STABILITY AND INVARIANCE OF FUNCTIONAL DIFFERENTIAL EQUATIONS

Dissertation for the Degree of Ph. D.

MICHIGAN STATE UNIVERSITY

RENG - SONG LO

1975

This is to certify that the

thesis entitled

"Stability and Invariance of
Functional Differential Equations"

presented by

Mr. Reng Song Lo

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

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Date June 5, 1975

O-7639

ABSTRACT

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By

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In recent years, Liapunov's method has been successively generalized to functional differential equations of retarded type by using Liapunov functionals. However, in many cases the problems are still open. For example, we already had a complete characterization of integral stability for ordinary differential equations by a Lipschitz Liapunov function which obeys certain bounds. But the problem in functional differential equations is still open. Another fundamental problem in differential equations is the characterization of invariance sets. In ordinary differential equations, it is known that invariance of a closed set is equivalent to a notion called subtangent. But the corresponding result in functional differential equations was not known.

In this thesis, we investigate the above two open problems. First in the case of integral stability, we found the usual approach of Liapunov's method is not very useful. Although one can easily get a lower semi-continuous functional which obeys certain bounds, but "continuity" and "Lipschitz" properties are extremely difficult to obtain. On the other hand, in the case of an invariance set, the hereditary nature of the equation also prevents one from doing a straightforward generalization to functional differential equations. For this purpose, a new Liapunov's theorem based on a class of lower semi-continuous non-Lipschitz functionals was developed. In particular, the usual Laipunov comparison principle hold true for this class of Liapunov functionals. Complete characterizations of integral stability and sets of invariance are obtained using the Laipunov theory developed earlier. As an application to the invariance characterization we give an invariance principle for a class of asymptotically autonomous systems.

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By

Reng-Song Lo

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1975

ACKNOWLEDGEMENTS

I would like to express my deepest appreciation to my thesis advisors, Professor P.K Wong and Professor S.N. Chow, for their patient guidance, constructive suggestions and numerous discussions during the preparation of this thesis.

I also would like to express my hearty thanks to those members in the department for their kindness, generosity, helpfulness that made my stay here a very pleasant one.

My thanks also go to Mrs. Noralee Burkhardt for her excellent typing of this thesis.

TABLE OF CONTENTS

INTRODUCTION

A relation of the form

$$x'(t) = f(t, x(t)), \tag{E}$$

where $x = x(t)$ is a d-dimensional vector value function defined on a real interval and $f(t,x)$ is a function from a certain region of $R \times R^d$ into R^d, is called an ordinary differential equation. The function f is called a vector field and the solution of (E) are integral curves whose tangent is prescribed by the vector field f. In most classical applications, the behavior of many phenomena are assumed to be governed by such ordinary differential equations. Implicit in this assumption is that the future behavior is uniquely determined by the present state of the system alone and is independent of its past history.

There is another type of differential equations, known as functional differential equations (FDE), in which the past history influences in a significant way on the future behavior of the system. It is known [see [6], [7], [19]] that such equations arise in many areas of application. The systems under study are better represented by FDE than by ordinary differential equations. Historically FDE was first encountered in the late eighteenth century, however, very little was done during the nineteenth and early twentieth century.

For the last forty years and especially the last twenty years, the subject has developed into one of the most active branches of differential equations. Much of the stimulus for this was due to the work of Volterra [17], who was interested in certain ecological models, and Krasovskii [10], who was interested in the theory of control, and other mathematicians who had encountered the problem in several different fields.

A good reference for FDE is Hale [8]. In this thesis we shall study three fundamental problems of functional differential equations of retarded type, namely, Liapunov theory, the characterization of invariance of a set and integral stability.

0.1. Definition of FDE of Retarded Type and Initial Value Problems

Let R^d be the real Euclidean d-space and $|x|$ be any norm. Let $\gamma > 0$ and $C = C[-\gamma,0]$ be the Banach space of all continuous functions $\varphi: [-r,0] \to R^d$ with the usual sup norm $\|\varphi\| = \sup\{|\varphi(\theta)|: -\gamma \le \theta \le 0\}$. Given a continuous function $x: [-\gamma + \sigma, \sigma + A) \to R^d$, $\sigma \in R$, $A > 0$, we define for each $t \in [\sigma, \sigma + A)$ an element $x_t \in C$ by $x_t(\theta) = x(t + \theta)$, $-\gamma \le \theta \le 0$.

Let $D \subset R \times C$ be open and $f: D \to R^d$ be continuous. A functional differential equation of retarded type is a functional relation of the form

$$x'(t) = f(t, x_t) \quad . \tag{0.1}$$

Let $(\sigma,\varphi) \in D$. A solution $x = x(t, \sigma, \varphi)$ of (0.1) through (σ,φ) is an absolutely continuous function defined on $[-\gamma + \sigma, \sigma + A)$ for some $A > 0$ such that

(0.2) $$x_\sigma = \varphi ,$$

(0.3) $$x'(\sigma^+) = f(\sigma, x_\sigma) ,$$

and

(0.4) $$x'(t) = f(t, x_t), \sigma < t < A,$$

where $x'(\sigma^+)$ denotes the right hand derivative of x at $t = \sigma$.

After defining FDE of retarded type, the immediate questions that one may ask are:

(i) When does a solution exist?

(ii) When do the equations have uniqueness property?

(iii) Does the family of solutions have certain properties concerning convergence and continuous dependence with respect to the initial condition?

The answer to the above questions may be summarized by the following theorems whose proofs are found in most standard references, c.f. [8].

Theorem 0.1 (Existence). Suppose U is an open set in $R \times C[-\gamma,0]$ and $f: U \to R^d$ is continuous. If $(\sigma,\varphi) \in U$, then there is a solution of (0.1) passing through (σ,φ).

A function $f(t,\varphi)$ defined on $R \times C[-\gamma,0]$ is called Lipschitzian in φ on $U \subset R \times C[-\gamma,0]$, if there exists a constant $L > 0$ such that

$$|f(t,\varphi) - f(t,\Psi)| \le L \cdot \|\varphi - \Psi\|$$

for all (t,φ), $(t,\Psi) \in U$.

Theorem 0.2 (Uniqueness). Suppose U is an open set in $R \times C[-\gamma,0]$, $f: U \to R^d$ is continuous, and $f(t,\varphi)$ is Lipschitzian in φ on each compact set in U. If $(\sigma,\varphi) \in U$, then there is a unique solution of (0.1) with initial value (σ,φ).

Definition 0.1. A continuous function $x: [-\gamma + t, b) \to R^d$ which is absolutely continuous for $t < s < b$ is said to be non-continuable with respect to an open set $D \subset R \times C[-\gamma,0]$ if, for each $-\gamma + t < s < b$, $(s, x_s) \in D$ and for each closed bounded set $U \subset D$, there exists $t < t_U < b$ such that

$$(s, x_s) \notin U \quad \text{for all} \quad t_U < s < b .$$

For a function x, let D_x denote the domain of this function. Then we have the following:

Theorem 0.3 (Convergence). Let $f: U \to R^d$ be continuous on $U \subset R \times C[-\gamma,0]$ and let $|f(t,\varphi)|$ be bounded on each closed and bounded subset of U. Suppose $\{x^n(\cdot)\}$ is a sequence of non-continuable functions on U such that $D_{x^n} \supset [t_o - \gamma, t_o + a_n)$, for some $a_n > 0$. And $\varphi^n(\cdot)$, $\varphi(\cdot)$ are continuous functions such that $\varphi^n(\cdot) \to \varphi(\cdot)$ uniformly on $[-\gamma,0]$, where $\varphi^n(t - t_o) = x^n(t)$, $t_o - \gamma \le t \le t_o$. Define a sequence of functions $\{G_n(\cdot)\}$ by

$$G_n(t) = x^n(t) - \varphi^n(o) - \int_{t_o}^{t} f(x, x_s^n)\,ds \quad \text{for} \quad [t_o, \infty) \cap D_{x^n}.$$

Assume that for each closed, bounded set $B \subset U$, there exists a sequence $\{\beta_n(B)\}$, $\beta_n \to 0$ as $n \to \infty$, such that if $(t, x_t^n) \in B$ for t between t_o and V_n, then $|G_n(V_n)| \le \beta_n(B)$. Then there exists a non-continuable function $x(\cdot)$ and a subsequence $\{x^{n_j}(\cdot)\}$ of $\{x^n(\cdot)\}$ such that

(i) $x_t^{n_j} \to x_t$ uniformly on compact subset of D_{x_t},

and

(ii) $x(t) = \varphi(o) + \int_{t_o}^{t} f(s, x_s)ds$ for $t \geq t_o$

$x_{t_o} = \varphi$.

Proof: See [4].

0.2. Liapunov Functions

Let $f: I \times D \to R^d$ be continuous, where D is an open set in R^d, and let I denote the interval $0 \leq t \leq \infty$. As in the usual Liapunov theory, see [22], we consider a continuous scalar function $V(t,x)$ defined on an open set S in $I \times D$. Furthermore, we assume that $V(t,x)$ satisfies locally a Lipschitz condition with respect to x. That is, for each point (t_o, x_o) in S, there exists a neighborhood $U = U(t_o, x_o)$ and a positive number $L(U)$ such that

$$|V(t,x) - V(t,y)| \leq L(U)|x - y|$$

for any $(t,x) \in U$, $(t,y) \in U$. We shall denote by $V \in Lip_o(x)$ for this fact.

Corresponding to $V(t,x)$, we define the function

$$\dot{V}_{(E)}(t,x) = \overline{\lim_{h \to o^+}} \frac{1}{h} \{V(t + h, x + hf(t,x)) - V(t,x)\} . \tag{0.5}$$

Let $x(t)$ be a continuous and differentiable function defined for $S \geq t$, denote by $V'(t, x(t))$ the upper right-hand derivative of $V(t, x(t))$, that is,

$$V'(t, x(t)) = \overline{\lim_{h \to o^+}} \frac{1}{h} \{V(t + h, x(t + h)) - V(t,x)\} \tag{0.6}$$

then we have the following, c.f. [22]:

Lemma 0.1. Let $x = x(t)$ be a solution of (0.1) which stays in S. Then

$$V'(t, x(t)) = \dot{V}_{(E)}(t,x) \quad .$$

As is well known, if $\dot{V}_{(E)}(t,x) \leq 0$ then by Lemma 0.1 $V'(t, x(t)) \leq 0$. The function $V(t, x(t))$ is therefore a non-increasing function of t along a solution of (E). Conversely, if $V(t,x)$ is nonincreasing along a solution of (E), then we have

$$\dot{V}_{(E)}(t,x) \leq 0 \ .$$

By a Liapunov function, in usual Liapunov theory, we always mean a continuous scalar-valued function such that $V \in Lip_o(x)$. The following is one of the simplest forms of a very general comparison principle, c.f. [22].

Definition 0.2. For the case $d = 1$ in the equation (E), if φ_M is a solution of (E) passing through (τ,ξ), existing on some interval I containing τ, with the property that every other solution φ of (E) passing through (τ,ξ) and existing on I is such that

$$\varphi(t) \leq \varphi_M(t) \qquad (t \in I)$$

then φ_M is called a maximum solution of (E) on I passing through (τ,ξ).

Theorem 0.1. Let $V(t,x)$ be a Liapunov function for (E). Suppose there exists a real-valued continuous function $w(t,u)$

defined for $0 \le t < \infty$, $|\mu| < \infty$, where u is a scalar, such that for all $(t,x) \in I \times D$

$$(0.7) \qquad \dot{V}_{(E)}(t,x) \le w(t, V(t,x)) .$$

Let $U(t, t_o, u_o)$ be the maximal solution of

$$(0.8) \qquad \mu' = w(t, \mu) , \qquad \mu_o = V(t_o, x_o)$$

and $x(t, t_o, x_o)$ be a solution of (E). Then

$$(0.9) \qquad V(t, x(t, t_o, x_o)) \le U(t, t_o, x_o)$$

for all $t \ge t_o$ for which both $x(t, t_o, x_o)$ and $U(t, t_o, x_o)$ are defined.

The comparison principle has been widely used in dealing with a variety of qualitative problems. It is a very important tool as it reduces the problem of determining the behavior of solution of (E) to the solution of a scalar equation (0.8) and properties of the Liapunov function V.

In this thesis, we shall first develop the Liapunov theory by using lower semi-continuous Liapunov functionals for FDE of the retarded type. In the later chapters we shall investigate the problems of integral stability and invariance of a set in which the Liapunov theory developed earlier will play an important role.

In Chapter I, we shall define the notion of "derivative along a solution" for a class of functionals which are assumed only to be lower semi-continuous and prove a comparison theorem analogous to Theorem 0.1 for FDE.

In Chapter II, we shall deal with the problem of the integral stability for FDE.

In Chapter III, we shall initiate another type of derivative for the same class of functionals as in Chapter I, and we shall also prove another comparison theorem analogous to Theorem 0.1 for FDE.

In Chapter IV, we shall deal with the characterization of invariance of a set for FDE.

For reading convenience, a hollow square □ is used to signal the end of a proof.

CHAPTER I

THE FIRST COMPARISON THEOREM FOR FDE

1.1. Liapunov Functional and Lower semi-continuity

In this chapter, we shall give the definition of "derivative along a solution" for a Liapunov functional, which is a natural extension of (0.6), and prove a comparison principle similar to Theorem 0.1.

Consider the functional differential equation (0.1). Let $x(\cdot)$ be a solution of (0.1) through (σ,φ) and let $V\colon R \times C[-\gamma,0] \to R^*$ where R^* denotes the extended real numbers, we shall refer to V as a Liapunov functional. Throughout this thesis, the term "Liapunov functional" means only a functional substantially different from the usual sense of Liapunov function in which both continuity and local Lipschitz condition are assumed. Define

$$(1.1) \qquad V'(\sigma, x_\sigma) = \limsup_{h\to 0^+} \frac{1}{h}\,[V(\sigma + h, x_{\sigma+h}) - V(\sigma, x_\sigma)] .$$

This extended real-valued function is well-defined for arbitrary V and f.

Let

$$C_A[-\gamma,0] = A\,C\,[-\gamma,0]$$
$$= \{\text{the set of all absolutely continuous function on } [-\gamma,0]\} .$$

Definition 1.1. For $a \geq 0$, a functional $V: [a,\infty) \times C_A[-\gamma,0] \to R^*$ is called lower-semi-continuous if, for every $(t,\varphi) \in [a,\infty) \times C_A[-\gamma,0]$, we have

$$V(t,\varphi) \leq \liminf_{(s,\Psi)\to(t,\varphi)} V(s,\Psi) \tag{1.2}$$

$$\text{for } (s,\Psi) \in [a,\infty) \times C_A[-\gamma,0]$$

where $(s,\Psi) \to (t,\varphi)$ means $|s - t| + \|\Psi - \varphi\| \to 0$.

