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INTEGRAL AVERAGING
FOR NONAUTONOMOUS EQUATIONS

By

Robert George White

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
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ABSTRACT

INTEGRAL AVERAGING
FOR NONAUTONOMOUS EQUATIONS

By

Robert George White

We consider the method of averaging and its application to bifurcation problems involving nonautonomous equations.

First, a two-dimensional nonautonomous ordinary differential equation is considered. This system, written in polar coordinates, admits a change of variables which reduces the search for periodic orbits and invariant manifolds to the study of a certain canonical form. Properties such as the existence, amplitude and stability of such structures bifurcating from an equilibrium can be determined from this canonical form of the equation. An illustration of the method is offered by investigating the well known Van der Pol equation.

Higher dimensional and infinite dimensional systems can be treated in essentially the same manner by restricting the equation to the center manifold. The method of averaging is used to approximate the equation of the center manifold.

A bifurcation problem in a forced Wright's equation is included in order to illustrate the application of the method to infinite dimensional systems.

In memory of my parents,
Robert George and Hazel Caroline.

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INTRODUCTION

In [4] Chow and Mallet-Paret showed how the classical method of averaging (see [1], [4], [8], [9], [17]) can be applied to bifurcation problems involving ordinary differential equations (ODE's), partial differential equations (PDE's) and functional differential equations (FDE's). They restricted themselves mainly to the investigation of autonomous differential systems. However, many nonautonomous systems arise naturally. For example, the bifurcation of an invariant torus from a periodic orbit (see [13], [15]) and periodic forcing problems introduce nonautonomous terms into the equation. This dissertation discusses how the method of averaging can be utilized to demonstrate the existence of invariant structures when nonautonomous terms are present in the equation.

In chapter 1, the method of averaging is described in general. Conditions are determined under which the method can be applied to reduce an ODE to a canonical form. In chapter 2 a Hopf bifurcation problem (see [4], [5], [6]) for a nonautonomous ODE is treated. The canonical form of this equation makes many properties (direction, amplitude and stability) of a bifurcating manifold virtually transparent. A method is described which reduces an n-dimensional problem to a two-dimensional one on the center manifold (see [11]). Also, a proof of the existence and periodicity of the bifurcating manifold is given. The forced Van der Pol equation (see [8], [9]) is offered as an illustration of the method. In chapter 3 FDE's are considered. Since finite dimensional space cannot be considered as the phase space for such equations (see [4], [10]), a suitable setting for the averaging to be carried out is defined. A generic bifurcation (see [3], [4], [5], [6]) for a forced Wright's equation (see [4], [10], [18]) is shown to exist when a parameter crosses certain critical values. An appendix which outlines the basic theory of almost periodic functions completes the work.

1. THEORY OF INTEGRAL AVERAGING

1.1. Introduction.

Consider the two dimensional system given by

$$\dot{x} = f(x,t) = Ax + g(x,t),\ x \in R^2, \cdot = d/dt \tag{1.1}$$

where A is a constant 2×2 matrix, $g(x,t) = o(|x|^2)$ uniformly in t as $|x| \to 0$ and $g(x,t)$ is almost periodic or P-periodic in t. Suppose that the linearized system

$$\dot{x} = Ax \tag{1.2}$$

is purely rotational, that is A has pure imaginary eigenvalues, $\pm i\omega$, with ω real and nonzero. Then by making the change of variable $x \to Rx$, where R is an appropriate 2×2 matrix we can assume that A is in Jordan form

$$A = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix},\ \omega \neq 0$$

Now all solutions of (1.2) are periodic (of period $2\pi/\omega$) and are represented by circles in the phase space $x = (x_1, x_2)$. One may consider these solutions to lie on cylinders in (x,t)-space with axis the line $x = 0$. These cylinders are invariant manifolds for (1.2) since any solution of (1.2) that is on one of these cylinders at any $t = t_0$ remains on the same cylinder for all $t \in (-\infty,\infty)$. Now if

$|x|$ is small then (1.1) is a perturbation of (1.2), so one may expect that (1.1) will also have invariant manifolds which are "cylinder like".

To see this more clearly scale by $x \to \varepsilon x$ in (1.1) with $|\varepsilon| \ll 1$ to get

$$\dot{x} = Ax + h(x,t,\varepsilon)$$

where $h(x,t,\varepsilon) = \varepsilon^{-1} g(\varepsilon x,t) = O(\varepsilon)$. Then switch to polar coordinates by $x = rN_\theta$, $N_\theta = (\cos\theta, \sin\theta)'$ (' denotes transposition) to obtain

$$\dot{r} = \varepsilon G(r,\theta,t,\varepsilon)$$

$$\dot{\theta} = \omega + \varepsilon H(r,\theta,t,\varepsilon)$$

where

$$\varepsilon G(r,\theta,t,\varepsilon) = N_\theta' h(rN_\theta,t,\varepsilon)$$

$$\varepsilon H(r,\theta,t,\varepsilon) = T_\theta' h(rN_\theta,t,\varepsilon)$$

with $T_\theta = (-\sin\theta, \cos\theta)'$. Expanding G and H in powers of ε yields

$$\text{(1.3)} \quad \begin{aligned} \dot{r} &= \varepsilon R_1(r,\theta,t) + \varepsilon^2 R_2(r,\theta,t) + \cdots \\ \dot{\theta} &= \omega + \varepsilon W_1(r,\theta,t) + \varepsilon^2 W_2(r,\theta,t) + \cdots \end{aligned}$$

where R_j and W_j are homogeneous trigonometric polynomials in $\sin\theta$ and $\cos\theta$ of degree $j + 1$ with coefficients depending on r and t, almost periodic or P-periodic in t. That is R_j and W_j have the form

$$\sum_{\substack{n+m=j+2 \\ n,m\geq 0}} \alpha_{n,m}(r,t)\cos^n\theta\sin^m\theta$$

where $\alpha_{n,m}(r,t)$ are almost periodic or P-periodic in t. We note here that by expanding $\cos^n\theta$ and $\sin^m\theta$ in powers of $\exp(ik\theta)$ for $|k| \leq j + 1$ we see that R_j and W_j have the form

$$\sum_{\substack{|k|\leq j+2 \\ k=j \pmod 2}} a_k(r,t)e^{ki\theta}; \; a_{-k} = \bar{a}_k$$

where the a_k are linear combinations of the $\alpha_{n,m}$. Further (1.3) may be viewed as a finite Taylor development with remainder, since we need only consider a finite number of these terms in the sequel.

Now if all the R_j are independent of θ and t then the periodic solutions of (1.1) are on those cylinders of radius r_0 where

$$\varepsilon R_1(r_0) + \varepsilon^2 R_2(r_0) + \cdots = 0.$$

However if $R_1, R_2, \cdots, R_k$ are independent of θ and t and R_{k+1}, $R_{k+2}, \cdots$ depend on θ and t, then one still expects an invariant manifold near the cylinder of radius r_0 where

$$\varepsilon R_1(r_0) + \cdots + \varepsilon^k R_k(r_0) = 0.$$

That is there is a function $g(\theta,t,\varepsilon)$ which is almost periodic (P-periodic) in t and 2π-periodic in θ so that

$$r = r_0 + \varepsilon^{k+1}g(\theta,t,\varepsilon)$$

defines an integral (invariant) manifold of (1.3), in the sense that if $(r^*(t),\theta^*(t))$ is any solution with $r^*(t_0) = r_0 + \varepsilon^{k+1}g(\theta^*(t_0),t_0,\varepsilon)$ then $r^*(t) = r_0 + \varepsilon^{k+1}g(\theta^*(t),t,\varepsilon)$ for all $t \in (-\infty,\infty)$.

The aim of the method of averaging is to make enough of the R_j (and W_j) in (1.3) independent of θ and t by means of coordinate changes $r \to \bar{r}$, $\theta \to \bar{\theta}$ so that the approximate amplitude of any such invariant manifold can be determined.

Now if we consider the higher dimensional system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} f_1(x,y,t) \\ f_2(x,y,t) \end{bmatrix}$$

where $x \in R^2$, $y \in R^{n-2}$, A is as before, B has no pure imaginary eigenvalues, $|f_i(x,y,t)| = O(|(x,y)|^2)$ uniformly in t as $|(x,y)| \to 0$ for $i = 1,2$ and $f_i(x,y,t)$ is almost periodic or P-periodic in t. Then after a scaling $x \to \varepsilon x$, $y \to \varepsilon y$ the system in the coordinates (r,θ,y) has the form

$$\begin{aligned} \dot{r} &= \varepsilon R(r,\theta,y,t,\varepsilon) \\ \dot{\theta} &= \omega + \varepsilon W(r,\theta,y,t,\varepsilon) \qquad (1.4) \\ \dot{y} &= By + \varepsilon g(r,\theta,y,t,\varepsilon)y + \varepsilon h(r,\theta,t,\varepsilon) \end{aligned}$$

where the $\dot{r}$ and $\dot{\theta}$ equations have the same form as before.

For $\varepsilon = 0$ (1.4) decouples and the plane $y = 0$ is an invariant manifold on which all solutions are periodic. If $0 < |\varepsilon| << 1$ then there is an invariant manifold, the center manifold, defined by $y = y^*(r,\theta,t,\varepsilon)$ tangent to the (r,θ) plane for all t and ε, 2π-periodic in θ, and almost periodic or P-periodic in t so that any solution of (1.4) which is bounded for all

$t \in (-\infty,\infty)$ lies on this manifold. On this surface (1.4) becomes a two dimensional system which can be treated as before. If $r = r^*(\theta,t)$ defines an invariant manifold of this two dimensional system then $(r,y) = (r^*(\theta,t), y^*(r^*(\theta,t),\theta,t))$ defines a two dimensional invariant manifold for (1.4) with the desired periodicity properties.

In 1.3 a procedure is described through which the manifold $y = y^*(r,\theta,t)$ can be approximated to any order of ε as desired provided the equation is smooth enough.

1.2. The Method of Averaging.

Consider a two dimensional system in polar coordinates (r,θ) given by

$$\begin{aligned} \dot{r} &= \varepsilon R_1(r,\theta,t) + \varepsilon^2 R_2(r,\theta,t) + \cdots \\ \dot{\theta} &= \omega + \varepsilon W_1(r,\theta,t) + \varepsilon^2 W_2(r,\theta,t) + \cdots \end{aligned} \tag{2.1}$$

where $\varepsilon \in R$, ω is a nonzero constant and R_j and W_j are 2π-periodic in θ, almost periodic or P-periodic in t and have the form

$$\sum_{|n|\le N_j} a_n(r,t)e^{ni\theta}, \quad a_{-n} = \bar{a}_n \tag{2.2}$$

where n and N_j are integers and $a_n(r,t)$ is almost periodic or P-periodic in t. The differential equation (2.1) is assumed to be smooth enough for the following calculations to be carried out. Also R_j and W_j may depend on additional parameters which are omitted since they will play no role in the following procedure, but will become important when bifurcation problems are encountered.

The goal here is to describe a change of variables $r \to \bar{r}$, $\theta \to \bar{\theta}$ so that in the $(\bar{r},\bar{\theta})$ coordinates $R_1, R_2, \cdots, R_k$ and $W_1, W_2, \cdots, W_k$ are independent of $\bar{\theta}$ and t. Proceeding by induction, suppose that the coefficients of ε^j for $1 \le j \le k - 1$ are independent of θ and t, so that

$$\text{(2.3)} \quad \begin{aligned} \dot{r} &= \varepsilon R_1(r) + \cdots + \varepsilon^{k-1} R_{k-1}(r) + \varepsilon^k R_k(r,\theta,t) + \cdots \\ \dot{\theta} &= \omega + \varepsilon W_1(r) + \cdots + \varepsilon^{k-1} W_{k-1}(r) + \varepsilon^k W_k(r,\theta,t) + \cdots \end{aligned}$$

Then consider a change of variables of the form

$$\text{(2.4)} \quad \begin{aligned} \bar{r} &= r + \varepsilon^k u(r,\theta,t) \\ \bar{\theta} &= \theta + \varepsilon^k v(r,\theta,t). \end{aligned}$$

Its inverse satisfies

$$\text{(2.5)} \quad \begin{aligned} r &= \bar{r} - \varepsilon^k u(\bar{r},\bar{\theta},t) + O(\varepsilon^{k+1}) \\ \theta &= \bar{\theta} - \varepsilon^k v(\bar{r},\bar{\theta},t) + O(\varepsilon^{k+1}). \end{aligned}$$

Now

$$\text{(2.6)} \quad \begin{aligned} \dot{\bar{r}} &= \dot{r} + \varepsilon^k \left[\frac{\partial u}{\partial r}\dot{r} + \frac{\partial u}{\partial \theta}\dot{\theta} + \frac{\partial u}{\partial t}\right] \\ \dot{\bar{\theta}} &= \dot{\theta} + \varepsilon^k \left[\frac{\partial v}{\partial r}\dot{r} + \frac{\partial v}{\partial \theta}\dot{\theta} + \frac{\partial v}{\partial t}\right] \end{aligned}$$

and the right hand side is evaluated at (r,θ,t). To evaluate at $(\bar{r},\bar{\theta},t)$, (2.5) must be inserted into (2.6) and (2.3). Then (2.3) becomes

$$\begin{aligned} \dot{r} &= \varepsilon R_1(\bar{r}) + \cdots + \varepsilon^{k-1} R_{k-1}(\bar{r}) + \varepsilon^k R_k(\bar{r},\bar{\theta},t) + O(\varepsilon^{k+1}) \\ \dot{\theta} &= \omega + \varepsilon W_1(\bar{r}) + \cdots + \varepsilon^{k-1} W_{k-1}(\bar{r}) + \varepsilon^k W_k(\bar{r},\bar{\theta},t) + O(\varepsilon^{k+1}) \end{aligned}$$

and

$$\frac{\partial u}{\partial r}(r,\theta,t) = \frac{\partial u}{\partial r}(\bar{r},\bar{\theta},t) + O(\varepsilon^k)$$

with similar expressions for $\frac{\partial u}{\partial \theta}, \frac{\partial u}{\partial t}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$, and $\frac{\partial v}{\partial t}$. So (2.6) evaluated at $(\bar{r},\bar{\theta},t)$ is written as

$$\dot{\bar{r}} = R_1(\bar{r}) + \cdots + \varepsilon^{k-1}R_{k-1}(\bar{r}) + \varepsilon^k\bar{R}_k(\bar{r},\bar{\theta},t) + O(\varepsilon^{k+1})$$

$$\dot{\bar{\theta}} = \omega + \varepsilon W_1(\bar{r}) + \cdots + \varepsilon^{k-1}W_{k-1}(\bar{r}) + \varepsilon^k\bar{W}_k(\bar{r},\bar{\theta},t) + O(\varepsilon^{k+1})$$

where

$$\text{(2.7a)} \qquad \bar{R}_k(\bar{r},\bar{\theta},t) = R_k(\bar{r},\bar{\theta},t) + \omega\frac{\partial u}{\partial \theta}(\bar{r},\bar{\theta},t) + \frac{\partial u}{\partial t}(\bar{r},\bar{\theta},t)$$

$$\text{(2.7b)} \qquad \bar{W}_k(\bar{r},\bar{\theta},t) = W_k(\bar{r},\bar{\theta},t) + \omega\frac{\partial v}{\partial \theta}(\bar{r},\bar{\theta},t) + \frac{\partial v}{\partial t}(\bar{r},\bar{\theta},t)$$

Now u and v must be chosen so that $\bar{R}_k$ and $\bar{W}_k$ are independent of $\bar{\theta}$ and t. Consider only (2.7a) and choose u so that $\bar{R}_k(\bar{r},\bar{\theta},t) = \bar{R}_k(\bar{r})$ since choosing v will follow similarly.

Let u have the same form as R_k, namely

$$u(r,\theta,t) = \sum_{|n|\le N_k} u_n(r,t)e^{ni\theta}, \quad u_{-n} = \bar{u}_n.$$

Inserting this expression and (2.2) into (2.7a) yields

$$\bar{R}_k(\bar{r}) = \sum_{|n|\le N_k} \left(a_n + in\omega u_n + \frac{\partial u_n}{\partial t}\right)e^{ni\theta}$$

where $a_n = a_n(\bar{r},t)$, $u_n = u_n(\bar{r},t)$. So we must solve

$$\text{(2.8a)} \qquad a_n + in\omega u_n + \frac{\partial u_n}{\partial t} = 0 \quad \text{for} \quad 0 < |n| \le N_k$$

$$\text{(2.8b)} \qquad a_0 + \frac{\partial u_0}{\partial t} = \bar{R}_k(\bar{r})$$

Let us first examine the case where all the a_n are P-periodic in t. The following lemma holds.

Lemma 1.2.1. Consider the differential equation

$$a(t) + i\omega b(t) + \dot{b}(t) = 0 \tag{2.9}$$

where $a(t)$ is P-periodic and ω is a constant. Then the following are equivalent.

(A) (2.9) has a P-periodic solution.

(B) Either ωP is not an integer multiple of 2π or
$\int_0^P e^{i\omega s} a(s)ds = 0$ if ωP is an integer multiple of 2π.

(C) $\int^t e^{i\omega s} a(s)ds$ is bounded.

Proof. The Fredholm alternative theorem implies that (2.9) has a P-periodic solution if and only if

$$\int_0^P b^*(t)a(t)dt = 0$$

for all P-periodic solutions, $b^*(t)$, of the adjoint equation

$$\dot{y} = i\omega y.$$

Thus

$$b^*(t) = \begin{cases} 0 & \text{if } \omega P \text{ is not an integer multiple of } 2\pi \\ e^{i\omega t} & \text{if } \omega P \text{ is an integer multiple of } 2\pi \end{cases}$$

so (A) is equivalent to (B).

