HOPF'S BIFURCATION FOR NON-LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH APPLICATIONS TO EPIDEMIC MODELS

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ABSTRACT

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By

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In this thesis we consider systems of functional differential equations with several parameters. Assuming a generic condition on the rate of change of the real part of eigenvalues of the linearized problem, we are able to show the existence of the bifurcating oscillations for the system of equations. The general theorem is then applied to the equations of epidemics to obtain the existence of bifurcating solutions.

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In Memory of My

Mother and Father

Mr. David Green, Sr.

and

Mrs. Rose Bud Green

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CHAPTER I

INTRODUCTION

The spread of communicable diseases through society involves many disease related factors such as incubation period, susceptibility, infectious period, mode of transmission, and resistance to the disease. Also social, cultural, economic, and geographic factors may be considered in any model describing the spread of the disease. To analyze communicable diseases, differential equation models which incorporate some of these factors are useful. In this thesis, deterministic models are employed to study the spread of a communicable disease through society. We use deterministic models rather than stochastic models so that differential equations are used to describe changes in the population. Stochastic models are necessary when dealing with diseased population which have very few individuals sick at various times, but these models are very difficult to analyze since there is no single solution $x:[t_0,t_1] \to R$, where $x(t)$ is the number of individuals in the population at time t. Both deterministic and stochastic models are described in the book by N.T. Bailey [1]. Deterministic threshold models are considered in the monograph by P. Waltman [11]. These models assume that a susceptible individual does not become infectious upon first exposure to an infectious individual, but only after repeated exposure to infectious individuals has broken down the susceptible individual's resistance.

The population or community under consideration in these models is divided into four disjoint classes which change with time t. The susceptible class, $S(t)$, consists of those individuals who can incur the disease but who are not yet infected. Infectious individuals will be referred to as infectives. The infective class, $I(t)$, consists of those who are transmitting the disease to others. The removed class, $R(t)$, consists of those individuals who are removed from the susceptible infective interaction by recovery. The exposed class, $E(t)$, consists of those individuals exposed to the infection, who will as a result become infectious (but are not yet infectious).

It is customary to scale the functions $I(t)$, $S(t)$, $E(t)$ and $R(t)$ so that they represent the corresponding fractions of the total populations: $I(t) + S(t) + E(t) + R(t) = 1$. Deterministic models treat each of these functions as being continuously varying.

In the sections that follow we present some communicable disease models and give explicit equations that govern the spread of the disease through society. An analysis of the equations derived from these models is given in Chapter III.

§1. Infectious Disease Model for Gonorrhea

In this section we present an infectious disease model for gonorrhea. The model presented here is by Cooke and Yorke [3].

Individuals who become infected with gonorrhea recover only after drug therapy and do not develop any observable resistance to the disease. After recovery, they immediately become susceptible. The incubation period is from 3 to 7 days and can be ignored when looking for long term oscillations. We therefore assume that an exposed individual immediately becomes infectious so that $E(t) \equiv 0$. Also we assume that there is no immunity from the disease so that $R(t) \equiv 0$. The population for the gonorrhea model is composed of two types of individuals, the susceptibles and the infectives. We call these the active population. We assume that the active population remains constant. Let $x(t)$ denote the size of the infectious population and $S(t)$ the number of susceptibles. The rate of new infection depends only on contacts between susceptibles and infectious individuals. We assume that there is a small time lag σ, $0 < \sigma < 1$, between contacts with a susceptible and an infective, before new infectives are observed. Since $S(t)$ equals the constant total minus $x(t)$, this rate in effect depends only on $x(t)$ and can be written $g(x(t - \sigma))$ for some continuous function g. This model also assumes that there is a single infectious period L, $0 < \sigma < L$ (the time it takes an individual to seek out and receive treatment). Therefore people are infected at the rate $g(x(t - \sigma))$ and are cured at the rate at which they contacted the disease L time units ago, $g(x(t - L))$. The function $x(t)$ then satisfies the differential equation

$$\frac{dx}{dt} = g(x(t - \sigma)) - g(x(t - L)) \tag{1.1}$$

Equation (1.1) can be written in integral form as

$$x(t) = \int_{t-L}^{t-\sigma} g(x(s))ds + c \tag{1.2}$$

for some constant c. The integral term in (1.2) is the number of individuals infected over the time period $[t - L, t - \sigma]$. To have a correct biological interpretation we must have $c = 0$, since $x(t)$ is the infected population.

Assume instead of a single infectious period that there is a distribution of times until cure. Let $P(a)$ be the probability of having the disease continuously for at least time "a" after infection, and let L be the maximum cure time so that $P(L) = 0$. As before, let $g(x(t))$ be the rate of new infection. Then the number of persons cured per unit time at t is

$$-\int_0^L g(x(t-s))P'(s)ds$$

since $-P'(s)ds$ is the probability of being cured in $[s, s+ds]$. In this model the function $x(t)$ satisfies

$$\frac{dx}{dt} = g(x(t-\sigma)) - \int_0^L g(x(t-s))p(s)ds \tag{1.3}$$

where for $0 \leq a \leq L$,

$$P(a) = \int_0^L p(s)ds, \quad P(0) = 1, \quad p(s) \geq 0. \tag{1.4}$$

The existence of the density $p(s)$ is assumed. Equation (1.4) has integral form

$$x(t) = \int_{t-L}^{t-\sigma} P(t-s)g(x(s))ds + c \tag{1.5}$$

The constant c is chosen to be zero for the same reason as in (1.2). Note that if $c > 0$, this is interpreted as meaning that there is a constant subpopulation of size c of incurable infectious carriers of the disease.

The active population in the case of gonorrhea is composed of two subpopulations, the infected males and the infected females. This decomposition into subpopulations is necessary for a more accurate and detail analysis in studying the spread of the disease. For males, gonorrhea is easily detected since pain usually develops a couple days after initial infection, whereas with females infection can go undetected for longer periods of time. Thus females can be infectious and is able to transmit the disease without knowing they have it. To study the disease when both the female and male population are considered as separate sub-populations of the active population, we assume that the number of males with the disease is directly proportional to the number of females with the disease. We call the female population the main reservior for spreading the disease. This assumption seems quite reasonable when one considers the social behavior of society at large. In this model, $x(t)$ is the size of the total population. Let c_m and c_f be the proportions of the population which are male and female respectively. Let $P_m(s)$ and $P_f(s)$ be respectively the fraction of the infected male and female population which takes longer than time s to be cured after infection begins. Thus $P_m(0) = P_f(0) = 1$. Let L be the maximum cure time. Choose L large enough so that $P_m(L) = P_f(L) = 0$. Let $P(s) = c_m P_m(s) + c_f P_f(s)$, then the function $x(t)$ satisfies (1.5). We will delay analyzing the solutions of the gonorrhea model and the following models until Chapter III.

§2. An Economic Interpretation

We now consider an economic interpretation of the second model for gonorrhea presented above.

Let $x(t)$ denote the value of a capital stock at time t. Assume that the rate of production of new capital depends only on $x(t)$, and that this rate is given by $g(x(t))$ for some continuous function g. We assume equipment depreciates over a time L to value 0. L is the lifetime of the equipment. We further assume that the depreciation is independent of the type of equipment and at time "a" after production, the value of a unit of capital equipment has decreased in value to $P(a)$ times its original value (so $P(0) = 1$ and $P(L) = 0$). Thus at any time t, $x(t)$ equals the sum of the capital product over the period $[t - L, t]$ plus some constant c, where c denotes the value of non-depreciating assets. The function $x(t)$ satisfies

$$x(t) = \int_0^L P(a)g(x(t - a))da + c \tag{2.1}$$

Letting $s = t - a$, we obtain

$$x(t) = \int_{t-L}^{t} P(t - s)g(x(s))ds + c \tag{2.2}$$

Equation (2.2) is the same as (1.5) with $\sigma = 0$. It is easy to see that (1.3) is the differentiated form of (2.2) where $p(a) = \frac{-d}{da} P(a)$ is the rate of decrease in the value at age "a" for a capital unit whose value at the beginning of production is 1.

§3. Infectious Disease Model

In this section we present an epidemic model by Hoppensteadt and Waltman [7]. This particular model is a generalization of the model of Cooke and Yorke [3] presented in section 1.

At time $t = 0$, I_0 infectious individuals, called infectives, are introduced in a homogeneous way into a population of S_0 susceptibles. At any time $t > 0$, the population is divided into four disjoint subpopulations, $S(t)$, $I(t)$, $R(t)$ and $E(t)$. These functions have been defined previously. We assume the disease spreads through the population according to the following rules:

i) The rate of exposure of susceptibles to infectives at time t is $-r(t)I(t)S(t)$, where r is a positive continuous function.

ii) An individual exposed at time τ becomes infective at time t if $\int_\tau^t [\rho_1(x) + \rho_2(x)I(x)]dx = m$ where $\rho_1(x)$, $\rho_2(x)$ are given non-negative continuous functions and m is a non-negative constant.

iii) An individual infected at time t recovers and becomes immune at time $t + \sigma$, σ a positive constant.

iv) An individual first immune at time t becomes susceptible at time $t + \omega$, ω a positive constant.

v) The population remains constant.

Let $I_0(t)$, $-\sigma \leq t \leq 0$ be the function which describes the past history of the infectives. $I_0(t)$ is monotone and satisfies $I_0(0) = I_0$, $I_0(-\sigma) = 0$. Assuming that condition (iii) applies to these initial infectives, the future of these infectives is then known. We take this into account by defining an extension of $I_0(t)$ to the real line by

$$I_0(t) = \begin{cases} 0, & |t| \geq \sigma \\ I_0(t), & -\sigma \leq t \leq 0 \\ I_0(0) - I_0(t - \sigma), & 0 \leq t \leq \sigma \end{cases}$$

We also assume the existence of $t_0 < \sigma$ such that

$$\int_0^{t_0} [\rho_1(x) + \rho_2(x) I_0(x)]dx = m.$$

This condition expresses the fact that some of the initial susceptibles must become infectious before time σ. Any monotone function $I_0(t)$ whose extension to the reals satisfies the above integral equation is called an admissible function.

We can describe the spread of the infection by three functional equations in the unknowns τ, S, and I. Equations for $R(t)$ and $E(t)$ can be obtained from these. The equations for the model will be derived from the rate at which individuals are leaving the susceptible class. To do this, it is necessary to account for those initially infective individuals who are in classes $S(t)$ and $R(t)$ for the first time. These will be denoted by $I_1(t)$ and $I_2(t)$ respectively. The earliest possible entry for an initial infective into class $S(t)$ is $t = \omega$, and for $t > \omega$, the number of initially infective individuals who are in class $S(t)$ for the first time is the number who recover before $t - \omega$. Therefore,

$$I_1(t) = \begin{cases} 0, & t \leq \omega \\ I_0(0) - I_0(t-\omega), & \omega \leq t \end{cases}$$

For $0 \leq t \leq \omega$, those initially infective individuals who are now in class $R(t)$ are those who were initially infective at time $t = 0$ minus those who are still infective from this initial infective population. For $t \geq \omega$, those initially infective who are now in class $R(t)$ are those who are in class $R(t)$ for $0 \leq t \leq \omega$ minus those in class $S(t)$ for $t \geq \omega$. Therefore,

$$I_2(t) = \begin{cases} I_0(0) - I_0(t), & 0 \leq t \leq \omega \\ I_0(t-\omega) - I_0(t), & \omega \leq t \end{cases}$$

The equations for I, S and τ that describe the spread of the disease are as follows:

(3.1) $\int_{\tau(t)}^{t} [\rho_1(x) + \rho_2(x) I(x) dx = m, \ \tau(t) \equiv 0$ if $t \leq t_0$.

The equation $\tau(t) \equiv 0$ is just a convenience for eliminating special cases in the equations that follow. Also, to be infective at time t, a susceptible must be exposed before $\tau(t)$. Therefore, we require that the susceptible population at time t consists of all those individuals who have not been exposed in the interval $(\tau(t - \sigma - \omega), t)$. Any individual exposed in this interval is eliminated from class $S(t)$; exposure before $\tau(t - \sigma - \omega)$ results in infection before time $t - \sigma - \omega$, recovery before $t - \omega$ and readmission to class $S(t)$ before time t. It follows that

$$(3.2) \qquad S(t) = I_1(t) + S_0 - \int_{\tau(t-\sigma-\omega)}^{t} r(x)I(x)S(x)\,dx$$

In a similar manner, the class of infectives at time t consists of the initial infectives who are still infective at time t plus those individuals who were exposed between $\tau(t - \sigma)$ and $\tau(t)$. Those who were exposed before $\tau(t - \sigma)$ have recovered by time t and those exposed after $\tau(t)$ are not yet infective. Therefore, we have

$$(3.3) \qquad I(t) = I_0(t) + \int_{\tau(t-\sigma)}^{\tau(t)} r(x)I(x)S(x)\,dx$$

Equations (3.1)-(3.3) constitute the basic equations that describe the spread of the disease through society.

The questions of the existence, uniqueness, and continuous dependence of solutions for these models have been resolved [3,7]. Also all the models presented above are very similar in that the oscillatory behavior of their solutions is similar. Each of these models has solutions which exhibit behavior that indicate the solutions may be periodic. The numerical work of J. Mosevich [9], and the work of Greenberg [4] indicates that the solutions of the functional differential equations which arise in the model by Hoppensteadt and Waltman are periodic for certain values of ω. In Chapter II, we will develop some general theory for the existence of nonzero periodic solutions to functional differential equations with a real parameter γ. Using this theory we show in Chapter III that these models presented above all have nonzero periodic solutions.

CHAPTER II

A BIFURCATION THEOREM FOR NONLINEAR FDE'S

§1. Preliminaries

Let E^n denote the real or complex Euclidean n-space. For $r > 0$, let $C = C[-r,0]$ be the space of continuous functions from $[-r,0]$ to E^n with the usual supremum norm. If x is a continuous function defined on $[\sigma - r, \sigma + A)$, $\sigma \in R$, $A > 0$, with values in E^n, let x_t, $t \in [\sigma, \sigma + A)$ be the element in $C[-r,0]$ defined by

$$x_t(\theta) = x(t + \theta), \quad -r \leq \theta \leq 0. \tag{1.1}$$

We will denote by $BV[-r,0]$, the space of vector-valued functions, η, on $[-r,0]$ which are of bounded variation and are normalized such that

$$\begin{aligned} \eta(\theta) &= 0 \qquad && \theta \geq 0 \\ \eta(\theta) &= \eta(-r) && \theta \leq -r \end{aligned}$$

and η is left-continuous on $[-r,0)$.

