# A STUDY OF A CLASS OF FUNCTIONS HOLOMORPHIC IN THE UNIT DISK

Thesis for the Degree of Ph. D.

MICHIGAN STATE UNIVERSITY

PHILIP JAMES PRATT

1971

This is to certify that the

thesis entitled

A STUDY OF A CLASS OF FUNCTIONS

HOLOMORPHIC IN THE UNIT DISK

presented by

Philip James Pratt

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

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Date May 4, 1971

O-7639

Jan —— 185

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ABSTRACT

# A STUDY OF A CLASS OF FUNCTIONS HOLOMORPHIC IN THE UNIT DISK

By

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The non-constant holomorphic function f is in class $\mathcal{B}$ * if and only if for each point of a dense subset of the unit circle f is bounded on an arc which ends at the point. The class $\mathcal{B}$ * is investigated with respect to its closure properties under certain elementary operations.

An approximation technique of Bagemihl and Seidel is used to show that any holomorphic function can be written as the sum and product of two functions each having radial limit zero on a dense subset of C. This result is used to show $\mathcal{B}$* is not closed under addition or multiplication.

Approximation techniques involving a repeated use of Mergelyan's Approximation Theorem and modifications of a technique of Barth and Schneider are used to show the existence of a function f which is not in $\mathcal{B}$ * such that $e^f$ is in $\mathcal{B}$ * and a function g which is in $\mathcal{B}$ * such that $\int_0^z g(t)dt$ is not in $\mathcal{B}$ *.

It is also shown that if f is in $\mathcal{B}$ *, $e^f$ is in $\mathcal{B}$ *. If f is in $\mathcal{B}$ * and f omits the finite value a, then $1/(f(z)-a)$ is in $\mathcal{B}$ *. The fact that f is in $\mathcal{B}$ * does not

imply that f' is in $\mathfrak{B}$ *. There are no sufficient slow or fast (infinite) growth conditions for a function to be in $\mathfrak{B}$ * or not in $\mathfrak{B}$ *.

Finally, the possibility of extending theorems from normal functions to functions in $\mathfrak{B}$ * is discussed.

# A STUDY OF A CLASS OF FUNCTIONS HOLOMORPHIC IN THE UNIT DISK

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A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirments
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1971

## ACKNOWLEDGEMENTS

I wish to express my appreciation to Professor Peter Lappan for his help and guidance throughout the preparation of this thesis and to Mr. Edwin F. Smith for drawing the figures that I used.

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# I. INTRODUCTION

Throughout this paper we shall let D denote the unit disk, $\{z: |z| < 1\}$, and let C denote the unit circle, $\{z: |z| = 1\}$.

By a <u>boundary path</u> in D is meant a simple continuous curve $z = z(t)$ $(0 \leq t < 1)$ where $\lim_{t \to 1} |z(t)| = 1$. If $\lim_{t \to 1} z(t) = \zeta$, where $\zeta \in C$, we say that the path ends at $\zeta$. If there is a path which ends at $\zeta$ and along which $f(z)$ tends to the complex number c, we say that f has <u>asymptotic value</u> c at $\zeta$.

If $\{\gamma_n\}_{n=1}^{\infty}$ is a sequence of continuous compact curves in D and $\gamma$ is an arc of C, $\{z \in C: \alpha \leq \arg z \leq \beta\}$, we say that $\gamma_n$ converges to $\gamma$ and use the notation that $\gamma_n \longrightarrow \gamma$, if for each $\epsilon > 0$ there exists a positive integer N such that

(1) $\gamma_n \subseteq \{z: 1 - \epsilon < |z| < 1\}$

(2) $\left| \inf_{z \in \gamma_n} \arg z - \alpha \right| < \epsilon$

(3) $\left| \sup_{z \in \gamma_n} \arg z - \beta \right| < \epsilon$

if $n > N$. We then call $\{\gamma_n\}_{n=1}^{\infty}$ a sequence of <u>Koebe arcs</u> and $\gamma$ the <u>end</u> of the sequence.

If, for a complex number c, there exists an arc $\gamma$ contained in C and a sequence of arcs $\{\gamma_n\}_{n=1}^{\infty}$ contained in D that converges to $\gamma$ such that

$$\lim_{n\to\infty} \sup_{z\in\gamma_n} |f(z) - c| = 0 \text{ if } c \text{ is finite and}$$

$$\lim_{n\to\infty} \inf_{z\in\gamma_n} |f(z)| = \infty \text{ if } c = \infty ,$$

then we say that f has c as a <u>Koebe value</u>.

If f is a holomorphic function in D and S is the family of all one-to-one conformal mappings of D onto itself, then f is said to be <u>normal</u> if the family $\{f(s(z))\}_{s\in S}$ is normal in the sense of Montel [17, p. 53]. We define $\mathcal{N}$ to be the class of all non-constant normal holomorphic functions defined in D.

In [19], MacLane defined some subsets of C as follows:

$A_a = \{\zeta \in C$: f has asymptotic value a at $\zeta\}$

$B^* = \{\zeta \in C$: there is a boundary path which ends at $\zeta$ and on which f is bounded$\}$

$A = \bigcup A_a$ where the union is taken over all complex numbers a.

$B = B^* \cup A_\infty$

He then used these to define two classes of functions, $\mathcal{A}$ and $\mathcal{B}$. We say the non-constant holomorphic function f is in $\mathcal{A}$ ($\mathcal{B}$) if and only if A (B) is dense in C. MacLane showed $\mathcal{A} = \mathcal{B}$ and proved several properties about class $\mathcal{A}$.

These results have been generalized in various ways. Barth [6] generalized many of them to meromorphic functions. In [16], Lappan defined a similar class of functions which he called $\mathcal{B}^*$ by saying the non-constant holomorphic function f is in $\mathcal{B}^*$ if and only if B* is dense in C. He then proved it was possible to characterize $\mathcal{B}^*$ in a slightly different way.

Koebe's Lemma [13] may be stated in the following way: A non-constant bounded holomorphic function has no Koebe values. This has been extended to normal functions [5, Theorem 1, p. 10] and Lappan showed it could also be extended to functions in $\mathcal{B}^*$. He actually showed more than this; he showed that it could not be extended to any larger class of functions. That is, $\mathcal{B}^*$ can be characterized as the class of all holomorphic functions having no Koebe values.

It is the purpose of this paper to study class $\mathcal{B}^*$. Before beginning, it should be noted that our definition is not precisely the same as Lappan's, although it certainly is equivalent to it. When he defined the set B* he required the paths on which the function is bounded to begin at the origin. Doing this enabled him to prove some theorems about the set B*. For our purposes, however, it is simpler to drop this restriction.

Lappan gave another characterization for $\mathcal{B}^*$ which is more useful to us in this paper and that is the fact that a non-constant holomorphic function is in $\mathcal{B}^*$ if and only if it does not have $\infty$ as a Koebe value. In general, in the results in this paper, when we know a non-constant holomorphic function is not in $\mathcal{B}^*$ we use the fact that it has $\infty$ as a Koebe value.

Before getting to the results in this paper, it would be useful to point out the relationship between $\mathcal{B}^*$ and two very well known classes, $\mathcal{N}$ and $\mathcal{A}$. Since non-constant normal holomorphic functions have no Koebe values, $\mathcal{N} \subseteq \mathcal{B}^*$ [5]. In [14], Lappan constructed a non-normal holomorphic function that was a product of a normal holomorphic function and a bounded holomorphic function. Since the product of a bounded holomorphic function and a function in $\mathcal{B}^*$ is obviously in $\mathcal{B}^*$, this function is in $\mathcal{B}^*$ and the inclusion is proper. Turning our attention to class $\mathcal{A}$, we note that if B* is dense in C, then so is B ( $= B^* \cup A_\infty$ ), and this shows that $\mathcal{B}^* \subseteq \mathcal{B}$. Then $\mathcal{B}^* \subseteq \mathcal{A}$, since $\mathcal{A} = \mathcal{B}$. In [19, Example 3, p. 57], MacLane presented a function in class $\mathcal{A}$ which has $\infty$ as a Koebe value and thus is not in $\mathcal{B}^*$. This shows that this inclusion is also proper.

