GRADIENT ESTIMATES FOR SOLUTIONS TO DIVERGENCE FORM ELLIPTIC EQUATIONS WITH PIECEWISE CONSTANT COEFFICIENTS IN DIMENSION N.

By

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ABSTRACT

GRADIENT ESTIMATES FOR SOLUTIONS TO DIVERGENCE FORM ELLIPTIC EQUATIONS WITH PIECEWISE CONSTANT COEFFICIENTS IN DIMENSION N.

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In a bounded C^{1,α_0} domain $\Omega \subset \mathbb{R}^n$, $0 < \alpha_0 \leq 1$, contains two simply connected and strictly convex C^{1,α_0} subdomains (inclusions) D_1 and D_2 that satisfy $D_1 \cup D_2 \subset\subset \Omega$ and $\overline{D_1} \cap \overline{D_2} = \{0\}$, we study the following elliptic differential equation

$$\begin{cases} \operatorname{div}(a(x)\nabla u) = 0 & \text{in } \Omega, \\ \partial_\nu u(x) = g & \text{on } \partial\Omega, \\ \int_{\partial\Omega} u = 0, \end{cases} \tag{1}$$

where

$$a(x) = 1 + (k-1)\chi_{(D_1 \cup D_2)}(x), \quad 0 < k < \infty,\ k \neq 1,$$

and

$$g \in L_0^2(\partial\Omega) := \{g \in L^2(\partial\Omega) : \int_{\partial\Omega} g = 0\}.$$

The problem arises from studying fiber re-inforced composite media with closely spaced inclusions in dimension $n = 2$. For dimension $n = 3$, the problem also appear in the study of steady state solutions to Maxwell's equations.

The bound on the gradient yields the boundedness of the strain in fiber re-inforced materials and electrical fields when the two inclusions are touching each other.

Following the ideas of [1], we assume that D_1 lies in the half space $x_n < 0$, and D_2 in

the half space $x_n > 0$. Then we begin by separating the inclusions by a distance $\delta > 0$, *that is*, we set

$$D_1^\delta = D_1 - \frac{\delta}{2}\mathbf{e_n}, \text{ and } D_2^\delta = D_2 + \frac{\delta}{2}\mathbf{e_n},$$

where $\mathbf{e_n} = (0', 1)$. Then we study the approximate differential equation corresponding to the separated inclusions which is

$$\begin{cases} \operatorname{div}(a_\delta(x)\nabla u_\delta) = 0 & \text{in } \Omega, \\ \partial_\nu u_\delta(x) = g & \text{on } \partial\Omega, \\ \int_{\partial\Omega} u_\delta = 0, \end{cases} \tag{2}$$

where $a_\delta(x) = 1 + (k-1)\chi_{(D_1^\delta \cup D_2^\delta)}(x)$. The solution of the elliptic equation (2) has an integral representation in terms of potential functions defined on the boundary of each subdomain. From the representation formula, we derive uniform piecewise $C^{1,\alpha}$, $0 < \alpha < \alpha_0$, estimates for this solution which are independent of the distance between the subdomains. That is, we find the estimate

$$\|u_\delta\|_{C^{1,\alpha}\left(\Omega_{\tilde\epsilon} \setminus \overline{D_1^\delta \cup D_2^\delta}\right)} + \|u_\delta\|_{C^{1,\alpha}(\overline{D_1^\delta})} + \|u_\delta\|_{C^{1,\alpha}(\overline{D_2^\delta})} \le C \|g\|_{L^2(\partial\Omega)},$$

where $\Omega_{\tilde\epsilon} = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \tilde\epsilon\}$ and C is independent of δ. Our result extends the earlier result for dimension $n = 2$ [1], but the analysis is much more complicated. Final estimates rely on detailed analysis near the touching point and collective compactness of some integral operators.

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Chapter 1

Introduction

The purpose of this work is to study gradient estimates for solutions of divergence form elliptic equations with piecewise constant coefficients in dimension $n \geq 2$. The problem arises from studying fiber re-inforced composite media with closely spaced inclusions. A composite medium described by a bounded domain Ω in $\mathbb{R}^n$ ($n \geq 2$), which includes finitely many inclusions (subdomains) D_j, $j = 1, \cdots, m$. The physical characteristics of the medium are smooth in each inclusion D_j as well as in $\Omega \setminus (\overline{D_1 \cup \cdots \cup D_m})$, but they are possibly discontinuous across their boundaries ∂D_j, $j = 1, \cdots m$.

We use the example from the introductions of [8, 30] to motivate the problem for dimension $n = 2$. The bounded domain $\Omega \subset \mathbb{R}^2$ models the cross-section of a fiber-reinforced composite. Let D_1 and D_2 be two subdomains represent the cross-section of the fibers, and $\Omega \setminus (\overline{D_1 \cup D_2})$ represent the region surrounding the fibers. Assume that the shear modulus (modulus of rigidity) of the fibers is $1 < k < \infty$, and is 1 for the materials in the surrounding area. Using a standard model of anti-plane shear, we obtain the following differential equation

$$\operatorname{div}\left(\left(1 + (k-1)\chi_{\left(D_1 \cup D_2\right)}\right)\nabla u\right) = 0 \quad \text{in } \Omega, \tag{1.1}$$

with appropriate boundary data, for instance,

$$u = g \quad \text{on } \partial\Omega.$$

The function u represents the out of plane elastic displacement and ∇u represents the strain.

For dimension $n = 3$, the problem also appear in the study steady state solutions to Maxwell's equations.

Let $\mathbf{E}$ be the electric field and $\mathcal{D}$ be the electric displacement in conducting isotropic medium $\Omega \subset \mathbb{R}^3$. The fields $\mathbf{E}$ and $\mathcal{D}$ satisfy the differential equations

$$\text{curl}\, \mathbf{E} = 0, \quad \mathcal{D} = \epsilon\, \mathbf{E}, \quad \text{and} \quad \text{div}(\mathcal{D}) = 0 \quad \text{in} \quad \Omega,$$

where the real valued function ϵ represents the conductivity (dielectricity) of the materials. Therefore, there exists a potential function u (representing the voltage) such that

$$\mathbf{E}(x) = -\nabla u(x) \quad \text{in} \quad \Omega,$$

where Ω assumed to be simply connected domain in $\mathbb{R}^3$. Hence,

$$\mathcal{D}(x) = -\epsilon(x)\, \nabla u(x), \quad \text{and} \quad \text{div}\big(\epsilon \nabla u\big) = 0 \quad \text{in} \quad \Omega.$$

Now suppose that a conductor filling the region Ω that consists of different materials, say, D_1, D_2 and $\Omega \setminus (D_1 \cup D_2)$. We assume that conductivity $\epsilon(x)$ equals to k in $D_1 \cup D_2$ and $\epsilon(x)$ equals to 1 in $\Omega \setminus \overline{D_1 \cup D_2}$. The current flux on $\partial\Omega$ is represented by the normal derivative $\frac{\partial u}{\partial \nu}$. Therefore, for a given Neumann boundary data g on $\partial\Omega$, we obtain the following differential equation

$$\begin{cases} \operatorname{div}\left(\left(1+(k-1)\chi_{\left(D_1\cup D_2\right)}\right)\nabla u\right)=0 & \text{in } \Omega, \\ \partial_\nu u(x)=g & \text{on } \partial\Omega, \\ \int_{\partial\Omega} g=0. \end{cases} \tag{1.2}$$

At this point, we can emphasize the aim of studying such class of divergence elliptic equations, which is to study the behavior of the strain (electric field) ∇u when the inclusions approach each other (touch) as well as when the shear modulus (conductivity) of the inclusions k degenerate ($k = 0$ or $k = \infty$).

E. Bonnetier and M. Vogelius [8] have shown that the solution u of problem (1.1) in dimension $n = 2$ is in $W^{1,\infty}(\Omega)$ for any fixed $0 < k < \infty$ when the inclusions D_1 and D_2 are two disks.

Y. Y. Li and M. Vogelius [30] studied more general class of elliptic differential equations, where Ω is a bounded domain in $\mathbb{R}^n$ with a $C^{1,\alpha}$ boundary, $0 < \alpha < 1$, containing m disjoint subdomains D_k, $1 \le k \le m$, each with a $C^{1,\alpha}$ boundary such that $\overline{\Omega} = \cup_{k=1}^m \overline{D}_k$ (for precise geometric picture of the subdomains see [30]). The matrix $A(x) = (a_{ij}(x))$ is uniformly elliptic matrix and $A(x)$ is C^μ in each subdomain D_k, $k = 1, \cdots, m$ (but possibly discontinuous across the boundaries ∂D_k). For a vector valued function $g = (g_1, \cdots, g_n)$ that is C^μ in each subdomain D_k, $k = 1, \cdots, m$ (but also possibly discontinuous across the boundaries ∂D_k), denote g^k to be the function g restricted to the subdomain D_k. They found that for $h \in L^\infty(\Omega)$, if $u \in H^1(\Omega)$ is a solution to

$$\partial_i\big(a_{ij}\,\partial_j u\big) = h + \partial_i g_i \quad \text{in } \Omega, \tag{1.3}$$

then the following estimate holds

$$\max_{1\le k\le m} \|u\|_{C^{1,\alpha'}(\overline{D_k}\cap\Omega_\epsilon)} \le C\Big(\|u\|_{L^\infty(\Omega)} + \|h\|_{L^\infty(\Omega)} + \max_{1\le k\le m} \left\|g^k\right\|_{C^{\alpha'}(\overline{D_k})} \Big), \qquad (1.4)$$

where C is independent of the distance between the inclusions, $\Omega_\epsilon = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \epsilon\}$, and $0 < \alpha' \le \mu$ and $\alpha' < \frac{\alpha}{n(1+\alpha)}$. They also have studied special case in $\mathbb{R}^2$ when Ω is a disk centered at the origin with radius R and it contains two unit disks D_1 and D_2 centered, respectively, at $(0,-1)$ and $(0,1)$. That is, the boundaries ∂D_1 and ∂D_2 touch at the origin. They considered the following differential equation

$$\begin{cases} \partial_i\big(a(x)\, \partial_i u\big) = 0 \quad \text{in } \Omega, \\ u = g \quad \text{on } \partial\Omega, \end{cases}$$

where $g \in H^{\frac{1}{2}}(\partial\Omega)$ and

$$a(x) = \begin{cases} 1 & \text{if } x \in \Omega \setminus \overline{D_1 \cup D_2} \\ 0 < a_0 < \infty & \text{if } x \in D_1 \cup D_2. \end{cases}$$

For $R > 2$, they found by using the conformal mapping the following bounds

$$\|D^\gamma u\|_\infty \le C(l) \qquad \text{in } \ D_1 \ \text{ and } \ D_2, \forall l, \ |\gamma| \le l,$$

$$\|D^\gamma u\|_\infty \le C(k,\epsilon) \qquad \text{in } \ D_3 \cap \Omega_\epsilon, \forall l, \ |\gamma| \le l,$$

where $D_3 = \Omega \setminus \overline{D_1 \cup D_2}$.

Y.Y. Li and L. Nirenberg [29] generalized the result of [30] into systems of elliptic differential equations and they found that u satisfies the same bound in (1.4) with slightly better regularity α' where

$$0 < \alpha' \leq \min\{\mu, \frac{\alpha}{2(1+\alpha)}\}.$$

G. Citti and F. Ferrari [9] followed [30] with slightly different modifications and they found the optimal regularity for the problem (1.3) when the inclusions are strictly separated. That is, they showed that the gradient ∇u is $C^{\alpha'}$, in each component, for $\alpha' \leq \min\{\mu, \alpha\}$ when the inclusions are strictly disjoint.

J. Mateu, J. Orobitg and J. Verdera [22] approached the problem $\Big(\text{div}(A\nabla u){=}0, \det(\text{A}){=}1\Big)$ via Beltrami equation and they used Calderon-Zygmund operators and quasiconformal mapping. If the inclusions D_m do not touch, they showed that ∇u is $C^{\alpha'}$, for $\alpha' < \min\{\mu, \alpha\}$ in each inclusion and their $C^{\alpha'}$ norm is independent of the distance between the inclusions.

Recently, H. Ammari, E. Bonnetier, F. Triki, M.S. Vogelius [1] have studied problem (1.5) in dimension $n = 2$ and they found that ∇u is piecewise uniformly bounded in $C^{\alpha'}$ norm for any $\alpha' < \alpha$ independent of the distance between the subdomains. In this work, we will generalize the results of [1] into higher dimensions $n \geq 2$ by mimicking their ideas with necessary modifications. Before we start outline our work, we briefly mention some results when the conductivity k degenerates.

In the case of $k = \infty$ (perfect conductivity) or $k = 0$ (insulated conductivity) the gradient may blow-up as the inclusions approach each other (touch). In [12, 13] it has been shown that the rate of the blow-up for the perfect conductivity problem is $\delta^{-1/2}$ for dimension $n = 2$, $(\delta\,|\ln \delta|)^{-1}$ for dimension $n = 3$ and δ^{-1} for higher dimensions $n \geq 4$, where δ is the

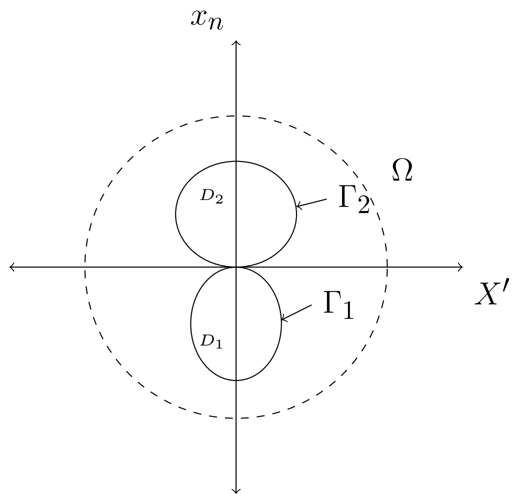

Figure 1.1: The touching subdomains

distance between the inclusions. For the insulated conductivity problem they have found $\delta^{-1/2}$ as an upper bound of the gradient.

H. Kang, M. Lim and K. Yun [20] studied problem (1.1) in dimension $n = 2$ when the inclusions are two disks separated by a distance δ. They decomposed the solution u to (1.1) as

$$u = g + b,$$

where ∇g is singular function depending on the distance between the inclusions while ∇b is bounded function regardless of the distance δ. They found an explicit formula for the singular part ∇g that describes the behavior of the ∇u which may blow-up at rate $\delta^{-1/2}$ when $k = 0$ or $k = \infty$.

H. Ammari, G. Ciraolo, H. Kang, H. Lee and K. Yun [17] generalized the results of [20] to any strictly convex simply connected $C^{1,\alpha}$ subdomains D_1 and D_2 in R^2.

For more results in dimension $n = 2$ when the inclusions are circles, see [16, 18, 19].

Now, we return to our problem in dimension $n \geq 2$. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain containing the origin 0. For $0 < \alpha_0 \leq 1$, let D_1 and D_2 be two C^{1,α_0} simply connected and strictly convex subdomains (inclusions) contained in Ω satisfying $D_1 \cup D_2 \subset\subset \Omega$ and $\overline{D_1} \cap \overline{D_2} = \{0\}$. Furthermore, we assume that D_1 lies in the half space $x_n < 0$, and D_2 in the half space $x_n > 0$. We denote by Γ_j to be the boundary of D_j, $j = 1, 2$. We study the following differentail equation

$$\begin{cases} \operatorname{div}(a(x)\nabla u) = 0 \quad \text{in } \Omega, \\ \partial_\nu u(x) = g \qquad\qquad \text{on } \partial\Omega, \\ \int_{\partial\Omega} u = 0, \\ \int_{\partial\Omega} g = 0, \end{cases} \tag{1.5}$$

where

$$a(x) = 1 + (k-1)\chi_{(D_1 \cup D_2)}(x).$$

Clearly, Lax-Milgram theorem shows that the problem (1.5) has a unique solution $u \in H^1(\Omega)$ for a given $g \in L^2(\partial\Omega)$.

Following the ideas of [1], we begin by separating the inclusions by a distance $\delta > 0$, *that is*, we set

$$D_1^\delta = D_1 - \frac{\delta}{2}\mathbf{e_n}, \text{ and } D_2^\delta = D_2 + \frac{\delta}{2}\mathbf{e_n},$$

where $\mathbf{e_n} = (0', 1)$. Then we study the approximate differential equation corresponding to the separated inclusions which is

$$
\begin{cases}
\operatorname{div}\left(a_{\delta}(x)\nabla u_{\delta}\right)=0 & \text{in } \Omega, \\
\partial_{\nu} u_{\delta}(x)=g & \text{on } \partial\Omega, \\
\int_{\partial\Omega} u_{\delta}=0, \\
\int_{\partial\Omega} g=0,
\end{cases}
\tag{1.6}
$$

where a_{δ} is the corresponding piecewise constant coefficients, *that is*,

$$
a_{\delta}(x)=1+(k-1)\chi_{(D_1^{\delta}\cup D_2^{\delta})}(x).
$$

It is well known that equation (1.6) has a unique solution u_{δ} in $H^1(\Omega)$ for a given Neumann boundary data $g \in L^2_0(\partial\Omega) := \{g \in L^2(\partial\Omega) : \int_{\partial\Omega} g \, d\sigma(x)=0\}$. We prove in Theorem 2.5 the solution to (1.6), u_{δ}, approaches uniformly in $H^1(\Omega)$ to u, the solution of equation (1.5), as δ approaches 0.

Now we state the main result of our work in the following theorem which coincides with the earlier result for dimension $n=2$ [1].

Theorem 1.1. *Let $\tilde{\epsilon}>0$ and $0<\alpha<\alpha_0$. The solution to (1.6) satisfies*

$$
\|u_{\delta}\|_{C^{1,\alpha}\left(\Omega_{\tilde{\epsilon}}\setminus\overline{D_1^{\delta}\cup D_2^{\delta}}\right)}+\|u_{\delta}\|_{C^{1,\alpha}(\overline{D_1^{\delta}})}+\|u_{\delta}\|_{C^{1,\alpha}(\overline{D_2^{\delta}})}\leq C\|g\|_{L^2(\partial\Omega)},
$$

where $\Omega_{\tilde{\epsilon}}=\{x\in\Omega : dist(x,\partial\Omega)>\tilde{\epsilon}\}$ and C is independent of δ.

In order to prove the main result, we define the single and double layer potentials for the

Laplacian operator, respectively, by

$$\mathbf{S}_{\partial\Omega}\varphi(x) := \int_{\partial\Omega} \Phi(x,y)\, \varphi(y)\, d\sigma(y), \qquad x \in \mathbb{R}^n, \tag{1.7}$$

and

$$\mathbf{D}_{\partial\Omega}\varphi(x) := \int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial\nu(y)}\, \varphi(y)\, d\sigma(y), \qquad x \in \mathbb{R}^n \setminus \partial\Omega. \tag{1.8}$$

where $\Phi(x,y)$ is the fundamental solution to Laplace equation $\Delta u(x) = 0$ in $\mathbb{R}^n$.

It is known from [23], the solution of problem (1.6) can be uniquely represented (see Theorem 2.14) as

$$u_\delta(x) = H_\delta(x) + \mathbf{S}_1\varphi_1^\delta(x) + \mathbf{S}_2\varphi_2^\delta(x), \quad x \in \Omega, \tag{1.9}$$

where $\mathbf{S}_j$ is the single layer potential on the surface Γ_j, $j = 1, 2.$ and H_δ is a harmonic function in Ω that has the following form

$$H_\delta(x) = -\mathbf{S}_{\partial\Omega}g(x) + \mathbf{D}_{\partial\Omega}\left(u_\delta|_{\partial\Omega}\right)(x), \qquad x \in \Omega.$$

While the potentials φ_1^δ and φ_2^δ solve the following system of integral equations

$$\begin{cases} \left(\lambda I - K_1^*\right)\varphi_1^\delta(x) - \partial_\nu \mathbf{S}_2\varphi_2^\delta(x - \delta\mathbf{e_n}) = \partial_\nu H_\delta\big(x - \frac{\delta}{2}\mathbf{e_n}\big), & x \in \Gamma_1, \\ -\partial_\nu \mathbf{S}_1\varphi_1^\delta\big(x + \delta\mathbf{e_n}\big) + \left(\lambda I - K_2^*\right)\varphi_2^\delta(x) = \partial_\nu H_\delta\big(x + \frac{\delta}{2}\mathbf{e_n}\big), & x \in \Gamma_2, \end{cases} \tag{1.10}$$

where in this system $\lambda = \dfrac{k+1}{2(k-1)}$, and K_i^* denotes the operator

$$K_i^*\varphi_i^\delta(x) = \frac{1}{\omega_n}\int_{\Gamma_i} \frac{(x-y)\cdot\nu(x)}{|x-y|^n}\, \varphi_i^\delta(y)\, d\sigma(y), \quad i = 1, 2.$$

From the classic potential theory, we easily show that K_j^*, $j = 1, 2.$ are compact operators from $C^\alpha(\Gamma_1)$ into $C^\alpha(\Gamma_2)$ for any $\alpha < \alpha_0$.

If we define for $\left(\varphi_1^\delta, \varphi_2^\delta\right) \in C^\alpha(\Gamma_1) \times C^\alpha(\Gamma_2)$ the operator

$$T^\delta \begin{pmatrix} \varphi_1^\delta \\ \\ \varphi_2^\delta \end{pmatrix} := \begin{pmatrix} \lambda I - K_1^* & L_2^\delta \\ \\ L_1^\delta & \lambda I - K_2^* \end{pmatrix} \begin{pmatrix} \varphi_1^\delta \\ \\ \varphi_2^\delta \end{pmatrix}, \tag{1.11}$$

where

$$L_2^\delta \varphi_2^\delta(x) = -\partial\nu \mathbf{S}_2 \varphi_2^\delta\big(x - \delta \mathbf{e_n}\big), \qquad x \in \Gamma_1, \tag{1.12}$$

and

$$L_1^\delta \varphi_1^\delta(x) = -\partial\nu \mathbf{S}_1 \varphi_1^\delta\big(x + \delta \mathbf{e_n}\big), \qquad x \in \Gamma_2. \tag{1.13}$$

Then we can rewrite the system (1.10) in the following form

$$T^\delta \begin{pmatrix} \varphi_1^\delta \\ \\ \varphi_2^\delta \end{pmatrix} = \begin{pmatrix} \partial_\nu H_\delta\big(y_1(x,\delta)\big) \\ \\ \partial_\nu H_\delta\big(y_2(x,\delta)\big) \end{pmatrix}, \tag{1.14}$$

where

$$y_1(x,\delta) = \big(x - \frac{\delta}{2}\mathbf{e_n}\big), \quad x \in \Gamma_1,$$

and

$$y_2(x,\delta) = \big(x + \frac{\delta}{2}\mathbf{e_n}\big), \quad x \in \Gamma_2.$$

Since H_δ is harmonic function on $\mathbb{R}^n$, then we easily prove that H_δ is uniformly bounded in

C^{α_0} norm on the surfaces Γ_j, j=1,2. That is, for $m = 1, 2, \cdots$, the following bounds hold

$$\|\partial_\nu H_\delta\|_{C^{\alpha_0}(\Gamma_j)} \leq C \|g\|_{L^2(\partial\Omega)}, \tag{1.15}$$

where C is independent of δ.

From the regularity of the single layer potential (see Lemma 2.18) and the representation formula (1.9), we have for any $0 < \alpha' < \alpha < \alpha_0$ and $\delta > 0$ the following bound

$$\sum_{j=1}^{2} \|u_\delta\|_{1,\alpha\left(\overline{D_j^\delta}\right)} + \|u_\delta\|_{1,\alpha\left(\Omega_{\tilde{\epsilon}} \setminus D_1^\delta \cup D_2^\delta\right)} \leq C \left(\sum_{i=1}^{2} \left\|\varphi_i^\delta\right\|_{\alpha'(\Gamma_i)} + \|g\|_{L^2(\partial\Omega)} \right), \tag{1.16}$$

where $\Omega_{\tilde{\epsilon}} = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \tilde{\epsilon}\}$ and C is depending on α, α', α_0, Ω, k, and $\tilde{\epsilon}$ but independent of δ.

We clearly observe from (1.16) the gradient ∇u_δ is C^α in each component $\overline{D_1^\delta}$, $\overline{D_2^\delta}$ and $\Omega_{\tilde{\epsilon}} \setminus D_1^\delta \cup D_2^\delta$ if the potentials φ_j^δ are uniformly bounded in $C^\alpha(\Gamma_j)$, $j = 1, 2.$, for any $\alpha < \alpha_0$. In other words, In order to obtain piecewise Hölder continuity of ∇u_δ, we need to show

$$\sum_{i=1}^{2} \left\|\varphi_i^\delta\right\|_{\alpha(\Gamma_i)} \leq C \|g\|_{L^2(\partial\Omega)}, \; \forall \, \alpha < \alpha_0, \tag{1.17}$$

where C is independent of δ.

By involving (1.14), the bound (1.17) follows if we prove the operator T^δ is invertible in $C^\alpha(\Gamma_1) \times C^\alpha(\Gamma_2)$ and its inverse is unifromly bounded on $C^\alpha(\Gamma_1) \times C^\alpha(\Gamma_2)$, for any $\alpha < \alpha_0$.

From elementary potential theory when $\delta > 0$, the operator T^δ is invertible as an operator on $C^\alpha(\Gamma_1) \times C^\alpha(\Gamma_2)$. To be more precise, T^δ can be written as $T^\delta = \lambda \mathbf{I} + \mathbf{K}_\delta$, where $\mathbf{K}_\delta$ is a compact operator on $C^\alpha(\Gamma_1) \times C^\alpha(\Gamma_2)$. Then we use Fredholm theory to show the invertibility of the operators T^{δ}.

Recall that, if we have a family of bounded linear operators, say, $\mathcal{T}^{\delta}$ defined on a Banach space, say, $\mathbf{X}$. Let us assume that $\mathcal{T}^{\delta}$ converges in **norm** to an operator $\mathcal{T}^{0} \in \mathcal{L}(\mathbf{X})$ (the space of all bounded linear operators on $\mathbf{X}$). Notice that, if $(\mathcal{T}^{0})^{-1} \in \mathcal{L}(\mathbf{X})$, then there exist $\delta_0 > 0$ be such that for any $\delta \leq \delta_0$ there exist uniformly bounded $(\mathcal{T}^{\delta})^{-1} \in \mathcal{L}(\mathbf{X})$, in which case

$$\left\|(\mathcal{T}^{\delta})^{-1} - (\mathcal{T}^{0})^{-1}\right\| \longrightarrow 0.$$

Naively, we conclude from this observation that we would obtain the existence of uniformly bounded $(T^{\delta})^{-1} \in C^{\alpha}(\Gamma_1) \times C^{\alpha}(\Gamma_2)$ if we proved the existence of $(T^{0})^{-1} \in C^{\alpha}(\Gamma_1) \times C^{\alpha}(\Gamma_2)$ (we consider T^{0} as a limiting operator corresponding to T^{δ}) and T^{δ} converges in **norm** into T^{0}. Therefore, we ask whether we can have convergence in **norm** for the family T^{δ} into their limiting operator T^{0}. Ammari, H. and Bonnetier, E. and Triki, F. and Vogelius,M.S. [1] have proved that such convergence in norm cannot hold. In fact, they showed for their case in dimension $n = 2$ the limiting operators $L_j^{0}, j = 1, 2.$ (the limiting operators corresponding to the **compact operators** $L_j^{\delta}, j = 1, 2.$) are not compact operators on C^{α} for any $\alpha < \alpha_0$. Thus $L_j^{\delta}, j = 1, 2.$ cannot converge in **norm** to their limits. Therefore, the above observation cannot be used to obtain the uniformly bounded operators $(T^{\delta})^{-1}$. In dimension $n = 2$ [1] the authors used the notion of collectively compact operators that require just pointwise convergence [3]. That is, ***For collectively compact operators*** K_n***,*** $n = 1, 2, ...$ ***such that*** $K_n \longrightarrow K$ ***(pointwise convergence) and*** $(I - K)^{-1}$ ***exists. Then for some*** $n \geq N$ ***the operators*** $(I - K_n)^{-1}$ ***exist and are bounded uniformly, in such a case***

$$(I - K_n)^{-1} \longrightarrow (I - K)^{-1}.$$

Therefore, in order to obtain the existence of uniformly bounded operators $(T^\delta)^{-1}$ in $C^\alpha(\Gamma_1)\times C^\alpha(\Gamma_1)\times$ for $\alpha < \alpha_0$ by using the notion of collectively compact operators, we need to show two major things which are:

1. The operator T^δ can be written as $T^\delta = I - \tilde{\Lambda}_\delta$ where $\tilde{\Lambda}_\delta$ are collectively compact and $\tilde{\Lambda}_\delta$ converges pointwise to some limiting operator $\tilde{\Lambda}_0$.

