This is to certify that the

thesis entitled

THE RATIONAL EXPECTATIONS HYPOTHESIS:
A FRAMEWORK FOR SOLUTIONS
WITH ECONOMETRIC IMPLICATIONS

presented by

Dennis Lee Hoffman

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Economics

Major professor

Date September 1, 1978

O-7639

THE RATIONAL EXPECTATIONS HYPOTHESIS: A FRAMEWORK FOR SOLUTIONS WITH ECONOMETRIC IMPLICATIONS

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A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Economics

1978

ABSTRACT

THE RATIONAL EXPECTATIONS HYPOTHESIS: A FRAMEWORK FOR SOLUTIONS WITH ECONOMETRIC IMPLICATIONS

By

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This study investigates the impact of the rational expectations hypothesis (REH) on economic models by constructing a framework for analyzing rational expectations solutions, outlining an approach to estimation and tests of hypothesis regarding structures which incorporate the REH, and surveying the recent literature to evaluate the use or misuse of the theory of rational expectations in previous studies.

The need for an analysis of the REH is derived from two distinct factors. First, the expanding role of expectations in economic models warrants the development of an explanation for how these generally unobservable perceptions of future events are formed. Second, the REH, which is one explanation, is not based upon a well developed theoretical foundation. Specifically, neither the guidelines for incorporating the REH into general models nor the econometric implications of applying the theory to economic models have been explicitly stated in previous studies.

In an effort to eliminate these deficiencies in the theoretical development of the REH, the present study adopts the

following format. First, a framework for the implementation of rational expectations in general models is constructed. Particular emphasis is centered upon the conditions under which one may obtain an observable expression for the expectation terms. This expression is designated as a rational expectations solution (RES). Second, the study reveals the econometric implications of replacing the expectations in the original structure with the RES suggested by the theory of rational expectations. Finally, a literature review compares the methodology employed in some recent treatments of the REH with that adopted in the present study.

The pursuit of this format yields a number of significant contributions. In the first place, the framework provides guidelines for the application of the REH to general models and accentuates the major complications a researcher is likely to encounter. Specifically, the analysis reveals that the RES depends upon the specification of stability conditions and the nature of the processes assumed to generate the exogenous variables contained in the original structure under consideration.

A second contribution of this study is the examination of the econometric significance of replacing the expectations terms with a relevant RES to generate a structure which is void of unobservable variables. In this econometric analysis the restrictions implied by the particular functional form of this reformulated system are revealed and a method for testing their validity is outlined. Consequently, this study provides a procedure for testing the validity of the REH as an explanation for individual's perceptions of future events.

Finally, the literature review reveals that many recent studies have avoided the significant aspects of the REH which are emphasized in the present study. This neglect generally stems from either a misinterpretation of the REH or the use of special cases which enable researchers to avoid many of the complexities inherent in the application of the theory of rational expectations to general economic models.

to my mother

Bea Hoffman

and my wife

Cindy

ACKNOWLEDGEMENTS

I would like to thank my thesis committee; Robert H. Rasche, Peter Schmidt, James Johannes, and Mark Ladenson for their helpful comments in regard to earlier drafts of this study. I am especially grateful to Peter Schmidt for his consultation in regard to the econometric section. Special thanks goes to my chairman, Robert H. Rasche, who provided constant guidance and encouragement at all stages of my research and for all the evenings and weekends he spent reading the manuscript.

Also, I wish to acknowledge the graduate students at Michigan State University for providing an atmosphere in which it was easy to exchange ideas as we expanded our knowledge of economics.

Noralee Burkhardt and Linda Morrone deserve the credit for the careful typing of both the preliminary and final drafts.

Finally, I wish to thank my wife Cindy for her patience during the hours I spent doing research as well as her expertise in editing a number of earlier drafts.

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CHAPTER I

INTRODUCTION

1.1 The Problem

This study investigates the impact of the rational expectations hypothesis (REH) on economic models by constructing a framework for analyzing rational expectations solutions in general models, outlining an approach to the estimation and testing structures which incorporate the REH and surveying the recent literature to evaluate the use or misuse of the theory of rational expectations in previous studies.

A study of rational expectations requires the consideration of models containing explanatory variables which appear as expectations of future endogenous variables. Since data is, for the most part, unavailable, one needs to determine how agents formulate these perceptions of future events. The theory of rational expectations, originally advanced by Muth (1961), is one explanation. This theory suggests that the rational agent equates expectations with the conditional forecasts of the "relevant economic theory".[1] The extensive application of this theory to models in current literature is the primary motivation of this analysis.

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[1] See Muth (1961), p. 316.

1.2 Need for the Study

The need for an analysis of the role of the REH is derived from two distinct factors. The first concerns the increasing emphasis placed upon expectations in recent studies coupled with an ad hoc explanation for how expectations are formed. Second, few studies have recognized the theoretical and econometric implications of the REH in general models.

Since the analysis of Fisher (1930), the significance of expectations of price changes has been understood. Within the past decade new emphasis on price expectations has accompanied the natural rate hypothesis (NRH) which renders long run stabilization policy impotent [Friedman (1968), Lucas (1972)]. Sargent and Wallace (1975, 1976) demonstrate that the addition of the REH to the NRH models preempts the role of short term stabilization policy, enabling the results of the long run natural rate proposition to hold in the short run as well. This controversial result is most responsible for the movement away from the ad hoc notion of extrapolative or adaptive expectations first employed by Cagan (1956).[2]

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[2]Extrapolative expectations pertains to the idea that expectations are adjusted according to the amount realized values deviate from previously formulated expectations: For example let ${}_{t-1}y_t^*$ represent the expectation of the value of y in period t formulated on the basis of information in period $t-1$; then

$$\{{}_{t-1}y_t^* - {}_{t-2}y_{t-1}^*\} = (1 - \alpha)\{y_{t-1} - {}_{t-2}y_{t-1}^*\} \qquad 0 < \alpha < 1$$

describes an extrapolative scheme. This leads to an expression for expectations in terms of past values of y alone;

$$ {}_{t-1}y_t^* = (1 - \alpha) \textstyle\sum_{i=1}^{\infty} \alpha^{i-1} y_{t-i}.$$

This framework has often been criticized for its lack of theoretical foundation as well as its suggestion that only knowledge of past values of the variable in question enter into the formation of expectations.

However, price expectations are not the only concern of rational expectations theorists. Theories of consumption and income, for example, the permanent income hypothesis; and interest rates, for example, the expectations theory of the term structure, place considerable emphasis upon expectations. An examination of the proper employment and validity of the REH is essential if rational expectations is to serve as a viable alternative to the extrapolative schemes employed in the previous analysis of these concepts.

The second problem area which this study addresses is the deficiency of theoretical developments applying the REH to general models, specifically models with both multi-period future (lead) expectations and lagged endogenous variables. An outline of the solution procedure applicable to general models and an analysis of the conditions under which rational expectations solutions may be obtained would clearly aid both previous and forthcoming studies. A method of estimating and testing models which incorporate these general solutions is essential in augmenting the theoretical development of rational expectations. Finally, an investigation of the use of rational expectations in recent studies is warranted to alleviate the discord generated by the Muth article and assist in developing a uniform interpretation of the REH.[3]

1.3 Format

To meet these problem areas, this study applies the REH to models of increasing generality in chapters II, III, and IV. In

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[3]The simple examples used by Muth in illustrating the REH have been the source of some confusion. This is discussed in chapter VI.

each chapter the model under consideration is specified and the steps leading to the reformulated structure suggested by the REH are outlined in detail. The individual treatment accentuates the different procedures required to obtain rational expectations solutions and outlines the various restrictions under which stable solutions are obtained in different models.[4]

Chapter V offers an approach to estimation and hypothesis testing which is consistent with the REH according to the general framework outlined in the introductory chapters.

Chapter VI supplies a discussion of a number of applications of the REH in current literature, highlighting both the improper interpretations of Muth's theory and the various assumptions which serve to circumvent the complexities of coping with the REH in more general specifications.

1.4 Limitations

Although this study provides an extensive treatment of rational expectations in economic models, it is not without limitations. It provides no explanation of how agents obtain the information required to form perceptions of future events which are rational in the sense of Muth. This difficult problem requires a general theory of rational expectations which has yet to be developed and is outside the scope of this study.

Furthermore, the analysis offers no mechanism for how rational expectations adjust when the relevant theory changes or is

[4]Stability is referred to, not in the probabilistic sense, but in the difference equation context.

expected to change. Some advances in this direction have been made in recent studies by Shiller (1978) and Taylor (1975). However, these endeavors overlook the significant problems encountered when the relevant theory is static. These problems are revealed in the present analysis.

Finally, although the test outlined in chapter V may prove useful in future applied work, the present study offers no motive for employing rational expectations in economic models, but examines the implications of this choice for most of the models a researcher is likely to encounter.

CHAPTER II

MODELS WITH CURRENT PERIOD ENDOGENOUS EXPECTATIONS

This chapter examines a specific class of models which contain expectations of endogenous variables. The expectations are expressed, following the REH, as functions of observable variables. Then, the reformulated version of the original model, incorporating the REH, is derived.

2.1 An Outline of Model I Structures

Perhaps the most common usage of expectations in economic models involves the assumption that present period (t) levels of endogenous variables are explained by perceptions of those variables formed by agents on the basis of information available one period earlier (t-1). Examples of this class are prevalent in the current literature and the general implications of the REH for these simple models are discussed in some of these studies. Nevertheless, these structures will be examined in the present analysis to provide a complete framework for analyzing the impact of the REH upon economic models. A model which contains the characteristics of these simple structures will be designated, Model I, and may be represented:

$$Ay_t = B(L)x_t + \Psi(L)y_t + \theta_{t-1}y_t^e + \varepsilon_t \qquad t = 1,2,\ldots,T \qquad (2.1.1)$$

where;

i) A is an $m \times m$ matrix of structural coefficients,

ii) y_t is an $m \times 1$ vector of endogenous variables,

iii) $B(L)$ is an $m \times n$ matrix with elements consisting of finite polynomials in the lag operator L. Hence, typical elements are:

$$B_{ij} = \sum_{g=0}^{q_{ij}} B_{ijg} L^g; \; L^g x_t = x_{t-g}$$

$i = 1,2,\ldots,m;\ j = 1,2,\ldots,n,$

a. q_{ij} is the order of the lag of the j^{th} exogenous variable in the i^{th} equation,

b. B_{ijg} is the g^{th} coefficient in the polynomial lag of order q_{ij} for the j^{th} exogenous variable in the i^{th} equation,

iv) x_t is an $n \times 1$ vector of exogenous variables,[1]

v) $\Psi(L)$ is an $m \times m$ matrix with elements consisting of finite polynomials in the lag operator L. Similarly:

[1]The term exogenous pertains to those variables which are determined outside the system (2.1.1) while endogenous variables are determined by the simultaneous interaction of the system. Including expectations of exogenous variables would be a trivial extension of (2.1.1). The ensuing analysis would be unaltered and the problem created by their unobservability would be eliminated by an assumption comparable to (2.2.6) in the following section.

$$\Psi_{ij} = \sum_{h=1}^{r_{ij}} \Psi_{ijh} L^h; \; L^h y_t = y_{t-h}$$

$i,j = 1,2,\ldots,m,$

a. r_{ij} and Ψ_{ijh} are analogous to q_{ij} and B_{ijg} above,

b. $\Psi_{ijo} = 0$ for all i,j,

vi) θ is an $m \times m$ matrix of structural coefficients,

vii) ${}_{t-1}y_t^e$ is an $m \times 1$ vector of the values the endogenous variables are expected to take on in period t formulated on the basis of all information available as of period $t-1$,

viii) ε_t is an $m \times 1$ vector of structural disturbances which follow a stationary, multivariate, ARMA process:

$$\xi(L)\varepsilon_t = \delta(L)u_t,$$

where both $\xi(L)$ and $\delta(L)$ are $m \times m$ matrices whose elements are finite order polynomials in the lag operator L. u_{ti} is independently and identically distributed $N\{0, \sigma_{u_i}^2 I_T\}$ for each $i = 1,\ldots,m$. Following Zellner and Palm (1974) this may be expressed as an infinite order moving average process:

$$\varepsilon_t = \xi^{-1}(L)\delta(L)u_t$$

$$\equiv \omega(L)u_t,$$

provided the roots of $|\xi(L)| = 0$ lie outside the unit circle.

The final assumption (viii) accommodates any order of autocorrelation in the vector of disturbances ε_t and is amenable to the moving average structures examined by Muth in his seminal article.

Incorporating this general expression for disturbances, (2.1.1) may be written:

$$Ay_t = B(L)x_t + \Psi(L)y_t + \theta_{t-1}y_t^e + \omega(L)u_t, \; t = 1,2,\ldots,T. \qquad (2.1.2)$$

The system described by (2.1.2) is the most general representation of a simultaneous equations system that includes expectations of current period endogenous variables formed from information available last period.

2.2 A Rational Expectation Solution for Model I

A rational expectations solution (RES) is an expression for rational expectations in terms of observable variables obtained by the application of the REH to a specific model. To obtain the RES for a general Model I structure, consider the reduced form for (2.1.2):

$$y_t = A^{-1}B(L)x_t + A^{-1}\Psi(L)y_t + A^{-1}\theta_{t-1}y_t^e + A^{-1}\omega(L)u_t. \qquad (2.2.1)$$

This system represents the "relevant economic theory" for the m endogenous variables in the system.

The theory of rational expectations suggests:

$$ {}_{t-1}y_t^e = \underset{t-1}{E}\{y_t | I_{t-1}\} \equiv \underset{t-1}{E}\, y_t, $$

where I_{t-1} pertains to the set of information from which expectations are formulated. Hence, rational expectations are the conditional forecasts of the relevant theory.

Making conditional forecasts from a Model I structure assumed to contain rational expectations yields:

$$ \underset{t-1}{E}\, y_t = \underset{t-1}{E}\, A^{-1}B(L)x_t + \underset{t-1}{E}\, A^{-1}\Psi(L)y_t + \underset{t-1}{E}\, A^{-1}\theta \underset{t-1}{E}\, y_t + \underset{t-1}{E}\, A^{-1}\omega(L)u_t . \quad (2.2.2) $$

This expression may be simplified by noting;

i) $\underset{t-1}{E}\, z_{t-i} = z_{t-i}$ for any variable z_{t-i}, $i = 1,2,\ldots,$

ii) Lag operators affect realization dates, not expectation formation dates, therefore,

$$ \underset{t-1}{E}\, L^k z_t = \underset{t-1}{E}\, z_{t-k},\ k = 1,2,\ldots $$

for any variable z_t,

iii) $\Psi_{ijo} = 0$ for all i,j;

to obtain:

$$ [I - A^{-1}\theta] \underset{t-1}{E}\, y_t = \underset{t-1}{E}\, A^{-1}B(L)x_t + A^{-1}\Psi(L)y_t + \underset{t-1}{E}\, A^{-1}\omega(L)u_t . \quad (2.2.3) $$

Assuming $A^{-1}\theta$ is of full rank yields:

$$ \underset{t-1}{E}\, y_t = \{I - A^{-1}\theta\}^{-1}\{\underset{t-1}{E}\, A^{-1}B(L)x_t + A^{-1}\Psi(L)y_t + \underset{t-1}{E}\, A^{-1}\omega(L)u_t\}. \quad (2.2.4) $$

Therefore, the application of REH to Model I yields an expression for the rational expectations of endogenous variables which contains expectations of both current exogenous variables and the disburbance term, plus all the lagged variables in the original structure. Particular functions of the original structural parameters form the coefficients of this expression.

However, (2.2.4) becomes a RES only when the expectation terms on the right side of (2.2.4) are expressed in terms of observable variables.[2] Assumption viii, section 2.1, insures that:

$$\underset{t-1}{E}\ u_t = 0,$$

while all other values of $\underset{t-1}{E} A^{-1}\omega(L)u_t$ are observable reduced form disturbances realized in previous periods. Additional assumptions are required to deal with the expectations of exogenous variables. Assume all exogenous variables follow known deterministic rules. This allows current exogenous variables to be predicted with certainty on the basis of last period's information. This assumption allows (2.2.4) to be written as:[3]

$$\underset{t-1}{E} y_t = \{I - A^{-1}\theta\}^{-1}\{A^{-1}B(L)x_t + A^{-1}\Psi(L)y_t + A^{-1}\omega'(L)u_{t-1}\}, \qquad (2.2.5)$$

which is a (RES) to a general Model I structure subject to the stated assumptions.

A more comprehensive assumption is that the exogenous variables follow some identifiable, stable, stochastic process:

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[2]This is the definition of a rational expectation solution employed in the present context.

[3]Writing $\omega(L) = \omega_0 L^0 + \omega_1 L^1 + \ldots$ where ω_i is the matrix of coefficients on the lag of order i on u_t in (2.1.2) then

$$\omega'(L) \equiv \omega_1 L^0 + \omega_2 L^1 + \ldots$$

i.e. $x_t = \Gamma(L)x_{t-1} + \eta_t$, (2.2.6)

where $\Gamma(L)$ is a diagonal matrix whose elements are finite polynomials in the lag operator L and $\eta_t \sim N\{0,\Sigma_\eta\}$.

Re-expressing the expectations term in (2.2.4), and following (2.2.6) yields:[4]

$$\underset{t-1}{E} A^{-1}B(L)x_t =$$

$$= A^{-1}B_0 \underset{t-1}{E} x_t + A^{-1}B'(L)x_{t-1}$$

$$= A^{-1}B_0\Gamma(L)x_{t-1} + A^{-1}B'(L)x_{t-1}$$

$$= A^{-1}\{B_0\Gamma(L) + B'(L)\}x_{t-1} . \qquad (2.2.7)$$

The substitution of (2.2.7) into (2.2.4) yields:

$$\underset{t-1}{E} y_t = \{I - A^{-1}\theta\}^{-1}\{A^{-1}(B_0\Gamma(L) + B'(L))x_{t-1} + A^{-1}\Psi(L)y_t$$
$$+ A^{-1}\omega'(L)u_{t-1}\}, \qquad (2.2.8)$$

which is a RES for a Model I structure conditional upon (2.2.6).

Hence, a valid RES for systems which have the characteristics of Model I, as defined above, depends upon two distinct factors. First, it demands the exact specification of the relevant theory, including the nature of the reduced form disturbances. Second, the nature of the process from which agents form expectations of the exogenous variables is required.

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[4] Writing $B(L) = B_0L^0 + \ldots + B_qL^q$, $q = \text{Max}(q_{ij})$ over all i,j where B_i is the matrix of coefficients on the lag of order i on x_t in (2.1.2), then:

$$B'(L) = B_1L^0 + B_2L^1 + \ldots + B_qL^{q-1}.$$

2.3 Observable Reduced Form for Model I

Having derived the RES, the observable reduced form is obtained by substituting (2.2.8) for ${}_{t-1}y_t^e$ in (2.2.1) to obtain:[5]

$$\begin{aligned} y_t &= A^{-1}B(L)x_t + A^{-1}\Psi(L)y_t \\ &+ A^{-1}\theta\{I - A^{-1}\theta\}^{-1}\{A^{-1}(B_0\Gamma(L) + B'(L))\}x_{t-1} \\ &+ A^{-1}\theta\{I - A^{-1}\theta\}^{-1}\{A^{-1}\Psi(L)y_t + A^{-1}\omega'(L)u_{t-1}\} \\ &+ A^{-1}\omega(L)u_t . \end{aligned} \tag{2.3.1}$$

This expression represents the reformulated, reduced form structure obtained by replacing the expectations with the RES derived from the application of the REH to the original reduced form system. Consequently, adding the REH to (2.2.1) yields an alternative structure which is void of the unobservable variables.

2.4 Summary

The analysis of Model I structures reveals several important factors. First, when the only expectations terms encountered in a model are perceptions of present period endogenous variables formed

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[5]Similarly an observable reduced form form (2.1.2) when the exogenous variables follow a known deterministic rule is obtainable, - making (2.3.1),

$$\begin{aligned} y_t &= A^{-1}B(L)x_t + A^{-1}\Psi(L)y_t + A^{-1}\theta\{I - A^{-1}\theta\}^{-1}\{A^{-1}B(L)x_t \\ &+ A^{-1}\Psi(L)y_t + A^{-1}\omega'(L)u_{t-1}\} \\ &+ A^{-1}\omega(L)u_t . \end{aligned}$$

one period earlier (Model I), the RES and consequent observable reduced form are computationally easy to obtain regardless of the magnitudes of structural parameters.[6] The investigation of models which contain expectations of endogenous variables for future periods, pursued in ensuing chapters, reveals that this is not always the case.

In addition, section 2.2 demonstrates that the REH alone is insufficient to obtain expressions for expectations of endogenous variables in terms of observable variables. Hence, the RES is conditional upon assumptions about the nature of the processes generating the exogenous variables and disturbances.

Finally, the application of the REH to a Model I structure yields a reformulation which is a function of a particular set of variables whose coefficients are, in turn, specific functions of the original structural parameters. Thus, the observable reduced form suggested by the REH is distinguishable from any other reformulation of (2.2.1) obtained under an alternative assumption about how expectations are formed.

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[6]Recall that the only restriction imposed upon structural parameters to obtain a RES was that $A^{-1}\theta$ be of full rank.

CHAPTER III

MODELS WITH MULTI-PERIOD FUTURE EXPECTATIONS

This chapter investigates a class of models which contain expectations of endogenous variables for a finite number of future periods. The RES and resulting observable reduced form are derived according to the format outlined in the previous chapter.

3.1 An Outline of Model II Structures

In the event that economic agents base current decisions upon expectations (formed in period $t-1$) of endogenous variables in future time periods $t+1, t+2, \ldots,$ the Model I framework may be broadened to include these multi-period future expectations. A few simple examples of these structures appear in the current literature.[1] However, a general treatment of these structures is not explicitly stated in any of the previous studies of rational expectations. Models which possess these characteristics will be designated Model II structures and may be represented as:[2]

$$Ay_t = B(L)x_t + \theta(F)_{t-1}y_t^e + \omega(L)u_t, \quad t = 1,2,\ldots,T \qquad (3.1.1)$$

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[1]Some of these are discussed in Chapter VI.

[2]The effect of introducing lagged endogenous variables into (3.1.1) is examined in Chapter III.

where;

i) A, y_t, $B(L)$, x_t, ${}_{t-1}y_t^e$, $\omega(L)$, u_t are defined as in Chapter II,

ii) $\theta(F)$ is an $m \times m$ matrix with elements consisting of finite polynomials in the lead operator F; hence, typical elements are,

$$\theta_{ij} = \sum_{k=0}^{s_{ij}} \theta_{ijk} F^k; \; F^k \, {}_{t-1}y_t^e = {}_{t-1}y_{t+k}^e,$$

$i,j = 1,2,\ldots,m,$

a) s_{ij} is the order of the lead of the expectations of the j^{th} endogenous variable in the i^{th} equation,

b) θ_{ijk} is the k^{th} coefficient in the lead polynomial equation for the expectation of the j^{th} endogenous variable in the i^{th} equation,

c) Since no information is available for periods later than (t-1), lead operators apply to realization dates and not expectation formation dates.

The system described by (3.1.1) is more general than (2.1.1) in the sense that it allows future expectation terms to enter as explanatory variables. However, it cannot accommodate lagged endogenous variables which could appear in Model I.

3.2 A Rational Expectation Solution for Model II

As in Model I, the construction of the reduced form is the first step toward replacing the expectations terms with expressions which are free of unobservables. Hence, consider:

$$y_t = A^{-1}B(L)x_t + A^{-1}\theta(F)_{t-1}y_t^e + A^{-1}\omega(L)u_t. \tag{3.2.1}$$

This system represents the "relevant economic theory" for the m endogenous variables in the system.