The first question we answer in Chapter II is whether or not the sum or product of two functions in $\mathcal{B}^*$, is necessarily in $\mathcal{B}^*$. This question has been answered in the negative for class $\mathcal{N}$ by Lappan [14] and for $\mathcal{A}$ by Ryan and Barth [22]. The answer for $\mathcal{B}^*$ is negative as well as is the answer to the next logical question: must the sum or product of two functions in $\mathcal{B}^*$ be in class $\mathcal{A}$?

We answer both of these in Theorem 1, Corollary 1.1 and Corollary 1.2 by showing that any holomorphic function can be expressed as the sum or as the product of two functions each having radial limit zero on a dense subset of C. (We say that f has the radial limit a at $\zeta \in C$ if $\lim_{r \to 1} f(r\zeta) = a$). In Theorem 2 we show a corresponding result for meromorphic functions.

We show in Theorem 3 that if f is in $\mathcal{B}$ * and f omits the finite value a, then 1 / (f (z) - a) is in $\mathcal{B}$ *. In Theorem 4, we show that if f is in $\mathcal{A}$ and f omits a finite value, then f is in $\mathcal{B}$ *. This generalizes a result in [17] which says that a holomorphic function which omits two finite values is in $\mathcal{N}$.

Theorem 5 shows that if f is in $\mathcal{B}$ * then $e^f$ is in $\mathcal{B}$ *. This is not true for $\mathcal{A}$ as Barth and Schneider showed in [8]. We cannot always define a logarithm but a related question is: if f is in $\mathcal{B}$ * and we can define a single valued holomorphic function log f, then is log f in $\mathcal{B}$ *? We answer this question negatively in Theorem 6 by constructing a holomorphic function f which is not in $\mathcal{B}$ * but for which $e^f$ is in $\mathcal{B}$ *.

We define $I^n(f)$ by $I(f(z)) = \int_0^z f(t)\,dt$ and $I^n(f) = I(I^{n-1}(f))$. We show in Theorem 7 that $f \in \mathcal{B}$ * does not imply $I(f) \in \mathcal{B}$ * and in Theorem 8 that $f \in \mathcal{B}$ * does not imply $f' \in \mathcal{B}$ *. The same results were proved for class $\mathcal{A}$ by Barth and Schneider [9]. A related question is: if f is in $\mathcal{B}$ * can we get out of $\mathcal{B}$ * by integrating or differentiating f enough times or, conversely, if f is outside of $\mathcal{B}$ * can we get into $\mathcal{B}$ * by integrating or differentiating f enough times? We answer both of these negatively. In Theorem 9, we construct a function f in $\mathcal{B}$ * such that $f^{(n)}$ and $I^{n}(f)$ are also in $\mathcal{B}$ * for all positive integers n. In Theorem 10 we exhibit a function g such that $g^{(n)}$ is not in $\mathcal{B}$ * for any positive integer n and in Theorem 11 we exhibit a function g where $I^{n}(g)$ is not in $\mathcal{B}$ * for any positive integer n.

In Chapter III we turn our attention first to growth functions with the two growth functions

$$M(r) = \sup_{|z| = r} |f(z)|$$

and

$$m(r) = (1/2\pi)\int_0^{2\pi} \log^{+} |f(re^{i\theta})|\, d\theta,$$

the maximum modulus and the Nevanlinna characteristic function. We show there is no sufficient condition involving either slow or fast growth for a function to be in $\mathcal{B}$ * or not to be in $\mathcal{B}$ *. By way of contrast, there is a sufficient slow growth condition for a function to be in $\mathcal{A}$ [ 19, pp. 36-38 ] and a sufficient fast growth condition for a function not to be in $\mathcal{N}$ [17, section 18, pp.57-58 ].

There are uniqueness theorems for bounded functions and functions in class $\mathcal{N}$ that say if f - g has zero as a Koebe value, then f $\equiv$ g. We show in Corollary 1.4 that these do not extend to $\mathcal{B}^*$.

Finally, we investigate the possibility of extending certain types of results from the class $\mathcal{N}$ to the class $\mathcal{B}^*$. We show in Theorem 14 the impossibility of extending theorems based on properties of normal functions along sequences.

## II. OPERATIONS IN CLASS $\mathcal{B}^*$

Let $\mathcal{R}$ denote the class of all holomorphic functions defined in D that have readial limit zero on a dense subset of C. Let $\mathcal{R}_m$ be the corresponding class for meromorphic functions.

THEOREM 1: <u>Let</u> h <u>be</u> <u>a</u> <u>holomorphic</u> <u>function</u> <u>in</u> D. <u>Then</u> <u>there</u> <u>exist</u> <u>functions</u> $f_o$, $f_1$, $g_o$, <u>and</u> $g_1$ <u>in</u> <u>class</u> $\mathcal{R}$ such that $h = f_o + g_o$ and $h = f_1 \cdot g_1$.

<u>Proof</u>: Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two disjoint dense subsets of C. Let $H_n$ be the radius to $\alpha_n$ intersected with $\{ z : 1 - (1/2^n) \leq |z| < 1 \}$ and $K_n$ be the radius to $\beta_n$ intersected with the same set $(n = 1, 2, \cdots)$. Let $H = \bigcup_{n=1}^{\infty} H_n$ and $K = \bigcup_{n=1}^{\infty} K_n$. We define a continuous function s on $H \cup K$ by

$$s(z) = h(z) \quad \text{if } z \in H$$
$$s(z) = 0 \quad \text{if } z \in K.$$

Since s is continuous on $H \cup K$ and $H \cup K$ is a closed subset of D, we can extend s to be a continuous function in all of D by Tietze's extension theorem [12, pp. 149-151].

The collection of radii to the $\alpha_n$'s and $\beta_n$'s form a tress [2] so that by [2, Theorem 1, p. 187] there exists a holomorphic function f such that

$$(1) \quad \lim_{r \to 1} \left| f(r\alpha_n) - s(r\alpha_n) \right| = 0$$
$$(2) \quad \lim_{r \to 1} \left| f(r\beta_n) - s(r\beta_n) \right| = 0$$

Remembering what s is equal to, these become

$$(1') \quad \lim_{r \to 1} \left| f(r\alpha_n) - h(r\alpha_n) \right| = 0$$
$$(2') \quad \lim_{r \to 1} \left| f(r\beta_n) \right| = 0$$

Thus, if we let $f_o = f$ and $g_o = h - f$, $h = f_o + g_o$ and $f_o \in R$ by (2'), $g_o \in R$ by (1').

The result for the product has been proved by Brannan and Hornblower in [10].

COROLLARY 1.1: <u>There exist functions</u> $f_o$, $f_1$, $g_o$, <u>and</u> $g_1$ <u>in class</u> $\mathcal{B}^*$ <u>such that</u> $f_o + g_o$ <u>and</u> $f_1 \cdot g_1$ <u>are not in</u> $\mathcal{B}^*$.

Rather than prove this corollary, we prove this more general result.

COROLLARY 1.2: <u>There exist functions</u> $f_o$, $f_1$, $g_o$ and $g_1$ <u>in class</u> $\mathcal{B}^*$ <u>such that</u> $f_o + g_o$ <u>and</u> $f_1 \cdot g_1$ <u>are not in</u> $\mathcal{A}$.

<u>Proof</u>: Let h be a holomorphic function that is not in $\mathcal{A}$. (For the existence of such a function see [20, Example 16, p. 80]. By Theorem 1, there exist functions

$f_o$ , $f_1$, $g_o$ , and $g_1$ in class $\mathcal{R}$ such that $h = f_o + g_o$ and $h = f_1 \cdot g_1$ . Since $\mathcal{R}$ is certainly contained in $\mathcal{B}$ *, the result follows.

COROLLARY 1.3: <u>There exist functions</u> f <u>and</u> g <u>in</u> $\mathcal{R}$ <u>such that</u> f - g <u>has a zero Koebe value, but</u> $f \neq g$.

<u>Proof</u>: Let h be any non-constant holomorphic function that has a zero Koebe value. Using Theorem 1, we get functions $f_o$ and $g_o$ in class $\mathcal{R}$ such that $h = f_o + g_o$ . Letting $f = f_o$ and $g = - g_o$ , the conclusion follows.

Since $\mathcal{R} \subseteq \mathcal{B}$ *, we also have the following trivial extension.