2. The limiting operator $T^0 = I - \tilde{\Lambda}_0$ is invertible.

Notice that, when δ approaches 0, the kernel of the operator T^δ becomes singular at $x = 0$. In fact, the off-diagonal operators L_2^δ and L_1^δ become singular at $x = 0$ when δ approaches 0.

To overcome the singularity, we follow [1] and decompose the operators L_j^δ, $j = 1, 2$. We only show it for L_2^δ but the decomposition of L_1^δ follows similarly.

Fix ϵ_0 and let $0 \le \epsilon \le \epsilon_0$. We define two auxiliary functions $\psi_{1,\epsilon}$ and $\psi_{2,\epsilon}$ globally on $\mathbb{R}^{n-1}$ (see Lemma 3.4) such that

$$\begin{cases} \psi_{j,\epsilon} = \psi_j, \quad |x'| \le \epsilon, \quad j = 1, 2, \\ \left\|\psi_{j,\epsilon}\right\|_{1,\beta;\mathbb{R}^{n-1}} \le C\, \epsilon^\nu \left\|\psi_j\right\|_{1,\alpha_0}, \quad j = 1, 2. \end{cases} \tag{1.18}$$

for any $\beta < \alpha_0$, $\nu = \alpha_0 - \beta$, and $x' = (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1}$. For the definition of $\psi_j, j = 1, 2$ see the Remark 2.1.

Let $\delta > 0$ and $\varphi \in C^\alpha(\Gamma_2)$. For $x \in \Gamma_1$, $|x| < R_0$, we set $x_\epsilon = (x', \psi_{1,\epsilon}(x'))$ and then we define the approximate surface

$$\Gamma_{2,\epsilon} = \{y_\epsilon = (y', \psi_{2,\epsilon}(y')) \mid y' \in \mathbb{R}^{n-1}\}.$$

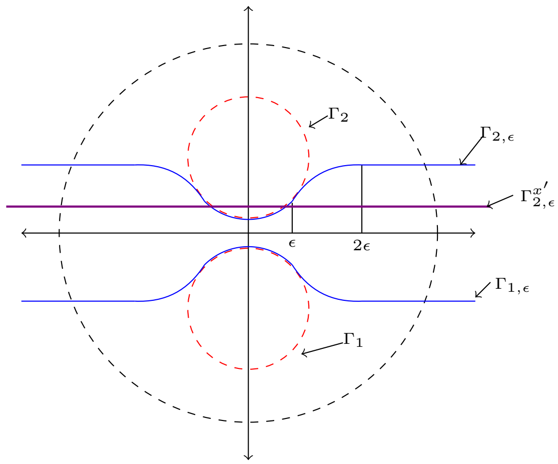

Figure 1.2: The approximate surfaces

Let $\mathbf{S}_{\mathbf{2},\epsilon}\mathcal{E}$ be the single layer potential defined on the approximate surface $\Gamma_{2,\epsilon}$ for some potential $\mathcal{E}$ that defined on the surface $\Gamma_{2,\epsilon}$ be such that

$$\mathcal{E} \equiv \varphi \quad \text{on} \;\; \Gamma_2 \cap \mathcal{B}(\epsilon).$$

We also define for the point $x_\epsilon = (x', \psi_{1,\epsilon}(x')) \in \Gamma_{1,\epsilon}$ the hyper-plane

$$\Gamma_{2,\epsilon}^{x'} = \{y \in \mathbb{R}^n \mid y_n = \psi_{2,\epsilon}(x')\}. \tag{1.19}$$

Again, let $\mathbf{S}_{\mathbf{2},\epsilon}^{\mathbf{x}'}\mathcal{E}$ be the single layer potential defined on the hyper-surface $\Gamma_{2,\epsilon}^{x'}$ for the potential $\mathcal{E}$ (we remark here that the potential $\mathcal{E}$ is also well defined on the hyper-plane $\Gamma_{2,\epsilon}^{x'}$ see Section 3.3).

For a cut-off function η defined on $\mathbb{R}^n$ that is supported in the ball $\mathcal{B}(R_0)$ and identically equals to 1 in the ball $\mathcal{B}(\epsilon_0)$, we write the off-diagonal operator L_2^δ in the following form

$$L_2^\delta\varphi(x) = \eta(x)L_2^\delta\varphi(x) + \big(1 - \eta(x)\big)L_2^\delta\varphi(x).$$

Since the singular part of the operator L_2^δ is ηL_2^δ, we decompose it as follows

$$L_2^\delta\varphi(x) = K_2^{\epsilon,\delta}\varphi(x) + \frac{1}{\omega_n\sqrt{1+\left|\nabla\psi_{1,\epsilon}(x')\right|^2}}\left(J_2^{\epsilon,\delta}\varphi(x) + I_2^{\epsilon,\delta}\varphi(x)\right). \tag{1.20}$$

For

$$\tilde{\mathbf{x}} = x_\epsilon - \delta\mathbf{e_n},$$

the operators $K_2^{\epsilon,\delta}$, $J_2^{\epsilon,\delta}$ and $I_2^{\epsilon,\delta}$ are defined as follows

$$K_2^{\epsilon,\delta}\varphi(x) = -\partial\nu\mathbf{S_2}\varphi(x-\delta\mathbf{e_n}) + \partial\nu_\epsilon\mathbf{S_{2,\epsilon}}\mathcal{E}(\tilde{\mathbf{x}}), \tag{1.21}$$

$$J_2^\delta\varphi(x) = \left(\omega_n\sqrt{1+\left|\nabla\psi_{1,\epsilon}(x')\right|^2}\right)\left(\partial\nu_\epsilon\mathbf{S}^{\mathbf{x}'}_{\mathbf{2},\epsilon}\mathcal{E}(\tilde{\mathbf{x}}) - \partial\nu_\epsilon\mathbf{S_{2,\epsilon}}\mathcal{E}(\tilde{\mathbf{x}})\right),$$

and

$$I_2^\delta\varphi(x) = \left(\omega_n\sqrt{1+\left|\nabla\psi_{1,\epsilon}(x')\right|^2}\right)\partial\nu_\epsilon\mathbf{S}^{\mathbf{x}'}_{\mathbf{2},\epsilon}\mathcal{E}(\tilde{\mathbf{x}}),$$

where $\nu_\epsilon(x_\epsilon)$ is the normal unit vector on the approximate surface $\Gamma_{1,\epsilon}$ at the point x_ϵ, *that is,*

$$\nu_\epsilon(x_\epsilon) = \frac{1}{\sqrt{1+\left|\nabla\psi_{1,\epsilon}(x')\right|^2}}\begin{pmatrix}-\nabla\psi_{1,\epsilon}(x')\\ 1\end{pmatrix}. \tag{1.22}$$

Remark 1.2. *Since $\psi_2 = \psi_{2,\epsilon}$ around the origin, then we see the operators $K_2^{\epsilon,\delta}$ are not singular in the neighborhood of the origin. Also we notice that away from the origin, the kernel functions that are corresponding to the operators $K_2^{\epsilon,\delta}$ are uniformly bounded. Then we can easily show that the operators $\{K_2^{\epsilon,\delta}\}$ form a family of collectively compact operators from $C^\alpha(\Gamma_2)$ into $C^\alpha(\Gamma_1)$.*

Remark 1.3. *In the definition of the operator J_2^δ, we have involved the operator $\partial\nu_\epsilon\mathbf{S}^{\mathbf{x}'}_{\mathbf{2},\epsilon}\mathcal{E}(\tilde{\mathbf{x}})$*

in order to take the advantage of the difference $\psi_{2,\epsilon}(x')-\psi_{2,\epsilon}(y')$ *to get the Hölder continuity for the operator* $J_2^{\epsilon,\delta}$.

After decomposing the off-diagonal operators L_j^δ, the operator T^δ defined in (1.11) may now be decomposed as follows

$$T^\delta \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \Lambda_{\epsilon,\delta} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} + C_{\epsilon,\delta} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{1.23}$$

where

$$\Lambda_{\epsilon,\delta} = \begin{pmatrix} \lambda I & \frac{\eta}{\omega_n \sqrt{1+\left|\nabla\psi_{1,\epsilon}(x')\right|^2}} \left(J_2^{\epsilon,\delta} + I_2^{\epsilon,\delta}\right) \\ \frac{\eta}{\omega_n \sqrt{1+\left|\nabla\psi_{2,\epsilon}(x')\right|^2}} \left(J_1^{\epsilon,\delta} + I_1^{\epsilon,\delta}\right) & \lambda I \end{pmatrix}, \tag{1.24}$$

and

$$C_{\epsilon,\delta} = \begin{pmatrix} -K_1^* & \eta K_2^{\epsilon,\delta} \\ \eta K_2^{\epsilon,\delta} & -K_2^* \end{pmatrix} + (1-\eta) \begin{pmatrix} 0 & L_2^\delta \\ L_1^\delta & 0 \end{pmatrix}. \tag{1.25}$$

We prove in Theorems (4.1) and (5.1) for any $0 \le \delta \le \delta_0$ and $\alpha < \alpha_0$ the following bound

$$\left\| \frac{\eta}{\omega_n \sqrt{1+\left|\nabla\psi_{1,\epsilon}\right|^2}} \left(J_2^{\epsilon,\delta} + I_2^{\epsilon,\delta}\right) \right\|_{\mathcal{L}(C^\alpha(\Gamma_2), C^\alpha(\Gamma_1))} \le \frac{1}{2}\Big(1 + C(\epsilon)\Big)\left(1+\epsilon_0\right).$$

Similarly, we will get the bound

$$\left\| \frac{\eta}{\omega_n \sqrt{1+\left|\nabla\psi_{2,\epsilon}\right|^2}} \left(J_1^{\epsilon,\delta} + I_1^{\epsilon,\delta}\right) \right\|_{\mathcal{L}(C^\alpha(\Gamma_1), C^\alpha(\Gamma_2))} \le \frac{1}{2}\Big(1 + C(\epsilon)\Big)\left(1+\epsilon_0\right).$$

Since

$$\frac{1+\epsilon_0}{2} < |\lambda|\,,$$

then we readily see the invertibility of the operators $\Lambda_{\epsilon,\delta}$ and the uniform bound

$$\left\|\Lambda_{\epsilon,\delta}^{-1}\right\| \leq \frac{C}{|\lambda| - \frac{1}{2}\Big(1+C(\epsilon)\Big)(1+\epsilon_0)}, \quad 0 \leq \delta \leq \delta_0. \tag{1.26}$$

As a consequence of the uniform bound (1.26), we have the pointwise convergence

$$\Lambda_{\epsilon,\delta}^{-1}\begin{pmatrix}\varphi_1\\ \\ \varphi_2\end{pmatrix} \longrightarrow \Lambda_{\epsilon,0}^{-1}\begin{pmatrix}\varphi_1\\ \\ \varphi_2\end{pmatrix} \quad \text{in } \; C^{\alpha'}(\Gamma_1)\times C^{\alpha'}(\Gamma_2), \quad \alpha' < \alpha.$$

We show in Theorem 6.12 that the operators $\{C_{\epsilon,\delta}\;\Lambda_{\epsilon,\delta}^{-1}\}$ are collectively compact on $C^{\alpha}(\Gamma_1)\times C^{\alpha}(\Gamma_2)$, and $C_{\epsilon,\delta}\;\Lambda_{\epsilon,\delta}^{-1} \longrightarrow C_{\epsilon,0}\;\Lambda_{\epsilon,0}^{-1}$ pointwise in $\mathcal{L}\big(C^{\alpha}(\Gamma_1)\times C^{\alpha}(\Gamma_2)\big)$ as δ approaches 0. Consequently, we obtain the compactness of the operator $\{C_{\epsilon,0}\;\Lambda_{\epsilon,0}^{-1}\}$ on $C^{\alpha}(\Gamma_1)\times C^{\alpha}(\Gamma_2)$.

In Theorem 6.13 we show that the operator T^0 is invertible. Thus, since T^0 and $\Lambda_{\epsilon,0}$ are invertible and

$$T^0 = \Big(I + C_{\epsilon,0}\Lambda_{\epsilon,0}^{-1}\Big)\,\Lambda_{\epsilon,0}.$$

Then we see that

$$\Lambda_{\epsilon,0}\;{T^0}^{-1} = \Big(I + C_{\epsilon,0}\Lambda_{\epsilon,0}^{-1}\Big)^{-1}.$$

Therefore, we get the pointwise convergence

$$I + C_{\epsilon,\delta}\Lambda_{\epsilon,\delta}^{-1} \longrightarrow I + C_{\epsilon,0}\Lambda_{\epsilon,0}^{-1} \;\text{ in }\; C^{\alpha}(\Gamma_1)\times C^{\alpha}(\Gamma_2).$$

Thus from Lemma (6.7) we have that $\Big(I + C_{\epsilon,\delta}\Lambda_{\epsilon,\delta}^{-1}\Big)^{-1}$ exist when δ is sufficiently small and they are uniformly bounded with respect to δ in the operator norm. Therefore, we obtain the pointwise convergence

$$\left(I + C_{\epsilon,\delta}\Lambda_{\epsilon,\delta}^{-1}\right)^{-1} \longrightarrow \left(I + C_{\epsilon,0}\Lambda_{\epsilon,0}^{-1}\right)^{-1}. \tag{1.27}$$

Since $T^{\delta} = \left(I + C_{\epsilon,\delta}\Lambda_{\epsilon,\delta}^{-1}\right)\Lambda_{\epsilon,\delta}$, then by using the bound (1.26) we obtain

$$(T^{\delta})^{-1} = \Lambda_{\epsilon,\delta}^{-1}\left(I + C_{\epsilon,\delta}\Lambda_{\epsilon,\delta}^{-1}\right)^{-1},$$

are uniformly norm bounded and satisfy

$$(T^{\delta})^{-1} \longrightarrow T^{0^{-1}} \;\text{ in }\; C^{\alpha'}(\Gamma_1) \times C^{\alpha'}(\Gamma_2),\;\; \alpha' < \alpha.$$

Therefore, the bound (1.17) has been proved and the Theorem 1.1 follows. That is, we have proved

$$\sum_{j=1}^{2} \|u_\delta\|_{C^{1,\alpha}(\overline{D_j^\delta})} + \|u_\delta\|_{C^{1,\alpha}\left(\Omega_{\tilde{\epsilon}}\setminus\overline{D_1^\delta \cup D_2^\delta}\right)} \leq C\,\|g\|_{L^2(\partial\Omega)}. \tag{1.28}$$

Since

$$u_\delta \longrightarrow u_0 \quad \text{in } \; H^1(\Omega), \quad \text{(see Theorem 2.5)},$$

and

$$(T^{\delta})^{-1} \longrightarrow T^{0^{-1}} \;\text{ in }\; C^{\alpha'}(\Gamma_1) \times C^{\alpha'}(\Gamma_2),\;\; \alpha' < \alpha.$$

Then we have that the solution u_0, to (1.5), can be represented by

$$u_0(x) = H_0(x) + \mathbf{S}_1\varphi_1^0(x) + \mathbf{S}_2\varphi_2^0(x), \quad x \in \Omega, \tag{1.29}$$

where H_0 is harmonic inside Ω, and defined by

$$H_0(x) = -\mathbf{S}_{\partial\Omega}g(x) + \mathbf{D}_{\partial\Omega}\left(u_0|_{\partial\Omega}\right)(x), \quad x \in \Omega, \tag{1.30}$$

and the pair $(\varphi_1^0, \varphi_2^0) \in C^\alpha(\Gamma_1) \times C^\alpha(\Gamma_2)$ is the unique solution to the limiting system corresponding to (1.14).

The dissertation is organized as follows: In Chapter 2, we introduce all necessary ingredients for the representation formula to the solution of equation (1.6). Chapter 3 is devoted to the decomposition of the operators L_j^δ, $j = 1, 2.$ as well as the operator T^δ. In Chapter 4, we prove for any $0 \le \delta \le \delta_0$ and $\alpha < \alpha_0$ the following bound holds

$$\left\| \frac{\eta}{\omega_n\sqrt{1 + \left|\nabla\psi_{1,\epsilon}\right|^2}}\, J_2^{\epsilon,\delta} \right\|_{\mathcal{L}(C^\alpha(\Gamma_2), C^\alpha(\Gamma_1))} \le C(\epsilon).$$

While in Chapter 5 we prove that

$$\left\| \frac{\eta}{\omega_n\sqrt{1 + \left|\nabla\psi_{1,\epsilon}\right|^2}}\, I_2^{\epsilon,\delta} \right\|_{\mathcal{L}(C^\alpha(\Gamma_2), C^\alpha(\Gamma_1))} \le \frac{1}{2}\Big(1 + C(\epsilon)\Big)\left(1 + \epsilon_0\right).$$

Finally, in Chapter 6 we show the invertibility of the operator T^0 and then the existence of uniformly boundedness of the operators $(T^\delta)^{-1}$.

Chapter 2

Layer Potentials for system of two inclusions

2.1 Notations and assumptions

Let $n \geq 2$. We denote by $\mathcal{B}(r)$ the open ball in $\mathbb{R}^n$ centered at the origin of radius r and by $\mathcal{B}'(r)$ the open ball in $\mathbb{R}^{n-1}$. For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, let $x' = (x_1, \cdots, x_{n-1})$ where we often regard x' as a point in $\mathbb{R}^{n-1}$. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain containing the origin 0. For $0 < \alpha_0 \leq 1$, let D_1 and D_2 be two C^{1,α_0} simply connected and strictly convex domains (inclusions) contained in Ω such that $D_1 \cup D_2 \subset\subset \Omega$ and $\overline{D_1} \cap \overline{D_2} = \{0\}$. Recall that we say D_j is a C^{1,α_0} domain if each point of the ∂D_j has a neighborhood in which ∂D_j is the graph of C^{1,α_0} function of $n-1$ variables $x_1, ..., x_{n-1}$, $j = 1, 2$. Furthermore, assume that D_1 lies in the half space $x_n < 0$, and D_2 in the half space $x_n > 0$.

Remark 2.1. *Since D_1 and D_2 are both C^{1,α_0} domains, then around the touching point $x = 0$, Γ_1 and Γ_2 would be parametrized by $\big(x', \psi_1(x')\big)$ and $\big(x', \psi_2(x')\big)$ respectively, where ψ_j are C^{1,α_0} functions and $\Gamma_j := \partial D_j$, $j = 1, 2$. The graph of ψ_1 lies below the x'-hyperplane, while the graph of ψ_2 lies above the x'-hyperplane.*

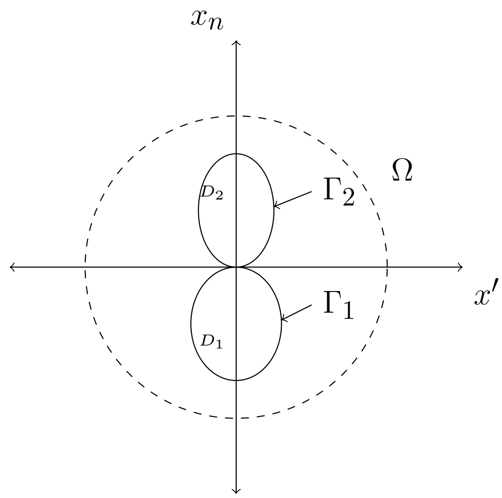

Figure 2.1: The touching inclusions

Let $H^1(\Omega) := W^{1,2}(\Omega)$ be the usual Sobolev space and $C_0^\infty(\Omega)$ is the space of smooth functions with compact support in Ω. We often use the space $L_0^2(\partial\Omega) := \{\varphi \in L^2(\partial\Omega) : \int_{\partial\Omega} \varphi \, d\sigma(x) = 0\}$.

2.2 Existence and Uniqueness Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary and consider a C^{1,α_0} bounded domain $D \subset\subset \Omega$. Let u be a solution to the Neumann problem

$$\begin{cases} \operatorname{div}\left(1+(k-1)\chi_D\right)\nabla u = 0 & \text{in } \Omega, \\ \partial_\nu u(x) = g & \text{on } \partial\Omega, \\ \int_{\partial\Omega} u \, d\sigma(x) = 0, \end{cases} \tag{2.1}$$

where $g \in L_0^2(\partial\Omega)$ and $\chi(D)$ is the characteristic function of D.

Definition 2.2. *We say that $u \in H^1(\Omega)$ is a weak solution to (2.1) if $\int_{\partial\Omega} u \, d\sigma(x) = 0$ and* *the following identity holds:*

$$\int_{\Omega} a_0(x) \nabla u(x)\ \nabla \eta(x)\ dx = \int_{\partial\Omega} g(x)\ \eta(x)\ d\sigma(x), \quad \forall\, \eta \in C^{\infty}(\Omega). \tag{2.2}$$

The existence of the solution to (2.1) is very basic and we show it in the following theorem.

Theorem 2.3. *For any $g \in L^2_0(\partial\Omega)$, there exists a unique $u \in H^1(\Omega)$ solves (2.1).*

Proof. Let $H^1_*(\Omega) = \{u \in H^1(\Omega) : \int_{\partial\Omega} u\ d\sigma(x) = 0\}$. Then $H^1_*(\Omega)$ is a closed subspace of $H^1(\Omega)$. In fact, let u_n be a sequence in $H^1_*(\Omega)$ such that $u_n \to u$ in $H^1(\Omega)$. Then by using the trace theorem it follows that

$$\left| \int_{\partial\Omega} u\ d\sigma \right| = \left| \int_{\partial\Omega} (u_n - u)\ d\sigma \right| \leq \int_{\partial\Omega} |u_n - u|\ \ d\sigma \leq C \left\| u_n - u \right\|_{H^1(\Omega)}.$$

Therefore $\int_{\partial\Omega} u\ d\sigma = 0$. Thus $H^1_*(\Omega)$ is a closed subspace of the Hilbert space $H^1(\Omega)$. Consequently, $H^1_*(\Omega)$ is a Hilbert space.

We define a bilinear form on $H^1_*(\Omega)$ as follows

$$B : H^1_*(\Omega) \times H^1_*(\Omega) \longrightarrow \mathbb{R}.$$

$$B[u, v] = \int_{\Omega} a_k(x)\ Du\ Dv\ dx \qquad \text{for } u, v \in H^1_*(\Omega),$$

where $a_k = 1 + (k-1)\chi(D)$ be such that $m \leq \|a_k\|_{L^\infty}(\Omega) \leq M$ for some positive constants m and M. Let ℓ be a bounded linear functional on H^1_* defined by

$$\ell(v^*) = \int_{\partial\Omega} g\ v^*\ d\sigma \quad \text{for } v^* \in H^1_*(\Omega),$$

It is clear that $|B[u,v]| \leq \alpha \|u\|_{H^1(\Omega)} \ \|v\|_{H^1(\Omega)}$ for some $\alpha > 0$.

We show that $B[u,u] \geq C \|u\|^2_{H^1(\Omega)}$ for some $C > 0$.

$$B[u,u] = \int_\Omega a_k |Du|^2 \ dx \geq m \|\nabla u\|^2_{L^2(\Omega)}.$$

By using *Poincaré's inequality* (2.7), it follows that

$$B[u,u] \geq C \|u\|^2_{H^1(\Omega)}. \tag{2.3}$$

By applying the Lax-Milgram Theorem, we find a unique function $u \in H^1_*(\Omega)$ satisfying

$$B[u, v^*] = \ell(v^*)$$

for all $v^* \in H^1_*(\Omega)$.

Now we show

$$B[u,v] = \int_{\partial\Omega} g \ v \ d\sigma \qquad \text{for } v \in H^1(\Omega).$$

Let $v \in H^1(\Omega)$, by projection theorem on Hilbert spaces, there are unique $v^* \in H^1_*(\Omega)$ and $c \in \mathbb{R}$ such that $v = v^* + c$.

Since $\int_{\partial\Omega} g \ d\sigma = 0$, then it follows that

$$\int_{\partial\Omega} g \ v = \int_{\partial\Omega} g \ v^* + c \int_{\partial\Omega} g = \int_{\partial\Omega} g \ v^* = B[u, v^*].$$

That is,

$$B[u,v] = B[u,v^*].$$

Thus there exists unique $u \in H^1_*(\Omega)$ such that

$$B[u, v] = \ell(v)$$

for all $v \in H^1(\Omega)$. Consequently, u is the unique weak solution to (2.1). $\square$

2.3 Approximation of differential equation.

For $0 < k < +\infty$ and $k \neq 1$, we study the differential equation

$$\begin{cases} \operatorname{div}(a_0(x)\nabla u_0) = 0 & \text{in } \Omega, \\ \partial_\nu u_0(x) = g & \text{on } \partial\Omega, \\ \int_{\partial\Omega} u_0 = 0, \end{cases} \tag{2.4}$$

where

$$a_0(x) = 1 + (k-1)\chi_{(D_1 \cup D_2)}(x),$$

χ is the characteristic function of $D_1 \cup D_2$ and $g \in L^2_0(\partial\Omega)$. The differential equation (2.4) has been introduced and very well studied in [1] for the dimension $n = 2$. We will use the idea of [1] to study the behavior of the solution to the differential equation (2.4) near the touching point for any dimension $n \geq 2$.

For $\delta > 0$, we set

$$D_1^\delta = D_1 - \frac{\delta}{2}\mathbf{e_n}, \;\; D_2^\delta = D_2 + \frac{\delta}{2}\mathbf{e_n},$$

where $\mathbf{e_n} = (0', 1)$ and we denote by a_δ the corresponding piecewise constant coefficients, *that is,*

$$a_\delta(x) = 1 + (k-1)\chi_{(D_1^\delta \cup D_2^\delta)}(x).$$

Let u_δ be the solution to the following modified differential equation

$$\begin{cases} \operatorname{div}(a_\delta(x)\nabla u_\delta) = 0 & \text{in } \Omega, \\ \partial_\nu u_\delta(x) = g & \text{on } \partial\Omega, \\ \int_{\partial\Omega} u_\delta = 0. \end{cases} \tag{2.5}$$

We clearly see that the function u_δ is harmonic inside and outside the inclusions D_1^δ, D_2^δ and satisfies the jump conditions

$$u_\delta^+ = u_\delta^-, \qquad \frac{\partial u_\delta^+}{\partial \nu} = k\frac{\partial u_\delta^-}{\partial \nu} \quad \text{on} \quad \partial D_i^\delta, \quad j = 1, 2. \tag{2.6}$$

Where u_δ^+ denotes the solution to (2.5) outside the inclusion $D_1^\delta \cup D_2^\delta$, while u_δ^- denotes the solution to (2.5) inside $D_1^\delta \cup D_2^\delta$ and ν is the outward unit normal to D_j^δ, $j = 1, 2$.

