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AN EXPLORATORY STUDY OF THE EFFECTS OF INSTRUCTION IN HEURISTICS UPON THE MATHEMATICAL PROBLEM-SOLVING ABILITY AND ATTITUDES OF PRESERVICE ELEMENTARY TEACHERS

By

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ABSTRACT

AN EXPLORATORY STUDY OF THE EFFECTS OF INSTRUCTION IN HEURISTICS UPON THE MATHEMATICAL PROBLEM-SOLVING ABILITY AND ATTITUDES OF PRESERVICE ELEMENTARY TEACHERS

By

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The purpose of this study was fourfold. The first goal was to examine what heuristics preservice elementary teachers tend to use in solving non-routine mathematics problems before taking any required college mathematics. The second purpose was to design and implement a required mathematics course for preservice elementary teachers in the structure of elementary school mathemtics that had a problem-solving component as a unifying theme. This problem-solving component involved the use of problems to motivate new concepts, provided instruction in the use of heuristics in solving problems, and provided students with practice in solving non-routine mathematics problems. The third purpose was to examine the effects of such a course on the heuristic usage of students, on their problem-solving ability, and on their attitude toward mathematics and problem solving. The fourth purpose was to identify features of verbal problems that are troublesome for preservice teachers.

The method of investigation involved the use of one section of a six-quarter-hour course in the structure of

elementary school mathematics containing twenty preservice elementary school teachers. This section was created from two intact sections during the spring quarter of 1975-76 academic year at Concordia Teachers College, River Forest, Illinois. Students were selected so as to have two ability-level groups with the same number of subjects in each group. The first and last week of the course the subjects participated in individual interviews with the investigator in which they were asked to think aloud as they solved twelve non-routine mathematics problems. The interviews were tape recorded for later analysis.

Two basic instruments were used to analyze the subjects' problem-solving behavior. First, a checklist was completed on each problem for each subject during the course of the interviews. The second instrument was a sequence of symbols, called a process-sequence code, which gave a sequential picture of the processes and heuristics the subjects used as they solved each problem. The audio tapes were used to write one process-sequence code for each problem worked by the subjects. The frequency of the usage of checklist items, processes, and heuristics gave rise to variables which were then analyzed. The attitudinal component of the study centered on the use of various self-report questionnaires developed by the investigator.

Major findings included the following: Before taking a required course in mathematics, the low-ability preservice teachers had a tendency to use only two heuristics--"drawing figures" and "setting up equations," while the high-ability subjects tended to use these two and "simplifies the problem" as well. There was some evidence that the course of instruction may have caused subjects to use the heuristic "setting up equations" considerably less on the posttest. Additional strategies employed on the posttest included "restates problem in his own words" and "separates/summarizes data."

Both at the beginning and at the end of the study, low-ability subjects tended to use trial and error in preference to deduction from condition as a basic process in solving non-routine problems while the high-ability group preferred to use deduction. A consideration of the change in usage of these two processes within each group reveals both groups showed an increase in the use of deduction over the period of instruction and the low-ablity group had the greatest increase.

In terms of performance as measured by scores on the problem-solving inventories, the high-ability group demonstrated the greatest improvement. However, the low-ability group seemed to make the greatest gains in persistence. On both tests the time subjects spent "looking back" at problems was negligible.

The course effected very little change in subjects' attitude toward mathematics. However, a large majority of the participants ended the course with either a neutral feeling or a positve attitude toward problem solving and felt they were more confident in their ability to solve non-routine problems.

Dedicated to,

My Wife,
Kathy, whose encouragement, support, and love have been limitless, and

My Children,
Kristin and Nicole, who have been constant delightful distractions.

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Chapter 1

INTRODUCTION

PURPOSE OF THE STUDY

The purpose of this study was fourfold. The first goal was to examine what heuristics preservice elementary teachers tend to use in solving non-routine mathematics problems before taking any required college mathematics and to determine if they have a preference for either "trial and error" or "deduction from condition" as a basic process in solving problems. The second purpose was to design and implement a required mathematics course for preservice elementary teachers in the structure of elementary school mathematics that has a problem-solving component as a unifying theme. This problem-solving component involves the use of problems to motivate new concepts, provides instruction in the use of heuristics in solving problems, and provides students with practice in solving non-routine mathematics problems. The third purpose was to examine the effects of such a course on the heuristic usage of students, on their problem-solving ability, and on their attitude toward mathematics and problem solving. The fourth and final purpose was to identify features of verbal non-routine problems that are especially troublesome for preservice teachers. Two groups of preservice teachers were included in the study. One group was slightly above average in mathematical ability and the other group below average. The experimenter suggests answers to the following questions for each of the groups.

1. Before taking a required course in mathematics, (a) what heuristics do preservice elementary teachers tend to use in solving non-routine problems, and (b) do preservice elementary teachers tend to show a preference for either "trial and error" or "deduction" as a basic technique in solving non-routine problems?

2. Does an extended learning experience in a mathematics course for preservice elementary teachers which emphasizes the use of heuristics in the solution of non-routine mathematics problems: (a) result in an increase in the use of heuristics by subjects and result in an increase in the variety of heuristics employed by subjects; (b) cause subjects to be more deductive in their problem-solving behavior; c) result in better performance on non-routine problem-solving inventories; (d) cause subjects to be more confident problem solvers; (e) result in a better attitude toward mathematics; (f) result in a better attitude toward problem solving; (g) result in a positive attitude toward

the course?

3. Can certain features of verbal non-routine problems be identified which are especially troublesome for preservice teachers?

BACKGROUND AND NEED FOR THE STUDY

The development of problem-solving ability of students has long been a basic goal of mathematics instruction at all levels. However, a perusal of textbooks used in schools today reveals that little material appears that directly relates to problem-solving processes and to techniques for solving problems. The problems that do appear generally are used to provide practice in developing some computational skill or in reinforcing some previously taught concept. Few problems are presented whose solution requires that the student employ high-order thinking strategies.

For the past thirty years mathematician George Polya has been trying to bring this regretable situation to light and has been extremely vocal in encouraging greater interest in the teaching of problem solving in mathematics at all levels. Polya (1962a) maintains that perhaps the greatest value to be gained from the study of mathematics is the ability to solve problems.

In his book How To Solve It, Polya outlines four familiar steps in the problem-solving process and suggests questions which a problem solver may ask himself as he goes about his task. He refers to this outline, parts of which follow, as his "list."

> 1. Understanding the problem. What is the unknown? What are the data? What is the condition? . . . Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory? . . .
>
> 2. Devising a plan. Have you seen the problem before? . . . Do you know a related problem? . . . Could you imagine a more accessible related problem? . . . Could you solve a part of the problem? . . .
>
> 3. Carrying out the plan . . . Check each step. Can you see clearly that the step is correct? Can you prove that it is correct?
>
> 4. Looking back. Can you check the result? Can you check the argument? Can you derive the result differently? Can you see it at a glance? . . . (1954:xvi-xvii)

Polya's philosophy of problem solving and of teaching mathematics has greatly influenced the direction of this study. It is probably best expressed by this paragraph.

> The teacher who wishes to develop his students' ability to do problems must instill some interest for problems into their minds and give them plenty of opportunity for imitation and practice. If the teacher wishes to develop in his students the mental operations which correspond to the questions and suggestions of our list, he puts these questions and suggestions to the students as often as he can do so naturally. Moreover, when the teacher solves a problem before the class, he should dramatize his ideas a little and he should put to himself the same questions which he uses when helping the students. Thanks to such guidance, the student will eventually discover the right use of these questions and suggestions, and doing so he will acquire something that is more important than the knowledge of any particular mathematical fact. (1954:5)

Paul Halmos (1975:467-68), another well-known mathematician, also encourages teacher trainers to teach our future teachers how to teach problem solving. He suggests creating a "problem-solving attitude" in students. He personally does this by starting each course with a problem which does not involve too much technical language, is different enough to capture interest, and is non-trivial. Obviously, with some careful thought and planning this idea can be extended by trying to introduce each new topic or unit in a course by some non-trivial interest-catching problem. Such an approach should go a long way in developing a "problem-solving attitude" in students.

If trainers of teachers are to instill in tomorrow's teachers a "problem-solving attitude" and an inclination toward emphasizing problem solving in their elementary and secondary school classrooms, then it is clear that we need a comprehensive model or description of the levels of mathematical performance. For without this, how are teachers to know if the problems they are presenting are appropriate for the age and ability of their students.

B. S. Bloom (1956) and others have developed such a comprehensive model in their book Taxonomy of Educational Objectives. Their "taxonomy" classifies educational objectives into six major categories or cognitive levels: Knowledge, Comprehension, Application, Analysis, Synthesis, and Evaluation. Objectives from each of these categories are ordered sequentially from low-cognitive level objectives to high-cognitive level objectives. If there is a shortcoming in this "taxonomy" it is that it is so general and very few of the examples presented of the various categories are from the field of mathematics. Thus, Shmuel Avital and Sara Shettleworth do the mathematics education community a real service in their monograph Objectives of Mathematics Learning, Some Ideas for the Teacher where they describe a way of categorizing mathematics teaching objectives, based on the categories of Bloom's "taxonomy" but adapted in certain ways to describe more accurately the various levels of mathematical performance. (1968:5)

A major theme of the monograph is that if teachers at all levels would carefully examine their instructional objectives, they would find that most of them are from the recognition or recall level or from the algorithmic thinking or generalization level. Little emphasis is placed on open-search (analysis and synthesis) objectives. To help teachers see how such objectives might be tested, the authors present numerous examples of multiple choice items designed to test objectives from the open-search category.

Many of the important conferences held during the past fifteen years to discuss mathematics programs at all levels have supported the views of people such as Polya, Halmos, and Avital and Shettleworth. The Cambridge Conference on School Mathematics, convened in 1963 in Cambridge, Massachusetts, had as participants twenty-five mathematicians and mathematics-users. Their task was to look into the future and design mathematics programs for grades K-12 which from a mathematician's point of view would be ideal curriculums. In so doing, they gave no consideration to teaching components such as teacher preparation for such curriculums, pedagogical considerations, and cognitive readiness factors. Nevertheless, it stands as a document of considerable importance in the mathematics and mathematics education community.

With regard to problem solving, the report of the Cambridge Conference on School Mathematics, Goals for School Mathematics (1963), makes three main points. First, attempts should be made to construct courses which use problems to motivate many of the new concepts. Secondly, mathematics should be seen by students as a discipline that is alive and requires original and creative thought. Presenting open-ended and unsolved problems contributes to the development of this view of mathematics. Thirdly, desirable or not, mathematics instruction in the United States centers on textbooks and as far as the student is concerned, centers on the exercise sets in textbooks. This being the case, authors should develop exercise sets with considerable care and incorporate problems of all kinds.

A conference was held on "The Role in High School of Interdisciplinary Learning Through Investigation and Action on Real Problems" in January, 1973, at Estes Park, Colorado. The report of this conference, Comprehensive Problem Solving in Secondary Schools (1975), lists two trends in secondary education that are indeed regretable and in need of changing.

> . . . The first is the virtual elimination of the learning-by-doing process that characterized the adolescent years of earlier times, and the substitution for it of learning by reading, by watching, and by listening. The second is a shift from education as a preparation for coping with problems presented by the culture to education as a process of acquiring a prescribed body of knowledge for its own sake or because an accreditation agency deems that knowledge valuable. Our educational system has expanded quantitatively to provide the extra years of education that the complexity of our society requires. But it has done so at great cost in the loss of appreciation by students of the value and power of learning. Together these losses have caused intense disaffection among large numbers of today's students. (1975:2)

What has been the effect of all the thought and consideration given to problem solving and in general to mathematics curricula? William Fitzgerald sums it up quite nicely when he states:

> In short, in spite of all the attention which has been paid to the mathematics curriculum during the past two decades, most of the mathematics teaching occuring today in schools in the United States continues to be mechanistic, skill oriented and motivated principally by the supposed need for those skills in the next mathematics class. The results of these efforts can been seen when one looks at the population who come out of schools. With the exception of a small percentage, most students leave school with very little conceptual basis of understanding of mathematics. They are not very skillful at using mathematical ideas and have rather negative attitudes about mathematics. (1975:40)

Thus in spite of many excellent attempts by individual educators to improve the quality of their students' problem-solving ability and in spite of extensive endorsement by most educators of an increased emphasis on problem solving in schools, there is still widespread dissatisfaction on the part of classroom teachers, mathematics educators, and mathematicians with the extent to which this is being done. Numerous reasons could be suggested for our failure to see improvement in problem-solving skills. First, teachers at all levels oftentimes lack confidence in their own problem-solving ability and thus are reticent about providing extensive problem-solving opportunities for their students. Secondly, over the years students tend to develop a negative attitude toward problems that are anything other than routine. This attitude is no doubt largely due to their lack of success in early problem-solving experiences and to a general feeling on the part of teachers and students that mathematics in general and problem solving in particular is difficult. Thirdly, educators know very little about how problem solving is learned and thus know even less about the kind of experiences that should be provided for a given student whom we wish to become a better problem solver. Scandura (1974) underscores this point when he says:

> . . . very little is known about the basic mechanisms which govern mathematical problem solving. It seems safe to say that apart from the insightful descriptions of mathematical problem solving by Polya, Hadamard, and a few other outstanding mathematician-philosopher-educators, there is very little in the research literature on problem solving which provides anything of potential value to mathematics education. (1974:273)

Finally, as Pollak (1966) points out, courses of instruction in mathematics for preservice elementary and secondary school teachers seldom provide extensive opportunities for solving problems in mathematics. How then can we expect these prospective teachers to emphasize problem solving in their teaching?

It is clear that there are many mathematicians and mathematics educators who are advocating a more central role in curriculums at all levels tor complex and challenging problems. As this happens there is a simultaneous need for research which, if possible, will develop a general theory that will adequately explain problem-solving behavior, will predict which problem-solving programs are best for what type of student, and will be able to measure the effects of such programs in both the cognitive and affective domain.

A survey of research done in years past shows a wide variety of approaches, subjects, materials, and techniques. Studies emphasizing functional relationships within problem situations (Wertheimer, 1945), cognitive strategies (Bruner, Goodnow, and Austin, 1956), artificial intelligence (Newell and Simon, 1972), and structural and linguistic variables (Beardslee and Jerman, 1974) have provided valuable information on problem solving from several theoretical viewpoints. However, as yet they have not been integrated into a theory general enough to adequately explain problem-solving behavior.

Whether this is an attainable goal remains to be seen. Since problem solving is an art, it defies easy analysis. Thus, if one wishes to attempt an analysis using statistical hypothesis and statistical tests he must of necessity examine one small component of the problem-solving process while controlling all other components. In so doing he misses the Gestalt of the problem-solving process. On the other hand, if one wishes to analyze a broader portion of the problem-solving process, he must take into account such things as: (1) the past problem-solving experiences of the subject, (2) the fact that challenging mathematics problems often take a long time to solve and this time is a function of many variables, (3) certain personality characteristics such as self-confidence and, (4) the fact that subjects exhibit little observable behavior while solving problems.

The aim of the present study then is not to make and substantiate any statistical tests of hypothesis but rather to attempt to take into account some of the factors delineated above and then design a course of instruction for preservice elementary teachers which has a problem-solving theme running throughout. The analysis of the course and the participants' problem-solving behavior in the course is through clinical methods. Such an approach will lead to a supply of conjectures for further study.

DEFINITION OF TERMS

The type of problem which serves as the focus for this study is referred to as a non-routine mathematics problem. It obviously is contrasted with a routine mathematics problem which in schools is often called an exercise. A routine mathematics problem or an exercise is designed to reinforce the mastery of a skill or concept and can usually be solved by following a relatively set pattern or algorithm. A non-routine mathematics problem has the following characteristics (Dodson, 1972):

1. It is challenging, i.e., it is a situation or question for which the student does not already possess a method of obtaining the solution or answer.

2. It is solvable using previous learnings.

3. It cannot be solved by a simple recall from memory or the standard use of a computational algorithm.

A second term that needs defining for this study is heuristics. Polya states that:

> . . . Heuristics is the study of the ways and means of discovery and invention. It studies especially such steps of problem solving that occur often and naturally and have some chance to bring us nearer to the solution. (1966:128)

Often we call "such steps" or procedures that "occur often" heuristics (e.g., looking for a pattern, recalling a similar problem, etc.).

A third idea that needs definition is that of heuristic teaching. J. W. A. Young states that "the heuristic method is dominated by the thought that the general attitude of the pupil is to be that of a discoverer, not that of a passive recipient of knowledge." (1906:69) However, Jon Higgins sees a shortcoming with this definition for he writes:

> It is not sufficient to simply consider heuristic teaching as a fancy phrase for discovery teaching. Most of what passes for discovery teaching could be more carefully described as <u>uncovery teaching</u>. A preselected inductive sequence of steps is constructed around a particular problem solution or concept organizations. If the teacher knows of only one means to the end, the task of the student becomes one of uncovering this particular solution or organization. The option of unusual solutions or organizations is not entertained, and the discovering lesson degenerates into a game of "Guess What's on My Mind."
>
> But the freedom to consider alternate possibilities lies at the heart of heuristics. (1971:492)

Higgins later concludes with four general characteristics of heuristic teaching. He states:

> We have seen that heuristic teachings--
>
> 1. approaches content through problems;
> 2. reflects problem-solving techniques in the logical construction of instructional procedures;
> 3. demands the flexibility for uncertainty and alternate approaches;
> 4. seeks to maximize student action and participation in the teaching-learning process. (1971:494)

We will call teaching that meets all of these criteria heuristic teaching.

Finally, a definition is needed of a tool used in the analysis of this study. Process-Sequence Codes are sequences of alphabetic letters, numerals, and other symbols that describe the problem-solving process of a subject on a given problem.

PROCEDURE

The subjects for this study were twenty of the Freshmen and Sophomores enrolled during the Spring Quarter, 1976, in Math-301, Structure of Elementary Mathematics, at Concordia Teachers College, River Forest, Illinois. This is a required six-quarter-hour math course designed for pre-service elementary teachers. Since two sections of this course were offered during the same time period, it was possible to form two homogeneous groups of eleven students each. One of these groups was below the average with respect to A.C.T. math scores, number of years of high school math, and scores on a pretest of mathematical ability. The other group was slightly above the average with respect to each of these variables.

The ten-week course was designed and taught following these guidelines:

1. New concepts and topics in the structure of elementary mathematics (i.e., numeration, whole numbers, integers, rational numbers, real numbers, and geometry and measurement) were motivated by the posing of non-routine problems which, if solved, show the importance or utility of

the concept or topic or would simply catch the imagination of the student.

2. Early in the course a period of one week was devoted to a general discussion of problem solving and to the definitions of heuristics and the applications of these to problem solving. At all times, as problems were solved in class by the instructor or by students, an emphasis was placed on Polya's scheme for solving problems (Polya, 1945) which consists of the four stages: understanding the problem, devising a plan, carrying out the plan, and looking back.

3. Eight handouts (Appendix H) which apply Polya's problem-solving scheme and which demonstrate the use of various heuristics on certain problems were written by the experimenter and provided for the students periodically during the ten-week term.

4. In working problems on their own, students were asked to be conscious of and to analyze their own problem-solving behavior. To encourage this, problems that were handed in were graded on the following basis: 0 to 2 points for approach, 0 to 3 points for plan, 0 to 2 points for result, and 0 to 2 points for looking back.

5. Although students were responsible for mastering the structure of elementary mathematics and were aided in this endeavor through the use of the required text Contemporary Mathematics by Bruce Meserve and Max Sobel, the majority of the time actually spent in class was devoted to

problem-solving activities such as those already described.

To provide means for answering the questions stated earlier, a pre and post-course interview was conducted with each of the twenty students. The pretest interview, held during the first week of the course, consisted of a problem-solving inventory (see Appendix B) of twelve problems which the student was asked to solve out loud, that is he was asked to verbalize his thoughts as he solved the problems. Each interview was recorded and analyzed using a checklist of problem-solving behavior (see Appendix D) and the process-sequence codes (see Appendix F). The problems chosen for the inventory were non-routine but required no more mathematics background than some algebra and intuitive geometry. Many of the problems were solvable in a variety of ways.

The posttest interview, held during the tenth week of the course, was similar to the pretest in structure and analysis. Some of the problems in the Posttest Problem-Solving Inventory (Appendix C) were similar in nature to the pretest problems. The others were general non-routine problems.

The problem-solving inventories, checklist, and process-sequence coding scheme used in the study were pilot tested to assure their applicability to the type of students involved in this study.

ASSUMPTIONS AND LIMITATIONS OF THE STUDY

The assumptions of the study are that:

1. The subjects did not differ significantly from the preservice teachers population with respect to the two variables attitude toward mathematics and general ability in mathematics.

2. Observing students as they solve non-routine mathematics problems while "thinking-aloud" is a reasonable and useful technique for determining how they arrive at their solutions.

The limitations of the study are that:

1. Ten weeks, the length of the treatment, is not sufficient to effect any dramatic changes in problem-solving ability. However, in most teacher training institutions mathematics educators are usually bound by this or some other similar time limitation for a period of instruction.

2. No comparison was made with some other method of teaching problem solving or with a control group. This was because of the great amount of time required for pretest and posttest interviews thus making it impossible to increase substantially the number of subjects involved in the study. Kilpatrick would maintain that such a limitation is not totally inappropriate since he writes:

> . . . Rather than comparing methods, researchers interested in instruction in heuristics should put their energies into devising the best instructional program they can and then demonstrating in detail how the program functions and how effective it is in the classroom. The creation, tryout, and

> revision of program components and instruments for measuring effectiveness are research activities of far greater potential than the comparison ot methods. (1975:8)

3. No sex differences were studied. Again the size of the sample limited study of this factor. This limitation was not serious because of the nature ot the research techniques used in the study.

4. People differ in their ability to verbalize what they are doing when they solve problems thinking out-loud. As a result the researcher had to be sensitive to this and be more insistent with some people that they describe what they are thinking.

5. No attempt was made to control "teacher effect." This limitation is again consistent with a thrust of the study which is to determine the researcher's effectiveness in teaching heuristics. Additionally, many of the techniques often used to control for teacher effect, such as written instructional programs, are not representative of the way most teachers structure the classroom routine. Thus, studies that use such techniques to control teacher effect lack applicability to the "typical" classroom situation.

6. No attempt was made to keep the students of one ability group from interacting with those of the other group. In fact students were encouraged to work together outside of class on problems. Thus, the performance on the posttest of those subjects who worked with other participants may have been influenced by such interaction.

The investigator had no way of measuring the extent of the effect of this interaction on posttest performance.

Chapter 2

REVIEW OF THE LITERATURE

Any survey of the research related to problem-solving processes reveals that it is much easier to talk about problem solving than it is to investigate it. However, in order for challenging non-routine mathematics problems to become an integral part of school curriculums, more research on the effective teaching and use of such problems is needed. Some research of the past has provided directions for future investigations. But, as in many areas of educational research, much of the problem-solving research has been disjointed and as a result has not contributed greatly to a general theory of problem solving. Also, problem solving requires a most complex form of human thought, but yet investigators doing traditional statistical studies have of necessity tried to isolate only one or two of the many variables involved in the process. Any survey of the literature finds that a multitude of such studies have examined numerous aspects of the problem-solving process. This is reflected in the first part of this review where studies are reported which have influenced the current investigation and yet have examined problem solving from narrow points of view.

The second part of this survey considers psychological research in human information processing. In such studies investigators attempt to learn more about how humans solve complex problems by simulating their behavior by computer programs. Such research requires the use of the "thinking aloud" technique for determining how the subjects are solving problems.

Psychological studies conducted by Soviet researchers are reported on next. The Soviet emphasis on researching the processes involved in solving mathematical problems stands in contrast to many studies conducted in the United States where the emphasis is on the product or the answer, a situation that to some extent is dictated by the empirical research techniques that we have traditionally employed. Because of the vast differences in research philosophy between American and Soviet researchers, much can be learned by examining their studies.

It is hard to imagine that any one person could influence an area of concern in mathematics education as much as George Polya has influenced the ways in which problem solving is taught and the ways in which it is researched. The fourth part of this chapter examines studies which are based on his theory of problem solving and which closely examine the problem-solving process using clinical techniques. One sees a Soviet influence in these studies as researchers have attempted to develop a coding procedure which adequately describes the problem-solving processes of subjects and which admits thorough analysis.

Finally, attitudinal studies that relate to problem solving are considered. The sparsity of such studies has influenced the investigator to include an attitudinal component in the present study.

INTRODUCTION

In an omnibus study conducted by Dodson (1972) an attempt was made to determine the discriminating characteristics of the successful insightful problem solver. Using the data for eleventh-grade students from the National Longitudinal Study of Mathematical Abilities (NLSMA), a lognitudinal study conducted by the School Mathematics Study Group, Dodson was able to examine the relationship between seventy-seven measures for each student and his problem-solving ability. The sample consisted of 10 percent of the NLSMA Z-population, a population that took NLSMA exams in the fifth, eighth, and eleventh grade.

Needless to say, with so many variables being considered numerous conclusions were reached. Many of these simply confirm the expected. For example, Dodson found that successful insightful mathematics problem solvers scored higher on all of the mathematics achievement variables than those who were unsuccessful problem solvers. Also:

> All of the variables measuring reasoning ability, both verbal and non-verbal, were strongly related to problem-solving ability with the two mathematically oriented reasoning variables being the strongest discriminators among ability groups. It is difficult to determine any causal relationship between reasoning ability and problem-solving ability; however, the results of the present study affirm a strong relationship between the two. Certainly mathematics educators should not ignore any opportunities to enhance the development of reasoning ability or to encourage teachers to provide experiences which challenge and demand their students' reasoning ability. (1972:127)

Further, Dodson suggests that insightful mathematics problem-solving ability is developed when students are: (1) exposed to advanced algebraic topics incl[illegible] the algebra of inequalities, the solution of quadratic inequalities, and systems of inequalities that involve a considerable amount of synthesis; and (2) exposed to geometry problems which require the synthesis of a large number of seemingly unrelated geometric ideas as opposed to problems which only require the simple application of some familiar theorem. This point should be heeded in constructing the geometry portion of a basic mathematics course for preservice teachers. Most of the geometry sections in the current textbooks used in such courses tend to present topics in such a way that little synthesis of ideas is required.

Finally, there are two other conclusions that have greatly influenced the present study. Dodson suggests that:

> The three cognitive factors, divergent thinking, spatial relationships, and field dependence, as

> measured by Anagrams, Card Rotations, and Hidden Figures, respectively, were shown in the present study to be strongly related to problem-solving ability. The students' possession of the traits measured by these scales contribute to their potential to develop problem-solving ability. In seeking to help students who are exhibiting difficulty in solving problems teachers may do well to employ the use of an approach to problem solving similar to that prescribed by Polya in his book, How To Solve It. In using such an approach, students could be directed to consider questions (concerning problem situations) that are directly related to their divergent thinking ability, their ability to resist distractions, their ability to identify the critical elements, and their ability to visualize spatial relationships. For example, a question that suggests students should consider alternate methods of solving the problem may stimulate the students to exploit divergent thinking. (1972:127)

Also Dodson not unexpectedly concludes that "The development of positive student attitudes toward mathematics should be a primary objective of teachers who are concerned with the development of problem-solving ability." (1972:128) He found that the variable attitude was a good discriminator of problem-solving ability.

There exists an abundance of studies that have tried to determine the relationship of reading ability to problem-solving ability. In one such study Knifong and Holtan (1976) had thirty-five sixth-grade students attempt the word-problem section of a standarized achievement test which contained thirty problems. An analysis of procedures was performed on problems that were not solved correctly (470 out of 1,050). This analysis showed that 52 percent of the incorrectly solved problems were missed because of either clerical errors (errors in copying numerals and so forth) or because of computational errors. Thus, there was little or no possibility of reading factors hindering the successful completion of 52 percent of the incorrectly solved problems. Reading difficulty may have been a factor in the other 48 percent of the incorrectly solved problems along with factors such as using the wrong operations or not knowing a simple computational formula. Thus the researchers conclude that, "the data neither confirm nor deny that improvement of reading skills will lead to word-problem success." (1976:111) However, the data does show that "improved computational skills could have eliminated nearly half of the word-problem errors and can be strongly recommended as a teaching strategy." (1976:111)

A more involved study conducted by Jerman (1973) substantiates the findings of Knifong and Holtan and adds others. Jerman compared two instructional strategies. The first strategy, called The Productive Thinking Program, does not include mathematical tasks as part of its instructional sequence. In comic book format the program attempts to develop general problem-solving skills using characters who act as amateur private detectives. It is intended that students learn how to think about things, how to look for clues, and how to enjoy using the mind.

The second strategy also uses a comic book format but characters encounter and solve arithmetic problems in a variety of situations using wanted-given heuristics. Clearly the investigator was trying to determine if a method of instruction which teaches general problem-solving skills would be more effective in enhancing mathematical problem-solving skills than a program designed specifically to teach problem solving in mathematics using a mathematical context.

Jerman administered each experimental program to three classes of fifth-grade students and had two classes which acted as controls. The treatment lasted sixteen days for forty minutes per day.

It comes as no surprise that Jerman concludes that:

> On the basis of the data presented and within their limitations, it would appear that teaching problem solving in mathematics to fifth graders can best be done in a mathematical context using a wanted-given approach. On the other hand, if the transfer of problem-solving skills to many subject areas out weigh the importance of a higher level of problem-solving achievement in mathematics, The Productive Thinking Program might be the better choice. (1973:17)

Jerman also determined that

> . . . students in the experimental groups who were using a correct procedure to solve problems often failed to obtain significantly more mathematically correct answers due to computational errors, than did students in the control group. (1973:18)

Thus, on the basis of this and other studies, it seems that computational difficulties are a major hinderance to success in problem solving. Taking note of this Jerman goes on to say:

> It may be that students are often taught how to proceed to solve problems correctly, but their new knowledge of how to solve a particular set of problems may be masked by the presence of computational errors. Computation and problem-solving ability may be highly correlated as many studies show, but the best measure of skill in problem solving will be one that brings to light the procedures used by students to solve problems as well as to give correct answers. (1973:18)

It is the intent of the present study to "bring to light" these procedures.

Another class of problem-solving studies is exemplified by investigations of Beardslee and Jerman (1974) or Barnett (1974), where attempts are made to identify structural or linguistic variables which account for errors students make on problem-solving inventories and which examine the effects of instruction in problem solving on these variables. In these studies a structural variable is any quantifiable parameter that may be observed as a part of the structure of a problem. For example, the parameter may be the number of words in the statement of the problem or the number of occurrences of the operation of multiplication or the total number of different operations required to find a solution. These latter two reflect the operational complexity of word problems.

In particular, Barnett (1974) attempted to answer the following general questions:

1. Can problem-solving ability of college students be improved by instruction specifically aimed at overcoming the amount of difficulty accounted for by specific structural variables?

2. What is the extent of the effect of instruction on each of the identified variables, with respect to the amount of variance in observed probability correct accounted for? (1974:3)

The specific structural variables referred to and their definitions are:

Operations: The sum of the following:

a. The number of occurrences of the operation of addition.

b. 2 times the number of occurrences of the operation of subtraction.

c. 5 times the number of occurrences of the operation of multiplication.

d. The number of occurrences of the operation of division.

e. The number of different operations.

f. 2 times the number of displacements of data.

Special Procedures: The sum of the following:

a. The number of ratios and proportions.

b. The number of different formulas.

c. The number of conversions of units.

d. 10 times the number of operations involving percents.

Linguistics: The sum of the following:

a. The total number of words in the statement of the problem, (each symbol, such as a "$" or "%", or number, such as 347 or 1/2 is considered one word);

b. The average number of words in each sentence in the problem statement, (i.e., the total number of words in the problem statement divided by the total number of sentences);

c. The total number of "mathematics" words in the problem statement, (a word is considered to be a "mathematics" word if it indicates an operation or formula, or specifies the conversion of units. These words are content specific in the sense that they may or may not be considered "mathematics" words, depending on the context of the problem in which they occur.). (1974:6)

Instruction in "Operations" included emphasis on such things as changing verbal phrases into mathematical sentences. For instruction in "Special Procedures" students were taught to recognize problems which require such things as ratios or the use of percent. Instructional methods in "Linguistics" included techniques to reduce the linguistic complexity of word problems such as eliminating distractors or underlining mathematical words, and so forth.

The study used as subjects 150 college students enrolled in an elementary mathematics methods course. These students were in one of three treatment groups. The experimental group received instruction and practice problems on the specific variables discussed above as well as supplementary material on problem solving. The first control group or the practice group received the same practice problems and supplementary material as the experimental group but did not receive instruction in problem solving. The second control group received the same tests as did the other two groups, but did not receive any material related to problem solving.

The important findings of Barnett are that the experimental group made significantly greater scores on all tests, than did the control groups and that the amount of variance accounted for by each variable following instruction on that variable remained consistent. Thus, he concludes that ". . . it would seem that structural variables can be used as a basis for instruction designed to help students become better problem solvers." (1974:31) Further, the lack of change in the amount of variance accounted for by the individual variables following instruction indicates:

> . . . that structural variables (at least of the type used in this study) may be good predictors regardless of the educational background of the subject, and are invarient when tested before and after instruction. This provides supporting evidence to the continued research into the use of structural variables, as predictors in designing problem sets with specified levels of difficulty. (1974:31)

Another study using preservice teachers as subjects was conducted by Milton Eisner (1974) at Michigan State University. Eisner interviewed fifteen preservice elementary school teachers every two weeks during the course of a term in which they were enrolled in a required mathematics course. During the interviews, students solved problems aloud and also discussed with the interviewer their perceptions about the course. Using this technique, Eisner was able to identify misconceptions students had about mathematical ideas they were taught in the course.

Among other things, he discovered that the students were very poor in solving word problems. He suggested that a good research study would be one that focuses on the thought pattern of prospective teachers while they are solving problems as well as on the features of the problem themselves which cause difficulty for the problem solver. (Eisner, 1974)

INFORMATION PROCESSING

Benjamin Kleinmuntz has edited a very interesting and useful volume titled <u>Problem Solving: Research, Method and Theory</u> (1966) which contains papers on problem solving delivered to the first annual symposium on cognition held at the Carnegie Institute of Technology in 1965. The papers were presented by well-known psychologists and they demonstrate a wide spectrum of views on problem solving.

In his epilogue Garlie Forehand (1966) suggests that most contemporary psychological studies of problem solving can be classified into four approaches. The first is also the oldest having developed from classical Gestalt psychology. The holistic Gestalt approach emphasizes an analysis of internal processes in problem solving. However, often in their efforts to avoid behaviorism, they have also avoided some of the detailed analysis of cognitive events necessary to get an accurate picture of the problem-solving processes. (1966:356-7)

The second approach to problem-solving research in psychology has been the psychometric.

> . . . The classic concern of this approach has been to understand the nature of intelligence, and its most characteristic method has been the attempt to discover and interpret additive components of general mental ability. . . . The major concern is with traits and the approach has had little to say about processes of or conditions for problem solving. The study of intellectual traits, however, must concern itself to some extent with the nature of problem solving <u>per se</u> at least in the initial (item construction) and final (factor interpretation) phases. (1966:357-8)

The third approach has been derived from the laboratory study of learning. This approach maintains that problem-solving behavior can be accounted for by examining constituent elementary relationships between stimuli and responses. Learning theorists have attempted to analyze complex processes and in doing so:

> . . . they have, in general, remained faithful to their heritage by postulating internal processes cautiously, and, when doing so, by expressing them as mediating S-R processes, obeying the same laws as elementary behavioral processes. (1966:358-9)

The fourth approach to problem-solving research done by psychologists is the information processing approach which uses computer programs to represent as accurately as possible the behavior of the human problem solver. Information processing should not be confused with "artificial intelligence" where the objective ". . . is to construct a program which will solve problems well, never mind how people solve them." (Taylor, 1966:121)

Perhaps it is the information processing approach which gives us the best of all worlds. For like the Gestalt approach it is interested in the total view of the problem-solving process. In addition, like behaviorism, it is elementistic. Finally, it can be used to demonstrate those elements of the problem-solving process that are sufficient for arriving at solutions.