COROLLARY 1.4: <u>There exist functions</u> f <u>and</u> g <u>in</u> $\mathcal{B}$ * <u>such that</u> f - g <u>has a zero Koebe value, but</u> $f \neq g$.

The result of Theorem 1 is also valid for meromorphic functions, with the use of some functions from $\mathcal{R}_m$.

THEOREM 2: <u>Let</u> h <u>be a meromorphic function in</u> D. <u>Then there exist functions</u> $f_o$ <u>and</u> $f_1$ <u>in</u> $\mathcal{R}$ <u>and functions</u> $g_o$ <u>and</u> $g_1$ <u>in</u> $\mathcal{R}_m$ <u>such that</u> $h = f_o + g_o$ <u>and</u> $h = f_1 \cdot g_1$ .

<u>Proof</u>: We observe that in the proof of Theorem 1 the only properties of h that we used in constructing $f_o$ were that it was continuous and finite valued on the set H. If we pick $\{\alpha_n\}_{n=1}^{\infty}$ so that none of the poles of h lie on H, this will still be true. We now get $f_o$ just as before and let $g_o = h - f_o$ . Now $g_o$ is meromorphic rather than

holomorphic, but it still has radial limit zero on a dense subset of C. Thus, $h = f_o + g_o$, $f_o \in \mathcal{R}$, $g_o \in \mathcal{R}_m$ and the theorem is proved for the sum.

For the product, we pick $\{\alpha_n\}_{n=1}^{\infty}$ in the same way but we define our continuous function s a little differently. We still define s (z) to be zero if $z \in K$. For each positive integer n, we now define s (z) on $H_n$ to be a real-valued continuous function that tends monotonically to $\infty$ as $|z|$ tends to 1 and which satisfies

$$|s(r\alpha_n)| > \left(\sup_{1-(1/2^n) \leq t \leq r} |h(t\alpha_n)|\right)^2$$

for $1 - (1/2^n) \leq r < 1$. We again have that s is continuous on $H \cup K$. By the same argument as in Theorem 1, we obtain a holomorphic function f such that

$$(3) \quad \lim_{r \to 1} |f(r\alpha_n) - s(r\alpha_n)| = 0$$

$$(4) \quad \lim_{r \to 1} |f(r\beta_n) - s(r\beta_n)| = 0$$

By the definition of s, these become

$$(3') \quad \lim_{r \to 1} |h(r\alpha_n) / f(r\alpha_n)| = 0$$

$$(4') \quad \lim_{r \to 1} |f(r\beta_n)| = 0$$

If we let $f_1 = f$ and $g_1 = h / f$, then $h = f_1 \cdot g_1$, where $f_1 \in \mathcal{R}$ by (3') and $g_1 \in \mathcal{R}_m$ by (4').

THEOREM 3: <u>If</u> f $\in \mathcal{B}$ * <u>and if</u> f <u>omits the finite</u>

<u>value</u> a <u>and</u> g (z) = 1 / (f (z) - a), then g $\in \mathcal{B}$ *.

<u>Proof</u>: If g $\notin \mathcal{B}$*; then g has $\infty$ as a Koebe value so that there is an arc $\gamma$ contained in C, a sequence of arcs $\{\gamma_n\}_{n=1}^{\infty}$ in D and a sequence of positive real numbers $\{\mu_n\}_{n=1}^{\infty}$ such that $\gamma_n \to \gamma$, $\mu_n \to \infty$, and $|g(z)| \geq \mu_n$ on $\gamma_n$, n = 1, 2, $\cdots$. But then $|f(z) - a| \leq 1/\mu_n$ on $\gamma_n$, n = 1, 2, $\cdots$, and $1/\mu_n \to 0$. This says that a is a Koebe value for f so that f $\notin \mathcal{B}$*. This contradiction proves the theorem.

THEOREM 4: <u>Let</u> f $\in \mathcal{A}$. <u>If</u> f <u>omits</u> <u>a</u> <u>finite</u> <u>value</u>, <u>then</u> f $\in \mathcal{B}$*.

<u>Proof</u>: Let c be the omitted finite value so that g (z) = 1 / (f (z) - c) is holomorphic in D. If f has an asymptotic value at a point, then so does g. (If f $\to \infty$, g $\to$ 0 ; if f $\to$ c, g $\to \infty$; if f $\to$ b and b $\neq$ c, $\infty$, g $\to$ 1 / (b-c)). Thus g $\in \mathcal{A}$.

If f $\notin \mathcal{B}$*, f has $\infty$ as a Koebe value. A similar argument to the previous theorem shows that g has zero as a Koebe value. By [19, Theorem 4, p. 18], this shows that g $\notin \mathcal{A}$. This contradiction proves the theorem.

Putting Theorems 3 and 4 together gives the following corollary.

COROLLARY 4.1: <u>If</u> f $\in \mathcal{A}$ <u>and</u> <u>if</u> f <u>omits</u> <u>the</u> <u>finite</u> <u>value</u> c, <u>then</u> g (z) = 1 / (f (z)-c) $\in \mathcal{B}$ *.

THEOREM 5: <u>If</u> $f \in \mathcal{B}^*$ <u>and</u> $g(z) = e^{f(z)}$, <u>then</u> $g \in \mathcal{B}^*$.

<u>Proof</u>: Let $u(z)$ be the real part of $f(z)$ and suppose $\zeta \in B^*$ for the function $f$. Then there is an arc $\Gamma$ that ends at $\zeta$ and a constant M such that $|f(z)| < M$ on $\Gamma$. Thus $|g(z)| = e^{u(z)}$ so that $|g(z)| < e^M$ on $\Gamma$. Thus $\zeta \in B^*$ for g and $B^*$ for f is contained in $B^*$ for g. Since $B^*$ for f is dense in C, $B^*$ for g must also be dense in C and $g \in \mathcal{B}^*$.

THEOREM 6: <u>There exists a function</u> h <u>not in</u> $\mathcal{B}^*$ <u>such that if</u> $g(z) = e^{h(z)}$, <u>then</u> $g \in \mathcal{B}^*$.

<u>Proof</u>: We pick two subsets of C as follows:

$$H_1^1 = \{1\}, \qquad H_1^2 = \{-1\}$$

$$H_2^1 = \{1, e^{2\pi i/3}, e^{4\pi i/3}\}, \qquad H_2^2 = \{e^{\pi i/3}, -1, e^{5\pi i/3}\},$$

$$H_3^1 = \{e^{2k\pi i/9}\}_{K=0}^{8} \qquad H_3^2 = \{e^{(2K+1)\pi i/9}\}_{K=0}^{8}$$

$$\vdots \qquad\qquad \vdots$$

$$H_n^1 = \{e^{2K\pi i/(3^{n-1})}\}_{K=0}^{3^{n-1}-1} \qquad H_n^2 = \{e^{(2K+1)\pi i/(3^{n-1})}\}_{K=0}^{3^{n-1}-1}$$

$$\vdots \qquad\qquad \vdots$$

Let $A_i = \bigcup_{n=1}^{\infty} H_n^i$, $i = 1, 2$, so that $A_1$ and $A_2$ are countable dense disjoint subsets of C. (See Figure 1).