Next, we show that the approximate solution u_δ to (2.5) converges to u_0, the solution to (2.4), in $H^1(\Omega)$ as δ approaches 0. For this purpose, we need the following modified *Poincaré's inequality.* The proof is similar to the usual *Poincaré's inequality* [28].

Proposition 2.4. *Let Ω be a bounded and connected domain in $\mathbb{R}^n$, with a C^1 boundary $\partial\Omega$. For $u \in H^1(\Omega)$ there exists a constant C, depending on n and Ω, such that*

$$\|u\|_{L^2(\Omega)}^2 \leq C\left(\|\nabla u\|_{L^2(\Omega)}^2 + \left(\int_{\partial\Omega} u \, d\sigma\right)^2\right). \tag{2.7}$$

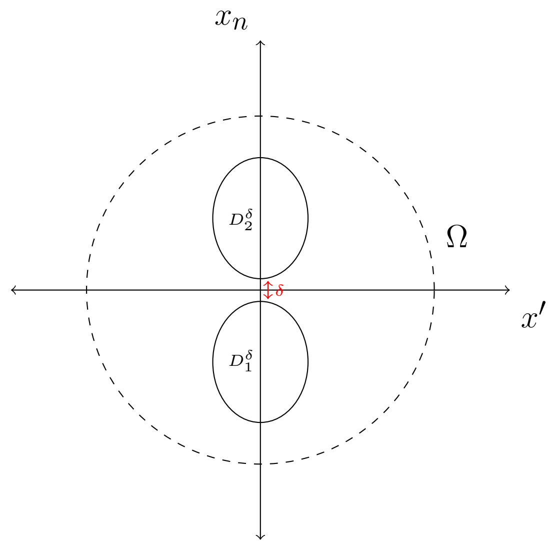

Figure 2.2: The separated inclusions

Proof. We argue by contradiction. If the inequality (2.7) were false, then one would find a sequence of functions $\{u_j\} \in H^1(\Omega)$ be such that

$$\|u_j\|^2_{L^2(\Omega)} > j\left(\|\nabla u_j\|^2_{L^2(\Omega)} + \left(\int_{\partial\Omega} u_j \, d\sigma\right)^2\right). \tag{2.8}$$

We normalize u_j by defining

$$w_j = \frac{u_j}{\|u_j\|_{L^2(\Omega)}} \quad (j = 1, 2, \cdots). \tag{2.9}$$

Then (2.8) implies

$$\|\nabla w_j\|^2_{L^2(\Omega)} + \left(\int_{\partial\Omega} w_j \, d\sigma\right)^2 < \frac{1}{j} \quad (j = 1, 2, \cdots). \tag{2.10}$$

Clearly we see that the functions $\{w_j\}$ are bounded in $H^1(\Omega)$. Then by Sobolev embedding theorem, there exists a sequence $\{w_{j_k}\} \subset \{w_j\}$ and a function $w \in L^2(\Omega)$ such that

$$w_{j_k} \longrightarrow w \quad \text{in } L^2(\Omega). \tag{2.11}$$

To show that w has weak derivative in $L^2(\Omega)$, let $\phi \in C_0^\infty(\Omega)$, then we use (2.10) to obtain the following

$$\int_\Omega w \; \nabla\phi \; dx = \lim_{j_k \to \infty} \int_\Omega w_{j_k} \; \nabla\phi \; dx = - \lim_{j_k \to \infty} \int_\Omega \nabla w_{j_k} \; \phi \; dx = 0.$$

Consequently $w \in H^1(\Omega)$, with $\nabla w = 0$. Thus w is constant because Ω is connected. On the other hand, the map

$$w' \longmapsto \left(\int_{\partial\Omega} w' \; d\sigma \right)^2,$$

is continuous on $H^1(\Omega)$. Then (2.10) implies that $w = 0$; in which case $\|w\|_{L^2(\Omega)} = 0$. This contradiction proves the inequality (2.7).

□

Now we are ready to state and prove the convergence of u_δ in $H^1(\Omega)$.

Theorem 2.5. *The solution of (2.5) approaches to the solution of (2.4) in $H^1(\Omega)$ as δ approaches zero. That is*

$$\lim_{\delta \to 0} \|u_\delta - u_0\|_{H^1(\Omega)} = 0. \tag{2.12}$$

Proof. First, we prove that $a_\delta = 1 + (k-1)\chi_{D_1^\delta \cup D_2^\delta} \to a_0$ in $L^p(\Omega)$ for any $p < \infty$. Let $\Omega^\delta = \cup_{j=1}^4 \Omega_j^\delta$, where $\Omega_1^\delta = D_1^\delta \setminus \left(D_1^\delta \cap D_1\right)$, $\Omega_2^\delta = D_1 \setminus \left(D_1^\delta \cap D_1\right)$, $\Omega_3^\delta = D_2^\delta \setminus \left(D_2^\delta \cap D_2\right)$, and

$\Omega_4^\delta = D_2 \setminus \left(D_2^\delta \cap D_2\right)$. Then

$$\begin{aligned}
\|a_\delta - a_0\|_{L^p(\Omega)}^p &= \int_\Omega |a_\delta - a_0|^p \, dx = \int_{\Omega^\delta} |a_\delta - a_0|^p \, dx \\
&= \int_{\Omega^\delta} |k-1|^p \, dx = |k-1|^p \, \mu(\Omega^\delta) \to 0 \quad \text{as} \quad \delta \to 0,
\end{aligned}$$

where μ is the $n-$dimensional Lebesgue measure. Thus $a_\delta \to a_0$ in $L^p(\Omega)$ for any $p < \infty$.

From the definition of the weak solution we have the following

$$\int_\Omega (a_\delta \nabla u_\delta - a_0 \nabla u_0) \nabla \eta = 0, \qquad \forall \eta \in H^1(\Omega).$$

Then we have

$$\int_\Omega \Big(a_0(\nabla u_\delta - \nabla u_0)\Big) \nabla \eta \, dx = -\int_\Omega (a_\delta - a_0) \nabla u_\delta \nabla \eta \, dx. \tag{2.13}$$

Choosing $\eta = u_\delta - u_0$ in (2.13), we get the following

$$\begin{aligned}
\int_\Omega a_0 \left|\nabla u_\delta - \nabla u_0\right|^2 \, dx &= -\int_\Omega (a_\delta - a_0) \nabla u_\delta (\nabla u_\delta - \nabla u_0) \, dx \\
&\leq \|a_\delta - a_0\|_{L^p(\Omega)} \|\nabla u_\delta\|_{L^q(\Omega)} \|\nabla u_\delta - \nabla u_0\|_{L^2(\Omega)} .
\end{aligned}$$

The last inequality makes sense if $u_\delta \in W^{1,q}(\Omega)$ for some $q > 2$. We will prove this later.

Since $C_1 < \|a_0\|_{L^\infty(\Omega)} < C_2$ for some positive constants C_1 and C_2, we obtain the following

$$\|\nabla u_\delta - \nabla u_0\|_{L^2(\Omega)} \leq C \|a_\delta - a_0\|_{L^p(\Omega)} \|\nabla u_\delta\|_{L^q(\Omega)} ,$$

Also since $\|a_\delta - a_0\|_{L^p(\Omega)} \to 0 \quad \text{as} \quad \delta \to 0$, then we have

$$\|\nabla u_\delta - \nabla u_0\|_{L^2(\Omega)} \to 0 \quad \text{as} \quad \delta \to 0. \tag{2.14}$$

By applying *Poincare' inequality* (2.7) for $u_\delta - u_0$, we have

$$\|u_\delta - u_0\|^2_{L^2(\Omega)} \leq C \, \|\nabla u_\delta - \nabla u_0\|^2_{L^2(\Omega)} \, .$$

Thus

$$\|u_\delta - u_0\|_{L^2(\Omega)} \to 0 \quad \text{as} \quad \delta \to 0. \tag{2.15}$$

Then (2.14) and (2.15) imply

$$\|u_\delta - u_0\|_{H^1(\Omega)} \to 0 \quad \text{as} \quad \delta \to 0.$$

Now we prove our claim that $u_\delta \in W^{1,q}(\Omega)$ for some $q > 2$. To do this, it suffices to prove $u_\delta \in W^{1,q}(V)$ for a subset V that satisfying $D_1^\delta \cup D_2^\delta \subset\subset V \subset\subset \Omega$. Let U be a subset so that $V \subset\subset U \subset \Omega$ and φ be a cut-off function such that $\varphi \equiv 1$ in V and $\operatorname{supp} \varphi \subset U$. Define $\tilde{u} = \varphi \, u_\delta$, then we see that

$$\begin{cases} \operatorname{div}\big(a_\delta \nabla \tilde{u}\big) = f & \text{in } \; U, \\ \tilde{u} = 0 & \text{on } \partial U, \end{cases} \tag{2.16}$$

where $f = \operatorname{div}(u_\delta \nabla\varphi) + (\nabla u_\delta \nabla \varphi)$ and we have used that $a_\delta \equiv 1 \quad \text{on} \quad U \setminus V$. Since $(u_\delta \nabla \varphi) \in L^q$ and $(\nabla u_\delta \nabla \varphi) \in L^2 \subset W^{-1,q}$, then $f \in W^{-1,q}(U)$ for some $q > 2$. By using

Meyer's theorem [7], we have $\tilde{u} \in W_0^{1,q}(U)$ and

$$\|\tilde{u}\|_{W_0^{1,q}(U)} \leq C \|f\|_{W^{-1,q}(U)}.$$

Consequently, $u_\delta \in W^{1,q}(V)$.

2.4 Layer Potentials

In this section we study two integral operators that called *layer potentials*. These operators play an important role in the derivation of the decomposition theorem for the solution of the transmission problem (2.4). These operators are very well studied for C^2-domains in [10, 24, 25, 26, 27]. Also [11, 15, 31] gave some basic results for these operators for $C^{1,\alpha}$ domains. Invertiblity and comapctness for Lipschitz domains are given in [2, 32]. We begin by defining the fundamental solution to the Laplace equation in dimension n. We denote by ω_n the area of unit sphere in dimension n.

Lemma 2.6. *[14] A fundamental solution to the Laplace equation $\Delta u = 0$ is given by*

$$\Phi(x, y) = \begin{cases} \frac{1}{2\pi} \ln |x - y| & \text{if} \quad n = 2, \\ \frac{1}{(2-n)\omega_n} |x - y|^{2-n} & \text{if} \quad n > 2. \end{cases} \tag{2.17}$$

We define the layer potentials for L^2 density functions. Given a bounded $C^{1,\alpha}$ domain Ω in $\mathbb{R}^n$, $n \geq 2$, we denote respectively the single layer and double layer potentials of a function $\varphi \in L^2(\partial\Omega)$ as $\mathbf{S}_{\partial\Omega}\varphi$ and $\mathbf{D}_{\partial\Omega}\varphi$, where

$$\mathbf{S}_{\partial\Omega}\varphi(x) := \int_{\partial\Omega} \Phi(x, y)\, \varphi(y)\, d\sigma(y), \qquad x \in \mathbb{R}^n, \tag{2.18}$$

and

$$\mathbf{D}_{\partial\Omega}\varphi(x) := \int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial\nu(y)}\,\varphi(y)\,d\sigma(y), \qquad x \in \mathbb{R}^n \setminus \partial\Omega. \tag{2.19}$$

For a function u defined on $\mathbb{R}^n \setminus \partial\Omega$, we denote

$$u^{\pm}(x) = \lim_{t\to 0^+} u\left(x \pm t\nu(x)\right), \qquad x \in \partial\Omega, \tag{2.20}$$

and

$$\frac{\partial}{\partial\nu^{\pm}}u(x) = \lim_{t\to 0^+} \nabla u\left(x \pm t\nu(x)\right)\cdot\nu(x), \qquad x \in \partial\Omega, \tag{2.21}$$

if the limit exists. Here $\nu(x)$ is the outward unit normal to $\partial\Omega$ at x. For simplicity, sometimes we use $\partial\nu u(x)$ instead of $\frac{\partial}{\partial\nu}u(x)$ and ν_x instead of $\nu(x)$.

We state the jump relations for the double and single layer potentials.

Lemma 2.7. *[2] Let Ω be a bounded $C^{1,\alpha}$ domain in $\mathbb{R}^n$. For $\varphi \in L^2(\partial\Omega)$*

$$\mathbf{S}_{\partial\Omega}\varphi^{+}(x) = \mathbf{S}_{\partial\Omega}\varphi^{-}(x) \quad x \in \partial\Omega, \tag{2.22}$$

$$\partial\nu^{\pm}\mathbf{S}_{\partial\Omega}\varphi(x) = \left(\pm\frac{1}{2}I + K^*_{\partial\Omega}\right)\varphi(x) \quad x \in \partial\Omega, \tag{2.23}$$

$$\mathbf{D}_{\partial\Omega}\varphi^{\pm}(x) = \left(\mp\frac{1}{2}I + K_{\partial\Omega}\right)\varphi(x) \quad x \in \partial\Omega, \tag{2.24}$$

where $K_{\partial\Omega}$ is defined by

$$K_{\partial\Omega}\varphi(x) = \frac{1}{\omega_n}\int_{\partial\Omega} \frac{(y-x)\cdot\nu(y)}{|x-y|^n}\varphi(y)\,d\sigma(y), \tag{2.25}$$

and $K^*_{\partial\Omega}$ *is the* L^2 *adjoint of* $K_{\partial\Omega}$*,i.e.,*

$$K^*_{\partial\Omega}\varphi(x) = \frac{1}{\omega_n}\int_{\partial\Omega} \frac{(x-y)\cdot\nu(x)}{|x-y|^n}\varphi(y)\, d\sigma(y). \tag{2.26}$$

The compactness of the operators $K_{\partial\Omega}$ and $K^*_{\partial\Omega}$ for C^2 domains has been shown in [10, 25], with some modifications we show the compactness for $C^{1,\alpha}$ domains.

Theorem 2.8. *Let* Ω *be a bounded* $C^{1,\alpha}$ *domain in* $\mathbb{R}^n$*, the operators*

$$K_{\partial\Omega},\ K^*_{\partial\Omega} : C^\beta(\partial\Omega) \longrightarrow C^\beta(\partial\Omega)$$

are compact operators for any $0 < \beta < \alpha \leq 1$.

Proof. First, we show that the operator

$$K^*_{\partial\Omega} : C(\partial\Omega) \longrightarrow C^\beta(\partial\Omega),$$

is bounded operator for any $\beta < \alpha$. That is, for any $\varphi \in C(\partial\Omega)$, we show

$$\|K^*_{\partial\Omega}\varphi\|_\beta \leq C\,\|\varphi\|_\infty\,,$$

where C is depending on $\partial\Omega$, β, α and n.

It is easy to show the following inequalities hold for $C^{1,\alpha}$ domain,

$$|(x-y)\cdot\nu(x)| \leq C\,|x-y|^{1+\alpha} \quad \forall\, x, y \in \partial\Omega, \tag{2.27}$$

and

$$|\nu(x) - \nu(y)| \leq C\,|x-y|^{\alpha} \quad \forall\, x, y \in \partial\Omega, \tag{2.28}$$

where C is depending on $\partial\Omega$, α and n. Also by using the mean value theorem, it is not difficult to show that for any $x_1, x_2, y \in \partial\Omega$ with $2\,|x_1 - x_2| \leq |x_1 - y|$, the following inequality holds

$$\left| \frac{1}{|x_1 - y|^n} - \frac{1}{|x_2 - y|^n} \right| \leq \frac{C\ |x_1 - x_2|}{|x_1 - y|^{n+1}}, \tag{2.29}$$

where C is depending on n only.

For the uniform boundedness of the operator $K^*_{\partial\Omega}$, we easily see that

$$|K^*_{\partial\Omega}\varphi(x)| \leq C\,\|\varphi\|_\infty \int_{\partial\Omega} |x-y|^{1+\alpha-n}\ d\sigma(y).$$

Therefore we have

$$|K^*_{\partial\Omega}\varphi(x)| \leq C\,\|\varphi\|_\infty\,.$$

That is

$$\|K^*_{\partial\Omega}\varphi\|_\infty \leq C\,\|\varphi\|_\infty\,. \tag{2.30}$$

To establish Hölder continuity, let us define for $x \in \partial\Omega$ and $r > 0$, the portion $S_{x,r}$ as follows

$$S_{x,r} := \{y \in \partial\Omega : |x-y| < r\}. \tag{2.31}$$

By using (2.28), it follows that

$$\left|\nu(x)\cdot\big(\nu(x)-\nu(y)\big)\right| \leq C\left|x-y\right|^{\alpha}.$$

Notice that,

$$\nu(x)\cdot\nu(y) = 1-\nu(x)\cdot\big(\nu(x)-\nu(y)\big).$$

Thus we can find $R \in (0,1]$ be such that for any $x, y \in \partial\Omega$ with $|x-y| \leq R$, the following holds

$$\nu(x)\cdot\nu(y) \geq \frac{1}{2}. \tag{2.32}$$

We assume R is sufficiently small be such that $S_{x,R}$ is connected for each $x \in \partial\Omega$. Thus by the help of (2.32), $S_{x,R}$ can be bijectively projected into the tangent plane to $\partial\Omega$ at the point x. The surface element $d\sigma(y)$ on $S_{x,R}$ and the surface element $d\tilde{\sigma}(y)$ on the tangent plane are related by

$$d\sigma(y) = \frac{d\tilde{\sigma}(y)}{\nu(x)\cdot\nu(y)} \leq \frac{1}{2}d\tilde{\sigma}(y). \tag{2.33}$$

Let $x_1, x_2 \in \partial\Omega$ be such that $|x_1-x_2| \leq \frac{R}{4}$ and let $r = 4\left|x_1-x_2\right|$. We estimate over the portion $S_{x,r}$ to obtain

$$\left|K^*_{S_{x_1,r}}\varphi(x_1) - K^*_{S_{x_1,r}}\varphi(x_2)\right| \leq C\left\|\varphi\right\|_\infty \left(\int_{S_{x_1,r}} \left|x_1-y\right|^{1+\alpha-n}\, d\sigma(y) + \int_{S_{x_2,2r}} \left|x_2-y\right|^{1+\alpha-n}\, d\sigma(y)\right).$$

Thus by using polar coordinates, we obtain

$$\left|K^*_{S_{x_1,r}}\varphi(x_1) - K^*_{S_{x_1,r}}\varphi(x_2)\right| \leq C\left\|\varphi\right\|_\infty \int_0^{2r} \rho^{\alpha-1}\,d\rho.$$

That is,

$$\left|K^*_{S_{x_1,r}}\varphi(x_1) - K^*_{S_{x_1,r}}\varphi(x_2)\right| \leq C\left\|\varphi\right\|_\infty \left|x_1 - x_2\right|^\alpha. \tag{2.34}$$

Now, we estimate over the portion $S_{x_1,R} \setminus S_{x_1,r}$. Notice that for $y \in S_{x_1,R} \setminus S_{x_1,r}$ and $4\left|x_1 - x_2\right| = r$, we have $2\left|x_1 - x_2\right| < \left|x_1 - y\right|$. Observe the following

$$\begin{aligned}\frac{(x_1-y)\cdot\nu(x_1)}{\left|x_1-y\right|^n} - \frac{(x_2-y)\cdot\nu(x_2)}{\left|x_2-y\right|^n} &= \frac{(x_1-y)\cdot\nu(x_1)-(x_2-y)\cdot\nu(x_1)}{\left|x_1-y\right|^n} + \frac{(x_2-y)\cdot\big(\nu(x_1)-\nu(x_2)\big)}{\left|x_1-y\right|^n} \\ &\quad + \frac{(x_2-y)\cdot\nu(x_2)}{\left|x_1-y\right|^n} - \frac{(x_2-y)\cdot\nu(x_2)}{\left|x_2-y\right|^n}.\end{aligned}$$

Therefore by using (2.27), (2.28), (2.29) and noticing that

$$\left|x_2 - y\right| \leq \left|x_2 - x_1\right| + \left|x_1 - y\right| < \frac{1}{2}\left|x_1 - y\right| + \left|x_1 - y\right| = \frac{3}{2}\left|x_1 - y\right|,$$

we have

$$\left|\frac{(x_1-y)\cdot\nu(x_1)}{\left|x_1-y\right|^n} - \frac{(x_2-y)\cdot\nu(x_2)}{\left|x_2-y\right|^n}\right| \leq \frac{C\left|x_1-x_2\right|^\alpha}{\left|x_1-y\right|^{n-1}} + \frac{C\left|x_1-x_2\right|}{\left|x_1-y\right|^{n-\alpha}}.$$

Thus after converting to the polar coordinates, we have the following

$$\int_{\tilde{S}_{R,r}} \left|\frac{(x_1-y)\cdot\nu(x_1)}{\left|x_1-y\right|^n} - \frac{(x_2-y)\cdot\nu(x_2)}{\left|x_2-y\right|^n}\right|\,d\sigma(y) \leq C\left|x_1-x_2\right|^\beta \int_{\frac{r}{4}}^{R} \rho^{(\alpha-\beta)-1}\,d\rho,$$

where $\tilde{S}_{R,r} = S_{x_1,R} \setminus S_{x_1,r}$. Thus we have

$$\int_{\tilde{S}_{R,r}} \left| \frac{(x_1 - y) \cdot \nu(x_1)}{|x_1 - y|^n} - \frac{(x_2 - y) \cdot \nu(x_2)}{|x_2 - y|^n} \right| \, d\sigma(y) \leq C \left| x_1 - x_2 \right|^{\beta}.$$

That is

$$\left| K^*_{\tilde{S}_{R,r}} \varphi(x_1) - K^*_{\tilde{S}_{R,r}} \varphi(x_2) \right| \leq C \left\| \varphi \right\|_\infty \left| x_1 - x_2 \right|^{\beta}. \tag{2.35}$$

Similarly, estimating over the portion $\partial\Omega \setminus S_{x_1,R}$, we have

$$\int_{\partial\Omega \setminus S_{x_1,R}} \left| \frac{(x_1 - y) \cdot \nu(x_1)}{|x_1 - y|^n} - \frac{(x_2 - y) \cdot \nu(x_2)}{|x_2 - y|^n} \right| d\sigma(y) \leq C \int_{\partial\Omega \setminus S_{x_1,R}} \left(\frac{|x_1 - x_2|}{|x_1 - y|^{n-\alpha}} + \frac{|x_1 - x_2|^{\alpha}}{|x_1 - y|^{n-1}} \right) d\sigma(y).$$

Therefore

$$\int_{\partial\Omega \setminus S_{x_1,R}} \left| \frac{(x_1 - y) \cdot \nu(x_1)}{|x_1 - y|^n} - \frac{(x_2 - y) \cdot \nu(x_2)}{|x_2 - y|^n} \right| d\sigma(y) \leq C \left| x_1 - x_2 \right|^{\alpha}.$$

That is

$$\left| K^*_{\partial\Omega \setminus S_{x_1,R}} \varphi(x_1) - K^*_{\partial\Omega \setminus S_{x_1,R}} \varphi(x_2) \right| \leq C \left\| \varphi \right\|_\infty \left| x_1 - x_2 \right|^{\alpha}. \tag{2.36}$$

By combining (2.34), (2.35), and (2.36), we obtain

$$\left| K^*_{\partial\Omega} \varphi(x_1) - K^*_{\partial\Omega} \varphi(x_2) \right| \leq C \left\| \varphi \right\|_\infty \left| x_1 - x_2 \right|^{\beta}, \ \ \forall \ \beta < \alpha. \tag{2.37}$$

Then by combining (2.30) and (2.37), it follows that

$$\|K^*_{\partial\Omega}\varphi\|_\beta \leq C\,\|\varphi\|_\infty\,.$$

For compactness of the operator $K^*_{\partial\Omega}$, let $\{\varphi_n\}$ be a bounded sequence in $C^\beta(\partial\Omega)$. That is

$$\|\varphi_n\|_\beta \leq \tilde{C}. \tag{2.38}$$

Then clearly we see that

$$|\varphi_n(x)| \leq \tilde{C},\ \ \forall\, x \in \partial\Omega,$$

and

$$|\varphi_n(x) - \varphi_n(y)| \leq \tilde{C}\,|x-y|^\beta\,,\ \ \forall\, x, y \in \partial\Omega,$$

Thus by Arzelà- Ascoli Theorem, $\{\varphi_n\}$ has convergent subsequence $\{\varphi_{n_j}\}$. That is,

$$\varphi_{n_j} \longrightarrow \varphi_0 \ \text{ in }\ C(\partial\Omega).$$

Thus we have

$$K^*_{\partial\Omega}\varphi_{n_j} \longrightarrow K^*_{\partial\Omega}\varphi_0 \ \text{ in }\ C^\beta(\partial\Omega).$$

Therefore we conclude that $K^*_{\partial\Omega}$ is compact operator. The compactness of the operator $K_{\partial\Omega}$ follows similarly. $\square$

Theorem 2.9. *[11] Let Ω be a bounded $C^{1,\alpha}$ domain in $\mathbb{R}^n$, then we have $K_{\partial\Omega}(1) = \frac{1}{2}$. That is*

$$\frac{1}{\omega_n} \int_{\partial\Omega} \frac{(y-x)\cdot\nu(y)}{|x-y|^n}\, d\sigma(y) = \frac{1}{2}.$$

Remark 2.10. *Since $K^*_{\partial\Omega}$is the L^2 adjoint operator of $K_{\partial\Omega}$, then we have*

$$\int_{\partial\Omega} K^*_{\partial\Omega}\varphi\, d\sigma(x) = \int_{\partial\Omega} \varphi K_{\partial\Omega}(1)\, d\sigma(x) = \frac{1}{2}\int_{\partial\Omega} \varphi\, d\sigma(x) \quad \forall\varphi \in L^2(\partial\Omega).$$

Theorem 2.11. *[2, 32] Let Ω be a bounded $C^{1,\alpha}$ domain in $\mathbb{R}^n$, the operator*

$$\lambda I - K^*_{\partial\Omega} : C^\beta(\partial\Omega) \longrightarrow C^\beta(\partial\Omega)$$

is invertible on for $|\lambda| > \frac{1}{2}$ and $0 < \beta < \alpha \leq 1$.

The normal derivative of the double layer potential is continuous across the boundary as we see in the following theorem.

Theorem 2.12. *[15] Let Ω be a bounded $C^{1,\alpha}$ domain in $\mathbb{R}^n$, then $\partial\nu^+\mathbf{D}_{\partial\Omega}\varphi$ and $\partial\nu^-\mathbf{D}_{\partial\Omega}\varphi$ exist. Moreover,*

$$\partial\nu^+\mathbf{D}_{\partial\Omega}\varphi = \partial\nu^-\mathbf{D}_{\partial\Omega}\varphi, \quad \forall\varphi \in C^\alpha(\partial\Omega).$$

2.5 Representation formula.

Before we give the representation formula to (2.1), we show that the Harmonic functions in an unbounded domain $\mathbb{R}^n \setminus \overline{\mathcal{B}(R)}$ with decay of order $|x|^{1-n}$ satisfy the Green first identity as we see in the following Lemma.

