

THE BERRY CONNECTION AND OTHER ASPECTS OF THE GINZBURG-LANDAU
THEORY IN DIMENSION 2

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A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

Mathematics — Doctor of Philosophy

2016

ABSTRACT

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In the first chapter, we analyze the 2-dimensional Ginzburg-Landau vortices at critical coupling, and establish asymptotic formulas for the tangent vectors of the vortex moduli space using theorems of Taubes and Bradlow. We then compute the corresponding Berry curvature and holonomy in the large area limit.

In the second chapter, we generalize Bradlow's theorem about existence of irreducible absolute minimizers of the Ginzburg-Landau functional.

To my parents, Ella and Lajos.

ACKNOWLEDGMENTS

First and foremost, I wish to thank my advisor Tom Parker for guiding me through graduate school. I was extremely lucky to study geometry from the excellent teachers of MSU, like Matt Hedden, Ron Fintushel, and Jon Wolfson. I learnt a great deal of math from my “academic brother”, Manos Maridakis. Finally, I also benefited a lot from discussions with David Duncan, Tim Nguyen, and Luke Williams.

I would also like to thank my friends all around the world, but most importantly the ones in, and around East Lansing. You guys have been my family on this continent for the past 5 years.

Finally I thank my parents, Ella and Lajos, for their constant, unquestionable, and selfless support, not just throughout graduate school, but my whole life.

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Chapter 1

Ginzburg-Landau vortices

1.1 Introduction

The Ginzburg-Landau theory is a phenomenological model for superconductivity, introduced in [GL50]; for a more modern review see [AK02]. The theory gives variational equations – the Ginzburg-Landau equations – for an Abelian gauge field and a complex scalar field. The gauge field is the EM vector potential, while the norm of the scalar field is the order parameter of the superconducting phase. The order parameter can be interpreted as the wave function of the so-called BCS ground state, a single quantum state occupied by a large number of Cooper pairs.

This paper focuses on certain static solutions of the 2-dimensional Ginzburg-Landau equations called τ -vortices. Physicists regard the number τ as a coupling constant, sometimes called the vortex-size. Mathematically, τ is a scaling parameter for the metric. The geometry of τ -vortices have been studied since [JT80, B90], and there is a large literature on the subject; cf. [MN99, MS03, CM05, B06, B11, DDM13, BR14, MM15].

Families of operators in quantum physics carry canonical connections. This idea was introduced by Berry in [B84], and generalized by Aharonov and Anandan in [AA87]. These so-called Berry connections were used, for example, to understand the Quantum Hall Effect [K85].

In gauge theories – including the Ginzburg-Landau theory – the Berry connection can be understood geometrically as follows: the space \mathcal{P} of solutions of gauge invariant equations is an infinite dimensional principal bundle over the part of moduli space \mathcal{M} where the action of the gauge group \mathcal{G} is free. Thus if all solutions are irreducible, then \mathcal{P} is an infinite dimensional principal \mathcal{G} -bundle

$$\begin{array}{c} \mathcal{P} \\ \downarrow \mathcal{G} \\ \mathcal{M} \end{array}$$

over \mathcal{M} . The canonical L^2 -metric of \mathcal{P} defines a horizontal distribution – the orthogonal complement of the gauge directions – which defines the Berry connection.

A connection on a principal G -bundle $P \rightarrow X$ defines parallel transport: for each smooth map $\Gamma : [0, 1] \rightarrow X$, parallel transport around Γ is a G -equivariant isomorphism from the fiber at $\Gamma(0)$ to the fiber at $\Gamma(1)$. If Γ is a closed loop, then $\Gamma(0) = \Gamma(1)$, and the corresponding parallel transport is called holonomy. If G is Abelian, holonomy is given by the action of an element in G .

Holonomies of the Berry connection are gauge transformations, which have a physical interpretation: they describe the adiabatic evolution of the state of the system, that is its behavior under slow changes in the physical parameters such as external fields, or coupling constants.

This dissertation investigates the Berry connection of the τ -vortex moduli space associated to a degree d hermitian line bundle over a closed, oriented Riemannian surface Σ . The Berry holonomy assigns a gauge transformation g_τ to each closed curve Γ in \mathcal{M}_τ . These gauge transformations are $U(1)$ -valued smooth functions on Σ . When d is positive and τ is greater than the geometry-dependent constant $\tau_0 = \frac{4\pi d}{\text{Area}(\Sigma)}$, then the moduli space, \mathcal{M}_τ ,

is identified with the d -fold symmetric power of the surface $\text{Sym}^d(\Sigma)$. A closed curve Γ in $\text{Sym}^d(\Sigma)$ defines a 1-cycle in Σ , called $\text{sh}(\Gamma)$, the shadow of Γ , which is defined by choosing a lift of Γ to $\Sigma^d \rightarrow \text{Sym}^d(\Sigma)$, and taking the union of the non-constant curves appearing in the lift (see equation (1.50) for precise definition). The main theorem of this paper gives a complete topological and analytical description of these gauge transformations in terms of the shadow:

Main Theorem of Chapter 1. [The Berry holonomy of the τ -vortex principal bundle] *Let $g_\tau \in \mathcal{G}$ be the Berry holonomy of a smooth curve Γ in the τ -vortex moduli space \mathcal{M}_τ , and $\text{sh}(\Gamma)$ be the closed 1-cycle in Σ defined in equation (1.50). Then the following properties hold as $\tau \rightarrow \infty$:*

1. [Convergence] $g_\tau \rightarrow 1$ in the C^1 -topology on compact sets of $\Sigma - \text{sh}(\Gamma)$.
2. [Crossing] Let $j : [0, 1] \rightarrow \Sigma$ be a smooth path that intersects $\text{sh}(\Gamma)$ transversally and positively once, and write $g_\tau \circ j = \exp(2\pi i \varphi_\tau)$. Then $\varphi_\tau(1) - \varphi_\tau(0) \rightarrow 1$.
3. [Concentration] As a 1-current, $\frac{1}{2\pi i} g_\tau^{-1} dg_\tau$ converges to the 1-current defined by $\text{sh}(\Gamma)$.

The map $\Gamma \mapsto g_\tau$ induces a pairing

$$\text{hol}_\star : H_1(\Sigma; \mathbb{Z}) \rightarrow H^1(\Sigma; \mathbb{Z}),$$

defined in (1.53).

- (4) [Duality] For all $\tau > \tau_0$, the homomorphism hol_\star is Poincaré duality.

When Γ is a positively oriented, bounding single vortex loop, or a positively oriented vortex interchange (see Section 1.6 for precise definitions) our main theorem implies that

the corresponding holonomy can be written as $g_\tau = \exp(2\pi i f_\tau)$, for a real function f_τ on Σ . Moreover, f_τ can be chosen so that it converges to 1 on the inside of the curve, and to 0 on the outside. This makes physicists' intuition about the holonomy precise; cf. [I01].

As mentioned above, τ is a scaling parameter for the metric: one can look at the Ginzburg-Landau theory with $\tau = 1$ fixed, but the Kähler form ω scaled as $\omega_t = t^2\omega$. Our results, including the Main Theorem above, can be reinterpreted as statements about the large area limit (i.e. $t \rightarrow \infty$), which can be more directly related to physics (see Section 1.7 for details).

This chapter is organized as follows. In Section 1.2, we give a brief introduction to the geometry of the τ -vortex equations on a closed surface, derive the tangent space equations of the τ -vortex moduli space, and then recast them in a compact form. In Section 1.3, we use theorems of Taubes and Bradlow to prove a technical result, Theorem 1.8, that establishes asymptotic formulas for the tangent vectors of the of the τ -vortex moduli space. In Section 1.4, we introduce the Berry connection associated to this problem. We then prove asymptotic formulas for the Berry curvature in Section 1.5. In Section 1.6, we prove our Main Theorem; the proofs are applications of Theorem 1.8. Section 1.7 discusses the large area limit.

1.2 Ginzburg-Landau theory on closed surfaces

1.2.1 The τ -vortex equations

Let Σ be a closed surface with Kähler form ω , compatible complex structure J , and Riemannian metric $\omega(-, J(-))$. Let $L \rightarrow \Sigma$ be a smooth complex line bundle of positive degree d with hermitian metric h . For each unitary connection ∇ , and smooth section ϕ , consider

the *Ginzburg-Landau free energy*:

$$\mathcal{E}_{\lambda,\tau}(\nabla, \phi) = \int_{\Sigma} \left(|F_{\nabla}|^2 + |\nabla\phi|^2 + \lambda w^2 \right) \omega, \quad (1.1)$$

where $\lambda, \tau \in \mathbb{R}_+$ are coupling constants, F_{∇} is the curvature of ∇ , and

$$w = \frac{1}{2}(\tau - |\phi|^2). \quad (1.2)$$

The Euler-Lagrange equations of the energy (1.1) are the *Ginzburg-Landau equations*:

$$d^* F_{\nabla} + i \operatorname{Im}(h(\phi, \nabla\phi)) = 0 \quad (1.3a)$$

$$\nabla^* \nabla \phi - \lambda w \phi = 0. \quad (1.3b)$$

When $\lambda = 1$, the energy (1.1) can be integrated by parts and rewritten as different sum of non-negative terms, and get the lower bound $2\pi\tau d$. The minimizers satisfy the τ -*vortex equations*:

$$i\Lambda F_{\nabla} - w = 0 \quad (1.4a)$$

$$\bar{\partial}_{\nabla} \phi = 0, \quad (1.4b)$$

where ΛF_{∇} is the inner product of the Kähler form ω and the curvature of ∇ , and $\bar{\partial}_{\nabla} = \nabla^{0,1}$ is the Cauchy-Riemann operator corresponding to ∇ . Solutions (∇, ϕ) to the first order equations (1.4a) and (1.4b) automatically satisfy the second order equations (1.3a) and (1.3b).

1.2.2 The τ -vortex moduli space

As is standard in gauge theory, we work with the Sobolev $W^{k,p}$ -completions of the space of connections and fields. Let \mathcal{C}_L be the $W^{1,2}$ -closure of the affine space of smooth unitary connections on L and Ω_L^0 be the $W^{1,2}$ -closure of the vector space of smooth sections of L . Similarly, let Ω^k , and Ω_L^k be the $W^{1,2}$ -closure of k -forms, and L -valued k -forms respectively. The corresponding gauge group \mathcal{G} is the $W^{2,2}$ -closure of $\text{Aut}(L)$ in the $W^{2,2}$ -topology. The gauge group is canonically isomorphic to the infinite dimensional Abelian Lie group $W^{2,2}(\Sigma, \text{U}(1))$, whose Lie algebra is $W^{2,2}(\Sigma; i\mathbb{R})$. Elements $g \in \text{Aut}(L)$ act on $\mathcal{C}_L \times \Omega_L^0$ as $g(\nabla, \phi) = (g \circ \nabla \circ g^{-1}, g\phi)$, and this defines a smooth action of \mathcal{G} on $\mathcal{C}_L \times \Omega_L^0$. Finally, the energy (1.1) extends to a smooth function on $\mathcal{C}_L \times \Omega_L^0$. The space \mathcal{P}_τ of all critical points of the extended energy is an infinite dimensional submanifold of $\mathcal{C}_L \times \Omega_L^0$. Due to the gauge invariance of energy (1.1), \mathcal{G} acts on \mathcal{P}_τ , and every critical point is gauge equivalent to a smooth one by elliptic regularity. The τ -vortex moduli space is the quotient space $\mathcal{M}_\tau = \mathcal{P}_\tau/\mathcal{G}$. Elements of \mathcal{P}_τ are called τ -vortex fields, while elements of \mathcal{M}_τ (gauge equivalence classes of τ -vortex fields) are called τ -vortices. For brevity, we sometimes write τ -vortex fields as $v = (\nabla, \phi) \in \mathcal{P}_\tau$ and the corresponding τ -vortices as $[v] = [\nabla, \phi] \in \mathcal{M}_\tau$.

There is a geometry-dependent constant $\tau_0 = \frac{4\pi d}{\text{Area}(\Sigma)}$, called the *Bradlow limit*, with the property that if $\tau < \tau_0$, then the moduli space is empty and if $\tau > \tau_0$, then there is a canonical bijection between \mathcal{M}_τ and the space of effective, degree d divisors; cf. [B90, Theorem 4.6]. This space is also canonically diffeomorphic to the d -fold symmetric product of the surface $\text{Sym}^d(\Sigma)$, which is the quotient of the d -fold product $\Sigma^{\times d} = \Sigma \times \dots \times \Sigma$ by the action of the permutation group S_d . Although this action is not free, the quotient is a smooth Kähler manifold of real dimension $2d$. For each value of $\tau > \tau_0$, there is a

canonical L^2 -Kähler structure (see Section 1.2.3). In the borderline $\tau = \tau_0$ case, the ϕ -field vanishes everywhere and the moduli space is in one-to-one correspondence with the moduli space holomorphic line bundles of degree d [B90, Theorem 4.7]. Accordingly, we focus on the $\tau > \tau_0$ case in this paper.

When $\tau > \tau_0$, Bradlow's map from \mathcal{M}_τ to $\text{Sym}^d(\Sigma)$ is easy to understand: By integrating equation (1.4a), one sees that the L^2 -norm of ϕ is positive. On the other hand, ϕ is a holomorphic section of L by equation (1.4b). Since ϕ is a non-vanishing holomorphic section it defines an effective, degree d divisor, giving us the desired map. The inverse of this map is much harder to understand and involves non-linear elliptic theory.

Much of this picture carries over to open surfaces, even with infinite area (for example $\Sigma = \mathbb{C}$), if one imposes proper integrability conditions; cf [T84]. For simplicity we will always assume that Σ is compact. Furthermore the moduli space is empty for $d < 0$, and a single point for $d = 0$. Thus we will always assume that $d > 0$ in this paper.

