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ABSTRACT

ON THE SYMMETRIC DERIVATIVE

By

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A class of functions, σ^*, is defined and is shown to contain all known symmetrically differentiable functions. It is proved that if $f \in \sigma^*$, then f is in the first Baire class. Using this result, it is shown that there is associated with each $f \in \sigma^*$ another function, g, which retains the symmetric differentiation properties of f while at the same time "maximizing" many of the more desirable properties of f such as differentiability, continuity and upper semi-continuity. Such a function, g, is in Baire class one and is uniquely determined up to its values on a set with countable closure. We call g the "nice copy" of f.

Using the properties of the nice copy, many of the standard theorems of ordinary differentiation can be reformulated in terms of the symmetric derivative. In particular, analogues of the mean value theorem and the Darboux property are presented. The methods also give simplified proofs of several well-known theorems. These results are then applied to develop an abstract Zahorski class structure for symmetric derivatives.

In addition, several structure theorems for completely arbitrary symmetric derivatives are proved.

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INTRODUCTION

If f is a real-valued function defined on $\mathbb{R}$, then the symmetric derivative of f at x (often called the first Schwarz derivative of f) is

$$f^s(x)=\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}.$$

The symmetric derivative arises naturally in studies of the pointwise convergence of Fourier and Taylor series as well as other areas of harmonic analysis. In this work, however, we do not consider these applications of symmetric differentiation, but rather, we investigate the symmetric derivative viewed as a generalization of the ordinary derivative. Specifically, our goal is to expose similarities between the well-known structure of ordinary derivatives and the structure of symmetric derivatives.

We begin in Chapter I by presenting much of the terminology used throughout this work and by stating the fundamental theorems needed in the succeeding chapters. In particular, we define a class of symmetrically differentiable functions, σ^*, which is the "domain" for most of the later theorems. It is shown that σ^* contains all measurable, symmetrically differentiable functions and therefore all known symmetrically differentiable functions, since the question of the measurability of such functions remains unresolved. Chapter I is concluded with the proof of a partitioning theorem which was first stated in a slightly weaker form by B. S. Thomson [26].

One of the most useful theorems available for the study of ordinary derivatives, due to Zahorski [29], is that any ordinary derivative belongs to the first class of Baire ($\mathfrak{B}_1$). It was proved by Filipczak [7] that the symmetric derivative of an approximately continuous function is in $\mathfrak{B}_1$. The main theorem of Chapter II is that this result can be extended to the more general case of σ^*. In Chapter II, we also examine the question of whether there are any symmetrically differentiable functions which are not contained in σ^*. While no answer to this question is reached, several results are obtained which strongly suggest that if any such function, f, exists, then f^s is in $\mathfrak{B}_1$.

It is well-known that if f is a finite-valued ordinary derivative, then any primitive function for f is determined up to an additive constant. That this is not the case with a symmetric derivative can be seen by considering the following two functions. Let f(x)=0 everywhere and let

$$g(x) = \begin{cases} x^{-2} & \text{for } x=\pm 1, \pm 1/2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Then it is easy to see that $f^s(x)=g^s(x)=0$ everywhere, but f(x)−g(x) is not constant. Because of this lack of a unique primitive, many of the standard theorems of ordinary differentiation are either false or much harder to prove with the symmetric derivative.

A solution to this uniqueness problem is presented in Chapter III with the introduction of the "nice copy" of a function in σ^*. The nice copy of $f \in \sigma^*$ is a function, g, which in some sense "maximizes" several of the desirable properties of f such as differentiability and continuity, while at the same time retaining the symmetric differentiability properties of f. In particular, it is shown that there is a set, A, with countable closure, such that $g^s(x)$ agrees with $f^s(x)$ on $\bar{A}^c$ and further, that g is uniquely determined and upper semicontinuous on $\bar{A}^c$.

The existence of the nice copy for any f in σ^* leads at once to the existence of a "nice primitive" which is uniquely determined up to an additive constant and its values on a set, A, with countable closure. This nice primitive solves the uniqueness problem presented above, and thus gives us a means of establishing many of the classical theorems of ordinary differentiation in terms of the symmetric derivative. For example, the quasi-mean value theorems of Aull [1] and Evans [5] and the monotonicity theorems of Weil [28] and Evans [5] can be generalized to σ^*. Another consequence of the methods employed in Chapter III is a simplification of a proof due to Charzynski [4] showing that the set of discontinuities of any $f \in \sigma^*$ such that f^s is finite-valued must be countable with no dense in itself subset.

Finally, in Chapter IV, we extend the results of Zahorski [29] on the associated sets of derivatives to symmetric derivatives. In so doing, the results of Kundu [16] are considerably strengthened. In particular, it is shown that if f^s is the symmetric derivative of an $f \in \sigma^*$ such that f^s has the Darboux property, then $f^s \in \mathcal{M}_2$. Kundu's theorems, in the cases of $\mathcal{M}_3$ and $\mathcal{M}_4$, are proved without his assumptions that f is continuous and f^s has the Darboux property. Examples are given to show that certain of the proved Zahorski class containments are proper with symmetric derivatives.

CHAPTER I

NOTATION, DEFINITIONS AND BASIC THEOREMS

Section 1.1: Notation

In this section we introduce most of the basic definitions and notation which will be used in later chapters. Several of the "classical" theorems concerning symmetric derivatives are also stated to motivate some of the new concepts.

Throughout this work the real numbers will be denoted by $\mathbb{R}$ and the extended reals, $[-\infty, \infty]$, will be denoted by $\mathbb{R}^*$. Z will stand for the integers and Z^+ will represent the positive integers.

If $A \subset \mathbb{R}$ is Lebesgue measurable, then the measure of A will be denoted by $|A|$. In fact, the only measure we shall have occasion to use is Lebesgue measure, so terms such as "measurable," "almost everywhere," etc., should be interpreted accordingly. A^c will stand for the complement of A.

Let f be a real-valued function defined on an open interval, I. If $x \in I$, we define the upper (lower) symmetric derivative of f at x to be

$$\overline{f}^s(x) = \limsup_{h\to 0} \frac{f(x+h)-f(x-h)}{2h}$$

$$\left(\underline{f}^s(x) = \liminf_{h\to 0} \frac{f(x+h)-f(x-h)}{2h}\right).$$

When $\overline{f}^s(x)=\underline{f}^s(x)$, whether finite or infinite, we call their common value the symmetric derivative of f at x and denote it by $f^s(x)$. If $f^s(x)$ exists at every point of the domain of f, then f is said to be symmetrically differentiable.

$D^+f(x_0)$, $D_-f(x_0)$, etc. stand for the Dini derivatives of f at x_0; $f^+(x_0)$, $f^-(x_0)$ and $f'(x_0)$ denote the ordinary right, left and bilateral derivatives of f at x_0, respectively. If both of the sums, $D_+f(x)+D^-f(x)$ and $D^+f(x)+D_-f(x)$, make sense, then it is easy to see that

$$\frac{1}{2}(D_+f(x)+D^-f(x)) \leq \underline{f}^s(x) \leq \overline{f}^s(x) \leq \frac{1}{2}(D^+f(x)+D_-f(x)).$$

Therefore, if both $f^+(x)$ and $f^-(x)$ exist, then so does $f^s(x)$, and $f^s(x)=\frac{1}{2}(f^+(x)+f^-(x))$. Further, if $f'(x)$ exists, finite or infinite, then $f^s(x)=f'(x)$. Thus, the symmetric derivative is a generalization of the ordinary derivative. To see that it is a strict generalization, consider $f(x)=|x|$ for which $f^s(0)=0$, but $f'(0)$ does not exist.

f is said to be symmetrically continuous at x_0 if

$$\lim_{h\to 0} (f(x_0+h)-f(x_0-h))=0.$$

As usual, a function which is symmetrically continuous at each point of its domain is called symmetrically continuous. It is clear that if $f^s(x_0)$ exists and is finite, then f is symmetrically continuous at x_0.

It easily follows from the definitions that if f is continuous at x_0, then f is also symmetrically continuous at x_0. That the converse is not true can be seen from the function $f(x)=\cos\frac{1}{x}$ which is symmetrically continuous (even symmetrically differentiable) at x=0, but certainly not continuous there. Therefore, just as symmetric differentiability is an extension of ordinary differentiability, so is symmetric continuity an extension of ordinary continuity.

We shall denote, for any function, f,

$$D(f)=\{x:\ f'(x) \text{ exists and is finite}\}$$

and

$$C(f)=\{x:\ f \text{ is continuous at } x\}.$$

The following proposition will prove useful in later chapters and will be employed without constant reference to this section. A proof of it may be found in [12].

<u>Proposition 1.1</u>. Let I be an interval and f a function defined on I. Then C(f) is a G_δ set.

Let $A \subset \mathbb{R}$ and x be a limit point of A. If there exists at least one sequence from A increasing to x, we define

$$A\text{-}\limsup_{t\to x-} f(t)=\lim_{\delta\to 0} \sup\{f(t):\ t\in(x-\delta,\ x)\cap A\}$$

and

$$A\text{-}\liminf_{t\to x-} f(t)=\lim_{\delta\to 0} \inf\{f(t):\ t\in(x-\delta,\ x)\cap A\}.$$

If both of the above limits agree, their common value will be denoted by $A\text{-}\lim_{t\to x-} f(t)$. The right-hand limits through A are defined analogously. The meanings of $A\text{-}\limsup_{t\to x} f(t)$, $A\text{-}\lim_{t\to x} f(t)$, etc., are now obvious. If, in the above definitions, $A=\mathbb{R}$, then it is omitted from the above expressions to conform to standard notation.

Suppose $A \subset \mathbb{R}$ and $x_0\in\mathbb{R}$. We denote the reflection of A through x_0 by $\mathcal{R}_{x_0}(A)$. For example, $\mathcal{R}_1([0,4))=(-2,2]$. Further suppose that I is an open interval and f is a function defined on I such that C(f) is dense on I. Let $x_0 \in I$ and $\{\delta_i: i \in Z^+\}$ be a sequence of positive numbers decreasing to 0 such that $(x_0-\delta_i, x_0+\delta_i) \subset I$ for all $i \in Z^+$. Choose any $k \in Z^+$. Since any G_δ set which is dense in an interval is residual in that interval, we see that C(f) is residual in both $(x_0-\delta_k, x_0)$ and $(x_0, x_0+\delta_k)$. It is then clear that $R_{x_0}(C(f) \cap (x_0-\delta_k, x_0))$ is residual in $(x_0, x_0+\delta_k)$ and thus

$$C(f) \cap R_{x_0}(C(f) \cap (x_0-\delta_k, x_0)) \neq \emptyset.$$

Choose x_k to be an element of the above intersection and let $y_k = R_{x_0}(x_k)$. This procedure can be followed for each $k \in Z^+$ to generate two sequences, $\{x_i: i \in Z^+\}$ and $\{y_i: i \in Z^+\}$, which satisfy

(1) $$R_{x_0}(x_i) = y_i \text{ for all } i \in Z^+,$$

(2) $$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x_0$$

and

(3) $$\{x_i: i \in Z^+\} \cup \{y_i: i \in Z^+\} \subset C(f).$$

Given a function, f, we say two sequences, $\{x_i: i \in Z^+\}$ and $\{y_i: i \in Z^+\}$, satisfying (1)-(3) converge C(f)-symmetrically to x_0. From the above considerations, the following proposition is clear.

Proposition 1.2. Let f be a function defined on an open interval, I, such that C(f) is dense on I. Then each $x_0 \in I$ has a pair of C(f)-symmetric sequences converging to it.

With f and I as in the proposition, we define $f^{sc}(x_0)$ to be

$$\lim_{n\to\infty}\frac{f(y_n)-f(x_n)}{y_n-x_n}$$

if the limit exists and is the same for all C(f)-symmetric sequences, $\{x_n: n\in Z^+\}$ and $\{y_n: n\in Z^+\}$, converging to x_0. For example, if $f^s(x_0)$ exists and C(f) is dense in a neighborhood of x_0, then $f^{sc}(x_0)$ exists and equals $f^s(x_0)$.

The following theorems, which motivated several of the definitions given above, are fundamental to the results in Chapters II and III.

Theorem 1.3. (Fried [10]) Suppose the set of points at which f is symmetrically continuous is residual on an open interval, I. Then C(f) is also residual on I.

Theorem 1.4. (Preiss [21]) If f is symmetrically continuous on an interval, I, then it is continuous a. e. on I.

Theorem 1.5. (Khintchine [13]) Let f be a measurable function defined on an open interval, I. Then f has a finite ordinary derivative at almost all points for which $\underline{f}^s(x)>-\infty$.

Suppose f is a function defined on an open interval, I, such that $f^s(x)$ is finite everywhere on I. Then f is symmetrically continuous on I and theorems 1.3 and 1.4 both imply that C(f) is dense. Further, if f is measurable on I and symmetrically differentiable (infinite values allowed), then by considering f and -f, we see from theorem 1.5 that f has a finite ordinary derivative a. e. and thus C(f) is again dense. The common thread which seems to bind all three of these theorems is that if f is a reasonably behaved function which is symmetrically differentiable, then C(f) is dense. This observation motivates the following definition.

Definition. Let I be an open interval. Define $\sigma^*(I)$ to be the class of all functions, f, such that C(f) is dense on I and $f^s(x)$ exists, finite or infinite, everywhere on I. Define $\sigma(I)$ to be the class of all functions, $f \in \sigma^*(I)$, such that $f^s(x)$ is finite at each $x \in I$.

Analogously, we denote by $\Delta^*(I)$ the class of all functions, f, such that $f'(x)$ exists everywhere on I and by $\Delta(I)$ the class of all functions, $f \in \Delta^*(I)$, such that $f'(x)$ is finite everywhere on I.

Using the above notation, we write

$$\sigma^{*s}(I) = \{f^s : f \in \sigma^*(I)\} \text{ and } \Delta^{*\prime}(I) = \{f' : f \in \Delta^*(I)\}.$$

$\sigma^s(I)$ and $\Delta'(I)$ are defined similarly.

In order to make the notation slightly less cumbersome, we denote $\sigma^*(\mathbb{R})$, $\Delta^*(\mathbb{R})$, etc., as just σ^*, Δ^*, etc.. Most propositions will be stated with this simplification, it being clear in all such cases that the restriction of the statement to an arbitrary open interval is valid. As a further notational convenience, if $\mathfrak{X}$ and $\mathfrak{U}$ happen to be classes of functions, we denote $\mathfrak{X} \cap \mathfrak{U}$ by $\mathfrak{X}\mathfrak{U}$.

The following corollaries are easy consequences of the definitions and the three theorems.

Corollary 1.6. Let f be a measurable function such that $f^s(x)$, finite or infinite, exists everywhere. Then

(a) $|\{x: |f^s(x)|=\infty\}|=0$

(b) $f'(x)$ exists and is finite a. e.

(c) $f\in\sigma^*$.

Proof. By applying theorem 1.5 to f and -f, we see that $|\{x: f^s(x)=\infty\}|=0$ and $|\{x: f^s(x)=-\infty\}|=0$ and (a) follows. (a) and another application of theorem 1.5 yield (b). Since $f'(x)$ exists and is finite a. e., it follows that f is continuous a. e. so C(f) is dense and $f\in\sigma^*$.

Corollary 1.7. $f\in\sigma$ if and only if $f^s(x)$ exists and is finite everywhere. In this case, f is measurable.

Proof. If $f\in\sigma$, then $f^s(x)$ exists and is finite everywhere by the definition of σ. On the other hand, if $f^s(x)$ is finite everywhere, then f is symmetrically continuous and theorem 1.3 implies that f is continuous a. e.. Thus, C(f) is dense and $f\in\sigma$. We note that any function which is continuous a.e. is measurable.

Putting these results together, we see that if $\mathcal{L}$ is the class of all measurable symmetrically differentiable functions, then $\Delta\subset\sigma\subset\mathcal{L}\subset\sigma^*$. The question of whether there are any symmetrically differentiable functions which are not measurable is unanswered and will be examined further in Chapter II. Using a theorem of Zahorski [29] that any $f\in\Delta^*$ is discontinuous on at most a countable set, we see that $\Delta\subset\Delta^*\subset\mathcal{L}\subset\sigma^*$.

Section 1.2: Some Preliminary Function Theory

Let f be a function defined in an open interval, $I \subset \mathbb{R}$, taking on values in $\mathbb{R}^*$. f is said to be in Baire class one if there exists a sequence, $\{f_n: n \in Z^+\}$, of functions continuous on I such that $f(x)=\lim_{n\to\infty} f_n(x)$ for each $x \in I$. The class of Baire one functions defined on I is denoted by $\mathcal{B}_1(I)$. If $I=\mathbb{R}$, we write $\mathcal{B}_1(I)=\mathcal{B}_1$.

Following is the fundamental theorem characterizing $\mathcal{B}_1$. Proofs of it can be found in Goffman [12] or Natensen [20].

Theorem 1.8. The following statements are equivalent:

(a) $f \in \mathcal{B}_1$;

(b) For all $a \in \mathbb{R}$, the sets $\{x: f(x) \le a\}$ and $\{x: f(x) \ge a\}$ are G_δ sets;

(c) For all $a \in \mathbb{R}$, the sets $\{x: f(x) < a\}$ and $\{x: f(x) > a\}$ are F_σ sets;

(d) If P is a perfect subset of $\mathbb{R}$, then the restriction of f to P has a point of continuity.