1.2. A Comparison Principle for FDE of Retarded Type

Assume $\varphi \in C_A[-\gamma,0]$, and let $x: [-\gamma, t + a) \to R$ be absolutely continuous, where $a > 0$, such that $x_t = \varphi$. We define the upper right-hand derivative of $V(t,\varphi)$ along the function $x(t)$ by

$$V'_x(t,\varphi) = \overline{\lim_{h\to o^+}} \frac{1}{h} \{V(t + h, x_{t+h}) - V(t,\varphi)\}$$

then we have the following theorem.

Theorem 1.1. Let $p(t)$ be a continuous function on $[t_o, a]$, where $a > t_o$, and let $V(t,\varphi)$ be a lower semi-continuous Liapunov functional on $[0,\infty) \times C_A[-r,0]$. If $x: [t_o - \gamma, a) \to R^d$ is an absolutely continuous function such that

(a) $x_{t_o} = \varphi$,

(b) $V'_x(t, x_t) \leq p(t)$, $\forall\, t_o \leq t < a$,

then

$$V(T, x_T) - V(t_o, x_{t_o}) \leq \int_{t_o}^{T} p(t)dt \qquad \forall\, t_o \leq T < a .$$

Proof: For fixed T, $t_o \leq T < a$, and a given positive integer n we define $A^n(t, x_t)$ for $t_o \leq t \leq T$ as follows:

$$(1.3) \qquad A^n(t, x_t) = \{\tau \mid t < \tau < t + \frac{1}{n} \text{ and } \frac{V(\tau, x_\tau) - V(t, x_t)}{\tau - t} < p(t) + \frac{1}{n}\} .$$

Since $V'_x(t, x_t) \le p(t)\ \forall\ t_o \le t \le T$, we see that $A^n(t, x_t)$ is not empty. Next we define recursively a sequence $A^n = \{t^n_k\}_{k=1}^{\infty}$ by taking

$$t^n_1 \in A^n(t_o, x_{t_o})$$

and

$$(1.4) \qquad t^n_k \in A^n(t^n_{k-1}, x_{t^n_{k-1}})$$

so that

$$t^n_k - t^n_{k-1} \ge \frac{1}{2} \sup \{\tau - t^n_{k-1} \mid \tau \in A^n(t^n_{k-1}, x_{t^n_{k-1}})\} .$$

Hence the sequence $\{t^n_k\}_{k=1}^{\infty}$ is monotone increasing. Furthermore, from Lemma 1.2, we know there exists an integer $j(n)$ such that $T \le t^n_{j(n)} \le T + \frac{1}{n}$. Thus we have

$$(1.5) \qquad \lim_{n\to\infty} t^n_{j(n)} = T ,$$

and

$$(1.6) \qquad \lim_{n\to\infty} x_{t^n_{j(n)}} = x_T .$$

Finally, it follows from (1.2), (1.5) and (1.6) that

$$(1.7) \qquad V(T, x_T) - V(t_o, x_{t_o})$$
$$\le \liminf_{n\to\infty} V(t^n_{j(n)}, x_{t^n_{j(n)}}) - V(t_o, x_{t_o})$$
$$\le \liminf_{n\to\infty} \sum_{i=1}^{j(n)} [V(t^n_i, x_{t^n_i}) - V(t^n_{i-1}, x_{t^n_{i-1}})] .$$

On the other hand we see from (1.3) and (1.4) that

$$V(t_i^n, x_{t_i^n}) - V(t_{i-1}^n, x_{t_{i-1}^n})$$

$$\leq [p(t_{i-1}^n) + \frac{1}{n}] \quad (t_i^n - t_{i-1}^n) .$$

Substituting into (1.7), we get

$$(1.8) \qquad V(T, x_T) - V(t_o, x_{t_o})$$

$$\leq \liminf_{n\to\infty} \sum_{i=1}^{j(n)} (p(t_{i-1}^n) + \frac{1}{n})(t_i^n - t_{i-1}^n)$$

$$\leq \liminf_{n\to\infty} \sum_{i=1}^{j(n)} p(t_{i-1}^n)(t_i^n - t_{i-1}^n) + \limsup_{n\to\infty} \frac{1}{n} (t_{j(n)}^n - t_o) .$$

Again, from (1.3), (1.4) and (1.5), we have

$$\lim_{n\to\infty} t_{j(n)}^n = T , \quad t_i^n - t_{i-1}^n < \frac{1}{n} .$$

Substituting into (1.8), we conclude that

$$V(T, x_T) - V(t_o, x_{t_o})$$

$$\leq \int_{t_o}^T p(t)dt + \limsup_{n\to\infty} \frac{1}{n} (T - t_o)$$

$$\leq \int_{t_o}^T p(t)dt . \qquad \square$$

<u>Remark</u>. In the proof of Theorem 1.1, only the assumption that x is continuous was used, absolutely continuous is not required.

<u>Lemma</u> 1.2. For the sequence $\{t_k^n\}_{k=1}^{\infty}$ that was defined in (1.4), there exists an integer $j(n)$ such that

$T \le t^n_{j(n)} \le T + \frac{1}{n}$.

Proof. First we claim $\lim_{i\to\infty} t^n_i \ge T$. Suppose not. Let $b_n = \lim_{i\to\infty} t^n_i$ so that $b_n < T$. If $\tau \in A^n(b_n, x_{b_n})$, then we assert that $\tau \in A^n(t^n_i, x_{t^n_i})$ for infinitely many values of i. To see this, we note that

$$\text{(1.9)} \quad \liminf_{i\to\infty} \frac{V(\tau, x_\tau) - V(t^n_i, x_{t^n_i})}{\tau - t^n_i}$$

$$\le \liminf_{i\to\infty} \frac{V(\tau, x_\tau) - V(b_n, x_{b_n})}{\tau - t^n_i} + \limsup_{i\to\infty} \frac{V(b_n, x_{b_n}) - V(t^n_i, x_{t^n_i})}{\tau - t^n_i}$$

$$\le \frac{V(\tau, x_\tau) - V(b_n, x_{b_n})}{\tau - b_n} + \limsup_{i\to\infty} \frac{V(b_n, x_{b_n})}{\tau - t^n_i}$$

$$- \liminf_{i\to\infty} \frac{V(t^n_i, x_{t^n_i})}{\tau - t^n_i} .$$

But, $b_n = \lim_{i\to\infty} t^n_i$, so it follows from (1.2) and (1.9), that we have

$$\text{(1.10)} \quad \liminf_{i\to\infty} \frac{V(\tau, x_\tau) - V(t^n_i, x_{t^n_i})}{\tau - t^n_i} \le \frac{V(\tau, x_\tau) - V(b_n, x_{b_n})}{\tau - b_n}$$

$$< p(b_n) + \frac{1}{n} .$$

Furthermore, p is a continuous function and $t^n_i \to b_n$, so that from (1.10), we see that there must be a J_1 such that

$$\liminf_{i\to\infty} \frac{V(\tau, x_\tau) - V(t^n_i, x^n_{t_i})}{\tau - t^n_i} < p(t^n_j) + \frac{1}{n} \quad \text{whenever} \quad j \ge J_1 .$$

Therefore there exists infinitely many values of $i \geq J_1$ such that

$$\frac{V(\tau, x_\tau) - V(t_i^n, x_{t_i^n})}{\tau - t_i^n} < p(t_i^n) + \frac{1}{n} \quad ,$$

and

$$t_i^n < \tau < t_i^n + \frac{1}{n} \quad .$$

Thus $\tau \in A^n(t_i^n, x_{t_i^n})$ for infinitely many values of i.

And hence there exists a t_k^n such that

$$\tau \in A^n(t_k^n, x_{t_k^n}) \quad ,$$

and

$$b_n - t_k^n < \frac{1}{2}(\tau - b_n) < \frac{1}{2}(\tau - t_k^n) \quad .$$

But this contradicts the choice of t_{k+1}^n, since $t_{k+1}^n - t_k^n \geq \frac{1}{2}(\tau - t_k^n) > b_n - t_k^n$ so that $t_{k+1}^n > b_n$. Consequently we have

$$\lim_{i \to \infty} t_i^n \geq T \quad . \qquad \square$$

1.3. C_1 Locally Lipschitzian

For $\varphi \in C_A[-\gamma, 0]$, define the C_1 norm of φ by

$$\|\varphi\|_{C_1} = \|\varphi\| + \int_{-\gamma}^{0} |\varphi'| d\theta \quad .$$

The following property of the functional $V(t,\varphi)$ is important, especially in the study of the behavior of solutions of perturbed systems.

Definition 1.3. Let $V(t,\varphi)$ be a functional as before. We say V is C_1 locally Lipschitzian if for every $(t_o, \varphi_o) \in D$, where $\varphi \in C_A[-\gamma, 0]$, there exists a neighborhood $N(t_o, \varphi_o)$ of (t,φ) and a constant $L = L(N(t_o, \varphi_o)) \geq 0$ such that

$$(1.11) \qquad |V(\bar{t}, \varphi_1) - V(\bar{t}, \varphi_2)| \leq L\|\varphi_1 - \varphi_2\|_{C_1}$$

for all $(\bar{t}, \varphi_1), (\bar{t}, \varphi_2) \in N(t_o, \varphi_o)$ and $\varphi_1, \varphi_2 \in C_A[-\gamma, 0]$.

Lemma 1.3. For two continuous functions $x(t,\varphi)$, $y(\tau,\varphi)$ with the right-hand derivatives such that $x_t = y_\tau = \varphi$, we have

$$\overline{\lim_{\delta\to o^+}} \frac{1}{\delta} \|x_{t+\delta}(t,\varphi) - y_{\tau+\delta}(\tau,\varphi)\| = |x'(t^+) - y'(\tau^+)| .$$

Proof. See [22], page 187.

The following lemma is immediate.

Lemma 1.4. If $V(t,\varphi)$ is C_1 locally Lipschitzian and $x(t)$, $y(t)$ are two absolutely continuous functions defined on $\sigma - \gamma \leq t \leq a + \sigma$, $a > 0$, such that $x_\sigma = y_\sigma = \varphi$; then

$$V_x'(\sigma,\varphi) \leq V_y'(\sigma,\varphi) + 2L|x'(\sigma^+) - y'(\sigma^+)|,$$

where $x'(\sigma^+)$, $y'(\sigma^+)$ denote the right hand derivative at $t = \sigma$, and L is the constant in (1.11) at the point (σ,φ).

Proof.

$$\begin{aligned} V_x'(\sigma,\varphi) &= \limsup_{h\to o^+} \frac{1}{h} \{V(\sigma + h, x_{\sigma+h}) - V(\sigma,\varphi)\} \\ &\leq \limsup_{h\to o^+} \frac{1}{h} \{V(\sigma + h, x_{\sigma+h}) - V(\sigma + h, y_{\sigma+h})\} \\ &\quad + \limsup_{h\to o^+} \frac{1}{h} \{V(\sigma + h, y_{\sigma+h}) - V(\sigma,\varphi)\} \end{aligned}$$

$$\leq V_y'(\sigma,\varphi) + \limsup_{h\to o^+} \frac{1}{h} \{L \cdot \|x_{\sigma+h} - y_{\sigma+h}\|_{C_1}\}$$

(1.12)

$$\leq V_y'(\sigma,\varphi) + \limsup_{h\to o^+} \frac{1}{h} \{L\|x_{\sigma+h} - y_{\sigma+h}\|\}$$

$$+ \limsup_{h\to o^+} \frac{1}{h} \{L \int_\sigma^{\sigma+h} |x'(\theta) - y'(\theta)|\, d\theta\} .$$

But from Lemma 1.3 we have

$$(1.13) \quad \limsup_{h\to o^+} \frac{1}{h} \{L\|x_{\sigma+h} - y_{\sigma+h}\|\} = L|x'(\sigma^+) - y'(\sigma^+)| .$$

Substitute (1.13) into (1.12), we conclude that

$$V_x'(\sigma,\varphi) \leq V_y'(\sigma,\varphi) + L|x'(\sigma^+) - y'(\sigma^+)|$$

$$+ L|x'(\sigma^+) - y'(\sigma^+)|$$

$$\leq V_y'(\sigma,\varphi) + 2L|x'(\sigma^+) - y'(\sigma^+)| . \qquad \square$$

CHAPTER II

INTEGRAL STABILITY OF FDE

2.1. Definition of Integral Stability

Consider the vector ordinary differential equation

$$x'(t) = f(t, x), \tag{2.1}$$

for which the identically zero function is a solution, i.e. $f(t,0) = 0$ for all time t; we denote this special solution simply by 0. Now suppose one knows that (2.1) is stable, i.e. all the solutions of (2.1) which start near 0 remain near 0 for all future time. If the differential equation (2.1) is subject to certain small perturbations, the above property concerning the solutions near 0 may or may not remain true. A more precise formulation of this problem is as follows: If 0 is stable for (2.1) and if the function $p(t)$ is small in some sense, give condition on f so that 0 is stable for the perturbed equation

$$x'(t) = f(t, x) + p(t) .$$

A great deal of work has been done in an attempt to provide positive answers to this problem. Historically, there have been two approaches. One approach is to impose conditions on f, such as being uniformly Lipschuitz, and find out what kind of perturbations $p(t)$ preserve stability [e.g. [3], Chapter 13]. The second approach is to restrict the type of perturbations $p(t)$ that will be allowed, e.g. $\int_0^\infty |p(t)|dt < \infty$, and find out which differential equations (2.1) will have their stability preserved by all such $p(t)$.

As to the second approach, Vrkoč [18], Okamura [15], Yozhizawa [22], Chow and Yorke [2], etc. all have made tremendous contributions to our understanding of this problem.

It is our intention in this chapter to consider the corresponding problem for FDE in the spirit of the second approach.

We shall consider the FDE

$$x' = f(t, x_t) , \tag{2.2}$$

where (i) $f: D \to R^d$ is continuous and

$$D \subset R \times C[-\gamma, 0] ,$$

(ii) f takes closed bounded sets into bounded sets,

(iii) f is uniformly continuous in φ for all of t, i.e. $\forall\ \epsilon > 0\ \ \exists\ \delta > 0$ such that $|f(t, \Psi) - f(t, \varphi)| < \epsilon$ whenever $\|\varphi - \Psi\| < \delta$, and

(iv) $f(t, 0) \equiv 0$,

(v) The solution of (2.2) is unique.

<u>Definition</u> 2.1. The zero solution of (2.2) is integral stable if for any $\epsilon > 0$, any $t_o \geq 0$ and any continuous function $p: [t_o, \infty) \to R$, there exists a $\delta(\epsilon) > 0$ such that $\|\varphi_o\| < \delta(\epsilon)$ and $\int_{t_o}^\infty |p(t)|dt < \delta(\epsilon)$ imply $|y(t, t_o, \varphi_o)| < \epsilon$ for all $t \geq t_o$, where $y(t, t_o, \varphi_o)$ denotes a solution of

(2.3) $$y'(t) = f(t, y_t) + p(t)$$

that passes through (t_o, φ_o).