1.2.3 The horizontal subspaces

The tangent space at any point of the affine space $\mathcal{C}_L \times \Omega_L^0$ is the underlying vector space $i\Omega^1 \oplus \Omega_L^0$. The tangent space of \mathcal{P}_τ is described in the next lemma.

Lemma 1.1. *The tangent space of \mathcal{P}_τ at the τ -vortex field $v = (\nabla, \phi)$ is the vector space of pairs $(a, \psi) \in i\Omega^1 \oplus \Omega_L^0$ that satisfy*

$$i\Lambda da + \text{Re}(h(\psi, \phi)) = 0 \tag{1.5a}$$

$$\bar{\partial}_\nabla \psi + a^{0,1} \phi = 0. \tag{1.5b}$$

Proof. The linearization of equations (1.4a) and (1.4b) in the direction of (a, ψ) is:

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{1}{t} \left(i\Lambda F_{\nabla+t a} - \frac{1}{2} \left(\tau - |\phi + t\psi|^2 \right) \right) &= i\Lambda da + \operatorname{Re}(h(\psi, \phi)) \\ \lim_{t \rightarrow 0} \frac{1}{t} \left(\bar{\partial}_{\nabla+t a}(\phi + t\psi) \right) &= \bar{\partial}_{\nabla} \psi + a^{0,1} \phi, \end{aligned}$$

where we used that v is a τ -vortex field. This completes the proof, since the tangent space is the kernel of the linearization of equations (1.4a) and (1.4b). \square

The affine space $\mathcal{C}_L \times \Omega_L^0$ has a canonical L^2 -metric given by

$$\langle (a, \psi) | (a', \psi') \rangle = \int_{\Sigma} (a \wedge \bar{*} a' + \operatorname{Re}(h(\psi, \psi')) \omega) \quad (1.6)$$

where $\bar{*}$ is the (conjugate-linear) Hodge operator of the Riemannian metric of Σ . One can check that the restriction of the L^2 -metric (1.6) to the solutions of equations (1.5a) and (1.5b) makes \mathcal{P}_{τ} a smooth, weak Riemannian manifold, and that gauge transformations act isometrically on \mathcal{P}_{τ} .

The pushforward of the tangent space $T_1 \mathcal{G}$ by the gauge action is called the *vertical subspace* of $T_v \mathcal{P}_{\tau}$. We define the *horizontal subspace* of $T_v \mathcal{P}_{\tau}$ to be the orthogonal complement of the vertical subspace by the L^2 -metric (1.6). Since $\mathcal{M}_{\tau} = \mathcal{P}_{\tau} / \mathcal{G}$, the horizontal subspace is canonically isomorphic to the tangent space $T_{[v]} \mathcal{M}_{\tau}$ of the moduli space. The next lemma shows that the horizontal subspace is also the kernel of a first order linear elliptic operator.

Lemma 1.2. *The horizontal subspace of $T_v \mathcal{P}_{\tau}$, at the τ -vortex field $v = (\nabla, \phi) \in \mathcal{P}_{\tau}$, is the*

vector space of pairs $(a, \psi) \in i\Omega^1 \oplus \Omega_L^0$ that satisfy

$$(i\Lambda d + d^*)a + h(\psi, \phi) = 0 \quad (1.7a)$$

$$\bar{\partial}_{\nabla}\psi + a^{0,1}\phi = 0. \quad (1.7b)$$

Proof. The real part of equation (1.7a) is equation (1.5a) and equation (1.7b) is equation (1.5b); thus solutions of equations (1.7a) and (1.7b) are in $T_v\mathcal{P}_\tau$. To finish the proof, we must check that a pair (a, ψ) in $T_v\mathcal{P}_\tau$ is orthogonal to the vertical subspace at v exactly if equations (1.7a) and (1.7b) hold. The pushforward of $if \in \text{Lie}(\mathcal{G})$ at v is given by $X_f(v) = (-idf, if\phi)$, hence horizontal vectors are pairs (a, ψ) , that satisfy the following equation for every $f \in C^\infty(\Sigma; \mathbb{R})$:

$$0 = \langle (a, \psi) | (-idf, if\phi) \rangle = \int_{\Sigma} (a \wedge \bar{*}(-idf) + \text{Re}(h(\psi, if\phi)))\omega.$$

Integrating the right-hand side by parts yields

$$0 = \int_{\Sigma} (d^*a + i \text{Im}(h(\psi, \phi)))if\omega.$$

Because this holds for all f , we conclude that (a, ψ) is orthogonal to the vertical subspace at v exactly if $d^*a + i \text{Im}(h(\psi, \phi)) = 0$ holds. Adding this (purely imaginary) equation to the (purely real) equation (1.5a) gives equation (1.7a). \square

Equations (1.7a) and (1.7b) depend on the choice of v , but if (a, ψ) is a solution of equations (1.5b) and (1.7a) for v and $g \in \mathcal{G}$, then $(a, g\psi)$ is a solution of equations (1.5b) and (1.7a) for $g(v) = (\nabla + gdg^{-1}, g\phi)$. Since gauge transformations act isometrically, the

L^2 -metric on the horizontal subspaces of $T\mathcal{P}_\tau$ descends to a Riemannian metric on \mathcal{M}_τ .

Let \bar{K} be the anti-canonical bundle of Σ , and $\Omega^{0,1} = \Omega_{\bar{K}}^0$, the $W^{1,2}$ -completion of the space of smooth sections of \bar{K} . We recast equations (1.7a) and (1.7b) in a more geometric way in the next lemma.

Lemma 1.3. *Equations (1.7a) and (1.7b) are equivalent to the following pair of equations on $(\alpha, \psi) \in \Omega^{0,1} \oplus \Omega_L^0$:*

$$\sqrt{2} \bar{\partial}^* \alpha - h(\phi, \psi) = 0 \tag{1.8a}$$

$$\sqrt{2} \bar{\partial}_\nabla \psi + \alpha \phi = 0. \tag{1.8b}$$

Moreover, the unitary bundle isomorphism

$$(a, \psi) \mapsto \left(\frac{1}{\sqrt{2}}(a + i\bar{*}a), \psi \right) \tag{1.9}$$

interchanges solutions of equations (1.7a) and (1.7b) with solutions of equations (1.8a) and (1.8b).

Proof. A complex 1-form α is in $\Omega^{0,1}$ exactly if $\bar{\alpha} = i\bar{*}\alpha$. For $a \in i\Omega^1$, define the unitary map u by

$$u(a) = \frac{1}{\sqrt{2}}(a + i\bar{*}a). \tag{1.10}$$

Using $\bar{*}^2 a = -a = \bar{a}$, we see that $\bar{*}u(a) = -i\overline{u(a)}$, and thus $u(a) \in \Omega^{0,1}$. Set $\alpha = u(a)$. With this notation $a^{0,1} = \frac{\alpha}{\sqrt{2}}$, which proves the equivalence of equations (1.7b) and (1.8b). The Kähler identities yield $(i\Lambda d + d^*)a = -\sqrt{2} \bar{\partial}^* \bar{\alpha}$, which is equivalent to equation (1.8a). \square

The vector space of solutions to equations (1.8a) and (1.8b) has a canonical almost

complex structure coming from the complex structures of \overline{K} and L , and this defines an almost complex structure for \mathcal{M}_τ . Mundet i Riera [R00] showed that this structure is integrable, and together with the L^2 -metric it makes \mathcal{M}_τ a Kähler manifold.

To put equations (1.8a) and (1.8b) in a more compact form, note that they are equivalent to the single equation

$$\mathcal{L}_v(a, \psi) = 0,$$

where $\mathcal{L}_v = \mathcal{D}_\nabla + \mathcal{A}_\phi$ is defined as

$$\begin{aligned} \mathcal{D}_\nabla : \Omega^{0,1} \oplus \Omega_L^0 &\rightarrow \Omega^0 \oplus \Omega_L^{0,1}; & (\alpha, \psi) &\mapsto \left(\sqrt{2} \bar{\partial}^* \alpha, \sqrt{2} \bar{\partial}_\nabla \psi \right) \\ \mathcal{A}_\phi : \Omega^{0,1} \oplus \Omega_L^0 &\rightarrow \Omega^0 \oplus \Omega_L^{0,1}; & (\alpha, \psi) &\mapsto (-h(\phi, \psi), \alpha\phi), \end{aligned}$$

The operator \mathcal{D}_∇ is a first order elliptic differential operator, and the operator \mathcal{A}_ϕ is a bundle map. Straightforward computation shows that for all $Z \in \Omega^{0,1} \oplus \Omega_L^0$

$$\mathcal{A}_\phi \mathcal{A}_\phi^*(Z) = |\phi|^2 Z. \tag{1.11}$$

Thus \mathcal{A}_ϕ^* is non-degenerate on the complement of the divisor of ϕ . Note that \mathcal{D}_∇ , \mathcal{A}_ϕ , and hence $\mathcal{L}_{(\nabla, \phi)}$ make sense for any pair $(\nabla, \phi) \in \mathcal{C}_L \times \Omega_L^0$.

Lemma 1.4. *Let $v = (\nabla, \phi) \in \mathcal{C}_L \times \Omega_L^0$ be a pair such that $\bar{\partial}_\nabla \phi = 0$. Then the operator $\mathcal{D}_\nabla \mathcal{A}_\phi^* + \mathcal{A}_\phi \mathcal{D}_\nabla^*$ is identically zero.*

Proof. The adjoint operators are

$$\mathcal{D}_{\nabla}^*(f, \xi) = \left(\sqrt{2} \bar{\partial} f, \sqrt{2} \bar{\partial}_{\nabla}^* \xi \right) \quad (1.12a)$$

$$\mathcal{A}_{\phi}^*(f, \xi) = (h(\phi, \xi), -f\phi) \quad (1.12b)$$

for any $(f, \xi) \in \Omega^0 \oplus \Omega_L^{0,1}$. The lemma follows from equations (1.12a) and (1.12b), the holomorphicity of ϕ , and the definitions of \mathcal{D}_{∇} and \mathcal{A}_{ϕ} . \square

Corollary 1.5. *Let $v = (\nabla, \phi) \in \mathcal{C}_L \times \Omega_L^0$ be a pair such that $\bar{\partial}_{\nabla} \phi = 0$. Then $\ker(\mathcal{L}_v^*)$ is trivial.*

Proof. From Lemma 1.4 and equation (1.11), we obtain $\mathcal{L}_v \mathcal{L}_v^* = \mathcal{D}_{\nabla} \mathcal{D}_{\nabla}^* + \mathcal{A}_{\phi} \mathcal{A}_{\phi}^*$. Hence if Z is in the kernel of \mathcal{L}_v^* , then

$$0 = \|\mathcal{L}_v^*(Z)\|_{L^2}^2 = \|\mathcal{D}_{\nabla}^*(Z)\|_{L^2}^2 + \|\mathcal{A}_{\phi}^*(Z)\|_{L^2}^2,$$

which implies that both terms on the right vanish. By equation (1.11), Z vanishes where ϕ does not, which is the complement of a finite set. But then Z vanishes everywhere by continuity. Hence the kernel of \mathcal{L}_v^* is trivial. \square

1.3 The asymptotic form of horizontal vectors

In this section we will use of the following results of Taubes and Bradlow about the large τ behavior of τ -vortex fields. Recall from equation (1.2) that

$$w = \frac{1}{2}(\tau - |\phi|^2).$$

Theorem 1.6. [Bradlow and Taubes] *There is a positive number c such that each τ -vortex field $v = (\nabla, \phi) \in \mathcal{P}_\tau$ satisfies*

$$|\phi|^2 \leq \tau \tag{1.13a}$$

$$w + |\nabla\phi| \leq c\tau \exp\left(-\frac{\sqrt{\tau}\text{dist}_D}{c}\right), \tag{1.13b}$$

where dist_D is the distance from the divisor $D = \phi^{-1}(0)$, and w is defined in equation (1.2).

Proof. In [B90, Proposition 5.2] Bradlow showed inequality (1.13a), using the fact that τ -vortex fields satisfy the elliptic equation

$$\left(\Delta + |\phi|^2\right)w = |\partial_{\nabla}\phi|^2. \tag{1.14}$$

The right-hand side is positive away from a finite set, so the maximum principle and equation (1.2) implies inequality (1.13a). Inequality (1.13b) was proved in [T99, Lemma 3.3]. \square

We call a divisor *simple* if the multiplicity of every divisor point is 1.

Lemma 1.7. *Fix a simple divisor $D \in \text{Sym}^d(\Sigma)$ and a corresponding τ -vortex field $v = (\nabla, \phi)$ with $\tau > \tau_0$. The smooth function*

$$h_{D,\tau} = \frac{1}{2\pi\tau} \left(|\partial_{\nabla}\phi|^2 + 2w^2 \right) \tag{1.15}$$

depends only on D and τ , but not on the choice of v . Moreover,

$$\lim_{\tau \rightarrow \infty} h_{D,\tau} = \sum_{p \in D} \delta_p, \tag{1.16}$$

in the sense of measures, where δ_p is the Dirac measure concentrated at the point $p \in \Sigma$.

