mathbb{D}} + \|\varphi_2\|_{r\partial\mathbb{D}}$$

$$\le \|f\|_{A(r)} + 2\|\varphi_2\|_{r\partial\mathbb{D}}$$

$$\le \|f\|_{A(r)}(3 + 4\pi^{-1}\ln(16(1-r^2))^{-1}) \quad \text{if} \quad r_0 < r < 1 .$$

If $0 < r \le r_0$ then the constant $2 + \sqrt{(1+r)/(1-r)}$ works. Q.E.D.

The estimate above is better than $2 + \sqrt{(1+r)/(1-r)}$ in the sense that $\lim\limits_{r\to 1^-} [(1-r)^{1/2}\ln(16(1-r^2))^{-1}] = 0.$ This estimate is still "bad", however, in the sense that $\lim\limits_{r\to 1^-} (3 + 4\pi^{-1}\ln(16(1-r^2))^{-1}) = \infty.$ We now give three results for operators $T \in \mathcal{B}(H)$. The first result is obtained by placing restrictions on the operators. The other two results are obtained by considering specific types of functions and investigating the norm inequality for this type of function.

<u>Proposition 6.2</u>: Let $\{T_r : r_0 < r < 1\} \subset \mathcal{B}(H)$ be a net of operators such that $\|T_r\| = 1$ and $\|T_r^{-1}\|^{-1} = r.$ Suppose there is a constant $M > 0$ such that $\|T_r^{-n}\| \le M$ for all $n \ge 0$ and for all r between r_0 and $1.$ Then for $r_0 < r < 1,$ $\|f(T_r)\| \le M\|f\|_{A(r)}$ for all $f \in \mathrm{Rat}(A(r)).$

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<u>Proof</u>: The proof of this proposition is easy. It is known that $\{q \in \mathrm{Rat}(A(r)) : q(z) = \sum_{k=-N}^{N} \hat{q}(k)z^k\}$ is dense in $\mathrm{Rat}(A(r))$. For $q(z) = \sum_{k=-N}^{N} \hat{q}(k)z^k$ we let $p(z) = z^n q(z)$. By von Neumann's theorem

$$\|p(T_r)\| \leq \|p\|_{\partial\mathbb{D}} = \|q\|_{\partial\mathbb{D}} \leq \|q\|_{A(r)} \quad .$$

Hence

$$\|q(T_r)\| = \|T_r{}^{-N}p(T_r)\| \leq \|T_r{}^{-N}\|\|p(T_r)\| \leq M\|q\|_{A(r)} \quad .$$

Q.E.D.

One could also prove this theorem by noting that if T_r and $T_r{}^{-1}$ are power bounded, then Tr is similar to a unitary operator U_r. The only difficulty would then come from computing the norms of the operators providing the similarity. We now have the two results concerning types of functions.

<u>Proposition 6.3</u>: Let $q(z) = \sum_{k=-\infty}^{\infty} \hat{q}(k)z^k \in \mathrm{Rat}(A(r))$ be such that $\hat{q}(k) \geq 0$ for all k. Then $\|q(T)\| \leq 2\|q\|_{A(r)}$ for all $T \in \mathcal{B}(H)$ with $\|T\| = 1$ and $\|T^{-1}\|^{-1} = r$.

<u>Proof</u>: Since $q(1) \geq \sum_{k=0}^{\infty} \hat{q}(k)$ and $q(r) \geq \sum_{k=-\infty}^{-1} \hat{q}(k)r^k$ we have

$$\|q(T)\| \leq \sum_{k=-\infty}^{\infty} |\hat{q}(k)| \|T^k\|$$

$$\leq \sum_{k=-\infty}^{-1} \hat{q}(k) r^k + \sum_{k=0}^{\infty} \hat{q}(k)$$

$$\leq q(r) + q(1)$$

$$\leq 2\|q\|_{A(r)} \quad . \qquad \text{Q.E.D.}$$

<u>Proposition 6.4</u>: Let $q(z) = z^{-N} + \alpha z^M$ where $M,N \geq 0$ and $\alpha \in \mathbb{C}$. If $T \in \mathcal{B}(H)$ with $\|T\| = 1$ and $\|T^{-1}\|^{-1} = r$ then $\|q(T)\| \leq 2\|q\|_{A(r)}$.