(B) $\Rightarrow$ (C). If $\omega P \neq 2\pi k$ for all integers k, let n_t be the integer such that $Pn_t \leq t < P(n_t + 1)$, then

$$\int_0^t e^{i\omega s}a(s)ds = \sum_{\nu=1}^{n_t}\int_{(\nu-1)P}^{\nu P} e^{i\omega s}a(s)ds + \int_{Pn_t}^{t} e^{i\omega s}a(s)ds$$

$$= \sum_{\nu=1}^{n_t} e^{i\omega(\nu-1)P}\int_0^P e^{i\omega s}a(s)ds + e^{i\omega n_t P}\int_0^{t-Pn_t} e^{i\omega s}a(s)ds$$

$$= \frac{1-e^{i\omega n_t P}}{1-e^{i\omega P}} \cdot \int_0^P e^{i\omega s}a(s)ds + e^{i\omega n_t P}\int_0^{t-Pn_t} e^{i\omega s}a(s)ds$$

which is easily seen to be bounded since $0 \le t - Pn_t < P$ and ωP is not an integer multiple of 2π. On the other hand if $\omega P = 2\pi k$ for some integer k and

$$\int_0^P e^{i\omega s}a(s)ds = 0$$

then by what has just been done, we have

$$\int_0^t e^{i\omega s}a(s)ds = \int_0^{t-Pn_t} e^{i\omega s}a(s)ds$$

which again is bounded.

(C) $\Rightarrow$ (B). If ωP is an integer multiple of 2π then again by what has just been done, we have

$$\int_0^t e^{i\omega s}a(s)ds = n_t\int_0^P e^{i\omega s}a(s)ds + \int_0^{t-Pn_t} e^{i\omega s}a(s)ds$$

which will not be bounded unless

$$\int_0^P e^{i\omega s}a(s)ds = 0.$$

This completes the proof of Lemma (1.2.1).

Thus (2.8a) has a P-periodic solution if and only if

$$\int_0^t e^{i\omega s} a_n(s)ds$$

is bounded for all $n \neq 0$ that appear in (2.2). To solve (2.8b) we need

$$\int_0^t a_0(\bar{r},s) - \bar{R}_k(\bar{r})ds$$

to be bounded. This will be the case if and only if

$$\bar{R}_k(\bar{r}) = \underset{t}{\text{mean}}[a_0].$$

Since

$$a_0(\bar{r},t) = \underset{\theta}{\text{mean}}[R_k]$$

we have

$$\bar{R}_k(\bar{r}) = \underset{\theta,t}{\text{mean}}[R_k].$$

We have proved the following theorem.

<u>Theorem 1.2.1</u>. Consider the differential equation

$$\dot{r} = \varepsilon R_1(r) + \cdots + \varepsilon^{k-1} R_{k-1}(r) + \varepsilon^k R_k(r,\theta,t) + O(\varepsilon^{k+1})$$

$$\dot{\theta} = \omega + \varepsilon W_1(r) + \cdots + \varepsilon^{k-1} W_{k-1}(r) + \varepsilon^k W_k(r,\theta,t) + O(\varepsilon^{k+1})$$

where R_k and W_k are 2π-periodic in θ and P-periodic in t and have the form

$$R_k(r,\theta,t) = \sum_{|n|\le N_k} a_n(r,t)e^{ni\theta}$$

$$W_k(r,\theta,t) = \sum_{|n|\le M_k} b_n(r,t)e^{ni\theta}$$

where $\bar{a}_n = a_{-n}$ and $\bar{b}_n = b_{-n}$ and

$$\int^t e^{in\omega s} a_n(s)ds, \int^t e^{in\omega s} b_n(s)ds$$

are bounded for all $n \neq 0$ that appear in these expansions of R_k and W_k respectively.

Then there exist functions $u(r,\theta,t)$ and $v(r,\theta,t)$ which are 2π-periodic in θ and P-periodic in t so that if

$$\bar{r} = r + \varepsilon^k u(r,\theta,t)$$

$$\bar{\theta} = \theta + \varepsilon^k v(r,\theta,t)$$

then

$$\dot{\bar{r}} = \varepsilon R_1(\bar{r}) + \cdots + \varepsilon^{k-1} R_{k-1}(\bar{r}) + \varepsilon^k \bar{R}_k(\bar{r}) + O(\varepsilon^{k+1})$$

$$\dot{\bar{\theta}} = \omega + \varepsilon W_1(\bar{r}) + \cdots + \varepsilon^{k-1} W_{k-1}(\bar{r}) + \varepsilon^k \bar{W}_k(\bar{r}) + O(\varepsilon^{k+1})$$

where

$$\bar{R}_k(r) = \underset{t,\theta}{\text{mean}}[R_k(r,\theta,t)]$$

$$\bar{W}_k(r) = \underset{t,\theta}{\text{mean}}[W_k(r,\theta,t)].$$

Now if the $a_n(r,t)$ in (2.8) are almost periodic in t then again we must solve equations of the form

$$a(t) + i\omega b(t) + \dot{b}(t) = 0$$

where $b(t)$ must be chosen to be almost periodic with $m[b] \subset m[a]$, (see Appendix). The variation of constants formula yields

$$b(t) = e^{-i\omega t}[c - \int_0^t e^{i\omega s}a(s)ds]$$

which is seen to be almost periodic in t if and only if

$$(2.10) \qquad \int_0^t e^{i\omega s}a(s)ds$$

is bounded.

Note that $-\omega$ cannot be a frequency of $a(t)$, since if this were the case then

$$\underset{t}{\text{mean}}[e^{i\omega t}a(t)] \neq 0$$

which implies that the integral in (2.10) is unbounded. Thus c must be chosen so that $-\omega$ is not a frequency of $b(t)$. Taking

$$c = \underset{t}{\text{mean}}[\int_0^t e^{i\omega s}a(s)ds]$$

yields

$$\underset{t}{\text{mean}}[e^{i\omega t}b(t)] = 0.$$

Further we must show that $b(t)$ possesses no frequency which is not a frequency of $a(t)$. To this end suppose $\lambda \neq -\omega$ is not a frequency of $a(t)$. Then

$$\underset{t}{\text{mean}}[e^{-i\lambda t}b(t)] = \lim_{T\to\infty}[\frac{c}{T}\int_0^T e^{-i(\lambda+\omega)t}dt + \frac{1}{T}\int_0^T e^{-i(\lambda+\omega)t}\int_0^t e^{i\omega s}a(s)dsdt]$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_0^T e^{-i(\lambda+\omega)t}\int_0^t e^{i\omega s}a(s)dsdt$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_0^T e^{i\omega s}a(s)\int_s^T e^{-i(\lambda+\omega)t}dtds$$

$$= \frac{i}{\lambda+\omega} \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{i\omega s} a(s)[e^{-i(\lambda+\omega)T} - e^{-i(\lambda+\omega)s}]ds$$

$$= \frac{i}{\lambda+\omega} [e^{-i(\lambda+\omega)T} \underset{t}{\text{mean}}[e^{i\omega t}a(t)] - \underset{t}{\text{mean}}[e^{-i\lambda t}a(t)]]$$

$$= 0.$$

Thus $\lambda \notin m[b]$ if $\lambda \notin m[a]$. The following lemma has now been proved.

Lemma 1.2.2. The equation

$$a(t) + i\omega b(t) + \dot{b}(t) = 0$$

where $a(t)$ is almost periodic and ω is a real constant, has an almost periodic solution, if and only if

$$\int_0^t e^{i\omega s} a(s)ds$$

is bounded, in which case $b(t)$ can be chosen so that $m[b] \subset m[a]$.

Unlike the case where the a_n are P-periodic in t, a_0 having mean value K does not imply that

$$\int_0^t a_0(s) - K\, ds$$

is bounded. Thus the boundedness of this integral is required for the averaging to be carried out. We have proved the following theorem.

Theorem 1.2.2. Consider the differential equation in polar coordinates

$$\dot{r} = \varepsilon R_1(r) + \cdots + \varepsilon^{k-1}R_{k-1}(r) + \varepsilon^k R_k(r,\theta,t) + O(\varepsilon^{k+1})$$

$$\dot{\theta} = \omega + \varepsilon W_1(r) + \cdots + \varepsilon^{k-1}W_{k-1}(r) + \varepsilon^k W_k(r,\theta,t) + O(\varepsilon^{k+1})$$

where R_k and W_k are 2π-periodic in θ and almost periodic in t and have the form

$$R_k(r,\theta,t) = \sum_{|n|\le N_k} a_n(r,t)e^{ni\theta}$$

$$W_k(r,\theta,t) = \sum_{|n|\le M_k} b_n(r,t)e^{ni\theta}$$

where $a_{-n} = \bar{a}_n$, $b_{-n} = \bar{b}_n$. Suppose that the following integrals are bounded in t.

$$\int_0^t e^{in\omega s}a_n(r,s)ds, \text{ for } 0 < |n| \le N_k$$

$$\int_0^t e^{in\omega s}b_n(r,s)ds, \text{ for } 0 < |n| \le M_k$$

$$\int_0^t (a_0(r,s) - \bar{R}_k(r),\ b_0(r,s) - \bar{W}_k(r))ds$$

where

$$\bar{R}_k(r) = \operatorname*{mean}_{\theta,t}[R_k(r,\theta,t)]$$

$$\bar{W}_k(r) = \operatorname*{mean}_{\theta,t}[W_k(r,\theta,t)].$$

Then there exist functions $u(r,\theta,t)$ and $v(r,\theta,t)$ which are 2π-periodic in θ, almost periodic in t with $m[u] \subset m[R_k]$, $m[v] \subset m[W_k]$ so that if

$$\bar{r} = r + \varepsilon^k u(r,\theta,t)$$

$$\bar{\theta} = \theta + \varepsilon^k v(r,\theta,t)$$

then

$$\dot{\bar{r}} = \varepsilon R_1(\bar{r}) + \cdots + \varepsilon^{k-1} R_{k-1}(\bar{r}) + \varepsilon^k \bar{R}_k(\bar{r}) + O(\varepsilon^{k+1})$$

$$\dot{\bar{\theta}} = \omega + \varepsilon W_1(\bar{r}) + \cdots + \varepsilon^{k-1} W_{k-1}(\bar{r}) + \varepsilon^k \bar{W}_k(\bar{r}) + O(\varepsilon^{k+1})$$

and explicitly

$$u(r,\theta,t) = \sum_{|n|\le N_k} u_n(r,t)e^{ni\theta}, \quad u_{-n} = \bar{u}_n$$

$$v(r,\theta,t) = \sum_{|n|\le M_k} v_n(r,t)e^{ni\theta}, \quad v_{-n} = v_n$$

where

$$u_n(r,t) = e^{-in\omega t}[c_n - \int_0^t e^{in\omega s} a_n(r,s)ds], \; 0 < |n| \le N_k$$

$$v_n(r,t) = e^{-in\omega t}[D_n - \int_0^t e^{in\omega s} b_n(r,s)ds], \; 0 < |n| \le M_k$$

where

$$c_n = \underset{t}{\text{mean}}[\int_0^t e^{in\omega s} a_n(r,s)ds] \quad 0 < |n| \le N_k$$

$$D_n = \underset{t}{\text{mean}}[\int_0^t e^{in\omega s} b_n(r,s)ds] \quad 0 < |n| \le N_k$$

and

$$u_0(r,t) = \int_0^t \bar{R}_k(r) - a_0(r,s)ds$$

$$v_0(r,t) = \int_0^t \bar{W}_k(r) - b_0(r,s)ds.$$

1.3. Higher Dimensional Considerations.

Consider the n-dimensional system in cylindrical coordinates (r,θ,y) given by

$$\dot{y} = By + \varepsilon^{\nu} g(r,\theta,y,t,\varepsilon)y + \varepsilon^{k} h(r,\theta,t,\varepsilon)$$

$$(3.1) \qquad \dot{r} = \varepsilon R(r,\theta,y,t,\varepsilon)$$

$$\dot{\theta} = \omega + \varepsilon W(r,\theta,y,t,\varepsilon)$$

where ν and k are positive integers, B is an $(n-2) \times (n-2)$ matrix with no pure imaginary eigenvalues and all functions are 2π-periodic in θ, almost periodic or P-periodic in t, and smooth enough for the following computations to be carried out.

Let $(r_\varepsilon(t),\theta_\varepsilon(t),y_\varepsilon(t))$ be a solution of (3.1) for which $y_\varepsilon(t)$ and $r_\varepsilon(t)$ are bounded, then $|y_\varepsilon(t)| = O(\varepsilon^L)$ for some $L \geq 0$ uniformly in t. Then decomposing y as $y = (y^s,y^u)$ corresponding to the subspaces where B is stable or unstable. Then it is clear that

$$y_\varepsilon^s(t) = \int_{-\infty}^{t} e^{B^s(t-s)}(\varepsilon^{\nu} g(r_\varepsilon,\theta_\varepsilon,y_\varepsilon,s,\varepsilon)y_\varepsilon(s) + \varepsilon^k h(r_\varepsilon,\theta_\varepsilon,s,\varepsilon))ds$$

$$y_\varepsilon^u(t) = \int_{t}^{\infty} e^{B^u(t-s)}(\varepsilon^{\nu} g(r_\varepsilon,\theta_\varepsilon,y_\varepsilon,s,\varepsilon)y_\varepsilon(s) + \varepsilon^k h(r_\varepsilon,\theta_\varepsilon,s,\varepsilon))ds$$

where $By = (B^s y^s, B^u y^u)$ where B^s is a stable matrix and B^u is unstable.

Then it is clear that

$$|y_\varepsilon(t)| = O(\varepsilon^{\nu+L}) + O(\varepsilon^k).$$

Thus $L = k$ and $|y_\varepsilon| = O(\varepsilon^k)$. We have proved the following.

Theorem 1.3.1. If $(r(t),\theta(t),y(t))$ is a solution of (3.1) with $r(t)$ and $y(t)$ bounded then $|y(t)| = O(\varepsilon^k)$.

Thus if k in (3.1) is large enough one can essentially ignore the presence of y in the $\dot{r}$ and $\dot{\theta}$ equations. Since in this case we have

$$\dot{r} = \varepsilon R(r,\theta,0,t,\varepsilon) + O(\varepsilon^{k+1})$$

$$\dot{\theta} = \omega + \varepsilon W(r,\theta,0,t,\varepsilon) + O(\varepsilon^{k+1})$$

and these equations can be averaged to the order of ε^k by proceeding as in section 1.2.

The following theorem shows that a change of coordinates $y \to \bar{y}$ can be made so that in the new coordinates, k in (3.1) is as large as we wish.

Theorem 1.3.2. Consider (3.1). There exists a function $U = U(r,\theta,t,\varepsilon)$ having the same periodicity properties as $h(r,\theta,t,\varepsilon)$ in θ and t so that if $y = \bar{y} + \varepsilon^k U(r,\theta,t,\varepsilon)$ then

$$\dot{\bar{y}} = By + \varepsilon^m \hat{g}(r,\theta,\bar{y},t,\varepsilon)\bar{y} + \varepsilon^{k+1}\hat{h}(r,\theta,t,\varepsilon)$$

$$\dot{r} = \bar{R}(r,\theta,\bar{y},t,\varepsilon)$$

$$\dot{\theta} = \omega + \varepsilon\bar{W}(r,\theta,\bar{y},t,\varepsilon)$$

where $\hat{g}$, $\hat{h}$, $\bar{R}$, $\bar{W}$ have the same periodicity properties in θ and t as g, h, R, W respectively, and $m = \min\{\nu,k+1\}$, further U is the unique bounded solution of

$$h + BU - \omega \frac{\partial U}{\partial \theta} - \frac{\partial U}{\partial t} = 0$$

<u>Proof</u>. First decompose $y = (y_s, y_u)$ corresponding to the subspaces where B is stable and unstable respectively. Then $By = (B^s y_s, B^u y_u)$ where B^s and B^u are respectively stable and unstable matrices. (3.1) is then written as

$$\dot{y}_s = B^s y_s + \varepsilon^\nu g_{11} y^s + \varepsilon^\nu g_{12} y^u + \varepsilon^k h_1$$

$$\dot{y}_u = B^u y_u + \varepsilon^\nu g_{21} y^s + \varepsilon^\nu g_{22} y^u + \varepsilon^k h_1$$

$$\dot{r} = \varepsilon R(r,\theta,y_s,y_u,t,\varepsilon)$$

$$\dot{\theta} = \omega + \varepsilon W(r,\theta,y_s,y_u,t,\varepsilon)$$

where

$$g(r,\theta,y,t,\varepsilon)y = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} y_s \\ y_u \end{bmatrix}$$

and

$$h(r,\theta,t,\varepsilon) = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

with

$$g_{ij} = g_{ij}(r,\theta,y_s,y_u,t,\varepsilon)$$

$$h_i = h_i(r,\theta,t,\varepsilon).$$

Here the dimensions are determined by those of y_s and y_u.