It is the purpose of this chapter to give a necessary and sufficient condition for the zero solution of (2.2) to be integral stable.

2.2. Definition and Properties of V_L Function

For an open set $U \subset R \times C_A[-\gamma,0]$ let $V: U \to R$ and denote by

$$N((\tau,\Psi), \delta) = \{(t,\varphi) \in U: |t - \tau| + \|\varphi - \Psi\| \le \delta\}$$

for all $(\tau, \Psi) \in U$ and $\delta > 0$. Then the following is immediate.

Lemma 2.1. $\liminf_{(t,\varphi)\to(\tau,\Psi)} V(t,\varphi) \ge A$ if, and only if, for each $\varepsilon > 0$ there is a $\delta > 0$ such that $V(t,\varphi) \ge A - \varepsilon$, whenever $(t,\varphi) \in N((\tau,\Psi), \delta)$.

Next we define the function $V_L: U \to R$ by

(2.4) $$V_L(t,\varphi) = \liminf_{(\tau,\Psi)\to(t,\varphi)} V(\tau, \Psi) ,$$

where $(\tau,\Psi), (t,\varphi) \in U$.

Remark: Since U is an open subset of $[o,\infty) \times C_A$, therefore for any $(t,\varphi) \in U$, there exists a sequence $\{(t_n, \varphi_n)\}_{n=1}^{\infty} \subset U$ such that

$$\lim_{n\to\infty} V(t_n, \varphi_n) = V_L(t, \varphi) .$$

Then we have the following lemma.

Lemma 2.2. The function V_L is a lower semi-continuous function on U.

Proof. Let $(t,\varphi) \in U$. We would like to prove

$$\liminf_{(\tau,\Psi)\to(t,\varphi)} V_L(\tau,\Psi) \geq V_L(t,\varphi) \qquad \forall\ (t,\varphi) \in U ,$$

i.e., for any given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$V_L(\tau,\Psi) \geq V_L(t,\varphi) - \varepsilon \text{ whenever } (\tau,\Psi) \in N((t,\varphi), \delta) .$$

It follows from (2.4) that there must exist a $\delta_1 > 0$ such that

$$V(\tau,\Psi) \geq V_L(t,\varphi) - \varepsilon/2 \text{ whenever } (\tau,\Psi) \in N((t,\varphi), \delta_1).$$

Choosing $\delta = \frac{\delta_1}{2}$ we will now show that

$$V_L(\tau,\Psi) \geq V_L(t,\varphi) - \varepsilon/2 \text{ whenever } (\tau,\Psi) \in N((t,\varphi), \delta) .$$

Since $(\tau,\Psi) \in N((t,\varphi), \delta)$, we can find a $\delta_3 > 0$ such that

$$N((\tau, \Psi), \delta_3) \subset N((t, \varphi), \delta_1) .$$

Hence

$$V(\bar{\tau},\bar{\Psi}) \geq V_L(t,\varphi) - \varepsilon/2 \text{ whenever } (\bar{\tau},\bar{\Psi}) \in N((\tau,\Psi), \delta_3),$$

i.e.

$$V_L(\tau, \Psi) \geq V_L(t, \varphi) - \varepsilon/2 .$$

Hence we have

$$V_L(\tau,\Psi) \geq V_L(t,\varphi) - \varepsilon \text{ whenever } (\tau,\Psi) \in N((t,\varphi), \delta),$$

i.e.,

$$\liminf_{(\tau,\Psi)\to(t,\varphi)} V_L(\tau,\Psi) \geq V_L(t,\varphi)$$

so V_L is lower semi-continuous. □

Theorem 2.1. Let $U \subset R \times C_A[-\gamma,0]$ be open and let $W_1, W_2 \colon U \to R$ be continuous. Suppose $V \colon U \to R$ is any arbitrary function such that

$$(2.5) \qquad W_1(t,\varphi) \le V(t,\varphi) \le W_2(t,\varphi)$$

for all $(t,\varphi) \in U$. Then the function $V_L \colon U \to R$ defined by (2.4) is lower semi-continuous and satisfies

$$(2.6) \qquad W_1(t,\varphi) \le V_L(t,\varphi) \le W_2(t,\varphi)$$

for all $(t,\varphi) \in U$.

Proof. The semi-continuity of V_L follows directly from Lemma 2.2. Next we note from (2.5) that

$$\liminf_{(\tau,\Psi)\to(t,\varphi)} W_1(\tau,\Psi) \le \liminf_{(\tau,\Psi)\to(t,\varphi)} V(\tau,\Psi) \le \liminf_{(\tau,\Psi)\to(t,\varphi)} W_2(\tau,\Psi)$$

$$\forall\ (t,\varphi) \in U .$$

Since W_1 and W_2 are continuous, (2.6) follows. □

2.3. The V Functional and its Special Properties

Let

$$C_A^H = \{\varphi \in C_A[-\gamma,0] : \|\varphi\| \le H\} ,$$

$$C_A^H(I) = \{\varphi \in AC(I) : \sup_{t\in I}|\varphi(t)| \le H\} ,$$

$$C^H = \{\varphi \in C[-\gamma,0] : \|\varphi\| \le H\} ,$$

$$C^H(I) = \{\varphi \in C(I) : \sup_{t\in I}|\varphi(t)| \le H\} .$$

For $(t,\varphi) \in [2\gamma,\infty) \times C_A^H$, we set

$$A_H(t,\varphi) = \{\Psi \in C_A[-r,t] \cap C_A^H[0,t] : \Psi \equiv \varphi \text{ on } [t-\gamma, t],$$

$$\Psi \equiv 0 \text{ on } [-\gamma, 0]\} ,$$

and define

$$(2.7) \qquad V(t,\varphi) = \inf_{\Psi \in A_H(t,\varphi)} \int_0^t |\Psi' - f(\mu, \Psi_\mu)| d\mu .$$

We have the following:

Lemma 2.3. Suppose $x(t, \sigma, \varphi)$, $(\sigma, \varphi) \in [0,\infty) \times C^H$ is a solution of (2.2). Then for $t \geq 2\gamma$ and $\sup_{s \in [\sigma,t]} |x(s,\sigma,\varphi)| \leq H$, $V(t, x_t(\sigma,\varphi))$ is a nonincreasing function of t.

Proof. For $t > s \geq 2\gamma \geq \sigma$, $\sup_{s \in [\sigma,t]} |x(t,\sigma,\varphi)| \leq H$, we want to prove $V(t, x_t(\sigma,\varphi)) \leq V(s, x_s(\sigma,\varphi))$. It follows from (2.7) that there exists a sequence of functions $\{\Psi_n\}$ in $A_H(s, x_s(\sigma,\varphi))$ such that

$$V(s, x_s(\sigma,\varphi)) = \lim_{n\to\infty} F_n$$

$$= \lim_{n\to\infty} \int_0^s |\Psi_n'(\mu) - f(\mu,\Psi_n)| d\mu .$$

Next for each $\Psi_n \in A_H(s, x_s(\sigma,\varphi))$, we define

$$\begin{aligned} \eta_n(\mu) &= \Psi_n(\mu) \qquad && -\gamma \leq \mu \leq s \\ &= x(\mu) && s \leq \mu \leq t . \end{aligned}$$

Then we see that $\eta_n \in A_H(t, x_t(\sigma,\varphi))$ and

$$\eta_n'(\mu) = f(\mu, \eta_{n,\mu}) , \qquad s \leq \mu \leq t,$$

so that

$$\int_0^t |\eta_n'(\mu) - f(\mu, \eta_{n,\mu})| d\mu$$

$$= \int_0^s |\Psi_n'(\mu) - f(\mu, \Psi_{n,\mu})| d\mu = F_n .$$

Consequently,

$$V(t, x_t(\sigma,\varphi)) = \inf_{\Psi \in A(t, x_t(\sigma,\varphi))} \int_0^t |\Psi'(\mu) - f(\mu, \Psi_n)| d\mu$$

$$\leq \lim_{n\to\infty} F_n \leq V(s, x_s(\sigma,\varphi)) . \qquad \square$$

<u>Lemma</u> 2.4. Let $U \subset [o, \infty) \times C_A^H$ be open and $(t_n, \varphi_n) \to (\tau, \Psi)$ in U. If $x^n(t, t_n, \varphi_n)$, $x(t, \tau, \Psi)$ are solutions of (2.2) that pass through (t_n, φ_n), (τ, Ψ) respectively; then for small $h > 0$ there exist a subsequence of $\{(t_n, \varphi_n)\}$, which we denote also by $\{(t_n, \varphi_n)\}$, such that

$$(t_n + h, x^n_{t_n+h}) \to (\tau + h, x_{\tau+h}) .$$

Proof. See [4], or [8] . $\square$

<u>Lemma</u> 2.5. Suppose $x(t, \sigma, \varphi)$, $(\sigma,\varphi) \in [o,\infty) \times C^H$ is a solution of (2.2). Then for $t \geq 2\gamma$ and $\sup_{s\in[\sigma,t]} |x(s,\sigma,\varphi)| \leq H$, $V_L(t, x_t(\sigma,\varphi))$ is a nonincreasing function of t.

Proof. Suppose $t_1 > t_2 \geq 2\gamma$ and $\sup_{s\in[\sigma,t]} |x(s,\sigma,\varphi)| \leq H$. Let $\{(t^n, \varphi^n)\}$ be a sequence in $[o,\infty) \times C_A^H$ such that

$$V_L(t_2, x_{t_2}) = \lim_{n\to\infty} V(t^n, \varphi^n) , \tag{2.8}$$

and

$$(t^n, \varphi^n) \to (t_2, x_{t_2}) .$$

Let $x^n(t, t^n, \varphi^n)$ be a solution of (2.2) that passes through (t^n, φ^n). If $|t_1 - t_2|$ small enough, then by Lemma 2.4, there will exist a subsequence of $\{(t^n, \varphi^n)\}$, which we denote also by $\{(t^n, \varphi^n)\}$ such that

$$(t^n + t_1 - t_2, x^n_{t^n+t_1-t_2}) \to (t_1, x_{t_1}) .$$

Thus from (2.4) and (2.8), we have

$$\begin{aligned} & V_L(t_1, x_{t_1}) - V_L(t_2, x_{t_2}) \\ (2.9) \qquad & \le \liminf_{n\to\infty} V(t^n + t_1 - t_2, x^n_{t^n+t_1-t_2}) - \lim_{n\to\infty} V(t^n, x^n_{t^n}) \\ & \le \liminf_{n\to\infty} [V(t^n + t_1 - t_2, x^n_{t^n+t_1-t_2}) - V(t^n, x^n_{t^n})] . \end{aligned}$$

But from Lemma 2.3 we have

$$(2.10) \qquad V(t^n + t_1 - t_2, x^n_{t^n+t_1-t_2}) - V(t^n, x^n_{t^n}) \le 0 .$$

Substitute (2.10) into (2.9). We conclude that

$$V_L(t_1, x_{t_1}) - V_L(t_2, x_{t_2}) \le 0 . \qquad \square$$

Lemma 2.6. For $\tau > 2\gamma$ and $\varphi \in C_A^H$, there exists a solution $x(t)$ of (2.2) such that $x_0 \equiv 0$, $x_\tau = \varphi$ and $|x(t)| \le H$ for $0 \le t \le \tau$ if, and only if, $V(\tau,\varphi) = 0$.

Proof. First suppose $x(t)$ is a solution of (2.2) such that $x_\tau = \varphi$, then by definition of $V(\tau,\varphi)$ we have $V(\tau,\varphi) = 0$. Next assume $V(\tau,\varphi) = 0$. We would like to show that there exists a solution of (2.2) such that $x_0 \equiv 0$, $x_\tau = \varphi$ and $\|x_t\| \le H$ for $0 \le t \le \tau$. Now since $V(\tau,\varphi) = 0$, it follows from the definition of V, that there exists a sequence of absolutely continuous functions $\{x_k(t)\}$ where $x_k(t) \in A_H(\sigma,\varphi)$ and such that

$$\lim_{k\to\infty} \int_0^\tau |x_k' - f(\mu, x_{k,\mu})| d\mu = 0 \ . \tag{2.11}$$

Set $\varphi_k(t) = x_k(t) - \int_0^t f(\mu, x_{k,\mu}) d\mu$ for $0 \le t \le \tau$. Since

$$|\varphi_k(t)| = |x_k(t) - \int_0^t f(\mu, x_{k,\mu}) d\mu|$$

$$\le \int_0^\tau |x_k' - f(\mu, x_{k,\mu})| d\mu \ ,$$

it follows from (2.11) that $\varphi_k(t)$ must converge uniformly to zero on $[o,\tau]$.

Setting $z_k(t) = x_k(t) - \varphi_k(t)$, then for t_1, t_2 such that $0 \le t_1 \le t_2 \le \tau$, we have

$$z_k(t_2) - z_k(t_1) = \int_{t_1}^{t_2} f(\mu, x_{k,\mu}) d\mu$$

and

$$|z_k(t_2) - z_k(t_1)| \le M(\tau)(t_2 - t_1)$$

where $M(\tau) = \text{Max}\{|f(t, x_t)| : 0 \le t \le \tau, \text{ and } |x| \le H\}$. Thus $\{z_k(t)\}$ is uniformly bounded and equi-continuous. By Ascoli's theorem, there exists a uniformly convergent subsequence, which we denote by $\{z_k(t)\}$ again such that

$$x(t) = \lim_{k\to\infty} z_k(t) \quad \text{uniformly on} \quad -\gamma \le t \le \tau \ .$$

Then clearly $x_o \equiv 0$ and $x_\tau = \varphi$ and also $x(t) = \int_0^t f(\mu, x_\mu) d\mu$, since $\varphi_k(t) \to 0$ uniformly as $k \to \infty$. Thus $x_k(t) \to x(t)$ uniformly as $k \to \infty$. This shows the existence of a solution $x(t)$ such that $x_o \equiv 0$, $x_\tau = \varphi$ and $|x(t)| \le H$ since $\varphi_k(t)$ converge uniformly to zero on $[o,\tau]$, and $|x_k(t)| \le H$. □

Lemma 2.7. Let $x(t), y(t) \in C_H^A[\tau - \gamma, \tau + a]$, where $a > o$, $\tau \geq 2\gamma$ such that $x(\theta) = y(\theta)$ $\forall\ \tau - \gamma \leq \theta \leq \tau$. Then

$$
\begin{aligned}
&|V(s, x_s) - V(s, y_s)| \\
(2.12) \qquad &\leq \|x_s - y_s\|_{c_1} + M(x,y,s) \quad \forall\ \tau \leq s < a + \tau ,
\end{aligned}
$$

where $M(x,y,s)$ is a positive number depending on x, y and s such that $\lim_{s \to \tau} \frac{M(x,y,s)}{s - \tau} = 0$.