<u>Proof</u>: We see that $q(z) = z^{-N}(1 + \alpha z^{M+N})$. By the maximum modulus theorem, we have

$$\|q\|_{A(r)} = \max\{\|q\|_{\partial \mathbb{D}}, \|q\|_{r\partial \mathbb{D}}\}$$

$$= \max\{1 + |\alpha|, r^{-N}(1 + |\alpha| r^{M+N})\}$$

$$\geq \tfrac{1}{2}((1 + |\alpha|) + (r^{-N} + |\alpha| r^M)) \; .$$

Now

$$\|q(T)\| \leq \|T^{-N}\| + |\alpha| \|T^M\| \leq r^{-N} + |\alpha| \leq 2\|q\|_{A(r)} \; . \qquad \text{Q.E.D.}$$

In the next part of this chapter we restrict our attention to invertible bilateral weighted shifts. The question of whether we can take $c_r = 1$ is question 7 in Shields, [23]. He actually has two parts in his question. The first part asks whether $\sup\{c_r : 0 < r < 1\} < \infty$

when we are only considering weighted shifts. We answer the second part of question 7 using the following example. We state beforehand that the answer is no; one cannot take $c_r = 1$ for weighted shifts.

Example 6.1: Let T be the weighted shift with weight sequence as below.

i) $w_n = 1$ if $n \geq 0$

ii) $w_n = \frac{1}{2}$ if $n < 0$

Then

$$\|T\| = \sup\{w_n : n \in \mathbb{Z}\} = 1 \quad \text{and} \quad \|T^{-1}\|^{-1} = \inf\{w_n : n \in \mathbb{Z}\} = \frac{1}{2}$$

For $m > 0$ let $f_m(z) = z^{-1}(2 - z^m)^{-1}$. Then by the maximum modulus theorem

$$\begin{aligned} \|f_m\|_{A(\frac{1}{2})} &= \max[\|f_m\|_{\partial \mathbb{D}}, \|f_m\|_{\frac{1}{2}\partial \mathbb{D}}] \\ &= \max[1, 2(2 - 2^{-m})^{-1}] \\ &= (1 - 2^{-m-1})^{-1}. \end{aligned}$$

From this computation we see that $\lim\limits_{m\to\infty} \|f_m\|_{A(\frac{1}{2})} = 1$.

Now since $\|e_0\| = 1$ we have $\|f_m(T)\| \geq \|f_m(T)e_0\|$. We compute $\|f_m(T)e_0\|$ as follows. We note first of all that

$$f_m(z) = \frac{1}{2}(z^{-1} + \sum_{k=1}^{\infty} 2^{-k} z^{mk-1}) \quad .$$

It is easy to see that $T^{-1}e_0 = 2e_{-1}$ and $T^{mk-1}e_0 = e_{mk-1}$ since $mk - 1 \geq 0$ for $k \geq 1$ and $m \geq 1$. Thus

$$f_m(T)e_0 = e_{-1} + \sum_{k=1}^{\infty} 2^{-k-1} e_{mk-1} .$$

Hence $\|f_m(T)e_0\| = (1 + \sum_{k=1}^{\infty} 4^{-k-1})^{1/2} = \sqrt{13/12}$. Thus there exists an $M_0 > 0$ such that $\|f_m(T)\| > \|f_m\|_{A(\frac{1}{2})}$ if $m > M_0$. So it is not possible for
$\|q(T)\| \leq \|q\|_{A(\frac{1}{2})}$ for all $q \in \mathrm{Rat}(A(\frac{1}{2}))$.

<u>Proposition 6.5</u>: Let T be the weighted shift with weights as below:

i) $w_n = 1$ if $n \geq 0$

ii) $w_n = r$ if $n < 0$ where $0 < r < 1$.

Then $\|q(T)\| \leq \sqrt{2}\ \|q\|_{A(r)}$ for all $q \in \mathrm{Rat}(A(r))$.

<u>Proof</u>: For the weighted shift above $\beta(n) = 1$ if $n \geq 0$ and $\beta(n) = r^n$ if $n < 0$. For $f \in L^2(\beta)$ we have the following:

$$\|q(M_z)f\|_2^2 = \|M_q f\|_2^2$$

$$= \sum_{n=0}^{\infty} |(\widehat{qf})(n)|^2 + \sum_{n=-\infty}^{-1} |\widehat{qf}(n)|^2 r^{2n} .$$

It is known, however, that $\sum_{n=0}^{\infty} |\widehat{qf}(n)|^2 \leq \|q\|_{\partial\mathbb{D}}^2 \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$ and $\sum_{n=-\infty}^{-1} |\widehat{qf}(n)|^2 r^{2n} \leq \|q\|_{r\partial\mathbb{D}}^2 \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 r^{2n}$.