Following the REH, the conditional forecasts from (3.2.1) are equated with the expectations in (3.2.1) to obtain:

$$\underset{t-1}{E}\, y_t = \underset{t-1}{E}\, A^{-1}B(L)x_t + \underset{t-1}{E}\, A^{-1}\theta(F)\, \underset{t-1}{E}\, y_t + \underset{t-1}{E}\, A^{-1}\omega(L)u_t. \tag{3.2.2}$$

Combining the expectations terms yields:

$$\{I - A^{-1}\theta(F)\}\underset{t-1}{E}\, y_t = \underset{t-1}{E}\, A^{-1}B(L)x_t + \underset{t-1}{E}\, A^{-1}\omega(L)u_t. \tag{3.2.3}$$

At first glance, this expression resembles the Model I analogue (2.2.3) which is a system of m equations in m unknown, current period, rational expectations. However, closer investigation reveals that the Model II expression, (3.2.3) above, contains m equations and up to $m(s+1)$, $s = \max\ s_{ij}$, unknown rational expectations. This structure, which represents current period rational expectations $(\underset{t-1}{E}\, y_t)$ as functions of rational expectations in later periods $(\underset{t-1}{E}\, y_{t+1}, \underset{t-1}{E}\, y_{t+2},\ldots)$ contains m, finite order, difference equations in leads as opposed to lags.

The typical equation of this system may be obtained by applying Wold's Chain Rule of forecasting to (3.2.3)[3], yielding:

$$\{I - A^{-1}\theta(F)\} \underset{t-1}{E}\, y_{t+j} = A^{-1} \underset{t-1}{E}\, B(L)x_{t+j} + A^{-1} \underset{t-1}{E}\, \omega(L)u_{t+j}, \qquad j = 0,1,\ldots \tag{3.2.4}$$

The relevant solution procedure for this system is analogous to the calculation of the "final form" of a simultaneous equations model.[4] In a "final form" all lagged endogenous variables are eliminated by substituting recursively. The lead expectation terms in (3.2.3) may be eliminated analogously, by recursive substitutions, utilizing the structures defined in (3.2.4) to obtain:[5]

$$\underset{t-1}{E}\, y_t = \underset{t-1}{E}\, \{I - A^{-1}\theta(F)\}^{-1}\{A^{-1}B(L)x_t + A^{-1}\omega(L)u_t\}\,. \qquad (3.2.5)^3$$

The condition which guarantees the stability of this solution is that the roots (with respect to F) of the determinantal

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[3]Wold (1938), Chapter 3.

[4]See Theil and Boot (1962), p. 136-152.

[5]The substitution procedure implied by (3.2.5) insures that all terms in periods t, t+1,..., later periods, appear as expectations terms while those in t-1, t-2,..., and earlier represent actual realized values. This interpretation is unambiguously maintained when lag and lead operators are manipulated prior to expectation operations. Hence, the expectation term is positioned outside of the inverted lead coefficient.
$\therefore\; F^k \underset{t-1}{E}\, z_t = \underset{t-1}{E}\, F^k z_t$ for all variables z_t in this analysis.

equation, $|I - A^{-1}\theta(F)| = 0$, lie outside the unit circle.[6] When this condition is satisfied, the system (3.2.5) expresses current period rational expectations as functions of all the lagged exogenous variables and disturbances in the original structure, plus expectations of exogenous variables and disturbances for all future periods.

Following the definition employed in Chapter II, (3.2.5) becomes a RES when the expectations terms on the right hand side of (3.2.5) are replaced by observable variables. As in the Model I analysis, assumption viii, section 2.1, insures that all future disturbances have zero expectation. Consequently:

$$\underset{t-1}{E}\ \{I - A^{-1}\theta(F)\}^{-1}A^{-1}\omega(L)u_t = D\{I - A^{-1}\theta(F)\}^{-1}A^{-1}\omega(L)u_t$$

where,

$$D\{I-A^{-1}\theta(F)\}^{-1}A^{-1}\omega(L)u_t = \begin{cases} \{I-A^{-1}\theta(F)\}^{-1}A^{-1}\omega(L)u_t & \text{for periods } t-1,t-2,\ldots,\text{earlier,} \\ 0 & \text{for periods } t,t+1,t+2,\ldots,\text{later.} \end{cases}$$

This notation accentuates the fact that (3.2.5) contains no unobservable expectations of disturbances.[7]

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[6] This condition is derived from considering an analogous example in Zeller and Palm (1974), p. 19.

[7] The lag and lead operators affect the time period for the disturbance term. Therefore if $\{I-A^{-1}\theta(F)\}^{-1}A^{-1}\omega(L)u_t$ is depicted as:

The expectations of exogenous variables are eliminated by utilizing the assumptions of Chapter II. As discussed in the Model I analysis, when exogenous variables follow known deterministic rules, they may be predicted with certainty for all future periods. The RES subject to this assumption is obtained by equating actual and expected future exogenous variables:

$$\underset{t-1}{E}\, y_t = \{I - A^{-1}\theta(F)\}^{-1}\{A^{-1}B(L)x_t\} + D\{I - A^{-1}\theta(F)\}^{-1}\omega(L)u_t \,. \tag{3.2.6}$$

An alternative assumption is that the exogenous variables follow the process described by (2.2.6). Taking conditional forecasts on (2.2.6) and leading ℓ periods obtains expressions for the ℓ-period forecasts of the n exogenous variables:

$$\begin{aligned}
\underset{t-1}{E}\, x_{it} &\equiv \hat{x}_{it}(0) = \gamma_{i1}x_{i-t-1} + \gamma_{i2}x_{it-2} + \ldots + \gamma_{ip}x_{it-p} \\
\underset{t-1}{E}\, x_{it+1} &\equiv \hat{x}_{it}(1) = \gamma_{i1}\hat{x}_{it}(0) + \gamma_{i2}x_{it-1} + \ldots + \gamma_{ip}x_{it-p+1} \\
&\;\;\vdots \qquad\qquad \vdots \\
\underset{t-1}{E}\, x_{it+\ell} &\equiv \hat{x}_{it}(\ell) = \gamma_{i1}\hat{x}_{it}(\ell-1) + \gamma_{i2}\hat{x}_{it}(\ell-2) + \ldots + \gamma_{ip}\hat{x}_{it}(\ell-p)
\end{aligned}$$

$$i = 1,2,\ldots,n$$

$$\sum_{i=-\infty}^{\infty} C_i u_{t+i} \;;\quad C_i \quad m \times m \text{ matrix for all } i.$$

Then;

$$D \sum_{i=-\infty}^{\infty} C_i u_{t+i} = \sum_{i=-\infty}^{-1} C_i u_{t+i} \,.$$

where;

i) ℓ is arbitrary,

ii) P is the order of the autoregressive process which generates all exogenous variables,

iii) $\hat{x}_{it}(-j) = x_{it-j}$; j = 1,2,...

Following Box and Jenkins (1976), the forecasts for lead times $\ell \leq 0$ may be expressed in terms of the observable lagged exogenous variables to obtain:[8]

$$\underset{t-1}{E}\, x_{i,t+\ell} \equiv \hat{x}_{i,t}(\ell) = \sum_{j=1}^{P} \gamma_{i,j}^{(\ell)} x_{i,t-j} \tag{3.2.7}$$

$$\gamma_{i,j}^{(\ell)} = \gamma_{i,j+\ell} + \sum_{h=0}^{\ell} \gamma_{i,h+1}\gamma_{i,j}^{(\ell-h)}$$

$$\gamma_{i,j}^{(0)} = \gamma_{i,j}.$$

Substituting (3.2.7) into (3.2.5) obtains the RES when exogenous variables adhere to (2.2.6):

$$\underset{t-1}{E}\, y_t = \{I - A^{-1}\theta(F)\}^{-1}\{A^{-1}B(L)\hat{x}_t(\ell)\} +$$

$$+ D\{I - A^{-1}\theta(F)\}^{-1}\{\omega(L)u_t\} \tag{3.2.8}$$

where;

i) $\hat{x}_t(\ell)$ represents the n-dimensional vector of ℓ-period forecasts of exogenous variables when $\ell \geq 0$, and realized, lagged, exogenous variables when $\ell < 0$,

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[8]These expressions are obtained by Box and Jenkins (1976), pp. 141-142. The notation is altered in (3.2.7) to accommodate $\ell = 0$ period forecasts.

ii) $F^k\hat{x}_t(\ell) = \hat{x}_t(\ell + k)$,

iii) $L^k\hat{x}_t(\ell) = \hat{x}_t(\ell - k)$.

In retrospect, the RES obtained for models which exhibit the characteristics of Model II, depend upon the same factors required in the Model I solution; namely the exact specification of both the structural model and process generating the exogenous variables. However, the present analysis reveals that models with lead endogenous expectations warrant the use of extremely intricate substitution procedures to obtain the weights on lagged variables which appear in the RES, as well as consideration for the conditions which guarantee the stability of the solution.

3.3 Observable Reduced Form for Model II

Following the format of the previous chapter, the observable reduced form is obtained by the substitution of the RES (3.2.8) into (3.2.1) to obtain:[9]

$$\begin{aligned} y_t = A^{-1}B(L)x_t &+ A^{-1}\theta(F)\{I - A^{-1}\theta(F)\}^{-1}\{A^{-1}B(L)\hat{x}_t(\ell)\} \\ &+ A^{-1}\theta(F)D\{I - A^{-1}\theta(F)\}^{-1}\{A^{-1}\omega(L)u_t\} \\ &+ A^{-1}\omega(L)u_t \, . \end{aligned} \tag{3.3.1}$$

This expression denotes the reformulated reduced form obtained by the application of the REH to a general Model II structure. This reformulation suggests that the original endogenous variables may be expressed as a function of all the predetermined variables in the

[9] The observable reduced form for Model II when all exogenous variables follow known deterministic rules is obtained by substituting $x_{t+\ell}$ for $\hat{x}_t(\ell)$.

system (3.1.1) plus the lagged terms inherent in the processes which generate the exogenous variables. This result is not unlike that obtained for Model I, in (2.3.1), in regard to the designation of the menu of variables which appear in the observable reduced form. However, the coefficients of (3.3.1) are considerably more complex functions of structural parameters than those in the observable reduced form (2.3.1) obtained for Model I.

3.4 An Example of a Model II Structure

A simple example exhibiting the characteristics of a Model II structure illustrates the steps leading to a Model II RES. Consider the single equation model:

$$y_t = \beta' x_t + \theta F_{t-1}y_t^e + \varepsilon_t \qquad t = 1,2,\ldots,T$$

where from (3.1.1);

i) y_t is a scalar,

ii) $B(L) = \beta'$ is a $1 \times n$ vector,

iii) x_t is a $n \times 1$ vector,

iv) $\theta(F) = \theta F$ is a scalar,

v) ${}_{t-1}y_t^e$ is a scalar,

vi) $\omega(L)u_t = \varepsilon_t \sim N\{0,\Sigma_\varepsilon\}$ scalar.

The structure (3.4.1) describes the "relevant economic theory" for the variable y_t. Therefore, following the procedure outlined in 3.2, the application of the REH yields an expression analogous to (3.2.5):

$$\underset{t-1}{E}\, y_{t+1} = \underset{t-1}{E}\{1 - \theta F\}^{-1}\, \beta'\, x_{t+1}$$

$$= \underset{t-1}{E} \sum_{i=0}^{\infty} \theta^{i} F^{i} \beta'\, x_{t+1}$$

$$= \underset{t-1}{E} \sum_{i=0}^{\infty} \theta^{i} \beta'\, x_{t+1+i}. \qquad (3.4.2)$$

Assume all exogenous variables are generated by first order, autoregressive processes. Hence, $\Gamma(L)$, in (2.2.6), is a diagonal, n-dimensional, matrix of zero order polynomials, Γ; where;

$$\Gamma_{ii} = \gamma_i \qquad i = 1,2,\ldots,n.$$

Therefore:[10]

$$\underset{t-1}{E}\, x_{t+1+i} = \Gamma^{i+2} x_{t-1} \qquad i = 0,1,2,\ldots, \qquad (3.4.3)$$

The RES for this example may be obtained by the substitution of (3.4.3) into (3.4.2); yielding:

$$\underset{t-1}{E}\, y_{t+1} = \sum_{i=0}^{\infty} \theta^{i} \beta' \Gamma^{i+2} x_{t-1}, \qquad (3.4.4)$$

---

[10] In the notation of (3.2.7),

$$\underset{t-1}{E}\, x_{i,t+\ell} = \hat{x}_{i,t}(\ell) = \gamma_{i,j}^{\ell+1}\, x_{i,t-1}$$

Since; $\gamma_{i,j}^{(\ell)} = \gamma_{i,j+\ell} + \sum_{h=0}^{\ell} \gamma_{i,h+1} \gamma_{i,j}^{(\ell-h)} = \gamma_{i,j}^{\ell+1}$ for all i if the x's are generated by first order processes. Hence,

$\gamma_{i,j} = \gamma_{i,h+1} = 0$ for $j = 2,3,\ldots$ and $h = 1,2,3,\ldots$

provided the stability condition, $|\theta| < 1$, is satisfied.[11] Since (3.4.4) is a geometric progression, it may be simplified to obtain:[12]

$$\underset{t-1}{E}\, y_{t+1} = \beta'\Gamma^2\{I - \theta\Gamma\}^{-1}x_{t-1}. \qquad (3.4.5)$$

Finally, the observable reduced form for this simple example is obtained by the substitution of (3.4.5) into (3.4.1):

$$y_t = \beta' x_t + \theta\beta'\Gamma^2\{I - \theta\Gamma\}^{-1}x_{t-1} + \varepsilon_t \;. \qquad (3.4.6)$$

This example reveals two aspects of the application of the REH to Model II structures. First, the RES requires consideration of all future expectations of the exogenous variables, even when only one period lead expectations appear in the original model. Also, both the coefficients and stability conditions for the RES may, in some cases, be simple functions of the original structural parameters; regardless of the complicated expressions obtained in the analysis of a general Model II RES.

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[11] Recall the condition defined in 3.2 requires that the roots of $|I - A^{-1}\theta(F)| = 0$ lie outside the unit circle. In this example the stability condition becomes,

$$|1 - \theta F| = 0 \quad ; \quad |F| > 1$$

or $$|F| = \frac{1}{|\theta|} \quad ; \quad |F| > 1$$

Therefore the single root lies outside the unit circle when $|\theta| < 1$.

[12] $\lim_{i\to\infty} \Gamma^i = \emptyset$ null matrix since $|\gamma_{i,1}| < 1$ for all $i = 1,\ldots,n$ because (2.2.6) is assumed to be a stable, stochastic process.

3.5 Summary

The implications of the REH for Model II structures are revealed by a comparison of the results obtained in Chapters II and III. First, the existence of a RES in models with lead expectations depends upon specific stability conditions which are, in turn, satisfied when structural parameters lie within particular intervals. This result differs from the Model I analysis which leads to a RES for all possible values of the original structural parameters.

Second, as in the Model I analysis, the particular RES obtained depends upon the assumption made about the process which generates the exogenous variables. However, the RES for Model II demands more extensive forecasting of future exogenous variables since the solution contains expectations of all future exogenous variables.

Finally, the arguments of the observable reduced form for models with lead expectations correspond with those obtained in the study of Model I. Also, the REH suggests a functional form for the weights on these variables that allows one to distinguish the observable reduced form, implied by the REH, from that obtained by employing an alternative expectations generating scheme. However, these weights are generally more complicated functions of the original structural parameters than those obtained in the observable reduced form for Model I. This complexity stems from the substitution procedure required to obtain a Model II RES.

CHAPTER IV

MODELS WITH MULTI-PERIOD FUTURE EXPECTATIONS AND LAGGED ENDOGENOUS VARIABLES

This chapter examines a class of structures which allow lagged endogenous variables to accompany multi-period future endogenous expectations as explanatory variables, thereby avoiding the simplification employed in the Model II analysis.

4.1 An Outline of Model III Structures

In an effort to analyze the impact of the REH on the most general models that a researcher is likely to encounter, lagged endogenous variables are added to the models with multi-period lead expectations to obtain a class of structures denoted as Model III. The general expression for this class is:

$$Ay_t = B(L)x_t + \Psi(L)y_t + \theta(F)_{t-1}y_t^e + \omega(L)u_t, \qquad (4.1.1)$$

where all terms have been defined in previous analyses.

This class is the most general representation of models which contain expectations formed from information available in period t-1.[1] Unlike Model I and Model II, which may be expressed

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[1]Shiller (1978), p. 29, deals with a more general version of (4.1.1) by including period t-2,t-3,... expectations of endogenous variables. The import of his analysis is discussed in Chapter VI.

as special cases of Model III, the details for obtaining a RES for models with both lagged endogenous variables and future endogenous expectations have yet to be explicitly stated in the literature. The present analysis intends to fill this void.

4.2 A Rational Expectations Solution for Model III

Following the format employed in previous chapters, the first step toward a RES is to obtain the reduced form:

$$y_t = A^{-1}B(L)x_t + A^{-1}\Psi(L)y_t + A^{-1}\theta(F)\,{}_{t-1}y_t^e + A^{-1}\omega(L)u_t. \qquad (4.2.1)$$

According to the REH, the expectations in (4.2.1) are equated with the conditional forecasts from the "relevant theory" [in this case (4.2.1)]. These forecasts are generated by taking expectations on (4.2.1) to obtain:

$$\underset{t-1}{E}\, y_t = \underset{t-1}{E}\, A^{-1}B(L)x_t + \underset{t-1}{E}\, A^{-1}\Psi(L)y_t + \underset{t-1}{E}\, A^{-1}\theta(F)\underset{t-1}{E}\, y_t^e$$
$$+ \underset{t-1}{E}\, A^{-1}\omega(L)u_t\ . \qquad (4.2.2)$$

Combining the expectations terms with the lagged endogenous variables and noting; $\underset{t-1}{E}\;\underset{t-1}{E}\, y_{t+s} = \underset{t-1}{E}\, y_{t+s}$ for all y_{t+s}:

$$\underset{t-1}{E}\, \{I - A^{-1}\theta(F) - A^{-1}\Psi(L)\}y_t = \underset{t-1}{E}\, A^{-1}B(L)x_t + \underset{t-1}{E}\, A^{-1}\omega(L)u_t. \qquad (4.2.3)$$

As with Model II, rational expectations solutions for the expected endogenous variables (up to $m(s+1)$ in number) may be obtained by leading (4.2.3) j-periods and substituting to obtain expressions for $\underset{t-1}{E}\, y_t,\ \underset{t-1}{E}\, y_{t+1},\ldots,\ \underset{t-1}{E}\, y_{t+s}$ in terms of observable variables. The typical lead equation for (4.2.3) may be obtained by using the chain rule:[2]

$$\underset{t-1}{E}\, \{I - A^{-1}\theta(F) - A^{-1}\Psi(L)\}y_{t+j-1} = \underset{t-1}{E}\, B(L)x_{t+j-1} + \underset{t-1}{E}\, \omega(L)u_{t+j-1}. \qquad j = 1,2,\ldots \qquad (4.2.4)$$

Inspection of (4.2.4) reveals that the Model III rational expectation for period $t+j$ depends upon rational expectations of endogenous variables in both later $(t+j+1,\ t+j+2,\ldots)$ and earlier $(t+j-1,\ t+j-2,\ldots)$ periods. Furthermore, rational expectations for periods $t+j$: $j \leq r$ depend upon lagged endogenous variables.[3] Therefore, there is simultaneous feedback through time among the expressions for $\underset{t-1}{E}\, y_t,\ldots,\underset{t-1}{E}\, y_{t+j}$ in (4.2.4), i.e.

$$\underset{t-1}{E}\, y_{t+j-1} \text{ depends upon } \underset{t-1}{E}\, y_{t+j}$$

and $\quad \underset{t-1}{E}\, y_{t+j}$ depends upon $\underset{t-1}{E}\, y_{t+j-1}$.

This result is not obtained in the Model II analog (3.2.4) where the relationship among rational expectations expressions is shown to be strictly recursive through time, i.e.

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[2]Wold, Chapter 3.

[3]This result follows from section 2.2, assumption i, expectations of variables in period $t-1$ and before equal actual values of those variables.

$\underset{t-1}{E}\, y_{t+j-1}$ depends upon $\underset{t-1}{E}\, y_{t+j}$,

but, $\underset{t-1}{E}\, y_{t+j}$ is independent of $\underset{t-1}{E}\, y_{t+j-1}$.

As a result, the substitution procedure required to obtain a RES for Model III may not be described, by the inversion of lead operator processes, the mechanism employed in the Model II analysis.

Even though the simultaneous substitutions involved in obtaining a Model III RES from (4.2.4) may not be characterized by convenient notation, the solution may be expressed in general terms as:

$$\begin{aligned}
\underset{t-1}{E} y_t &= R_1\{y_{t-r},\ldots,y_{t-1};x_{t-q},\ldots,x_{t-1};\underset{t-1}{E} x_t,\ldots,\underset{t-1}{E} x_{t+n-1};K_0,\ldots,K_{s-1}\} \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
\underset{t-1}{E} y_{t+s} &= R_{s+1}\{y_{t-r},\ldots,y_{t-1};x_{t-q},\ldots,x_{t-1};\underset{t-1}{E} x_t,\ldots,\underset{t-1}{E} x_{t+n-1};K_0,\ldots,K_{s-1}\}
\end{aligned} \tag{4.2.5}$$

where;

i) R_j, $j = 1,\ldots,s+1$, are linear functions which describe the RES for $\underset{t-1}{E}\, y_{t+j-1}$. These may be obtained by solving the system of lead equations in (4.2.4) for $\underset{t-1}{E} y_t,\ldots,\underset{t-1}{E}\, y_{t+s}$,

ii) K_k; $k = 0,1,\ldots,s-1$, are the values of distant future endogenous expectations which invariably appear in $R_1,\ldots,R_{s-1}$,

iii) the effect of $K_0,\ldots,K_{s-1}$ in the solution for $\underset{t-1}{E}\, y_{t+j-1}$ in (4.2.5), $j = 1,2,\ldots,s+1$, diminishes as n increases; hence, the solution is stable.[4]

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[4]An analogous assumption is imposed in the Model II analysis. The condition which insures that the effect of distant future values of endogenous expectations attenuate as the time horizon lengthens,

Two separate issues will be addressed in the analysis of this general RES for Model III structures. The first concerns the determination of the functional form for R_j, $j = 1,\ldots,s+1$, in (4.2.5) by analyzing the nature of the substitution procedure required to obtain a RES for Model III. The second is the specification of conditions which guarantee that the effect of distant future endogenous variables $K_0,\ldots,K_{s-1}$ in (4.2.5) declines as n increases, thereby insuring the stability of the solutions. Both of these objectives may be achieved by considering a simplified version of (4.2.1). Assume (4.2.1) is a single equation model; $A = 1$, $B(L) = \beta(L)$, $\beta(L)$ and x_t are $n \times 1$ vectors, all other variables are scalars, and $\varepsilon_t \sim N\{0,\Sigma_\varepsilon\}$.[5] Therefore (4.2.1) becomes:

$$y_t = \beta(L)'x_t + \Psi(L)y_t + \theta(F)_{t-1}y_t^e + \varepsilon_t \; . \qquad (4.2.6)$$

The functional form of the solutions described in (4.2.5), when the relevant theory follows (4.2.6), may be obtained by the following procedure.

i) Assume initially that $\underset{t-1}{E}\, y_{t+n}, \underset{t-1}{E}\, y_{t+n+1},\ldots,$ $\underset{t-1}{E}\, y_{t+n+s-1}$ are known. Denote these as $K_0,\ldots,K_{s-1}$ respectively.[6]

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is that the roots of the characteristic equation $|I - A^{-1}\theta(F)| = 0$ lie outside the unit circle.

[5] $\Psi(L)$ is a polynomial of order r in the lag operator L. $\theta(F)$ is a polynomial of order s in the lead operator F.

[6] Models with s lead endogenous expectations require s of these assumptions.

ii) Construct a system of n-equations in n-"unknowns," $\underset{t-1}{E} y_t, \ldots, \underset{t-1}{E} y_{t+n-1}$, by making j-period forecasts outlined in (4.2.4) where, j = 1,...,n.

iii) Solve this system of equations by inverting the n^{th}-order coefficient matrix for this system.