Proof. For $\tau < s < a + \tau$, let $\{\varphi_k\}$ be a sequence of functions, such that

(1) $\varphi_k \in A_H(s, x_s)$,

and

(2) $\lim_{n \to \infty} F_n = V(s, x_s)$, where $F_n = \int_o^s |\varphi_k'(\mu) - f(\mu, \varphi_{k,\mu})| d\mu$.

For each φ_k define $\overline{\varphi_k}$ as follows

$$
\begin{aligned}
\overline{\varphi}_k(\theta) &= \varphi_k(\theta) , && \theta \leq \tau , \\
&= y(\theta) , && \tau \leq \theta \leq s .
\end{aligned}
$$

Then

$$
\begin{aligned}
V(s, y_s) &\leq \int_o^s |\overline{\varphi}_k' - f(\mu, \overline{\varphi}_{k,\mu})| d\mu \\
&\leq \int_o^s |\varphi_k' - f(\mu, \varphi_{k,\mu})| d\mu \\
&+ \int_\tau^s |\overline{\varphi}_k' - f(\mu, \overline{\varphi}_{k,\mu}) - \varphi_k' + f(\mu, \varphi_{k,\mu})| d\mu \\
&\leq \int_o^s |\varphi_k' - f(\mu, \varphi_{k,\mu}) d\mu + \int_\tau^s |\overline{\varphi}_k' - \varphi_k'| d\mu \\
&+ \int_\tau^s |f(\mu, \overline{\varphi}_{k,\mu}) - f(\mu, \varphi_{k,\mu})| d\mu \\
&\leq \int_o^s |\varphi_k' - f(\mu, \varphi_{k,\mu})| d\mu + \int_\tau^s |x' - y'| d\mu \\
&+ \int_\tau^s |f(\mu, \overline{\varphi}_{k,\mu}) - f(\mu, \varphi_{k,\mu})| d\mu .
\end{aligned}
$$

Hence

$$V(s, y_s) \le V(s, x_s) + \int_\tau^s |x' - y'| d\mu + \sup_k \int_\tau^s |f(\mu, \bar{\varphi}_{k,\mu})$$

(2.13)
$$- f(\mu, \varphi_{k,\mu})| d\mu$$

$$\le V(s, x_s) + \|x_s - y_s\|_{c_1} + M(x, y, s)$$

where

$$M(x,y,s) = \sup_k \int_\tau^s |f(\mu, \bar{\varphi}_{k,\mu}) - f(\mu, \varphi_{k,\mu})| d\mu .$$

It remains to show that

$$\lim_{s\to\tau} \frac{M(x,y,s)}{s-\tau} = 0 .$$

Since f is uniformly continuous in φ for all of t, it follows that for each $\epsilon > 0$, there exists a $\delta > 0$ such that

(2.14) $$|f(\mu, \bar{\varphi}_{k,\mu}) - f(\mu, \varphi_{k,\mu})| < \epsilon \quad \text{whenever} \quad 0 < \mu - \tau < \delta .$$

From (2.13) and (2.14), we then have

$$M(x,y,s) = \sup_k \int_\tau^s |f(\mu, \bar{\varphi}_{k,\mu}) - f(\mu, \varphi_{k,\mu})| d\mu$$

$$\le (s - \tau) \cdot \epsilon \quad \text{whenever} \quad |s - \tau| < \delta ,$$

and the result follows. □

<u>Lemma</u> 2.8. Let $x(t), y(t) \in C_A^H[\tau - \gamma, \tau + a)$ where $a > 0$, $\tau \ge 2\gamma$ such that $x(\theta) = y(\theta)$ $\forall\ \tau - \gamma \le \theta \le \tau$. Then $|V_L(s, x_s) - V_L(s, y_s)| \le \|x_s - y_s\|_{c_1} + M(x,y,s)$ $\forall\ \tau \le s \le a + \tau$, where $M(x,y,s)$ is a positive number depends on x, y and s such that $\lim_{s\to\tau} \frac{M(x,y,s)}{s-\tau} = 0$.

Proof. Assume

$$V_L(s, x_s) = \lim_{n\to\infty} V(t_n, \varphi_n),$$

where

$$(t_n, \varphi_n) \to (s, x_s) \quad \text{in} \quad [o, \infty) \times C_A^H .$$

Set

$$\Psi(\theta) = y(s+\theta) - x(s+\theta) \quad \forall\ -\gamma \le \theta \le 0 ,$$

and

$$\bar{\phi}_n = \phi_n + \Psi$$

$$x_n(t) = \varphi(t - t_n) \quad \text{for} \quad t_n - \gamma \le t \le t_n$$

$$= x(t) + \varphi_n(-\gamma) - x(t_n - \gamma) \quad \text{for} \quad \tau - \gamma \le t \le t_n - \gamma,$$

$$y_n(t) = \bar{\phi}(t - t_n) \quad \text{for} \quad t_n - \gamma \le t \le t_n$$

$$= x(t) + \phi_n(-\gamma) - x(t_n - \gamma) \quad \text{for} \quad \tau - \gamma \le t \le t_n - \gamma.$$

Then

$$(t_n, \bar{\varphi}_n) \to (s, y_s)$$

and

$$V_L(s, y_s) - V_L(s, x_s)$$

(2.15)
$$\le \liminf_{n\to\infty} V(t_n, \bar{\varphi}_n) - \lim_{n\to\infty} V(t_n, \varphi_n)$$

$$\le \liminf_{n\to\infty} [V(t_n, \bar{\varphi}_n) - V(t_n, \varphi_n)] .$$

But from (2.12) we have

(2.16)
$$V(t_n, \bar{\varphi}_n) - V(t_n, \varphi_n)$$

$$\le \int_{t_n - s + \tau}^{t_n} |x_n' - y_n'|\, d\mu + M(x_n, y_n, t_n) .$$

Substituting (2.16) into (2.15) we have

$$V_L(s, y_s) - V_L(s, x_s) \le \liminf_{n\to\infty} \{\int_{t_n - s+\tau}^{t_n} |x_n' - y_n'| d\mu + M(x_n, y_n, t_n)\}$$

$$\le \int_\tau^s |x' - y'| d\mu + M(x, y, s)$$

where

$$M(x, y, s) = \limsup_{n\to\infty} M(x_n, y_n, t_n) .$$

By using the same technique as in Lemma 2.7, we can prove that

$$\lim_{s\to\tau} \frac{M(x,y,s)}{s-\tau} = 0 . \qquad \square$$

Lemma 2.9. Let $x(t), y(t) \in C_A^H[\tau - \gamma, \tau + a)$ where $a > 0$, $\tau \ge 2\gamma$ such that

$$x(\theta) = y(\theta) \quad \forall\, \tau - \gamma \le \theta \le \tau ,$$

then

$$V_y'(\sigma, y_\sigma) \le V_x'(\sigma, x_\sigma) + 2|x'(\sigma^+) - y'(\sigma^+)| .$$

Proof. From (1.1) we have

$$V_y'(\sigma, y_\sigma) = \limsup_{h\to o^+} \frac{1}{h} \{V(\sigma+h, y_{\sigma+h}) - V(t, y_t)\}$$

$$\text{(2.17)} \qquad \le \limsup_{h\to o^+} \frac{1}{h} \{V(\sigma+h, x_{\sigma+h}) - V(\sigma, x_\sigma)\}$$

$$+ \limsup_{h\to o^+} \frac{1}{h} \{|V(\sigma+h, y_{\sigma+h}) - V(\sigma+h, x_{\sigma+h})|\} .$$

On the other hand, from Lemma 2.7 we have

$$\text{(2.18)} \qquad |V(\sigma + h, y_{\sigma+h}) - V(\sigma + h, x_{\sigma+h})|$$

$$\le \|y_{\sigma+h} - x_{\sigma+h}\|_{c_1} + M(x, y, \sigma + h)$$

where

$$\lim_{h\to o^+} \frac{M(x, y, \sigma + h)}{h} = 0 .$$

Substituting (2.18) into (2.17) we get

$$V_y'(\sigma, y_\sigma) \le V_x'(\sigma, x_\sigma) + \limsup_{h\to o^+} \frac{\|y_{\sigma+h} - x_{\sigma+h}\|_{c_1}}{h}$$

$$(2.19) \qquad \le V_x'(\sigma, x_\sigma) + \limsup_{h\to o^+} \frac{\|y_{\sigma+h} - x_{\sigma+h}\|}{h}$$

$$+ \limsup_{h\to o^+} \frac{\int_\sigma^{\sigma+h} |y'_{\sigma+h} - x'_{\sigma+h}|\, d\theta}{h} .$$

But, we see from Lemma 1.3 that

$$(2.20) \quad \limsup_{h\to o^+} \frac{\|y_{\sigma+h} - x_{\sigma+h}\|}{h} = |x'(\sigma^+) - y'(\sigma^+)| .$$

Substituting (2.20) into (2.19), we conclude that

$$V_y'(\sigma, y_\sigma) \le V_x'(\sigma, x_\sigma) + 2|x'(\sigma^+) - y'(\sigma^+)| . \qquad \square$$

<u>Lemma</u> 2.10. For fixed $\tau \ge 2\gamma$; $(\tau, \varphi) \in [o,\infty) \times C_A^H$, then $V(\tau, \varphi) \to 0$ as $\|\varphi\|_{c_1} \to 0$.

Proof. Set $\bar\varphi(t) = 0 \qquad -\gamma \le t \le 0$

$= \text{linear from } 0 \text{ to } \varphi(-\gamma) \qquad 0 \le t \le \tau - \gamma$

$= \varphi(t - \tau) \qquad \tau - \gamma \le t \le \tau .$

Then $V(\tau, \varphi) \le \int_o^\tau |\bar\varphi'(\mu) - f(\mu, \bar\varphi_\mu)|\, d\mu$

$$\le \int_o^\tau |\bar\varphi'(\mu)| + \int_o^\tau |f(\mu, \bar\varphi_\mu)|\, d\mu$$

$$\le \tau \cdot \mathrm{Max}(\|\varphi\|_{c_1}, \varphi(-\gamma)/\tau-\gamma) + \int_o^\tau |f(\mu, \bar\varphi_\mu)|\, d\mu .$$

Since f is uniformly continuous in φ for all of t and $f(t, o) = 0$ we have

$$\int_o^\tau |f(\mu, \bar{\varphi}_\mu)| d\mu \to 0 \quad \text{as} \quad \|\varphi\|_{c_1} \to 0 .$$

Hence $\quad V(\tau, \varphi) \to 0 \quad \text{as} \quad \|\varphi\|_{c_1} \to 0$.

Lemma 2.11. For fixed $\tau \geq 2\gamma$, if $(\tau, \varphi) \in [o, \infty) \times C_A^H$, then $V_L(\tau, \varphi) \to 0$ as $\|\varphi\|_{c_1} \to 0$.

Proof. It follows from Lemma 2.10 that

$$V(\tau, \varphi) \to 0 \quad \text{as} \quad \|\varphi\|_{c_1} \to 0 .$$

On the other hand $V(\tau, \varphi) \geq 0 \quad \forall\ (\tau, \varphi) \in U$ so that

$$V_L(\tau, \varphi) \to 0 \quad \text{as} \quad \|\varphi\|_{c_1} \to 0 . \qquad \square$$

Lemma 2.12. Let $y(t) \in C[\alpha - \gamma, \beta] \cap C_A^H[\alpha, \beta]$, then for given $\varepsilon > 0$, there exists a function $x(t)$ with its derivative $x'(t)$ continuous on $\alpha \leq t \leq \beta$ and $x(t) \equiv y(t)$ on $\alpha - \gamma \leq t \leq \alpha$ such that

$$\left| \int_\alpha^\beta |x'(t) - f(t, x_t)| dt - \int_\alpha^\beta |y'(t) - f(t, y_t) dt| \right| < \varepsilon$$

and $\|x_\beta - y_\beta\|_{c_1} < \varepsilon$.

Proof. Given $\varepsilon > 0$, choose $0 < \delta(\varepsilon) < \varepsilon$ such that

(2.21) $\quad |f(t,\varphi) - f(t,\Psi)| < \frac{\varepsilon}{2(\beta-\alpha)}$ for all $\alpha \leq t \leq \beta$

$$\text{whenever} \quad \|\varphi(t) - \Psi(t)\| < \delta(\varepsilon) .$$

Since $y'(t)$ is integrable, there exists a continuous function

$\mu(t)$ such that

$$\int_{\alpha}^{\beta} |y'(t) - \mu(t)| dt < 1/2\ \delta(\epsilon) .$$

Set

$$x(t) = y(\alpha) + \int_{\alpha}^{t} \mu(s) ds, \qquad \text{for} \quad \alpha \le t \le \beta,$$

$$= y(t), \qquad \text{for} \quad \alpha - \gamma \le t \le \alpha .$$

Since

$$y(t) = y(\alpha) + \int_{\alpha}^{t} y'(s) ds \qquad \text{for} \quad \alpha \le t \le \beta$$

we see that

$$(2.22) \qquad |y(t) - x(t)| \le \int_{\alpha}^{t} |y'(s) - \mu(s)| ds < \frac{1}{2} \delta(\epsilon), \ \alpha \le t \le \beta ,$$

and

$$(2.23) \qquad \int_{\beta-\gamma}^{\beta} |y'(t) - x'(t)| dt = \int_{\beta-\gamma}^{\beta} |y'(t) - \mu(t)| dt < \frac{1}{2} \delta(\epsilon) .$$

Thus by using (2.21), (2.22) and (2.23), we conclude that

$$\|y_{\beta} - x_{\beta}\|_{C_1} \le \delta(\epsilon) < \epsilon$$

and

$$\left| \int_{\alpha}^{\beta} |x'(t) - f(t, x_t)| dt - \int_{\alpha}^{\beta} |y'(t) - f(t, y_t)| dt \right|$$

$$\le \int_{\alpha}^{\beta} |x'(t) - y'(t)| dt + \int_{\alpha}^{\beta} |f(t, x_t) - f(t, y_t)| dt$$

$$< \frac{1}{2} \delta(\epsilon) + (\beta - \alpha) \cdot \frac{\epsilon}{2(\beta-\alpha)}$$

$$< \frac{1}{2} \epsilon + \frac{1}{2} \epsilon = \epsilon . \qquad \square$$

<u>Lemma</u> 2.13. Suppose the zero solution of (2.2) is integral stable. Then there exists a monotonic increasing function

$b: [o, \infty) \to [o, \infty)$ such that

(1) $b(\|\varphi\|) \le V(t,\varphi)$, $\forall\ (t, \varphi) \in [2\gamma, \infty) \times C_A^H$;

(2) $\lim_{\gamma \to o} b(\gamma) = 0$; and

(3) $b(\gamma) = 0$ if, and only if, $\gamma = 0$.