Thus

$$\|q(M_z)f\|_2^2 \leq \|q\|_{\partial \mathbb{D}}^2 \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 + \|q\|_{r\partial \mathbb{D}}^2 \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 r^{2n}$$

$$\leq 2\|q\|_{A(r)}^2 \|f\|_2^2 \quad \text{since} \quad \|f\|_2^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 + \sum_{n=-\infty}^{-1} |\hat{f}(n)|^2 r^{2n}.$$

Thus $\|q(M_z)\| \leq \sqrt{2}\ \|q\|_{A(r)}$ and the result follows easily. Q.E.D.

So for the weighted shift T in Example 6.1, $A(\frac{1}{2})$ is a $\sqrt{2}$-spectral set for T. The result of Proposition 6.5 says this type of idea will not be helpful in getting an example where $\sup\{c_r : 0 < r < 1\}$ is infinite. There is a generalization of Proposition 6.5 which we now prove.

<u>Proposition 6.6</u>: Let T be an invertible weighted shift with $\|T\| = 1$ and $\|T^{-1}\|^{-1} = r$ where $0 < r < 1$. Also, suppose the weight sequence $\{w_n\}$ of T satisfies the following two conditions:

i) $\{w_n : n \in \mathbb{Z}\} \subset \{\alpha_1, \cdots \alpha_N\}$

ii) $r \leq w_n \leq w_{n+1} \leq 1$ for all n

Then $\|q(T)\| \leq \sqrt{N}\ \|q\|_{A(r)}$ for all $q \in \mathrm{Rat}(A(r))$.

Proof: Let $0 = n_1 < n_2 < \cdots < n_{N-1}$. We will assume without loss of generality that:

a) $w_n = r = \alpha_1$ if $n < n_1$

b) $w_n = \alpha_k$ if $n_{k-1} \leq n < n_k$ $k = 2,3,\cdots,N-1$

c) $w_n = \alpha_n$ if $n_{N-1} \leq n$.

This implies, of course, that $r \leq \alpha_k \leq \alpha_{k+1} \leq 1$. From the above we have the following:

a′) $\beta(n) = \alpha_1^n$ if $n < n_1$

b′) $\beta(n) = \left(\prod_{\ell=2}^{k-1} \alpha_\ell^{n_\ell - n_{\ell-1}}\right)\left(\alpha_k^{n-n_{k-1}}\right)$ if $n_{k-1} \leq n < n_k$

c′) $\beta(n) = \left(\prod_{\ell=2}^{N-1} \alpha_k^{n_\ell - n_{\ell-1}}\right)\alpha_N^{n-n_{N-1}}$ if $n \geq n_{N-1}$.

Now let $P_1 = 1 - P$ where P is the orthogonal projection of $L^2(\beta)$ onto $H^2(\beta)$. For $k = 2,\cdots,N-1$ let P_k be the orthogonal projection of $L^2(\beta)$ onto the span of $\{z^n : n_{k-1} \leq n < n_k\}$. Finally, let P_N be the orthogonal projection onto the closed linear span of $\{z^n : n \geq n_{N-1}\}$.

Then for $q \in \mathrm{Rat}(A(r))$ and $f \in L^2(\beta)$ we have

$$(*) \qquad \|q(M_z)f\|_2^2 = \|M_q f\|^2 = \sum_{k=1}^{N} \|P_k(qf)\|_2^2 .$$

Now

$$\|P_1(qf)\|_2^2 = \sum_{n=-\infty}^{-1} |\widehat{qf}(n)|^2 \alpha_1^{2n}$$

$$\leq \|q\|^2_{\alpha_1 \partial \mathbb{D}} \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \alpha_1^{2n} \quad \text{as in Proposition 6.5}$$

$$\leq \|q\|^2_{A(r)} \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \beta^2(n) \quad \text{since}$$

$$\beta^2(n) \geq r^{2n} \quad \text{for all} \quad n$$

$$\leq \|q\|^2_{A(r)} \|f\|^2_2 \quad .$$

$$\|P_2(qf)\|^2_2 = \sum_{n=0}^{n_2-1} |\widehat{qf}(n)|^2 \alpha_2^{2n}$$

$$\leq \|q\|^2_{\alpha_2 \partial \mathbb{D}} \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \alpha_2^{2n}$$

$$\leq \|q\|^2_{A(r)} \|f\|^2_2 \quad \text{since} \quad \alpha_2^{2n} \leq \beta^2(n) \quad \text{for all } n .$$