Proof. Every term in equation (1.15) is gauge invariant, which proves the independence of the choice of v for D . Using equation (1.14) we get $h_{D,\tau} = \frac{1}{2\pi\tau}(\Delta w + \tau w)$, hence for any smooth function f :

$$\begin{aligned} \int_{\Sigma} h_{D,\tau} f \omega &= \frac{1}{2\pi\tau} \int_{\Sigma} (\Delta w + \tau w) f \omega \\ &= \frac{1}{2\pi\tau} \int_{\Sigma} w (\Delta f) \omega + \frac{1}{2\pi} \int_{\Sigma} w f \omega. \end{aligned}$$

By [HJS96, Theorem 1.1], w converges to $2\pi\delta_D$ in the sense of measures as $\tau \rightarrow \infty$. Thus the first term converges to 0, and the second term converges to $\sum_{p \in D} f(p)$, which completes the proof. \square

The space of simple divisors, $\text{Sym}_s^d(\Sigma)$, is an open dense set in $\text{Sym}^d(\Sigma)$, and its complement is called the *big diagonal*. When D is simple, a tangent vector in $T_D \text{Sym}^d(\Sigma)$ can be given by specifying a tangent vector to Σ at each divisor point. Thus the rank d complex vector bundle $\bar{\mathcal{K}} \rightarrow \text{Sym}_s^d(\Sigma)$ defined by

$$\bar{\mathcal{K}}_D = \bigoplus_{p \in D} \bar{K}_p$$

is isomorphic to $T^{0,1} \text{Sym}_s^d(\Sigma)$. We next use ideas of [T99, Lemma 3.3] to construct an almost unitary isomorphism from $\bar{\mathcal{K}}$ to $T^{0,1} \text{Sym}_s^d(\Sigma)$.

Fix a simple divisor D , and let v be a corresponding τ -vortex field. Define

$$\rho_D = \min(\{\text{dist}(p, q) | p, q \in D \text{ \& } p \neq q\} \cup \{\text{inj}(\Sigma, \omega)\}), \quad (1.17)$$

where $\text{inj}(\Sigma, \omega)$ is the injectivity radius of the metric. Let χ be a smooth function on $[0, \infty)$ that satisfies $0 \leq \chi \leq 1$, $\chi|_{[0,1]} = 1$, and $\chi|_{[2,\infty)} = 0$, and set

$$\chi_p = \chi\left(\frac{2\text{dist}_p}{\rho_D}\right). \quad (1.18)$$

For each $\Theta = \{\theta_p\}_{p \in D} \in \overline{\mathcal{K}}_D$ let $\widehat{\theta}_p$ be the extension of θ_p to the open ball of radius $\text{inj}(\Sigma, \omega)$ centered at p using the exponential map. Define a smooth section σ_Θ of \overline{K} supported in neighborhood of D by setting

$$\sigma_\Theta = \sum_{p \in D} \chi_p \widehat{\theta}_p \quad (1.19)$$

and extending by 0 to all of Σ . Note that σ_Θ satisfies

$$\sigma_\Theta(p) = \theta_p \quad \& \quad |\nabla \sigma_\Theta| = O(\text{dist}_D) \quad \forall p \in D. \quad (1.20)$$

Finally, for each such v and Θ define $Y_{v,\Theta}$ as

$$Y_{v,\Theta} = \frac{1}{\sqrt{2\pi\tau}} \left(\sqrt{2} w \sigma_\Theta, i\Lambda(\sigma_\Theta \wedge \partial_{\nabla} \phi) \right) \in \Omega^{0,1} \oplus \Omega_L^0, \quad (1.21)$$

where again $w = \frac{1}{2}(\tau - |\phi|^2)$. Note that $Y_{v,\Theta}$ is gauge equivariant, that is, for every $g \in \mathcal{G}$:

$$g_* Y_{v,\Theta} = Y_{g(v),\Theta}. \quad (1.22)$$

The following analytic result is the key ingredient needed to compute the asymptotic curvature in Theorem 1.11 and holonomies in the Main Theorem.

Theorem 1.8. [The asymptotic form of horizontal vectors] *For every $v \in \mathcal{P}_\tau$ and $\Theta \in \overline{\mathcal{K}}_D$*

as above, there is a unique $Z_{v,\Theta} \in \Omega^0 \oplus \Omega_L^{0,1}$ such that

$$X_{v,\Theta} = Y_{v,\Theta} - \mathcal{L}_v^*(Z_{v,\Theta}) \quad (1.23)$$

is a horizontal tangent vector at v . Moreover, the following asymptotic estimates hold:

1. [L^2 -estimate] $\|Y_{v,\Theta}\|_{L^2(\Sigma)}^2 \rightarrow \sum_{p \in D} |\theta_p|^2$ as $\tau \rightarrow \infty$.
2. [Pointwise bound] $|X_{v,\Theta} - Y_{v,\Theta}| = O\left(\tau^{-1/2} \exp\left(-\frac{\sqrt{\tau} \text{dist}_D}{c}\right)\right)$, where dist_D is the distance from D , and c is the positive number from Theorem 1.6.

Equation (1.23) defines a bundle map from $\bar{\mathcal{K}}$ to $T^{0,1}\text{Sym}_s^d(\Sigma)$ by

$$(D, \Theta) \mapsto (D, \Pi_*(X_{v,\Theta})), \quad (1.24)$$

where v is any τ -vortex field corresponding to the divisor D , and Π is the projection from \mathcal{P}_τ to $\mathcal{M}_\tau \cong \text{Sym}^d(\Sigma)$. Equation (1.22) implies that (1.24) does not depend on the choice of v . Furthermore, this map is almost unitary by Statements (1) and (2). Similar results have only been known for flat metrics [T99, Lemma 3.3].

Proof of Theorem 1.8: Fix $Y_{v,\Theta}$ as in equation (1.21). Since \mathcal{L}_v is elliptic, and $\ker(\mathcal{L}_v^*) = \{0\}$, by Corollary 1.5, the operator $\mathcal{L}_v \mathcal{L}_v^*$ is has a bounded inverse $(\mathcal{L}_v \mathcal{L}_v^*)^{-1}$. Thus the equation

$$\mathcal{L}_v(Y_{v,\Theta} - \mathcal{L}_v^*(Z_{v,\Theta})) = 0 \quad (1.25)$$

has a unique solution for $Z_{v,\Theta}$ given by

$$Z_{v,\Theta} = (\mathcal{L}_v \mathcal{L}_v^*)^{-1}(\mathcal{L}_v(Y_{v,\Theta})) \quad (1.26)$$

Consequently $X_{v,\Theta}$ in equation (1.23) is horizontal.

The pointwise norm of $Y_{v,\Theta}$ satisfies

$$|Y_{v,\Theta}|^2 = h_{D,\tau} |\sigma_\Theta|^2,$$

where $h_{D,\tau}$ is defined in equation (1.15). Using equation (1.16), one gets Statement (1).

In order to prove Statement (2), we put $Z_{v,\Theta} = (f_{v,\Theta}, \xi_{v,\Theta}) \in \Omega^0 \oplus \Omega_L^{0,1}$ in equation (1.25) to obtain the equations

$$\left(\frac{1}{2}\Delta + \frac{1}{2}|\phi|^2\right) f_{v,\Theta} = \frac{1}{\sqrt{8\pi\tau}} w \bar{\partial}^* \sigma_\Theta, \quad (1.27a)$$

$$\left(\bar{\partial}_\nabla \bar{\partial}_\nabla^* + \frac{1}{2}|\phi|^2\right) \xi_{v,\Theta} = \frac{i}{\sqrt{4\pi\tau}} \Lambda \left(\nabla^{0,1} \sigma_\Theta \wedge \partial_\nabla \phi \right). \quad (1.27b)$$

Let $G_{[\phi]}$ be the Green's operator of the non-degenerate elliptic operator $H_{[\phi]} = \frac{1}{2}\Delta + \frac{1}{2}|\phi|^2$ on Ω^0 . Both $H_{[\phi]}$ and $G_{[\phi]}$ depend only on the gauge equivalence class of ϕ . By an abuse of notation $G_{[\phi]}$ will also denote the corresponding Green's function, which is a positive, symmetric function on $\Sigma \times \Sigma$ with a logarithmic singularity along the diagonal. With this definitions, we can write $f_{v,\Theta}$ as

$$f_{v,\Theta} = \frac{1}{\sqrt{8\pi\tau}} \int_{\Sigma} G_{[\phi]} w \bar{\partial}^* \sigma_\Theta \omega. \quad (1.28)$$

Standard elliptic theory gives the following bounds on the Green's function:

$$G_{[\phi]}(x, y) \leq c \left(1 + \left| \ln(\sqrt{\tau} \text{dist}(x, y)) \right| \right) \exp\left(-\frac{\sqrt{\tau} \text{dist}(x, y)}{c}\right) \quad (1.29a)$$

$$\left| dG_{[\phi]}(x, y) \right| \leq \frac{c}{\sqrt{\tau} \text{dist}(x, y)} \exp\left(-\frac{\sqrt{\tau} \text{dist}(x, y)}{c}\right) \quad (1.29b)$$

for some $c \in \mathbb{R}_+$ independent of τ or D (see [T99, Equation (6.10)]). Using equation (1.28) and inequalities (1.29a) and (1.29b), together with the bound on w in inequality (1.13b) and on $|\nabla\sigma_\Theta|$ in equation (1.20) we get (after possibly increasing c)

$$|f_{v,\Theta}| \leq \frac{c}{\tau} \exp\left(-\frac{\sqrt{\tau}\text{dist}_D}{c}\right) \quad (1.30a)$$

$$|\bar{\partial}f_{v,\Theta}| \leq \frac{c}{\sqrt{\tau}} \exp\left(-\frac{\sqrt{\tau}\text{dist}_D}{c}\right). \quad (1.30b)$$

Before turning our attention to equation (1.27b), note that we have the following two scalar identities

$$\Delta|\xi_{v,\Theta}|^2 = 2\text{Re}(\langle \xi_{v,\Theta} | \nabla^* \nabla \xi_{v,\Theta} \rangle) - 2|\nabla \xi_{v,\Theta}|^2. \quad (1.31a)$$

$$\Delta|\xi_{v,\Theta}|^2 = 2|\xi_{v,\Theta}|\Delta|\xi_{v,\Theta}| - 2|d|\xi_{v,\Theta}||^2, \quad (1.31b)$$

Using the Kähler identity on $\Omega_L^{0,1}$

$$\nabla^* \nabla = 2\bar{\partial}_\nabla \bar{\partial}_\nabla^* - i\Lambda F_\nabla = 2\bar{\partial}_\nabla \bar{\partial}_\nabla^* - \frac{1}{2}(\tau - |\phi|^2), \quad (1.32)$$

in equation (1.27b), together with the Cauchy-Schwarz inequality in equation (1.31a), and Kato's inequality (cf. [FU91, Equation (6.20)])

$$|d|\xi_{v,\Theta}|| \leq |\nabla \xi_{v,\Theta}| \quad (1.33)$$

gives us

$$\left(\Delta + \frac{1}{2}\tau\right)|\xi_{v,\Theta}| \leq \frac{c}{\sqrt{\tau}}|\nabla\sigma_\Theta||\bar{\partial}_\nabla\phi|. \quad (1.34)$$

Equation (1.34), together with the bound on $|\bar{\partial}_{\nabla}\phi|$ in inequality (1.13b) and the bound on $|\nabla\sigma_{\Theta}|$ in equation (1.20) gives us (again after possibly increasing c)

$$|\xi_{v,\Theta}| \leq \frac{c}{\tau} \exp\left(-\frac{\sqrt{\tau}\text{dist}_D}{c}\right). \quad (1.35)$$

Applying $\bar{\partial}_{\nabla}^*$ to equation (1.27b) gives an elliptic equation on $\bar{\partial}_{\nabla}^*\xi_{v,\Theta}$. Similarly to the previous computation we get the following inequality:

$$\left(\Delta + \frac{1}{2}\tau\right) \left|\bar{\partial}_{\nabla}^*\xi_{v,\Theta}\right| \leq c\sqrt{\tau} \exp\left(-\frac{\sqrt{\tau}\text{dist}_D}{c}\right).$$

Thus (after possibly increasing c one last time)

$$\left|\bar{\partial}_{\nabla}^*\xi_{v,\Theta}\right| \leq \frac{c}{\sqrt{\tau}} \exp\left(-\frac{\sqrt{\tau}\text{dist}_D}{c}\right). \quad (1.36)$$

Finally, inequalities (1.30a), (1.30b), (1.35) and (1.36) give us

$$\begin{aligned} |X_{v,\Theta} - Y_{v,\Theta}| &= |\mathcal{L}_v^*(Z_{v,\Theta})| \\ &\leq \sqrt{2}|\bar{\partial}f_{v,\Theta}| + |\phi||f| + \sqrt{2}\left|\bar{\partial}_{\nabla}^*\xi\right| + |\phi||\xi| \\ &= O\left(\tau^{-1/2} \exp\left(-\frac{\sqrt{\tau}\text{dist}_D}{c}\right)\right), \end{aligned}$$

which completes the proof of Statement (2). □

1.4 The Berry connection

The τ -vortex principal bundle is the principal \mathcal{G} -bundle $\Pi : \mathcal{P}_\tau \rightarrow \mathcal{M}_\tau$ described in Section 1.2, with $\Pi(v) = [v]$. In Lemma 1.1 we constructed a horizontal distribution on the τ -vortex principal bundle, which is the orthogonal complement of the kernel of Π_* . This distribution is \mathcal{G} -invariant, so is a connection in the distributional sense (cf. [KN63, Chapter II]), which we call the *Berry connection*. The corresponding connection 1-form is the unique $\text{Lie}(\mathcal{G})$ -valued 1-form A that satisfies the three conditions:

1. $\ker(A_v)$ is the horizontal subspace at $v \in \mathcal{P}_\tau$,
2. $(g^*A)_{g(v)} = \text{ad}_g(A_v)$, for all $g \in \mathcal{G}$,
3. $A(X_f) = if$, for all $if \in W^{2,2}(\Sigma; i\mathbb{R}) \cong \text{Lie}(\mathcal{G})$, where $X_f(v) = (-idf, if\phi)$, as defined in Lemma 1.2.