The primary importance of the class of Baire one functions lies in the fact that it contains all ordinary derivatives. If f is a function such that $f'(x)$ exists and is finite everywhere, then evidently f is continuous, and it is easy to see that $f' \in \mathcal{B}_1$. If infinite derivatives are allowed, the situation is not as clear because f need not be continuous. For proofs that $f' \in \mathcal{B}_1$, even in this case, see Zahorski [29] or corollary 2.6 of this work. In fact, in Chapter II, we will prove that $\sigma^{*s} \subset \mathcal{B}_1$.

A function, f, defined on an interval, I, is said to have the Darboux (intermediate value) property if whenever x and y are in I and α is any number between f(x) and f(y), then there exists a number, z, between x and y such that $f(z)=\alpha$. We shall denote the class of all functions defined on I which have the Darboux property by $\mathcal{D}(I)$. As usual, $\mathcal{D}(\mathbb{R})$ is just written as $\mathcal{D}$.

We shall rarely have use of $\mathcal{D}$ by itself, but rather, will make much use of the properties of the class $\mathcal{DB}_1$. There are more than a dozen known ways of characterizing $\mathcal{DB}_1$. The following theorem contains the ones we will need.

<u>Theorem 1.9</u>. Let $f \in \mathcal{B}_1$. The following are equivalent:

(a) $f \in \mathcal{DB}_1$;

(b) For each $x \in \mathbb{R}$ there is a sequence, $\{y_n: n \in Z^+\}$, increasing to x, and a sequence, $\{z_n: n \in Z^+\}$, decreasing to x, such that $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f(z_n) = f(x)$;

(c) For each $x \in \mathbb{R}$,

$$f(x) \in [\liminf_{t\to x-} f(t), \limsup_{t\to x-} f(t)] \cap$$
$$\cap [\liminf_{t\to x+} f(t), \limsup_{t\to x+} f(t)].$$

For a proof of theorem 1.9, as well as many other of the characterizations of $\mathcal{DB}_1$, see Bruckner [3, p. 9].

It is well-known that the derivative of a continuous function has the Darboux property. That the same is not true of the symmetric derivative can be seen from the function $f(x)=|x|$, where $f^s(x)=1$ for $x>0$, $f^s(0)=0$ and

$f^s(x)=-1$ when $x<0$. The Darboux property is clearly violated at $x=0$. In section 3.2 and in Chapter IV we will explore some consequences of this lack of the Darboux property for symmetric derivatives.

Before proceeding much further, the following proposition is probably worth noting to avoid some possible misconceptions.

Proposition 1.10. $\mathscr{B}_1$ is closed under addition and multiplication by constants. $\mathscr{DB}_1$ is closed under multiplication by constants, but not addition.

Proof. The assertion for $\mathscr{B}_1$ follows easily from the definition of $\mathscr{B}_1$. That $\mathscr{DB}_1$ is closed under constant multiplication can be seen from theorem 1.9(b).

Let

$$f(x)=\begin{cases}\sin\frac{1}{x} & x\neq 0\\ \frac{1}{2} & x=0\end{cases}$$

and

$$g(x)=\begin{cases}\sin\frac{-1}{x} & x\neq 0\\ \frac{1}{2} & x=0.\end{cases}$$

It is easy to see from theorem 1.8(d) that f and g are in $\mathscr{B}_1$. Now, use theorem 1.9(c) to show that f and g are in $\mathscr{DB}_1$. But,

$$f(x)+g(x)=\begin{cases}0 & x\neq 0\\ 1 & x=0\end{cases}$$

so $f+g\notin\mathscr{DB}_1$.

One of the main uses we will have for $\mathscr{DB}_1$ is contained within the following theorem.

<u>Theorem 1.11</u>. (Weil [28]) Let $f \in \mathcal{DB}_1$ such that $\underline{f}^s(x) > -\infty$ everywhere and $\underline{f}^s(x) \geq 0$ a. e.. Then f is nondecreasing.

Evans [5] extended this theorem from $\mathcal{DB}_1$ to the larger class of all measurable functions, f, satisfying

(1) $$\liminf_{t\to x} f(t) \leq f(x) \leq \limsup_{t\to x} f(t)$$

at each x. (Note the similarity to theorem 1.9(c).) In fact, he showed that this class is the largest class of measurable functions for which a statement like theorem 1.11 is true.

For our purposes, we will need a slightly more general version of theorem 1.11.

<u>Theorem 1.12</u>. Let f be a function satisfying (1) such that C(f) is dense, $\underline{f}^s(x) > -\infty$ everywhere and $\underline{f}^s(x) \geq 0$ a. e.. Then f is nondecreasing.

Proof. We pattern our proof after that of Weil [28].

First, let $\underline{f}^s(x) > 0$ everywhere. Suppose f is not nondecreasing. Then there exist a_o and b_o with $a_o < b_o$ such that $f(a_o) > f(b_o)$. Choose any $\alpha \in (f(b_o), f(a_o))$ and define $E_\alpha = \{x \in [a_o, b_o]: f(x) \leq \alpha\}$ and $E^\alpha = \{x \in [a_o, b_o]: f(x) \geq \alpha\}$. Suppose that neither E_α nor E^α contains an interval. Then, if $x \in (a_o, b_o) \cap C(f)$, it is clear that $f(x) = \alpha$. According to proposition 1.2, if $x_o \in (a_o, b_o)$ we can choose C(f)-symmetric sequences, $\{x_n: n \geq 1\}$ and $\{y_n: n \geq 1\}$, converging to x_o. Then

$$\underline{f}^s(x_o) \leq \lim_{n\to\infty} \frac{f(x_n) - f(y_n)}{x_n - y_n} = \lim_{n\to\infty} \frac{\alpha - \alpha}{x_n - y_n} = 0$$

which contradicts our assumption that $\underline{f}^s(x_o) > 0$. Thus, either E^α or E_α contains an interval.

Suppose, for example, that E_α contains an interval. (If E^α contains an interval, the argument is similar.) We can then choose an interval $(c,d)\subset E^\alpha$ such that

(2) $$c = \inf\{x: (x,d)\subset E_\alpha\}.$$

$c>a_0$, for otherwise (2), (1) and $\alpha<f(a_0)$ would imply that $\underline{f}^s(a_0)=-\infty$. Since $\underline{f}^s(c)>0$, it follows that there is a δ with $0<\delta<d-c$ such that $f(c-h)<f(c+h)\leq\alpha$ whenever $0<h<\delta$, which implies that $(c-\delta,c)\subset E_\alpha$. Thus, $\limsup_{t\to c} f(t)\leq\alpha$, so that by (1), $f(c)\leq\alpha$ and $c\in E_\alpha$. From this, it follows that $(c-\delta,d)\subset E_\alpha$, which contradicts (2). This contradiction shows that the points a_0 and b_0 cannot exist and f must therefore be nondecreasing.

Next, suppose $\underline{f}^s(x)\geq 0$ everywhere. Let $\epsilon>0$ and define $f_\epsilon(x)=f(x)+\epsilon x$. Then $\underline{f}^s_\epsilon(x)\geq\epsilon>0$ everywhere and f_ϵ satisfies (1), so according to the above argument, f_ϵ is nondecreasing. Since f_ϵ is nondecreasing for every $\epsilon>0$, we can take the limit as $\epsilon\to 0$ to see that f is nondecreasing.

Finally, let f be as in the statement of the theorem. In Zahorski [29], it is shown that for any set, A, such that $|A|=0$, there exists a continuous and nondecreasing function, g, which is differentiable everywhere and for which $g'(x)=\infty$ whenever $x\in A$. Let $\epsilon>0$ and $A=\{x: \underline{f}^s(x)<0\}$. Since $|A|=0$, there is a function, g, as above. Define $f_\epsilon(x)=f(x)+\epsilon g(x)$. Then, clearly, $\underline{f}^s_\epsilon(x)\geq 0$ everywhere and $f_\epsilon(x)$ satisfies (1), so f_ϵ is nondecreasing. Again, letting $\epsilon\to 0$ shows that f is nondecreasing, and the theorem follows.

Following Evans [5], we define the class $\mathcal{M}_{-1}$ to consist of all functions, f, satisfying (1) such that C(f) is dense. Then from theorem 1.12, the following is clear.

Corollary 1.13. Let $f\in\mathcal{M}_{-1}\sigma^*$ such that $f^s(x)>-\infty$ everywhere and $f^s(x)\geq 0$ a. e.. Then f is nondecreasing.

A function, f, defined on an open interval, I, is upper semicontinuous at $x\in I$ if

$$\limsup_{t\to x} f(t)\leq f(x).$$

f is lower semicontinuous at x if -f is upper semicontinuous at x. If f is upper (lower) semicontinuous at each point of its domain, then it is said to be upper (lower) semicontinuous.

Note that this definition appears at first glance to be slightly different than that in some common books because of the way we defined the upper and lower limits of a function. The following theorem shows that our definition is the same.

Theorem 1.14. Let f be a function defined on an open interval, I. The following statements are equivalent:

(a) f is upper semicontinuous;

(b) For each $x\in I$, $f(x)\geq\limsup_{t\to x} f(t)$;

(c) For each $a\in\mathbb{R}$, $\{x: f(x)\geq a\}$ is closed relative to I;

(d) For each $a\in\mathbb{R}$, $\{x: f(x)<a\}$ is open.

Proof. (a) is obviously equivalent to (b) and (c) is obviously equivalent to (d). Let $a\in\mathbb{R}$ and $A=\{x: f(x)\geq a\}$. Suppose that (b) is true. $x\in\bar{A}$ iff there is a sequence,

$\{x_n: n\geq 1\}\subset A$, such that $\lim_{n\to\infty} x_n=x$. Then, (b) implies that $f(x)\geq a$ so that $x\in A$. Therefore, A is closed and (c) is true. Suppose (c) is true and $f(x)<\limsup_{t\to x} f(t)$. Then there is a sequence, $\{x_n: n\geq 1\}\subset I$, such that $\lim_{n\to\infty} x_n=x$ and $\lim_{n\to\infty} f(x_n)= \limsup_{t\to x} f(t)$. Let $f(x)<a<\limsup_{t\to x} f(t)$ and $A=\{x: f(x)\geq a\}$. A is closed. There exists an $N\in Z^+$ such that for all $n\geq N$, $x_n\in A$. Since A is closed, this clearly implies that $x\in A$. This is impossible, and the contradiction shows that (b) must be true.

Note that theorem 1.14(c) and theorem 1.8(b) imply that if f is upper semicontinuous, then $f\in\mathcal{B}_1$. Analagous results can clearly be established for a lower semicontinuous function, f.

Section 1.3: A Covering Theorem

Suppose that (a, b) is an open interval such that there is a $\delta(x)>0$ associated with each $x\in(a, b)$. For each x, we define

$$S_x=\{[x-h, x+h] : 0<h<\delta(x)\}$$

and

$$\mathcal{C}=\cup_{x\in(a, b)} S_x.$$

$\mathcal{C}$ is called a full symmetric cover for (a, b).

The above definition is due to Thomson [26], who used it to prove a slightly weaker version of the following theorem.

<u>Theorem 1.15</u>. Let (a, b) be an interval and $c=(a+b)/2$. Suppose $\mathcal{C}$ is a full symmetric cover for (a, b). Then there is a set, $D\subset(0, (b-a)/2)$ such that D has countable closure and $\mathcal{C}$ contains a partition of $[c-x, c+x]$ for every $x\in D^c$. Further, each of these partitions can be chosen to contain an element of S_c. (Where S_c is as in the above definition.)

Proof. To simplify notation, we assume that $c=0$ and that $\mathcal{C}$ is a full symmetric cover of the interval $(-b, b)$. With the assumptions that $\delta(x)=\delta(-x)$ and $\delta(|x|)<|x|$ for $x\neq 0$, we lose no generality because $\mathcal{C}$ is at worst made smaller. Define

$$D=\{x\in(0, b): \mathcal{C} \text{ contains no partition of } [-x, x]\}$$

and let

$$\alpha= \sup \{x\in(0, b): \overline{D}\cap(0, x) \text{ is countable}\}.$$

We must show that $\alpha=b$. Suppose this is false; i. e., suppose $\alpha<b$. First, note that $\alpha>\delta(0)$. By the definition of α, for every $\epsilon>0$, $\overline{D}\cap(\alpha-\epsilon, \alpha]$ is countable and $\overline{D}\cap(\alpha,\alpha+\epsilon)$ is uncountable. But, if $x\in\overline{D}^c\cap(\alpha-\delta(\alpha), \alpha)$, then $\mathcal{C}$ contains a partition of $[R_0(R_\alpha(x)), R_\alpha(x)]$, from which it follows that $\overline{D}\cap(\alpha, \alpha+\delta(\alpha))\subset R_\alpha(\overline{D}\cap(\alpha-\delta(\alpha), \alpha))$, which is countable. This contradiction shows that $\alpha=b$.

Now, let $x_0\in(0, b)$ and let

$$-x_0=a_0<a_1< \dots <a_n=x_0$$

be a partition of $[-x_0, x_0]$ from $\mathcal{C}$. There is a $k\in\{0,1,\dots,n\}$ such that $a_k\leq 0$ and $a_{k+1}\geq 0$; i. e., $0\in[a_k, a_{k+1}]\in\mathcal{C}$. Because

$\delta(x) < |x|$ when $x \neq 0$, it is clear that whenever $x \neq 0$ and $I \in S_x$, then $0 \notin I$. This implies that $[a_k, a_{k+1}] \in S_0$.

CHAPTER II

THE CLASS OF ARBITRARY SYMMETRIC DERIVATIVES

Section 1.1: Comparison with Baire Class One

It is an easy matter to prove that a finite-valued ordinary derivative is a member of $\mathcal{B}_1$. The situation would appear to be more complex in the case of the symmetric derivative because the primitive function may have many discontinuities. Nevertheless, we have the following theorem, which will prove very useful in the succeeding chapters.

<u>Theorem 1.1</u>. $\sigma^{*S} \subset \mathcal{B}_1$.

Proof. According to theorem 1.8(b) it suffices to show that for any $a \in \mathbb{R}$, $\{x: f^s(x) \geq a\}$ and $\{x: f^s(x) \leq a\}$ are both G_δ sets. To do this, define $g(x)=f(x)-ax$ so that $C(f)=C(g)$, g is symmetrically differentiable everywhere with

$$g^s(x)=f^s(x)-a$$

and

(1) $$\{x: f^s(x) \geq a\} = \{x: g^s(x) \geq 0\}.$$

Choose a $\delta>0$. Using proposition 1.2, we see that for each $x \in \mathbb{R}$, there is an $h \in (0, \delta)$ such that $x+h$ and $x-h$ are both elements of $C(g)$. From this observation, it makes sense to define

$A=\{x: \sup_{0<h<\delta} g(x+h)-g(x-h)>0,$ where $x+h\in C(g)$ and $x-h\in C(g)\}$.

If $x\in A$, then there are h, α and ϵ, each positive, satisfying the following inequalities:

(2) $$x-h,\ x+h\in C(g)\cap(x-\delta,\ x+\delta);$$

(3) $$g(x+h)-g(x-h)>2\alpha;$$

(4) $$|x+h-y|<\epsilon \text{ implies } |g(x+h)-g(y)|<\alpha;$$

(5) $$|x-h-y|<\epsilon \text{ implies } |g(x-h)-g(y)|<\alpha;$$

(6) $$\epsilon+h<\delta.$$

Choose any $x_0\in\mathbb{R}$ such that $0<|x-x_0|<\epsilon$ and choose $\beta>0$ with $|x-x_0|+\beta<\epsilon$. Let

(7) $$R_1=(x_0-h-\beta,\ x_0-h+\beta) \text{ and } R_2=(x_0+h-\beta,\ x_0+h+\beta).$$

Since $|x-x_0|+\beta<\epsilon$, it is clear that

(8) $$R_1\subset(x-h-\epsilon,\ x-h+\epsilon) \text{ and } R_2\subset(x+h-\epsilon,\ x+h+\epsilon).$$

Since $C(g)$ is a dense G_δ set, it is residual. From this, it follows that $\mathcal{R}_{x_0}(C(g)\cap R_1)$ and $C(g)\cap R_2$ are both residual in R_2 and cannot be disjoint. So, there is an $h'>0$ such that $x_0-h'\in C(g)\cap R_1$ and $x_0+h'\in C(g)\cap R_2$. From (7) and (8) it follows that $h'<h+\beta<h+\epsilon<\delta$. From (4) and (5) it follows that

$$|g(x+h)-g(x_0+h)|<\alpha \text{ and } |g(x-h)-g(x_0-h')|<\alpha.$$

Combining these two inequalities with (3), it is seen that

$$g(x_0+h')-g(x_0-h')>0$$

so that $x_0\in A$. Since the only requirement on x_0 was that $|x-x_0|<\epsilon$, it follows that $(x-\epsilon,\ x+\epsilon)\subset A$. Thus, for each $x\in A$, there is an $\epsilon>0$ such that $(x-\epsilon,\ x+\epsilon)\subset A$. Therefore, A is open.

Similarly, if we define, for each $n \in Z^+$, the set A_n to be

$\{x: \sup_{0<h<\frac{1}{n}} g(x+h)-g(x-h) > \frac{2h}{n}$ where $x+h \in C(g)$ and $x-h \in C(g)\}$,

then A_n is also open.

It is clear that

$$\{x: g^s(x) \geq 0\} = \bigcap_{n=1}^{\infty} A_n,$$

so the set in (1) is a G_δ set.

The set $\{x: f^s(x) \leq a\}$ can also be shown to be a G_δ set by considering $-f$ instead of f.

Using theorem 1.3 with corollary 1.6 and theorem 2.1, we arrive at the following corollaries.

Corollary 2.2. Let f be a symmetrically differentiable function such that $f^s(x)$ is finite on a residual set. Then $f^s \in \mathcal{B}_1$.