Lemma 2.13. *Let $R >> 1$ be large and u be a harmonic function in $\mathbb{R}^n \setminus \overline{B_R(0)}$ that has decay of order $|x|^{1-n}$ in $\mathbb{R}^n \setminus B_\rho(0)$, where $\rho > 4R$. Then the following identity holds*

$$\int_{\partial B_R(0)} u\,\partial_\nu u\, d\sigma(x) = -\int_{\mathbb{R}^n\setminus\overline{B_R(0)}} |\nabla u|^2\ dx.$$

Proof. Since u is harmonic in the annulus $B_\rho(0) \setminus \overline{B_R(0)}$, then we have

$$\int_{B_\rho(0)\setminus\overline{B_R(0)}} |\nabla u|^2\ dx = \int_{\partial\left(B_\rho(0)\setminus\overline{B_R(0)}\right)} u\,\partial_\nu u\, d\sigma(x),$$

that is,

$$\int_{B_\rho(0)\setminus\overline{B_R(0)}} |\nabla u|^2\ dx = \int_{\partial B_\rho(0)} u\,\partial_\nu u\, d\sigma(x) - \int_{\partial B_R(0)} u\,\partial_\nu u\, d\sigma(x). \tag{2.39}$$

From the assumptions, we know that $u = O(\rho^{1-n})$ on $\partial B_\rho(0)$. Now we show that $\partial_\nu u = O(\rho^{-n})$ on $\partial B_\rho(0)$. Choose $x_0 \in \partial B_\rho(0)$, then clearly we see that u is harmonic in $B_{\frac{\rho}{4}}(x_0)$. Let v be a harmonic function in the ball $B_1(x_0)$, then by Poisson's formula we have that

$$v(x) = \frac{1}{\omega_n}\int_{\partial B_1(x_0)} \frac{1-|x-x_0|^2}{|x-y|^n} v(y)\, d\sigma(y), \quad x \in B_1(x_0).$$

By differentiation, we obtain the following

$$\nabla v(x)|_{x=x_0} = \frac{1}{\omega_n}\int_{\partial B_1(x_0)} \nabla\left(\frac{1-|x-x_0|^2}{|x-y|^n}\right)|_{x=x_0} v(y)\, d\sigma(y), \quad x \in B_1(x_0).$$

Therefore,

$$|\nabla v(x_0)| \leq C\ \|v\|_{L^\infty\left(\overline{B_1(x_0)}\right)}. \tag{2.40}$$

Assuming $v(x) = u(\frac{\rho}{4}x)$ and using (2.40), it follows that

$$|\nabla u(x_0)| \leq \frac{C}{\rho} \ \|u\|_{L^\infty\left(\overline{B_{\rho/4}(x_0)}\right)} . \tag{2.41}$$

From (2.41) and the decay condition of u in $\overline{B_{\rho/4}(x_0)}$, it follows that $\partial_\nu u = O(\rho^{-n})$ on $\partial B_\rho(0)$. Then we conclude that

$$\int_{\partial B_\rho(0)} u \, \partial_\nu u \; d\sigma(x) = O(\rho^{1-2n}).$$

Thus the lemma follows by letting $\rho \longrightarrow \infty$ in (2.39). $\square$

We give a representation formula for the solution to (2.1). This formula depends on the subdomain D and the pair $(u|_{\partial\Omega}, g)$. The proof was given in [2, 23].

Theorem 2.14. *Let Ω be a bounded domain with smooth boundary and let D be a subdomian compactly embedded in Ω with C^{1,α_0} boundary and conductivity $0 < k \neq 1 < \infty$. The solution to (2.1) can be uniquely represented as*

$$u(x) = H(x) + \mathbf{S}_{\partial D}\varphi(x), \quad x \in \Omega, \tag{2.42}$$

where H is a harmonic function given by

$$H(x) = -\mathbf{S}_{\partial\Omega} g(x) + \mathbf{D}_{\partial\Omega} f(x), \quad x \in \Omega, \quad f := u|_{\partial\Omega}, \tag{2.43}$$

and $\varphi \in L^2_0(\partial D)$ satisfies the integral equation

$$\left(\frac{k+1}{2(k-1)} I - K^*_{\partial D}\right)\varphi = \frac{\partial H}{\partial \nu}|_{\partial D} \qquad on \quad \partial D. \tag{2.44}$$

Proof. Consider the following problem

$$\begin{cases} \nabla \cdot \Big(a_k(x)\, \nabla h \Big) = 0 & \text{in } \mathbb{R}^n \setminus \partial\Omega, \\ h^- - h^+ = f & \text{on } \partial\Omega, \\ \partial_{\nu^-} h - \partial_{\nu^+} h = g & \text{on } \partial\Omega, \\ h(x) = O(|x|^{1-n}) & \text{if } |x| \to \infty, \end{cases} \tag{2.45}$$

where $a_k = 1 + (k-1)\chi(D)$.

Let

$$V_1(x) := -\mathbf{S}_{\partial\Omega} g(x) + \mathbf{D}_{\partial\Omega} f(x) + \mathbf{S}_{\partial D}\varphi(x) \quad \text{for} \quad x \in \mathbb{R}^n.$$

We begin by showing V_1 is a weak solution to (2.45) in the sense of the following definition:

Definition 2.15. *We say v is a weak solution to (2.45) if the following identities hold:*

$$\int_\Omega a_k(x)\, \nabla v\, \nabla \eta\, dx = 0 \quad \forall\, \eta \in C_0^\infty(\Omega),$$

and

$$\int_{\mathbb{R}^n \setminus \overline{\Omega}} \nabla v\, \nabla \tilde{\eta}\, dx = 0 \quad \forall\, \tilde{\eta} \in C_0^\infty(\mathbb{R}^n \setminus \overline{\Omega}).$$

Let us first verify the jump and the decay conditions for V_1. By the continuity of the single layer potential $\mathbf{S}_{\partial\Omega} g$ across $\partial\Omega$, smoothness of $\mathbf{S}_{\partial D}\varphi$ on $\partial\Omega$ and the jump condition

$$\mathbf{D}_{\partial\Omega} f^{\pm}(x) = \left(\mp \frac{1}{2} I + K_{\partial\Omega} \right) f(x), \quad x \in \partial\Omega,$$

it follows that

$$V_1^-(x) - V_1^+(x) = f(x) \ \text{ for } x \in \partial\Omega.$$

Similarly, by the continuity of the normal derivative of the double layer potential $\mathbf{D}_{\partial\Omega} f$ across $\partial\Omega$, smoothness of $\mathbf{S}_{\partial D}\varphi$ on $\partial\Omega$ and the jump relation for the normal derivative of the single layer potential

$$\partial\nu^{\pm}\mathbf{S}_{\partial\Omega} g(x) = \left(\pm\frac{1}{2}I + K^*_{\partial\Omega}\right) g(x), \quad x \in \partial\Omega,$$

we have

$$\partial_{\nu^-} V_1 - \partial_{\nu^+} V_1 = g \quad \text{on} \quad \partial\Omega.$$

From the definition of single layer potential, we have

$$\mathbf{S}_{\partial\Omega} g(x) = \int_{\partial\Omega} \Phi(x,y)\ g(y)\ d\sigma(y).$$

Since $g \in L^2_0(\partial\Omega)$, we get

$$\mathbf{S}_{\partial\Omega} g(x) = \int_{\partial\Omega} \left[\Phi(x,y) - \Phi(x,y_0)\right]\ g(y)\ d\sigma(y),$$

for fixed $y_0 \in \Omega$. Since

$$|\Phi(x,y) - \Phi(x,y_0)| \leq C\,|x|^{1-n} \ \text{ when } \ |x| \to \infty \ \text{ and } \ y \in \Omega$$

for some constant C. Therefore

$$\mathbf{S}_{\partial\Omega} g(x) = O(|x|^{1-n}) \ \text{ as } \ |x| \to \infty.$$

Similarly, we have

$$\mathbf{S}_{\partial D}\varphi(x) = O(|x|^{1-n}) \text{ as } |x| \to \infty.$$

Obviously from the definition of the double layer potential, we see that

$$\mathbf{D}_{\partial\Omega} f(x) = O(|x|^{1-n}) \text{ as } |x| \longrightarrow \infty.$$

Therefore $V_1(x) = O\big(\,|x|^{1-n}\,\big)$ when $|x| \longrightarrow \infty$.

Now we show V_1 is a weak solution to (2.45) in the sense of the definition 2.15. Since V_1 is harmonic in $\mathbb{R}^n \setminus \overline{\Omega}$, then it is clear that

$$\int_{\mathbb{R}^n\setminus\overline{\Omega}} \nabla V_1 \nabla \tilde{\eta}\; dx = 0, \quad \forall \tilde{\eta} \in C_0^\infty\big(\mathbb{R}^n \setminus \overline{\Omega}\big).$$

For $\eta \in C_0^\infty(\Omega)$, we have

$$\int_\Omega a_k(x)\; \nabla V_1 \nabla \eta\; dx = \int_{\Omega\setminus D} \nabla V_1 \nabla \eta\; dx + k \int_D \nabla V_1 \nabla \eta\; dx.$$

Since V_1 is harmonic inside $\Omega \setminus D$ and also is harmonic inside D, then it follows that

$$\int_\Omega a_k(x)\; \nabla V_1 \nabla \eta\; dx = k \int_{\partial D} \partial\nu^- V_1\; \eta\; d\sigma - \int_{\partial D} \partial\nu^+ V_1\; \eta\; d\sigma.$$

That is,

$$\int_\Omega a_k(x)\; \nabla V_1 \nabla \eta\; dx = k \int_{\partial D} \Big(\partial_\nu H + \partial_{\nu^-} \mathbf{S}_{\partial D}\varphi\Big)\eta\; d\sigma - \int_{\partial D} \Big(\partial_\nu H + \; \partial_{\nu^+} \mathbf{S}_{\partial D}\varphi\Big)\eta\; d\sigma.$$

Note that if

$$k\Big(\partial_\nu H(x) + \partial_{\nu^-} S_{\partial D}\varphi(x)\Big) = \partial_\nu H(x) + \ \partial_{\nu^+}\mathbf{S}_{\partial D}\varphi(x) \ \text{ for } x \in \partial D, \tag{2.46}$$

then we have

$$\int_\Omega a_k(x)\ \nabla V_1 \nabla \eta \ dx = 0, \quad \forall \eta \in C_0^\infty(\Omega).$$

By substituting the jump conditions (2.23) in equation (2.46), we obtain the following

$$(k-1)\partial_\nu H(x) + k\left(-\frac{1}{2}I + K_{\partial D}^*\right)\varphi(x) - \left(\frac{1}{2}I + K_{\partial D}^*\right)\varphi(x) = 0 \ \text{ for } x \in \partial D,$$

that is

$$\partial_\nu H = \left(\frac{k+1}{2(k-1)}I - K_{\partial D}^*\right)\varphi \ \text{ on } \partial D. \tag{2.47}$$

Thus we have shown that

$$\int_\Omega a_k(x)\ \nabla V_1 \nabla \eta \ dx = 0, \quad \forall \eta \in C_0^\infty(\Omega),$$

if and only if (2.47) holds. Therefore V_1 is a weak solution to (2.45).

Now, we define

$$V_2(x) = \begin{cases} u(x) & \text{if} \quad x \in \Omega, \\ 0 & \text{if} \quad x \in \mathbb{R}^n \setminus \bar{\Omega}. \end{cases}$$

Then V_2 is also a weak solution to (2.45). Therefore in order to prove the representation formula (2.42), it suffices to show (2.45) has unique solution in $W^{1,2}_{\text{loc}}(\mathbb{R}^n \setminus \partial\Omega)$. For that, suppose $w \in W^{1,2}_{\text{loc}}(\mathbb{R}^n \setminus \partial\Omega)$ is a weak solution to (2.45) with $f = g = 0$. Thus we clearly see that w is continuous on $\mathbb{R}^n$ and harmonic in $\mathbb{R}^n \setminus \overline{D}$. Thus w is a weak solution to (2.45) in the entire domain $\mathbb{R}^n$. For a large R, we have

$$\begin{aligned}
\int_{B_R(0)} |\nabla w|^2 \ dx &\leq \frac{1+k}{k} \int_{B_R(0)} \Big(1 + (k-1)\chi(D)\Big) \ |\nabla w|^2 \ dx \\
&= \frac{1+k}{k} \int_{\partial B_R(0)} w \ \partial_\nu w \ d\sigma \quad \text{(by definition of weak solution)} \\
&= -\frac{1+k}{k} \int_{\mathbb{R}^n \setminus \overline{B_R(0)}} |\nabla w|^2 \ dx \leq 0 \quad \text{(by Lemma (2.13))}.
\end{aligned}$$

This inequality holds for all R and hence w is constant. Since $w(x) \to 0$ at infinity, we conclude that $w \equiv 0$.

To prove the uniqueness of the representation formula (2.42), suppose that H' is harmonic in Ω and $H + S_{\partial D}\varphi = H' + S_{\partial D}\varphi'$ in Ω. Then $S_{\partial D}(\varphi - \varphi')$ is harmonic in Ω and hence

$$\partial_{\nu^-} S_{\partial D}(\varphi - \varphi') = \partial_{\nu^+} S_{\partial D}(\varphi - \varphi') \ \text{ on } \partial D.$$

It follows from (2.23) that $\varphi - \varphi' = 0$ on ∂D and then $H = H'$. □

By using Theorem 2.14, the harmonic parts H_0 and H_δ of u_0 and u_δ can be respectively represented as

$$H_0(x) = -\mathbf{S}_{\partial\Omega} g(x) + \mathbf{D}_{\partial\Omega} \left(u_0|_{\partial\Omega}\right)(x), \quad x \in \Omega, \tag{2.48}$$

$$H_\delta(x) = -\mathbf{S}_{\partial\Omega} g(x) + \mathbf{D}_{\partial\Omega} \left(u_\delta|_{\partial\Omega}\right)(x), \quad x \in \Omega. \tag{2.49}$$

The following lemma shows that H_δ is uniformly bounded independently of δ in any norm as well as H_δ approaches H_0 when δ approaches 0 in any compact subset of Ω. It was proved in [1] for dimension $n = 2$ and the proof is still valid for any dimension $n \geq 2$.

Lemma 2.16. *Let $\delta_0 > 0$ and $W \subset\subset \Omega$, such that $D_1^\delta \cup D_2^\delta \subset W$, for all $\delta < \delta_0$. Then for all $m = 0, 1, 2, 3, ...$ there exists $C = C(n, m, k, \Omega, d)$ where $d = dist(\partial\Omega, W)$ such that*

$$\|H_\delta\|_{C^m(\overline{W})} \le C \|g\|_{L^2(\partial\Omega)} \quad \forall\, \delta < \delta_0, \tag{2.50}$$

and,

$$\lim_{\delta \to 0} \|H_\delta - H_0\|_{C^m(\overline{W})} = 0. \tag{2.51}$$

Proof. From (2.48) and (2.49), we see that

$$H_\delta - H_0 = \mathbf{D}_{\partial\Omega}\left(u_\delta|_{\partial\Omega}\right) - \mathbf{D}_{\partial\Omega}\left(u_0|_{\partial\Omega}\right),$$

where

$$\mathbf{D}_{\partial\Omega}\left(u_\delta|_{\partial\Omega}\right) = \frac{1}{\omega_n}\int_{\partial\Omega} \frac{(y-x)\cdot\nu_y}{|x-y|^n}\, u_\delta(y)\, d\sigma\,(y)\,,$$

and

$$\mathbf{D}_{\partial\Omega}\left(u_0|_{\partial\Omega}\right) = \frac{1}{\omega_n}\int_{\partial\Omega} \frac{(y-x)\cdot\nu_y}{|x-y|^n}\, u_0(y)\, d\sigma\,(y)\,.$$

Since $\overline{W} \subset\subset \Omega$, then it is clear that

$$\|H_\delta - H_0\|_{C^m\left(\overline{W}\right)} \le C \|u_\delta - u_0\|_{L^2(\partial\Omega)}\,.$$

By trace theorem, we have

$$\|H_\delta - H_0\|_{C^m\left(\overline{W}\right)} \le C \|u_\delta - u_0\|_{H^1(\Omega)}\,. \tag{2.52}$$

From Theorem 2.5, it follows that

$$\lim_{\delta \to 0} \|H_\delta - H_0\|_{C^m\left(\overline{W}\right)} = 0.$$

Now we show the uniform bound (2.50). From the definition of H_δ, we clearly see that

$$\|H_\delta\|_{C^m\left(\overline{W}\right)} \le C\left(\|g\|_{L^2(\partial\Omega)} + \|u_\delta\|_{L^2(\partial\Omega)}\right). \tag{2.53}$$

By applying trace theorem, we get

$$\|H_\delta\|_{C^m\left(\overline{W}\right)} \le C\left(\|g\|_{L^2(\partial\Omega)} + \|u_\delta\|_{H^1(\Omega)}\right). \tag{2.54}$$

To prove $\|u_\delta\|_{H^1(\Omega)}$ is uniformly bounded by $\|g\|_{L^2(\partial\Omega)}$, we use the bilinear form inequality (2.3) for H_δ to deduce

$$\begin{aligned}
\|u_\delta\|^2_{H^1(\Omega)} &\le C\int_{\partial\Omega} g\; u_\delta \; d\sigma(x) \\
&\le C\,\|g\|_{L^2(\partial\Omega)}\,\|u_\delta\|_{H^1(\Omega)} \\
&\le C\left(\frac{1}{4\epsilon}\,\|g\|^2_{L^2(\partial\Omega)} + \epsilon\,\|u_\delta\|^2_{H^1(\Omega)}\right).
\end{aligned}$$

Choosing $C\epsilon < 1$ in the last inequality, it follows that

$$\|u_\delta\|_{H^1(\Omega)} \le C\,\|g\|_{L^2(\partial\Omega)} \tag{2.55}$$

where C is independent of δ. Thus by substituting (2.55) in (2.54), the uniform bound follows. □

2.6 Piecewise Hölder continuity for the modified differential equation.

For $\delta > 0$, we define potential functions φ_1^δ and φ_2^δ respectively on the boundaries of D_1 and D_2 by

$$\varphi_1^\delta(x) = \left(\partial_\nu u_\delta^+ - \partial_\nu u_\delta^-\right)\left(x - \frac{\delta}{2}\mathbf{e_n}\right) \quad \text{for } x \in \Gamma_1, \tag{2.56}$$

and

$$\varphi_2^\delta(x) = \left(\partial_\nu u_\delta^+ - \partial_\nu u_\delta^-\right)\left(x + \frac{\delta}{2}\mathbf{e_n}\right) \quad \text{for } x \in \Gamma_2. \tag{2.57}$$

To simplify the notation of the single layer potential on the boundaries of D_1 and D_2, we write

$$\mathbf{S}_1\varphi_1^\delta(x) = \int_{\Gamma_1} \Phi(x-y)\varphi_1^\delta(y)\, d\sigma(y),$$

and

$$\mathbf{S}_2\varphi_2^\delta(x) = \int_{\Gamma_2} \Phi(x-y)\varphi_2^\delta(y)\, d\sigma(y),$$

It was given in [1] the representation formula for the approximated solution u_δ to (2.5) which works for any dimension $n \geq 2$.

Theorem 2.17. *The solution of problem (2.5) can be uniquely represented as*

$$u_\delta(x) = H_\delta(x) + \mathbf{S}_1\varphi_1^\delta(x) + \mathbf{S}_2\varphi_2^\delta(x), \quad x \in \Omega, \tag{2.58}$$

where

$$H_\delta(x) = -\mathbf{S}_{\partial\Omega}g(x) + \mathbf{D}_{\partial\Omega}\left(u_\delta|_{\partial\Omega}\right)(x), \qquad x \in \Omega,$$

and φ_1^δ and φ_2^δ solve the following system of integral equations

$$
\begin{aligned}
&\left(\lambda I - K_1^*\right) \varphi_1^\delta(x) - \partial_\nu \mathbf{S}_2 \varphi_2^\delta(x - \delta \mathbf{e_n}) = \partial_\nu H_\delta\big(x - \frac{\delta}{2}\mathbf{e_n}\big), \qquad x \in \Gamma_1,\\
&- \partial_\nu \mathbf{S}_1 \varphi_1^\delta\big(x + \delta \mathbf{e_n}\big) + \left(\lambda I - K_2^*\right) \varphi_2^\delta(x) = \partial_\nu H_\delta\big(x + \frac{\delta}{2}\mathbf{e_n}\big), \quad x \in \Gamma_2,
\end{aligned}
\tag{2.59}
$$

in this system $\lambda = \dfrac{k+1}{2(k-1)}$, and K_i^* denotes the operator

$$
K_i^* \varphi_i^\delta(x) = \frac{1}{\omega_n} \int_{\Gamma_i} \frac{(x-y)\cdot \nu(x)}{|x-y|^n}\, \varphi_i^\delta(y)\, d\sigma(y), \quad i = 1, 2.
$$

Proof. For $\eta \in C_0^\infty\big(\Omega\big)$, we have

$$
\begin{aligned}
\int_\Omega \Big(1 + (k-1)\chi_{D_1^\delta \cup D_2^\delta}\Big) \nabla\big(H_\delta + \mathbf{S}_1\varphi_1^\delta + \mathbf{S}_2\varphi_2^\delta\big)\nabla\eta \, dx &= \int_{\Omega \setminus \left(D_1^\delta \cup D_2^\delta\right)} \nabla\big(H_\delta + \mathbf{S}_1\varphi_1^\delta + \mathbf{S}_2\varphi_2^\delta\big)\nabla\eta \, dx\\
&+ k \int_{\left(D_1^\delta \cup D_2^\delta\right)} \nabla\big(H_\delta + \mathbf{S}_1\varphi_1^\delta + \mathbf{S}_2\varphi_2^\delta\big)\nabla\eta \, dx\\
&= \int_{\partial D_1^\delta \cup \partial D_2^\delta} \Big(- \frac{\partial}{\partial \nu^+}\big(H_\delta + \mathbf{S}_1\varphi_1^\delta + \mathbf{S}_2\varphi_2^\delta\big) + k \frac{\partial}{\partial \nu^-}\big(H_\delta + \mathbf{S}_1\varphi_1^\delta + \mathbf{S}_2\varphi_2^\delta\big)\Big)\eta \, d\sigma\\
&= \int_{\partial D_1^\delta} \Big(- \frac{\partial}{\partial \nu^+}\big(H_\delta + \mathbf{S}_1\varphi_1^\delta + \mathbf{S}_2\varphi_2^\delta\big) + k \frac{\partial}{\partial \nu^-}\big(H_\delta + \mathbf{S}_1\varphi_1^\delta + \mathbf{S}_2\varphi_2^\delta\big)\Big)\eta \, d\sigma\\
&+ \int_{\partial D_2^\delta} \Big(- \frac{\partial}{\partial \nu^+}\big(H_\delta + \mathbf{S}_1\varphi_1^\delta + \mathbf{S}_2\varphi_2^\delta\big) + k \frac{\partial}{\partial \nu^-}\big(H_\delta + \mathbf{S}_1\varphi_1^\delta + \mathbf{S}_2\varphi_2^\delta\big)\Big)\eta \, d\sigma\\
&= (k-1) \int_{\partial D_1^\delta} \Big(\partial\nu H_\delta - \Big(\frac{k+1}{2(k-1)} I + K_{\delta_1}^*\Big)\, \varphi_1^\delta + \partial\nu \mathbf{S}_2 \varphi_2^\delta\Big)\eta \, d\sigma\\
&+ (k-1) \int_{\partial D_2^\delta} \Big(\partial\nu H_\delta - \Big(\frac{k+1}{2(k-1)} I + K_{\delta_2}^*\Big)\, \varphi_2^\delta + \partial\nu \mathbf{S}_1 \varphi_1^\delta\Big)\eta \, d\sigma \quad \big(\text{by } (2.23)\big).
\end{aligned}
$$

If $\big(\varphi_1^\delta, \varphi_2^\delta\big)$ solves (2.59), then we get

$$
\int_\Omega \Big(1 + (k-1)\chi_{D_1^\delta \cup D_2^\delta}\Big) \nabla\big(H_\delta + \mathbf{S}_1\varphi_1^\delta + \mathbf{S}_2\varphi_2^\delta\big)\nabla\eta \, dx = 0, \quad \forall\, \eta \in C_0^\infty(\Omega).
$$

Therefore u_δ is weak solution of (2.5). $\square$

The following Lemma is very useful and it will be used to give the piecewise Hölder estimate for the modified differential equation (2.5). The proof can be found in [15, 21].

Lemma 2.18. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $D \subset\subset \Omega$ be a C^{1,α_0} subdomain. The first derivatives of a single layer potential*

$$\mathbf{S}_{\partial D}\varphi(x) = \int_{\partial D} \Phi(x,y)\varphi(y)\, d\sigma(y), \quad x \in \Omega,$$

with Hölder density function $\varphi \in C^{\alpha}(\partial D)$, $0 < \alpha < \alpha_0$ can be uniformly extended in a Hölder continuous fashion from $\Omega \setminus \overline{D}$ into $\Omega \setminus D$ and from D into $\overline{D}$ with limiting values

$$\partial_\nu \mathbf{S}_{\partial D}\varphi|_{\pm}(x) = \left(\pm\frac{1}{2}I + K^*_{\partial D}\right)\varphi(x), \quad x \in \partial D. \tag{2.60}$$

Furthermore, we have for $\alpha' < \alpha$ the estimate

$$\|\mathbf{S}_{\partial D}\varphi\|_{1,\alpha'\left(\overline{D}\right)} + \|\mathbf{S}_{\partial D}\varphi\|_{1,\alpha'\left(\Omega\setminus D\right)} \leq C\,\|\varphi\|_{\alpha\left(\partial D\right)}. \tag{2.61}$$

Applying Lemma 2.18 for the potentials φ_1^δ and φ_2^δ that solve (2.59), we immediately have for any $0 < \alpha' < \alpha < \alpha_0$ the following estimate

$$\left\|\mathbf{S}_i\varphi_i^\delta\right\|_{1,\alpha'\left(\overline{D_i^\delta}\right)} + \left\|\mathbf{S}_i\varphi_i^\delta\right\|_{1,\alpha'\left(\overline{\Omega\setminus D_i^\delta}\right)} \leq C\left\|\varphi_i^\delta\right\|_{\alpha(\Gamma_i)}, \quad i = 1, 2, \tag{2.62}$$

where C is independent of δ. From the representation formula (2.58), Lemma (2.16) and the estimate (2.62), we obtain

Lemma 2.19. *Let u_δ be the solution to (2.5) and let $\left(\varphi_1^\delta, \varphi_2^\delta\right)$ be the solution to (2.59). For any small $\tilde{\epsilon} > 0$, let $\Omega_{\tilde{\epsilon}}$ denotes the set $\Omega_{\tilde{\epsilon}} = \{x \in \Omega : dist(x, \partial\Omega) > \tilde{\epsilon}\}$. Then for any $0 < \alpha' < \alpha < \alpha_0$ we have the bound*

$$\|u_\delta\|_{1,\alpha'\left(\overline{D_1^\delta}\right)} + \|u_\delta\|_{1,\alpha'\left(\overline{D_2^\delta}\right)} + \|u_\delta\|_{1,\alpha'\left(\Omega_{\tilde{\epsilon}} \setminus D_1^\delta \cup D_2^\delta\right)} \leq C \left(\sum_{i=1}^{2} \left\|\varphi_i^\delta\right\|_{\alpha\left(\Gamma_i\right)} + \|g\|_{L^2(\partial\Omega)} \right), \tag{2.63}$$

for some constant C depending on α, α', α_0, Ω, k, and $\tilde{\epsilon}$ but independent of δ.

As a consequence of this Lemma, we obtain the desired piecewise Hölder continuity for ∇u_δ on each component of Ω if the right hand side of (2.63) can be shown to be uniformly bounded independently of δ. That is, if we prove

$$\sum_{i=1}^{2} \left\|\varphi_i^\delta\right\|_{\alpha(\Gamma_i)} \leq C \|g\|_{L^2(\partial\Omega)}, \tag{2.64}$$

where C is independent of δ, then we obtain the desired bound for ∇u_δ. The uniform bound (2.64) depends on the solvability of the system (2.59) and this will be our task in the next chapters.