When the relevant theory is (4.2.6), the particular system of n-equations obtained by leading (4.2.3) may be described as:

$$\underset{t-1}{E} \{1 - \theta(F)\} y_{t+j-1} - \sum_{i=1}^{j<r} \Psi_i \underset{t-1}{E} y_{t+j-i} = \underset{t-1}{E} \beta(L)' x_{t+j-1} + \sum_{i=j\leq r}^{r} \Psi_i y_{t+j-i-1}$$

for j = 1,...,r;

$$\underset{t-1}{E} \{1 - \theta(F) - \Psi(L)\} y_{t+j-1} = \underset{t-1}{E} B(L)' x_{t+j-1}$$

for j = r+1,...,n-s,

$$\underset{t-1}{E} \{1-\theta_0 - \Psi(L)\} y_{t+j-1} = \underset{t-1}{E} B(L)' x_{t+j-1} + \sum_{i=n-j+1}^{s} \theta_i K_{i-(n-j+1)}$$

for j = n-(s-1),...,n.

The j^{th} equation j = 1,...,n is obtained by the analysis of the j-period forecast from (4.24). This system may be expressed in matrix notation as:

$$\{{}_{(r,s)}A^{(n)}\} Z = b + c \qquad (4.2.7)$$

where;

i) r and s refer to the order of $\Psi(L)$ and $\theta(F)$ respectively,

ii) $\{_{(r,s)}A^{(n)}\}$ is an $n \times n$ "band" matrix which may be described:

$$
\begin{aligned}
{}_{(r,s)}A^{(n)}_{i,j} &= 0 && \text{for } i-j > r \\
&= -\Psi_g && \text{for } i-j = g;\ g = 1,2,\ldots,r \\
&= 1-\theta_o && \text{for } i-j = 0 \\
&= -\theta_h && \text{for } i-j = h;\ h = -1,-2,\ldots,-s \\
&= 0 && \text{for } i-j < s,
\end{aligned}
$$

iii) Z is an $n \times 1$ vector of rational expectations of endogenous variables for period t through period $t+n-1$,

iv) b is an $n \times 1$ vector of predetermined variables, and future expected exogenous variables where;

$$
\begin{aligned}
b_i &= \underset{t-1}{E}\ \beta(L)'x_{t+i-1} + \sum_{g=i}^{r} \Psi_g y_{t+i-g+1} && \text{for } i = 1,2,\ldots,r \\
&= \underset{t-1}{E}\ \beta(L)x_{t+i-1} && \text{for } i = r+1,\ldots,n-s \\
&= \underset{t-1}{E}\ \beta(L)x_{t+i-1} && \text{for } i = n-(s-1),\ldots,n.
\end{aligned}
$$

v) c is an $n \times 1$ vector of distant future endogenous expectations where;

$$
\begin{aligned}
c_i &= 0 && \text{for } i = 1,2,\ldots,n-s \\
&= \sum_{h=n-i+1}^{s} \theta_h K_{h-(n-i+1)} && \text{for } i = n-(s-1),\ldots,n.
\end{aligned}
$$

Following the procedure for obtaining a Model III RES outlined above, the coefficient matrix from (4.2.7) is inverted to obtain:

$$Z = \{{}_{(r,s)}A^{(n)}\}^{-1}(b + c)$$

$$= \left\{\frac{\text{adj}\{{}_{(r,s)}A^{(n)}\}}{|{}_{(r,s)}A^{(n)}|}\right\}(b + c)$$

where;

i) $\text{adj}\{{}_{(r,s)}A^{(n)}\}$ is the adjoint matrix of $\{{}_{(r,s)}A^{(n)}\}$,

ii) $|{}_{(r,s)}A^{(n)}|$ is the determinant of the coefficient matrix.

Letting $\alpha_{i,j}^{(n-1)}$ be the cofactor of the i^{th} row and j^{th} column of the matrix $\{{}_{(r,s)}A^{(n)}\}$, the adjoint matrix is the transpose of the matrix of cofactors:[7]

$$\{\text{adj}\ {}_{(r,s)}A^{(n)}\} = \begin{Bmatrix} \alpha_{1,n}^{(n-1)} & \cdots & \alpha_{n,1}^{(n-1)} \\ \vdots & & \vdots \\ \alpha_{1,n}^{(n-1)} & \cdots & \alpha_{n,n}^{(n-1)} \end{Bmatrix}$$

Therefore, the solutions for the first (s+1) elements of the vector Z in (4.2.7) may be expressed:

$$Z_j = \frac{\sum_{i=1}^{n} \alpha_{i,j}^{(n-1)}(b_i + c_i)}{|{}_{(r,s)}A^{(n)}|} \qquad j = 1,\ldots,s+1. \tag{4.2.8}$$

---

[7]The cofactor, $\alpha_{i,j}^{(n-1)}$, is the determinant of the $(n-1)^{st}$-order minor obtained by deleting the i^{th} row and j^{th} column from $\{{}_{r,s}A^{(n)}\}$, multiplied by $(-1)^{i+j}$.

Hence,(4.2.8) describes the particular functional form of R_j, $j = 1,\ldots,s+1$, in (4.2.5) when the relevant theory is (4.2.6).

The isolation of stability conditions for this particular solution may be accomplished by considering only the portion of R_j in (4.2.5) which contains K_k, $k = 0,1,\ldots,s-1$. This portion, denoted Q_j, may be expressed by multiplying the last s columns of A^{-1} by the elements of c to obtain:

$$Q_j = \sum_{i=n-(s-1)}^{n} \sum_{h=n-i+1}^{s} \frac{\alpha_{i,j}^{(n-1)} \theta_h K_{h-(n-i+1)}}{|{}_{(r,s)}A^{(n)}|} \tag{4.2.9}$$

Summing over i and h yields:

$$\begin{aligned} Q_j = {} & |{}_{(r,s)}A^{(n)}|^{-1} \alpha_{n-(s-1),j}^{(n-1)} \theta_s K_o + \\ & |{}_{(r,s)}A^{(n)}|^{-1} \alpha_{n-(s-2),j}^{(n-1)} \{\theta_{s-1} K_o + \theta_s K_1\} + \\ & \quad \vdots \qquad\qquad\qquad \vdots \\ & |{}_{(r,s)}A^{(n)}|^{-1} \alpha_{n-1,j}^{(n-1)} \{\theta_2 K_o + \ldots + \theta_s K_{s-2}\} + \\ & |{}_{(r,s)}A^{(n)}|^{-1} \alpha_{n,j}^{(n-1)} \{\theta_1 K_o + \ldots + \theta_s K_{s-1}\}, \quad j = 1,2,\ldots,s+1. \end{aligned}$$

Combining terms in the K_k, $k = 0,1,\ldots,s-1$, yields:

$$\begin{aligned} Q_j = {} & |{}_{(r,s)}A^{(n)}|^{-1} \sum_{h=1}^{s} \theta_h \alpha_{n-(h-1),j}^{(n-1)} K_o + \\ & |{}_{(r,s)}A^{(n)}|^{-1} \sum_{h=2}^{s} \theta_h \alpha_{n-(h-2),j}^{(n-1)} K_1 + \\ & \quad \vdots \qquad\qquad\qquad \vdots \\ & |{}_{(r,s)}A^{(n)}|^{-1} \sum_{h=s-1}^{s} \theta_h \alpha_{n-(h-(s-1)),j}^{(n-1)} K_{s-2} + \\ & |{}_{(r,s)}A^{(n)}|^{-1} \theta_s \alpha_{n,j}^{(n-1)} K_{s-1} \qquad j = 1,2,\ldots,s+1 . \end{aligned} \tag{4.2.10}$$

Hence, from (4.2.10), the coefficients on the distant future values $K_0,\ldots,K_{s-1}$, in the RES for Z_j ($\underset{t-1}{E}\, y_{t+j-1}$), are ratios with the determinant of $\{{}_{(r,s)}A^{(n)}\}$ being the common denominator. The weighted sums of cofactors, which comprise the numerators of these coefficients, may be expressed as single determinants for each K_k. For example, the numerator of the coefficient on K_0 in (4.2.10) equals the determinant of the matrix formed by deleting the j^{th} column of $\{{}_{(r,s}A^{(n)}\}$ and augmenting it with a new n^{th} column,

$$(0,\ldots,0\ \theta_s \theta_{s-1},\ldots,\theta_1)'\ .$$

This new determinant must be weighted by $(-1)^{n+j}$ to account for the sign carried by the sum of cofactors.[8] The numerators of the remaining K_k, $k = 1,\ldots,s-1$, equal determinants which differ only with respect to the augmented column which becomes

$$(0,\ldots,0\ \theta_s,\ldots,\theta_{k+1})' \quad \text{for } K_k\ .$$

These newly formed determinants may, in turn, all be expressed as cofactors of a common matrix: $\{{}_{(r,s)}A^{(n+s)}\}$. This is illustrated by the following system which links the numerators of the coefficients on K_k from (4.2.10) to cofactors of $\{{}_{(r,s)}A^{(n+s)}\}$.[9]

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[8]The sign must correspond to that carried by the $(n,j)^{th}$ cofactor which is the first element in the weighted sum of cofactors for the coefficients on all K_k.

[9]The distinction between cofactors from $\{{}_{(r,s)}A^{(n)}\}$ and $\{{}_{(r,s)}A^{(n+s)}\}$ will be emphasized by letting $\rho_{i,j}$ denote the i,j^{th} cofactor from $\{{}_{(r,s)}A^{(n+s)}\}$.

when

$$k = 0;\ \sum_{h=1}^{s} \theta_h \alpha^{(n-1)}_{n-(h-1),j} = (-1)^{o} \rho^{(n)}_{(n+1,j)\cdot(n+2,n+2)\cdots\ (n+s,n+s)},$$

$$k = 1;\ \sum_{h=2}^{s} \theta_h \alpha^{(n-1)}_{n-(h-2),j} = (-1)^{1} \rho^{(n)}_{(n+1,j)\cdot(n+2,n+1)\cdot(n+3,n+3)\cdots\ (n+s,n+s)},$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$k = s-2;\ \sum_{h=s-1}^{s} \theta_h \alpha^{(n-1)}_{n-(h-(s-1)),j} = (-1)^{s-2} \rho^{(n)}_{(n+1,j)\cdot(n+2,n+1)\cdots\ (n+s-1,n+s-2)\ (n+s,n+s)},$$

$$k = s-1;\ \theta_s \alpha^{n-1}_{n,j} = (-1)^{s-1} \rho^{(n)}_{(n+1,j)\cdot(n+2,n+1)\cdots\ (n+s,n+s-1)}.$$

The subscript on $\rho^{(n)}$ denotes the s pairs of rows and columns deleted from $\{_{(r,s)}A^{(n+s)}\}$ to form the particular cofactor (n^{th} order determinant) that equals the weighted sum of cofactors ($(n-1)^{st}$ order determinants from $\{_{(r,s)}A^{(n)}\}$) which constitute the numerators of the coefficients on K_k, $k = 0,\ldots,s-1$ in (4.2.10). The sign $(-1)^k$ is necessary since the sign included in the cofactor,

$$\rho^{(n)}_{(n+1,j)(n+2,n+1)\cdots(n+k+1,n+k)(n+k+2,n+k+2)\cdots(n+s,n+s)},\ k = 0,1,\ldots,s-1,$$

corresponds to $(-1)^{n+j}$ only when k is even.[10]

Turning to the denominators of the coefficients in (4.2.10), the determinant of $\{_{(r,s)}A^{(n)}\}$ may also be expressed as a cofactor of $\{_{(r,s)}A^{(n+s)}\}$:

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[10]By definition, all cofactors are weighted by $(-1)^v$; where v = the sum of the numbers of all the deleted rows and columns.

$$|{}_{(r,s)}A^{(n)}| = \rho^{(n)}_{(n+1,n+1)\cdots(n+s,n+s)}.$$

Therefore, by isolating the coefficients on the distant future endogenous expectations in the RES for $Z_j\,(\underset{t-1}{E}\,y_{t+j-1})$, it is clear that the effect of K_k, $k = 0,\ldots,s-1$, in (4.2.8) dissipates iff:

$$\lim_{n\to\infty}(-1)^k \frac{\rho^{(n)}_{(n+1,j)(n+2,n+1)\cdots(n+k+1,n+k)(n+k+2,n+k+2)\cdots(n+s,n+s)}}{\rho^{n}_{(n+1,n+1)\cdots(n+s,n+s)}} = 0 \qquad (4.2.11)$$

for $k = 0,1,\ldots,s-1$

$j = 1,2,\ldots,s+1$

The stability condition (4.2.11) is expressed in terms of the ratios of n^{th} order determinants. These are determinants of matrices which have $(n-1)$ common columns. This fact is useful in expressing the stability conditions in terms of structural parameters for certain values of r and s (shown in the example in 4.4). However, no restrictions which are necessary to satisfy (4.2.11) are apparent when r and s are arbitrary.

An alternative approach to simplifying (4.2.11) is to write the determinants as the infinite products of their latent roots. The stability condition, (4.2.11), becomes:

$$(-1)^k \lim_{n\to\infty} \frac{\prod_{i=1}^{n} \lambda_{ijk}}{\prod_{i=1}^{n} \lambda_i} = 0 \qquad (4.2.12)$$

where;

i) λ_{ijk} is the i^{th} latent root of the $n \times n$ matrix obtained by deleting the following s pairs of rows and columns from $\{_{(r,s)}A^{(n+s)}\}$:

$(n+1,j)\cdot(n+2,n+1)\cdots\cdot(n+k-1,n+k)(n+k+2,n+k+2)$

$(n+s,n+s)$,

ii) λ_i is the i^{th} latent root of the $n \times n$ matrix $\{_{r,s}A^{(n)}\}$ obtained by deleting $(n+1,n+1)\cdots\cdot(n+s,n+s)$ from $\{_{(r,s)}A^{(n+s)}\}$.

Since the matrices which generate λ_{ijk} and λ_i have n-1 common columns, the relationship between λ_{ijk} and λ_i may be exploited to simplify (4.2.12) for certain values of r and s. This idea is pursued in 4.4 for $r = s = 1$. However, no general simplification may be achieved unless knowledge of the root formulas for arbitrary r and s is obtained.

The RES of (4.2.8) may now be expressed in terms of observable variables by invoking the stability conditions (4.2.11) and adopting an assumption about how expectations of exogenous variables are formed, thereby yielding :

$$Z_j = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\alpha_{i,j}^{(n-1)}\hat{b}_i'}{|_{(r,s)}A^{(n)}|} \qquad j = 1,2,\ldots,s+1 \qquad (4.2.13)$$

where ;

i) the expectations in b_i have been replaced by the ℓ-period forecasts of (3.27) generated by assuming the exogenous variables follow a stable stochastic process as in (2.2.6); hence b_i becomes $\hat{b}_i$,

ii) the stability conditions insure that the effect of distant future endogenous expectations is negligible when n is sufficiently large; hence,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\alpha_{i,j}^{(n-1)} c_i}{|{}_{r,s}A^{(n)}|} = 0 .$$

Consequently, the application of the REH to (4.2.6) yields an expression which depends upon all the predetermined variables in the original structure, plus any additional lagged exogenous variables which are needed to forecast the exogenous variables.

This RES compares quite closely with that obtained in Chapter III in regard to the specification of relevent variables. The complexity of the coefficients stems from the complicated substitution procedure required to eliminate future expectations of endogenous variables from the Model III RES.

4.3 An Observable Reduced Form for Model III

Following the procedure employed in the analysis of both Model I and Model II, the observable reduced form for Model III is obtained by substituting the RES, (4.2.13), into the original reduced form (4.2.6) to obtain:

$$y_t = \beta(L)'x_t + \Psi(L)y_t + \theta(F)Z_1 + \varepsilon_t \qquad (4.3.1)$$

where Z_1 corresponds to the RES for ${}_{t-1}E\, y_t$; therefore:

$$\theta(F)Z_1 = \theta_0 Z_1 + \ldots + \theta_s Z_{s+1} .$$

This observable reduced form may be simplified by considering:

$$\theta(F)Z_1 = \sum_{j=1}^{s+1} \theta_{j-1} \sum_{i=1}^{n} \frac{\alpha_{i,j}^{(n-1)} \hat{b}_i'}{|_{r,s}A^{(n)}|} \qquad \text{n-large}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{s+1} \frac{\theta_{j-1}\alpha_{i,j}^{(n-1)} \hat{b}_i'}{|_{r,s}A^{(n)}|} \quad .$$

Combining terms in $\hat{b}_i'$ yields:

$$\theta(F)Z_1 = |_{(r,s)}A^{(n)}|^{-1} \sum_{i=1}^{n} \{\theta_0\alpha_{i,1}^{(n-1)} + \ldots + \theta_s\alpha_{i,s+1}^{(n-1)}\}\hat{b}_i' \qquad (4.3.2)$$

Analogous to the procedure in section (4.2), the linear combinations of cofactors, which form the coefficients on $\hat{b}_i'$ in (4.3.2) may be expressed as single determinants for each $\hat{b}_i'$. The coefficient, $\{\theta_0\alpha_{i,1}^{(n-1)} + \ldots + \theta_s\alpha_{i,s+1}^{(n-1)}\}$, equals the determinant of the matrix obtained by replacing the i^{th} row of $\{_{(r,s)}A^{(n)}\}$ with

$$\{\theta_0, \theta_1, \ldots, \theta_s, 0\ 0\ \ldots\ 0\}.$$

For example, assume $i = 1$, then:

Assume $i = 2$, then:

$$\theta_o\alpha_{2,1}^{(n-1)} + \ldots + \theta_s\alpha_{2,s+1}^{(n-1)} = \begin{vmatrix} (1-\theta_o) & -\theta_1 & \ldots & -\theta_s & 0 & \ldots & 0 \\ \theta_o & \theta_1 & \ldots & \theta_s & 0 & \ldots & 0 \\ -\Psi_2 & -\Psi_1 & & & & & \\ \vdots & \vdots & & & & & \\ -\Psi_r & -\Psi_{r-1} & \{{}_{(r,s)}A^{(n-2)}\} & & & & \\ 0 & -\Psi_r & & & & & \\ \vdots & 0 & & & & & \\ & \vdots & & & & & \\ 0 & 0 & & & & & \end{vmatrix}$$

$$= \begin{vmatrix} (1-\theta_o) & -\theta_1 & \ldots & -\theta_s & 0 & \ldots & 0 \\ -\theta_o & -\theta_1 & \ldots & -\theta_s & 0 & \ldots & 0 \\ -\Psi_2 & -\Psi_1 & & & & & \\ \vdots & \vdots & & & & & \\ -\Psi_r & -\Psi_{r-1} & \{{}_{(r,s)}A^{(n-2)}\} & & & & \\ 0 & -\Psi_r & & & & & \\ \vdots & 0 & & & & & \\ & \vdots & & & & & \\ 0 & 0 & & & & & \end{vmatrix}$$

Expanding down the first column, this determinant may be expressed,[11]

$$+(1 - \theta_o)\alpha_{2,1}^{(n-1)} + \theta_o\alpha_{2,1}^{(n-1)} = \alpha_{2,1}^{(n-1)} .$$

Hence,

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[11]The elements of the Laplace expansion of the 3rd through nth terms in the first column equal zero since they contain determinants of singular matrices.

$$\frac{\theta_o \alpha_{2,1}^{(n-1)} + \ldots + \theta_s \alpha_{2,s+1}^{(n-1)}}{|_{(r,s)}A^{(n)}|^{-1}} = \frac{\alpha_{2,1}^{(n-1)}}{|_{(r,s)}A^{(n)}|} .$$

Assume $i = 3$, then:

$$\theta_o \alpha_{3,1}^{(n-1)} + \ldots + \theta_s \alpha_{3,s+1}^{n-1} = - \begin{vmatrix} (1-\theta_o) & -\theta_1 & -\theta_2 & \cdots & -\theta_s & & 0 \cdots\cdots 0 \\ -\Psi_1 & (1-\theta_o) & -\theta_1 & \cdots & -\theta_{s-1} & -\theta_s & 0 \cdots 0 \\ -\theta_o & -\theta_1 & -\theta_2 & \cdots & -\theta_s & & 0 \cdots\cdots 0 \\ -\Psi_3 & -\Psi_2 & -\Psi_1 & & & & \\ \vdots & \vdots & \vdots & & & & \\ -\Psi_r & -\Psi_{r-1} & -\Psi_{r-2} & & \{_{(r,s)}A^{(n-3)}\} & & \\ 0 & -\Psi_r & -\Psi_{r-1} & & & & \\ \vdots & 0 & -\Psi_r & & & & \\ \vdots & \vdots & 0 & & & & \\ \vdots & \vdots & \vdots & & & & \\ 0 & 0 & 0 & & & & \end{vmatrix}$$

Interchanging the second and third row yields:

$$\theta_o\alpha_{3,1}^{(n-1)} +\ldots+\theta_s\alpha_{3,s+1}^{(n-1)} = \begin{vmatrix} (1-\theta_o) & -\theta_1 & \cdots\cdots\cdots\cdots & \theta_s & 0 \cdots\cdots & 0 \\ -\theta_o & -\theta_1 & -\theta_2 \cdots & -\theta_s & 0 \cdots\cdots & 0 \\ -\Psi_1 & (1-\theta_o) & -\theta_1 \cdots & -\theta_{s-1}-\theta_s & 0 \cdots & 0 \\ -\Psi_3 & -\Psi_2 & -\Psi_1 & & & \\ \vdots & \vdots & \vdots & \{{}_{(r,s)}A^{(n-s)}\} & & \\ -\Psi_r & -\Psi_{r-1} & -\Psi_{r-2} & & & \\ 0 & -\Psi_r & -\Psi_{r-1} & & & \\ \vdots & 0 & -\Psi_r & & & \\ \vdots & \vdots & 0 & & & \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & & & \end{vmatrix}$$

$$\ddagger \quad \frac{(1-\theta_o)\alpha_{3,1}^{(n-1)} + \theta_o\alpha_{3,1}^{(n-1)}}{|{}_{(r,s)}A^{(n)}|} = \frac{\alpha_{3,1}^{(n-1)}}{|{}_{(r,s)}A^{(n)}|}$$

Therefore, continuing the analysis would yield:

$$|{}_{(r,s)}A^{(n)}|^{-1}\{\theta_o\alpha_{i,1}^{(n-1)} +\ldots+ \theta_s^{(n-1}\alpha_{i,s+1}\} = |{}_{(r,s)}A^{(n)}|^{-1}\alpha_{i,1}^{(n-1)}$$

for $i = 2,\ldots,n.$

Hence;[12]

$$\theta(F)Z_1 = -\hat{b}_1' + \sum_{i=1}^{n} \frac{\alpha_{i,1}^{(n-1)}}{|{}_{r,s}A^n|}\hat{b}_i'. \qquad (4.3.3)$$

---

[12]This type of simplification allows the conditions for

$$\lim_{n\to\infty} \theta(F)Q_1 = 0$$

The observable reduced form may now be re-expressed by substituting (4.3.3) into (4.3.1) to obtain:

$$y_t = \beta(L)'x_t - \underset{t-1}{E}\ \beta(L)'\hat{x}_t(0)$$

$$+ \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\alpha_{i,1}^{(n-1)}}{|{}_{(r,s)}A^{(n)}|}\{\beta(L)'\hat{x}_t(i-1) + \sum_{\substack{j=i \\ i\leq r}}^{r} \Psi_j y_{t-(i-j+1)}\}$$

$$+ \varepsilon_t. \qquad (4.3.4)$$

---

to be stated in simpler terms than considering each Q_j separately as in section 4.2. Consider the coefficient on K_0 in $\theta(F)Q_1$:

$$\{\rho^{(n)}_{(n+1,n+1)\cdots(n+s,n+s)}\}^{-1} \sum_{j=1}^{s+1} \theta_{j-1}\rho^{n}_{(n+1,j)(n+2,n+2)\cdots(n+s,n+s)}$$

But this equals the determinant of the matrix formed by replacing the first row of the $(n+1)^{st}$ order minor, obtained by eliminating $(n+2,1)\cdot(n+3,n+3)\cdots(n+s,n+s)$ from $\{{}_{(r,s)}A^{(n+s)}\}$, with $\{\theta_0\ \theta_1\ \cdots\ \theta_s\ 0\cdots 0\}$. Expanding down the first column as in the present analysis this new determinant may be evaluated as:

$$\{\rho^{(n)}_{(n+1,n+1)\cdots(n+s,n+s)}\}^{-1}\{-\theta_0-(1-\theta_0)\}\rho^{(n)}_{(n+1,1)\cdot(n+2,n+2)\cdots(n+s,n+s)}$$

$$= \frac{\rho^{(n)}_{(n+1,1)\cdot(n+2,n+2)\cdots(n+s,n+s)}}{\rho^{(n)}_{(n+1,n+1)\cdots(n+s,n+s)}}.$$

Similarly all other coefficients of K_k could be simplified to obtain

$$\theta(F)Q_1 =$$

$$\sum_{k=0}^{s+1}(-1)^k \frac{\rho^{(n)}_{(n+1,1)\cdot(n+2)(n+1)\cdots(n+k-1)(n+k)(n+k+2,n+k+2)\cdots(n+s,n+s)}}{\rho^{n}_{(n+1,n+1)\cdots(n+s,n+s)}}$$

Hence the stability of the weighted sum of rational expectations solutions which appear in the observable reduced form is insured if the limit of the above expression approaches zero for all values of $k = 0,\ldots,s-1$.