Consider the autonomous linear functional differential equation

$$\dot{x}(t) = L(x_t) \tag{1.2}$$

where $L(\cdot)$ is a continuous linear operator mapping $C[-r,0]$ into E^n. By the Reisz representation theorem, there exists a unique $n \times n$ normalized matrix $\eta(\cdot) \in BV[-r,0]$ such that

(1.3) $$L(\varphi) = \int_{-r}^{0} [d\eta(\theta)]\varphi(\theta), \quad \varphi \in C[-r,0].$$

Let $x(\varphi)$ be the unique solution of (1.2) through $(0,\varphi)$ and let $T(t)$ be the bounded linear operator mapping $C[-r,0]$ into $C[-r,0]$ defined by

(1.4) $$T(t)\varphi = x_t(\varphi)$$

It is well-known that $\{T(t): t \geq 0\}$ is a strongly continuous semi-group of bounded linear operators on $C[-r,0]$ (see Hale [5, p.94]). The infinitesimal generator A of the semi-group $T(t)$, $t \geq 0$ is given by

(1.5) $$A\varphi(\theta) = \begin{cases} \dot{\varphi}(\theta) & -r \leq \theta < 0 \\ L(\varphi) = \int_{-r}^{0} [d\eta(\theta)]\varphi(\theta), & \theta = 0 \end{cases}$$

It is known that the domain of A consists of all continuously differentiable functions $\varphi(\theta)$, $-r \leq \theta \leq 0$, with

(1.6) $$\lim_{\theta \to 0^-} \frac{\varphi(\theta) - \varphi(0)}{\theta - 0} = \int_{-r}^{0} [d\eta(\theta)]\varphi(\theta)$$

It is well-known that if A is defined by (1.5), then the spectrum of A, $\sigma(A)$ is the same as the point spectrum of A and possibly $\lambda = 0$. Moreover, $\lambda \in \sigma(A)$ if and only if

(1.7) $$\det \Delta(\lambda) = \det[\lambda I - \int_{-r}^{0} e^{\lambda\theta}\, d\eta(\theta)] = 0$$

where $\Delta(\lambda) = \lambda I - \int_{-r}^{0} e^{\lambda\theta} d\eta(\theta)$. Equation (1.7) is called the characteristic equation of (1.2) and its roots are the eigenvalues of (1.2). For any real β, the set $\{\lambda : \det \Delta(\lambda) = 0, \ \mathrm{Re}\,\lambda > \beta\}$ is finite.

Given a fixed eigenvalue λ_0 of (1.2), the null spaces $\eta(A - \lambda_0 I)^j$ of $(A - \lambda_0 I)^j$, $j = 1,2,\ldots,$ satisfy the following nested relations: $\eta(A - \lambda_0 I) \subset \eta(A - \lambda_0 I)^2 \subset \cdots$. There exists a smallest integer $k \geq 1$ such that $\eta(A - \lambda_0 I)^m$ is a subspace of $\eta(A - \lambda_0 I)^k$ for all $1 \leq m \leq k$, and $\eta(A - \lambda_0 I)^m = \eta(A - \lambda_0 I)^k$ for all $m \geq k$. The generalized eigenspace P_{λ_0} belonging to λ_0 is defined to be $\eta(A - \lambda_0 I)^k$. If Q_{λ_0} is the complementary subspace of P_{λ_0}, the following properties are known:

i) $\dim P_{\lambda_0} < \infty$, where $\dim P_{\lambda_0}$ = algebraic multiplicity of λ_0.

ii) P_{λ_0} and Q_{λ_0} are invariant under $T(t)$.

iii) $Q_{\lambda_0} = R(A - \lambda_0 I)^k$, the range of $(A - \lambda_0 I)^k$.

iv) $C = P_{\lambda_0} \oplus Q_{\lambda_0}$, where $\oplus$ denotes direct sum.

§2. Space Decomposition

In this section we consider the projection Π_P on the space $C = C[-r,0]$.

Definition 2.1. An eigenvalue of (1.2) is called simple if its algebraic multiplicity is one.

<u>Theorem 2.1</u>. Let $\lambda_0 = i\nu_0$, $\nu_0 \neq 0$ be a purely imaginary simple eigenvalue of (1.2) and let P_{λ_0} be the generalized eigenspace belonging to λ_0 with projection

$$\Pi_{P_{\lambda_0}} : C[-r,0] \to P_{\lambda_0},$$

then

(2.1) $\quad \Pi_{P_{\lambda_0}}(\varphi) = \varphi_0\langle\psi_0,\varphi\rangle$ for any $\varphi \in C[-r,0]$, $\psi_0 \in C[0,r]$.

(2.2) $$\varphi_0(\theta) = e^{\lambda_0\theta} a_0, \quad -r \leq \theta \leq 0$$

(2.3) $$\psi_0(s) = b_0^T e^{-\lambda_0 s}(b_0^T\Delta'(\lambda_0)a_0)^{-1}, \quad 0 \leq s \leq r$$

and $\langle\cdot,\cdot\rangle$ is a bilinear form defined by

(2.4) $$\langle\psi,\varphi\rangle = \psi_0^T\varphi(0) - \int_{-r}^{0}\int_{0}^{\theta} \psi^T(\xi-\theta)d\eta(\theta)\varphi(\xi)d\xi,$$

where a_0 is the unique solution of $\Delta(\lambda_0)a = 0$ with $|a| = 1$, b_0^T is the unique solution of $b^T\Delta(\lambda_0) = 0$ with $|b| = 1$. ("T" denotes the transpose).

<u>Proof</u>: Clearly $\varphi_0 \in \eta(A - \lambda_0 I)$ since $A\varphi_0 = \lambda_0\varphi_0$ implies $(A - \lambda_0 I)\varphi_0 = 0$. Thus by definition of a simple eigenvalue P_{λ_0} consists of exactly those scalar multiples of φ_0. This shows the uniqueness of a_0. Similarly we have the uniqueness of b_0^T. Let $\psi_0^*(s) = b_0^T e^{-\lambda_0 s}$, $0 \leq s \leq r$, then

$$\langle \psi_0^*, \varphi_0 \rangle = b_0^T [I - \int_{-r}^{0} \int_0^{\theta} e^{\lambda_0(\theta-\xi)} d\eta(\theta) e^{\lambda_0 \xi} d\xi] a_0$$

$$= b_0^T [I - \int_{-r}^{0} d\eta(\theta)\, \theta e^{\lambda_0 \theta}] a_0$$

$$= b_0^T \Delta'(\lambda_0) a_0$$

where

$$\Delta'(\lambda_0) = I - \int_{-r}^{0} d\eta(\theta)\, \theta e^{\lambda_0 \theta}.$$

It follows from the proof of Lemma 21.2 in [5, p.109] that $\langle \psi_0^*, \varphi_0 \rangle \neq 0$. Therefore $\psi_0(s)$ is defined. $\langle \psi_0^*, \varphi_0 \rangle \neq 0$ also follows from more general results in functional analysis (see, for example [10]).

$$\langle \psi_0, \varphi_0 \rangle = \langle b_0^T e^{-\lambda_0 s} (b_0^T \Delta'(\lambda_0) a_0)^{-1}, \varphi_0 \rangle$$

$$= (b_0^T \Delta'(\lambda_0) a_0)^{-1} \langle b_0^T e^{-\lambda_0 s}, \varphi_0 \rangle$$

$$= (b_0^T \Delta'(\lambda_0) a_0)^{-1} \langle \psi_0^*, \varphi_0 \rangle$$

$$= (b_0^T \Delta'(\lambda_0) a_0)^{-1} (b_0^T \Delta'(\lambda_0) a_0) = 1.$$

Therefore,

$$\Pi_{P_{\lambda_0}}(\varphi) = \varphi_0 \langle \psi_0, \varphi \rangle.$$

In applications, system (1.2) is often given in the real Euclidean n-space R^n. For this reason, it is often desirable to obtain the projections and the constant of variation formula in R^n.

Suppose (1.2) is real. All complex eigenvalues of (1.2) appear in complex conjugate pairs. Assume $\lambda_0 = i\nu_0$ is a simple eigenvalue of (1.2). Hence $\bar{\lambda}_0 = -i\nu_0$ is also an eigenvalue of (1.2). Let P be the generalized eigenspace belonging to $\{i\nu_0, -i\nu_0\}$. Then, $\dim P = 2$ and $\Phi_0 = (\mathrm{Re}\ \varphi_0, \mathrm{im}\ \varphi_0)$ is a basis for P (where φ_0 is as in Theorem 2.1).

<u>Theorem 2.2</u>. Let $\lambda_0 = i\nu_0$, $\nu_0 > 0$ be a simple eigenvalue of the real system (1.2) and P be the generalized eigenspace belonging to $\{i\nu_0, -i\nu_0\}$. Then the projection Π_P onto P is given by

$$\Pi_P(\varphi) = \Phi_0\langle\Psi_0, \varphi\rangle \tag{2.5}$$

where

$$\varphi_0(\theta) = e^{\lambda_0\theta} a_0, \quad -r \le \theta \le 0 \tag{2.6}$$

$$\psi_0(s) = b_0^T(b_0^T\Delta'(\lambda_0)a_0)^{-1} e^{-\lambda_0 s}, \quad 0 \le s \le r \tag{2.7}$$

where

$$\begin{aligned} \Phi_0 &= (\mathrm{Re}\ \varphi_0, \mathrm{Im}\ \varphi_0) \\ \Psi_0 &= \begin{pmatrix} \mathrm{Re}\ \psi_0 \\ \mathrm{Im}\ \psi_0 \end{pmatrix} \end{aligned} \tag{2.8}$$

$a_0 \in E^n$ is the unique solution of $\Delta(\lambda_0)a = 0$ with $|a| = 1$, b_0^T is the unique solution of $b_0^T\Delta(\lambda_0) = 0$ with $|b| = 1$.

Proof: $\Phi_0 = (\mathrm{Re}\ \varphi_0, \mathrm{Im}\ \varphi_0)$ is a basis for P. By Lemma 21.4, Hale [5, p.109], we have $\langle\psi_0, \varphi_0\rangle = 1$, $\langle\psi_0, \bar{\varphi}_0\rangle = 0$, $\langle\bar{\psi}_0, \varphi_0\rangle = 0$ and $\langle\bar{\psi}_0, \bar{\varphi}_0\rangle = 1$. These imply that $\langle\Psi_0, \Phi_0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This proves the theorem.

Corollary 2.1. For any $\varphi \in C[-r,0]$,

$$\Pi_P(\varphi) = \langle \mathrm{Re}\ \psi_0, \varphi\rangle \mathrm{Re}\ \varphi_0 + \langle \mathrm{Im}\ \psi_0, \varphi\rangle \mathrm{Im}\ \varphi_0 \tag{2.9}$$

Proof: Since $\langle \mathrm{Re}\ \psi_0, \mathrm{Re}\ \varphi_0\rangle = 1$, $\langle \mathrm{Re}\ \psi_0, \mathrm{Im}\ \varphi_0\rangle = 0$, $\langle \mathrm{Im}\ \psi_0, \mathrm{Im}\ \varphi_0\rangle = 1$ and $\langle \mathrm{Im}\ \psi_0, \mathrm{Re}\ \varphi_0\rangle = 0$, writing out (2.5) gives the corollary.

§3. Linear Autonomous FDE's With Real Parameter

In this section we consider (1.2) with a real parameter γ. With real parameter γ, system (1.2) becomes

$$\dot{x}(t) = L(\gamma, x_t) \tag{3.1}$$

where for each γ, $L(\gamma, \cdot)$ is a continuous linear operator mapping $C[-r,0]$ into E^n. We assume that there exists a unique $n \times n$ normalized matrix $\eta(\gamma, \cdot) \in BV[-r,0]$ such that

$$L(\gamma, \varphi) = \int_{-r}^{0} -d\eta(\gamma, \theta)]\varphi(\theta), \quad \varphi \in C[-r,0], \tag{3.2}$$

and the dependence of η on γ is smooth. By smoothness, we mean the following:

$$\eta(\gamma, \theta) = \eta_0(\theta) + \gamma\eta_1(\theta) + \gamma^2\eta_2(\theta) + O(|\gamma|^2) \tag{3.3}$$

where η_0, η_1 and $\eta_2 \in BV[-r,0]$. The characteristic equation of (3.1) has eigenvalues which also depend on γ.

Let $\gamma = \gamma_0$ be fixed. If $\mathrm{Re}\,\lambda < 0$ for every eigenvalue λ of (3.1) with $\gamma = \gamma_0$, then the zero solution of (3.1) for $\gamma = \gamma_0$ is exponentially stable. Suppose for a different value of γ, say $\gamma = \gamma_1$, there exists a pair of complex conjugate eigenvalues of (3.1) with real part greater than zero, then the zero solution of (3.1) for $\gamma = \gamma_1$ is unstable.

We assume that there exist such a γ_0 and γ_1. $\mathrm{Re}\,\lambda$ is a continuous function of the real parameter γ and therefore by the intermediate value property for continuous functions there is a γ, say $\gamma = \tilde{\gamma}$ with $\gamma_0 < \tilde{\gamma} < \gamma_1$ and a $\lambda(\tilde{\gamma})$ such that $\mathrm{Re}\,\lambda(\tilde{\gamma}) = 0$. Without loss of generality we may assume $\tilde{\gamma} = 0$. We further assume that the eigenvalue, $\lambda(\gamma) = \mu(\gamma) + i\nu(\gamma)$, of the characteristic equation of (3.1) satisfies near $\gamma = 0$,

$$\mu(\gamma) = \mu_0 + \mu_1\gamma + \mu_2\gamma^2 + o(|\gamma|^2) \tag{3.4}$$

$$\nu(\gamma) = \nu_0 + \nu_1\gamma + \nu_2\gamma^2 + o(|\gamma|^2) \tag{3.5}$$

where $\mu_i, \nu_i \in R$ for $i = 0,1,2$. The condition $\mathrm{Re}\,\lambda = 0$ then implies $\mu(0) = \mu_0 = 0$ and $\nu(0) = \nu_0 \neq 0$. Also for $\gamma = 0$ in (3.1) there exist periodic solutions of period $2\pi\nu_0^{-1}$. For non-linear systems we expect periodic solutions to bifurcate from $\gamma = 0$. In the next sections we discuss such possibilities.

