FINITE CYCLIC GROUP ACTIONS ON $S^1 X S^n$

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ABSTRACT

FINITE CYCLIC GROUP ACTIONS ON $S^1 \times S^n$

By

Richard Lanham Fremon

This thesis is a study of the ways in which a homeomorphism of finite period can act on $S^1 \times S^n$.

In chapter I we show by an elementary argument that the cohomology groups of the fixed point set of such a homeomorphism are quite restricted. It is shown that they must be either those of a sphere, two spheres, or of $S^1 \times S^k$. From this it follows that there are only ten possible fixed point sets of dimension two or less.

In chapter II we classify those actions on $S^1 \times S^2$ with two dimensional fixed point sets: there are only four.

FINITE CYCLIC GROUP ACTIONS ON $S^1 \times S^n$

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TABLE OF CONTENTS

LIST OF TABLES AND FIGURES

LIST OF SYMBOLS AND ABBREVIATIONS

clc_L	Cohomology locally connected [1]
HLC	Homology locally connected [2]
H_*	Singular homology
$_\Delta H^*$	Singular cohomology
H^*	Sheaf cohomology [2] (closed supports)
∂X	Boundary of the manifold X
$A \cup_f B$	The adjunction of A to B along f
$[X,A,x;Y,B,y]$	The set of homotopy classes of maps $f : x \to y$ satisfying $fA \subset B$ and $f(x) = y$
$[\omega]$	The homotopy class of ω
I	$[0,1]$
Z	The infinite cyclic group of integers
Z_k	The integers modulo k

INTRODUCTION

The study of finite period transformations began in 1938 with the discovery by P. A. Smith that when a prime period homeomorphism acts on the three-sphere the fixed point set must be S^0, S^1, or a (Cech) homology two-sphere. Since then various authors have made improvements on the methods and results. Most of the work has been concentrated on actions on spheres, projective spaces, and products of these. In 1964 J. C. Su extended a technique of R. G. Swan to characterize the cohomology groups of the fixed point sets of Z_p actions on $S^m \times S^n$. Recently K. W. Kwun proved that a piecewise linear involution on $S^1 \times S^2$ is unique up to equivalence if it fixes a torus or two two-spheres. In this thesis we use very basic tools to simplify Su's result for the case $m = 1$ and we prove that when there is a two dimensional component in the fixed point set of an involution on $S^1 \times S^2$ the action is determined by that fixed point set.

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CHAPTER I

FIXED POINT SETS

BASIC CONSTRUCTIONS

We shall begin by defining certain basic tools which will be useful throughout. Let $h : S^1 \times S^n \circlearrowleft$ $(n > 1)$ be the transformation satisfying $h^p = I$ (p prime) and $F = F_h = \{x \in S^1 \times S^n \mid h(x) = x\} \neq \emptyset$ or $S^1 \times S^n$. As most of the work is done in the universal covering space we define $p = \mathrm{Exp} \times I : \mathbb{R} \times S^n \to S^1 \times S^n$ where $\mathrm{Exp}(x) = e^{2\pi i x}$; and $p_k : \mathbb{R} \times S^n \circlearrowleft$ by $p_k(x,z) = (x+k,z)$.

We lift h to $\mathbb{R} \times S^n$ as follows. Pick $x_0 \in F$ and $y_0 \in p^{-1}(x_0)$. Then let $\tilde{h}_{y_0} : (\mathbb{R} \times S^n, y_0) \circlearrowleft$ be the unique lifting of ph through p fixing y_0. Thus $ph = \tilde{h}_{y_0} p$. When there is no ambiguity "$\tilde{h}$" and "$\tilde{F}$" will be used to denote $\tilde{h}_{y_0}$ and $F_{\tilde{h}_{y_0}}$. The uniqueness property guarantees that $\tilde{h}^p = I$.