Proof. For any $\gamma \ge 0$, we define

$$(2.24) \qquad b(\gamma) = \inf_{\substack{(t,\varphi)\in[2\gamma,\infty)\times C_A^H \\ \|\varphi\| \ge \gamma}} V(t, \varphi) .$$

Then we claim $b(\gamma) > 0$ for $\gamma > 0$. Suppose not, then there exists sequences $\{t_k\}$ and $\{\varphi_k\}$ such that

$$(2.25) \qquad \|\varphi_k\| \ge \gamma \ , \ (t_k, \varphi_k) \in [2\gamma, \infty) \times C_A^H$$

and

$$V(t_k, \varphi_k) \to 0 \quad \text{as} \quad k \to \infty .$$

Let $\delta(\gamma/2)$ be the number in Definition 2.1 of integral stability that corresponds to $\gamma/2$, choose n so large that $V(t_n, \varphi_n) < \frac{\delta(\gamma/2)}{4}$ and let $\mu_n \in A_H(t_n, \varphi_n)$ be so chosen that

$$\int_o^{t_n} |\mu_n'(t) - f(t, \mu_{n,t})|\, d\mu < \frac{\delta(\gamma/2)}{4} .$$

Then it follows from Lemma 2.12 that there exists a function $\mu(t)$ with continuous derivative such that

$$(2.26) \qquad \int_o^{t_n} |\mu'(t) - f(t, \mu_t)|\, dt < \delta(\tfrac{\gamma}{2}), \text{ and } \|\mu_{t_n} - \varphi_n\| < \frac{\gamma}{2} .$$

Next we define

$$(2.27)\qquad p(t) = \begin{cases} \mu'(t) - f(t, \mu_t) & \text{for } t \in [o, t_n] \\ 0 & \text{for } t \in [t_n, \infty) . \end{cases}$$

By changing $p(t)$ slightly if necessary, we may further assume that $p(t)$ is continuous. From (2.26), (2.27) we therefore have

$$\int_o^\infty |p(t)|dt < \delta(\tfrac{\gamma}{2}) .$$

On the other hand, we see from (2.27) that $\mu(t)$ is a solution of

$$\dot{x}(t) = f(t, x_t) + p(t) \quad \text{on} \quad -\gamma \le t \le t_n$$

so that, by using (2.25), (2.26), we have

$$\begin{aligned} \|\mu_{t_n}\| &\ge \|\varphi_n\| - \|\varphi_n - \mu_{t_n}\| \\ &> \|\varphi_n\| - \gamma/2 \\ &\ge \gamma - \gamma/2 \ge \gamma/2 , \end{aligned}$$

which contradicts the fact that zero solution of (2.2) is integral stable. Hence

$$(2.28)\qquad b(\gamma) > 0 \quad \text{for} \quad \gamma > 0 .$$

It follows from (2.24) that $b(\gamma)$ is a monotonic increasing function and satisfies (1). Next, from Lemma 2.6 and (2.28), we see (3) is satisfied. Finally, combining Lemma 2.10 and (2.24) we obtain (2). □

<u>Lemma</u> 2.14. Suppose zero solution of (2.2) is integral stable. Then there exists a monotonic increasing function $b: [o, \infty) \to [o, \infty)$ such that

(1) $b(\|\varphi\|) \le V_L(t, \varphi)$, $\forall\ (t, \varphi) \in [2\gamma, \infty) \times C_A^H$;

(2) $\lim_{\gamma \to o} b(\gamma) = 0$; and

(3) $b(\gamma) = 0$ if, and only if, $\gamma = 0$.

Proof. It follows from Lemma 2.13 that there exists a monotonic increasing function $a: [o, \infty) \to [o, \infty)$ such that

(i) $a(\|\varphi\|) \le V(t, \varphi)$, $\forall\ (t, \varphi) \in [2\gamma, \infty) \times C_A^H$;

(ii) $\lim_{\gamma \to o} a(\gamma) = 0$; and

(iii) $a(\gamma) = 0$ if, and only if, $\gamma = 0$.

Next we define the function $b: [o, \infty) \to [o, \infty)$ by

$$b(\gamma) = \liminf_{t \to \gamma} a(\gamma) \quad \forall\ \gamma \in [o, \infty) .$$

Then the function $b(\gamma)$ is a monotonic function. Also, we see from (i) that

$$\liminf_{(\tau,\Psi)\to(t,\varphi)} a(\|\Psi\|) \le \liminf_{(\tau,\Psi)\to(t,\varphi)} V(\tau, \Psi) ,$$

i.e. $b(\|\varphi\|) \le V_L(t, \varphi)$ $\forall$ $(t, \varphi) \in [o, \infty) \times C_A^H$.

It is clear from (ii) that $\lim_{\gamma \to o} b(\gamma) = 0$. Finally since $a(\gamma) = 0$ if, and only if, $\gamma = 0$, we conclude that $b(\gamma) = 0$ if, and only if, $\gamma = 0$. $\square$

2.4. Characterization of Integral Stability

Combining Lemma 2.2, Lemma 2.5, Lemma 2.8, Lemma 2.11, and Lemma 2.14, we obtain the following theorem.

Theorem 2.2. Suppose the zero solution of (2.2) is integral stable. Then there exists a lower semi-continuous function

$$V_L: \ [2\gamma, \infty) \times C_A^H \to [o, \infty)$$

Having the following properties:

(1) $V_L(t, \varphi) \geq b(\|\varphi\|) \ \forall \ (t, \varphi) \in [2\gamma, \infty) \times C_A^H$, where $b(\gamma)$ is a monotonic increasing function such that $\lim_{\gamma \to o} b(\gamma) = 0$ and $b(\gamma) = 0$ if, and only if, $\gamma = 0$

(2) $V_L(t, \varphi) \to 0$ as $\|\varphi\|_{c_1} \to 0$ for each fixed $t \geq 2\gamma$

(3) For any solution $x(t, \sigma, \varphi)$ of (2.2) with $(\sigma, \varphi) \in [o, \infty) \times C^H$, we have for $t \geq 2\gamma$ and $\sup_{s \in [\sigma, t]} |x(s, \sigma, \varphi)| \leq H$, $V(t, x_t(\sigma, \varphi))$ is a non-increasing function of t.

(4) Let $x, y \in C_A^H[\tau - \gamma, \tau + a)$, where $a > 0$, $\tau \geq 2\gamma$ such that $x(\theta) = y(\theta) \ \forall \ \tau - \gamma \leq \theta \leq \tau + a$. Then

$$|V_L(s, x_s) - V_L(s, y_s)|$$
$$\leq \|x_s - y_s\|_{c_1} + M(x, y, s) \ \forall \ \tau < s < \tau + a,$$

where $M(x,y,s)$ is a positive number which depends on x, y and s such that

$$\lim_{s \to \tau} \frac{M(x,y,s)}{s - \tau} = 0 .$$

Now we consider the converse part of Theorem 2.2.

Lemma 2.15. Suppose $V(t, \varphi)$ is a lower semi-continuous functional on $[2\gamma, \infty) \times C_A^H$. Let x,y be two solutions of (2.2) passing through (σ, φ) Then $V_x'(\sigma, \varphi) = V_y'(\sigma, \varphi)$.

Proof. It follows from Lemma 2.9 that

$$V_x'(\sigma,\varphi) \le V_y'(\sigma,\varphi) + 2|x'(\sigma^+) - y'(\sigma^+)|$$

and

$$V_y'(\sigma,\varphi) \le V_x'(\sigma,\varphi) + 2|x'(\sigma^+) - y'(\sigma^+)| \ .$$

Since $f(t, \varphi)$ is uniformly continuous in φ for all of t, we have $x'(\sigma^+) = y'(\sigma^+)$ so that

$$V_x'(\sigma, \varphi) = V_y'(\sigma, \varphi) \ . \qquad \square$$

Lemma 2.16. Let V be a functional on $[2\gamma, \infty) \times C_A^H$, which satisfies (2.12) and the conclusion of Lemma 2.3. Then for any continuous function $p: [2\gamma, \infty) \to R$ and for any solution $y(t, \sigma, \varphi)$ of

$$x' = f(t, x_t) + p(t) \ , \tag{2.29}$$

we have

$$V_y'(t, y_t(\sigma, \varphi)) \le 2|p(t)|$$

for all t that lie in the domain of $y(t, \sigma, \varphi)$.

Proof. It follows from Lemma 2.9 that

$$V_y'(t, y_t(\sigma, \varphi)) \le V_x'(t, y_t(\sigma, \varphi)) + 2|x'(\sigma^+) - y'(\sigma^+)| \ . \tag{2.30}$$

Next, we see from Lemma 2.3 that

$$(2.31) \qquad V_x'(t, y_t(\sigma, \varphi)) \le 0 .$$

On the other hand, from (2.2), (2.29) we have

$$(2.32) \qquad |x'(\sigma^+) - y'(\sigma^+)| = |p(t)| .$$

Substituting (2.31), (2.32) into (2.30) we have

$$V_y'(t, y_t(\sigma, \varphi)) \le 2|p(t)|$$

for all t in the domain of $y(t, \sigma, \varphi)$. □

<u>Lemma</u> 2.17. Let $x(\sigma, \varphi)$ be a solution of (2.2) passing through (σ, φ). Then for any $\delta > 0$, there exists η such that

$$\|x_{\sigma+\gamma}\|_{c_1} \le \delta \quad \text{whenever} \quad \|\varphi\| < \eta .$$

Proof. Since $f(t, \varphi)$ is uniformly continuous in φ for all of t and $f(t, 0) = 0$, there exists for given $\delta > 0$ an $\eta_1 < \delta/2$ such that

$$(2.33) \qquad |f(s, \varphi)| < \delta/2\gamma \quad \text{whenever} \quad \|\varphi\| < \eta_1 .$$

Also it follows from the fact that f takes a bounded set into a bounded set that there exists an $\eta_2 > 0$ such that

$$(2.34) \qquad \|x_{\tau+\gamma}\| < \eta_1 \quad \text{for all} \quad \|\varphi\| < \eta_2 .$$

Choosing

$$(2.35) \qquad \eta = \mathrm{Min}(\eta_1, \eta_2) ,$$

then for $\|\varphi\| < \eta$ we have from (2.33), (2.34), (2.35) that

$$\begin{aligned} |x'(\mu)| &\le |f(\mu, x_n)| \qquad \forall\, \mu \in [\sigma, \sigma + \gamma] \\ &< \delta/2\gamma . \end{aligned}$$

Consequently

$$\int_{\sigma}^{\sigma+\gamma} |x'(\mu)| < \frac{1}{2\gamma} \int_{t}^{t+\gamma} \delta \, du = \frac{\delta}{2} ,$$

and

$$\|x_{\sigma+\gamma}\| \le \eta_1 < \frac{\delta}{2} .$$

Hence we conclude that

$$\|x_{\sigma+\gamma}\|_{c_1} \le \delta \quad \text{whenever} \quad \|\varphi\| < \eta . \qquad \square$$

Now we are ready to state and prove the following theorem.

<u>Theorem</u> 2.3. For the equation (2.2) suppose there exists a lower semi-continuous functional

$$V: \ [2\gamma, \infty) \times C_A^H \to [o, \infty)$$

such that the following four conditions are satisfied.

(1) $V(t, \varphi) \ge b(\|\varphi\|) \ \forall \ (t, \varphi) \in [2\gamma, \infty) \times C_A^H$, where $b(\gamma)$ is a monotonic increasing function such that $\lim_{\gamma \to o} b(\gamma) = 0$ and $b(\gamma) = 0$ if, and only if, $\gamma = 0$.

(2) $V(t, \varphi) \to 0$ as $\|\varphi\|_{c_1} \to 0$ for each fixed $t \ge 2\gamma$.

(3) For any solution $x(t, \sigma, \varphi)$ of (2.2), where $(\sigma, \varphi) \in [o, \infty) \times C^H$, we have for $t \ge 2\gamma$ and $\sup_{s \in [\sigma, t]} |x(s, \sigma, \varphi)| \le H$, $V(t, x_t(\sigma, \varphi))$ is a non-increasing function of t.

(4) Let $x, y \in C_A^H[\tau - \gamma, \tau + a)$, where $a > 0$, $\tau \ge 2\gamma$ such that $x(\theta) = y(\theta)$, $\forall \ \tau - \gamma \le \theta \le \tau + a$, then

$$|V_L(s, x_s) - V_L(s, y_s)|$$

$$\leq \|x_s - y_s\|_{c_1} + M(x, y, s) \quad \forall\, \tau < s < \tau + a$$

where $M(x,y,s)$ is a positive number depends on x,y and s such that

$$\lim_{s \to \tau} \frac{M(x,y,s)}{s-\tau} = 0 .$$

Then the zero solution of (2.2) is integral stable.

Proof. Suppose not, then for any $\varepsilon > 0$, and $\delta > 0$ there exists a continuous function $p(t)$, $t_o \geq 0$ such that $\int_{t_o}^{\infty} |p(t)| < \delta$ and a $\varphi_o \in C[-\gamma, o]$ such that $\|\varphi_o\| < \delta$, for which the equation

$$x' = f(t, x_t) + p(t)$$

will have a solution $x_p = x_p(t, t_o, \varphi_o)$ such that

$$\|x_p(t_2, t_o, \varphi_o)\| \geq \varepsilon, \text{ for some } t_2 > t_o + \gamma . \tag{2.36}$$

It follows from (1), (2), that we can choose $N > t_2$ so large that

$$\frac{b(\varepsilon)}{N} < \varepsilon$$

and

$$|V(t_o + \gamma, \varphi)| < \frac{b(\varepsilon)}{2} \text{ whenever } \|\varphi\|_{c_1} < \frac{b(\varepsilon)}{N} . \tag{2.37}$$

On the other hand, in view of Lemma 2.17, we may assume $\delta > 0$ to be so small that

$$2\delta < b(\varepsilon)$$

and

$$(2.38) \qquad \|x_{t_o+\gamma}(\cdot, t_o, \varphi_o)\|_{c_1} < \frac{b(\varepsilon)}{N} < \varepsilon \quad \text{whenever} \quad \|\varphi_o\| < \delta .$$

Furthermore, from Lemma 2.16, we have

$$V'_{x_p}(t, x_t) \le 2|p(t)| \quad \forall\, t_o \le t \le t_2 .$$

Thus by Theorem 1.1 we have

$$(2.39) \qquad V(t_2, x_{t_2}) \le V(t_o + \gamma, x_{t_o+\gamma}) + \int_{t_o+\gamma}^{t_2} |p(t)|\,dt .$$

But then (1), (2.36), (2.37) and (2.38) together imply

$$b(\varepsilon) \le b(\|x_{t_2}\|) \le V(t_2, x_{t_2}) < \frac{b(\varepsilon)}{2} + \delta$$

$$< \frac{b(\varepsilon)}{2} + \frac{b(\varepsilon)}{2} = b(\varepsilon) .$$

From the above contradiction, we conclude that the zero solution of (2.2) is integral stable. □

Combining Theorem 2.1 and Theorem 2.2, we have the desired characterization of integral stability for the zero solution of (2.2).