Chapter 3

Decomposition of the system of integral equations.

Our goal is to solve the system (2.59) with bounds uniform in δ. To that end, let us define the operator on $C^{\alpha}(\Gamma_1) \times C^{\alpha}(\Gamma_2)$ by

$$T^{\delta}\begin{pmatrix}\varphi_1^{\delta}\\ \varphi_2^{\delta}\end{pmatrix} := \begin{pmatrix}\lambda I - K_1^* & L_2^{\delta}\\ L_1^{\delta} & \lambda I - K_2^*\end{pmatrix}\begin{pmatrix}\varphi_1^{\delta}\\ \varphi_2^{\delta}\end{pmatrix},$$

where

$$L_2^{\delta}\varphi_2^{\delta}(x) = -\partial\nu \mathbf{S}_2\varphi_2^{\delta}\big(x - \delta \mathbf{e_n}\big), \qquad x \in \Gamma_1, \tag{3.1}$$

and

$$L_1^{\delta}\varphi_1^{\delta}(x) = -\partial\nu \mathbf{S}_1\varphi_1^{\delta}\big(x + \delta \mathbf{e_n}\big), \qquad x \in \Gamma_2. \tag{3.2}$$

Thus the system (2.59) can be written as

$$T^{\delta}\begin{pmatrix}\varphi_1^{\delta}\\ \varphi_2^{\delta}\end{pmatrix} = \begin{pmatrix}\partial_{\nu}H_{\delta}\big(y_1(x,\delta)\big)\\ \partial_{\nu}H_{\delta}\big(y_2(x,\delta)\big)\end{pmatrix}, \tag{3.3}$$

where

$$y_1(x,\delta) = \big(x - \frac{\delta}{2}\mathbf{e_n}\big), \quad x \in \Gamma_1,$$

and

$$y_2(x,\delta) = \big(x + \frac{\delta}{2}\mathbf{e_n}\big), \quad x \in \Gamma_2.$$

From Lemma 2.16 we have that the right hand side of (3.3) is uniformly bounded in C^{α_0} norm on the surfaces Γ_j, $j = 1, 2$. That is

$$\|\partial_\nu H_\delta\|_{C^{\alpha_0}(\Gamma_j)} \leq C \|g\|_{L^2(\partial\Omega)}.$$

Therefore to show the uniform bound (2.64), we need to show that the operator T^δ is invertible as an operator on $C^\alpha(\Gamma_1) \times C^\alpha(\Gamma_2)$ and its inverse is uniformly bounded in $C^\alpha(\Gamma_1) \times C^\alpha(\Gamma_2)$, for any $\alpha < \alpha_0$. Thus the purpose of this chapter is to study the behavior of the T^δ and how to overcome the singularity of its kernel when δ approaches 0.

3.1 Behavior of T^δ, $\delta > 0$.

Recall that $\lambda = \frac{k+1}{2(k-1)}$, where $0 < k < \infty$ and $k \neq 1$. That is, we conclude that $|\lambda| > \frac{1}{2}$.

Theorem 3.1. *For $\delta > 0$, T^δ is continuous linear operator on $C^\alpha(\Gamma_1) \times C^\alpha(\Gamma_2)$, invertible with bounded inverse for any $0 < \alpha < \alpha_0$ and for any $|\lambda| > \frac{1}{2}$.*

Proof. For $\delta > 0$, we evidently see that T^δ is bounded linear operator on $C^\alpha(\Gamma_1) \times C^\alpha(\Gamma_2)$.

For

$$\lambda\mathbf{I} = \begin{pmatrix} \lambda I & O \\ O & \lambda I \end{pmatrix} \text{ and } \mathbf{K}_\delta = \begin{pmatrix} -K_1^* & L_2^\delta \\ L_1^\delta & -K_2^* \end{pmatrix}, T^\delta \text{ can be written as } T^\delta = \lambda\mathbf{I} + \mathbf{K}_\delta.$$

Since $\mathbf{K}_\delta$ is a compact operator on $C^\alpha(\Gamma_1)\times C^\alpha(\Gamma_2)$, then T^δ is Fredholm operator. Therefore the invertibility of T^δ follows if we show that T^δ is injective. Let $(\varphi_1^\delta, \varphi_2^\delta) \in C^\alpha(\Gamma_1) \times C^\alpha(\Gamma_2)$ be such that $T^\delta \begin{pmatrix} \varphi_1^\delta \\ \varphi_2^\delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. *That is,*

$$\begin{cases} \left(\lambda I - K_1^*\right) \varphi_1^\delta(x) - \partial\nu \mathbf{S}_2 \varphi_2^\delta(x - \delta \mathbf{e_n}) = 0, & x \in \Gamma_1, \\ -\partial\nu \mathbf{S}_1 \varphi_1^\delta(x + \delta \mathbf{e_n}) + \left(\lambda I - K_2^*\right) \varphi_2^\delta(x) = 0, & x \in \Gamma_2. \end{cases} \tag{3.4}$$

Or equivalently,

$$\begin{cases} \left(\lambda I - K_1^*\right) \varphi_1^\delta\left(x + \frac{\delta}{2}\mathbf{e_n}\right) - \partial\nu \mathbf{S}_2 \varphi_2^\delta(x - \frac{\delta}{2} \mathbf{e_n}) = 0, & x \in \partial D_1^\delta, \\ -\partial\nu \mathbf{S}_1 \varphi_1^\delta(x + \frac{\delta}{2} \mathbf{e_n}) + \left(\lambda I - K_2^*\right) \varphi_2^\delta(x - \frac{\delta}{2}\mathbf{e_n}) = 0, & x \in \partial D_2^\delta. \end{cases} \tag{3.5}$$

Consider the function w_δ defined on $\mathbb{R}^n$ by

$$w_\delta = \mathbf{S}_1 \varphi_1^\delta + \mathbf{S}_2 \varphi_2^\delta. \tag{3.6}$$

We claim that w_δ is a weak solution of the following problem

$$\begin{cases} \operatorname{div}\left(\left(1 + (k-1)\chi_{(D_1^\delta \cup D_2^\delta)}\right) \nabla w_\delta\right) = 0, & x \in \mathbb{R}^n, \\ w_\delta(x) = O(|x|^{1-n}), & |x| \to \infty. \end{cases} \tag{3.7}$$

To prove the claim, let $\eta \in C_0^\infty\left(B_R(0)\right)$ where $B_R(0) \supset\supset \Omega$. For $a_k = 1 + (k-1)\chi(D_1^\delta \cup D_2^\delta)$, we see the following

$$\begin{aligned}\int_{B_R(0)} a_k \nabla w_\delta \nabla\eta\, dx &= k\int_{D_1^\delta} \nabla\left(\mathbf{S}_1\varphi_1^\delta(x+\frac{\delta}{2}\mathbf{e_n}) + \mathbf{S}_2\varphi_{\Gamma_2}^\delta(x-\frac{\delta}{2}\mathbf{e_n})\right)\nabla\eta\, dx \\ &+ k\int_{D_2^\delta} \nabla\left(\mathbf{S}_1\varphi_1^\delta(x+\frac{\delta}{2}\mathbf{e_n}) + \mathbf{S}_2\varphi_2^\delta(x-\frac{\delta}{2}\mathbf{e_n})\right)\nabla\eta\, dx \\ &+ \int_{B_R(0)\setminus(D_1^\delta\cup D_2^\delta)} \nabla\left(\mathbf{S}_1\varphi_1^\delta(x+\frac{\delta}{2}\mathbf{e_n}) + \mathbf{S}_2\varphi_2^\delta(x-\frac{\delta}{2}\mathbf{e_n})\right)\nabla\eta\, dx.\end{aligned}$$

Using the divergence theorem, it follows

$$\begin{aligned}\int_{B_R(0)} a_k \nabla w_\delta \nabla\eta\, dx &= k\int_{\partial D_1^\delta} \left(\partial\nu^-\mathbf{S}_1\varphi_1^\delta(x+\frac{\delta}{2}\mathbf{e_n}) + \partial\nu\mathbf{S}_2\varphi_{\Gamma_2}^\delta(x-\frac{\delta}{2}\mathbf{e_n})\right)\eta\, d\sigma(x) \\ &+ k\int_{D_2^\delta} \left(\partial\nu\mathbf{S}_1\varphi_1^\delta(x+\frac{\delta}{2}\mathbf{e_n}) + \partial\nu^-\mathbf{S}_2\varphi_2^\delta(x-\frac{\delta}{2}\mathbf{e_n})\right)\eta\, d\sigma(x) \\ &- \int_{\partial D_1^\delta} \left(\partial\nu^+\mathbf{S}_1\varphi_1^\delta(x+\frac{\delta}{2}\mathbf{e_n}) + \partial\nu\mathbf{S}_2\varphi_2^\delta(x-\frac{\delta}{2}\mathbf{e_n})\right)\eta\, d\sigma(x) \\ &- \int_{D_2^\delta} \left(\partial\nu\mathbf{S}_1\varphi_1^\delta(x+\frac{\delta}{2}\mathbf{e_n}) + \partial\nu^+\mathbf{S}_2\varphi_2^\delta(x-\frac{\delta}{2}\mathbf{e_n})\right)\eta\, d\sigma(x).\end{aligned}$$

Utilizing the jump conditions for the single layer potential (2.23), we have

$$\begin{aligned}\int_{B_R(0)} a_k \nabla w_\delta \nabla\eta\, dx &= (k-1)\int_{\partial D_1^\delta} (\lambda I - K_1^*)\ \varphi_1^\delta(x+\frac{\delta}{2}e_n) - \partial\nu\mathbf{S}_2\varphi_2^\delta(x-\frac{\delta}{2}e_n)\, d\sigma(x) \\ &+ (k-1)\int_{\partial D_2^\delta} (\lambda I - K_2^*)\ \varphi_2^\delta(x-\frac{\delta}{2}e_n) - \partial\nu\mathbf{S}_1\varphi_1^\delta(x+\frac{\delta}{2}e_n)\, d\sigma(x).\end{aligned}$$

By applying (3.4), we obtain the following

$$\int_{B_R(0)} a_k \nabla w_\delta \nabla\eta\, dx = 0.$$

In order to get the decay condition for w_δ, it is enough to show the following identity

$$\int_{\Gamma_1} \varphi_1^\delta(\cdot + \delta \mathbf{e_n})\, d\sigma = \int_{\Gamma_2} \varphi_2^\delta(\cdot - \delta \mathbf{e_n})\, d\sigma = 0, \tag{3.8}$$

since, if $\int_{\Gamma_j} \varphi_j^\delta \, d\sigma(y) = 0$, then

$$\mathbf{S}_j \varphi_j^\delta = \int_{\Gamma_j} \Phi(x,y)\varphi_j^\delta \, d\sigma(y) = \int_{\Gamma_j} \Big(\Phi(x,y) - \Phi(x,y_0)\Big)\varphi_j^\delta \, d\sigma(y) \le C_j \left|x\right|^{1-n}, \tag{3.9}$$

where $y_0 \in D_j$. Since

$$|\Phi(x,y) - \Phi(x,y_0)| \le C \left|x\right|^{1-n} \quad \text{if} \quad |x| \longrightarrow \infty \;\text{ and }\; y \in \Gamma_j.$$

We only show $\int_{\Gamma_1} \varphi_1^\delta(\cdot + \delta \mathbf{e_n})\, d\sigma = 0$ and same work would hold for $\int_{\Gamma_2} \varphi_2^\delta(\cdot - \delta \mathbf{e_n})\, d\sigma = 0$. Since $S_{\Gamma_2}\varphi_2^\delta(\cdot - \delta \mathbf{e_n})$ is harmonic in D_1, therefore we get

$$\int_{\Gamma_1} \partial\nu \mathbf{S}_2 \varphi_2^\delta(x - \delta \mathbf{e_n})\, d\sigma(x) = 0.$$

Therefore by using (3.4), it follows that

$$0 = \int_{\Gamma_1} (\lambda I - K_1^*)\, \varphi_1^\delta(x + \delta \mathbf{e_n}) - \partial\nu \mathbf{S}_2 \varphi_2^\delta(x - \delta \mathbf{e_n})\, d\sigma(x).$$

That is,

$$\int_{\Gamma_1} (\lambda I - K_1^*)\, \varphi_1^\delta(x + \delta \mathbf{e_n})\, d\sigma(x) = 0.$$

Thus by the duality (remark 2.10), we obtain the following

$$\int_{\Gamma_1} \Big(\lambda\varphi_1^\delta(x+\delta\mathbf{e_n}) - K_1(1)\,\varphi_1^\delta(x+\delta\mathbf{e_n})\Big)\,d\sigma(x) = (\lambda-\frac{1}{2})\int_{\Gamma_1} \varphi_1^\delta(x+\delta\mathbf{e_n})\,d\sigma(x) = 0.$$

Since $|\lambda| > \frac{1}{2}$, then it follows that

$$\int_{\Gamma_1} \varphi_1^\delta(x+\delta\mathbf{e_n})\,d\sigma(x) = 0.$$

We show that $w_\delta \equiv 0$ in $\mathbb{R}^n$. For $R >> 1$, we have

$$\begin{aligned}
\int_{B_R(0)} |\nabla w_\delta|^2\,dx &\leq \frac{1+k}{k}\int_{B_R(0)} \Big(1+(k-1)\chi(D_1^\delta\cup D_2^\delta)\Big)\,|\nabla w_\delta|^2\;dx \\
&= \frac{1+k}{k}\int_{\partial B_R(0)} w_\delta\,\partial_\nu w_\delta\;d\sigma(x)\;\;\text{(by definition of weak solution)} \\
&= -\frac{1+k}{k}\int_{\mathbb{R}^n\setminus\overline{B_R(0)}} |\nabla w_\delta|^2\;dx\;\;\text{(by lemma 2.13)} \\
&\leq 0.
\end{aligned}$$

Thus w_δ is constant in $\mathbb{R}^n$. Using the decay condition at infinity, it follows that $w_\delta \equiv 0$ in $\mathbb{R}^n$, that is

$$\mathbf{S}_1\varphi_1^\delta(x+\frac{\delta}{2}\mathbf{e_n}) + \mathbf{S}_2\varphi_2^\delta(x-\frac{\delta}{2}\mathbf{e_n}) = 0, \quad x\in\mathbb{R}^n.$$

Consequently, we conclude that $\mathbf{S}_1\varphi_1^\delta$ is smooth across Γ_1 and $\mathbf{S}_2\varphi_2^\delta$ is smooth across Γ_2. Therefore applying the jump conditions for the single layer potential (2.23), we have

$$\begin{aligned}
\varphi_1^\delta(x) &= \partial\nu^+\mathbf{S}_1\varphi_1^\delta(x) - \partial\nu^-\mathbf{S}_1\varphi_1^\delta(x) = 0, \quad x\in\Gamma_1, \\
\varphi_2^\delta(x) &= \partial\nu^+S_{\Gamma_2}\varphi_2^\delta(x) - \partial\nu^-\mathbf{S}_2\varphi_2^\delta(x) = 0, \quad x\in\Gamma_2.
\end{aligned}$$

Thus T^δ is injective. $\square$

3.2 Auxiliary functions and Basic properties.

As in [1], we study the behavior of T^δ and its inverse as δ approaches 0 and we use the same notations that they used in their paper. When the subdomains D_1^δ and D_2^δ approach the touching point $x = 0$, the kernel of the operators

$$\partial_\nu \mathbf{S}_2 \varphi_2^\delta (x - \delta \mathbf{e_n}) = \int_{\Gamma_2} \frac{(x - \delta \mathbf{e_n} - y) \cdot \nu}{|x - \delta \mathbf{e_n} - y|^n} \varphi_2^\delta(y)\, d\sigma(y), \quad x \in \Gamma_1,$$

and

$$\partial_\nu \mathbf{S}_1 \varphi_1^\delta (x - \delta \mathbf{e_n}) = \int_{\Gamma_2} \frac{(x - \delta \mathbf{e_n} - y) \cdot \nu}{|x - \delta \mathbf{e_n} - y|^n} \varphi_1^\delta(y)\, d\sigma(y), \quad x \in \Gamma_2,$$

become singular at $x = 0$.

For dimension $n = 2$ [1] the singularity was overcome by decomposing T^δ as a sum $\Lambda_{\epsilon,\delta} + C_{\epsilon,\delta}$ where, for a fixed $\epsilon > 0$ sufficiently small, the operator $\Lambda_{\epsilon,\delta}$ contains the singular part of T^δ (i.e, the identity plus a piece of the off-diagonal terms), and $C_{\epsilon,\delta}$ is compact. Their idea works for any dimension $n \geq 2$ with some necessary technical modifications. So we are going to mimic their idea to overcome the singularity for general dimension $n \geq 2$.

We fix a small parameter $0 < \epsilon_0 < 1$ so that

$$\frac{1}{2} < \frac{1 + \epsilon_0}{2} < |\lambda|, \tag{3.10}$$

where $\lambda = \frac{k+1}{2(k-1)}$ and recalling that the constant k is the conductivity in the subdomains D_1 and D_2 while the constant 1 is the conductivity in $\Omega \setminus (D_1 \cap D_2)$. Let $R_0 = 2(1 + \frac{1}{\epsilon_0})$,

$r_1 = \mathrm{diam}(D_1)$, $r_2 = \mathrm{diam}(D_2)$ and $\varrho = 20(1 + r_1 + r_2)R_0$ where $\mathrm{diam}(D_i)$ is the diameter of the subdomain D_i, $i = 1, 2$. We rescale the domain Ω by assuming $x = \varrho x$ to make sure that the surface $\Gamma_1 \cup \Gamma_2$ within the ball $\mathcal{B}(2R_0)$ meets x_n-axis at 0 only.

Let η be a smooth cut-off function in $\mathbb{R}^n$ that supported in the ball $\mathcal{B}(R_0)$ be such that

$$\begin{cases} 0 \leq \eta \leq 1, \\ \eta \equiv 1, \quad x \in \mathcal{B}(\epsilon_0), \\ \|\nabla \eta\|_\infty \leq \epsilon_0. \end{cases} \tag{3.11}$$

We also assume that ϵ_0 is sufficiently small so that around the touching point $x = 0$, the surfaces Γ_1 and Γ_2 can be parametrized by

$$\begin{cases} |x'| \leq \epsilon_0 \longrightarrow x = \big(x', \psi_1(x')\big) \in \Gamma_1, \\ |y'| \leq \epsilon_0 \longrightarrow y = (y', \psi_2(y')) \in \Gamma_2, \end{cases} \tag{3.12}$$

for some C^{1,α_0} parametric functions ψ_1 and ψ_2.

Definition 3.2. *[1] A closed and bounded surface $\Gamma \subset \mathbb{R}^n$ is called of regularity $C^{1,\alpha}$ if it can be covered by a local set of charts*

$$\psi_j : x \in \mathcal{B}'_j \subset \mathbb{R}^{n-1} \longrightarrow \big(\psi_{j,1}(x), \cdots, \psi_{j,n}(x)\big) \subset \mathbb{R}^n,$$

where $\mathcal{B}'_j, 1 \leq j \leq m$, are open balls in $\mathbb{R}^{n-1}$ and $\psi_{j,i}, 1 \leq i \leq n$ are $C^{1,\alpha}\big(\overline{\mathcal{B}_j}\big)$ functions with $\mathbf{Rank}(\nabla \psi_j) = \mathbf{Rank}\big(\nabla \psi_1, \cdots, \nabla \psi_n\big) = n - 1$. We say that a continuous function f

is of regularity $C^{0,\alpha}(\Gamma)$ *if for any of the local charts*

$$\left|f\circ\psi_j\right|_{C^{0,\alpha}\left(\overline{\mathcal{B}_j}\right)} := \sup_{x',y'\in\overline{\mathcal{B}_j},|x'-y'|<1} \frac{\left|f\Big(\psi_{j,1}(x'),\cdots,\psi_{j,n}(x')\Big)-f\Big(\psi_{j,1}(y'),\cdots,\psi_{j,n}(y')\Big)\right|}{|x'-y'|^{\alpha}} \le C.$$

The norm on $C^{0,\alpha}(\Gamma)$ *is defined by*

$$\|f\|_{0,\alpha} = \max\left(\|f\|_{L^{\infty}(\Gamma)}, \max_{1\le j\le m}\left|f\circ\psi_j\right|_{C^{0,\alpha}\left(\overline{\mathcal{B}_j}\right)}\right).$$

Lemma 3.3. *[1] Given* $0<\epsilon_0<1$ *for which (3.12) holds with (3.12) as one of local coordinate chart, and given* $0<\alpha<1$*; there exists an operator*

$$E : C^{\alpha}(\Gamma_2) \longrightarrow C^{\alpha}(\mathbb{R}^{n-1}),$$

such that for any $\varphi\in C^{\alpha}(\Gamma_2)$*, we have*

$$\begin{cases} \|E\varphi\|_{\alpha,\mathbb{R}^{n-1}} \le (1+\epsilon_0)\,\|\varphi\|_{\alpha,\Gamma_2} \\ \Big(E\varphi\Big)(y') = \varphi\Big(y',\psi_2(y')\Big),\ \left|y'\right|\le\epsilon_0, \\ supp(E\varphi)\subset\mathcal{B}\Big(\frac{2}{\epsilon_0}\Big). \end{cases} \tag{3.13}$$

Proof. For $\varphi\in C^{\alpha}(\Gamma_2)$, let $\tilde\varphi$ be a function on $\mathbb{R}^{n-1}$ that defined by

$$\tilde\varphi(y') = \begin{cases} \varphi(y',\psi_2(y')), & \text{if} \quad \left|y'\right|\le\epsilon_0, \\ \varphi\Big(y^*,\psi_2(y^*)\Big), & \text{if} \quad \left|y'\right|>\epsilon_0, \end{cases}$$

where $y^* = \epsilon_0\ \nu(y')$and $\nu(y')$ is the normal unit vector $\frac{y'}{|y'|}$. We show $\tilde\varphi$ is C^{α} Hölder continuous on $\mathbb{R}^{n-1}$.

Let $y_1', y_2' \in \mathbb{R}^{n-1}$. **Case 1:** When $\left|y_1'\right| \le \epsilon_0$ and $\left|y_2'\right| \le \epsilon_0$. We trivially see that

$$\frac{\left|\tilde{\varphi}(y_1') - \tilde{\varphi}(y_2')\right|}{\left|y_1' - y_2'\right|^{\alpha}} = \frac{\left|\varphi(y_1', \psi_2(y_1')) - \varphi\left(y_2', \psi_2(y_2')\right)\right|}{\left|y_1' - y_2'\right|^{\alpha}} \le \|\varphi\|_{\alpha}.$$

Case 2: When $\left|y_1'\right| \le \epsilon_0$ and $\left|y_2'\right| > \epsilon_0$. We have

$$\frac{\left|\tilde{\varphi}(y_1') - \tilde{\varphi}(y_2')\right|}{\left|y_1' - y_2'\right|^{\alpha}} = \frac{\left|\varphi(y_1', \psi_2(y_1')) - \varphi\left(y_2^*, \psi_2\left(y_2^*\right)\right)\right|}{\left|y_1' - y_2'\right|^{\alpha}}.$$

Since $\left|y_1' - y_2^*\right| \le \left|y_1' - y_2'\right|$, then we obtain

$$\frac{\left|\tilde{\varphi}(y_1') - \tilde{\varphi}(y_2')\right|}{\left|y_1' - y_2'\right|^{\alpha}} \le \frac{\left|\varphi(y_1', \psi_2(y_1')) - \varphi\left(y_2^*, \psi_2\left(y_2^*\right)\right)\right|}{\left|y_1' - y_2^*\right|^{\alpha}} \le \|\varphi\|_{\alpha}.$$

Case 3: When $\left|y_1'\right| > \epsilon_0$ and $\left|y_2'\right| > \epsilon_0$. We first show that

$$|y_1^* - y_2^*| \le \left|y_1' - y_2'\right|. \tag{3.14}$$

The above inequality would be trivial if the dot product $y_1' \cdot y_2' \le 0$. Then it is enough to show (3.14) when $y_1' \cdot y_2' > 0$. Without loss of generality we may assume that $\left|y_2'\right| \ge \left|y_1'\right|$ and let $\tilde{y}_2 = \left|y_1'\right| \nu_2(y')$. We clearly see that

$$|y_1^* - y_2^*| = \frac{\epsilon_0}{\left|y_1'\right|}\left|y_1' - \tilde{y}_2\right| < \left|y_1' - \tilde{y}_2\right| \le \left|y_1' - y_2'\right|.$$

Where we naively can see that validity of last inequality if we assume after rotation that

$y_1' = (0, 0, \cdots, 0, y_{1,n-1})$. Now we form the following

$$\frac{\left|\tilde{\varphi}(y_1') - \tilde{\varphi}(y_2')\right|}{\left|y_1' - y_2'\right|^{\alpha}} = \frac{\left|\varphi\Big(y_1^*, \psi(y_1^*)\Big) - \varphi\Big(y_2^*, \psi(y_2^*)\Big)\right|}{\left|y_1' - y_2'\right|^{\alpha}}.$$

By taking account of (3.14), we obtain

$$\frac{\left|\tilde{\varphi}(y_1') - \tilde{\varphi}(y_2')\right|}{\left|y_1' - y_2'\right|^{\alpha}} \leq \frac{\left|\varphi\Big(y_1^*, \psi(y_1^*)\Big) - \varphi\Big(y_2^*, \psi(y_2^*)\Big)\right|}{\left|y_1^* - y_2^*\right|^{\alpha}} \leq \|\varphi\|_{\alpha}.$$

Thus we have shown that

$$\|\tilde{\varphi}\|_{\alpha} \leq \|\varphi\|_{\alpha}. \tag{3.15}$$

Next, let $\rho \in C^1(\mathbb{R}^{n-1})$ be such that $0 \leq \rho \leq 1$ and

$$\begin{cases} \rho(y') = 1, \ \text{ if } \left|y'\right| \leq \epsilon_0, \\ \|\nabla \rho\| \leq \epsilon_0, \\ \text{supp}(\rho) \in \mathcal{B}\Big(\frac{2}{\epsilon_0}\Big). \end{cases} \tag{3.16}$$

Define $E\varphi(y') = \rho(y')\tilde{\varphi}(y')$. It is clear that $\|E\varphi\|_{\infty} \leq \|\varphi\|_{\infty}$.

Now we show C^{α}-norm for $E\varphi$. For $y_1^{'}, y_1^{'} \in \mathbb{R}^{n-1}$ we have

$$\begin{aligned}
\sup_{\substack{|y_1'-y_2'|<1\\ y_1'\neq y_2'}} \frac{\left|E\varphi(y_1') - E\varphi(y_2')\right|}{\left|y_1' - y_2'\right|^{\alpha}} &= \sup_{\substack{|y_1'-y_2'|<1\\ y_1'\neq y_2'}} \frac{\left|\rho(y_1')\tilde{\varphi}(y_1') - \rho(y_2')\tilde{\varphi}(y_2')\right|}{\left|y_1' - y_2'\right|^{\alpha}} \\
&\leq \sup_{\substack{|y_1'-y_2'|<1\\ y_1'\neq y_2'}} \left[\|\rho\|_{\infty} \frac{\left|\tilde{\varphi}(y_1') - \tilde{\varphi}(y_2')\right|}{\left|y_1' - y_2'\right|^{\alpha}} + \|\tilde{\varphi}\|_{\infty} \frac{\left|\rho(y_1') - \rho(y_2')\right|}{\left|y_1' - y_2'\right|^{\alpha}}\right] \\
&\leq \|\varphi\|_{\alpha} + \epsilon_0 \;\|\varphi\|_{\alpha} \\
&\leq (1+\epsilon_0)\,\|\varphi\|_{\alpha}\,.
\end{aligned}$$

Therefore $\|E\varphi\|_{\alpha,\mathbb{R}^{n-1}} \leq (1+\epsilon_0)\,\|\varphi\|_{\alpha,\Gamma_2}$. □

Next we define two $C^{1,\beta}$ auxiliary functions $\psi_{1,\epsilon}$ and $\psi_{2,\epsilon}$ that defined globally on $\mathbb{R}^{n-1}$ be such that $\left\|\psi_{j,\epsilon}\right\|_{1,\beta;\mathbb{R}^{n-1}} = O(\epsilon^{\nu}),\ j=1,2$ for every $\beta < \alpha_0$ and $\nu = \alpha_0 - \beta$.