We will also make use of the usual construction, $P(X,x) = [I,1,0;X,X,x]$, of the universal covering space of a connected, locally path connected, semilocally 1-connected space X. Details of the topology on $P(X,x)$ will not be needed in the sequel. Observe that $P(X,x)$ is a functor and let $f_\#$ denote $P(f)$. Let $\varepsilon : P(X,x) \to X$ be the projection defined by $\varepsilon([\omega]) = \omega(1)$. Then, as $\mathbb{R} \times S^n$ is simply connected, $\varepsilon : P(\mathbb{R} \times S^n, y_0) \to \mathbb{R} \times S^n$ is a bijection. As $p : \mathbb{R} \times S^n \to S^1 \times S^n$ is a covering projection $p_\# : P(\mathbb{R} \times S^n, p_k(y_0)) \to P(S^1 \times S^n, x_0)$ is

also a bijection for any k. Thus we can define $q_k = p_{\#}^{-1}$: $P(S^1 \times S^n, x_0) \to P(\mathbb{R} \times S^n, p_k(y_0))$. Then $p\tilde{h} = hp$ yields the relation $\tilde{h}_{\#} = q_0 h_{\#} p_{\#} : P(\mathbb{R} \times S^n, y_0) \circlearrowleft$ and $\tilde{h} = \epsilon \tilde{h}_{\#} \epsilon^{-1} = \epsilon q_0 h_{\#} p_{\#} \epsilon^{-1}$.

Using this fact we can show that $\tilde{h}$ and p_k commute up to sign : $\tilde{h} p_k = p_{\pm k} \tilde{h}$. Pick $y \in \mathbb{R} \times S^n$ and let $[\alpha] = \epsilon^{-1}(y)$. That is $\alpha(0) = y_0$ and $\alpha(1) = y$. Also let $[\omega] = \epsilon^{-1} p_k(y_0)$. Then $[\omega \cdot (p_k \circ \alpha)] = \epsilon^{-1}(p_k(y))$ and $\tilde{h} p_k(y) = \epsilon q_0 h_{\#} p_{\#} [\omega \cdot (p_k \circ \alpha)] = \epsilon q_0 (h_{\#} p_{\#} [\omega] \cdot h_{\#} p_{\#} [p_k \circ \alpha]) = \epsilon q_0 (h_{\#}[p \circ \omega] \cdot h_{\#} p_{\#} [\alpha])$. As $p\omega(0) = p\omega(1) = x_0$ $h_{\#}[p \circ \omega]$ depends upon how h acts on $\pi_1(S^1 \times S^n) \cong Z$. There are two cases : $h_{\#} = \pm I$: $\pi_1(S^1 \times S^n) \circlearrowleft$.

CASE I: INVOLUTIONS WITH $h_{\#} = -I$

Assume $h_{\#}[p \circ \omega] = -[p \circ \omega]$. We will need $\beta \in P(\mathbb{R} \times S^n, y_0)$ satisfying $\beta(1) = p_{-k}(y_0)$. $\omega \cdot (p_k \circ \beta)$ is null homotopic; so $0 = p_{\#}[\omega \cdot (p_k \circ \beta)] = p_{\#}[\omega] \cdot p_{\#}[p_k \circ \beta] = p_{\#}[\omega] \cdot p_{\#}[\beta]$. Hence $p_{\#}[\beta] = -p_{\#}[\omega] = h_{\#}[p \circ \omega]$. Consider the path $\beta \cdot (p_{-k} \circ \gamma)$ where $[\gamma] = q_0 h_{\#} p_{\#}[\alpha] = q_0 h_{\#}[p \circ \alpha]$. $p_{\#}[\beta \cdot (p_{-k} \circ \gamma)] = p_{\#}[\beta]$. $p_{\#}[p_{-k} \circ \gamma] = h_{\#}[p \circ \omega] \cdot h_{\#}[p \circ \alpha]$. Therefore $\tilde{h} p_k(y) = \epsilon q_0 p_{\#}[\beta \cdot (p_{-k} \circ \gamma)] = \epsilon[\beta \cdot (p_{-k} \circ \gamma)] = \epsilon[p_{-k} \circ \gamma] = p_{-k} \gamma(1) = p_{-k} \epsilon[\gamma] = p_{-k} \epsilon q_0 h_{\#} p_{\#}[\alpha] = p_{-k} \tilde{h}(y)$. In short $\tilde{h} p_k = p_{-k} \tilde{h}$.