As an example of this latter point, Simon and Newell (1970) have described a program called The Logic Theorist (LT) that is capable of solving certain logic problems that are difficult for humans. They conclude that:

> . . . LT was, first and foremost, a demonstration of sufficiency. The program's ability to discover proofs for theorems in logic showed that, with no more capabilities than it possessed--capabilities for reading, writing, storing, erasing, and comparing patterns--a system could perform tasks, that in humans, require thinking. To anyone with a taste for parsimony, it suggested (but, of course, did not prove) that only those capabilities, and no others, should be postulated to account for the magic of human thinking. (1970:146-7)

Thus, in summary, in many ways information processing provides the best model for the psychological study of problem-solving processes.

In observing humans solve difficult well-structured problems, researchers (for example, Simon, 1975 or Newell and Simon, 1972) have observed certain broad characteristics of the underlying human information processing system (IPS) that supports the problem-solving processes. In simple terms, they have discovered that the human information system generally operates serially, one process at a time. It seems that a problem solver always searches his problem space (a tree-like network or maze which outlines various approaches to a problem) sequentially, adding small successive bits of information to the total of what he knows about the problem and its solution.

Secondly, it has been discovered that elementary processes take tens or hundreds of milliseconds. The inputs and outputs of these processes are held in a small short-term memory with a capacity of only a few symbols. The time required for moving symbols into and out of a man's short-term memory has been found to be around two-hundred milliseconds in some cases.

In addition, a human IPS has access to an almost infinite long-term memory, but the time required to store a symbol in that memory is of the order of seconds or tens of seconds. Items stored in this long-term memory are recalled by the recognition of stimuli which serve as a sort of index of the memory. (Simon, 1975)

Everything that is known about the human IPS indicates that it meets the specifications indicated above. However, it is possible to present a more formal description. Simon constructs such a description along these lines:

1. There is a set of elements called symbols, which are capable of denoting, or pointing to, objects.

2. There are symbol structures, consisting of organizations of symbols connected by a set of relations.

3. A memory is a component of an IPS capable of storing and retaining symbol structures.

4. An information process is a process that takes symbol structures as its inputs or outputs.

5. A processor is a component of an IPS consisting of: (a) a set of elementary information processes; (b) a short-term memory that holds the input and output structures of the information processes; and (c) an interpreter that selects the order in which the information processes will be executed.

6. The symbol structures that determine for the interpreter the order in which it will execute processes are its program. (1975:4)

It is clear from this description that there is a close connection between human information processing and the way in which a modern computer works. Thus, computers can be programmed to model human IPS strategies. The computer programs that model human strategies can solve the same problems that humans solve. The output of the computer program then simulates the sequence of problem-solving efforts of the human subject.

In discussing this process, Simon states:

> It cannot be emphasized too strongly that in this application of the computer it is not being used as a super-fast "number cruncher," nor is it competing in speed or accuracy with the human subject. The computer is being used, as its very general capabilities enable it to be, as a general purpose information processor. It is programed to imitate as closely as possible the actual processes used by humans, including their foibles, and it avoids entirely taking advantage of its powerful arithmetic capabilities, which are patently unlike those of a human. If the computer programed for simulation solves a problem either more skillfully or less skillfully than do human subjects, then the program is a poor simulation--a poor theory of the human processes. (1975:5)

Besides being an extremely fascinating and exciting area of psychological research, human information processing systems investigations have importance for the present study largely because of the way in which such investigations often analyze human problem-solving protocols. Oftentimes "thinking aloud" techniques are used to discover and test information processing theories of complex human cognitive behavior.

In a monumental volume, titled Human Problem

Solving, Newell and Simon (1972) have summarized seventeen years of research regarding human information processing. The book is divided into five parts. The first part consists of definitions and technical groundwork necessary for discussing an IPS. The next three parts consist of discussion and analysis of three different task environments (the problem situation as the experimenter views it). The three task environments are a type of puzzle called cryptarithmetic, a form of elementary symbolic logic problems, and finally chess. The fifth part of the book states a theory of human problem solving from an IPS point of view.

It is impossible to give a complete description of Newell and Simon's research techniques without devoting considerable space to it. In like fashion it took them eighty-one pages to describe their theory of human problem solving. However, they maintain that the general shape of their theory can be captured by four propositions. First, "A few, and only a few, gross characteristics of the IPS are invariant over task and problem solver." (1972:788) Those which are invariant are the size, access characteristics, and read and write times for the various memories in the human IPS. Others are the serial character of the information processing and the rate at which the elementary information processes can be performed.

A second proposition states that, "These characteristics are sufficient to determine that a task environment is represented (in the IPS) as a problem space, and that problem solving takes place in a problem space." (1972:789) As mentioned earlier, a task environment is the problem situation as the experimenter sees it. This notion is contrasted with "problem space" which is the problem situation as the problem solver sees it. In short, it is the space in which the problem-solving activities take place.

Thirdly, "The structure of the task environment determines the possible structures of the problem space." (1972:789) Further, the structural information of the task environment can be defined without reference to any properties of a problem solver who might use it. (1972:824) This is of vital importance to the investigations of Newell and Simon.

A final proposition states that, "The structure of the problem space determines the possible programs that can be used for problem solving." (1972:789) It is this proposition which confronts the elusive question of individual differences.

Paige and Simon (1966) report on a study dealing with algebra word problems which differs in methodology from those presented in Human Problem Solving. Approximately sixteen years ago a Massachusetts Institute of Technology student, Daniel Bobrow wrote a computer program called STUDENT which was designed to solve a considerable range of algebra word problems. STUDENT was constructed in an investigation of artificial intelligence. Bobrow did not intend that the program simulate human problem solving (information processing). Nevertheless, since the processes used in the program do parallel human processes in important respects, Paige and Simon thought it might be productive to use the STUDENT program as a theory of human problem solving and then in turn test that theory by observing the problem-solving protocols of humans as they solve algebra word problems and comparing these protocols with those used by STUDENT.

Among other things, Paige and Simon found that contradictory problems that represent physically impossible situations proved to be useful tools for detecting the relative uses of direct (where the words of the problem are transformed directly into an equation) and auxiliary (where the subject tends to use a mental or physical picture or some other information known about the situation defined in the problem) cues. One such contradictory problem states:

> The number of quarters a man has is seven times the number of dimes he has. The value of the dimes exceeds the value of the quarters by two dollars and fifty cents. How many has he of each coin. (1966:84)

Paige and Simon found that in attacking this and similar problems subjects would either: (1) set up an equation for the problem as it is stated literally (for example, 10D=25(7D)+$2.50); (2) set up an equation for a related, physically possible, situation (for example, 10D=25(7D)-$2.50); or (3) recognize the physical impossibility of the situation. They found that if their high-school and college-age subjects used primarily direct cues, they would make the first response; if primarily physical cues, the second; if he attended to both, the problem description will be redundant, and he may detect the contradiction. (1966:117)

Discussing this experiment nine years later Simon (1975) states:

> Of course there is no contradiction in the problem statement. There is a contradiction only if we add to the statement some semantic knowledge that an American student might be expected to have stored in his long-term memory: a quarter is worth more than a dime, and the number of both quarters and dimes must be non-negative integers. The difference in the processes of the three groups of students now becomes rather clear. The students in the first group proceeded purely syntactically (except for recovering from memory the fact that dime equals ten cents and a quarter, twenty-five cents). The students in the second group used their semantic knowledge to infer that the total value of the quarters must be greater than the total value of the dimes, and that therefore the $2.50 must be added to the latter or subtracted from the former. Evidently they never checked this inference against the syntactical detail of the sentence (after all, something was to be added to something), but used the semantic knowledge to construct the "correct" equation. The students of the third group processed the input sentences both syntactically and semantically, thereby discovering the "contradiction." (1975:15-6)

So this problem illustrates alternative ways in which a given problem may be processed. Obviously the richer the semantic context of the problem becomes, the greater the number of alternatives.

It is difficult to survey the thousands of pages reporting what is known about information processing and to derive straight-forward suggestions for improving instruction in problem solving. Fortunately, Simon has attempted to do so and four of eight suggestions are especially applicable to the classroom teacher and are included here. First:

> . . . An important component of problem-solving skill lies in being able to recognize salient problem features rapidly, and to associate with those features promising solution steps. Much current instruction probably gives inadequate attention to explicit training of these perceptual skills and the kind of understanding that is associated with them. (1975:21-2)

Secondly:

> . . . It is often possible to substitute syntactic for semantic processing and vice versa. Awareness of these alternatives, and skill in employing both of them can enhance accuracy of understanding by exploiting redundancies in the problem information. (1975:21-2)

Thirdly:

> . . . understanding processes include the processes of constructing representations of problem situations. Most problems are capable of being represented in a variety of ways, and problem difficulty may be greatly affected by the representation chosen. The skills of searching for effective problem representations are probably learnable and teachable skills, but they are not now generally taught in a systematic way. (1975:21-2)

And finally:

> . . . It is becoming increasingly possible to determine in detail what is involved in understanding any specific subject matter area to the point of writing computer programs that specify what a person who understands knows, what processes he has available for solving problems and acquiring new knowledge in that domain, and how his knowledge is organized in memory. (1975:22)

SOVIET STUDIES

Soviet research in problem solving is easily contrasted with that traditionally done in the United States. Whereas most of the problem-solving research in the United States has been concerned with the product, in the Soviet Union researchers have attempted to better understand the process aspect of human problem solving. Their research methods tend to follow the pattern of devising an instructional method based upon preliminary observations, trying the method in intact classrooms, and studying the effects of instruction by careful analysis of records of student behavior. Often these records take the form of tape-recorded interviews conducted on a one-to-one basis. Thus, Soviet studies prove to be a good resource for American researchers who are attempting to observe and analyze the process aspect of human problem solving and who are attempting to develop courses of instruction which will enhance the problems-solving ability of students.

A ready source of reports of Soviet research has been provided by the School Mathematics Study Group and the Survey of Recent East European Mathematical Literature which have jointly produced a fourteen volume work titled Soviet Studies in the Psychology of Learning and Teaching Mathematics, edited by Jeremy Kilpatrick, Izaak Wirszup, Edward Begle, and James Wilson. These volumes contain both expository articles on problem solving and the psychology of learning and reports on Soviet studies dealing with problem solving.

One such study conducted by E. N. Ivanitsyna (1970) serves as an example here both because of the nature of the questions the study was to answer and because of the clinical technique employed to gather data to arrive at answers. Ivanitsyna suggested that, ". . . the role of knowledge in the thinking process has still not been studied." (1970:149) Thus, his purpose was to determine what knowledge is recalled when geometry problems are being solved. This knowledge is then classified either (1) as knowledge of problem elements, or, (2) as knowledge of appropriate operations plus the ability to carry them out.

An example of one problem used reads as follows: "The diagonals of a trapezoid divide it into four triangles. Prove that the triangles on the lateral sides are equivalent (i.e., have the same area)." (1970:149) Knowledge of problem elements includes knowledge about trapezoids, diagonals of trapezoids, relations between triangles (for example, congruent, equivalent, similar), and methods for proving triangles equivalent. Knowledge of appropriate operations includes the ability to analyze the problem, to isolate the problem's question, to correlate one's prior knowledge with the problem's condition and question and on this basis to select or discard the knowledge brought to mind, to determine the problem type, and so forth. (1970:149-50)

Clearly the questions that are posed in the study are neither well-defined nor precise. In turn the analysis lacks objectivity and is to some extent based on intuition. Nevertheless, insights into the relationship between "knowledge" and success in solving geometry problems are gained that would probably not be achieved if a more carefully controlled factor analysis were performed.

As stated earlier, a second unique feature of this study was the clinical techniques employed to gather data. Ivanitsyna presented a series of geometry problems such as the one given above to eighteen subjects. Students were recorded as they solved the problem thinking aloud. Then transcripts were made of the interviews and analyzed for the knowledge, both necessary and unnecessary, that was recalled for solving the given problem.

In analyzing his study Ivanitsyna reports that during the whole course of the solutions of problems,

> . . . there is continuous actualization of particular knowledge, which helps the entire thought process; this knowledge may be correct, incorrect, necessary, or unnecessary for the solution. (1970:153)

Thus the study provides some evidence that even in the initial stages of solution a subject's progress and direction are a function of the interplay between the knowledge he brings to the problem and his ability to analyze the problem. (1970:153)

Just as Allen Newell and Herbert Simon summarized seventeen years of their research on human information processing in <u>Human Problem Solving</u>, the Russian psychologist V. A. Krutetskii summarized eleven years of his research in the volume <u>The Psychology of Mathematical Abilities in Schoolchildren</u>. The purpose of his program of research, stated generally, was to explore the nature and structure of mathematical abilities.

Whereas Newell and Simon basically used adults as subjects, Krutetskii used students from the ages of six to seventeen who were classified according to their ability level as being either very capable, relatively capable, average, or relatively incapable in mathematics. The clinical methods used for obtaining data regarding the nearly 200 subjects in his study included individual interviews by Krutetskii and his associates as subjects solved series of problems aloud, home visits, and classroom observations. Several longitudinal studies were done with these 200 participants. In addition he collected data on more than 1,000 pupils in Moscow secondary schools to compare their progress in mathematics with their progress in other school subjects. He also conducted longitudinal investigations of nine children who were especially gifted in mathematics but who were not part of the main study.

In attempting to establish a theoretical framework for his study, Krutetskii developed a hypothetical scheme of the components of the structure of mathematical abilities. The components include:

1. An ability to formalize mathematical material, to isolate form from content, to abstract oneself from concrete numerical relationships and spatial forms, and to operate with formal structure--with structures of relationships and connections.

2. An ability to generalize mathematical material, to detect what is of chief importance, abstracting oneself from the irrelevant, and to see what is common in what is externally different.

3. An ability to operate with numerals and other symbols.

4. An ability for sequential, properly segmented logical reasoning, . . . which is related to the need for substantiation, and deductions. . . .

5. An ability to shorten the reasoning process, to think in curtailed structures.

6. An ability to reverse a mental process (to transfer from a direct to a reverse train of thought).

7. Flexibility of thought--an ability to switch from one mental operation to another; freedom from the binding influence of the commonplace and the hackneyed. This characteristic of thinking is important for the creative work of a mathematician.

8. A mathematical memory. It can be assumed that its characteristics also arise from the specific features of the mathematical sciences, that this is a memory for generalizations, formalized structures, and logical schemes.

9. An ability for spatial concepts, which is directly related to the presence of a branch of mathematics such as geometry (especially the geometry of space). (1976:87-8)

One of the most important features of Krutetskii's work is his development of twenty-six series of problems used in the main study. Each series is a set of problems of the same type and should be a ready source of material for researchers who are examining some of the same components of mathematical ability. Just a few of the twenty-six series are: (1) problems with incomplete information (whose purpose is to evaluate a subject's perception of relations and concrete facts in a problem); (2) problems on proof (whose purpose is to determine reasoning methods, use of logic, and curtailment of reasoning processes); (3) problems with several solutions (which enables one to examine flexibility of thinking); (4) mathematical sophisms (which indicate logic in reasoning); (5) problems with terms that are hard to remember (whose purpose is to evaluate mathematical memory and generalization); (6) problems with varying degrees of visuality in their solution; and (7) direct and reverse problems. Examples of two problems in this last mentioned series are:

> Direct: The distance between cities A and B is x km. Two trains departed, traveling toward each other. One went at a speed of a km per hour, the other at b km per hour. How long did it take before they met? Reverse: The distance between two points is y km. Two trains departed from them, going toward each other, and they met in n hours. One train traveled at a speed of 40 km per hour. What was the speed of the second train? (1976:145)

Krutetskii has developed an outline of the structure of mathematical ability which shows the interrelationships of the components which he discovered to be integral to quality problem solving. This outline acts as a good summary of his findings from his main study.

Of the twenty-six series of problems used in the main study, the first three (problems with an unstated question, problems with incomplete information, and problems with surplus information) were used to form the first stage of Krutetskii's outline of mental activity in solving a mathematical problem. That stage involves "obtaining mathematical information." He has determined that the more capable mathematics students are better able to grasp the essence of a problem and to gather the necessary information for solving it.

The second stage involves "processing mathematical information." Abilities or components included in this stage are the ability to think in mathematical symbols; the ability to manipulate these in a logical fashion; an aptitude for logical thought in the sphere of quantitative and spatial relations, number, and letter symbols; the ability to make rapid and broad mathematical generalizations; the ability to curtail the process of mathematical thinking; flexibility in mental processes; the striving for clarity and simplicity of solutions; and the ability for rapid and free switching from a direct to a reverse train of thought.

The third stage of mental activity involves the "retention of mathematical information." The better problem solvers have a memory for mathematical relationships, type characteristics, schemes of arguments and proofs, methods of problem solving, and principles of approach.

The fourth stage of his outline is simply a component of mathematical ability which Krutetskii calls the "mathematical cast of mind." In short, a mathematical cast of mind is a tendency to interpret the world mathematically. (1970:350-1) As with other Soviet research, the study just reported lacks objectivity and to some extent is based on intuition. However, it still is of considerable worth to the mathematics education community. For as Kilpatrick and Wirszup say in the introduction to the book:

> . . . It is no exaggeration to say that this work could have the same sort of impact on mathematics education that Piaget's work has had. Just as Piaget's tasks have been adapted and used by teachers and researchers alike, so Krutetskii's tasks--which are more closely related to the school mathematics curriculum--could be adopted and used in the same fashion. Just as Piaget's notions of intellectual growth have made mathematics educators aware of differences in children's thinking at various ages, so Krutetskii's notions on the structure of mathematical abilities could make them aware of different components of ability and how they might function together. And just as Piaget has broadened our conception of what are appropriate research techniques, so Krutetskii may broaden this conception still further. (1976:xv)

STUDIES OF HEURISTICS

Studies evaluating the effect of instruction in heuristics on problem-solving ability have been conducted with students of all ages. For examples of studies using college-age subjects, see Goldberg (1974) or Leggette (1973). Mendoza (1976), Vos (1976), and Basseler, Beers, and Richardson (1975) have conducted such studies using high-school-age youth; studies using elementary-school children have been conducted by Shields (1976) and Smith and

Padilla (1975).

The research studies to be discussed in this section are of particular importance for the present investigation because they provide a frame of reference for the research design and for many of the techniques of analysis used here. Jeremy Kilpatrick's doctoral dissertation from Stanford University is of notable importance because in it is found a unique research instrument and technique designed and evaluated by Kilpatrick. This research technique has been used by Kilpatrick and recently by other researchers to achieve significant information concerning problem-solving processes. In examining these studies one sees the Soviet influence in the clinical methods used and in the breadth of the studies.

In his dissertation, Kilpatrick (1967) developed a unique instrument which when used in an interview setting with students allows for a careful analysis of problem-solving behavior. The instrument, which includes both heuristic and non-heuristic categories, can be used to code protocols of subjects solving mathematical word problems aloud. Once successful in developing this coding-system his aims were (1) to investigate the relationships of variables derived from the coding-system categories to one another, and (2) to determine whether coding system variables could be used to group subjects into meaningful clusters, where "meaning" is exhibited through associations with independent behavioral measures. In line with this second aim several more specific questions were asked, only one of which has great importance for the development of the present study. It asked which dimensions from the coding system seem most promising for grouping subjects into meaningful clusters of problem-solving behavior? (1967:31) Kilpatrick attempted to answer these questions by the use of a creative blend of clinical and statistical techniques.

Kilpatrick interviewed fifty-six above-average students who had just completed the eighth grade. Each of these students had been participating for two years in the National Logitudinal Study of Mathematical Abilities (NLSMA), a five-year study supported by the National Science Foundation and administered through the School Mathematics Study Group. Data from these NLSMA tests were used to compare a subject's behavior in the problem-solving interview with his performance on selected ability, achievement, and attitude measures from the previous two years. The single interview with each child had as its main focus a problem-solving inventory although a test of impulsivity, two spatial tests, two numerical tests, and a questionnaire were also administered individually to each student during the interview.

The problem-solving inventory consisted of twelve verbal mathematics problems which were chosen because

> . . . 1) they are relatively difficult to solve and yet do not require special tricks or techniques; 2) most of them can be solved in more than one way; 3) they span a diversity of topics and

> "problem types"; and 4) several can be solved with the aid of a geometric representation. (1967:26)

In the tape recorded interview the subjects were asked to solve the problem "out loud." The interviewer marked a "checklist" during the course of the interview which provided a record of certain observed behavior. Later the tapes of the interview were analyzed using the coding system and a "process-sequence code" was developed to describe the problem-solving process of each subject on each given problem. Quantative data were derived from the check list and from the process-sequence codes and analyzed for each subject.

Consistent with one of his main objectives, Kilpatrick was able to group students into meaningful clusters of problem-solving behavior with respect to certain coding system variables. The variables which seemed to lend themselves best to this grouping were "deduction" and "trial and error." On the basis of these he was able to identify three disjoint groups of students. Group A consisted of subjects characterized by a tendency to begin problems with trial and error plus a related tendency to use trial and error frequently. Group C subjects were characterized by infrequent use of trial and error and a tendency to begin with some other process. Group B subjects tended to be somewhere between Group A and C with respect to the two processes. A fourth group, Group E, consisted of subjects who tended to set up equations to solve problems. It had a non-empty intersection with Group B and with Group C. Its intersection with Group A was empty.

Two of the more pertinent results of this study were:

1. On the average, the subjects in Group C spent the least time on the problem-solving inventory and got the lowest scores. However, students who set up equations (Group E) and were also in Group C tended to get higher scores for correct solutions. Thus, infrequent use of trial and error appears to be an incapacity unless the student employes the use of an equation.

2. It seems that Groups A, B, and C maintained the same order on a number of coding variables and that Group E fell outside the order, resembling each of them on some variables. For example, Group E members were like Group A in making a moderate number of structural errors, having less difficulty with processes, and achieving high scores; Group B in making a moderate number of requests for assistance, admitting confusion more often, and making a moderate number of checks; and Group C in drawing more figures, using successive approximation less often, rereading the problem more times, and stopping more often without a solution. (1967:85-6)

Considering these and other findings of his study, Kilpatrick concludes by remarking that:

> . . . Our findings with regard to trial and error need further thought and exploration. What are the causes of incapacity or reluctance to choose trial values in solving a problem? What cues in a problem cause certain subjects to resort to trial and error immediately? . . . How do superior

> students use it in solving mathematical problems? Can low ability students be taught to use it too? (1967:168)

It is hoped that the present study will provide partial answers to some of these questions as applied to preservice teachers.

John Lucas, in his doctoral dissertation at the University of Wisconsin, Madison, conducted an exploratory study to investigate heuristic usage and problem-solving performance and to analyze the influence of heuristic-oriented teaching on a group of first-year university calculus students (1972, 1974). His study is an important extension of the work started by Kilpatrick. Kilpatrick was basically looking for problem-solving modes and examining heuristic usage of students after eight years of schooling. His subjects were chosen from different schools and had not been subjected to an experimental treatment. In contrast, Lucas, while using many of the same research techniques as Kilpatrick, taught two intact sections of freshman calculus and provided one with an experimental treatment. While both groups received inquiry-style instruction, the control group was given only expository treatment of problem solutions with hardly any attention to heuristic strategies. In the inquiry approach of the experimental group heuristic strategies were emphasized and problems were discussed for the sake of learning problem solving rather than simply to reinforce concepts as was the case in the control group. Also, twelve papers were distributed to the experimental class which demonstrated various heuristics and how these are used to aid in solving problems. The control class did not receive these. Finally, the problems handed in by the experimental group were graded in such a way as to reward heuristic usage.

Lucas used a four-group design, similar to the Solomon Four-Group Design of Campbell and Stanley (1963), so as to control for pretest by treatment interactions. However, he did not have the randomization required in the Solomon Four-Group Design. His pretest and posttest consisted of a problem-solving inventory in which students were asked to solve the seven problems out loud. The interviews, approximately two hours in length, were taped and then analyzed using a checklist and coding system similar to that developed by Kilpatrick.

Some of the pertinent findings of Lucas follow. The researcher was able to code reliabily thirty-five behavior variables and analyze them using chi-square tests for main effects due to treatment. Examples of some of these variables are uses mnemonic notation, uses method of related problem, check of manipulations, computation error, score (approach, plan, result, total), synthetic deduction, analytic deduction, trial and error, etc. He found that first:

> Heuristic-oriented instruction . . . has a positive effect on:
>
> (1) the kind of notation used,
>
> (2) applying the method of a related problem,
>
> (3) applying the result of a related problem,
>
> (4) reasoning by analysis,
>
> (5) organizing problem data (separating: summarizing),
>
> (6) certain performance measures: effectiveness of approach, devising workable plans, obtaining accurate results,
>
> (7) certain measures of difficulty: decrease in frequency of reading the problem and in amount of hesitation,
>
> (8) time spent looking back. (1972:427-8)

Also:

> Heuristic-oriented instruction . . . has no effect on:
>
> (1) drawing or modifying diagrams,
>
> (2) the overall frequency of checking,
>
> (3) the kind of checking, except possibly for checking to see if a result is reasonable or if it satisfies the conditions of the problem,
>
> (4) the number or kinds of executive errors committed,
>
> (5) the number of structural errors committed,
>
> (6) reasoning by synthesis,
>
> (7) productivity of relationships, equations, or algorithmic processes,
>
> (8) using trial and error as a solution method,
>
> (9) stopping without solution. (1972:428)

Secondly, inductive reasoning, varying the process of solution, and varying the problem were observed least often in the problem-solving protocols of college students. Thirdly,

> . . . The data for treatment effects on all performance scores i.e., time score, frequency of errors, and amount of difficulty provide clear evidence--at least on the basis of an existence argument--that heuristic instruction can result in improved problem-solving performance. (1972:370)

In his study, Lucas admits that all of his conclusions are tentative. This follows since (1) subjects were not randomly assigned to treatment; (2) sample size was too small for some of the statistical tests employed; (3) teacher effect was not controlled; and finally, as with Kilpatrick's study (1967), (4) the outcomes are highly dependent on the types of problems used in the pretest and posttest.

A serious difficulty with the clinical technique used by both Kilpatrick and Lucas in their studies is the great amount of time needed to analyze the audio tapes of the problem-solving inventories for each subject. Zalewski (1974) thought that if he could construct a written test such that performance on it would be highly correlated with performance measures of a corresponding interview test, then he would have the concurrent validity necessary to establish the feasibility of using the written test as a substitute for the complex interview and coding procedure. (1974:49) Thus, he constructed two types of non-routine problem-solving tests for seventh graders. One test involved the use of six mathematical problems which were to be solved by students in a taped interview setting. Their problem-solving protocols were then evaluated using process-sequence codes similar to those employed by Kilpatrick and Lucas. A paper and pencil problem-solving test was also constructed for comparison. Unfortunately, his written test did not provide the high correlation he had hoped to achieve.

Perhaps it is impossible to attain Zalewski's goal. Nevertheless, the coding techniques used in the present study still provide a useful, though inefficient, way of analyzing problem-solving behavior.

Norman Webb (1975) has done a follow-up of the studies of Kilpatrick and Lucas in his dissertation at Stanford University. He examined the problem-solving processes of thirty-seven students who were taking their second year of high school algebra. His research techniques were again clinical in nature. Like Kilpatrick but unlike Lucas, he did not provide a group with instruction in heuristics. Rather in examining problem-solving behavior at exactly one point in time he attempted to answer these three questions:

> 1. What is the separate and cummulative relationship of the pretest (cognitive and affective) variables and the process variables to the problem-solving scores of second year high school algebra students? 2. Of the heuristic strategies considered, which are problem-specific and which are general strategies? 3. What problem-solving modes can be identified as being used by groups of students? (1975:16)

Thirty-seven second year high school algebra students participated in Webb's study. They took a series of pretests which measured such things as mathematical achievement and different cognitive abilities such as verbal ability, spatial ability, reasoning ability and so forth. Then in an interview setting each took an eight problem verbal problem-solving test called the problem-solving inventory. The students thought aloud as they solved the problems and were tape-recorded. Coding and analysis techniques similar to those of Kilpatrick and Lucas were used.

Using the total score for problems correct on the problem-solving inventory as the dependent variable, Webb found that 50 percent of the variance of this score was accounted for by mathematics achievement as measured by the pretests. Only 13 percent of the variance was accounted for by all the heuristic strategies combined. Of all of these "pictorial representation" contributed 8 percent of the total 13 percent. Thus mathematics achievement had the highest relation to problem-solving ability.

Secondly, Webb not surprisingly found that heuristic strategies were problem-specific. Students who used a wide variety of strategies, however, were on the average better problem solvers.

Thirdly, three modes of problem-solving behavior were identified when clustering was done on heuristic strategies, and when clustering was done on the process-sequence variables. Some modes that resulted from this analysis are (1) those who used a medium amount of trial and error and equations also had a low frequency of structural errors; (2) those with high use of equations tended to be low in the use of trial and error; and (3) those high in the use of trial and error tended to be low in the use of equations.

Finally, Mary Grace Kantowski (1977) conducted a longitudinal exploratory study with the purpose of uncovering information about and gaining insight into the processes involved in solving complex, non-routine problems. (1977:164)

Eight high-ability ninth-grade algebra students participated in the study which consisted of four phases. The first was simply a pretest where subjects were asked to think aloud as they solved eight non-routine problems. Verbal protocols were recorded and analyzed, again using techniques similar to those first used by Kilpatrick. (1967) The second phase was a readiness instruction phase which lasted four weeks with three lessons per week. Here subjects received instruction in using heuristics in solving problems. A test followed this phase.

The third phase involved instruction in geometry using heuristic instructional techniques and lasted four months. The fourth phase was a posttest consisting of geometry and verbal problems and "prerequisite knowledge" tests which considered facts and concepts necessary for solution of the posttest problems. (1977:164-5)

Kantowski defined a "process-product score" which was assigned to each problem. This score was calculated for each problem by giving one point for each of the following: suggesting a plan of solution, persistence (a student lacked persistence if he stopped without a solution or if he decides upon a solution on an irrelevant basis), looking back, absence of structural errors, absence of executive errors, absence of superfluous deductions, and correctness of result. (1977:165)

Kantowsi's analysis is lengthy and covers a broad spectrum of results. However, some of the pertinent findings are: (1) students with higher process-product scores tended to use heuristics more consistently; (2) the use of heuristics increased as problem-solving ability developed; (3) regular patterns of analysis and synthesis are observed in successful problem solvers; (4) the failure to use a heuristic often led to analyses that were not germane to the solution of the problem; (5) "looking back" did not increase as problem-solving ability developed. (1977:165-9)

ATTITUDINAL STUDIES

In 1969 Lewis Aiken (1969) published a comprehensive review of research on attitudes toward mathematics in which he suggests that before 1960 very little research was done in this area. However, he observes that from 1960 to 1969 the number of dissertations and published articles dealing with attitudes toward mathematics has increased geometrically. (1969:1) His review covers that span of time.

In another comprehensive review conducted by Ronald Trimmer in 1974 and reported on in an unpublished document of the United States Department of Health, Education and Welfare, it was observed that no research was found ". . . directly relating *problem solving and attitude toward mathematics*" [italics not in the original]. (1974:9) His review considered the relationship of problem solving to attitude and other psychological variables.

Thus it seems that although there has been an abundance of study on attitudes and other related psychological variables and their relationship to mathematics achievement, we have learned very little about the relationship of attitudes to problem-solving ability. This observation provides some of the motivation for the parts of the present study that deal with attitude and its connection with problem-solving ability.

There have been studies conducted in which attempts are made to improve attitudes. One such study was conducted by Bonnie Litwiller (1970) and deals with the same sort of population as the present study.

Litwiller wanted to determine if attitudes of preservice teachers could be improved. The design of her study was simple. A control group and an experimental group were used. The control group consisted of two sections (fifty students) of a general mathematics course for preservice teachers at Indiana University. The experimental group consisted of four sections (ninety-five students) of the same course.

Each day students in the experimental group received some sort of enrichment experience. Often times this was a problem not necessarily related to the concepts being studied at the time. The control group had no such experiences.

Dutton's Attitude Scale was used for a pretest and a posttest. A uniform final exam provided a measure of mathematical achievement.

Litwiller determined that there was a rather dramatic improvement in the attitudes of the students in the experimental sections and that on the attitude posttest there was a significant difference (at the .001 level) between the means of the experimental group and the control group. She also determined that there was no significant difference between the two groups on mathematical achievement in the course.

One must look at these conclusions however with some suspicion since she did not control for teacher effect. She reports that four people taught the six sections but does not describe how they were assigned or who they were.

SUMMARY

This survey has shown a vast number of research approaches to the problem of studying problem-solving processes. One cannot help but be overwhelmed by the complexity of the task that lies before the mathematics education community as it attempts to better understand the problem-solving process and as it attempts to find ways of improving instruction in problem solving.

In spite of the great number of studies done that have examined the various aspects of problem solving, there are certain areas of concern that have not been addressed. In particular, in clinical studies where emphasis is on the process of problem solving no one has (1) examined the problem-solving styles of preservice teachers and ways of improving their problem-solving ability, (2) examined the interrelationship of attitudes with problem-solving processes, and (3) paid much attention to the processes employed by low-ability students as they solve non-routine mathematics problems or to ways of improving their problem-solving ability. The present study attempts to examine parts of these heretofore unaddressed questions.

Chapter 3

METHOD OF INVESTIGATION

INTRODUCTION

The primary purpose of this study was to design, implement, and evaluate a required course in the structure of elementary school mathematics for preservice teachers. The unifying thread of the course was a problem-solving component which uses problems to motivate new concepts, provides instruction in the use of heuristics in solving problems, and provides students with practice in solving non-routine problems. Secondly, within this context the study was to (1) examine what heuristics preservice elementary teachers tend to use in solving non-routine mathematics problems before taking any required mathematics; (2) examine the effects of such a course on the heuristic usage of students and on their problem-solving ability; (3) determine the effects such a course has on student attitude toward problem solving and mathematics; and (4) identify features of verbal non-routine problems that are especially troublesome for preservice teachers.

The method of investigation involved the use of one section of a six-quarter-hour course in the structure of elementary school mathematics containing twenty preservice elementary school teachers. This section was created from two intact sections during the spring quarter of 1975-76 academic year at Concordia Teachers College, River Forest, Illinois. Students were selected so as to have two ability-level groups with the same number of subjects in each group. The first week of the course the subjects participated in individual interviews with the investigator in which they were asked to think aloud as they solved twelve non-routine mathematics problems. The interviews were tape recorded for later analysis.

The course of instruction covered the usual topics in such a first course for preservice teachers and had an extensive laboratory component. In addition, the course had a problem-solving component which was integrated into all units of the course except those covered the first and last week of the ten-week quarter when pretest and posttest interviews were being conducted. The second full week of the course was totally devoted to problem-solving activities which emphasized the use of certain heuristics. Other strategies were discussed as the course proceeded. In total, twenty-three non-routine problems were solved in class discussions and thirty-four were assigned for students to solve as homework. Most of these were also discussed when they were collected. During the last week of the course subjects were again tape recorded as they worked through a twelve problem posttest problem-solving inventory.

Two basic instruments were used to analyze the subjects' problem-solving behavior. A checklist was completed on each problem for each subject during the course of the interviews. The items on the checklist then became variables for the analysis. These items or variables were grouped into the following seven categories: time, score, approach, production, looking back, comments, and executive errors.