This observable reduced form differs very little from that obtained for Model I and Model II in terms of specifying the observable variables which must appear. The complexity of the coefficients stem from the intricate substitution procedure required to obtain a Model III RES. However, since the adjoints $(\alpha_{i,1}^{(n-1)})$ are determinants of $(n-1)^{st}$ order minors from $\{_{(r,s)}A^n\}$, an investigation of the relationship between $\alpha_{i,1}^{(n-1)}$ and $|_{(r,s)}A^{(n)}|$ may simplify (4.3.4) for certain values of r and s. This is demonstrated in the following section.

4.4 An Example of a Model III Structure

A simple example exhibiting the characteristics of a Model III structure illustrates the steps leading to a Model III RES and resulting observable reduced form. Consider the single equation model:

$$y_t = \beta(L)'x_t + \Psi Ly_t + \theta F_{t-1}y_{t+1}^e + \varepsilon_t \quad t = 1,2,\ldots,T \qquad (4.4.1)$$

where from (4.1.1),

i) y_t is a scalar,

ii) $\beta(L)'$ is a $(1 \times n)$ vector of polynomials in the lag operator L,

iii) x_t is a $n \times 1$ vector of exogenous variables,

iv) $\Psi(L) = \Psi L$ is a scalar,

v) $\theta(F) = \theta F$ is a scalar,

vi) $\varepsilon_t \sim N\{0, \Sigma_\varepsilon\}$.

The structure (4.4.1) describes the "relevant economic theory" for the variable y_t. Therefore, following the procedure outlined in 4.2, the application of the REH to (4.4.1) yields an expression analogous to (4.2.4):

$$\underset{t-1}{E}\ \{1 - \theta F - \Psi L\}y_t = \underset{t-1}{E}\ \beta(L)'x_t \ . \qquad (4.4.2)$$

The RES for (4.4.1) is obtained by leading (4.4.2) and following the solution procedure suggested in section 4.2, hence:

$$\underset{t-1}{E}\ y_{t+j-1} = Z_j = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \alpha_{i,j}^{(n-1)}\ \hat{b}_i}{|{}_{(1,1)}A^{(n)}|},\ j = 1,2,\ldots \qquad (4.4.3)$$

where, in this simple example:

i) $\{{}_{(1,1)}A^{(n)}\}$ is a tri-diagonal matrix with

$$\begin{aligned} {}_{(1,1)}A^{(n)}{}_{i,j} &= 1 && \text{for} \quad i = j \\ &= -\theta && \text{for} \quad i-j = -1 \\ &= -\Psi && \text{for} \quad i-j = +1 \\ &= 0 && \text{for} \quad \text{elsewhere,} \end{aligned}$$

ii) $$\begin{aligned} \hat{b}_i' &= \beta(L)'\hat{x}_t(0) + y_{t-1} && \text{for} \quad i = 1 \\ &= \beta(L)'\hat{x}_t(i-1) && \text{for} \quad i = 2,\ldots,n. \end{aligned}$$

The stability conditions for the RES (4.4.3) are obtained by applying (4.2.11) to this example; hence the RES is stable iff:

$$\lim_{n\to\infty} \frac{\rho^{(n)}_{(n+1,2)}}{\rho^{(n)}_{(n+1,n+1)}} = 0; \text{ since } \ s = 1.$$

The numerator of this expression is the determinant of the matrix formed by eliminating the $(n+1)^{st}$ row and 2^{nd} column of $\{{}_{(1,1)}A^{(n+1)}\}$. Since $\{{}_{(1,1)}A^{(n+1)}\}$ is tridaigonal, any minor formed by eliminating the last row will be a lower trianglar matrix. Hence, the cofactor is expressed as the product of the diagonal elements:

$$\rho^{(n)}_{(n+1,2)} = (-1)^{n+3}(-\theta)^{n-1} = (-1)^n\theta^{n-1}.$$

Focusing on the denominator of this expression, the latent roots of the minor which yields $\rho^{(n)}_{(n+1,n+1)}$: $\{_{(1,1)}A^{(n)}\}$, may be expressed:[13]

$$\lambda_i = (1 + 2\sqrt{\Psi\theta}\ \text{Cos}\ \frac{i\pi}{n+1}) \qquad i = 1,2,\ldots,n.$$

Hence $\rho^{(n)}_{(n+1,n+1)} = (-1)^n \prod_{i=1}^{n} (1 + 2\sqrt{\Psi\theta}\ \text{Cos}\ \frac{i\pi}{n+1})$. The stability condition may now be explicitly stated in terms of the structural parameters as:

$$\lim_{n\to\infty} \frac{\theta^{n-1}}{\prod_{i=1}^{n}\{1 + 2\sqrt{\Psi\theta}\ \text{Cos}\ \frac{i\pi}{n+1}\}} = 0\ .$$

Clearly, $\lim_{n\to\infty} \theta^{n-1}$ converges when $|\theta| < 1$. The infinite product limit in the denominator may be examined by noting:

i) $$\text{Cos}\ \frac{i\pi}{n+1} \begin{cases} \simeq 1 & \text{for } i\text{-small} \\ \simeq 0 & \text{for } i \simeq \frac{n+1}{2} \\ \simeq -1 & \text{for } i\text{-large} \end{cases},$$

ii) $\text{Cos}\ \theta = -\text{Cos}(\pi - \theta) \qquad 0 \leq \theta \leq \pi.$

Therefore, if n is even;

$$\begin{aligned}\text{Cos}(\frac{i\pi}{n+1}) &= -\text{Cos}(\pi - \frac{i\pi}{n+1}) \\ &= -\text{Cos}(\pi[\frac{n+1-i}{n+1}]) \qquad i = 1,2,\ldots,n/2,\end{aligned}$$

if n is odd;

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[13] S.J. Hammarling, (1970), pp. 153-154.

$$\text{Cos}\,\frac{i\pi}{n+1} = -\,\text{Cos}(\pi[\frac{n+1-i}{n+1}]) \qquad i = 1,2,\ldots,\frac{n-1}{2}$$

$$= 0 \qquad i = \frac{n+1}{2}\ .$$

Therefore;

$$\lim_{n\to\infty} \rho^{(n)}_{(n+1,n+1)} = \prod_{i=1}^{n} \{1 + 2\sqrt{\theta\Psi}\ \text{Cos}\,\frac{i\pi}{n+1}\}$$

$$= \prod_{i=1}^{m} \{1 - 4\theta\Psi\ \text{Cos}^2\,\frac{i\pi}{n+1}\},$$

$m = n/2$ for n-even

$m = (n-1)/2$ for n-odd.

Hence, $\lim_{n\to\infty} \rho^{(n)}_{(n+1,n+1)}$ diverges to infinity when $\theta\Psi < 0$. The sufficient conditions for the stability of the RES for this example may now be stated:

$$|\theta| < 1,$$

$$\theta\Psi < 0\ .$$

Therefore, knowledge of the formulas for the infinite determinants of (4.2.11), as provided in this particular example, will allow the researcher to obtain sufficient conditions for the stability of the RES, which may be stated in terms of the original structural parameters.

The observable reduced form for (4.4.1) may be obtained from (4.3.4). Since $r = s = 1$;

$$y_t = \beta(L)'x_t - \beta(L)'\hat{x}_t(0)$$

$$+ \lim_{n\to\infty} \frac{\alpha_{1,1}^{(n-1)}\{\beta(L)'\hat{x}_t(0) + \Psi y_{t-1}\}}{|_{(1,1)}A^{(n)}|}$$

$$+ \lim_{n\to\infty} \sum_{i=2}^{n} \frac{\alpha_{i,1}^{(n-1)}\beta(L)'\hat{x}_t(i,1)}{|_{(1,1)}A^{(n)}|} .$$

However, since $\{_{(1,1)}A^{(n)}\}$ is tridiagonal;

$$\alpha_{i,1} = (-1)^{i+1}\theta^{i-1}|_{(1,1)}A^{(n-i)}|,$$

the observable reduced form becomes:

$$y_t = \beta(L)'\{x_t - \hat{x}_t(0)\}$$

$$+ \lim_{n\to\infty} \frac{|_{(1,1)}A^{(n-1)}|}{|_{(1,1)}A^{(n)}|}\{\beta(L)'\hat{x}_t(0) - \Psi y_{t-1}\}$$

$$+ \lim_{n\to\infty} \sum_{i=2}^{n}(-1)^{i+1}\theta^{i-1}\frac{|_{(1,1)}A^{(n-i)}|}{|_{(1,1)}A^{(n)}|}\beta(L)'\hat{x}_t(i-1)$$

$$+ \varepsilon_t . \qquad (4.4.4)$$

The analysis of this simple example of a Model III structure emphasizes two distinct characteristics of the application of the REH to models with both lead endogenous expectations and lagged endogenous variables. First, when expressions for the infinite determinants in (4.2.11) exist, the stability conditions may be stated in terms of the original structural parameters. Also, inspection of (4.4.4) reveals that, in the simple one lead-one lag case, the coefficients in the Model III observable reduced form may be simplified to obtain ratios which are different ordered determinants from identical band matrices.

4.5 An Outline of the Multivariate Extension of Model III Analysis

The procedure which generates a RES for the single equation model (4.2.6) may be extended to multi-equation structures with general disturbance assumptions as described by (4.2.1). The coefficient matrix $\{_{(r,s)}A^{(n)}\}$ becomes mn-dimensional since (4.2.1) contains up to $m\cdot(s+1)$ rational expectations which need to be replaced with expressions of observable variables. The RES is obtained by inverting this coefficient matrix and post-multiplying it by an mn-dimensional vector analogous to $\hat{b}'$ in (4.2.13). The elements in $\hat{b}'$ depend upon the nature of the relevant theory and assumptions about the processes which generate the exogenous variables.[14]

The stability conditions require that the coefficients on up to $s\cdot m$ values of distant future endogenous expectations (K_k in (4.2.5)) converge to zero for each of the $(s+1)m$ rational expectations solutions.

Finally, the RES may be substituted into (4.2.1) to obtain a reformulated reduced form which is free of unobservable variables.

4.6 Summary

Despite the complex substitution procedured required to obtain a RES for Model III, the resulting observable reduced form is quite similar to that obtained in Model II. The RES depends upon

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[14] If $\omega(L) \neq I$ in (4.4.1), b' will contain lagged disturbance terms.

stability conditions which insure that the coefficients on distant future endogenous variables converge to zero when the number of substitutions is large. In Model II, the restrictions required for this condition are stated in terms of the roots of a characteristic equation. The Model III analysis obtains this condition in terms of limits of latent roots of infinite order determinants.

A second similarity between the application of the REH to Model II and Model III structures is the dependence of the solutions on the process assumed to generate the exogenous variables. The RES for both models contains expectations of exogenous variables for all future periods. Hence, assumptions analogous to (2.2.6) are required to obtain reformulated, reduced forms, void of unobservable variables.

Finally, the observable reduced form for Model III is generally indistinguishable from that obtained in the analyses of Models I and II in regard to specifying the set of explanatory variables which must appear. However, the particular functional form of the coefficients is more complex than that obtained for either Model I or Model II. This complexity stems from the degree of simultaneity in the system of equations (4.2.4) which gives rise to the Model III RES.

CHAPTER V

THE ECONOMETRIC IMPLICATIONS OF MODELS WITH RATIONAL EXPECTATIONS

This chapter examines three major topics in the econometric analysis of models with rational expectations of endogenous variables. First, the identification of reduced form (RF) parameters from knowledge of those in the observable reduced form (ORF) is considered. Second, a method of estimating structural parameters is introduced. Finally, a procedure for testing the restrictions implied by the REH is developed.

5.1 The Approach

The point of departure for the econometric analysis will be the models analyzed in Chapters II, III, and IV. It will be assumed that the structural parameters of the models in question may be identified from knowledge of those in the corresponding reduced form (RF) expression.[1]

The analysis in previous chapters reveals that the application of the REH to these RF structures yields expressions for the expectations terms which are free of unobservable variables. These expressions may be substituted into the original RF to obtain an ORF. This ORF structure is advantageous for applied work since it

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[1]This is the standard notion of identification.

no longer contains unobservable variables. However, the implications of employing the particular ORF suggested by the REH as opposed to any alternative reformulation of the original RF structure (for example, that obtained by assuming adaptive expectations) cannot be realized without considering three econometric questions. First, under what conditions are the RF parameters identified relative to those in the ORF? Second, what estimation procedure is appropriate for a model which incorporates the REH? Third, is the nature of the ORF, obtained by following the procedure outlined above, conducive to a test of the REH? The following analysis reveals the econometric implications of the REH by responding to these three questions.

The first two questions are pursued in some detail for simple Model I structures in a paper by Wallis (1977).[2] The reference made to a test of the REH in Wallis' analysis is undetailed.[3] Consequently, the following analysis contains two primary contributions. First, it extends the identification and estimation analysis pioneered by Wallis to more general rational expectations models.[4] At the same time, some of Wallis' results are modified to account for general specifications. Second, it provides the details for a test based upon the particular functional form of the ORF obtained by following the theory of rational expectations.

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[2]Wallis considers models which do not contain lagged variables.

[3]Wallis, p. 25-26.

[4]Wallis makes no reference to structures which may be categorized as Model III.

5.2 Identification

When the parameters of the original RF can be obtained from knowledge of the ORF coefficients the application of the REH to models with expectations, generates useful results. In this case, when the structural parameters are also identified in the standard sense, from knowledge of RF coefficients, estimates of structural parameters may be obtained from those in the ORF.[5] Hence, by applying the REH, the researcher is able to perform tests of hypotheses regarding the structural parameters of models which originally contained unobservable expectations terms.

Generally, identification will depend upon the number of independent parameter estimates which are available from the ORF, relative to the number of original RF coefficients. The following analysis concentrates on Model I structures and outlines an approach for determining the identification conditions for models with lead expectations.

Consider the following Model I reduced form expression:

$$y_t = \pi_1(L)x_t + \pi_2(L)y_t + \pi_{3t-1}y_t^e + v_t, \qquad (5.2.1)$$

where, from (2.2.1);

i) $\pi_1(L) = A^{-1}B(L)$ - an $m \times n$ matrix of lag polynomials,

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[5]In some cases structural parameters may be identified from knowledge of ORF coefficients even when RF parameters are not. Wallis points out, p. 25, that it may be possible to "tradeoff" the over-identification of structural parameters from RF against the under-identification of RF parameters from the ORF. Nevertheless, the separate treatments of the two concepts of identification is significant since no general solution for this "tradeoff" procedure has yet been obtained.

ii) $\pi_2(L) = A^{-1}\Psi(L)$ - an $m \times m$ matrix of lag polynomials, recall $\Psi_0 \equiv \emptyset$

iii) $\pi_3 = A^{-1}\theta$ - an $m \times m$ matrix of reduced form coefficients,

iv) $v_t = A^{-1}\varepsilon_t$; $\varepsilon_t \sim N\{0,\Sigma_\varepsilon\}$. Σ_ε is unrestricted. The analysis is intended to accommodate any assumption about the distribution of error terms.

Following (2.3.1), the corresponding ORF obtained by applying the REH to (5.2.1) is:

$$\begin{aligned} y_t &= \pi_1(L)x_t + \pi_2(L)y_t \\ &\quad + \pi_3(I - \pi_3)^{-1}\{\pi_1^0\Gamma(L) + \pi_1'(L)\}x_{t-1} \\ &\quad + \pi_3(I - \pi_3)^{-1}\{\pi_2(L)y_t\} + v_t \end{aligned} \tag{5.2.2}$$

where;

i) $\pi_1^0 = A^{-1}B_0$ - an $m \times n$ matrix of RF coefficients containing all the coefficients on those exogenous variables in period t which occur in (5.2.1),

ii) $\pi_1'(L) = A^{-1}B'(L)$ - an $m \times n$ matrix of lag polynomials containing all the coefficients on lagged exogenous variables in (5.2.1),[6]

iii) $\hat{x}_t = \Gamma(L)x_{t-1}$ from (2.2.6) where $\Gamma(L)$ is an $n \times n$ diagonal matrix of lag polynomials. For convenience each of these polynomials is of order P.

$\therefore \Gamma_{jj}(L) = \gamma_{j1} + \gamma_{j2}L + \ldots + \gamma_{jp}L^{p-1}$.

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[6]Recall $B'(L) = B_1 + B_2L + \ldots + B_qL^{q-1}$ from (2.2.7).

Re-express (5.2.2) by combining all lagged terms, which enter (5.2.2) from the RES, with those appearing in (5.2.1) to obtain:

$$\begin{aligned} y_t &= \pi_1^0 x_t + \pi_2(L)y_t + \pi_3(I - \pi_3)^{-1}\pi_2(L)y_t \\ &\quad + \pi_3(I - \pi_3)^{-1}\, \pi_1^0 \Gamma(L)x_{t-1} \\ &\quad + \pi_1'(L)x_{t-1} + \pi_3(I - \pi_3)^{-1}\pi_1'(L)x_{t-1} + v_t \end{aligned}$$

$$\begin{aligned} y_t &= \pi_1^0 x_t + [I + \pi_3(I - \pi_3)^{-1}]\pi_2(L)y_t \\ &\quad + \pi_3(I - \pi_3)^{-1}\pi_1^0 \Gamma(L)x_{t-1} \\ &\quad + [I + \pi_3(I - \pi_3)^{-1}]\pi_1'(L)x_{t-1} + v_t \end{aligned}$$

$$\begin{aligned} y_t &= \pi_1^0 x_t + (I - \pi_3)^{-1}\pi_2(L)y_t \\ &\quad + \pi_3(I - \pi_3)^{-1}\pi_1^0 \Gamma(L)x_{t-1} \\ &\quad + (I - \pi_3)^{-1}\pi_1'(L)x_{t-1} + v_t\,. \end{aligned} \tag{5.2.3}$$

The analysis of (5.2.3) is facilitated by noting that each equation of a reduced form system contains all the predetermined variables in the system. Hence, the explanatory variables in each equation of (5.2.3) may be categorized;

n_1 ≡ the number of period t exogenous variables in each equation of (5.2.1). Without loss of generality, let these be the first n_1 of the n^{th} order vector x_t.

k_1 ≡ the total number of lagged variables in (5.2.1) which can be obtained by lagging members of n_1. Note, Max $k_1 = (n_1 \cdot q)$; $q = \text{Max } q_{ij}$ from 2.1 assumption iii.

k_2 ≡ the total number of lagged variables that do not have corresponding period t values in (5.2.1). Note, Max $k_2 = (n - n_1)q$.

k_3 ≡ the total number of lagged endogenous variables in (5.2.1). Note, Max $k_3 = m \cdot r$

$r = \text{Max } r_{ij}$ from 2.1 assumption v.

$\Gamma^*(L)$ an $n \times n$ diagonal matrix of polynomials in the lag operator L which contain the coefficients on only those lagged exogenous variables from the autoregressive process (2.2.6) which do not also appear in (5.2.1). Hence, the j^{th} diagonal element of $\Gamma^*(L)$ is:

$$\Gamma^*_{jj}(L) = \gamma^*_{j,1} + \gamma^*_{j,2}L + \ldots + \gamma^*_{j,p}L^{p-1}, \quad j = 1,\ldots,n_1$$

where;

$$\gamma^*_{j,g} = \begin{cases} \gamma_{j,g} & \text{when the } g^{th} \text{ lag on the } j^{th} \text{ exogenous variable does not appear in (5.2.1)}, g = 1,2,\ldots,p \\ 0 & \text{when the } g^{th} \text{ lag on the } j^{th} \text{ exogenous variables does appear in (5.2.1)} \end{cases}$$

$\Gamma^{**}(L) \equiv \Gamma(L) - \Gamma^*(L)$.[7]

Employing these assumptions, (5.2.3) may be written:

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[7]The question of locating the last $n - n_1$ diagonal elements in $\Gamma^*(L)$ or $\Gamma^{**}(L)$ is immaterial since the last $n - n_1$ columns of π_1^0 are null vectors.

$$y_t = \pi_1^0 x_t + (I - \pi_3)^{-1}\pi_2(L)y_t$$
$$+ \pi_3(I - \pi_3)^{-1}\pi_1^0\Gamma^*(L)x_{t-1}$$
$$+ \pi_3(I - \pi_3)^{-1}\pi_1^0\Gamma^{**}(L)x_{t-1}$$
$$+ (I - \pi_3)\pi_1'(L)x_{t-1} + v_t . \qquad (5.2.4)$$

Equation (5.2.4) illustrates that the $m \cdot n_1$ elements of π_1^0 are immediately obtainable from knowledge of the ORF coefficients. The parameters of the autoregressive processes, $\Gamma(L)$ from (2.2.6), are identified outside the system (5.2.4). The condition which insures that the remaining RF coefficients can be obtained from those in the ORF is that the individual elements of π_3 are identified from knowledge of:

i) π_1^0,
ii) $\Gamma^*(L)$,
iii) $\pi_3(I - \pi_3)^{-1}\pi_1^0\Gamma^*(L)$.

This may be demonstrated by equating the ORF coefficients of (5.2.4) with those obtained in the simple linear regression of y_t on the set of variables which appear in (5.2.4). Denoting the coefficients from this unrestricted version of (5.2.4) by $\varphi(L)$, define the

$m \times n$ matrix, $\varphi_0 \equiv \pi_1^0$,

the $m \times m$ matrix, $\varphi_1(L) \equiv (I - \pi_3)^{-1}\pi_2(L)$,

the $m \times n$ matrix, $\varphi_2(L) \equiv \pi_3(I - \pi_3)^{-1}\pi_1^0\Gamma^*(L)$,

the $m \times n$ matrix, $\varphi_3(L) = \pi_3(I - \pi_3)^{-1}\pi_1^0\Gamma^{**}(L) + (I - \pi_3)^{-1}\pi_1'(L)$.

Since π_3 is an $m \times m$ matrix of full rank;

$$\pi_2(L) = (I - \pi_3)\varphi_1(L) \quad \text{and}$$

$$\pi_1'(L) = (I - \pi_3)\,\varphi_3(L) - \pi_3(I - \pi_3)^{-1}\varphi_0\Gamma^{**}(L)\ .$$

Therefore, when the elements of π_3 may be obtained from φ_0, $\Gamma^*(L)$, and $\varphi_1(L)$; then $\pi_2(L)$ and $\pi_1'(L)$ may be expressed in terms of the φ_i and Γ. Hence, when the conditions for the identification of the elements of π_3 are satisfied, the $m(k_1 + k_2 + k_3)$ remaining in the ORF provide the information to solve for the $m(k_1 + k_2 + k_3)$ remaining unidentified parameters in $\pi_1'(L)$ and $\pi_2(L)$.[8] Consequently, the following analysis concentrates on the coefficients in the third term in (5.2.4):

$$\pi_3(I - \pi_3)^{-1}\pi_1^0\Gamma^*(L). \tag{5.2.5}$$

By construction, the last $n - n_1$ columns of π_1^0 are null vectors. The jj^{th} diagonal element of $\Gamma^*(L)$ will be non-zero if at least one lagged value of the j^{th} type of exogenous variable $j = 1,2,\ldots,n_1$ appears in the autoregressive process for the j^{th} variable but not as an explanatory variable in (5.3.1). Let n_2 be the number of null diagonal elements in the first n_1 rows and columns of $\Gamma^*(L)$. Then a necessary and sufficient condition for identifying the elements of π_3 from knowledge of ORF coefficients is:

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[8] A simple example which illustrates this point is contained in the Appendix; Example III.

$$\rho\{\pi_1^0\Gamma^*(L)\} = m. \qquad (5.2.6)$$

Hence, only when the rank of $\pi_1^0\Gamma^*(L)$ equals m will there be m^2 independent equations for the m^2 unknown elements of π_3.