$H^*(\mathbb{R} \times S^n; Z_2) = H^*(S^n; Z_2)$. So, from Smith Theory [p. 43, 1] we know that $H^*(\tilde{F}; Z_2) \cong H^*(S^k; Z_2)$ for some $0 \leq k \leq n$; also [p. 76, 1] $\tilde{F}$ must be a Z_2-orientable cohomology manifold. $\tilde{h} p_m = p_{-m} \tilde{h}$ tells us that when $\tilde{h}$ is extended to the two point compactification of $R \times S^n$ the resulting map does not

fix the added points. Hence $\tilde{F}$ is a compact k-cohomology manifold. It is also known [p. 72, 1] that $\tilde{F}$ is clc_{Z_2} which implies it is HLC. Hence ${}_{\Delta}H^*(\tilde{F};Z_2) \cong H^*(\tilde{F};Z_2)$ and $\tilde{F}$ must be path connected or S^0.

Let C_{x_0} be the path component of x_0 in F. Clearly $p(\tilde{F}) \supset C_{x_0}$. Suppose p were not 1-1 on $\tilde{F}$. Let ω be a path in $\tilde{F}$ with $p\omega(0) = p\omega(1)$. Then $[p\circ\omega]$ is an element of the image of $\pi_1 F \to \pi_1 S^1 \times S^n$. Consider the following diagram.

$$\begin{array}{ccc} \pi_1 F & \xrightarrow{i_\#} & \pi_1 S^1 \times S^n \\ \downarrow I & & \downarrow h_\# = -I \\ \pi_1 F & \xrightarrow{i_\#} & \pi_1 S^1 \times S^n \end{array}$$

Its commutativity yields $[p\circ\omega] = -[p\circ\omega]$: $p\circ\omega \simeq 0$ and $\omega(0) = \omega(1)$. Therefore every path component of $\tilde{F}$ projects homeomorphically and every component of F is a cohomology sphere or a point.

We would like to show that the discrete points of F pair off to make F the union of cohomology spheres. We will say that two such points are related, $x_1 R x_2$, if $\tilde{h}_{y_1}(y_2) = y_2$ for some choice of $y_i \in p^{-1}(x_i)$. As each lifting $\tilde{h}_{y_i}$ is uniquely determined by the points it fixes $\tilde{h}_{y_1} = \tilde{h}_{y_2}$ and R is symmetric. Suppose $x_1 R x_2 R x_3$. Then $\tilde{h}_{y_1} = \tilde{h}_{y_2}$ and $\tilde{h}_{p_k(y_2)} = \tilde{h}_{y_3}$. $\tilde{h}_{y_1} = \tilde{h}_{y_2} = p_{-k}\tilde{h}_{p_k(y_2)}p_k = p_{-k}\tilde{h}_{y_3}p_k = \tilde{h}_{p_{-k}(y_3)}$. So $\tilde{h}_{y_1}$ fixes y_1, y_2 and $p_{-k}(y_3)$.

If these points were distinct $F_{\tilde{h}_{y_1}}$ and C_x would be cohomology spheres of positive dimension. Hence two of the y_i must be identical; the same is true for the x_i. Thus the equivalence classes contain at most two points.

Suppose that one of the equivalence classes was a singleton. As $\tilde{F}$ is a cohomology sphere $\tilde{h}_{y_0}$ must fix some point, $p_k(y_0)$; but $\tilde{h}_{y_1}(p_k(y_1)) = p_{-k}(y_1)$. Therefore the discrete points come in pairs and we could redefine C_{x_0} to include both equivalent points.

Suppose now that F includes three distinct cohomology spheres $C_{x_1}, C_{x_2}, C_{x_3}$. Choose $y_i \in p^{-1}(x_i)$ as before and define k_{ij} by $\tilde{h}_{y_i}(y_j) = p_{k_{ij}}(y_j)$. If k_{ij} were even $p_{\frac{1}{2}k_{ij}}(y_j) = p_{-\frac{1}{2}k_{ij}}\tilde{h}_{y_i}(y_j) = \tilde{h}_{y_i}(p_{\frac{1}{2}k_{ij}}(y_j))$. This would put $p_{\frac{1}{2}k_{ij}}(y_j)$ in $F_{\tilde{h}_{y_i}}$ and $x_j \in C_{x_i}$. Hence k_{ij} must be odd. By uniqueness of liftings $\tilde{h}_{y_1} = p_{k_{12}}\tilde{h}_{y_2}$. So $p_{k_{13}}(y_3) = \tilde{h}_{y_1}(y_3) = p_{k_{12}}\tilde{h}_{y_2}(y_3) = p_{k_{12}}p_{k_{23}}(y_3) = p_{(k_{12}+k_{23})}(y_3)$. This is impossible as it would require that $k_{13} = k_{12} + k_{23}$.