Lemma 3.4. *Let $\beta < \alpha_0$, $\nu = \alpha_0 - \beta$ and fix $0 < 2\epsilon < \epsilon_0$. There exist $C^{1,\beta}$-functions $\psi_{1,\epsilon}$ and $\psi_{2,\epsilon}$ defined on $\mathbb{R}^{n-1}$ so that*

$$\begin{cases} \psi_{j,\epsilon} = \psi_j, \quad \left|x'\right| \leq \epsilon, \quad j=1,2, \\ \left\|\psi_{j,\epsilon}\right\|_{1,\beta;\mathbb{R}^{n-1}} \leq C\,\epsilon^{\nu}\left\|\psi_j\right\|_{1,\alpha_0}, \quad j=1,2. \end{cases} \tag{3.17}$$

where C is independent of ϵ.

Proof. Recall that the functions $\psi_j \in C^{1,\alpha_0}(\mathcal{B}'(\epsilon_0))$ are defined locally to satisfy $\psi_j(x') = \left|\nabla\psi_j(x')\right| = 0$ only at $x'=0$, $j=1,2$ and ψ_2 is non-negative while ψ_1 is non-positive. Then for any $x' \in \mathcal{B}'(\epsilon_0)$, we have

$$\left|\psi_j(x')\right| \leq C\;\left|x'\right|^{1+\alpha_0}\left\|\psi_j\right\|_{1,\alpha_0}, \quad j=1,2. \tag{3.18}$$

For $\epsilon < \frac{1}{2}\epsilon_0$, let $\eta_\epsilon \in C_c^\infty(\mathbb{R}^{n-1})$ be a cut-off function that satisfying the following

$$\begin{cases} 0 \le \eta_\epsilon \le 1, \\ \eta_\epsilon(x') = 1, & \text{if} \quad \left|x'\right| \le \epsilon, \\ \eta_\epsilon(x') = 0, & \text{if} \quad \left|x'\right| \ge 2\epsilon, \\ \|\nabla \eta_\epsilon\|_\infty = O(\epsilon^{-1}), \\ \left\|\nabla^2 \eta_\epsilon\right\|_\infty = O(\epsilon^{-2}), \end{cases} \tag{3.19}$$

where $\nabla^2 \eta_\epsilon$ is the Hessian matrix of η_ϵ.

For any $0 < \beta < \alpha_0$, we define

$$\begin{aligned} \psi_{2,\epsilon}(x') &= \psi_2(x')\eta_\epsilon(x') + \Big(1 - \eta_\epsilon(x')\Big)\epsilon^{1+\alpha_0} \|\psi_2\|_{1,\alpha_0}, \\ \psi_{1,\epsilon}(x') &= \psi_1(x')\eta_\epsilon(x') - \Big(1 - \eta_\epsilon(x')\Big)\epsilon^{1+\alpha_0} \|\psi_1\|_{1,\alpha_0}. \end{aligned} \tag{3.20}$$

Clearly we see that $\psi_{j,\epsilon}$ are globally C^{1,α_0} and $\psi_{j,\epsilon}(x') = \psi_j(x')$ for $x' \in \mathcal{B}'(\epsilon)$, $j = 1, 2$. Then we only need to show the bound

$$\left\|\psi_{j,\epsilon}\right\|_{1,\beta;\mathbb{R}^{n-1}} \le C\, \epsilon^\nu \|\psi_2\|_{1,\alpha_0}, \quad \forall \beta < \alpha_0, \quad j = 1, 2. \tag{3.21}$$

We prove (3.21) for $j = 2$ and the same arguments would hold when $j = 1$. First, we show the $L^\infty-$norm for $\psi_{2,\epsilon}$. In the case $\left|x'\right| \le \epsilon$, we have $\psi_{2,\epsilon}(x') = \psi_2(x')$ and then by using (3.18) we obtain the following

$$\left|\psi_{2,\epsilon}(x)\right| \le C \|\psi_2\|_{1,\alpha_0} |x|^{1+\alpha_0} \le C \|\psi_2\|_{1,\alpha_0} \epsilon^{1+\alpha_0} \le C \|\psi_2\|_{1,\alpha_0} \epsilon^\nu.$$

That is

$$\left|\psi_{2,\epsilon}(x')\right| \leq C \left\|\psi_2\right\|_{1,\alpha_0} \epsilon^{\nu}. \tag{3.22}$$

For the case $\epsilon < \left|x'\right| < 2\epsilon$, we have

$$\psi_{2,\epsilon}(x') = \psi_2(x')\eta_\epsilon(x') + (1 - \eta_\epsilon(x'))\epsilon^{1+\alpha_0} \left\|\psi_2\right\|_{1,\alpha_0}.$$

Again by using (3.18) it follows that

$$\left|\psi_{2,\epsilon}(x')\right| \leq C \left\|\psi_2\right\|_{1,\alpha_0} \left|x'\right|^{1+\alpha_0} + 2\epsilon^{1+\alpha_0} \left\|\psi_2\right\|_{1,\alpha_0}.$$

Thus we have

$$\left|\psi_{2,\epsilon}(x')\right| \leq C\epsilon^{\nu} \left\|\psi_2\right\|_{1,\alpha_0}. \tag{3.23}$$

For the last case $\left|x'\right| \geq 2\epsilon$, we have $\psi_{2,\epsilon}(x') = \epsilon^{1+\alpha_0} \left\|\psi_2\right\|_{1,\alpha_0}$. Then trivially we get

$$\left|\psi_{2,\epsilon}(x')\right| \leq C\ \epsilon^{\nu} \left\|\psi_2\right\|_{1,\alpha_0}. \tag{3.24}$$

Combining (3.22), (3.23), and (3.24), it follows that

$$\left\|\psi_{2,\epsilon}\right\|_{\infty} \leq C\ \epsilon^{\nu} \left\|\psi_2\right\|_{1,\alpha_0}. \tag{3.25}$$

We proceed to show the boundedness of $L^{\infty}-$norm for $\nabla\psi_{2,\epsilon}$. For that we have

$$\nabla\psi_{2,\epsilon}(x') = \nabla\psi_2(x)\eta_\epsilon(x') + \psi_2(x)\nabla\eta_\epsilon(x') - \epsilon^{1+\alpha_0}\left\|\psi_2\right\|_{1,\alpha_0}\nabla\eta_\epsilon(x).$$

When $\left|x'\right| \leq \epsilon$, we have $\nabla\psi_{2,\epsilon}(x') = \nabla\psi_2(x')\eta_\epsilon(x')$ and since $\nabla\psi_{2,\epsilon}(0) = 0$, then we get

$$\left|\nabla\psi_{2,\epsilon}(x')\right| = \left|\nabla\psi_{2,\epsilon}(x') - \nabla\psi_{2,\epsilon}(0)\right| \leq \left\|\psi_2\right\|_{1,\alpha_0}|x|^{\alpha_0} \leq \left\|\psi_2\right\|_{1,\alpha_0}\epsilon^{\alpha_0}.$$

Therefore

$$\left|\nabla\psi_{2,\epsilon}(x')\right| \leq C\,\epsilon^{\nu}\ \left\|\psi_2\right\|_{1,\alpha_0}. \tag{3.26}$$

For the case $\epsilon < \left|x'\right| < 2\epsilon$, we see the following

$$\left|\nabla\psi_{2,\epsilon}(x')\right| \leq \left|\nabla\psi_2(x')\right| + \frac{C}{\epsilon}\left|\psi_2(x')\right| + C\epsilon^{\alpha_0}\left\|\psi_2\right\|_{1,\alpha_0}. \tag{3.27}$$

Since $\psi_{2,\epsilon}(0) = \left|\nabla\psi_{2,\epsilon}(0)\right| = 0$, then it follows that

$$\left|\nabla\psi_{2,\epsilon}(x')\right| \leq C\,\epsilon^{\nu}\left\|\psi_2\right\|_{1,\alpha_0}. \tag{3.28}$$

Finally, when $|x| \geq 2\epsilon$, then $\nabla\psi_{2,\epsilon}(x) = 0$ and trivially we get the bound

$$\left|\nabla\psi_{2,\epsilon}(x')\right| \leq C\,\epsilon^{\nu}\left\|\psi_2\right\|_{1,\alpha_0}. \tag{3.29}$$

Thus we have shown the bound

$$\left\|\nabla\psi_{2,\epsilon}\right\|_{\infty} \leq C\,\epsilon^{\nu}\left\|\psi_2\right\|_{1,\alpha_0}. \tag{3.30}$$

For the β-Hölder estimate of $\nabla\psi_{2,\epsilon}$, we have seven cases and we elaborate these cases as follows:

1. If $\left|x'\right| \leq \epsilon$ and $\left|y'\right| \leq \epsilon$, we have

$$\frac{\left|\nabla\psi_{2,\epsilon}(x') - \nabla\psi_{2,\epsilon}(y')\right|}{|x'-y'|^{\beta}} = \frac{\left|\nabla\psi_{2}(x') - \nabla\psi_{2,\epsilon}(y')\right|}{|x'-y'|^{\beta}} \leq \left|x'-y'\right|^{\alpha_0-\beta}\left\|\psi_2\right\|_{1,\alpha_0}.$$

Thus we conclude that

$$\frac{\left|\nabla\psi_{2,\epsilon}(x') - \nabla\psi_{2,\epsilon}(y')\right|}{|x'-y'|^{\beta}} \leq C\epsilon^{\nu}\left\|\psi_2\right\|_{1,\alpha_0}. \tag{3.31}$$

2. If $\epsilon < \left|x'\right| < 2\epsilon$ and $\epsilon < \left|y'\right| < 2\epsilon$, we have

$$\begin{aligned}\frac{\left|\nabla\psi_{2,\epsilon}(x') - \nabla\psi_{2,\epsilon}(y')\right|}{|x'-y'|^{\beta}} \leq{}& \frac{\left|\nabla\psi_{2}(x')\eta_{\epsilon}(x') - \nabla\psi_{2}(y')\eta_{\epsilon}(y')\right|}{|x'-y'|^{\beta}} + \frac{\left|\psi_{2}(x')\nabla\eta_{\epsilon}(x') - \psi_{2}(y')\nabla\eta_{\epsilon}(y')\right|}{|x'-y'|^{\beta}} \\ &+ \frac{\epsilon^{1+\alpha_0}\left\|\psi_2\right\|_{1,\alpha_0}\left|\nabla\eta_{\epsilon}(x') - \nabla\eta_{\epsilon}(y')\right|}{|x'-y'|^{\beta}}.\end{aligned}$$

We estimate each term in the right hand side of the above inequality separately. Let

$$|A| = \frac{\left|\nabla\psi_{2}(x')\eta_{\epsilon}(x') - \nabla\psi_{2}(y')\eta_{\epsilon}(y)\right|}{|x'-y'|^{\beta}}.$$

Then we see the following

$$
\begin{aligned}
|A| &\le \frac{\left|\nabla\psi_2(x')\eta_\epsilon(x') - \nabla\psi_2(y')\eta_\epsilon(x')\right|}{|x'-y'|^\beta} + \frac{\left|\nabla\psi_2(y')\eta_\epsilon(x') - \nabla\psi_2(y')\eta_\epsilon(y)\right|}{|x'-y'|^\beta} \\
&\le \left|x'-y'\right|^{\alpha_0-\beta} \frac{\left|\nabla\psi_2(x') - \nabla\psi_2(y')\right|}{|x'-y'|^{\alpha_0}} + \frac{\left|\nabla\psi_2(y')\right| \left|\eta_\epsilon(x') - \eta_\epsilon(y')\right|}{|x'-y'|^\beta}.
\end{aligned}
$$

Thus for some ξ' lies on the line that joining x' and y', and since $\nabla\psi_{2,\epsilon}(0) = 0$, we have

$$
|A| \le C\,\epsilon^\nu \left\|\psi_2\right\|_{1,\alpha_0} + \left\|\psi_2\right\|_{1,\alpha_0} \left|y'\right|^{\alpha_0} \left|x'-y'\right|^{1-\beta} \left|\nabla\eta_\epsilon(\xi')\right|.
$$

Therefore

$$
|A| \le C\epsilon^\nu \left\|\psi\right\|_{1,\alpha_0}. \tag{3.32}
$$

Similarly, let

$$
|B| = \frac{\left|\psi_2(x')\nabla\eta_\epsilon(x') - \psi_2(y')\nabla\eta_\epsilon(y')\right|}{|x'-y'|^\beta}.
$$

Then we have

$$
|B| \le \frac{\left|\psi_2(x')\nabla\eta_\epsilon(x') - \psi_2(y')\nabla\eta_\epsilon(x')\right|}{|x'-y'|^\beta} + \frac{\left|\psi_2(y')\nabla\eta_\epsilon(x') - \psi_2(y')\nabla\eta_\epsilon(y')\right|}{|x'-y|^\beta}.
$$

For some ξ_1' that joining x' and y', we obtain the following bound

$$
|B| \le \frac{C}{\epsilon}\frac{\left|\psi_2(x') - \psi_2(y')\right|}{|x'-y'|^\beta} + \frac{\left|\psi_2(y')\right| \left|x'-y'\right| \left|\nabla^2\eta_\epsilon(\xi_1')\right|}{|x'-y'|^\beta}.
$$

Again for some ξ_2' that joining x' to y', we obtain the following bound

$$|B| \leq \frac{C}{\epsilon} \frac{\left|\nabla\psi_2(\xi_2')\right| \left|x'-y'\right|}{\left|x'-y'\right|^{\beta}} + \frac{C}{\epsilon^2} \; \|\psi_2\|_{1,\alpha_0} \left|y'\right|^{1+\alpha_0} \left|x'-y'\right|^{1-\beta}.$$

That is

$$|B| \leq C\,\epsilon^{\nu}\,\|\psi_2\|_{1,\alpha_0}. \tag{3.33}$$

For the last term, we have for some ξ' that joining x' and y' the following bound

$$\frac{\epsilon^{1+\alpha_0}\,\|\psi_2\|_{1,\alpha_0}\left|\nabla\eta_\epsilon(x') - \nabla\eta_\epsilon(y')\right|}{\left|x'-y'\right|^{\beta}} \leq \epsilon^{1+\alpha_0}\,\|\psi\|_{1,\alpha_0}\left|x'-y'\right|^{1-\beta}\left|\nabla^2\eta_\epsilon(\xi')\right|.$$

That is

$$\frac{\epsilon^{1+\alpha_0}\,\|\psi_2\|_{1,\alpha_0}\left|\nabla\eta_\epsilon(x') - \nabla\eta_\epsilon(y')\right|}{\left|x'-y'\right|^{\beta}} \leq C\,\epsilon^{\nu}\,\|\psi_2\|_{1,\alpha_0}. \tag{3.34}$$

3. For the case $\left|x'\right| \geq 2\epsilon$ and $\left|y'\right| \geq 2\epsilon$, we have $\nabla\psi_{2,\epsilon}(x') = 0$ and $\nabla\psi_{2,\epsilon}(y') = 0$. The desired bound follows trivially.

4. If $\left|x'\right| \leq \epsilon$ and $\epsilon < \left|y'\right| < 2\epsilon$, we have $\nabla\psi_{2,\epsilon}(x') = \nabla\psi_2(x')$ and

$$\nabla\psi_{2,\epsilon}(y') = \nabla\psi_2(y')\eta_\epsilon(y') + \psi_2(y')\nabla\eta_\epsilon(y') - \epsilon^{1+\alpha_0}\,\|\psi_2\|_{1,\alpha_0}\,\nabla\eta_\epsilon(y').$$

Then we have the following bound

$$\frac{\left|\nabla \psi_{2,\epsilon}(x') - \nabla \psi_{2,\epsilon}(y')\right|}{|x'-y'|^{\beta}} \leq \frac{\left|\nabla \psi_2(x') - \nabla \psi_2(y')\eta_\epsilon(y')\right|}{|x'-y'|^{\beta}} + \frac{\left|\psi_2(y')\nabla \eta_\epsilon(y')\right|}{|x'-y'|^{\beta}}$$
$$+ \frac{\epsilon^{1+\alpha_0}\ \|\psi_2\|_{1,\alpha_0} \left|\nabla \eta_\epsilon(y')\right|}{|x'-y'|^{\beta}}$$

Since $\nabla \eta_\epsilon(x') = 0$, then we clearly see that the last two terms on the right hand side of the above inequality satisfy the desired bound. For the first term we have $\eta_\epsilon(x') = 1$ then we form the following bound

$$\frac{\left|\nabla \psi_2(x') - \nabla \psi_2(y')\eta_\epsilon(y')\right|}{|x'-y'|^{\beta}} \leq \frac{\left|\nabla \psi_2(x')\right|\ \left|\eta_\epsilon(x') - \eta_\epsilon(y')\right|}{|x'-y'|^{\beta}} + \frac{\left|\eta_\epsilon(y')\right|\ \left|\nabla \psi_2(x') - \nabla \psi_2(y')\right|}{|x'-y'|^{\beta}}.$$

Thus for some ξ' lies on the line segment that joining x' and y', we obtain the bound

$$\frac{\left|\nabla \psi_2(x') - \nabla \psi_2(y')\eta_\epsilon(y')\right|}{|x'-y'|^{\beta}} \leq \left|x'-y'\right|^{1-\beta} \left|x'\right|^{\alpha_0} \|\psi_2\|_{1,\alpha_0} \left|\nabla \eta_\epsilon(\xi')\right| + \left|x'-y'\right|^{\alpha_0-\beta} \|\psi_2\|_{1,\alpha_0}.$$

Therefore we obtain the bound

$$\frac{\left|\nabla \psi_2(x') - \nabla \psi_2(y')\eta_\epsilon(y')\right|}{|x'-y'|^{\beta}} \leq C\ \epsilon^{\nu}\ \|\psi_2\|_{1,\alpha_0}.$$

5. When $\left|x'\right| \leq \epsilon$ and $\left|y'\right| \geq 2\epsilon$, we have $\nabla \psi_{2,\epsilon}(x') = \nabla \psi_2(x')$ and $\nabla \psi_{2,\epsilon}(y') = 0$. Thus we have the bound

$$\frac{\left|\nabla \psi_{2,\epsilon}(x') - \nabla \psi_{2,\epsilon}(y')\right|}{|x'-y'|^{\beta}} = \frac{\left|\nabla \psi_2(x')\right|}{|x'-y'|^{\beta}} \leq C\ \epsilon^{-\beta} \left|x'\right|^{\alpha_0} \|\psi_2\|_{1,\alpha_0}.$$

That is

$$\frac{\left|\nabla\psi_{2,\epsilon}(x') - \nabla\psi_{2,\epsilon}(y')\right|}{|x'-y'|^{\beta}} \leq C\,\epsilon^{\nu}\,\|\psi_2\|_{1,\alpha_0}\,. \tag{3.35}$$

6. When $\epsilon < \left|x'\right| < 2\epsilon$ and $2\epsilon < \left|y'\right| < 3\epsilon$, we have $\nabla\psi_{2,\epsilon}(y') = 0$. Then we have

$$\frac{\left|\nabla\psi_{2,\epsilon}(x') - \nabla\psi_{2,\epsilon}(y')\right|}{|x'-y'|^{\beta}} = \frac{\left|\nabla\psi_2(x')\eta_\epsilon(x') + \psi_2(x)\nabla\eta_\epsilon(x') - \epsilon^{1+\alpha_0}\,\|\psi_2\|_{1,\alpha_0}\,\nabla\eta_\epsilon(x')\right|}{|x'-y'|^{\beta}}.$$

Since $\eta_\epsilon(y') = \left|\nabla\eta_\epsilon(y')\right| = 0$, then we obtain the following bound

$$\begin{aligned}\frac{\left|\nabla\psi_{2,\epsilon}(x') - \nabla\psi_{2,\epsilon}(y')\right|}{|x'-y'|^{\beta}} &\leq \frac{\left|\nabla\psi_2(x')\eta_\epsilon(x') - \nabla\psi_2(x')\eta_\epsilon(y')\right|}{|x'-y'|^{\beta}} + \frac{\left|\psi_2(x')\right|\left|\nabla\eta_\epsilon(x') - \nabla\eta_\epsilon(y')\right|}{|x'-y'|^{\beta}} \\ &+ \frac{\epsilon^{1+\alpha_0}\,\|\psi_2\|_{1,\alpha_0}\left|\nabla\eta_\epsilon(x') - \nabla\eta_\epsilon(y')\right|}{|x'-y'|^{\beta}}.\end{aligned}$$

Thus for some ξ_1' and ξ_2' that lie on the line segment joining x' and y' we obtain the following bound

$$\begin{aligned}\frac{\left|\nabla\psi_{2,\epsilon}(x') - \nabla\psi_{2,\epsilon}(y')\right|}{|x'-y'|^{\beta}} &\leq \frac{\left|x'\right|^{\alpha_0}\|\psi_2\|_{1,\alpha_0}\left|x'-y'\right|\left|\nabla\eta_\epsilon(\xi_1')\right|}{|x'-y'|^{\beta}} \\ + \frac{C\left|x'\right|^{1+\alpha_0}\|\psi_2\|_{1,\alpha_0}\left|x'-y'\right|\left|\nabla^2\eta_\epsilon(\xi_2')\right|}{|x'-y'|^{\beta}} &+ \frac{\epsilon^{1+\alpha_0}\,\|\psi_2\|_{1,\alpha_0}\left|x'-y'\right|\left|\nabla^2\eta_\epsilon(\xi_2')\right|}{|x'-y'|^{\beta}}.\end{aligned}$$

Therefore we have the desired bound

$$\frac{\left|\nabla\psi_{2,\epsilon}(x') - \nabla\psi_{2,\epsilon}(y')\right|}{|x'-y'|^{\beta}} \leq C\,\epsilon^{\nu}\,\|\psi_2\|_{1,\alpha_0}\,. \tag{3.36}$$

7. If $\epsilon < \left|x'\right| < 2\epsilon$ and $\left|y'\right| \geq 3\epsilon$, then $\nabla\psi_{2,\epsilon}(y') = 0$ and we have the following

$$\frac{\left|\nabla\psi_{2,\epsilon}(x') - \nabla\psi_{2,\epsilon}(y')\right|}{|x'-y'|^{\beta}} \leq \frac{\left|\nabla\psi_2(x')\right|}{|x'-y'|^{\beta}} + \frac{\left|\psi_2(x')\right|\left|\nabla\eta_\epsilon(x')\right|}{|x'-y'|^{\beta}} + \frac{\epsilon^{1+\alpha_0}\left\|\psi_2\right\|_{1,\alpha_0}\left|\nabla\eta_\epsilon(x')\right|}{|x'-y'|^{\beta}}.$$

That is

$$\frac{\left|\nabla\psi_{2,\epsilon}(x') - \nabla\psi_{2,\epsilon}(y')\right|}{|x'-y'|^{\beta}} \leq C\,\epsilon^{\nu}\left\|\psi_2\right\|_{1,\alpha_0}. \tag{3.37}$$

Thus we have shown the following bound

$$\sup_{\substack{x',y'\in\mathbb{R}^{n-1}\\ x'\neq y'}} \frac{\left|\nabla\psi_{2,\epsilon}(x') - \nabla\psi_{2,\epsilon}(y')\right|}{|x'-y'|^{\beta}} \leq C\,\epsilon^{\nu}\left\|\psi_2\right\|_{1,\alpha_0}. \tag{3.38}$$

Therefore the Lemma follows by combining (3.25), (3.30) and (3.38). □

The following Lemma will be needed for the decomposition of the operator L_2^{δ} and it will be used as a tool to extend the integral over the surface Γ_2 to the whole space $\mathbb{R}^{n-1}$. It was stated in [1] for dimension $n = 2$. The proof is elementary.

Lemma 3.5. *Let $0 < \alpha < \alpha_0$, for $\varphi \in C^{\alpha}(\Gamma_2)$ we define*

$$\phi(y') = E\varphi(y')\sqrt{1 + \left|\nabla\psi_{2,\epsilon}(y')\right|^2}, \quad y' \in \mathbb{R}^{n-1}.$$

Then $\phi \in C^{\alpha}(\mathbb{R}^{n-1})$ and

$$\left\|\phi\right\|_{\alpha;\mathbb{R}^{n-1}} \leq (1 + C(\epsilon))(1+\epsilon_0)\left\|\varphi\right\|_{\alpha;\Gamma_2}, \tag{3.39}$$

where α_0 is the regularity of the surface Γ_2, $E\varphi$ is defined in Lemma 3.3 and $C(\epsilon) \longrightarrow 0$ as

ϵ *approaches* 0.

Proof. First, we see that ϕ is well-defined on $\mathbb{R}^{n-1}$ because $E\varphi$ has compact support in $\mathcal{B}(\frac{2}{\epsilon_0})$. From Lemma 3.3, we have the bound

$$\|E\varphi\|_{\alpha;\mathbb{R}^{n-1}} \leq (1+\epsilon_0)\|\varphi\|_{\alpha;\Gamma_2}. \tag{3.40}$$

Also from Lemma 3.4 we have for any $\alpha < \alpha_0$ that $\|\psi_{2,\epsilon}\|_{1,\alpha} = O(\epsilon^\nu)$ where $\nu = \alpha_0 - \beta$, then for any $x' \in \mathbb{R}^{n-1}$, it follows that

$$\left|\phi(x')\right| \leq \left|E\varphi(x')\right|\Big(1 + C(\epsilon)\Big).$$

That is

$$\|\phi\|_\infty \leq \Big(1+C(\epsilon)\Big)(1+\epsilon_0)\|\varphi\|_\alpha. \tag{3.41}$$

Next for $x', y' \in \mathbb{R}^{n-1}$, we estimate

$$\begin{aligned}
\frac{\left|\phi(x')-\phi(y')\right|}{|x'-y'|^\alpha} &= \frac{\left|E\varphi(x')\sqrt{1+\left|\nabla\psi_{2,\epsilon}(x')\right|^2} - E\varphi(y')\sqrt{1+\left|\nabla\psi_{2,\epsilon}(y')\right|^2}\right|}{|x'-y'|^\alpha} \\
&\leq \frac{\left|E\varphi(x')-E\varphi(y')\right|\sqrt{1+\left|\nabla\psi_{2,\epsilon}(x')\right|^2}}{|x'-y'|^\alpha} + \frac{\left|E\varphi(y')\right|\left|\sqrt{1+\left|\nabla\psi_{2,\epsilon}(x')\right|^2}-\sqrt{1+\left|\nabla\psi_{2,\epsilon}(y')\right|^2}\right|}{|x'-y'|^\alpha}.
\end{aligned}$$

By using (3.40), we easily see that

$$\frac{\left|E\varphi(x')-E\varphi(y')\right|\sqrt{1+\left|\nabla\psi_{2,\epsilon}(x')\right|^2}}{|x'-y'|^\alpha} \leq \Big(1+C(\epsilon)\Big)(1+\epsilon_0)\|\varphi\|_\alpha. \tag{3.42}$$

For the second term, let

$$\mathcal{I}_2 = \left|E\varphi(y')\right| \ \frac{\left|\sqrt{1+\left|\nabla\psi_{2,\epsilon}(x')\right|^2} - \sqrt{1+\left|\nabla\psi_{2,\epsilon}(y')\right|^2}\right|}{|x'-y'|^{\alpha}}.$$

Then we have

$$\mathcal{I}_2 \le \left|E\varphi(y')\right| \ \frac{\left|\nabla\psi_{2,\epsilon}(x') - \nabla\psi_{2,\epsilon}(y')\right| \left(\left|\nabla\psi_{2,\epsilon}(x')\right| + \left|\nabla\psi_{2,\epsilon}(y')\right| \right)}{|x'-y'|^{\alpha} \left(\sqrt{1+\left|\nabla\psi_{2,\epsilon}(x')\right|^2} + \sqrt{1+\left|\nabla\psi_{2,\epsilon}(y')\right|^2} \right)}.$$

That is,

$$\mathcal{I}_2 \le \left|E\varphi(y')\right| \ \frac{\left|\nabla\psi_{2,\epsilon}(x')\right| + \left|\nabla\psi_{2,\epsilon}(y')\right|}{\sqrt{1+\left|\nabla\psi_{2,\epsilon}(x')\right|^2} + \sqrt{1+\left|\nabla\psi_{2,\epsilon}(y')\right|^2}} \frac{\left|\nabla\psi_{2,\epsilon}(x') - \nabla\psi_{2,\epsilon}(y')\right|}{|x'-y'|^{\alpha}}.$$

Therefore we obtain the following bound

$$\mathcal{I}_2 \le C\epsilon^{\nu} \ (1+\epsilon_0) \ \|\varphi\|_{\alpha} \left\|\psi_{2,\epsilon}\right\|_{1,\alpha_0}. \tag{3.43}$$

Thus the bound (3.39) follows by combining (3.41), (3.42) and (3.43). □

3.3 Decomposition of L_2^{δ}

In this section we elaborate a method to decompose the operator L_2^{δ} in order to overcome the singularity when $x = 0$ and $\delta = 0$. The method was introduced in [1] for dimension $n = 2$. We start with the following Lemma which will be used frequently during our work.