The next lemma gives a formula for A_v . Recall that for each τ -vortex field $v = (\nabla, \phi)$, the Green's operator $G_{[\phi]}$ is the inverse of the non-degenerate elliptic operator $H_{[\phi]} = \frac{1}{2}\Delta + \frac{1}{2}|\phi|^2$.

Lemma 1.9. *The $\text{Lie}(\mathcal{G})$ -valued 1-form on \mathcal{P}_τ defined as*

$$A_v(a, \psi) = -\frac{1}{2}G_{[\phi]}(d^*a + i \text{Im}(h(\psi, \phi))) \quad (1.37)$$

is the connection 1-form corresponding to the Berry connection.

Proof. The right-hand side of equation (1.37) is the composition of the non-degenerate Green's operator and a $\text{Lie}(\mathcal{G})$ -valued 1-form. In the proof of Lemma 1.1 we saw that the kernel of this 1-form is exactly the horizontal subspace. This proves Condition (1) above.

Because \mathcal{G} is Abelian, the adjoint representation of \mathcal{G} is trivial, and hence the Condition (2) reduces to $(g^*A)_{g(v)} = A_v$. Since $g_*(a, \psi) = (a, g\psi)$, we have

$$\begin{aligned} g^*A_{g(v)}(a, \psi) &= \frac{-1}{2}G_{[\phi]}(d^*a + i \operatorname{Im}(h(g\psi, g\phi))) \\ &= \frac{-1}{2}G_{[\phi]}(d^*a + i \operatorname{Im}(h(\psi, \phi))), \end{aligned}$$

thus $g^*A_{g(v)} = A_v$. This proves Condition (2).

Finally, we show that A is the canonical isomorphism between the fibers of the vertical bundle and the Lie algebra of \mathcal{G} , that is $A(X_f) = if$ for every $f \in C^\infty(\Sigma; \mathbb{R})$:

$$\begin{aligned} A_v(X_f) &= \frac{-1}{2}G_{[\phi]}(d^*(-idf) + i \operatorname{Im}(h(if\phi, \phi)))\omega \\ &= iG_{[\phi]} \left(\left(\frac{1}{2}\Delta f + \frac{1}{2}|\phi|^2 \right) f \right) \\ &= if, \end{aligned}$$

thus Condition (3) holds. □

We can use Lemma 1.9 to compute the curvature 2-form of the Berry connection. Since \mathcal{G} is Abelian, the curvature, called the *Berry curvature*, is a $\operatorname{Lie}(\mathcal{G})$ -valued 2-form which descends to the base space \mathcal{M}_τ .

Theorem 1.10. *The curvature 2-form of the Berry connection at $[v] \in \mathcal{M}_\tau$ is*

$$\Omega_{[v]}(X, Y) = G_{[\phi]}(i \operatorname{Im}(h(\psi_X, \psi_Y))) \tag{1.38}$$

where (a_X, ψ_X) and (a_Y, ψ_Y) are the horizontal lifts of X and Y , respectively, at v . Moreover, equation (1.38) does not depend on the choice of the τ -vortex field v representing $[v]$.

Proof. The claim about the independence of the choice v is immediate since everything on the right-hand side is gauge invariant.

The curvature is the unique $\text{Lie}(\mathcal{G})$ -valued 2-form Ω on \mathcal{M}_τ that satisfies $\Pi^*(\Omega) = dA$, where Π is the projection from \mathcal{P}_τ to \mathcal{M}_τ . Thus it is enough to compute the expression $dA_v((a_X, \psi_X), (a_Y, \psi_Y))$, and compare it with equation (1.38). Recall, that the formula for the exterior derivative

$$dA(\tilde{X}, \tilde{X}) = \tilde{X}(A(\tilde{Y})) - \tilde{Y}(A(\tilde{X})) - A([\tilde{X}, \tilde{Y}]), \quad (1.39)$$

where \tilde{X} and \tilde{Y} are smooth local extensions of (a_X, ψ_X) and (a_Y, ψ_Y) respectively. Choose the extensions so that their Lie bracket vanishes at v . Let Υ_t be the local flow generated by \tilde{X} , so $\Upsilon_t(v) = v + t(a_X, \psi_X) + O(t^2)$. Since $A(\tilde{Y}) = 0$ at $\Upsilon_0(v) = v$, we have

$$\tilde{X}_v(A(\tilde{Y})) = \lim_{t \rightarrow 0} \frac{1}{t} A_{\Upsilon_t(v)}(\tilde{Y}(\Upsilon_t(v))).$$

Note that $\tilde{Y}(\Upsilon_t(v)) = ((\Upsilon_t)_*(a_Y, \psi_Y)) + O(t^2)$, because $[\tilde{X}, \tilde{Y}]_v = 0$. Finally, let us write

$$G_{[\phi + t\psi_X + O(t^2)]} = G_{[\phi]} + t G_{[\phi]}^X + O(t^2).$$

Keeping only the linear terms, we obtain

$$\begin{aligned}
\tilde{X}_v\left(A\left(\tilde{Y}\right)\right) &= \lim_{t \rightarrow 0} \frac{1}{t} A_{\Upsilon_t(v)}\left(\tilde{Y}\left(\Upsilon_t(v)\right)\right) \\
&= \lim_{t \rightarrow 0} -\frac{1}{2t} G\left[\phi+t \psi_X+O\left(t^2\right)\right]\left(d^* a_Y+i \operatorname{Im}\left(h\left(\psi_Y, \phi+t \psi_X\right)\right)+O\left(t^2\right)\right) \\
&= \lim_{t \rightarrow 0} -\frac{1}{2t}\left(i t G_{[\phi]}(\operatorname{Im}\left(h\left(\psi_Y, \psi_X\right)\right))+t G_{[\phi]}^X\left(d^* a_Y+i \operatorname{Im}\left(h\left(\psi_Y, \phi\right)\right)\right)\right) \\
&= -\frac{i}{2} G_{[\phi]}(\operatorname{Im}\left(h\left(\psi_Y, \psi_X\right)\right)),
\end{aligned}$$

where we used the fact that $d^* a_Y + i \operatorname{Im}(h(\psi_Y, \phi)) = 0$ for tangent vectors. Interchanging \tilde{X} and \tilde{Y} changes sign, since $\operatorname{Im}(h(\psi_Y, \psi_X))$ is skew. Substituting these into equation (1.39), and noting that the commutator vanishes, gives equation (1.38). \square

1.5 The asymptotic Berry curvature

In this section we use Theorems 1.8 and 1.10 to analyze the Berry curvature in the large τ limit.

As before, let D be a simple divisor, and $v = (\nabla, \phi)$ be the corresponding τ -vortex field. For each $p \in D$, choose $\Theta_p = \{\theta_{p,q}\}_{q \in D} \in \bar{\mathcal{K}}_D$, so that $|\theta_{p,q}| = \delta_{p,q}$. Let $\sigma_p = \sigma_{\Theta_p}$ be the corresponding section defined by equation (1.19), and let $X_{v, \Theta_p} = (a_p, \psi_p)$, as defined in Theorem 1.8. By equation (1.23),

$$X_{v, \Theta_p} = Y_{v, \Theta_p} - \mathcal{L}_v^*(Z_{v, \Theta_p}). \quad (1.40)$$

where $Z_{v, \Theta_p} = (f_p, \xi_p) \in \Omega^0 \oplus \Omega_L^{0,1}$. It is easy to see that in Statement (2) of Theorem 1.8 we can now replace dist_D with dist_p , the distance from the single point p .

The set $\{X_p\}_{p \in D}$, where $X_p = \Pi_* \left(X_{\nu, \Theta_p} \right) \in T_{[v]} \mathcal{M}_\tau$, is an asymptotically orthonormal basis for the horizontal subspace at ν , in the sense that as $\tau \rightarrow \infty$

$$\langle X_p | X_q \rangle = \delta_{p,q} + O\left(\exp\left(-\frac{\sqrt{\tau} \rho_D}{c}\right)\right) \rightarrow \delta_{p,q}.$$

Finally, for each tangent vector X , let $X^\flat = \langle X | - \rangle$ be the metric-dual covector.

Theorem 1.11. [The asymptotic Berry curvature] *There is a positive number c such that if*

$\tau > \tau_0 = \frac{4\pi d}{\text{Area}(\Sigma)}$ and $[v]$ is a simple τ -vortex, then the Berry curvature satisfies

$$\Omega_{[v]} = \sum_{p,q \in D} \left(\left(\chi_p \delta_{p,q} \frac{iw}{\pi\tau} + iA_\tau^{p,q} \right) X_p^\flat \wedge (iX_q)^\flat + iB_\tau^{p,q} X_p^\flat \wedge X_q^\flat + iC_\tau^{p,q} (iX_p)^\flat \wedge (iX_q)^\flat \right), \quad (1.41)$$

where χ_p as defined in Equation (1.18), and $A_\tau^{p,q}$, $B_\tau^{p,q}$, and $C_\tau^{p,q}$ are real functions, with

$$|A_\tau^{p,q}| + |B_\tau^{p,q}| + |C_\tau^{p,q}| = O\left(\tau^{-1} \exp\left(-\frac{\sqrt{\tau} \rho_D}{c}\right)\right). \quad (1.42)$$

Proof. For $p \neq q$, Theorems 1.6 and 1.8 imply that

$$|h(\psi_p, \psi_q)| = |\psi_p| |\psi_q| \leq c \exp\left(-\frac{\sqrt{\tau}(\text{dist}_p + \text{dist}_q)}{c}\right) \leq c \exp\left(-\frac{\sqrt{\tau} \rho_D}{c}\right).$$

This inequality together with the fact that $G_{[\phi]}(1) = O(\tau^{-1})$ from inequality (1.29a), gives equation (1.42) in this case.

In general, for every $p \in D$, Theorem 1.10 shows that

$$\frac{1}{i} \Omega_{[v]}(X_p, iX_p) = G_{[\phi]}(|\psi_p|^2). \quad (1.43)$$

This is non-negative, because $G_{[\phi]}$ is given by convolutions with the positive Green's function.

By equation (1.40), we can write

$$\psi_p = \frac{1}{\sqrt{2\pi\tau}} i\Lambda(\sigma_p \wedge \partial_{\nabla}\phi) - \sqrt{2} \bar{\partial}_{\nabla}^* \xi_p + f_p \phi. \quad (1.44)$$

Applying the bounds in Theorems 1.6 and 1.8 to equations (1.43) and (1.44), we obtain

$$\frac{1}{i} \Omega_{[v]}(X_p, iX_p) = \frac{1}{2\pi\tau} G_{[\phi]}(|\sigma_p|^2 |\partial_{\nabla}\phi|^2) + G_{[\phi]} \left(O \left(\exp \left(-\frac{\sqrt{\tau} \text{dist}_p}{c} \right) \right) \right). \quad (1.45)$$

By inequality (1.29a), and the positivity of the Green's function the last term is

$$G_{[\phi]} \left(O \left(\exp \left(-\frac{\sqrt{\tau} \text{dist}_p}{c} \right) \right) \right) = O \left(\tau^{-1} \exp \left(-\frac{\sqrt{\tau} \text{dist}_p}{c} \right) \right). \quad (1.46)$$

Theorem 1.6 and equation (1.14) gives us

$$H_{[\phi]}(\chi_p w) = \frac{1}{2} \chi_p |\partial_{\nabla}\phi|^2 + O \left(\tau \exp \left(-\frac{\sqrt{\tau} \text{dist}_p}{c} \right) \right). \quad (1.47)$$

Thus we can write the main term in equation (1.45) as

$$\begin{aligned} G_{[\phi]} \left(\frac{|\sigma_p|^2 |\partial_{\nabla}\phi|^2}{2\pi\tau} \right) &= G_{[\phi]} \left(\frac{\chi_p |\partial_{\nabla}\phi|^2}{2\pi\tau} \right) + O \left(G_{[\phi]} \left(\tau \text{dist}_p^2 \exp \left(-\frac{\sqrt{\tau} \text{dist}_p}{c} \right) \right) \right) \\ &= \chi_p \frac{w}{\pi\tau} + O \left(\tau^{-1} \exp \left(-\frac{\sqrt{\tau} \text{dist}_p}{c} \right) \right), \end{aligned} \quad (1.48)$$

since $|\sigma_p|^2 - \chi_p = O(\text{dist}_p^2)$ by (1.20). Combining equations (1.45), (1.46) and (1.48) yields

$$\frac{1}{i} \Omega_{[v]}(X_p, iX_p) = \chi_p \frac{w}{\pi\tau} + O \left(\tau^{-1} \exp \left(-\frac{\sqrt{\tau} \text{dist}_p}{c} \right) \right).$$

This completes the proof of equations (1.41) and (1.42). □

1.6 The asymptotic Berry holonomy

A connection on a principal G -bundle $P \rightarrow X$ defines the notion of parallel transport; cf. [KN63, Chapter II]. Parallel transport around a loop is called holonomy. Holonomy can be viewed as a map from the loop space of X to the space of conjugacy classes of G . For Abelian G , the later space is canonically isomorphic to G .

In our case, the τ -vortex principal bundle, $\mathcal{P}_\tau \rightarrow \mathcal{M}_\tau$, is a principal \mathcal{G} -bundle equipped with the Berry connection. The physical interpretation is that if one adiabatically moves the divisor points along a curve Γ in $\text{Sym}^d(\Sigma)$, then the corresponding τ -vortex field evolves by the parallel transport defined the Berry connection (cf. [K50], and [B84]). In particular, when Γ is a loop, the holonomy of the Berry connection, called the *Berry holonomy*, is a gauge transformation. In this section, we give analytic and topological descriptions of the gauge transformations that arise as Berry holonomies.