§4. Non-linear Autonomous FDE's With Real Parameters

Consider the real non-linear system

$$\dot{x}(t) = L(\gamma, x_t) + N(\gamma, x_t) \tag{4.1}$$

where γ is a real parameter, L is as in section 3, N is the non-linear part and is smooth in its arguments. Assume that for φ near zero, $N(\gamma, \varphi) = O(|\varphi|)$ uniformly on bounded sets of γ.

The unique solution of (4.1) with initial condition $x_\sigma = \varphi$ is given by the constant of variation formula

$$x_t = T(\gamma)(t - \sigma)\varphi + \int_\sigma^t T(\gamma)(t - s)X_0 N(\gamma, x_s)ds \tag{4.2}$$

where $T(\gamma)(t)$ is the semi-group operator associated with (3.1) and

$$X_0(\theta) = \begin{cases} 0 & -r \leq \theta < 0 \\ I & \theta = 0 \end{cases} \tag{4.3}$$

where I is the identity matrix.

We seek to study the solutions of (4.1) on generalized eigenspaces of the linear part of (4.1). In fact on generalized eigenspaces, the solutions of (4.1) behave as solutions of an ordinary differential equation [5, p.101].

Suppose $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ is a finite set of eigenvalues of (3.1) and $C = C[-r,0]$ is decomposed by Λ as $C = P \oplus Q$

where P is the generalized eigenspace belonging to Λ and Q is the complementary subspace of P.

Let $\Phi(\gamma)$ be a basis for the generalized eigenspace P associated with Λ and $\Psi(\gamma)$ be a basis for the generalized eigenspace of the adjoint equation of (3.1) associated with Λ. As in Hale [5, p.101], the operator A defined by (1.5), satisfies $AP \subset P$. Since P is finite dimensional, there is a $n \times n$ matrix $B(\gamma)$ such that $A\Phi(\gamma) = \Phi^T(\gamma)B(\gamma)$. Here n is the dimension of the subspace P.

Let A^* be the adjoint operator of the operator A defined by (1.5). Also define $C^* = C([0,r],R^{n^*})$ where R^{n^*} is the n-dimensional vector space of row vectors. Then for any $\psi \in C^*$, the operator A^* is defined by (see [5], p.105),

$$A^*\psi = \begin{cases} -\dot{\psi}(s), & 0 < s \leq r \\ \int_{-\gamma}^{0} \psi(-\theta)d\eta(\theta), & s = 0 \end{cases} \tag{4.4}$$

It is known that if the domain of A^* consists of all functions ψ in C^* which have a continuous first derivative such that

$$-\dot{\psi}(0) = \int_{-r}^{0} \psi(-\alpha)d\eta(\theta) \tag{4.5}$$

then A^* maps the domain of A^* into C^* and the domain of A^* is dense in C^* [5, p.104].

For $\varphi \in C = C([-r,0],R^n)$ and $\psi \in C^* = C([0,r],R^{n^*})$ define the bilinear form $\langle\cdot,\cdot\rangle$ by

$$(4.6) \qquad \langle\psi,\varphi\rangle = \psi(0)\varphi(0) - \int_{-r}^{0}\int_{0}^{\theta} \psi(s-\theta)[d\eta(\theta)]\varphi(\theta)ds$$

We may assume that $\langle\Psi(\gamma),\Phi(\gamma)\rangle = I$, where I is the identity matrix. Also if the decomposition of any element φ in C is written $\varphi = \varphi^P + \varphi^Q$ where φ^P is in P and φ^Q is in Q, then $\varphi^P = \Phi(\gamma)\langle\Psi(\gamma),\varphi(\gamma)\rangle$. We now consider the following situation.

Let γ be fixed, $\lambda_0 = \mu + i\nu$ and $\bar{\lambda}_0 = \mu - i\nu$ be simple eigenvalues of (3.1). Let P be the generalized eigenspace associated with $\lambda_0 = \mu + i\nu$ and $\bar{\lambda}_0 = \mu - i\nu$. Then the dimension of P is 2, and the operator A defined by (1.5) satisfies $AP \subset P$. Therefore there is a 2×2 matrix B such that $A\Phi = \Phi^T B$, where Φ is the basis for the generalized eigenspace P. The subspace P is spanned by eigenfunctions of the form $e^{\lambda\theta}$, $-r \leq \theta \leq 0$. We will now determine the form of the matrix B.

Let φ_0 be the eigenfunction corresponding to λ_0. Since the system is real, $\bar{\varphi}_0$ is the eigenfunction corresponding to $\bar{\lambda}_0$. Moreover, $\varphi_0(\theta) = e^{\lambda_0\theta} a_0$, $a_0 \in E^n$. Let

$$\varphi_1(\theta) = \frac{1}{2}[\varphi_0(\theta) + \bar{\varphi}_0(\theta)]$$

$$\varphi_2(\theta) = \frac{1}{2i}[\varphi_0(\theta) - \bar{\varphi}_0(\theta)]$$

The eigenfunctions φ_1 and φ_2 are real and are a basis for P. We will consider this basis because the Banach space $C = C[-r,0]$ is real. Applying the operator A defined by (1.5) to these basis elements, we have

$$A\varphi_1(\theta) = \frac{1}{2}\frac{d}{d\theta}[\varphi_0(\theta) + \bar{\varphi}_0(\theta)] \tag{4.7}$$
$$= \frac{1}{2}[\lambda_0\varphi_0(\theta) + \bar{\lambda}_0\bar{\varphi}_0(\theta)] = \mu\varphi_1(\theta) - \nu\varphi_2(\theta)$$

$$A\varphi_2(\theta) = \frac{1}{2i}\frac{d}{d\theta}[\varphi_0(\theta) - \bar{\varphi}_0(\theta)] \tag{4.8}$$
$$= \frac{1}{2i}[\lambda_0\varphi_0(\theta) - \bar{\lambda}_0\bar{\varphi}_0(\theta)] = \mu\varphi_2(\theta) + \nu\varphi_1(\theta)$$
$$= \nu\varphi_1(\theta) + \mu\varphi_2(\theta)$$

Therefore,

$$A\begin{pmatrix}\varphi_1(\theta)\\ \varphi_2(\theta)\end{pmatrix} = (\varphi_1(\theta),\varphi_2(\theta))\begin{pmatrix}\mu & \nu\\ -\nu & \mu\end{pmatrix} \tag{4.9}$$

The matrix B is given by

$$B = \begin{pmatrix}\mu & \nu\\ -\nu & \mu\end{pmatrix} \tag{4.10}$$

Thus the matrix B is strictly determined by the basis for the generalized eigenspace P. The operator A^* defined by (4.4) has eigenfunctions of the form $e^{-\lambda\theta}$, $0 \leq \theta \leq r$. We will now prove the following.

<u>Theorem 4.1</u>. Let γ be fixed. Let $\lambda(\gamma) = \mu(\gamma) + i\nu(\gamma)$ and $\bar{\lambda}(\gamma) = \mu(\gamma) - i\nu(\gamma)$ be simple eigenvalues of (3.1), and P be the generalized eigenspace associated with $\lambda(\gamma)$ and $\bar{\lambda}(\gamma)$ with projection

$$\Pi_P : C[-r,0] \to P$$

Consider the real non-homogeneous system (4.1). If $x(\gamma,\varphi)$ is the unique solution of (4.1) through $(0,\varphi)$ and

$$w(\gamma)(t) = \langle \Psi_0(\gamma), x_t(\gamma) \rangle$$

then

$$(4.11) \qquad \frac{d}{dt} w(\gamma)(t) = B(\gamma)w(\gamma)(t) + \Psi_0(\gamma)(0)N(\gamma, x_t(\gamma))$$

$$(4.12) \qquad z_t = \Pi_Q T(\gamma)(t-\sigma)\varphi + \Pi_Q \int_\sigma^t T(\gamma)(t-s)X_0 N(\gamma, x_s(\gamma))ds$$

where $\Pi_Q = 1 - \Pi_P$ and $z_t = \Pi_Q x_t$

$$(4.13) \qquad B(\gamma) = \begin{pmatrix} \mu(\gamma) & \nu(\gamma) \\ -\nu(\gamma) & \mu(\gamma) \end{pmatrix}$$

<u>Proof</u>: Let $x(\gamma)(t)$ be a solution of (4.1), and $y(\gamma)(t)$ be a solution of the adjoint equation to (4.1) on $(-\infty,\sigma]$. For each $t \in (-\infty,\sigma]$, let $y(\gamma)^t$ be the element in $C^* = C[0,r]$ defined by $y(\gamma)^t(\alpha) = y(\gamma)(t+\alpha)$ for $\alpha \in [0,r]$. Then by Theorem 17.1, [5, p.90] we have for all $t \geq \sigma$,

$$\langle y(\gamma)^t, x_t(\gamma) \rangle = \int_\sigma^t y(\gamma)(s)N(\gamma, x_s(\gamma))ds + \langle y(\gamma)^\sigma, x_\sigma(\gamma) \rangle.$$

Each row y the matrix $e^{-B(\gamma)\theta}\,\Psi_0(\gamma)(0)$, $0 \leq \theta \leq s$ is a solution of the adjoint equation on $(-\infty,\infty)$. Therefore

$$\langle e^{-B(\gamma)t}\,\Psi_0(\gamma) x_t(\gamma) \rangle = \int_\sigma^t e^{-B(\gamma)s}\,\Psi_0(\gamma)(0)N(\gamma, x_s(\gamma))ds$$
$$+ \langle e^{-B(\gamma)\sigma}\,\Psi_0(\gamma), \varphi(\gamma) \rangle$$

$$\langle\Psi_0(\gamma), x_t(\gamma)\rangle = \int_\sigma^t e^{B(\gamma)(t-s)}\ \Psi_0(\gamma)(0)N(\gamma, x_s(\gamma))ds$$
$$+ \langle e^{B(\gamma)(t-\sigma)}\ \Psi_0(\gamma), \varphi(\gamma)\rangle.$$

Let

$$\omega(\gamma)(t) = \langle\Psi_0(\gamma), x_t(\gamma)\rangle$$

then

$$\omega(\gamma)(\sigma) = \langle\Psi_0(\gamma), \varphi(\gamma)\rangle$$

Therefore,

$$\omega(\gamma)(t) = \int_\sigma^t e^{B(\gamma)(t-s)}\ \Psi_0(\gamma)(0)N(\gamma, x_s(\gamma))ds$$
$$+ e^{B(\gamma)(t-\sigma)}\ \omega(\gamma)(\sigma)$$

Differentiating with respect to t, we get

$$\frac{d}{dt}\ \omega(\gamma)(t) = B(\gamma)e^{B(\gamma)(t-\sigma)}\ \omega(\gamma)(\sigma)N(\gamma, x_t(\gamma))$$

Let $\sigma = t$, then

$$\frac{d}{dt}\ \omega(\gamma)(t) = B(\gamma)\omega(\gamma)(t) + \Psi_0(\gamma)(0)N(\gamma, x_t(\gamma))$$

Equation (4.10) follows immediately from (4.2) be taking projection Π_Q on both sides. This completes the proof of the theorem.

§5. The Bifurcation Theorem

In this section the main theorem is proved. We shall show the existence of non-zero periodic solutions using a technique employed by Hopf for ordinary differential equations with a real parameter γ [2,6].

<u>Theorem 5.1</u>. Consider the non-linear system

$$\dot{x}(t) = L(\gamma, x_t) + N(\gamma, x_t) \tag{5.1}$$

where γ is a real parameter, $L(\gamma, \cdot)$ is a continuous linear operator mapping $C[-r,0]$ into E^n, and the associated linear system

$$\dot{x}(t) = L(\gamma, x_t) \tag{5.2}$$

Assume

i) N is Fréchet differentiable and $N = O(|\varphi|)$ uniformly on bounded sets of γ.

ii) For $\gamma = 0$, there exists a unique pair of simple purely imaginary eigenvalues $i\nu_0$, $-i\nu_0$, $\nu_0 \neq 0$ and no other purely imaginary roots that are integral multiples of $i\nu_0$.

iii) $\text{Re}\,\lambda'(0) \neq 0$.

Then there exist non-zero periodic solutions bifurcating from $\gamma = 0$.

To prove Theorem 5.1, we need the following lemmas.

Let φ_0 and ψ_0 be the eigenfunctions corresponding to λ_0 and $\bar{\lambda}_0$. It is known that (see Section 2)

$$\varphi_0(\theta) = e^{\lambda_0 \theta} a_0, \quad -r \leq \theta \leq 0$$

$$\psi_0(s) = b_0^T e^{-\lambda_0 s} (b_0^T \Delta'(\lambda_0) a_0)^{-1}, \quad 0 \leq s \leq r$$

with

$$\langle\psi_0,\varphi_0\rangle = 1, \qquad \langle\bar{\psi}_0,\varphi_0\rangle = 0$$

$$\langle\psi_0,\bar{\varphi}_0\rangle = 0, \qquad \langle\bar{\psi}_0,\bar{\varphi}_0\rangle = 1$$

Let

$$\varphi_1(\theta) = \frac{1}{2}[\varphi_0(\theta) + \bar{\varphi}_0(\theta)]$$

$$\varphi_2(\theta) = \frac{1}{2i}[\varphi_0(\theta) - \bar{\varphi}_0(\theta)]$$

$$\psi_1(\theta) = \frac{1}{2}[\psi_0(\theta) + \bar{\psi}_0(\theta)]$$

$$\psi_2(\theta) = \frac{1}{2i}[\psi_0(\theta) - \bar{\psi}_0(\theta)]$$

$$T(\gamma,t) = T(\gamma)(t), \quad T(0,t) = T(t)$$

$$\omega_0 = \frac{2\pi}{\nu_0}, \qquad \lambda_0 = i\nu_0$$

We will now prove the following lemma.