For $k = 3,4,5,\cdots,N-1$

$$\|P_k(qf)\|^2_2 = \sum_{n=n_{k-1}}^{n_k-1} |\widehat{qf}(n)|^2 \beta^2(n)$$

$$\leq \left(\prod_{\ell=2}^{k-1} \alpha_\ell^{n_\ell - n_{\ell-1}}\right)^2 \alpha_k^{-2(n_{k-1})} \sum_{n=n_{k-1}}^{n_k-1} |\widehat{qf}(n)|^2 \alpha_k^{2n}$$

$$\leq \left(\prod_{\ell=2}^{k-1} \alpha_\ell^{n_\ell - n_{\ell-1}}\right)^2 \alpha_k^{-2(n_{k-1})} \|q\|^2_{\alpha_k \partial \mathbb{D}} \cdot \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \alpha_k^{2n}$$

However $\left[\prod_{\ell=2}^{k-1} \alpha_\ell^{n_\ell - n_{\ell-1}}\right]^2 \alpha_k^{2n-2n_{k-1}} \leq \beta^2(n)$ for all n.

Hence $\|P_k(qf)\|^2 \leq \|q\|^2_{A(r)} \|f\|^2_2$ for $k = 1,2,3,\cdots,N-1$. Similarly, one can show $\|P_N(qf)\|^2 \leq \|q\|^2_{A(r)} \|f\|^2_2$. Now, going back to formula (*) we get $\|q(M_z)f\|_2 \leq \sqrt{N}\, \|q\|_{A(r)} \|f\|$ for all $f \in L^2(\beta)$. Hence $\|q(T)\| \leq \sqrt{N}\, \|q\|_{A(r)}$ for all $q \in \mathrm{Rat}(A(r))$. Q.E.D.

Again, it may not be that $\sqrt{N}$ is the best possible constant. In fact, if we pick N large enough so that $\sqrt{N} > 2 + \sqrt{(1+r)/(1-r)}$, then $\sqrt{N}$ is not the best constant by Theorem 6.2. However, $\sqrt{N}$ does give a better estimate in certain cases.

One open question in the area of c-spectral sets is whether every operator whose spectrum is a c-spectral set is similar to an operator whose spectrum is a spectral set. It is possible that Example 6.1 might answer this question in the negative. However, the following example may indicate that it might not answer the question.

Example 6.2: Let T be the weighted shift with weights as given below:

i) $w_n = 1$ for $n \neq -1$

ii) $w_n = \frac{1}{2}$ for $n = -1$.

For this weighted shift $r(T) = \|T\| = 1 = r(T^{-1})^{-1}$ and $\|T^{-1}\|^{-1} = \frac{1}{2}$. Using the same functions as in Example 6.1, we see that $\sigma(T) = \partial \mathbb{D}$ is not a spectral set for T. (Note: the part that makes the computation work is that $T^{-1}_{e_0} = 2e_{-1}$). However, by verifying the requirements of Theorem 2 in Shields [23], it is easy to see that T is similar to the unweighted shift. The unweighted shift is normal and hence its spectrum is a spectral set. We also note that since T is similar to a normal operator, its spectrum is a c-spectral set.

Suppose we could find a collection of weighted shifts T_r for $0 < r < 1$ satisfying:

i) $\|T_r\| = 1$ and $\|T_r^{-1}\|^{-1} = r$

ii) $\sigma(T_r) = A(r)$

iii) $\sup\{\|\mathcal{P}_r\| : 0 < r < 1\} = M < \infty$ where $\mathcal{P}_r$ is the "analytic" projection on $L^\infty(\beta_r)$.

Then if $q(z) = \sum_{k=-N}^{N} \hat{q}(k)z^k$ we would have

$$\|\sum_{k=0}^{N} \hat{q}(k)z^k\|_{\partial\mathbb{D}} \leq \|\mathcal{P}_r q\|_{L^\infty(\beta_r)} \qquad \text{(the spectral mapping theorem)}$$

$$\leq \|\mathcal{P}_r\|\|q\|_{L^\infty(\beta_r)} \qquad \text{(assumption)}$$

$$\leq Mc_r\|q\|_{A(r)} \qquad \text{(Theorem 6.2)} .$$

We note that $\lim_{r\to 1^-} \|q\|_{A(r)} = \|q\|_{\partial\mathbb{D}}$. Thus if $\sup\{c_r : 0 < r < 1\}$ was finite, the "analytic" projection on $C(\partial\mathbb{D})$ would be bounded. This is not true, however, as stated before. Hence, it would not be possible for c_r to be bounded as a function of r. Unfortunately, it is not possible to find weighted shifts T_r satisfying the condition above. The proposition below verifies this.