We have shown that when $h_{\#} = -I : \pi_1(S^1 \times S^n) \circlearrowleft$ F_h is the disjoint union of at most two cohomology spheres. Furthermore, as the codimension of $\tilde{F}$ can be odd if and only if $\tilde{h}$ and h reverse orientations [p. 58, 1], when F has more than one component they must be both odd or both even dimensional cohomology spheres.

CASE II: $h_{\#} = I$

Just as in case I we can use Smith Theory to conclude that $H^*(\tilde{F};Z_p) \cong H^*(S^{\tilde{r}};Z_p)$ for some $0 \le \tilde{r} \le n$; but in this case the dimension of $\tilde{F}$ is not so easily determined.

Let Σ be the two point compactification of $\mathbb{R} \times S^n$ and let $\hat{h}$ extend $\tilde{h}$ to Σ. Using an argument like the one above one can show that $\tilde{h}p_k = p_{+k}\tilde{h}$ when $h_{\#} = I$. Hence $\tilde{h}(\pm\infty) = \pm\infty$ and $\hat{F} = \tilde{F} \cup \{\pm\infty\}$. As Σ is an $(n+1)$-sphere we can again use Smith theory to characterize $\hat{F}$: $H^*(\hat{F};Z_p) \cong H^*(S^{\hat{r}};Z_p)$. As $\hat{F}$ must be Z_p-orientable [p. 76, 1] $\hat{F}$ and $\tilde{F}$ must be $\hat{r}$ cohomology manifolds. By Poincaré duality [p. 210, 2] we have

$$H^q(\hat{F},\tilde{F};Z_p) \cong H_{\hat{r}-q}(\hat{F}-\tilde{F};Z_p) \cong \begin{cases} Z_p^2 & \text{when } q = \hat{r} \\ 0 & \text{otherwise.} \end{cases}$$

When this information is combined with the long exact sequence for the cohomology of a pair it is seen that

$$H^q(\tilde{F};Z_p) \cong \begin{cases} Z_p & \text{for } q = 0, \hat{r}-1 \\ 0 & \text{otherwise} \end{cases}$$

when $\hat{r} > 2$. Otherwise $\hat{F} \approx S^1$ or S^2 and $\tilde{F} \approx \mathbb{R} \times S^0$ or $\mathbb{R} \times S^1$. Thus, in general, $\tilde{F}$ is an $\hat{r}$ cohomology manifold and an $(\hat{r}-1)$-(co)homology sphere.

Now we can characterize F. Consider $y \in p^{-1}F$ and define k by $\tilde{h}(y) = p_k(y)$. $y = \tilde{h}^{p-1}p_k(y) = p_{pk}(y)$ implies $k = 0$ and $y \in \tilde{F}$. Hence $\tilde{F} = p^{-1}F$ and F is the orbit space of $\{p_k\}_{k \in Z}$ acting on $\tilde{F}$. If $\tilde{F} \approx \mathbb{R} \times S^0$ F must be either one S^1 or the disjoint union of two. For $\hat{r} \ge 2$ we

will use a spectral sequence argument [p. 343, 10].