For each non-negative integer n, let $s_n = 1 - 1/(n+1)$ and let $\gamma_n^1$ be the line segment from the point $s_n e^{\pi i/4^n}$ to the point $s_{n+1} e^{\pi i/4^{n+1}}$ and let $\gamma_n^2$ be the line

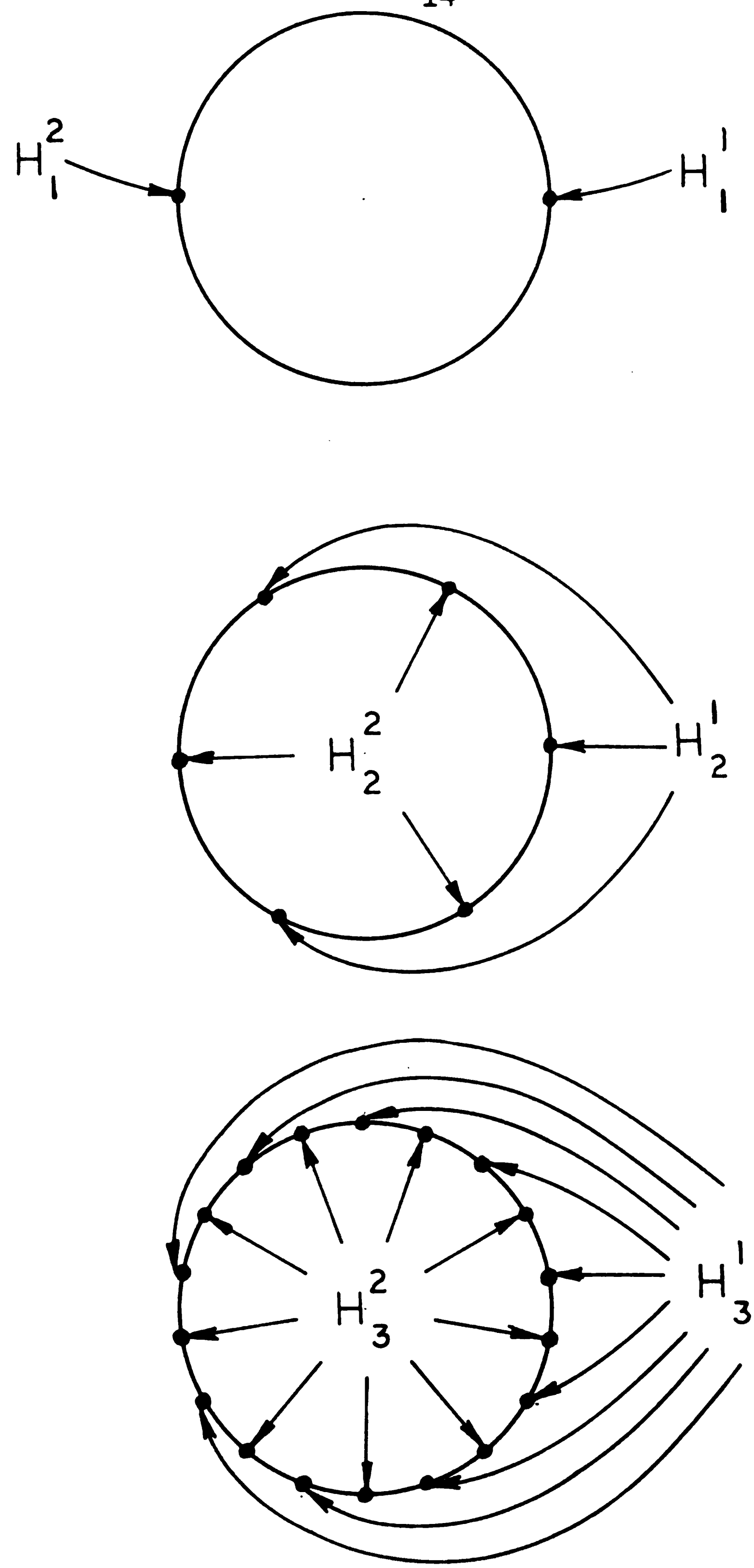


Figure 1.

segment joining the conjugates of these points. Let $\Gamma_i = \bigcup_{h=0}^{\infty} \gamma_n^i$, $i = 1, 2$, and let T be the domain bounded by $\Gamma_1$, $\Gamma_2$, and $\{1\}$. Let $t_1 > t_2 > t_3 > \cdots \downarrow 0$ be a sequence picked so that

$$t_n < 1/2 \min \{s_n - s_{n-1}, s_{n+1} - s_n\} \quad \text{and}$$

$$F_n = \{z = x + iy : |y| \leq t_n, (s_{n-1} + s_n)/2 \leq x \leq (s_n + s_{n+1})/2\} \subseteq T.$$

Let $b_n$ = upper boundary $F_n \cap \{z : |z| = s_n\}$

$c_n$ = lower boundary $F_n \cap \{z : |z| = s_n\}$

$\Delta_n$ = line segment from $b_n$ to $c_n$

$a_n = \Delta_n \cap \{z : I_m z = 0\}$

$\alpha_n = \{z = x + iy : x = a_n, t_n/2 \leq y \leq t_n\}$

$\beta_n = \{z = x + iy : x = a_n, -t_n \leq y \leq -t_n/2\}$

$\sigma_n = \{z : |z - a_n| = t_n/2, \pi/2 \leq \arg(z-a) \leq 3\pi/2\}$

(See Figure 2)

Let $T_n = T \cap \{z : |z| \geq s_{n-1}\}$, $n = 1, 2, \cdots$. If $p = e^{i\theta} \in A_2$, there exists N such that $p \in H_n^2$ but $p \notin H_{n-1}^2$. Define $T^p$ to be $T_n$ rotated through the angle $\theta$. For $n \geq N$, let $F_n^p$, $\alpha_n^p$, $\beta_n^p$, $\sigma_n^p$, $a_n^p$ be the rotations of $F_n$, $\alpha_n$, $\beta_n$, $\sigma_n$, $a_n$ respectively through angle $\theta$.

Let $\tau_n = \{z: |z| = s_n\} \cap C(\bigcup_{p \in H_n^2} F_n^p)$

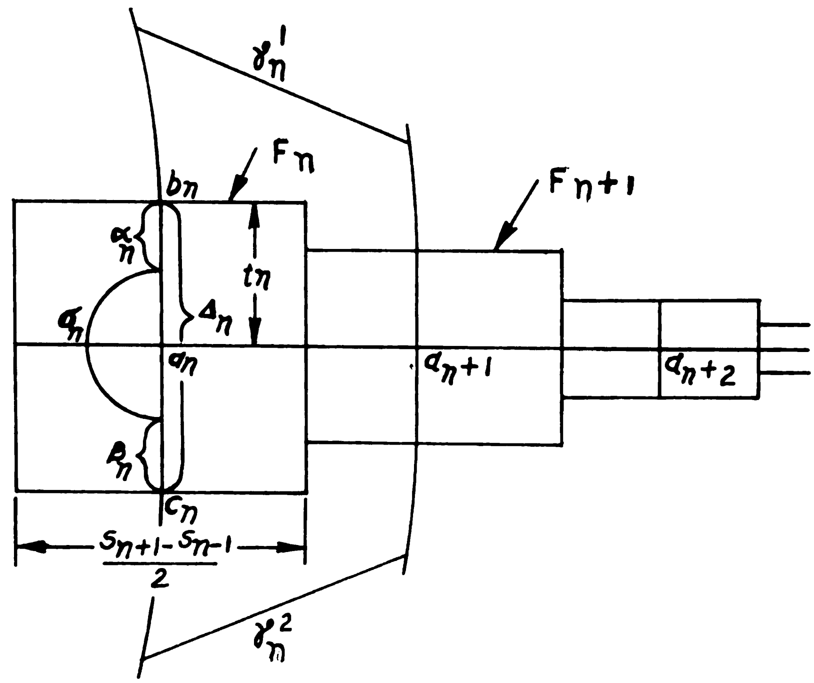


Figure 2

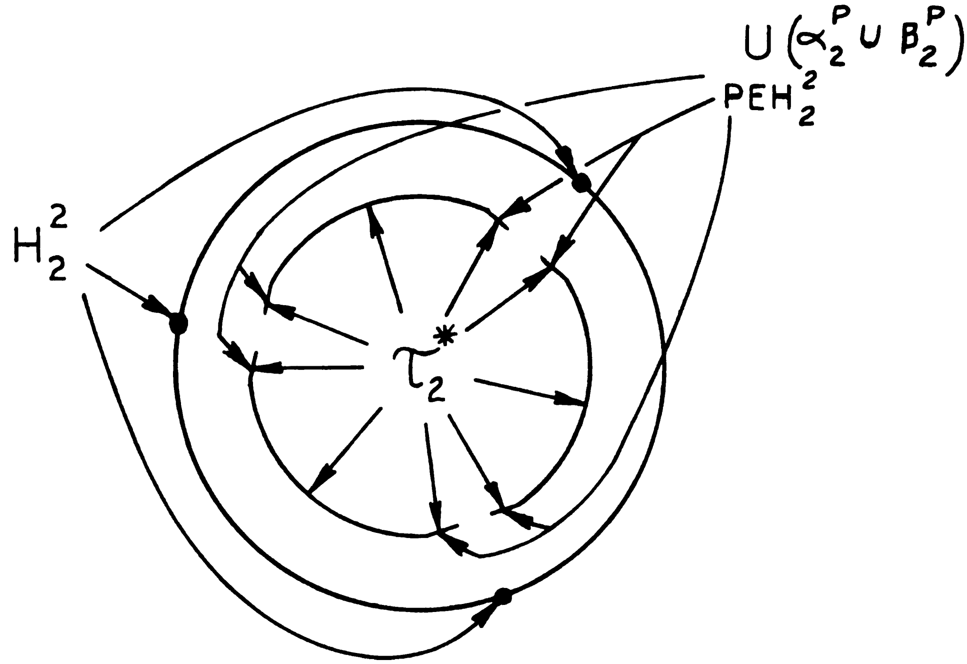


Figure 3

and $\tau_n^* = \tau_n \cup ( \bigcup_{p \in H_n^2} ( \alpha_n^p \cup \beta_n^p ) )$.

where $\mathcal{C}$ represents complementation with respect to the disk. (See Figure 3).