Proof. This follows easily from theorem 1.3 and the observation that if $f^s(x)$ is finite, then f is symmetrically continuous at x.

Corollary 2.3. If f is measurable and symmetrically differentiable, then $f^s \in \mathcal{B}_1$.

Proof. By corollary 1.6, $f \in \sigma^*$.

The above statements represent improvements over the strongest previously known similar result, due to Filipczak [7], who proved that if f is approximately continuous and symmetrically differentiable, then $f^s \in \mathcal{B}_1$.

It is not known whether every symmetric derivative is in Baire class one. Consideration of corollaries 2.2 and 2.3 shows that such a function would have to be very badly behaved; at the very least, it would have to be nonmeasurable and have an infinite symmetric derivative on a second category set. The question of the existence of a nonmeasurable symmetric derivative was posed as long ago as 1928 by Sierpinski [24] and still remains open. We explore these questions further in the next section.

Section 2.2: Arbitrary Symmetric Derivatives

As noted above, it is perhaps possible that an arbitrary symmetric derivative could be a very badly behaved function. However, a few statements concerning the behavior of such a function can be established.

Theorem 2.4. Let f be a symmetrically differentiable function and $-\infty\le\alpha<\beta\le\infty$. Suppose $A=\{x:\ f^s(x)\le\alpha\}$, $B=\{x:\ f^s(x)\ge\beta\}$ and I is an interval such that $I\subset A\cup B$. Then both A and B cannot be dense in I.

Proof. It may be assumed without loss of generality that $\alpha<0<\beta$, for otherwise, we just consider $g(x)=f(x)-ax$, where $\alpha<a<\beta$, as in the proof of theorem 2.1.

Suppose both A and B are dense in I. Since $I\subset A\cup B$, at least one of the sets, A or B, must be of the second category in I; suppose B is of the second category. Define for $n\in Z^+$

$$B_n=\{x\in I:\ f(x+h)-f(x-h)>0 \text{ for } 0<h<\tfrac{1}{n}\}.$$

Since $B \subset \bigcup_{n=1}^{\infty} B_n$ and B is of the second category, there is an $n_0 \in Z^+$ and an open interval $J \subset I$ such that B_{n_0} is dense in J.

Because A is dense in J also, we may choose an $a \in A \cap J$ and two sequences, $\{x_n\}$ and $\{y_n\}$, from B_{n_0} with $\{x_n\}$ increasing to a and $\{y_n\}$ decreasing to a such that $y_n - x_n < \frac{1}{n_0}$ for each $n \in Z^+$. Then

$$f(a+(y_n-x_n))-f(a-(y_n-x_n))=f(y_n+(a-x_n))-f(x_n-(y_n-a))$$
$$=f(y_n+(a-x_n))-f(y_n-(a-x_n))+f(y_n-(a-x_n))-f(x_n-(y_n-a))$$
$$=(f(y_n+(a-x_n))-f(y_n-(a-x_n))+(f(x_n+(y_n-a))-f(x_n-(y_n-a)))>0$$

because $y_n - x_n < \frac{1}{n_0}$ and $x_n < a < y_n$ imply that $y_n - a < \frac{1}{n_0}$ and $a - x_n < \frac{1}{n_0}$ with both x_n and y_n elements of B_{n_0}.

Thus, if $h_n = y_n - x_n$, we see that h_n decreases to 0 and $f(a+h_n)-f(a-h_n)>0$ for each n. This implies that $f^s(a) \geq 0$, because f is symmetrically differentiable at a. But, $f^s(a) \leq \alpha < 0$ because $a \in A$. This contradiction shows the supposition to be false, so both A and B cannot be dense in I.

Theorem 2.4 shows that the associated sets for an arbitrary symmetric derivative behave much like the associated sets for a function in Baire class one. In view of theorem 2.1, this is hardly surprizing.

Theorem 2.5 rules out another form of pathological behavior for symmetric derivatives.

<u>Theorem 2.5</u>. Let f be any function. Then $\{x: |f^s(x)| = \infty\}$ contains no interval.

Proof. Suppose, to the contrary, that there is an interval $I\subset\{x: |f^s(x)|=\infty\}$. According to theorem 2.4, both of the sets, $A=\{x: f^s(x)=-\infty\}$ and $B=\{x: f^s(x)=\infty\}$, cannot be dense in I. So assume there are $\alpha,\beta\in I$ such that $\alpha<\beta$ and $(\alpha, \beta)\subset B$.

For each $x\in(\alpha, \beta)$ and each $\rho>0$, there is a $\delta(x,\rho)>0$ such that if $0<h<\delta(x,\rho)$, we have $[x-h, x+h]\subset(\alpha, \beta)$ and

(9) $$f(x+h)-f(x-h)>2h\rho.$$

For each $n\in Z^+$, define

(10) $$\mathcal{J}_n=\{[x-h, x+h] : x\in(\alpha, \beta) \text{ and } 0<h<\delta(x,n)\}.$$

Each $\mathcal{J}_n$ is a full symmetric cover of (α, β), so by theorem 1.15, there is a set $D_n\subset((\alpha+\beta)/2, \beta)$ with $|D_n|=0$ such that $\mathcal{J}_n$ contains a partition of $[(\alpha+\beta)/2-x, (\alpha+\beta)/2+x]$ for every $x+(\alpha+\beta)/2\in((\alpha+\beta)/2, \beta)-D_n$. Let

(11) $$D=\bigcup_{n=1}^{\infty}D_n$$

and

(12) $$E=((\alpha+\beta)/2,\beta)-D.$$

Since $|D|=0$, $E\neq\emptyset$, so we may choose an $x+(\alpha+\beta)/2\in E$.

Let

(13) $$d=f(\frac{\alpha+\beta}{2}+x)-f(\frac{\alpha+\beta}{2}-x)$$

and choose any $n\in Z^+$. By (11) and (12), there is a partition of $[(\alpha+\beta)/2-x, (\alpha+\beta)/2+x]$ in $\mathcal{J}_n$. Denote the intervals in the partition by $[\alpha_i, \beta_i]$, $i=1,\ldots,m$. Then, using (13), (10) and (9), we see

(14) $$d=\Sigma_{i=1}^{m}f(\beta_i)-f(\alpha_i)>\Sigma_{i=1}^{m}(\beta_i-\alpha_i)n=2xn.$$

Since n is arbitrary and $x>0$, (14) clearly leads to a contradiction of (13). Thus, B can contain no interval. Similarly, A can contain no interval.

<u>Corollary 2.6</u>. $\Delta^{*'} \subset \mathcal{B}_1$.

Proof, According to theorem 2.5, the set on which f' is finite is dense. Whenever f'(x) is finite, f is continuous at x, so C(f) is dense. Now apply theorem 2.1.

Corollary 2.6 was apparently first proved by Zahorski [29, p. 14].

CHAPTER III

THE STRUCTURE OF FUNCTIONS IN σ^*

Section 3.1: Nice Copies of Functions in σ^*

The primary goal of this section is to prove the following theorem.

Theorem 3.1. Let $f \in \sigma^*$. Then there are two sets, A_1 and A_2, each with countable closure, and two functions, g_1 and g_2, each in Baire class one satisfying:

(a) $g_i^{sc}(x) = f^s(x)$ everywhere, $i=1,2$;

(b) $g_i^s(x) = f^s(x)$ everywhere on $\overline{A_i}^c$, $i=1,2$;

(c) g_1 (g_2) is upper (lower) semicontinuous on $\overline{A_1}^c$ ($\overline{A_2}^c$);

(d) $C(f) \subset C(g_i)$ and $f(x) = g_i(x)$ for each $x \in C(f)$, $i=1,2$;

(e) $D(f) \subset D(g_i)$ and $f'(x) = g_i'(x)$ for each $x \in D(f)$, $i=1,2$;

(f) If I is a component of $\overline{A_i}^c$, then $g_i \in \mathcal{M}_{-1}(I)$, $i=1,2$.

The proof is accomplished with the aid of the following series of lemmas, some of which are interesting in their own right.

Lemma 3.2. Let I be an open interval, C a dense subset of I and f any function defined on I. Define

$\mu(x)=C\text{-}\lim\sup_{t\to x} f(t)$ and $\ell(x)=C\text{-}\lim\inf_{t\to x} f(t)$.

Then μ is upper semicontinuous and ℓ is lower semicontinuous.

Proof. Let $a\in\mathbb{R}$ and $A=\{x\in I: \mu(x)\geq a\}$. If $A=\emptyset$, then A is closed. Otherwise, we may choose a sequence, $\{x_n: n\in Z^+\}\subset A$ such that $\lim_{n\to\infty} x_n=x$. From the definition of μ, for each $n\in Z^+$, there is a $t_n\in C$ satisfying $|t_n-x_n|<\frac{1}{n}$ and $f(t_n)+\frac{1}{n}>a$. Then, clearly $\lim_{n\to\infty} t_n=x$ and

$$C\text{-}\lim\sup_{t\to x} f(t)\geq\lim\sup_{n\to\infty} f(t_n)\geq\lim\sup_{n\to\infty}(a-\tfrac{1}{n})=a$$

so $\mu(x)\geq a$ and $x\in A$. Thus A is closed and it follows that μ is upper semicontinuous.

The proof that ℓ is lower semicontinuous follows by noting that $-\ell$ is μ for $-f$.

<u>Lemma 3.3</u>. Let f, C, μ and ℓ be as in lemma 3.2. Then

(a) $C(f)\subset C(\mu)$ $(C(f)\subset C(\ell))$ and $f(x)=\mu(x)$ $(f(x)=\ell(x))$ for each $x\in C(f)$.

(b) $D(f)\subset D(\mu)$ $(D(f)\subset D(\ell))$ and $f'(x)=\mu'(x)$ $(f'(x)=\ell'(x))$ for each $x\in D(f)$.

Proof. We may suppose without loss of generality that $0\in C(f)$ and $f(0)=0$. Then, given an $\epsilon>0$, there is a $\delta>0$ satisfying $|f(h)|<\epsilon$ whenever $|h|<\delta$. Fix an h such that $|h|<\delta$ and choose a sequence, $\{x_n: n\in Z^+\}\subset C$, such that $|x_n|<\delta$ for each n and $\lim_{n\to\infty} f(x_n)=\mu(h)$. It is clear that $\mu(0)=0$, so that

$$|\mu(h)-\mu(0)|=\lim_{n\to\infty}|f(x_n)|\leq\epsilon$$

and it follows that $0\in C(\mu)$ and $\mu(0)=0$. Therefore (a) follows.

Now, suppose $0\in D(f)$. We may assume without loss of generality that $f(0)=0=f'(0)$. (Otherwise we just add an appropriate linear function to f.) Then, given an $\epsilon>0$, there is a $\delta>0$ such that when $|h|<\delta$, $|f(h)|<\epsilon|h|$. Fix an h with $0<|h|<\delta$ and a sequence, $\{x_n: n\in Z^+\}\subset C$, as above. Then

$$|\mu(h)|=\lim_{n\to\infty}|f(x_n)|\leq\lim_{n\to\infty}|x_n|\epsilon=|h|\epsilon.$$

Because ϵ may be chosen arbitrarily small, we see that $0\in D(\mu)$ and $\mu'(0)=0=f'(0)$. Therefore, $D(f)\subset D(\mu)$ and $f'(x)=\mu'(x)$ for each $x\in D(f)$.

The assertions for ℓ follow by noting that $-\ell$ is μ for $-f$.

<u>Lemma 3.4</u>. Let $f\in\sigma^*$ and define μ and ℓ as above with $C=C(f)$. If μ (ℓ) is finite in a neighborhood of x_0, then $\mu^s(x_0)$ ($\ell^s(x_0)$) exists and equals $f^s(x_0)$.

Proof. By translating f, we may assume without loss of generality that $x_0=0$ and there is an $\varkappa>0$ such that $|\mu(x)|<\infty$ whenever $|x|<\varkappa$.

First, suppose $f^s(0)=0$. Then, given $\epsilon>0$, there is a $\delta\in(0,\varkappa)$ such that when $0<t<\delta$,

$$(1)\qquad |f(t)-f(-t)|<2t\epsilon.$$

Fix a $t\in(0,\delta)$ and choose a sequence, $\{s_n: n\in Z^+\}\subset C(f)$, such that $\lim_{n\to\infty}s_n=t$, $0<s_n<\delta$ and

$$(2)\qquad \lim_{n\to\infty}f(s_n)=C(f)\text{-}\limsup_{x\to t}f(x)=\mu(t).$$

Since $\{s_n: n\in Z^+\}\subset C(f)$, for each $n\in Z^+$ there is a $\rho_n>0$ such that when $|s_n-x|<\rho_n$, then $|f(s_n)-f(x)|<\frac{1}{n}$. ρ_n may also be chosen small enough so that $s_n+\rho_n<\delta$ and $\lim_{n\to\infty}\rho_n=0$.

For each $n\in Z^+$, let

$$G_n=\mathcal{R}_0(C(f)\cap(s_n,s_n+\rho_n)).$$

$C(f)$ being residual in $\mathbb{R}$ clearly implies that G_n is residual in $(-s_n-\rho_n,-s_n)$, so $C(f)\cap G_n\neq\emptyset$ for each $n\in Z^+$. Form a sequence, $\{t_n: n\in Z^+\}$, by choosing $t_n\in(s_n,s_n+\rho_n)$ such that $\mathcal{R}_0(t_n)\in C(f)\cap G_n$. It follows then that $\lim_{n\to\infty}t_n=t$, $0<t_n<\delta$ for each n and $|f(s_n)-f(t_n)|<\frac{1}{n}$. Using (2), we see that

(3) $$\lim_{n\to\infty}f(t_n)=\mu(t).$$

From the definition of μ, (3) and then (1), we see that

(4) $\mu(-t)\geq\lim\sup_{n\to\infty}f(-t_n)\geq\lim\sup_{n\to\infty}(f(t_n)-2t_n\epsilon)=\mu(t)-2t\epsilon$.

Similarly, it follows that

(5) $$\mu(t)\geq\mu(-t)-2t\epsilon.$$

(4) and (5) imply

(6) $$|\mu(t)-\mu(-t)|\leq 2t\epsilon.$$

Since ϵ may be chosen arbitrarily small, we conclude from (6) that $\mu^s(0)=0=f^s(0)$.

If $f^s(0)=\alpha\in\mathbb{R}$, we consider the above argument applied to $g(x)=f(x)-\alpha x$ to see that $\mu^s(0)=\alpha$.

Now suppose $f^s(0)=\infty$. Then, given $\alpha\in\mathbb{R}$, we can choose a $\delta\in(0,\varkappa)$ such that whenever $0<t<\delta$,

(7) $$f(t)-f(-t)>2t\alpha.$$

Fix a $t\in(0,\delta)$ and choose a sequence, $\{s_n: n\in Z^+\}\subset C(f)\cap(-\delta,0)$ such that $\lim_{n\to\infty}s_n=-t$ and

$$\lim_{n\to\infty}f(s_n)=C(f)-\lim\sup_{x\to t}f(-x)=\mu(-t).$$

In the same manner as above, we may choose a new sequence, $\{t_n: n\in Z^+\}\subset C(f)\cap(-\delta,0)$ such that $\mathcal{R}_0(\{t_n: n\in Z^+\})\subset C(f)$,

$\lim_{n\to\infty} t_n = -t$ and

(8) $$\lim_{n\to\infty} f(t_n) = \mu(-t).$$

Then, using (8) and (7),

$$\mu(t)-\mu(-t) = C(f)\text{-}\limsup_{x\to t} f(x) - C(f)\text{-}\limsup_{x\to t} f(-x)$$
$$= C(f)\text{-}\limsup_{x\to t} f(x) - \lim_{n\to\infty} f(t_n)$$
(9) $$\geq \limsup_{n\to\infty} f(-t_n) - \lim_{n\to\infty} f(t_n)$$
$$\geq \limsup_{n\to\infty} (f(-t_n) - f(t_n))$$
$$> \limsup_{n\to\infty} 2|t_n|\alpha = 2t\alpha.$$

By choosing α arbitrarily large in (9), we see that $\mu^s(0)=\infty$.

The case when $f^s(0)=-\infty$ succumbs to a similar argument.

Therefore, $\mu^s(0)=f^s(0)$.

The assertion that $\ell^s(x_0)=f^s(x_0)$ follows by noting that $-\ell$ is μ for $-f$.

It should perhaps be noted that some condition such as requiring μ and ℓ to be finite in a neighborhood of x_0 is necessary. To see this, for $n\in Z^+$, let

$$f_n(x) = \begin{cases} \dfrac{\sin 4^{-n}\pi(x-2^{-n})^{-1}}{x-2^{-n}} & x\in(2^{-n}-4^{-n}, 2^{-n}+4^{-n})-\{2^{-n}\} \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_n(x) = \sum_{n=1}^{\infty} f_n(x) + \sum_{n=1}^{\infty} f_n(-x).$$

Then $f\in\sigma$, but $\mu(x)=\infty=-\ell(x)$ whenever $x\in\{\pm 2^{-n}: n\in Z^+\}$, so the difference quotients

$$\frac{\mu(h)-\mu(-h)}{2h} \text{ and } \frac{\ell(h)-\ell(-h)}{2h}$$

are undefined whenever $h=2^{-n}$ for some $n\in Z^+$. Thus, $\mu^s(0)$ and $\ell^s(0)$ are also undefined.

<u>Lemma 3.5</u>. Let $f \in \sigma^*$ with

$A=\{x: \limsup_{t\to x} f(t)=\infty\}$ and $B=\{x: \liminf_{t\to x} f(t)=-\infty\}$.

Then both A and B are countable closed sets.

Proof. We prove the lemma in the case of A. The assertion for B then follows by considering $-f$.