When Z operates properly on a pathwise connected space, X, there is a first quadrant spectral sequence, E, with

$$E^2_{ij} = H_i(Z; H_j(X; Z_p)) \Rightarrow H(X/Z; Z_p).$$

Because Z is free $H_i(Z;-) = 0$ for all $i > 1$ [p. 458, 6]; so $E^2_{ij} = 0$ unless $i = 0,1$ and $j = 0, \tilde{r}$. It is known [p. 457, 6] that when the action of Z on the coefficient group is trivial, for example when $j = 0$ or $p = 2$, $H_i(Z; H_j(\tilde{F}; Z_p)) \cong H_j(\tilde{F}; Z_p)$. When this action is not trivial $H_i(Z; H_j(\tilde{F}; Z_p)) \cong H_i(S^1; \mathfrak{A})$ where $\mathfrak{A}$ is a local system of coefficients with stalks $\mathfrak{A}_x = H_j(\tilde{F}; Z_p)$ and the action of $\pi_1(S^1, 1)$ on $H_j(\tilde{F}; Z_p)$ is determined by $\{p_k\}_{k \in Z}$. Hence (see Hilton and Wylie — p. 351 for the construction) $E^2_{1\tilde{r}} = E^2_{0\tilde{r}} = 0$.

$$E^{s+1}_{ij} = H_0(E^s_{i-*, j+*-1}) = \frac{\ker(E^s_{ij} \xrightarrow{\partial^s} E^s_{i-s, j+s-1})}{\operatorname{im}(E^s_{i+s, j-s-1} \xrightarrow{\partial^s} E^s_{ij})} = E^s_{ij}.$$

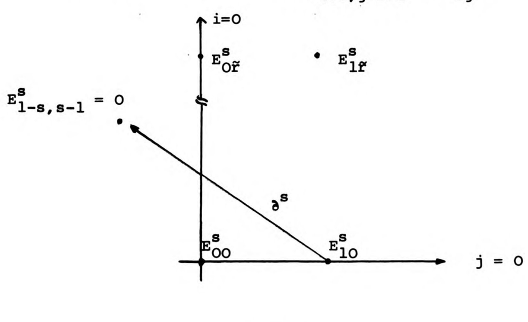

FIGURE I

THE E^s LEVEL

Therefore $E^{\infty}_{ij} = E^2_{ij}$.

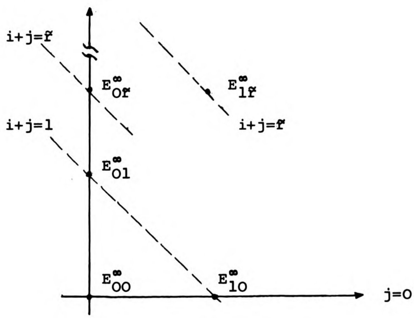

FIGURE II

THE E^{∞} LEVEL

As $E^2_{pq} \Rightarrow H(F;Z_p)$ E^{∞} is the graded module associated with some filtration of $H_*(F;Z_p)$. That is $E^{\infty}_{i,k-i} = H_k(F;Z_p)_i/H_k(F;Z_p)_{i-1}$. Hence $H_{\hat r}(F;Z_p) \cong E^2_{1\tilde r}$. This group must be non-zero as F is Z_p-orientable; so the action induced by $\{p_k\}$ on $H_j(\tilde F;Z_p)$ must be trivial. In this case $H_{\hat r}(F;Z_p) \cong E^2_{1\tilde r} \cong H_{\tilde r}(\tilde F;Z_p) \cong Z_p$; clearly $H_0(F;Z_p) \cong Z_p$. When $\tilde r > 1$ the complexes $E^{\infty}_{*,\tilde r-*}$ and $E^{\infty}_{1+*,-*}$ have only one non-zero group: $H_1(F;Z_p) \cong E^2_{10} \cong Z_p$ and $H_{\tilde r}(F;Z_p) \cong E^2_{1\tilde r} \cong Z_p$. When $\tilde r = 1$ we have the short exact sequence $0 \to E^2_{10} \to H_1(F;Z_p) \to E^2_{01} \to 0$. Hence, in every case $H_*(F;Z_p) \cong H_*(S^1 x S^{\tilde r};Z_p)$. We have proven the following.

THEOREM: Suppose $h : S^1 \times S^n \circlearrowleft$ so that $h^p = I$ and $\emptyset \neq F \neq S^1 \times S^n$ then the cohomology of F is either that of S^k, $S^{k_1} \dot{\cup} S^{k_2}$, or $S^1 \times S^{k-1}$ where $0 \leq k$, k_1, and $k_2 \leq n$ and $k_1 \equiv k_2 \pmod 2$.