<u>Proposition 6.7</u>: Suppose that $\{T_r : 0 < r < 1\}$ is a collection of weighted shifts with $\|T_r\| = 1$ and $\|T_r^{-1}\|^{-1} = r$. Then $\sup\{\|\mathcal{P}_r\| : 0 < r < 1\} = \infty$.

Proof: Suppose $\sup\{\|\mathcal{O}_r\| : 0 < r < 1\} = M < \infty$. Then for $q(z) = \sum_{k=-N}^{N} \hat{q}(k)z^k$ we have

$$\|\sum_{k=0}^{N} \hat{q}(k)z^k\|_{\partial\mathbb{D}} = \|\sum_{k=0}^{N} \hat{q}(k)T_r{}^k\| \quad \text{(von Neumann's theorem)}$$

$$= \|\mathcal{O}_r q\|_{L^\infty(\beta_r)}$$

$$= M\|q\|_{L^\infty(\beta_r)}$$

However, I claim $\lim_{r\to 1^-} \|q\|_{L^\infty(\beta_r)} = \|q\|_{\partial\mathbb{D}}$. To see this, let T be the unweighted shift. Then $\|q(T)\| = \|q\|_{\partial\mathbb{D}}$ since T is normal and $\sigma(T) = \partial\mathbb{D}$.

Now

$$\|q(T_r) - q(T)\| \leq \sum_{k=-N}^{N} |\hat{q}(k)|\,\|T_r{}^k - T^k\|$$

$$\leq \sum_{k=1}^{N} |\hat{q}(k)|(1-r^k) + \sum_{k=1}^{N} |\hat{q}(-k)|(r^{-k}-1) .$$

Since the sums above are finite $\lim_{r\to 1^-}\|q(T_r) - q(T)\| = 0$. Thus $\lim_{r\to 1^-}\|q\|_{L^\infty(\beta_r)} = \lim_{r\to 1^-} \|q(T_r)\| = \|q\|_{\partial\mathbb{D}}$. The claim is now verified. Using this, we have

$$\|\sum_{k=0}^{N} \hat{q}(k)z^k\|_{\partial\mathbb{D}} \leq M\|q\|_{\partial\mathbb{D}} .$$

As before, this inequality says the "analytic" projection on $C(\partial\mathbb{D})$ is bounded. This is false and so $\sup\{\|\mathcal{O}_r\| : 0 < r < 1\} = \infty$. Q.E.D.

This is all we will say about weighted shifts. We will end this chapter by considering linear operators T on $\mathbb{C}^2$ with $\|T\| = 1$ and $\|T^{-1}\|^{-1} = r$. Any linear operator T on $\mathbb{C}^2$ has a matrix representation

$$T = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$$

where $\alpha_{ij} \in \mathbb{C}$. By unitary equivalence, we may assume that T has the form $\begin{bmatrix} \lambda_1 & \alpha \\ 0 & \lambda_2 \end{bmatrix}$ with respect to some orthonormal basis for $\mathbb{C}^2$. The norm of a 2×2 matrix can be computed easily by noting that $\|T\|^2 = \|T^*T\|$. Since T^*T is selfadjoint, its norm is $\sup\{|\lambda| : \lambda \in \sigma(T^*T)\}$. By similar reasoning, $\|T^{-1}\|^{-2}$ will be $\inf\{|\lambda| : \lambda \in \sigma(T^*T)\}$. One can find the eigenvalues of T^*T by finding the solutions to the quadratic equation $\det(tI - T^*T) = 0$. If one does this for

$$T = \begin{bmatrix} \lambda_1 & a \\ 0 & \lambda_2 \end{bmatrix}$$

with $\|T\| = 1$, $\|T^{-1}\|^{-1} = r$, we get the following restrictions on λ_1, λ_2, and a.

i) $|\lambda_1||\lambda_2| = r$

ii) $|a|^2 = (1 - |\lambda_1|^2)(1 - |\lambda_2|^2)$

We will use these, but first we will find out what $q(T)$ is for $q \in \mathrm{Rat}(A(r))$.