If $p \in A_1$, there exists M such that $p \in H_m^1$ but $p \notin H_{m-1}^1$. Define $R_p$ to be the intersection of the radius to p and $\{ z : s_m \leq z < 1 \}$ and let $R = \bigcup_{p \in A_1} R_p$. Then $R \cup ( \bigcup_{n=1}^{\infty} \tau_n^* )$ is a network that does not disconnect the plane ( i. e. its complement is connected ) , so that we can repeatedly apply Mergelyan's Approximation Theorem [22, Theorem 20.5, p. 386] to the function $i / (1 - |z| )$ to get a holomorphic function f that satisfies

(1) $|f (z) - i n| < 1$ if $z \in \tau_n^*$

(2) $f (z) \to \infty$ as $|z| \to 1$ along $R_p$ in such a way that Re $f (z) < 1$ for all $p \in A_1$. (For examples of such a use of Mergelyan's Theorem see Theorem 14 in this paper, [2, Theorem 1, p. 187], [7], and [8]).

We now construct another holomorphic function g as follows. Enumerate $A_2$ as $\{ p_m \}_{m=1}^{\infty}$. If m is a positive integer, there exists N such that $p_m \in H_N^2$ but $p_m \notin H_{N-1}^2$. The constructions have been virtually identical to those of Barth and Schneider [ 7 ] so that by the same technique that they used we get a holomorphic function $g_m$ such that

(1) $| g_m (z)| < 1 / 2^{m+1}$ on $\mathcal{C} ( \bigcup_{j \geq N} F_j^{p_m} )$

(2) $\operatorname{Im}\ (g_m(z)) > -1/2$ on $\bigcup_{j \geq N} (\alpha_j^{p_m} \cup \beta_j^{p_m})$

(3) $|g_m(z)| > \sup_{z \in \sigma_j^{p_m}} (|f(z)|) + j$ on $\sigma_j^{p_m}$

if $j \geq N$.

Let $g(z) = \sum_{m=1}^{\infty} g_m(z)$. If K is any compact subset of D, K intersects only finitely many $F_m^{p}$'s, so that for all but finitely many m's we have $|g_m(z)| < 1/2^{m+1}$ for all z in K. The series thus converges uniformly on each compact subset and g is holomorphic in D.

Set $h(z) = f(z) + g(z)$ and $\nu_n = \tau_n^* \cup \bigcup_{p \in H_n^2} \sigma_n^p$, so that $\{\nu_n\}_{n=1}^{\infty}$ is a sequence of closed curves tending to C.

On $\sigma_n^{p_m}$, $|g_m(z)| \geq |f(z)| + n$ and $\sum_{n=1, n \neq m}^{\infty} |g_n(z)|$

$$\leq \sum_{n=1}^{\infty} 1/2^n = 1 \text{ so that}$$

$$|h(z)| = \left|f(z) + g_m(z) + \sum_{n=1, n \neq m}^{\infty} g_n(z)\right|$$

$$\geq |g_m(z)| - |f(z)| - \sum_{n=1, n \neq m}^{\infty} |g_n(z)|$$

$$\geq |f(z)| + n - |f(z)| - 1$$

$$\geq n - 3$$

On $\alpha_n^{p_m} \cup \beta_n^{p_m}$, $|f(z) - in| < 1$, $\operatorname{Im} g_m(z) > -1/2$ so that we have

$$h(z) = \left| f(z) + g_m(z) + \sum_{n=1,\, n \neq m}^{\infty} g_n(z) \right|$$

$$\geq \left| f(z) + g_m(z) \right| - \sum_{n=1,\, n \neq m}^{\infty} \left| g_n(z) \right|$$

$$\geq (n-1) - 1/2 - 1$$

$$\geq n - 3$$

On $\tau_n$, $|f(z) - i\,n| < 1$, $|g(z)| \leq \sum_{n=1}^{\infty} 1/2^n = 1$, so that

$$|h(z)| \geq f(z) - g(z)$$

$$\geq (n-1) - 1$$

$$\geq n - 3$$

Thus, on $\nu_n$, $|h(z)| \geq n - 3$ so that h has $\infty$ as a Koebe value and is not in $\mathcal{B}$ *.

If $p \in A_1$ and $z \in R_p$, Re $(f(z)) < 1$ and $|g(z)| \leq \sum_{n=1}^{\infty} 1/2^n = 1$ so that Re $(h(z)) < 2$ and thus $|e^{h(z)}| = e^{\text{Re}\, h(z)} < e^2$. Since $A_1$ is dense in C, this shows that $e^{h(z)} \in \mathcal{B}$ * and the result is proved.

For notational purposes, let $I^0(f) = f$, $I(f)(z) = \int_0^3 f(t)\,dt$, and $I^n(f) = I(I^{n-1}(f))$.

THEOREM 7: There exists a function f in $\mathcal{B}$ * such that I (f) is not in $\mathcal{B}$ *.

<u>Proof</u>: Let $A_1$, $A_2$, $H_n^1$, $H_n'$, T, $T^p$, $F_n^p$, $\alpha_n^p$, $\beta_n^p$, $\sigma_n^p$, $a_n^p$, $\tau_n$, $\tau_n^*$, and $\nu_n$ be defined exactly as in the proof of Theorem 6. ( $n = 1, 2, \cdots$)

Let $\eta$: $z = z(t)$ ( $0 \leq t < 1$ ) be an unrectifiable curve lying in T and joining the points 0 and 1 such that $t_1 < t_2 \Rightarrow |z(t_1)| < |z(t_2)|$, $\lim_{t \to 1} |z(t)| = 1$, and let $\eta$ be sufficiently smooth so that if $\eta$ is parameterized with respect to arc length s, then $\frac{dz}{ds}$ is continuous and $\arg \frac{dz}{ds}(0) = 0$. Let $\theta(s) = \arg \frac{dz}{ds}(s)$ and define a continuous function r by $r(z(s)) = e^{-i\theta(s)}$. If L (z) is the arc length along $\eta$ from 0 to z, then

$$\begin{aligned}
\int_0^z r(w)\,dw &= \int_0^{L(z)} r(z(s)) \frac{dz}{ds}(s)\,ds \\
&= \int_0^{L(z)} e^{-i\theta(s)} \frac{dz}{ds}(s)\,ds \\
&= \int_0^{L(z)} \left| \frac{dz}{ds} \right| ds \\
&= \int_0^{L(z)} |dz| = L(z)
\end{aligned}$$

so that I (r) (z) is a real valued function that tends monotonically to $\infty$ along $\eta$. Further, $|r(z)| = 1$ for all z in $\eta$. We also require that the length of the intersection of $\eta$ and $\{z : s_n \leq |z| \leq s_{n+1}\}$ be at least one for each positive integer n.

If $p = e^{i\varphi} \in A_1$, let $\eta_p$ be the rotation of $\eta$ through angle $\varphi$. Define r on $\eta_p$ in exactly the same manner that we defined r on $\eta$. (r is well defined since in both cases $\arg \frac{dz}{ds}(0) = 0$ and thus $r(0) = e^{-i \cdot 0} = 1$ and 0 is the only point of intersection of $\eta_p$ and $\eta$ ). Let $S = \bigcup_{p \in A_1} \eta_p$. We observe that any component of

$\bigcup_{n=1} \tau_n{}^*$ intersects S at exactly one point. We define r to be constant on each component (namely equal to the value of r at the point of intersection). We now have r continuous on $S \cup ( \bigcup_{n=1}^{\infty} \tau_n{}^* )$ which is a network that does not disconnect the plane so that we can again use the approximation techniques we used in the proof of Theorem 6, this time being careful to keep the integral of our holomorphic function f "reasonably close" to the integral of r on this network. We thus obtain a holomorphic function f such that

(1) For each p in $A_1$, f is bounded on $\eta_p$.