Lemma 5.1. Let $T(t) = T(0,t)$ and $g_i(t) = \langle\Psi_i,T(t)\varphi_1\rangle$. Then

$$\begin{pmatrix} \dfrac{\partial g_1}{\partial t} \\ \\ \dfrac{\partial g_2}{\partial t} \end{pmatrix}_{t=\omega_0} = \begin{pmatrix} 0 \\ \\ \dfrac{\nu_0}{2} \end{pmatrix}$$

Proof: Applying the solution operator $T(t)$ to φ_0 and $\bar{\varphi}_0$ we have

$$T(t)\varphi_0 = e^{\lambda_0 t}\varphi_0$$

$$T(t)\bar{\varphi}_0 = e^{\bar{\lambda}_0 t}\bar{\varphi}_0$$

Therefore

$$\langle \Psi_1, T(t)\varphi_1\rangle = \frac{1}{4}\langle \psi_0 + \bar{\psi}_0, T(t)(\varphi_0 + \bar{\varphi}_0)\rangle$$
$$= \frac{1}{4}\langle \psi_0, e^{\lambda_0 t}\varphi_0\rangle + \frac{1}{4}\langle \bar{\psi}_0, e^{\bar{\lambda}_0 t}\bar{\varphi}_0\rangle$$

It follows that

$$\frac{\partial g_1}{\partial t}\Big|_{t=\omega_0} = \frac{\partial}{\partial t}\langle \Psi_1, T(t)\varphi_1\rangle\Big|_{t=\omega_0}$$
$$= \{\frac{1}{4}\lambda_0\langle \psi_0, e^{\lambda_0 t}\varphi_0\rangle + \frac{1}{4}\bar{\lambda}_0\langle \bar{\psi}_0, e^{\bar{\lambda}_0 t}\bar{\varphi}_0\rangle\}_{t=\omega_0}$$
$$= \frac{1}{4}(\lambda_0 + \bar{\lambda}_0) = 0.$$

Also

$$\langle \Psi_2, T(t)\varphi_1 = \frac{1}{4}\langle\frac{1}{i}(\psi_0 - \bar{\psi}_0), T(t)(\varphi_0 + \bar{\varphi}_0)\rangle$$
$$= \frac{1}{4}\langle\frac{1}{i}\psi_0, e^{\lambda_0 t}\varphi_0\rangle - \frac{1}{4}\langle\frac{1}{i}\bar{\psi}_0, e^{\bar{\lambda}_0 t}\bar{\varphi}_0\rangle$$

Differentiating with respect to t, we obtain

$$\frac{\partial g_2}{\partial t}\Big|_{t=\omega_0} = \frac{\partial}{\partial t}\langle \Psi_2, T(t)\varphi_1\rangle\Big|_{t=\omega_0} = \frac{1}{4i}(\lambda_0 - \bar{\lambda}_0)$$
$$= \frac{1}{4i}(2i\nu_0) = \frac{\nu_0}{2}$$

This proves Lemma 5.1.

<u>Lemma 5.2</u>. Let $T(\gamma,t)\varphi_0 = x_t(\gamma)$ where φ_0 is complex and x_t satisfies (3.1). Let

$$v(t) = \frac{\partial}{\partial\gamma}x(\gamma)(t)$$

then

$$(5.3) \qquad \dot{v}(t) = \begin{cases} L_0(v_t) + L_1(\varphi_0)e^{\lambda_0 t} \\ v_0 = 0 \end{cases}$$

where λ_0 is a complex eigenvalue of the characteristic equation of (3.1) and

$$(5.4) \qquad L_0(\varphi) = \int_{-r}^{0} [d\eta_0(\theta)]\varphi(\theta)$$

$$(5.5) \qquad L_1(\varphi) = \int_{-r}^{0} [d\eta_1(\theta)]\varphi(\theta).$$

<u>Proof</u>: Since $x(0) = \varphi(0)$ and $x(\gamma)(\theta) = \varphi(\theta)$ for all $\theta \in [-r,0]$, then $v(\theta) = \frac{\partial}{\partial\gamma} x(\gamma)(\theta) = \frac{\partial}{\partial\gamma} \varphi(\theta) = 0$ for all θ in $[-r,0]$. Therefore $v_0 = 0$. We also have

$$\dot{x}(\gamma)(t) = L(\gamma, x_t) = \int_{-r}^{0} [d\eta(\gamma,\theta)]x(\gamma)(s + \theta)$$

Therefore

$$x(\gamma)(t) = x(\gamma)(0) + \int_0^t\int_{-r}^{0} [d\eta(\gamma,\theta)]x(\gamma)(s + \theta)ds$$

Since the initial conditions for any solution are the same, we have

$$\begin{aligned} x(\gamma)(t) - x(0)(t) &= \int_0^t\int_{-r}^{0} [d\eta(\gamma,\theta)]x(\gamma)(s + \theta)ds \\ &\quad - \int_0^t L_0(x_s(0))ds \\ &= \int_0^t\int_{-r}^{0} [d\eta_0(\theta) + \gamma d\eta_1(\theta) \\ &\quad + O(|\gamma|^2)]x(\gamma)(s + \theta)ds - \int_0^t L_0(x_s(0))ds \\ &= \int_0^t L_0(x_s(\gamma) - x_s(0))ds \\ &\quad + \gamma\int_0^t L_1(x_s(\gamma))ds + O(|\gamma|^2) \end{aligned}$$

Therefore

$$\frac{x(\gamma)(t) - x(0)(t)}{\gamma} = [\frac{1}{\gamma}\int_0^t L_0(x_s(\gamma) - x_s(0))ds]$$
$$+ \int_0^t L_1(x_s(\gamma))ds + O(|\gamma|^2)$$

Letting $\gamma \to 0$ and recalling that $x_t(0) = \varphi_0 e^{\lambda_0 t}$, we get

$$v(t) = \int_0^t L_0(v_s)ds + \int_0^t L_1(\varphi_0)e^{\lambda_0 s}\, ds.$$

Differentiating with respect to t, we get

$$\dot{v}(t) = L_0(v_t) + L_1(\varphi_0)e^{\lambda_0 t}.$$

This proves Lemma 5.2.

Lemma 5.3. Let $\lambda_1 = \lambda'(0)$. Assume $\operatorname{Re}\lambda_1 \neq 0$. Then $\lambda_1 b_0^T \Delta'(\lambda_0)a_0 = b_0^T L_1(e^{\lambda_0 \cdot})a_0$.

Proof: Let $\lambda(\gamma)$ be an eigenvalue of the characteristic equation $\det \Delta(\lambda(\gamma)) = 0$ with corresponding eigenvector $a(\gamma)$. By a result in Hale [5, p.99] we have

$$[\lambda(\gamma)I - \int_{-r}^{0} [d\eta(\gamma,\theta)]e^{\lambda(\gamma)\theta}]a(\gamma) = 0.$$

Differentiating with respect to γ at $\gamma = 0$, we have

$$[\lambda'(0)I - \int_{-r}^{0} [d\eta_1(\theta)]e^{\lambda_0\theta} - \lambda'(0)\int_{-r}^{0} [d\eta_0(\theta)]\theta e^{\lambda_0\theta}]a_0$$
$$+ [\lambda_0 I - \int_{-r}^{0} [d\eta_0(\theta)]e^{\lambda_0\theta}]a'(0) = 0.$$

Therefore

$$[\lambda_1 I - \lambda_1 \int_{-r}^{0} [d\eta_0(\theta)]\theta e^{\lambda_0\theta}]a_0 - [\int_{-r}^{0} [d\eta_1(\theta)]e^{\lambda_0\theta}]a_0$$
$$+ [\lambda_0 I - \int_{-r}^{0} [d\eta_0(\theta)]e^{\lambda_0\theta}]a'(0) = 0$$

Since b_0^T annihilates the range of $\Delta(\lambda_0)$, we have $b_0^T[\lambda_0 I - \int_{-r}^{0} [d\eta_0(\theta)]e^{\lambda_0\theta}]a'(0) = 0$. If we multiply the above expression by b_0^T we get, $\lambda_1 b_0^T\Delta'(\lambda_0)a_0 = b_0^T L_1(e^{\lambda_0 \cdot})a_0$ where $\Delta'(\lambda_0) = \int_{-r}^{0} [d\eta_0(\theta)]\theta e^{\lambda_0\theta}$. This completes the proof of Lemma 5.3.

<u>Lemma 5.4</u>. Let $v(t)$ be as in Lemma 5.2. Then

$$\langle\psi_0, v(\omega_0)\rangle = \frac{2\pi}{\nu_0}\lambda_1.$$

<u>Proof</u>: Let

$$y(t) = \langle\psi_0, v_t\rangle$$

By Theorem 4.1 (in the complex form) and Lemma 5.2,

$$\dot{y}(t) = \lambda_0 y(t) + \psi_0(0)L_1(\varphi_0)e^{\lambda_0 t}$$

Since $v_0 \equiv 0$, $y(0) = 0$. Therefore

$$\begin{aligned}
y(t) &= \int_0^t e^{\lambda_0(t-s)} \psi_0(0)L_1(\varphi_0)e^{\lambda_0 s}\,ds \\
&= e^{\lambda_0 t}\int_0^t \psi_0(0)L_1(\varphi_0)ds \\
&= e^{\lambda_0 t}\int_0^t b_0^T(b_0^T\Delta'(\lambda_0)a_0)^{-1} L_1(e^{\lambda_0 \cdot})a_0\,ds \\
&= \lambda_1 e^{\lambda_0 t}\int_0^t ds \quad \text{(by Lemma 5.3)} \\
&= \lambda_1 t e^{\lambda_0 t}.
\end{aligned}$$

Therefore

$$\langle \psi_0, v(\omega_0) \rangle = \frac{2\pi}{\nu_0}\,\lambda_1 .$$

This follows from the fact that $\omega_0 = \dfrac{2\pi}{\nu_0}$ and $\lambda = \lambda_0 = i\nu_0$ at $t = \omega_0$. This proves Lemma 5.4.

<u>Lemma 5.5</u>. $\dfrac{\partial}{\partial\gamma}\langle \psi_1, T(\gamma)(\omega_0)\varphi_1 \rangle = \dfrac{2\pi}{\nu_0}\,\mathrm{Re}\ \lambda_1$ and $\dfrac{\partial}{\partial\gamma}\langle \psi_2, T(\gamma)(\omega_0)\varphi_1 \rangle = \dfrac{2\pi}{\nu_0}\,\mathrm{Im}\ \lambda_1$.

<u>Proof</u>:

$$\begin{aligned}
\frac{\partial}{\partial\gamma}\langle \psi_1, T(\gamma)(\omega_0)\varphi_1 \rangle &= \frac{1}{4}\,\frac{\partial}{\partial\gamma}\langle \psi_0 + \bar{\psi}_0, T(\gamma)(\omega_0)\varphi_0 + T(\gamma)(\omega_0)\bar{\varphi}_0 \rangle \\
&= \frac{1}{4}\langle \psi_0, \frac{\partial}{\partial\gamma}\,T(\gamma)(\omega_0)\varphi_0 \rangle \\
&\quad + \frac{1}{4}\langle \bar{\psi}_0, \frac{\partial}{\partial\gamma}\,T(\gamma)(\omega_0)\bar{\varphi}_0 \rangle \\
&= \frac{1}{4}\langle \psi_0, \frac{\partial}{\partial\gamma}\,x(\gamma)(w_0) \rangle \\
&\quad + \frac{1}{4}\langle \bar{\psi}_0, \frac{\partial}{\partial\gamma}\,\bar{x}(\gamma)(\omega_0) \rangle \quad \text{by Lemma 5.2} \\
&= \frac{1}{4}\langle \psi_0, v(\omega_0) \rangle + \frac{1}{4}\langle \bar{\psi}_0, \bar{v}(\omega_0) \rangle \\
&= \frac{1}{4}\,\lambda_1\,\frac{2\pi}{\nu_0} + \frac{1}{4}\,\bar{\lambda}_1\,\frac{2\pi}{\nu_0} \qquad \text{by Lemma 5.4} \\
&= \frac{\pi}{2\nu_0}\,\mathrm{Re}\ \lambda_1 .
\end{aligned}$$

Similarly

$$\begin{aligned}
\frac{\partial}{\partial\gamma}\langle \psi_2, T(\gamma)(\omega_0)\varphi_1 \rangle &= \frac{1}{4i}\langle \psi_0 - \bar{\psi}_0, \frac{\partial}{\partial\gamma}\,T(\gamma)(\omega_0)\varphi_0 \\
&\quad + \frac{\partial}{\partial\gamma}\,T(\gamma)(\omega_0)\bar{\varphi}_0 \rangle \\
&= \frac{1}{4i}\langle \psi_0, \frac{\partial}{\partial\gamma}\,x(\gamma)(\omega_0) \rangle \\
&\quad - \frac{1}{4i}\langle \bar{\psi}_0, \frac{\partial}{\partial\gamma}\,\bar{x}(\gamma)(\omega_0) \rangle
\end{aligned}$$

$$= \frac{1}{4i} \langle \psi_0, v(\omega_0) \rangle - \frac{1}{4i} \langle \bar{\psi}_0, \bar{v}(\omega_0) \rangle$$

$$= \frac{1}{4i} \lambda_1 \frac{2\pi}{\nu_0} - \frac{1}{4i} \lambda_1 \frac{2\pi}{\nu_0} = \frac{2\pi}{\nu_0} \operatorname{Im} \lambda_1$$

We will now prove Theorem 5.1.