Using $C=\mathbb{R}$ in lemma 3.2, we see that A is closed and because $C(f)$ is dense, A must be nowhere dense. Being closed, A may be written as $A=P\cup N$, where P is perfect and N is countable. Suppose $P\neq\emptyset$ and let (α,β) be a component of P^c. (α or β could be infinite.) Since $P\neq\emptyset$, α or β must be finite. Suppose β is finite. Then, since P is closed, $\beta\in P$. P being perfect and $(\alpha,\beta)\subset P^c$, we see that for each $\delta>0$, $[\beta,\beta+\delta)\cap P$ is uncountable. Since $A\cap(\alpha,\beta)\subset N$ is countable, we may choose a sequence, $\{\beta_n: n\in Z^+\}\subset P$, such that β_n decreases to β and $R_\beta(\{\beta_n: n\in Z^+\})\cap A=\emptyset$. Using the facts that $\beta_n\in A$ for each n and

$$\limsup_{t\to R_\beta(\beta_n)} f(t)\in[-\infty,\infty)$$

we may choose a $t_n>\beta$ for each $n\in Z^+$ such that $|t_n-\beta_n|<\frac{1}{n}$ and $f(t_n)-f(R_\alpha(t_n))>n$. Clearly, $\lim_{n\to\infty} t_n=\beta$ and

$$\liminf_{n\to\infty}\frac{f(t_n)-f(R_\alpha(t_n))}{2(t_n-\beta)}\geq\liminf_{n\to\infty}\frac{n}{2(t_n-\beta)}=\infty$$

so $f^s(\beta)=\infty$. Similarly, if $\alpha>-\infty$, then $f^s(\alpha)=-\infty$.

Now, we note that

$$P^c=\bigcup_{n=1}^{\infty}(\alpha_n,\beta_n)$$

where (α_n,β_n) is a component of P^c for each n. Clearly, then since P is a nowhere dense set in $\mathbb{R}$, the sets

$\{\alpha_n: n\in Z^+\}$ and $\{\beta_n: n\in Z^+\}$ are both dense in P. This implies, from the above, that the sets

$$I^+=\{x: f^s(x)=\infty\} \text{ and } I^-=\{x: f^s(x)=-\infty\}$$

are disjoint dense subsets of P.

P, being closed, is a G_δ set and according to theorem 2.1, $I^+$and I^- are G_δ sets, so $P\cap I^+$ and $P\cap I^-$ are dense G_δ subsets of P. As such, both $P\cap I^+$ and $P\cap I^-$ are residual in P. Since P is a Baire space, $P\cap I^+\cap I^-\neq\emptyset$, which contradicts the fact that $I^+\cap I^-=\emptyset$.

Therefore, we conclude that $P=\emptyset$ and A=N, a countable set.

Enough machinery has now been developed to accomplish our primary goal.

Proof. (Theorem 3.1) Let $f\in\sigma^*$, μ and ℓ be as in lemma 3.4 and A and B be as in lemma 3.5. Define

$$A_1=\{x: |\mu(x)|=\infty\} \text{ and } A_2=\{x: |\ell(x)|=\infty\}$$

and let

$$g_1(x)=\begin{cases}\mu(x) & x\in A_1^c\\ f(x) & x\in A_1\end{cases} \quad\text{and}\quad g_2(x)=\begin{cases}\ell(x) & x\in A_2^c\\ f(x) & x\in A_2\end{cases}$$

Since $A_1\cup A_2\subset A\cup B$, the countable closure of both A_1 and A_2 follows from lemma 3.5. (b) follows from lemma 3.4. (c) follows from lemma 3.2. (d) and (e) follow from lemma 3.3. (a) follows from the definitions of μ and ℓ and (d).

The rest of the theorem will be proved in the case i=1, the proof in the case i=2 being similar.

Choose an $a\in\mathbb{R}$. The upper semicontinuity of g_1 on

$\overline{A}_1^c$ and theorem 1.14(c) imply that for each $n\in Z^+$,

$$E_n=\{x\in\overline{A}_1^c:\ g_1(x)\geq a+\frac{1}{n}\}$$

is closed relative to $\overline{A}_1^c$ and so is an F_σ set relative to $\mathbb{R}$. Since

$$F_1=\{x\in\overline{A}_1^c:\ g_1(x)>a\}=\bigcup_{n=1}^{\infty}E_n,$$

we see that F_1 is also an F_σ set relative to $\mathbb{R}$. It is clear that

(10) $$F_2=\{x\in\overline{A}_1:\ g_1(x)>a\}$$

is an F_σ set, because $\overline{A}_1$ is countable. Thus,

$$\{x\in\mathbb{R}:\ g_1(x)>a\}=F_1\cup F_2$$

is also an F_σ set.

Similarly, the upper semicontinuity of g_1 implies that

$$F_3=\{x\in\overline{A}_1^c:\ g_1(x)<a\}$$

is open in $\overline{A}_1^c$ and so is open in $\mathbb{R}$. Thus, F_3 is an F_σ set relative to $\mathbb{R}$. As above,

$$F_4=\{x\in\overline{A}_1:\ g_1(x)<a\}$$

is also an F_σ set relative to $\mathbb{R}$. Therefore,

(11) $$\{x\in\mathbb{R}:\ g_1(x)<a\}=F_3\cup F_4$$

is also an F_σ set relative to $\mathbb{R}$. Since a was chosen arbitrarily, (10),(11) and theorem 1.8(c) imply that $g_1\in\mathcal{B}_1$.

Using the definition of g_1 and (d), we see that if $x\in\overline{A}_1^c$, then

$$\liminf_{t\to x}g_1(t)\leq C(f)\text{-}\liminf_{t\to x}g_1(t)=$$

(12) $$=C(f)\text{-}\liminf_{t\to x}f(t)\leq C(f)\text{-}\limsup_{t\to x}f(t)=$$

$$=g_1(x)=C(f)\text{-}\limsup_{t\to x}g_1(t)\leq\limsup_{t\to x}g_1(t).$$

Comparing the above relations, we obtain (f).

An examination of the statement of theorem 3.1 shows that we can slightly improve on the semicontinuity relations in (c) and (f). In particular, (c) implies

(13) $$g_1(x) \geq \limsup_{t \to x} g_1(t)$$

whenever $x \in \overline{A_1^c}$, and

(14) $$g_2(x) \leq \liminf_{t \to x} g_2(t)$$

whenever $x \in \overline{A_2^c}$. (f) implies that

(15) $$g_1(x) \leq \limsup_{t \to x} g_1(t)$$

whenever $x \in \overline{A_1^c}$ and

(16) $$g_2(x) \geq \liminf_{t \to x} g_2(t)$$

for all $x \in \overline{A_2^c}$. Combining (13)-(16), we see

Corollary 3.6. Let f, A_1, A_2, g_1 and g_2 be as in theorem 3.1. Then

(17) $$g_1(x) = \limsup_{t \to x} g_1(t)$$

for all $x \in \overline{A_1^c}$ and

(18) $$g_2(x) = \liminf_{t \to x} g_2(t)$$

for all $x \in \overline{A_2^c}$.

Definition. Let f, A_1, A_2, g_1 and g_2 be as in theorem 3.1. We call g_1 (g_2) the upper (lower) semicontinuous nice copy of f. A_1 (A_2) will be called the upper (lower) essential set for f.

In the following, we shall adopt the convention that if g is said to be the nice copy of f, it will be the upper semicontinuous nice copy unless it is specifically noted otherwise. Similarly, an essential set will be an upper essential set unless contrary mention is made. These conventions will cause no problems because of the similarities in the behavior of the upper and lower semicontinuous nice copies of a function.

Corollary 3.7. If $f\in\sigma^*$, then $|\{x: |f^s(x)|=\infty\}|=0$.

Proof. Let g be the nice copy of f and A the essential set for f. Then, from theorem 3.1, $g\in\mathcal{B}_1$ and is thus measurable. Now apply corollary 1.6(a).

The following theorem answers some natural questions concerning the uniqueness of the nice copy of a function.

Theorem 3.8. Let f and g be functions in σ^* and suppose that $D\subset\mathbb{R}$ is any dense set. If $f(x)=g(x)$ for every $x\in D$, then the essential sets for f and g are equal and the nice copies of f and g are equal up to their values on the essential set.

Proof. Suppose $x_0\in\mathbb{R}$ and

(21) $$C(f)\text{-}\lim\sup_{t\to x_0} f(t)=\alpha$$

where α may be infinite. Then, there is a sequence,

$$\{t_n: n\in Z^+\}\subset C(f)$$

such that $\lim_{n\to\infty} t_n = x_0$ and

(22) $$\lim_{n\to\infty} f(t_n) = \alpha.$$

Since $\{t_n: n\in Z^+\}\subset C(f)$, for each $n\in Z^+$, there is a $\delta_n\in(0,\frac{1}{n})$ such that when $|t_n-y|<\delta_n$,

(23) $$|f(t_n)-f(y)|<\frac{1}{n}.$$

Because $g\in\sigma^*$, $C(g)$ is dense, so for each $n\in Z^+$, there is a $\mu_n\in C(g)\cap(t_n-\frac{\delta_n}{2},t_n)$. Then, for each $n\in Z^+$, there is an $\eta_n\in(0,\frac{\delta_n}{2})$ such that when $|y-\mu_n|<\eta_n$,

(24) $$|g(y)-g(\mu_n)|<\frac{1}{n}.$$

Finally, D being dense, we may choose, for each $n\in Z^+$, a $\nu_n\in(\mu_n-\eta_n,\mu_n)\cap D$. Thus, clearly

$$\lim_{n\to\infty}\mu_n=\lim_{n\to\infty}\nu_n=x_o,$$

and using (24), the fact that $f(\nu_n)=g(\nu_n)$ for each $n\in Z^+$, (22) and (21) we see

$$C(g)\text{-}\lim\sup_{t\to x_o}g(t)\geq\lim\sup_{n\to\infty}g(\mu_n)\geq$$
$$\geq\lim\sup_{n\to\infty}(g(\nu_n)-\frac{1}{n})\geq\lim\sup_{n\to\infty}f(\nu_n)\geq$$
$$\geq\lim\sup_{n\to\infty}f(t_n)-\frac{1}{n}=C(f)\text{-}\lim\sup_{t\to x_o}f(t).$$

The reverse inequality can be shown by interchanging f and g in the above argument. Therefore, at each $x_o\in\mathbb{R}$,

$$C(f)\text{-}\lim\sup_{t\to x_o}f(t)=C(g)\text{-}\lim\sup_{t\to x_o}g(t).$$

From this, the theorem easily follows using the definition of the essential set and the nice copy.

As a consequence of this theorem, we see that the nice copy of f is in some definite ways the "best" representation of f. For instance, suppose that in a nontechnical manner, we consider a "copy" of f to be any function which has the same symmetric differentiation properties as f and agrees with f on a "large" set. Then, g of theorem 3.8 is a copy of f. The theorem says that both f and g have the same nice copies. Moreover, from (d) and (e) of theorem 3.1, we see that if h is a nice copy of f and g, then $C(g)\subset C(h)$, $C(f)\subset C(h)$, $D(g)\subset D(h)$ and $D(f)\subset D(h)$. This can be interpreted to mean that of all the copies of f, its nice copy is the "most continuous" and the "most differentiable." Using similar arguments, the nice copy can be shown to "maximize" many other such properties such as upper semicontinuity, monotonicity and higher order differentiability. Thus, it does seem appropriate to call it the "nice copy" of f.

Section 3.2: Nice Copies of Functions in σ

If we restrict our considerations in theorem 3.1 to functions in σ, some of the properties of the nice copy can be considerably sharpened. Before we state and prove this new version of theorem 3.1, we first establish the following interesting lemma, which will be a useful tool in several of the following proofs.

Lemma 3.9. Let f and g be elements of σ such that $f^s(x)=g^s(x)$ everywhere. Then there exists a constant, $c\in\mathbb{R}$, such that the set

$$M=\{x: f(x)\neq g(x)+c\}$$

is countable and has no nonempty subset which is dense in itself.

Proof. Let $r>0$. It suffices to show that the set $M\cap(-r,r)$ satisfies the lemma.

To do this, we define $\Psi(x)=f(x)-g(x)$. Since f^s and g^s are finite everywhere, $\Psi^s(x)=0$ everywhere, so by corollary 1.7, $\Psi\in\sigma$ and $C(\Psi)$ is dense. We may suppose without loss of generality that $0\in C(\Psi)$ and $\Psi(0)=0$. We will show that $c=0$ satisfies the lemma.

Let $\epsilon>0$. Because $0\in C(\Psi)$, there is a $\delta_1>0$ such that when $0\leq|h|<\delta_1$, $|\Psi(h)|<\frac{\epsilon}{2}$. Since $\Psi^s(x)=0$ everywhere, for each $x \in \mathbb{R}$, there is a $\delta(x) \in (0, \delta_1)$ such that when $0 < h < \delta(x)$,

(25) $$|\Psi(x+h) - \Psi(x-h)| < \frac{h\epsilon}{r}.$$

Using this $\delta(x)$, we form a full symmetric cover for $\mathbb{R}$ by defining

$$\mathcal{J}_x = \{[x-h, x+h] : 0 < h < \delta(x)\} \text{ and } \mathcal{J} = \bigcup_{x \in \mathbb{R}} \mathcal{J}_x.$$

Then, by theorem 1.15, there is a set, $D \subset (0,r)$, with countable closure, such that $\mathcal{J}$ contains a partition of $[-x,x]$ for every $x \in D^c \cap (-r,r)$, and further, each such partition contains an element of $\mathcal{J}_0$. Choose any $x_\circ \in (0,r) \cap D^c$ and let

(26) $$-x_\circ = \alpha_1 < \alpha_2 < \ldots < \alpha_{k-1} < 0 < \alpha_k < \ldots < \alpha_n = x_\circ$$

be a partition of $[-x_\circ, x_\circ]$ from $\mathcal{J}$. Using (25) and the facts that $\alpha_k < \delta_1$ and $\Psi(0) = 0$, we see that

$$|\Psi(x_\circ)| = |\Psi(x_\circ) - \Psi(0)| \le \Sigma_{i=k}^{n-1} |\Psi(\alpha_i) - \Psi(\alpha_{i+1})| + |\Psi(\alpha_k) - \Psi(0)| <$$
$$< \frac{\epsilon}{2r} \Sigma_{i=k}^{n-1} (\alpha_{i+1} - \alpha_i) + \frac{\epsilon}{2} < \frac{\epsilon x_\circ}{2r} + \frac{\epsilon}{2} < \epsilon.$$

A similar argument shows that $|\Psi(-x_\circ)| < \epsilon$.

Therefore, for each $x \in (-r,r) - (D \cup \mathcal{R}_0(D))$, we see that $|\Psi(x)| < \epsilon$. Since D has countable closure, this implies that

$$\{x \in (-r,r) : |\Psi(x)| \ge \epsilon\}$$

also has countable closure. If for each $n \in Z^+$, we define $E_n = \{x : |\Psi(x)| \ge \frac{1}{n}\}$, then, for each $n \in Z^+$, E_n is countable. Since

$$M \cap (-r,r) = \bigcup_{n=1}^{\infty} E_n,$$

it follows that $M \cap (-r,r)$ is countable.

Define $S = \{x \in \mathbb{R} : M \cap \mathcal{R}_x(M) \neq \emptyset\}$. Thus, if $x \in S$, then there are μ and ν in M such that $x = \frac{\mu+\nu}{2}$. Since M is countable, there are at most a countable number of such pairs, so S

must be countable. Let $S=\{s_n: n\in Z^+\}$.

Now, suppose M has a subset, $N\neq\emptyset$, which is dense in itself. We inductively choose two sequences, $\{x_n: n\in Z^+\}$ and $\{\epsilon_n: n\in Z^+\}\subset(0,1)$, as follows: First, choose any $x_1\in N$ such that $x_1\neq s_1$. Then, using this x_1, choose ϵ_1 such that $0<\epsilon_1<|x_1-s_1|$ and $\epsilon_1<|\Psi(x_1)|$. Suppose $x_2,\ldots,x_{n-1}$ and $\epsilon_1,\ldots,\epsilon_{n-1}$ have been chosen. Select $x_n\in N$ such that $x_n\neq s_n$, $x_n\neq x_i$ for $i<n$ and $|x_n-x_{n-1}|<\epsilon_{n-1}$. (This selection is possible because N is dense in itself.) $\epsilon_n\in(0,\frac{1}{n})$ is chosen such that $[x_n-\epsilon_n,x_n+\epsilon_n]\subset[x_{n-1}-\epsilon_{n-1},x_{n-1}+\epsilon_{n-1}]$, $\epsilon_n<|x_n-s_n|$ and $\epsilon_n<|\Psi(x_n)|$.

From this selection procedure, it is clear that there is an $x\in\mathbb{R}$ such that $\lim_{n\to\infty}x_n=x$. For each $n\in Z^+$,

$$|x_n-x|<\epsilon_n<|x_n-s_n|,$$

so $x\neq s_n$ for any n. Therefore, $x\notin S$, and by the definition of S, it must be true that $\Psi(R_x(x_n))=0$ for each n. Since $\Psi^s(x)=0$, we see

$$0=\lim_{n\to\infty}\frac{|\Psi(x_n)-\Psi(R_x(x_n))|}{2|x-x_n|}=\lim_{n\to\infty}\frac{|\Psi(x_n)|}{2|x-x_n|}>\lim_{n\to\infty}\frac{\epsilon_n}{2\epsilon_n}=\frac{1}{2}$$

This contradiction shows that M has no nonempty dense in itself subset.