Now let

$$\bar{y}_s = y_s - \varepsilon^k u(r,\theta,t,\varepsilon)$$

$$\bar{y}_u = y_u - \varepsilon^k v(r,\theta,t,\varepsilon).$$

Then not writing dependence on (r,θ,t,ε), we have

$$g_{11}(y_s,y_u)y_s = g_{11}(\bar{y}_s + \varepsilon^k u,\ \bar{y}_u + \varepsilon^k v)(\bar{y}_s + \varepsilon^k u)$$

$$= g_{11}(\bar{y}_s + \varepsilon^k u,\ \bar{y}_u + \varepsilon^k v)\bar{y}_s + \varepsilon^k[g_{11}(\varepsilon^k u,\ \varepsilon^k v)$$

$$+ G(u,v,\bar{y}_s,\bar{y}_u,\varepsilon)\bar{y}_s + H(u,v,\bar{y}_s,\bar{y}_u,\varepsilon)\bar{y}_u]u$$

$$\overset{\text{def}}{=} {}_1\hat{g}_{11}(\bar{y}_s,\bar{y}_u)\bar{y}_s + {}_2\hat{g}_{11}(\bar{y}_s,\bar{y}_u)\bar{y}_u + \varepsilon^k G_{11}.$$

Similarly

$$g_{12}(y_s,y_u)y_u = {}_1\hat{g}_{12}(\bar{y}_s,\bar{y}_u)\bar{y}_s + {}_2\hat{g}_{12}(\bar{y}_s,\bar{y}_u)\bar{y}_u + \varepsilon^k G_{12}$$

$$g_{21}(y_s,y_u)y_s = {}_1\hat{g}_{21}(\bar{y}_s,\bar{y}_u)\bar{y}_s + {}_2\hat{g}_{21}(\bar{y}_s,\bar{y}_u)\bar{y}_u + \varepsilon^k G_{21}$$

$$g_{22}(y_s,y_u)y_u = {}_1\hat{g}_{22}(\bar{y}_s,\bar{y}_u)\bar{y}_s + {}_2\hat{g}_{22}(\bar{y}_s,\bar{y}_u)\bar{y}_u + \varepsilon^k G_{22}.$$

Also

$$R(y_s,y_u) = R(\varepsilon u,\varepsilon v) + \hat{R}_1(\bar{y}_s,\bar{y}_u)\bar{y}_s + \hat{R}_2(y_s,y_u)\bar{y}_u$$

$$\overset{\text{def}}{=} \hat{R} + \hat{R}_1(\bar{y}_s,\bar{y}_u)\bar{y}_s + \hat{R}_2(\bar{y}_s,\bar{y}_u)\bar{y}_u$$

$$W(y_s,y_u) = \hat{W} + \hat{W}_1(\bar{y}_s,\bar{y}_u)\bar{y}_s + \hat{W}_2(\bar{y}_s,\bar{y}_u)\bar{y}_u.$$

Now we have

$$\dot{\bar{y}}_s = \dot{y}_s - \varepsilon^k[\frac{\partial u}{\partial r}\dot{r} + \frac{\partial u}{\partial \theta}\dot{\theta} + \frac{\partial u}{\partial t}]$$

$$= B^s(\bar{y}_s + \varepsilon^k u) + \varepsilon^\nu[{}_1\hat{g}_{11}y_s + {}_2\hat{g}_{11}\bar{y}_u + \varepsilon^k G_{11}]$$

$$+ \varepsilon^\nu[{}_1\hat{g}_{12}\bar{y}_s + {}_2\hat{g}_{12}\bar{y}_u + \varepsilon^k G_{12}] + \varepsilon^k[h_1 - \omega\frac{\partial u}{\partial \theta} - \frac{\partial u}{\partial t}]$$

$$- \varepsilon^{k+1}\frac{\partial u}{\partial r}(\hat{R} + \hat{R}_1\bar{y}_s + \hat{R}_2\bar{y}_u) - \varepsilon^{k+1}\frac{\partial u}{\partial \theta}(\hat{W} + \hat{W}_1 y_s + \hat{W}_2\bar{y}_u)$$

$$= B^s\bar{y}_s + \varepsilon^m\hat{g}_1\bar{y} + \varepsilon^{k+1}\hat{h}_1 + \varepsilon^k[h_1 + B^s u - \omega\frac{\partial u}{\partial \theta} - \frac{\partial u}{\partial t}]$$

where

$$\bar{y} = (\bar{y}_s, \bar{y}_u)$$

$$m = \min\{\nu, k+1\}$$

$$\hat{g}_1 = \hat{g}_1(r,\theta,\bar{y},t,\varepsilon)$$

$$\hat{h}_1 = \hat{h}_1(r,\theta,t,\varepsilon).$$

Similarly, one obtains

$$\dot{\bar{y}}_u = B^u\bar{y}_u + \varepsilon^m\hat{g}_2\bar{y} + \varepsilon^{k+1}\hat{h}_2 + \varepsilon^k[h_2 + B^u v - \omega\frac{\partial v}{\partial \theta} - \frac{\partial v}{\partial t}].$$

To obtain the desired result the following equations must be solved.

$$h_1 + B^s u - \omega\frac{\partial u}{\partial \theta} - \frac{\partial u}{\partial t} = 0$$

$$h_2 + B^u v - \omega\frac{\partial v}{\partial \theta} - \frac{\partial v}{\partial t} = 0$$

The bounded solutions are easily seen to be

$$u(r,\theta,t,\varepsilon) = \int_{-\infty}^{t} e^{B^s(t-s)} h_1(r,\omega(s-t) + \theta,s,\varepsilon)ds$$

$$v(r,\theta,t,\varepsilon) = \int_{t}^{\infty} e^{B^u(t-s)} h_2(r,\omega(s-t) + \theta,s,\varepsilon)ds.$$

Since h_1 and h_2 are 2π-periodic in θ, so are u and v. We also have

$$\begin{aligned} u(r,\theta,t+P,\varepsilon) &= \int_{-\infty}^{t+P} e^{B^s(t+P-s)} h_1(r,\omega(s-t-P) + \theta,s,\varepsilon)ds \\ &= \int_{-\infty}^{t} e^{B^s(t-\sigma)} h_1(r,\omega(\sigma-t) + \theta,\sigma+P,\varepsilon)d\sigma \\ &= u(r,\theta,t,\varepsilon). \end{aligned}$$

The last equality holds since h_1, is P-periodic in t.

If h_1 is almost periodic in t, let $\{\tau_j\}$ be a sequence so that $h_1(r,\theta,t+\tau_j,\varepsilon) - h(r,\theta,t,\varepsilon) \to 0$ as $j \to \infty$, uniformly in (r,θ,t,ε). Now one has

$$\begin{aligned} u(r,\theta,t+\tau_j,\varepsilon) &= \int_{-\infty}^{t+\tau_j} e^{B^s(t+\tau_j-s)} h_1(r,\omega(s-t-\tau_j) + \theta,s,\varepsilon)ds \\ &= \int_{-\infty}^{t} e^{B^s(t-\sigma)} h_1(r,\omega(\sigma-t) + \theta, \sigma + \tau_j,\varepsilon)d\sigma \\ &\xrightarrow{j\to\infty} \int_{-\infty}^{t} e^{B^s(t-\sigma)} h_1(r,\omega(\sigma-t) + \theta,\sigma,\varepsilon)d\sigma \\ &= u(r,\theta,t,\varepsilon). \end{aligned}$$

Thus u is almost periodic in t with $m[u] \subset m[h_1]$. A similar argument applied to v establishes that v has the same periodicity properties in θ and t as h_2.

Then letting

$$\hat{g}(r,\theta,\bar{y},t,\varepsilon)\bar{y} = (g,\bar{y},g_2\bar{y})'$$

$$\hat{h}(r,\theta,t,\varepsilon) = (\hat{h}_1,\hat{h}_2)'$$

$$\bar{R}(r,\theta,\bar{y},t,\varepsilon) = \hat{R} + \hat{R}_1 y_s + \hat{R}_2 y_u$$

$$\bar{W}(r,\theta,\bar{y},t,\varepsilon) = \hat{W} + \hat{W}_1 y_s + \hat{W}_2 y_u$$

$$U(r,\theta,t,\varepsilon) = (u,v)'$$

the theorem is established.

2. APPLICATION TO BIFURCATION PROBLEMS

2.1. Hopf Bifurcation for an O.D.E. in R^2

Let us consider the nonautonomous O.D.E. in R^2

$$\dot{x} = f(x,t,\alpha),\ x \in R^2,\ t \in R,\ \alpha \in (-\alpha_0, \alpha_0) \tag{1.1}$$

where $f(0,t,\alpha) \equiv 0$ and f is P-periodic or almost periodic on t. Suppose that the linear part of the equation linearized about $x = 0$ at $\alpha = 0$ is independent of t and has the pure imaginary eigenvalues $\pm i\omega_0$, with ω_0 real and non zero. Then expanding $f(x,t,\alpha)$ in powers of x and α (1.1) can be written as

$$\dot{x} = Ax + \alpha B(t,\alpha)x + G(x,t,\alpha) \tag{1.2}$$

where $|G(x,t,\alpha)| = O(|x|^2)$ uniformly in t and α as $|x| \to 0$ and $B(t,\alpha)$, $G(x,t,\alpha)$ are P-periodic or almost periodic in t. By the change of coordinates $x \to Px$, where P is an appropriate 2×2 matrix we can assume that A is in Jordan form

$$A = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix} .$$

Now write

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} , \quad G(x,t,\alpha) = \begin{bmatrix} G_1(x_1,x_2,t,\alpha) \\ G_2(x_1,x_2,t,\alpha) \end{bmatrix}$$

where for $j = 1,2$ we have

$$(1.3)\qquad G_j(x_1,x_2,t,\alpha) = \sum_{k=2}^{\infty} B_{j,k}(x_1,x_2,t,\alpha)$$

$$B_{j,k}(x_1,x_2,t,\alpha) = \sum_{\substack{n+m=k\\ n,m\geq 0}} b^{j,k}_{n,m}(t,\alpha)x_1^n x_2^m \quad .$$

An infinite sum is indicated here only for convenience. Since only a finite number of terms will be considered (1.3) may be viewed as a finite Taylor development with remainder.

Passing to polar coordinates in (1.2) by letting $x = rN_\theta$, yields

$$(1.4)\qquad \begin{aligned} \dot{r} &= \alpha N_\theta' B(t,\alpha)N_\theta r + \cos\theta\ G_1 + \sin\theta\ G_2 \\ \dot{\theta} &= \omega_0 + \alpha T_\theta' B(t,\alpha)N_\theta + \frac{1}{r}(\cos\theta\ G_2 - \sin\theta\ G_1) \end{aligned}$$

where for $j = 1,2$

$$G_j = \sum_{k=2}^{\infty} r^k \sum_{\substack{n+m=k\\ n,m\geq 0}} b^{j,k}_{n,m}(t,\alpha)\cos^n\theta\ \sin^m\theta$$

$$\underset{=}{\mathrm{def}} \sum_{k=2}^{\infty} r^k \sum_{\substack{n=k(\mathrm{mod}\ 2)\\ |n|\leq k}} \beta^{j,k}_n(t,\alpha)e^{ni\theta}$$

with $\beta^{j,k}_n(t,\alpha)$ linear combinations of the $b^{j,k}_{n,m}(t,\alpha)$ with complex coefficients and satisfying

$$\beta^{j,k}_{-n}(t,\alpha) = \overline{\beta^{j,k}_n(t,\alpha)} \ .$$

Then

$\cos\theta\, G_1 + \sin\theta\, G_2 =$

$$= \frac{1}{2}\sum_{k=2}^{\infty} r^k \sum_{\substack{n=k(\mathrm{mod}\ 2)\\ |n|\le k}} \{(e^{i\theta} + e^{-i\theta})\beta_n^{1,k} - i(e^{i\theta} - e^{-i\theta})\beta_n^{2,k}\}e^{ni\theta}$$

$$= \frac{1}{2}\sum_{k=2}^{\infty} r^k \sum_{\substack{n=k(\mathrm{mod}\ 2)\\ |n|\le k}} \{e^{i(n+1)\theta}(\beta_n^{1,k} - i\beta_n^{2,k}) + e^{i(n-1)\theta}(\beta_n^{1,k} + i\beta_n^{2,k})\}$$

$$\stackrel{\text{def}}{=} \frac{1}{2}\sum_{k=2} r^k \sum_{\substack{n=k+1(\mathrm{mod}\ 2)\\ |n|\le k+1}} \gamma_n^{k+1}(t,\alpha)e^{ni\theta}$$

$$\stackrel{\text{def}}{=} \sum_{k=2}^{\infty} r^k\, C_{k+1}(\theta,t,\alpha)\ .$$

Similarly, we have

$\cos\theta\, G_2 - \sin\theta\, G_1 =$

$$= \frac{1}{2}\sum_{k=2}^{\infty} r^k \sum_{\substack{n=k(\mathrm{mod}\ 2)\\ |n|\le k}} \{e^{i(n+1)\theta}(\beta_n^{2,k} + i\beta_n^{1,k}) + e^{i(n-1)\theta}(\beta_n^{2,k} - i\beta_n^{1,k})\}$$

$$\stackrel{\text{def}}{=} \frac{1}{2}\sum_{k=2}^{\infty} r^k \sum_{\substack{n=k+1(\mathrm{mod}\ 2)\\ |n|\le k+1}} \delta_n^{k+1}(t,\alpha)e^{ni\theta}$$

$$\stackrel{\text{def}}{=} \sum_{k=2}^{\infty} r^k\, D_{k+1}(\theta,t,\alpha)\ .$$

Since both these expressions are real we must have $\gamma_{-n}^k = \overline{\gamma_n^k}$, $\delta_{-n}^k = \overline{\delta_n^k}$ and also note that $i\gamma_{k+1}^{k+1} = i(\beta_{k+1}^{1,k} - i\beta_{k+1}^{2,k}) = \delta_{k+1}^{k+1}$ so that we have

$$i\gamma_k^k = \delta_k^k\ , \quad -i\gamma_{-k}^k = \delta_{-k}^k\ .$$

Further note that C_k and D_k are homogenous trigonometric polynomials of degree k with coefficients depending on α and t, P-periodic or almost periodic in t.

Inserting these expressions into (1.4) we obtain

$$\dot{r} = \alpha r C_2 + r^2 C_3 + r^3 C_4 + \ldots$$

(1.5)

$$\dot{\theta} = \omega_0 + \alpha D_2 + r D_3 + r^2 D_4 + \ldots \quad .$$

Then scaling by $r \to \varepsilon r$, $\alpha \to \varepsilon\alpha$ in (1.5) yields

$$\dot{r} = \varepsilon(\alpha r C_2 + r^2 C_3) + \varepsilon^2 r^3 C_4 + O(\varepsilon^3)$$

$$\dot{\theta} = \omega_0 + \varepsilon(\alpha D_2 + r D_3) + \varepsilon^2 r^2 D_4 + O(\varepsilon^3)$$

where C_k and D_k are functions of $(\theta, t, \varepsilon\alpha)$. Expanding all functions in powers of $\varepsilon\alpha$ yields

$$\dot{r} = \varepsilon(\alpha r C_2 + r^2 C_3) + \varepsilon^2 r^3 C_4 + \text{h.o.t.}$$

(1.6)

$$\dot{\theta} = \omega_0 + \varepsilon(\alpha D_2 + r D_3) + \varepsilon^2 r^2 D_4 + \text{h.o.t.}$$

where $\text{h.o.t.} = O(\varepsilon^3) + O(\varepsilon^2\alpha)$, and C_k, D_k are evaluated at $(\theta, t, 0)$.