CHAPTER II

ACTIONS ON $S^1 \times S^2$

When we restrict our attention to $S^1 \times S^2$ the distinction between cohomology manifold and manifold disappears: each component of F is locally Euclidean. Hence we are left with the following possible fixed point sets

	h preserves orientation	h reverses orientation
$h_\# = I$	$S^1 \dot{\cup} S^1$	$S^1 \times S^1$
	S^1	Klein bottle
$h_\# = -I$	$S^1 \dot{\cup} S^1$	$S^2 \dot{\cup} S^2$
	S^1	$S^0 \dot{\cup} S^2$
		$S^0 \dot{\cup} S^0$
		S^0

TABLE I

POSSIBLE FIXED POINT SETS

To construct examples of the above actions we need the following maps. Define $C : I \circlearrowleft$ by $C(x) = 1 - x$. This induces conjugation on $S^1 \subset \mathbb{C}$ when 0 and 1 are identified. Use the standard embedding of $S^2 \subset E^3$ to define R_1, R_2, $T : S^2 \circlearrowleft$ by $R_1(x_1,x_2,x_3) = (-x_1,-x_2,x_3)$, $R_2(x_1,x_2,x_3) = (x_1,-x_2,-x_3)$, and $T(x_1,x_2,x_3) = (x_1,x_2,-x_3)$.

Examples of $F \approx S^1$ or $S^1 \dot{\cup} S^1$ with $h_\# = I$ can be induced on $S^1 \times S^2$ by $I \times R_1 : I \times S^2 \circlearrowleft$ when $0 \times S^2$ is identified

with $1 \times S^2$ by either I or R_2. Similarly we get $F \approx S^1 \times S^1$ and K from $I \times T : I \times S^2 \circlearrowleft$. Examples of the other two-dimensional fixed point sets are found by considering $C \times I : I \times S^2 \circlearrowleft$. $C \times R_1$ induces an action with fixed point set $S^0 \dot{\cup} S^0$. It is suspected, but not known, that involutions with fixed point set S^1 and $h_\# = -I$ are non-existant. A simple application of the Lefshitz fixed point theorem to the compliment of any $S^0 \subset S^1 \times S^2$ shows that no homeomorphism leaves just S^0 fixed.

It is known [8] that in the piecewise linear category actions with $F \approx S^1 \times S^1$ or $F \approx S^2 \dot{\cup} S^2$ are unique up to equivalence: any involution with one of these fixed point sets must be a conjugate of the standard action above (i.e. $h = fh_s f^{-1}$ for some $f : S^1 \times S^2 \circlearrowleft$). We will now show that this is also the case for the other two-dimensional fixed point sets K and $S^2 \dot{\cup} S^2$.

Consider first the case of $F \approx K$. Suppose that $S^1 \times S^2$ has been triangulated so that h is simplicial. Let X be the orbit space of h with the induced triangulation and let $q : S^1 \times S^2 \to X$ be the identification map: $q(x) = qh(x)$. The triangulation of $S^1 \times S^2$ also induces a local polyhedral structure on $\mathbb{R} \times S^2$, the covering space. As F is two dimensional h must be an involution and $q | S^1 \times S^2 - F$ must be a two to one covering projection. Recall that in this case $\hat{F} \approx S^2$. When h is piecewise linear $\hat{F}$ is locally polyhedral except at $\pm\infty$. J. C. Cantrell has shown [4] that the complimentary domains, A and B, of $\hat{F} \subset \Sigma \approx S^3$ are open

three cells. Thus either component, A, will be a universal covering space for $S^1 \times S^2 - F$ and $X - qF$. The translations of $qp|A : A \to X - qF$ form a subgroup of those of $p : \mathbb{R} \times S^2 \to S^1 \times S^2$; hence, as the latter is infinite cyclic, the former and $\pi_1(X) \cong \pi_1(X - qF) \cong Z$. This suggests that X fibers over S^1. All that needs to be shown is that each tame two sphere in X bounds a three cell (i.e. X is irreducible). As X is homeomorphic to the compliment of an open collar of the boundary we need only worry about spheres interior to X. These can be lifted to A which is irreducible. Hence [13] X fibers over S^1 with fiber D^2. As h reverses orientation X must be the non-orientable disc bundle over S^1.