<u>Proposition 6.8</u>: Let $T = \begin{bmatrix} \lambda_1 & a \\ 0 & \lambda_2 \end{bmatrix}$ and let q be analytic on a neighborhood of $\sigma(T) = \{\lambda_1, \lambda_2\}$. Then

$$q(T) = \begin{bmatrix} q(\lambda_1) & a[q(\lambda_1)-q(\lambda_2)]/(\lambda_1-\lambda_2) \\ 0 & q(\lambda_2) \end{bmatrix} .$$

Note: if $\lambda_1 = \lambda_2$, then $(q(\lambda_1)-q(\lambda_2))/(\lambda_1-\lambda_2)$ is replaced by $q'(\lambda_1)$ in the above formula.

<u>Proof</u>: First suppose $\lambda_1 = \lambda_2 = \lambda$ and let q be analytic on $U = \{z : |z-\lambda| < \varepsilon\}$. Let $\Gamma = \{z : |z-\lambda| = \varepsilon/2\}$ and $q(z) = \sum_{n=0}^{\infty} (q^{(n)}(\lambda)/n!)(z-\lambda)^n$ for $z \in U$. Then by the Riesz functional calculus

$$q(T) = \frac{1}{2\pi i} \int_\Gamma q(z)(z-T)^{-1} dz$$

Thus $(q(T)e_i,e_j) = \frac{1}{2\pi i} \int_\Gamma q(T)((z-T)^{-1}e_i,e_j)dz$ for $i,j = 1,2$. One easily computes that $(z-T)^{-1} = \begin{bmatrix} (z-\lambda)^{-1} & a(z-\lambda)^{-2} \\ 0 & (z-\lambda)^{-1} \end{bmatrix}$. Thus $(q(T)e_i,e_i) = \frac{1}{2\pi i} \int_\Gamma q(z)((z-T)^{-1}e_i,e_i)dz = q(\lambda)$ by the Cauchy integral formula. Also $(q(T)e_1,e_2) = \frac{1}{2\pi i} \int_\Gamma q(z)a(z-\lambda)^{-2}dz = aq'(\lambda)$ by the residue theorem.

If $\lambda_1 \neq \lambda_2$ and q is analytic on a neighborhood of $\{\lambda_1, \lambda_2\}$ we choose two disjoint circles Γ_1 and Γ_2 in U surrounding λ_1 and λ_2 respectively. Then

$$(q(T)e_i, e_j) = \frac{1}{2\pi i}\int_{\Gamma_1} q(z)((z-T)^{-1}e_i, e_j)dz$$

$$+ \frac{1}{2\pi i}\int_{\Gamma_2} q(z)((z-T)^{-1}e_i, e_j)dz \,.$$

If $\lambda_1 \neq \lambda_2$ then $(z-T)^{-1} = \begin{bmatrix} (z-\lambda_1)^{-1} & a(z-\lambda_1)^{-1}(z-\lambda_2)^{-1} \\ 0 & (z-\lambda_2)^{-1} \end{bmatrix}.$

Hence $(q(T)e_i, e_i) = \frac{1}{2\pi i}\int_{\Gamma_1} q(z)(z-\lambda_i)^{-1}dz + \frac{1}{2\pi i}\int_{\Gamma_2} q(z)(z-\lambda_i)^{-1}dz$

$= q(\lambda_i)$ since for $i \neq j$, $\int_{\Gamma_j} q(z)(z-\lambda_i)^{-1}dz = 0$

because $q(z)(z-\lambda_i)^{-1}$ is analytic inside of Γ_j. Now

$$(q(T)e_1, e_2) = \frac{1}{2\pi i}\int_{\Gamma_1} q(z)a((z-\lambda_1)^{-1}(z-\lambda_2)dz$$

$$+ \frac{1}{2\pi i}\int_{\Gamma_2} q(z)a((z-\lambda_1)^{-1}(z-\lambda_2)dz$$

$$= a[q(\lambda_1)/(\lambda_1-\lambda_2) + q(\lambda_2)/(\lambda_2-\lambda_1)]$$

$= a[(q(\lambda_1)-q(\lambda_2))/(\lambda_1-\lambda_2)]$ as desired.

Q.E.D.

We end this chapter with the following proposition. It says that for two-dimensional linear operators $\sup\{c_r : 0 < r < 1\}$ is finite. A conjecture might be that this is true for operators on n-dimensional space. The constants c_r would however perhaps depend upon the dimension of the space.