Since the Berry holonomy is a map from the loop space of τ -vortex moduli space, we recall some well-known properties of loops in $\mathcal{M}_\tau \cong \text{Sym}^d(\Sigma)$.

We call a loop Γ in $\text{Sym}^d(\Sigma)$ a *single vortex loop* if only one of the divisor points moves, and all other divisor points are fixed. In other words, single vortex loops are induced by loops in Σ that are based at one of the divisor points. Every loop in $\text{Sym}^d(\Sigma)$ can be decomposed up to homotopy (and thus homology) to a product of single vortex loops. Moreover,

$$H_1(\mathcal{M}_\tau; \mathbb{Z}) \cong H_1(\text{Sym}^d(\Sigma); \mathbb{Z}) \cong H_1(\Sigma; \mathbb{Z}), \tag{1.49}$$

where the last isomorphism is given by sending single vortex loops to their homology classes by the Hurewicz homomorphism.

Recall that the complement of $\text{Sym}_s^d(\Sigma)$ is called the big diagonal. A loop in $\text{Sym}^d(\Sigma)$ is *regular* if it is a smooth, embedded (immersed, if $d = 1$) loop that does not intersect the big diagonal. The big diagonal is empty when $d = 1$. When $d > 1$ the big diagonal is a subvariety of codimension at least 2, thus every smooth loop in $\text{Sym}^d(\Sigma)$ can be made regular after a small smooth perturbation. Now consider the canonical covering map $\Sigma_s^{\times d} \rightarrow \text{Sym}_s^d(\Sigma)$, where $\Sigma_s^{\times d}$ is the space of ordered d -tuples in Σ without repetition. Given a regular loop Γ that starts at the simple divisor $D = \Gamma(0) \in \text{Sym}_s^d(\Sigma)$, each lift $\tilde{D} \in \Sigma_s^{\times d}$ of D determines a unique lift $\tilde{\Gamma}$ of Γ . The lift $\tilde{\Gamma}$ can be regarded as a d -tuple $(\gamma_1, \dots, \gamma_d)$ of curves (not necessarily loops) in Σ . The *shadow of Γ* , $\text{sh}(\Gamma) \subset \Sigma$ is

$$\text{sh}(\Gamma) = \bigcup \text{image}(\gamma_i), \tag{1.50}$$

where the union is over all non-constant γ_i . The set $\text{sh}(\Gamma)$ has a natural orientation coming from the orientation of Γ . Since Γ is regular, $\text{sh}(\Gamma)$ is a union of immersed, oriented loops, hence it is an integer 1-cycle in Σ . The homology class in $H_1(\Sigma; \mathbb{Z})$ represented by $\text{sh}(\Gamma)$ is independent of the choice of the lift \tilde{D} . We denote this class by $[\Gamma]$. It is easy to check the homotopy class of Γ in $\text{Sym}^d(\Sigma)$ is sent to the homology class $[\Gamma]$ by the isomorphism (1.49). In general, for a single vortex loop, only one of the γ_i 's is not constant, say γ , and $[\Gamma] = [\gamma] \in H_1(\Sigma; \mathbb{Z})$.

Example 1.12. *An example of $\text{sh}(\Gamma)$ is seen on Figure 1.1, where*

$$[\Gamma] = [\gamma_1] + [\gamma_2] + [\gamma_3] = [\gamma_3],$$

since both γ_1 and γ_2 are null-homologous. Thus Γ is homologous to a single vortex loop.

We call a loop a (positively oriented) vortex interchange if, as in Figure 1.2, only two γ_i 's, say γ_1 and γ_2 , are not constant, and the composition $\Gamma = \gamma_1 * \gamma_2$ is the (oriented) boundary of a disk.

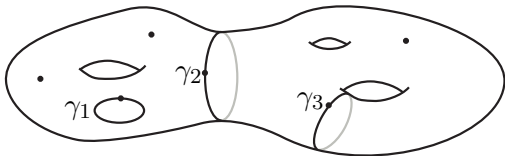


Figure 1.1: Single vortex loops: One divisor point moves along one of the γ_i 's. All other divisor points are fixed.

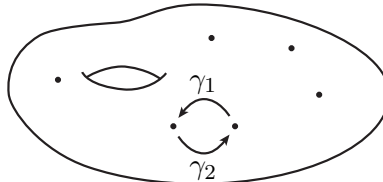


Figure 1.2: Vortex interchange: One divisor point moves along γ_1 and another divisor point moves along γ_2 . All other divisor points are fixed.

Since the Berry holonomy has values in the gauge group \mathcal{G} , we also recall a couple well-known properties of gauge transformations. Elements $g \in \mathcal{G}$ represent classes in $H^1(\Sigma; \mathbb{Z})$ as follows: For a closed manifold X and a finitely generated Abelian group G , $H^n(\Sigma; G)$ is canonically isomorphic to the space $[X, K(G, n)]$ of homotopy classes of continuous maps from X to the Eilenberg-MacLane space $K(G, n)$ (cf. [H02, Theorem 4.57]). Since $K(\mathbb{Z}, 1) \cong \mathbb{U}(1)$ and \mathcal{G} is homotopy equivalent to $[\Sigma, \mathbb{U}(1)]$, we get that $H^1(\Sigma; \mathbb{Z})$ is canonically isomorphic to $\pi_0(\mathcal{G})$, which is also a group, because \mathcal{G} is. In fact, if \mathcal{G}_0 is the identity component of \mathcal{G} , then $\pi_0(\mathcal{G}) \simeq \mathcal{G}/\mathcal{G}_0$, and the short exact sequence

$$\{0\} \rightarrow \mathcal{G}_0 \hookrightarrow \mathcal{G} \twoheadrightarrow H^1(\Sigma; \mathbb{Z}) \rightarrow \{0\} \quad (1.51)$$

is non-canonically split.

The isomorphism between $\pi_0(\mathcal{G})$ and $H^1(\Sigma; \mathbb{Z})$ can be understood on the (co)cycle level; since Σ is a closed, oriented surface, $H^1(\Sigma; \mathbb{Z})$ is canonically isomorphic to $\text{Hom}(H_1(\Sigma; \mathbb{Z}), \mathbb{Z})$.

An element $g \in \mathcal{G}$ defines an element $[g] \in \text{Hom}(H_1(\Sigma; \mathbb{Z}), \mathbb{Z})$ via

$$[g]([\gamma]) = g(\gamma) = \frac{1}{2\pi i} \int_{\gamma} g^{-1} dg \in \mathbb{Z}. \quad (1.52)$$

The Berry holonomy can be viewed as a map from the loop space $\Omega\mathcal{M}_\tau$ of \mathcal{M}_τ to \mathcal{G} . It then induces a map hol_* on the connected components:

$$\pi_1(\mathcal{M}_\tau) \xrightarrow{\sim} \pi_0(\Omega\mathcal{M}_\tau) \xrightarrow{\text{hol}_*} \pi_0(\mathcal{G}) \xrightarrow{\sim} H^1(\Sigma; \mathbb{Z}).$$

Since cohomology groups are Abelian, the above map factors down to the homology, and thus defines a homomorphism:

$$\text{hol}_* : H_1(\Sigma; \mathbb{Z}) \xrightarrow{\sim} H_1(\mathcal{M}_\tau; \mathbb{Z}) \xrightarrow{\text{hol}_*} H^1(\Sigma; \mathbb{Z}), \quad (1.53)$$

where the first isomorphism is from (1.49). Using equation (1.52), an explicit formula for hol_* can be given as follows: if $g = \text{hol}(\Gamma)$, then $\text{hol}_*([\Gamma])$ evaluates on any 1-cycle γ by

$$\text{hol}_*([\Gamma])([\gamma]) = \frac{1}{2\pi i} \int_{\gamma} g^{-1} dg. \quad (1.54)$$

Finally, recall that a k -current is a continuous linear functional on Ω^k . A 1-form $a \in \Omega^1$ defines a 1-current by

$$C_a(b) = \int_{\Sigma} a \wedge b. \quad (1.55)$$

Similarly, a smooth 1-chain γ defines a 1-current by

$$C_\gamma(b) = \int_\gamma b. \quad (1.56)$$

We say that the 1-currents in equations (1.55) and (1.56) are the *1-currents defined by a and γ* respectively.

Now we are ready to prove our main theorem about the Berry holonomy, stated in the introduction.

The proof of the Main Theorem. Since every smooth path can be made regular by an arbitrarily small smooth perturbation, it is enough to check regular loops, Γ .

We prove Statement (1) first: Let v be a τ -vortex field corresponding to D , and $\widehat{\Gamma}$ be the horizontal lift of Γ starting at $v \in \mathcal{P}_\tau$. Since Γ is regular, $\Gamma(t)$ is simple for all t , thus we can apply Theorem 1.8 to the velocity vector $\widehat{\Gamma}'(t) = (a_t, \psi_t)$, which is horizontal, and hence obtain

$$|a_t| + |\psi_t| \leq c\tau \sum_{i=1}^d |\gamma_i'(t)| \exp\left(-\frac{\sqrt{\tau} \text{dist}_{\gamma_i}(t)}{c}\right). \quad (1.57)$$

By definition of the holonomy,

$$g_\tau(v) = v + \int_0^1 \widehat{\Gamma}'(t) dt = \left(\nabla + \int_0^1 a_t dt, \phi + \int_0^1 \psi_t dt \right).$$

On the other hand, by the definition of the gauge action (for Abelian groups),

$$g_\tau(v) = \left(\nabla + g_\tau dg_\tau^{-1}, g_\tau \phi \right).$$

Thus we have

$$g_\tau dg_\tau^{-1} = \int_0^1 a_t dt, \quad (1.58)$$

and

$$(g_\tau - 1)\phi = \int_0^1 \psi_t dt. \quad (1.59)$$

Let $V \subset \Sigma$ be any compact set in the complement of $\text{sh}(\Gamma)$. Since $\text{dist}(\text{sh}(\Gamma), V) > 0$, Theorem 1.6 shows that $|\phi| \geq \frac{\sqrt{\tau}}{2}$ on V for all large τ . Hence equations (1.57) and (1.59) imply that for $x \in V$

$$|\phi||g_\tau - 1|_x \leq \frac{\sqrt{\tau}}{2}|g_\tau - 1|_x \leq \int_0^1 |\psi_t(x)| dt \leq c\tau \sum_{i=1}^d \int_0^1 |\gamma'_i(t)| \exp\left(-\frac{\sqrt{\tau}\text{dist}(\gamma_i(t), x)}{c}\right) dt. \quad (1.60)$$

Using $|g_\tau| = 1$, $|dg_\tau^{-1}| = |dg_\tau|$, and equations (1.57) and (1.58), we also obtain

$$|dg_\tau|_x \leq \sum_{i=1}^d \int_0^1 |a_t(x)| dt \leq c\tau \sum_{i=1}^d \int_0^1 |\gamma'_i(t)| \exp\left(-\frac{\sqrt{\tau}\text{dist}(\gamma_i(t), x)}{c}\right) dt. \quad (1.61)$$

Combining the last two inequalities gives

$$\sqrt{\tau}|g_\tau - 1|_x + |dg_\tau|_x \leq c'\tau \exp\left(-\frac{\sqrt{\tau}\text{dist}(\text{sh}(\Gamma), V)}{c}\right), \quad (1.62)$$

for all large τ , which implies that $|g_\tau - 1|$ and $|dg_\tau|$ converge to 0, uniformly on V , as $\tau \rightarrow \infty$.

This proves Statement (1).

In order to prove Statement (2), we first assume that Γ is a single vortex loop, for which $\text{sh}(\Gamma)$ is an embedded loop, that bounds an embedded disk B in Σ , and $D = \Gamma(0) = \Gamma(1)$ has no divisor points in the interior B , and let p be the divisor point in D that is moved by

Γ . There is a canonical embedding of B into \mathcal{M}_τ that sends a point $x \in B$ to the divisor $x + (D - p)$. The image of this map, \widehat{B} , is an (oriented) disk in \mathcal{M}_τ , whose (oriented) boundary is Γ . We will denote this embedding by $\pi_B : B \rightarrow \widehat{B}$.

Since Γ is null-homotopic, g_τ is in the identity component of \mathcal{G} , and so can be written as $g_\tau = \exp(2\pi i f_\tau)$, where f_τ is a smooth, real function on Σ . By Stokes' Theorem,

$$f_\tau = \frac{1}{2\pi i} \int_{\widehat{B}} \Omega. \quad (1.63)$$

If j is a path as in Statement (2) of the Main Theorem, then $\varphi_\tau = f_\tau \circ j$. Using equation (1.63) we see that

$$\varphi_\tau(1) - \varphi_\tau(0) = f_\tau(j(1)) - f_\tau(j(0)) = \frac{1}{2\pi i} \int_{\widehat{B}} (\Omega(j(1)) - \Omega(j(0))),$$

where $j(1)$ is in B , and $j(0)$ is not. To evaluate this integral, we reparametrize using π_B . By Theorem 1.8, if ω_τ is the pullback of the Kähler class of \mathcal{M}_τ from \widehat{B} to B using π_B , then

$$\omega_\tau = \pi^* \omega + O(1).$$

For each $x \in B$, let w_x be the function w , defined in equation (1.2), corresponding to the divisor $D = \pi_B(x)$, and let $d_x = \text{dist}(x, \{j(0), j(1)\})$. By [HJS96, Lemma 1.1], $w_x|_B$ converges to $2\pi\delta_x$, in measure, since $D \cap B = \{x\}$. Now using Theorem 1.11, the last integral

above equals to

$$\begin{aligned}
\varphi_\tau(1) - \varphi_\tau(0) &= \frac{1}{2\pi} \int_B \left(\frac{w_x(j(1)) - w_x(j(0))}{\pi\tau} + O\left(\tau^{-1} \exp\left(-\frac{2\sqrt{\tau}dx}{c}\right)\right) \right) (\pi\tau + O(1)) \omega_x \\
&= \int_B \left(\delta(x, j(1)) + O\left(\exp\left(-\frac{\sqrt{\tau}dx}{c}\right)\right) \right) \omega_x + O\left(\tau^{-1}\right) \\
&= 1 + O\left(\tau^{-1}\right).
\end{aligned}$$

This implies Statement (2) in the case where Γ is a simple vortex loop that bounds a disk.