The second instrument was a sequence of symbols, called a process-sequence code, which gave a sequential picture of the processes and heuristics the subjects used as they solved each problem. The audio tapes were used to write one process-sequence code for each problem worked by the subjects. The frequency of the usage of given processes and heuristics gave rise to additional variables called process-sequence variables. The attitudinal component of the study centered on the use of Aiken's "Revised Mathematics Attitude Scale" (1970, 1972) and various self-report questionnaires developed by the investigator.

The research design was of necessity clinical in nature so as to analyze the "process" of problem solving as well as the "product" of problem solving. Thus, informal techniques were used in the analysis of the data derived from the study.

The next section of this chapter describes the pilot study that was conducted to determine which non-routine mathematics problems would be acceptable for use in the testing instruments of the study as well as to evalute the coding system which was to be used to describe problem-solving behavior and to provide the investigator with practice in using the system. Section three describes that coding system. Section four provides background concerning the setting in which the study was done. The fifth section describes in detail the students who participated in the study.

Section six addresses the task of selecting problems for use in the pretest and posttest interviews and section seven details the procedures used in those interviews. The interviews were used to gain information about the processes the subjects used in solving problems but could not be used to measure student attitude toward problem solving and mathematics. The instruments used to measure these are described in section eight, "Attitudinal Instruments." Finally, the last section describes the content of the course and the techniques of instruction.

PILOT STUDY

A pilot study was conducted to accomplish three things. First, it was necessary to determine what problems would be appropriate for inclusion on the pretest problem-solving inventory and the posttest problem-solving inventory. Second, the checklist variables of the coding form and the process-sequence variables which were to be used to describe subjects' problem-solving behavior needed to be devised. These were modifications of the forms and variables used by Kilpatrick (1967) and Lucas (1972, 1974). Third, it provided the investigator with the opportunity to develop skill in using the coding forms and in conducting the interviews.

In deciding which problems should be included on the tests it would have been convenient and instructive to have used verbatim the tests employed in the studies of Kilpatrick or Lucas, studies which have served as a theoretical basis for this study. However, each of these studies was conducted with subjects whose levels of mathematical development were different from the levels of the subjects of this study. The subjects in Kilpatrick's study were gifted eighth graders and those in Lucas' study were college freshmen enrolled in Calculus. Also many of the problems in the study done by Lucas were course specific. Nevertheless, it was possible to include several of the same problems used by Kilpatrick in the pilot study tests. A report on the source of all problems, a report on the research previously conducted on certain problems, and a rationale for including the problems is included in a later section of this chapter which deals specifically with the selection of problems for the two problem-solving inventories.

In the pilot study, ten problems were initially selected for the pretest and the posttest. Those in the pretest were:

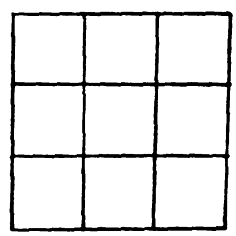
1. A two-digit number is added to a number formed by reversing its digits. What is the largest number that is always a divisor or factor of this sum?

2. What is the smallest number having 4, 6, 9, and 15 as divisors or factors?

3. Is 3688 a perfect square?

4. What is the remainder when 2^{89} is divided by 3?

5. Fill in the nine small boxes below with the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 so that the sums of the numbers in the 3 rows, the 3 columns, and the 2 diagonals are all the same.



6. A wooden cube is painted black on all faces. It is then cut into 27 equal smaller cubes. How many of the <u>smaller</u> cubes are found to be painted on a) three faces, b) two faces, c) one face, and d) no face?

7. A board is sawed into two pieces. One piece is two-thirds as long as the whole board and is exceeded in length by the second piece by 4 feet. How long was the board before it was cut?

8. If a brick balances evenly with three-quarters of a pound and three-quarters of a brick, what is the weight of a whole brick?

9. A barrel of honey weighs 50 pounds. The same barrel with kerosene in its weighs 35 pounds. If honey is twice as heavy as kerosene, how much does the empty barrel weigh?

10. A carpenter makes only three-legged stools and four-legged tables. One day when he looked at his day's output he counted 31 legs. How many stools and how many tables did he complete that day?

Those problems initially selected for the posttest were:

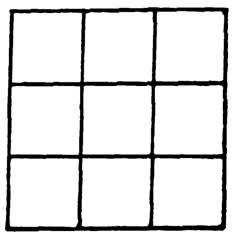
1. Take any three digit number (e.g., 638), reverse the digits, and find the non-negative difference between these two numbers (e.g., 836 - 638 = 198). What is the largest number that is always a divisor or factor of this difference?

2. What is the smallest number having 9, 12, 22, and 24 as divisors or factors?

3. Millionnaire Jones wants to build a large patio behind his mansion. He has 692,482 square tiles (1 ft. x 1 ft.) to use for the patio. Can he construct a <u>square</u> patio which will use all of the tiles?

4. What is the remainder when 3^{197} is divided by 7?

5. Fill in the nine small boxes below with the digits 5, 6, 7, 8, 9, 10, 11, 12, 13 so that the sums of the numbers in the 3 rows, the 3 columns, and the 2 diagonals are all the same.



6. Three people play a game in which one person loses and two people win each game. The one who loses must double the amount of money that each of the other two players has at that time. The three players agree to play three games. At the end of the three games, each player has lost one game and each person has $8. What was the original stake of each player?

7. A man has 7 times as many quarters as he has dimes. The value of the dimes exceeds the value of the quarters by $2.50. How many has he of each coin?

8. If a chicken and a half lay an egg and a half in a day and a half, then how many eggs do three chickens lay in one day?

9. An airline passenger fell asleep when he had traveled halfway. When he awoke, the distance remaining to go was half the distance he had traveled while asleep. For what part of the way did he sleep?

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10. A farmer has hens and rabbits. These animals have 50 heads and 140 feet. How many hens and how many rabbits has the farmer?

The participants in the pilot study were seven unpaid volunteers enrolled in the investigator's winter-quarter section of the course, Structure of Elementary Mathematics, offered at Concordia Teachers College, River Forest, Illinois, during the 1975-76 academic year. The volunteers tended to be above average in mathematical ability. The interviews were scheduled over a period of three weeks from January 29, 1976, to February 20, 1976. This scheduling allowed time for the investigator to code each volunteer's problem-solving protocol and make appropriate improvements both in the tests and in the coding scheme between interviews.

Each volunteer agreed to meet with the investigator for a two-hour period of time in a room specifically arranged to accommodate the interviews. Most of the interviews lasted considerably less than two hours. The students were asked to think aloud as they solved the word problems presented to them in a booklet at the time of the interview. Their protocols were tape recorded. The interview procedure used was identical to that used in the main study and is described in more detail in a later section of this chapter.

After the first three interviews, in each of which the students worked through the pretest, it was decided to increase the length of the tests to twelve problems. It seemed most subjects would still be able to complete the test in two hours or less and the additional problems would provide the investigator with more data for analysis. Also, problems five and six of the pretest and problem five of the posttest were deleted. It was determined that if in problem five of the pretest and posttest a subject did not have any insights to streamline his solution, then he would resort to random trial and error for determining a solution and, depending on his persistence may spend an inordinate amount of time on those problems. Problem six of the pretest was deleted because it proved to be too simple and thus resulted in little data of interest when protocols of students solving the problem were analyzed. In addition, problem eight of the pretest was rewritten to promote clarity.

The subjects participating in the next two interviews took the revised pretest. Editorial changes in the pretest and some minor changes in the coding forms resulted.

The last two volunteers in the pilot study took the posttest that had been revised to include twelve problems. Minor editorial changes were suggested by the analysis of those interviews.

Appendix A (Pretest and Posttest in Main Study) contains the pretest and posttest as they were developed for use in the main study. The problems in the tests are grouped according to topic when possible (for example, number theory, or geometry) and are ordered the same in both tests so as to allow for comparison of the two tests. When the problem-solving test booklets were constructed for use in the main study, the problems were put in random order. The pretest booklet is found in Appendix B (Pretest Problem-Solving Inventory) and the posttest booklet is found in Appendix C (Posttest Problem-Solving Inventory).

The second major purpose of the pilot study was to revise the checklist variables of the coding form and the process-sequence variables that the investigator constructed based on the studies of Kilpatrick and Lucas. There were only minor changes made in these and they can be described quite simply by examining Appendix D (Checklist Variables and Coding Form for Problem-Solving Protocols) and Appendix F (Process-Sequence Coding Variables). In the checklist variables, two categories were added that were not in the original checklist. Under "Looking Back," "Checks that all information is used" and "Attempts to find other solutions" were not in the original checklist. In the process-sequence variables only "N-nonsense process or behavior not classifiable" was added to the original listing.

Finally, the pilot study was to provide the investigator with the opportunity to develop skill in using the coding forms and in conducting the interviews. It was felt that the pilot study achieved both of these goals since first of all, seventy-eight separate problem-solving protocols were coded. Also, the interviews themselves were quite simple in nature and seven different interviews were sufficient to make the investigator feel comfortable with the techniques employed.

THE CODING SYSTEM

The coding system employed in this study centered on two parts, the checklist (see Appendix D) and the process-sequence codes (see Appendix F). During the course of the interview the investigator filled out one checklist coding form (see Appendix D) for every problem attempted by the subject. This procedure included, (1) recording a tape reading off the cassette tape recorder which not only indicated the location of the protocol on the tape but was also used later when the reading was converted into the number of fifteen-second units that the subject worked on the problem, both in trying to find a solution and in looking back over a solution, and (2) checking once those variables on the checklist under "Approach," "Production," "Looking Back," and "Comments" that were exhibited by the subjects at least once. In other words if under "Comments" a subject "Requests assistance, more information" more than once on any given problem, the checklist is still only checked once. Under "Executive Errors" tallies are kept for each error made of the three types listed. "Interviewer comments" were made both during the taping of the protocol and later during the analysis of the tapes. A complete description of the checklist variables and description of when a subject is assumed to have demonstrated the various behaviors listed is found in Appendix E (Description of Checklist Variables).

The second major part of the coding system was the process-sequence codes. A process-sequence code is a series of letters, numerals, punctuation marks, and other symbols taken from the list in Appendix F (Process-Sequence Coding Variables) which when written in the proper order presents a sequential picture of the subjects problem-solving behavior. By looking at the codes one can determine immediately what methods the subject used to solve the problem, the order in which he used them, whether he had great difficulty, whether he made a lot of executive or structural errors, whether he checked his work, and so forth. These codes were determined in the three months following the spring quarter in which the study was conducted. The investigator listened to the recorded protocol of each subject on each problem until he could write the process-sequence code for that protocol. The code was then written at the bottom of the checklist coding form (Appendix D) in the space set aside for it. A total of four hundred thirty-two protocols were determined for later analysis. A comprehensive description of the process-sequence variables and of the guidelines used in writing the process-sequence codes is found in Appendix G (Description of Process-Sequence Coding Variables).

After writing the process-sequence codes the investigator scored the protocol and wrote the points for

"Approach," "Plan," "Result," "Looking Back" and "Total" on the checklist coding form (Appendix D). The analysis of the tapes also resulted in some changes in the notations on the checklist which were made during the course of the interview. More often than not such changes involved checking behaviors demonstrated by the subjects that went unnoticed at the time of the interview largely because the interviewer was not only looking for instances of checklist behaviors but was also following the train of thought of the subject.

For each protocol the investigator also wrote a brief description of the subject's approach. Particular attention was paid to bizarre happenings both in reasoning and in computation.

Three examples of the application of the coding system are found in Appendix I (Three Examples of Applying the Coding Scheme). Three protocols are transcribed and the corresponding coding forms are there for comparison.

BACKGROUND

The study was conducted at Concordia Teachers College in River Forest, Illinois, a western suburb of Chicago. The college is owned and maintained by The Lutheran Church--Missouri Synod. Undergraduate programs are offered in liberal arts or in teacher education. Recently, approximately 90 percent of the undergraduate student body has been enrolled in teacher education programs. The undergraduate enrollment has averaged approximately 1,050 students over recent years.

The teacher education program prepares students desiring to become elementary or secondary school teachers, directors of Christian education, or directors of parish music. The program is accredited by the North Central Association of Colleges and Secondary Schools and the National Council for the Accreditation of Teacher Education. Approximately 95 percent of the graduates of the teacher education program take positions in Lutheran elementary or secondary schools or Lutheran parishes. Masters degrees are offered in education and church music.

The mathematics department at Concordia offers students in elementary education a concentration (sixteen quarter hours) in mathematics and offers a minor in computer science and a minor or a major in mathematics for students in secondary education. Required of all students in the elementary education program is eight quarter hours of mathematics. At the time of the study this eight hours was packaged as a six-quarter-hour course in mathematical content titled MATH-301, Structure of Elementary Mathematics and one of two two-quarter hour methods courses titled MAE-380, Teaching Math in the Lower Grades or MAE-381, Teaching Math in the Upper Grades. All of these courses were taught by members of the mathematics department. The content course was typically taken by freshmen or sophomores and the methods courses by juniors or seniors.

THE SAMPLE

The students who participated in the study were taken from two regularly scheduled sections of the six-quarter-hour course MAT-301, Structure of Elementary Mathematics, during the ten-week spring quarter of the 1975-76 academic year. Structure of Elementary Mathematics is a required mathematics content course for preservice teachers.

Peculiarities in the scheduling of freshmen in this course at Concordia influenced the nature of the sample chosen from these two sections. Concordia's calendar consists of a ten-week fall quarter from mid-September to Thanksgiving, a winter quarter that is divided into a three-week interim from Thanksgiving to Christmas and a seven-week term from early January through February during which an average credit load for a student is twelve hours, and a ten-week spring quarter. Freshmen preservice teachers with a mathematics ACT score of thirty or greater are prescheduled into a special interim mathematics course for talented students which substitutes for Structure of Elementary Mathematics in their program. Freshmen preservice teachers who have mathematics ACT scores in the high twenties are typically scheduled into a section of Structure of Elementary Mathematics that is offered during the seven-week part of the winter quarter; the rationale being that it is not advisable to have students with low ability in mathematics take this six-quarter hour course during such a short period of time as seven weeks.

As a result of these standard procedures, students enrolled in Structure of Elementary Mathematics during the fall or spring quarters tend to be average or lower ability freshmen or upper classmen who for some reason did not take the course as freshmen. Oftentimes the spring quarter sections are populated by students who are doing their best to avoid the course as long as possible.

Since both scheduled sections of Structure of Elementary Mathematics during the spring quarter of the 1975-76 academic year were to be offered during the same time period, the investigator was able to choose subjects for participation in the study from both sections. Since the study was to be done with freshmen and sophomores, this was one easily identified criteria for selection. In addition, subjects were selected so as to assure the existence of two distinct ability groups within the class. Twenty-two students were placed in the investigators section. The remainder of the students, largely juniors and seniors, were placed in the other section which was scheduled to be taught by a different instructor.

The class was divided into two ability groups according to (1) a score on a test on the structure of elementary mathematics administered to all incoming freshmen

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during their orientation week, (2) ACT mathematics scores, and (3) number ot years of high school mathematics. One group will be referred to as the Low-Ability Group and the other as the High-Ability Group with the understanding that this latter description is a bit of a misnomer because of the scheduling considerations discussed earlier.

Of the initial twenty-two subjects, eighteen were considered in the final analysis. Subject D, a low-ability student, failed to appear for the initial interview and failed to appear a second time when the interview was rescheduled. She did not attend class more than few times during the quarter and got an F in the course. Subject Q, a high-ability student, participated in the early part of the investigation but because of personal problems quit coming to class after about two weeks. Since he missed a major part of the treatment, he was dropped from the analysis. Subject P, a high-ability student, participated in all aspects of the study. However, due to a malfunction in the recording equipment during her pretest, her interview could not be analyzed and thus she was also dropped from the analysis. In order to maintain an equal number of subjects in each ability group, a low-ability student, subject N, was randomly selected. Although she participated in all aspects of the study, the data obtained for her was dropped from the analysis.

Table 1 provides background data on the eighteen participants who served as the sample in the investigation.

Table 1

Table 1

Background Data on Participants in Study

	Subject	Sex	Class	Years of High-School Math	High School Grade-Point Average	ACT Math	ACT Composite	Structure of Elementary School Mathematics Test	College Grade-Point Average
Low-ability students	C	F	So.	2.0	2.05	12	15	17	2.24
	F	F	So.	2.0	2.95	17	20	20	3.15
	H	F	Fr.	2.0	2.84	07	15	15	2.32
	I	M	So.	2.0	2.21	22	23	10	2.11
	L	F	So.	2.0	2.54	15	19	14	2.20
	M	F	Fr.	2.0	2.72	17	17	11	2.71
	O	F	Fr.	2.5	3.22	20	21	17	3.00
	R	F	Fr.	2.0	3.11	13	15	13	2.63
	T	F	So.	2.0	3.22	25	27	14	3.17
High-ability students	A	F	So.	3.0	3.64	28	23	17	3.28
	B	M	Fr.	3.0	2.85	24	26	16	2.68
	E	F	Fr.	3.0	3.24	33	26	29	3.36
	G	F	So.	3.0	3.35	29	25	26	3.61
	J	F	Fr.	2.0	3.70	19	25	22	3.87
	K	F	Fr.	4.0	2.52	29	23	22	2.16
	S	F	Fr.	4.0	3.12	28	25	25	3.25
	U	F	Fr.	3.0	3.88	25	24	22	3.87
	V	M	Fr.	2.5	2.64	24	22	21	2.96

All of the students were freshmen or sophomores in teacher education programs. All were female except for one male in the low-ability group and two in the high-ability group. A subject is regarded as having two years of high school mathematics if he had one year of algebra and a year of geometry. Two students had a year of general math and then a year of algebra and a year of geometry. This is regarded as two years of mathematics in the table. The college grade-point averages listed for each student was for courses taken up to the spring quarter, 1975-76.

Table 2 lists the means of the high school grade-point averages, ACT mathematics scores, ACT composite scores, structure of elementary school mathematics tests, and college grade-point averages for the two ability-level groups. Clearly, there is a distinct difference between the two groups on all of these measures.

SELECTION OF PROBLEMS FOR TESTS

One of the real difficulties in doing research on problem solving lies in the fact that it is so hard to classify non-routine mathematics problems. If one has a set of problems that cannot be classified into any subgroup, then it is impossible to observe a person's behavior as he solves one problem and then make any sort of generalization as to what his behavior will be on another problem from the set. This was a problem that had to be addressed in the

Table 2

Means of Background Data for Two Ability-Level Groups

	High School Grade-Point Average	ACT Math[a]	ACT Composite[b]	Structure of Elementary School Mathematics Test[c]	College Grade-Point Average
Low-ability students	2.76	16.4	19.1	14.6	2.61
High-ability students	3.22	26.6	24.3	22.2	3.23

[a]Mean at Concordia Teachers College, (1975-76) was 21.1. National Mean (1975-76) was 18.9.

[b]Mean at Concordia Teachers College, (1975-76) was 20.8. National Mean (1975-76) was 19.4.

[c]Mean for Freshmen entering Concordia Teachers College in 1974 and 1975 was 19.3.

present study.

There were four main considerations that guided the selection of problems. They were:

1. The questions had to be non-routine mathematics problems as defined in Chapter 1 since such problems are the center of the study.

2. The problems were not restricted to any one area of mathematics. However, their solutions in general required at least some knowledge of algebra and geometry.

3. The problems had to be solvable in two or more ways. This prompted diversity of student approaches and made analysis of student protocols more interesting and productive.

4. As much as possible, certain of the problems were classified. But this had to be done without betraying the studies emphasis on non-routine problems. In many cases problems between the pretest and posttest were paired according to problem type. This was done to contribute to a more interesting analysis.

In the description of the pretest and posttest that follows, numerical reference to the problems coincides with the numerical listing of the problems in Appendix A (Pretest and Posttest in Main Study). It is in this appendix that the problems are grouped or classified to some extent according to content area and type. When the tests were used in the study the questions were put in random order and the Pretest Problem-Solving Inventory and the Posttest

Problem-Solving Inventory are given in Appendix B and Appendix C respectively.

Questions one through five on both the pretest and the posttest are number theory problems, an area of mathematics that is rich in fascinating problems of any level of difficulty. Number theory is also an important component of any course in the structure of elementary mathematics for preservice teachers.

The problems were obviously paired. Problem one on the posttest is slightly more difficulty than number one on the pretest. Nevertheless, they can both be solved by similar techniques.

Problems two on both tests are almost identical except the numbers used in the posttest question are larger. The course of instruction did include some practice with the various methods for finding the least common multiple of two or three numbers.

Questions three on both tests are again identical in structure. However, the numbers used are different and the posttest question is not stated as directly as the pretest question. Its geometric setting may make it more difficult for certain people. During the course students were exposed to two techniques for determining if a number is a perfect square. First, by looking at its prime factorization and second, by noting if the last digit is a last digit of a perfect square, namely, zero, one, four, five, six, or nine.

The fourth questions on each test are of the same structure. The posttest problem is probably more difficult because the divisor is larger. No instruction on ways to do this specific type of problem was given in the course.

Questions five on the pretest and posttest seem to be quite different but both require some knowledge of factors and both require some counting. Other than the usual introduction to factors and multiples, no instruction was given in the course on problems of this type.

Problems six on both tests are geometric in nature and both measure to some degree a subject's geometric perception. During the course of instruction students performed a laboratory experience related to this question. They were given the pattern of a box and asked to visualize the construction of a box that is half as wide, half as long, and half as high as the original. Then they were asked to guess how many of the smaller boxes would fit into the larger box. After making a guess they were directed to do the construction of the smaller box and check their guess. Students also had various other laboratory experiences with volume using one-inch cubes.

Problem seven on the pretest is geometric in nature and usually requires the drawing of a figure for solution. Problem seven on the posttest also usually requires the construction of a diagram for solution. The pretest problem was included in the studies of Lucas (1972, 1974) and Webb (1975) and the posttest problem was included in the study of Kilpatrick (1967) and Krutetskii (1976).

Both questions eight have no solution. Both problems were studied in depth in an interview setting by Jeffrey Paige and Herbert Simon (1966). They compared human behavior in solving these problems with the technique of solution used by the computer program called STUDENT which was discussed in Chapter 2. During the course of instruction the participants in the study were faced with several other problems with no solution.

Questions nine on both tests are somewhat difficult to understand at a first reading. In most cases if a problem solver understands what is being asked in each problem, then he gets the problem correct. Both problems were included in Kilpatrick's study (1967).

Problems ten on each test are "typical" questions involving two unknowns. Students versed in algebra may write two equations in two unknowns to describe the problems. Others may use trial and error for arriving at solutions. The course included no instruction in algebra per se. Occasionally, simple linear equations were used to solve other problems. Problem ten of the posttest was discussed at length by Polya (1962:I,23-26) where he examined the many approaches there are to solving the problem.

The remaining two pairs of problems are not paired according to type but nevertheless they are interesting non-routine problems. Problem eleven of the pretest can be approached in several ways and often people do not recognize that part of the information given in the problem is not necessary for the solution. Problem eleven of the posttest is of interest because the initial guess of many people as to what the answer is, namely, fifteen minutes, is wrong. Many different strategies can be employed in solving the problem.

Number twelve on the pretest is also a problem involving two unknowns. It was previously studied by Kilpatrick (1967) and Webb (1975). Problem twelve on the posttest is rather difficult and the easiest way to solve it is to "work backwards." This heuristic was only briefly covered in the course. Nevertheless, the investigator thought it would be interesting to include such a problem on the posttest. The problem is discussed at length by Wickelgren (1974).

INTERVIEW PROCEDURES

The pretest interviews were held during the first week of the spring quarter of the 1975-76 academic year. The first day of class the students were told that they would be expected to participate in two interviews in the course, one during the first week of the course and one during the last week. It was explained that during the interviews they would be asked to solve some problems. The interviews would be recorded so that I could examine them more carefully at a later time. They were assured that their performance in the interviews would in no way influence their grade in the course. Since typically such interviews are not a part of the course they were told they would be paid a total of eight dollars for the two interviews. That same day students signed up for pretest interviews during two-hour blocks of time. It was anticipated that most subjects would not require two hours to complete the tests but that a few might. The twenty-one pretest interviews were held from March 2, 1976, to March 8, 1976. Two-hour blocks of time were also set aside for posttest interviews. These twenty interviews were held from May 4, 1976, to May 12, 1976.

All interviews had the same format. Participants were asked to come to a small seminar room, reserved for this purpose, near the investigator's office at the agreed upon time. The subject was seated at one corner of a large seminar table. The interviewer sat at the corner perpendicular to the subject so that he could see the written work of the subject.

At the start of each pretest interview the investigator talked casually with the subject in an attempt to put him at ease. The student was then handed the Pretest Problem-Solving Inventory (Appendix B). He was asked to first fill in the "general information" and then to read aloud the page of "instructions." It was explained that the two sample problems, Sample Problem A and Sample Problem B, were there for the subject to practice solving a problem while thinking aloud. The interviewer was free to explain in detail what the subject was to verbalize and to explain parts of the directions that the subject may not have digested from one reading. It was felt that two sample problems were sufficient to prepare the subject to start the pretest.

As the subject turned the page to start working on the problems of the pretest, the interviewer started the cassette recorder which together with the microphone was in full view of the student. Thus, the investigator had an audio recorded record of the subject's problem-solving protocol as well as a written record in the test booklet. On the rare occasions when the subject did not have enough room in the test booklet to complete the problem he was provided with an additional sheet of paper that was then inserted into the booklet.

The "thinking aloud" technique for determinig the processes used in solving problems has been shown to be effective. Flaherty (1975) has determined that the use of the thinking aloud technique has little influence on problem-solving success. However, the fact remains that some subjects are more verbal than others. Thus, during the interviews if a subject fell silent for any length of time, usually fifteen to thirty seconds, the interviewer would ask the subject, "What are you thinking now?" or "Can you describe your thoughts to me?". If a subject wrote something and did not describe why he did it he was asked, "What thoughts led you to do that?" or "Can you verbalize your thoughts which led you to write that?".

The only other comments made by the investigator during the interview were either responses to inquiries made by the subjects or reminders to the subjects about certain aspects of the directions. Questions that were answered directly by the interviewer were those that had to do with procedure (for example, "Do I have to verbalize this long division process?", or "Can I leave this problem for now?") or with definitions. If a student asked, "Does a cube have equal sides?", the interviewer would respond "Yes." If a student would ask, "What is a factor?", the interviewer would provide definition by means of examples. If a subject would ask, "Do I need to use all of the information given in the problem?", the response would be, "I'm sorry I cannot answer that." If he would write or say something and ask, "Is that right?", the response would be, "I cannot answer that" or "I cannot answer that; you do what you think is right."

It was not uncommon for subjects to forget parts of the directions. Besides not verbalizing their thoughts, a problem discussed above, on occasion subjects would erase part of their work. If so they were asked to rewrite it and draw a line through it. In this way, the written as well as the verbal record was complete.

Finally, at times a subject would draw a diagram or write some computational problem without verbalizing what he was doing. At such times the interviewer would make a comment such as, "So you're drawing a figure" or "So you're multiplying twelve times eleven." This would give a record on tape as to when these things were done.

Students were free to work as long as they chose and could return to a given problem as frequently as they wanted. Only one subject on the pretest was somewhat pressed by the two-hour time limit. She also happened to be the subject dropped from the analysis because of a malfunction of the recording equipment on the pretest. No one had difficulty completing the work they wanted to do on the posttest in the two-hour time limit.

The posttest interviews were scheduled in the same way and held in the same place as the pretest interviews. They were conducted during the last week of the course. The Posttest Problem-Solving Inventory (Appendix C) was presented to the students when they arrived for the interview. After writing their name, age, and the date they were asked to read aloud the instructions which were identical to the instructions for the pretest. Then they were asked to solve the single sample problem that appears. This again was intended to provide them with practice in thinking aloud. When they turned the page to start the problems, the cassette recorder was turned on. The other procedures followed during the course of the interview were identical to those used during the pretest. The students were paid for the interviews during the last class period before final exams.

ATTITUDINAL INSTRUMENTS

The problem-solving inventories, both pretest and posttest were used to evaluate the problem-solving performance of the participants. A second important feature of this study was an attitudinal one where an attempt was made to examine the attitudes of the subjects toward problem solving and toward mathematics in general and to determine if the course of instruction had any effect on their attitudes.

To accomplish this purpose five instruments were used. First, Aiken's "Revised Mathematics Attitude Scale" (1970, 1972) was used to measure anxiety toward mathematics (see Appendix J, Attitudinal Measures). The scale was administered the first and last day of the course. Second, a self-report sheet with the three questions, "How do you feel about mathematics?", "What do you like best about it?", and "What do you like least about it?" was administered the last day of the course (Appendix J, Form A). This was to measure student attitude toward mathematics. Third, a sheet with the three questions, "Has this course influenced your feelings toward mathematics?", "What did you like about this course?", and "What would you do to change this course to make it more interesting?" was also administered the last day of the course (Appendix J, Form B). This was to measure student attitude toward the course. Fourth, in like manner a sheet with the three questions, "After a few minutes reflection, try to describe below your attitudes toward problem solving at the present time.", "Describe any changes that may have taken place in your attitude toward problem solving since the beginning of the course.", and "Describe how you see the usefulness of the eight course handouts as an aid in learning to become better problem solvers." was administered the last day of the course (Appendix J, Form C). Clearly, this form was intended to measure student attitude toward problem solving.

The fifth instrument had two components. During the second week of the course a two-page handout consisting of thirteen non-routine mathematics "Supplemental Problems" was distributed (see Appendix J). The students were told simply that these are problems they may enjoy working. They would not be asked to hand any in, they would not be discussed in class, and they would not appear on tests. However, the instructor would be happy to discuss any of them outside of class with anyone who is interested.

The last day of class the students received a questionnaire in which they were to indicate whether or not they attempted to solve any of the "Supplemental Problems." If they had not, they were asked to indicate why and if they had they were to indicate how many they attempted, how many they think they got right and approximately how many hours in total they worked on them (Appendix J, Form D). Forms A, B, and C were submitted unsigned. However, they were coded so the investigator could distinguish the responses of the low-ability group from those of the high-ability group. Form D was signed. Students were assured that their responses to this signed self-report would in no way influence their grade. Obviously, this fifth instrument attempted to measure student attitude toward mathematics by evaluating how interested they were in solving problems when they were not required to do them and turn them in for a grade.

METHOD OF INSTRUCTION

Four basic principles guided the development of the course of instruction for this study. They were:

1. There should be a period of instruction when the emphasis is totally on the stages of problem solving, as set forth by Polya (1954), on heuristics or strategies which aid the problem-solving process, and on practice in solving problems.

2. As new topics in the structure of elementary mathematics are introduced, non-routine and interest-catching problems should be used to stimulate student interest whenever possible.

3. In addition to the usual "textbook problems" which are intended to reinforce the basic concepts of the course, students should be required to hand in solutions to

numerous non-routine mathematics problems. Their solutions should include an analysis of how they arrived at their answer.

4. There should be an emphasis on concrete laboratory experiences. Most preservice teachers gain a great deal of insight into the structure of elementary mathematics by participating in such activities.

A six-quarter-hour credit course such as Structure of Elementary Mathematics at Concordia Teachers College requires three hundred contact minutes per week. This requirement was met during the term the present study was conducted by meeting for seventy-five minutes on Monday and Thursday and for fifty minutes on Tuesday, Wednesday, and Friday. Typically the longer periods have been used for laboratory experiences.

The required textbook for the course was Contemporary Mathematics by Bruce Meserve and Max Sobel (1972). In terms of topics considered, this text is typical of the many on the market that are intended to be used for a mathematics content course for preservice elementary teachers. Students were also required to purchase a laboratory manual entitled "A Packet of Mathematics Laboratory Experiences" written by members of the mathematics faculty at Concordia. The laboratory experiences were designed to coincide with the topics presented in the textbook.

Since the pretest interview was being conducted during the first week of the course, no special problem-solving instruction was given to the students. "Set Theory" was the topic for the week. The method of instruction included lecture and discussion, the working of problems from the text by students, and one laboratory period on attribute games.

The second week of the course was devoted to problem-solving activities. Discussion on problems centered on Polya's four steps in solving problems which he outlines and discusses in How To Solve It (1954) and on heuristics which aid in the problem-solving process.

Early in the week students were handed eight pages of non-routine mathematics problems that they would be working on through the term (Appendix K, Non-routine Mathematics Problems for the Course). For eight of these fifty-seven problems (numbers 1, 2, 3, 4, 27, 33, 54, and 56) the investigator had written handouts which describe in great detail one solution to the given problem (Appendix H, Course Handouts). Each handout considers the four stages of problem solving; "Understanding the Problem," "Devising a Plan," "Carrying Out the Plan," and "Looking Back," and applies them to the given problem. Also each handout is designed to emphasize at least one basic heuristics. Thus, through the handouts the students were exposed to these and other heuristics: simplification, finding a pattern, contradiction, consideration of cases, making inferences from the goal, making inferences from the givens, and working backwards.

As stated before, many mathematics educators maintain that an important component of any problem-solving instruction is to provide learners with the opportunity to see someone else go through the solutions of problems step by step. The course handouts as well as instructor directed solutions of some problems in class discussions served this purpose.

The problem-solving handouts were never distributed until the subjects had first worked on or solved the given problem. When distributed, the handouts were discussed with an emphasis on the heuristic demonstrated.

Since the second week of the course was devoted to problem solving, four of the eight handouts were used then. The others were spaced through the remainder of the term.

Of the fifty-seven non-routine mathematics problems (Appendix K) distributed to the students, eight were, as mentioned, topics for the course handouts, fifteen were solved in class discussions led by either the instructor or a student, and thirty-four were solved by students and handed in to be graded. Students were usually given one week to work on problems and they were collected at the start of the period on Mondays. Then most of that period was spent discussing the problems. The students were asked to include as part of their solutions an analysis of their thought processes as they arrived at an answer. They were asked to use Polya's four stages of the problem-solving process as an outline for their write-up. Their problems were in turn graded according to the following point scheme: 0 - 2 points for "Understanding the Problem," 0 - 3 points for "Devising a Plan" and "Carrying it Out," 0 - 2 points for "Result" and 0 - 2 points for "Looking Back." Their total performance on the non-routine problems accounted for 30 percent of their grade in the course.

When non-routine problems were assigned, an attempt was made to coordinate the problem type with the topic being studied at the time. For example, when numeration was the topic being considered in class, students were also working "numeration type" problems from their list (for example, Appendix K, numbers 9, 27, 28, 29, 30, 31, and 32). When number theory was the topic being considered in class, students were also working "number theory type" problems from their list of non-routine problems.

The third week of the course centered on abstract mathematical systems and included two laboratory experiences. During week four work on numeration was started and again students worked through two labs and as usual participated in class discussions and worked problems assigned from the required textbook.

Numeration was completed during the fifth week as was number theory. The system of whole numbers and the system of integers were covered during week six. Also, since number theory had just been completed week six was a good time to concentrate on the abundance of non-routine number theory problems.

Non-metric geometry was the topic for week seven and metric geometry was considered during week eight and the start of week nine. When considering geometric topics class time was almost completely devoted to laboratory experiences. The students were expected to digest the textual material on their own.

At the time that the posttest interviews started during the ninth week of the course there was no more in-class emphasis on non-routine problems and no more non-routine problems were assigned to be handed in. The class topics considered during this period of time were the system of rational numbers and the system of real numbers.

Throughtout the course, an attempt was made to introduce and motivate new topics by the use of interest-catching problems. For example, before class time was devoted to place-value numeration systems other than base ten the Base two "Magic Cards" were introduced (Appendix K, Problem 27). Before a unit on number theory was started a game called "The Factor Game" was played and analyzed. Before studying "The Fundamental Theorem of Arithmetic" this question was posed: "How could you determine if a whole number is a perfect square? or a perfect cube?". It was hoped that this tactic would help to improve student attitude toward mathematics and help them see mathematics as an alive and exciting discipline.