We are now ready to average the ε and ε^2 terms. Let

$$\bar{r} = r + u_1(r,\theta,t,\alpha) + \varepsilon^2 u_2(r,\theta,t,\alpha)$$

$$\bar{\theta} = \theta + \varepsilon v_1(r,\theta,t,\alpha) + \varepsilon^2 v_2(r,\theta,t,\alpha)$$

with inverses satisfying

$$r = \bar{r} - \varepsilon u_1(\bar{r},\bar{\theta},t,\alpha) + O(\varepsilon^2)$$

$$\theta = \bar{\theta} - \varepsilon v_1(\bar{r},\bar{\theta},t,\alpha) + O(\varepsilon^2) \quad .$$

Then

$$\dot{\bar{r}} = \dot{r} + \varepsilon \frac{\partial u_1}{\partial r}\dot{r} + \frac{\partial u_1}{\partial \theta}\dot{\theta} + \frac{\partial u_1}{\partial t}$$

$$+ \varepsilon^2 \frac{\partial u_2}{\partial r}\dot{r} + \frac{\partial u_2}{\partial \theta}\dot{\theta} + \frac{\partial u_2}{\partial t}$$

(1.7)

$$\dot{\bar{\theta}} = \dot{\theta} + \varepsilon \frac{\partial v_1}{\partial r}\dot{r} + \frac{\partial v_1}{\partial \theta}\dot{\theta} + \frac{\partial v_1}{\partial t}$$

$$+ \varepsilon^2 \frac{\partial v_2}{\partial r}\dot{r} + \frac{\partial v_2}{\partial \theta}\dot{\theta} + \frac{\partial v_2}{\partial t} \quad .$$

In terms of the new coordinates $\bar{r}$ and $\bar{\theta}$, we have

$$\dot{r} = \varepsilon[\alpha\bar{r}C_2 + \bar{r}^2C_3] + \varepsilon^2\bar{r}^3C_4$$

$$- \varepsilon^2(u_1 \frac{\partial}{\partial \bar{r}} + v_1 \frac{\partial}{\partial \bar{\theta}})(\alpha\bar{r}C_2 + \bar{r}^2C_3) + \text{h.o.t.}$$

$$\dot{\theta} = \omega_0 + \varepsilon[\alpha D_2 + \bar{r}D_3] + \varepsilon^2\bar{r}^2D_4$$

$$- \varepsilon^2(u_1 \frac{\partial}{\partial \bar{r}} + v_1 \frac{\partial}{\partial \theta})(\alpha D_2 + \bar{r}D_3) + \text{h.o.t.}$$

where the right hand side is evaluated at $\bar{r}$ and $\bar{\theta}$. Also neglecting dependence on t and α, we have

$$\frac{\partial u_1}{\partial r}(r,\theta) = \frac{\partial u_1}{\partial \bar{r}} + O(\varepsilon)$$

$$\frac{\partial u_1}{\partial \theta}(r,\theta) = \frac{\partial u_1}{\partial \bar{\theta}} - \varepsilon(u_1 \frac{\partial}{\partial \bar{r}} + v_1 \frac{\partial}{\partial \bar{\theta}}) \frac{\partial u_1}{\partial \bar{\theta}} + O(\varepsilon^2)$$

$$\frac{\partial u_1}{\partial t}(r,\theta) = \frac{\partial u_1}{\partial t} - \varepsilon(u_1 \frac{\partial}{\partial \bar{r}} + v_1 \frac{\partial}{\partial \bar{\theta}}) \frac{\partial u_1}{\partial t} + O(\varepsilon^2)$$

where again the right hand side is evaluated at $\bar{r}$ and $\bar{\theta}$. Similar expressions hold for $\partial u_2/\partial r$, $\partial u_2/\partial\theta$, $\partial u_2/\partial t$, $\partial v_1/\partial r$, $\partial v_1/\partial\theta$, $\partial v_1/\partial t$, $\partial v_2/\partial r$, $\partial v_2/\partial\theta$ and $\partial v_2/\partial t$. Inserting these expressions into (1.7) and dropping the bars yields

$$\begin{aligned}
\dot{r} &= \varepsilon[\alpha r C_2 + r^2 C_3 + \omega_0 \frac{\partial u_1}{\partial\theta} + \frac{\partial u_1}{\partial t}] \\
&\quad + \varepsilon^2[r^3 C_4 + r^2 \frac{\partial u_1}{\partial r} C_3 + r \frac{\partial u_1}{\partial\theta} D_3 + \omega_0 \frac{\partial u_2}{\partial\theta} + \frac{\partial u_2}{\partial t}] \\
&\quad - \varepsilon^2(u_1 \frac{\partial}{\partial r} + v_1 \frac{\partial}{\partial\theta})(\alpha r C_2 + r^2 C_3 + \omega_0 \frac{\partial u_1}{\partial\theta} + \frac{\partial u_1}{\partial t}) \\
&\quad + \text{h.o.t.}
\end{aligned}$$

(1.8)

$$\begin{aligned}
\dot{\theta} &= \omega_0 + \varepsilon[\alpha D_2 + r D_3 + \omega_0 \frac{\partial v_1}{\partial\theta} + \frac{\partial v_1}{\partial t}] \\
&\quad + \varepsilon^2[r^2 D_4 + r^2 \frac{\partial v_1}{\partial r} C_3 + r \frac{\partial v_1}{\partial\theta} D_3 + \omega_0 \frac{\partial v_2}{\partial\theta} + \frac{\partial v_2}{\partial t}] \\
&\quad - \varepsilon^2(u_1 \frac{\partial}{\partial r} + v_1 \frac{\partial}{\partial\theta})(\alpha D_2 + r D_3 + \omega_0 \frac{\partial v_1}{\partial\theta} + \frac{\partial v_1}{\partial t}) \\
&\quad + \text{h.o.t.}
\end{aligned}$$

First choose u_1 and v_1 so that the coefficient of ε in (1.8) is independent of θ and t. The following equations must be solved

$$\alpha r C_2 + r^2 C_3 + \omega_0 \frac{\partial u_1}{\partial\theta} + \frac{\partial u_1}{\partial t} = \underset{\theta,t}{\text{mean}}[\alpha r C_2 + r^2 C_3] \overset{\text{def}}{=} K_1(r,\alpha)$$

(1.9)

$$\alpha D_2 + r D_3 + \omega_0 \frac{\partial v_1}{\partial\theta} + \frac{\partial v_1}{\partial t} = \underset{\theta,t}{\text{mean}}[\alpha D_2 + r D_3] \overset{\text{def}}{=} L_1(r,\alpha) .$$

Now C_3 and D_3 being homogenous trignometric in θ of odd degree (i.e. 3) have mean value zero, thus

$$K_1(r,\alpha) = \alpha r \operatorname*{mean}_{\theta,t} [C_2] \overset{\text{def}}{=} \alpha r K_1$$

$$L_1(r,\alpha) = \alpha \operatorname*{mean}_{\theta,t} [D_2] \overset{\text{def}}{=} \alpha L_1 \,.$$

Then Theorem 1.2.2 implies that (1.9) can be solved so that u_1 and v_1 have the same periodicity properties as C_2, C_3, D_2, D_3 in θ and t if and only if the following integrals are bounded

$$I_1: \int_0^t e^{in\omega_0 s} (\gamma_n^3(s), \delta_n^3(s))ds \quad \text{for} \quad |n| = 1,3$$

$$I_2: \int_0^t e^{2i\omega_0 s} (\gamma_2^2(s), \delta_2^2(s))ds$$

$$I_3: \int_0^t (\gamma_0^2(s) - K_1, \delta_0^2(s) - L_1)ds$$

where

$$C_k = \sum_{\substack{n=k(\text{mod } 2)\\ |n|\leq k}} \gamma_n^k(t)e^{ni\theta}, \quad D_k = \sum_{\substack{n=k(\text{mod } 2)\\ |n|\leq k}} \delta_n^k(t)e^{ni\theta}$$

and

$$(u_1(r,\theta,t,\alpha),v_1(r,\theta,t,\alpha)) = (r^2u_{1,3}(\theta,t), rv_{1,3}(\theta,t)) + O(\alpha)$$

where

$$(u_{1,3}(\theta,t),v_{1,3}(\theta,t)) = \sum_{|n|\leq 3} (u_{1,3}^n(t),v_{1,3}^n(t))e^{ni\theta} \,.$$

So (1.8) can now be written as

$$\dot{r} = \varepsilon\alpha r K_1 + \varepsilon^2[r^3 C_4 + 2r^3 u_{1,3} C_3 + r^3 \frac{\partial u_{1,3}}{\partial\theta} D_3 + \omega_0 \frac{\partial u_2}{\partial\theta} + \frac{\partial u_2}{\partial t}] + \text{h.o.t.}$$

(1.10)

$$\dot{\theta} = \omega_0 + \varepsilon\alpha L_1 + \varepsilon^2[r^2 D_4 + r^2 v_{1,3} C_3 + r^2 \frac{\partial v_{1,3}}{\partial\theta} D_3 + \omega_0 \frac{\partial v_2}{\partial\theta} + \frac{\partial v_2}{\partial t}] + \text{h.o.t.}$$

Now u_2 and v_2 must be chosen. The coefficient of ε^2 in (1.10) may be written as

$$(r^3 h_6(\theta,t,\alpha),\ r^2 g_6(\theta,t,\alpha)) + \omega_0 \frac{\partial}{\partial\theta}(u_2,v_2) + \frac{\partial}{\partial t}(u_2,v_2)$$

where

$$(h_6,g_6) = \sum_{|n|=0,2,4} (\gamma_n^4,\delta_n^4)e^{ni\theta} + \sum_{|n|,|k|=1,3} ((2\gamma_k^3 + in\delta_k^3)u_{1,3}^n,(\gamma_k^3 + in\delta_k^3)v_{1,3}^n)e^{i(n+k)\theta}$$

$$\overset{\text{def}}{=} \sum_{|n|=0,2,4,6} (h_6^n,\ g_6^n)e^{ni\theta}\ .$$

The following equations must be solved

(1.11)

$$r^3 h_6 = \omega_0 \frac{\partial u_2}{\partial\theta} + \frac{\partial u_2}{\partial t} = \underset{\theta,t}{\text{mean}}\,[r^3 h_6] \overset{\text{def}}{=} r^3 K_2$$

$$r^2 g_6 + \omega_0 \frac{\partial v_2}{\partial\theta} + \frac{\partial v_2}{\partial t} = \underset{\theta,t}{\text{mean}}\,[r^2 g_6] \overset{\text{def}}{=} r^2 L_2\ .$$

Again Theorem 1.2.2 implies that (1.11) can be solved so that u_2 and v_2 have the desired periodicity properties in θ and t if and only if the following integrals are bounded.

$$I_4: \int_0^t e^{in\omega_0 s} (h_6^n(s), g_6^n(s))ds \qquad |n| = 2,4$$

$$I_5: \int_0^t (h_6^0(s) - K_2, g_6^0(s) - L_2)ds$$

$$I_6: \int_0^t e^{in\omega_0 s} (h_6^n(s), g_6^n(s))ds \qquad |n| = 6 .$$

Now for $n = 6$ in I_6 we have

$$(h_6^6, g_6^6) = ((2\gamma_3^3 + 3i\delta_3^3)u_{1,3}^3, (\gamma_3^3 + 3i\delta_3^3)v_{1,3}^3)$$

$$(u_{1,3}^3, v_{1,3}^3) = e^{-3i\omega_0 t}[(c_1,c_2) - \int_0^t e^{3i\omega_0 s}(\gamma_3^3(s),\delta_3^3(s))ds]$$

where (c_1,c_2) are appropriately chosen constants. Then using the fact that $i\gamma_3^3 = \delta_3^3$ this integral may be written as

$$[\int_0^t e^{3i\omega_0 s}\gamma_3^3(s)ds]^2(1,2i) - \int_0^t e^{3i\omega_0 s} \gamma_3^3(s)ds\ (c_1,2c_2)$$

which is bounded because I_1 is assumed to be bounded. Since $n = -6$ in I_6 is just the complex conjugate of this integral, I_6 is always bounded.

In the case where $f(x,t,\alpha)$ in (1.1) is P-periodic in t then the boundedness of $I_1 - I_5$ is equivalent to $n\omega_0 P \neq 2\pi k$ for all integers k with $|n| = 3,4$ ($|n| = 1,2$ is redundant).

If $I_1 - I_5$ are bounded and u_1,u_2,v_1,v_2 are chosen according to Theorem 1.2.2, (1.6) becomes

$$\text{(1.12)}\qquad \begin{aligned} \dot{r} &= \varepsilon\alpha r K_1 + \varepsilon^2 r^3 K_2 + \text{h.o.t.} \\ \dot{\theta} &= \omega_0 + \varepsilon\alpha L_1 + \varepsilon^2 r^2 L_2 + \text{h.o.t.} \end{aligned}$$

The following theorem summarizes the above results.

Theorem 2.1.1. Consider the differential equation

$$\begin{aligned} \dot{r} &= \varepsilon[\alpha r C_2 + r^2 C_3] + \varepsilon^2 r^3 C_4 + \text{h.o.t.} \\ \dot{\theta} &= \omega_0 + \varepsilon[\alpha D_2 + r D_3] + \varepsilon^2 r^2 D_4 + \text{h.o.t.} \end{aligned}$$

where h.o.t. $= O(\varepsilon^3) + O(\varepsilon^2\alpha)$ uniformly in θ and t as $|\varepsilon| + |\alpha| \to 0$. $\omega_0 \neq 0$ is real. Assume that C_k and D_k are related by $C_k = N_\theta' G_k$, $D_k = T_\theta' G_k$ with $G_k = G_k(\theta,t) \in R^2$ a homogenous trignometric polynomial of degree $k-1$ with coefficients almost periodic (P-periodic) in t. Then there exist functions u_1, v_1, u_2, v_2 which are 2π-periodic in θ, almost periodic in t so that if

$$r = \bar{r} + \varepsilon u_1 + \varepsilon^2 u_2; \quad \theta = \bar{\theta} + \varepsilon v_1 + \varepsilon^2 v_2$$

and the integrals $I_1 - I_5$ are bounded ($n\omega_0 P \neq 2\pi k$ for $n = 3,4$ and all integers k, if the functions are P-periodic in t) then

$$\begin{aligned} \dot{\bar{r}} &= \varepsilon\alpha\bar{r}K_1 + \varepsilon^2\bar{r}^3 K_2 + \text{h.o.t.} \\ \dot{\bar{\theta}} &= \omega_0 + \varepsilon\alpha L_1 + \varepsilon^2\bar{r}^2 L_2 + \text{h.o.t.} \end{aligned}$$

where

$$K_1 = \underset{\theta,t}{\text{mean}} [C_2] \ , \ L_1 = \underset{\theta,t}{\text{mean}} [D_2]$$

$$K_2 = \underset{\theta,t}{\text{mean}} [C_4 + \partial u C_3 + \frac{\partial u}{\partial \theta} D_3]$$

$$L_2 = \underset{\theta,t}{\text{mean}} [D_4 + v C_3 + \frac{\partial v}{\partial \theta} D_3]$$

with u and v defined by

$$C_3 + \omega_0 \frac{\partial u}{\partial \theta} + \frac{\partial u}{\partial t} = 0$$

$$D_3 + \omega_0 \frac{\partial v}{\partial \theta} + \frac{\partial v}{\partial t} = 0 \ .$$

If $K_1 \cdot K_2 < 0$ the choice of $\alpha = \varepsilon$ brings (1.12) to

$$\dot{r} = \varepsilon^2 [r K_1 + r^3 K_2] + O(\varepsilon^3)$$

$$\dot{\theta} = \omega_0 + \varepsilon^2 [L_1 + r^2 L_2] + O(\varepsilon^3).$$

So the presence of a "cylinder like" invariant manifold with radius approximately:

$$r_0 = (-K_1 \cdot K_2^{-1})^{\frac{1}{2}}$$

is suggested. The existence, uniqueness and periodicity properties of this structure are proved in 2.3, in a more general setting and are not treated here.

If $r = r(\theta,t,\alpha)$ defines an invariant manifold of (1.1), 2π-periodic in θ, almost periodic (P-periodic) in t, which bifurcates from $r = 0$ at $\alpha = 0$, we need to show that if $(r^*(t),\theta^*(t))$ is any solution lying on this manifold that this solution satisfies (1.12) (i.e. no manifolds of the desired type are lost in scaling). Since we scaled by $r \to \varepsilon r$, $\alpha \to \varepsilon\alpha$ and then took $\alpha = \varepsilon$, we must show that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} r(\theta,t,\alpha)$$

exists and is finite.

Because of the periodicity properties of $r(\theta,t,\alpha)$ in θ and t its gradient must vanish some place (see Appendix), say

$$\nabla r(\theta_0,t_0,\alpha) = (0,0) .$$

Suppose that

$$(r^*(t_0),\theta^*(t_0)) = (r(\theta_0,t_0,\alpha),\theta_0)$$

then by scaling by $\varepsilon = r^*(t_0)/r_0$ we can assume that $r^*(t_0) = r_0$ and since

$$\dot{r}^*(t_0) = \nabla r(\theta_0,t_0,\alpha)(\dot{\theta}^*(t_0), 1) = 0$$

we have

$$0 = \varepsilon r_0 K_1(\alpha - \varepsilon + O(\varepsilon\alpha) + O(\varepsilon^2)) .$$

Where upon scaling again by $\alpha \to \varepsilon\alpha$ gives $\alpha = 1 + O(\varepsilon)$. This scaling brings (1.12) to

$$\dot{r} = \varepsilon^2 r K_1(1 - r_0^{-2} r^2) + O(\varepsilon^3)$$
$$\dot{\theta} = \omega_0 + O(\varepsilon^2) .$$

Now consider the thin cylindrical shell given by

$$C:\ r_1 \overset{\text{def}}{=} (1 - \gamma)r_0 \le r \le (1 + \gamma)r_0 \overset{\text{def}}{=} r_2$$

where for appropriate $\gamma \to 0$ as $\varepsilon \to 0$ we have at $r = r_j$, $j = 1,2$, if ε is small enough $\dot{r}_1\dot{r}_2 = \varepsilon^4 r_1 r_2 K_1^2(1 - (1 - \gamma)^2)(1 - (1 + \gamma)^2) + O(\varepsilon^5) < 0$ and

$$\operatorname{sgn}(\dot{r}_1) = \operatorname{sgn}(K_1) .$$

Thus the cylindrical shell, C, is positively invariant if $K_1 > 0$ and negatively invariant if $K_1 < 0$. In unscaled coordinates we have

$$\varepsilon r_1 \le r(\theta,t,\alpha) \le \varepsilon r_2$$

so that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} r(\theta,t,\alpha) = r_0 \quad , \quad \alpha = \varepsilon^2 + O(\varepsilon^3) .$$

Hence no manifold of the desired type is lost in scaling.

<u>Theorem 2.1.2</u>. Suppose that the integrals in $I_1 - I_5$ are bounded, ($n\omega_0 P$ is not an integer multiple of 2π for $n = 3,4$ and all integers k, if f in (1.1) is P-periodic in t). Let K_1, K_2 be defined in (1.9) and (1.11). Suppose $K_1 \cdot K_2 < 0$. Then if $r = r(\theta,t,\alpha)$ defines an invariant manifold for (1.1) bifurcating from $r = 0$ at $\alpha = 0$, which is 2π-periodic in θ and almost periodic (P-periodic) in t, then any solution lying on this manifold may be obtained by the scaling $r \to \varepsilon r$, $\alpha \to \varepsilon\alpha$ and averaging so that in the scaled and averaged coordinates (1.1) becomes

$$\dot{r} = \varepsilon\alpha r K_1 + \varepsilon^2 r^3 K_2 + O(\varepsilon^3)$$

$$\dot{\theta} = \omega_0 + \varepsilon\alpha L_1 + \varepsilon^2 r^2 L_2 + O(\varepsilon^3) .$$

Then letting $\alpha = \varepsilon$, $r_0 = (-K_1 \cdot K_2^{-1})^{1/2}$ so that $\alpha = \varepsilon^2$ (unscaled) and

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} r(\theta,t,\alpha) = r_0 \quad \text{(unscaled)}.$$

If $K_1 \cdot K_2 > 0$ the same result holds except we now let $\alpha = -\varepsilon$, $r_0 = (K_1 \cdot K_2^{-1})^{1/2}$ so that $\alpha = -\varepsilon^2$ (unscaled).