Lemma 3.6. *For $n \geq 2$, we have*

$$\int_{\mathbb{R}^{n-1}} \frac{dz'}{\left(|z'|^2+1\right)^{\frac{n}{2}}} = \frac{\omega_n}{2}, \tag{3.44}$$

where ω_n is the surface area of unit sphere $\partial\mathcal{B}(1)$ in $\mathbb{R}^n$.

Proof. The surface area ω_n is represented in polar coordinates by the following form

$$\omega_n = \int_{\phi_1=0}^{\pi} ... \int_{\phi_{n-2}=0}^{\pi} \int_{\phi_{n-1}=0}^{2\pi} (\sin\phi_1)^{n-2} ... (\sin\phi_{n-3})^2 \,(\sin\phi_{n-2})\, d\phi_{n-1}\, d\phi_{n-2} \cdots\, d\phi_1.$$

Then we can see that

$$\omega_n = \omega_{n-1} \int_0^{\pi} (\sin\phi_1)^{n-2}\, d\phi_1, \tag{3.45}$$

where ω_{n-1} is the surface area of unit sphere $\partial\mathcal{B}(1)$ in $\mathbb{R}^{n-1}$. Thus by using the polar coordinates, the left hand side of (3.44) becomes

$$\int_{\mathbb{R}^{n-1}} \frac{dz'}{\left(|z'|^2+1\right)^{\frac{n}{2}}} = \omega_{n-1} \int_0^{\infty} \frac{r^{n-2}}{\left(r^2+1\right)^{\frac{n}{2}}}\, dr. \tag{3.46}$$

Substituting $r = \tan\phi_1$ in the right hand side of (3.46), we have

$$\int_{\mathbb{R}^{n-1}} \frac{dz'}{\left(|z'|^2+1\right)^{\frac{n}{2}}} = \omega_{n-1} \int_0^{\frac{\pi}{2}} (\sin\phi_1)^{n-2}\, d\phi_1 = \frac{\omega_n}{2}$$

$\square$

To begin, let $\delta > 0$ and $\varphi \in C^{\alpha}(\Gamma_2)$. For $x \in \Gamma_1$, $|x| < R_0$, we set $x_\epsilon = (x', \psi_{1,\epsilon}(x'))$ and we define the approximate surface

$$\Gamma_{2,\epsilon} = \{y_\epsilon = (y', \psi_{2,\epsilon}(y')) \mid y' \in \mathbb{R}^{n-1}\}.$$

The off-diagonal operator L_2^δ is defined by

$$L_2^\delta \varphi(x) = -\partial\nu \mathbf{S_2} \varphi(x - \delta \mathbf{e_n}).$$

Let η be the cut-off function defined in (3.11). Then we decompose the operator L_2^δ as follows

$$L_2^\delta \varphi(x) = \eta(x) L_2^\delta \varphi(x) + \big(1 - \eta(x)\big) L_2^\delta \varphi(x). \tag{3.47}$$

Notice that the term $(1 - \eta(x)) L_2^\delta \varphi(x)$ has smooth kernel. Whereas the term $\eta(x) L_2^\delta \varphi(x)$ is singular when $x = 0$ and $\delta = 0$. To simplify the notation, we define

$$\mathcal{L}_2^\delta(x) := \eta(x) L_2^\delta \varphi(x), \quad x \in \Gamma_1, \;\; \varphi \in C^\alpha(\Gamma_2).$$

Next we decompose the operator $\mathcal{L}_2^\delta$ as follows:

$$\mathcal{L}_2^\delta(x) = \Big(\mathcal{L}_2^\delta(x) + \eta(x)\partial\nu_\epsilon \mathbf{S_{2}}_{,\epsilon} E\varphi \circ P(x_\epsilon - \delta \mathbf{e_n})\Big) - \eta(x)\partial\nu_\epsilon \mathbf{S_{2}}_{,\epsilon} E\varphi \circ P(x_\epsilon - \delta \mathbf{e_n}), \tag{3.48}$$

where P is the projection map from $\Gamma_{2,\epsilon}$ onto $\mathbb{R}^{n-1}$, *that is*

$$\begin{cases} P: \quad \Gamma_{2,\epsilon} \longrightarrow \mathbb{R}^{n-1}, \\ \qquad P\Big(y', \psi_{2,\epsilon}(y')\Big) = y', \end{cases} \tag{3.49}$$

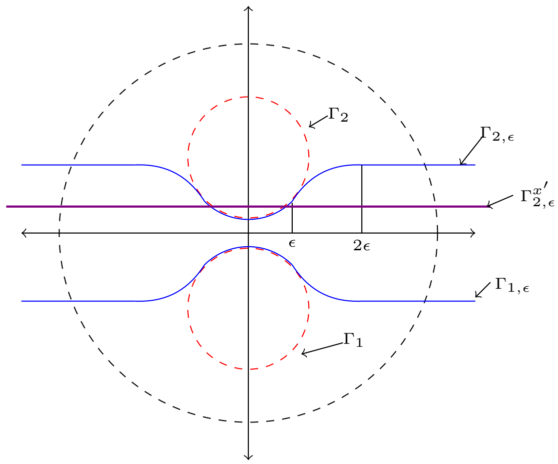

Figure 3.1: The approximate surfaces

and $\mathbf{S}_{\mathbf{2},\epsilon}$ is the single layer potential on the approximate surface $\Gamma_{2,\epsilon}$. From the smoothness of $\mathbf{S}_{\mathbf{2},\epsilon}$ on $\Gamma_{1,\epsilon}$ we have

$$\eta(x)\partial\nu_\epsilon \mathbf{S}_{\mathbf{2},\epsilon} E\varphi \circ P(x_\epsilon - \delta\mathbf{e_n}) = \frac{\eta(x)}{\omega_n}\int_{\Gamma_{2,\epsilon}} \frac{(x_\epsilon - y_\epsilon - \delta\mathbf{e_n})\cdot \nu_\epsilon(x_\epsilon)}{|x_\epsilon - y_\epsilon - \delta\mathbf{e_n}|^n}\, E\varphi \circ P(y_\epsilon)\, d\sigma(y_\epsilon), \tag{3.50}$$

where $\nu_\epsilon(x_\epsilon)$ is the normal unit vector on the approximate surface $\Gamma_{1,\epsilon}$ at the point x_ϵ, *that is*,

$$\nu_\epsilon(x_\epsilon) = \frac{1}{\sqrt{1 + |\nabla\psi_{1,\epsilon}(x')|^2}} \begin{pmatrix} -\nabla\psi_{1,\epsilon}(x') \\ 1 \end{pmatrix}. \tag{3.51}$$

We rewrite $\partial\nu_\epsilon \mathbf{S}_{\mathbf{2},\epsilon}$ for the potential $\mathcal{E} = E\varphi \circ P$ at the point $\tilde{\mathbf{x}} = x_\epsilon - \delta\mathbf{e_n}$ as follows

$$\eta(x)\partial\nu_\epsilon \mathbf{S}_{\mathbf{2},\epsilon}\mathcal{E}(\tilde{\mathbf{x}}) = \tilde{C}(x')\eta(x)\int_{\mathbb{R}^{n-1}} \frac{(y'-x')\cdot\nabla\psi_{1,\epsilon}(x') + \Big(\psi_{1,\epsilon}(x') - \psi_{2,\epsilon}(y') - \delta\Big)}{\Big(|x'-y'|^2 + (\psi_{1,\epsilon}(x') - \psi_{2,\epsilon}(y') - \delta)^2\Big)^{n/2}}\,\phi(y')\,dy', \tag{3.52}$$

where ϕ is defined in Lemma 3.5 by $\phi(y') = E\varphi(y')\sqrt{1+\left|\nabla\psi_{2,\epsilon}(y')\right|^2}$ and $\tilde{C}(x') = \frac{1}{\omega_n\sqrt{1+\left|\nabla\psi_{1,\epsilon}(x')\right|^2}}$.
Now, we define

$$\eta(x)K_2^{\epsilon,\delta}\varphi(x) = -\eta(x)\partial\nu\mathbf{S_2}\varphi(x-\delta\mathbf{e_n}) + \eta(x)\partial\nu_\epsilon\mathbf{S}_{\mathbf{2},\epsilon}\mathcal{E}(\tilde{\mathbf{x}}). \tag{3.53}$$

We will prove later that operators in (3.53) form a family of compact operators from $C^\alpha(\Gamma_2)$ to $C^\alpha(\Gamma_1)$. Also by using the definition of the function ϕ we can rewrite the operators $\eta K_2^{\epsilon,\delta}$ as follows

$$\begin{aligned}\eta(x)K_2^{\epsilon,\delta}\varphi(x) = &\frac{-\eta(x)}{\omega_n}\int_{\Gamma_2}\frac{(x-y-\delta\mathbf{e_n})\cdot\nu(x)}{|x-y-\delta\mathbf{e_n}|^n}\,\varphi(y)\,d\sigma(y),\\ &+\tilde{C}\eta(x)\int_{\mathbb{R}^{n-1}}\frac{(y'-x')\nabla\psi_{1,\epsilon}(x') + \Big(\psi_{1,\epsilon}(x') - \psi_{2,\epsilon}(y') - \delta\Big)}{\Big(|x'-y'|^2 + (\psi_{1,\epsilon}(x') - \psi_{2,\epsilon}(y') - \delta)^2\Big)^{n/2}}\,\phi(y')\,dy'.\end{aligned}$$

From the parameterization of the surfaces Γ_1 and Γ_2 around the origin , we have that $x = (x', \psi_1(x')) \in \Gamma_1$ for $\left|x'\right| < \epsilon$, and similarly, $y = (y', \psi_2(y')) \in \Gamma_2$ when $\left|y'\right| < \epsilon$. Therefore we conclude that for $\left|x'\right| < \epsilon$ and $\left|y'\right| < \epsilon$ the two integrands in the definition of $\eta(x)K_2^{\epsilon,\delta}$ coincide. We further define for each $x_\epsilon = (x', \psi_{1,\epsilon}(x')) \in \Gamma_{1,\epsilon}$ the hyper-plane

$$\Gamma_{2,\epsilon}^{x'} = \{y \in \mathbb{R}^n \mid y_n = \psi_{2,\epsilon}(x')\}. \tag{3.54}$$

Thus we rewrite $\eta(x)L_2^{\epsilon,\delta}\varphi(x)$ as follows

$$\eta(x)L_2^{\delta}\varphi(x) = \eta(x)K_2^{\epsilon,\delta}\varphi(x) - \eta(x)\partial\nu_\epsilon \mathbf{S_{2}}_{,\epsilon}\mathcal{E}(\tilde{\mathbf{x}}) + \eta(x)\partial\nu_\epsilon \mathbf{S}^{\mathbf{x}'}_{\mathbf{2},\epsilon}\mathcal{E}(\tilde{\mathbf{x}}) - \eta(x)\partial\nu_\epsilon \mathbf{S}^{\mathbf{x}'}_{\mathbf{2},\epsilon}\mathcal{E}(\tilde{\mathbf{x}}), \tag{3.55}$$

where $\mathbf{S}^{\mathbf{x}'}_{\mathbf{2},\epsilon}$ is the single layer potential on the hyper-surface $\Gamma^{x'}_{2,\epsilon}$ and its normal derivative for the potential $\mathcal{E}$ is defined by

$$\eta(x)\partial\nu_\epsilon \mathbf{S}^{\mathbf{x}'}_{\mathbf{2},\epsilon}\mathcal{E}(\tilde{\mathbf{x}}) = \frac{\eta(x)}{\omega_n}\int_{\Gamma^{x'}_{2,\epsilon}} \frac{\Big(x_\epsilon - y(x') - \delta\mathbf{e_n}\Big)\cdot\nu_\epsilon(x_\epsilon)}{|x_\epsilon - y(x') - \delta\mathbf{e_n}|^n}\,\mathcal{E}(y(x'))\,d\sigma(y(x')), \tag{3.56}$$

where $y(x') = (y', \psi_{2,\epsilon}(x'))$. To be more explicit,

$$\eta(x)\partial\nu_\epsilon \mathbf{S}^{\mathbf{x}'}_{\mathbf{2},\epsilon}\mathcal{E}(\tilde{\mathbf{x}}) = \tilde{C}\eta(x)\int_{\mathbb{R}^{n-1}} \frac{(y'-x')\cdot\nabla\psi_{1,\epsilon}(x') + \Big(\psi_{1,\epsilon}(x') - \psi_{2,\epsilon}(x') - \delta\Big)}{\Big(|x'-y'|^2 + \big(\psi_{1,\epsilon}(x') - \psi_{2,\epsilon}(x') - \delta\big)^2\Big)^{\frac{n}{2}}}\,\phi(y')\,dy'. \tag{3.57}$$

Note that we have perturbed by the operator $\eta(x)\partial\nu_\epsilon \mathbf{S}^{\mathbf{x}'}_{\mathbf{2},\epsilon}\mathcal{E}(\tilde{\mathbf{x}})$ in order to get the Hölder continuity for the operator $-\eta(x)\partial\nu_\epsilon \mathbf{S_{2}}_{,\epsilon}\mathcal{E}(\tilde{\mathbf{x}})$. That is, we will use the advantage of the difference $\psi_{2,\epsilon}(x') - \psi_{2,\epsilon}(y')$ to get the Hölder continuity for the term

$$\eta(x)\partial\nu_\epsilon \mathbf{S}^{\mathbf{x}'}_{\mathbf{2},\epsilon}\mathcal{E}(\tilde{\mathbf{x}}) - \eta(x)\partial\nu_\epsilon \mathbf{S_{2}}_{,\epsilon}\mathcal{E}(\tilde{\mathbf{x}}).$$

While the term $\eta(x)\partial\nu_\epsilon \mathbf{S}^{\mathbf{x}'}_{\mathbf{2},\epsilon}\mathcal{E}(\tilde{\mathbf{x}})$ will be easily proved is Hölder continuous because it is produced from the single layer potential over the hyper-plane $\Gamma^{x'}_{2,\epsilon}$.

For $\Gamma_1 \ni x \neq 0$ or $\delta \neq 0$ and $\varphi \in C^{\alpha}(\Gamma_2)$ we define the operator

$$I_2^{\epsilon,\delta}\varphi(x) = \int_{\mathbb{R}^{n-1}} \frac{(\delta + \psi_{2,\epsilon}(x') - \psi_{1,\epsilon}(x')) - \nabla\psi_{1,\epsilon}(x') \cdot (y' - x')}{\left(|x' - y'|^2 + \left(\psi_{1,\epsilon}(x') - \psi_{2,\epsilon}(x') - \delta\right)^2\right)^{\frac{n}{2}}} \phi(y')\, dy'. \tag{3.58}$$

whereas for $x = 0$ and $\delta = 0$, we define $I_2^{\epsilon,0}\varphi(0)$ as follows

$$I_2^{\epsilon,0}\varphi(0) = \lim_{\delta\to 0} \int_{\mathbb{R}^{n-1}} \frac{\delta}{\left(|y'|^2 + \delta^2\right)^{\frac{n}{2}}} \phi(y')\, dy'. \tag{3.59}$$

Using the change of variable $z' = \frac{y'}{\delta}$, we have

$$I_2^{\epsilon,0}\varphi(0) = \lim_{\delta\to 0} \int_{\mathbb{R}^{n-1}} \frac{1}{\left(|z'|^2 + 1\right)^{\frac{n}{2}}} \phi(\delta z')\, dz'. \tag{3.60}$$

From the dominated convergence theorem, it follows that

$$I_2^{\epsilon,0}\varphi(0) = \int_{\mathbb{R}^{n-1}} \frac{1}{\left(|z'|^2 + 1\right)^{\frac{n}{2}}} \phi(0)\, dz' = \frac{\omega_n}{2}\varphi(0), \tag{3.61}$$

where $\phi(0) = \varphi(0)$ and ω_{n-1} is the surface area of unit sphere $\partial\mathcal{B}(1)$ in $\mathbb{R}^{n-1}$. Again to simplify the notation, let

$$\mathcal{J}_2(x) = -\eta(x)\partial\nu\mathbf{S}_{\mathbf{2},\epsilon}\mathcal{E}(\tilde{\mathbf{x}}) + \eta(x)\partial\nu\mathbf{S}^{\mathbf{x}'}_{\mathbf{2},\epsilon}\mathcal{E}(\tilde{\mathbf{x}}).$$

From (3.52) and (3.57), we have

$$\mathcal{J}_2(x) = \tilde{C}(x')\,\eta(x)\int_{\mathbb{R}^{n-1}} \frac{\Big(\delta+\psi_{2,\epsilon}(y')-\psi_{1,\epsilon}(x')\Big)-\nabla\psi_{1,\epsilon}(x')\cdot(y'-x')}{\Big(|x'-y'|^2+\big(\psi_{1,\epsilon}(x')-\psi_{2,\epsilon}(y')-\delta\big)^2\Big)^{\frac{n}{2}}} - \frac{\Big(\delta+\psi_{2,\epsilon}(x')-\psi_{1,\epsilon}(x')\Big)-\nabla\psi_{1,\epsilon}(x')\cdot(y'-x')}{\Big(|x'-y'|^2+\big(\psi_{1,\epsilon}(x')-\psi_{2,\epsilon}(x')-\delta\big)^2\Big)^{\frac{n}{2}}}\;\phi(y')\,dy'.$$

For $x \neq 0$ or $\delta \neq 0$ and $\varphi \in C^{\alpha}(\Gamma_2)$, we define the operator

$$J_2^{\epsilon,\delta}\varphi(x) = \int_{\mathbb{R}^{n-1}} \frac{(\delta+\psi_{2,\epsilon}(y')-\psi_{1,\epsilon}(x'))-\nabla\psi_{1,\epsilon}(x')\cdot(y'-x')}{\Big(|x'-y'|^2+\big(\psi_{1,\epsilon}(x')-\psi_{2,\epsilon}(y')-\delta\big)^2\Big)^{\frac{n}{2}}} - \frac{(\delta+\psi_{2,\epsilon}(x')-\psi_{1,\epsilon}(x'))-\nabla\psi_{1,\epsilon}(x')\cdot(y'-x')}{\Big(|x'-y'|^2+\big(\psi_{1,\epsilon}(x')-\psi_{2,\epsilon}(x')-\delta\big)^2\Big)^{\frac{n}{2}}}\;\phi(y')\,dy', \tag{3.62}$$

whereas, when $x = 0$ and $\delta = 0$, we set

$$J_2^{\epsilon,0}\varphi(0) = \int_{\mathbb{R}^{n-1}} \frac{\psi_{2,\epsilon}(y')}{\Big(y_1^2+\ldots+y_{n-1}^2+\psi_{2,\epsilon}^2(y')\Big)^{\frac{n}{2}}}\;\phi(y')\,dy', \tag{3.63}$$

where this integral is well- defined because when $\left|y'\right| < \epsilon$, we have $\left|\psi_{2,\epsilon}(y')\right| \leq C\left|y'\right|^{1+\alpha_0}$. Thus we write $\eta(x)L_2^{\delta}$ as

$$\eta(x)L_2^{\delta}\varphi(x) = \eta(x)K_2^{\epsilon,\delta}\varphi(x) + \frac{\eta(x)}{\omega_n\sqrt{1+\left|\nabla\psi_{1,\epsilon}(x')\right|^2}}\Big(J_2^{\epsilon,\delta}\varphi(x)+I_2^{\epsilon,\delta}\varphi(x)\Big). \tag{3.64}$$

Using the above definitions, we have

$$L_2^\delta\varphi(x) = \eta(x)\left(K_2^{\epsilon,\delta} + \frac{1}{\omega_n\sqrt{1+\left|\nabla\psi_{1,\epsilon}(x')\right|^2}}\left(J_2^{\epsilon,\delta}+I_2^{\epsilon,\delta}\right)\right)\varphi(x)$$
$$+ (1-\eta(x))L_2^\delta\varphi(x). \tag{3.65}$$

In a similar manner, we define operators $\eta(x)K_1^{\epsilon,\delta}$, $\eta(x)J_1^{\epsilon,\delta}$ and $\eta(x)I_1^{\epsilon,\delta}$ from $C^\alpha(\Gamma_1)$ into $C^\alpha(\Gamma_2)$, that help to decompose the operator L_1^δ. That is, we decompose the operator L_1^δ as follows: For $\delta > 0$, $x \in \Gamma_2$, $|x| < R_0$ and $\varphi \in C^\alpha(\Gamma_1)$ we form the following:

$$L_1^\delta\varphi(x) = \eta(x)L_1^\delta\varphi(x) + (1-\eta(x))L_1^\delta\varphi(x) \tag{3.66}$$
$$= \eta(x)\left(K_1^{\epsilon,\delta} + \frac{1}{\omega_n\sqrt{1+\left|\nabla\psi_{2,\epsilon}(x')\right|^2}}\left(J_1^{\epsilon,\delta}+I_1^{\epsilon,\delta}\right)\right)\varphi(x)$$
$$+ (1-\eta(x))L_1^\delta\varphi(x). \tag{3.67}$$

3.4 Decomposition of T^δ.

After decomposing the off-diagonal operators L_2^δ and L_1^δ, we may now decompose the operator T^δ for $\delta > 0$ as follows:

$$T^\delta\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \Lambda_{\epsilon,\delta}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} + C_{\epsilon,\delta}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \tag{3.68}$$

where

$$\Lambda_{\epsilon,\delta} = \begin{pmatrix} \lambda I & \frac{\eta}{\omega_n\sqrt{1+\left|\nabla\psi_{1,\epsilon}(x')\right|^2}}\left(J_2^{\epsilon,\delta}+I_1^{\epsilon,\delta}\right) \\ \frac{\eta}{\omega_n\sqrt{1+\left|\nabla\psi_{2,\epsilon}(x')\right|^2}}\left(J_1^{\epsilon,\delta}+I_1^{\epsilon,\delta}\right) & \lambda I \end{pmatrix}, \tag{3.69}$$

and

$$C_{\epsilon,\delta} = \begin{pmatrix} -K_1^* & \eta K_2^{\epsilon,\delta} \\ \eta K_2^{\epsilon,\delta} & -K_2^* \end{pmatrix} + (1-\eta)\begin{pmatrix} 0 & L_2^\delta \\ L_1^\delta & 0 \end{pmatrix}. \tag{3.70}$$

We clearly notice that, for $\varphi \in C^\alpha(\Gamma_2)$ and $x \in \Gamma_1$ the normal derivative of the single layer potential

$$\partial_\nu \mathbf{S_2}\varphi(x) = \frac{1}{\omega_n}\int_{\Gamma_2}\frac{(x-y)\cdot\nu(x)}{|x-y|^n}\varphi(y)\,d\sigma(y)$$

is α_0-Hölder continuous whenever $|x| > \epsilon_0$. Therefore, it is convenient to define the limit of the compact operator $(1-\eta)L_2^\delta$ as an operator from $C^\alpha(\Gamma_2)$ into $C^\alpha(\Gamma_1)$ as follows

$$\mathbf{L}_2^0\varphi = \lim_{\delta\to 0} L_2^\delta\varphi \quad \text{in} \quad C^\alpha(\Gamma_1 \cap \{|x| > \epsilon_0\}). \tag{3.71}$$

We highlight at this point that the operator $\mathbf{L}_2^0$ is defined as an operator from $C^\alpha(\Gamma_2)$ into $C^\alpha(\Gamma_1 \cap \{|x| > \epsilon_0\})$. Furthermore, the operator $\mathbf{L}_2^0$ is compact operator because it is the norm limit of compact operators.

Now we turn to define the limit of the operators $L_2^\delta\varphi(x)$, when $|x| \le \epsilon_0$ by using the operators $\eta(x)K_2^{\epsilon,\delta}$, $\eta(x)J_2^{\epsilon,\delta}$ and $\eta(x)I_2^{\epsilon,\delta}$ that defined in the previous section. In other words, for $\varphi \in C^{\alpha}(\Gamma_2)$ and $x \in \Gamma_1 \cap \{|x| \leq \epsilon_0\}$, we define

$$\left(K_2^{\epsilon,0} + \frac{1}{\omega_n \sqrt{1 + \left|\nabla \psi_{1,\epsilon}(x')\right|^2}} \left(J_2^{\epsilon,0} + I_2^{\epsilon,0} \right) \right) \varphi(x) \tag{3.72}$$

as a limit operator of $L_2^{\delta}\varphi(x)$. Globally, we use (3.71) and (3.72) to define the limiting operator corresponding to L_2^{δ}. That is, for $\varphi \in C^{\alpha}(\Gamma_2)$ and $x \in (\Gamma_1)$, we define

$$\begin{aligned} L_2^0\varphi(x) = \eta(x) &\left(K_2^{\epsilon,0} + \frac{1}{\omega_n \sqrt{1 + \left|\nabla \psi_{1,\epsilon}(x')\right|^2}} \left(J_2^{\epsilon,0} + I_2^{\epsilon,0} \right) \right) \varphi(x) \\ &+ \big(1 - \eta(x)\big)\mathbf{L_2^0}\varphi(x). \end{aligned} \tag{3.73}$$

Similarly, we define the operator L_1^0. That is, for $\varphi \in C^{\alpha}(\Gamma_1)$ and $x \in (\Gamma_2)$, we define

$$\begin{aligned} L_1^0\varphi(x) = \eta(x) &\left(K_1^{\epsilon,0} + \frac{1}{\omega_n \sqrt{1 + \left|\nabla \psi_{1,\epsilon}(x')\right|^2}} \left(J_1^{\epsilon,0} + I_1^{\epsilon,0} \right) \right) \varphi(x) \\ &+ \big(1 - \eta(x)\big)\mathbf{L_1^0}\varphi(x). \end{aligned} \tag{3.74}$$

We will show later the operator L_2^0 is a pointwise limit of L_2^{δ} as an operator from $C^{\alpha}(\Gamma_2)$ into $C^{\alpha}(\Gamma_1)$ and similary we will obtain that L_1^0 is a pointwise limit of L_1^{δ} as an operator from $C^{\alpha}(\Gamma_1)$ into $C^{\alpha}(\Gamma_2)$. However, in [1] the authors have proven the following Theorem

Theorem 3.7. *[1] The operators L_1^0 and L_2^0 are not compact on C^{α} for any $0 < \alpha < \alpha_0$.*

Consequently, we conclude that the compact operators $\mathbf{L}^{\delta} := \left(L_1^{\delta}, L_2^{\delta}\right)$ do not converge in norm to $\mathbf{L^0} := \left(L_1^0, L_2^0\right)$.