Chapter 4

ANALYSIS AND DISCUSSION

INTRODUCTION

The analysis for this study centered on four main measures. First twenty-five checklist variables (see Appendix D and Appendix E) were identified from four categories. The variables were:

Approach

Restates problem in his own words
Performs exploratory manipulations
Draws figure

Production

Recalls same or related problem
Uses method of related problem
Uses result of related problem
Misinterprets problem

Looking Back

Checks solution by substitution in equation
Checks that solution satisfies condition
Checks solution by retracing steps
Checks if solution is reasonable/realistic
Derives the solution by another method
Generalizes the problem
Considers extreme cases
Checks that all information is used
Attempts to find other solutions

Comments

Requests assistance, more information
Questions existence of solution
Questions uniqueness of solution
Indicates uncertainty about final solution
Says he does not know how to solve the problem
Expresses enjoyment, liking for the problem
Expresses distaste, dislike for the problem
Admits confusion
Expresses confidence that he can arrive at a solution.

Secondly, thirteen process-sequence variables (see Appendix F and Appendix G) were identified from four categories. The variables were:

Preparation Symbols

R Reads the problem
G Separates/summarizes data
M Introduces model by means of a diagram or figure.

Process Symbols

D Deduction from condition
T Random trial and error
A Reasoning by analogy
C Reasoning by contradiction
E Setting up equations
S Simplifies the problem
L Looks back
N Nonsense process or behavior not classifiable.

Punctuation Marks

/ Stops without solution

Errors

$\bar{X}$ Structural errors in process X.

Other checklist variables and process-sequence variables were combined into a third main category which could be called a "performance measures" category.

Performance measures included time variables (time excluding looking back, time spent looking back, and total time); score variables (approach, plan, result, looking back, and total); difficulty, as measured by the process-sequence coding variables--Reads the problem (R), Hesitation of approximately 1 unit (-), Stops without solution (/), and Difficulty with process X (X); Executive errors (Numerical computation, Algebraic manipulation, and Other); and Structural errors.

The fourth and final category for analysis dealt with attitudinal measures (see Appendix J). Five different instruments were used in an attempt to examine the attitudes of the subjects toward problem solving and toward mathematics in general and to determine if the course of instruction had any effect on their attitudes.

Some of the variables listed in category one and two above did not occur with sufficient frequency to be included for further analysis. The first section of this chapter discusses which were deleted from further consideration in the study. The next section discusses the correlation analysis which was performed on the remaining variables from these first two categories.

Finally, variables from all four categories were used to answer the following questions which were originally posed in Chapter 1 for both ability-level groups.

1. Before taking a required course in mathematics, (a) what heuristics do preservice elementary teachers tend

to use in solving non-routine problems, and (b) do preservice elementary teachers tend to show a preference for either "trial and error" or "deduction" as a basic technique in solving non-routine problems?

2. Does an extended learning experience in a mathematics course for preservice elementary teachers which emphasizes the use of heuristics in the solution of non-routine mathematics problems: (a) result in an increase in the use of heuristics by subjects and result in an increase in the variety of heuristics employed by subjects; (b) cause subjects to be more deductive in their problem-solving behavior; (c) result in better performance on non-routine problem-solving inventories; (d) cause subjects to be more confident problems solvers; (e) result in a better attitude toward mathematics; (f) result in a better attitude toward problem solving; (g) result in a positive attitude toward the course?

3. Can certain features of verbal non-routine problems be identified which are especially troublesome for preservice teachers?

SELECTION OF VARIABLES FOR ANALYSIS

An initial step in the anlaysis of the data obtained from the interviews and the recorded problem-solving protocols simply involved the construction of a histogram of the frequency of occurrence of each of the twenty-five checklist variables and each of the seventeen process-sequence variables for the eighteen subjects on both the pretest and the posttest. These frequencies were then totaled within the low-ability group and the high-ability group. Thus, for each variable a total frequency was determined for each of the following four categories: (1) low-ability group on the pretest, (2) high-ability group on the pretest, (3) low-ability group on the posttest, and (4) high-ability group on the posttest.

For some variables the totals in each of these categories were uniformly low. Thus, the behavior represented by these variables either did not occur very often, or if it did occur, it was not revealed to the researcher in what the subject said or did. Such variables were excluded from further analysis.

The checklist variable (see Appendix D) excluded under Approach was "Performs exploratory manipulations." Those excluded under Production were "Uses method of related problem" and "Uses result of related problem." Eight variables, "Checks solution by substitution into equation," "Checks solution by retracing steps," "Checks if solution is reasonable/realistic," "Derives the solution by another method," "Generalizes the problem," "Considers extreme cases," "Checks that all information is used," and "Attempts to find other solutions" were deleted from Looking Back. Variables under Comments excluded were "Questions existence of solution," "Questions uniqueness of solution," "Expresses enjoyment, liking for the problem," "Expresses distaste, dislike for the problem," and "Expresses confidence that he can arrive at a solution."

Only one process-sequence coding variable (see Appendix F) was excluded. "Contradiction" (C) did not occur with sufficient frequency to merit its inclusion for further analysis.

CORRELATION ANALYSIS

Before considering the questions originally posed in Chapter 1, a descriptive analysis of each of the remaining twelve checklist variables and twelve process-sequence coding variables was performed. These variables proved to be of three main types, either (1) continuous, symmetrical, and unimodal, or (2) continuous but asymmetrical, or (3) could be regarded as dichotomous.

_T_ scaling was performed on those variables which were continuous but asymmetrical. This procedure normalizes distributions. After _T_ scaling a variable has a mean of 50 and a standard deviation of 10.

Variables were regarded as dichotomous when the frequency of occurrence of the behavior represented by the variable was small. For example, for the checklist variable "Restates problem in his own words," the frequencies of occurrence on the pretest for the eighteen subjects were 0, 0, 2, 2, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 3, 0 (median is

0). These datum seem to indicate that a subject either demonstrated the behavior or did not. Thus, the variable was dichotomized and assigned the value zero if the behavior was observed zero times and the value one if the behavior was observed one or more times.

Thus, in general, a dichotomized variable was assigned the value zero if the original variable was below the median and the value one if the original variable was above the median. Variables equal to the median were assigned the value of zero or one, depending on which choice would divide the subjects most evenly into two groups.

Of the twelve checklist variables included in the correlation analysis, the original data for Total score and Draws figure were continuous, symmetrical, and unimodal on both the pretest and the posttest. The data for Total time and Misinterprets problem were _T_ scaled on both tests. The remaining eight variables, Restates problem in his own words, Recalls same or related problem, Checks that solution satisfies condition, Requests assistance, more information, Indicates uncertainty about final solution, Says he does not know how to solve the problem, Admits confusion, and Total executive errors were regarded as dichotomous on both the pretest and posttest.

Of the twelve process-sequence coding variables the original data were continuous, symmetrical, and unimodal for Introduces model by means of a diagram or figure (M) on both the pretest and posttest and for Separates/summarizes data

(G) on the posttest. Variables that were T scaled on both tests were Reads the problem (R), Deduction from condition (D), Random trial and error (T), Stops without solution (/), and Structural error in process X ($\overline{X}$). Separates/summarizes data (G) was dichotomized on the pretest and Reasoning by analogy (A), Setting up equations (E), Simplifies the problem (S), Looks back (L), and Nonsense process or behavior not classifiable (N) were dichotomized on both the pretest and posttest.

Three product-moment coefficients were used to correlate the three types of variables. When both correlated variables were continuous and normal or at least symmetrical and unimodal, the Pearson product-moment coefficient of correlation was used (Guilford:79-96). When both of the variables were dichotomous, the phi coefficient was used (Guilford:306-10). When one variable was continuous and the other was dichotomous, the point-biserial coefficient was calculated (Guilford:297-300). Tables 3, 4, 5, and 6 contain these correlation coefficients for the checklist variables on the pretest, checklist variables on the posttest, process-sequence variables on the pretest, and process-sequence variables on the posttest respectively.

An examination of the statistically significant correlation coefficients in these tables reveals some obvious outcomes. For example, it is not surprising that in Table 3, Total score is significantly correlated with Checks that solution satisfies condition. On the other hand there

Table 3

Product-moment Coefficients for Checklist Variables on Pretest

	2	3	4	5	6	7	8	9	10	11	12
1. Total time	.25	.43	.43[c]	-.02	-.16	.00	.33	-.26	-.36	.17	.42
2. Total score		.06	.38	-.16	-.77[b]	.54[c]	-.36	.06	-.64[b]	-.37	-.01
3. Restates problem in his own words[a]			.21	.00	.21	.09	.16	.63[b]	.32	.08	.08
4. Draws figure				-.11	-.21	.28	.03	.24	-.07	-.19	.03
5. Recalls same or related problem[a]					.08	.38	.16	.08	.40	.32	.32
6. Misinterprets problem						-.27	.39	-.17	.32	.31	.67[b]
7. Checks that solution satisfies condition[a]							.33	.21	.33	.40	.40
8. Requests assistance[a]								.35	.10	.43	.03
9. Indicates uncertainty about final solution[a]									.13	.25	.03
10. Says he does not know how to solve problem[a]										.20	.03
11. Admits confusion[a]											.06
12. Total executive errors[a]											

[a]Dichotomous variable.

[b]Correlation significant at the .01 level.

[c]Correlation significant at the .05 level.

Table 4

Product-moment Coefficients for Checklist Variables on Posttest

	2	3	4	5	6	7	8	9	10	11	12
1. Total time	-.12	-.09	.70[b]	.12	-.12	.02	.42	.50[c]	.18	.15	.17
2. Total score		.01	-.08	.14	.12	.34	-.44	.08	-.38	-.45	.11
3. Restates problem in his own words[a]			-.05	.32	.04	.19	.33	.16	.03	.16	.08
4. Draws figure				.03	-.52[c]	.15	.39	.64[b]	.45	-.18	.08
5. Recalls same or related problem[a]					.15	.18	.08	.00	.16	.25	.50
6. Misinterprets problem						-.09	-.45	-.42	-.49[c]	.12	.01
7. Checks that solution satisfies condition[a]							.44	.44	.01	.35	.09
8. Requests assistance[a]								.32	.43	.32	.08
9. Indicates uncertainty about final solution[a]									.40	.00	.00
10. Says he does not know how to solve problem[a]										.16	.08
11. Admits confusion[a]											.00
12. Total executive errors[a]											

[a]Dichotomous variable.

[b]Correlation significant at the .01 level.

[c]Correlation significant at the .05 level.

are others that are difficult to explain and may be spurious as in the case of the positive correlatioin between Separates/summarizes data (G) with Structural errors ($\overline{X}$) in Table 6. Still others suggest relationships that are of considerable interest and that require further study.

The checklist variables Total time and Draws figure have significant positive correlation on both the pretest and posttest (Tables 3 and 4). However, the figures drawn are typically not of enough detail or sophistication to require a lot of time to draw. Thus, it seems that the time such people spent on problems was devoted to some activity other than just drawing figures. The correlations then seem to suggest a possible relationship between subjects who tend to use a visual representation of a problem and who spend a lot of time on problems, possibly because they have difficulty in solving the problems or possibly because they are confident they can solve problems, even if they are difficult.

The checklist variable Draws figure is negatively correlated with Misinterprets problem on the posttest (Table 4) which again may suggest that visual representation of a problem is important for the "understanding the problem" stage of the problem-solving process. Other correlations with Misinterprets problem are a positive significant correlation with Total executive errors on the pretest (Table 3) and a negative significant correlation with Says he does not know how to solve problem on the posttest (Table 4).

The process-sequence variable Separates/summarizes data (G) has significant negative correlation with Stops without solution (/) on the pretest (Table 5) and a significant positive correlation with Deduction from condition (D) on the posttest (Table 6). It is noteworthy that there is no similar positive correlation of Separates/summarizes data (G) with Random trial and error (T) on either test.

On the pretest Deduction from condition (D) and Random trial and error (T) are both negatively correlated with Stops without solution (/) (Table 5). On the posttest Deduction from condition (D) is also negatively correlated with Stops without solution (/) but Random trial and error (T) is positively correlated with Stops without solution (/) (Table 6). Thus, on the pretest and posttest subjects who tended to use the process deduction also tended to complete problems. Also on the pretest, subjects who tended to use the T process were inclined to complete problems. However, on the posttest this situation was reversed. Those who tended to use the T process there also tended to stop working on a problem without achieving a solution.

The significant negative correlation of D with T on the posttest is also interesting (Table 6). Subjects who used one of these processes tended not to use the other on the posttest. There was no such relationship on the pretest.

A final significant correlation of interest is the

Table 5

Product-moment Coefficients for Process-Sequence Variables on Pretest

	2	3	4	5	6	7	8	9	10	11	12
1. Reads the problem (R)	-.19	.00	-.01	-.09	.30	.34	.08	-.02	.39	.31	.00
2. Separates/summarizes data (G)[a]		.28	.38	.14	.15	.11	.34	.15	.00	-.47[c]	-.36
3. Introduces model (M)			.31	.14	.21	.11	.09	-.04	-.03	-.33	-.11
4. Deduction from condition (D)				.02	-.15	.46	.19	.48[c]	-.16	-.61[b]	.11
5. Random trial and error (T)					.68[b]	-.09	-.10	-.31	.31	-.44[c]	.06
6. Reasoning by analogy (A)[a]						.05	.05	.20	.32	-.13	-.08
7. Setting up equations (E)[a]							.40	.36	.08	-.24	-.40
8. Simplifies the problem (S)[a]								.25	.32	-.47[c]	-.34
9. Looks back (L)[a]									.32	-.16	.03
10. Nonsense process (N)[a]										-.05	.14
11. Stops without solution (/)											.01
12. Structural errors ($\overline{X}$)											

[a]Dichotomous variable.

[b]Correlation significant at the .01 level.

[c]Correlation significant at the .05 level.

Table 6

Product-moment Coefficients for Process-Sequence Variables on Posttest

	2	3	4	5	6	7	8	9	10	11	12
1. Reads the problem (R)	-.20	.21	.17	.16	-.21	.25	.13	-.09	.02	.33	-.26
2. Separates/summarizes data (G)		.12	.47[c]	-.08	.23	-.34	.01	.01	.23	-.14	.44[c]
3. Introduces model (M)			.11	-.02	.17	.07	.17	-.03	.55[c]	.26	-.05
4. Deduction from condition (D)				-.48[c]	-.06	.01	.51[c]	-.40	-.35	-.44[c]	.37
5. Random trial and error (T)					-.21	-.19	-.52[c]	-.33	.16	.57[b]	.03
6. Reasoning by analogy (A)[a]						.32	.06	.53[c]	.40	-.44	-.38
7. Setting up equations (E)[a]							.32	.56[c]	.16	-.16	-.63[b]
8. Simplifies the problem (S)[a]								.53[c]	.06	-.38	-.19
9. Looks back (L)[a]									.40	-.56[c]	-.43
10. Nonsense process (N)[a]										.45	.38
11. Stops without solution (/)											.05
12. Structural errors ($\overline{X}$)											

[a] Dichotomous variable.

[b] Correlation significant at the .01 level.

[c] Correlation significant at the .05 level.

students who were able to restate a problem in the form of an equation tended to make fewer structural errors, that is errors that stem from a misunderstanding of the problem or of some principle necessary for its solution.

In summary, one must be careful not to imply that any of the foregoing significant correlations suggest cause-effect relationships. However, they do pose questions that could possibly be answered by further research.

PROBLEM-SOLVING BEHAVIOR OF PRESERVICE TEACHERS BEFORE THEY TAKE A COLLEGE COURSE IN MATHEMATICS

The first question that was posed in this study deals with the problem-solving behavior of preservice teachers before they took a required course in college mathematics.

Question 1 (a): Before taking a required course in mathematics, what heuristics do preservice elementary teachers tend to use in solving non-routine problems?

The heuristics the investigator thought the subjects might use were:

Restates problem in his own words,
Draws figure,
Recalls same or related problem,
Uses method of related problem,
Uses result of related problem,
G Separates/summarizes data,
A Reasoning by analogy,
C Reasoning by contradiction,

E Setting up equations, and

S Simplifies the problem.

The first five of these were checklist variables (see Appendix D) and the second five were process-sequence variables (see Appendix F).

As mentioned earlier in this chapter, three of these behaviors were excluded from analysis because they either did not happen very frequently on either the pretest or posttest or if they did occur, it was not revealed to the investigator by what the subject said or did. Thus the heuristics:

Restates problem in his own words,
Draws figures,
Recalls same or related problem,
G Separates/summarizes data,
A Reasoning by analogy,
E Setting up equation, and
S Simplifies the problem

were observed frequently enough on both tests to merit further consideration. If a heuristic was observed a total of nine or more times for each ability group, then the heuristic was considered to be one that preservice teachers tend to use. The number nine was chosen since the number of subjects in each ability group was nine and a total frequency of nine indicated that on the average the strategy was used at least once by each subject. Using this criteria it can be seen from the pretest scores of Table 7 that the only strategies that the preservice teachers included in this study tended to use before completing a required course in mathematics were:

Table 7

Frequency of Occurrence of Heuristics

Heuristics	Pretest Frequencies Low-Ability Group	Pretest Frequencies High-Ability Group	Posttest Frequencies Low-Ability Group	Posttest Frequencies High-Ability Group
Restates problem in his own words	5	5	10	15
Draws figure	26	31	24	22
Recalls same or related problem	6	2	9	9
G Separates/summarizes data	3	8	25	26
A Reasoning by analogy	3	1	2	8
E Setting up equations	16	48	3	26
S Simplifies the problem	0	14	9	26

Draws figure, and

E Setting up equations

for both ability-level groups; and

S Simplifies the problem

for the high-ability group.

It is not surprising that the subjects made use of the strategies Draws figure, and E Setting up equations. The former is one of the first and one of the most basic heuristics taught and retaught in schools. The abundant use of equations in attempting to solve problems on the pretest is more than likely the result of the great emphasis in high-school algebra on solving word problems by translating the problem into at least one equation. Since for each of these subjects a high-school algebra and high-school geometry class were their most recent involvements with formal mathematics, it seems reasonable that they would try to write an equation to describe a problem whenever possible.

It is interesting that only the high-ability group tended to use any other heuristic, namely, S Simplifies the problem. The lack of use of the other heuristics indicates that the preservice teachers employed but a fraction of the many tools that are available to aid them in the problem-solving process.

<u>Question 1 (b)</u>: Before taking a required course in mathematics do preservice elementary teachers tend to show a preference for either "trial and error" or "deduction" as a basic technique in solving non-routine problems?

There are two approaches that can be used to answer this question. The first simply involves comparing total frequencies of occurrence on the pretest of the two processes D Deduction from condition and T Random trial and error. These totals for the low-ability group were 148 (or 48.8 percent) and 155 (or 51.2 percent) respectively. For the high-ability group the frequencies were 190 (or 56.1 percent) and 149 (or 43.9 percent) respectively. This analysis shows that the low-ability group had a marginal preference for T Random trial and error (51.2 percent compared with 48.8 percent) while the high-abiliy group had a preference for D Deduction from condition (56.1 percent compared with 43.9 percent).

A shortcoming of the foregoing analysis may be that in considering only the total frequency of occurrence of the D and T processes, the order in which the processes were performed is ignored. One of the advantages of describing a problem-solving protocol using a process-sequence code is that by looking at the code one gets a sequential picture of the subjects problem-solving behavior. Under the assumption that if a subject had a preference for using either deduction or trial and error in solving a problem then he would probably use that preferred process first in attacking a problem, it was decided that an examination of the first two processes used by the subjects would be appropriate. A subject was regarded as having a preference for deduction from condition if he only used one process and it was deduction or if he used more than one process and the first two used were deduction. Similarly, a subject was regarded as having a preference for random trial and error if he only used one process and it was trial and error or if he used more than one process and the first two used were trial and error. This analysis was restricted to four pretest problems--numbers one, two, three and ten as listed in Appendix A. These problems were chosen since they were the ones where the process trial and error was used on both tests and in the opinion of the investigator, they lent themselves to solution by trial and error. Hence, if a student would choose to solve such problems using a deductive process then that would certainly indicate his preference.

This analysis revealed that for the low-ability group there were 4 instances of preference for deduction (or 13.8 percent) and 25 instances of a preference for trial and error (or 86.2 percent). For the high-ability group the corresponding numbers were 15 (or 53.6 percent) and 13 (or 46.4 percent). Note that for both groups the instances do not sum to 36 (4 problems times 9 subjects) since some subjects used neither process in attempting a solution and in fact did not arrive at a solution, and others used deduction followed by trial and error or vice versa as their first two processes.

Thus, the second method of analysis reinforces what was observed before. The low-ability group had a preference for using trial and error while the high-ability group demonstrated a preference for deduction from condition on the pretest.

PROBLEM-SOLVING BEHAVIOR OF PRESERVICE TEACHERS AFTER THEY TAKE A COLLEGE COURSE IN MATHEMATICS

Question 2 (a): Does an extended learning experience in a mathematics course for preservice elementary teachers which empahsizes the use of heuristics in the solution of non-routine mathematics problems result in an increase in the use of heuristics and result in an increase in the variety of heuristics employed by subjects?

As determined in question 1 (a), the strategies Draws figure and E Setting up equations were used by subjects in both ability groups and the strategy S Simplifies the problem was used by the high-ability group on the pretest. An examination of Table 7 reveals that there was a slight and inconsequential reduction for both groups in the use of the heuristic Draws figure on the posttest and a rather large decrease for both groups in the use of E Setting up equations (from 16 to 3 for the low-ability group and from 48 to 26 for the high-ability group).

It was suggested earlier that a possible reason why the subjects made considerable use of equations in solving problems on the pretest was that one of their most recent formal involvements with mathematics was either in an Algebra I or an Algebra II class. In such classes students sometimes get the impression that all problems are solvable using equations. The dramatic reduction in the use of equations on the posttest in this study may reveal that the mathematics course they were in helped them to realize that such is not the case and it is not always desirable to attempt to solve a problem by describing it with an equation.

Finally, for the high-ability group there was a sizeable increase in the use of S Simplifies the problem (from 14 to 26). Thus, for two of the three heuristics that the subjects used on the pretest, there were interesting and important changes in the usage pattern on the posttest.

It can also be seen from Table 7 that there was an increase in the variety of heuristics employed by subjects. Using the same criteria that was used in answering question 1 (a), namely that if a heuristic was observed a total of nine or more times for each ability group then the heuristic was considered to be one the subjects tended to use, it can be said that the strategies Restates problem in his own words, Recalls same or related problem, G Separates/ summarizes data, and S Simplifies the problem were employed by both groups on the posttest and E Setting up equations was used only by the high-ability group. The increase in the usage of Recalls same or related problem is probably artifical since it is more than likely dependent on the fact that several of the problems on the pretest and posttest are so similar to one another.

Thus, it can be concluded that there was an increase in the variety of heuristics employed by the students in this study. Those used on the posttest but not on the pretest included:

Restates problem in his own words, and

G Separates/Summarizes data

for both groups, and

S Simplifies the problem

for the low-ability group.

Question 2 (b): Does an extended learning experience in a mathematics course for preservice elementary teachers which emphasizes the use of heuristics in the solution of non-routine mathematics problems cause subjects to be more deductive in their problem-solving behavior?

As in question 1 (b), there are two approaches that can be used in answering this question. Table 8 reveals that when the total frequency of occurrence of the deduction (D) and trial and error (T) processes was used, there was a very slight decrease in the use of deduction (48.8 percent to 45.8 percent) with a corresponding increase in the use of trial and error (51.2 percent to 54.2 percent) for the low-ability group. For the high-ability group there was a rather substantial increase in the use of deduction (56.1 percent to 64 percent).

The second technique for answering this question involves determining the first two processes subjects used in their problem-solving protocols from the four problems which lend themselves to solution by trial and error, namely, problems one, two, three, and ten as listed in the pretest and posttest in Appendix A. Table 9 contains the

Table 8

Total Frequency of Usage of Deduction (D) vs.
Trial and Error (T)

	Pretest		Posttest	
	Low-Ability Group	High-Ability Group	Low-Ability Group	High-Ability Group
Deduction (D)	148 (48.8%)	190 (56.1%)	185 (45.8%)	208 (64%)
Trial and Error (T)	155 (51.2%)	149 (43.9%)	219 (54.2%)	117 (36%)

Table 9

Preference for Deduction (D) vs. Trial and Error (T)
Using First Two Processes of Protocols on
Four Selected Problems

	Pretest		Posttest	
	Low-Ability Group	High-Ability Group	Low-Ability Group	High-Ability Group
Deduction (D)	4 (13.8%)	15 (53.6%)	11 (37.9%)	18 (64.3%)
Trial and Error (T)	25 (86.2%)	13 (46.4%)	18 (62.1%)	10 (35.7%)

results of this analysis.

The changes in behavior between tests is much more pronounced in this table. The use of deduction for the low-ability group went from 13.8 percent to 37.9 percent and for the high-ability group from 53.6 percent to 64.3 percent. Although there was a rather large increase in deductive behavior in this analysis for the low-ability group, after the mathematics course was over they still tended to prefer trial and error as a basic technique for starting to solve problems that lend themselves to solution by trial and error. The increase in the use of deduction by the high-ability group (53.6 percent to 64.3 percent) is not as great as the increase for the low-ability group but nevertheless, after the course is over the high-ability group had a much stronger preference for deduction than did the low-ability group (64.3 percent to 37.9 percent).

In comparing the two tables we see what at first seems to be an inconsistency. For the low-ability group the use of deduction went down from pretest to posttest (48.8 percent to 45.8 percent) when total frequencies were considered (Table 8), and it went up (13.8 percent to 37.9 percent) when just the first two processes used were considered on four selected problems (Table 9). This phenomena can be explained, however, by examining Table 10. This table contains the total frequency of occurrence for the low-ability group of the deduction and trial and error processes on the same four problems considered in Table 9.

It can be seen that on those problems there was an increase in the use of deduction (from 13.5 percent to 26.3 percent).

Table 10

Total Frequency of Usage of Deduction (D) vs. Trial and Error (T) for Low-Ability Group on Four Selected Problems

	Pretest	Posttest
Deduction (D)	22 (13.5%)	67 (26.3%)
Trial and Error (T)	141 (86.5%)	188 (73.7%)

Thus on problems that lent themselves to solution by trial and error, low-ability students demonstrated an increase in usage of deduction, both when just the first two processes were considered and when the total frequency of occurrence was considered on those problems. The slight decrease in percentage of total frequency for the low-ability group (Table 8) simply means that on the other eight problems of the posttest, subjects tended to make less use of the deduction process.

In summary an examination of Table 9 reveals that the course had the most pronounced effect on the deductive behavior of the low-ability group when change in percentage of usage is the criteria (a 24.1 percent increase vs. a 10.7 percent increase for the high-ability group). However, after the course was over the low-ability group still had a preference for trial and error (62.1 percent vs. 37.9 percent for deduction) and the high-ability group which started with a preference for deduction simply increased that preference over the ten-week period of the course (64.3 percent vs. 35.7 percent for trial and error).

Question 2 (c): Does an extended learning experience in a mathematics course for preservice elementary teachers which emphasizes the use of heuristics in the solution of non-routine mathematical problems result in better performance on non-routine problem-solving inventories?

Performance on the non-routine problem-solving inventories was initially measured by sixteen variables which were grouped into five categories--Time, Score, Difficulty, Executive Errors, and Structural Errors. The time it takes to do a given task in mathematics is often used as a measure of performance. Oftentimes faster is considered better. On problem tasks of the kind considered in this study, a student who spends less time working than many others may indeed be a very capable problem solver and may be working through the task quickly and with a high degree of competence. However, he may also be a very poor problem solver without much desire to stick with a difficult task and thus may give up very quickly. Conversely, a subject who spends more time than average on a problem-solving inventory may be either slow and not a very gifted problem solver but nevertheless tries hard, or he may be a capable problem solver who has considerable drive and will put forth considerable effort on all problems, especially the difficult ones. Thus, the data for the three "Time" variables--Excluding looking back, Looking back, and Total--must be regarded in this light.

The category "Score" was an obvious choice for measuring performance. The five Score variables were Approach, Plan, Result, Looking back and Total.

The category "Difficulty" was measured by four variables--R Reads the problem, - Hesitation of unit, / Stops without solution, and X Difficulty with process X. Since "Reads the problem" was coded not only at the first reading of a problem but also if a subject rereads a problem, a high occurrence of this variable indicates that the subject needed to read problems numerous times to fully understand them or to arrive at plans of attack. Obviously hesitations were a good indicator of difficulty. Also, at times hesitations simply meant that the subject was rereading the problem silently. If a subject arrived at a solution, whether it was right or wrong, then in his mind the problem was not so difficult that a solution could not be attained. On the other hand, if he did not arrive at a solution, then probably as far as he was concerned the problem was too difficult to solve.

Finally, the three "Executive Errors" variables were Numerical computation, Algebraic manipulation, and Other. The category "Structural Errors" was also a variable name. The presence of executive errors indicated the subject had difficulty with computation, manipulation of expressions, or with copying numbers and information correctly. The presence of structual errors indicated a difficulty with understanding problems or certain necessary principles.

Table 11 presents the totals for each of these sixteen variables for all problems on the pretest and posttest. The totals in the category "Time" reveal that each group spent more time on the posttest problem-solving inventory than on the pretest inventory. The low-ability group spent 21.1 more minutes on the posttest and the corresponding number for the high-ability group was 5.6 minutes. The increase in the total time spent for both groups is probably explained by several things. The posttest must be considered more difficult than the pretest and thus students would be expected to work a longer time on it. Also, the increase may reflect a greater desire on the part of the subjects to solve the problems presented on the posttest. This observation may be especially true for the low-ability group where there was a rather dramatic increase in total time (21.1 minutes).

A disappointing fact exposed by the data in this category is that there was no change in the subjects' behavior with regard to the "looking back" stage of the problem-solving process. As mentioned in Chapter 3, considerable emphasis was placed on this aspect of problem solving in the course of instruction. Thus, it seems that either ten weeks of instruction and drill which emphasizes this stage is not enough to inculate the practice of looking back or else the subjects felt that looking back is a

Table 11

Performance Variables for All Problems

	Pretest Low-Ability Group	Pretest High-Ability Group	Posttest Low-Ability Group	Posttest High-Ability Group
Time[a]				
Excluding looking back	1769.2 (49.1)	1881.1 (52.3)	2528.7 (70.2)	2068.6 (57.5)
Looking back	.3 (0.0)	29.1 (0.8)	0.0 (0.0)	44.4 (1.2)
Total	1769.5 (49.1)	1910.2 (53.1)	2528.7 (70.2)	2113.0 (58.7)
Score[b]				
Approach (108)	55	82	80	88
Plan (324)	85	165	101	196
Result (216)	57	116	52	115
Looking back (216)	0	12	0	9
Total (864)	197 (22.8%)	375 (43.4%)	233 (26.9%)	408 (47.2%)
Difficulty				
R Reads the problem	184	179	179	187
- Hesitation of 1 unit (15 seconds)	678	694	1010	696
/ Stops without solution	34	17	31	11
X Difficulty with process X	22	24	12	7
Executive Errors				
Numerical computation	16	12	26	26
Algebraic manipulation	0	8	0	3
Other	3	3	5	4
Structural Errors[c]	86	56	108	65

[a] Time is expressed in 15 second units and in minutes.

[b] The numbers in parentheses are the maximum possible points that could be in each cell.

[c] Entries are total structural errors minus structural errors corrected.

behavior you engage in only when you have an ample amount of time and certainly not one you pursue when taking a "test." The investigator feels that the former reason probably has the most credibility since during the inventories none of the subjects asked the interviewer if they should take time to "look back." It was as if it did not even occur to them to do so.

When performance is measured by increase in total score from pretest to posttest on all twelve problems, it is clear from the "Score" category of Table 11 that there is very little improvement. The increase for each group is approximately 4 percent (4.1 percent for the low-ability group and 3.8 percent for the high-ability group). However, it is interesting to note where this increase came from. There was essentially no change in performance between tasks and within groups on the variables Result and Looking back. Most of the increase within groups was a result of increases on the variables Approach (55 to 80 for the low-ability group and 82 to 86 for the high-ability group) and Planning (85 to 101 for the low-ability group and 165 to 196 for the high-ability group). Thus, it seems that the subjects made improvement in working their way through these two stages of the problem-solving process but yet they had difficulty in carrying out the plan to get the correct result.

The 4 percent increase in total score needs further analysis in light of the observation made earlier, namely that the posttest must be considered more difficult than the pretest. The variance in difficulty level is largely because of the difference in five problems between the two tests, namely numbers 5, 7, 9, 11 and 12 (see Appendix A). The other seven problems, numbers 1, 2, 3, 4, 6, 8, and 10 (Appendix A), are comparable. In some cases they are the same type of problem but with different data given (also see Chapter 3, SELECTION OF PROBLEMS FOR TESTS). It would be appropriate to conduct an analysis of performance similar to that outlined in Table 11 but to restrict attention to the seven comparable problems of the two problem-solving inventories. Such an analysis was performed and will be discussed shortly.

A further examination of Table 11 shows that in the category "Difficulty" there is little difference in the variable R Reads the problem within either group. However, on the variable (-) Hesitation of 1 unit, the low-ability group showed a marked increase (from 678 to 1010). This increase corresponds with the increase noted earlier in total time spent on the inventories by the low-ability group. Again, it seemed that this group may have been demonstrating perseverence that was not present in taking the pretest.

For the final two variables in this category, / Stops without solution and X Difficulty with process X, there were no differences of any consequence between tests. As would be expected, however, the high-ability group stopped without solutions considerably less often than did the low-ability group.

There were only small differences between groups on the variables Numerical computation, Algebraic manipulation, Other, and Structural errors. The most curious differences here lie in the fact that there were greater incidences of errors in numerical computation and greater incidences of structural errors on the posttest for both groups. These differences are difficult to explain. They may be related to the fact that for both groups there was marked improvement in the approach and planning stage of the problem-solving process but not in getting correct answers (the variable, Result). A possible explanation is that subjects made either computational or structural errors which prevented them from achieving correct results.

As mentioned above seven pairs of problems between tests could be considered comparable in nature (numbers 1, 2, 3, 4, 6, 8, and 10 of Appendix A). For the reasons previously stated, an analysis similar to that just conducted on all twelve problems was performed on these seven. That analysis is summarized in Table 12.

In the category "Time," the variables Excluding looking back and Looking back were eliminated from Table 12 since the time spent looking back was almost negligible (see Table 11). Further study of the variable Total time in Table 12 shows nothing that would give rise to an analysis different from what was said before.

This is not true in the category "Score." Whereas

Table 12

Performance Variables for Seven Similar Problems

	Pretest		Posttest	
	Low-Ability Group	High-Ability Group	Low-Ability Group	High-Ability Group
Total time[a]	28.5	30.2	43.2	34.4
Score[b]				
Approach (63)	38	49	49	56
Plan (189)	55	84	76	149
Result (126)	39	59	39	90
Looking back (126)	0	4	0	7
Total (504)	132 (26.2%)	196 (38.9%)	164 (32.5%)	302 (59.9%)
Difficulty				
R Reads the problem	108	107	97	99
- Hesitation of 1 unit (15 seconds)	394	400	620	373
/ Stops without solution	17	12	23	5
X Difficulty with process X	11	13	5	2
Executive Errors				
Numerical computation	13	7	14	16
Algebraic manipulation	0	4	0	2
Other	2	1	3	3
Structural Errors[c]	42	39	68	26

[a]Time is expressed in minutes.

[b]The numbers in parentheses are the maximum possible points that could be in each cell.

[c]Entries are total structural errors minus structural errors corrected.

the increase in total score for each group was approximately 4 percent when all twelve problems were considered, when only the seven similar problems were taken into consideration the increase was 6.3 percent for the low-ability and 21.0 percent for the high-ability group. While again the low-ability group did not demonstrate a substantial gain, the high-ability group did. In fact, their total score on those problems on the posttest was a respectable 59.9 percent.