<u>Proposition 6.9</u>: Let $T \in \mathcal{B}(\mathbb{C}^2)$ satisfy $\|T\| = 1$ and $\|T^{-1}\|^{-1} = r$. Then $\|q(T)\| \leq 64r^{-1/2}\|q\|_{A(r)}$ for all $q \in \mathrm{Rat}(A(r))$.

<u>Proof</u>: We let T have the form $\begin{bmatrix} \lambda_1 & a \\ 0 & \lambda 2 \end{bmatrix}$ and thus have i) $|\lambda_1||\lambda_2| = r$ and ii) $|a|^2 = (1-|\lambda_1|^2)(1-|\lambda_2|^2)$. We also note that if $A = (a_{ij})$ is any operator on $\mathbb{C}^2$ then $\|A\| \leq 4\max\{|a_{ij}| : i,j = 1,2\}$. By Proposition 6.8

we know that if $\lambda_1 \neq \lambda_2$ Then $q(T) =$

$$\begin{bmatrix} q(\lambda_1) & a(q(\lambda_1)-q(\lambda_2))/(\lambda_1-\lambda_2) \\ 0 & q(\lambda_2) \end{bmatrix}.$$ Since $\{\lambda_1,\lambda_2\} = \sigma(T) \subset A(r)$

we have $|q(\lambda_i)| \leq \|q\|_{A(r)}$ for $i = 1,2$. Thus we need only investigate $|a||q(\lambda_1)-q(\lambda_2)||\lambda_1-\lambda_2|^{-1}$. We will use three cases. The first case will also indicate what to do if $\lambda_1 = \lambda_2$.

<u>Case 1</u>: Assume $|\lambda_1| = |\lambda_2| = \sqrt{r}$ and let

$$f(z) = \begin{cases} \dfrac{q(z)-q(\lambda_1)}{z-\lambda_1} & \text{if } z \in A(r),\ z \neq \lambda_1 \\ q'(\lambda_1) & \text{if } z = \lambda_1 \end{cases}$$

Then f is a function analytic on a neighborhood of $A(r)$. Thus $\|f\|_{A(r)} = \max\{\|f\|_{\partial \mathbb{D}}, \|f\|_{r\partial \mathbb{D}}\}$ by the maximum modulus theorem. It is easy to see that $\|f\|_{\partial\mathbb{D}} \leq 2\|q\|_{A(r)}/(1-\sqrt{r})$ and $\|f\|_{r\partial\mathbb{D}} \leq 2\|q\|_{A(r)}/(\sqrt{r}-r)$ and $|a|^2 = (1-|\lambda_1|^2)(1-|\lambda_2|^2) = (1-r)^2$.

Hence $|a|\,|q(\lambda_1)-q(\lambda_2)|\,|\lambda_1-\lambda_2|^{-1} \leq 4r^{-1/2}\|q\|_{A(r)}$.

<u>Case 2</u>: Assume $\sqrt{r} < |\lambda_1| \leq \sqrt[4]{r}$. Then $\sqrt[4]{r^3} \leq |\lambda_2| < \sqrt{r}$. In this case we let $f(z)$ be defined as in Case 1.

Then $|f(z)| \leq 2\|q\|_{A(r)} \max\{(1-r^{1/4})^{-1}, (\sqrt{r}-r)^{-1}\}$ and $|a|^2 \leq (1-r)(1-r^2) \leq 2(1-r)^2$. Thus $|a|\,|q(\lambda_2)-q(\lambda_1)|\,|\lambda_2-\lambda_1|^{-1} \leq 8\sqrt{2}\ r^{-1/2}\ \|q\|_{A(r)}$ since

$$(1-r)/(1-r^{1/4}) = (1+\sqrt{r})(1+r^{1/4}) \text{ and } (1-r)/(\sqrt{r}-r) = (1+\sqrt{r})/\sqrt{r}\ .$$

<u>Case 3</u>: Assume $\sqrt[4]{r} < |\lambda_1| \leq 1$. Then $r \leq |\lambda_2| < \sqrt[4]{r^3}$. In this case we let $f(z) = (q(z)-q(\lambda_1))/(z-\lambda_1)$ for $z \in \{z \in \mathbb{C} : r \leq |z| \leq \sqrt{r}\}$. Then $|q(\lambda_2)-q(\lambda_1)|\,|\lambda_2-\lambda_1|^{-1} \leq 2\|q\|_{A(r)}(\sqrt[4]{r}-\sqrt{r})^{-1}$ and $|a|^2 \leq (1-\sqrt{r})(1-r^2) \leq 4(1-\sqrt{r})^2$.