In the general case, let $I(2\varepsilon)$ be the tubular 2ε -neighborhood of $I = \text{image}(j)$, where ε is small enough so that $I(2\varepsilon) \cap \text{sh}(\Gamma)$ is a single embedded arc. Let Γ° be a single, embedded bounding loop in Σ , as in the previous case, for which $\text{sh}(\Gamma)$ and $\text{sh}(\Gamma^\circ)$ coincide on $I(2\varepsilon)$. Let Γ_{out} and $\Gamma_{\text{out}}^\circ$ denote the parts of Γ and Γ° , respectively, for which $\text{sh}(\Gamma_{\text{out}})$ and $\text{sh}(\Gamma_{\text{out}}^\circ)$ lie in the complement of $I(2\varepsilon)$. Similarly $\Gamma_{\text{in}} = \Gamma_{\text{in}}^\circ$ is their common part. Now one can join $\Gamma_{\text{out}}^\circ$ with the reverse of Γ_{out} to get a piecewise smooth loop Γ^{new} , such that $\text{sh}(\Gamma^{\text{new}})$ is disjoint from $I(2\varepsilon)$, and furthermore, the loop sum $\Gamma * \Gamma^{\text{new}}$ and Γ° differ only by Γ_{out} and its reverse, thus

$$g_\tau \text{hol}_{\Gamma^{\text{new}}} = \text{hol}_{\Gamma^\circ}.$$

On $I(\varepsilon)$ we have that $\text{hol}_{\Gamma^{\text{new}}}$ converges to 1 in the C^1 -topology as $\tau \rightarrow \infty$. Since the crossing formula holds for Γ° , it must hold for Γ as well. This establishes the general case of Statement (2).

In order to prove Statement (3) we pick a finite cover \mathcal{U} of Σ by coordinate charts such that for every $(U, \Psi) \in \mathcal{U}$ the preimage of the intersection $\Psi^{-1}(\text{sh}(\Gamma) \cap U) \subset \mathbb{R}^2$ is either (i) empty, (ii) the x -axis, or (iii) the union of the two axes. By using a subordinate partition of

unity, it is enough to prove Statement (3) for 1-forms that are supported inside one of these charts. Fix one such chart $(U, \Psi) \in \mathcal{U}$, and let $b \in \Omega^1$ a 1-form with $\text{supp}(b) \subset U$. Let us also write

$$\Psi^* a_\tau = df_\tau \tag{1.64a}$$

$$\Psi^* b = A dx + B dy, \tag{1.64b}$$

where A and B are compactly supported functions on \mathbb{R}^2 .

If $\Psi^{-1}(\text{sh}(\Gamma) \cap U)$ is empty, then the support of b and $\text{sh}(\Gamma)$ are disjoint, hence

$$C_{\text{sh}(\Gamma)}(b) = \int_{\text{sh}(\Gamma)} b = \int_{\text{sh}(\Gamma) \cap \text{supp}(b)} b = \int_{\emptyset} b = 0.$$

On the other hand, as $\tau \rightarrow \infty$, $a_\tau \rightarrow 0$ on $\text{supp}(b)$ by Statement (1), and hence

$$|C_{a_\tau}(b)| \leq \max_{\text{supp}(b)} \{|a_\tau|\} \|b\|_{L^1} \rightarrow 0,$$

which proves Statement (3) in the case (i).

Next, if $\Psi^{-1}(\text{sh}(\Gamma) \cap U)$ is the x -axis, then by equation (1.64b),

$$C_{\text{sh}(\Gamma)}(b) = \int_{\Psi^{-1}(\text{sh}(\Gamma) \cap U)} \Psi^* b = \int_{-\infty}^{\infty} A(x, 0) dx. \tag{1.65}$$

We also have

$$C_{a_\tau}(b) = \int_{\Sigma} a_\tau \wedge b = \int_U a_\tau \wedge b = \int_{\mathbb{R}^2} df_\tau \wedge \Psi^* b. \tag{1.66}$$

Again using equation (1.64b), this becomes

$$C_{a_\tau}(b) = \int_{\mathbb{R}^2} df_\tau \wedge (A dx + B dy) = \int_{\mathbb{R}^2} d(f_\tau A dx + f_\tau B dy) - \int_{\mathbb{R}^2} f_\tau \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx \wedge dy.$$

The first integral on the right-hand side is zero by Stokes' Theorem and the fact that A and B are compactly supported. By Statement (2), we choose f_τ so that it converges to 0 when y is positive and to 1 when y is negative. As $\tau \rightarrow \infty$, we then have

$$\begin{aligned} C_{a_\tau}(b) &= \int_{\mathbb{R}^2} f_\tau \left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x} \right) dx \wedge dy \\ &\rightarrow \int_{-\infty}^{\infty} \int_{-\infty}^0 \left(\frac{\partial A}{\partial y}(x, y) - \frac{\partial B}{\partial x}(x, y) \right) dy dx \\ &= \int_{-\infty}^{\infty} A(x, 0) dx. \end{aligned}$$

Together with equation (1.65) this proves Statement (3) for case (ii).

Finally, if $\Psi^{-1}(\text{sh}(\Gamma) \cap U)$ is the union of the two axes, then again by equation (1.64b),

$$C_{\text{sh}(\Gamma)}(b) = \int_{\Psi^{-1}(\text{sh}(\Gamma) \cap U)} \Psi^* b = \int_{-\infty}^{\infty} A(x, 0) dx + \int_{-\infty}^{\infty} B(0, y) dy. \quad (1.67)$$

As before, we also have

$$C_{a_\tau}(b) = \int_{\mathbb{R}^2} f_\tau \left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x} \right) dx \wedge dy. \quad (1.68)$$

But now Statement (2) shows that, as $\tau \rightarrow \infty$, f_τ converges to 0 in the upper left quadrant, to 1 in the upper right and the lower left quadrants, and to 2 in the lower right quadrant.

Hence by equation (1.68),

$$\begin{aligned}
C_{a_\tau}(b) &\rightarrow 2 \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} \left(\frac{\partial A}{\partial y}(x, y) - \frac{\partial B}{\partial x}(x, y) \right) dy dx \\
&+ \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \left(\frac{\partial A}{\partial y}(x, y) - \frac{\partial B}{\partial x}(x, y) \right) dy dx \\
&+ \int_{\mathbb{R}_-} \int_{\mathbb{R}_-} \left(\frac{\partial A}{\partial y}(x, y) - \frac{\partial B}{\partial x}(x, y) \right) dy dx \\
&= \int_{-\infty}^{\infty} A(x, 0) dx + \int_{-\infty}^{\infty} B(0, y) dy,
\end{aligned}$$

where the last step is an elementary computation. Together with (1.67), this proves Statement (3) for the case (iii).

To prove Statement (4), according to equation (1.54) and the definition of the Poincaré duality, we need to show that for any $[\gamma] \in H_1(\Sigma; \mathbb{Z})$,

$$\text{hol}_\star([\Gamma])([\gamma]) = \frac{1}{2\pi i} \int_{\gamma} g_\tau^{-1} dg_\tau = [\Gamma] \cdot [\gamma]. \tag{1.69}$$

Since everything in (1.69) is homotopy invariant, we have the freedom to change Γ by a homotopy. Recall from (1.49) that Γ can be decomposed, up to homotopy, to a product of single vortex loops; thus it suffices to check Statement (4) for a basis of $\pi_1(\text{Sym}^d(\Sigma)) \cong H_1(\Sigma; \mathbb{Z})$. We use a “symplectic” basis: a set of simple closed curves $\{\alpha_i, \beta_i\}_{1 \leq i \leq g(\Sigma)}$ such that α_i intersects β_i transversally and positively once, and $\alpha_i \cap \beta_j = \alpha_i \cap \alpha_j = \beta_i \cap \beta_j = \emptyset$ for $i \neq j$. There is always such a set, and $\{[\alpha_i], [\beta_i]\}_{1 \leq i \leq g(\Sigma)}$ is a basis of $H_1(\Sigma; \mathbb{Z})$, with

$$[\alpha_i] \cdot [\alpha_j] = 0, \quad [\beta_i] \cdot [\beta_j] = 0, \quad [\alpha_i] \cdot [\beta_j] = \delta_{i,j},$$

where \cdot is the homology intersection. Denote the corresponding single vortex loops in $\text{Sym}^d(\Sigma)$ by $\{\widehat{\alpha}_i, \widehat{\beta}_i\}_{1 \leq i \leq g(\Sigma)}$.

To prove Statement (4), we need only to verify (1.69) for every pair in the basis. When $\gamma \in \{\alpha_i, \beta_i\}$, and $\Gamma \in \{\widehat{\alpha}_j, \widehat{\beta}_j\}$ with $i \neq j$, we have by Statement (1) that

$$\int_{\gamma} i g_{\tau}^{-1} d g_{\tau} = O\left(\tau \exp\left(-\frac{\sqrt{\tau} \text{dist}(\gamma, \text{sh}(\Gamma))}{c}\right)\right). \quad (1.70)$$

When $\gamma = \alpha_i$ and $\Gamma = \widehat{\alpha}_i$, for some i , we can chose another representative α'_i for $[\gamma] = [\alpha_i]$ that is disjoint from α_i . Thus, again by Statement (1), we have

$$\int_{\gamma} i g_{\tau}^{-1} d g_{\tau} = O\left(\tau \exp\left(-\frac{\sqrt{\tau} \text{dist}(\alpha_i, \alpha'_i)}{c}\right)\right). \quad (1.71)$$

Thus all integrals in (1.70) and (1.71) converge to 0 as $\tau \rightarrow \infty$. On the other hand, these integrals are integer multiples of 2π , so they had to be 0 for all $\tau > \tau_0$.

Finally, assume that $i = j$ and $\gamma = \alpha_i$ and $\text{sh}(\Gamma) = \widehat{\beta}_i$, or $\gamma = \beta_i$ and $\text{sh}(\Gamma) = \widehat{\alpha}_i$. In order to prove the first of these cases, let us fix a small embedded segment j on $\gamma = \beta_i$ that intersects $\text{sh}(\Gamma) = \alpha_i$ once positively. Such paths exist by the construction of the basis. Write $g_{\tau}|_I = \exp(2\pi i \varphi_{\tau})$, and so $g_{\tau}^{-1} d g_{\tau}|_I = 2\pi i d \varphi_{\tau}$. Thus by equation (1.54),

$$\begin{aligned} \text{hol}_{\star}([\Gamma])([\gamma]) &= \int_I d \varphi_{\tau} + \frac{1}{2\pi i} \int_{\gamma-I} g_{\tau}^{-1} d g_{\tau} \\ &= \varphi_{\tau}(1) - \varphi_{\tau}(0) + O\left(\sqrt{\tau} \exp\left(-\frac{\sqrt{\tau} \text{dist}(\text{sh}(\Gamma), \{j(0), j(1)\})}{c}\right)\right). \end{aligned}$$

By Statement (2), this converges to 1 as $\tau \rightarrow \infty$. On the other hand, $\text{hol}_{\star}([\Gamma])([\gamma])$ is an integer, so it had to be 1 for all $\tau > \tau_0$. The same argument can be used in the case

of $\text{sh}(\Gamma) = \tilde{\beta}_i$ and $\gamma = \alpha_i$, which completes the proof of Statement (4) and the Main Theorem. □

Corollary 1.13. *For all $\tau > \tau_0$, the space \mathcal{P}_τ is an infinite-dimensional vector bundle over a connected, oriented and smooth manifold without boundary, $\widehat{\mathcal{M}}_\tau$. This manifold has real dimension $2d+1$ and is a $U(1)$ -principal bundle over the universal cover of \mathcal{M}_τ . In particular \mathcal{P}_τ is homotopy retracts to $\widehat{\mathcal{M}}_\tau$.*

Proof. First we will impose the Coulomb gauge: fix a τ -vortex field $v_0 = (\nabla, \phi) \in \mathcal{P}_\tau$. We say that $v = (\nabla', \phi')$ is in *Coulomb gauge with respect to v_0* if the 1-form $a = \nabla' - \nabla$ is satisfies:

$$d^*a = 0.$$

For each $v \in \mathcal{P}_\tau$ there is in fact a gauge transformation g that is in the identity component of \mathcal{G} , and is unique up to constant gauge transformations (factors in $U(1)$), such that $g(v)$ is in Coulomb gauge with respect to v_0 . A proof of this, which applies to our case too, can be found, for example, in [EN11, Lemma 2.1]. The set $\widehat{\mathcal{M}}_\tau \subset \mathcal{P}_\tau$, called the *Coulomb slice*, consisting the τ -vortex fields that are in Coulomb gauge with respect to v_0 intersects each fiber.

Fix then a point $x \in \Sigma$, and require $g(x) = 1$. Such a $g = g_v$ is then unique, moreover, can be written as $g_v = \exp(if_v)$, and f_v is also unique if one prescribes $f_v(x) = 0$. Set $g_{t,v} = \exp(itf_v)$. Then the map defined as $r(t, v) = g_{t,v}(v)$ is a homotopy retraction of \mathcal{P}_τ to the Coulomb slice.