A function, f, is said to be symmetric if for every $x\in\mathbb{R}$ there exists a $\delta(x)>0$ such that $f(x+h)=f(x-h)$ whenever $0<h<\delta(x)$. A set, A, is a symmetric set, if its characteristic function is symmetric.

It is clear from the definition that if f is symmetric, then $f^s(x)=0$ everywhere. We may apply the above lemma to simplify the proofs of theorems due to Ruzsa [22] and M. Foran [9].

Corollary 3.10. If f is a symmetric function, then there is a $c \in \mathbb{R}$ such that the set, $M=\{x: f(x) \neq c\}$ has countable closure.

Proof. From lemma 3.9, it follows that there exists a $c \in \mathbb{R}$ such that the set, $M=\{x: f(x) \neq c\}$, contains no nonempty dense in itself subset. It is easily seen from this that M must be nowhere dense. So, without loss of generality, we may assume there exists a $\delta_1 > 0$ such that $f(x)=c$ when $0<x<\delta_1$. Since f is symmetric, for each $x \in \mathbb{R}$, there is a $\delta(x) \in (0,\delta_1)$ such that when $0 \leq h < \delta(x)$, then $f(x+h)=f(x-h)$. Using this $\delta(x)$, we form a full symmetric cover, $\mathcal{S}$, for $\mathbb{R}$ as in the proof of lemma 3.9. The rest of the proof proceeds as in the first half of the proof of the lemma, where it can be shown that $M \subset D \cup \mathcal{R}_0(D)$, with D the set of theorem 1.15.

Corollary 3.11. If A is a symmetric set, then either A or A^c has countable closure.

Proof. This is immediate from the definition of a symmetric set and corollary 3.10.

We are now ready to state the restricted version of theorem 3.1.

Theorem 3.12. Let $f \in \sigma$ with A_1 (A_2) the upper (lower) essential set for f and g_1 (g_2) the upper (lower) semicontinuous nice copy of f. Then A_1 and A_2 are symmetric sets and g_1 and g_2 satisfy:

(a) $g_i^s(x)=f^s(x)$ everywhere, $i=1,2$;

(b) $g_1(g_2)$ is upper (lower) semicontinuous on $\overline{A_1^c}$ $(\overline{A_2^c})$;

(c) $C(f)\subset C(g_i)$ and $f(x)=g_i(x)$ for each $x\in C(f)$, $i=1,2$;

(d) $D(f)\subset D(g_i)$ and $f'(x)=g_i'(x)$ for each $x\in D(f)$, $i=1,2$;

(e) If I is a component of $\overline{A_i^c}$, then $g_i\in\mathcal{B}_1(I)$, $i=1,2$.

Further, g_i is determined up to an additive constant and its values on A_i by (a), (b) and (e), $i=1,2$.

Proof. (b), (c) and (d) follow from theorem 3.1. The rest of the theorem will be proved in the case $i=1$. The case when $i=2$ then follows by considering $-f$.

To prove that A_1 is symmetric, let $x_0\in\mathbb{R}$. Since $f^s(x_0)$ is finite, f is symmetrically continuous at x_0, so there is a $\delta>0$ such that if $0<h<\delta$, then

(27) $$|f(x_0+h)-f(x_0-h)|<1.$$

Suppose $y_0\in A_1\cap(x_0-\delta,x_0+\delta)$ and that $C(f)\text{-}\lim\sup_{t\to y_0}f(t)=\infty$. Then we may choose a sequence, $\{y_n\}\subset(x_0-\delta,x_0+\delta)\cap C(f)$ such that $\lim_{n\to\infty}y_n=y_0$ and $\lim_{n\to\infty}f(y_n)=\infty$. For each $n\in Z^+$, there exists a $\delta_n\in(0,\frac{1}{n})$ such that $(y_n-\delta_n,\ y_n+\delta_n)\subset(x_0-\delta,x_0+\delta)$ and $|y-y_n|<\delta_n$ implies that

(28) $$|f(y)-f(y_n)|<1.$$

Using the fact that $C(f)$ is residual, we may then choose, for each n, a new

$$z_n\in\mathcal{R}_{x_0}((y_n-\delta_n,y_n+\delta_n)\cap C(f))\cap C(f).$$

Then, using (27) and (28)

$$C(f)\text{-}\lim\sup_{t\to\mathcal{R}_{x_0}(y_0)}f(t)\geq\lim\sup_{n\to\infty}f(z_n)\geq$$

$$\geq\lim\sup_{n\to\infty}(f(\mathcal{R}_{x_0}(z_n))-1)\geq\lim\sup_{n\to\infty}(f(y_n)-2)=\infty$$

so that $\mathcal{R}_{x_0}(y_0)\in A_1$. A similar argument shows that if

$$C(f)\text{-}\lim\sup_{t\to y_0} f(t)=-\infty,$$

then $\mathcal{R}_{x_0}(y_0)\in A_1$ once again. Therefore,

$$\mathcal{R}_{x_0}(A_1\cap(x_0-\delta,x_0+\delta))=A_1\cap(x_0-\delta,x_0+\delta).$$

Since x_0 was chosen arbitrarily, it follows that A_1 is a symmetric set.

Again, let $x_0\in\mathbb{R}$. We may, without losing generality, assume $f^s(x_0)=0$. Then, for each $\epsilon>0$ there is a $\delta>0$ such that whenever $0<h<\delta$,

(29) $$|f(x_0+h)-f(x_0-h)|<2h\epsilon$$

and because A_1 is symmetric, we may choose δ small enough so that

(30) $$\mathcal{R}_{x_0}(A_1\cap(x_0-\delta,x_0+\delta))=(x_0-\delta,x_0+\delta)\cap A_1.$$

Fix an $h\in(0,\delta)$. If $x_0+h\in A_1$, then by (30), the definition of g_1 and (29),

(31) $$|g_1(x_0+h)-g_1(x_0-h)|=|f(x_0+h)-f(x_0-h)|<2h\epsilon.$$

If $x_0+h\notin A_1$, we can use the same argument as in the proof of lemma 3.4 to show that

(32) $$|g_1(x_0+h)-g_1(x_0-h)|<2h\epsilon.$$

If we let ϵ go to 0, we see from (31) and (32) that $g^s(x_0)=0$ and (a) follows.

Let I be a component of $\overline{A_1^c}$ and $x_0\in I$. Since $g_1\in\sigma$, g_1 is symmetrically continuous at x_0. According to corollary 3.6,

$$g_1(x_0)=\lim\sup_{t\to x_0} g_1(t).$$

Combining these two facts, we see that

(33) $$g_1(x_0)=\lim\sup_{t\to x_0+} g_1(t)$$

and

(34) $$g_1(x_0)=\limsup_{t\to x_0-} g_1(t)$$

so that

(35) $$g_1(x_0)\in[\liminf_{t\to x_0-} g_1(t),\limsup_{t\to x_0-} g_1(t)]\cap \cap[\liminf_{t\to x_0+} g_1(t),\limsup_{t\to x_0+} g_1(t)].$$

According to theorem 3.1, $g_1\in\mathscr{B}_1$. Now, we apply theorem 1.9(c) with (35) to see that $g_1\in\mathscr{DB}_1(I)$.

Now, let h_1 be a function satisfying (a), (b) and (e) and let I and x_0 be as above. Since $h_1\in\mathscr{DB}_1(I)$, it follows from theorem 1.9(b) that

(36) $$h_1(x_0)\le\limsup_{t\to x_0+} h_1(t)$$

and

(37) $$h_1(x_0)\le\limsup_{t\to x_0-} h_1(t).$$

Using the upper semicontinuity and the symmetric continuity of h_1 at x_0, we see that

(38) $$h_1(x_0)\ge\limsup_{t\to x_0-} h_1(t)$$

and

(39) $$h_1(x_0)\ge\limsup_{t\to x_0+} h_1(t).$$

Combining (36)-(39) yields

(40) $$h_1(x_0)=\limsup_{t\to x_0+} h_1(t)$$

and

(41) $$h_1(x_0)=\limsup_{t\to x_0-} h_1(t).$$

Define $\Psi(x)=g_1(x)-h_1(x)$. Through the addition of an appropriate constant (the constant of the theorem), we may assume without loss of generality that $\Psi(x_0)=0$.

According to (40) and (33), we may choose sequences,

$\{x_n: n\in Z^+\}$ and $\{y_n: n\in Z^+\}$, increasing to $x_\circ$ and satisfying

(42) $$\lim_{n\to\infty} h_1(x_n)=h_1(x_\circ)$$

and

(43) $$\lim_{n\to\infty} g_1(y_n)=g_1(x_\circ).$$

Then, from (42) and (34),

$$\liminf_{t\to x_\circ -}\Psi(t)=\liminf_{t\to x_\circ -}(g_1(t)-h_1(t))\le$$

(44) $$\le\liminf_{n\to\infty}(g_1(x_n)-h_1(x_n))=\liminf_{n\to\infty} g_1(x_n)-h_1(x_\circ)\le$$
$$\le g_1(x_\circ)-h_1(x_\circ)=\Psi(x_\circ).$$

Then, from (41) and (43),

$$\limsup_{t\to x_\circ -}\Psi(t)=\limsup_{t\to x_\circ -}(g_1(t)-h_1(t))\ge$$

(45) $$\ge\limsup_{n\to\infty}(g_1(y_n)-h_1(y_n))=g_1(x_\circ)-\liminf_{n\to\infty} h_1(y_n))\ge$$
$$\ge g_1(x_\circ)-h_1(x_\circ)=\Psi(x_\circ).$$

Combining (44) and (45) yields

(46) $$\liminf_{t\to x_\circ -}\Psi(t)\le\Psi(x_\circ)\le\limsup_{t\to x_\circ -}\Psi(t).$$

In a similar fashion, it follows that

(47) $$\liminf_{t\to x_\circ +}\Psi(t)\le\Psi(x_\circ)\le\limsup_{t\to x_\circ +}\Psi(t).$$

(46) and (47) imply

(48) $$\Psi(x_\circ)\in[\liminf_{t\to x_\circ -}\Psi(t),\limsup_{t\to x_\circ -}\Psi(t)]\cap$$
$$\cap[\liminf_{t\to x_\circ +}\Psi(t),\limsup_{t\to x_\circ +}\Psi(t)].$$

Since both h_1 and g_1 are in $\mathcal{B}_1(I)$, we see that $\Psi\in\mathcal{B}_1(I)$. Now apply theorem 1.9(c) and (48) to conclude that $\Psi\in\mathcal{DB}_1(I)$.

Clearly, from its definition, $\Psi^s(x)=0$ everywhere. By theorem 1.11, Ψ is seen to be constant on I. Since $\Psi(x_\circ)=0$, $\Psi(x)=0$ for each $x\in I$.

Using the same argument as above, if J is any other component of $\overline{A_1^c}$, then there is a $C(J)\in\mathbb{R}$ such that $\Psi(x)=C(J)$ for each $x\in J$. From lemma 3.9 and the fact that

Ψ is identically 0 on I, it easily follows that C(J)=0 for each component, J, of $\overline{A_1^c}$. Therefore $g_1(x)=h_1(x)$ on $\overline{A_1^c}$.

Again, the semicontinuity-type conditions in theorem 3.12(b) and (e) can be improved. Specifically, (33) and (34) imply the following corollary.

Corollary 3.13. Let f, g_1, g_2, A_1 and A_2 be as in theorem 3.12. Then

$$g_1(x_\circ)=\lim\sup_{t\to x_\circ -} g_1(t)=\lim\sup_{t\to x_\circ +} g_1(t)$$

for all $x_\circ \in \overline{A_1^c}$ and

$$g_2(y_\circ)=\lim\inf_{t\to y_\circ -} g_2(t)=\lim\inf_{t\to y_\circ +} g_2(t)$$

for all $y_\circ \in \overline{A_2^c}$.

The following propositions are meant to explore the relationship between an arbitrary $f\in\sigma$ and its nice copy.

Corollary 3.14. Let $f\in\sigma$ and let g be the nice copy of f. Then the set $\{x: f(x)\neq g(x)\}$ is countable and has no dense in itself subset.

Proof. This is clear from lemma 3.9 and theorem 3.12(a) and (c).

Corollary 3.15. Let f and g be elements of σ such that $f^s(x)=g^s(x)$ everywhere. Then the essential sets for f and g are equal. If A is this essential set and f_1 and g_1 are nice copies of f and g, respectively, then there is a $c\in\mathbb{R}$ such that $f_1(x)=g_1(x)+c$ on A^c.

Proof. By lemma 3.9, there is a $c\in\mathbb{R}$ such that $f(x)=g(x)+c$ n. e.. Now, apply theorem 3.8 to f and g(x)+c.

Theorem 3.16. If $f \in \sigma$, then $C(f)^c$ is countable and has no subset which is dense in itself.

Proof. Let A and B be as in lemma 3.5 with $C = A \cup B$. Then, according to lemma 3.5, C is closed and countable, so

$$C^c = \bigcup_{n=1}^{\infty} I_n$$

where the I_n are pairwise disjoint open intervals. (Some of them may be empty.) Let μ and ℓ be as in lemma 3.4. Then, by that lemma, $\mu^s(x) = \ell^s(x) = f^s(x)$ on C^c and $\mu(x) = \ell(x) = f(x)$ on $C(f)$. By lemma 3.10,

$$D_1 = \{x \in C^c : \ell(x) \neq f(x)\} \text{ and } D_2 = \{x \in C^c : \mu(x) \neq f(x)\}$$

both have no subset which is dense in itself. It is clear that if $x \in C(f)^c$, then either $f(x) \neq \mu(x)$ or $f(x) \neq \ell(x)$. From this, it follows that $C(f)^c = C \cup D_1 \cup D_2$. Since each set in this union has no subset which is dense in itself, it follows that $C(f)^c$ has no subset which is dense in itself.

Theorem 3.16 was first proved by Charzynski [4], building upon methods developed by Mazurkiewicz [17] and Sierpinski [24]. The proof given here is much easier. Szpilrajn [25] showed that if a set, B, has no dense in itself subset, then $B = C(f)^c$ for some $f \in \sigma$, thus characterizing C(f) for an arbitrary $f \in \sigma$.

Viewing theorem 3.16 as an extension of theorem 1.4, one might suspect that a similar extension of theorem 1.5 is also true. This was shown to be false by J. Foran [8], who constructed a continuous $f \in \sigma$ such that $D(f)^c$ is uncountable.

<u>Corollary 3.17</u>. Let $f \in \sigma^s$. Then there is a unique symmetric set, A, and a function, $F \in \sigma\mathcal{B}_1$, satisfying:

(a) $F^s(x)=f(x)$ everywhere;

(b) F is upper semicontinuous on $\overline{A}^c$;

(c) If I is a component of $\overline{A}^c$, then $F \in \mathcal{D}\mathcal{B}_1(I)$;

(d) F is unique up to an additive constant and its values on A.

Proof. Since $f \in \sigma^s$, there is a $g \in \sigma$ such that $g^s(x)=f(x)$ everywhere. Let F be the nice copy of g and A be the essential set for g. The uniqueness of A and (d) follow from corollary 3.14. Theorem 3.11 yields (a), (b) and (c).

<u>Definition</u>. Using the notation of corollary 3.17, we shall call F a nice primitive for f and A the singular set for f.

Corollary 3.17 cannot be extended to σ^{*s} because there is no guarantee of a unique primitive when f is allowed to attain infinite values. A discussion of the problems involved in this case can be found in Bruckner [3, p.80].

Section 3.3: Monotonicity and Mean Value Theorems

In this section, we will present present some standard theorems of ordinary differentiation in terms of the symmetric derivative.

<u>Theorem 3.18</u>. Let $f \in \sigma^*$ such that $f^s(x) \geq 0$ a. e. and $f^s(x)$ is never $-\infty$. Then the nice copy of f is nondecreasing and the essential set for f is empty.

Proof. Let A be the essential set for f and g be the nice copy of f. Suppose x is an isolated point of A and (x,β) is a component of $\overline{A}^c$. According to theorem 3.1(f), $g\in\mathcal{M}_{-1}(x,\beta)$. By supposition, $g^s(t)\geq 0$ a. e. on (x,β) and $g^s(t)$ is never $-\infty$ on (x,β), so theorem 1.12 implies that g is nondecreasing on (x,β). From this and theorem 3.1(d) it is clear that

$$\text{(49)} \qquad C(f)\text{-}\lim\sup_{t\to x+} g(t)<\infty.$$

Similarly, if (α,x) is a component of $\overline{A}^c$, we can see that g is nondecreasing on (α,x) and

$$\text{(50)} \qquad C(f)\text{-}\lim\sup_{t\to x-} g(t)>-\infty.$$

Since $x\in A$, either $C(f)\text{-}\lim\sup_{t\to x} g(t)=\infty$, or $C(f)\text{-}\lim\sup_{t\to x} g(t)=-\infty$. Assume the former. Using (49) and (50), we see

$$\text{(51)} \qquad C(f)\text{-}\lim\sup_{t\to x+} g(t)=a<\infty$$

and

$$\text{(52)} \qquad C(f)\text{-}\lim\sup_{t\to x-} g(t)=\infty.$$

According to (52), we may choose a sequence, $\{x_n: n\in Z^+\}\subset C(f)$ such that x_n increases to x and $\lim_{n\to\infty} g(x_n)=\infty$. Since $C(f)\subset C(g)$ by theorem 3.1(c), for each $n\in Z^+$ there is a $\delta_n\in(0,\frac{1}{n})$ such that $|y-x_n|<\delta_n$ implies that $|g(y)-g(x_n)|<1$. $C(f)$ being residual in $\mathbb{R}$ implies that for each n, there is a $z_n\in C(f)\cap\mathcal{R}_x((x_n-\delta_n,x_n)\cap C(f))$. Since $\lim_{n\to\infty}\delta_n=0$, z_n decreases to x. From the choice of z_n and (49),

$$\lim\sup_{n\to\infty}(f(x+(z_n-x))-f(x-(z_n-x))=\lim\sup_{n\to\infty}(f(z_n)-f(\mathcal{R}_x(z_n))\leq$$
$$\leq\lim\sup_{n\to\infty} f(z_n)-\lim\inf_{n\to\infty} f(\mathcal{R}_x(z_n))<$$
$$<a-\lim_{n\to\infty}(g(x_n)+1)=-\infty,$$

from which we see that $f^s(x)=-\infty$, a contradiction. Therefore,

(53) $$C(f)\text{-}\lim\sup_{t\to x-} g(t)<\infty.$$

Similarly, it can be shown that

(54) $$C(f)\text{-}\lim\sup_{t\to x+} g(t)>-\infty.$$

(49), (50), (53) and (54) imply that

$$|C(f)\text{-}\lim\sup_{t\to x} g(t)|<\infty$$

which contradicts the choice of $x\in A$. Therefore A has no isolated points.