But, $$\rho(\pi_1^0) \leq \min\{n_1, m\}$$

and $$\rho\left\{\Gamma^*(L)\begin{pmatrix} I_{n_1} & 0 \\ 0 & 0 \end{pmatrix}\right\} \leq n_1 - n_2{}^{9};$$

$$\therefore \quad \rho\{\pi_1^0\Gamma^*(L)\} \leq \min\{m, n_1 - n_2\}.$$

Clearly, a necessary condition for the identification of the elements of π_3, and hence, all RF parameters, from knowledge of ORF coefficients is:

$$n_1 - n_2 \geq m \ . \qquad (5.2.7)$$

Therefore, the condition for identification requires that the number of period t exogenous variables, for which at least one lagged value (lag $\leq$ P) does not appear in the original system, exceeds the number of equations. Clearly, the condition could be modified to account for the possibility that not all endogenous variables appear in expectation form. In this case (5.2.7) would be:

$$n_1 - n_2 \geq m_1$$

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[9] Only the rank of this submatrix is of concern since the last $n - n_1$ columns of π_1^0 are null vectors. Hence, independent rows and columns in the last $n - n_1$ rows and columns of $\Gamma^*(L)$ will not augment $\rho\{\pi_1^0\Gamma^*(L)\}$.

where, m_1 is the number of endogenous expectations in each equation.

This result generalizes Wallis' identification condition for models with lagged variables.[10] However, when lagged variables are considered, two conclusions reached by Wallis must be modified. First, his identification condition no longer applies to general models. Second, the nature of the autoregressive process does have a role in the identification analysis, contrary to Wallis' contention. The following analysis demonstrates these results.

In the simple models employed by Wallis, the condition for identification of RF coefficients from those in the ORF is that the number of exogenous variables exceed the number of expectations terms. This condition is a special case of (5.2.7). When no lagged terms exist, the elements of π_3 are the only terms which are not immediately identified from knowledge of the ORF coefficients. Also, $n_1 - n$, $\Gamma^*(L) = \Gamma(L)$ and $n_2 = 0$. Therefore;

$$\rho\{\pi_1^0\Gamma^*(L)\} \leq \min\{m, n\}.$$

Hence, Wallis' identification condition,

$$n \geq m, \tag{5.2.8}$$

is confirmed. However, expression (5.2.7) reveals that this statement is invalid when some of the explanatory variables in the original RF are lagged exogenous or lagged endogenous variables. Clearly, the ORF coefficients on lagged values provide no

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[10] Wallis, p. 25.

information for identifying the elements of π_3.[11]

An additional result obtained from the Wallis study must be modified in light of the present analysis. Wallis maintains that substituting the autoregressive processes for exogenous variables into the ORF (as opposed to merely calculating a single value for the optimum forecast) is of "no assistance" in identifying the elements of π_3.[12] This result is confirmed for Wallis' simple model by considering (5.2.8). The rank of $\Gamma^*(L)$ will be n, regardless of the orders of the individual autoregressive processes for the n exogenous variables. However, (5.2.7) reveals that n_2 corresponds to the number of current period exogenous variables which are generated by a particular set of lagged exogenous variables that also appear in the original RF equation. The previous analysis demonstrates that when autoregressive processes for these exogenous varialbes are substituted into the ORF, the form of the resulting coefficients is not conducive to identifying elements of π_3. Hence, the value of n_2 depends upon the particular lag structure of the autoregressive processes. As a result, when the model under investigation contains lagged variables, the condition for identification does depend upon the nature of the process generating the exogenous variables. However, when $m > n_1$, the additional

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[11]Example III in the Appendix offers an illustration of this fact.

[12]Wallis, p. 25.

information which may be gained by extending the order of the process assumed to generate the exogenous variables will never be great enough to identify the elements of π_3.

When the question of identifying RF parameters from those in the ORF is extended to models with lead expectations, the analysis is complicated by two factors. One stems from the increased number of expectations terms contained in models with lead expectations. As a result, the number of elements in π_3 increased from a maximum of m^2 to a maximum of $m^2\{s + 1\}$. Secondly, the ORF coefficients are much more complicated functions of the RF parameters. The ensuing analysis contains an outline of the general approach for determining the conditions for identification for Model II and Model III structures.

Initially, consider an unrestricted version of any Model II or Model III ORF as a function, linear in both parameters and variables, which contains the set of variables appearing in the ORF of the model under investigation. Hence, let:

$$y_t = \varphi(L)z_t + v_t \tag{5.2.9}$$

where;

i) $z_t = \{x_t \vdots y_{t-1}\}'$,

ii) $\varphi(L)$ is an $m \times n + m$ matrix whose elements are polynomials in the lag operator L.[13] The length

[13]The elements of $\varphi(L)$ are defined analogous to those of $B(L)$ in section 2.1, assumption ii.

of the lags will depend upon the lags in $\pi_1(L)$, $\pi_2(L)$, and $\Gamma(L)$,

iii) forecasts from $\hat{x}_t = \Gamma(L)x_{t-1}$ have been entered to eliminate all "future" values.

The general ORF structure obtained by an analysis of either Model II or Model III, illustrated in (3.3.1) and (4.3.4) respectively may be designated:

$$y_t = \pi(L)z_t + v_t \tag{5.2.10}$$

where;

i) z_t is defined in (5.2.9),

ii) $\pi(L)$ is an $m \times n + m$ matrix whose elements correspond to those in either (3.3.1) or (4.3.4) when expressions for the lead forecasts of $\hat{x}_t(\ell)$ have been substituted for the expected future values of exogenous variables.

Since (5.2.9) and (5.2.10) contain identical sets of explanatory variables, the conditions for identification may be examined by equating the coefficients from (5.2.9) with the functions of original RF parameters which constitute the coefficients of (5.2.10). The resulting system is a functional relation from the elements of $\pi(L)$ to the elements of $\varphi(L)$. If the rank of the Jacobian matrix for this transformation equals or exceeds the number of original RF parameters, one may solve for the elements of $\pi(L)$ in terms of those in $\varphi(L)$.[14,15] Therefore the condition for identification is:[16]

$$\rho\left\{\frac{\partial \varphi(L)_i}{\partial \pi(L)_j}\right\} = \text{the number of parameters in the RF system.}$$

where;

$\frac{\partial \varphi(L)_i}{\partial \pi(L)_j}$ are the gradients which comprise the Jacobian $\left\{\frac{\partial \varphi(L)_i}{\partial \pi(L)_j}\right\}$.

In conclusion, the conditions under which RF coefficients may be obtained from knowledge of ORF parameters becomes more obscure as one investigates more general models with rational expectations. Nevertheless, the issue is important if statements about values of the structural parameters must be made. Furthermore, the question of identification must be considered prior to that of structural estimation; the subject of the next section.

5.3 Estimation

This section outlines an approach to the estimation of the structural coefficients for any model with rational expectations. The analysis is somewhat abbreviated since it follows the procedure outlined by Wallis.[17] The estimators obtained by this method are

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[14]See Ramsey (1976), p. 170.

[15]See Hadley (1964), p. 48.

[16]Clearly, the Model I Condition is a special case of this formula. With simple models it is easier to isolate the analysis upon those terms which are crucial for identification.

[17]Wallis, p. 28-30.

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then compared with approaches to estimation employed in earlier studies.

The estimation of structural parameters in models with rational expectations is accomplished by the following procedure. Obtain a RES by applying the REH to the relevant reduced form system. Substitute this expression into the system of structural equations to eliminate the unobservable expectations terms. Full information maximum likelihood (FIML) estimators for the structural coefficients may be obtained from the resulting system which is nonlinear in the parameters but linear in the variables.[18]

This procedure yields consistent and asymptotically efficient estimates when two conditions are satisfied. The first requires that efficient estimates for the elements of $\Gamma(L)$, obtained from a Zellner's seemingly unrelated regressions technique, appear in the particular RES which serves as the observable expression for the expectations terms. Second, structural coefficients must be identified from knowledge of ORF coefficients.

Following Wallis, this procedure may be outlined by writing:

$$Ay_t = M(L;\ \alpha,\ \hat{\Gamma})z_t + \varepsilon_t \tag{5.3.1}$$

where;

i) A is the coefficient matrix employed in the descriptions of Models I, II, and III,

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[18]An analysis of this type of estimation procedure is contained in Bard (1974), p. 64.

ii) $M(L; \alpha, \hat{\Gamma})$ is an $m \times m + n$ matrix of structural coefficients that contains parameters corresponding to any model in which expectations terms have been replaced by a relevant RES. Forecasts from $\hat{x}_t = \hat{\Gamma}(L)x_{t-1}$ have been employed to eliminate all exogenous expectations,

iii) α represents the vector of structural parameters which comprise the coefficients of this reformulated system.

When the vector of disturbance terms $\varepsilon_t \sim N\{0, \Sigma_\varepsilon\}$, the FIML estimators are those which maximize:[19]

$$L = -\frac{1}{2} MT \log(2\pi) - \frac{T}{2} \log |\Sigma_\varepsilon|$$

$$+ T \log \| A \| - \frac{1}{2} \operatorname{tr} \Sigma_\varepsilon^{-1} \{AY - M(L; \alpha, \hat{\Gamma})Z\}'\{AY - M(L; \alpha, \hat{\Gamma})\}$$

where, Y and Z are matrices of observations for y_t and z_t.

Suggestions for the particular numerical optimization procedure for this problem may be found in Wallis.[20] Much of Wallis' analysis is based upon an earlier study by Sargan and Sylvwestrowicz (1976).

The properties of the estimators generated by this procedure may be compared to those obtained in previous studies by McCallum (1976) and Sargent and Wallace (1973). Briefly, McCallum

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[19] Wallis points out, p. 26, that estimation may require the sample period to be "appropriately truncated."

[20] Wallis, p. 29.

substitutes actual values of endogenous variables for the expectations to overcome the problem of unobservables. The analysis reduces to an "errors in variables" problem where it is assumed that unbiased forecast errors separate actual and expected values of the future endogenous variables. McCallum obtains consistent estimators by using the predetermined variables in the original structural model as instruments for the future endogenous variables. Sargent and Wallace (S-W) suggest replacing the unobservable expectations with the forecasts of endogenous variables formulated from a menu of variables chosen from the set of predetermined variables. A FIML or 3SLS technique applied to this reformulated structure results in consistent estimates for the structural parameters. When the information set used to forecast endogenous variables in the S-W approach equals the set of instruments employed by McCallum, the two procedures yield identical estimators.[21]

Although these procedures yield consistent estimates of structural parameters, the manner in which expectations are eliminated from the structural equations reduces to replacing expectations terms with arbitrary linear functions of the set of variables which comprise the RES. Hence, the McCallum and S-W approaches ignore the REH result which links expectations terms with expressions, obtained by making conditional forecasts on the relevant theory, whose coefficients are known functions of structural parameters. In contrast, the estimators described in the present

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[21]McCallum (1976), p. 45.

analysis reflect all the information revealed by the application of the REH to a specific model, including the particular functional form obtained in a RES. Hence, the FIML estimation procedure, outlined above, incorporates any constraints on the parameters of the reformulated structural equations generated by substituting a suitable RES for the expectations terms.[22] Therefore, by incorporating information about structural parameters which is ignored in the previous approaches, the estimators derived from the present analysis are necessarily more efficient than those suggested by McCallum or Sargent and Wallace.[23]

5.4 Testing the REH

The REH may be tested by examining the validity of the restrictions inherent in the ORF generated by applying the theory of rational expectations to a model which contains expectations. These restrictions arise since the coefficients of the ORF are functions of the RF parameters and there are generally more variables in the ORF than there are parameters in the RF. If the REH is a valid hypothesis, these restrictions on the ORF coefficients must be true. The restrictions may be tested by comparing the explanatory power of the ORF suggested by the REH with an alternative, unrestricted version, which is a simple function, linear in both parameters and

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[22] The nature of these constraints are the subject of the following section.

[23] This point is confirmed in Kmenta (1971), p. 450.

variables, of the set of variables which appear in the ORF.[24] Clearly, a test which rejects the hypothesis that these restrictions are true would cast doubt on the validity of the REH.

Two separate test procedures may be employed; Wald's test or the Likelihood Ratio test.[25] For Wald's test, the estimates of φ from (5.2.9) and of Γ from (2.2.6) are obtained.[26] The restrictions implied by the REH are of the form:[27]

$$f\{Vec(\hat{\varphi},\hat{\Gamma})\} = \emptyset \quad ; \quad \emptyset \equiv \text{null vector}$$

therefore, the test statistic is:

$$W = f\{Vec(\hat{\varphi},\hat{\Gamma})\}'Vf\{Vec(\hat{\varphi},\hat{\Gamma})\}$$

where;

i) $W \sim \chi^2_{(R)}$; $R \equiv$ total number of restrictions,

ii) V is the asymptotic covariance matrix of $f\{Vec(\hat{\varphi},\hat{\Gamma})\}$.

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[24] The test could also be undertaken with respect to the structural equations. In this case the restricted structure is (5.3.1) while an alternative, unrestricted specification is simple linear function containing the variables in (5.3.1). Emphasis is placed upon ORF analysis in the present analysis to avoid any confusion between standard overidentifying restrictions and those restrictions implied by the REH.

[25] An excellent discussion of these tests may be found in Silvey (1970), p. 108-118.

[26] Each of these estimators may be obtained by a Zellner's seemingly unrelated regressions approach. Furthermore, equally efficient estimates of φ could be obtained by applying O.L.S. to each equation in (5.2.9); since the regressors in each equation are identical. See Kmenta (1971), p. 521.

[27] "Vec" maps the elements of matrices into a single vector.

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The likelihood ratio test compares the values obtained by maximizing the likelihood function with, first, a restricted and then an unrestricted parameter space to obtain:

$$\text{Max } L|H_0 \quad ; \quad H_0 \Rightarrow \varphi \text{ restricted,}$$
$$\text{Max } L|H_A \quad ; \quad H_A \Rightarrow \varphi \text{ unrestricted.}$$

The value of Max $L|H_A$ is obtained by maximizing the likelihood function formed when the relationship between y_t and z_t is described by (5.2.9). Hence, the alternate hypothesis suggests:

$$y_t = \varphi(L)z_t + v_t \; .$$

The likelihood function in this case is maximized with respect to the elements of φ.

Alternatively, Max $L|H_0$ may be calculated by obtaining the restrictions imposed on the elements of φ by the REH, substituting these into (5.2.9) to eliminate R elements of φ and then maximize the modified likelihood function, containing the restrictions, with respect to the remaining elements of φ. However, the ORF incorporates the REH restrictions. Therefore, Max $L|H_0$ could also be obtained by maximizing the likelihood function formed by assuming the true relationship between y_t and z_t is (5.2.10):

$$y_t = \pi(L)z_t + v_t \; .$$

In this case, the maximization is undertaken with respect to the RF parameters when the RF coefficients can be obtained from knowledge of those in the ORF. When RF parameters are not identified, the

REH may still be a testable proposition; in this case, the value for Max $L|H_0$ may be obtained by maximizing with respect to a sutiable combination of RF parameters.[28]

Having obtained Max $L|H_0$ and Max $L|H_A$, the relevant test statistic is:

$$\lambda = \frac{\text{Max } L|H_0}{\text{Max } L|H_A}$$

where;

$$- 2 \log \lambda \sim \chi^2_{(R)}.$$

Both test procedures require enumeration of the restrictions implied by the REH. In the following analysis, the number of restrictions for general Model I structures is established and the procedure for counting restrictions for models with lead expectations is outlined. Examples of the form of these restrictions are contained in the Appendix for Chapter V.

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[28]For example, assume that the elements of π_3 cannot be obtained from knowledge of the ORF coefficients. In this case the maximum likelihood value is obtained by maximizing with respect to:

i) $m \cdot n_1$ elements of π_1^0,
ii) $m(n - n_1)$ elements of the product $\pi_3(I - \pi_3)\pi_1^0$,
iii) the $m \cdot k_1$ elements formed by summing the first n_1 columns of the product $(I - \pi_3)^{-1}\pi_1'(L)$ and the product $\pi_3(I - \pi_3)^{-1}\pi_0^1$,
iv) the $m \cdot k_2$ elements of the last $n - n_1$ columns of the product $(I - \pi_3)^{-1}\pi_1'(L)$,
v) the $m \cdot k_3$ elements of the product, $(I - \pi_3)\pi_2(L)$.

The following analysis demonstrates that restrictions may exist in the underidentified case and an example is provided by IId in the Appendix.

The number of restrictions implied by the REH, when the relevant theory is a Model I structure, may be revealed by reconsidering (5.2.5):

$$\pi_3(I - \pi_3)^{-1}\pi_1^0 \, \Gamma^*(L).$$

Since every set of coefficients, except (5.2.5), may be expressed in terms of a greater number of RF coefficients, (5.2.5) is the only source of restrictions.

It is convenient to divide the restrictions into two separate categories. One set, type I, may be attributed to the extent in which the elements of π_3 are overidentified by independent estimates from the ORF. By construction, the number of independent estimates for the terms in (5.2.5) is $m(n_1 - n_2)$ and the number of parameters to be estimated is m^2; π_1^0 and $\Gamma^*(L)$ are identified outside of (5.2.5). Hence, there are

$$m(n_1 - n_2) - m^2$$

overidentifying restrictions.[29,30]

A second class of restrictions, type II, may be revealed from (5.2.5) by comparing the ORF coefficients for each of the lagged values of the j^{th}, $j = 1,2,\ldots,n_1$, exogenous variable within each equation. This is facilitated by denoting the elements of the diagonal matrix $\Gamma^*(L)$ as $\Gamma_{jj}^*(L)$ and letting c_j^* equal the number of

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[29] This type of restrictions are noted by Wallis, p. 25.

[30] Examples of these type I restrictions are provided in Ib, IIb, and III in the Appendix.

lagged values, (lag $\leq$ P) that appear in the original RF (5.2.4) and which can be generated by lagging the j^{th} period t exogenous variable x_{jt}. Then, $P - c_j^*$ is the number of nonzero coefficients in the polynomial in L which constitutes the j^{th} diagonal element in $\Gamma^*(L)$. Since $\Gamma^*(L)$ is a diagonal matrix, the j^{th} element of the i^{th} row of the $m \times n$ matrix product (5.2.5) is obtained by multiplying the i^{th} row of $\pi_3(I - \pi_3)^{-1}$ by the j^{th} column of π_1^0 which, in turn, is weighted by the scalar, $\Gamma_{jj}^*(L)$. Therefore, the coefficients on the lagged values of the j^{th} exogenous variable in the i^{th} equation of the ORF share a common factor, namely the i, j^{th} element of $\pi_3(I - \pi_3)^{-1}\pi_1^0$. Hence, these coefficients differ only by factors of $\gamma_{j,1}^*, \gamma_{j,2}^*, \ldots, \gamma_{j,P-c_j^*}^*$; the terms in Γ_{jj}^*. Since the elements of $\Gamma^*(L)$ are identified by estimation of (2.2.6), the relation between these $P - c_j^*$ coefficients may be expressed in $P - c_j^* - 1$ restrictions. For example, let φ_{ijg} be the coefficient on the g^{th} lagged value of the j^{th} exogenous variable in the i^{th} equation of the ORF. Then the restrictions described above may be expressed:[31]

$$\frac{\varphi_{ij1}}{\varphi_{ijg}} = \frac{\gamma_{j1}^*}{\gamma_{jg}^*} \qquad \begin{array}{l} g = 2,3,\ldots,P - c_j^* \\ \text{for each individual } j; \\ j = 1,2,\ldots,n_1. \end{array}$$

Combining these n_1 sets of $P - c_j^* - 1$, $j = 1,\ldots,n_1$, restrictions yields

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[31]When $P = c_j^*$, no restrictions are implied by the nature of the coefficients on the lagged values of the j^{th} exogenous variable in the ORF.

$$\sum_{n=1}^{n_1} \text{Max}\{P - c_j^* - 1,0\}$$

total type II restrictions in the i^{th} equation.[32] Since each equation contains exactly the same set of variables, this is

$$m \sum_{j=1}^{n_1} \text{Max}\{P - c_j^* - 1,0\}$$

type II restrictions for the system.

Combining restrictions of both type I and II yields

$$\begin{aligned} R &= m \cdot \text{Max}\{n_1 - n_2 - m,0\} \\ &\quad + m \sum_{j=1}^{n_1} \text{Max}\{P - c_j^* - 1,0\} \end{aligned} \tag{5.4.1}$$

total independent restrictions.[33] But $P - c_j^* - 1 = -1$ only when $P = c_j^*$, or when every term in the process which generates the j^{th} exogenous variable also appears in the original RF. By construction, $P = c_j^*$ for n_2 exogenous variables. Hence;

$$\sum_{j=1}^{n_1} \text{Max}\{P - c_j^* - 1,0\} = \{\sum_{j=1}^{n_1} P - c_j^*\} - n_1 + n_2,$$

and R may be expressed as:[34]

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[32]Examples of these restrictions are provided by Ic, IIc, IId, and III in the Appendix.

[33]There are many ways to express the restrictions, but any additional ones are redundant — the $(R+1)^{st}$ may be obtained from the original R.

[34]Section A-3 of the Appendix for Chapter V compares these restriction rules with the number of restrictions which appear in the examples considered.

$$m\{\sum_{j=1}^{n_1} \{P - c_j^*\} - \text{Min}(n_1 - n_2, m)\}. \qquad (5.4.2)$$

This result may be verified by comparing the total number of variables in the ORF with the total number of identifiable parameters contained in the ORF coefficients. Initially, recall that the number of ORF parameters is less than or equal to the number of RF parameters for all the groups ORF coefficients defined above, except for those contained in $\varphi_2(L)$ (from section 5.2). Hence, all restrictions are isolated in (5.2.5).[35] The number of ORF terms described by (5.2.5) is $m \sum_{j=1}^{n_1}(P - c_j^*)$. When the RF parameters are identified from knowledge of those in the ORF, these $m \sum_{j=1}^{n_1}(P - c_j^*)$ unrestricted ORF coefficients can be expressed in terms of the m^2 elements of π_3, since π_1^0 and $\Gamma^*(L)$ are identified outside of (5.2.5). This suggests

$$m\left[\{\sum_{j=1}^{n_1} P - c_j^*\} - m\right]$$

restrictions. This result conforms to the rule, (5.4.2), since $m \leq n_1 - n_2$ when the RF parameters are identified. When the RF parameters are not identified, restrictions may still exist since the $m \sum_{j=1}^{n_1} (P - c_j^*)$ unrestricted ORF coefficients may be expressed in terms of $m(n_1 - n_2)$ identifiable functions of RF parameters; namely, the elements in the $n_1 - n_2$ nonzero columns of

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[35]Example III of the Appendix for Chapter V illustrates this point.

$\pi_3(I - \pi_3)^{-1}\pi_1^0$.[36] As a result this yields

$$m\left[\{\sum_{j=1}^{n_1} P - c_j^*\} - (n_1 - n_2)\right]$$

restrictions. This result also follows the rule, (5.4.2), since $m \geq n_1 - n_2$ when the RF parameters are not identified.

The present analysis reveals that testable REH restrictions may exist, even when the elements of the RF are not separately identified from knowledge of ORF coefficients.[37] Therefore, Wallis' contention that the overidentified case is "more interesting, since it permits a test of the REH...," is somewhat misleading.[38]

Having obtained the number of restrictions implied by the REH, it is a trivial matter to set up a critical region for the maximum likelihood ratio test at a chosen level of significance. The usefulness of Wald's test procedure depends upon the degree of difficulty encountered in determining the form of the restrictions. For simple Model I structures this is relatively easy; but it may be quite demanding for more complicated models with lead expectations. However, a test of the REH may be facilitated by noting that, regardless of the complexity of the model in question, the likelihood ratio test requires only the number of restrictions, since their

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[36]Example IId of the Appendix illustrates this point.