Now consider two involutions h and h' with $F \approx F' \approx K$ and the corresponding identifications $q : S^1 \times S^2 \to X$ and $q' : S^1 \times S^2 \to X$. Using the lifting theorem we get a homeomorphism $f : S^1 \times S^2 - F \to S^1 \times S^2 - F'$ satisfying $q = q'f$. $q'h'^{-1}fh = q'h'fh = q'fh = qh = q$; so, by uniqueness of liftings $f = h'^{-1}fh$ and $h' = fhf^{-1}$. f extends trivially to $S^1 \times S^2$ as $q'^{-1}q$ is singlevalued on F. Thus h and h' are equivalent.

Uniqueness of actions which fix $S^0 \dot{\cup} S^2$ is still simpler to prove. Consider two such actions, h_1 and h_2, and their fixed point sets F_1 and F_2. Cutting along the $S^2 \subset F_i$ yields two connected, compact manifolds X_i with boundary $S^2 \dot{\cup} S^2$. The X_i are connected because the two spheres can not separate; otherwise they would bound cells. As h_i

reverses orientation on $S^1 \times S^2$ it must switch the sides of the $S^2 \subset F_i$ and hence the induced map on X_i switches the boundary components. Thus when we cone over each $S^2 \subset \partial X_i$ to attach two balls B_i^1 and B_i^2 to X_i forming $Y_i = X_i \cup_{a_1} B_i^1 \cup_{a_2} B_i^2$ the extention, $h_i' : Y_i \circlearrowleft$, of this induced map switches the B_i^j : $h_i'B_i^1 = B_i^2$. As $Y_i \approx S^3$ [proof to prop. 4, 8] and $F_{h_i'} = S^0$, (and if G. R. Livesay's proof [9] is correct) h_i' is equivalent to $T : S^3 \circlearrowleft$ by $T(x_1, x_2, x_3, x_4) = (-x_1, -x_2, -x_3, x_4)$: $h_i = f_i^{-1} T f_i$.

If $f_1 B_1^1 = f_2 B_2^j$ for $j = 1$ or 2 we would be done; in this case $f_2^{-1} f_1 | X_i$ would induce the equivalence between h_1 and h_2. When $f_1 B_1^1 \subset f_2 B_2^j$ the proof is almost as easy. Suppose this is true for $j = 1$ and let $B_1 = f_1 B_1^1$, $B_2 = f_2 B_2^1$, and B_3 be a derived neighborhood [16] of B_2 in S^3. Define $g : S^3 \circlearrowleft$ as follows. On B_3 let g be a homeomorphism fixing ∂B_3 and carrying B_2 onto B_1; let $g | TB_3 = TgT | TB_3$; and let $g | S^3 - B_3 - TB_3 = I$. Then $gT = Tg$ and $h_1' = f_1^{-1} g f_2 h_2' f_2^{-1} g^{-1} f_1$. Furthermore $f_2^{-1} g^{-1} f_1 B_1^1 = f_2^{-1} g^{-1} B_1 = f_2^{-1} B_2 = B_2^1$ and $f_2^{-1} g^{-1} f_1$ induces the required equivalence between h_1 and h_2 on $S^1 \times S^2$.

For the general case we will construct a third pair of balls, N and TN, associated with an action equivalent to both h_1 and h_2. If $f_1 B_1^1 \cap f_2 B_2^j \neq \emptyset$ for $j = 1$ or 2 let N be a ball in the intersection; otherwise construct N as follows. Let N_1 be a derived neighborhood of $T(B_1 \cup B_2)$ and N_2 a three ball neighborhood of $B_1 \cup B_2$ in $S^3 - N_1$. To obviate $N \cap TN \neq \emptyset$ let N_3 be a derived neighborhood of TN_2 and N

a three ball neighborhood of $B_1 \cup B_2$ - this time in $S^3 - N_3$.