CHAPTER III

THE SECOND COMPARISON THEOREM FOR FDE

In this chapter, we shall present a new Liapunov theory for FDE of retarded type. The theory follows closely with that of Yorke's [20], in which he developed a Liapunov theory for ordinary differential equations. Due to the hereditary nature of the equations, we have to use somewhat different techniques.

3.1. Definition of V^* and $\bar{V}$

Let $D \subset R \times C$ be open and $f\colon D \to R^d$ be continuous. Considering the FDE of retarded type

$$x'(t) = f(t, x_t) \tag{3.1}$$

Since f is continuous in D, from Theorem 0.1 the initial value problem for (3.1) is solvable for every $(\sigma,\varphi) \in D$. If, in addition, f maps bounded sets into bounded sets, then every solution may be continuous to the boundary ∂D of D. For details, see [8] (p. 13-20). Thus, we assume throughout this thesis that f maps bounded sets into bounded sets.

Consider now the functional differential equation (3.1). Let $x(\cdot, \sigma,\varphi)$ be a solution of (3.1) through (σ,φ). Given a Liapunov functional $V\colon R \times C[-\gamma,o] \to R$, we define

(3.2) $\bar{V}_x(\sigma, x_\sigma) = \liminf_{h \to o^+} \frac{1}{h} [V(\sigma + h, x_{\sigma+h}) - V(\sigma,\varphi)]$.

This extended real-valued function is well-defined for arbitrary V and f.

Definition 3.1. A function $V: R \times C[-r,o] \to R$ is called lower semi-continuous if for every $(t,\varphi) \in R \times C[-r,o]$,

(3.3) $$V(t,\varphi) \le \liminf_{(s,\psi)\to(t,\varphi)} V(s,\psi) .$$

Definition 3.2. Let $(t,\varphi) \in D$ be given and $V: R \times C[-r,o] \to R$ be well-defined. Define the extended real-valued function,

(3.4) $$V^*(t,\varphi) = \liminf_{\substack{h\to o^+ \\ |\psi(o)|\to o}} \frac{1}{h} [V(t+h, \varphi+h\bar{f}(t,\varphi) + h\psi) - V(t,\varphi)]$$

where $\psi \in C[-r,o]$ satisfies the following restriction:

(3.5) $\psi = \frac{1}{h} [z_{t+h} - \varphi] - \bar{f}(t,\varphi)$, for some $z(\cdot) \in APX(t,\varphi,h; f)$

$= APX(t,\varphi,h)$

and $\bar{f}(t,\varphi) \in C[-r,o]$ is defined by

(3.6) $$\bar{f}(t,\varphi)(\theta) \equiv f(t,\varphi), \quad -r \le \theta \le 0 .$$

In (3.5), $APX(t,\varphi,h)$ denotes the set of approximate solutions of (3.1) through (t,φ) defined on $[-r+t, t+h]$. More precisely $APX(t,\varphi,h)$ consists of all continuous functions $z: [-r+t, t+h] \to R^d$ such that

(3.7) $$z_t = \varphi ,$$

(3.8) z is absolutely continuous on $[t, t+h]$,

(3.9) $$(t+s, z_{t+s}) \in D, \quad 0 \le s \le h \quad ,$$

(3.10) $$|z'(t+s)| \le M(t,h) + h, \quad \text{a.e.,} \quad 0 \le s \le h$$

where $M(t,h) = \sup\{|f(t+s, z_{t+s})| : 0 \le s \le h\}$

(3.11) $$z'(t+) = f(t,\varphi) .$$

Remark. It follows from Theorem 0.1 that for any $(t,\varphi) \in D$ the set $APX(t,\varphi,h)$ is non-empty for sufficiently small h. Hence $V^*(t,\varphi)$ is always well-defined. Moreover, this gives the following lemma.

Lemma 3.1. Let $x(\cdot,t,\varphi)$ be a solution of (3.1) that pass through (t,φ), then

(3.12) $$V^*(t,\varphi) \le \overline{V}_x(t,\varphi) \quad .$$

The next theorem says that if V satisfies a local Lipschitz condition, then the usual Liapunov theorems still hold true when $\overline{V}$ is replaced by V^*.

Theorem 3.1. Suppose that $V: R \times C[-r,o] \to R$ is well-defined and for every $(t,\varphi) \in D$ there exists a neighborhood $N(t,\varphi)$ of (t,φ) and a constant $L = L(t,\varphi) \ge 0$ such that

(3.13) $$|V(\bar{t},\varphi_1) - V(\bar{t},\varphi_2)| \le L\|\varphi_1 - \varphi_2\|$$

for all $(\bar{t},\varphi_1)$, $(\bar{t},\varphi_2) \in N(t,\varphi)$. Then

$$V^*(t,\varphi) = \overline{V}_x(t, x_t)$$

for any solution $x(\cdot)$ of (3.1) through (t,φ).

Proof. By Lemma 3.1, it suffices to show that $V^*(t,\varphi) \geq \overline{V}_x(t, x_t)$. Let the solution $x(\cdot)$ be fixed. By Definition 3.2, (3.2) and (3.13), we have

$$(3.14)\quad \overline{V}_x(t, x_t) = \liminf_{h\to o^+} \frac{1}{h} [V(t+h , x_{t+h}) - V(t,\varphi)]$$

$$\leq \liminf_{\substack{h\to o^+ \\ |\psi(o)|\to o}} \frac{1}{h} [V(t+h, \varphi+h\overline{f}(t,\varphi) + h\psi) - V(t,\varphi)]$$

$$+ \limsup_{h\to o^+} \frac{1}{h} L\|x_{t+h} - (\varphi+h\overline{f}(t,\varphi) + h\psi)\|$$

where ψ is the same as in Definition 3.2. Since $\varphi + h\overline{f}(t,\varphi) + h\psi = z_{t+h}$ for some $z(\cdot) \in APX(t,\varphi,h)$,
$\|x_{t+h} - (\varphi + h\overline{f}(t,\varphi) + h\psi)\| = |x(t+\overline{h}) - z(t+\overline{h})|$

where $0 < \overline{h} \leq h$. For each fixed $h > 0$,

$$\frac{1}{h} L\|x_{t+h} - (\varphi + h\overline{f}(t,\varphi) + h\psi)\|$$

$$\leq \frac{1}{h} L|x(t + \overline{h}) - x(t) - [z(t + \overline{h}) - x(t)]| \quad .$$

Substituting into (3.14),

$$\overline{V}_x(t, x_t) \leq V^*(t,\varphi) + \limsup_{h\to o^+} \frac{1}{h} |x(t+\overline{h}) - x(t) - [z(t+\overline{h}) - x(t)]|$$

$$= V^*(t,\varphi) + L|x'(t^+) - z'(t^+)|$$

$$= V^*(t,\varphi) \qquad \square$$

because of (3.11) and (3.1). This completes the proof.

Remark. In [22] (p. 186-188), it is shown that if V is continuous in (t,φ) and is Lipschitzian in φ, then $\overline{V}_x(t, x_t)$ is independent of any particular solution through (t,φ). Theorem 3.1 says that $\overline{V}_x(t, x_t)$ is in fact equal to $V^*(t,\varphi)$.

3.2. A Comparison Principle for FDE

The next theorem shows that the usual Liapunov comparison principle may be obtained for V^* derivative.

Theorem 3.2. Let $D \subset R \times C[-r,o]$ be open and $f\colon D \to R^d$ be continuous and map bounded sets into bounded sets. Suppose that $W\colon D \to R$ is continuous and $V\colon D \to R$ is lower semi-continuous. If

$$V^*(t,\varphi) \le W(t,\varphi), \quad (t,\varphi) \in D \tag{3.15}$$

along $x'(t) = f(t, x_t)$, then for every $(t,\varphi) \in D$ there exists a solution $x(t,\varphi)$ such that

$$V(t+s, x_{t+s}) - V(t, x_t) \le \int_t^{t+s} W(u, x_u)du \tag{3.16}$$

for all $s \ge 0$ such that x_{t+s} remains to be a solution.

Remark. In applications, it is often easier to use $\overline{V}$ to conclude that $\overline{V}_x(t,\varphi) \le W(t,\varphi)$. Lemma 3.1 will show that (3.15) is true.

3.3. Proof of the Comparison Theorem

In this section the proof of Theorem 3.2 is given. The following lemmas are needed. For simplicity, all the assumptions in Theorem 3.2 are assumed in these lemmas.

Lemma 3.2. Let $(t,\varphi) \in D$ be given and $A_n(t,\varphi)$ denote the set of $(\tau,\psi) \in D$ such that

(3.17) $$t < \tau < t + \frac{1}{n}$$

(3.18) $$\left|\frac{\psi(o) - \varphi(o)}{\tau - t} - f(t,\varphi)\right| < \frac{1}{n}$$

(3.19) $$\frac{V(\tau,\psi) - V(t,\varphi)}{\tau - t} < W(t,\varphi) + \frac{1}{n}$$

(3.20) $$\psi = z_\tau \quad \text{for some} \quad z(\cdot) \in APX(t,\varphi, \tau - t).$$

Then $A_n(t,\varphi)$ is non-empty for $n = 1,2,\ldots$.

Proof. It follows from Definition 3.2 and (3.15) that for any given n there exist $h > 0$ and $\psi_1 \in C[-r,o]$ such that

(3.21) $$h + |\psi_1(o)| < \frac{1}{2n} \text{ , and}$$

(3.22) $$\frac{1}{h}\,[V(t+h,\ \varphi+h\bar{f}(t,\varphi) + h\psi_1) - V(t,\varphi)] < W(t,\varphi) + \frac{1}{n} \quad .$$

Let $z_1(\cdot) \in APX(t,\varphi,h)$ be such that (3.5) is satisfied. Let $\tau = t+h$ and $\psi = z_{1,t+h}$. It is easy to see that (3.17), (3.19) and (3.20) are satisfied for this choice of (τ,ψ). Moreover, by (3.5)

$$\psi(o) - \varphi(o) = z_{1,t+h}(o) - \varphi(o) = hf(t,\varphi) + h\psi_1(o) \ .$$

Hence

$$\left|\frac{\psi(o) - \varphi(o)}{\tau - t} - f(t,\varphi)\right| \le |\psi_1(o)| < \frac{1}{n} \quad .$$

This proves that $(\tau,\psi) \in A_n(t,\varphi)$. □

For each n sufficiently large, we now construct an approximate solution $x_n(\cdot)$ of (3.1) through (t,φ).

Since $A_n(t,\varphi)$ is non-empty,

$$\sup\{h : (t + h,\psi) \in A_n(t,\varphi)\} > 0.$$

We may therefore find $(t_1, \varphi_1) \in A_n(t,\varphi)$ such that

$$t_1 - t > \frac{1}{2} \sup\{h : (t+h, \psi) \in A_n(t,\varphi)\} .$$

Now, for each $i = 1,2,3,\ldots$ there exists inductively $(t_{i+1}, \varphi_{i+1}) \in A_n(t_i, \varphi_i)$ such that

$$(3.23) \qquad t_{i+1} - t_i > \frac{1}{2} \sup \{h : (t_i + h,\psi) \in A_n(t_i, \varphi_i)\}.$$

Let

$$b_n = \sup_{i\geq 1} t_i .$$

Define $x_n: [-r + t, b_n) \to R^d$ by

$$(3.24) \qquad \begin{aligned} x_{n,t} &= \varphi \\ x_{n,t_i} &= \varphi_i \qquad i = 1,2,3,\ldots . \end{aligned}$$

The following is immediate from the definition of $A_n(t,\varphi)$.

Lemma 3.3. $x_n: [-r + t, b_n) \to R^d$ is continuous and is absolutely continuous for $t \leq s < b_n$. Moreover, $(s, x_{n,s}) \in D$ for all $t \leq s < b_n$.

Lemma 3.4. For each n, $x_n(\cdot)$ is non-continuable with respect to D.

Proof. We may assume that $b_n < \infty$. If $x_n(\cdot)$ is not non-continuable with respect to D, then there exists a sequence $t_k \to b_n$ as $k \to \infty$ such that

$$(t_k, x_{n,t_k}) \in U \quad \text{for all} \quad k = 1,2,\ldots$$

where $U \subset D$ is some closed bounded subset. This implies that $x_n(s)$, $-r + t \le s < b_n$, is bounded. If $M > 0$ denotes the bound of $|f(\tau,\psi)|$ for (τ,ψ) in the closure of $\{(s, x_{n,s}) : t \le s < b_n\}$, then it follows from (3.20) and (3.10) that

$$|x_n'(s)| \le M + \frac{1}{n}\ , \text{ a.e., } t \le s < b_n\ .$$

Thus, x_n is uniformly continuous on $[t - r, b_n)$. This implies $\{(s, x_{n,s}) : t \le s < b_n\}$ belongs to a compact set in D. Hence, x_{n,b_n} is well-defined and $(b_n, x_{n,b_n}) \in U$. We now claim that $(b_n, x_{n,b_n}) \notin U$. This contradiction will prove the lemma.

Proof of claim. $(b_n, x_{n,b_n}) \notin U$. If $(\tau,\psi) \in A_n(b_n, x_{n,b_n})$, and if $(\tau,\psi) \in A_n(t_i,x_{n,t_i})$ for all sufficiently large i, then for all sufficiently large i

$$t_{i+1} - t_i < \frac{1}{2}(\tau - b_n) < \frac{1}{2}(\tau - t_i)\ .$$

This contradicts the choice of t_{i+1} (3.23). Hence, it suffices to show that if $(\tau,\psi) \in A_n(b_n, x_{n,b_n})$, then $(\tau,\psi) \in (t_i, x_{n,t_i})$ for all large i.