A having no isolated points implies that $\bar{A}$ has no isolated points. A closed set with no isolated points is perfect, and all nonempty perfect sets are uncountable. Since $\bar{A}$ is countable by theorem 3.1, we conclude $A=\emptyset$.

Thus, $\bar{A}^c=\mathbb{R}$, so by theorem 3.1(f), $g\in\mathcal{M}_{-1}$. Now, apply theorem 1.12 to see that g is nondecreasing on $\mathbb{R}$.

Corollary 3.19. Let $f\in\sigma^s$ such that $f(x)\geq 0$ a. e.. Then any nice primitive for f is continuous and nondecreasing.

Proof. Let F be a nice primitive for f. Since $F\in\sigma$, F clearly satisfies the conditions of corollary 3.18. Hence, we may conclude that the essential set for F is empty and F is nondecreasing. According to theorem 3.12(f), $F\in\mathcal{DB}_1$. Any monotone function satisfying the Darboux condition must be continuous.

Corollary 3.19 is also true in the more general case of the parametric derivative, as shown in [6].

Corollary 3.20. Let $f\in\sigma^s$ such that f is bounded above or below a. e.. Then any nice primitive for f is continuous.

Proof. Suppose there is an $M \in \mathbb{R}$ such that $f(x) \geq M$ a. e.. Then $g(x)=f(x)-M \geq 0$ a. e. and $g \in \sigma^s$. Apply corollary 3.19 to g. If f is bounded above, then consider -f.

Corollary 3.21. Let f and g be elements of σ^s such that $f(x)=g(x)$ a. e.. Then $f(x)=g(x)$ everywhere.

Proof. Let $h(x)=f(x)-g(x)$. It is clear that $h \in \sigma^s$ and $h(x)=0$ a. e.. By corollary 3.19, any nice primitive of h is constant. Therefore, $h(x)=0$ everywhere and $f(x)=g(x)$ everywhere.

In the above corollary it is necessary that $f(x)=g(x)$ a. e.. If the two functions are not equal a. e., the conclusion is not even true for the ordinary derivative. A proof of this may be found in Bruckner [3, p. 202].

When considering symmetrically differentiable functions the ordinary mean value theorem is not true because symmetric derivatives need not satisfy the Darboux condition. However, some replacements in the same spirit as the mean value theorem can be established.

Theorem 3.22. Let $f \in \sigma^*$ and $\alpha, \beta \in C(f)$ with $\alpha < \beta$. Then there are nonempty G_δ sets, A and B, both contained in (α, β), such that

$$f^s(a) \leq \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \leq f^s(b)$$

for all $a \in A$ and $b \in B$.

Proof. Let F be the nice copy of f. Theorem 3.1(c) implies that $F(\alpha)=f(\alpha)$ and $F(\beta)=f(\beta)$. Through the addition of an appropriate linear term, we may suppose

(55) $$F(\alpha)=F(\beta).$$

Define

$$A=\{x\in(\alpha,\beta):\ f^s(x)\le 0\} \text{ and } B=\{x\in(\alpha,\beta):\ f^s(x)\ge 0\}.$$

According to theorem 2.1, A and B are G_δ sets. Suppose $B=\emptyset$. Then $f^s(x)>0$ for each $x\in(\alpha,\beta)$ and applying corollary 3.18, we see that F is strictly increasing on (α,β). Since $\alpha,\beta\in C(f)$, we see from theorem 3.1(c) that F is strictly increasing on $[\alpha,\beta]$. This implies that $F(\alpha)<F(\beta)$, whcih contradicts (55). Therefore, $B\neq\emptyset$. A similar contradiction is reached if we assume $A=\emptyset$.

<u>Corollary 3.23</u>. Let f,α,β,A and B be as in theorem 3.22. If $f^s(x)>-\infty$ for all $x\in(\alpha,\beta)$, then $|A|>0$.

Proof. If we proceed as in the proof of theorem 3.22 and assume that $|A|=0$, we arrive at a contradiction in the same way via theorem 3.18.

<u>Corollary 3.24</u>. Let $f\in\sigma$ with α,β, A and B as in theorem 3.22. Then both A and B have positive measure.

Proof. Apply corollary 3.23 to f and -f.

Propositions of the type of theorem 3.22, corollary 3.23 and corollary 3.24 are often called quasi-mean value theorems. Theorem 3.22 was apparently first proved by Aull [1] for continuous functions. It was later extended by Evans [5] and Kundu [14] to functions satisfying certain monotonicity conditions.

As mentioned above, theorem 3.20 cannot take on the form of the usual mean value theorem because f^s may not satisfy the Darboux condition. For example, let $f(x)=|x|$, $\alpha=-1$ and $\beta=2$. Then

$$\frac{f(\beta)-f(\alpha)}{\beta-\alpha}=\frac{1}{3}$$

and $f^s(\mathbb{R})=\{-1,0,1\}$. However, if $f^s\in\mathcal{D}$, then we do arrive at the usual statement of the mean value theorem.

<u>Corollary 3.25</u>. Let $f\in\sigma^*$ such that f^s has the Darboux property. Suppose $\alpha,\beta\in C(f)$ such that $\alpha<\beta$. Then there is a $\gamma\in(\alpha,\beta)$ such that $f(\beta)-f(\alpha)=f^s(\gamma)(\beta-\alpha)$.

Proof. This is immediate from theorem 3.22.

Even though f^s need not satisfy the Darboux condition, there is a weaker "Darboux-like" condition which it must satisfy at every point.

<u>Theorem 3.26</u>. Let $f\in\sigma^s$. Then, for each $x\in\mathbb{R}$,

(56) $$\liminf_{h\to 0}\frac{f(x+h)+f(x-h)}{2}\le f(x)\le\limsup_{h\to 0}\frac{f(x+h)+f(x-h)}{2}.$$

Proof. We may assume, without losing generality, that $x=0$ in (56). Suppose the right-hand inequality in (56) is false. Through the addition of an appropriate constant, we may assume that there is an $\alpha\in\mathbb{R}$ such that

(57) $$f(0)>\alpha>0>\limsup_{h\to 0}\frac{f(h)+f(-h)}{2}.$$

It is clear that $f\in\sigma^s$ implies that $f(-x)\in\sigma^s$ and

$$g(x)=\frac{f(x)+f(-x)}{2}\in\sigma^s.$$

(57) may be rewritten as

(58) $$g(0)>\alpha>0>\limsup_{h\to 0}g(h).$$

(58) implies that there is a $\delta>0$ such that $g(h)<0$ whenever $0<|h|<\delta$. By corollary 3.20, there is a primitive, G, for g such that G is continuous and decreasing on $(-\delta,\delta)$. This implies that $g(0)=G^s(0)\leq 0$, which contradicts (58). Thus, we conclude that (57) is impossible and the right-hand inequality in (56) is true.

The left-hand inequality is established similarly.

Corollary 3.27. Let $f\in\sigma^{*s}$ and F be a nice copy of a primitive for f. Then $C(f)\subset D(F)$.

Proof. Let $x\in C(f)$ and $\epsilon>0$. Then there is a $\delta>0$ such that $|f(x)-f(y)|<\frac{\epsilon}{2}$ whenever $|y|\leq\delta$. Let $-\delta<h<\delta$. By corollary 3.20, F is continuous at x and x+h. Theorem 3.22 can then be applied to see that

$$f(x)-\epsilon\leq\frac{F(x+h)-F(x)}{h}\leq f(x)+\epsilon.$$

Letting $\epsilon\to 0$, we see that $F'(x)$ exists and equals $f(x)$.

Corollary 3.27 was first proved by Aull [1] for continuous F and then extended by Evans [5] to measurable F in $\mathcal{M}_{-1}$.

Corollary 3.28. Let $f\in\sigma^{*s}$ and F the nice copy of a primitive for f. Then $F'(x)$ exists and is finite on a residual set of full measure.

Proof. Since $f\in\mathcal{B}_1$, $C(f)$ is residual (see [20]). Corollary 3.27 then implies that $D(f)$ is residual. By theorem 3.1, $F\in\mathcal{B}_1$, so F is measurable. Now apply corollary 1.6(b).

Corollary 3.28 extends results of Evans [5] and

Muckhopadhyay [18] to σ^*.

Corollary 3.29. Let $f\in\sigma^{*s}$ such that f is continuous. Then $f\in\Delta'$.

Proof. This is immediate from corollary 3.28, or even the fundamental theorem of calculus.

Finally, we conclude this section with an extension of a result due to Kundu [16] which will be useful in Chapter IV.

Lemma 3.30. Let $f\in\sigma^s$ and $M\in\mathbb{R}$ such that $f(x)\le M$ a. e.. If F is a nice primitive for f and $a<b$, then

$$F(b)-F(a)\le M|\{x: f(x)>0\}\cap(a,b)|.$$

Proof. According to corollary 3.20, F is continuous. For each $n\in Z^+$ and each $x\in\mathbb{R}$, define

(59) $$F_n(x)=n(F(x+\tfrac{1}{n})-F(x)).$$

F_n is continuous for all $n\in Z^+$ and if $x\in D(f)$, then

(60) $$\lim_{n\to\infty}F_n(x)=f(x).$$

According to corollary 3.28, (60) is true a. e.. Now

$$\liminf_{n\to\infty}\int_a^b F_n(x)\,dx=\liminf_{n\to\infty}n\int_a^b(F(x+\tfrac{1}{n})-F(x))\,dx=$$

$$=\liminf_{n\to\infty}n\left(\int_{a+\frac{1}{n}}^{b+\frac{1}{n}}F(x)\,dx-\int_a^b F(x)\,dx\right)=$$

(61) $$=\liminf_{n\to\infty}n\left(\int_b^{b+\frac{1}{n}}F(x)\,dx-\int_a^{a+\frac{1}{n}}F(x)\,dx\right)\ge$$

$$\ge\liminf_{n\to\infty}n\int_b^{b+\frac{1}{n}}F(x)\,dx-\limsup_{n\to\infty}n\int_a^{a+\frac{1}{n}}F(x)\,dx=$$

$$=F(b)-F(a)$$

by the fundamental theorem of calculus.

Using theorem 3.22 with (59) it follows that

$F_n(x) \leq M$ for all n and x. Define, for each $n \in Z^+$,

$$G_n(x) = \max(F_n(x), 0).$$

Then, G_n is continuous for each n with $0 \leq G_n(x) \leq M$ and from (60), $\lim_{n\to\infty} G_n(x) = \max(f(x), 0)$ a. e.. Applying the dominated convergence theorem, we see

$$\liminf_{n\to\infty} \int_a^b F_n(x)\,dx \leq \lim_{n\to\infty} \int_a^b G_n(x)\,dx = \tag{62}$$

$$= \int_{\{x:\ f(x)>0\}} f(x)\,dx \leq M \left| \{x:\ f(x)>0\} \cap (a,b) \right|$$

A combination of (61) and (62) yields the lemma.

CHAPTER IV

SYMMETRIC DERIVATIVES AND THE ZAHORSKI CLASSES

Section 4.1: The Abstract Zahorski Classes

In 1950, Z. Zahorski [29] began a classification of derivatives based upon the structure of their associated sets. In the course of this work, he defined a descending sequence of subclasses of $\mathcal{DB}_1$ which he called $\mathcal{M}_i$, i=0, ...,5. If we represent the classes of functions which are, respectively, approximately continuous, bounded in Δ' and both bounded and approximately continuous by $\mathcal{A}$, bΔ' and b$\mathcal{A}$, then Zahorski's conclusions can be represented schematically.

(1)
$$\begin{array}{ccccccccccc} \mathcal{M}_0 = & \mathcal{M}_1 & \supset & \mathcal{M}_2 & \supset & \mathcal{M}_3 & \supset & \mathcal{M}_4 & \supset & \mathcal{M}_5 = \mathcal{A} \\ & \cup & & \cup & & \cup & & \cup & & \cup \\ & \mathcal{DB}_1 & \supset & \mathcal{D}\Delta^{*\prime} & \supset & \Delta' & \supset & b\Delta' & \supset & b\mathcal{A} \end{array}$$

Kundu [16], in 1976, defined abstract Zahorski classes and succeeded in demonstrating a similar structure for continuous functions, f, such that $f^s \in \mathcal{D}$. In the following sections, we will extend Kundu's theorems to larger subclasses of σ^{*s}.

Definitions. Let $A \subset \mathbb{R}$.

$M_0(A)$ is the collection of all F_σ sets, F, such that for all $x \in A \cap F$, x is a bilateral limit point of F.

$M_1(A)$ is the collection of all F_σ sets, F, such that for all $x \in A \cap F$, x is a bilateral condensation point of F.

$M_2(A)$ is the family of all F_σ sets, F, such that for all $x \in A \cap F$ and all $\delta > 0$, $|(x-\delta, x) \cap F| > 0$ and $|(x, x+\delta)| > 0$.

$M_3(A)$ is the collection of all F_σ sets, F, such that if $x \in A \cap F$ and $\{I_n: n \in Z^+\}$ is any sequence of closed intervals converging to x (i. e., any neighborhood of x contains all but a finite number of the I_n) such that $I_n \cap F = \emptyset$ for all n, then

$$\lim_{n\to\infty} \frac{|I_n|}{d(x, I_n)} = 0.$$

$M_4(A)$ is the collection of all F_σ sets, F, such that there is a sequence of closed sets, $\{F_n: n \in Z^+\}$, and a sequence of numbers, $\{\eta_n: n \in Z^+\} \subset (0,1)$, such that $F = \bigcup_{n=1}^{\infty} F_n$ and for every $c > 0$ and any $x \in A \cap F_n$ there is an $\epsilon(x,c) > 0$ such that for any two real numbers, h and h_1, satisfying $hh_1 > 0$, $h < ch_1$ and $|h+h_1| < \epsilon(x,c)$, the relation

$$\frac{|F \cap J|}{|J|} > \eta_n$$

is true, where J is the interval with endpoints $x+h$ and $x+h+h_1$.

$M_5(A)$ is the collection of all F_σ sets, F, such that for all $x \in A \cap F$, x is a density point of F.

For $i = 0, 1, \ldots, 5$, define the abstract Zahorski class, $\mathcal{M}_i(A)$, to be the collection of all functions, f, such that for any $a \in \mathbb{R}$, $\{x: f(x) > a\}$ and $\{x: f(x) < a\}$ are both in $M_i(A)$.

Finally, define the class, $Z(A)$, to be the collection of all functions, $f\in\mathcal{B}_1$, such that for each $x\in A$, each $\varepsilon>0$ and each sequence, $\{I_n: n\in Z^+\}$, of closed intervals converging to x such that for each n, $f(y)\geq f(x)$ on I_n, or $f(y)\leq f(x)$ on I_n,

$$\lim_{n\to\infty}\frac{|\{y\in I_n: |f(y)-f(x)|\geq\varepsilon\}|}{|I_n|+d(x,I_n)}=0.$$

If $A=\mathbb{R}$ in any of the above definitions, we omit the reference to A; e. g., $M_i(\mathbb{R})=M_i$, $\mathcal{M}_i(\mathbb{R})=\mathcal{M}_i$ and $Z(\mathbb{R})=Z$.

It follows easily from the definitions that for any $A\subset\mathbb{R}$, $M_{i+1}(A)\subset M_i(A)$, $i=0,\ldots,4$. Therefore, $\mathcal{M}_{i+1}(A)\subset\mathcal{M}_i(A)$, $i=0,\ldots,4$. The following lemma is less obvious.

<u>Lemma 4.1</u>. Let $A\subset\mathbb{R}$. Then $Z(A)\subset\mathcal{M}_3(A)$.

Proof. Let $f\in Z(A)$ and $a\in\mathbb{R}$. It must be shown that the sets $\{x: f(x)>a\}$ and $\{x: f(x)<a\}$ are in $M_3(A)$. We will prove that $\{x: f(x)>a\}\in M_3(A)$. The proof of the other inclusion is similar.