[37]A value for Max $L|H_0$ is obtained with respect to a combination of sums and products of RF parameters in this case.

[38]Wallis, p. 25. Wallis hints at the existence of Type II restrictions on p. 7, but ignores them in his brief statements about testing the REH.

form is implied by the nature of the nonlinear functions of RF parameters, which constitute the restricted ORF coefficients.

Analyzing the test procedure for general Model I structures provides some insight for testing the REH when the relevant theory contains lead expectations. For special cases one may be able to dichotomize the restrictions into type I and type II; however, the nature and actual form of the restrictions, for general specifications, will be obscured by the complexity of the coefficients obtained in the Model II and Model III ORF^s. Nevertheless, it is relatively easy to compare the number of variables in the ORF with the number of identifiable RF parameters. Given the experience of the Model I analysis, it is reasonable to assume that when the RF parameters are identified, the number of restrictions is the difference between the total number of unrestricted ORF coefficients and the total number of RF parameters that form the coefficients in the restricted version of the ORF. Therefore, let:

$Q_{ORF} \equiv$ # of unrestricted ORF parameters,

$Q_{RF} \equiv$ # of RF parameters.

Then, total $R = Q_{ORF} - Q_{RF}$ in the identified case. When the RF elements are not identified, the Q_{ORF} coefficients cannot be expressed by Q_{RF} individual RF coefficients but may be represented by $\rho\left\{\frac{\partial \rho_i}{\partial \pi_j}\right\}$ identifiable functions of the RF parameters. Therefore, a general rule for counting the restrictions implied by the REH is:

$$R = Q_{ORF} - \text{Min}(Q_{RF}, Q_J), \qquad (5.4.3)$$

where, Q_J equals the rank of the Jacobian matrix of transformation for the relation between the elements of π and $\wp$, described in section 5.2.

The similarity between (5.4.3) and (5.4.2), the rule for Model I restrictions, is not surprising since the Model I result is a special case of (5.4.3) obtained by isolating the analysis on only those terms in the ORF in which Q_{ORF} exceeds Q_{RF}.

5.5 Summary

The econometric implications of applying the REH to models with expectations have been investigated by exploring the three questions posed at the outset of this chapter. The answer to these questions, provided in the above analysis may be briefly summarized.

In regard to the identification of RF parameters from knowledge of those in the ORF, a general condition for identification is determined, placing specific emphasis on Model I structures. When no lagged variables appear, this condition requires that the number of exogenous variables exceed the number of expectations terms in each equation. Analysis of models with lagged variables reveals that this simple condition, originally suggested by Wallis, is not applicable for all specifications. The current study reveals that lagged variables may not be treated like period-t exogenous variables in establishing the condition for identification. Moreover, the appearance of period t exogenous variables contributes to the identification of RF parameters only when the entire generating process for that particular variable does not appear in the original RF system. Finally, the complex nature of the ORF coefficients will often make it difficult to actually obtain expressions of RF coefficients in terms of those in the ORF, but the procedures outlined in 5.2 provide guidelines for determining those situations in which the RF are simply not identified and consequently, when the endeavor will be futile.

The estimation procedure outlined in section (5.3) serves as the response to the second inquiry broached at this chapter's inception. Estimators, obtained by following the technique outlined above, account for the restrictions implied by the REH and, consequently, are more efficient than those obtained by procedures which ignore this additional information. The major disadvantage of this approach to estimation is the complex numerical solution procedures required to maximize the likelihood function. Nevertheless, given the nonlinear nature of the particular RES which is substituted for the expectations terms, the FIML approach which incorporates the REH restrictions, is the appropriate estimation procedure.

Finally, the test developed in section (5.4) provides an answer to the most important question posed in Chapter V. By employing this test procedure in various models with expectations, the applied researcher will be able to judge the durability of the REH restrictions. In this manner, the validity of the REH, as a proxy for the perceptions of economic agents, may be determined.

CHAPTER VI

AN ANALYSIS OF THE RECENT LITERATURE ON THE THEORY OF RATIONAL EXPECTATIONS

This chapter compares the use of rational expectations in previous studies with the guidelines for employing the REH, provided in the current analysis. Particular emphasis is centered upon past researchers' concerns about three issues highlighted in the current study; namely, the conditions which insure the stability of a RES, the role of specifying generating mechanisms for exogenous variables, and the particular functional form of the observable structure obtained by applying the REH to the model in question.

The following analysis considers several of the numerous treatments of the REH contained in recent studies. These range from applied studies to strictly theoretical analyses. First, this survey considers studies which employ the REH contrary to the guidelines of the current framework. Second, examples of simple models with lead expectations are explored. Third, the result of applying the REH to a relevant theory which reflects the natural rate hypothesis is analyzed. Finally, the alternative solution procedures employed by Muth (1961) and Shiller (1978) are considered.

A common theme of many of these recent studies is that researchers often fail to discuss the three issues listed above. The reasons for the de-emphasis of these seemingly important topics is revealed in the ensuing analysis.

6.1 Examples which Deviate from the Approach Employed in the Current Study

This section explores the manner in which the REH is applied to models investigated by McCallum (1977) and Haley (1976). These results are compared with those obtained by following the guidelines of the current approach under similar circumstances. Finally, possible explanations for the methodologies employed by these authors are provided.

McCallum, in an analysis of foreign exchange rates, proposes the following structural model:[1]

$$F_t = \gamma_0 + \gamma_1 F_t^* + \gamma_2 S_{t+3}^e + u_{1t}$$

$$F_t^* = S_t + (R_t - 1)$$

$$F_t^* = F_t - (\beta/\alpha)S_t^e + (\beta/\alpha)F_{t-3} - (\delta/\alpha)R_t + \text{constant},$$

where;

i) F_t is the three-period forward rate as of period t,

ii) F_t^* is the interest parity forward rate,

iii) S_{t+3}^e is the expected spot rate three periods hence,

iv) S_t is the period t spot rate,

v) $R_t \equiv \dfrac{1 + ic_t}{1 + iu_t}$;

ic_t and iu_t are the 90-day interest rates in Canada and the U.S., respectively,

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[1]This model is obtained from equations 7, 8, and A-5, McCallum (1977), p. 147.

vi) γ, β, α are structural parameters,

vii) u_{1t} is a structural disturbance,

viii) F_t, F_t^* and S_t are the endogenous variables in the system.

McCallum obtains a reduced form for the interest parity forward rate, designated (9), from his model:

$$F_t^* = \pi_0 + \pi_1 R_t + \pi_2 F_{t-3} + \pi_3 S_{t+3}^e + V_t,$$

where the form of the π_i, i = 0,1,2,3 are not explicitly specified.

At this point, McCallum invokes the REH to obtain the following expression, labeled (13), for the expectations terms:

$$E\{S_{t+3}|\emptyset_t\} = \delta_0 + \delta_{11} R_t + \delta_{12} F_{t-3} + \delta_{21} R_{t-1} + \delta_{22} F_{t-4} + \ldots$$

where;

i) $\emptyset_t$ represents the relevant information set as of period-t,

ii) δ_0, δ_{11}, δ_{12}, δ_{22},... are unspecified parameters.

The basis for McCallum's RES is clearly stated,

> "...since S_t is an endogenous variable in the system consisting of equstions (7), (8) and (9), our model implies that spot prices are 'actually' determined by the predetermined variables of that system. Thus we see that $\emptyset_t$ will consist of the values of these variables -- and perhaps additional lagged endogenous variables -- that are known to market participants at time t. Consequently, $E\{S_{t+3}|\emptyset_t\}$ is a function -- which we assume to be linear -- of current and past values of the system's predetermined variables (now excluding S_{t+3}^e)."[2]

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[2]McCallum, p. 147.

Therefore, McCallum presumes the information set contains knowledge of the variables appearing in the relevant theory for spot prices, but not its particular functional form. The details of his solution procedure are left unexplained.

Following the guidelines of the present analysis, the reduced form (relevant theory) for S_t could have been obtained by substituting (9) into the identity (8). This is identical to a RF equation for S_t from McCallum's model. Since (8) is an identity and (9) is the RF for F_t^* from 7, 8, and A-5, then the RF for S_t written in McCallum's notation may be designated as 9':

$$S_t = (\pi_0 + 1) + (\pi_1 - 1)R_t + \pi_2 F_{t-3} + \pi_3 S^e_{t+3} + V_t \ .$$

Since F_t is an endogenous variable, (9') is one equation of a three equation Model III structure. By making conditional forecasts on (9') a Model III RES following (4.2.13) may be obtained.

Hence, the McCallum study contains a number of theoretical deficiencies. It fails to account for the constraints on the parameters of 9' (analogous to (4.2.11)) which insure the stability of the solution. Also, it ignores the generating mechanism for R_t required by the solution. And finally, McCallum obtains a RES containing coefficients which bear no relation to those in the original reduced form. Moreover, McCallum's estimates of the reduced form parameters are obtained by substituting actual spot prices, S_{t+3}, into (7) to eliminate the expectations terms. Consistent estimates obtained by using the variables in (13) as instruments. However, the estimators obtained in this approach assume less information than those obtained by following the FIML procedure of section 5.3. Consequently, the latter are more efficient.

In another example, Haley constructs a simple model of the cherry market using expectations as a determinant of the level of inventories:

$$p^*_{t+1} - p_t = \beta I_t$$

$$q^d_t = I_{t-1} + q^s_t - I_t$$

$$q^d_t = \alpha_0 + \alpha_1 p_t + \alpha_2 p^a_t + \alpha_3 p^I_t + \alpha_4 y_t$$

$$q^s_t = q_t + u_t$$

where;

i) p^*_{t+1} is the expected price one period hence,

ii) p_t is the current price,

iii) I_t (I_{t-1}) is the current (last period) level of inventories,

iv) p^a_t is the price of a substitute,

v) p^I_t is an index of all prices,

vi) y_t is income,

vii) supply (q_t) is assumed to be exogenous,

viii) q^d_t and q^s_t are the quantities demanded and supplied, respectively,

ix) the endogenous variables are p_t, I_t, q^d_t, q^s_t, and p^*_{t-1},

x) u_t is the single structural disturbance.

Appealing to Muth's definition of rational expectations, Haley closes the model by assuming expectations are rational:

$$p^{*}_{t+1} = \delta p_t \quad , \quad \delta \in (0,1).$$

This RES is designated as equation (4). The desired reduced form expression for cherry prices, void of unobservables, is then obtained:

$$p_t = \pi_0 + \pi_1 p^a_t + \pi_2 p^I_t + \pi_3 y_t + \pi_4 q_t + \pi_5 I_{t-1} + V_t \; .$$

The similarities between the Haley and McCallum results is not surprising since earlier McCallum studies are specifically referenced by Haley.[3] However, Haley's RES reflects an information set which is even more restricted than that employed by McCallum. The foundation for (4) is unexplained by Haley; except that it is obtained "according to Muth's rational expectations hypothesis."[4]

Alternatively, the guidelines provided by the present analysis would suggest obtaining the relevant theory for cherry prices from the RF of the equation for prices,

$$p_t = -(\alpha_1 - \beta^{-1})^{-1}\{\alpha_0 + \alpha_2 p^a_t + \alpha_3 p^I_t + \alpha_4 y_t - I_{t-1} + q_t + \beta^{-1} p^{*}_{t+1}\} + (\alpha_1 - \beta^{-1})^{-1} u_t,$$

and apply the Model III solution procedure applicable to this structure. In this case the three issues ignored by Haley, stability conditions; exogenous variable generating processes; and the specific functional form of the RES, would all be considered.

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[3]For example McCallum (1972) and (1974).

[4]Haley, p. 58.

The degree in which the Haley and McCallum applications of the REH diverge from the approach suggested by the framework of Chapters II, III and IV warrants an investigation of the probable motivation for their particular solution procedures. Most of McCallum's numerous applications of the REH, including the one cited in the present analysis, reference an article by Lucas (1972) as the source of the particular solution procedure employed.[5] Lucas initiates his analysis by clearly specifying the "relevant theory" under consideration. His example takes the form of a simple Model II single equation structure with one period lead expectations. He then obtains an ORF subject to two stated conditions; a second order process assumed to generate the exogenous variables, and "reasonable parameter values" -- insuring the existence of the solution.[6,7]

This reformulation is identical to that which can be obtained by applying the Model II RES under similar circumstances. However, when McCallum appeals to this analysis as a justification for his own, neither the particular functional form of the solution nor the conditions which must be satisfied for its existence are discussed. Omission of these details allows McCallum to treat Model III structures as lightly as Lucas' simple Model II Example. The analyses of previous chapters demonstrates that this uniform treatment of models with rational expectations is not justified.

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[5]For example McCallum (1974), (1975), and (1976).

[6]Lucas treats the problem in a difference equation context and solves it using the method of undetermined coefficients.

[7]Lucas, p. 56.

The Haley article references the studies of McCallum (1972), (1974) and Muth but not the paper by Lucas. Therefore, the probable source of Haley's solution is Muth's original paper. The RES obtained by Haley is identical to that obtained in Muth's inventory example. However, casual inspection reveals that the two inventory models contain very different sets of explanatory variables. Muth's model contains no exogenous variables and his RES requires knowledge of the weights in the process assumed to generate the disturbance term. This clearly is not characteristic of Haley's structural model. Moreover, Nelson (1975) has demonstrated that Muth's RES is a particular result of the structure of his relevant theory. Surely, Muth's simple solution is not a general rule applicable to all inventory models.

In summary, the different approaches required to obtain RESs when the relevant theory appears in various forms is ignored by researchers whose interpretation of the REH follows the two cited examples. Whether these interpretations reflect erroneous generalizations from Muth's simple examples (Haley), or the omission of assumptions required to fill the steps from the relevant theory to the resulting ORF generated by the REH (McCallum), the resulting studies will contain a number of theoretical deficiencies. The solution's dependence upon stability conditions and processes assumed to generate exogenous variables will be overlooked. Furthermore, the estimation techniques employed will yield estimators which are ineffecient relative to those suggested in 5.3 since the particular functional form of the reformulated observable structures is ignored.

6.2 Examples of Models with Lead Expectations

The following discussion compares the simple models with one period lead expectations of Sargent and Wallace (1973), Turnovsky (1977), and Wickens (1976), with the Model II solution procedures outlined in Chapter III.

Sargent and Wallace apply the REH to Cagan's (1956) model of hyperinflation. This single equation structure is:

$$X_t = u_t - \alpha \underset{t}{E} X_{t+1} + \alpha \underset{t-1}{E} X_t - U_t + U_{t-1}$$

where;

$$X_t \equiv \log P_t/P_{t-1} \quad ; \quad P_t \equiv \text{price level},$$

$$u_t \equiv \log M_t/M_{t-1} \quad ; \quad M_t \equiv \text{money stock},$$

$$U_t \equiv \text{random disturbance.}$$

Sargent and Wallace derive a RES for this model:

$$\underset{t}{E} X_{t+1} = \frac{1}{1-\alpha} \sum_{j=1}^{\infty} \{\frac{-\alpha}{1-\alpha}\}^{j-1} \underset{t}{E} u_{t+j}$$

$$- \frac{1}{1-\alpha} \sum_{j=1}^{\infty} \{\frac{-\alpha}{1-\alpha}\}^{j-1} \{\underset{t}{E} U_{t+j} - \underset{t}{E} U_{t+j-1}\},$$

subject to the terminal condition:

$$\lim_{n\to\infty} \{\frac{-\alpha}{1-\alpha}\}^{n-1} \underset{t}{E} X_{t+n} = 0.$$

This solution could be verified by comparing it to one obtained using the Model II framework. Initially lead the structure one period to obtain:

$$X_{t+1} = u_{t+1} - \alpha \underset{t}{E} X_{t+2} + \alpha \underset{t}{E} X_{t+1} - U_{t+1} + U_t \; .$$

Utilizing this expression as the "relevant theory" for X_{t+1}, the Model II solution would be:

$$\underset{t}{E} X_{t+1} = \underset{t}{E} \{1 - \alpha - \alpha F\}^{-1} \{u_{t+1} - U_{t+1} + U_t\},$$

with the stability condition:

roots of $\{1 - \alpha - \alpha F\} = 0$

lie outside the unit circle;

$$\Rightarrow |F| = \left|\frac{1-\alpha}{\alpha}\right| > 1 \quad -1 < \frac{\alpha}{1-\alpha} < 1,$$

since $\alpha < 0$ (restriction of Cagan's model),

$$\Rightarrow 0 < \frac{-\alpha}{1-\alpha} < 1 \; .$$

Clearly, this stability condition is equivalent to the terminal condition obtained by Sargent and Wallace.

Sargent and Wallace confront another problem emphasized in the Model II framework: the specification of the process which generates the exogenous variables. They demonstrate that the adaptive scheme employed by Cagan will be "rational" only under specific restrictions on both the disturbances and the stochastic process governing the growth of the money stock.[8]

Hence, Sargent and Wallace obtain a reformulated structure which corresponds to the form of the RES suggested in the Model II framework. Furthermore, they state the conditions under which stable

[8]Sargent and Wallace (1973), p. 336.

solutions may be obtained and accentuate the role of specifying the nature of the sequence of exogenous expectations generated by applying the REH to models with lead expectations.

The recent studies by Wickens and Turnovsky each contain theoretical sections which outline the procedure for dealing with rational expectations in a simultaneous equation model that contains one period lead endogenous expectations but no lagged endogenous variables.

Turnovsky employs a solution procedure which is identical, except for the omission of lead operator notation, to that outlined in the Model II framework. An analogous stability condition and corresponding ORF are obtained.

An innovative aspect of Turnovsky's analysis is the computation of the effects of "incremental" and sustained" changes in exogenous (policy) variables on the endogenous variables in the system.[9] This is accomplished through a comparative static analysis of Turnovsky's system of equations.

In principle, Wickens obtains identical results, but the methodology employed varies somewhat. Wickens' solution procedure may be outlined as follows;

i) eliminate expectations from the model by utilizing:

$$Y_{t+1} = Y^e_{t+1,t} + \eta_{t+1,t}$$

where;

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[9]Turnovsky, p. 855.

a. Y_{t+1} is a vector of endogenous variables,

b. $Y^e_{t+1,t}$ is the forecast for period $t+1$ for these variables based upon knowledge of the structure in period t,

c. $\eta_{t+1,t}$ is the forecast error, observed in period $t+1$, for a forecast made in period t,

ii) solve this original structure as a system of difference equations, by "successive substitution or utilizing a lead operator," to obtain current Y_t as a function of future exogenous variables, disturbances and forecast errors under a specified stability condition,[10]

iii) make conditional forecasts on this reformulated structure and simplify to obtain the desired RES.

Therefore, Wickens' approach involves purging the lead endogenous variables from the original structure prior to making conditional forecasts.

Nevertheless, the solution obtained by Wickens is similar to that suggested by the Model II framework.[11] The presence of expected forecast errors $(\eta^e_{t+1,t})$ in the Wickens solution could be accounted for in the Model II RES by allowing the existence of forecast errors in the conditional expectations on the original reduced form. This is a trivial extension of the Model II solution procedure. Hence,

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[10]Wickens, p. 8.

[11]Wickens, p. 8.

the Model II, Turnovsky and Wickens approaches yield identical results when the conditional forecasts are set equal to the rational expectations.

Therefore, the Sargent and Wallace, Turnovsky, and Wickens studies compare closely with the methodology and emphasis of the Model II framework. However, they do contain several limitations. The omission of lead operator notation is not conducive to extending the results of these studies to multi-lead models (i.e. more general Model II structures). Also, the solution procedures offer no guidelines for obtaining a Model III RES. Finally, none of the studies suggest estimation procedures comparable to section 5.3, which account for the specific functional form of the RES.

6.3 The Natural Rate Hypothesis

Perhaps the most notable application of the REH is Sargent and Wallace's (1975) simple "textbook" macro model which demonstrates that the level of output is independent of the choice of the deterministic money supply rule. This controversial result warrants a comparison of Sargent and Wallace's treatment of the REH with the framework outlined in the current analysis.

Initially, Sargent and Wallace specify the "relevant theory" for prices from their structural model to obtain:

$$p_t = J_0 \underset{t-1}{E} p_t + J_1 \underset{t-1}{E} p_{t+1} + J_2 M_t + X_t$$

where;

i) $p_t \equiv$ price level

ii) $M_t \equiv$ money stock

iii) $\underset{t-1}{E}\, p_t \equiv$ expectation of prices based upon information as of period t-1; assumed to be rational.

iv) $X_t \equiv$ reduced form disturbances.

Clearly, this single equation reduced form designated equation (15) by Sargent and Wallace, is a scalar Model II. Therefore the rational expectation for this structure may be expressed as a function of current and future expected M_t and X_t.

However, output in this model is presumed to be affected by the difference between expected and actual prices (the natural rate hypothesis). This is reflected in the structural equation for real output designated by Sargent and Wallace as equation (1):

$$y_t = a_1 K_{t-1} + a_2(p_t - {}_t p^*_{t-1}) + u_{1t}$$

where;

i) $y_t \equiv$ real output,

ii) $K_{t-1} \equiv$ capital stock,

iii) $u_{1t} \equiv$ structural disturbance,

iv) ${}_t p^*_{t-1} \equiv$ agent's expectations of the price level for period t formulated as of period t-1.

Following the theory of rational expectations, conditional forecasts obtained from the relevant theory (15) are equated with the expectations in (1). However, the natural rate hypothesis allows an alternative to the standard solution procedure. By taking expectations on (15) and subtracting the resulting forecast from (15), one obtains Sargent and Wallace's equation (16):

$$p_t - \underset{t-1}{E}\, p_t = J_2\{M_t - \underset{t-1}{E}\, M_t\} + X_t - \underset{t-1}{E}\, X_t.$$

Since the intent of the analysis is to obtain an observable expression for the expectations terms in (1), casual inspection reveals that an observable expression for the entire difference,

$$p_t - {}_tp^*_{t-1},$$

would be equally valuable for this model. But this is supplied by (16). Assuming the rule that generates the money supply is deterministic

$$M_t - \underset{t-1}{E} M_t = 0, \quad \text{and (16)}$$

is substituted into (1) to obtain:

$$y_t = a_1K_{t-1} + a_2\ X_t - \underset{t-1}{E}\ X_t \ + u_{1t}.$$

Since K_{t-1} is exogenous in this model, Sargent and Wallace conclude that output is independent of the money supply rule.[12]

Therefore, when actual and expected values appear in difference form, as in the structure which displays the natural rate hypothesis, the issues emphasized in the current study are no longer significant. Clearly, the question of stability is no longer crucial, and, when exogenous variables are initially assumed to be deterministic, Sargent and Wallace demonstrate that the solution is independent of the exogenous variable generating rule.

This result agrees with that obtained by following the guidelines of the current study when actual and expected values appear

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[12]Sargent and Wallace (1975), p. 247.

as a difference. Moreover, this simplification may be employed even when the relevant theory for p_t, (15), appears as a Model III structure. Hence, the examination of the Sargent and Wallace article reveals that a researcher may avoid the complex problems of obtaining an RES by imposing certain restrictions on the coefficients of the model in question.

A brief comment on Sargent and Wallace's policy prescriptions is noteworthy in light of the current examination of the natural rate hypothesis and the role of rational expectations. Many critics have attacked the use of the REH in Sargent and Wallace's study since policy rules regain their potency if expectations are assumed to be generated by an autoregressive scheme. However, the natural rate hypothesis is equally culpable since Sargent and Wallace's result hinges upon the occurance of $p_t - {}_tp^*_{t-1}$ in the original structural equation. For example, assume output is affected by the individual levels of prices and expected prices (the long run Phillips Curve is not vertical).[13] Therefore, (1) becomes:

$$y_t = a_1K_{t-1} + a_2p_t - a_3\ {}_tp^*_{t-1} + u_{1t}.$$

Applying a Model II solution procedure from a reformulated version of (15) obtains:

$$y_t = f\{K_{t-1},\ p_t,\ X_t,\ {}_{t-1}E\,X_t, \ldots,\ M_t,\ {}_{t-1}E\,M_t, \ldots,\ U_{1t}\}\ ,$$

where the deterministic money supply rule now obviously affects the

[13]No theoretical basis is provided for this alternative structure. It is constructed only for expositional purposes.

level of output.[14] Clearly, the REH alone is not responsible for the controversial policy conclusions derived in the Sargent and Wallace investigation.