Since $t_i \to b_n$ as $i \to \infty$, by (3.17) we have

$$(3.25) \qquad t_i < \tau < t_i + \frac{1}{n} \quad \text{for all large } i.$$

By the continuity of f and x_n and by

$$(3.26)\qquad \lim_{i\to\infty} \left|\frac{\psi(o) - \varphi_i(o)}{\tau - t_i} - f(t_i,\varphi_i)\right|$$

$$= \left|\frac{\psi(o) - x_{n,b_n}(o)}{\tau - b_n} - f(b_n, x_{n,b_n})\right| < \frac{1}{n}$$

we have

$$\left|\frac{\psi(o) - \varphi_i(o)}{\tau - t_i} - f(t_i,\varphi_i)\right| < \frac{1}{n} \quad \text{for all large } i .$$

Next,

$$(3.27)\qquad \limsup_{i\to\infty} \frac{V(\tau,\psi) - V(t_i,\varphi_i)}{\tau - t_i} \le \frac{V(\tau,\psi) - V(b_n,x_{n,b_n})}{\tau - b_n}$$

$$+ \limsup_{i\to\infty} \frac{V(b_n,x_{n,b_n}) - V(t_i,\varphi_i)}{\tau - t_i} \le W(b_n,x_{n,b_n})$$

$$+ \frac{1}{n} + \frac{V(b_n,x_{n,b_n}) - V(t_i,\varphi_i)}{\tau - b_n} - \liminf_{i\to\infty} \frac{V(t_i,\varphi_i)}{\tau - t_i} .$$

From the definition of lower semi-continuity of V we have

$$- \liminf_{i\to\infty} V(t_i,\varphi_i) \le -V(b_n,x_{n,b_n}) \quad .$$

This inequality together with (3.27) implies

$$(3.28)\qquad \limsup_{i\to\infty} \frac{V(\tau,\psi) - V(t_i,\varphi_i)}{\tau - t_i} < W(b_n,x_{n,b_n}) + \frac{1}{n}$$

$$= \lim_{i\to\infty} W(t_i,\varphi_i) + \frac{1}{n} .$$

The strict inequality in (3.28) says that

$$(3.29)\qquad \frac{V(\tau,\psi) - V(t_i,\varphi_i)}{\tau - t_i} < W(t_i,\varphi_i) + \frac{1}{n} \quad \text{for all large } i.$$

Since $(\tau,\psi) \in A_n(b_n, x_{n,b_n})$, $\psi = z_\tau$ for some $z(\cdot) \in APX(b_n, x_{n,b_n}, \tau - b_n)$. Let $z_i : [t_i - r, \tau] \to R^n$ be defined by

$$z_i(s) = z(s),\ b_n - r \le s \le \tau$$

$$z_i(s) = x_n(s),\ t_i - r \le s \le b_n - r .$$

It is not difficult to see that $z_i(\cdot) \in APX(t_i, \varphi_i, \tau - t_i)$ for all large i. Since $\psi = z_{i,\tau}$, we have from (3.25), (3.26) and (3.29) that $(\tau,\psi) \in A_n(t_i, \varphi_i)$ for all large i. This proves the claim. □

<u>Lemma</u> 3.5. Let

(3.30) $$G_n(s) = x_n(s) - \varphi(o) - \int_t^s f(u, x_{n,u})du, \quad t \le s < b_n .$$

Suppose that for each closed bounded subset $U \subset D$ there exists a sequence $\{\beta_n\}$, $\beta_n = \beta_n(U)$, such that

(3.31) $$\beta_n \to 0 \text{ as } n \to \infty$$

(3.32) if $(s, x_{n,s}) \in U$ for all $t \le s \le v_n$, then

$$|G_n(v_n)| \le \beta_n .$$

Then there exists a non-continuable function $x: [t - r, b) \to R^d$ and a subsequence $\{x_{n_k}(\cdot)\}$ such that

(3.33) $x_{n_k}(\cdot) \to x(\cdot)$ uniformly on compact subsets of $[t - r, b)$ as $n_k \to \infty$.

(3.34) $$x(s) = \varphi(o) + \int_t^s f(u, x_u)du$$

$$x_t = \varphi .$$

Proof. See [4]. We remark that in [4] it is assumed that the projection of D onto $C[-r, o]$ is bounded. However, the same proof may be used in our case with almost no changes. □

Lemma 3.6.

$$(3.35)\quad \left[\frac{x_n(t_{i+1}) - x_n(t_i)}{t_{i+1} - t_i} - f(t_i, x_{n,t_i})\right] \cdot (t_{i+1} - t_i)$$

$$= \int_{t_i}^{t_{i+1}} [x_n'(s) - f(s, x_{n,s})]ds - \int_{t_i}^{t_{i+1}} [f(t_i, x_{n,t_i}) - f(s, x_{n,s})]ds .$$

Lemma 3.7. Let $U \subset D$ be closed and bounded and

$$(3.36)\quad U_1 = \{(s, x_{n,s}) : n = 1,2,\ldots; (u, x_{n,u}) \in U \text{ for all } t \le u \le s\} .$$

Then U_1 is relatively compact.

Proof. For each n, we obtain from (3.10) that if $(s, x_{n,s}) \in U_1$,

$$(3.37)\quad |x_n'(s)| \le M + \frac{1}{n} \le (1 + M), \text{ a.e.},$$

where $M = \sup\{|f(s,\psi)| : (s,\psi) \in U\}$. We also note that $x_{n,t} = \varphi$ for all n. For each $x_{n,s}$ such that $(s, x_{n,s}) \in U_1$, $x_{n,s} \in C[-r, o]$ may be broken into two parts. Namely, one part is some portion of φ and the other part is absolutely continuous and satisfies (3.37). Since φ is uniformly continuous and the bound in (3.37) is independent of n, the set $\{x_{n,s} : (s, x_{n,s}) \in U_1\}$ is equi-continuous. Now, an argument using Ascoli theorem will complete the proof. □

Proof of Theorem 3.2. Let U, U_1 and M be as in Lemma 3.7, and $L = \sup\{s - t : (s,\psi) \in U \text{ for some } \psi \in C[-r, o]\}$. Let

$$\alpha_n(U) = \sup|f(s_1, \psi_1) - f(s_2, \psi_2)|$$

where sup is taken over the set of (s_1, ψ_1), $(s_2, \psi_2) \in U_1$ such that

$$|s_1 - s_2| < \frac{1}{n}$$

$$\|\psi_1 - \psi_2\| < \max\{(M\cdot n+1)/n^2, \gamma(n)\}$$

where $\gamma(n)$ is determined from the uniform continuity of x_n on $[t - r, t + \frac{1}{n}]$, i.e., if $|s_1 - s_2| < \frac{1}{n}$, then $|x_n(s_1) - x_n(s_2)| < \gamma(n)$. It follows from the uniform continuity of φ and (3.37) that $\gamma(n) \to o$ as $n \to \infty$. Let

$$\beta_n(U) = (\alpha_n(U) + \frac{1}{n})L + \frac{1}{n}(2M + 1) .$$

Since U_1 has compact closure, $\alpha_n(U) < \infty$. Moreover, the uniform continuity of f on U_1 yields that $\alpha_n(U) \to o$ and $\beta_n(U) \to o$ as $n \to \infty$. Let $G_n(t)$ be as in Lemma 3.5. We claim that the condition (3.32) is satisfied by the above choice of β_n. Let $v_n > t$ and $(s, x_{n,s}) \in U$ for all $t \le s \le v_n$. We have

$$(3.38) \qquad |G_n(v_n)| = |\int_t^{v_n}[x_n'(s) - f(s, x_{n,s})]ds|$$

$$= |\int_t^{t_1}[x_n'(s) - f(s, x_{n,s})]ds + \ldots + \int_{t_i}^{t_{i+1}}[x_n'(s) - f(s, x_{n,s})]ds + \ldots + \int_{t_j}^{v_n}[x_n'(s) - f(s, x_{n,s})]ds|$$

where t_i's are from the definition of $x_n(\cdot)$ and t_j is such that $t_j \le v_n < t_{j+1}$. It follows from Lemma 3.6 and (3.18) that

$$(3.39)\quad \left|\int_{t_i}^{t_{i+1}}[x_n'(s) - f(s, x_{n,s})]ds\right| \le \frac{1}{n}(t_{i+1} - t_i) + \int_{t_i}^{t_{i+1}}|f(t_i, x_{n,t_i}) - f(s, x_{n,s})|ds .$$

Since for $-r \le \theta \le 0$, $t_i \le s \le t_{i+1} < t_i + \frac{1}{n}$ and $t_i + \theta \ge t$

$$|x_{n,s}(\theta) - x_{n,t_i}(\theta)| = |x_n(s+\theta) - x_n(t_i+\theta)| \le \int_{t_i+\theta}^{s+\theta}|x_n'(u)|du,$$

we obtain from (3.10) and the uniform continuity of $x_n(\cdot)$ on $[t - r, t + \frac{1}{n}]$,

$$\|x_{n,s} - x_{n,t_i}\| \le \max\{(Mn+1)/n^2, \gamma(n)\} .$$

This inequality, the definition $\alpha_n(U)$ and (3.39) yield

$$\left|\int_{t_i}^{t_{i+1}}[x_n'(s) - f(s, x_{n,s})]ds\right| \le [\alpha_n(U) + \frac{1}{n}](t_{i+1} - t_i) .$$

Substituting this into (3.38),

$$|G_n(v_n)| \le [\alpha_n(U) + \frac{1}{n}][t_j - t] + \int_{t_j}^{v_n}|x_n'(s) - f(s, x_{n,s})|ds$$

$$\le [\alpha_n(U) + \frac{1}{n}]L + \frac{1}{n}[2M + 1] = \beta_n(U) .$$

This proves the claim. Hence, from Lemma 3.5 there exists a solution $x(\cdot)$ of (3.1) defined on $[t - r, b)$, $b > t$, through (t,φ) which is non-continuable with respect to D and there exists a subsequence $x_{n_j}(\cdot)$ such that

$$x_{n_j}(\cdot) \to x(\cdot) \quad \text{uniformly on compact subsets of} \quad [t - r, b) \quad \text{as} \quad n \to \infty .$$

We now prove that this solution $x(\cdot)$ satisfies (3.16). For simplicity, denote the subsequence $x_{n_j}(\cdot)$ by $x_j(\cdot)$ and the sequence $\{t_i\}$ used to define $x_j(\cdot)$ by $t_i(j)$. Let $s \in [t,b)$. For each large j let $i = i(j)$ be so chosen that

$$t_i(j) \le s \le t_{i+1}(j) < t_i(j) + \frac{1}{n_j} .$$

Thus, $t_i(j) \to s$ as $j \to \infty$. Since $x_j(\cdot) \to x(\cdot)$ uniformly on $[t - r, s]$, $\|x_{j,u} - x_u\| \to 0$ uniformly for $t \le u \le s$ as $j \to \infty$. Let $U_s = \{(u, x_{j,u}) : t \le u \le s,\ j = 1,2,\ldots\}$. It is shown by the same proof of Lemma 3.7 that U_s is relatively compact. The uniform continuity of W on U_s yields

$$W(u, x_{j,u}) \to W(u, x_u) \quad \text{uniformly on} \quad [t,s] \quad \text{as} \quad j \to \infty.$$

Thus, if $w_j = \sup\{|W(u, x_{j,u}) - W(u, x_u)| : t \le u \le s\}$, then $w_j \to 0$ as $j \to \infty$. Now,

$$\text{(3.40)} \quad V(s, x_s) - V(t,\varphi) \le \liminf_{j\to\infty} V(t_i(j), x_{j,t_i(j)}) - V(t,\varphi)$$

$$= \liminf_{j\to\infty} \sum_{k=0}^{i(j)-1} [V(t_{k+1}(j), x_{j,t_{k+1}(j)}) - V(t_k(j), x_{j,t_k(j)})]$$

where $t_o(j) = t$ for all $j = 1,2,\ldots$. By the definition of $x_n(\cdot)$ and by (3.19), we have

$$V(t_{k+1}(j), x_{j,t_{k+1}(j)}) - V(t_k(j), x_{j,t_k(j)})$$

$$\le [W(t_k(j), x_{j,t_k(j)}) + \frac{1}{n_j}][t_{k+1}(j) - t_{k+(j)}] \ .$$

Substituting into (3.40),

$$V(s, x_s) - V(t,\varphi) \le \liminf_{j\to\infty} \sum_{k=0}^{i(j)-1} [W(t_k(j), x_{j,t_k(j)}) + \frac{1}{n_j}]$$

$$[t_{k+1}(j) - t_k(j)]$$

$$\le \liminf_{j\to\infty} \sum_{k=0}^{i(j)-1} [W(t_k(j), x_{j,t_k(j)})]$$

$$[t_{k+1}(j) - t_k(j)]$$

$$\le \liminf_{j\to\infty} \sum_{k=0}^{i(j)-1} W(t_k(j), x_{t_k(j)})[t_{k+1}(j) - t_k(j)]$$

$$+ \limsup_{j\to\infty} \sum_{k=0}^{i(j)-1} [W(t_k(j), x_{t_k(j)}) - W(t_k(j),$$

$$x_{j,t_k(j)})][t_{k+1}(j) - t_k(j)]$$

$$\le \int_t^s W(u, x_u)du + \limsup_{j\to\infty} W_j[s - t]$$

$$= \int_t^s W(u, x_u)du \ .$$

This completes the proof. □

CHAPTER IV

SEMI-INVARIANCE OF FDE OF RETARDED TYPE

One of the fundamental problems in differential equations is the characterization of invariance of a set. In ordinary differential equations, it is known that invariance of a closed set is equivalent to a notion called subtangent. This theorem was first obtained by Nagumu [14], and was later rediscovered by Yorke [21]. Recently, it was again proven by Hartman [9] and Crandall [5]. In this chapter, we shall give a complete generalization of this theorem to functional differential equations. The proof is different from those given in [14], [21], [9] and [5]. The Liapunov theory developed earlier in Chapter III is our main tool.

The notion, subtangent, in FDE is more complicated than that in ordinary differential equations. However, this does not limit its applicabilities. In section 2 of this chapter, we shall give an invariance principle for an asymptotically autonomous system as an application.

4.1. Semi-Invariance

Definition 4.1. Let $Q \subset D$. Q is said to be invariant with respect to (3.1) if for each $(t,\varphi) \in Q$, then any solution $x(\cdot)$ of (3.1) through (t,φ) satisfies $(s, x_s) \in Q$ for all $s \geq t$ and $(s, x_s) \in D$. Q is said to be semi-invariant if for each $(t,\varphi) \in Q$ there exists a solution $x(\cdot)$ of (3.1) through (t,φ) such that $(s, x_s) \in Q$ for all $s \geq t$ and $(s, x_s) \in D$.

The concept of invariance for ordinary differential equations has been discussed by many authors (see, for example, [23], [11]). In particular, the necessary and sufficient condition for a set to be invariant has been given in [14], [21], [9], [5]. In this section, we present a similar theorem for functional differential equations. First, we give a definition of $f(t,\varphi)$ to be subtangential to a set $Q \subset D$.