Let $B=\{x: f(x)>a\}$. By the definition of $Z(A)$, we see that $f\in\mathcal{B}_1$, so theorem 1.8(c) implies that B is an F_σ set. If $A\cap B=\emptyset$, it follows vacuously that $B\in M_3(A)$. So, suppose that $A\cap B\neq\emptyset$ and choose an $x\in A\cap B$. Let $\varepsilon=f(x)-a$ and choose a sequence of closed intervals, $\{I_n: n\in Z^+\}$, converging to x such that $I_n\cap B=\emptyset$ for each $n\in Z^+$. Then, since $f\in Z(A)$ and $f(y)\leq f(x)$ for each $y\in I_n$ and each $n\in Z^+$,

$$0=\lim_{n\to\infty}\frac{|\{y\in I_n: |f(y)-f(x)|\geq\varepsilon\}|}{|I_n|+d(x,I_n)}=$$

$$=\lim_{n\to\infty}\frac{|I_n|}{|I_n|+d(x,I_n)}=\lim_{n\to\infty}\frac{1}{1+\frac{d(x,I_n)}{|I_n|}}$$

from which it easily follows that

$$\lim_{n\to\infty}\frac{|I_n|}{d(x,I_n)}=0.$$

The classes M_i and $\mathcal{M}_i$, i=0, ... ,5 are due to Zahorski [29]. The generalized Zahorski classes, $M_i(A)$ and $\mathcal{M}_i(A)$, for i=0, ... ,4, are apparently due to Kundu [16], although he states several of them in a different, but equivalent way. $M_5(A)$ and $\mathcal{M}_5(A)$ are extensions of Zahorski's original classes in the spirit of Kundu's generalizations. The class, Z, was defined by Weil [27], who proved that $\Delta\subset Z\subset\mathcal{M}_3$ and that the containment is strict.

A function, f, is said to be nonangular iff

(2) $$D^+f(x)\geq D_-f(x)$$

and

(3) $$D^-f(x)\geq D_+f(x)$$

for all $x\in\mathbb{R}$. f is said to be angular at x if either (2) or (3) fails to be true. (This definition is due to Garg [11].)

A typical example of a function which is not nonangular is $f(x)=|x|$, which is angular at x=0 because

$$D^-f(0)=-1<1=D_+f(0)$$

violates (3). It is clear from the definition that any $f\in\Delta^*$ is nonangular. In general, it is an easy consequence of the proof of the Denjoy-Saks-Young theorem [23] that the set of points at which any function is angular is at most countable.

Our interest in nonangular functions is motivated by the fact that much of the rest of this chapter concerns functions in $\mathcal{D}\sigma^{*s}$. Using nonangularity, we can state a sufficient condition for f^s to be in $\mathcal{D}\sigma^{*s}$ when $f\in\sigma^*$.

<u>Theorem 4.2</u>. Let $f\in\sigma^*$ such that f is nonangular and symmetrically continuous. Then $f^s\in\mathcal{D}\mathcal{B}_1$.

Proof. According to theorem 1.1, $f^s\in\mathcal{B}_1$, so by theorem 1.9(c), it suffices to show that for any $x\in\mathbb{R}$,

$$f^s(x)\in[\liminf_{t\to x-}f^s(t),\limsup_{t\to x-}f^s(t)]\cap \tag{4}$$
$$\cap[\liminf_{t\to x+}f^s(t),\limsup_{t\to x+}f^s(t)].$$

Suppose (4) is false. Then, for example, we may suppose that there is an $x\in\mathbb{R}$ such that

$$f^s(x)<\liminf_{t\to x+}f^s(t). \tag{5}$$

Then there must be an $\epsilon>0$ and a $\delta>0$ such that $f^s(t)>f^s(x)+\delta$ when $t\in(x,x+\epsilon)$. Through the addition of an appropriate linear term to f, we may assume

$$f^s(x)<0<f^s(t) \tag{6}$$

whenever $t\in(x,x+\epsilon)$. (6) implies, via theorem 3.18, that the nice copy of f is strictly increasing on $(x,x+\epsilon)$.

Suppose

$$f(x)>\lim_{t\to x+}f(t), \tag{7}$$

where the limit on the right exists because of the monotonicity of f on $(x,x+\epsilon)$. Then the symmetric continuity of f implies that

$$f(x)>\limsup_{t\to x-}f(t). \tag{8}$$

From (7) and (8), we see that $D^+f(x)=-\infty$ and $D_-f(x)=\infty$, which is a violation of (2). Therefore, $f(x)\le\lim_{t\to x+}f(t)$.

In a similar manner it can be shown that $f(x)\geq\lim_{t\to x+}f(t)$. Thus, $f(x)=\lim_{t\to x+}f(t)$. Because f is increasing on $(x,x+\epsilon)$, we can now see that

(9) $$D_+f(x)\geq 0.$$

Since $f^s(x)<0$ and $f(t)>f(x)$ on $(x,x+\epsilon)$, it is easy to see that there is an $\eta>0$ such that $f(t)>f(x)$ on $(x-\eta,x)$. Therefore,

(10) $$D^-f(x)\leq 0.$$

(3), (9) and (10) imply that $D_+f(x)=D^-f(x)=0$.

$D^-f(x)=0$ implies that there is a sequence, $\{x_n:n\in Z^+\}$, increasing to x such that

$$\lim_{n\to\infty}\frac{f(x)-f(x_n)}{x-x_n}=0.$$

$D_+f(x)=0$ implies that

$$\liminf_{n\to\infty}\frac{f(R_x(x_n))-f(x)}{x-x_n}\geq 0.$$

Now, consider,

$$f^s(x)=\lim_{n\to\infty}\frac{f(R_x(x_n))-f(x_n)}{2(x-x_n)}=$$

$$=\lim_{n\to\infty}\left(\frac{f(R_x(x_n))-f(x)}{2(x-x_n)}+\frac{f(x)-f(x_n)}{2(x-x_n)}\right)=$$

$$=\liminf_{n\to\infty}\frac{f(R_x(x_n))-f(x)}{2(x-x_n)}+\lim_{n\to\infty}\frac{f(x)-f(x_n)}{2(x-x_n)}\geq 0.$$

This is a contradiction of (6), so we are forced to conclude that (5) never occurs.

The impossibility of the other assumptions which violate (4) is established similarly.

The converse of this theorem is false. To see this, consider

$$f(x)=\begin{cases}\frac{x}{3}\sin\frac{1}{x} & x>0\\ 0 & x=0\\ \frac{x}{3}\sin\frac{1}{x}+x & x<0.\end{cases}$$

It is clear that f has a finite derivative on $(-\infty,0)\cup(0,\infty)$, so there is no problem with angularity or the Darboux property on either of these intervals. It is also easy to show that $f^s(0)=\frac{1}{2}$, so $f\in\sigma$. Since f is continuous and

$$f'(x)=\begin{cases}\frac{1}{3}\sin\frac{1}{x}-\frac{1}{3x^2}\cos\frac{1}{x} & x>0\\ \frac{1}{3}\sin\frac{1}{x}-\frac{1}{3x^2}\cos\frac{1}{x}+1 & x<0,\end{cases}$$

theorem 1.9(c) can be used to see that $f\in\mathcal{DB}_1$. But, $D^+f(0)=\frac{1}{3}$ and $D_-f(0)=\frac{2}{3}$ shows that (2) is violated and f is angular at 0.

Section 4.2: Symmetric Derivatives and the Class $\mathcal{M}_2$

The main purpose of this section is to prove the following theorem.

<u>Theorem 4.3</u>. $\mathcal{D}\sigma^{*s}\subset\mathcal{M}_2$.

To prove this theorem, we use the following lemma, which should be viewed in light of theorem 3.18.

<u>Lemma 4.4</u>. Let $f\in\mathcal{D}\sigma^{*s}$ with F a primitive for f. If $f(x)\geq 0$ a. e., then any nice copy of F is nondecreasing.

Proof. Without loss of generality, we may assume that F is the nice copy of itself. It then suffices to show that F is nondecreasing. To do this, according to theorem 3.18, we must show that

$$A=\{x:\ f(x)=-\infty\}$$

is empty.

Define

$$B=\{x:\ f(x)\leq -1\} \text{ and } C=\{x:\ f(x)=-2\}.$$

Since $f\in\mathcal{D}\sigma^{*s}$, theorem 2.1 implies that $f\in\mathcal{D}\mathcal{B}_1$ from which it follows, using theorem 1.8(b), that A, B and C are G_δ sets. We claim that A is relatively dense in B. To see this, suppose it is not. Then there is an open interval, I, such that $I\cap A=\emptyset$ and $I\cap B\neq\emptyset$. An application of theorem 3.18 shows that F is nondecreasing on I, so $I\cap B=\emptyset$, a contradiction. Therefore, A is relatively dense in B.

We now claim that C is also relatively dense in B. To see this, let $x\in B$. Since A is relatively dense in B, we may choose a sequence, $\{x_n:\ n\in Z^+\}\subset A$, such that $\lim_{n\to\infty}x_n=x$. By assumption, the set $\{x:\ f(x)\geq 0\}$ is dense in $\mathbb{R}$, so we may choose, for each $n\in Z^+$, a y_n such that $|x_n-y_n|<\frac{1}{n}$ and $f(y_n)\geq 0$. Since $f\in\mathcal{D}$, for each $n\in Z^+$, there is a z_n between x_n and y_n such that $f(z_n)=-2$. Because $|z_n-x_n|<|y_n-x_n|<\frac{1}{n}$, we see that $\lim_{n\to\infty}z_n=\lim_{n\to\infty}x_n=x$. Therefore, the claim has been established.

But, B, being a G_δ set, is a Baire space. Any dense G_δ subset of a Baire space is residual in that space. So, A and C must be disjoint residual subsets of a Baire space, which is impossible. This contradiction forces us to conclude that $A=\emptyset$, and the lemma follows.

Proof. (Theorem 4.3) Let $f\in\mathcal{D}\sigma^{*s}$ and $A=\{x:\ f(x)>0\}$.

Suppose there exists an $x\in A$ and an $\epsilon>0$ such that

(11) $$|(x,x+\epsilon)\cap A|=0.$$

Then $f(x)\leq 0$ a. e. on $(x,x+\epsilon)$. By the lemma, it follows that $f(t)\leq 0$ everywhere on $(x,x+\epsilon)$. Now, theorem 1.9(b) implies that $f(x)\leq 0$ so that $x\notin A$. This contradiction shows that (11) is false for every $x\in A$ and every $\epsilon>0$.

In a similar manner, it can be shown that

$$|(x-\epsilon,x)\cap A|>0$$

for each $x\in A$ and each $\epsilon>0$.

Therefore, $A\in M_2$.

Through the addition of an appropriate linear term to a primitive of f, it may be shown that

$$\{x:\ f(x)>a\}\in M_2$$

for any $a\in[-\infty,\infty)$. By considering $-f$, we see that

$$\{x:\ f(x)<a\}\in M_2$$

for any $a\in(-\infty,\infty]$.

Therefore, $f\in\mathcal{M}_2$ and the theorem follows.

<u>Corollary 4.5</u>. If $f\in\sigma^*$ such that f is nonangular and symmetrically continuous, then $f^s\in\mathcal{M}_2$.

Proof. This follows at once from theorems 4.2 and 4.3.

<u>Corollary 4.6</u>. $\mathcal{D}\Delta^{*\prime}\subset\mathcal{M}_2$.

In particular, the derivative of any continuous function is in $\mathcal{M}_2$. That theorem 4.3 cannot be improved to include $\mathcal{M}_3$ instead of $\mathcal{M}_2$, even for the ordinary derivative of a continuous function, was shown by Zahorski [29]. The Darboux condition is also necessary because $\mathcal{M}_2\subset\mathcal{DB}_1$.

Theorem 4.3 improves a result of Kundu [16], who proved that if $f\in\sigma^*$ is continuous, $\{x: |f^s(x)|=\infty\}$ is countable and $f^s\in\mathcal{D}$, then $f^s\in\mathcal{M}_2$. Corollary 4.6 was first shown by Zahorski [29].

Section 4.3: Symmetric Derivatives and the Class $\mathcal{M}_3$

Comparing theorem 4.2 and (1), one might be tempted to conclude that $\mathcal{D}\sigma^s\subset\mathcal{M}_3$. Unfortunately, the situation is a bit more complicated, as can be seen from the following example.

<u>Example</u>. There is a continuous and nonangular $f\in\sigma$ such that $f^s\in\mathcal{DB}_1$ and $|f^s(x)|\leq 3$ for every x, but $f^s\notin\mathcal{M}_3$.

To construct such a function, for $n\in Z^+$, let $I_n=(3^{-n-1},3^{-n}]$. For each $x\in I_n$, define

$$r_n(x)=\begin{cases} 2x-\dfrac{2}{3^{n+1}} & \dfrac{1}{3^{n+1}}<x<\dfrac{2}{3^{n+1}}-\dfrac{2}{3^{2n+1}} \\ -\dfrac{3^{2n+1}}{2}\left(x-\dfrac{2}{3^{n+1}}\right)^2+\dfrac{2}{3^{n+1}}-\dfrac{2}{3^{2n+1}} & \dfrac{2}{3^{n+1}}-\dfrac{2}{3^{2n+1}}\leq x\leq\dfrac{2}{3^{n+1}}+\dfrac{2}{3^{2n+1}} \\ -2x-\dfrac{2}{3^n} & \dfrac{2}{3^{n+1}}+\dfrac{2}{3^{2n+1}}<x\leq\dfrac{1}{3^n} \end{cases}$$

Using these functions, we define a function, f, with domain $\mathbb{R}$ by:

$$r_n(x)=\begin{cases} (-1)^n r_n(x) & x\in I_n \\ 0 & x=0 \text{ or } |x|>\frac{1}{3} \\ r_n(-x)+x & x\in\mathcal{R}_0(I_n) \end{cases}$$

It is an easy calculation to show that f is differentiable on $(-\infty,0)\cup(0,\infty)$ with $|f'(x)|\leq 3$ whenever $x\neq 0$. It is also evident from the symmetry of its definition that $f^s(0)=\frac{1}{2}$. Therefore, $f\in\sigma$.

For each $n\in Z$, $r_n(x)$ attains its maximum value on I_n at $\frac{2}{3^{n+1}}$. From this, we see that

$$D^+f(0) = \lim_{n\to\infty}\frac{3^{2n+1}}{2}r_{2n}\left(\frac{2}{3^{2n+1}}\right)=\lim_{n\to\infty}\left(1-\frac{1}{3^{2n}}\right) = 1$$

and similarly, $D_+f(0)=-1$. From this and the definition of f, it follows that $D^-f(0)=0$ and $D_-f(0)=-2$. Thus, f satisfies (2) and (3) at $x=0$. f satisfies (2) and (3) everywhere else because $f'(x)$ exists when $x\neq 0$. Therefore, f is nonangular.

Since each Dini derivative of f is finite everywhere, f is continuous. Using the facts that f is continuous and nonangular, theorem 4.2 is applied to show that $f^s\in\mathcal{DB}_1$.

Let $A=\{x: f^s(x)>0\}$. Since $f^s(0)=\frac{1}{2}$, we see that $0\in A$. From the definition of $r_n(x)$, it is clear that $r_n'(x)\leq 0$ whenever $x\in J_n=[\frac{2}{3^{n+1}},\frac{1}{3^n}]$. Observe that $J_{2n}\cap A=\emptyset$ for all $n\in Z$ and $\{J_n: n\in Z^+\}$ converges to 0. In addition,

$$\lim_{n\to\infty}\frac{|J_{2n}|}{d(0,J_{2n})}=\lim_{n\to\infty}\frac{3^{-2n-1}}{2\cdot 3^{-2n-1}}=\frac{1}{2}$$

Therefore, $f\notin\mathcal{M}_3$.

Notice that this example also invalidates the next natural assumption from (1), that a bounded symmetric derivative with the Darboux property is in $\mathcal{M}_4$.

The following theorem somewhat clarifies the situation.

Theorem 4.7. Let $f\in\sigma^{*s}$ with F any primitive for f. Then $f\in Z(D(F))$.

Proof. According to theorems 3.1(d) and 3.8, there is no generality lost in assuming that F is the nice copy of a primitive of f, because $D(F)$ is, at worst, made larger. Let $x\in D(F)$. Since $f(x)\in\mathbb{R}$, through the addition of a linear function, it may be assumed that $f(x)=0=F(x)$. Because $x\in D(F)\subset C(F)$, there is a $\delta>0$ such that $|F(t)|<1$ whenever $|t-x|<\delta$. Let $[a,b]\subset(x,x+\delta)$ be such that $f(t)\geq 0$ on $[a,b]$ and let $\epsilon>0$. Define

$$A=\{t\in[a,b]:\ f(t)\geq\epsilon\}.$$

We claim that

$$\epsilon|A|\leq\lim_{t\to b-}F(t)-\lim_{t\to a+}F(t). \tag{12}$$

To see this, first note that the limits in (12) make sense because theorem 3.18 guarantees that F is nondecreasing on (a,b). Define

$$F_n(t)=n(F(t+\tfrac{1}{n})-F(t)).$$

It is clear that if $y\in D(F)$, then $\lim_{n\to\infty}F_n(y)=f(y)$, so that by corollary 3.28, F_n converges to f a. e.. Choose any $[c,d]\subset(a,b)$ such that $\{c,d\}\subset C(F)$. Then, since f is non-negative and measurable on $[c,d]$,

$$\epsilon|\{x\in[c,d]:\ f(x)\geq\epsilon\}|\leq\int_c^d\liminf_{n\to\infty}F_n(t)\,dt\leq$$
$$\leq\liminf_{n\to\infty}\int_c^d F_n(t)\,dt\leq\lim_{n\to\infty}\int_d^{d+\frac{1}{n}}nF(t)\,dt-\lim_{n\to\infty}\int_c^{c+\frac{1}{n}}nF(t)\,dt=$$
$$=F(d)-F(c)$$

because F is continuous at c and d. Now, choose two sequences, $\{c_n:\ n\in Z^+\}$ and $\{d_n:\ n\in Z^+\}$, contained in $C(F)$, such that c_n decreases to a and d_n increases to b. Then

$$\epsilon|A|=\lim_{n\to\infty}\epsilon|\{x\in[c_n,d_n]:\ f(x)\geq\epsilon\}|\leq$$

$\leq \limsup_{n\to\infty}(F(d_n)-F(c_n)) \leq \lim_{t\to b-}F(t)-\lim_{t\to a+}F(t)$

which is (12).