2.2. Higher Dimensional Hopf Bifurcation

Consider the O.D.E.

(2.1) $$\dot{z} = f(z,t,\alpha), \quad z \in R^n, \; n \geq 3$$

where $f(0,t,\alpha) \equiv 0$, f is almost periodic (P-periodic) in t uniformly for z and α in compact sets. Further suppose that

$$\frac{\partial f}{\partial z}(0,t,0) \overset{\text{def}}{=} A = \text{independent of } t$$

and that A has the pair of pure imaginary eigenvalues $\pm i\omega_0$, with ω_0 real and non zero and all other eigenvalues of A have non zero real parts. Then (2.1) can be written as

(2.2) $$\dot{z} = Az + \alpha B(t,\alpha)z + F(z,t,\alpha)$$

where $|F(z,t,\alpha)| = O(|z|^2)$ uniformly in t and α as $|z| \to 0$, and $B(t,\alpha)$, $F(z,t,\alpha)$ are almost periodic (P-periodic) in t. By the change of variable $z \to Pz$ where P is an $n \times n$ matrix we can assume that A is in the form

$$A = \begin{bmatrix} A_P & 0 \\ 0 & A_Q \end{bmatrix}, \qquad A_P = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix}$$

and A_Q has no eigenvalues with zero real part.

Now let $z = (x,y) \in R^2 \times R^{n-2}$ and

$$F(z,t,\alpha) = (F_1(x,t,\alpha),\ F_2(x,t,\alpha)) \in R^2 \times R^{n-2}$$

$$B(t,\alpha) = \begin{bmatrix} B_{11}(t,\alpha) & B_{12}(t,\alpha) \\ B_{21}(t,\alpha) & B_{22}(t,\alpha) \end{bmatrix}$$

where $B_{11}(t,\alpha)$ is a 2×2 matrix. Then (1.2) becomes

$$\begin{aligned} \dot{x} &= A_P x + \alpha B_{11}(t,\alpha)x + \alpha B_{12}(t,\alpha)y + F_1(x,y,t,\alpha) \\ \dot{y} &= A_Q y + \alpha B_{21}(t,\alpha)x + \alpha B_{22}(t,\alpha)y + F_2(x,y,t,\alpha) \end{aligned} \tag{2.3}$$

where $|F_i(x,y,t,\alpha)| = O((|x| + |y|)^2)$, $i = 1,2$. Then expanding F_1 and F_2 in powers of x and y, we have

$$F_1(x,y,t,\alpha) = F_1^{2,0}(t,\alpha)x^2 + F_1^{1,1}(t,\alpha)xy + F_1^{0,2}(t,\alpha)y^2 + \ldots$$

$$F_2(x,y,t,\alpha) = F_2^{2,0}(t,\alpha)x^2 + F_2^{1,1}(t,\alpha)xy + F_2^{0,2}(t,\alpha)y^2 + \ldots$$

where the notation indicates that $F_1^{0,2}(t,\alpha)y^2$ is a bilinear map from $R^{n-2} \times R^{n-2}$ into R^n by $(y_1,y_2) \to F_1^{0,2}(t,\alpha)(y_1,y_2)$, and we have let $F_1^{0,2}(t,\alpha)(y,y) = F_1^{0,2}(t,\alpha)y^2$ with similar interpretations for $F_k^{n,m}(t,\alpha)x^n y^m$.

Then passing to polar coordinates in x by setting $x = rN_\theta$, (2.3) becomes

$$\dot{r} = \alpha C_2(\theta,t,\alpha)r + C_3(\theta,t,\alpha)r^2 + C_4(\theta,t,\alpha)r^3$$
$$+ (\alpha C_1(\theta,t,\alpha) + \hat{C}_2(\theta,t,\alpha)r + \hat{C}_3(\theta,t,\alpha)r^2)y$$
$$+ O(r^4) + O(|y|^2)$$

(2.4)
$$\dot{\theta} = \omega_0 + \alpha D_2(\theta,t,\alpha) + D_3(\theta,t,\alpha)r + D_4(\theta,t,\alpha)r^2$$
$$+ r^{-1}(\alpha D_1(\theta,t,\alpha) + \hat{D}_2(\theta,t,\alpha)r + \hat{D}_3(\theta,t,\alpha)r^2)y$$
$$+ O(r^3) + r^{-1}O(|y|^2)$$

$$\dot{y} = A_Q y + \alpha B_{22}(t,\alpha)y + G(r,\theta,y,t,\alpha)y^2$$
$$+ \alpha E_1(\theta,t,\alpha)r + E_2(\theta,t,\alpha)r^2 + O(r^3) \quad .$$

where C_k, D_k, E_k, $\hat{C}_k$, $\hat{D}_k$ are homogenous trignometric polynomials of degree k with coefficients depending on t and α.

Then scale by $r \to \varepsilon r$, $\alpha \to \varepsilon\alpha$, $y \to \varepsilon y$ in (1.4) to obtain

$$\dot{r} = \varepsilon[\alpha r C_2 + r^2 C_3] + \varepsilon r\hat{C}_2 y + \varepsilon^2 r^3 C_4 + ((|\varepsilon| + |\alpha| + |y|)^3)$$

(2.5)
$$\dot{\theta} = \omega_0 + \varepsilon[\alpha D_2 + rD_3] + \varepsilon\hat{D}_2 y + \varepsilon^2 r^2 D_4 + O((|\varepsilon| + |\alpha| + |y|)^2)$$

$$\dot{y} = A_Q y + \varepsilon Hy + \varepsilon[\alpha r E_1 + r^2 E_2] + O(\varepsilon^2)$$

where $Hy = \alpha B_{22}y + Gy^2$, and $C_k, D_k, E_k, \hat{C}_k, \hat{D}_k$ are evaluated at $(\theta,t,\varepsilon\alpha)$. So that by expanding these functions in powers of $\varepsilon\alpha$ does not change the form of (2.5), we can assume that they are evaluated at $(\theta,t,0)$.

We are now in position to apply Theorem 1.3.2 to decouple the $\dot{r}$ and $\dot{\theta}$ equations from the $\dot{y}$ equation up to the cubic order. Let $y = \bar{y} + \varepsilon U$ where $U = U(r,\theta,t,\alpha)$ is the unique bounded solution of

$$\alpha r E_1 + r^2 E_2 + A_Q U - \omega_0 \frac{\partial u}{\partial \theta} - \frac{\partial u}{\partial t} = 0 .$$

Then Theorem 1.3.2 implies that (2.5) becomes

$$\begin{aligned} \dot{r} &= \varepsilon[\alpha r C_2 + r^2 C_3] + \varepsilon^2[r\hat{C}_2 U + r^3 C_4] + O((|\varepsilon| + |\alpha| + |\bar{y}|)^3) \\ \dot{\theta} &= \omega_0 + \varepsilon[\alpha D_2 + r D_3] + \varepsilon^2[\hat{D}_2 U + r^2 D_4] + O((|\varepsilon| + |\alpha| + |\bar{y}|)^3) \qquad (2.6) \\ \dot{\bar{y}} &= A_Q \bar{y} + \varepsilon \bar{H} \bar{y} + O(\varepsilon^2) \end{aligned}$$

where $\bar{H} = \bar{H}(r,\theta,\bar{y},t,\alpha,\varepsilon)$. And Theorem 1.3.1 gives $|\bar{y}| = O(\varepsilon^2)$, so $O((|\varepsilon| + |\alpha| + |\bar{y}|)^3) = O((|\varepsilon| + |\alpha|)^3)$. Further note that $U = \alpha r V + r^2 W$ so that (1.6) can be written as

$$\begin{aligned} \dot{r} &= \varepsilon[\alpha r C_2 + r^2 C_3] + \varepsilon^2 r^3[\hat{C}_2 W + C_4] + O((|\varepsilon| + |\alpha|)^3) \\ \dot{\theta} &= \omega_0 + \varepsilon[\alpha D_2 + r D_3] + \varepsilon^2 r^2[\hat{D}_2 W + D_4] + O((|\varepsilon| + |\alpha|)^3) \qquad (2.7) \\ \dot{y} &= A_Q \bar{y} + O(\varepsilon^2) \end{aligned}$$

as long as r and $\bar{y}$ remain in a bounded region as $\varepsilon \to 0$ and r is considered to be away from 0.

<u>Theorem 2.2.1</u>. Consider the differential equation in $(r,\theta,y) \in R \times R \times R^{n-2}$

$$\begin{aligned} \dot{r} &= \varepsilon[\alpha r C_2 + r^2 C_3] + \varepsilon r \hat{C}_2 y + \varepsilon^2 r^3 C_4 + \text{h.o.t.} \\ \dot{\theta} &= \omega_0 + \varepsilon[\alpha D_2 + r D_3] + \varepsilon \hat{D}_2 y + \varepsilon^2 r^2 D_4 + \text{h.o.t.} \\ \dot{y} &= A_Q y + \varepsilon H y + \varepsilon[\alpha r E_1 + r^2 E_2] + O(\varepsilon^2) \end{aligned}$$

where h.o.t. $= O((|\varepsilon| + |\alpha| + |y|)^3)$, $O(\varepsilon^2)$ are uniform in t and

θ as $|\varepsilon| + |\alpha| + |y| \to 0$ and all functions are 2π-periodic in θ, almost periodic (P-periodic) in t, A_Q has no pure imaginary eigenvalues, $\omega_0 \neq 0$ is real, subscripted functions depend on (θ,t), $H = H(r,\theta,y,t,\alpha,\varepsilon)$. Then there exists a unique function $U = U(r,\theta,t,\alpha)$, 2π-periodic in θ almost periodic (P-periodic) in t so that if $y = \bar{y} + \varepsilon U$ then $|\bar{y}| = O(\varepsilon^2)$ and

$$\dot{r} = \varepsilon[\alpha r C_2 + r^2 C_3] + \varepsilon^2 r^3 [\hat{C}_2 W + C_4] + O((|\varepsilon| + |\alpha|)^3)$$

$$\dot{\theta} = \omega_0 + \varepsilon[\alpha D_2 + r D_3] + \varepsilon^2 r^2 [\hat{D}_2 W + D_4] + O((|\varepsilon| + |\alpha|)^3)$$

$$\dot{\bar{y}} = A_Q \bar{y} + O(\varepsilon^2)$$

where $W = W(\theta,t)$ is the unique bounded solution of

$$E_2 + A_Q W - \omega_0 \frac{\partial W}{\partial \theta} - \frac{\partial W}{\partial t} = 0$$

and W is 2π-periodic in θ, almost periodic (P-periodic) in t.

Now $\hat{C}_2 W$ and $\hat{D}_2 W$ are homogenous trignometric polynomials of degree 4 with coefficients which are almost periodic (P-periodic) in t. So set

$$\hat{C}_4 = \hat{C}_2 W + C_4 \ , \quad \hat{D}_4 = \hat{D}_2 W + D_4 \ .$$

Thus, if the integrals in $I_1 - I_5$ defined in 2.1 are bounded when C_4, D_4 are replaced by $\hat{C}_4$, $\hat{D}_4$, we may apply Theorem 2.1.1 to (2.7) to obtain the averaged equation

$$\begin{aligned} \dot{r} &= \varepsilon \alpha r K_1 + \varepsilon^2 r^3 K_2 + O((|\varepsilon| + |\alpha|)^3) \\ \dot{\theta} &= \omega_0 + \varepsilon \alpha L_1 + \varepsilon^2 r^2 L_2 + O((|\varepsilon| + |\alpha|)^3) \qquad (2.8) \\ \dot{\bar{y}} &= A_Q \bar{y} + O(\varepsilon^2) \end{aligned}$$

where K_1, K_2, L_1, L_2 are the constants defined in Theorem 2.1.1 with C_4, D_4 replaced by $\hat{C}_4$, $\hat{D}_4$.

If $K_1 \cdot K_2 < 0$ taking $\alpha = \varepsilon$ yields

$$\dot{r} = \varepsilon^2(rK_1 + r^3K_2) + O(\varepsilon^3)$$

$$\dot{\theta} = \omega_0 + \varepsilon^2(L_1 + r^2L_2) + O(\varepsilon^3) \tag{2.9}$$

$$\dot{\bar{y}} = A_Q\bar{y} + O(\varepsilon^2) .$$

This equation suggests the presence of an invariant manifold of the form

$$(r,y) = (r(\theta,t,\alpha),\ y(\theta,t,\alpha))$$

which is 2π-periodic in θ, almost periodic (P-periodic) in t, so that in unscaled coordinates

$$r = \varepsilon r_0 + O(\varepsilon^2)$$

$$y = \varepsilon r_0^2 W(\theta,t) + O(\varepsilon^2)$$

where

$$r_0 = (-K_1,\ K_2^{-1})^{\frac{1}{2}} , \quad \alpha = \varepsilon^2 .$$

This is actually the case as is proved in the next section.

On the other hand if $(r,y) = (r(\theta,t,\alpha),\ y(\theta,t,\alpha))$ defines an invariant manifold bifurcating from $(r,y) = (0,0)$ at $\alpha = 0$ with the desired periodicity properties, let $(r(t),\ y(t))$ be a solution lying on this manifold. Now $\dot{r}(t_0) = 0$ for some t_0. Take ε to be the supremum of $|r(t)| + |y(t)|$, so that after scaling, $|r(t)| + |y(t)|$ has 1 as its supremum. Now we have shown that $|y| = O(\varepsilon)$ (scaled) as long as (r,y) remain bounded as $\varepsilon \to 0$. Set $R = r(t_0)/r_0$ (scaled). Then at $t = t_0$

$$0 = \varepsilon r(t_0)K_1(\alpha - \varepsilon R^2 + O(\varepsilon\alpha) + O(\varepsilon^2))$$

so that $\alpha = \varepsilon R^2 + O(\varepsilon^2)$ (scaled). Now for appropriate $\gamma \to 0$ as $\varepsilon \to 0$ if $r = (1 - \gamma)r(t_0)$, then

$$\dot{r} = \varepsilon^2 R^2 r(t_0)K_1(1 - (1 - \gamma)^2 + O(\varepsilon))$$

is of constant sign as $\varepsilon \to 0$. Thus $(1 - \gamma)r(t_0) \leq r \leq 1$ and so as $\varepsilon \to 0$, $r/r(t_0) \to 1$ but for some t, $1 - O(\varepsilon) \leq r(t) \leq 1$, which implies that $r(t_0) \to 1$ (scaled). Hence by replacing ε by εR, we have in scaled coordinates, $|y| = O(\varepsilon)$, $\alpha = \varepsilon + O(\varepsilon^2)$, $r = O(1)$, uniformly in t.

Theorem 2.2.2. Suppose that the integrals in $I_1 - I_5$ defined in 2.1 are bounded when C_4, D_4 are replaced with $\hat{C}_4$, $\hat{D}_4$ ($n\omega_0 P \neq 2\pi k$ for $|n| = 1,2,3,4$, k an integer if f in (2.1) is P-periodic). If K_1, K_2 are defined as in Theorem 2.1.1 with C_4, D_4 replaced with $\hat{C}_4$, $\hat{D}_4$ and $K_1 \cdot K_2 < 0$, then any solution lying on an invariant manifold for (1.1) which is 2π-periodic in θ, almost periodic (P-periodic) in t, which bifurcates from $(r,y) = (0,0)$ at $\alpha = 0$, can be obtained by the scaling

$$r \to \varepsilon r,\ y \to \varepsilon y \quad \text{with} \quad \alpha = \varepsilon^2$$

then letting $y = \bar{y} + U(r,\theta,t,\alpha)$ with $U(r,\theta,t,\alpha)$ as in Theorem 2.2.1. So that $|\bar{y}| = O(\varepsilon^2)$ and averaging the $\dot{r}$ and $\dot{\theta}$ equations as in Theorem 2.1.1 to obtain

$$\dot{r} = \varepsilon^2(rK_1 + r^3K_2) + O(\varepsilon^3)$$

$$\dot{\theta} = \omega_0 + \varepsilon^2(L_1 + r^2L_2) + O(\varepsilon^3)$$

$$\dot{\bar{y}} = A_Q\bar{y} + O(\varepsilon^2) \ .$$

If $K_1 \cdot K_2 > 0$ then $\alpha = -\varepsilon^2$ and

$$\dot{r} = \varepsilon^2(-rK_1 + r^3K_2) + O(\varepsilon^3)$$

$$\dot{\theta} = \omega_0 + \varepsilon^2(-L_1 + r^2L_2) + O(\varepsilon^3)$$

$$\dot{\bar{y}} = A_Q\bar{y} + O(\varepsilon^2) \ .$$

2.3. Existence of the Manifold

In the previous section it was shown that the system (2.1) after scaling and averaging can be written as

$$\begin{aligned} \dot{r} &= \varepsilon^2(\pm\, rK_1 + r^3K_2) + O(\varepsilon^3) \\ \dot{\theta} &= \omega_0 + \varepsilon^2(L_1 + r^2L_2) + O(\varepsilon^3) \qquad (3.1) \\ \dot{y} &= A_Q y + O(\varepsilon^2) \end{aligned}$$

where $\pm = -\text{sgn}(K_1 \cdot K_2)$, $K_1 \cdot K_2 \neq 0$ and A_0 has no pure imaginary eigenvalues.