Proof: The existence of ω-periodic solutions of (5.1) is equivalent to solving

$$\varphi = T(\gamma)(\omega)\varphi + \int_0^{\omega} T(\gamma)(\omega - s)X_0 N(\gamma, x_s(\gamma,\varphi))ds$$

$$= T(\omega_0)\varphi + [T(\gamma)(\omega)\varphi - T(\omega_0)\varphi$$

$$+ \int_0^{\omega} T(\gamma)(\omega - s)X_0 N(\gamma, x_s(\gamma,\varphi)]ds.$$

Let P be the generalized eigenspace associated with $\Lambda = \{i\nu_0, -i\nu_0\}$ and Q be the complementary subspace of P such that $C = P \oplus Q$. We scale the above equation by letting $\varphi = \epsilon(\varphi_1 + \psi)$ where ϵ belongs to R, $\psi \in Q$, and $\varphi_1 \in P$. Let Π_P and Π_Q be the projections of $C[-r,0]$ onto P and Q respectively. The projection Π_P on the scaled equation for $\varphi(\varphi^P = \Pi_P(\varphi))$ gives

$$\Pi_P\{T(\gamma)(\omega)(\epsilon\varphi_1 + \epsilon\psi) - T(\omega_0)(\epsilon\varphi_1 + \epsilon\psi)$$

$$+ \int_0^{\omega} T(\gamma)(\omega - s)X_0 N(\gamma, x_s(\gamma, \epsilon\varphi_1 + \epsilon\psi))ds\} = 0$$

where $T(\omega_0)\varphi^P = \varphi^P$. Let $\tilde{\varphi} \in C[-r,0]$. By Theorem 2.2, $\Pi_P\tilde{\varphi} = \langle\psi_1, \tilde{\varphi}\rangle\varphi_1 + \langle\psi_2, \tilde{\varphi}\rangle\varphi_2$ where φ_1 and φ_2 span the subspace P and ψ_1, ψ_2 belong to the span of the generalized eigenspace P^* of the adjoint equation of (5.1) associated with Λ. Therefore $\Pi_P \tilde{\varphi} = 0$ if and only if $\langle \psi_1, \tilde{\varphi} \rangle = 0$ and $\langle \psi_2, \tilde{\varphi} \rangle = 0$. Thus we have

$$\langle \psi_1, T(\gamma)(\omega)(\varphi_1 + \psi) - T(\omega_0)(\varphi_1 + \psi) + \frac{1}{\epsilon}\int_0^\omega T(\gamma)(\omega - s)X_0 N(\gamma, x_s(\gamma, \epsilon\varphi_1 + \epsilon\psi))ds \rangle = 0$$

$$\langle \psi_2, T(\gamma)(\omega)(\varphi_1 + \psi) - T(\omega_0)(\varphi_1 + \psi) + \frac{1}{\epsilon}\int_0^\omega T(\gamma)(\omega - s)X_0 N(\gamma, x_s(\gamma, \epsilon\varphi_1 + \epsilon\psi))ds \rangle = 0$$

Because there are no other eigenvalues that are integral multiples of $i\nu_0$, the operator $(I - T(\omega_0))$ is invertible. Thus the projection $\Pi_Q(\varphi^Q = \Pi_Q(\varphi))$ defined by (4.12), on the scaled equation for φ gives,

$$\psi - (I - T(\omega_0))^{-1}\{\Pi_Q T(\gamma)(\omega)(\varphi_1 + \psi) - T(\omega_0)(\varphi_1 + \psi) + \frac{1}{\epsilon}\Pi_Q\int_0^\omega T(\gamma)(\omega - s)X_0 N(\gamma, x_s(\epsilon\varphi_1 + \epsilon\psi))ds\} = 0$$

Let G be the map from $R^3 \times Q$ into $R^2 \times Q$ defined by $G = G(\epsilon, \gamma, \omega, \psi) = (G_1, G_2, G_3)$ where

$$G_1 = \langle \psi_1, T(\gamma)(\omega)(\varphi_1 + \psi) - T(\omega_0)(\varphi_1 + \psi) + \frac{1}{\epsilon}\int_0^\omega T(\gamma)(\omega - s)X_0 N(\gamma, x_s(\gamma, \epsilon\varphi_1 + \epsilon\psi))ds \rangle$$

$$G_2 = \langle \psi_2, T(\gamma)(\omega)(\varphi_1 + \psi) - T(\omega_0)(\varphi_1 + \psi) + \frac{1}{\epsilon}\int_0^\omega T(\gamma)(\omega - s)X_0 N(\gamma, x_s(\gamma, \epsilon\varphi_1 + \epsilon\psi))ds \rangle$$

$$G_3 = \psi - (I - T(\omega_0))^{-1}\{\Pi_Q T(\gamma)(\omega)(\varphi_1 + \psi) - T(\omega_0)(\varphi_1 + \psi) + \frac{1}{\epsilon}\Pi_Q\int_0^\omega T(\gamma)(\omega - s)X_0 N(\gamma, x_s(\gamma, \epsilon\varphi_1 + \epsilon\psi))ds\}.$$

Note that G is well defined for $\epsilon = 0$ since $N = o(|\varphi|)$. Also G is continuously differentiable in a neighborhood of $(0,0,\omega_0,0)$ since each G_i, $i = 1,2,3$ is continuously differentiable. By Theorem 2.2, the zeros of G are solutions of (5.1). Let $\epsilon = \gamma = \psi = 0$. Since $T(0)(\omega_0) = T(\omega_0)$ and $N = o(|\varphi|)$, we have $G_1 = G_1(0,0,\omega_0,0) = 0$ and $G_2 = G_2(0,0,\omega_0,0) = 0$. Also for $\gamma = 0$, $\Pi_Q T(0)(\omega_0)(\varphi_1 + \psi) = T(\omega_0)(\varphi_1 + \psi)^Q = \psi$. Therefore for $\epsilon = \gamma = \psi = 0$, $G_3 = G_3(0,0,\omega_0,0) = 0$. Consider the matrix D defined by

$$D = \begin{bmatrix} \frac{\partial G_1}{\partial \omega} & \frac{\partial G_1}{\partial \gamma} & \frac{\partial G_1}{\partial \psi} \\ \frac{\partial G_2}{\partial \omega} & \frac{\partial G_2}{\partial \gamma} & \frac{\partial G_2}{\partial \psi} \\ \frac{\partial G_3}{\partial \omega} & \frac{\partial G_3}{\partial \gamma} & \frac{\partial G_3}{\partial \psi} \end{bmatrix}_{(\epsilon,\gamma,\omega,\psi) = (0,0,\omega_0,0)}$$

To prove that D is an isomorphism from $R^2 \times Q$ onto $R^2 \times Q$, it suffices to show that the matrix A defined by

$$A = \begin{bmatrix} \frac{\partial G_1}{\partial \omega} & \frac{\partial G_1}{\partial \gamma} \\ \frac{\partial G_2}{\partial \omega} & \frac{\partial G_2}{\partial \gamma} \end{bmatrix}_{(\epsilon,\gamma,\omega,\psi) = (0,0,\omega_0,0)}$$

is non-singular and that

$$\left.\frac{\partial G_3}{\partial \psi}\right|_{(\epsilon,\gamma,\omega,\psi) = (0,0,\omega_0,0)}$$

is an isomorphism on Q. We compute the entries of A as follows:

$$G_1 = G_1(0,0,\omega,0) = \langle \psi_1, T(\omega)\varphi_1 - T(\omega_0)\varphi_1 \rangle$$

Therefore,

$$\left.\frac{\partial G_1}{\partial \omega}\right|_{(0,0,\omega_0,0)} = \frac{\partial}{\partial \omega} \langle \psi_1, T(\omega)\varphi_1 \rangle \Big|_{\omega=\omega_0}$$

$$= 0 \quad \text{by Lemma 5.1.}$$

Similarly,

$$G_2 = G_2(0,0,\omega,0) = \langle \psi_2, T(\omega)\varphi_1 - T(\omega_0)\varphi_1 \rangle$$

and

$$\left.\frac{\partial G_2}{\partial \omega}\right|_{(0,0,\omega_0,0)} = \frac{\partial}{\partial \omega} \langle \psi_2, T(\omega)\varphi_1 \rangle \Big|_{\omega=\omega_0}$$

$$= \frac{\nu_0}{2} \quad \text{by Lemma 5.1}$$

$$G_1 = G_1(0,\gamma,\omega_0,0) = \langle \psi_1, T(\gamma)(\omega_0)\varphi_1 - T(\omega_0)\varphi_1 \rangle$$

Computing the partial derivative of G_1 with respect to γ, we obtain

$$\left.\frac{\partial G_1}{\partial \gamma}\right|_{(0,\gamma,\omega_0,0)} = \langle \psi_1, \frac{\partial}{\partial \gamma} T(\gamma)(\omega_0)\varphi_1 \rangle$$

$$= \langle \psi_1, \frac{\partial}{\partial \gamma} x(\gamma,\varphi_1)(\omega_0) \rangle$$

$$= \frac{\pi}{2\nu_0} \operatorname{Re} \lambda_1 \quad \text{by Lemma 5.5.}$$

Similarly,

$$G_2 = G_2(0,\gamma,\omega_0,0) = \langle \psi_2, T(\gamma)(\omega_0)\varphi_1 - T(\omega_0)\varphi_1 \rangle$$

Therefore

$$\begin{aligned} \left.\frac{\partial G_2}{\partial \gamma}\right|_{(0,\gamma,\omega_0,0)} &= \langle \psi_2, \frac{\partial}{\partial\gamma} T(\gamma)(\omega_0)\varphi_1 \rangle \\ &= \langle \psi_2, \frac{\partial}{\partial\gamma} x(\gamma,\varphi_1)(\omega_0) \rangle \\ &= \frac{\pi}{2\nu_0} \operatorname{Im} \lambda_1 \quad \text{by Lemma 5.5} \end{aligned}$$

Thus the matrix A is given by

$$A = \begin{bmatrix} 0 & \frac{\pi}{2\nu_0} \operatorname{Re} \lambda_1 \\ \\ \frac{\nu_0}{2} & \frac{\pi}{2\nu_0} \operatorname{Im} \lambda_1 \end{bmatrix}$$

Since $\operatorname{Re} \lambda_1 \neq 0$, A is clearly non-singular. We now compute $\frac{\partial G_3}{\partial \psi}$.

$$G_3 = G_3(0,0,\omega_0,\psi) = \psi$$

Therefore, $\frac{\partial G_3}{\partial \psi} = I$, where $I:Q \to Q$ is the identity map. We therefore have that D is a linear isomorphism from $R^2 \times Q$ onto $R^2 \times Q$. Therefore, by the implicit function theorem in Banach spaces ([8, p.17]), there exist C' maps $\omega(\epsilon)$, $\gamma(\epsilon)$ and $\psi(\epsilon)$ defined for ϵ in $(-\epsilon_0,\epsilon_0)$ where $\epsilon_0 > 0$ such that $\omega(0) = \omega_0 = \frac{2\pi}{\nu_0}$, $\gamma(0) = 0$, $\psi(0) = 0$ and $G(\epsilon,\gamma(\epsilon),\omega(\epsilon),\psi(\epsilon)) = 0$, $-\epsilon_0 < \epsilon < \epsilon_0$. This gives a one parameter family of non-zero periodic orbits bifurcating from $(0, \lambda_0)$. This completes the proof of Theorem 5.1.

<u>Definition 5.1</u>. Let $I: C \times R \to R$ where $C = C[-r, 0]$. I is called a first integral of

$$\dot{x}(t) = L(\gamma, x_t) + N(\gamma, x_t) \tag{5.1}$$

if, for every solution x_t, $I(x_t, \gamma) =$ constant for all $t \geq 0$.

<u>Theorem 5.2</u>. Consider the non-linear FDE (5.1). Assume (5.1) has first integral I, and $I: C \times R \to R$ is smooth. Assume that $M = \{\varphi : I(\varphi, \gamma) = c,\ c = \text{constant},\ \gamma \in R\}$ is a smooth manifold of co-dimension one in C. Let Φ be the smooth vector field defined on M induced by the solution of (5.1), and $\tilde{\varphi} \in M$ be a critical point of Φ. By using local coordinates around $\tilde{\varphi}$, we obtain a FDE in the form of (5.1). Then the conclusions of Theorem 5.1 is valid for the vector field Φ.

CHAPTER III

EXAMPLES

§1. Infectious Disease Model

As an application of the theory presented in Chapter II, we consider the infectious disease model of Hoppensteadt and Waltman [7] given in Chapter I. In this model the spread of the disease is governed by three functional differential equations in the unknowns τ, I and S. These equations are:

$$(1.1)\qquad \int_{\tau(t)}^{\tau} [\rho_1(x) + \rho_2(x)I(x)]dx = m, \quad \tau(t) \equiv 0,\ t \le t_0$$

$$(1.2)\qquad S(t) = I_1(t) + S_0 - \int_{\tau(t-\sigma-\omega)}^{t} r(x)S(x)I(x)dx$$

$$(1.3)\qquad I(t) = I_0(t) + \int_{\tau(t-\sigma)}^{\tau(t)} r(x)S(x)I(x)dx$$

The functions I_1 and I_0 are as defined in Chapter I:

$$I_0(t) = \begin{cases} 0, & |t| \ge \sigma \\ I_0(t), & -\sigma \le t \le 0 \\ I_0(0) - I_0(t-\sigma), & 0 \le t \le \sigma \end{cases}$$

$$I_1(t) = \begin{cases} 0, & t \le \omega \\ I_0(0) - I_0(t-\omega), & \omega \le t \end{cases}$$

The function I_1 gives the fraction of those initially infected at time $t = 0$ who are now in class $S(t)$ for the first time, while for $t \in [-\sigma, 0]$, $I_0(t)$ describes the past history of the disease. The function $I_0(t)$ has been extended to all reals so that the future behavior of the initial infectives is known. The value t_0 in (1.1) satisfies,

$$\int_0^{t_0} [\rho_1(x) + \rho_2(x) I_0(x)]dx = m.$$

In our study we are interested in the behavior of I and S for large time t, so without loss of generality we consider the differentiated forms of (1.2) and (1.3) for $t \geq \sigma + \omega$. (σ is the fixed time during which an infective remains infected, and ω is the period of immunity). For $t \geq \sigma + \omega$, the functions $I_0(t)$, $I_0'(t)$, and $I_1'(t)$ are all zero ("$'$" denotes derivative with respect to time). The differentiated forms are: (taking $\tau(t) = t$)

(1.4) $\qquad S'(t) = -r(t)\{I(t)S(t) - I(t - \sigma - \omega)S(t - \sigma - \omega)\}$

(1.5) $\qquad I'(t) = r(t)\{I(t)S(t) - I(t - \sigma)S(t - \sigma)\}$

Every pair of constants is a solution of (1.4) and (1.5). It is customary in the stability theory of non-linear differential equations to study solutions which are "close" to constant solutions. To study the stability of (1.4) and (1.5) we choose a particular pair of constants