(2) $|I(f)(z)| > n$ and $\operatorname{Im}(I(f)(z)) < 1$ on $\tau_n{}^*$ $n = 1, 2, \cdots$

We again construct another holomorphic function g as follows. The constructions have been virtually the same as those Barth and Schneider used in the proof of [9, Example 1, pp. 15-22] so that, just as they did in that proof, for each positive integer m we get a holomorphic function $g_m$ such that

(1) $|g_m(z)| < 1/2^{m+1}$ on $\zeta( \bigcup_{j \geq N} F_j{}^{p_m} )$.

(2) $|I(g_m)(z)| < 1/2^{m+1}$ on $\zeta( \bigcup_{j \geq N} F_j{}^{p_m} )$

(3) $|I(g_m)(z)| > \sup_{z \in \sigma_j} p_m(I(f)(z)) + j$ on $\sigma_j{}^{p_m}$, $j \geq N$.

(4) $\operatorname{Re}(I(g_m)(z)) > -1/2$ on $\bigcup_{j \geq N} (\alpha_j^{p_m} \cup \beta_j^{p_m})$

where N is the integer such that $p_m \in H_N^2$ but $p_m \notin H_{N-1}^2$. Let $g(z) = \sum_{m=1}^{\infty} g_m(z)$ so that, as before, g is holomorphic in D. Also, if $p \in A_1$, $|g(z)| \leq \sum_{m=1}^{\infty} (1/2^n) = 1$ so that, if $h = f+g$, h is bounded on $\eta_p$ (both f and g are) and hence $h \in \mathcal{B}^*$.

We now show $I(h)$ has $\infty$ as a Koebe value along $\nu_n$'s so that $I(h) \notin \mathcal{B}^*$.

On $\sigma_n^{p_m}$, $|I(g_m)(z)| \geq |I(f)(z)| + n$ and $\sum_{n=1, n \neq m}^{\infty} |I(g_n)(z)| \leq \sum_{n=1}^{\infty} 1/2^n = 1$ so that

$$|I(h)(z)| = \left|I(f)(z) + I(g_m)(z) + \sum_{n=1, n \neq m}^{\infty} I(g_n)(z)\right|$$
$$\geq |I(g_m)(z)| - |I(f)(z)| - \sum_{n=1, n \neq m}^{\infty} |I(g_n)(z)|$$
$$\geq |I(f)(z)| + n - |I(f)(z)| - 1$$
$$\geq n - 3$$

On $\alpha_n^{p_m} \cup \beta_n^{p_m}$, $|I(f)(z)| > n$, $|\operatorname{Im}(I(f)(z))| < 1$, and $\operatorname{Re}(I(g_m)(z)) > -1/2$ and $\sum_{n=1, n \neq m}^{\infty} I(g_n)(z) \leq \sum_{n=1}^{\infty} 1/2^n = 1$ so that

$$|I(h)(z)| = \left|I(f)(z) + I(g_m)(z) + \sum_{n=1, n \neq m}^{\infty} I(g_n)(z)\right|$$

$$\geq \left| I(f)(z) + I(g_m)(z) \right| - \sum_{n=1, n \neq m}^{\infty} \left| I(g_n)(z) \right|$$

$$\geq ((n-1) - \tfrac{1}{2}) - 1$$

$$\geq n - 3$$

On $\tau_n$, $\left| I(f)(z) \right| > n$, $\left| I(g)(z) \right| \leq \sum_{n=1}^{\infty} (1/2^n) = 1$

so that

$$\begin{aligned} \left| I(h)(z) \right| &= \left| I(f)(z) + I(g)(z) \right| \\ &\geq \left| I(f)(z) \right| - \left| I(g)(z) \right| \\ &\geq n - 1 \\ &\geq n - 3 \end{aligned}$$

Thus on $\nu_n$, $\left| I(h)(z) \right| \geq n - 3$ and the theorem is proved.

Lappan, in [15], constructed a function that had $\infty$ as a Koebe value, but whose integral was uniformly normal, hence normal. Since normal functions are in $\mathcal{B}$ *, we note the following result.

THEOREM 8: <u>There exists a function</u> f <u>in</u> $\mathcal{B}$ * <u>such that</u> f' <u>is not in</u> $\mathcal{B}$ *.

THEOREM 9: <u>There exists a function</u> f <u>in</u> $\mathcal{B}$ * <u>such that</u> $I^n(f)$ <u>and</u> $f^{(n)}$ <u>are in</u> $\mathcal{B}$ * <u>for all positive integers</u> n.

<u>Proof</u>: If $|f(z)| \leq M$ in D, then $|I(f)(z)| = \left| \int_0^z f(t)\,dt \right| \leq M \cdot |z| \leq M$ in D. Thus, for the integral any bounded function will suffice. We need a little more care for the case of the derivative, but this is easily rectified by taking a function holomorphic in $\{ z$ :

$|z| < 2\}$. Then each of its derivatives is also holomorphic in $\{z : |z| < 2\}$ so that they are **all** bounded in $\overline{D}$ which is a compact subset of this domain.

THEOREM 10: <u>There exists a function</u> g <u>not in</u> $\mathcal{B}$ * <u>such that</u> $g^{(j)}$ <u>is not in</u> $\mathcal{B}$ * <u>for any non-negative integer</u> j.

<u>Proof</u>: For each positive integer n we define three subsets of D as follows.

$$T_n = \{re^{i\theta} : r = 1 - 1/(n+1),\ -\pi/4 \leq \theta \leq \pi/4\}$$

$$S_n = \{x : (1 - 1/n) + 1/(4 \cdot n \cdot (n+1)) < x < (1 - 1/n) + 3/(4 \cdot n \cdot (n+1))\}$$

$$G_n = \{z : |z| < (1 - 1/n) + 1/(8 \cdot n \cdot (n+1))\}$$

We now construct a sequence of functions inductively. Let $V_1$ be the continuous function defined on $\overline{G_1} \cup T_1$ which is equal to zero on $\overline{G_1}$ and $1\frac{1}{2}$ on $T_1$. We approximate $V_1$ to within $\frac{1}{2}$ by Mergelyan's theorem [22, Theorem 20.5, p. 386] to get a holomorphic function $g_1(z)$ with the property that

$$|g_1(z)| < 1/2 \quad \text{if } z \in \overline{G_1}$$
$$|g_1(z)| > 1 \quad \text{if } z \in T_1 .$$

If $g_1, g_2, \cdots, g_{n-1}$ have been defined let

$$M_n = n + 1 + \sum_{j=1}^{n-1} \max_{0 \leq i \leq n} \left( \sup_{z \in T_n} \left( \left| g_j^{(i)}(z) \right| \right) \right).$$

Let $V_n(z)$ be a continuous function which is zero in $\overline{G_n}$, sufficiently small and real on $T_n$, sufficiently large and real on $S_n$, and linear on the two parts of the real axis from $\overline{G_n}$ to $T_n$ that are not in $S_n$ and approximate $V_n(z)$ to within $1 / 2^n$ by Mergelyan's theorem to get a holomorphic function $f_n(z)$ with the property that

$$\left| I^n (f_n)(z) \right| < 1 / 2^n \text{ if } z \in \overline{G_n}$$

$$\min_{0 \leq j \leq n} \left( \inf_{z \in T_n} \left( \left| I^j (f_n)(z) \right| \right) \right) \geq M_n$$

Then, setting $g_n = I^n (f_n)$ we see that

$$\left| g_n(z) \right| < 1 / 2^n \quad \text{if} \quad z \in \overline{G_n}$$

$$\min_{0 \leq j \leq n} \left( \inf_{z \in T_n} \left( \left| g_n^{(j)}(z) \right| \right) \right) \geq M_n$$

Let $g(z) = \sum_{n=1}^{\infty} g_n(z)$. Since $\left| g_n(z) \right| < 1/2^n$ in $\overline{G_n}$ this series converges uniformly on each compact subset so that $g$ is holomorphic.