We now prove that if ε is small enough there is a unique two dimensional manifold parametrized by θ and t, 2π-periodic in θ, almost periodic (P-periodic) in t so that solutions of (3.1) which begin on M will remain on M for all time. More generally the following theorem holds.

Theorem 2.3.1. Consider the differential equation in the coordinates $(r,\theta,y) \in R \times R \times R^{n-2}$ given by

$$\begin{aligned} \dot{r} &= \varepsilon r + \varepsilon^a R(r,\theta,y,t,\varepsilon) \\ \dot{\theta} &= \omega(\varepsilon) + \varepsilon^b W(r,\theta,y,t,\varepsilon) \qquad (3.2) \\ \dot{y} &= Ay + \varepsilon^c Y(r,\theta,y,t,\varepsilon) \end{aligned}$$

where R, W, Y are 2π-periodic in θ, almost periodic or P-periodic in t. A has no pure imaginary eigenvalues. All functions are continuously differentiable, $a > 1$, $b > 1$, $c > 0$. Then there exists an ε_0 so that if $0 < \varepsilon \leq \varepsilon_0$ then there are unique functions $r^*(\theta,t,\varepsilon)$, $y^*(\theta,t,\varepsilon)$, which are 2π-periodic in θ, almost periodic (P-periodic) in t. So that for a fixed ε in $(0,\varepsilon_0]$ the two dimensional manifold M defined by

$$M: \quad (r,y) = (r^*(\theta,t,\varepsilon),\ y^*(\theta,t,\varepsilon))$$

is invariant under the flow induced by solutions of (3.2).

Proof. Decompose $y = (y_1,y_2) \in R^k \times R^\ell$ according to the subspaces of R^{n-2} where A_0 is respectively stable and unstable. Let $A^s y_1$ denote Ay restricted to the stable subspace and $A^u y_2$ for Ay on the unstable subspace. Then (3.2) has the form

$$\dot{r} = \varepsilon r + \varepsilon^a R(r,\theta,y_1,y_2,t,\varepsilon) \qquad (3.3a)$$

$$\dot{\theta} = \omega(\varepsilon) + \varepsilon^b W(r,\theta,y_1,y_2,t,\varepsilon) \qquad (3.3b)$$

$$\dot{y}_1 = A^s y_1 + \varepsilon^c Y_1(r,\theta,y_1,y_2,t,\varepsilon) \qquad (3.3c)$$

$$\dot{y}_2 = A^u y_2 + \varepsilon^c Y_2(r,\theta,y_1,y_2,t,\varepsilon) \ . \qquad (3.3d)$$

Define X to be the space of triples of functions $(f(\theta,t),\ g_1(\theta,t),\ g_2(\theta,t))$ taking values in $R \times R^k \times R^\ell$, where $(y_1, y_2) \in R^k \times R^\ell$, which are 2π-periodic in θ, almost periodic (P-periodic) in t.

$$\|f\| = \sup_{\theta,t} |f(\theta,t)| \leq D,\ \|g_1\| \leq D,\ \|g_2\| \leq D,$$

$$|f(\theta,t) - f(\bar{\theta},t)| \leq \delta|\theta - \bar{\theta}|,$$

$$|g_1(\theta,t) - g_1(\bar{\theta},t)| \leq \delta|\theta - \bar{\theta}|,$$

$$|g_2(\theta,t) - g_2(\bar{\theta},t)| \leq \delta|\theta - \bar{\theta}| .$$

X is clearly complete in the norm $\|\cdot\|$. We will define a mapping F: $X \to X$ so that the unique fixed point of F in X will define an invariant manifold of (3.2) with the desired periodicity properties.

Let $(f,g_1,g_2) \in X$, and $(r,y_1,y_2) = (f,g_1,g_2)$ in (3.3b). We have

$$\dot{\theta} = \omega(\varepsilon) + \varepsilon^b W(f(\theta,t),\theta,g_1(\theta,t),g_2(\theta,t),t,\varepsilon)$$

which has a unique solution passing through (ξ,τ) denoted by

$$\theta^* = \theta^*(t;\ \tau,\xi;\ f_1,g_1,g_2) .$$

Substituting θ^* for θ, (f_1,g_1,g_2) for (r,y_1,y_2) into the remaining equations in (3.3) yields

$$\dot{r} = \varepsilon r + \varepsilon^a R(f(t,\theta^*),\theta^*,\ g_1(t,\theta^*),\ g_2(t,\theta^*),t,\varepsilon)$$

$$\dot{y}_1 = A^s y_1 + \varepsilon^c Y_1(f(t,\theta^*),\theta^*,\ g_1(t,\theta^*),\ g_2(t,\theta^*),t,\varepsilon)$$

$$\dot{y}_2 = A^u y_2 + \varepsilon^c Y_2(f(t,\theta^*),\theta^*, g_1(t,\theta^*), g_2(t,\theta^*),t,\varepsilon).$$

Then the variation of constants formula gives the unique bounded solutions as

$$r(t;\ \xi;\ f,g_1,g_2) = \varepsilon^a \int_{-\infty}^{t} e^{\varepsilon(s-t)} R(f,\theta^*,g_1,g_2,s)ds$$

$$y_1(t;\ \xi;\ f,g_1,g_2) = \varepsilon^c \int_{-\infty}^{t} e^{A^s(t-s)} Y_1(f,\theta^*,g_1,g_2,s)ds$$

$$y_2(t;\ \xi;\ f,g_1,g_2) = -\varepsilon^c \int_{t}^{\infty} e^{A^u(t-s)} Y_2(f,\theta^*,g_1,g_2,s)ds$$

define $F(f,g_1,g_2) = (r(t,\theta),\ y_1(t,\theta),\ y_2(t,\theta))$. We will show that $F\colon\ X \to X$ is a contraction in $\|\cdot\|$.

To accomplish this, first define

$$L(R) = \max\{\|R\|,\ \|\tfrac{\partial R}{\partial r}\|,\ \|\tfrac{\partial R}{\partial \theta}\|,\ \|\tfrac{\partial R}{\partial y_1}\|,\ \|\tfrac{\partial R}{\partial y_2}\|,\ \|\tfrac{\partial R}{\partial t}\|\}$$

with similar definitions for $L(W)$, $L(Y_1)$, $L(Y_2)$.

Then

$$|r(t,\theta)| \leq \varepsilon^a L(R) \int_{-\infty}^{t} e^{\varepsilon(s-t)}dt = \varepsilon^{a-1}L(R)$$

$$|y_1(t,\theta)| \leq \varepsilon^c L(Y_1) \int_{-\infty}^{t} |e^{A^s(t-s)}|ds$$

but there are constants γ_s, K_s so that $|e^{A^s\sigma}| \leq K_s e^{-\gamma_s\sigma}$, thus $|y_1(t,\theta)| \leq \varepsilon^c K_s \gamma_s^{-1} L(Y_1)$, and similarly $|y_2(t,\theta)| \leq \varepsilon^c K_u \gamma_u^{-1} L(Y_2)$. Thus if ε is small enough we have $\|r\| \leq D$, $\|y_1\| \leq D$, $\|y_2\| \leq D$.

Now let $(\bar{r},\bar{y}_1,\bar{y}_2) = (r(t,\bar{\theta}),\ y_1(t,\bar{\theta}),\ y_2(t,\bar{\theta}))$ then

$$|(r,y_1,y_2) - (\bar{r},\bar{y}_1,\bar{y}_2)| \leq |r - \bar{r}| + |y_1 - \bar{y}_1| + |y_2 - \bar{y}_2|$$

but

$$|r - \bar{r}| \leq \varepsilon^a \int_{-\infty}^{t} e^{\varepsilon(s-t)}|R - \bar{R}|ds$$

$$|y_1 - \bar{y}_1| \leq \varepsilon^c K_s \int_{-\infty}^{t} e^{\gamma_s(s-t)}|Y_1 - \bar{Y}_1|ds$$

$$|y_2 - \bar{y}_2| \leq \varepsilon^c K_u \int_{t}^{\infty} e^{\gamma_u(t-s)}|Y_2 - \bar{Y}_2|ds$$

where

$$|R - \bar{R}| = |R(f,\theta^*,g_1,g_2) - R(\bar{f},\bar{\theta}^*,\bar{g}_1,\bar{g}_2)|$$

$$\leq (3\delta L(R) + 1)|\theta^* - \bar{\theta}^*|$$

$$|Y_1 - \bar{Y}_1| \leq (3\delta L(Y_1) + 1)|\theta^* - \bar{\theta}^*|$$

$$|Y_2 - \bar{Y}_2| \leq (3\delta L(Y_2) + 1)|\theta^* - \bar{\theta}^*|$$

and

$$\theta^* = \theta^*(t,\theta) = \theta + \varepsilon^b \int_{\tau}^{t} W(f,\theta^*,g_1,g_2)ds + \omega(\varepsilon)(t - \tau)$$

$$\bar{\theta}^* = \theta^*(t,\bar{\theta}) \ .$$

Let $M = \max\{L(R),\ L(W),\ L(Y_1),\ L(Y_2)\}$ and $B = 3\delta M + 1$. Then

$$|\theta^* - \bar{\theta}^*| \leq |\theta - \bar{\theta}| + \varepsilon^b B \int_{\tau}^{t} |\theta^* - \bar{\theta}^*||ds|$$

so the Gronwall inequality yields

$$|\theta^* - \bar{\theta}^*| \leq |\theta - \bar{\theta}|\exp(\varepsilon^b B|t - \tau|) \ .$$

Thus

$$|r - \bar{r}| \leq \varepsilon^a B|\theta - \bar{\theta}|\int_{-\infty}^{t} \exp((\varepsilon - \varepsilon^b B)(s - t))ds$$

since $b > 1$ choose ε so small that $\varepsilon - \varepsilon^b B > 0$, then

$$|r - \bar{r}| \leq \frac{\varepsilon^a B}{\varepsilon - \varepsilon^b B}|\theta - \bar{\theta}| = \frac{\varepsilon^{a-1} B}{1 - \varepsilon^{b-1} B}|\theta - \bar{\theta}|$$

also

$$|y_1 - \bar{y}_1| \leq \varepsilon^c K_s B|\theta - \bar{\theta}| \int_{-\infty}^{t} \exp((\gamma_s - \varepsilon^b B)(s - t))ds$$

$$|y_2 - \bar{y}_2| \leq \varepsilon^c K_u B|\theta - \bar{\theta}| \int_{t}^{\infty} \exp((\gamma_u - \varepsilon^b B)(t - s))ds$$

so that ε must be small enough to have $\gamma_s - \varepsilon^b B > 0$, $\gamma_u - \varepsilon^b B > 0$ in which case

$$|y_1 - \bar{y}_1| \leq \frac{\varepsilon^c K_s B}{\gamma_s - \varepsilon^b B}, \quad |y_2 - \bar{y}_2| \leq \frac{\varepsilon^c K_u B}{\gamma_u - \varepsilon^b B}$$

and

$$|(r,y_1,y_2) - (\bar{r},\bar{y}_1,\bar{y}_2)| \leq Q(\varepsilon)|\theta - \bar{\theta}|$$

where

$$Q(\varepsilon) = \frac{\varepsilon^a B}{\varepsilon - \varepsilon^b B} + \frac{\varepsilon^c K_s B}{\gamma_s - \varepsilon^b B} + \frac{\varepsilon^c K_u B}{\gamma_u - \varepsilon^b B}$$

thus if ε is small enough $Q(\varepsilon) < \delta$.

Next (r,y_1,y_2) must be shown to have the same periodicity properties as the functions in X. If (f,g_1,g_2) are 2π-periodic in θ, then by uniqueness of solution we have

$$\theta^*(t;\ \tau,\ \xi + 2\pi) = \theta^*(t;\tau,\xi) + 2\pi$$

and since R, Y_1, Y_2 are 2π-periodic in θ we must have (r,y_1,y_2) 2π-periodic in θ.

If (f,g_1,g_2) are P-periodic in t then again by uniqueness of solution we have

$$\theta^*(t + P; \tau + P,\xi) = \theta^*(t; \tau,\xi)$$

and then

$$r(t + P,\theta) = \varepsilon^a \int_{-\infty}^{t+P} e^{\varepsilon(s-t-P)} R(f,\theta^*,g_1,g_2)ds$$

where f, g_1, g_2 are evaluated at $(\theta^*(s; t + T,\theta), s)$ so that after the change of variable $\sigma = s - P$

$$r(t + P,\theta) = \varepsilon^a \int_{-\infty}^{t} e^{\varepsilon(\sigma-t)} R(f,\theta^*,g_1,g_2)ds$$

where now f, g_1, g_2 are evaluated at $(\theta^*(\sigma; t,\theta), \sigma)$ and we have $r(t + P,\theta) = r(t,\theta)$. Similarly y_1, y_2 are P-periodic in t.

If the functions in X are almost periodic in t, let $\{\tau_j\}$ be a sequence so that

$$(f,g_1,g_2)(t + \tau_j) - (f,g_1,g_2)(t) \to 0$$

$$(R,W,Y_1,Y_2)(t + \tau_j) - (R,W,Y_1,Y_2)(t) \to 0$$

uniformly in t and the remaining variables as $j \to \infty$. Then since $\theta^*(t + \tau_j; \tau + \tau_j,\theta) = \theta^*(t; \tau,\theta)$ we have

$$r(t + \tau_j,\theta) = \varepsilon^a \int_{-\infty}^{t+\tau_j} e^{\varepsilon(s-t-\tau_j)} R(f,\theta^*,g_1,g_2)ds$$

where f, g_1, g_2 are evaluated at $(\theta^*(s; t + \tau_j,\theta), s)$, after the change of variable $\sigma = s - \tau_j$

$$r(t + \tau_j,\theta) = \varepsilon^a \int_{-\infty}^{t} e^{\varepsilon(\sigma-t)} R(f,\theta^*,g_1,g_2)d\sigma$$

where f, g_1, g_2 are evaluated at $(\theta^*(\sigma;\, \tau,\theta),\ \sigma + \tau_j)$, passing to the limit we have established that $r(t,\theta)$ is almost periodic in t. Similar arguments establish that $y_1(t,\theta)$ and $y_2(t,\theta)$ are also almost periodic. Thus $(r,y_1,y_2) \in X$ whenever $(f,g_1,g_2) \in X$, so if ε is small enough

$$F\colon\ X \to X\ .$$

Finally, it is shown that F is a uniform contraction in the supremum norm on X.

Let $\bar{r} = r(t,\theta;\ \bar{f},\bar{g}_1,\bar{g}_2)$, similarly for $\bar{y}_1$, $\bar{y}_2$. We have

$$|(r,y_1,y_2) - (r,\bar{y}_1,\bar{y}_2)| \le |r - \bar{r}| + |y_1 - \bar{y}_1| + |y_2 - \bar{y}_2|.$$

Now

$$|r - \bar{r}| \le \varepsilon^a \int_{-\infty}^{t} e^{\varepsilon(s-t)} |R - \bar{R}|ds$$

$$|y_1 - \bar{y}_1| \le \varepsilon^c K_s \int_{-\infty}^{t} e^{\gamma_s(s-t)} |Y_1 - \bar{Y}_1|ds$$

$$|y_2 - \bar{y}_2| \le \varepsilon^c K_s \int_{t}^{\infty} e^{\gamma_u(t-s)} |Y_2 - \bar{Y}_2|ds$$

where

$$\begin{aligned} |R - \bar{R}| &= |R(f,\theta^*,g_1,g_2) - R(\bar{f},\bar{\theta}^*,\bar{g}_1,\bar{g}_2)| \\ &\le M[\|f - \bar{f}\| + \|g_1 - \bar{g}_1\| + \|g_2 - \bar{g}_2\|] + M|\theta^* - \bar{\theta}^*| \\ &\overset{\text{def}}{=} M\Delta + M|\theta^* - \bar{\theta}^*|. \end{aligned}$$

With the identical inequality holding for $|W - \overline{W}|$, $|Y_1 - \overline{Y}_1|$, $|Y_2 - \overline{Y}_2|$. Then

$$|\theta^* - \overline{\theta}^*| \leq \varepsilon^b \int_\tau^t |W - \overline{W}||ds|$$

$$\leq \varepsilon^b M\Delta|t - \tau| + \varepsilon^b M \int_\tau^t |\theta^* - \theta^*||ds|$$

and by the generalized Gronwall inequality

$$|\theta^* - \overline{\theta}^*| \leq \varepsilon^b M\Delta|t - \tau| + \varepsilon^b M^2 \int_\tau^t e^{\varepsilon^b M|s-t|}|s - \tau||ds|$$

$$= [e^{\varepsilon^b M|t-\tau|} - 1]\Delta$$

so that

$$|R - \overline{R}| \leq M\Delta e^{\varepsilon^b M|t-\tau|}$$

with identical inequalities holding for $|Y_1 - \overline{Y}_1|$, $|Y_2 - \overline{Y}_2|$ which implies that

$$|r - \bar{r}| \leq \varepsilon^a M\Delta \int_{-\infty}^t e^{(\varepsilon - \varepsilon^b M)(s-t)} ds = \frac{\varepsilon^a M\Delta}{\varepsilon - \varepsilon^b M}$$

provided $\varepsilon - \varepsilon^b M > 0$. Similarly if $\gamma_s - \varepsilon^b M > 0$, $\gamma_u - \varepsilon^b M > 0$ we have

$$|y_1 - \bar{y}_1| \leq \frac{\varepsilon^c M K_s \Delta}{\gamma_s - \varepsilon^b M}; \quad |y_2 - \bar{y}_2| \leq \frac{\varepsilon^c M K_u \Delta}{\gamma_u - \varepsilon^b M}$$

thus

$$|(r, y_1, y_2) - (\bar{r}, \bar{y}_1, \bar{y}_2)| \leq T(\varepsilon)\Delta$$

where

$$T(\varepsilon) = \frac{\varepsilon^{a}M}{\varepsilon - \varepsilon^{b}M} + \frac{\varepsilon^{c}MK_s}{\gamma_s - \varepsilon^{b}M} + \frac{\varepsilon^{c}MK_u}{\gamma_u - \varepsilon^{b}M}$$

so that if ε is small enough $T(\varepsilon) < 1$ and F is a uniform contraction on X. This completes the proof of the theorem.