The intersection of each fiber with the Coulomb slice is a collection of circles, due to the $U(1)$ ambiguity mentioned above. Moreover, these circles are in bijection with $\pi_0(\mathcal{G}) \cong$

$\pi_1(\mathcal{M}_\tau)$. Thus $\widetilde{\mathcal{M}}_\tau = \widehat{\mathcal{M}}_\tau/\mathrm{U}(1)$ is a $\pi_1(\mathcal{M}_\tau)$ -cover of \mathcal{M}_τ , which is the universal cover if connected.

Our Main Theorem implies that \mathcal{P}_τ is connected, by the following argument: let v and v' be two arbitrary τ -vortex fields. Since simple divisors are dense in $\mathrm{Sym}^d(\Sigma)$, we can assume, that the corresponding divisors are simple. Join the two divisors by a regular path Γ_0 . Then $\mathrm{hol}_{\Gamma_0}(v)$ is equal to $g(v')$ for some $g \in \mathcal{G}$. If g is not in the identity component of \mathcal{G} , then it represents a non-zero cohomology class $[g] \in H^2(\Sigma; \mathbb{Z})$. Let γ be a smooth loop based at a divisor point of v that represents the Poincaré dual of $[g]$; let $\widehat{\gamma}$ be the induced single vortex loop, and set $\Gamma = \widehat{\gamma}^{-1} * \Gamma_0$. Now v and $\mathrm{hol}_\Gamma(v) = v''$ are connected by the path in \mathcal{P}_τ given by parallel transport. On the other hand, v'' and v are gauge equivalent, and the connecting gauge transformation is in the identity component of \mathcal{G} , which means that there is a path from v'' to v . Thus \mathcal{P}_τ is connected, but then so is $\widehat{\mathcal{M}}_\tau$, which completes the proof. \square

1.7 The large area limit

Consider the energy (1.1) for the critical coupling constant $\lambda = 1$ and with $\tau = 1$. Bradlow's criterion for the existence of irreducible vortices in this case becomes

$$\tau_0 = \frac{2\pi d}{\mathrm{Area}(\Sigma)} < 1, \tag{1.72}$$

using the area with respect to the given area 2-form ω . Even when inequality (1.72) does not hold for ω , it still holds for $\omega_t = t^2\omega$ if $t > t_0 = \sqrt{\tau_0}$.

Let P_t be the space of all solutions of the 1-vortex equations with Kähler form ω_t for

$t > t_0$. A pair $(\nabla, \phi) \in \mathcal{C}_L \times \Omega_L^0$ is in P_t if

$$i\Lambda_t F_\nabla = \frac{1}{2}(1 - |\phi|^2) \quad (1.73a)$$

$$\bar{\partial}_\nabla \phi = 0, \quad (1.73b)$$

where $\Lambda_t = \Lambda/t^2$. Let M_t be the corresponding moduli space P_t/\mathcal{G} . Bradlow's Theorem still holds, hence $M_t \cong \text{Sym}^d(\Sigma)$, where the diffeomorphism is again given by the divisor of the ϕ -field.

By [B90, Proposition 5.1], the following diagram is commutative when $t^2 = \tau$:

$$\begin{array}{ccc} P_t & \xrightarrow{\Phi_t} & \mathcal{P}_\tau, \\ & \searrow & \swarrow \\ & \text{Sym}^d(\Sigma) & \end{array}$$

where Φ_t is the isomorphism of principal bundles given by $\Phi_t(\nabla, \phi) = (\nabla, t\phi)$.

The L^2 -metric on P_t is defined by equation (1.6), but with Hodge operator and area form given by ω_t . For all $X \in TP_t$, we now have:

$$\|(\Phi_t)_* X\|_{\mathcal{P}_\tau} = t\|X\|_{P_t}. \quad (1.74)$$

Thus the L^2 -metric of P_t is conformally equivalent to the pullback of the L^2 -metric of \mathcal{P}_τ via the bundle isomorphism Φ_t .

The Berry connection on $P_t \rightarrow M_t$ is again defined as the orthogonal complement of the vertical subspaces, hence it is the same as the pullback of the Berry connection on \mathcal{P}_τ via Φ_t . Thus the results of Theorems 1.8 and 1.11 and our Main Theorem hold in the *large area*

limit ($t \rightarrow \infty$):

Main Theorem for the large area limit. *The conclusions of the Main Theorem in the introduction hold for the principal \mathcal{G} -bundle $P_t \rightarrow M_t$ with τ replaced everywhere by t^2 .*

Chapter 2

Irreducible Ginzburg-Landau fields

In this chapter we show that the Ginzburg-Landau equations (1.3a) and (1.3b) admit irreducible solutions under certain conditions. This generalizes Bradlow's result for the critical $\lambda = 1$ case [B90, Theorem 4.3]. We use the hypotheses and notation of Chapter 1.

As mentioned in the introduction of Chapter 1, the Ginzburg-Landau theory is a mathematical model for superconductivity. The reducible solutions correspond to insulating states, while irreducible solutions model superconducting states.

The results of this chapter are summarized in our second main theorem.

Main Theorem of Chapter 2. *The Ginzburg-Landau equations (1.3a) and (1.3b) admit irreducible solutions if and only if $\lambda\tau > \frac{4\pi|d|}{\text{Area}(\Sigma)}$.*

The content of the main theorem is illustrated by the following phase-diagram:

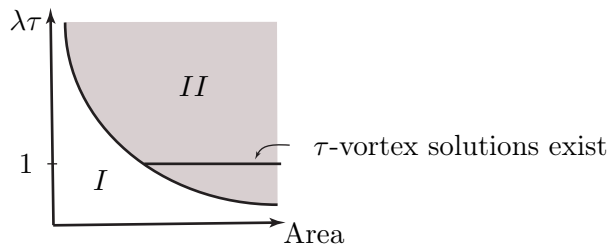


Figure 2.1: Phase Diagram: In Region I, defined by $\lambda\tau \leq \frac{4\pi|d|}{\text{Area}(\Sigma)}$, only reducible solutions exist (Insulator Phase). In the complementary Region II (shaded), there exist irreducible solutions (Superconductor Phase).

For simplicity, we assume that d , the degree of the line bundle, is non-negative. The results for negative degrees can be proven similarly.

This chapter is organized as follows. In Section 2.1 we use standard geometric analytic methods to show that there are only reducible solutions for parameters in Region I in Figure 2.1. In Section 2.2, we prove the existence of irreducible solutions for parameters in Region II. This is done using a gauged version of the Palais-Smale compactness property.

2.1 Non-existence

First of all, any critical point of the energy (1.1) is gauge equivalent to a smooth critical point, which satisfies equations (1.3a) and (1.3b) (see [JT80]). Furthermore, if a smooth connection $\nabla \in \mathcal{C}_L$ satisfies the vacuum Maxwell equations:

$$d^*F_\nabla = 0. \tag{2.1}$$

We call such a connection a *Maxwell connection*. Note that a pair $(\nabla, 0)$, consisting a Maxwell connection and the identically zero section of L , is always a solution of equations (1.3a) and (1.3b). We call such a solution a *Maxwell solution*.

A solution (∇, ϕ) is called *reducible*, if there is a gauge transformation $g \in \mathcal{G}$ such that $g(\nabla, \phi) = (\nabla, \phi)$, otherwise *irreducible*. Maxwell solutions are all reducible, because constant gauge transformations act ineffectively on them. On the other hand, if (∇, ϕ) is a pair with non-vanishing ϕ -field, then the only way a non-trivial $U(1)$ -valued gauge transformation act ineffectively on this pair is if ϕ vanishes on an open subset of Σ . Since equation (1.3b) is elliptic, by unique continuation, any such solution would vanish everywhere. Thus Maxwell solutions are the only reducible ones.

To prove that there are no irreducible solutions – and hence the only solutions are the

Maxwell solutions – when $\lambda\tau \leq \frac{4\pi d}{\text{Area}(\Sigma)}$, we first prove the following technical lemma.

Lemma 2.1. *If $(\nabla, \phi) \in \mathcal{C}_L \oplus \Omega_L^0$ is a critical point of $\mathcal{E}_{\lambda, \tau}$, then*

$$\left(\frac{4\pi d}{\text{Area}(\Sigma)} - \lambda\tau\right) \|\phi\|_{L^2}^2 \leq -\frac{\lambda}{2} \|\phi\|_{L^4}^4. \quad (2.2)$$

Proof. Fix a Maxwell connection ∇^0 , and write $(\nabla, \phi) = (\nabla^0 + a, \phi)$, where $a \in i\Omega^1$. With this notation, equation (1.3a) becomes

$$(d^*d + |\phi|^2)a = i \operatorname{Im}\left(h(\nabla^0\phi, \phi)\right). \quad (2.3)$$

Take the L^2 -inner product of both sides with a , use the fact that a is imaginary-valued, and integrate by parts to get

$$0 \leq \|da\|_{L^2}^2 + \|a\phi\|_{L^2}^2 = \operatorname{Re}\left(\langle a\phi | \nabla^0\phi \rangle_{L^2}\right).$$

In particular, this shows that the right hand side is non-negative.

Next, rewrite equation (1.3b) in terms of ∇^0 , a , and ϕ :

$$\left(\nabla^0\right)^* \nabla^0\phi + a^*\left(\nabla^0\phi\right) + \left(\nabla^0\right)^*(a\phi) + |a|^2\phi = \frac{\lambda}{2}\left(\tau\phi - |\phi|^2\phi\right). \quad (2.4)$$

Take the L^2 -inner product of both sides with ϕ , and integrate by parts to get

$$\|\nabla^0\phi\|_{L^2}^2 + 2\operatorname{Re}\left(\langle a\phi | \nabla^0\phi \rangle_{L^2}\right) + \|a\phi\|_{L^2}^2 = \frac{\lambda\tau}{2}\|\phi\|_{L^2}^2 - \frac{\lambda}{2}\|\phi\|_{L^4}^4. \quad (2.5)$$

To obtain a third equation, note that that all Maxwell connections have the same curvature,

given by

$$F_{\nabla} = \frac{2\pi d}{i\text{Area}(\Sigma)}\omega, \quad (2.6)$$

or equivalently that $i\Lambda F_{\nabla}0 = \frac{2\pi d}{\text{Area}(\Sigma)}$. Combining this with the Kähler identity yields

$$\|\nabla^0\phi\|_{L^2}^2 = 2\|\bar{\partial}_{\nabla}0\phi\|_{L^2}^2 + \frac{2\pi d}{\text{Area}(\Sigma)}\|\phi\|_{L^2}^2. \quad (2.7)$$

Using equation (2.7), equation (2.5) becomes

$$2\|\bar{\partial}_{\nabla}0\phi\|_{L^2}^2 + \frac{2\pi d}{\text{Area}(\Sigma)}\|\phi\|_{L^2}^2 + 2\text{Re}\left(\langle a\phi|\nabla^0\phi\rangle_{L^2}\right) + \|a\phi\|_{L^2}^2 = \frac{\lambda\tau}{2}\|\phi\|_{L^2}^2 - \frac{\lambda}{2}\|\phi\|_{L^4}^4. \quad (2.8)$$

Every term on the left hand side is non-negative. Leaving the first, the third, and the fourth term gives the inequality

$$\frac{2\pi d}{\text{Area}(\Sigma)}\|\phi\|_{L^2}^2 \leq \frac{\lambda\tau}{2}\|\phi\|_{L^2}^2 - \frac{\lambda}{2}\|\phi\|_{L^4}^4,$$

which is equivalent to inequality (2.2). □

The non-existence of irreducible solutions is an immediate consequence of Lemma 2.1.

Corollary 2.2. *If $\lambda\tau \leq \frac{4\pi d}{\text{Area}(\Sigma)}$, and $(\nabla, \phi) \in \mathcal{C}_L \oplus \Omega_L^0$ is a critical point of $\mathcal{E}_{\lambda,\tau}$, then ϕ is identically zero.*

Proof. If $\lambda\tau \leq \frac{4\pi d}{\text{Area}(\Sigma)}$ and ϕ is not identically zero, then the left hand side of inequality (2.2) is non-negative, whereas the right hand side is strictly negative, which is a contradiction. □

2.2 Existence

We now turn to the proof of the existence of irreducible solutions of equations (1.3a) and (1.3b) when $\lambda\tau > \frac{4\pi d}{\text{Area}(\Sigma)}$. We will show two things: (i) absolute minimizers of the energy (1.1) exist, and (ii) that when $\lambda\tau > \frac{4\pi d}{\text{Area}(\Sigma)}$, the Maxwell solutions are unstable, in the sense that there are configurations arbitrarily close with lower energy. It follows that the absolute minimizers cannot be Maxwell solutions, and therefore are irreducible.

First we prove that the functional $\mathcal{E}_{\lambda,\tau}$ satisfies a gauged version of the Palais-Smale compactness condition.

Lemma 2.3. *Let $(\nabla_n, \phi_n) \in \mathcal{C}_L \oplus \Omega_L^0$ be a sequence with the following properties:*

1. *The sequence $\mathcal{E}_{\lambda,\tau}(\nabla_n, \phi_n)$ is bounded,*
2. *The sequence of derivatives $\mathcal{E}'_{\lambda,\tau}(\nabla_n, \phi_n)$ converges to zero in the $(W^{1,2})^*$.*

Then there is a subsequence $(\nabla_{n_k}, \phi_{n_k}) \in \mathcal{C}_L \oplus \Omega_L^0$, together with a sequence of gauge transformations $g_{n_k} \in \mathcal{G}$, such that $g_{n_k}(\nabla_{n_k}, \phi_{n_k})$ converges in $\mathcal{C}_L \oplus \Omega_L^0$.