Further it can be seen that while the bulk of the increase in the total score for each group in Table 11 and for the low-ability group in Table 12 was the result of an improvement in performance on the approach and planning stage of the problem-solving process, for the high-ability group in Table 12, there was also a substantial increase from 59 to 90 points (or 46.8 percent to 71.4 percent) for work in the result stage of the process. Thus, the high-ability group improved considerably in their ability to achieve correct answers on the similar problems while the low-ability group did not.

For the remaining variables in Table 12, only the data for Structural errors are significantly different from the data of Table 11. When all problems were considered, the incidence of Structural errors went up for both groups. This was also the case on the seven similar problems for the low-ability group. However, for the high-ability group, the incidence of Structural errors went down (from 39 to 26) on the seven similar problems. This coincides with this group's improved performance on the "Result" stage of the problem-solving process.

In summary, it seems that both groups spent more time working on the posttest than on the pretest and that the difference was greatest for the low-ability group. This may reflect that both groups, but especially the low-ability group, had a greater desire to solve problems on the posttest. Secondly, neither group improved in the use of the looking back stage of the problem-solving process. This is more than likely true because ten weeks of instruction is not sufficient to make the looking back stage a part of their problem-solving routine. Thirdly, when all twelve problems were considered, there was only slight increase in total score on the posttest and any positive change that was observed was largely in the approach and planning stage of the problem-solving process. However, when only the seven similar problems were considered, the high-ability group showed a striking increase in total points with a substantial part of this increase coming from the result stage of the problem-solving process. Fourthly, on all problems and on the seven similar problems, the low-ability group demonstrated a significant increase in hesitations. This may be because of increased perseverance on the part of the low-ability students. Finally, it seems that the commission of Executive and Structural errors may have impeded the achievement of correct answers.

Question 2 (d): Does an extended learning experience in a mathematics course for preservice teachers which emphasizes the use of heuristics in the solution of non-routine mathematics problems cause subjects to be more confident problem solvers?

In general the notion of "confidence" can be very difficult to measure, partly because it is such a task to define it as it applies to given contexts and partly because an individual's confidence may change dramatically from moment to moment depending on his circumstances. For the purposes of this study, confidence was that trait of the subjects that was measured by their performance on the three variables--Total time, Requests assistance, more information, and / Stops without solution (Table 13).

It was mentioned earlier when addressing question 2 (c) that the substantial increase between pretest and posttest in the total time spent by the low-ability group (49.1 minutes to 70.2 minutes) may have reflected greater desire on their part to solve the problems of the posttest. Or in other words, this could indicate that the low-ability subjects felt more confident in their problem-solving ability on the posttest and as a result they worked a little harder at attaining solutions. Although there was also an increase in total time for the high-ability group, it was not as great (53.1 minutes to 58.7 minutes).

If a subject requested assistance or more information about problems that had been carefully and clearly written and that were appropriate for his academic level, then this may indicate that the subject lacked

Table 13

Three Confidence Variables

Variable	Pretest Low-Ability Group	Pretest High-Ability Group	Posttest Low-Ability Group	Posttest High-Ability Group
Total time[a]	49.1	53.1	70.2	58.7
Requests assistance more information (108)[b]	32	25	28	20
/ Stops without solution (108)[b]	34	17	31	11

[a]Time is expressed in minutes.

[b]The total possible for each cell in the row is 108.

confidence in his own interpretation of the problem and in his ideas for solving the problem. Table 13 reveals that for both tests, the low-ability group requested more assistance than the other group and that each group demonstrated a decrease in the number of times they requested assistance from pretest to posttest. However, these decreases are very small (a decrease of 4 for the low-ability group and a decrease of 5 for the high-ability group) and thus little significance can be attached to them.

It may be that a subject who stopped working on a problem without attaining a solution lacked confidence in his ability to attain a solution through further efforts. Thus, in using the third variable in Table 13 as a measure of confidence it can again be seen that on both tests the high-ability group had greater confidence by a margin of 17 to 34 for the pretest and a margin of 11 to 31 on the posttest. Further, within groups and between tests there was a decrease in the number of problems on which the subjects stopped without a solution (34 to 31 for the low-ability group and 17 to 11 for the high-ability group) reflecting an increase in confidence. However, these changes are again slight and not much weight can be attached to them.

Thus, it seems that the low-ability group started and ended with less confidence in their problem-solving ability than the other group. Also it seems that the course of instruction did not have a detrimental effect on subjects' confidence. If anything, it may have increased their confidence. However, there is not sufficient evidence to support this.

Question 2 (e): Does an extended learning experience in a mathematics course for preservice elementary teachers which emphasizes the use of heuristics in the solution of non-routine mathematics problems result in a better attitude toward mathematics?

Two instruments were used to try to answer this question. The first was Aiken's "Revised Mathematics Attitude Scale" (1970, 1972) which was used to measure anxiety toward mathematics (Appendix J). It was administered the first and last class day of the quarter. Although both ability groups showed a decrease in their anxiety toward mathematics, neither decrease could be considered significant. For the low-ability group the mean scores on this scale went from 45.6 percent on the first day of class (with a standard deviation of 16.5) to 50.7 percent on the last day of class (with a standard deviation of 10.5). For the high-ability group the means score went from 65.9 percent (with a standard deviation of 21.3) to 66.7 percent (with a standard deviation of 18.1). Two interesting observations can be made concerning this data. First, although the improvement in attitude for both groups is slight, that of the low-ability group is greatest as measured by the sample means. Secondly, for both groups, the standard deviations became smaller between the first and last day of class.

It should be noted that the number of subjects included in the analysis of Aiken's attitude survey was twenty, ten low-ability subjects and ten high-ability subjects. These same twenty students also filled out Forms A, B, C, and D (see Appendix J) the last class day of the quarter. This number includes the eighteen subjects whose data provided the basis for the analysis on the preceeding questions and subjects P and N. These latter two subjects participated in all aspects of the study however, as discussed in Chapter 3, a malfunction in the recording equipment during the pretest of subject P prevented the acquisition of data on her performance. Thus subject P, a high-ability subject, had to be dropped from part of the study. Likewise the recorded protocols of subject N, a randomly selected low-ability subject were, not included in the analysis of the preceeding questions.

The second instrument used to answer this question was an unsigned self-report questionnaire containing three questions concerning how the students felt about mathematics (Form A, Appendix J). This questionnaire was administered to the subjects during the last class day of the course. They were not told before hand that they would be asked to complete the form. Thus, it was assumed that their responses were spontaneous and independent.

Because of the open-ended nature of the questions on this instrument as well as those on subsequent questionnaires, it was difficult to quantify the results. The technique that was used to attempt to summarize the results consisted of identifying certain key words or phrases with which subjects responded, categorizing these as much as possible, and then determining the percent of responses within each category. The results of this analysis follow. The first column of percentages is for the low-ability group and the second column is for the high-ability group.

a) How do you feel about mathematics?

Key words or Phrases	Percentages	
I enjoy it; I like it; my favorite subject; I like it when I understand it; I like working with numbers; never had much trouble with it	33.3	64.3
I don't like it; I don't especially like it; my worst subject; don't always feel I know how it all fits together; it frustrates me, makes me feel stupid; it makes me nervous and confused	58.3	21.4
It's necessary for life; it's useful	8.3	7.1
I'm indifferent about it	0.0	7.1
I wish they wouldn't teach me things I don't need like geometry and trig	8.3	0.0

b) What do you like best about it?

Key words or Phrases	Percentages	
Geometry and proofs	10.0	30.8
The set way of doing things to arrive at correct answers; its logicalness; knowing why you do what you do	30.0	15.4
Story problems	0.0	23.1
Intuitive geometry, concrete geometry	20.0	0.0

The good feeling you get when you work a problem correctly	20.0	0.0
Algebra	10.0	7.7
Doing addition, subtraction, multiplication, and division	10.0	7.7
It's challenging; it's stimulating; it's fun	0.0	7.7
Everything	0.0	7.7

c) What do you like least about it?

Key Words or Phrases	Percentages	
Working a long time on what seems to be an easy problem and not getting it; the frustration of working on hard problems; spending a lot of time on problems; doing problems that are too hard	15.4	30.0
Never doing well on tests; wanting to do well but just not able to	23.1	0.0
Properties and proofs; notation and sets	23.1	0.0
Problem solving	15.4	20.0
Geometry	7.7	20.0
Algebra	7.7	10.0
Nothing	7.7	10.0
Addition, subtraction, multiplication, division; number bases	0.0	10.0

Perhaps the most significant bit of information gained from this instrument is found under the question a) "How do you feel about mathematics?" It is there revealed that after experiencing the course of instruction 58.3 percent of the responses of the low-ability group indicated some sort of dislike for mathematics or at least a frustration with it. Much of this dislike and frustration was no doubt related to the problem-solving component of the course. So although there was some improvement in problem-solving ability for this group, there was still considerable room for betterment of attitudes toward mathematics.

<u>Question 2 (f)</u>: Does an extended learning experience in a mathematics course for preservice elementary teachers which emphasizes the use of heuristics in the solution of non-routine mathematics problems result in a better attitude toward problem solving?

Two instruments were also used to investigate this question. The first was again an unsigned self-report questionnaire which was also administered during the last day of the quarter and the subjects were not warned ahead of time that they would be asked to complete such a questionnaire. The three questions posed related to how the subjects viewed their attitude toward problem solving and toward the eight problem-solving handouts (Form C, Appendix J).

The second instrument consisted of two parts. During the second week of the course a two-page handout consisting of thirteen non-routine mathematics "Supplemental Problems" was distributed (see Appendix J). At that time the students were simply told that these are problems they may enjoy working. They would not be asked to hand any in, they would not be discussed in class, and they would not appear on tests. However, the instructor would be happy to discuss any of them outside of class with anyone who is interested.

The second part of this instrument consisted of a signed questionnaire administered the last class day which quized the students as to how many of the supplemental problems they worked, how much time they spent on them and so forth (see Form D, Appendix J).

For the first instrument, the key words or phrases method was again used to analyze the subjects' responses. The first column of percentages is for the low-ability group and the second column is for the high-ability group.

a) After a few minutes of reflection, try to describe below your attitudes toward problem solving at the present time.

Key words or Phrases	Percentages	
Neutral: One of frustration, but at least now there is less (frustration); don't like story problems but at least I've gotten better at them; story problems are still not my favorite types of problems but I do understand better how to go about doing them; I feel I can get many of them if I work long enough; I'm better at solving problems but they still frustrate me; don't enjoy story problems but I like skill-type problems; still don't like them and still get upset with them but now I would stick to a problem longer before giving up; it's ok, I don't dislike it; they can be interesting but so frustrating	50.0	50.0
Positive: It is challenging and I enjoy solving a problem if		

I know I'm going in the right direction; it is very hard for me but I enjoy it; I enjoy some non-routine problems but they sometimes frustrate me; I like the non-routine problems since it takes thought to do them; each problem is unique; problem solving is the main function of math, it is the most enjoyable part of math, I look forward to doing problems; I enjoy problem solving and I feel that it is a challenge to me; it doesn't upset me because I look at it as sort of a game	8.3	40.0
Negative: I want to avoid them as much as possible; I don't like it at all; YUKKO; I hate it but I know it's necessary	25.0	10.0
Other: Wish I had more math knowledge to solve problems; many problems involve skills I don't have	16.7	0.0

b) Describe any changes that may have taken place in your attitude toward problem solving since the beginning of the course.

Key words or Phrases	Percentages	
Better: I used to hate it, now I don't mind it, I know how to go about solving a problem, I'm not as afraid of them; at least now I am willing to try to solve word problems; I have a much more positive attitude to doing story problems, they helped me to think and solve problems systematically; at the beginning I absolutely detested them, now it's just dislike; I stick with a problem longer; I gained confidence; I feel more at ease with them	70.0	60.0

No Change: I still hate them; I have always enjoyed it and this class has revitalized this enjoyment in me;	10.0	40.0
Worse: I feel stupid when I can't solve a problem; I dislike them since they counted so much for our grade	20.0	0.0

c) Describe how you see the usefulness of the eight course handouts as an aid in learning to become better problem solvers.

Key words or Phrases	Percentages	
Postive: Very useful, explained the hows and whys, served as "similar problems"; they helped me very much, the steps were clearly given, showed how to write out all the steps; they were fun; they were helpful in arranging a logical way to attack the problem; useful, but I rarely used them after we went over them in class; they were an excellent idea; they really helped you understand the problem and the ways to figure them out; they were easy to follow and understand	90.0	88.9
Indifferent: I really didn't think they helped me extremely much	10.0	0.0
Negative: I didn't see any usefulness for them, if anything they confused me	0.0	11.1

The analysis of the first question seems to reveal that very few of the responses indicate a strong negative attitude toward problem solving (25 percent for the low-ability group vs. 10 percent for the high-ability group). On the other hand, there was not a strong feeling in support of it either (8.3 percent for the low-ability group vs. 40 percent for the high-ability group). In both cases the differences between groups are predictable. It is interesting that so many subjects regret being so frustrated by non-routine problems, but yet, if there is no frustration to accompany the solution of a problem, then that problem is probably not a non-routine problem (as defined in Chapter 1) for that person.

In examining the second question, it is disappointing that 20 percent of the responses of the low-ability group indicated a more negative attitude toward problem solving because of the course of instruction. Also at first glance it may be surprising that the low-ability group had a greater percentage (70 percent) of their responses indicating an improvement in their attitude toward problem solving (vs. 60 percent for the high-ability group). However, one must also note that 40 percent of the high-ability group responses indicated no change in their attitude and some of the subjects in that category had a positive attitude toward problem solving before they started the course.

As mentioned above the second instrument for evaluating question 2 (f) consisted of the list of "Supplemental Problems" (see Appendix J) distributed the second week of the course and the signed self-report sheet indicating what the subjects did with these problems (see Form D, Appenddix J).

In response to the question, "Did you attempt to solve any of the problems?" three of the ten low-ability students said yes and two of the ten high-ability students said yes. These three low-ability students recalled working on an average of two to three problems, felt they got about two correct on the average, and on the average spent almost two hours on the supplemental problems. The two high-ability students who did some of the problems recalled working on an average of five problems, felt they get about three or four correct on the average, and on the average spent almost two or three hours working.

Those who did not work on any of the supplemental problems were asked on Form D to explain ". . . why you did not attempt any." The majority for both groups said that they felt the regular problem assignments in class were sufficient. A few others (two in each ability group) simply stated that they did not care to work any more problems and two low-ability group students admitted that they had forgotten about the problems.

This data provides little or no information on the subject's attitudes toward problem solving. If most of the subjects had worked on a goodly number of the supplemental problems then it could have been concluded that they were really interested in problem solving and had a very positive attitude toward it. The data included here does not indicate any such overwhelming excitement about problem solving. The class was indeed required to do a goodly number of non-routine problems during the course of the quarter and was probably not very enthused about the idea of working additional problems.

Question 2 (g): Does an extended learning experience in mathematics course for preservice elementary teachers which emphasizes the use of heuristics in the solution of non-routine mathematics problems result in a positive attitude toward the course?

In order to attempt to answer this question another unsigned self-report questionnaire was administered during the last class day of the course (Form B, Appendix J). The key word or phrase technique was again used to analyze the results.

a) Has this course influenced your feelings toward mathematics?

Key Words or Phrases	Percentages	
Yes-Positive: I don't mind math anymore, now I understand it better; I feel more comfortable with it now; I feel better about mathematics; it has helped me feel better about math; it has helped me understand math better, I understand problems better now that I have had to pick them apart; have grown to enjoy math more; I feel more confident about math because of the concrete approach; have better understanding; when I figured a problem out by myself I felt triumphant	50.0	50.0
No-No change: I still dislike the parts I don't understand; I put up with it; No, I've always liked math; I'm still in the middle of the road	30.0	50.0
Yes-Negative: I like it less largely because of the word problem	20.0	0.0

b) What did you like about this course?

Key Words or Phrases	Percentages	
Labs, I learned that not all math is impossible; they helped me understand better; it helps me to see how to show kids; experimenting with concrete objects, first concrete experiences I had ever had	56.3	42.9
Functions and negatives; metric system and bases; review of grade-school topics; review of elementary topics in a way different from how we learned in grade school	6.3	28.6
Seeing many ways to solve a given problem; the way the homework was handled	12.6	7.1
The way we learned the "why" of everything	12.6	7.1
Textbook activities; class activities, games etc.	12.6	0.0
Varied content, no time to get bored	0.0	7.1
Nothing, it was too easy	0.0	7.1

c) What would you do to change this course to make it more interesting?

Key words of phrases	Percentages	
Do more labs; less lecture	33.3	22.2
Have everyone work on problems in class but don't assign them and grade them; don't make the problems so much an important part of the grade, do problems more for the experience; less emphasis on word problems, they are fun and good for experience but shouldn't be much of the grade except for a grade in		

"trying hard"; I disliked the problems we had to turn it	33.3	22.2
Not assign story problems to be written out in such detail; not so many story problems	11.1	22.2
Make it harder, including the labs; some of the labs are really stupid and should be changed or thrown out completely	0.0	22.2
Good the way it is now; nothing, fine the way it is except go through more story problems in class	11.1	11.1
Spend more time on fewer things	11.1	0.0

It is noteworthy that question a) reveals that 50 percent of the responses indicated the course was influential in causing subjects to feel better about mathematics. Another 30 percent of the responses of the low-ability students and 50 percent of the responses of the high-ability subjects indicated they liked mathematics initially and the course did not affect this attitude.

The non-routine problems did not rate very high as something about the course the students liked. The laboratory experiences were the overwhelming favorite.

The change in the course that relates to problem solving and was recommended in 33.3 percent of the responses of the low-ability subjects and 22.2 percent of the responses of the high-ability subjects dealt with not basing so much of the course grade on student performance on the non-routine problems. As mentioned in Chapter 3, the problems accounted for about 30 percent of the course grade.

FEATURES OF VERBAL NON-ROUTINE PROBLEMS WHICH ARE TROUBLESOME

As discussed in Chapter 3, Eisner (1974) conducted a research project at Michigan State University with preservice teachers enrolled in a course similar to the course in the present study. He suggested that an appropriate follow-up study to his would be one that tries to determine the features of word problems that cause difficulty for the problem solver. In the present study, the investigator wrote a short verbal description of the problem-solving procedure used by each subject on each problem on both the pretest and posttest. These descriptions were written after each process-sequence code was written in the analysis of the audio tapes. It was these descriptions that were used to answer the final question of the study.

<u>Question 3</u>: Can certain features of verbal non-routine problems be identified which are especially troublesome for preservice teachers?

The problems from both tests that are very similar are paired for the following discussion. Corresponding problems that are not similar in nature are discussed separately. The numbering scheme used is the same as the one found in Appendix A.

Pretest 1. If you take any two-digit number, form a new number by reversing its digits, and then add the two numbers together, what is the largest number that is <u>always</u> a divisor or factor of the sum?

Posttest 1. If you take <u>any</u> three-digit number (e.g., 638), reverse the digits, and find the non-negative difference between these two numbers (e.g., 836 - 638 = 198), what is the largest number that is always a divisor or factor of this difference?

All students worked these problems by trial and error. In some cases they guessed an answer after just one or two trials. Only one student tried to derive an answer in general and that was on the posttest question. Her attempt at a generalization however, was made after an answer was first proposed by guessing. It seems that the troublesome feature of these problems was that they required more analysis than the subjects were inclined to attempt. In general, subjects were too quick to make a guess and more often than not, very little reasoning accompanied the guess.

Pretest 2. What is the smallest number having 4, 6, 9, and 15 as divisors or factors?

Posttest 2. What is the smallest number having 9, 12, 22, and 24 as divisors or factors?

No single aspect of these problems were particularly troublesome except for the computation involved in a trial and error solution which was the approach used by most subjects on the pretest question. It was curious that on the pretest question, most subjects found successive integral multiples of 15 by repeatedly multiplying 15 by successive integers instead of using repeated addition. Also, of the twelve subjects who on the pretest found successive multiples of 15, only three reasoned that because of the presence of the even numbers 4 and 6 on the list, it was not necessary to consider the odd multiples of 15 as trial values.

Pretest 3. Is 3688 a perfect square?

Posttest 3. Millionnaire Jones wants to build a large patio behind his mansion. He has 692,482 square tiles (1 ft. x 1 ft.) to use for the patio. Can he construct a square patio which will use all of the tiles?

On the pretest question four subjects revealed a lack of understanding of place value notions since they either noticed that 36 is a perfect square and 88 is not or simply noted that 88 is not a perfect square. From these observations they concluded that 3688 is not a perfect square.

In addition, one person on the pretest stated that since 4 does not divide 3688, it cannot be a perfect square. Perhaps she was visualizing the picture of a square and was in turn confusing perimeter with area. Or, she could have been saying that since 2 divides 3688 and 4 does not, 3688 is not a perfect square since its prime factorization contains one prime (namely 2) that is not raised to an even power. It is unlikely however that this low-ability subject would make this latter observation on the pretest. On the posttest question which is couched in a geometric setting three subjects confused the concept of perimeter with the area. Each one of these three stated that since 4 (where the 4 clearly stood for the number of sides of the patio) is not a divisor of 692,482, then a square patio cannot be constructed.

Pretest 4. What is the remainder when 2^{89} is divided by 3?

Posttest 4. What is the remainder when 3^{197} is divided by 5?

Many students who understood the exponential expression were overwhelmed by the magnitude of the two dividends and simply remarked that if they had time or if they had a calculator they would multiply out and divide. On the pretest only one subject considered a simpler problem and looked for a pattern in the remainders. Many others on both tests performed nonsensical manipulations with the given numbers to arrive at various answers (such as dividing 89 by 3 and taking the remainder as the answer).

Pretest 5. How many numbers in the set $\{1, 2, 3, 4, \ldots, 800\}$ have either 2 or 5 but not 10 as a factor?

For most of the few subjects who did not correctly answer this question, the reason seems to be that they did not consider all conditions of the problem. They either counted those that had 2 or 5 as a factor or those that had 10 as a factor. One subject even counted the number of elements of the set that had 2 and 5 but not 10 as factors.

Posttest 5. There are no zeros at the end of 3! = 6, one zero at the end 5! = 120, and 2 zeros at the end of 10! = 3,628,800. How many zeros are there at the end of 100!?

This is a posttest problem and most subjects decided that with a number such as 100! in the problem they needed to consider simpler problems and then look for patterns. However, oftentimes the pattern search was not very organized. For example, from the data in the problem, they often set up a table with the numbers 3!, 5!, 10! and then typically 15! in one column and the number of zeros, 0, 1, 2, and 3 in the corresponding column. This arrangement may not have been best for gaining insight into what causes zeros to appear at the end of a factorial. Also, the calculation of 15! or any other factorial provided many opportunities for computational errors. It is fascinating that of all the subjects who calculated some factorial just one considered only the last few decimal places of each product where the zeros appear. The others calculated each product completely.

Since the data given in the problem tended to suggest to students that they needed to look for a pattern, this may have diverted their attention from a basic question that leads to a solution rather easily, namely, "What makes a zero appear at the end of a product?". To get the problem solver to consider that question, perhaps only the last sentence of the problem should be posed.

Pretest 6. Cube A holds 5 quarts of water. Cube B is twice as long, twice as wide, and twice as high as Cube A. How many quarts of water does Cube B hold?

Posttest 6. Cube X holds 8 times as much sand as Cube Y does. How tall is Cube X compared to Cube Y?

In short, the students who had difficulty with these questions did not create in their mind or on paper a picture of what the problem was asking. Ten quarts was a common answer on the pretest question since "Cube B is twice as big." The corresponding answer on the posttest question was "eight times as high." One "interesting" answer on the pretest question proceeded as follows: the subject took the original 5 quarts, added 5 quarts for "twice as long," added

5 quarts for "twice as wide," and added 5 quarts for "twice as high." He then wrote, "5 + 5 + 5 + 5 = 25 quarts."

Pretest 7. Find the maximum possible area for a rectangle with a perimeter of 72 inches.

Of the eight subjects who did not correctly answer this question, three couldn't even get started on the problem, and it was clear that two others were confusing the concepts of perimeter and area. The other three were able to decide upon a rectangle with a perimeter of 72 inches, however, they stated that the rectangle also had the greatest area. It was clear from their comments that these three were working under the assumption that all rectangles with a given perimeter have the same area. Thus, it appears that a major troublesome feature of this problem involves inadequate understanding of perimeter and area.

Posttest 7. An airline passenger fell asleep when he had traveled halfway. When he awoke, the distance remaining to go was half the distance he had traveled while asleep. For what part of the way did he sleep?

Of the ten subjects who failed to get the correct answer for this problem, seven misinterpreted the problem and arrived at the answer one fourth. They interpreted the problem as saying, "When he awoke, the distance remaining to go was half the length of the trip from the point where he fell asleep." Thus, the somewhat complex syntax of the second sentence of the problem seemed to be the most troublesome feature here.

Pretest 8. A board is sawed into two pieces. One piece is two-thirds as long as the whole board and is exceeded in length by the second piece by 4 feet. How long was the board before it was cut?

Posttest 8. A man has 7 times as many quarters as he has dimes. The value of the dimes exceeds the value of the quarters by $2.50. How many has he of each coin?

On the pretest question six subjects saw the inconsistency in the conditions of the problem but yet proceeded to find an answer or at least tried to find an answer. Two additional subjects each wrote three different correct equations to describe the problem, correctly solved the equations, and got answers that indicated there was no solution to the problem. Nevertheless, they failed to recognize that there was no solution.

The most common answer was twelve feet. Then one piece of the board was eight feet and the other four feet.

Clearly, the most bothersome feature of this problem was that it has no solution. The subjects had simply not seen enough such problems and tended to believe that every mathematics problem must have a solution.

During the period of the ten-week course, the students had the opportunity to deal with other questions that had no solution. As a result, the performance on the posttest question was considerably better. Nevertheless, there were still four subjects who attempted to find or found solutions.

Pretest 9. If on a balance a brick balances evenly with three-quarters of a pound and three-quarters of a brick, what is the weight of a whole brick?

Again, it seems that on this problem subjects had the most difficulty in interpreting the conditions of the problem. Seven people missed the problem because they interpreted the problem as saying that three-quarters of a pound balances evenly with three-quarters of a brick. Thus, they determined that the weight of a brick was one pound.

Posttest 9. If a chicken and a half lay an egg and a half in a day and a half, then how many eggs do three chickens lay in one day?

Six subjects who missed this problem reasoned along the following lines. First, they incorrectly reasoned that from the conditions of the problem it follows that one half chicken lays one half an egg in one half a day. From this they reasoned correctly that three chickens would lay three eggs in one day. So although the two conditions that gave rise to the two conclusions were very similar in structure and meaning, the subjects conclusions were incorrect in one case and correct in the other.

The same pattern of inconsistency in reasoning existed for the three subjects who arrived at the answer one egg. Typically, these people erroneously reasoned first that three chickens must lay three eggs in three days. From this they legitimately concluded that three chickens would lay one egg in one day.

Why these nine subjects could draw a correct conclusion in one situation but not in a similar situation is a mystery. Since they each demonstrated some ability at arriving at a proper conclusion in this problem, perhaps their errors can simply be attributed to lack of care and thoughtfulness in the problem-solving process.

Pretest 10. A carpenter makes only three-legged stools and four-legged tables. One day when he looked at his day's output he counted 31 legs. If he made 9 stools and tables that day, then how many of these 9 were stools and how many were tables?

Posttest 10. A farmer has hens and rabbits. These animals have 50 heads and 140 feet. How many hens and how many rabbits has the farmer?

Of the ten subjects who missed the pretest question seven failed to take into consideration one datum of the problem, namely, that there were a total of nine stools and tables made. Each of these seven people arrived by trial and error at the answer seven tables and one stool which provided the correct number of legs but only eight stools and tables altogether.

On the posttest question only three students failed to achieve the correct answer and none of these failed to take into account all of the data of the problem. Thus, before taking the course it again seems that subjects were a bit careless in reading problems and in taking into account all the given information.

Pretest 11. Two missiles speed directly toward each other, one at 9,000 miles per hour and the other at 21,000 miles per hour. They start 1,317 miles apart. How far apart are they one minute before they collide?

Generally, the subjects who correctly answered this question where not bothered by the unnecessary datum provided in the problem. However, many who missed the problem tried to involve the 1,317 miles in some calculations to arrive at a solution. Five of those who missed it wanted to somehow use the formula distance equals rate times time and generally for those people the value for distance in this formula was 1,317.

In general it seems that the most troublesome feature of this problem was that it involves rates. Most of the twelve subjects who missed the problem either did nothing or if they did perform calculations, they were often nonsensical. It is the view of the investigator that many of these subjects were simply afraid of such problems or they felt they had to use a formula of which they were not sure. They were not able or inclined to approach the problem from a common sense point of view.

Posttest 11. The leavening agent in an unusual bread dough makes the dough double itself in size every minute. If it takes 30 minutes for one loaf to just fill an oven, how long will it take two loaves to just fill the same oven?

Only two subjects answered this question correctly. In each case they considered simpler problems and then generalized to the given problem. Two others started out on the right track but abandoned what they were doing. These two plus thirteen more concluded that the answer was fifteen minutes since ". . . it should take half as long." Oftentimes this answer was provided rather quickly and without much thought. Thus, although there was some emphasis in the course on three dimensional geometry, students in general did not relate such considerations to this problem and were eager to treat it as a one dimensional problem.

Subjects did try to apply a laboratory experience from the course to the problem. In that laboratory activity the students were asked to think of a box. Then mentally they were to consider a box that is half as long, half as wide, and half as high as the original. They were then asked to guess how many of the smaller boxes would fit into the larger box. After making a guess they do the construction to check their answer. Subject S used the answer from that activity, namely eight, in this problem by taking the thirty minutes and dividing by eight to get an answer of three and three-fourth minutes.

Pretest 12. A barrel of honey weighs 50 pounds. The same barrel with kerosene in it weighs 35 pounds. If honey is twice as heavy as kerosene, how much does the empty barrel weigh?

Several types of errors were made as this problem was attempted. Four subjects used incorrect equations which either were not solved or when solved resulted in nonsensical answers that were not checked; two additional subjects performed meaningless operations on the data of the problem and got wrong answers which were not checked. Two other subjects gave up in short order.

If subjects could not use equations to solve this problem it seems that at least they could use trial and error. However, it again seems that on the pretest subjects had an aversion for this technique. Additionally, on this problem it is especially easy to check ones answer and yet this was hardly ever done.

Posttest 12. Three people play a game in which one person loses and two people win each game. The one who loses must double the amount of money that each of

> the other two players has at that time. The three players agree to play three games. At the end of the three games, each player has lost one game and each person has $8. What was the original stake of each player?

The most troublesome feature of this problem is that it is not explicitly stated that the payoff must be made from the loser's stake. Of course if one assumes the payoffs come from some source other than the original stakes, then there really is no problem and obviously each player started with two dollars. This was the approach taken by nine subjects. Four other students realized that the original stakes totaled twenty-four dollars and they unsuccessfully used trial and error to attempt to determine those values.

Since twenty-four different problems were considered in analyzing question 3, it was difficult if not impossible, to categorize and summarize the troublesome features of all problems. However, it was possible to identify certain features of the subjects' behavior which gave rise to trouble as they went about solving problems. It seemed that often subjects were not willing to carefully analyze a problem. Often they were satisfied with a superficial examination of the given conditions and applied what they thought they read to arrive at answers that were unreasonable. Many times these unreasonable answers were accepted without a simple check against the given conditions.

Lack of knowledge of certain concepts (for example, place value, area, and perimeter) and weaknesses in executing certain operations (for example, multiplication and exponentiation) also adversely affected performance. Finally, on the pretest subjects seemed to have an aversion for the use of trial and error. Many problems on that test that were left undone could have been solved without too much additional effort if the subjects had been more willing to use this problem-solving process.

Chapter 5

CONCLUSIONS, DISCUSSION, AND RECOMMENDATIONS

INTRODUCTION

The purpose of this study was fourfold. The first goal was to examine what heuristics preservice elementary teachers tend to use in solving non-routine mathematics problems before taking any required college mathematics and to determine if they have a preference for either "trial and error" or "deduction from condition" as a basic process in solving problems. The second purpose was to design and implement a required mathematics course for preservice elementary teachers in the structure of elementary school mathematics that has a problem-solving component as a unifying theme. This problem-solving component involves the use of problems to motivate new concepts, provides instruction in the use of heuristics in solving problems, and provides students with practice in solving non-routine mathematics problems. The third purpose was to examine the effects of such a course on the heuristic usage of students, on their problem-solving ability, and on their attitude toward mathematics and problem solving. The fourth and final purpose was to identify features of verbal non-routine problems that are especially troublesome for preservice teachers.

CONCLUSIONS AND DISCUSSION

Because of the clinical nature of this study and the associated informal evaluation techniques, the conclusions reached are tentative. However, they do give rise to a considerable number of ideas and proposals that can be used by instructors of future mathematics teachers. They provide such professionals with ideas as to what problem-solving behavior they might expect their students to demonstrate and they suggest what behaviors and attitudes are subject to change by a course of instruction such as the one used in this study.

This study determined that it is possible to use a problem-solving component as a unifying thread in a course in the structure of elementary mathematics for preservice teachers without sacrificing the usual content of such a course. The problem-solving component included instruction in the use of specific heuristics, practice in applying these heuristics in solving non-routine mathematics problems, and using non-routine problems to motivate new concepts introduced in the course. Thus, if improving preservice teachers' problem-solving ability is an objective of a teacher-training program, then the course introduced in this study could serve as a model for other institutions.

Other significant findings are as follows:

1. Before taking a required course in mathematics the low-ability preservice teachers included in this study had a tendency to use only two heuristics--Draws figure and Setting up equations, while the high-ability subjects tended to use these two and Simplifies the problem as well. This seems to reveal that the subjects were well versed in and used to employing these two common heuristics (Draws figure and Setting up equations) that are emphasized so much in high-school mathematics. However, they were using only a fraction of the abundance of problem-solving tools or heuristics available to them.

There was some evidence in this study that the course of instruction may have caused subjects to use the heuristic Setting up equations considerably less. Thus, subjects may have come to realize that it is not always necessary or desirable to attempt to solve problems by setting up equations.

Additional strategies employed on the posttest that were not used on the pretest were Restates problem in his own words and Separates/summarizes data for both groups and Simplifies the problem for the low-ability group. Thus the course of instruction was probably responsible for an increase in the variety of heuristics subjects employed.

2. It was demonstrated that before taking a mathematics course, preservice teachers of low-ability tended to prefer to use trial and error over against deduction as a basic process in solving non-routine problems. On the other hand, high-ability subjects tended to prefer to use deduction from condition. This was also the situation after the subjects completed the course of instruction. However, a consideration of the change in usage of these two processes within each group reveals both groups showed an increase in the use of deduction over the period of instruction and the low-ability group had the greatest increase in usage of deduction.

The preference of the low-ability group for trial and error on the pretest is not surprising. Such subjects typically either lack or have not been trained to use thinking skills that can be classified as deduction from condition. For them it is often easier to simply make wild guesses and arrive at an answer correct or incorrect. Likewise, it is not surprising that the ten-week course of instruction was not long enough to change this preference. However, it is encouraging that of the two groups, the low-ability group demonstrated the greatest increase in usage of deduction from condition.

It is interesting to note that although the low-ability group tended to prefer to use trial and error, there were instances on specific problems of the pretest when subjects of both groups left problems undone when in fact they could have been solved without too much additional effort if the subjects had resorted to trial and error. This was especially true of problems that lent themselves to solution by setting up equations. It was as if some subjects felt they had to solve the problems in one way and if they could not solve them that way, then they could not resort to any other technique such as trial and error. It seems that the course of instruction was effective in convincing subjects that there may be many ways to solve some problems and in helping them to learn when to apply various strategies or processes.