Since $x\in D(F)$ and $f(x)=0$, we see that given an $\eta>0$ there is a $\zeta\in(0,\delta)$ such that $0<t-x<\zeta$ implies that

$$|F(t)-F(x)|\leq|t-x|\epsilon\eta.$$

Therefore, if $[a,b]\subset(x,x+\zeta)$, then

$$|\lim_{t\to b-}F(t)-\lim_{t\to a+}F(t)|=$$
$$\leq|\lim_{t\to b-}F(t)-F(x)+F(x)-\lim_{t\to a+}F(t)|\leq$$
$$\leq|\lim_{t\to b-}(F(t)-F(x))|+|\lim_{t\to a+}(F(t)-F(x))|\leq$$
$$\leq\lim_{t\to b-}\epsilon\eta|t-x|+\lim_{t\to a+}\epsilon\eta|t-x|=\epsilon\eta|b-x|+\epsilon\eta|a-x|=$$
$$\epsilon\eta(2(a-x)+(b-a))\leq 2\epsilon\eta(d(x,[a,b])+|[a,b]|).$$

Combining this with (12), we see that

(13) $$|A|\leq 2\eta(d(x,[a,b])+|[a,b]|)$$

In a similar manner, (13) can be established if $f(t)\leq f(x)$ for all $t\in[a,b]$ or if $[a,b]\subset(x-\delta,x)$. Since η can be chosen arbitrarily, the theorem follows.

The following corollaries are immediate from theorem 4.7 and lemma 4.1.

<u>Corollary 4.8</u>. Let f and F be as in theorem 4.7. Then $f\in\mathcal{M}_3(D(F))$.

<u>Corollary 4.9</u>. $\Delta'\subset\mathcal{M}_3$.

Theorem 4.7 is a generalization of Weil's original theorem [27], which was that $\Delta'\subset\mathcal{Z}$. Corollary 4.8 is an improvement on Kundu's theorem [16] which required f to have a continuous primitive and to be in $\mathcal{B}\sigma^s$. Corollary 4.9 is Zahorski's original result [29].

Section 4.4: Symmetric Derivatives and the Class $\mathcal{M}_4$

Theorem 4.7 and Zahorski's original results motivate the following theorem.

Theorem 4.10. Let $f\in\sigma^s$ such that f is bounded. If F is any primitive for f, then $f\in\mathcal{M}_4(D(F))$.

The proof is immediate from the following lemma.

Lemma 4.11. Let $f\in\sigma^{*s}$ such that f is bounded above and let F be a primitive for f. If $\alpha\in\mathbb{R}$, then $\{x: f(x)>\alpha\}\in M_4(D(F))$.

Proof. We may suppose that F is the nice copy of a primitive for f because this at worst makes D(F) larger. Let $\alpha=0$ and define $E=\{x: f(x)>\alpha\}$. According to theorems 2.1 and 1.8(c), E is an F_σ set. Let $x_o\in E\cap D(F)$ and $f(x_o)=a$. Using the fact that $x_o\in D(F)$, we may write, for h sufficiently close to 0

$$F(x_o+h)=F(x_o)+ah+h\eta(h)$$

where $\lim_{h\to 0}\eta(h)=0$. If $hh_1>0$ and $|h+h_1|$ is sufficiently small, then

$$\text{(14)}\qquad \frac{F(x_o+h+h_1)-F(x_o+h)}{h_1}=a+\frac{h(\eta(h+h_1)-\eta(h))}{h_1}+\eta(h+h_1).$$

Choose $c>0$ with $0<\frac{h}{h_1}<c$. For h small enough, say $h<\varepsilon(x_o,c)$, (14) implies that

$$|\eta(h)|<\frac{a}{2c+1}.$$

Then, $|h+h_1|<\varepsilon(x_o,c)$ implies

$$\text{(15)}\qquad 0<\frac{a}{2}<\frac{F(x_o+h+h_1)-F(x_o+h)}{h_1}$$

Now, let $M>1$ be an upper bound for f. Then lemma 3.28 implies

(16) $F(x_0+h+h_1)-F(x_0+h) \leq M|\{x: f(x_0)>0\}\cap(x_0+h, x_0+h+h_1)|$.

It then follows from (15) and (16) that

(17) $$\frac{|\{x: f(x)>0\}\cap(x_0+h, x_0+h+h_1)|}{h_1} > \frac{a}{2M} > 0.$$

Theorem 2.1 implies that $\{x: f(x)>\frac{1}{n}\}$ is an F_σ set for each $n\in Z^+$, so we may choose, for each $n\in Z^+$, a sequence of closed sets, $\{E_{n,m}: m\in Z^+\}$, such that

$$\{x: f(x)>\tfrac{1}{n}\} = \cup_{m=1}^{\infty} E_{n,m}.$$

It is clear that

$$E = \cup_{n=1}^{\infty}\{x: f(x)>\tfrac{1}{n}\} = \cup_{n=1}^{\infty}\cup_{m=1}^{\infty} E_{n,m}$$

and since $x_0\in E$, there are integers, n and m, such that $x_0\in E_{n,m}$. Then $a>\frac{1}{n}$ and from (17) we see

$$\frac{|\{x: f(x)>0\}\cap(x_0+h, x+h+h_1)|}{h_1} > \frac{1}{2nM} > 0.$$

Therefore, if we choose $\eta_n=\frac{1}{2nM}\in(0,\frac{1}{2})$, the definition of $M_4(D(F))$ is satisfied.

Corollary 4.12. $b\Delta\subset\mathcal{M}_4$.

Theorem 4.10 improves on a result of Kundu [16] which was that if $f\in\sigma$ is continuous such that $f^s\in\mathcal{B}\sigma^s$ is bounded, then $f^s\in\mathcal{M}_2(D(f))$. The corollary is due to Zahorski [29].

From (1), we see that $b\mathcal{A}\subset b\Delta'\subset\mathcal{M}_4$. It is easy to see that $b\Delta'\subset b\mathcal{B}\sigma^s$ and the example in section 4.3 shows that $b\mathcal{B}\sigma^s$ is not contained in $\mathcal{M}_3$. .Therefore $b\Delta'$ is properly contained in $b\mathcal{B}\sigma^s$. The next two examples show that even the containments $b\Delta' \subset b\sigma^s \mathcal{M}_4$ and $\Delta'\mathcal{A} \subset \sigma^s\mathcal{A}$ are proper.

Example. There is a bounded symmetric derivative, $f \in \mathcal{M}_4$, which is not a derivative.

Let $I_n = [2^{-n}, 2^{-n+1}]$ for $n \in Z^+$. Partition each I_n into 2^n equal subintervals, I_n^k, $k=1, \ldots, 2^n$. If we write $[\alpha, \beta] = I_n^k$, for some k and n, then we may define

$$g_n^k(x) = \begin{cases} \frac{4}{\alpha-\beta}(x-\alpha) & x \in [\alpha, \alpha+\frac{\beta-\alpha}{4}) \\ \frac{4}{\beta-\alpha}(x-\frac{\alpha+\beta}{2}) & x \in [\alpha+\frac{\beta-\alpha}{4}, \alpha+\frac{3(\beta-\alpha)}{4}] \\ \frac{4}{\alpha-\beta}(x-\beta) & x \in (\alpha+\frac{3(\beta-\alpha)}{4}, \beta] \\ 0 & x \in [\alpha,\beta]^c. \end{cases}$$

Using these functions, we define

$$f_1(x) = \begin{cases} \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} (g_n^k(x) - g_n^k(-x)) + \chi_{(-\infty,0)}(x) & x \neq 0 \\ \frac{1}{2} & x=0 \end{cases}$$

We must show that $f_1 \in \mathcal{M}_4$. Let $\alpha \in \mathbb{R}$ and define

$$F = \{x \in \mathbb{R} : f_1(x) > \alpha\}.$$

If $\alpha \geq \frac{1}{2}$, then it is clear from the continuity of each g_n^k that F is open and consequently $F \in M_4$. So, we suppose that $\alpha < \frac{1}{2}$. Then we may write

$$F = \bigcup_{m=1}^{\infty} \{x \in \mathbb{R} : f_1(x) \geq \alpha + \frac{1-2\alpha}{m}\}.$$

Using the definition of f_1, we see that for each $n \in Z^+$, the set

$$F_n = \{x \in \mathbb{R} : f_1(x) \geq \alpha + \frac{1-2\alpha}{n}\}$$

consists of the set $\{0\}$ and a sequence of disjoint closed intervals converging bilaterally to zero. Thus, F_n is closed for each n, which implies that F is an F_σ set. We choose the sequence, $\{F_n\}$, to be the sequence of sets in the definition of M_4.

Now, let $A=\{x>0: f_1(x)\geq\frac{1}{2}\}$. In the same manner as above, we see that A is closed and also that $A\subset F_n$ for each $n\in Z^+$.

Choose a $c>0$ and let $k_\circ=\min\{n\in Z^+: c<2^{n-3}\}$. Pick $\varepsilon(0,c)\in(0,2^{-k_\circ})$ and let h and h_1 be positive numbers such that $h<ch_1$ and $h+h_1<\varepsilon(0,c)$. Then $h\in[2^{-n_\circ},2^{-n_\circ+1})$ for some $n_\circ>k_\circ$.

We claim that there exist integers, k and n, such that $I_n^k\subset(h,h+h_1)$. To see this, suppose not. Then $(h,h+h_1)$ can intersect at most two if the intervals $\{I_n^k: n\in Z^+, 1\leq k\leq 2^n\}$, because otherwise it must contain one of them. Using the fact that $h<2^{-n_\circ+1}$, we see

$$h_1\leq|I_{n_\circ}^{2^{n_\circ}}|+|I_{n_\circ-1}^{1}|=5\cdot 2^{-2n_\circ}<2^{-2n_\circ+3}.$$

This implies that

$$\frac{h}{h_1}>\frac{2^{-n_\circ}}{2^{-n_\circ+3}}>2^{k_\circ-3}>c$$

by the choice of $k_\circ$. This violates the assumption that $h<ch_1$. Therefore we are forced to conclude that the claim is true.

Let $\{J_1, \ldots, J_m\}$ be the set of all intervals contained in $\{I_n^k: n\in Z^+, 1\leq k\leq 2^n\}$ such that $J_i\cap(h,h+h_1)\neq\emptyset$ and such that the J_i are arranged in increasing distance from zero. It is clear that $m\geq 3$, for otherwise $(h,h+h_1)$ contains no I_n^k.

Thus, $h\in J_1$, $h+h_1\in J_m$ and $J_i\subset(h,h+h_1)$ for $2\le i\le m-1$. From the definition g_n^k, we see that for $2\le i\le m-1$,

$$|A\cap J_i\cap(h,h+h_1)|=|A\cap J_i|=\frac{1}{4}|J_i|$$

so that

$$(18)\qquad |A\cap\bigcup_{i=2}^{m-1}J_i|=\frac{1}{4}|\bigcup_{i=2}^{m-1}J_i|.$$

From the definition of the I_n^k, it follows that $|J_1|\le|J_2|$ and $|J_m|<4|J_{m-1}|$ so that

$$(19)\qquad |\bigcup_{i=1}^{m}J_i|=|J_1|+|\bigcup_{i=2}^{m-1}J_i|+|J_m|\le 6|\bigcup_{i=2}^{m-1}J_i|.$$

(18) and (19) imply that

$$(20)\qquad \frac{|A\cap(h,h+h_1)|}{h_1}\ge\frac{|A\cap\bigcup_{i=2}^{m-1}J_i|}{|\bigcup_{i=1}^{m-1}J_i|}\ge\frac{\frac{1}{4}|\bigcup_{i=2}^{m-1}J_i|}{6|\bigcup_{i=2}^{m-1}J_i|}=\frac{1}{24}.$$

Choose η such that $0<\eta<\frac{1}{24}$. Then (19) and (20) imply

$$\frac{|F\cap(h,h+h_1)|}{h_1}>\frac{|A\cap(h,h+h_1)|}{h_1}>\eta.$$

If h and h_1 are chosen to be negative such that $h<ch_1$ and $h+h_1>-\varepsilon(0,c)$, we note that $R_0(A\cap(0,\infty))\subset A$ to establish

$$\frac{|F\cap(h+h_1,h)|}{-h_1}\ge\frac{|A\cap(h+h_1,h)|}{-h_1}\ge$$

$$\ge\frac{|R_0(A\cap(0,\infty))\cap(h+h_1,h)|}{-h_1}=\frac{|A\cap(-h,-h-h_1|}{-h_1}>\eta.$$

Therefore, in the definition of the class M_4, if we let $\eta_n=\eta$ for each $n\in Z^+$, then the definition is satisfied at 0 with the set F. If $x\in F$ such that $x\neq 0$, then there is a neighborhood $(x-\rho,x+\rho)\subset F$. From this it is evident that the criteria of the definition of M_4 are satisfied at x. Therefore, $F\in M_4$.

If $E=\{x: f_1(x)<\alpha\}$, then it is similarly established that $E\in M_4$. Therefore, $f_1\in\mathcal{M}_4$.

Now let

$$F(x)=\int_0^x f_1(t)\,dt.$$

Since f_1 is continuous on $(-\infty,0)\cup(0,\infty)$, F is differentiable with $F'(x)=f_1(x)$ whenever $x\neq 0$. Also

$$F^s(0)=\lim_{x\to 0}\frac{F(x)-F(-x)}{2x}=$$

$$=\lim_{x\to 0}\frac{1}{2x}\left(\int_0^x f_1(t)\,dt-\int_0^{-x} f_1(t)\,dt\right)=$$

$$=\lim_{x\to 0}\frac{1}{2x}\int_0^x (f_1(t)-f_1(-t))\,dt=$$

$$=\lim_{x\to 0}\frac{1}{2x}\int_0^x dt=\frac{1}{2}=f_1(0).$$

Thus, $F\in\sigma$ with $f_1(x)=F^s(x)$ everywhere and $f\in b\sigma^s\mathcal{M}_4$.

Define the function

$$f_2(x)=\begin{cases}\sum_{n=0}^{\infty}\sum_{k=1}^{2^n}(-g_n^k(x)-g_n^k(-x))+\chi_{(-\infty,0)}(x) & x\neq 0\\ \frac{1}{2} & x=0\end{cases}$$

Then, in the same manner as above, it can be established that $f_2\in b\sigma^s\mathcal{M}_4$. But,

$$f_1(x)+f_2(x)=\begin{cases}0 & x>0\\ 1 & x=0\\ 2 & x<0\end{cases}$$

which violates the Darboux condition at $x=0$. Thus, either f_1 or f_2 cannot be a derivative. Let f be either f_1 or f_2 such that f is not a derivative.

<u>Example</u>. There exists an $f\in\mathcal{M}_5$ which is a symmetric derivative, but not a derivative.

For $n\in Z^+$ define $I_n=[2^{-n},\ 2^{-n}+2^{-2n}]$ and let g_n be a nonnegative continuous function supported in I_n such that

$$\int_{I_n} g_n = 2^{-n-1}.$$

Then, let $g(x)=\sum_{n=1}^{\infty} g_n(x)$ and

$$f(x)=\begin{cases} g(x) & x>0 \\ 0 & x=0 \\ -g(-x) & x<0. \end{cases}$$

f is continuous on $(-\infty,0)\cup(0,\infty)$. We must show that f is approximately continuous at 0. To do this, let $c>0$,

$$k=\max\{n\in Z^+ :\ 2^{-n}\geq c\} \text{ and } N=\{x\in\mathbb{R} :\ f(x)=0\}.$$

Then

$$|N\cap(-c,c)|\geq 2c-2\left|\bigcup_{i=k}^{\infty} I_n\right|=2c-2\sum_{i=k}^{\infty} 2^{-i}+2^{-2i}=$$

$$=2c-2(2^{-k+1}+\tfrac{1}{3}2^{-2k+2})\geq 2c-2^{-k}.$$

From this it is clear that

$$\lim_{c\to 0}\frac{|N\cap(-c,c)|}{2c}=1.$$

Therefore, 0 is a density point of N. Since $f(0)=0$, we see that f is approximately continuous at 0.

Let $F(x)=\int_0^x f(t)\,dt$. Then $F'(x)=f(x)$ whenever $x\neq 0$ because of the continuity of f. Also,

$$F^s(0)=\lim_{h\to 0}\frac{F(h)-F(-h)}{2h}=$$

$$=\lim_{h\to 0}\frac{\int_0^h f(t)\,dt-\int_0^{-h} f(t)\,dt}{2h}=$$

$$=\lim_{h\to 0}\frac{\int_0^h g(t)\,dt-\int_0^h g(t)\,dt}{2h}=0.$$

Therefore, $F\in\sigma$ and $F^s=f\in\sigma^s\mathcal{A}$.

To see that $f \notin \Delta'$, we first note that

$$D^+F(0) \geq \lim_{n\to\infty} 2^n \int_0^{2^{-n}} f(t)\,dt = 2^n \sum_{m=n}^{\infty} 2^{-m-1} > 1 > f(0)$$

so F is not differentiable at $x=0$ and therefore $F \notin \Delta$. Since F is absolutely continuous, it must be the nice copy of itself. Suppose $G \in \Delta$ is an ordinary primitive for f. Then $F^s(x) - G^s(x) = 0$ everywhere and by corollary 3.19 there must be a $c \in \mathbb{R}$ such that $F(x) = G(x) + c$. But, this implies that $F \in \Delta$, which is a contradiction. Therefore f has no ordinary primitive and is therefore not an ordinary derivative.

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