Let $j$ be a non-negative integer and $m > j$ and look at $g^{(j)}$ on $T_m$.

$$\left| g^{(j)}(z) \right| = \left| \sum_{n=1}^{\infty} g_n^{(j)}(z) \right|$$

$$= \left| \sum_{n=1}^{m-1} g_n^{(j)}(z) + g_m^{(j)}(z) + \sum_{n=m+1}^{\infty} g_n^{(j)}(z) \right|$$

$$\geq \left| g_m^{(j)}(z) \right| - \sum_{n=1}^{m-1} \left| g_n^{(j)}(z) \right| - \sum_{n=m+1}^{\infty} 1/2^n$$

$$\geq (m+1+\sum_{n=1}^{m-1} \left| g_n^{(j)}(z) \right|) - \sum_{n=1}^{m-1} \left| g_n^{(j)}(z) \right| - 1$$

$$= m$$

Thus, for all non-negative integers $j$, $g^{(j)}$ has $\infty$ as a Koebe value and hence is not in $\mathcal{B}^*$. This proves the theorem.

THEOREM 11: <u>There exists a function</u> $g$ <u>not in</u> $\mathcal{B}^*$ <u>such that</u> $I^j(g)$ <u>is not in</u> $\mathcal{B}^*$ <u>for any non-negative integer</u> $j$.

<u>Proof</u>: Let $T_n$, $S_n$, and $G_n$ be as before. Again, we shall define a sequence of functions inductively.

Let $g_1$ be as in the proof of Theorem 10 and suppose $g_1, \ldots, g_{n-1}$ have been defined. This time we will let

$$M_n = n+1+\sum_{j=1}^{n-1} \max_{0 \leq i \leq n} \left( \sup_{z \in T_n} \left( \left| I^{(i)}(g_j)(z) \right| \right) \right)$$

and, in exactly the same manner, construct a holomorphic function $g_n$ such that

$$\left| g_n(z) \right| < 1/2^n \quad \text{if } z \in \overline{G_n}$$

$$\min_{0 \leq j \leq n} \left( \inf_{z \in T_n} \left( \left| I^j(g_n)(z) \right| \right) \right) > M_n$$

We set $g(z) = \sum_{n=1}^{\infty} g_n(z)$ and again the series converges uniformly on each compact subset so that $g$ is holomorphic.

Let j be a non-negative integer and m $>$ j and look at $I^j(g)$ on $T_m$.

$$\begin{aligned}
\left|I^j(g)(z)\right| &= \left|\sum_{n=1}^{\infty} I^j(g_n)(z)\right| \\
&= \left|\sum_{n=1}^{m-1} I^j(g_n)(z) + I^j g_m(z) + \sum_{n=m+1}^{\infty} I^j(g_n)(z)\right| \\
&\geq \left|I^j(g_n)(z)\right| - \sum_{n=1}^{m-1}\left|I^j(g_n)(z)\right| - \sum_{n=m+1}^{\infty} 1/2^n \\
&\geq (m+1 + \sum_{n=1}^{m-1}\left|I^j(g_n)(z)\right|) - \sum_{n=1}^{m-1}\left|I^j(g_n)(z)\right| - 1 \\
&\geq m
\end{aligned}$$

Thus $I^j(g)$ has $\infty$ as a Koebe value and is not in $\mathcal{B}^*$ for any non-negative integer j.

## III. FURTHER PROPERTIES OF FUNCTIONS IN CLASS $\mathcal{B}$ *

We now turn to growth conditions. As usual, we use the following two measures of growth

$$M(r) = \sup_{|z| = r} ( |f(z)| )$$

and

$$m(r) = (1 / 2\pi) \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta$$

THEOREM 12: <u>There exist functions in $\mathcal{B}$ * of arbitrarily slow (infinite) growth and functions in $\mathcal{B}$ * of arbitrarily fast (infinite) growth.</u>

<u>Proof</u>: We first observe that

$$\begin{aligned} m(r) &= (1 / 2\pi) \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta \\ &\leq (1 / 2\pi) \int_0^{2\pi} \log^+ |M(r)| \, d\theta \\ &= \log^+ M(r) \end{aligned}$$

so that for slow growth it is sufficient to look at M (r) and for fast growth it is sufficient to look at m (r).

In [19, Example 6, p. 60], MacLane constructed a function f for which M (r) grows arbitrarily slowly and for which $A_0$ is dense in C. This function is thus in $\mathcal{B}$ * and the result for slow growth is proved.

If there were an upper bound on the growth rate for functions in $\mathcal{B}$ *, then there would also be an upper bound on the growth rate for functions in $\mathcal{R}$ . Since any holomorphic function h can be expressed as the sum of two functions f and g, in class $\mathcal{R}$ and $m(r, h) \leq m(r, f) + m(r, g) +$ bounded terms, this would say there is an upper bound on the growth rate for all holomorphic functions. This contradiction proves the result.

THEOREM 13: <u>There exist functions not in $\mathcal{B}$ * of arbitrarily slow (infinite) growth and functions not in * of arbitrarily fast (infinite) growth.</u>

<u>Proof</u>: As in the proof of Theorem 12, for slow growth it is sufficient to look at $M(r)$ and for fast growth it is sufficient to look at $m(r)$.

In [1, Theorems 3 and 5], Bagemihl, Erdos, and Seidel constructed a function f for which $M(r)$ grows arbitrarily slowly and which has $\infty$ as a Koebe value and is thus not in $\mathcal{B}$ *. This proves the slow growth result.

For the fast growth, let $\mu(r)$ be a positive, strictly increasing function defined on $[0, 1)$ that tends to $\infty$ as r tends to 1. Let $r_n = 1 - 1/2^n$ and $S_n = \{r_n e^{i\theta}: -\pi/2 \leq \theta \leq \pi/2\}$, $n = 1, 2, \cdots$ . Through a slight variation in the technique of repeatedly using Mergelyan's Theorem we obtain a holomorphic function f such that

$$|f(z) - e^{2\mu(r_{n+1})} - 1| < 1 \quad \text{for all } z \in S_n$$

so that f has $\infty$ as a Koebe value and is thus not in $\mathcal{B}$ *.

Further

$$m(r_n) = (1 / 2\pi) \int_0^{2\pi} \log^+ |f(r_n e^{i\theta})| \, d\theta$$
$$\geq (1 / 2\pi) \int_{-\pi/2}^{\pi/2} \log^+ |f(r_n e^{i\theta})| \, d\theta$$
$$\geq (1 / 2\pi) \int_{-\pi/2}^{\pi/2} \log^+ |e^{2\mu(r_{n+1})}| \, d\theta$$
$$\geq (1 / 2) \cdot 2 \cdot \mu(r_{n+1})$$
$$= \mu(r_{n+1}) \qquad n = 1, 2, \cdots$$

But $\mu$ and m are increasing functions so that, if $r_n \leq r \leq r_{n+1}$, $m(r) \geq m(r_n) \geq \mu(r_{n+1}) \geq \mu(r)$. Since any r that is bigger than $r_1$ is in some interval $[r_n, r_{n+1})$, this shows that $m(r) \geq \mu(r)$ and the result is proved.

THEOREM 14: Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of points in D such that $0 < |z_1| < |z_2| < |z_3| < \cdots$, $\lim_{n\to\infty} |z_n| = 1$ and let $\{w_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. Then There exists a function f in $\mathcal{B}^*$ such that $\lim_{n\to\infty} |f(z_n) - w_n| = 0$.

Proof: Let $\{r_n\}_{n=0}^{\infty}$ be a sequence of real numbers such that $r_0 < |z_1| < r_1 < |z_2| < r_2 < |z_3| < \cdots$. Let

$\{\alpha_n\}_{n=1}^{\infty}$ be a dense subset of C such that the collection of radii to the $\alpha_n$'s do not contain any $z_i$'s. We denote $\{z : |z| = r\}$ by $C_r$ and $\{z : |z| < r\}$ by $D_r$. $R_n$ will denote the radius to $\alpha_n$, $n = 1, 2, \cdots$. We set

$$F_o = \overline{D_{r_o}} \cup (R_1 \cap C_{r_1}) \cup \{z_1\}$$

$$\vdots$$

$$F_n = \overline{D_{r_n}} \cup \left[\left(\bigcup_{j=1}^{n} R_j\right) \cap \left(\overline{D_{r_{n+1}}} - D_{r_n}\right)\right] \cup (R_{n+1} \cap C_{r_{n+1}}) \cup \{z_n\}$$

$$\vdots$$

Then each $F_n$ is a compact set which does not disconnect the plane.