3. In terms of performance on the problem-solving inventories it seems that both groups spent more time working on the posttest than on the pretest and that the difference was greatest for the low-ability group. This may reflect that both groups, and especially those of lower ability, demonstrated greater perseverance on the posttest. This observation was reinforced for the low-ability group by the substantial increase in hesitations between tests.

A disappointing outcome of the study where little improvement in performance was realized was in the area time spent looking back at a problem. Neither group improved in the use of the looking back stage of the problem-solving process. It appears that a ten-week period of instruction is not sufficient to make this stage a part of the subjects' problem-solving routine.

When performance was measured by scores achieved on all twelve problems of both tests, there was no improvement of any consequence in performance. However, when attention was restricted to the seven similar problems of the two tests, it was seen that the high-ability group showed a striking increase in total points with a substantial part of this increase coming from the result stage of the problem-solving process. Since the posttest was considered to be substantially more difficult than the pretest, little change in score between tests cannot be considered a failure of the course of instruction. In addition the dramatic increase in the scores of the high-ability group on the seven similar problems may reveal that the course of instruction was especially beneficial to them.

Finally, an analysis of performance as measured by the commission of Executive and Structural errors reveals that the commission of such errors may have impeded the achievement of correct answers. This observation comes as no surprise and has been supported in numerous studies on problem solving.

4. Although the low-ability group started and ended with less confidence in their problem-solving ability than the other group, it seems that neither group's confidence was diminished through the time period of the treatment. Because of the difficulty inherent in measuring confidence, it is not surprising that only slight differences were detected. It may be the case that there indeed was very little change in confidence or it may be that the variables

used to measure it were not appropriate.

5. The treatment in this study had little effect on the attitude of the subjects toward mathematics. The low-ability group started and ended with a less than positive attitude toward mathematics whereas the high-ability group consistently showed a better attitude toward mathematics, both at the start and at the end of the experiment.

6. Concerning attitude toward problem solving, it is noteworthy that a large majority of the subjects of both groups ended the course indicating either a neutral feeling toward problem solving or a positive attitude toward problem solving. When asked to describe any changes in their attitude toward problem solving over the period of the course of instruction, the overwhelming majority of responses indicated that subjects either felt better at the end about the prospect of working non-routine problems and felt greater confidence in their problem-solving ability or they had such attitudes at the start of the experiment and the course simply reinforced what they felt before. In general it was the opinion of most of the participants in the study that although the problem-solving component of the course was difficult, they had nevertheless grown in their knowledge of strategies available to them, in their ability to apply these strategies to specific problems, and in their perseverance.

7. With regard to attitude toward the course, half

of the responses of the participants indicated that the course was influential in causing them to feel better about mathematics. The majority of the rest of the responses indicated that subjects already felt good about mathematics and the course did not have an adverse effect on their feelings. As would probably be expected, the problem-solving component was not the favorite part of the course for most people. The component that was the overwhelming favorite was the laboratory component. Needless to say, many of the laboratory experiences are centered on problems that would fit our definition from Chapter 1 of non-routine mathematics problems. Thus, it seems that problems that can be solved using manipulatives have the greatest appeal to preservice teachers.

8. Certain features of subjects' behavior which gave rise to trouble as they went about solving problems were identified. It seems that often subjects were not willing to carefully analyze a problem. At times they would understand and pay heed to all the terms and data given in the problem statement, but would disregard the syntax of the sentences. Consequently they were satisfied with a superficial examination of the given conditions and applied what they thought they read to arrive at answers that were incorrect and often unreasonable. Oftentimes there was an absence of a simple check of an answer against the givens of the problem which would have uncovered incorrect answers. Lack of knowledge of certain concepts (for example, place

value, area, and perimeter) and weaknesses in executing certain operations (for example, multiplication and exponentiation) also adversely affected performance.

RESEARCH RECOMMENDATIONS

As indicated in Chapter 2, REVIEW OF THE LITERATURE, it seems that the number of different approaches to researching problem-solving behavior is limitless. The present study with preservice teachers has centered on an examination of the process of problem solving rather than the product. It would be productive to conduct a similar study using an experimental design which would allow for statistical analysis of the many factors or variables considered. However, such a study would of necessity require the full-time effort of an experimenter and much part time help, especially for the interviewing of large numbers of participants and for coding the problem-solving protocols.

It is noteworthy that at about the time the present study was conducted the Georgia Center for the Study of Learning and Teaching Mathematics was forming a group of mathematics educators from throughout the country who were interested in researching problem solving. This "working group" has been pooling their research efforts and hopefully cooperative research programs will be developed which may shed further light on the issues studied here.

The correlation analysis in this study provides many questions that need further examination. Some of them are listed here.

1. Is there an explanation for the significant positive correlations between the use of visual representation of problems and the time spent on problems? Do the visual thinkers tend to have more difficulty with problems and thus they work longer on them or is there a relationship between visual thinking and confidence or perseverance?

2. Does the negative correlation between drawing figures and misinterpreting problems on the posttest indicate that visual representation of a problem is important for the "understanding the problem" stage of the problem-solving process?

3. The correlation analysis revealed a significant negative correlation of the process deduction with the process trial and error on the posttest while there was no such relationship on the pretest. Does this suggest that the course of instruction caused subjects to fall into one of two classifications which indicates a problem-solving mode--one group tending to use deduction and the other tending to use trial and error?

There are other questions that are prompted by the analysis done in this study. Both on the pretest and on the posttest there was the tendency for the low-ability group to use trial and error and for the other group to use deduction. What other factors (for example, personality factors) contribute to this difference? Or what cues are there in problems that cause subjects to try deduction from condition or trial and error?

How does a teacher help subjects to see the necessity of considering a problem both from a semantic and from a syntactic point of view? Too often in this experiment students were observed paying too much attention to one at the exclusion of the other. As a result, they arrived at incorrect answers.

One follow-up question that can always be asked in conducting a study such as this is, "What are the long-term effects of the treatment?" However, since this investigation was conducted with preservice teachers, a more important research question would be, "What effect, if any, does a course such as the one conducted for this study have on the teaching of the participants in the course?" or even more importantly, "What effect, if any, does a course such as this have on the problem-solving behavior of the pupils of the subjects who participated in the course?"

This study has suggested one possible treatment that has been relatively effective in changing certain problem-solving behaviors of the participants. Further research questions are, "What other treatments would be effective?" or "Is there a most efficient way to teach heuristics or to teach problem solving to preservice teachers?"

Obviously, there is much to be learned about how problems are solved and about effective ways of teaching problems solving. There are enough questions to keep researchers busy for decades and although the task is immense, it is the responsibility and the privilege of the mathematics education community to do all within its power to continue to try to resolve the many issues and questions before them in the realm of problem solving.

ISSUES AND RECOMMENDATIONS IN
MATHEMATICS EDUCATION

Teachers at all levels have a tendency to believe that their students should be better prepared when they reacher their classroom. Mathematics teachers are no exception. Most college mathematics teachers will say that their students cannot solve non-routine problems because such nave not been emphasized in elementary school and secondary school. The poor performance of the subjects of this study on the pretest indicates that something was lacking in their training in solving problems. Thus, an obvious recommendation is that students of all ages be provided with opportunities to improve their problem-solving abilities. This study has demonstrated one teaching model in which heuristic teaching was integrated into a typical classroom setting for preservice teachers without sacrificing the usual content of the course. Such a model could be adapted to almost any level of mathematics instruction.

The strong endorsement of the eight course handouts by the preservice teachers in this study reflects the value they placed on them in helping them to become better problem solvers. Similar instruments could be written for pupils of any age. Also, having problems worked in class, either by the instructor or by a student, where attention is given to Polya's outline of the problem-solving process is an effective technique that can be used in most mathematics classes.

Since laboratory activities seemed to be the most popular part of the treatment in this study, perhaps more problems should have been posed which could have been imbedded in a laboratory setting and which would have required some sort of concrete materials for solution. This may not only cause preservice teachers to feel better about the "active learning" of mathematics but also better about problem solving.

For the sake of making comparisons between low and high-ability subjects this study had students of both ability levels receiving the same treatment. As a consequence, there is no doubt that in general the low-ability subjects experienced more frustration and less success than the high-ability students, especially in working the assigned problems. Certainly if this instructional model were to be employed in other settings, instructors would want to make differentiated assignments of problems if the class had a broad spectrum of ability levels represented.

This study reflects an urgent need for high school algebra teachers to allow and encourage their pupils to use techniques for solving algebra word problems other than by setting up equations. Their students should come to realize that oftentimes such problems can be solved by inspection or trial and error (successive approximation). Teachers at all levels need to make students aware that more often than not, there are many ways to solve a given problem and if one approach does not work, another should be tried.

It was determined in this investigation that a ten-week period of instruction was not sufficient for inculcating the practice of "looking back." Aside from having the potential of being a very exciting part of the problem-solving process, the "checking of answers" aspect of this stage is critical for increasing accuracy. Mathematics teachers at all levels need to use this stage themselves as they demonstrate problems and need to constantly encourage their pupils to do the same.

Students are often told that the concept of "function" is a unifying notion in mathematics since it is used in almost all branches of mathematics. In an analogous fashion, the heuristics of problem solving can act as a unifier in mathematics. For these heuristics, as outlined by Polya (1954), are applicable to problems of all branches of mathematics. In addition, the application of these heuristics to a problem encourages independent thinking and indeed makes the problem solver a mathematician for the duration of the time he is working on the problem.

Finally, teachers at all levels may find it useful to modify the instruments used in this study to make them appropriate for their students and then take time to interview some of their students as they solve problems aloud. In so doing they may gain understanding as to the thought processes these students are using and thus be better able to prescribe instructional activities in problem solving.

Mathematics education in the elementary school has undergone vast changes during the last two decades. At times the changes have been radical and perhaps too radical for the best of all concerned. Nevertheless as classroom teachers and mathematics educators have together sifted through all the changes, much good has resulted in the areas of adapting the mathematics taught to the developmental level of the child, of structuring mathematics around certain key principles, of teaching mathematics and not just arithmetic, and of integrating mathematics with other disciplines. Of course one basic area where very little has changed over the years is in the teaching of problem solving. It seems that the time is right for cultivating interest in this area. The present investigation has demonstrated one model for helping inservice teachers to become better problem solvers. But yet there are so many other needs; in particular, helping inservice teachers to become better problem solvers, developing teachers' ability to teach problem solving, and finally putting quality materials on problem solving in the hands of teachers and students. These needs will take years to satisfy and hopefully the mathematics education community will meet the challenge.

APPENDICES

APPENDIX A

PRETEST AND POSTTEST IN MAIN STUDY

Pretest

1. If you take any two-digit number, form a new number by reversing its digits, and then add the two numbers together, what is the largest number that is always a divisor or factor of the sum?

2. What is the smallest number having 4, 6, 9, and 15 as divisors or factors?

3. Is 3688 a perfect square?

4. What is the remainder when 2^{87} is divided by 3?

5. How many numbers in the set $\{1, 2, 3, 4, \ldots, 800\}$ have either 2 or 5 but not 10 as a factor?

6. Cube A holds 5 quarts of water. Cube B is twice as long, twice as wide, and twice as high as cube A. How many quarts of water does cube B hold?

7. Find the maximum possible area for a rectangle with a perimeter of 72 inches.

8. A board is sawed into two pieces. One piece is two-thirds as long as the whole board and is exceeded in length by the second piece by 4 feet. How long was the board before it was cut?

9. If on a balance a brick balances evenly with three-quarters of a pound and three-quarters of a brick, what is the weight of a whole brick?

10. A carpenter makes only three-legged stools and four-legged tables. One day when he looked at his day's output he counted 31 legs. If he made 9 stools and tables that day, then how many of these 9 were stools and how many were tables?

11. Two missiles speed directly toward each other, one at 9,000 miles per hour and the other at 21,000 miles per hour. They start 1,317 miles apart. How far apart are they one minute before they collide?

12. A barrel of honey weighs 50 pounds. The same barrel with kerosene in it weighs 35 pounds. If honey is twice as heavy as kerosene, how much does the empty barrel weigh?

Posttest

1. If you take any three-digit number (e.g., 638), reverse the digits, and find the non-negative difference between these two numbers (e.g., 836 - 638 = 198), what is the largest number that is always a divisor or factor of this difference?

2. What is the smallest number having 9, 12, 22, and 24 as divisors or factors?

3. Millionnaire Jones wants to build a large patio behind his mansion. He has 692,482 square tiles (1 ft. x 1 ft.) to use for the patio. Can he construct a square patio which will use all of the tiles?

4. What is the remainder when 3^{197} is divided by 5?

5. There are no zeros at the end of 3! = 6, one zero at the end of 5! = 120, and 2 zeros at the end of 10! = 3,628,800. How many zeros are there at the end of 100!?

6. Cube X holds 8 times as much sand as cube Y does. How tall is cube X compared to cube Y?

7. An airline passenger fell asleep when he had traveled halfway. When he awoke, the distance remaining to go was half the distance he had traveled while asleep. For what part of the way did he sleep?

8. A man has 7 times as many quarters as he has dimes. The value of the dimes exceeds the value of the quarters by $2.50. How many has he of each coin?

9. If a chicken and a half lay an egg and a half in a day and a half, then how many eggs do three chickens lay in one day?

10. A farmer has hens and rabbits. These animals have 50 heads and 140 feet. How many hens and how many rabbits has the farmer?

11. The leavening agent in an unusual bread dough makes the dough double itself in size every minute. If it takes

30 minutes for one loaf to just fill an oven, how long will it take two loaves to just fill the same oven?

12. Three people play a game in which one person loses and two people win each game. The one who loses must double the amount of money that each of the other two players has at that time. The three players agree to play three games. At the end of the three games, each player has lost one game and each person has $8. What was the original stake of each player?

APPENDIX B

PRETEST PROBLEM-SOLVING INVENTORY

Problem-Solving Inventory

General Information:

Name:______________________________Age:_____Date:___________

Name of High School and City:_______________________________

High School Math Courses (course name and year taken):

____

Your College Class (term, year; e.g., 2nd quarter Freshman):

____

Program at Concordia (Elem. Ed., Sec. Ed., D.C.E., etc.):

____

Major and Minor or concentrations __________________________
For how many quarter hours are you registered this term?____
For how many hours per week are you employed?_______________

Instructions:

The purpose of this interview is to obtain some information on the ways in which students enrolled in Structure of Elementary Mathematics solve problems. This is for my own information and will be used to improve instruction and assign problems in this course in the future. This is not a test and will have absolutely nothing to do with your grade in the course.

This booklet consists of a set of problems that you are to solve and to think aloud as you are solving each problem. This means that in addition to writing your scratch work, your steps to the solution, and the solution, you are being asked to say out loud everything that you are thinking about while working each problem. You will be tape recorded while you do this. The reason for recording your work is so that I can get a clearer picture of the thought processes used in arriving at solutions.

There are several other instructions that you should follow as you work through the inventory:

a) Read each problem aloud before beginning.

b) Talk in your usual tone of voice (but clear enough to be understandable).

c) If you need pencil and paper to solve a problem, please write down everything you would usually write down while solving a problem--including scratch work, diagrams, equations, calculations, etc. Be sure to talk as you write.

d) Do not erase anything. If you decide not to use something already written down, draw a line through it

e) If there is something about the problem statement that you do not understand, you may ask me for clarification. However, I will not tell you if you are "on the right track" or if the answer you obtain is right or wrong.

f) If a given problem seems too hard, you may leave it and come back to it later.

Since other students will be participating in these interviews, please do not discuss the problems or the interview with anyone. This would invalidate the results for the whole class. If someone has questions, tell him or her to see me. Your cooperation is greatly appreciated.

Sample Problem A:
What is the perimeter of a square whose area is 25 square inches?

Sample Problem B:
To number the pages of a book a printer used 131 digits. How many pages has the volume?

1. Cube A holds 5 quarts of water. Cube B is twice as long, twice as wide, and twice as high as cube A. How many quarts of water does cube B hold?

2. What is the smallest number having 4, 6, 9, and 15 as divisors or factors?

3. How many numbers in the set $\{1, 2, 3, 4, ... , 800\}$ have either 2 or 5 but not 10 as a factor?

4. A carpenter makes only three-legged stools and four-legged tables. One day when he looked at his day's output he counted 31 legs. If he made 9 stools and tables that day, then how many of these 9 were stools and how many were tables?

5. If on a balance, a brick balances evely with three-quarters of a pound and three-quarters of a brick, what is the weight of a whole brick?

6. A barrel of honey weighs 50 pounds. The same barrel with kerosene in it weighs 35 pounds. If honey is twice as heavy as kerosene, how much does the empty barrel weigh?

7. Is 3688 a perfect square?

8. Two missiles speed directly toward each other, one at 9,000 miles per hour and the other at 21,000 miles per hour. They start 1,317 miles apart. How far apart are they one minute before they collide?

9. If you take any two-digit number, form a new number by reversing its digits, and then add the two numbers together, what is the largest number that is always a divisor or factor of the sum?

10. What is the remainder when 2^{89} is divided by 3?

11. A board is sawed into two pieces. One piece is two-thirds as long as the whole board and is exceeded in length by the second piece by 4 feet. How long was the board before it was cut?

12. Find the maximum possible area for a rectangle with a perimeter of 72 inches.

APPENDIX C

POSTTEST PROBLEM-SOLVING INVENTORY

Problem-Solving Inventory

Name:_______________________________Age:_____Date:___________

Instructions:

The purpose of this interview, as with the earlier interview in which you participated, is to obtain some information on the ways in which students enrolled in Structure of Elementary Mathematics solve problems. This is for my own information and will be used to improve instruction and assign problems in this course in the future. This is not a test and will have absolutely nothing to do with your grade in the course.

As before, this booklet consists of a set of problems that you are to solve and to think aloud as you are solving each problem. This means that in addition to writing your scratch work, your steps to the solution, and the solution, you are being asked to say out loud everything that you are thinking about while working each problem. You will be tape recorded while you do this. The reason for recording your work is so that I can get a clearer picture of the thought processes used in arriving at solutions.

Again, there are several other instructions that you should follow as you work through the inventory.

a) Read each problem aloud before beginning.

b) Talk in your usual tone of voice (but clear enough to be understandable).

c) If you need pencil and paper to solve a problem, please write down everything you would usually write down while solving a problem--including scratch work, diagrams, equations, calculations, etc. Be sure to talk as you write.

d) Do not erase anything. If you decide not to use something already written down, draw a line through it.

e) If there is something about the problem statement that you do not understand, you may ask me for clarification. However, I will not tell you if you are "on the right track" or if the answer you obtain is right or wrong.

f) If a given problem seems too hard, you may leave it and come back to it later.

Since other students will be participating in these interviews, please do not discuss the problems or the interview with anyone. This would invalidate the results for the whole class. If someone has questions, tell him or her to see me. Your cooperation is greatly appreciated.

Sample Problem A

Which, if either of the following regions, has the greater area?

1. Millionnaire Jones wants to build a large patio behind his mansion. He has 692,482 square tiles (1 ft. x 1 ft.) to use for the patio. Can he construct a square which will use all of the tiles?

2. An airline passenger fell asleep when he had traveled halfway. When he awoke, the distance remaining to go was half the distance he had traveled while asleep. For what part of the way did he sleep?

3. What is the smallest number having 9, 12, 22, and 24 as divisors or factors?

4. Cube X holds 8 times as much sand as cube Y does. How tall is cube X compared to cube Y?

5. If you take <u>any</u> three-digit number (e.g., 638), reverse the digits, and find the non-negative difference between these two numbers (e.g., 836 - 638 = 198), what is the largest number that is always a divisor or factor of this difference?

6. A man has 7 times as many quarters as he has dimes. The value of the dimes exceeds the value of the quarters by $2.50. How many has he of each coin?

7. Three people play a game in which one person loses and two people win each game. The one who loses must double the amount of money that each of the other two players has at that time. The three players agree to play three games. At the end of the three games, each player has lost one game and each person has $8. What was the original stake of each player?

8. The leavening agent in an unusual bread dough makes the dough double itself in size every minute. If it takes 30 minutes for one loaf to just fill an oven, how long will it take two loaves to just fill the same oven?

9. If a chicken and a half lay an egg and a half in a day and a half, then how many eggs do three chickens lay in one day?

10. What is the remainder when 3^{197} is divided by 5?

11. There are no zeros at the end of 3! = 6, one zero at the end of 5! = 120, and 2 zeros at the end of 10! = 3,628,800. How many zeros are there at the end of 100!?

12. A farmer has hens and rabbits. These animals have 50 heads and 140 feet. How many hens and how many rabbits has the farmer?

APPENDIX D

CHECKLIST VARIABLES AND CODING FORM FOR PROBLEM-SOLVING PROTOCOLS

Subject No. ____
Problem No. ____
Date ____
Pretest-Posttest ____
Coder ____

	Tape Reading	Time
Time: Excluding looking back	____	____
looking back	____	____
Total	____	____

Score: Approach (0-1) ____
Plan (0-3) ____
Result (0-2) ____
Looking back (0-2) ____
Total ____

- -

Approach:

__Restates problem in his own words
__Performs exploratory manipulations
__Draws figure

Production:

__Recalls same or related problem
__Uses method of related problem
__Uses result of related problem
__Misinterprets problem

Looking Back:

__Checks solution by substitution into equation
__Checks that solution satisfies condition
__Checks solution by retracing steps
__Checks if solution is reasonable/realistic
__Derives the solution by another method
__Generalizes the problem
__Considers extreme cases
__Checks that all information is used
__Attempts to find other solutions

Comments:

__Requests assistance, more information
__Questions existence of solution
__Questions uniqueness of solution
__Indicates uncertainty about final solution
__Says he does not know how to solve the problem
__Expresses enjoyment, liking for the problem
__Expresses distaste, dislike for the problem
__Admits confusion
__Expresses confidence that he can arrive at a solution

Executive Errors:

Numerical computation ____
Algebraic manipulation ____
Other

Interviewer Comments:

PROCESS SEQUENCE

201

APPENDIX E

DESCRIPTION OF CHECKLIST VARIABLES

Approach

The Approach classification contains any behavior which occurs as the subject trys to understand the problem. The three checklist categories follow.

1. Restates problem in his own words is checked if the subject makes any attempt to clarify to himself what the problem says or is asking.

2. Performs exploratory manipulations refers to any calculations that are performed by the subject mainly just to see what he gets as opposed to those performed on the basis of some condition or clue in the problem. For example, a subject may simply add two numbers that appear in the problem just to see what results. If the calculation leads to a useful result or if it seems that the subject has some plan for the use of the result, the category is not checked.

3. Draws figure is checked if the subject makes any kind of figure, diagram, or sketch, no matter how primitive so long as it is intended to contribute to his understanding of the problem.

Production

Polya (1954:98) claims that ". . . We can scarcely imagine a problem absolutely new, unlike and unrelated to any formerly solved problem; but, if such a problem could exist, it would be insoluble. In fact, when solving a problem, we always profit from previously solved problems, using their result, or their method, or the experience we acquired solving them. And, of course, the problems from which we profit must be in some way related to our present problem."

The Production classification of the checklist provides for the discrimination of what sort of recall the subject is using to generate a solution to the problem and for the indication of a misinterpretation of the problem. The four checklist categories follow.

1. Recalls same or related problem is checked if the subject indicates that he has seen the problem before or one similar to it. It is not checked if the subject simply says "this is an algebra problem" or "I'll have to find the prime factorization of these numbers."

2. Uses method of related problem refers to a subjects indication that he recalls a similar problem whose method of solution may have some applicability to the present problem.

3. Uses result of related problem is checked when the subject recalls and uses the result of some related problem.

4. If the subject indicates by means of comments or by manipulations of problem elements that he has either misread or misinterpreted a crucial part of the problem, then the coder checks Misinterprets problem.

Looking Back

Looking back, Polya's fourth stage of the problem-solving process includes not only the checking of a solution but also the reconsideration and reexamination of the path that led to it, the consideration of other solutions, and the investigation of generalizations of the problem. The nine checklist categories follow.

1. Checks solution by substitution in equation is checked if the subject substitutes a solution for an unknown in an equation he derived for the problem. It is not checked if he substitutes trial values into the derived equation.

2. When a subject substitutes the value of his solution into the condition of the original problem to determine its correctness, the coder marks Checks that solution satisfies condition.

3. Checks solution by retracing steps is checked if the subject specifically repeats or goes over an operation or manipulation after attaining a solution. It is not coded if the subject simply rereads the problem since this usually indicates he is checking to see that the solution obtained satisfies the condition.

4. If the subject reflects on the reasonableness of his solution or checks in some way to see if the solution is consistent with what he knows about the "real world" then the category Checks if solution is reasonable/realistic is coded.

5. Derives the solution by another method is coded if after arriving at a solution the subject proceeds to try to solve the same problem using a different approach or technique. This is not coded if a subject attempts to solve the problem in a second way after the first approach failed.

6. When the subject gives consideration to a more general problem after solving the stated problem the category Generalizes the problem is checked.

7. Considers extreme cases is coded when an attempt is made to expand the meaning and implication of the problem and its solutions by considering extreme values for the data and substituting these into the conditions of the problem. It is conceivable that this behavior is observed with a check of the problem. So, for example, a subject may consider extreme cases and in so doing is checking to see if his solution is reasonable or realistic. In such a case both appropriate categories are checked.

8. Checks that all information is used is checked if after arriving at a solution the subject determines if all the data and conditions of the problem have been used. This is not checked if the subject does such a check during the process of solving the problem initially.

9. When some clue in the problem or its solution causes a subject to suspect that there may be more than one solution to the problem and he in turn trys to find it, then Attempts to find other solutions is coded.

Comments

Comments that a subject makes that are not directly related to the problem-solving process can provide insight into his feelings toward himself and his problem-solving ability. They can be a measure of his confidence and of his willingness to admit failure verbally. The nine checklist categories follow.

1. Oftentimes a subject will have questions regarding the meaning of certain phrases or words in the problem and the assumptions that he can make in attempting a solution. Such requests are coded as Requests assistance, more information. Requests concerning procedure are not coded. This category can be a measure of the student's self-sufficiency or independence in problem solving as well as an indication of the difficulty level of the problems.

2. When the subject suggests in some way that there

may not be a solution to the problem then Questions existence of solution is coded.

3. If for some reason the subject feels that there may be more than one solution, or there is not enough information to determine exactly one solution, then the coder checks Questions uniqueness of solution.

4. Indicates uncertainity about final solution is checked only if as leaving a problem a subject makes remarks which indicate that he doubts the correctness of his solution. It is not checked if he makes such remarks and then proceeds to rework the problem or if he has not achieved any solutions.

5. Says he does not know how to solve the problem is self-explanatory. However, it is only coded if the subject makes the remark as he is leaving the problem. This category is a measure of the student's tendency to admit defeat.

6&7. Expresses enjoyment, liking for the problem and Expresses distaste, dislike for the problem are categories which give a measure of the student's feelings toward given problems and thus to problem solving in general. They are indicated by comments such as "This is a neat problem" or "What a dumb question."

8. If a subject admits in some way that the problem, his manipulations of the givens or his solution causes him some amount of bewilderment, then Admits confusion is checked.

9. Expresses confidence that he can arrive at a solution is self-explanatory. The category attempts to measure a student's confidence in his problem-solving ability.

Executive Errors

Executive errors are those that are committed in carrying out manipulations, either computational or algebraic. They are usually the result of carelessness, or temporary memory lapse rather than from lack of understanding. Tallies are kept to represent the total number of executive errors of each category. An error is not tallied if it is immediately corrected. The three checklist categories follow.

1. Each time a subject makes an error in counting or in performing a numerical algorithm or computation, a tally

is made in the category Numerical computation.

2. Each error in the manipulation of an algebraic equation is tallied under Algebraic manipulation.

3. Other executive errors include such things as miscopying or misreading of a number or meaning one thing while writing or saying something else (e.g., while thinking "The carpenter makes one three-legged stool" he says "The carpenter makes three stools").

Interviewer Comments

Observations of interest to the coder are recorded under Interviewer Comments. Observations may concern such things as the student's state of mind, nervousness and how this may be affecting his performance, and so forth.

Time

The time a subject works on a problem is divided into three parts. Time Excluding looking back is measured from the point the subject starts the problem until he has arrived at a solution or stops working on the problem without having arrived at a solution. Time spent Looking back is measured from the point where the subject has arrived at a solution to the point where he completes the checking of his work or whatever other behavior which can be classified as "Looking back." Total time is then the sum of time excluding looking back plus time looking back.

Tape readings are recorded at the time of the interview. These in turn are converted into units of time where 15 seconds equals one unit of time.

Score

Points are used to give a numerical indication of the quality of a student's performance. The maximum possible number of points awarded on any given problem is eight points. The eight points are broken down into the following categories.

1. Approach (0-1): If a subject understands the problem and if his work is free of structural errors due to a misinterpretation of the problem, then he is awarded one point. Otherwise, he is given zero points.

2. Plan (0-3): When a problem has been approached in such a way that the correct result would follow if no executive errors were committed or when a sufficient number of trials have been carried out or equations written to arrive at the correct result, then three points are earned. Two, one, or zero points are awarded according to the extent to which these things are not done.

3. Result (0-2): This category is self-explanatory.

4. Looking back (0-2): Zero to two points are awarded in this category according to whether the subject checks his solution or his work in some way after having arrived at a solution. Points are also awarded if he attempts to solve the problem in another way, or if he tries to generalize the problem, or if he engages in any other activity which can properly be classified as "looking back" behavior.

APPENDIX F

PROCESS-SEQUENCE CODING VARIABLES

Preparation Symbols:

R Reads the problem

G Separates/summarizes data (e.g., "What is given," "What must I find")

M Introduces model by means of a diagram or figure

Process Symbols:

D Deduction from condition

T Random trial and error

A Reasoning by analogy

C Reasoning by contradiction

E Setting up equations

S Simplifies the problem

L Looks back

N Nonsense process or behavior not classifiable

Results of D and T Processes:

1 Abandons process

2 Impasse

3 Intermediate result (Correct or incorrect)

4 Incorrect result

5 Correct result

Punctuation Marks:

- (dash) Hesitation of approximately 1 unit (15 seconds)

() Scope of D and T processes

, Inserted between successive processes

/ Stops without solution

. Stops with solution (correct or incorrect)

<u>Errors</u>:

$\overline{X}$ Structural error in process X

X* Executive error in process X

$\overline{X}$' Previous structural error in process X corrected

X*'Previous executive error in process X corrected

<u>Difficulty</u>:

$\underline{X}$ Indicates subject had difficulty with process X

APPENDIX G

DESCRIPTION OF PROCESS-SEQUENCE CODING VARIABLES

Process-sequence codes are a sequence of letters, numerals, punctuation marks and other symbols that are written in order as the coder listens to the tape recording of a subject's problem-solving protocol. One process-sequence code is written for each problem that is attempted. It provides the researcher with a sequential picture of the subject's problem-solving behavior. The sequential nature of the process-sequence codes is what distinguishes these from the information tallied in the checklist; for the checklist deals with supplementary nonsequential information.

Preparation Symbols:

An R is recorded for Reads the problem when the subject reads the problem and thereafter when he rereads the problem or parts of it as he tries to better understand it. An R is not recorded if the subject rereads the problem to aid him in drawing a figure, writing an equation, or performing a check. The frequency of R's can be regarded as an indicator of the difficulty the subject has with the problem.

If a subject attempts to operate on the data in some way, for example, if he asks, "What is given" or "What must I find," then a G is written to denote Separates/summarizes data.

When a subject draws a figure or sketch, a M is written in the process-sequence code to denote Introduces model by means of a diagram or figure.

Process Symbols:

If a subject attempts to achieve a logical conclusion on the basis of a reasoning process on some condition or conditions given in the problem, then a D is recorded for Deduction from condition.

In contrast to deduction, a subject uses Random trial and error (T) if instead of operating on the data using the condition to guide his reasoning, he chooses a trial value to be substituted into the condition. T is often difficult to distinguish from D. When there is doubt about which is being used, D is used under the assumption that more than likely the subject is reasoning in some way from the condition.

An A, Reasoning by analogy is recorded if the subject recalls the same or what to him is a similar problem and attempts to solve the present problem on the basis of what he recalls concerning that previous problem.

A C, Reasoning by contradiction is written when the subject determines that a goal (or subgoal) cannot possibly be obtained from the givens and conditions of the problem since it is inconsistent with these.

When a subject writes any symbols in conjunction with an equality symbol, an E is recorded for Setting up equations (e.g., "x=age of father" is considered an equation).

An S, Simplifies the problem represents an attempt to gain insight to the solution of the problem by considering a simpler problem.

If a person checks his work or solution in some way, tries to solve the problem in a different way, or tries to generalize the problem or the solution, then L for Looks back is recorded. The type of behavior observed under this category is coded in the checklist.

A N, Nonsense process or behavior not classifiable is recorded for exploratory manipulations (i.e., operations lacking rationale), mumbling, and unclear or irrational statements.

Results of D and T Processes:

The D and T processes are always followed by a numeral which indicates the outcome of the performance of the given process. The numeral 1 (Abandons process) indicates that the subject has abandoned the process before completion. This is often done because the subject is not sure how to carry out the process or does not believe the process will be productive. The numeral 2 (Impasse) indicates that the process has led to a contradiction or absurdity which precludes further progress. The numeral 3 (Intermediate result or observation (correct or incorrect)) is used if the process used has led to a result which, if correct, may lead to a solution to the problem. The numeral 4 (Incorrect result) is recorded if the process leads to a wrong answer to the problem and the numeral 5 (Correct result) is used if the process leads to a correct solution to the problem.

Punctuation Marks:

Dashes (-) are inserted in the process-sequence codes where there are hesitations in the subject's protocol. One dash represents a hesitation of 15 seconds. The subject's behavior during such hesitations often includes silence, at which time the subject may be trying to come upon some tactic which will aid him in solving the problem or his mind may simply be wandering. He may also be orally or silently checking over the work he has already performed. Dashes are not recorded for time spent doing calculations or requesting assistance from the interviewer.

Parentheses (()) are used to indicate the scope of a D and T. Often times a subject, while arguing from the condition of a problem towards a solution (deduction), will do so by perhaps setting up an equation (E) or by reasoning by contradiction (C) and so forth. These steps that are a part of the deductive process are included in parentheses.

Commas (,) are used to separate successive process symbols.

A virgule (/) indicates that a subject has stopped without a solution. A count of the number of virgules for a given subject is a measure of the difficulty he had with the problem-solving inventory.

A period (.) indicates the subject stopped with a solution, either correct or incorrect.

Errors:

Structural errors stem from a misunderstanding of the problem or of some principle necessary for its solution. For example, a student may set up an equation for the solution of a problem incorrectly because he does not understand the problem or some phrase in the problem. This error would be a structural error. Structural errors are contrasted with executive errors, or slips in carrying out manipulations. For example, a student may set up an equation correctly for the conditions given in a problem but then may make a numerical computation error or an algebraic manipulation error. This type of error would be an executive error.

The codes for structural error or executive error in process X are clear. It should be remarked, however, that if a subject catches his error immediately after making it, then it is not coded. On the other hand, if he makes one of the two types of error and later discovers the error and corrects it, then an apostrophe (') is recorded behind the symbol that indicated the original error.

The total number of executive errors is kept by tallies on the checklist and among other things is a measure of the carelessness of the subject. The total number of structural errors is a measure of the ability of the subject to grasp the relationships and principles embodied in the problem-solving inventories.

Difficulty:

Difficulty is indicated by hesitation and stumbling on the part of the subject. If such behavior occurs as a subject is performing process X, then X is underscored.