6.4 Muth's Approach

The difficulties encountered in obtaining a RES by following the guidelines of the framework outlined above are mitigated by utilizing the solution procedure employed in Muth's original study. This is revealed by comparing Muth's approach with that employed in the current study.

Muth suggests the following RES procedure:

i) express endogenous variables, expectations, and disturbances as output of a white noise process,

ii) substitute these expressions into the "relevant theory" (reduced form),

iii) assume knowledge of the weights on the process generating the disturbances,

iv) since the "relevant theory" must hold for all shocks, obtain the weights on the white noise process for endogenous variables from those on the process generating disturbances,

v) solve for the "rational expectation" in terms of past disturbances.

This procedure contains a number of limitations. Initially, endogenous variables may be expressed as functions of white noise

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[14] The relevant theory for prices would no longer be (15) due to the change in (1). However, only the coefficients would be different.

only when the model is void of exogenous variables. Second, the solution obtained is totally dependent upon knowledge of the weights in the process assumed to generate the disturbances. And finally, the solution is expressed in terms of "past realizations as opposed to current state variables."[15]

In the unlikely event that Muth's solution procedure is applicable (when all elements of B(L) equal zero), assumption (viii) section 2.1 insures that the framework of Chapters II, III and IV is able to accommodate the assumptions required for Muth's RES. Furthermore, the solution procedures yield identical results under similar assumptions.

Therefore, Muth provides the definition for rational expectations in his seminal paper; but his suggested solution procedure, useful only for restricted structural models, offers little insight for coping with the issues raised in the current study. As a result, researchers who consider models which are restricted to accommodate Muth's solution procedure do not confront the problems of stability; specifying exogenous variable generating processes; and the particular function form of the solution, which are inherent in obtaining a RES for more general structures.

6.5 Shiller's Approach

Shiller provides an extensive treatment of the REH in his recent study. A solution procedure for general models is outlined.

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[15]The last point is attributed to Lucas (1970), p. 55.

It is suggested that the most general models (for example, Model III) may be handled by solving them as partial difference equation with variable coefficients.[16] Shiller emphasizes the difficulty in obtaining solutions in this case. A comparison of Shiller's approach with the framework outlined in the present study reveals that the overall scope of the two analyses is very similar but there is a distinct divergence in emphasis.

Shiller questions the uniqueness of a "rational expectations equilibrium."[17] Since agents gain information each period, additional knowledge on the structure of the relevant theory and the process generating the exogenous variables is obtained.[18] This prompts Shiller to examine models of the form.[19]

$$y_t = Bx_t + \Psi(L)y_t + \theta_1(F)\underset{t-1}{E}y_t + \ldots + \theta_K(F)\underset{t-K}{E}y_t + \varepsilon_t \quad (26)$$

Therefore, current endogenous variables are not only influenced by expectations of future values of endogenous variables based upon information available last period - (t-1), but also expectations of future endogenous variables formulated up to K periods in the past. Since there are innumerable ways to model the manner in which

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[16]Shiller (1978), p. 30.

[17]Shiller (1978), p. 4; this term is analogous to "rational expectation solution."

[18]Shiller incorporates the idea of Taylor (1975), noting that the RES is influenced by the influx of information.

[19]This is Shiller's equation (26) expressed in the notation outlined in Chapters II, III and IV.

agents alter their perceptions about the future in response to new information, Shiller argues that an infinite number of solutions, or rational expectations equilibria, exist for models with lead endogenous expectations. Therefore, he concludes that "the existence of so many solutions to the rational expectation model implies a fundamental indeterminacy for these models."[20]

In comparison with the Shiller approach, the framework outlined in the preceding chapters does not take issue with the notion that the economy will reach a unique rational expectations equilibrium. Instead, emphasis is centered upon the examination of the specific stability conditions required for a solution's existence, the specification of an exogenous variable generating scheme to insure that the RES is void of unobservable variables, and the econometric implications of the particular functional form of the reformulated structure suggested by the REH. These issues originate from simply extending Muth's definition to more general models. Therefore, while Shiller emphasizes the difficulty of obtaining unique rational expectations solutions when agents' perceptions change in light of new information, he fails to note the problems stressed in the current study which are significant even when the information set is fixed.[21]

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[20]Shiller (1978), p. 33.

[21]Shiller's point is analogous to the questions raised by Lucas (1976) for economic models that did not contain expectations. Lucas emphasizes the difficulty in monitoring the effects of policy actions due to the constant changes in the structure of the system in light of the new information precipitated by the policy action itself.

6.6 Summary

The applications of the REH which appear in recent literature have taken on a number of different forms. However, to varying degrees, all studies neglect the issues which were emphasized in the development of the framework constructed in Chapters II, III, IV and V. These issues involved the specification of stability conditions which guarantee the existence of the solutions derived in the Model II and Model III frameworks, the process assumed to generate the exogenous variables, and taking advantage of the specific functional form of the reformulated structure to fully realize the econometric implications of the REH.

In spite of this common theme, past researchers have been led to de-emphasize these issues by following diverse approaches to the theory of rational expectations. Some have overlooked these problems by failing to specify the steps leading to a RES (Haley, McCallum). Others have confined the scope of their analyses to simple models; thereby, not realizing the theoretical and econometric implications of applying the theory to more general models (Sargent and Wallace (1973), Lucas, Turnovsky, Wickens). Still others confront models with such severe parameter restrictions that all the difficulties of obtaining general solutions vanish (Sargent and Wallace (1975), Muth). Finally, some articles deal with such

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Since rational expectations are conditional forecasts of the relevant theory, this constant evolution of the structure of economic models poses complications for the theory of rational expectations. Shiller's analysis focuses upon some of these problems.

a broad scope that other difficulties of coping with the REH take precedence over the issues raised in the present framework (Shiller, Taylor).

Therefore, the general approach to the REH provided in the present framework provides insight which may not be gained by analyzing individual studies. This framework, which accommodates many types of specifications, provides the foundation for determining the validity of the theory of rational expectations as an explanation for the perceptions of economic agents.

CHAPTER VII

SUMMARY AND CONCLUDING COMMENTS

The goal of this study is to examine the implications of the theory of rational expectations as an explanation of individuals' perceptions of future events. The investigation concentrates upon construction of a framework for applying the REH to various economic models, examination of the econometric implications of incorporating the REH into a structural model, and briefly reviews the manner in which rational expectations have been employed in recent studies.

The framework provides an analysis of three categories of models (Model I, Model II, and Model III) which contain rational expectations formed on the basis of all the information available prior to the current period. Model I refers to those structures which contain expectations of current period values of endogenous variables. Models which allow expectations of a finite number of future periods to appear as explanatory variables, but omit lagged endogenous variables, are considered in the analysis of Model II. Model III structures possess both multi-period future expectations and lagged endogenous variables. The framework follows a uniform format in its investigation of all of these structures. First, the expectations are equated with conditional forecasts from the model in question to obtain observable expressions for the expectations terms. These expressions are referred to as rational expectations solutions (RESs). Second, these RESs are substituted into the original model to obtain a structure which no longer contains unobservable variables.

Several significant implications of the REH are revealed in this framework. First, restrictions on the parameters of the structural model are necessary to insure the stability of the solutions. Second, a RES will invariably be dependent upon the nature of the process assumed to generate the exogenous variables in the model under investigation. Finally, the observable structure obtained in the analysis is a function of the predetermined variables in the model and the variables inherent in the exogenous variable generating process. Moreover, the coefficients of this expression are known functions of the parameters in the original model under investigation.

The econometric implications of the REH are realized by pursuing three issues which arise due to the particular functional form of the observable structure suggested by the REH. The study examines the conditions which insure that the original reduced form coefficients can be identified from knowledge of those in the observable reduced form. An estimation procedure which accounts for the functional form of the observable structure is developed. Finally, the analysis reveals the restrictions implied by this reformulated observable structure and outlines a procedure for testing the theory of rational expectations.

The survey of current literature confirms the import of the present analysis since the significant issues discussed throughout this study and briefly summarized above are notably absent from many previous applications of the REH. The literature review accentuates this omission and offers explanations for the deemphasis of some of these important aspects of the theory of rational expectations.

In the final analysis, this study provides a number of significant contributions to the continuing study of the REH as an explanation for the expectations of economic agents. The framework of Chapters II, III, and IV accentuates the problems of specifying rational expectations solutions which must be considered in any application of rational expectations. Moreover, the complexity of these solutions illustrates how difficult it may be to formulate a general theory which explains the process whereby agents obtain enough information to form "rational" expectations. Also, the framework establishes guidelines for the uniform interpretation of the REH in all economic models. This may eliminate potential disagreements about what is meant by assuming expectations are "rational" in the sense of Muth. Finally, this study provides the details for a test of the theory of rational expectations, thereby establishing a method for determining the validity of the hypothesis as an explanation of individual's perceptions of future events.

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APPENDIX FOR CHAPTER V

The actual form of the restrictions generated by the REH may be examined by considering a number of simple examples. The distinction between type I and type II restrictions is noted and the number of restrictions obtained is compared with the restriction rule developed in Chapter V.

A-1 A Description of the Examples Considered

Following the notation of Chapter V:

i) m is the total number of equations and the number of expectations terms contained in each,

ii) n is the number of exogenous variables in each equation,

iii) n_1 is the number of period-t exogenous variables in each equation,

iv) n_2 is the number of these n_1 period-t exogenous variables for which every lagged value, lag $\leq P$, appears in the original RF,

v) P is the order of the auto-regressive process generating all of the exogenous variables,

vi) c_j^* is the number of lagged values of the j^{th} period t exogenous variable, $j = 1,\ldots,n_1$, lag $\leq P$, which appear in the original RF; i.e. $\Sigma_{j=1}^{n_1} c_j^* = k_1$,

vii) k_1 is the total number of lagged variables the original RF which can be obtained by lagging members of n_1,

viii) k_2 is the total number of lagged exogenous variables that do not have corresponding period t values in the original RF.

The examples which will be considered may be described:

Ia: $m = 1, n = n_1 = 1, \; P = 1, k_i = 0$ for all i,

Ib: $m = 1, n = n_1 = 2, \; P = 1, k_i = 0$ for all i,

Ic: $m = 1, n = n_1 = 1, \; P = 2, k_i = 0$ for all i,

IIa: $m = 2, n = n_1 = 2, \; P = 1, k_i = 0$ for all i,

IIb: $m = 2, n = n_1 = 3, \; P = 1, k_i = 0$ for all i,

IIc: $m = 2, n = n_1 = 2, \; P = 2, k_i = 0$ for all i,

IId: $m = 2, n = n_1 = 1, \; P = 2, k_i = 0$ for all i,

III: $m = 1, n = 4, n_1 = 3, n_2 = 1, k_1 = 3, k_2 = 1, k_3 = 1, P = 2.$

Two additional comments clarify the following analysis. First, the disturbances in all the models considered are assumed to be individually and identically distributed normal -- the standard disturbance assumption. Also, the notation employed in the single equation examples corresponds to the symbols used to denote structural parameters in Chapters II, III, IV, since the reduced form and structural model are indistinguishable in the single equation case.

A-2 REH Restrictions in Simple Theoretical Models

EXAMPLE I

Ia: Assume the "relevant theory" may be expressed as:

$$y_t = \beta x_t + \theta y_t^e + \varepsilon_t, \tag{Ia}$$

where; m = 1, n = 1, assume P = 1,

$$\therefore \qquad x_t = \gamma x_{t-1} + \eta_t. \tag{AR-1}$$

Applying the REH to (Ia), obtain the resulting RES:

$$\underset{t-1}{E}\, y_t = (1 - \theta)^{-1}\beta \underset{t-1}{E}\, x_t = (1 - \theta)^{-1}\beta\gamma x_{t-1}.$$

Hence, the ORF is:

$$y_t = \beta x_t + \theta(1 - \theta)^{-1}\beta\gamma x_{t-1} + \varepsilon_t. \tag{Ia-ORF}$$

The unrestricted version of this structure is:

$$y_t = \varphi_1 x_t + \varphi_2 x_{t-1} + \varepsilon_t. \tag{Ia-ORF-H_A}$$

Clearly there are no restrictions implied by the application of the REH to Ia since the number of RF parameters in (Ia-ORF) equals the number of coefficient estimates obtainable from (Ia-ORF-H_A). However, the RF parameters are identified from knowledge of (Ia-ORF-H_A) since:

$$\beta = \varphi_1$$
$$\theta = (\varphi_1\gamma + \varphi_2)^{-1}\varphi_2$$

where γ can be obtained from (AR-1).

Ib: Assume the relevant theory of Ia is altered by adding an exogenous variable:

$$y_t = \beta_1 x_{1t} + \beta_2 x_{2t} + \theta y_t^e + \varepsilon_t \qquad \text{(Ib)}$$

where; m = 1, n = 2, assume P = 1,

$$\therefore \qquad x_{1t} = \gamma_1 x_{1t-1} + \eta_{1t} \quad ; \quad x_{2t} = \gamma_2 x_{2t-1} + \eta_{2t} \, .$$

Applying the REH to Ia, obtain the resulting RES:

$$\underset{t-1}{E} y_t = (1 - \theta)^{-1}(\beta_1 \gamma_1 x_{1t-1} + \beta_2 \gamma_2 x_{2t-1}).$$

Hence, the ORF is:

$$y_t = \beta_1 x_{1t} + \beta_2 x_{2t} + \theta(1 - \theta)^{-1}(\beta_1 \gamma_1 x_{1t-1} + \beta_2 \gamma_2 x_{2t-1}) + \varepsilon_t. \qquad \text{(Ib-ORF)}$$

The unrestricted version of this structure is:

$$y_t = \varphi_1 x_{1t} + \varphi_2 x_{2t} + \varphi_3 x_{1t-1} + \varphi_4 x_{2t-1} + \varepsilon_t. \qquad (\text{Ib-ORF-H}_A)$$

The number of coefficient estimates from (Ib-ORF-H_A) exceeds the number of RF parameters. The restriction may be revealed by equating the coefficients in the two structures:

$$\varphi_1 = \beta_1,$$
$$\varphi_2 = \beta_2,$$
$$\varphi_3 = \theta(1 - \theta)^{-1}\beta_1 \gamma_1,$$
$$\varphi_4 = \theta(1 - \theta)^{-1}\beta_2 \gamma_2.$$

Clearly; $\dfrac{\varphi_3}{\varphi_4} = \dfrac{\varphi_1}{\varphi_2}\dfrac{\gamma_1}{\gamma_2}$.

There may be a number of ways to express this single restriction. The restriction is a type I, or overidentifying, restriction since it stems from the fact that this four equation system contains two independent solutions for θ in terms of the φ_i and γ_i:

i.e. $$\theta = (\varphi_1\gamma_1 + \varphi_3)^{-1}\varphi_3$$

and $$\theta = (\varphi_2\gamma_2 + \varphi_4)^{-1}\varphi_4.$$

Ic: Assume the relevant theory is (Ia) but the process generating the exogenous variables is changed:

$$y_t = \beta x_t + \theta y_t^e + \varepsilon_t$$

where; m = 1, n = 1; but now P = 2,

i.e. $$x_t = \gamma_{11}x_{1t-1} + \gamma_{12}x_{1t-2} + \eta_t. \qquad \text{(AR-2)}$$

Applying the REH to Ic, obtain the resulting RES:

$$\underset{t-1}{E}\, y_t = (1 - \theta)^{-1}(\beta_1\gamma_{11}x_{1t-1} + \beta_1\gamma_{12}x_{1t-2}).$$

Hence, the ORF is:

$$y_t = \beta_1x_{1t} + \theta(1 - \theta)^{-1}(\beta_1\gamma_{11}x_{1t-1} + \beta_1\gamma_{12}x_{1t-2}) + \varepsilon_t. \qquad \text{(Ic-ORF)}$$

The unrestricted version of this structure is:

$$y_t = \varphi_1x_{1t} + \varphi_2x_{1t-1} + \varphi_3x_{1t-2} + \varepsilon_t. \qquad \text{(Ic-ORF-}H_A\text{)}$$

Following the procedure outlined above, equate the coefficients in (Ic-ORF) and (Ic-ORF-H_A) to reveal the restriction:

$$\varphi_1 = \beta_1,$$
$$\varphi_2 = \theta(1 - \theta)^{-1}\beta_1\gamma_{11},$$
$$\varphi_3 = \theta(1 - \theta)^{-1}\beta_1\gamma_{12}.$$

Clearly; $\dfrac{\varphi_2}{\varphi_3} = \dfrac{\gamma_{11}}{\gamma_{12}}$.

This restriction may be classified as a type II since it is obtained from the fact that the ORF coefficients on the lagged values of the single exogenous variable differ only by a factor of γ_{1i}, i = 1,2.

EXAMPLE II

IIa: Assume the relevant theory is:

$$\begin{aligned} y_{1t} &= \pi_{11}x_{1t} + \pi_{12}x_{2t} + \pi_{13}y^e_{1t} + \pi_{14}y^e_{2t} + v_{1t}, \\ y_{2t} &= \pi_{21}x_{1t} + \pi_{22}x_{2t} + \pi_{23}y^e_{1t} + \pi_{24}y^e_{2t} + v_{2t} \end{aligned} \qquad \text{(IIa)}$$

where; m = 2, n = 2, P = 1,

i.e. $x_{1t} = \gamma_1 x_{1t-1} + \eta_{1t}$,

$x_{2t} = \gamma_2 x_{2t-1} + \eta_{2t}$.

Applying the REH to obtain a RES:

$$\begin{Bmatrix} {}_{t-1}E\, y_{1t} \\ {}_{t-1}E\, y_{2t} \end{Bmatrix} = D^{-1} \begin{Bmatrix} (1 - \pi_{24}) & \pi_{14} \\ \pi_{23} & (1 - \pi_{13}) \end{Bmatrix} \begin{Bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{Bmatrix} \begin{Bmatrix} \gamma_1 x_{1t-1} \\ \gamma_2 x_{2t-1} \end{Bmatrix}$$

where; $D^{-1} = (1 - \pi_{24})(1 - \pi_{13}) - \pi_{23}\pi_{14}$.

$$\therefore \quad \underset{t-1}{E} y_{1t} = D^{-1}\{(1 - \pi_{24})\pi_{11} + \pi_{14}\pi_{21}\}\gamma_1 x_{1t-1}$$
$$+ D^{-1}\{(1 - \pi_{24})\pi_{12} + \pi_{14}\pi_{22}\}\gamma_2 x_{2t-1},$$
$$\underset{t-1}{E} y_{2t} = D^{-1}\{\pi_{23}\pi_{11} + (1 - \pi_{13})\pi_{21}\}\gamma_1 x_{1t-1}$$
$$+ D^{-1}\{\pi_{23}\pi_{12} + (1 - \pi_{13})\pi_{22}\}\gamma_2 x_{2t-1}.$$

The resulting ORF is:

$$y_{1t} = \pi_{11}x_{1t} + \pi_{12}x_{2t}$$
$$+ D^{-1}\{\pi_{13}((1-\pi_{24})\pi_{11} + \pi_{14}\pi_{21}) + \pi_{14}(\pi_{23}\pi_{11} + (1-\pi_{13})\pi_{21})\}\gamma_1 x_{1t-1}$$
$$+ D^{-1}\{\pi_{13}((1-\pi_{24})\pi_{12} + \pi_{14}\pi_{22}) + \pi_{14}(\pi_{23}\pi_{12} + (1- \pi_{13})\pi_{22})\}\gamma_2 x_{2t-1}$$
$$+ v_{1t},$$

$$y_{2t} = \pi_{21}x_{1t} + \pi_{22}x_{2t}$$
$$+ D^{-1}\{\pi_{23}((1-\pi_{24})\pi_{11} + \pi_{14}\pi_{21}) + \pi_{24}(\pi_{23}\pi_{11} + (1-\pi_{13})\pi_{21}\}\gamma_1 x_{1t-1}$$
$$+ D^{-1}\{\pi_{23}((1-\pi_{24})\pi_{12} + \pi_{14}\pi_{22}) + \pi_{24}(\pi_{23}\pi_{12} + (1-\pi_{13})\pi_{22}\}\gamma_2 x_{2t-1}$$
$$+ v_{2t}.$$

The unrestricted version of this structure is:

$$y_{1t} = \varphi_{11}x_{1t} + \varphi_{12}x_{2t} + \varphi_{13}x_{1t-1} + \varphi_{14}x_{2t-1} + v_{1t},$$
$$y_{2t} = \varphi_{21}x_{2t} + \varphi_{22}x_{2t} + \varphi_{23}x_{1t-1} + \varphi_{24}x_{2t-1} + v_{1t}.$$

Equating the coefficients in the restricted and unrestricted versions obtains:

For;

x_{1t}: $\varphi_{11} = \pi_{11}$

x_{2t}: $\varphi_{12} = \pi_{12}$

x_{1t-1}: $\varphi_{13} = D^{-1}\{\pi_{13}((1-\pi_{24})\pi_{11} + \pi_{14}\pi_{21}) + \pi_{i4}(\pi_{23}\pi_{11} + (1-\pi_{13})\pi_{21})\}\gamma_1$

x_{2t-1}: $\varphi_{14} = D^{-1}\{\pi_{13}((1-\pi_{24})\pi_{12} + \pi_{14}\pi_{22}) + \pi_{14}(\pi_{23}\pi_{12} + (1-\pi_{13})\pi_{22})\}\gamma_2$,

x_{1t}: $\varphi_{21} = \pi_{21}$

x_{2t}: $\varphi_{22} = \pi_{22}$

x_{1t-1}: $\varphi_{23} = D^{-1}\{\pi_{23}((1-\pi_{24})\pi_{11} + \pi_{14}\pi_{21}) + \pi_{24}(\pi_{23}\pi_{11} + (1-\pi_{13})\pi_{21})\}\gamma_1$

x_{2t-1}: $\varphi_{24} = D^{-1}\{\pi_{23}((1-\pi_{24})\pi_{12} + \pi_{14}\pi_{22}) + \pi_{24}(\pi_{23}\pi_{12} + (1-\pi_{13})\pi_{22})\}\gamma_2$,

or;

$$\varphi_{11} = \pi_{11}, \quad \varphi_{12} = \pi_{12}$$

$$\varphi_{13} = \{c_{11}\pi_{11} = c_{12}\pi_{21}\}\gamma_1$$

$$\varphi_{14} = \{c_{11}\pi_{12} + c_{12}\pi_{22}\}\gamma_2$$

$$c_{11} = (\pi_{13}(1 - \pi_{24}) + \pi_{14}\pi_{23})D^{-1}$$

$$c_{12} = \pi_{14}D^{-1}$$

$$\varphi_{21} = \pi_{21}, \quad \varphi_{22} = \pi_{22}$$

$$\varphi_{23} = (c_{21}\pi_{11} + c_{22}\pi_{21})\gamma_1$$

$$\varphi_{24} = (c_{21}\pi_{12} + c_{22}\pi_{22})\gamma_2$$

$$c_{21} = \pi_{23}D^{-1}$$

$$c_{22} = (\pi_{23}\pi_{14} + \pi_{24}(1 - \pi_{23}))D^{-1}.$$

Hence, the eight unrestricted ORF coefficient estimates can be expressed in terms of eight RF parameters. Therefore, no restrictions eixst. However, the RF parameters can be identified from these eight equations; π_{13}, π_{14}, π_{23}, π_{24} can be obtained from the solutions for c_{11}, c_{12}, c_{21}, c_{22}.