Proof. Since smooth pairs in $\mathcal{C}_L \oplus \Omega_L^0$ are dense, it is enough to prove the statement for smooth sequences.

Fix a reference connection ∇^0 and define the $W^{k,p}$ norms, as

$$\|\psi\|_{W^{k,p}} = \sum_{n=0}^k \left\| \left(\nabla^0 \right)^n \psi \right\|_{L^p} \quad (2.9)$$

We pick ∇^0 to be a smooth Maxwell connection, and write $a_n = \nabla_n - \nabla^0$.

By the Hodge decomposition, the imaginary valued 1-forms a_n can be decomposed as

$$a_n = idf_n + id^*g_n + ih_n, \quad (2.10)$$

where f_n is a smooth function, g_n is a smooth 2-form, and h_n is a harmonic 1-form. We can eliminate the first term by applying the gauge transformations $\exp(if_n)$ to the pair (∇_n, ϕ_n) . We can also assume, after suitable gauge transformations, that the sequence h_n is a bounded sequence in L^2 . By elliptic regularity, the space of harmonic 1-forms is finite dimensional, so we can choose a subsequence, so that h_n is convergent. Because all h_n are harmonic, this implies convergence in the $W^{1,2}$ as well.

The Ginzburg-Landau free energy (1.1) can now be written as

$$\mathcal{E}_{\lambda,\tau}(\nabla^0 + a_n, \phi_n) = \mathcal{E}_{\lambda,\tau}^M + \|da_n\|_{L^2}^2 + \|(\nabla^0 + a_n)\phi_n\|_{L^2}^2 + \frac{\lambda}{4} \int_{\Sigma} (\tau - |\phi|^2)^2 \omega. \quad (2.11)$$

Since all of the terms on the right hand side are non-negative, and the energy of the sequence is bounded, we get the following:

- (1) The sequence $\|da_n\|_{L^2}^2$ is bounded. Because a_n has no exact part, and has convergent harmonic part, this means that a_n is bounded in $W^{1,2}$. By the Sobolev inequality the sequence a_n is also bounded in L^4 .
- (2) The sequence $\|(\nabla^0 + a_n)\phi_n\|_{L^2}^2$ is bounded, and thus so is $|\|\nabla^0\phi_n\|_{L^2}^2 - \|a_n\phi_n\|_{L^2}^2|$
- (3) The sequence $\int_{\Sigma} (\tau - |\phi|^2)^2 \omega$ is bounded. Jensen's inequality implies that

$$\left(\tau \text{Area}(\Sigma) - \|\phi\|_{L^2}^2\right)^2 \leq \int_{\Sigma} (\tau - |\phi|^2)^2 \omega,$$

so ϕ is bounded in L^2 . This in turn immediately implies that ϕ_n is bounded in L^4 as

well, because of the equality

$$\|\phi\|_{L^4}^4 = \int_{\Sigma} (\tau - |\phi|^2)^2 \omega - \tau^2 \text{Area}(\Sigma) + 2\|\phi_n\|_{L^2}^2. \quad (2.12)$$

(4) Combining (1) and (3), together with the Cauchy-Schwarz inequality, shows that $a_n \phi_n$ is bounded in L^2 .

(5) Combining (2) and (4), we get that $\nabla^0 \phi_n$ is bounded in L^2 , and thus ϕ_n is bounded in $W^{1,2}$.

Using that $\mathcal{E}'_{\lambda,\tau}(\nabla^0 + a_n, \phi_n)$ converges to zero, and the fact that $d^* a_n = 0$, we get

$$\Delta a_n = -|\phi_n|^2 a_n + i \text{Im} \left(h \left(\nabla^0 \phi, \phi \right) \right) + b_n, \quad (2.13)$$

where b_n converges to zero in $i\Omega^1$, and

$$\left(\nabla^0 \right)^* \left(\nabla^0 \right)^2 \phi = -a^* \left(\nabla^0 \phi \right) - \left(\nabla^0 \right)^* (a\phi) + |a|^2 \phi + \frac{\lambda}{2} \left(\tau \phi - |\phi|^2 \phi \right) + \psi_n, \quad (2.14)$$

where $\Delta = d^*d + dd^*$ is the Laplace-de Rahm operator, and ψ_n converges to zero in Ω_L^0 .

It is easy to see, using the previous observations, that the right hand sides are bounded in L^2 . Elliptic regularity then implies, (a_n, ϕ_n) is bounded in $W^{2,2}$. Since the embedding $W^{2,2} \hookrightarrow W^{1,2}$ is compact in dimension 2, we conclude that the sequence (a_n, ϕ_n) has a convergent subsequence in $W^{1,2}$. \square

Corollary 2.4. *The infimum of $\mathcal{E}_{\lambda,\tau}$ is achieved by some field $(\nabla, \phi) \in \mathcal{C}_L \oplus \Omega_L^0$.*

Proof. The functional $\mathcal{E}_{\lambda,\tau}$ non-negative, thus so is its infimum. Pick a minimizer sequence

$(\nabla_n, \phi_n) \in \mathcal{C}_L \oplus \Omega_L^0$. By Lemma 2.3, there is a subsequence, and gauge transformations g_n , such that the sequence $g_n(\nabla_n, \phi_n)$ converges to a limit. Since $\mathcal{E}_{\lambda, \tau}$ is gauge invariant, and analytic in the $W^{1,2}$ norm, the limit is an absolute minimizer. \square

Finally we show that the infimum cannot be a Maxwell solution when $\lambda\tau > \frac{4\pi d}{\text{Area}(\Sigma)}$. To this end, note that all Maxwell solutions have the same Ginzburg-Landau free energy, which we call the *Maxwell energy*:

$$\mathcal{E}_{\lambda, \tau}^{\text{M}} = \frac{4\pi^2 d^2}{\text{Area}(\Sigma)} + \frac{\lambda\tau^2 \text{Area}(\Sigma)}{4}. \quad (2.15)$$

The next lemma computes the second variation of the energy functional (1.1) around a Maxwell solution.

Lemma 2.5. *Let ∇^0 be a Maxwell connection. Then for any $(a, \phi) \in i\Omega^1 \oplus \Gamma(L)$ the following second variation formula holds:*

$$\begin{aligned} \mathcal{E}_{\lambda, \tau}(\nabla^0 + a, \phi) - \mathcal{E}_{\lambda, \tau}^{\text{M}} &= \|da\|_{L^2}^2 + 2\|\bar{\partial}\nabla\phi\|_{L^2}^2 \\ &+ \left(\frac{2\pi d}{\text{Area}(\Sigma)} - \frac{\lambda\tau}{2}\right)\|\phi\|_{L^2}^2 + O\left(\|(a, \phi)\|_{W^{1,2}}^3\right). \end{aligned} \quad (2.16)$$

Proof. In the expansion of $\mathcal{E}_{\lambda, \tau}(\nabla^0 + a, \phi)$ the linear terms in (a, ϕ) vanish, because $(\nabla^0, 0)$ is a critical point of $\mathcal{E}_{\lambda, \tau}$. Thus up to quadratic order we get

$$\mathcal{E}_{\lambda, \tau}(\nabla^0 + a, \phi) - \mathcal{E}_{\lambda, \tau}^{\text{M}} = \|da\|_{L^2}^2 + \|\nabla^0\phi\|_{L^2}^2 - \frac{\lambda\tau}{2}\|\phi\|_{L^2}^2 + O\left(\|(a, \phi)\|_{W^{1,2}}^3\right).$$

By the Kähler identity the middle term can be rewritten as

$$\begin{aligned}
\|\nabla^0 \phi\|_{L^2}^2 &= \langle \nabla^0 \phi | \nabla^0 \phi \rangle_{L^2} \\
&= \langle \phi | (\nabla^0)^* \nabla^0 \phi \rangle_{L^2} \\
&= \langle \phi | 2(\bar{\partial}_{\nabla^0})^* \bar{\partial}_{\nabla^0} \phi + \frac{2\pi d}{\text{Area}(\Sigma)} \phi \rangle_{L^2} \\
&= 2\|\bar{\partial}_{\nabla^0} \phi\|_{L^2}^2 + \frac{2\pi d}{\text{Area}(\Sigma)} \|\phi\|_{L^2}^2,
\end{aligned}$$

which concludes the proof. \square

Corollary 2.6. *When $\lambda\tau > \frac{4\pi d}{\text{Area}(\Sigma)}$ the minimum of the Ginzburg-Landau free energy, $\mathcal{E}_{\lambda,\tau}$ is not the Maxwell energy (2.15).*

Proof. In the formula (2.16) let $a = 0$ and ϕ be a non-zero holomorphic section with respect to the holomorphic structure induced by ∇^0 . If $\lambda\tau > \frac{4\pi d}{\text{Area}(\Sigma)}$, then for $t > 0$ small enough

$$\mathcal{E}_{\lambda,\tau}(\nabla^0, t\phi) = \mathcal{E}_{\lambda,\tau}^M + t^2 \left(\frac{2\pi d}{\text{Area}(\Sigma)} - \frac{\lambda\tau}{2} \right) \|\phi\|_{L^2}^2 + O(t^3) < \mathcal{E}_{\lambda,\tau}^M, \quad (2.17)$$

because the coefficient of the quadratic term is negative. \square

Lemma 2.3 and Corollary 2.6 imply the existence part of the main theorem of Chapter 2:

Corollary 2.7. *There exist irreducible solutions of the equations (1.3a) and (1.3b) when*

$$\lambda\tau > \frac{4\pi d}{\text{Area}(\Sigma)}.$$

REFERENCES

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- [AA87] Y. Aharonov and J. Anandan. Phase change during a cyclic quantum evolution. *Phys. Rev. Lett.*, 58:1593–1596, 1987.
- [AK02] I. S. Aranson and L. Kramer. The world of the complex Ginzburg-Landau equation. *Rev. Mod. Phys.*, 74:99–143, 2002.
- [B06] J. M. Baptista. Vortex equations in abelian gauged sigma-models. *Commun. Math. Phys.*, 261:161–194, 2006.
- [B11] J. M. Baptista. On the L^2 -metric of vortex moduli spaces. *Nucl. Phys.*, B, 844:308–333, 2011.
- [B84] M. V. Berry. Quantal phase factors accompanying adiabatic changes. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 392(1802):45–57, 1984.
- [B90] S. B. Bradlow. Vortices in holomorphic line bundles over closed Kähler manifolds. *Commun. Math. Phys.*, 135(1):1–17, 1990.
- [BR14] M. Bokstedt and N. M. Romao. On the curvature of vortex moduli spaces. *Math. Z.*, 277:549–573, 2014.
- [CM05] H. Y. Chen and M. S. Manton. The Kähler potential of abelian Higgs vortices. *J. Math. Phys.*, 46:052305, 2005.
- [DDM13] D. Dorigoni, M. Dunajski, and N. S. Manton. Vortex motion on surfaces of small curvature. *Annals Phys.*, 339:570–587, 2013.
- [EN11] G. Etesi and Á. Nagy. S-duality in abelian gauge theory revisited. *J. Geom. Phys.*, 61:693–707, 2011.
- [FU91] Daniel S. Freed and Karen K. Uhlenbeck. *Instantons and four-manifolds*, volume 1 of *Mathematical Sciences Research Institute Publications*. Springer-Verlag, New York, second edition, 1991.

- [GL50] V. L. Ginzburg and L. D. Landau. On the theory of superconductivity. *Zh. Eksp. Teor. Fiz.*, 20:1064–1082, 1950.
- [H02] A. Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.
- [HJS96] M. Hong, J. Jost, and M. Struwe. Asymptotic limits of a Ginzburg-Landau type functional. In *Geometric analysis and the calculus of variations*, pages 99–123. Int. Press, Cambridge, MA, 1996.
- [I01] D. A. Ivanov. Non-abelian statistics of half-quantum vortices in p -wave superconductors. *Phys. Rev. Lett.*, 86(2):268, 2001.
- [JT80] A. Jaffe and C. H. Taubes. *Vortices and monopoles*. Progress in Physics. Birkhäuser, Boston, Mass., 1980.
- [K50] T. Kato. On the Adiabatic Theorem of Quantum Mechanics. *Jour. Phys. Soc. Japan*, 5(6):435–439, 1950.
- [K85] M. Kohmoto. Topological invariant and the quantization of the Hall conductance. *Annals of Physics*, 160(2):343 – 354, 1985.
- [KN63] S. Kobayashi and K. Nomizu. *Foundations of differential geometry. Vol I*. Interscience Publishers, New York-London, 1963.
- [MM15] R. Maldonado and N. S. Manton. Analytic vortex solutions on compact hyperbolic surfaces. *J. Phys., A*, 48(24):245403, 2015.
- [MN99] N. S. Manton and S. M. Nasir. Volume of vortex moduli spaces. *Commun. Math. Phys.*, 199(3):591–604, 1999.
- [MS03] N. S. Manton and J. M. Speight. Asymptotic interactions of critically coupled vortices. *Commun. Math. Phys.*, 236:535–555, 2003.
- [R00] I. Mundet i Riera. A Hitchin-Kobayashi correspondence for Kähler fibrations. *J. Reine Angew. Math.*, 528:41–80, 2000.
- [T84] C. H. Taubes. On the Yang-Mills-Higgs equations. *Bull. Amer. Math. Soc. (N.S.)*, 10(2):295–297, 1984.
- [T99] C. H. Taubes. $GR = SW$: counting curves and connections. *J. Diff. Geom.*, 52(3):453–609, 1999.