Miscellaneous guidelines for coding:

1. If at some point in his problem-solving protocol, a subject elaborates on the thought processes he used earlier in the problem to such an extent that it causes the coder to change his initial impression of what symbols to use, then the coder should go back and make appropriate changes in the process-sequence code.

2. There are times when a subject will determine a provisional solution and not being sure of its correctness will try to solve the problem again. In such cases this latter activity should not be coded as "Looking Back." A subject is only to be regarded as looking back at a problem if his behavior and comments clearly show that he is attempting to check his work or engage in some other activity of this category.

3. Trial values that lead to a contradiction are coded as T2. In problems such as number 9 of the pretest (see Appendix B) or number 5 of the posttest (see Appendix C) where trial values typically lead to intermediate values on which a deduction will later be made, the trial values are coded as T3.

4. When equations are deduced from the conditions of a problem and are written incorrectly because of a structural error, then this is coded as $\overline{D}$(E...). That is the bar goes over the D and not the E.

5. If a subject in attempting to compute such expressions as 2^{π} shows that he does not understand what

2^{89} represents, then his error is recorded as a structural error.

6. If a subject repeats an observation that was made earlier, then this probably indicates that he is rethinking the problem and what he has done thus far and this is coded as a hesitation.

7. If a subject makes an impulsive guess concerning the solution of a problem and then proceeds to examine it more closely, then the initial guess is coded as D3 or $\overline{D}3$.

APPENDIX H

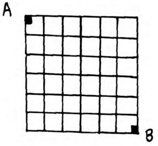
COURSE HANDOUTS

Adam Loves Betty

Problem:

At the right is the map of a city. Adam lives at A and his sweetheart Betty lives at B. Adam walks to Betty's house regularly and is curious as to:

a) The route or routes which are shortest and
b) How many different shortest routes he could take to her house?

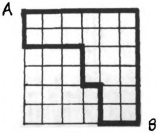


I. Understanding the Problem:

The most difficult thing about the statement of the problem could center on what is meant by "different routes." We would probably infer from the problem that only routes that follow the indicated streets are permissible (i.e., cutting through alleys, for example, would not be allowed) and that one route is different from the other if at least one block is included in the one that is not included in the other.

II. Devising a Plan:

With some trial and error and a little reflection, one decides rather quickly that to get from point A to point B using the shortest possible route, Adam must <u>in total</u> travel 6 blocks East and 6 blocks South. Below are <u>2</u> possible routes Adam could take.



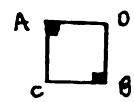
Clearly these are both examples of shortest possible routes and are 12 blocks long. Thus part (a) of the problem is easy.

In attempting to determine how many possible routes Adam could take we may try to simply count them. Try this! Do you find this to be very difficult or even impossible? Why?

Well, then, if we can't count the number of routes, how will we solve the problem? It seems that we have decided that what makes this problem so difficult is that it is so big or complex. Thus one might wonder if we could find a related problem that is much simpler and which could be solved rather easily. Can you formulate such a "simpler problem"?

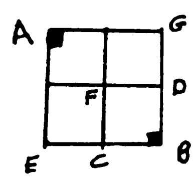
III. Carrying Out the Plan:

One such problem could be diagramed as follows, where A is Adam's house and B is Betty's house.



Clearly for this problem, Adam must first reach either point C or point D before reaching point B. So since there is only one way of reaching point C and one way of reaching point D, there are 1 + 1 = 2 ways of reaching point B.

Now we consider a slightly tougher problem. (Call it the 2 x 2 problem.)

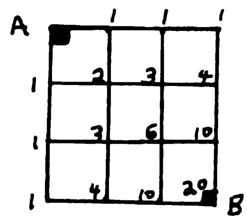


In order for Adam to reach point B, he must first reach points C or D; and to reach C, or D he must first reach E, F or G, and so forth. We see that:

> There is 1 way to reach point G
> There are 2 ways to reach point F and
> There is 1 way to reach point E.

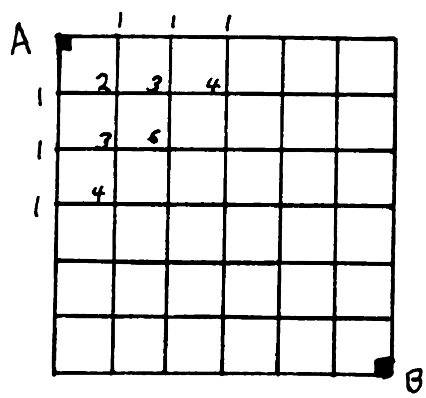
Since Adam gets to point D only by first getting to points F or G, there are 2 + 1 = 3 ways to get to point D. Similarly, since he only gets to point C by first getting to points E or F, there are 1 + 2 = 3 ways to get to point C. Finally, since there are 3 ways to get to C and 3 ways to get to D, there must be 3 + 3 = 6 ways to get to B.

This type of analysis can easily be applied to the 3 x 3 problem shown below. The numerals at the street intersections represent the number of possible different routes that Adam may take in going from A to the given intersection.



Thus there are 20 ways of getting from A to B in the 3 x 3 problem.

Now let's return to the original (6 x 6) problem. Can you fill in the appropriate numbers at the intersections left unmarked and in so doing determine the number of different routes from A to B?



The number of different "shortest routes" from point A to point B is ______.

IV. Looking Back:

The "array" of numbers you constructed in the 6 x 6 problem above may "ring a bell" for you. You may recall seeing it in the following form and calling it "Pascal's Triangle" in a course in elementary algebra.

1	0th row
1 1	1st row
1 2 1	2nd row
1 3 3 1	3rd row
1 4 6 4 1	4th row
1 5 10 10 5 1	5th row
⋮	⋮

The numbers in the nth row provide the coefficients in the algebraic expansion of $(a + b)^n$. So, for example:

$$(a + b)^0 = \underline{1}$$
$$(a + b)^1 = \underline{1}a + \underline{1}b$$
$$(a + b)^2 = \underline{1}a^2 + \underline{2}ab + \underline{1}b^2$$
$$(a + b)^3 = \underline{1}a^3 + \underline{3}a^2b + \underline{3}ab^2 + \underline{1}b^3$$
$$\vdots \qquad \vdots$$

A second interesting application of Pascal's Triangle occurs in counting combinations. You may recall that given any set containing n objects, any subset of that set containing k objects is said to be a combination of k elements chosen from n elements. The number of such combinations is given by the expression $\frac{n!}{k!\ (n-k)!}$.

As an example of this, consider the set of 5 elements $\{a,b,c,d,e\}$.

The combinations of 2 elements taken from this set are:

$\{a,b\}$	$\{b,d\}$
$\{a,d\}$	$\{b,e\}$
$\{a,c\}$	$\{c,d\}$
$\{a,e\}$	$\{c,e\}$
$\{b,c\}$	$\{d,e\}$

We note that there are 10 of them and this is the same number that appears as the 3rd element of the 5th row of Pascal's Triangle. We see that

The number of combinations of 0 objects taken from 5 objects is

$$\frac{5!}{0! \quad (5-0)!} = 1$$

(Can you list it?)

The number of combinations of 1 object taken from 5 objects is

$$\frac{5!}{1! \quad (5-1)!} = 5$$

(Can you list them?)

The number of combinations of 2 objects taken from 5 objects is

$$\frac{5!}{2! \quad (5-2)!} = 10$$

(See example above)

The number of combinations of 3 objects taken from 5 objects is

$$\frac{5!}{3! \quad (5-3)!} = 10$$

(Can you list them?)
And so forth.

Can you find any relationship between this first and second application of Pascal's Triangle?

Can you find at least 5 different number patterns in Pascal's Triangle?

V. Discussion:

The basic strategy that we employed in solving the original problem was "simplification." We solved a rather difficult problem by first solving a simpler problem. How can one tell if the "simplification" strategy will be useful on a specific problem? There are no hard and fast rules, but the following situations illustrate the types of problems on which it is natural to try simplification.

1. Given a problem involving an n x n array of some sort, consider an analogous problem with a smaller array. The "Adam loves Betty" problem above is an example of this type.

2. Given a problem involving n quantities of some sort, consider an analogous problem with fewer quantities.

3. Given a problem involving an arbritrary triangle, consider first the case of an isosceles or equilateral or right triangle.

4. Given a problem in 2 (or 3) dimensions, i.e., in a plane (or in space), consider an analogous problem in 1 (or 2) dimension(s), i.e., on a line (or plane).

Finally, a strategy very similar to the "simplification" strategy is the "related problem" strategy. At times a very difficult problem may be solved by first solving a related problem. The related problem may not actually be a simpler one but merely one to which a solution is known.

Cutting Up

Problem:

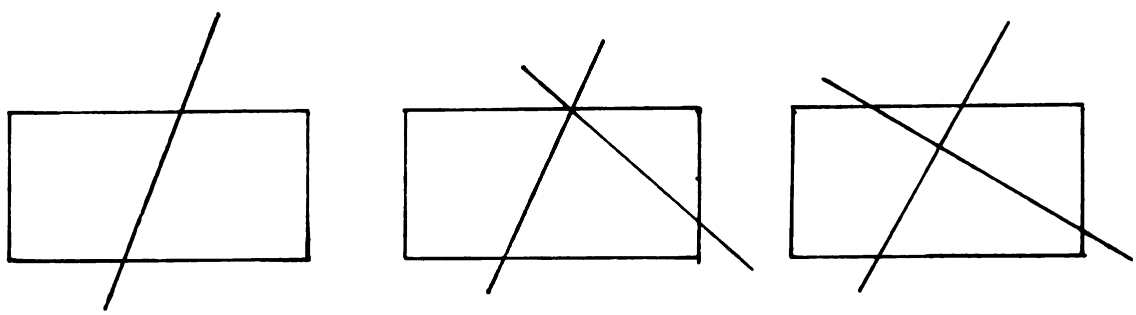
Given a 3-inch by 5-inch card, what is the largest number of pieces you can make by cutting through the card 8 times?

I. Understanding the Problem:

One question that may come to mind immediately is, "Is it permissible to stack the pieces of cards after making a cut?" Since such a procedure would make the problem much more difficult than it already is, let's assume that this is not permissible. We also want to pay special attention to the requirement that we are to achieve the maximum number of pieces.

II. Devising a Plan:

As with other problems we have seen, this problem seems too complex to solve outright. Thus we might want to start by considering a simpler problem; in fact the simplest possible problem, a 1-cut problem. Clearly, 1 cut makes 2 pieces. We also see that 2 cuts can be made in such a way that we get either 3 pieces or 4 pieces. Obviously we want the latter. (See figures below.)



1 cut 2 pieces | 2 cuts 3 pieces | 2 cuts 4 pieces

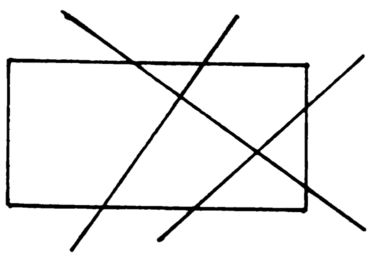
Can we proceed in this fashion? Can we make more cuts and eventually <u>see a pattern</u> which will aid us in determining the maximum number of pieces when using 8 cuts? Let's try it!

III. Carrying Out the Plan:

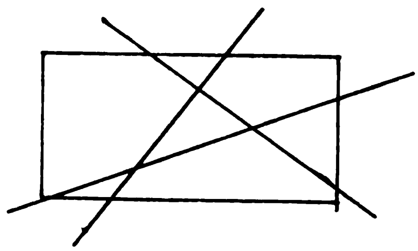
Since we hope to see a pattern which will aid us in achieving a solution we had better find some way to organize our data. With this in mind, we will fill in the following table.

number of cuts	maximum number of pieces
1	2
2	4
3	
4	
5	
6	
7	
8	

What is the maximum number of pieces with 3 cuts? Consider the diagrams below.



3 cuts
6 pieces



3 cuts
7 pieces

Is 7 the largest number of pieces one can get with 3 cuts?

Can you fill in the remainder of the table above?

IV. Looking Back:

In filling in the table did you see any procedure or rule which when followed would assure you of the maximum number of pieces for a given number of cuts? Can you state such a rule?

Did your rule say something to the effect that "for any given cut to result in the maximum number of pieces, the cut must pass through each of the preceeding cuts"? The discovery of this rule leads rather nicely to another discovery. If you have not already made this discovery the following table may help.

number of cuts made	number of lines newest cut passes through	number of add. pieces	total no. of pieces
1	0	2	2
2	1	2	4
3	2	3	7
4	3	4	11

We see that at each stage (with the exception of the first) the "number of additional pieces" is equal to the "number of cuts made." These discoveries aid one greatly in filling in the table on page 2.

Finally, anytime one has a problem whose solution is arrived at by means of a table such as that on page 1, one naturally wants to find a general solution. Can you find a formula which will result in the maximum number of pieces for any number of cuts n?

number of cuts	maximum number of pieces
1	2
2	4
.	.
.	.
.	.

Alphabetic Arithmetic

Problem:

The following three lines represent an addition problem.

```
  D O N A L D
+ G E R A L D
  -----------
  R O B E R T
```

Each letter represents a unique digit and a given digit is represented by only one letter. Determine the addition problem. (Hint D=5)

I. Understanding the Problem:

We should have little difficulty in understanding this problem. It is clear that each letter will represent only one digit and that letter represents that digit each time it appears in the problem. We can also assume that none of the three numbers start with 0, that is D $\neq$ 0, G $\neq$ 0, and R $\neq$ 0.

II. Devising a Plan:

We might first observe that there are exactly 10 distinct letters in the problem and thus we will "use up" all 10 digits. Also, since D = 5, we can rewrite the problem as follows:

```
          /
  5 O N A L 5
+ G E R A L 5
  -----------
  R O B E R 0
```

where clearly T = 0 and we have regrouped one group of ten in the ten's column.

At this point it may be helpful to agree on a simple way to refer to the different values of the problem. Let's agree to call the hundred-thousands place Column 1, the ten-thousands place Column 2, and so forth.

Now to proceed any further in solving the problem, we need some special insights. What special features of the problem may provide these insights? We may have noticed that:

(a) In column 1 there is no regrouping since 5 + G = R. Also it must be the case that G equals 1, 2, 3, or 4 and thus R = 6, 7, 8, or 9.

(b) In column 2 we have O + E = O. What sort of restrictions does this place on E?

(c) In columns 4 and 5 we have doubles, A + A and L + L. Does this observation help any?

III. Carrying Out the Plan:

From column 2, since O + E = O, it must be the case that E equals 0 or 9 (if E = 9, then 1 must be regrouped from column 3). But E cannot equal 0 since T = 0. Thus E = 9.

Now from column 5, since 1 + L + L = R, R must be odd (since L + L is even, no matter what L is). But from Column 1 we observed R = 6, 7, 8, or 9. Thus R = 7 or 9. But R cannot equal 9 since E does. Thus, R = 7. Now our problem looks like the following:

```
   ' '
  5 O N A L 5
+ G 9 7 A L 5
-------------
  7 O B 9 7 0
```

Clearly 1 is regrouped into column 1. Thus G = 1.

Now how about L? L = 3 or 8. To decide which we notice that in column 4 since 9 is odd, it must be the case that 1 was carried from column 5. Thus L = 8. Does this help us to determine A? A must equal 4 or 9. But A $\neq$ 9 since E = 9. Thus A = 4. Now our problem has this form.

```
  ' '   ' '
  5 O N 4 8 5
+ 1 9 7 4 8 5
-------------
  7 O B 9 7 0
```

We must yet determine O, N, and B and we have the digits 2, 3, and 6 left to assign. Since we must carry 1 to column 2, N must equal 3 or 6. But N cannot equal 3 because if so, then B = 0 which cannot be since T = 0. Thus N = 6. Hence, B = 3 and thus O must equal 2 since that is all that is left. Our completed problem is:

```
  5 2 6 4 8 5
+ 1 9 7 4 8 5
-------------
  7 2 3 9 7 0
```

IV. Looking Back:

Since this problem is quite close-ended, the "Looking Back" stage of the solution is not overly exciting. In this stage we should at least check our answer to be sure that it satisfies the conditions of the problem. A second component of this stage for this problem could be to formulate other problems of this type and present them to your friends to see if they can solve them. You might try this!

V. Discussion:

If there is any one heuristic that is exemplified in the solution to this problem it is the strategy or heuristic of "contradiction." This strategy is often useful in problems in which you must decide which of two or more goals could be reached from the givens of the problem. The effective use of contradiction results in your knowing which goals cannot be reached from the givens and thus also which goals can be reached. So for example, at one stage in our solution above we decided that N must equal 3 or 6. To assume that N = 3 would lead to the "contradiction," B = 0 and T = 0. Thus we must conclude that N = 6.

"Magic" Cards

The "Magic" Cards below work as follows: You, the "Magician," ask someone to name the letters that identify all the cards on which his age appears. You in turn will tell him his age by adding the numbers in the upper left hand corner of the cards he names. (For example, if he tells you his age is found on cards E, D, and A, then you can tell him with absolute confidence that his age is 16 + 8 + 1 = 25.)

E		D		C		B		A	
16	17	8	9	4	5	2	3	1	3
18	19	10	11	6	7	6	7	5	7
20	21	12	13	12	13	10	11	9	11
22	23	14	15	14	15	14	15	13	15
24	25	24	25	20	21	18	19	17	19
26	27	26	27	22	23	22	23	21	23
28	29	28	29	28	29	26	27	25	27
30	31	30	31	30	31	30	31	29	31

Question: Why does this "trick" work?

I. Understanding the Problem:

If you do not have a clear understanding of how the "trick" works, then take a couple examples and work them out. For example, does the trick work for your own age?

The question in this problem is "why does this trick work?" Obviously, the cards are not "magic." There has to be some logical explanation for why they work.

II. Devising a Plan:

Some students may be tempted, primarily out of desperation, to say that the cards were arrived at through trial and error. That is someone "played around" with the numbers until he found the right combinations to make the trick work. While this is certainly a possibility it seems highly unlikely. Additionally, it makes the prospect of extending the cards so that they will work for ages greater than 31 rather unappealing.

A second approach is the following. One might ask himself what is so important about the numbers in the upper left hand corner of the cards -- 16, 8, 4, 2, and 1? Clearly they form a pattern. We see at once that they are each powers of 2 -- 2^4, 2^3, 2^2, 2^1, 2^0. Perhaps we also realize that these powers are five of the place values of a base two system of numeration. Ah ha! Could it be that there is a relationship between the base two representation of a number and the cards that that number is on. This seems like a reasonable idea to pursue.

III. Carrying Out the Plan:

Let's take 25 for example. The base two representation of 25 is $25 = (11001)_{two}$. Recall that $(11001)_{two} = 1 \times 16 + 1 \times 8 + 0 \times 4 + 0 \times 2 + 1 \times 1$.

Also we note that 25 is on card E and D, is not on card C and B and is on card A. Thus, when I write a number as a base two numeral, a 1 in the sixteens place indicates the number should be on cards E and a 0 in the sixteens place indicates the number should not be on card E, and so forth.

So now it seems that I can take any number from 1 to 31 and explain why it is on some cards and not on others. Thus, I have explained why the "trick" works!

IV. Looking Back:

Now that we have answered the original question to our satisfaction, there are several related questions that we wish to pursue. First, can you add a card F and modify cards A through E above so that the cards will accommodate ages greater than 31? With the proper addition of a single card F, what is the largest age for which the cards will work?

Secondly, can you discover a number pattern of any sort on the cards? Is there some number pattern that when properly applied will aid you in creating a card F?

Thirdly, we have seen an integral relationship between the base two numeral representing the ages and the cards. Can we develop a set of cards based on a base three representation of the ages? Try your hand at this!

V. Discussion:

In analyzing how we arrived at our solution, it is clear that we first observed a pattern -- the 1, 2, 4, 8, and 16 on the upper left hand corner of each card. Seeing this pattern was the key in our arriving at a solution.

It is often the case in solving problems that the discovery of a pattern will play a key role. At times the pattern is obvious and at other times, it may take considerable effort to see it.

A second key to the solution of the problem involved making the connection between the pattern we observed and a base two system of numeration. In other words, we were able to make a connection between a pattern and some previous learning. The ability to make such connections aids the problem-solving process greatly.

Missionaries and Cannibals

Problem:

Three missionaries and three cannibals wish to cross a river. There is a boat which can carry three people and either a missionary or a cannibal can operate the boat. It is never permissible for cannibals to out number missionaries, neither in the boat nor on either shore. What is the smallest number of trips necessary to make the crossing?

I. Understanding the Problem:

Although the problem simply asks for the "smallest number of trips necessary," it is clear that to determine this number one must first answer the more interesting question of what sequence of trips with what passengers will achieve the goal of transfering all six people from one side to the other. Additionally, this sequence of trips must be the most efficient method possible so as to minimize the number of trips.

II. Devising a Plan:

For the first trip across the river one could send over either

(a) 1 cannibal (but then he would have to return with the boat and nothing would have been accomplished), or

(b) 1 cannibal and 1 missionary with the missionary making the return trip, or 2 cannibals with one of them making the return trip, or 2 missionaries (but this would not work since there would then be 3 cannibals and 1 missionary on the shore, or

(c) some combination of 3 missionaries and cannibals with the intent of leaving 2 on the other side and having 1 make the return trip.

Since it is plan (c) that maximizes the number of people who are left on the other bank and since we wish to accomplish the goal with the least number of trips, let us pursue this plan.

III. Carrying Out the Plan:

There are 4 ways that the first crossing of the river could be made with 3 people in the boat. They are

(1) MMM (meaning all three missionaries cross with one of them returning with the boat), or

(2) MMC or

(3) MCC or

(4) CCC

In case (1) the missionary who returns with the boat would be eaten by the cannibals. In case (2) the missonary left on the first bank of the river would be eaten by the 2 cannibals there. In case (3) the missionary on the boat would be eaten by the 2 cannibals. Thus only case (4) is possible.

Considering case (4) then, we will have 1 cannibal return to the first bank (2 trips). If he were to go back to the second bank with 2 missionaries, then the 3 cannibals would eat those 2 missionaries on the second bank. Thus, it is clear that the cannibal must remain on the first bank and the three missionaries must take the boat across to the second side (trip 3). Two missionaries remain on the second side and 1 returns to the first side (4 trips). He picks up the cannibal on the first side and together they cross to the second side (5 trips).

From our analysis it is clear that 5 is the least number of trips in which the goal can be accomplished. They can be symbolized by

$$\overrightarrow{CCC}$$
$$\overleftarrow{C}$$
$$\overrightarrow{MMM}$$
$$\overleftarrow{M}$$
$$\overrightarrow{MC}$$

IV. Looking Back:

Are there other sequences of 5 trips which can accomplish the same goal? See if you can find some.

V. Discussion:

If there is any one problem-solving strategy or heuristic that is apparent in this solution, it is that of "considering special cases." The first time this heuristic was applied was in Section II, "Devising a Plan." There we decided it might be productive to consider the 3 special cases.

Case (a) 1 person makes first crossing

Case (b) 2 people make first crossing

or Case (c) 3 people make first crossing

After deciding that case (c) would probably be the best case to pursue we broke it down into other special cases. Namely,

Case 1 MMM make first crossing

Case 2 MMC make first crossing

Case 3 MCC make first crossing

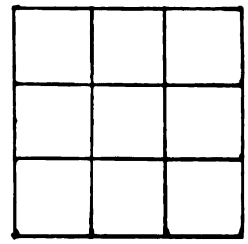
Case 4 CCC make first crossing

A simple analysis of thse cases told us that only case 4 would possibly lead to a solution. This sort of an approach was continued until we finally reached a solution. In summary, we see that the consideration of all special cases eventually led to a solution of the original problem.

Magic Squares

Problem:

Fill in the nine squares at the right with the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 in such a way that the sums of the three numbers in each of the three columns, each of the three rows, and each of the two diagonals are all the same.

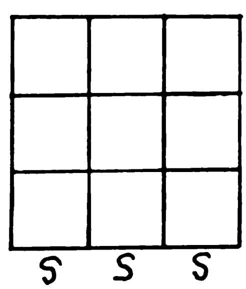


I. Understanding the Problem:

The statement of this problem is quite clear. Obviously, what makes the square "magic" is the fact that each of the 8 sums indicated in the problem are all equal to the same number. Let's agree to call this number the "magic sum."

II. Devising a Plan:

It surely would be helpful if we could determine the "magic sum." This would give us a "target" to shoot for as we try to fill in the boxes. A little reflection convinces us that this is not difficult to determine. For suppose we simply let S equal the magic sum. Then consider the 3 sums formed by the 3 columns of the magic square.



Since S + S + S represents the sum of all the numbers in the magic square we have

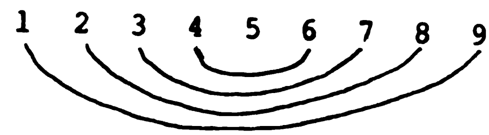
$$S + S + S = 1+2+3+4+5+6+7+8+9$$

$$\text{or} \quad 3S = 45$$

$$\text{or} \quad S = 15.$$

Knowing that our magic sum is 15 should aid us greatly in filling in the blanks. What other observations can we make that will aid us in reaching our goal?

We may note that 15 is odd and in order for 3 numbers to have that sum they must all be odd or 2 must be even and 1 odd. Secondly, we may note that there are 4 different ways to make a sum of 10 which do not use the number 5. These are indicated in the diagram below.



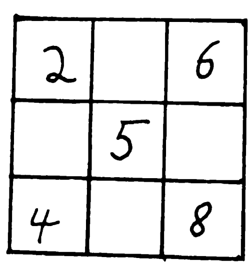
III. Carrying Out the Plan:

The fact that there are 4 even numbers between 1 and 9 suggests that these might be arranged in either of the two ways indicated by the X's below.

A B

But clearly the arrangement in square A would not work since we cannot add 2 odds and 1 even (as required by the first column for example) and get an odd sum. Thus arrangement B is the one we want to use.

Now since 2 + 8 = 10 and 4 + 6 = 10, let's put 5 in the center square and start our magic square as indicated below.



Complete the magic square.

IV. Looking Back:

There are many questions that could be posed at this time which are related to the original problem. Some examples are:

(a) Can you make a 3 x 3 magic square using 2, 3, 4, 5, 6, 7, 8, 9, 10 or 3, 4, 5, 6, 7, 8, 9, 10, 11, etc.?

(b) Can you make a 4 x 4 magic square using 1, 2, 3, 4, . . . , 16 or using 2, 3, 4, 5, . . . , 17, etc.?

(c) Can you find a form for a general 3 x 3 magic square?

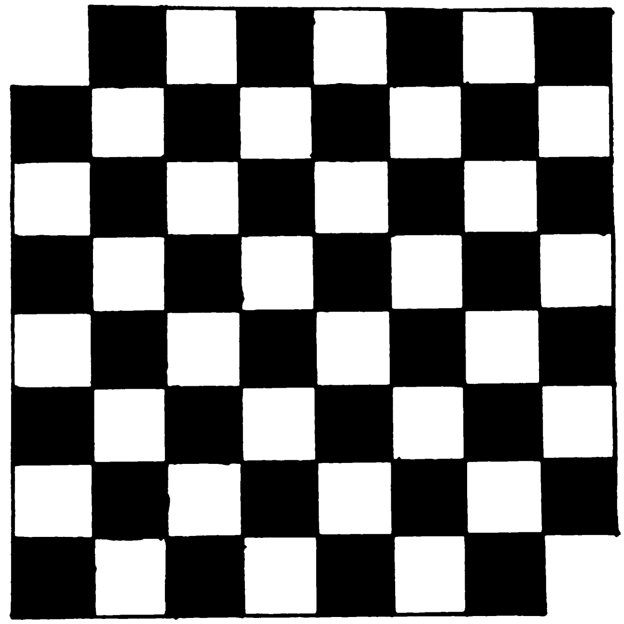
(d) Can you construct a magic square using only primes?

(e) Can you construct a "multiplicative magic square?"

You might want to try to state other magic square problems. Can you solve some to those above or some posed by yourself.

Two Missing Pieces

Problem:



You are given a checkboard and 32 dominoes. Each domino covers exactly two adjacent squares on the board. Thus, the 32 dominoes can cover all 64 squares of the checkboard. Now suppose two squares are cut off at diagonally opposite corners of the board (see the figure). Is it possible to place 31 dominoes on this board so that all the 62 remaining squares are covered? If so, show how it can be done. If not, show why not.

I. Understanding the Problem:

One should probably have very little trouble understanding what this problem is asking. One striking feature about the problem, however, is that it is open-ended, i.e., part of the solution involves determining whether or not something is possible. In our past experience, open-ended problems have probably been the exception rather than the rule.

II. Devising a Plan:

As a first attempt to solve this problem, many people will start mentally covering the grid above with 2 by 1 dominoes and thus will attempt to arrive at a solution by trial and error. One soon discovers that this approach is very time consuming and it is quite difficult to try to keep track of the "trials" one has made. Is there a better way to attack this problem than by trial and error? Is there something else stated in the problem that may aid us? For example, why does the problem talk about a checkboard instead of simply an 8 by 8 grid? A checkboard has different colored squares! Ah ha! Can we in some way use the colors of the squares to achieve a solution?

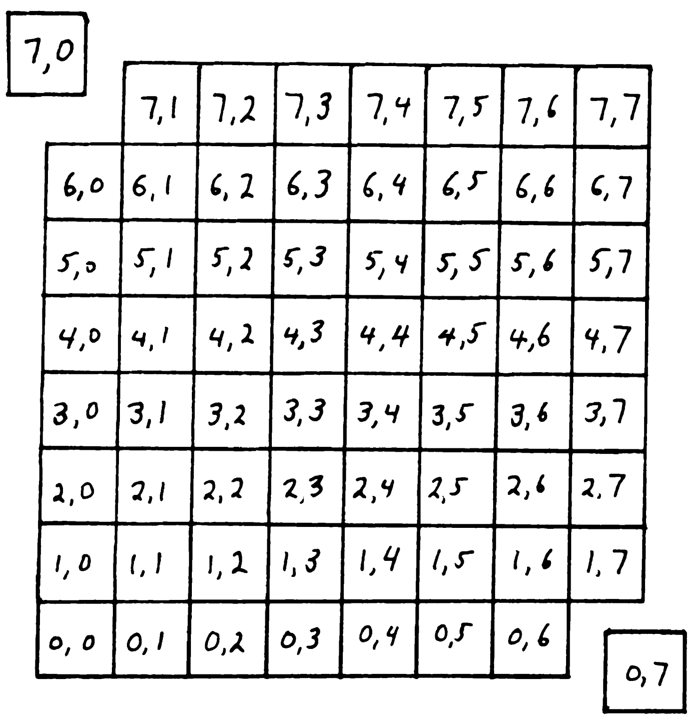
We notice that the two squares missing on our board are both white! That means that there are 32 black squares and 30 white squares left on the board. Can these be covered with 31 dominoes?

III. Carrying Out the Plan:

We note that anywhere we place a domino on the board, it must cover one black square and one white square. Thus if we place 31 dominoes on the board, they must cover 31 black squares and 31 white squares. Thus, it would be impossible to cover the 32 black squares and 30 white squares with 31 dominoes.

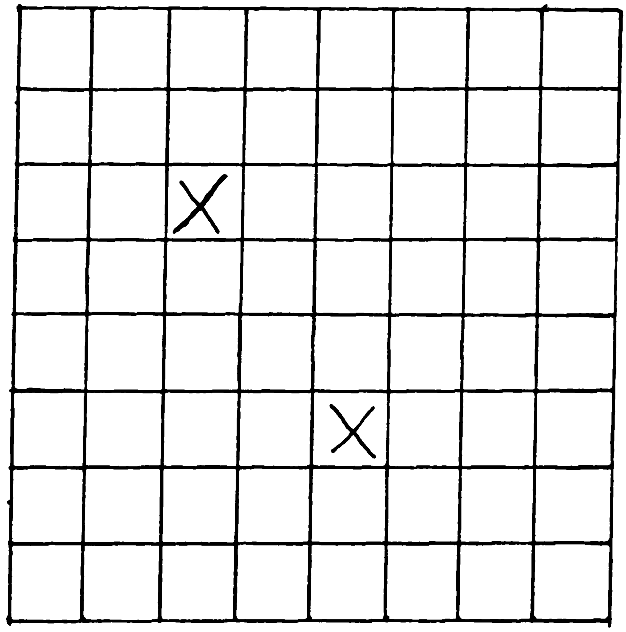
IV. Looking Back:

Suppose the statement of the original problem had not included the fact that the board was a checkboard. Suppose you were simply given an 8 x 8 grid with two opposite corners cut off. Instead of using colors of squares as a key for the solution, one might have instead tried to establish a coordinate system on the grid such as the one shown.



We then observe that a domino will only cover adjacent pairs of squares whose coordinates sum to an odd number. Since the sum of the coordinates of the 2 squares cut off from the board is 14 or even, we can conclude that we will not be able to cover the 62 squares with the 31 dominoes.

We have seen before that it is often productive to try to generalize a problem and its solution. How would we generalize this problem? One way would be to suggest that we delete any two little squares we wish from the 8 by 8 grid. For example, consider the grid below (the x on a square means that square is missing).



Can it be covered by 31 dominoes? Why or why not? What about other such grids with 2 squares labeled? Can you state a general solution?

I Win

Problem:

Fifteen pennies are placed on a table in front of two players. Each player must remove at least one penny on his move and not more than five pennies on his move. The players alternate turns, each removing from one to five pennies n number of turns until one player takes the last penny on the table, and wins all 15 pennies. Is there a method of play that will guarantee victory? If so, what is it?

I. Understanding the Problem:

Anytime one is asked to develop a winning strategy for a game, it is imperative that he first play the game a few times to develop a "feeling" for it. Having done this, one is then ready to attempt an indepth analysis of the game.

In this game it is important to notice that the last person to pick up pennies is the winner. This is noteworthy because there are other forms of this popular game in which the opposite is the case, the last person to pick up coins is the loser.

II. Devising a Plan:

To facilitate communication we will label the pennies 1, 2, 3, 4, . . . , 15 with 1 meaning the first penny removed, 2, the second penny removed, . . . , and 15 meaning the fifteenth penny removed.

In playing the game, participants quickly discover that they can tell who the winner will be at least on the second to last move, if not sooner. So, for example, if after a few rounds of the game the point is reached where player A takes

coin 10
or coins 10 and 11
or coins 10, 11, and 12
or coins 10, 11, 12, and 13
or coins 10, 11, 12, 13 and 14, then it

is clear to both players that player B will win the game by taking the rest of the coins that are on the table. Thus, it follows that if player B can take coin 9 or coins up to and including 9 on a given move, then he will be able to win the game no matter what player A does.

It appears that what we have done in the preceeding paragraph is to work backwards from the case where the person who takes coin 15 is the winner to the case where the person who takes coin 9 can be the winner. Can we use this same strategy and work backwards again? Will this process eventually lead to the development of a method of play which will guarantee victory? Let's try it!

III. Carrying Out the Plan:

If player B is to be the eventual winner, that is the person who is to take coin 9 or coins up to and including 9 on a given move. Then in order for him to achieve this enviable position, his opponent, player A must have taken either

coin 4
or coin 4 and 5
or coin 4, 5, and 6
or coin 4, 5, 6, and 7
or coin 4, 5, 6, 7, and 8 on a given

move.

And how can player B force player A into the position where he only has the choices listed immediately above? Clearly, he can do this by playing first and taking coins 1, 2, and 3 or if player A goes first and takes coins 1 or 1 and 2 then he can take coins 2 and 3 or coin 3 respectively. Clearly, if player B does not go first and player A takes coins 1, 2, and 3 or 1, 2, 3, and 4 or 1, 2, 3, 4, and 5, then player B must simply hope that player A does not know a winning strategy and will provide him with an opportunity to get an "upperhand" in the game. For example, if player A goes first and takes coins 1, 2, and 3, then player B might decide to take coin 4. If player A would take only coins 5 and 6, then player B should take coins 7, 8, and 9. He would then be in a position to win the game.