To apply Theorem (2.3.1) to (3.1) first suppose that $K_1 \cdot K_2 < 0$ and let $r = r_0 + \varepsilon^{1/2}\hat{r}$ where $r_0^2 = -K_1 \cdot K_2^{-1}$ so that

$$\dot{\hat{r}} = -2\varepsilon^2 K_1 \hat{r} + O(\varepsilon^{5/2})$$

$$\dot{\theta} = \omega_0 + \varepsilon^2(L_1 + r_0 L_2) + O(\varepsilon^{5/2})$$

$$\dot{y} = A_Q y + O(\varepsilon^2)$$

and then replacing ε^2 with ε, t with $-2K_1 t$, and dropping the hats yields

$$\dot{r} = r + O(\varepsilon^{5/4})$$

$$\dot{\theta} = \omega(\varepsilon) + O(\varepsilon^{5/4})$$

$$\dot{y} = A_Q y + O(\varepsilon)$$

with $\omega(\varepsilon) = -2K_1(\omega_0 + \varepsilon(L_1 + r_0 L_2))$ and Theorem 2.3.1 applies. Similarly if $K_1 \cdot K_2 > 0$.

Also the cylindrical shell given by

(3.4) $\quad C^*: \ (1 - \gamma)r_0 \leq r \leq (1 + \gamma)r_0; \quad |y| = O(\varepsilon)$

with $\gamma \to 0$ as $\varepsilon \to 0$ with be invariant only if A_Q is a stable matrix and $K_1 \cdot K_2 < 0$ with $K_1 > 0$ ($K_1 \cdot K_2 > 0$ with $K_1 < 0$). In which case the manifold is stable. If A_Q has at least one eigenvalue with a positive real part then solutions may enter C^*

through $r = (1 \pm \gamma)r_0$ but will leave along the eigendirection of this eigenvalue.

Theorem 2.3.2. Let K_1, K_2 be defined as in Theorem 2.2.2. If $K_1 \cdot K_2 \neq 0$ then the system (1.1) of section 2.2 has a unique invariant manifold M defined by

$$M: \quad (r,y) = (r(\theta,t,\varepsilon), y(\theta,t,\varepsilon))$$

where r and y are 2π-periodic in θ, almost periodic (P-periodic) in t, $(r,y) \to (0,0)$ as $\varepsilon \to 0$ with $\varepsilon^2 = -\mathrm{sgn}(K_1 \cdot K_2)\alpha$. So that in the original coordinates (x,y) with $x = (r \cos \theta, r \sin \theta)$

$$r(\theta,t,\varepsilon) = \varepsilon r_0 + O(\varepsilon^2), \quad r_0 = |K_1 \cdot K_2^{-1}|^{1/2}$$

$$y(\theta,t,\varepsilon) = O(\varepsilon) .$$

M is stable if and only if all the eigenvalues of A_Q have negative real parts and $K_2 < 0$.

2.4. An Example

Consider the forced Van Der Pol equation

$$\ddot{x} + x - \varepsilon(1 - x^2)\dot{x} = f(t) \tag{4.1}$$

where $x \in R$, $f(t)$ is almost periodic or P-periodic and

$$\int_0^t N_s f(s)ds, \quad N_\theta = (\cos \theta, \sin \theta)'$$

is bounded.

Let $u(t)$ be the unique almost periodic (P-periodic) solution of

$$\ddot{u} + u = f(t) \tag{4.2}$$

then

$$u(t) = N_t'c + \int_0^t \sin(t - s)f(s)ds \ .$$

And let $y = x - u$ so that after using (4.2) we have

$$\ddot{y} + y - \varepsilon g(y,\dot{y},t) = 0 \tag{4.3}$$

where $g(y,\dot{y},t) = (1 - u^2 - 2uy - y^2)(\dot{y} + \dot{u})$. Writing (4.3) as a system in R^2 yields

$$\dot{z} = Az + \varepsilon F(z,t) \tag{4.4}$$

where

$$z = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad F(z,t) = \begin{bmatrix} 0 \\ g(z,t) \end{bmatrix} .$$

Passing to polar coordinates by setting $z = rN_\theta$, gives

$$\begin{aligned} \dot{r} &= \varepsilon[C_1 + rC_2 + r^2C_3 + r^3C_4] \\ \dot{\theta} &= -1 - \varepsilon[r^{-1}D_1 + D_2 + rD_3 + r^2D_4] \end{aligned} \tag{4.5}$$

where

$$C_1 = (1 - u^2)\dot{u} \sin \theta$$

$$C_2 = (1 - u^2)\sin^2\theta - 2u\dot{u} \sin \theta \cos \theta$$

$$C_3 = -\dot{u} \cos^2\theta \sin \theta - 2u \sin^2\theta \cos \theta$$

$$C_4 = -\sin^2\theta \cos^2\theta$$

$$D_1 = (1 - u^2)\dot{u} \cos \theta$$

$$D_2 = (1 - u^2)\sin\theta\cos\theta - 2u\dot{u}\cos^2\theta$$

$$D_3 = -\dot{u}\cos^3\theta - 2u\cos^2\theta\sin\theta$$

$$D_4 = -\cos^3\theta\sin\theta\ .$$

So that (4.5) may be averaged provided the following integrals are bounded.

$$A_1:\quad \int_0^t e^{ins}u(s)ds \quad \text{for} \quad |n| = 1,3$$

$$A_2:\quad \int_0^t e^{ins}u^2(s)ds \quad \text{for} \quad |n| = 2$$

$$A_3:\quad \int_0^t e^{ins}u^3(s)ds \quad \text{for} \quad |n| = 1$$

$$A_4:\quad \int_0^t u^2(s) - \operatorname*{mean}_t\,[u^2]ds\ .$$

In which case the averaged form of (4.5) is

$$\text{(4.6)}\qquad \begin{aligned} \dot{r} &= \varepsilon[\tfrac{1}{2}Kr - \tfrac{1}{8}r^3] + O(\varepsilon^2) \\ \dot{\theta} &= -1 + O(\varepsilon^2) \end{aligned}$$

where $K = \operatorname*{mean}_t\,[1 - u^2]$.

Thus an invariant manifold near $r_0 = 2K^{\frac{1}{2}}$ is expected. In fact if $r \to r_0 + \varepsilon^{\frac{1}{2}}K^{-1}r$ and $t \to -t$ then

$$\dot{r} = r + O(\varepsilon^{3/2})$$

$$\dot{\theta} = 1 + O(\varepsilon^2)$$

and we may apply Theorem 2.3.1 to assure the existence of a manifold of the form

$$r = r(\theta,t,\varepsilon) = r_0 + O(\varepsilon)$$

which is invariant under the flow induced by solutions of (4.4) and r is 2π-periodic in θ, almost periodic (P-periodic) in t.

If $f(t)$ can be written in a finite Fourier series

$$f(t) = \sum_{\nu} a_{\nu} e^{\lambda_{\nu} t}, \quad \lambda_0 = 0$$

then if $|\lambda_{\nu}| \neq 1$ for all ν

$$u(t) = \sum_{\nu} u_{\nu} e^{\lambda_{\nu} t}, \quad u_{\nu} = a_{\nu}(1 - \lambda_{\nu}^2)^{-1}$$

so that $A_1 - A_4$ are bounded if

$$\begin{aligned} &n + \lambda_j \neq 0 && \text{for} \quad |n| = 1,3 \\ (4.7) \qquad &n + \lambda_j + \lambda_k \neq 0 && \text{for} \quad |n| = 2 \\ &n + \lambda_j + \lambda_k + \lambda_\ell \neq 0 && \text{for} \quad |n| = 1 \end{aligned}$$

3. FUNCTIONAL DIFFERENTIAL EQUATIONS

3.1. The Abstract Equation.

Consider the retarded functional differential equation (RFDE)

$$\dot{z}(t) = f(z_t, t, \alpha) \tag{1.1}$$

where

$$z_t \in C \overset{\text{def}}{=} C([-r,0], R^n)$$

$$z_t(\theta) = z(t+\theta) \quad \text{for} \quad \theta \in [-r,0]$$

and $f\colon\ C \times \dot{R} \times R \to R^n$ is almost periodic (P-periodic) in t and smooth enough for the following calculations to be carried out. Assume further that

$$f(0,t,\alpha) = 0, \quad \frac{\partial f(0,t,0)}{\partial z_t} = L$$

where $L\colon\ C \to R^n$ is bounded, linear and independent of t. Thus L has the Stieltjes integral representation

$$L\phi = \int_{-r}^{0} d\eta(\theta)\phi(\theta)$$

where $\eta(\theta)$ is an $n \times n$ matrix function of bounded variation. Then (1.1) becomes

$$\begin{aligned} \dot{z}(t) &= Lz_t + H(z_t, t, \alpha) \\ H(\phi, t, \alpha) &= \alpha M(t,\alpha)\phi + F(\phi, t, \alpha)\phi^2 \end{aligned} \tag{1.2}$$

where as before $F(\phi,t,\alpha)$: $C \times C \to R^n$ is a symmetric bilinear map. Upon scaling by $z \to \varepsilon z$, $\alpha \to \varepsilon\alpha$ (1.2) becomes (with a different H)

$$\dot{z}(t) = Lz_t + \varepsilon H(z_t,t,\alpha,\varepsilon). \tag{1.3}$$

Now $v(t,\cdot) \in C$ is a solution of (1.3) if and only if $v(t,\theta) = z_t(\theta)$ where $z(t)$ satisfies (1.3). This fact gives us the clue as to how (1.3) can be rewritten as an ODE in an appropriate Banach space.

<u>Lemma 4.1.1</u>. If $v(t,\theta)$ is C^1 in $t \in R$ and $\theta \in [-r,0]$ then a necessary and sufficient condition that $v(t,\theta) = u(t+\theta)$ for some u in C^1 is that $\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial t}$.

<u>Proof</u>. Necessity is obvious. On the other hand along $\theta + t = a$ we have $\frac{d}{dt} v(t,\theta) = 0$ so define $u(a) = v(0,a)$.

Thus $v(t,\theta)$ is a C^1 solution of (1.3) if and only if

$$\begin{aligned} \frac{\partial v(t,\theta)}{\partial t} &= \frac{\partial v(t,\theta)}{\partial \theta} \\ \frac{d}{dt} v(t,0) &= Lv(t,\cdot) + \varepsilon H(v(t,\cdot),t,\alpha,\varepsilon). \end{aligned} \tag{1.4}$$

Now define $E = C \oplus R^n$ and A: $C^1 \to E$ by $v \to (\dot{v}, Lv - \dot{v}(0))$. Then (1.4) may be written as

$$\frac{\partial}{\partial t}(v(t,\theta),0) = Av(t,\cdot) + (0,\varepsilon H). \tag{1.5}$$

Now suppose that A has a pair of pure imaginary simple eigenvalues $\pm i\omega_0$, $\omega_0 \neq 0$ and all other eigenvalues of A have nonzero real parts. It is well known that all eigenvalues of A

are isolated and of finite multiplicity and are determined by solving

$$\det(\lambda I - \int_{-r}^{0} d\eta(\theta)e^{\lambda\theta}) = 0$$

Let $C^* = C([0,r], R^n)$ (row vectors), $E^* = C^* \oplus R^n$. Define a bilinear form on $E^* \times E$ by

$$<(\psi,a),(\phi,b)> = a\cdot b + a\cdot\phi(0) + \psi(0)\cdot b + [\psi,\phi]$$

$$[\psi,\phi] = \psi(0)\cdot\phi(0) - \int_{-r}^{0}\int_{0}^{\theta}\psi(s - \theta)d\eta(\theta)\phi(s)ds$$

so that A^*, the adjoint of A, is given by

$$A^* = (-\dot{\psi}, L^*\psi - \dot{\psi}(0))$$

$$L^*\psi = \int_{-r}^{0}\psi(-\theta)d\eta(\theta).$$

Let $\Phi = (\phi_1,\phi_2)$ be a basis for $P = N(A\pm i\omega_0 I)$ and $\Psi = (\psi_1,\psi_2)$ a basis for $N(A^*\pm i\omega_0 I)$ chosen so that $[\Psi,\Phi] = I$. Then any $(v,b) \in E$ can be written as

$$(v,b) = (v^P,0) + (v^Q,b)$$

$$v^P = \Phi[\Psi,v]$$

$$v^Q = v - v^P.$$

Note that $[\Psi,v^Q] = 0$ and $E = P \oplus Q$. Also there is a unique 2×2 matrix A_p so that $A\Phi = \Phi A_p$ and A_p has eigenvalues $\pm i\omega_0$.

Now for any $(v(t,\theta),b) \in E$ we have

$$(v(t,\theta),b) = (v^P(t,\theta),0) + (v^Q(t,\theta),b)$$

$$= (\Phi[\Psi,v(t,\theta)],0) + (v^Q(t,\theta),b)$$

$$\overset{\text{def}}{=} (\Phi x(t),0) + (v^Q(t,\theta),b).$$

Then (1.5) becomes

$$(1.6) \qquad (\Phi\dot{x}(t) + \frac{d}{dt} v^Q(t,\theta),0) = (\Phi A_p x(t),0) + Av^Q(t,\cdot) + (0,\varepsilon H).$$

And since $E = P \dot{\oplus} Q$ we may decompose (1.6) as

$$(1.7) \qquad \begin{aligned} \dot{x} &= A_p x + \varepsilon\Psi(0)H \\ \frac{d}{dt} y_t &= Ay_t + \varepsilon(-\Phi\Psi(0)H,H) \end{aligned}$$

where

$$y_t = (v^Q(t,\cdot),0)$$

$$H = H(x,y_t,t,\varepsilon)$$

We may now pass to polar coordinates and average as before.

3.2. A Perturbed Wright's Equation.

Consider the RFDE

$$(2.1) \qquad \dot{z}(t) = -(a + \delta f(t))z(t - 1)(1 + z(t))$$

where a and δ are real parameters $f(t)$ is almost periodic or P-periodic. We wish to study the local behavior of (2.1) near the bifurcation points $(a,\delta) = (a_n,0)$ where

(2.2) $$a_n = (-1)^n(\pi/2 + n\pi) \overset{\text{def}}{=} (-1)^n b_n.$$

At these points the linear part of (2.1) near $z = 0$ has the eigenvalues $\lambda = \pm i b_n$. Set $\alpha = a - a_n$ and scale by $z \to \varepsilon z$, $\alpha \to \varepsilon\alpha$, $\delta \to \varepsilon\delta$ so that (2.1) becomes

(2.3) $$\begin{aligned} \dot{z}(t) &= -a_n z(t-1) - \varepsilon H(z_t,t,\alpha,\delta,\varepsilon) \\ H(\phi,t,\alpha,\delta,\varepsilon) &= (\alpha + \delta f(t))\phi(-1)(1 + \varepsilon\phi(0)) + a_n\phi(-1)\phi(0). \end{aligned}$$

Now a basis for the eigenspace corresponding to $\lambda = \pm i b_n$ is found to be

$$\Phi(\theta) = (\cos b_n\theta,\ \sin b_n\theta)$$

and the bilinear form

$$[\psi,\phi] = \psi(0)\phi(0) - a_n\int_{-1}^{0}\psi(\theta + 1)\phi(\theta)d\theta$$

gives the dual basis

$$\Psi(\theta) = \frac{2}{1+b_n^2}\begin{bmatrix} \cos b_n\theta - b_n \sin b_n\theta \\ \sin b_n\theta + b_n \cos b_n\theta \end{bmatrix}$$

so that $[\Psi,\Phi] = I = 2 \times 2$ identity matrix.