IIb: Assume Case a) is altered by the addition of one exogenous variable; $m = 2$, $n = n_1 = 3$, $P = 1$

$\therefore$ the relevant theory is:

$$y_{1t} = \pi_{10}x_{0t} + \pi_{11}x_{1t} + \pi_{12}x_{2t} + \pi_{13}y^e_{1t} + \pi_{14}y^e_{2t} + v_{1t}$$

$$y_{2t} = \pi_{20}x_{0t} + \pi_{21}x_{1t} + \pi_{22}x_{2t} + \pi_{23}y^e_{1t} + \pi_{24}y^e_{2t} + v_{2t}. \qquad \text{(IIb)}$$

Following the procedure from IIa, obtain a RES and resulting ORF. Then equate the coefficients with an unrestricted ORF. Hence, for the coefficient on;

x_{0t}: $\varphi_{10} = \pi_{10}$

x_{1t}: $\varphi_{11} = \pi_{11}$

x_{2t}: $\varphi_{12} = \pi_{12}$

x_{0t-1}: $\varphi_{13} = D^{-1}\{\pi_{13}(1-\pi_{24})\pi_{10} + \pi_{14}\pi_{20}) + \pi_{14}(\pi_{23}\pi_{10} + (1-\pi_{13})\pi_{20})\}\gamma_0$

x_{1t-1}: $\varphi_{14} = D^{-1}\{\pi_{13}(1-\pi_{24})\pi_{11} + \pi_{14}\pi_{21}) + \pi_{14}(\pi_{23}\pi_{11} + (1-\pi_{13})\pi_{21})\}\gamma_1$

x_{2t-1}: $\varphi_{15} = D^{-1}\{\pi_{13}(1-\pi_{24})\pi_{12} + \pi_{14}\pi_{22}) + \pi_{14}(\pi_{23}\pi_{12} + (1-\pi_{13})\pi_{22})\}\gamma_2$,

x_{0t}: $\varphi_{20} = \pi_{20}$

x_{1t}: $\varphi_{21} = \pi_{21}$

x_{2t}: $\varphi_{22} = \pi_{22}$

x_{0t-1}: $\varphi_{23} = D^{-1}\{\pi_{23}((1-\pi_{24})\pi_{10} + \pi_{14}\pi_{20}) + \pi_{24}(\pi_{23}\pi_{10} + (1-\pi_{13})\pi_{20})\}\gamma_0$

x_{1t-1}: $\varphi_{24} = D^{-1}\{\pi_{23}((1-\pi_{24})\pi_{11} + \pi_{14}\pi_{21}) + \pi_{24}(\pi_{23}\pi_{11} + (1-\pi_{13})\pi_{21})\}\gamma_1$

x_{2t-1}: $\varphi_{25} = D^{-1}\{\pi_{23}((1-\pi_{24})\pi_{12} + \pi_{14}\pi_{22}) + \pi_{24}(\pi_{23}\pi_{12} + (1-\pi_{13})\pi_{22})\}\gamma_2$,

or;

$$\varphi_{10} = \pi_{10}, \quad \varphi_{11} = \pi_{11}, \quad \varphi_{12} = \pi_{12}$$

$$\varphi_{13} = (c_{11}\pi_{10} + c_{12}\pi_{20})\gamma_0$$

$$\varphi_{14} = (c_{11}\pi_{11} + c_{12}\pi_{21})\gamma_1$$

$$\varphi_{15} = (c_{11}\pi_{12} + c_{12}\pi_{22})\gamma_2$$

$$c_{11} = D^{-1}(\pi_{13}(1 - \pi_{24}) + \pi_{14}\pi_{23})$$

$$c_{12} = D^{-1}(\pi_{14})$$

$$\varphi_{20} = \pi_{20},\ \varphi_{21} = \pi_{21},\ \varphi_{12} = \pi_{22}$$

$$\varphi_{23} = (c_{21}\pi_{10} + c_{22}\pi_{20})\gamma_0$$

$$\varphi_{24} = (c_{21}\pi_{11} + c_{22}\pi_{21})\gamma_1$$

$$\varphi_{25} = (c_{21}\pi_{12} + c_{22}\pi_{22})\gamma_2$$

where;

$$c_{21} = D^{-1}(\pi_{23}),$$

$$c_{22} = D^{-1}(\pi_{23}\pi_{14} + \pi_{24}(1-\pi_{13})).$$

Eliminating c_{11} and c_{12} from the first system reveals:

$$\frac{\varphi_{13}\ \varphi_{11}\ \gamma_1 - \varphi_{14}\ \varphi_{10}\ \gamma_0}{\varphi_{11}\ \varphi_{20}\ \gamma_1 - \varphi_{10}\ \varphi_{21}\ \gamma_1} = \frac{\varphi_{13}\ \varphi_{12}\ \gamma_2 - \varphi_{15}\ \varphi_{10}\ \gamma_0}{\varphi_{12}\ \varphi_{20}\ \gamma_2 - \varphi_{10}\ \varphi_{22}\ \gamma_2}\ .$$

Similarly, eliminating c_{21} and c_{22} from the second system yields:

$$\frac{\varphi_{23}\ \varphi_{11}\ \gamma_1 - \varphi_{24}\ \varphi_{10}\ \gamma_0}{\varphi_{11}\ \varphi_{20}\ \gamma_1 - \varphi_{10}\ \varphi_{21}\ \gamma_1} = \frac{\varphi_{23}\ \varphi_{12}\ \gamma_2 - \varphi_{25}\ \varphi_{10}\ \gamma_0}{\varphi_{12}\ \varphi_{20}\ \gamma_2 - \varphi_{10}\ \varphi_{22}\ \gamma_2}\ .$$

These two restrictions are type I, or overidentifying, since there are three independent equations for c_{11} and c_{12} and three independent equations for c_{21} and c_{22}. As a result there are six independent expressions for four of the RF parameters π_{13}, π_{14}, π_{23}, π_{24} -- resulting in two restrictions.

IIc: Assume the "relevant theory" is IIa but the processes generating the exogenous variables are altered:

$$\therefore \quad x_{1t} = \pi_{11}x_{1t-1} + \pi_{12}x_{1t-2} + \eta_{1t},$$
$$x_{2t} = \pi_{21}x_{2t-1} + \pi_{22}x_{2t-2} + \eta_{2t}.$$

Again, following the previous format, obtain a RES and ORF. Then equate the coefficients with an unrestricted ORF to obtain:

For;

x_{1t}: $\varphi_{11} = \pi_{11}$

x_{2t}: $\varphi_{12} = \pi_{12}$

x_{1t-1}: $\varphi_{131} = D^{-1}\{\pi_{13}((1-\pi_{24})\pi_{11}+\pi_{14}\pi_{21})+\pi_{14}(\pi_{23}\pi_{11}+(1-\pi_{13})\pi_{21})\}\gamma_{11}$

x_{1t-2}: $\varphi_{132} = D^{-1}\{\pi_{13}((1-\pi_{24})\pi_{11}+\pi_{14}\pi_{21})+\pi_{14}(\pi_{23}\pi_{11}+(1-\pi_{13})\pi_{21})\}\gamma_{12}$

x_{2t-1}: $\varphi_{141} = D^{-1}\{\pi_{13}((1-\pi_{24})\pi_{12}+\pi_{14}\pi_{22})+\pi_{14}(\pi_{23}\pi_{22}+(1-\pi_{13})\pi_{22})\}\gamma_{21}$

x_{2t-2}: $\varphi_{142} = D^{-1}\{\pi_{13}((1-\pi_{24})\pi_{12}+\pi_{14}\pi_{22})+\pi_{14}(\pi_{23}\pi_{12}+(1-\pi_{13})\pi_{22})\}\gamma_{22},$

x_{1t}: $\varphi_{21} = \pi_{21}$

x_{2t}: $\varphi_{22} = \pi_{22}$

x_{1t-1}: $\varphi_{231} = D^{-1}\{\pi_{23}((1-\pi_{24})\pi_{11}+\pi_{14}\pi_{21})+\pi_{24}(\pi_{23}\pi_{11}+(1-\pi_{13})\pi_{21})\}\gamma_{11}$

x_{1t-2}: $\varphi_{232} = D^{-1}\{\pi_{23}((1-\pi_{24})\pi_{11}+\pi_{14}\pi_{21})+\pi_{24}(\pi_{23}\pi_{11}+(1-\pi_{13})\pi_{21})\}\gamma_{12}$

x_{2t-2}: $\varphi_{241} = D^{-1}\{\pi_{23}((1-\pi_{24})\pi_{12}+\pi_{14}\pi_{22})+\pi_{24}(\pi_{23}\pi_{12}+(1-\pi_{13})\pi_{22})\}\gamma_{21}$

x_{2t-2}: $\varphi_{242} = D^{-1}\{\pi_{23}((1-\pi_{24})\pi_{12}+\pi_{14}\pi_{22})+\pi_{24}(\pi_{23}\pi_{12}+(1-\pi_{13})\pi_{22})\}\gamma_{22},$

or;

$\varphi_{11} = \pi_{11}$

$\varphi_{12} = \pi_{12}$

$\varphi_{131} = \{c_{11}\pi_{11} + c_{12}\pi_{21}\}\gamma_{11}$ c_{11}, c_{12} defined as in both

$\varphi_{132} = \{c_{11}\pi_{11} + c_{12}\pi_{21}\}\gamma_{12}$ IIa and IIb

$\varphi_{141} = \{c_{11}\pi_{12} + c_{12}\pi_{22}\}\gamma_{21}$

$\varphi_{142} = \{c_{11}\pi_{12} + c_{12}\pi_{22}\}\gamma_{22}$,

$\varphi_{21} = \pi_{21}$

$\varphi_{22} = \pi_{22}$

$\varphi_{231} = \{c_{21}\pi_{11} + c_{22}\pi_{21}\}\gamma_{11}$

$\varphi_{232} = \{c_{21}\pi_{11} + c_{22}\pi_{21}\}\gamma_{12}$ c_{21}, c_{22} defined as in both IIa and IIb.

$\varphi_{241} = \{c_{21}\pi_{12} + c_{22}\pi_{22}\}\gamma_{21}$

$\varphi_{242} = \{c_{21}\pi_{12} + c_{22}\pi_{22}\}\gamma_{22}$.

Clearly; $\frac{\varphi_{131}}{\varphi_{132}} = \frac{\gamma_{11}}{\gamma_{12}}$; $\frac{\varphi_{141}}{\varphi_{142}} = \frac{\gamma_{21}}{\gamma_{22}}$;

$$\frac{\varphi_{231}}{\varphi_{232}} = \frac{\gamma_{11}}{\gamma_{12}} ; \quad \frac{\varphi_{241}}{\varphi_{242}} = \frac{\gamma_{21}}{\gamma_{22}} .$$

As before there are numerous ways to express the four independent restrictions implied by this system. The restrictions originate from the similarity (up to a factor of γ_{ij}) of the coefficients for the lagged values of each of the exogenous variables in each equation. Analogous to Ic, these restrictions are not over-identifying. Although there are eight equations in the four unknows, (c_{11}, c_{12}, c_{21}, c_{22}), it is obvious that only four of these are independent. Consequently, the values for π_{13}, π_{14}, π_{23}, π_{24} may be obtained from exactly four expressions which link the c_{ij} with unrestricted ORF coefficients.

IId: Assume the relevant theory now contains only one exogenous variable -- but the process generating it is second order:

$$\therefore \quad y_t = \pi_{11}x_t + \pi_{13}y^e_{1t} + \pi_{14}y^e_{2t} + v_{1t},$$

$$y_t = \pi_{21}x_t + \pi_{23}y^e_{1t} + \pi_{24}y^e_{2t} + v_{2t}.$$

Once again, following the REH and comparing the resulting ORF with an unrestricted version obtains:

For;

x_t: $\varphi_{11} = \pi_{11}$

x_{t-1}: $\varphi_{121} = D^{-1}\{\pi_{13}((1-\pi_{24})\pi_{11}+\pi_{14}\pi_{21})+\pi_{14}(\pi_{23}\pi_{11}+(1-\pi_{13})\pi_{21})\}\gamma_{11}$

x_{t-2}: $\varphi_{122} = D^{-1}\{\pi_{13}((1-\pi_{24})\pi_{11}+\pi_{14}\pi_{21})+\pi_{14}(\pi_{23}\pi_{11}+(1-\pi_{13})\pi_{21})\}\gamma_{12}$,

x_t: $\varphi_{21} = \pi_{21}$

x_{t-1}: $\varphi_{221} = D^{-1}\{\pi_{23}((1-\pi_{24})\pi_{11}+\pi_{14}\pi_{21})+\pi_{24}(\pi_{23}\pi_{11}+(1-\pi_{13})\pi_{21})\}\gamma_{11}$

x_{t-2}: $\varphi_{222} = D^{-1}\{\pi_{23}((1-\pi_{24})\pi_{11}+\pi_{14}\pi_{21})+\pi_{24}(\pi_{23}\pi_{11}+(1-\pi_{13})\pi_{21})\}\gamma_{12}$.

Clearly; $\dfrac{\varphi_{121}}{\varphi_{122}} = \dfrac{\gamma_{11}}{\gamma_{12}} = \dfrac{\varphi_{221}}{\varphi_{222}}$.

These two restrictions, like those in IIc, are type II. This is especially clear in this case since the values of π_{13}, π_{14}, π_{23}, π_{24} cannot be identified from knowledge of the ORF coefficients in the above system. Despite the existence of six equations and six RF parameters, there are clearly only four independent equations in the above system. Nevertheless, the REH does generate testable restrictions in this case.

Example III

Assume the "relevant theory" contains lagged variables:

$$y_t = \beta_1x_{1t} + \beta_2x_{2t} + \beta_3x_{3t} + \beta_4x_{2t-1} + \beta_5x_{3t-1} + \beta_6x_{3t-2}$$
$$+ \beta_7x_{4t-1} + \Psi y_{t-1} + \theta y^e_t + \varepsilon_t.$$

$\therefore \quad m = 1,\ n_1 = 4,\ n_1 = 3,\ k_1 = 3,\ k_2 = 1,\ k_3 = 1.$

Assume all exogenous variables are generated by second order processes:

$$\therefore \quad P = 2; \quad \begin{aligned} x_{1t} &= \gamma_{11}x_{1t-1} + \gamma_{12}x_{1t-2} + \eta_{it}, \\ x_{2t} &= \gamma_{21}x_{2t-1} + \gamma_{22}x_{2t-2} + \eta_{it}, \\ x_{3t} &= \gamma_{31}x_{3t-1} + \gamma_{32}x_{3t-2} + \eta_{it}, \\ x_{4t} &= \gamma_{41}x_{4t-1} + \gamma_{42}x_{4t-2} + \eta_{it}. \end{aligned}$$

$\therefore \quad n_2 = 1$

Apply the REH to obtain the resulting RES for this model:

$$\underset{t-1}{E}\, y_t = (1-\theta)^{-1}\{\beta_1\gamma_{11}x_{1t-1} + \beta_1\gamma_{12}x_{1t-2} + \beta_2\gamma_{21}x_{2t-1} + \beta_2\gamma_{22}x_{2t-2} + \beta_3\gamma_{31}x_{3t-1} + \beta_3\gamma_{32}x_{3t-2} + \beta_4x_{2t-1} + \beta_5x_{3t-1} + \beta_6x_{3t-2} + \beta_7x_{4t-1} + \Psi y_{t-1}\}\ .$$

Combining terms and substituting the RES into the relevant theory yields the ORF:

$$\begin{aligned} y_t = {} & \beta_1x_{1t} + \beta_2x_{2t} + \beta_3x_{3t} \\ & + [\Psi + \theta(1 - \theta)^{-1}\Psi]y_{t-1} \\ & + \theta[1-\theta]^{-1}\{\beta_1\gamma_{11}x_{1t-1} + \beta_1\gamma_{12}x_{1t-2} + \beta_2\gamma_{22}x_{2t-2}\} \\ & + [\beta_4 + \theta(1 - \theta)^{-1}(\beta_2\gamma_{21} + \beta_4)]x_{3t-1} \\ & + [\beta_5 + \theta(1 - \theta)^{-1}(\beta_3\gamma_{31} + \beta_5)]x_{3t-1} \\ & + [\beta_6 + \theta(1 - \theta)^{-1}(\beta_3\gamma_{32} + \beta_6)]x_{3t-2} \\ & + [\beta_7 + \theta(1 - \theta)^{-1}\beta_7]x_{4t-1} + \varepsilon_t. \end{aligned}$$

In the notation of section 5.2, the coefficients on;

x_{1t}, x_{2t}, x_{3t} correspond to π_1^0;

y_{t-1} correspond to $(I - \pi_3)^{-1}\pi_2(L)$;

$x_{1t-1}, x_{1t-2}, x_{2,t-2}$ correspond to $\pi_3(I - \pi_3)^{-1}\pi_1^0\Gamma^*(L)$;

since

$$\Gamma^*(L) = \begin{Bmatrix} \gamma_{11} + \gamma_{12}L & 0 & 0 & 0 \\ 0 & \gamma_{22}L & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix}$$

and,

$x_{2t-1}, x_{3t-1}, x_{3t-2}, x_{4t-1}$ correspond to $\pi_3(I - \pi_3)^{-1}\pi_1^0\Gamma^{**}(L) + (I - \pi_3)^{-1}\pi_1'(L)$.

Therefore, following the notation used in the general example from Cahpter V, page 60, the (ORF-H_A) for this example is:

$$y_t = \varphi_{01}x_{1t} + \varphi_{02}x_{2t} + \varphi_{02}x_{3t} + \varphi_1 y_{t-1} + \varphi_{21}x_{1t-1} + \varphi_{22}x_{2t-2} + \varphi_{23}x_{3t-2} + \varphi_{31}x_{2t-1} + \varphi_{32}x_{3t-1} + \varphi_{33}x_{3t-2} + \varphi_{34}x_{4t-1} + \varepsilon_t$$

where; the unrestricted ORF coefficients; φ_{0i}, $i = 1,2,3$ apply to period t exogenous variables;

φ_1 apply to lagged endogenous variables;

φ_{2i}, $i = 1,2,3$ apply to lagged values whose coefficients appear in $\Gamma^*(L)$;

φ_{3i}, $i = 1,2,3,4$ apply to lagged values which appear in the original RF.

To examine the conditions for identification and reveal the REH restrictions, equate the restricted and unrestricted ORF coefficients:

$$\varphi_{01} = \beta_1$$
$$\varphi_{02} = \beta_2$$
$$\varphi_{03} = \beta_3$$
$$\varphi_1 = [\Psi + \theta(1 - \theta)^{-1}\Psi]$$
$$\varphi_{21} = \theta(1 - \theta)^{-1}\beta_1\gamma_{11}$$
$$\varphi_{22} = \theta(1 - \theta)^{-1}\beta_1\gamma_{12}$$
$$\varphi_{23} = \theta(1 - \theta)^{-1}\beta_2\gamma_{22}$$
$$\varphi_{31} = [\beta_4 + \theta(1 - \theta)^{-1}(\beta_2\gamma_{21} + \beta_4)]$$
$$\varphi_{32} = [\beta_5 + \theta(1 - \theta)^{-1}(\beta_3\gamma_{31} + \beta_5)]$$
$$\varphi_{33} = [\beta_6 + \theta(1 - \theta)^{-1}(\beta_3\gamma_{32} + \beta_6)]$$
$$\varphi_{34} = [\beta_7 + \theta(1 - \theta)^{-1}\beta_7].$$

The condition for the identification of the nine RF parameters from the relevant ORF coefficients is satisfied since;

$$n_1 - n_2 = 2 > 1 = m.$$

In the example above, θ corresponds to π_3. Clearly, a solution for θ, in terms of φ and γ, can be obtained from any of the expressions, φ_{21}, φ_{22}, φ_{23}. This solution may then be substituted into the expressions for φ_1, φ_{3i}, $i = 1,\ldots,4$ to identify the remaining RF parameters.

The restrictions implied by this structure are:

$$\frac{\varphi_{01}}{\varphi_{02}}\,\frac{\gamma_{11}}{\gamma_{22}} = \frac{\varphi_{21}}{23} \;;\; \frac{\varphi_{21}}{\varphi_{22}} = \frac{\gamma_{11}}{\gamma_{22}}\,.$$

The restrictions are obtained by analyzing the expressions for

φ_{21}, φ_{22}, φ_{23} which are the coefficients contained in the expression analogous to (5.2.5) for this example.

The first restriction is type I, or overidentifying, since the system contains ten independent equations and nine RF parameters. The second restriction is obtained by noting the similarities of the coefficients on lagged values of x_{1t}, hence, it is a type II.

A-3 Comparing the REH Restrictions with the Rule of Chapter V

The rule for counting restrictions is:

$$\text{Type I} = m \cdot \text{Max}\{n_1 - n_2 - m,\ 0\},$$

$$\text{Type II} = m \sum_{j=1}^{n_1} \text{Max}\{P - c_j^* - 1,\ 0\},$$

and

$$R = m\{\sum_{j=1}^{n_1} \{P - c_j^*\} - \text{Min}(n_1 - n_2,\ m)\}.$$

Both n_2 and c_j^* equal zero for examples I and II due to the absence of lagged variables.

Example Ia: $m = 1$, $n = n_1 = 1$, $P = 1$,

$$\text{type I} = 1\ \text{Max}\{0,\ 0\} = 0.$$

$$\text{type II} = 1\ \text{Max}\{0,\ 0\} = 0.$$

$$R = 1\{1 - \text{Min}(1,\ 1)\} = 0.$$

$\therefore$ Rule agrees with restrictions obtained.

Ib: $m = 1$, $n = n_1 = 2$, $P = 1$,

type I = 1{Max(1, 0)} = 1.

type II = 1{Max{0, 0} + Max{0, 0}} = 0.

R = 1{1 + 1 - Min(2, 1)} = 1.

∴ Rule agrees with restrictions obtained.

Ic: $m = 1$, $n = n_1 = 1$, $P = 2$,

type I = 1{Max(0, 0)} = 0.

type II = 1{Max(1, 0)} = 1.

R = 1{2 - Min(1, 1)} = 1.

∴ Rule agrees with restrictions obtained.

IIa: $m = 2$, $n = n_1 = 2$, $P = 1$,

type I = 2{Max(0, 0)} = 0.

type II = 2{Max(0, 0) + Max(0, 0)} = 0.

R = 2{1 + 1 - Min(2, 2)} = 0.

∴ Rule agrees with restrictions obtained.

IIb: $m = 2$, $n = n_1 = 3$, $P = 1$,

type I = 2 · Max{1, 0} = 2.

type II = 2(Max(0, 0) + Max(0, 0) + Max(0, 0)) = 0.

R = 2[1 + 1 + 1 - Min(3, 2)] = 2.

∴ Rule agrees with restrictions obtained.

IIc: $m = 2, n = n_1 = 2, P = 2,$

$$\text{type I} = 2 \cdot \text{Max}\{0, 0\} = 0.$$

$$\text{type II} = 2\{\text{Max}(1, 0) + \text{Max}(1, 0)\} = 4.$$

$$R = 2\{2 + 2 - \text{Min}(2, 2)\} = 4.$$

∴ Rule agrees with restrictions obtained.

IId: $m = 2, n = n_1 = 1, P = 2,$

$$\text{type I} = 2 \cdot \text{Max}\{-1, 0\} = 0.$$

$$\text{type II} = 2 \cdot \{\text{Max}(1, 0)\} = 2.$$

$$R = 2 \cdot \{2 - \text{Min}\{1, 2\}\} = 2.$$

∴ Rule agrees with restrictions obtained.

III: $m = 1; P \geq 2,$

$n = 4 : x_1, x_2, x_3, x_4.$

$n_1 = 3 : x_1, x_2, x_3.$

$n_2 = 1 : x_3.$

$c_1^* = 0.$

$c_2^* = 1.$

$c_3^* = 2.$

$$\text{type I} = 1 \cdot \text{Max}\{3 - 1 - 1, 0\} = 1,$$

$$\begin{aligned}\text{type II} = {} & 1 \cdot \text{Max}\{2 - 0 - 1, 0\} + \\ & 1 \cdot \text{Max}\{2 - 1 - 1, 0\} + \\ & 1 \cdot \text{Max}\{2 - 2 - 1, 0\} = 1.\end{aligned}$$

$$R = 1\{(2-0) + (2-1) + (2-2) - \text{Min}(2,1)\} = 2.$$

∴ Rule agrees with restrictions obtained.