In summary, a winning strategy has been developed in this game for the person who goes first. If one does not play first, then he must hope his opponent does not know the winning strategy and must wait for the opportunity to take the upper-hand on a mistake by his opponent.

IV. Looking Back:

There are many other ways to play this game. For example, suppose there are 50 pennies and a move consists of taking either 1 or 2 or 3 or 4 or 5 coins. Can a winning strategy be developed? What is it?

What if a move consists of taking either 1 or 2 or 3 or 4 coins. Can a winning strategy be developed? What is it?

Suppose the original game is modified so that the person who takes the last coin is the loser. Can a winning strategy be developed? What is it?

APPENDIX I

THREE EXAMPLES OF APPLYING THE CODING SCHEME

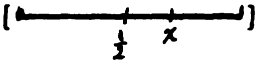
This appendix contains the transcription of the audio tapes of three subjects as they worked problems from the problem-solving inventories. Following the transcriptions are the corresponding coding forms which contain the checklist variables and the process-sequence codes for each example.

The transcriptions employ these symbols. Square brackets [ ] contain material written by students in their test booklets as they worked through a problem. The location of the brackets in the transcription corresponds as closely as possible to the time the student was writing it in his test booklet. Parentheses ( ) contain interviewer comments. A dash - indicates a pause by the subject of approximately fifteen seconds. The symbol ⊃ indicates where a subject leaves a problem with the intent of returning to it later.

<u>Pretest, Subject S, Problem 6</u>

A barrel of honey weighs 50 pounds. The same barrel with kerosene in it weighs 35 pounds. If honey is twice as heavy as kerosene, how much does the empty barrel weigh? Ok, honey is twice as heavy as kerosene [$H= 2K$], so the barrel of honey weighs 50 pounds; the barrel of kerosene weighs 35 pounds, so there's a difference of 15 pounds in there. [H - 50 / K - 35 / 15] - So, let's see, the difference of 15 pounds, - - So, that means honey is 15 pounds heavier than kerosene [$H = K + 15$]. So kerosene would equal 15 pounds, so 35 minus 15 is equal to 20 pounds [$35 - 15 = 20$]. So, let's see, that would leave the barrel weighing 20 pounds, the kerosene 15, and the honey, no wait. The kerosene weighing 35, that's right. Wait. The barrel is 20 pounds and the kerosene, uh, boy! Hum - - - I think that's right. I'll go on. (OK, so your conclusion is that the barrel is 20?) Yes. (OK) I suppose I should check it over, [Barrel 20lbs. K - ~~15~~ ~~35~~ H -] Huh, all right, well let's see. A barrel of honey is 50. The same barrel is 35, that's 15 difference [$50 - 35 = 15$]. The honey equals twice the kerosene. Hum. So, the honey, the kerosene is 30, no, that's not right. Uh. Let's see, 50 minus 20 is 30 [$50 - 20 = 30$]. Leave the kerosene at 15 pounds. Um, that's right, the barrel weighs, is 20 pounds. Ok.

<u>Posttest, Subject G, Problem 2</u>

An airline passenger fell asleep when he had traveled halfway. When he awoke, the distance remaining to go was half the distance he had traveled while asleep. For what part of the way did he sleep? Ok, I'm just going to kind of diagram this out. An airline passenger fell asleep when he had traveled halfway. When he awoke, the distance remaining, when awoke the distance remaining to go was half the distance he had traveled while asleep. For what part of the way did he sleep? [

] - - Ok, I would say that he had traveled halfway when he woke up, the distance remaining was half the distance he had traveled while asleep. So, um, just say, if I said hum, the distance he traveled while he was asleep was x, and then the distance remained to go was half the distance traveled while asleep so it would be one-half x [$\frac{1}{2}x$]. For what part of the way did he sleep? Do you mean like the time span or, (Huh, what fractional part of the total did he sleep.) Oh, Ok. Say like from one-half of the trip to whatever or just, (Huh, from one-half to whatever fractional part, and if you can determine that then you can name the fractional part that he slept of the whole.) Ok. Ok, so one-half x is what I said would be the distance remaining and then x would be what it was when he slept so that would have to equal one-half [$\frac{1}{2}x + x = \frac{1}{2}$] because he fell asleep when he was one-half way. So, just to determine that, so one and one-half x hum, equals one-half [$1\frac{1}{2}x = \frac{1}{2}$]. Let's see if that would be right. - - Hum, maybe, it, I'll just finish out this problem and see how it works, but I think I'm not on the right track. So x would equal one half divided by one and one half or one half times, well, that would be the same as saying 3 halves. [$x = \frac{1}{2} / 1\frac{1}{2} = \frac{3}{2}$] so one half times two thirds from inverting would be three fifths [$\frac{1}{2} \cdot \frac{2}{3} = \frac{3}{5}$] hum - - - Let's see, that, hum, - It's like, I guess since I have three fifths it would be three-fifths of the one-half, so if I found the common, let's see, I'm trying, this isn't coming to me like, the three-fifths of one half, [$\frac{3}{5}$ of the $\frac{1}{2}$], - hum, - - Ok, once he reached the one half, the halfway distance that he traveled, he slept for three-fifths of the second half and then he woke up for two-fifths of the last of the second half. Do you need it more specific than that, or (No, then that's your answer. Then if that's your answer, that is explicit enough for me.) Ok, I'll say from the halfway point, he went three-fifths of the remaining distance before waking up. Ok, [From the halfway point he went $\frac{3}{5}$'s of the remaining distance before waking up.]

Posttest, Subject J, Problem 2

An airline passenger fell asleep when he had traveled halfway. When he awoke the distance remaining to go was half the distance he had traveled while asleep? For what part of the way did he sleep. Ok. - Well, if he fell asleep when he had traveled halfway, then he was awake for half. [awake $\frac{1}{2}$, asleep $\frac{1}{2}$] - - Ok, so he was asleep for half the distance but then for half of that he woke up. - So, he was asleep for one-fourth of the way [$\frac{1}{4}$]. (Ok, will you describe how you arrived at that?) Ok, Well, if he fell asleep when he had traveled halfway, then he was awake for half and asleep for half. But, then it says he woke up, when he woke up the distance remaining to go was half the distance he had traveled while asleep. Oh, wait, - Ya, one fourth because, hum, - Ya, because if he was awake halfway, and then when he woke up he had half the distance to go of when he went to sleep. Hum - Ya, I think that's right, but I'd like to come back to that later. An airline passenger fell asleep when he had traveled halfway. When he awoke, the distance remaining to go was half the distance he had traveled while asleep. For what part of the way did he sleep? Hum. - Ok, the first time he was awake, it was half of the distance. Now, the distance remaining was half the distance he had traveled while asleep. [awake/_____]. Ok, if this is the distance he had to travel as he was awake for, part of this he slept for, - - He - Hum - The distance remaining to go was half the distance, - - How do I do this? - Ok, this isn't right. (One-fourth is not right?) No. (Ok) Because, if he slept for a fourth of the way, that means that when he awoke, the distance he had left would be an eighth and then if he slept a fourth, he'd be missing a fourth - Hum - - - (Can you tell me what you are thinking?) Well, like I'm just trying my different fractions, here, like I tried, like if he slept, if he was awake halfway and slept halfway, then when he awoke, the distance remaining to go was half the distance he had traveled while asleep. Well, that would be impossible if he slept halfway and was awake halfway. Then I tried like if he slept an eighth of the way then the distance remaining to go, huh, when he awoke would be one sixteenth; that doesn't add up to a whole trip. He traveled halfway - - - ok, I was just thinking if he slept a third of the way, ok, you'd have the half while he was awake plus the third while he was asleep, plus the distance remaining to go is half the distance he had traveled while asleep, would be one sixth [$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} =$], I think - ya, Ok, so then if you wanted to add these up, you have to change this to three sixths and this to two sixths, this one sixth [$\frac{3}{6} + \frac{2}{6} + \frac{1}{6} = \frac{6}{6}$] ya, so that would add up to a whole trip. It would have to be one third.

CHECKLIST VARIABLES AND CODING FORM FOR PROBLEM-SOLVING PROTOCOLS

Subject No. 5
Problem No. 6
Date 3-4-76
Pretest-Posttest (Pretest circled)
Coder G.K.

		Tape Reading	Time
Time:	Excluding looking back	219-270	11.8
	looking back	270-292	5.1
	Total		16.9

Score:	
Approach (0-1)	1
Plan (0-3)	3
Result (0-2)	2
Looking back (0-2)	2
Total	8

Approach:

- __Restates problem in his own words
- __Performs exploratory manipulations
- __Draws figure

Production:

- __Recalls same or related problem
- __Uses method of related problem
- __Uses result of related problem
- __Misinterprets problem

Looking Back:

- __Checks solution by substitution into equation
- ✓ Checks that solution satisfies condition
- __Checks solution by retracing steps
- __Checks if solution is reasonable/realistic
- __Derives the solution by another method
- __Generalizes the problem
- __Considers extreme cases
- __Checks that all information is used
- __Attempts to find other solutions

Comments:

- __Requests assistance, more information
- __Questions existence of solution
- __Questions uniqueness of solution
- __Indicates uncertainty about final solution
- __Says he does not know how to solve the problem
- __Expresses enjoyment, liking for the problem
- __Expresses distaste, dislike for the problem
- __Admits confusion
- __Expresses confidence that he can arrive at a solution

Executive Errors:

Numerical computation ____
Algebraic manipulation ____
Other

Interviewer Comments:

PROCESS SEQUENCE

R, D(E) 3, D3, — — — D3, D5, — — — . L

CHECKLIST VARIABLES AND CODING FORM FOR PROBLEM-SOLVING PROTOCOLS

Subject No. G
Problem No. 2
Date 5-11-76
Pretest-Posttest (circled: Posttest)
Coder G.K.

Time:	Tape Reading	Time
Excluding looking back	070 - 171	27.1
looking back		
Total		27.1

Score:	
Approach (0-1)	1
Plan (0-3)	3
Result (0-2)	1
Looking back (0-2)	0
Total	5

Approach:

- __ Restates problem in his own words
- __ Performs exploratory manipulations
- ✓ Draws figure

Production:

- __ Recalls same or related problem
- __ Uses method of related problem
- __ Uses result of related problem
- __ Misinterprets problem

Looking Back:

- __ Checks solution by substitution into equation
- __ Checks that solution satisfies condition
- __ Checks solution by retracing steps
- __ Checks if solution is reasonable/realistic
- __ Derives the solution by another method
- __ Generalizes the problem
- __ Considers extreme cases
- __ Checks that all information is used
- __ Attempts to find other solutions

Comments:

- ✓ Requests assistance, more information
- __ Questions existence of solution
- __ Questions uniqueness of solution
- __ Indicates uncertainty about final solution
- __ Says he does not know how to solve the problem
- __ Expresses enjoyment, liking for the problem
- __ Expresses distaste, dislike for the problem
- __ Admits confusion
- __ Expresses confidence that he can arrive at a solution

Executive Errors:

Numerical computation 1
Algebraic manipulation __
Other

Interviewer Comments:

$\frac{1}{2} \cdot \frac{2}{3} = \frac{3}{5}$

PROCESS SEQUENCE

R, M -- $\underline{D}^{*}$(E - - - - - - - - - - - -)4.

CHECKLIST VARIABLES AND CODING FORM FOR PROBLEM-SOLVING PROTOCOLS

Subject No. J
Problem No. 2
Date 5-5-76
Pretest-Posttest (Posttest circled)
Coder G.K.

Time:	Tape Reading	Time
Excluding looking back	128 - 175 II 029 - 151	12.6 32.7
looking back		
Total		45.3

Score:	
Approach (0-1)	1
Plan (0-3)	3
Result (0-2)	2
Looking back (0-2)	0
Total	6

Approach:

- __ Restates problem in his own words
- ✓ Performs exploratory manipulations
- ✓ Draws figure

Production:

- __ Recalls same or related problem
- __ Uses method of related problem
- __ Uses result of related problem
- ✓ Misinterprets problem

Looking Back:

- __ Checks solution by substitution into equation
- __ Checks that solution satisfies condition
- __ Checks solution by retracing steps
- __ Checks if solution is reasonable/realistic
- __ Derives the solution by another method
- __ Generalizes the problem
- __ Considers extreme cases
- __ Checks that all information is used
- __ Attempts to find other solutions

Comments:

- __ Requests assistance, more information
- __ Questions existence of solution
- __ Questions uniqueness of solution
- __ Indicates uncertainty about final solution
- __ Says he does not know how to solve the problem
- __ Expresses enjoyment, liking for the problem
- __ Expresses distaste, dislike for the problem
- __ Admits confusion
- __ Expresses confidence that he can arrive at a solution

Executive Errors:

Numerical computation ____
Algebraic manipulation ____
Other

Interviewer Comments:

PROCESS SEQUENCE

R - D3, -- D'(----)3 ⌒ R, - D(M ---- R ----------)3, T2, T2, --- T(-) 5.

APPENDIX J

ATTITUDINAL MEASURES

Aiken Revised Math Attitude Scale

Directions:

Each of the statements on the opinionnaire expresses a feeling which a person might have towards mathematics. Fill in the space on the answer card which best describes the extent of agreement between the feeling expressed in each statement and your own personal feeling. Fill in A--if you STRONGLY AGREE; B--if you AGREE; C--if you are UNDECIDED; D--if you DISAGREE; E--if you STRONGLY DISAGREE.

1. I do not like mathematics, and it scares me to have to take it.
2. I feel at ease in mathematics, and I like it very much.
3. Mathematics makes me feel as though I'm lost in a jungle of numbers and can't find my way out.
4. Mathematics is something which I enjoy a great deal.
5. I approach math with a feeling of hesitation, resulting from a fear of not being able to do math.
6. The feeling that I have towards mathematics is a good feeling.
7. Mathematics is a course in school which I have always enjoyed studying.
8. When I hear the word "math", I have a feeling of dislike.
9. I feel a definite positive reaction to mathematics; it is enjoyable.
10. I feel a sense of insecurity when attempting mathematics.
11. I am happier in a math class than in any other class.
12. I really like mathematics.

13. Mathematics makes me feel uncomfortable, restless, irritable, and impatient.

14. I have never liked math, and it is my most dreaded subject.

15. Mathematics is fascinating and fun.

16. My mind goes blank, and I am unable to think clearly when working math.

17. Mathematics makes me feel secure and at the same time it is stimulating.

18. I am always under a terrible strain in a math class.

19. Mathematics is very interesting to me, and I enjoy math courses.

20. It makes me nervous to even think about having to do a math problem.

FORM A

a) How do you feel about mathematics?

b) What do you like best about it?

c) What do you like least about it?

FORM B

a) Has this course influenced your feelings toward mathematics?

b) What did you like about this course?

c) What would you do to change this course to make it more interesting?

FORM C

a) After a few minutes of reflection, try to describe below your attitudes towards problem solving at the present time.

b) Describe any changes that may have taken place in your attitude toward problem solving since the beginning of the course.

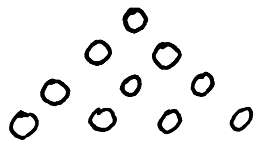
c) Describe how you see the usefulness of the eight course handouts as an aid in learning to beccome better problem solvers.

Supplemental Problems

This is a listing of supplemental problems that you may enjoy working. I am not asking you to hand these problems in; they will not appear on tests, and they will not be discussed in class. However, I would be happy to discuss any of them privately with you if you so desire.

1. A cat is at the bottom of a 30-foot well. Each day she climbs up 3 feet; each night she slides back 2 feet. How long will it take for the cat to get out of the well?

2. Ten coins are arranged to form a triangle as shown. By rearranging only three of the coins, form a new triangle that points in the opposite direction from the one shown.



3. A sailor lands on an island inhabited by two types of people. The A's always lie, and the B's always tell the truth. The sailor meets three inhabitants on the beach and asks the first of these: "Are you an A or a B?" The man answers, but the sailor doesn't understand him and asks the second person what he had said. The man replies: "He said that he was a B. He is, and so am I." The third inhabitant then says: "That's not true. The first man is an A and I'm a B." Can you tell who was lying and who was telling the truth?

4. Write the numbers from 1 through 10 using four (4's) for each. Here are the first three completed for you.

$$\frac{44}{44} = 1 \quad \frac{4 \times 4}{4 + 4} = 2 \quad \frac{4 + 4 + 4}{4} = 3 \quad \left(\sqrt{4} + \frac{\sqrt{4}\,\sqrt{4}}{4} = 3 \right)$$

5.
```
FORTY
  TEN
+ TEN
-----
SIXTY
```

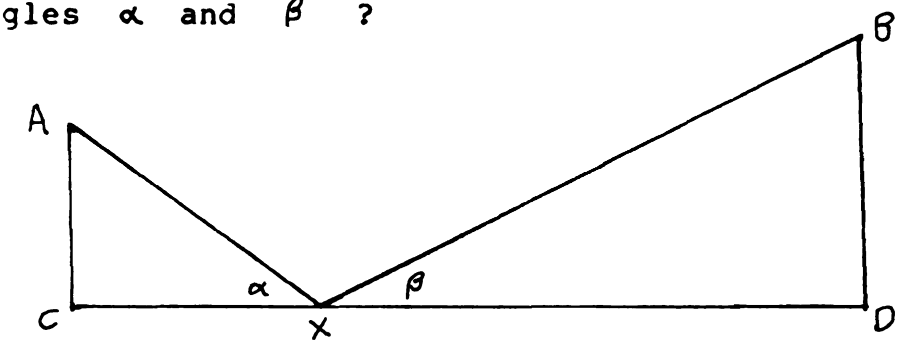
6. Put the digits 1, 2, . . . , 9 into a 3 x 3 matrix, one digit into each cell. Your assignment of digits to cells must satisfy two conditions: (1) Row 1 plus row 2 must equal row 3 (considering each row as a

three-digit number). (2) The digit i must be located immediately next to (above, below, to the right, or to the left) the digit i - 1, for i = 2, . . . , 9. This second condition means you may place the digit 1 anywhere, but 2 must be placed next to 1 along a row or column, 3 must be placed next to 2, and so on.

7. Given a jar that will hold exactly 7 quarts of water, a jar that will hold exactly 3 quarts of water, no other containers holding water, but an infinite supply of water, describe a sequence of fillings and emptyings of water jars that will result in achieving 5 quarts of water.

8. A man, a fox, a goose, and some corn are together on one side of the river with a boat. How would you transfer all of these entities to the other side of the river by means of the boat, which will carry the man and one other entity? The fox and the goose cannot be left alone together, nor can the goose and the corn.

9. A ray of light travels from point A to point B in the figure below by bouncing off a mirror represented by the line CD. Determine the point X on the mirror such that the distance traveled from point A to point B is a minimum. What is the relationship between the angles α and β ?



10. A poor hungry man wants to steal some oranges from an orchard, but the fence is too high, so he figures he can bribe the guards. There are 3 gates (3 guards). The first guard says, "I'll let you out if you give me 1/2 of what you have plus 1/2 an orange. Agreed, so the man proceeds. Coincidently the guard at the next (second) gate says the same. . . "I'll let you out if you give me 1/2 of what you have plus 1/2 an orange." Agreed again, and again the man proceeds and the third guard has the same request. "I'll let you out if you give me 1/2 of what you have, plus 1/2 an orange." Agreed again. On his way out, the man fulfills each guards bribe. When he is outside all 3 gates he finds that he is still hungry (he ate no oranges) and he has no oranges, and his hands aren't even sticky because they never split an orange. How many oranges did he steal?

11. A man goes to a well with three cans whose capacities are 3 gallons, 5 gallons, and 8 gallons. Explain how he can obtain exactly 4 gallons of water from the well.

12. What is the largest amount of money you can have in coins and still not be able to give change for a dollar? What are the coins? (Do not use silver dollars.)

13. A wooden cube whose edges are 3 inches long is to be cut into 27 one-inch cubes. If, after each cut with a saw, the resulting pieces may be lined up or piled up and the next cut made through the pile, what is the smallest number of cuts that will provide the desired dissection?

Name____________________

FORM D

During the second week of the course you were given a set of 13 "Supplemental Problems." At that time I said that these were not to be handed in, would not appear on tests, and would not be discussed in class.

Please answer the following questions concerning those 13 problems.

a) Did you attempt to solve any of the problems? Yes__ No__

b) If you answered "No", describe why you did not attempt any.

c) If you answered "Yes", then answer the following:

1) On how many problems did you work?

2) Of these, how many do you think you got right?

3) Approximately how long (in hours) did you work on these "Supplemental Problems"?

APPENDIX K

NON-ROUTINE MATHEMATICS PROBLEMS FOR THE COURSE

Problems

Structure of Elementary Mathematics

Directions:

Problems included on this sheet and subsequent sheets will be either

a) used as a basis for discussion and not handed in, or
b) solved by the student and handed in.

It will be made clear as we go along which problems are to be handed in. Normally these will be collected on Monday of each week (late work will not be accepted except for extenuating circumstances).

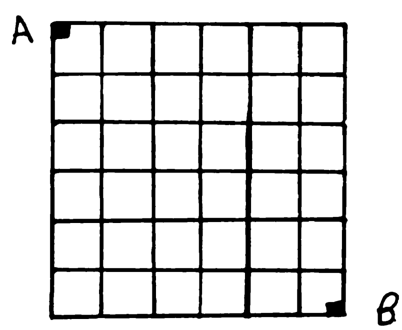
Problems will be graded on the following basis:

0 - 1 points for "Understanding the Problem"
0 - 3 points for "Devising a Plan" and "Carrying it Out"
0 - 2 points for "Result"
0 - 2 points for "Looking Back"

---

1. At the right is the map of a city. Adam lives at A and his sweetheart, Betty, lives at B. Adam walks to Betty's house regularly and is curious as to:

   a) The route or routes which are shortest and
   b) How many different shortest routes he could take to her house?



2. Given a 3-inch by 5-inch card, what is the largest number of pieces you can make by cutting through the card 8 times?

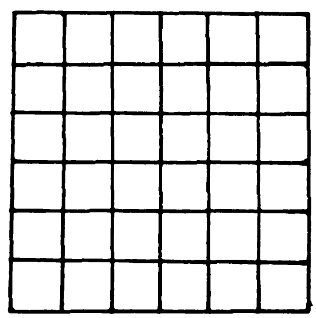
3. The following three lines represent an addition problem:

$$\begin{array}{r} \text{D O N A L D} \\ +\ \text{G E R A L D} \\ \hline \text{R O B E R T} \end{array}$$

Each letter represents a unique digit and a given digit is represented by only one letter. Determine the addition problem. (HINT: D=5)

4. Three missionaries and three cannibals wish to cross a river. There is a boat which can carry three people and either a missionary or a cannibal can operate the boat. It is never permissible for cannibals to outnumber missionaries, neither in the boat nor on either shore. What is the smallest number of trips necessary to make the crossing?

5. How many squares are there in the figure below?



6. Tower of Hanoi

7. Given n points in a plane, no three of which are collinear, how many line segments must be drawn to connect all pairs of points?

8. A number of marbles are stacked in a triangular pyramid. How many are in the nth layer? How many marbles are in the top n layers?

9. A sultan arranged his wives in order of increasing seniority and presented each with a gold ring. Next, every third wife, starting with the second, was given a second ring. Of these, every third one starting with the second received a third ring, and so on in this manner. His first and most cherished wife was the only one to receive ten rings. How many wives did the sultan have?

10. 100 mathematicians attended a recent convention. Each mathematician in attendance shook hands with every other mathematician in attendance. How many handshakes were there at the convention?

11. Determine the addition problem below.

```
   H O C U S
 + P O C U S
 -----------
   P R E S T O
```

Each letter represents a unique digit.

12. Determine all of the digits represented by X in the following problem. Also determine the remaining digits in the answer

```
                    8
          ------------------
X X X  |  X X X X X X X X X
            X X X
          -------
              X X X X
                X X X
              -------
                  X X X X
                  X X X X
                  -------
```

13. Determine the addition problem below.

```
       O N E
       T W O
   + F I V E
   ---------
   E I G H T
```

14. Determine the subtraction problem below.

```
     F I V E
   - F O U R
   ---------
       O N E
```

15. A goat is tied at the corner of a 20 ft. by 40 ft. barn with a 50 ft. rope. If it can graze everywhere outside of the barn to which its rope can reach, what is the area of its grazing region?

16. A family of four--two parents, a son, and a daughter--have a set of Christmas cups. One evening they observed that Mother has the cup decorated with candles, Father has the one decorated with holly; the son has the one decorated with carolers, and the daughter has the one decorated with angels. In how many ways can the cups be distributed the next evening so that no individual has the same cup?

17. Mr. Buick, Mr. Chrysler, and Mr. Ford owned a Buick, Chrysler, and a Ford (not necessarily in that order).

The Chrysler's owner often beat Ford at cards. Ford was the brother-in-law of the Buick's owner. Chrysler had more children than the Chrysler's owner. Who owned the Buick?

18. The Nelsons have gone out for the evening, leaving their four children with a new babysitter, Nancy Wiggens. Among the many instructions the Nelsons gave Nancy before they left was that three of their children were consistent liars and only one of them consistenly told the truth, and told her which one. But in the course of receiving so much other information Nancy forgot which child was the truar. As she was preparing dinner for the children, one of them broke a vase in the next room. Nancy rushed in and asked who broke the vase. These were the children's statements:

Betty: Steve broke the vase.
Steve: John broke it.
Laura: I didn't break it.
John: Steve lied when he said, "I broke it."

Knowing that only one of these statements was true, Nancy quickly determined which child broke the vase. Who was it?

19. On a certain basketball team the positions of center, guard, and forward are held by Bill, Lew, and Tom (though not necessarily in that order).

The forward is an only child.
The forward has the lowest free-throw average.
Tom's best friend is Bill's brother.
Tom has a higher free-throw average than the guard.

What position does each student play?

20. How many natural numbers less than or equal to 900 have either 3 or 5 as a factor?

21. If a man walks to work and rides back it takes him an hour and a half. When he rides both ways it takes him 30 minutes. How long will it take him to make the round trip by walking?

22. Ingrid brings a quantity of hats to sell at the Saturday market. In the morning she sells her hats for $3.00 each grossing $18. In the afternoon, she reduces her price to $2.00 each and sells twice as many. What was Ingrid's gross income for the day from the sale of hats?

23. A fireman stood on the middle rung of a ladder, directing water into a burning building. As the smoke lessened, he stepped up three rungs and continued his work from that point. A sudden flare-up forced him to go down 5 rungs. Later he climbed up seven rungs and worked there until the fire was out. Then he climbed the remaining six rungs and entered the building. How many rungs were there in the ladder?

24. Mr. Smith is 4 times older than his son Mark. Mrs. Smith is 3 times older than Mark. If Mr. Smith's age is exceeded by Mark's by 25 years, then how old is Mark?

25. Do the Missionaries and Cannibals problem (# 4) with the condition that the boat only holds two people.

26. A Chinese prince who was forced to flee his kingdom by his traitorous brother sought refuge in the hut of a poor man. The prince had no money, but he did have a very valuable golden chain with seven links. The poor man agreed to hide the prince, but because he was poor and because he risked considerable danger should the prince be found, he asked that the prince pay him one link of the golden chain for each day of hiding. Since the prince might have to flee at any time, he did not want to give the poor man the entire chain, and since it was so valuable, he did not want to open more links than absolutely necessary. What is the smallest number of links that the prince must open in order to be certain that the poor man hs one link on the first day, two links on the second day, etc.?

27. The "Magic" Cards below work as follows: You, the "Magician," ask someone to name the letters that identify all the cards on which his age appears. You in turn will tell him his age by adding the numbers in the upper left hand corner of the cards he names. (For example, if he tells you his age is found on cards E, D, and A, then you can tell him with absolute confidence that his age is 16 + 8 + 1 = 25.)

E		D		C		B		A	
16	17	8	9	4	5	2	3	1	3
18	19	10	11	6	7	6	7	5	7
20	21	12	13	12	13	10	11	9	11
22	23	14	15	14	15	14	15	13	15
24	25	24	25	20	21	18	19	17	19
26	27	26	27	22	23	22	23	21	23
28	29	28	29	28	29	26	27	25	27
30	31	30	31	30	31	30	31	29	31

Question: a) Why does this "trick" work?
b) Construct cards like these which will work for ages 1 to 53.

28. Wanda the Witch agrees to trade one of her magic broomsticks to Gasper the Ghost in exchange for one of his gold chains. Gasper is somewhat skeptical that the broomstick is in working order and insists on a guarantee equal in days to the number of links in his gold chain. To facilitate enforcement of the guarantee, he insists on paying by the installment plan, one gold link per day until the end of the 63-day period, with the balance to be forfeit if the broomstick malfunctions during the guarantee period. Wanda agrees to this request, but insists that the installment payment be effected by cutting no more than three links in the gold chain. Can this be done, and, if so, what links in the chain should be cut? The chain initially consists of 63 gold links arranged in a simple linear order (not closed into a circle).

29. The Boston Track Association makes shot-puts that weigh 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, or 15 pounds. They keep them in storage houses all over the United States. You have been hired to travel around the country and make sure that the weights stamped on the shot-puts are correct. (Assume that each shot-put weighs an exact number of pounds. You must check, for example, that no 4-pound shot has been mistakenly stamped "3 pounds"). What are the minimum number of weights you must carry so that you can check each

weight of shot-put?

30. A man had 40 sacks of gold dust which ranged in weight from 1 to 40 ounces. (No two sacks weighed the same and there were no fractional weights.) One day, upon returning from a long trip, he suspected that a thief had pilfered some of the gold, so he hires you to weigh each sack. He provides you with a balance scale and a box of 40 weights, rangng from 1 ounce to 40 ounces.

a. If you are allowed to place weights on only <u>one</u> side of the balance what is the <u>minimum</u> number of weights you'll take from the box in order to weigh the sacks of gold? How many ounces does each of these weights weigh?

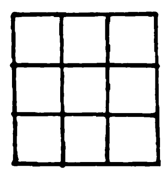
b. If you are given the same task but allowed to use <u>both</u> sides of the balance, what is the minimum number of weights you'd need to use? How heavy is each weight?

c. With the weights in (a) above, every counting number 1-40 can be represented. To what base are these weights related? What about the weights in (b) above.

31. You have 10 stacks of quarters with 10 quarters in each stack. One entire stack is composed of quarters, each of which weighs 2 grams less than it should. You know the correct weight of a quarter. You may weigh the coins on a pointer scale, which tells you how many grams a set of objects placed on it weighs. What procedure will determine the light stack in the smallest number of weighings?

32. A boy had 1,000 pennies which he partitioned into 10 bags in such a way that he could hand over any amount from 1c to $10.00 using some combination of the bags. How many pennies did each bag contain?

33. Fill in the nine squares at the right with the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 in such a way that the sums of the three numbers in each of the three columns, each of the three rows, and each of the two diagonals are all the same.



34. Are there two different prime numbers p and q such that p + q divides p x q? If so, give examples, if not, then explain why not.

35. Determine whether each of the following statements is true or false. For those which are true, construct an

argument which would convince someone else; for those which are false, find a counter example.

a) If $a \nmid b$ and $b \nmid c$, then $a \nmid c$.

b) If $a \nmid b$ and $a \nmid c$, then $a \nmid (b + c)$.

c) If $a | b$ and $c | d$, then $ac | bd$.

d) If $a | c$ and $b | c$, then $ab | c$.

e) If $a | b$, then $a^2 | b^2$.

36. Ignoring the initial primes 2, 3, and 5, is the product of each pair of twin primes divisible by 12? Is the sum divisible by 12? (In each case, if not, give a counter example, if so provide a logical argument to prove your response.)

37. An unusual parlor trick is performed as follows. Ask spectator A to jot down any three-digit number, and then to repeat the digits in the same order to make a six-digit number (e.g., 394,394). With your back turned so that you cannot see the number, ask A to pass the sheet of paper to spectator B, who is requested to divide the number by 7.

"Don't worry about the remainder," you tell him, "Because there won't be any. B is surprised to discover that you are right. Without telling you the result, he passes it on to spectator C who is told to divide it by 11. Once again you state that there will be no remainder, and this also proves correct.

With your back still turned, and no knowledge whatever of the figures obtained by these computations, you direct a fourth spectator, D, to divide the last result by 13. Again the division comes out even. This final result is written on a slip of paper, which is folded and handed to you. Without opening it you pass it on to spectator A. "Open this," you tell him, "and you will find your original three-digit number." Prove that the trick cannot fail to work regardless of the digits chosen by the first spectator.

38. Find the smallest (a) square (b) cube which has 180 as a factor. Explain your reasoning.
Do the same using 1,440 as a factor.

39. What is the smallest natural number greater than 1 which is simultaneously
(a) a square and a cube?
(b) a square, a cube, and a fifth power?

40. For what type of natural numbers n is n a factor of $(n - 1)!$? Justify.

41. Show that the product of the digits of any 2-digit natural number is always less than that natural number (e.g., $5 \times 7 < 57$).

42. Find all 2-digit natural numbers which equal twice the product of their digits.

43. Find all 2-digit natural numbers such that the product of that number and the number formed by reversing its digits is a 3-digit palindrome.

44. Guess the rule for squaring (without doing any computations on paper) suggested by the following examples: $25^2 = 625$, $65^2 = 4225$, and $45^2 = 2025$.

Compute 35^2, 55^2, 75^2, 85^2, 95^2, using your rule. Justify your rule.

45. What natural numbers can occur as the unit's digit of a perfect square?

46. Which numbers in the set $\{1, 11, 111, 1111, \ldots\}$ are perfect squares? Why?

47. If the GCF (a, b) = 3 what values are possible for

a) GCF (a^2, b)

b) GCF (a^3, b)

c) GCF (a^2, b^3)?

48. Find when a sequence of 9 consecutive natural numbers can be used to form a 3×3 magic square.

49. Construct a 4 by 4 magic square using 1, 2, 3, 4, . . . , 16.

50. Billy, a third grader, was practicing addition by addidng the numbers along each full week on the calendar. After a while, he saw the following pattern for finding the sum of the numbers in a week: "Take the first day, add 3, and multiply by 7." Try It! Explain <u>why</u> this rule works.

51. Find all primes p such that $5p + 1$ is a perfect square. Justify your anser.

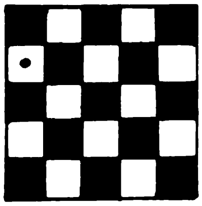
52. Does your shadow get shorter or longer as you walk away from a lamp post? If someone disagreed with you, how would you convince him you were right?

53. a) What would the path of the shadow look like if a person walked by a lamp post in a straight line?

b) What would the path of the shadow look like if the person walked in a circle

54. You are given a checkerboard and 32 dominoes. Each domino covers exactly two adjacent squares on the board. Thus, the 32 dominoes can cover all 64 squares of the checkerboard. Now suppose two squares are cut off at diagonally opposite corners of the board. Is it possible to place 31 dominoes on this board so that all the 62 remaining squares are covered? If so, show how it can be done. If not, show why not.

55. Given a 5 by 5 checkerboard, as shown, try to draw a line through all the squares of the checkerboard, starting from the square with the dot in it on the left side and passing through each box once and only once, without every lifting pencil from paper and without ever passing outside of the checkerboard. Show how to do this or prove it is impossible. (Diagonal lines are not allowed.) Can you generalize this problem and its solution?



56. Fifteen pennies are placed on a table in front of two players. Each player is allowed to remove at least one penny but not more than five pennies at his turn. The players alternate turns, each removing from one to five pennies n number of turns, until one player takes the last penny on the table, and wins all 15 pennies. Is there a method of play that will guarantee victory? If so, what is it?

57. Three squares A, B, C are constructed so that the area of square B is 20 more than that of A, and 28 less than that of C. Find the length of all sides of all three squares.

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