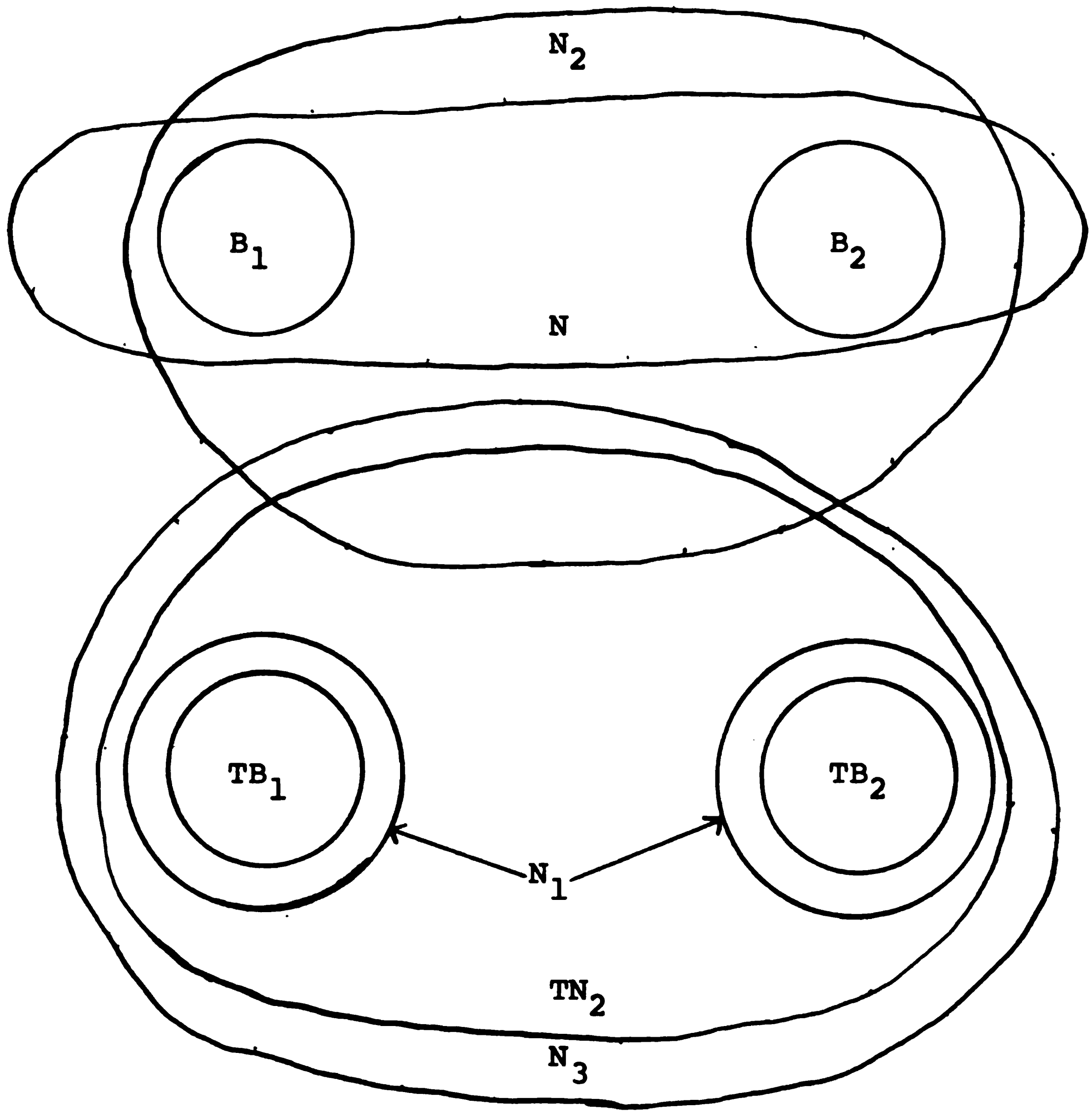

FIGURE III

CONSTRUCTION OF N

Now remove $\mathring{N}$ and $T\mathring{N}$ from S^3 and attach ∂N to $T\partial N$ by T to form a space $Z \approx S^1 \times S^2$. T induces an involution on Z which, by the above argument, is equivalent to h_1 and

h_2. Thus h_1 and h_2 are equivalent and we have proven the following.

THEOREM: Let h be a piecewise linear involution of $S^1 \times S^2$. If any component of the fixed point set of h is two-dimensional then it uniquely determines h up to equivalence.

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