α, β which solves (1.4) and (1.5) and linearize about this pair of constants. We make the following substitutions

$$I(t) = y(t) + \beta \tag{1.6}$$

$$S(t) = x(t) + \alpha \tag{1.7}$$

Substituting (1.6) and (1.7) into (1.4) and (1.5) respectively and for simplicity choosing $r(t) = c$, c is a constant, we obtain, after dropping non-linear terms,

$$\begin{aligned} \dot{x}(t) &= -c\beta\{x(t) - x(t - \sigma - \omega)\} \\ &\quad - c\alpha\{y(t) - y(t - \sigma - \omega)\} \end{aligned} \tag{1.8}$$

$$\dot{y}(t) = c\beta\{x(t) - x(t - \sigma)\} + c\alpha\{y(t) - y(t - \sigma)\} \tag{1.9}$$

Note that (1.8) and (1.9) are functional differential equations with real parameters c, β, α, σ and ω. In Chapter II we developed some general theory for non-linear functional differential equations with real parameter γ. We showed that if λ was a simple complex eigenvalue of the non-linear functional differential equation (II, 4.1) with dependence on the real parameter γ, such that $\operatorname{Re} \lambda'(\gamma = 0) \neq 0$, then (II,4.1), has non-zero periodic solutions. We showed such existence in part by studying the zeros of the characteristic equation associated with (II, 4.1). Therefore to examine the behavior of the solutions of (1.8) and (1.9), we shall study the solutions in terms of the characteristic roots of the characteristic equation associated with (1.8) and (1.9). To obtain the characteristic equations for (1.8) and (1.9) we let $x(t) = k_1 e^{\lambda t}$ and $y(t) = k_2 e^{\lambda t}$ where k_1 and k_2 are constants, λ is an eigenvalue of the characteristic equation associated with (1.8) and (1.9). Substituting these values of $x(t)$ and $y(t)$ into (1.8) and (1.9) we obtain

$$\lambda k_1 = -c\beta\{k_1 - k_1 e^{-\lambda(\sigma+\omega)}\} - c\alpha\{k_2 - k_2 e^{-\lambda(\sigma+\omega)}\} \tag{1.10}$$

$$\lambda k_2 = c\beta\{k_1 - k_1 e^{-\lambda\sigma}\} + c\alpha\{k_2 - k_2 e^{-\lambda\sigma}\} \tag{1.11}$$

Rearranging equations (1.10) and (1.11) we obtain,

$$\{\lambda + c\beta - c\beta e^{-\lambda(\sigma+\omega)}\}k_1 + \{c\alpha - c\alpha e^{-\lambda(\sigma+\omega)}\}k_2 = 0 \tag{1.12}$$

$$\{c\beta e^{-\lambda\sigma} - c\beta\}k_1 + \{\lambda - c\alpha + c\alpha e^{-\lambda\sigma}\}k_2 = 0 \tag{1.13}$$

The characteristic equation for system (1.12) and (1.13) is obtained by solving the determinant below:

$$\begin{vmatrix} \lambda + c\beta - c\beta e^{-\lambda(\sigma+\omega)} & c\alpha - c\alpha e^{-\lambda(\sigma+\omega)} \\ c\beta e^{-\lambda\sigma} - c\beta & \lambda - c\alpha + c\alpha e^{-\lambda\sigma} \end{vmatrix} = 0$$

Evaluating the determinant, we obtain

$$\begin{aligned} f(\lambda,\alpha,\beta,c,\omega,\sigma) = \lambda^2 &+ c\beta\lambda - c\beta\lambda e^{-\lambda(\sigma+\omega)} \\ &- c\alpha\lambda + \lambda c\alpha e^{-\lambda\sigma} = 0 \end{aligned} \tag{1.14}$$

From the model, we fix c, σ, α, and β, then (1.14) becomes

$$f(\lambda,\omega) = \lambda^2 + c\beta\lambda - c\beta\lambda e^{-\lambda(\sigma+\omega)} - c\alpha\lambda + \lambda c\alpha e^{-\lambda\sigma} = 0 \tag{1.15}$$

Note that $\lambda = 0$ is a solution of (1.15). This expresses the fact that every constant is a solution of (1.8) and (1.9). It is of interest to study the zeros of (1.15) for $\lambda \neq 0$. Applying Theorem 5.2 to (1.8) and (1.9) we obtain the equivalent form of (1.15) for $\lambda \neq 0$. This equivalence is given by

$$(1.16) \qquad f(\lambda, \omega) = \lambda + c\beta - c\beta e^{-\lambda(\sigma+\omega)} - c\alpha + c\alpha e^{-\lambda\sigma} = 0.$$

We examine the roots of the characteristic equation (1.16) with values of $c = 0.2$ and $\sigma = 1$. We also assume $\lambda = i\nu$ and $\lambda = -i\nu$ are purely imaginary eigenvalues of (1.16). With these values of c, σ and λ, (1.16) becomes

$$(1.17) \qquad f(i\nu, \omega) = i\nu + 0.2\beta - 0.2\beta e^{-i\nu(1+\omega)} - 0.2\alpha + 0.2\alpha e^{-i\nu} = 0$$

Equation (1.17) is equivalent to

$$(1.18) \qquad i\nu + 0.2\beta - 0.2\beta \cos \nu(1 + \omega) + 0.2i\beta \sin \nu(1 + \omega) - 0.2\alpha + 0.2\alpha \cos \nu - 0.2i\alpha \sin \nu = 0$$

Equating real and imaginary parts we obtain

$$(1.19a) \qquad 0.2\beta - 0.2\beta \cos \nu(1 + \omega) + 0.2\alpha \cos \nu - 0.2\alpha = 0$$

$$(1.19b) \qquad \nu + 0.2\beta \sin \nu(1 + \omega) - 0.2\alpha \sin \nu = 0$$

For fixed ν and ω we solve (1.19) for a particular pair of constants α and β. Table 3.1 shows values of α and β obtained by fixing values of ω in (1.19).

TABLE 3.1

ν	ω	α	β
$\pi/2$	1	7.85398	3.92699
$\pi/2$	2	3.92699	3.92699
π	2.25	3.25323	22.21441
$\pi/4$	3	5.75094	19.63495
$\pi/4$	4	.56270	4.74030
$\pi/4$	5	.81331	4.74030
$\pi/6$		5.23599	4.88524

For each fixed α and corresponding ω in Table 3.1, we plot β versus ν to find additional values of β and ν which solve (1.19). There are an infinite number of pairs (ν,β) which solve (1.19). For each choice of α and ω choose that pair (ν,β) for which $i\nu$ and $-i\nu$ are closest to the origin. Other pairs of intersections in the $\nu\beta$-plane will not be considered.

In Figure (3.1), we have $\alpha = 7.85398$ and $\omega = 1$. The graph is for β versus ν. There appear to be no intersections. However, there is an intersection at infinity since

$$\beta = \frac{\alpha \sin \nu - 5\nu}{\sin 2\nu}$$

by (1.19b).

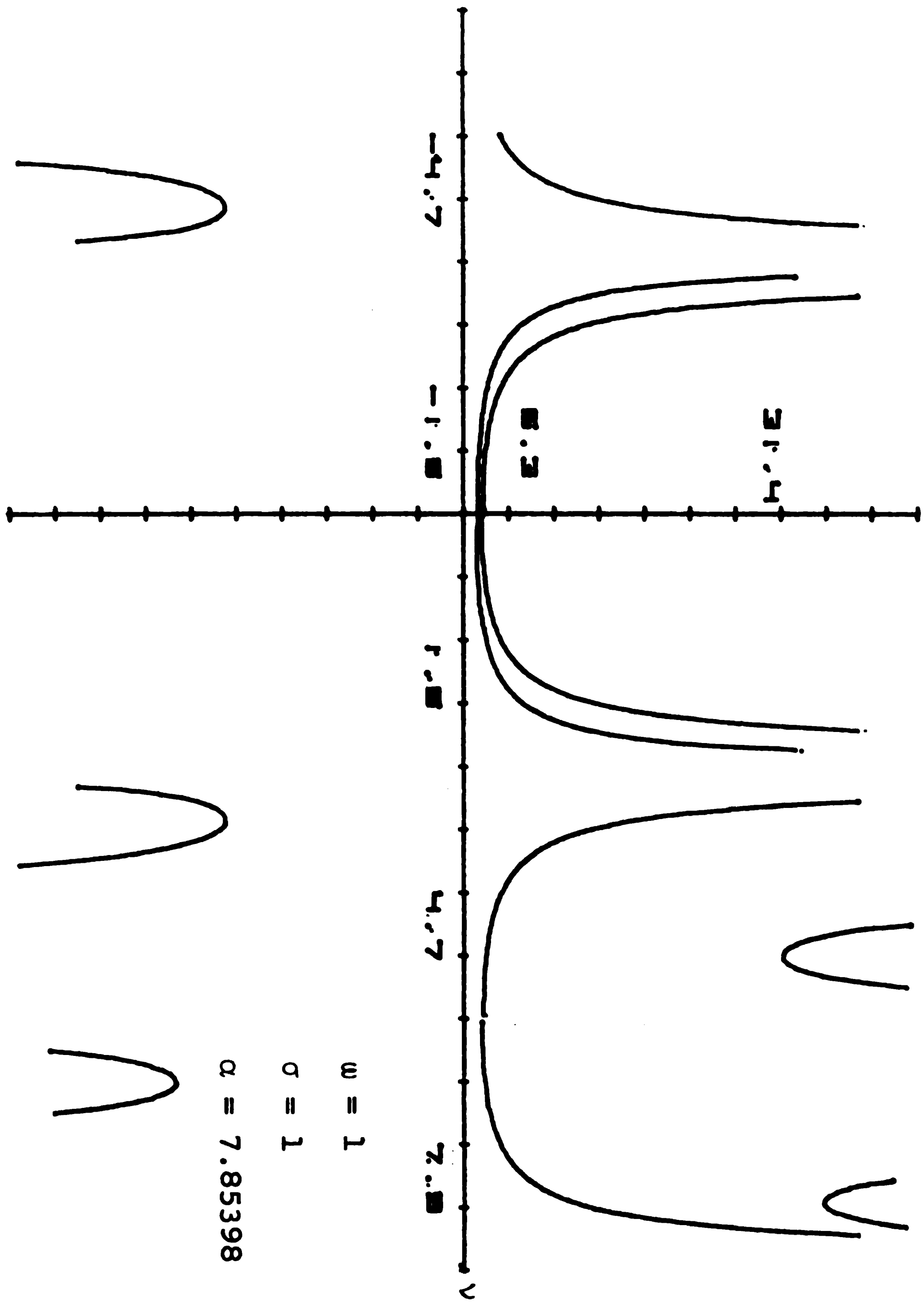

FIGURE 3.1

In Figures (3.2)-(3.7), there are several pairs of (ν,β) intersections shown. It appears from Figures (3.2)-(3.7) that for each such pair (ν,β) which solves (1.8) and (1.9), there are no integral multiples of ν such that $(k\nu,\hat{\beta})$ solves (1.8) and (1.9). At these intersections we will show first that Re $\lambda' \neq 0$, then, by applying Theorem 5.1, that the system (1.8) and (1.9) has non-zero periodic solutions, at least for the values of α, σ and ω used in each graph, Figure (3.2)-(3.7).

To show $\frac{d \text{ Re } \lambda(\beta)}{d\beta} \neq 0$, at β_i, $i = 0,1,2,\ldots,$ we differentiate (1.16) with respect to β and obtain (taking $\sigma = 1$)

$$\text{(1.20)} \qquad \frac{d\lambda}{d\beta} + c - c[-\beta(1+\omega)e^{-\lambda(1+\omega)}\frac{d\lambda}{d\beta} + e^{-\lambda(1+\omega)}] - c\alpha e^{-\lambda}\frac{d\lambda}{d\beta} = 0$$

Solving for $\frac{d\lambda}{d\beta}$, we have

$$\text{(1.21)} \qquad \frac{d\lambda}{d\beta} = \frac{ce^{-\lambda(1+\omega)} - c}{1 + c\beta(1+\omega)e^{-\lambda(1+\omega)} - c\alpha e^{-\lambda}}$$

The β's which solve system (1.19) in Figures (3.2)-(3.7) correspond to those eigenvalues λ which solve (1.16) with real part equal to zero. Thus for all such β's, $\lambda(\beta) = i\nu$. The pair (ν_j,β_j) such that $i\nu_j$ and $-i\nu_j$ are closest to the origin will be denoted by (ν_0,β_0). At $\beta = \beta_0$, (1.21) becomes