Thus $|a||q(\lambda_2)-q(\lambda_1)||\lambda_2-\lambda_1|^{-1} \leq 8r^{-1/4}\|q\|_{A(r)} \leq 8r^{-1/2}\|q\|_{A(r)}$.

Using the results of these three cases, we have

$$\|q(T)\| \leq 64r^{-1/2}\|q\|_{A(r)} \quad \text{for all} \quad q \in \mathrm{Rat}(A(r)) \ .$$

Q.E.D.

BIBLIOGRAPHY

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[1] Abrahamse, A.B., Toeplitz operators in multiply connected domains, Bull. Amer. Math. Soc. 77 (1971), 449-544.

[2] Axler, Berg, Jewell, Shields, Approximation by compact operators and the space $H^\infty + C$, Annals of Math. 109 (1979), 601-612.

[3] Berger, C.A. and Stampfli, J.G., Mapping Theorems for the Numerical Range, Amer. J. Math. 89 (1967), 1047-1055.

[4] Byrd, P. and Friedman, M., Handbook of Elliptic Integrals for Engineers and Physicists, Berlin, Springer-Verlag, 1954.

[5] Devinatz, A., Toeplitz operators on H^2 spaces, Trans. Amer. Math. Soc. 112 (1964), 304-317.

[6] Douglas, R., Banach Algebra Techniques in Operator Theory, New York, Academic Press, 1972.

[7] Gohberg, I.C. and Krein, M.G., Systems of integral equations on a half line with kernels depending on the difference of arguments, Amer. Math. Soc. Transl. 14 (2) (1960), 217-287.

[8] Halmos, P.R., A Hilbert Space Problem Book, New York, American Book Company, 1967.

[9] Halmos, P.R. and Brown, A., Algebraic Properties of Toeplitz operators, J. Reine Angew. Math. 213 (1964), 89-102.

[10] Hardy, Littlewood, and Polya, Inequalities, Cambridge, Cambridge University Press, 1959.

[11] Hartman, P. and Wintner, A., On the Spectra of Toeplitz's Matrices, Amer. J. Math. 72 (1950), 359-366.

[12] Hartman, P., On Completely Continuous Hankel Matrices, Proc. Amer. Math. Soc. 9 (1958), 862-866.

[13] Katznelson, Y., An Introduction to Harmonic Analysis, New York, Dover Publications, 1976.

[14] Lebow, A., On von Neumann's theory of spectral sets, J. Math. Anal. Appl. 7 (1963), 64-90.

[15] Nehari, Z., On bounded bilinear forms, Annals of Math. 65 (1957), 153-162.

[16] O'Donovan, D., Bounded Matrices and Some Algebras of Operators, Proc. Amer. Math. Soc. 58 (1976), 134-138.

[17] Rudin, W., Functional Analysis, New York, McGraw-Hill Book Co., 1973.

[18] __________, Real and Complex Analysis, New York, McGraw-Hill Book Co., 1974.

[19] __________, Spaces of Type $H^\infty + C$, Annales de l'Institut Fourier 25 (1)(1975), 99-125.

[20] Ryshik and Gradstein, Tafeln Tables, Berlin, VEB Deutscher Verlag Der Wissenschaften, 1963.

[21] Sarason, D., Function Theory on the Unit Circle, VPI Notes, Blacksburg, Virginia, 1978.

[22] __________, Generalized Interpolation on H^∞, Trans. Amer. Math. Soc. 127 (1967), 179-203.

[23] Shields, A., Weighted Shift Operators and Analytic Function Theory, Amer. Math. Soc. Surveys 13 (1974), 49-128.

[24] Sz.-Nagy, B., Harmonic Analysis of Operators on Hilbert Space, Budapest, Akademiai Kiado', 1970.

[25] __________, Appendix to Functional Analysis by Riesz and Sz.-Nagy, New York, Frederick Ungar Pub. Co., 1960.

[26] von Neumann, J., Spektral theorie für allgemeine operatoren einen unitaren raumes, Math. Nachr. 4 (1951), 258-281.

[27] Wadwha, B.L., A hyponormal operator whose spectrum is not a spectral set, Proc. Amer. Math. Soc. 38 (1973), 83-85.

[28] Williams, J.P., Minimal spectral sets of compact operators, Acta Sci. Math. (Szeged) 28 (1967), 93-106.