Definition 4.2. Let $Q \subset D$ and $(t,\varphi) \in Q$. We say that f is subtangential to Q at (t,φ) if

$$(4.1) \qquad \liminf_{h \to o^+} \frac{1}{h} d_o(\varphi + h\bar{f}(t,\varphi), Q_{t+h}) = 0$$

where Q_{t+h} denotes the set of all ψ such that $(t + h,\psi) \in Q$ and $\psi = z_{t+h}$ for some $z \in APX(t,\varphi, h)$ (see Definition 3.2), and

$$d_o(\varphi + h\bar{f}(t,\varphi), Q_{t+h}) = \inf\{|\varphi(o) + hf(t,\varphi) - \psi(o)| : \psi \in Q_{t+h}\} .$$

If Q_{t+h} is empty, we define $d_o(\varphi + h\bar{f}(t,\varphi), Q_{t+h}) = +\infty$.

Theorem 4.1. Let $Q \subset D$ be closed. Q is semi-invariant with respect to (3.1) if and only if for each $(t,\varphi) \in Q$, $f(t,\varphi)$ is subtangential to Q at (t,φ) .

Proof. Let Q be semi-invariant. For each $(\tau,\varphi) \in Q$, let $x(\cdot)$ be the solution of (3.1) through (t,φ) such that $(s, x_s) \in Q$ for all $s \geq t$ and $x(s)$ is defined. It is clear that $x(\cdot) \in APX(t,\varphi, h)$ and $x_{t+h} \in Q_{t+h}$ for every small $h > 0$.

Moreover,

$$\frac{1}{h}\,[\varphi(o) + hf(t,\varphi) - x_{t+h}(o)] = f(t,\varphi) - \frac{x(t+h) - x(t)}{h} .$$

But the right hand side tends to zero as $h \to o^+$. Hence, by definition $f(t,\varphi)$ is subtangential to Q at (t,φ).

Conversely, first define

$$(4.2) \qquad V_Q(t,\varphi) = \begin{cases} 0 & (t,\varphi) \in Q \\ 1 & (t,\varphi) \notin Q \end{cases} .$$

The closeness of Q implies V_Q is lower semi-continuous. By Definition 4.2, for each $\varepsilon > 0$ there exists h, $0 < h < 0$, such that

$$\frac{1}{h}\,|\varphi(o) + hf(t,\varphi) - \psi(o)| < \varepsilon$$

where $\psi = z_{t+h}$ for some $z(\cdot) \in APX(t,\varphi,h)$ such that $(t + h, z_{t+h}) \in Q_{t+h}$. Let

$$\psi_1 = \frac{1}{h}\,[z_{t+h} - \varphi - h\bar{f}(t,\varphi)] .$$

We have $\varphi + h\bar{f}(t,\varphi) + h\psi_1 = z_{t+h}$ and $|\psi_1(o)| < \varepsilon$. Hence

$$V_Q(t + h, \varphi + h\bar{f}(t,\varphi) + h\psi_1) - V_Q(t,\varphi) = 0 .$$

Letting $\varepsilon \to 0$, we have from the definition

$$V_Q^*(t,\varphi) = 0 .$$

By Theorem 3.2, there exists a non-continuable solution $x(\cdot)$ of (3.1) through (t,φ) such that

$$V_Q(s, x_s) - V_Q(t, x_t) \le 0, \quad s \ge t \text{ and } x(s) \text{ is defined.}$$

From (4.2), $(s, x_s) \in Q$ for $s \ge t$ and $x(s)$ is defined. □

Corollary. Let $Q \subset D$ be closed. Suppose that for each $(t,\varphi) \in D$ there exists a unique solution of (3.1) through (t,φ). Then Q is invariant if and only if $f(t,\varphi)$ is subtangential to Q at every $(t,\varphi) \in Q$.

Remark. In applications, it is often that (3.1) is autonomous. For this reason, we will state Theorem 4.1 separately for the autonomous case.

Let $E \subset C[-r, o]$ be open and $g : E \to R^d$ be continuous and map bounded sets into bounded sets. Consider the autonomous system

$$x'(t) = g(x_t) \ . \tag{4.3}$$

Definition 4.3. Let $P \subset E$. P is said to be semi-invariant with respect to (4.3) if for each $\varphi \in P$ there exists a solution $x(\cdot)$ of (4.3) through (o,φ) such that $x_t \in P$ for all $t \ge 0$ and $x(t)$ is defined. We say that $g(\varphi)$ is subtangential to P if

$$\liminf_{h \to o^+} \frac{1}{h} d_o(\varphi + h\bar{g}(\varphi), P_h) = 0 \tag{4.4}$$

where P_h denotes the set of all $\psi \in P$ such that $\psi = z_h$ for some $z(\cdot) \in APX(o,\varphi,h; g)$.

Theorem 4.2. Let $P \subset E$ be closed. P is semi-invariant with respect to (4.3) if and only if for each $\varphi \in P$, $g(\varphi)$ is subtangential to P at φ.

4.2. Asymptotically Autonomous Systems

In this section, we consider the autonomous system

$$(4.5) \qquad x'(t) = g(x_t)$$

and its perturbation

$$(4.6) \qquad y'(t) = g(y_t) + h(t, y_t) .$$

Let $E \subset C[-r, o]$ be open. We assume that $g : E \to R^d$ and $h : R \times E \to R^d$ are continuous and map closed bounded sets into bounded sets. When h tends to zero (in some sense), (4.6) is said to be asymptotically autonomous [12]. One is generally interested in knowing under what conditions on h, the limit sets of (4.6) are semi-invariant with respect to (4.5). This question has been studied by many authors (see [23], [11], [12], [17] and [1]). In [13], Miller extended the result to functional differential equations. We shall present an approach different from Miller's based on our previous work. Note that, the use of Liapunov theory is not new for ordinary differential equations [23], [1].

<u>Definition</u> 4.4. Let $y(\cdot)$ be a solution of (4.6). The limit set $L(y(\cdot))$ is the set of all $\psi \in C[-r, o]$ such that $Y_{t_n} \to \psi$ as $t_n \to \infty$ for some sequence $\{t_n\}$.

We shall assume the following smallness condition on h.

(H) There exists a decreasing function $\mu : [o,\infty) \to [o, \infty)$ such that $\mu(t) \to 0$ as $t \to \infty$ and for every continuous function $z : [-r,\infty) \to R^d$ we have

$$\left|\int_{T_o}^{T_1} h(t, z_t)dt\right| \le \mu(T_o)$$

for all $0 \le T_o \le T_1 \le T_o + 1$.

This condition on h is slightly more general than that given by Miller [13].

<u>Theorem</u> 4.3. If h satisfies condition (H) and if L is the limit set of a solution $y(\cdot)$ of (4.6), then L is semi-invariant with respect to (4.5).

We shall assume that L is non-empty. Let $\varphi \in L$ be fixed and $\delta > 0$ be so small that if $\|\varphi - \psi\| \le \delta$, then $\psi \in E$. Let $M = \sup\{|g(\psi)| : \|\varphi - \psi\| \le \delta\}$. By definition, there exists a sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and $y_{t_n} \to \varphi$ as $n \to \infty$. The following lemmas are needed.

<u>Lemma</u> 4.1. If $\alpha = \min\{1, \delta/3M\}$ and n is large, then

$$\|\varphi - y_{t_n+t}\| \le \delta, \quad 0 \le t \le \alpha \,. \tag{4.7}$$

Proof. For large t_n,

$$y(t_n + t + \theta) = y(t_n + \theta) + \int_{t_n+\theta}^{t_n+t+\theta} g(y_s)ds + \int_{t_n+\theta}^{t_n+t+\theta} h(s,y_s)ds \,.$$

We assume that t_n is so large that $\|\varphi - y_{t_n}\| < \delta/3$ and $\mu(t_n - r) \le \delta/3$. Let t_n be fixed. As long as $0 \le t \le 1$ and $\|y_{t_n+t} - \theta\| \le \delta$, we have for $-r \le \theta \le 0$

$$\begin{aligned} |\varphi(\theta) - y(t_n + t + \theta)| &\le \|\varphi - y_{t_n}\| + tM + \left|\int_{t_n+\theta}^{t_n+t+\theta} h(s,y_s)ds\right| \\ &\le \delta/3 + tM + \mu(t_n + \theta) \le \delta/3 + tM + \mu(t_n - r) \\ &\le \frac{2\delta}{3} + tM \,. \end{aligned}$$

A "suppose not " argument yields (4.7). □

Lemma 4.2. The set

$$Y = \{y_{t_n+t} : t_n \geq 2r,\ 0 \leq t \leq \alpha\}$$

is relatively compact.

Proof. It follows from (4.7) that Y is uniformly bounded in $C[-r, o]$. If $-r \leq \theta_1 < \theta_2 < \theta_1+1$, $\theta_2 < 0$, and $0 \leq t \leq \alpha$, then

$$(4.8) \quad |y(t_n + t + \theta_2) - y(t_n + t + \theta_1)| \leq \int_{t_n+t+\theta_1}^{t_n+t+\theta_2} |g(y_s)|ds$$

$$+ \left|\int_{t_n+t+\theta_1}^{t_n+t+\theta_2} h(s, y_s)ds\right|$$

$$\leq (\theta_2 - \theta_1)M + \mu(t_n + t + \theta_1)$$

$$\leq (\theta_2 - \theta_1)M + \mu(t_n - r) .$$

Let $\varepsilon > 0$. Choose $t_N = t_N(\varepsilon)$ so large that $\mu(t_n - r) < \varepsilon/2$ for all $t_n \geq t_N$. Let

$$M_n = M(\varepsilon) = \sup\{|g(\psi)| + |h(t,\psi)| : \psi \in B \text{ and}$$

$$0 \leq t \leq t_N + \alpha\} < \infty$$

where $B \subset D$ is a bounded set such that $y_{t_n+t} \in B$ for all t_n and $0 \leq t \leq \alpha$. (The existence of the set B is a consequence of Lemma 4.1.) We note that M_n depends only on $\varepsilon > 0$. Thus $|y'(t_n + t + \theta)| \leq M_n$, a.e., for all $2r \leq t_n \leq t_N$, $0 \leq t \leq \alpha$, and $-r \leq \theta \leq 0$. If $|\theta_2 - \theta_1| < \min\{1, \varepsilon/M_N, \varepsilon/2M\}$, we have

$$|y(t_n + t + \theta_2) - y(t_n + t + \theta_1)| < \varepsilon,\ 2r \leq t_n,\ 0 \leq t \leq 2 .$$

This proves the equi-continuity of Y and completes the proof by using Ascoli Theorem. □

<u>Corollary</u> 1. Let

$$\beta(\rho) = \sup\{|g(\varphi) - g(\psi)| : \psi \in Y, \|\varphi - \psi\| \le \rho\} .$$

Then $\beta(\rho)$ decreases monotonically to zero as $\rho \to 0$.

<u>Corollary</u> 2. For each t, $0 \le t \le \alpha$, $\{y_{t_n+t}\}$ has a limit point ψ. Such that $\|\varphi - \psi\| \le \delta$.

<u>Lemma</u> 4.3. Let ψ be as in Corollary 2 of Lemma 4.2. Define a function $z : [-r, t] \to R^d$ by $z_o = \varphi$, $z_t = \psi$. Then $z(\cdot) \in APX(o,\varphi,t; g)$.

Proof. Since $y(t_n + \theta) \to \varphi(\theta)$ uniformly in θ as $n \to \infty$ and $y(t_{n_j} + t + \theta) \to \psi(\theta)$ uniformly in θ as $t_{n_j} \to \infty$ (where $\{t_{n_j}\}$ is a subsequence of $\{t_n\}$), z is well-defined. For $0 \le s_1, s_2 \le t$, by (4.8)

$$\begin{aligned}(4.9)\quad |z(s_1) - z(s_2)| &= |\psi(s_1 - t) - \psi(s_2 - t)| \\ &\le |y(t_{n_j} + s_1) - \psi(s_1 - t)| \\ &+ |y(t_{n_j} + s_2) - \psi(s_2 - t)| + \\ &\quad |y(t_{n_j} + s_2) - y(t_{n_j} + s_1)| \\ &\le |y(t_{n_j} + s_1) - \psi(s_1 - t)| + \\ &\quad |y(t_{n_j} + s_2) - \psi(s_2 - t)| \\ &+ |s_2 - s_1|M_j + \mu(t_{n_j} - r)\end{aligned}$$

where

$$M_j = \sup\{|g(y_s)| : t_{n_j} \le s \le t_{n_j} + t\} .$$

Since g is continuous on the relatively compact set Y (Lemma 4.2), M_j is defined. Since $y(t_{n_j} + s) \to z(s)$ uniformly for $-r \le s \le t$ as $t_{n_j} \to \infty$, we have for large j,

$$M_j \le t + \sup\{|g(z_s)| : 0 \le s \le t\} = t + M_z , \quad \text{say} .$$

Letting $t_{n_j} \to \infty$ in (4.9),

$$|z(s_1) - z(s_2)| \le |M_z + t| \; |s_2 - s_1| \quad .$$

Similarly, we can show that $z'(o^+) = g(\varphi)$. Hence, $z(\cdot) \in APX(o,\varphi,t; g)$. □

Proof of Theorem 4.3. We will show that $g(\varphi)$ is subtangential to L at φ. From Lemma 4.3, $z(\cdot) \in APX(o,\varphi,t; g)$ and $\psi = z_t \in L$. Thus,

$$(4.10) \quad d_o(\varphi + t\bar{g}(\varphi), L_t) \le |\varphi(o) + tg(\varphi) - \psi(o)|$$

$$\le \limsup_{n\to\infty} |\varphi(o) + tg(\varphi) - y(t_n + t)|$$

$$\le \limsup_{n\to\infty} \{|\varphi(o) - y(t_n)| + |\textstyle\int_o^t g(\varphi)\, ds$$

$$- \textstyle\int_o^t g(y_{t_n+s})ds| + |\textstyle\int_{t_n}^{t_n+t} h(s, y_s)ds|\}$$

$$\le \limsup_{n\to\infty} |\varphi(o) - y(t_n)| + \limsup_{n\to\infty} |\textstyle\int_o^t [g(\varphi)$$

$$- g(y_{t_n+s})]ds| + \limsup_{n\to\infty} \mu(t_n) \le t\beta(\rho)$$

where β is from Lemma 4.2 and

$$\rho = \limsup_{n\to\infty} \{\|\varphi - y_{t_n+s}\| : 0 \le s \le t\} \quad .$$

For $0 \le s \le t$, by using (4.8)

$$|y_{t_n+s}(\theta) - \varphi(\theta)| \le |y(t_n + \theta) - \varphi(\theta)| + |y(t_n + \theta) - y(t_n + s + \theta)|$$

$$\le |y(t_n + \theta) - \varphi(\theta)| + tM + \mu(t_n - r), \ -r \le \theta \le 0 \ .$$

Hence, $\rho \le tM$. From (4.10)

$$\frac{1}{t} d_o(\varphi + t\bar{g}(\varphi), L_t) \le \beta(tM) \ .$$

This completes the proof by an application of Theorem 4.1. □

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