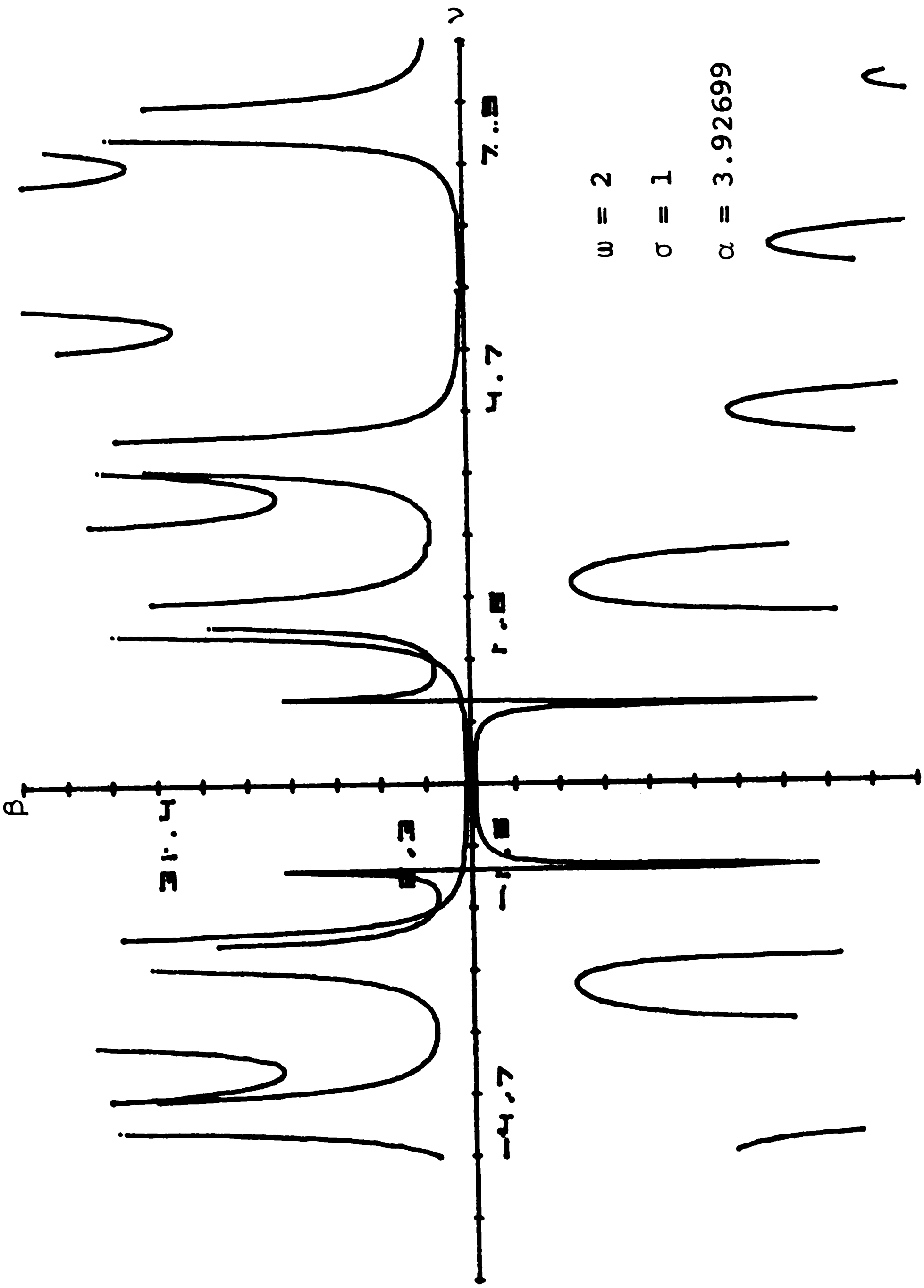

FIGURE 3.2

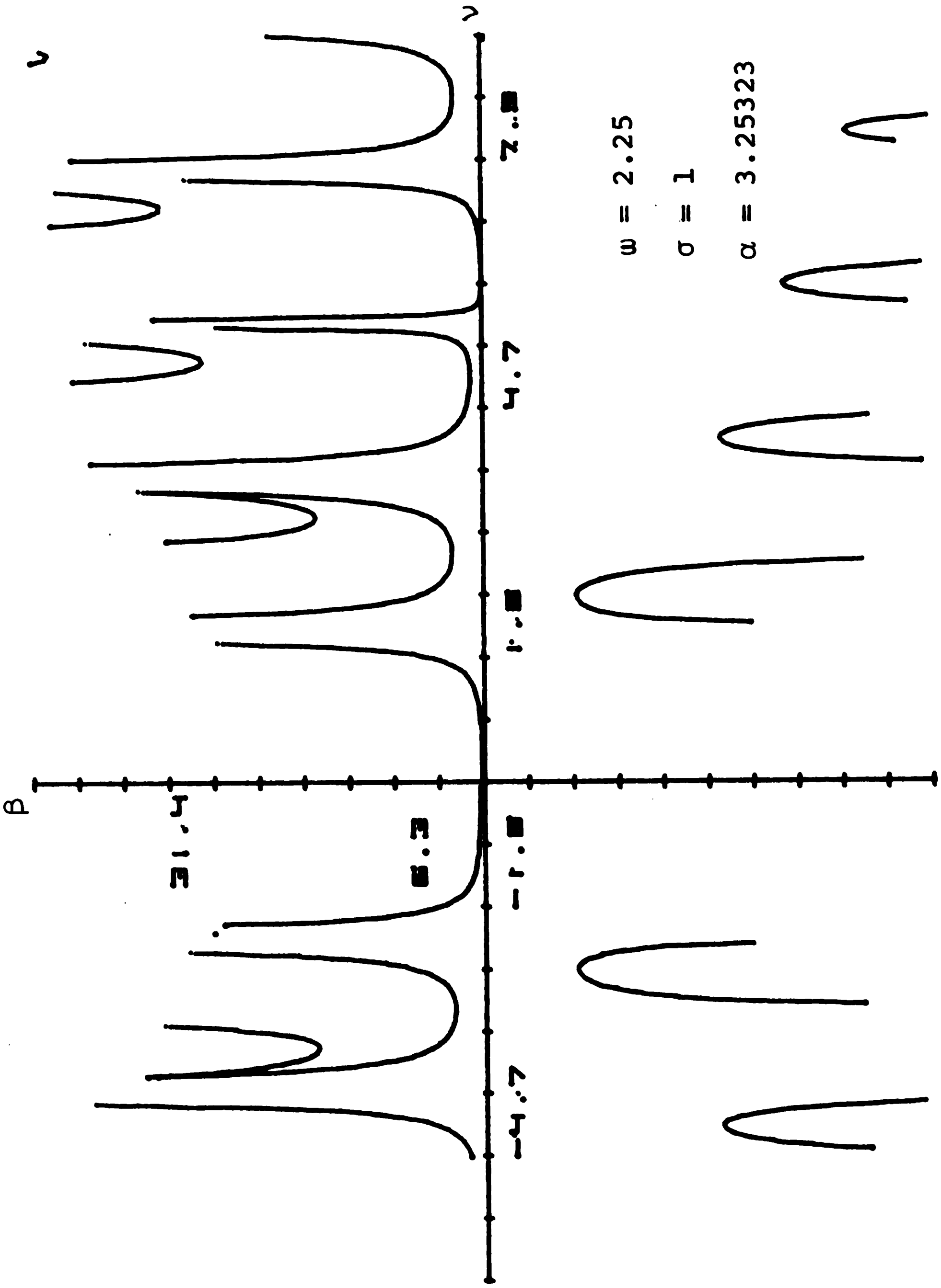

FIGURE 3.3

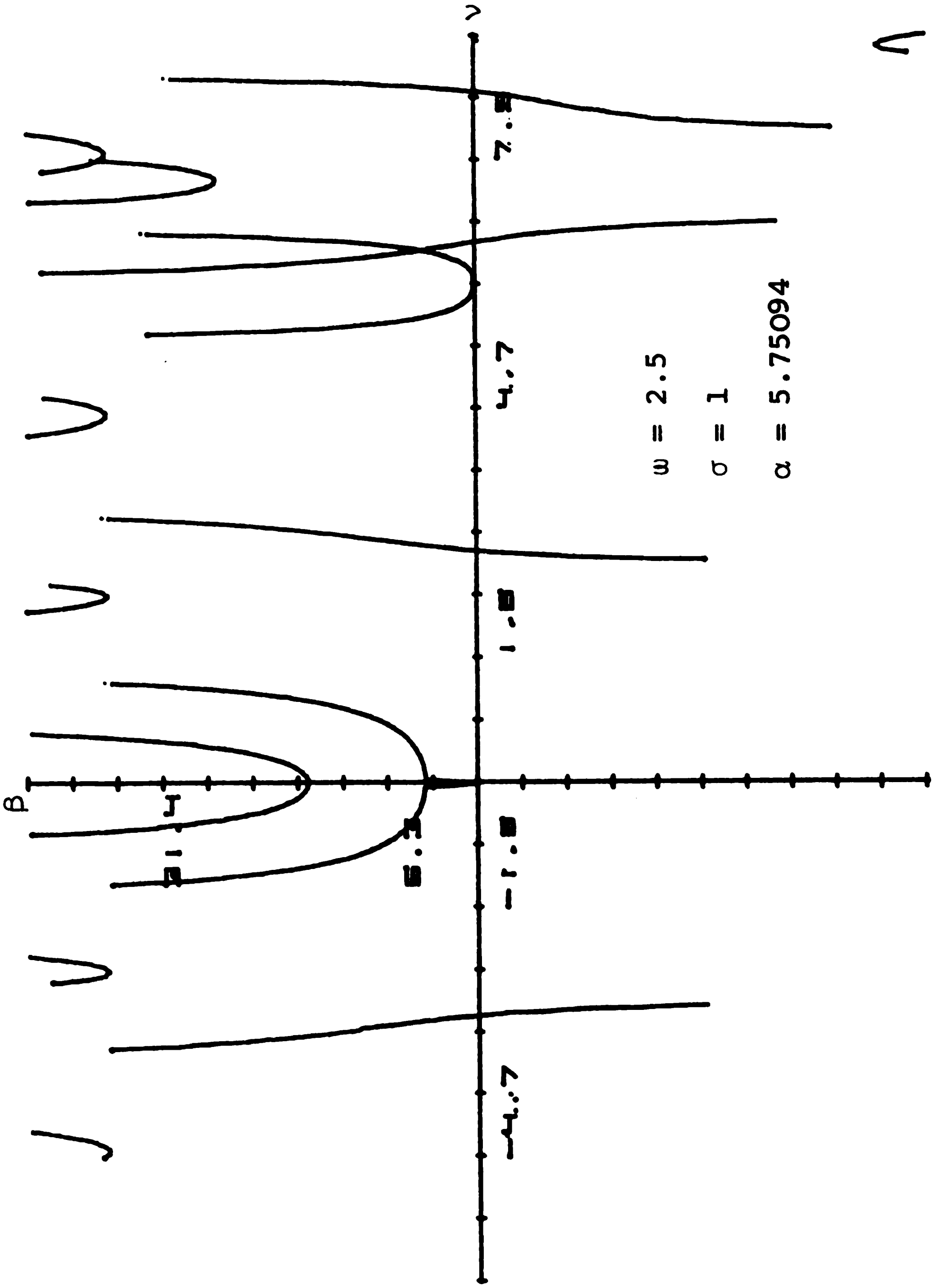

FIGURE 3.4

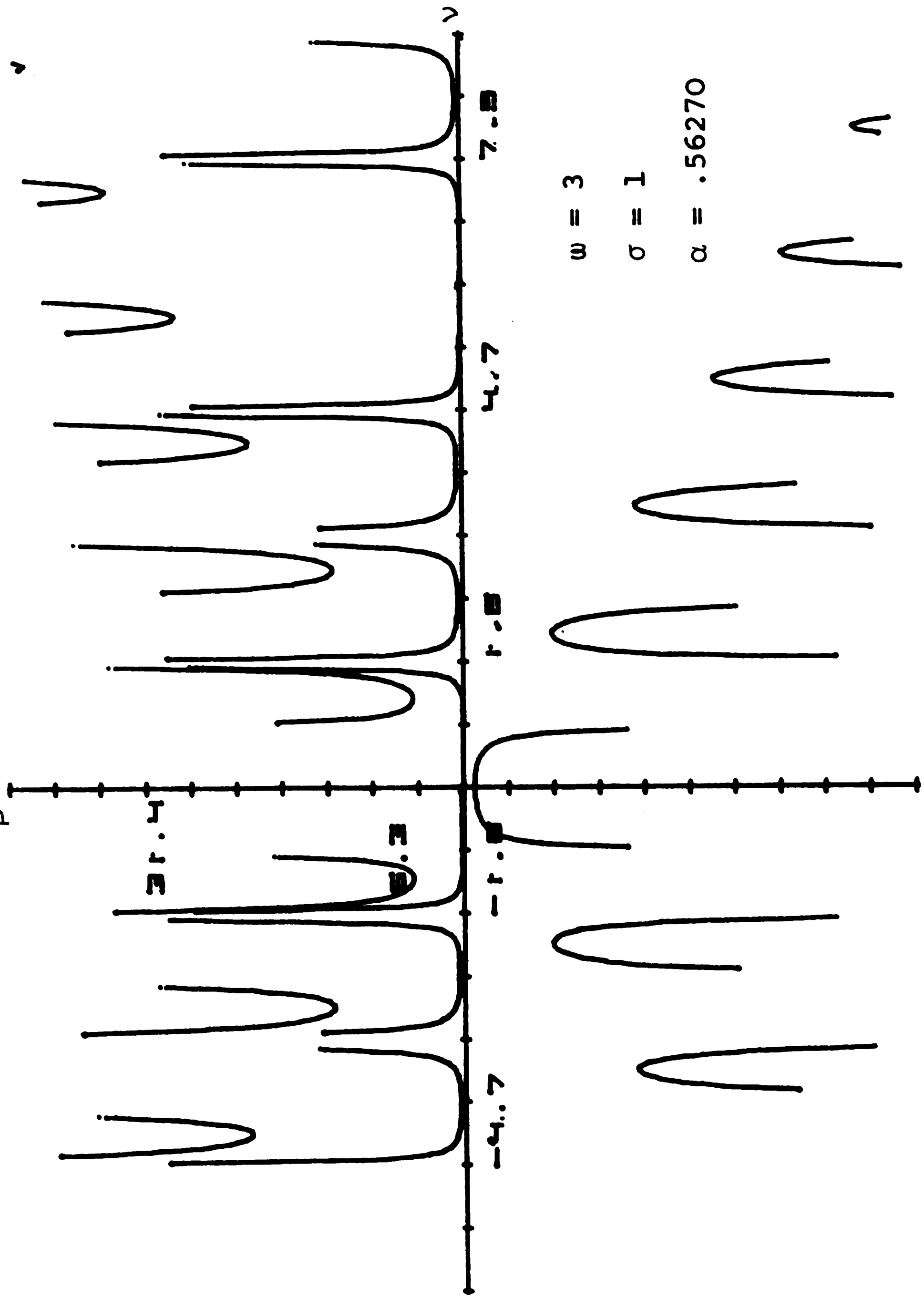

FIGURE 3.5

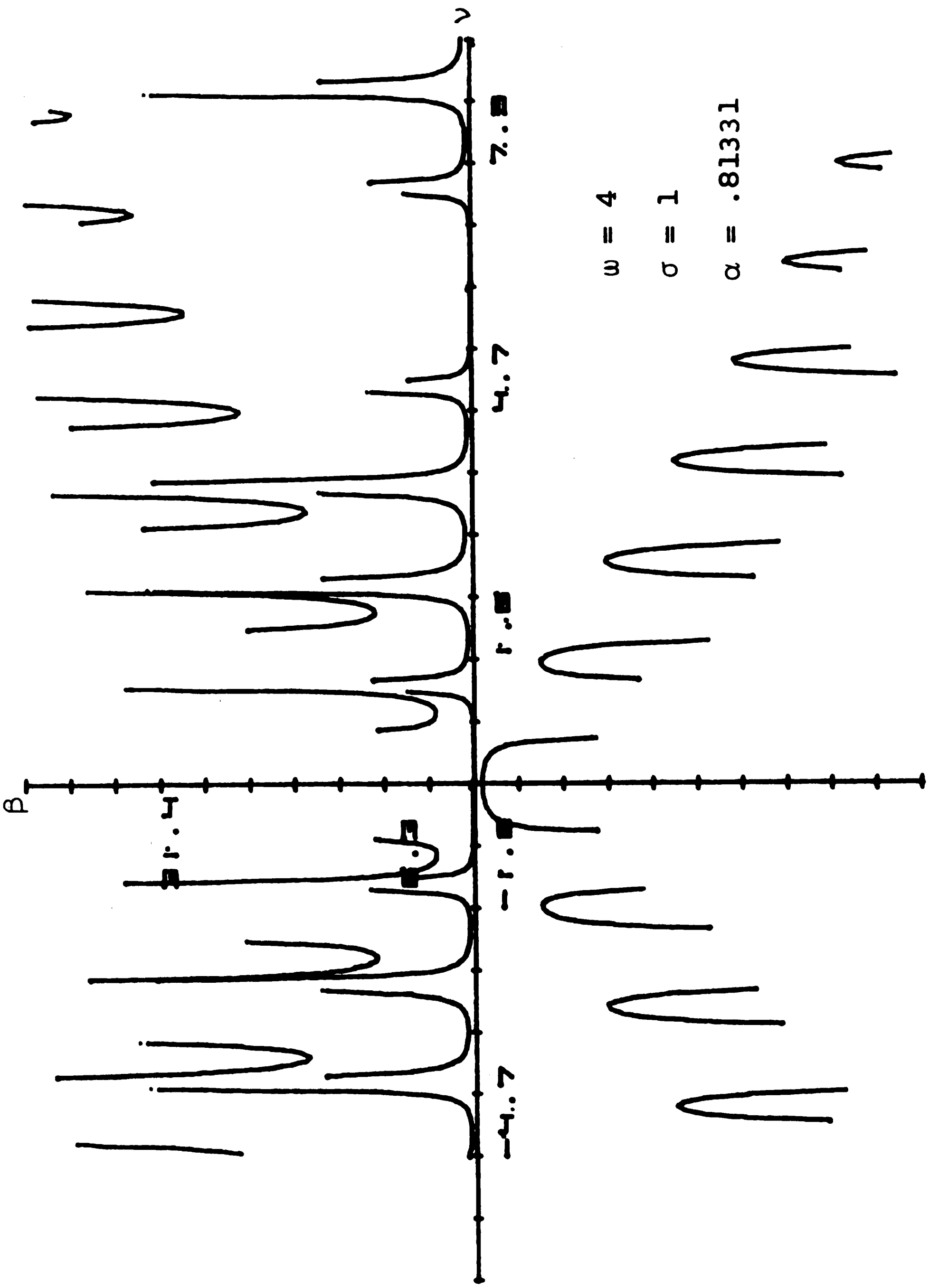

FIGURE 3.6

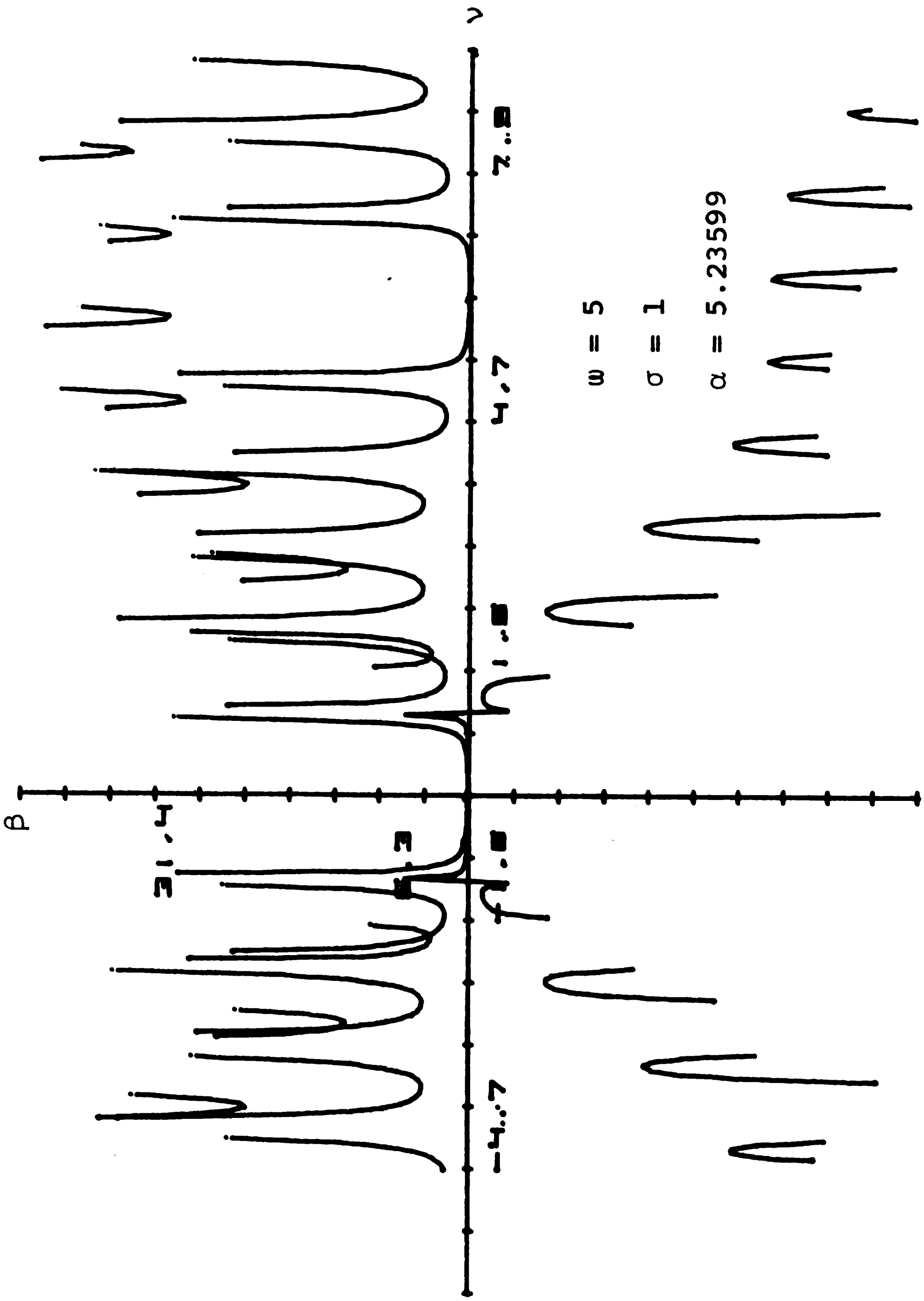

FIGURE 3.7

(1.22) $\frac{d\lambda}{d\beta}\Big|_{\beta=\beta_0}$

$$= \frac{c\{\cos \nu_0(1+\omega) - i \sin \nu_0(1+\omega) - 1\}}{1 + c\beta_0(1+\omega)(\cos \nu_0(1+\omega) - i \sin \nu_0(1+\omega) - c\alpha(\cos \nu_0 - \sin \nu_0)}$$

Writing (1.22) in the form $\beta + bi$ and considering only the real part, we obtain

(1.23)
$$\frac{d \operatorname{Re} \lambda(\beta)}{d\beta}\Big|_{\beta=\beta_0} = \{c \cos \nu_0(1 + \omega) - c + c^2\beta_0(1 + \omega) - c^2\beta_0(1 + \omega)\cos \nu_0(1 + \omega) - c^2\alpha \cos \nu_0(1 + \omega) + c^2\alpha \cos \nu_0 - c^2\alpha \sin \nu_0(1 + \omega)\sin \nu_0\} / (1 + c\beta_0(1 + \omega)\cos \nu_0(1 + \omega) - c\alpha \cos \nu_0)^2 + (c\alpha \sin \nu_0 - c\beta_0(1 + \omega)\sin \nu_0(1 + \omega))^2$$

Computed values of $\operatorname{Re} \lambda'(\beta_0)$ are given in Table 3.2. The calculations show that $\operatorname{Re} \lambda'(\beta_0) \neq 0$ for all values of β_0 considered. Thus for each pair (ν_0, β_0) we have $\operatorname{Re} \lambda(\beta_0) = 0$, $\operatorname{Re} \lambda'(\beta_0) \neq 0$ and $\lambda_0 = i\nu_0$.

Remarks. (1) λ_0 is a purely imaginary simple eigenvalue if λ_0 satisfies, $\Delta(i\nu_0) = 0$ and $\Delta'(i\nu_0) \neq 0$. These conditions are easy to check and are not considered here.

(2) To check that there are no integral multiples of $i\nu_0$, we note that there are finitely many roots of the characteristic equation (1.16) that are purely imaginary. If there exist integral multiples of $i\nu_0$, then there exists a largest integral multiple $ni\nu_0$. Let $\nu_1 = n\nu_0$.

Applying Theorem 5.1 to ν_1 we obtain the desired results. That is (1.8) and (1.9) has non-zero periodic solutions.

TABLE 3.2

α	β	ν	Re $\lambda'(\beta)$	ω
3.92699	3.92699	1.57080	.428318	2
18.9612	42.5261	9.21942	4.19972	2.25
5.75094	115.682	5.14383	5.26839	2.5
.562698	11.93520	2.77406	.32172	3
.81331	7.08452	2.18788	1.29186	4
5.23599	4.11505	1.74635	1.30180	5

§2. The Gonorrhea Model

As a second example, we consider the gonorrhea model proposed by Cooke and Yorke [3]. The model is described in Chapter I. The spread of the disease through society is given by

$$\dot{x}(t) = g(x(t - \sigma)) - g(x(t - L)) \tag{1.24}$$

where $0 < \sigma < L$. Without loss of generality we choose $L = 1$ by a change of time scale. The rate of new infection is given by $g(x(t - \sigma))$; $g(x(t - 1))$ is the rate at which the infectives are being cured assuming they contacted the disease 1 time unit ago. One choice for the function g is

$$g(u) = au(1 - u), \quad a > 0 \text{ constant.} \tag{1.25}$$