By Mergelyan's Theorem, there exists a polynomial $p_o(z)$ such that

$$|p_o(z)| < 1/2 \quad \text{if } z \in \overline{D_{r_o}} \cup (R_1 \cap C_{r_1})$$

$$|p_o(z_1) - w_1| < 1/2$$

To construct $p_1(z)$ we first define a continuous function $h_1(z)$ on $R_1 \cap (\overline{D_{r_2}} - D_{r_1})$ by

$$h_1(r\alpha_1) = p_o(r_1\alpha_1)(r_2 - r)/(r_2 - r_1)$$

Then, if we define a function $g_1(z)$ on $F_1$ by

$$g_1(z) = 0 \quad \text{if } z \in \overline{D_{r_1}}$$

$$= h_1(z) - p_o(z) \quad \text{if } z \in R_1 \cap (\overline{D_{r_2}} - D_{r_1})$$

$$= w_2 - p_o(z) \quad \text{if} \quad z = z_2$$

$$= - p_o(z) \quad \text{if} \quad z \in R_2 \cap C_{r_2}$$

$g_1(z)$ is continuous on $F_1$ and holomorphic at each interior point of $F_1$ so, by Mergelyan's Theorem, there exists a polynomial $p_1(z)$ such that

$$|p_1(z)| < 1/4 \quad \text{if } z \in \overline{D_{r_1}}$$

$$|w_2 - (p_o(z_2) + p_1(z_2))| < 1/4$$

$$|h_1(z) - (p_o(z) + p_1(z))| < 1/4 \text{ if}$$
$$z \in R_1 \cap (\overline{D_{r_2}} - D_{r_1})$$

$$|p_o(z) + p_1(z)| < 1/4 \quad \text{if } z \in R_2 \cap C_{r_2}$$

For the inductive step, suppose that polynomials $p_o$, $p_1, \cdots, p_n$ have been defined such that

$$|p_n(z)| < 1/2^{n+1} \quad \text{if} \quad z \in \overline{D_{r_n}}$$

$$|w_{n+1} - (p_o(z_{n+1}) + p_1(z_{n+1}) + \cdots + p_n(z_{n+1}))| < 1/2^{n+1}$$

$$|h_n(z) - (p_o(z) + p_1(z) + \cdots + p_n(z))| < 1/2^{n+1}$$
$$\text{if } z \in \left(\bigcup_{j=1}^{n} R_j\right) \cap (\overline{D_{r_{n+1}}} - D_{r_n})$$

$$|p_o(z) + p_1(z) + \cdots + p_n(z)| < 1/2^{n+1}$$
$$\text{if } z \in R_{n+1} \cap C_{r_{n+1}}$$

where $h_n(z)$ is defined by

$$h_n(r\alpha_j) = (p_o(r_n\alpha_j) + p_1(r_n\alpha_j) + \cdots + p_{n-1}(r_n\alpha_j)) \cdot (r_{n+1}-r) / (r_{n+1} - r_n) \quad j = 1,2,\cdots, n$$

We define a continuous function $h_{n+1}$ on $\left(\bigcup_{j=1}^{n+1} R_j\right) \bigcap \left(\overline{D_{r_{n+2}}} - D_{r_{n+1}}\right)$ by

$$h_{n+1}(r\alpha_j) = (p_o(r_{n+1}\alpha_j) + p_1(r_{n+1}\alpha_j) + \cdots + p_n(r_{n+1}\alpha_j))(r_{n+2} - r) / (r_{n+2}-r_{n+1})$$

$$j = 1, 2, \cdots, n + 1$$

and then a function $g_{n+1}$ on $F_{n+1}$ by

$$\begin{aligned} g_{n+1}(z) &= 0 \text{ if } z \in \overline{D_{r_{n+1}}} \\ &= h_{n+1}(z) - (p_o(z) + \cdots + p_n(z)) \\ &\qquad \text{if } z \in \left(\bigcup_{j=1}^{n+1} R_j\right) \bigcap \left(\overline{D_{r_{n+2}}} - D_{r_{n+1}}\right) \\ &= w_{n+2} - (p_o(z) + \cdots + p_n(z)) \quad \text{if } z = z_{n+2} \\ &= -(p_o(z) + \cdots + p_n(z)) \text{ if } z \in R_{n+2} \bigcap C_{r_{n+1}} \end{aligned}$$

Again, $g_{n+1}$ is continuous and holomorphic at each interior point so that we can approximate it to get a polynomial $p_{n+1}(z)$ such that

$$|p_{n+1}(z)| < 1/2^{n+2} \text{ if } z \in \overline{D_{r_{n+1}}}$$

$$|w_{n+2} - (p_o(z_{n+2}) + \cdots + p_{n+1}(z_{n+2}))| < 1/2^{n+2}$$

$$|h_{n+1}(z) - (p_o(z) + \cdots + p_{n+1}(z))| < 1/2^{n+2}$$

$$\text{if } z \in \left(\bigcup_{j=1}^{n+1} R_j\right) \cap \left(\overline{D_{r_{n+2}}} - D_{r_{n+1}}\right)$$

$$|p_o(z) + \cdots + p_{n+1}(z)| < 1/2^{n+2} \text{ if } z \in R_{n+2} \cap C_{r_{n+2}}$$

This completes the inductive step. We now define $f(z) = \sum_{j=0}^{\infty} p_j(z)$. If $z \in \overline{D_{r_n}}$ and $m \geq n$, then $|p_m(z)| < 1/2^{m+1}$ so the series converges on each compact subset and f is holomorphic.

If $z \in R_j$ and $|z| > r_j$, there exists $n \geq j$ such that $r_n \leq |z| < r_{n+1}$ so that $h_n$ is defined at z. Further

$$\begin{aligned} |h_n(z)| &= |p_o(r_n \alpha_j) + \cdots + p_{n-1}(r_n \alpha_j)| \\ &\quad \cdot (r_{n+1} - r) / (r_{n+1} - r_n) \\ &\leq 1/2^{n+1} \end{aligned}$$

We have

$$\begin{aligned} |h_n(z) - f(z)| &= \left|h_n(z) - \sum_{j=0}^{\infty} p_j(z)\right| \\ &= \left|h_n(z) - \sum_{j=0}^{n} p_j(z) - \sum_{j=n+1} p_j(z)\right| \\ &\leq \left|h_n(z) - \sum_{j=0}^{n} p_j(z)\right| + \sum_{j=n+1}^{\infty} |p_j(z)| \end{aligned}$$

$$\leq 1/2^{n+1} + \sum_{j=n+1} 1/2^{j+1}$$

$$= 1/2^{n+1} + 1/2^{n+1}$$

$$= 1/2^{n}$$

Putting these together gives

$$|f(z)| \leq 1/2^{n} + 1/2^{n+1}$$

$$= 3/2^{n+1}$$

$$\leq 1/2^{n-1}$$

As $z \longrightarrow 1$ the $r_n$ involved tends to 1 and $n \longrightarrow \infty$. Thus $\lim_{r \to 1} f(r\alpha_j) = 0$. This shows f has radial limit zero at each point of a dense subset of C and hence is in $\mathcal{B}$ *. (It is actually in $\mathcal{R}$ ).

Next

$$|f(z_j) - w_j| = \left|\sum_{n=0}^{\infty} p_n(z_j) - w_j\right|$$

$$\leq \left|\sum_{n=0}^{j-1} p_n(z_j) - w_j\right| + \sum_{n=j}^{\infty} |p_n(z_j)|$$

$$\leq 1/2^{j+1} + \sum_{n=j}^{\infty} 1/2^{n+1}$$

$$= 1/2^{j+1} + 1/2^{j}$$

$$1/2^{j-1}$$

Thus, $\lim_{j \to \infty} |f(z_j) - w_j| = 0$ and the result is proved.

Remark: This is an example of the technique of repeatedly using Mergelyan's Theorem. It is essentially the same as that used by Bagemihl and Seidel in [2, Theorem 1] and is generally attributed to them.

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