The abstract equation is then

(2.4) $$\begin{aligned} \dot{x} &= A_P x - \varepsilon\Psi(0)H(\Phi x + y_t,t,\alpha,\delta,\varepsilon) \\ \frac{d}{dt} y_t &= A_Q y_t + \varepsilon(-\Phi\Psi(0),1)H(\Phi x + y_t,t,\alpha,\delta,\varepsilon) \end{aligned}$$

where

$$A_P = \begin{bmatrix} 0 & b_n \\ -b_n & 0 \end{bmatrix}$$

$$A_Q\phi = (\dot{\phi},-a_n\phi(-1) - \dot{\phi}(0)).$$

In polar coordinates (2.4) becomes

$$\dot{r} = -\varepsilon[C_2 r + r^2 C_3] - \varepsilon r\hat{C}_2 y_t + \text{h.o.t}$$

(2.5) $$\dot{\xi} = -b_n - \varepsilon[D_2 + rD_3] - \varepsilon\hat{D}_2 y_t + \text{h.o.t}$$

$$\frac{d}{dt} y_t = A_Q y_t - \varepsilon r^2 E + O(\varepsilon(\alpha + \delta)) + O(\varepsilon|y_t|)$$

where $\text{h.o.t} = O(\varepsilon^2(\alpha + \delta)) + O(\varepsilon(\alpha + \delta)|y_t|) + O(\varepsilon|y_t|^2)$ and

$$C_2 = (\alpha + \delta f(t))N_\xi'\Psi(0)\Phi(-1)N_\xi$$

$$C_3 = a_n N_\xi'\Psi(0)\Phi(-1)N_\xi\Phi(0)N_\xi$$

$$\hat{C}_2\phi = a_n N_\xi'\Psi(0)[\Phi(-1)\phi(0) + \Phi(0)\phi(-1)]N_\xi$$

$$E = a_n\Phi(-1)N_\xi\Phi(0)N_\xi(-\Phi\Psi(0),1)$$

$$D_2 = (\alpha + \delta f(t))T_\xi'\Psi(0)\Phi(-1)N_\xi$$

$$D_3 = a_n T_\xi'\Psi(0)\Phi(-1)N_\xi\Phi(0)N_\xi$$

and note that only C_2 and D_2 depend on t, so that the averaging can be carried out if I_2 and I_3 (section 2.1) are bounded (kb_nP is not an integer multiple of 2π for $|k| = 2$). The averaged form of (2.5) is

$$\dot{r} = -\varepsilon K_1(\alpha,\delta)r - \varepsilon^2 K_2 r^3 + \text{h.o.t}$$

(2.6) $$\dot{\xi} = -b_n - \varepsilon L_1(\alpha,\delta) - \varepsilon^2 L_2 r^2 + \text{h.o.t}$$

$$\frac{d}{dt}\bar{y}_t = A_Q y_t + O(\varepsilon(\alpha + \delta)) + O(\varepsilon^2)$$

where

$$K_1(\alpha,\delta) = \underset{\xi,t}{\text{mean}}[C_2]$$

(2.7) $$K_2 = \underset{\xi}{\text{mean}}[b_n^{-1}C_3D_3 + \hat{C}_4]$$

where $\hat{C}_4 = \hat{C}_2 W$ and $W = (W,0)$ is the unique 2π-periodic in ξ solution of

(2.8) $$E + A_Q W + b_n \frac{\partial W}{\partial \xi} = 0$$

with similar expressions holding for $L_1(\alpha,\delta)$ and L_2 and $|y_t - \bar{y}_t| = O(\varepsilon)$.

Now straightforward computation yields

(2.9) $$K_1(\alpha,\delta) = \frac{-a_n}{1+a_n^2}(\alpha + \delta \underset{t}{\text{mean}}[f])$$

(2.10) $$\underset{\xi}{\text{mean}}[C_3D_3] = 0.$$

To compute K_2, write $E = E_1 + E_2$ where

$$E_1 = -\tfrac{1}{2}b_n \sin 2\xi \cdot (0,1) = \sum_{|k|=2} (0,h_k)e^{ki\xi}$$

$$E_2 = -\tfrac{1}{2}b_n \sin 2\xi(\Phi(\theta)\Psi(0),0) = \sum_{|k|=2} (g_k(\theta),0)e^{ki\xi}$$

$$h_2 = \frac{ib_n}{4},\quad h_{-2} = \bar{h}_2$$

$$g_2(\theta) = \frac{ib_n}{4(1+a_n^2)}[(1 - ib_n)e^{ib_n\theta} + (1 + ib_n)e^{-ib_n\theta}]$$

$$g_{-2} = \bar{g}_2$$

Now it is clear that W has the form

$$W = \sum_{|k|=2} (w_k(\theta)e^{ki\theta}, 0)$$

also $W = W_1 + W_2$ where for $j = 1,2$

$$E_j + A_Q W_j + b_n \frac{\partial W_j}{\partial \xi} = 0. \tag{2.11}$$

Also note that we have for $\phi \in C$

$$\hat{C}_2\phi = \sum_{|k|=0,2} (\gamma_k \phi(0) + \beta_k \phi(-1)) e^{ik\xi}$$

with $\gamma_{-k} = \bar{\gamma}_k$, $\beta_{-k} = \bar{\beta}_k$, so that

$$K_2 = \underset{\xi}{\text{mean}}[\hat{C}_2 W] = 2\text{Re}(\gamma_{-2} w_2(0) + \beta_{-2} w_2(-1))$$

$$\gamma_{-2} = \frac{b_n(b_n - i)}{2(1+a_n^2)}$$

$$\beta_{-2} = \frac{a_n(1+ib_n)}{2(1+a_n^2)} .$$

Now $w_k(\theta) = u_k(\theta) + v(\theta)$ corresponding to $W = W_1 + W_2$ so that (2.11) for $j = 1$ becomes

$$(0,0) = \sum_{|k|=2} (\dot{u}_k(\theta) + kib_n u_k(\theta),\ h_k - a_n u_k(-1) - \dot{u}_k(0)) e^{ki\xi}.$$

Hence

$$u_2(\theta) = \frac{-(2+i(-1)^n)}{20} e^{2ib_n\theta}$$

and then

$$\underset{\xi}{\text{mean}}[\hat{C}_2 W_1] = \frac{a_n(1-a_n)}{20(1+a_n^2)} .$$

For $j = 2$ in (2.11) we have

$$(0,0) = \sum_{|k|=2} (\dot{v}_k(\theta) + kib_n v_k(\theta) + g_k(\theta),\ -a_n v_k(-1) - \dot{v}_k(0)) e^{ki\xi},$$

which yields

$$v_2(\theta) = e^{-2ib_n\theta}[v_2(0) - P(\theta)]$$

where

$$P(\theta) = \int_0^\theta e^{2ib_n s} g_2(s)ds$$

$$u_2(0) = (a_n + 2ib_n)(a_nP(-1) - g_2(0)).$$

Explicitly it is found that

$$P(-1) = \frac{1}{6(1+a_n^2)} [2(a_n - 1) - i(b_n + (-1)^n)]$$

$$g_2(0) = \frac{ib_n}{2(1+a_n^2)}$$

so that

$$v_2(0) = \frac{-1}{6(1+a_n^2)} (2 + ib_n)$$

$$v_2(-1) = \frac{1}{6(1+a_n^2)} (2a_n - i(-1)^n)$$

and then

$$\underset{\xi}{\text{mean}}[\hat{C}_2W_2] = 0.$$

Thus we have

$$K_2 = \frac{a_n(1-3a_n)}{20(1+a_n^2)} < 0. \tag{2.12}$$

Now scaling again in (2.6) by $\alpha \to \varepsilon\alpha$, $\delta \to \varepsilon\delta$ the $\dot{r}$ equation becomes

$$\dot{r} = \varepsilon^2K_2r(\alpha M_1 + \delta M_2 - r^2) + O(\varepsilon^3)$$

where

$$\begin{aligned} M_1 &= 20(1 - 3a_n)^{-1} \\ M_2 &= M_1 \underset{t}{\text{mean}}[f] \end{aligned} \tag{2.13}$$

We do not compute $L_1(\alpha,\delta)$ or L_2 since no information about the direction, amplitude or stability of a bifurcating manifold will be gained.

Thus, if $\alpha M_1 + \delta M_2 > 0$, (2.1) will have an invariant manifold with amplitude approximately r_0 where

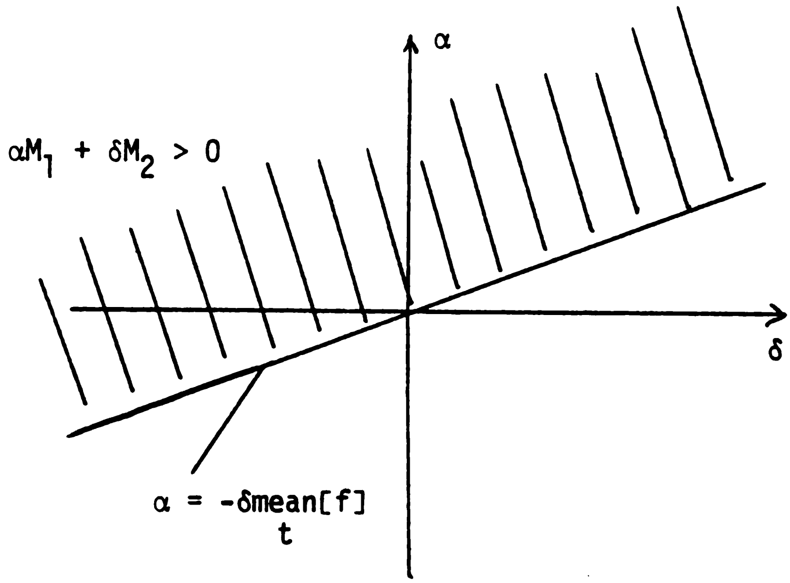
$$r_0 = (\alpha M_1 + \delta M_2)^{\frac{1}{2}}. \tag{2.14}$$

Since a_n is given by (2.2), if n is even $a_n \geq \pi/2$ so that $M_1 < 0$ and $M_1 > 0$ if n is odd (see Figure 1).

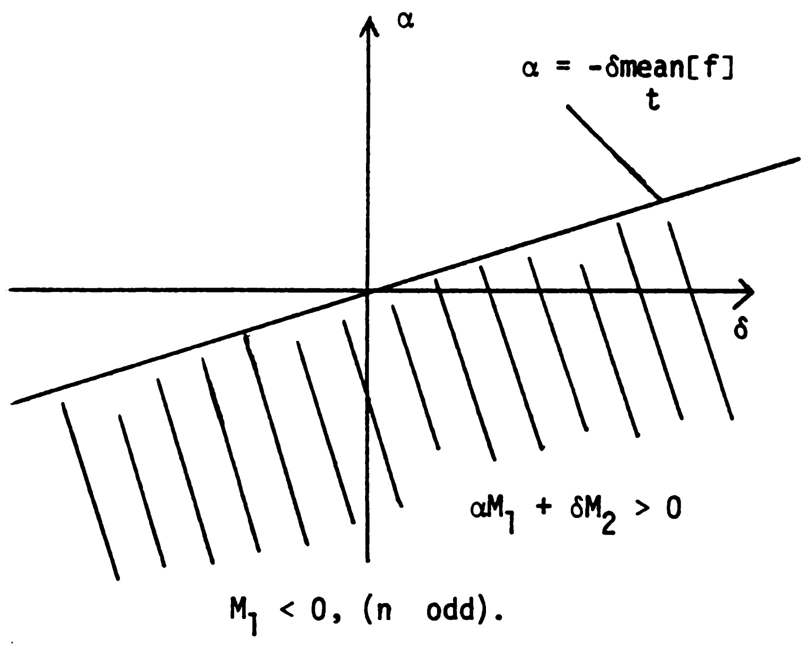
Now by Theorem A5 in Hale [8] all roots of the characteristic equation $(\dot{z}(t) = -az(t - 1))$

$$\lambda e^{\lambda} + a = 0 \tag{2.15}$$

have negative real parts if and only if $0 < a < \pi/2$. Since for $a = a_0 = \pi/2$, $\lambda = \pm i\pi/2$ are the only pure imaginary roots of (2.15) and these are not eigenvalues of A_Q, we must have all eigenvalues of A_Q with negative real parts. For $a = a_n$, $n \neq 0$, A_Q must have eigenvalues with nonnegative real parts. Thus, only the manifold bifurcating at $(a,\delta) = (\pi/2,0)$ is stable.



$M_1 > 0$, (n even).



$M_1 < 0$, (n odd).

Figure 1. Bifurcation diagrams for equation (2.1).

APPENDIX

APPENDIX

In this appendix the basic theory of almost periodic functions is outlined. No proofs are given for the classical results as they are readily available in standard texts on the subject, for example in Bohr [2], Favard [7], Hale [9].

Definition 1. A continuous $f: R \to R$ (or C) is said to be almost periodic if given $\varepsilon > 0$ there exist $\ell = \ell(\varepsilon) > 0$ so that for all $a \in R$ there is a $\tau \in [a, a + \ell]$ with $|f(t + \tau) - f(t)| < \varepsilon$ for all $t \in R$. τ is called an almost period of f in $[a, a + \ell]$ relative to ε.

A1. If $f(t)$ and $g(t)$ are almost periodic then so are $f(t + a)$, $f(at)$ for a real, $zf(t)$ for z complex, $f^n(t)$ for $n = 0,1,2,3,\cdots$, $|f(t)|$, $f(t)$, $\overline{f(t)} + g(t)$, $f(t)\cdot g(t)$. In fact, if $F: R^2 \to R$ (or C) is uniformly continuous then $F(f(t),g(t))$ is almost periodic.

A2. If $f(t)$ is almost periodic then so is $f'(t)$ provided it is uniformly continuous.

A3. If $f(t)$ is almost periodic then $\int^t f(s)ds$ is almost periodic if and only if it is bounded.

A4. If $f(t)$ is an almost periodic function then the following limit exists, is finite and independent of $a \in R$.

$$\lim_{T\to\infty} \frac{1}{T} \int_a^T f(t)dt \overset{\text{def}}{=} \underset{t}{\text{mean}}[f].$$

A5. If $f(t)$ is almost periodic then there are at most a countable number of λ so that $a(\lambda) \overset{\text{def}}{=} \underset{t}{\text{mean}}[E_\lambda f] \neq 0$ where $E_\lambda(t) = \exp(i\lambda t)$. Any λ for which $a(\lambda) \neq 0$ we call a frequency of f with Fourier coefficient $a(\lambda)$.

If $\{\lambda_n\}$ is a sequence of real numbers and $\sum r_n\lambda_n = 0$, where r_n is an integer implies that $r_n = 0$ for all n, then $\{\lambda_n\}$ is said to be rationally independent. The span of $\{\lambda_n\}(sp(\lambda_n))$ is the set of all linear combinations of the λ_n with integer coefficients. If $\{\lambda_n\}$ is rationally independent and $\{\alpha_n\} \subset sp(\lambda_n)$ then $\{\lambda_n\}$ is called a basis for $\{\alpha_n\} \subset R$.

Definition 2. If $f(t)$ is almost periodic with frequencies $\{\lambda_n\}$ then the module of f, $m[f] \overset{\text{def}}{=} sp(\lambda_n)$. If $\{\lambda_n\}$ has a finite base then $f(t)$ is called quasi-periodic.

The following result is very useful in showing that a function is almost periodic.

A6. If $f(t)$ is almost periodic and g: $R \to R$ (or C) and for any sequence $\{\tau_j\} \subset R$ with $f(t + \tau_j) - f(t) \to 0$ uniformly in t as $j \to \infty$ we have $g(t + \tau_j) - g(t) \to 0$ uniformly in t as $j \to \infty$ then $g(t)$ is almost periodic and $m[g] \subset m[f]$.

Definition 3. A continuous function $f(x,t) \in R^n$ (or C^n) is said to be almost periodic in t uniformly with respect to x in compact sets if given a compact set $K \subset R^n$ and $\varepsilon > 0$ there exists $\ell = \ell(\varepsilon,K)$ so that for any $a \in R$ there is a $\tau \in [a, a + \ell]$ with $|f(x,t + \tau) - f(x,t)| < \varepsilon$ for all $t \in R$ and $x \in K$. In what follows and in the text we will refer to such functions simply as almost periodic in t.

Now if $f(\theta,t)$ is 2π-periodic in θ and almost periodic in t then we define

$$\underset{\theta,t}{\text{mean}}[f] = \frac{1}{2\pi}\int_0^{2\pi} \underset{t}{\text{mean}}[f](\theta)d\theta.$$

The following result is used in the text. Since no proof seems to be accessible one is supplied.

A7. If $f(\theta,t) \in R$ is 2π-periodic in θ and almost periodic in t then there exist (θ_0,t_0) so that $\nabla f(\theta_0,t_0) = (0,0)$ (∇ = gradient).

<u>Proof</u>. If $\nabla f(\theta,t)$ never vanishes let $f(\bar{\theta},0) = \max_{\theta\in[0,2\pi]} f(\theta,0)$ and let $(\psi(s),\tau(s))$ define the steepest ascent curve originating at $(\bar{\theta},0)$. That is

$$\frac{d}{ds}(\psi(s), \tau(s)) = \nabla f(\psi,\tau)$$

$$(\psi(0), \tau(0)) = (\bar{\theta},0)$$

$\psi(s)$ and $\tau(s)$ exist for all $s \in (-\infty,\infty)$ since $\nabla f(\psi,\tau)$ is bounded. Note that $\lim_{s\to+\infty} \tau(s) = \infty$ for if $|\tau(s)| \leq M < \infty$ define $g(s) = f(\psi(s), \tau(s))$ and set $\eta = \min_{s\geq 0}|\nabla f(\psi(s), \tau(s))|^2 > 0$, then for $s \geq 0$, $g'(s) = |\nabla f(\psi(s), \tau(s))|^2 \geq \eta$ so that $g(s)$ is unbounded which is absurd.

Let $f(\psi(1), \tau(1)) - f(\bar{\theta},0) = a$, $a > 0$ by assumption. Let T be an almost period in $[\tau(1),\tau(1) + \ell(a/2)]$ relative to $\varepsilon = a/2$ and $s \geq 1$ so that $\tau(s) = T$ then

$$
\begin{aligned}
a/2 &> f(\psi(s),\tau(s)) - f(\psi(s),0) \\
&\geq a + f(\bar{\theta},0) - f(\psi(s),0) \\
&= a + f(\bar{\theta},0) - f(\psi(s) - 2\pi k,0) \\
&\geq a.
\end{aligned}
$$

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