This choice of g is reasonable since at $u = 0$ there are no new infections and at $u = 1$ all new infectives have been cured. With this choice of g, (1.25) becomes

$$(1.26) \qquad \dot{x}(t) = ax(t - \sigma) - ax(t - 1) - ax^2(t - \sigma) + ax^2(t - 1)$$

Linearizing (1.26) about zero, and considering only the linear part, we obtain

$$(1.27) \qquad \dot{x}(t) = ax(t - \sigma) - ax(t - 1).$$

The solution of the linear equation (1.27) can be studied in terms of its charactersitic roots. To obtain the characteristic equation, let $x(t) = e^{\lambda t}$ where λ is an eigenvalue of the characteristic equation of (1.27). Substituting $x(t)$ into (1.27) we obtain

$$(1.28) \qquad \lambda = ae^{-\lambda\sigma} - ae^{-\lambda}$$

Clearly $\lambda = 0$ is a characteristic root of (1.28). Applying Theorem 5.2, we eliminate the zero solution of (1.28). Assume $\lambda = i\nu$ is a purely imaginary simple eigenvalue of (1.28), then separating real and imaginary parts of (1.28) we obtain

$$(1.29a) \qquad a \cos \nu\sigma - a \cos \nu = 0$$

$$(1.29b) \qquad a \sin \nu - a \sin \nu\sigma = \nu, \quad 0 < \sigma < 1.$$

The first equation in (1.29) shows that if $\nu\sigma = -\nu + 2\pi m$, m an integer, then any positive value of "a" will satisfy (1.29a). We do not consider $a < 0$. Setting $\nu\sigma = -\nu + 2\pi m$ in (1.29b) gives

$$2a \sin \nu = \nu. \tag{1.30}$$

For fixed $\sigma \in (0,1)$ and any integer m, such that $\frac{2\pi m}{1+\sigma}$ is non-integral, we find values of "a" which satisfy (1.29).

By assumption, the pairs (ν_0, a_0) which satisfy (1.29) correspond to the eigenvalues $\lambda(a_0) = \mathrm{Re}\,\lambda(a_0) + i\, I_m \lambda(a_0)$ of (1.28) with $\mathrm{Re}\,\lambda(a_0) = 0$. For all pairs (ν_0, a_0) which satisfy (1.29) we first show that $\mathrm{Re}\,\lambda'(a_0) \neq 0$, then, applying Theorem 5.1, concludes that (1.27) has non-zero periodic solutions.

To show $\frac{d}{da} \mathrm{Re}\,\lambda\big|_{a=a_0} \neq 0$ for all pairs (ν_0, a_0) which satisfy (1.29) we differentiate (1.28) with respect to a. We obtain

(1.31) $\mathrm{Re}\,\lambda'(a = a_0)$

$$= \frac{\nu_0 \sin \nu_0 - \nu_0\sigma \sin \nu_0\sigma}{[(1 + a_0\sigma \cos \nu_0\sigma - a_0 \cos \nu_0)^2 - a_0^2(\sin \nu_0 - \sigma \sin \nu_0\sigma)^2]}$$

TABLE 3.3

σ	ν	a	Re $\lambda'(a)$
.25	15.0796	12.8275	11.0795
.25	20.1062	10.5705	23.9027
.25	40.2124	34.2067	29.545
.5	83.7758	48.3680	108.828
.5	71.2094	41.1128	92.5038
.5	58.6431	33.8576	76.1796
.5	46.0767	26.6024	59.8554
.5	33.5103	19.3472	43.5312
.5	20.9440	12.0920	27.2070
.5	8.3776	4.8368	10.8828
.75	7.18078	4.59228	9.82478
.75	14.3616	7.36545	24.5026
.75	21.5423	24.8250	16.3570
.75	32.3135	20.6653	44.2115

At each pair (ν_0, a_0) which satisfy (1.29), the computations in Table 3.3 show that Re $\lambda'(a_0) \neq 0$. The same remarks stated in Example 1 above applies here. Applying Theorem 5.1, (1.27) has non-zero periodic solutions.

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