Now we use the operators L_1^0 and L_2^0 to define the system

$$T^0 \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \lambda I - K_1^* & L_2^0 \\ L_1^0 & \lambda I - K_2^* \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \tag{3.75}$$

Also we will show later that T^0 is a pointwise limit of the compact operators T^δ as an operator from $C^\alpha(\Gamma_1) \times C^\alpha(\Gamma_2)$ into $C^\alpha(\Gamma_1) \times C^\alpha(\Gamma_2)$. By using (3.73) and (3.74), we may decompose T_0 as follows:

$$T^0 = \Lambda_{\epsilon,0} + C_{\epsilon,0},$$

where

$$\Lambda_{\epsilon,0} = \begin{pmatrix} \lambda I & \frac{\eta}{\omega n \sqrt{1+\left|\nabla\psi_{1,\epsilon}(x')\right|^2}} \left(J_2^{\epsilon,0} + I_1^{\epsilon,0}\right) \\ \frac{\eta}{\omega n \sqrt{1+\left|\nabla\psi_{2,\epsilon}(x')\right|^2}} \left(J_1^{\epsilon,0} + I_1^{\epsilon,0}\right) & \lambda I \end{pmatrix}, \tag{3.76}$$

and

$$C_{\epsilon,0} = \begin{pmatrix} -K_1^* & \eta K_2^{\epsilon,0} \\ \eta K_2^{\epsilon,0} & -K_2^* \end{pmatrix} + (1-\eta) \begin{pmatrix} 0 & \mathbf{L}_2^0 \\ \mathbf{L}_1^0 & 0 \end{pmatrix}. \tag{3.77}$$

As we have mentioned earlier that the operators (L_1^δ, L_1^δ) do not converge in norm to (L_1^0, L_2^0). Thus the operators T^δ do not converge in norm to T^0. Consequently, even though we show that the limiting system T^0 is invertable we can't get directly that the operators T^δ are invertible and their inverses $(T^\delta)^{-1}$ are uniformly bounded. Recall that, the uniform bounded (2.64) requires uniform boundedness for $(T^\delta)^{-1}$ in $C^\alpha(\Gamma_1) \times C^\alpha(\Gamma_2)$ for $\alpha < \alpha_0$. In the absence of norm convergence, our goal is to prove uniform boundedness for $(T^\delta)^{-1}$ in $C^\alpha(\Gamma_1) \times C^\alpha(\Gamma_2)$ for $\alpha < \alpha_0$ by using the pointwise convergence.

Chapter 4

Hölder estimate for the operator $J_2^{\epsilon,\delta}$

We devote this chapter to prove the uniform Hölder continuity of the operator $\eta J_2^{\epsilon,\delta}$ where η is the cut-off function defined in (3.11). The work, which is in any dimension $n \geq 2$, depends on the earlier work for dimension $n = 2$ [1], we use the same decomposition for the operator $J_2^{\epsilon,\delta}$ as well as we use same notations that presented in dimension $n = 2$ for the auxiliary operators with some modifications that are required for the higher dimension.

Theorem 4.1. *Given any $0 < \epsilon < \epsilon_0$ and any $0 \leq \delta < \delta_0$, then $\eta J_2^{\epsilon,\delta}$ is a continuous linear operator from $C^\alpha(\Gamma_2)$ to $C^\alpha(\Gamma_1)$, $\alpha < \alpha_0$. Moreover, we have*

$$\left\| \frac{\eta}{\omega_n \sqrt{1 + \left|\nabla \psi_{1,\epsilon}\right|^2}} \, J_2^{\epsilon,\delta} \right\|_{\mathcal{L}(C^\alpha(\Gamma_2), C^\alpha(\Gamma_1))} \leq C(\epsilon), \quad \forall \, \alpha < \alpha_0,$$

where α is the regularity of the auxilary functions $\psi_{j,\epsilon}$, α_0 is the regularity of the boundaries Γ_j, $j = 1, 2$, and $C(\epsilon)$ converges to 0 as ϵ converges to 0 uniformly in δ.

Before starting the proof, recall that the functions ψ_1 and ψ_2 are defined in a neighborhood of the origin and have regularity C^{1,α_0} for some $0 < \alpha_0 \leq 1$, whereas the auxiliary functions $\psi_{1,\epsilon}$ and $\psi_{2,\epsilon}$ are defined globally on $\mathbb{R}^{n-1}$ and satisfy the bound

$$\left\|\psi_{j,\epsilon}\right\|_{1,\alpha} \leq C\epsilon^\nu \left\|\psi_j\right\|_{1,\alpha_0}, \quad j = 1, 2,$$

where $\nu = \alpha_0 - \alpha$. *That is,*

$$\|\psi_{j,\epsilon}\|_{1,\alpha} \leq C(\epsilon), \quad j = 1, 2,$$

where $C(\epsilon)$ approaches 0 when ϵ approaches 0. The operator $J_2^{\epsilon,\delta}$ is defined explicitly in (3.62) and (3.63) but it is beneficiary to write it down here in order to see the behavior of the kernel function that corresponding to the operator near the origin. For $x \neq 0$ or $\delta \neq 0$ and $\varphi \in C^{\alpha}(\Gamma_2)$, the main operator $J_2^{\epsilon,\delta}$ is defined by

$$\begin{aligned} J_2^{\epsilon,\delta}\varphi(x) = \int_{\mathbb{R}^{n-1}} & \frac{\Big(\delta + \psi_{2,\epsilon}(y') - \psi_{1,\epsilon}(x')\Big) - \nabla\psi_{1,\epsilon}(x') \cdot (y' - x')}{\Big(|x' - y'|^2 + \big(\psi_{1,\epsilon}(x') - \psi_{2,\epsilon}(y') - \delta\big)^2\Big)^{\frac{n}{2}}} \\ & - \frac{\Big(\delta + \psi_{2,\epsilon}(x') - \psi_{1,\epsilon}(x')\Big) - \nabla\psi_{1,\epsilon}(x') \cdot (y' - x')}{\Big(|x' - y'|^2 + \big(\psi_{1,\epsilon}(x') - \psi_{2,\epsilon}(x') - \delta\big)^2\Big)^{\frac{n}{2}}} \; \phi(y')\, dy', \end{aligned} \tag{4.1}$$

and for $x = 0$ and $\delta = 0$, the operator is defined by

$$J_2^{\epsilon,0}\varphi(0) = \int_{\mathbb{R}^{n-1}} \frac{\psi_{2,\epsilon}(y')}{\Big(y_1^2 + ... + y_{n-1}^2 + \psi_{2,\epsilon}^2(y')\Big)^{\frac{n}{2}}} \; \phi(y')\, dy', \tag{4.2}$$

where $\phi(x') = E\varphi(x')\sqrt{1 + |\nabla\psi_2(x')|^2}$ has compact support in $\mathcal{B}'(R_0)$, $\phi(0) = \varphi(0)$ and bounded by the norm

$$\|\phi\|_{\alpha} \leq \Big(1 + C(\epsilon)\Big)(1 + \epsilon_0)\, \|\varphi\|_{\alpha}\,. \tag{4.3}$$

Furthermore, there exists a constant $M > 0$ such that for any $x = (x', x_n) \in \Gamma_1$, the function $\zeta' \to \phi(\zeta' + x')$ is supported in $\mathcal{B}'(M)$.

We start with some basic preliminaries that are needed for the proof of the Theorem 4.1.

4.1 Preliminaries

In this section we present some basic but useful bounds for the auxiliary functions $\psi_{2,\epsilon}$,$\psi_{1,\epsilon}$ and for some linear combinations of $\psi_{2,\epsilon}$ and $\psi_{1,\epsilon}$ with the perturbation constant δ. From the definition of the operator in (4.1) and in (4.2) we see that the denominator of the kernel function depends on the functions $\psi_{1,\epsilon}$ and $\psi_{2,\epsilon}$ so it is a good idea to start finding the lower bounds of these functions.

Lemma 4.2. *Suppose* $0 < \alpha < \alpha_0$. *Then there exists a constant* $C > 0$, *independent of* ϵ *(where* ϵ *is assumed to be sufficiently small), such that for any* $x' \in \mathbb{R}^{n-1}$,

$$\left|\nabla\psi_{i,\epsilon}(x')\right| \leq C\big(\alpha, \|\psi_i\|_{1,\alpha_0}\big)\left|\psi_{i,\epsilon}(x')\right|^{\frac{\alpha}{1+\alpha}}, \quad i = 1, 2. \tag{4.4}$$

Proof. We only focus on $\psi_{2,\epsilon}$, but the same arguments hold for $\psi_{1,\epsilon}$. Recall the definition of $\psi_{2,\epsilon}$ from (3.20)

$$\psi_{2,\epsilon}(x) = \psi(x)\eta_\epsilon(x) + (1 - \eta_\epsilon(x))\epsilon^{1+\alpha_0}\|\psi\|_{C^{1,\alpha_0}},$$

where η_ϵ is a cut-off function defined in (3.19). For any x', $\xi' \in \mathbb{R}^{n-1}$, we see that

$$\psi_{2,\epsilon}(x' + \xi') = \psi_{2,\epsilon}(x') + \nabla\psi_{2,\epsilon}(x')\cdot\xi' + \int_0^1 \frac{d}{dt}\psi_{2,\epsilon}(x' + t\xi') - \nabla\psi_{2,\epsilon}(x')\cdot\xi'\,dt,$$

and clearly we have the bound

$$\psi_{2,\epsilon}(x' + \xi') \leq \psi_{2,\epsilon}(x') + \nabla\psi_{2,\epsilon}(x')\cdot\xi' + \int_0^1 \left|\nabla\psi_{2,\epsilon}(x' + t\xi')\cdot\xi' - \nabla\psi_{2,\epsilon}(x')\cdot\xi'\right|\,dt.$$

Using the Hölder continuity of $\nabla\psi_{2,\epsilon}$, it follows that

$$
\begin{aligned}
\psi_{2,\epsilon}(x'+\xi') &\leq \psi_{2,\epsilon}(x') + \nabla\psi_{2,\epsilon}(x')\cdot\xi' + \left|\xi'\right|^{1+\alpha}\left\|\nabla\psi_{2,\epsilon}\right\|_{\alpha}\int_0^1 t^{\alpha}\,dt \\
&\leq \psi_{2,\epsilon}(x') + \nabla\psi_{2,\epsilon}(x')\cdot\xi' + \frac{1}{1+\alpha}\left|\xi'\right|^{1+\alpha}\|\psi_2\|_{1,\alpha_0}.
\end{aligned}
$$

Assuming $C = \frac{1}{1+\alpha}\|\psi_2\|_{1,\alpha_0}$ which is evidently independent of ϵ (sufficiently small), we obtain

$$
\psi_{2,\epsilon}(x'+\xi') \leq \psi_{2,\epsilon}(x') + \nabla\psi_{2,\epsilon}(x')\cdot\xi' + C\left|\xi'\right|^{1+\alpha}.
$$

That is,

$$
\psi_{2,\epsilon}(x'+\xi') - \psi_{2,\epsilon}(x') - \nabla\psi_{2,\epsilon}(x')\cdot\xi' \leq C\left|\xi'\right|^{1+\alpha}.
$$

Since $\psi_{2,\epsilon}(x'+\xi') \geq 0$, we get that

$$
-\nabla\psi_{2,\epsilon}(x')\cdot\xi' - C\left|\xi'\right|^{1+\alpha} \leq \left|\psi_{2,\epsilon}(x')\right|. \tag{4.5}
$$

Letting $\xi' = -\left(\frac{1}{C(1+\alpha)}\right)^{1/\alpha}\left|\nabla\psi_{2,\epsilon}(x')\right|^{\frac{1}{\alpha}-1}\nabla\psi_{2,\epsilon}(x')$ in (4.5), we have the following

$$
\left(\frac{1}{C(1+\alpha)}\right)^{1/\alpha}\left|\nabla\psi_{2,\epsilon}(x')\right|^{\frac{1}{\alpha}+1} - C\left(\frac{1}{C(1+\alpha)}\right)^{(1+\alpha)/\alpha}\left|\nabla\psi_{2,\epsilon}(x')\right|^{\frac{1+\alpha}{\alpha}} \leq \left|\psi_{2,\epsilon}(x')\right|.
$$

Therefore we get the desired bound

$$
\left|\nabla\psi_{2,\epsilon}(x')\right| \leq C\left|\psi_{2,\epsilon}(x')\right|^{\frac{\alpha}{1+\alpha}}.
$$

$\square$

The following lemma [1] is a straightforward consequence of Young's inequality and it has been used frequently for finding the lower bounds for the denominator of the kernel function that corresponding to the operator $J_2^{\epsilon,\delta}$.

Lemma 4.3. *For any $r, t \geq 0$ and for any $0 \leq \mu \leq 1$ we have*

$$r^2 + t^2 \geq r^{1+\mu}\, t^{1-\mu}.$$

Proof. By Young's inequality, we have the following

$$r^{1+\mu}\, t^{1-\mu} \leq \Big(\frac{1+\mu}{2}\Big) r^2 + \Big(\frac{1-\mu}{2}\Big) t^2 \leq r^2 + t^2$$

$\square$

To find appropriate uniform bounds for the kernel function, we need some basic uniform bounds for a linear combination of the auxiliary functions $\psi_{1,\epsilon}$ and $\psi_{2,\epsilon}$ with the perturbation constant δ. We compile the most frequently used bounds in the upcoming lemma. To simplify the notation and because this work is generalization of the earlier work for dimension $n = 2$, we use same notations for the combination of the auxiliary functions that has been given in [1]. For any x', y' and ζ' in $\mathbb{R}^{n-1}$, we define the following quantities

$$a := a(x') = \delta + \psi_{2,\epsilon}(x') - \psi_{1,\epsilon}(x'), \tag{4.6}$$

$$\hat{a} := a(y') = \delta + \psi_{2,\epsilon}(y') - \psi_{1,\epsilon}(y'), \tag{4.7}$$

$$b := b(x', \zeta') = \delta + \psi_{2,\epsilon}(\zeta' + x') - \psi_{1,\epsilon}(x'), \tag{4.8}$$

$$\hat{b} := b(y', \zeta') = \delta + \psi_{2,\epsilon}(\zeta' + y') - \psi_{1,\epsilon}(y'). \tag{4.9}$$

Lemma 4.4. *For any x', y' and ζ' in $\mathbb{R}^{n-1}$, the following bounds hold*

1. $|b-a| \leq \left\|\nabla\psi_{2,\epsilon}\right\|_{\alpha} \, \left|\zeta'\right|$, *alternatively;*

2. $|b-a| \leq \left\|\nabla\psi_{2,\epsilon}\right\|_{\alpha} \, \left|\zeta'\right|^{\alpha+1} + \left|\nabla\psi_{2,\epsilon}(x')\right| \left|\zeta'\right|$, *alternatively;*

3. $|b-a| \leq \left|\nabla\psi_{2,\epsilon}(\zeta'+x')\right| \left|\zeta'\right| + \left\|\nabla\psi_{2,\epsilon}\right\|_{\alpha} \, \left|\zeta'\right|^{\alpha+1}$.

4. $\left|b-\hat{b}\right| \leq d\left(\left\|\nabla\psi_{2,\epsilon}\right\|_{\alpha} + \left\|\nabla\psi_{1,\epsilon}\right\|_{\alpha}\right)$, *where* $d = \left|x'-y'\right|$, *alternatively;*

5. $\left|b-\hat{b}\right| \leq d\left|\nabla\psi_{2,\epsilon}(\zeta'+y')\right| + d\left|\nabla\psi_{1,\epsilon}(y')\right| + d^{\alpha+1}\left(\left\|\nabla\psi_{2,\epsilon}\right\|_{\alpha} + \left\|\nabla\psi_{1,\epsilon}\right\|_{\alpha}\right)$, *alternatively;*

6. $\left|b-\hat{b}\right| \leq d\left|\nabla\psi_{2,\epsilon}(\zeta'+x')\right| + d\left|\nabla\psi_{1,\epsilon}(x')\right| + d^{\alpha+1}\left(\left\|\nabla\psi_{2,\epsilon}\right\|_{\alpha} + \left\|\nabla\psi_{1,\epsilon}\right\|_{\alpha}\right)$,

7. $|a-\hat{a}| \leq d\left(\left\|\nabla\psi_{2,\epsilon}\right\|_{\alpha} + \left\|\nabla\psi_{1,\epsilon}\right\|_{\alpha}\right)$.

Proof. 1. From the definition of a and b, we have

$$|b-a| = \left|\psi_{2,\epsilon}(\zeta'+x') - \psi_{2,\epsilon}(x')\right|,$$

and by using the regularity of $\psi_{2,\epsilon}$, the desired bound follows

$$|b-a| \leq \left\|\nabla\psi_{2,\epsilon}\right\|_{\alpha} \, \left|\zeta'\right|.$$

2. Agian from the definition of a and b, we have

$$|b-a| = \left|\psi_{2,\epsilon}(\zeta'+x') - \psi_{2,\epsilon}(x')\right|.$$

Using the mean value theorem for some ξ' lies on the line segment that joining 0 and

ζ', it follows that

$$|b-a| \leq \left|\nabla\psi_{2,\epsilon}(x'+\xi')\right| \left|\zeta'\right|.$$

By adding and subtracting the quantity $\nabla\psi_{2,\epsilon}(x')$, we obtain

$$|b-a| \leq \Big(\left|\nabla\psi_{2,\epsilon}(x'+\xi') - \nabla\psi_{2,\epsilon}(x')\right| + \left|\nabla\psi_{2,\epsilon}(x')\right| \Big) \left|\zeta'\right|.$$

Using the Hölder continuity of $\nabla\psi_{2,\epsilon}$, the desired bound follows

$$|b-a| \leq \left\|\nabla\psi_{2,\epsilon}\right\|_{\alpha} \left|\zeta'\right|^{\alpha+1} + \left|\nabla\psi_{2,\epsilon}(x')\right| \left|\zeta'\right|.$$

3. For the alternative bound. If we add and subtract the quantity $\nabla\psi_{2,\epsilon}(\zeta'+x')$ instead of $\nabla\psi_{2,\epsilon}(x')$, it follows that

$$|b-a| \leq \Big(\left|\nabla\psi_{2,\epsilon}(x'+\xi') - \nabla\psi_{2,\epsilon}(\zeta'+x')\right| + \left|\nabla\psi_{2,\epsilon}(\zeta'+x')\right| \Big) \left|\zeta'\right|.$$

Noticing that $\left|\zeta'-\xi'\right| \leq \left|\zeta'\right|$, The desired bound follows

$$|b-a| \leq \left\|\nabla\psi_{2,\epsilon}\right\|_{\alpha} \left|\zeta'\right|^{\alpha+1} + \left|\nabla\psi_{2,\epsilon}(\zeta'+x')\right| \left|\zeta'\right|.$$

4. Follows trivially from the definition of b and $\hat{b}$.

5. For the alternative bound for $\left|b-\hat{b}\right|$, let $\psi(x') = \psi_{2,\epsilon}(\zeta'+x') - \psi_{1,\epsilon}(x')$. Then by

applying the mean value theorem for some $t \in (0,1)$, we have

$$b - \hat{b} = \psi(x') - \psi(y') = \nabla\psi(\xi') \cdot (x' - y'),$$

where $\xi' = tx' + (1-t)y'$. Next, we add and subtract the quantity $\nabla\psi(y')$ to get the following

$$b - \hat{b} = \Big(\nabla\psi(\xi') - \nabla\psi(y') + \nabla\psi(y')\Big) \cdot (x' - y'),$$

equivalently,

$$b-\hat{b} = \Big(\nabla\psi_{2,\epsilon}(\zeta'+\xi') - \nabla\psi_{2,\epsilon}(\zeta'+y') - \nabla\psi_{1,\epsilon}(\xi') + \nabla\psi_{1,\epsilon}(y') + \nabla\psi_{2,\epsilon}(\zeta'+y') - \nabla\psi_{1,\epsilon}(y')\Big) \cdot (x'-y').$$

Thus by the Hölder continuity of $\psi_{2,\epsilon}$ and $\psi_{1,\epsilon}$ and noticing that $\left|\xi' - y'\right| \leq \left|x' - y'\right|$, the desired bound follows

$$\left|b - \hat{b}\right| \leq d^{\alpha+1}\left(\left\|\nabla\psi_{2,\epsilon}\right\|_{\alpha} + \left\|\nabla\psi_{1,\epsilon}\right\|_{\alpha}\right) + d\left|\nabla\psi_{2,\epsilon}(\zeta' + y')\right| + d\left|\nabla\psi_{1,\epsilon}(y')\right|.$$

6. Follows similarly.

7. Follows trivially from the definition of a and $\hat{a}$.

□

4.2 $L^\infty - norm$ for $J_2^{\epsilon,\delta}$, $\delta > 0$

In this section we show the uniform boundedness for the operator$J_2^{\epsilon,\delta}$ as an operator from Γ_2 to Γ_1, *that is*, for $\varphi \in C^\alpha(\Gamma_2)$, and $x \in \Gamma_1 \cap B(0, R_0)$, we show that

$$\left| J_2^{\epsilon,\delta}\varphi(x) \right| \leq C(\epsilon) \left\| \varphi \right\|_\alpha ,$$

where $C(\epsilon)$ approaches 0 as ϵ approaches 0. Let $\mathbf{k}_2^{\epsilon,\delta}$ be the kernel function corresponding to the operator $J_2^{\epsilon,\delta}$,*that is*,

$$\begin{aligned} \mathbf{k}_2^{\epsilon,\delta}(y', x') = & \frac{\left(\delta + \psi_{2,\epsilon}(y') - \psi_{1,\epsilon}(x')\right) - \nabla\psi_{1,\epsilon}(x') \cdot (y' - x')}{\left(|x' - y'|^2 + \left(\psi_{1,\epsilon}(x') - \psi_{2,\epsilon}(y') - \delta\right)^2\right)^{\frac{n}{2}}} \\ & - \frac{\left(\delta + \psi_{2,\epsilon}(x') - \psi_{1,\epsilon}(x')\right) - \nabla\psi_{1,\epsilon}(x') \cdot (y' - x')}{\left(|x' - y'|^2 + \left(\psi_{1,\epsilon}(x') - \psi_{2,\epsilon}(x') - \delta\right)^2\right)^{\frac{n}{2}}}, \end{aligned}$$

and then in short $J_2^{\epsilon,\delta}$ can be written in a simple form

$$J_2^{\epsilon,\delta}\varphi(x) = \int_{\mathbb{R}^{n-1}} \mathbf{k}_2^{\epsilon,\delta}(y', x')\phi(y')\, dy'.$$

Using the change of variables $\zeta' = y' - x'$ and recalling that the function $\phi(\cdot + x')$ has compact support in $\mathcal{B}'(M)$, it follows that

$$J^{\epsilon,\delta}\varphi(x) = \int_{|\zeta'|<M} \mathbf{k}_2^{\epsilon,\delta}(\zeta', x')\phi(\zeta' + x')\, d\zeta'.$$

Involving the definitions of $a(x')$ and $b(x',\zeta')$ that defined in (4.6) and (4.8) respectively, the kernel function $\mathbf{k}_2^{\epsilon,\delta}$ has the equivalent form

$$\mathbf{k}_2^{\epsilon,\delta}(\zeta',x') = \frac{b(x',\zeta') - \zeta'\cdot\nabla\psi_{1,\epsilon}(x')}{\left[|\zeta'|^2 + b^2(x',y')\right]^{\frac{n}{2}}} - \frac{a(x') - \zeta'\cdot\nabla\psi_{1,\epsilon}(x')}{\left[|\zeta'|^2 + a^2(x')\right]^{\frac{n}{2}}}. \tag{4.10}$$

In naively way, we have the bound

$$\left|J_2^{\epsilon,\delta}\varphi(x)\right| \le \|\phi\|_\infty \int_{|\zeta'|<M} \left| \frac{b(x',\zeta') - \zeta'\cdot\nabla\psi_{1,\epsilon}(x')}{\left[|\zeta'|^2 + b^2(x',\zeta')\right]^{\frac{n}{2}}} - \frac{a(x') - \zeta'\cdot\nabla\psi_{1,\epsilon}(x')}{\left[|\zeta'|^2 + a^2(x')\right]^{\frac{n}{2}}} \right| d\zeta'.$$

We proceed to simplify the integrand by using the least common denominator. We note that

$$\begin{aligned}
&\left|J_2^{\epsilon,\delta}\varphi(x)\right| \\
&\le \|\phi\|_\infty \int_{|\zeta'|<M} \left| \frac{(|\zeta'|^2 + a^2)^{\frac{n}{2}}(b-a) + \left((|\zeta'|^2 + a^2)^{\frac{n}{2}} - (|\zeta'|^2 + b^2)^{\frac{n}{2}}\right)\left(a - \zeta'\cdot\nabla\psi_{1,\epsilon}(x')\right)}{(|\zeta'|^2 + b^2)^{\frac{n}{2}}(|\zeta'|^2 + a^2)^{\frac{n}{2}}} \right| d\zeta' \\
&\le \|\phi\|_\infty \int_{|\zeta'|<M} \frac{|b-a|}{(|\zeta'|^2 + b^2)^{\frac{n}{2}}} := A \\
&+ \|\phi\|_\infty \int_{|\zeta'|<M} \frac{\left|(|\zeta'|^2 + a^2)^{\frac{n}{2}} - (|\zeta'|^2 + b^2)^{\frac{n}{2}}\right| \left|\left(a - \zeta'\cdot\nabla\psi_{1,\epsilon}(x')\right)\right|}{(|\zeta'|^2 + b^2)^{\frac{n}{2}}(|\zeta'|^2 + a^2)^{\frac{n}{2}}} \, d\zeta' := B.
\end{aligned}$$

For A term, we use bound (3) from Lemma 4.4 to obtain the following inequality

$$A \le \|\phi\|_\infty \int_{|\zeta'|<M} \frac{\|\nabla\psi_{2,\epsilon}\|_\alpha \ |\zeta'|^{1+\alpha}}{\left(|\zeta'|^2 + b^2\right)^{n/2}} + \frac{|\nabla\psi_{2,\epsilon}(x'+\zeta')| \ |\zeta'|}{\left(|\zeta'|^2 + b^2\right)^{n/2}} \, d\zeta' := A_1 + A_2,$$

where

$$A_1 = \|\phi\|_\infty \int_{|\zeta'|<M} \frac{\|\nabla\psi_{2,\epsilon}\|_\alpha \ |\zeta'|^{1+\alpha}}{\left(|\zeta'|^2 + b^2\right)^{n/2}} \, d\zeta',$$

and

$$A_2 = \|\phi\|_\infty \int_{|\zeta'|<M} \frac{\left|\nabla\psi_{2,\epsilon}(x'+\zeta')\right| \, |\zeta'|}{\left(|\zeta'|^2 + b^2\right)^{n/2}} \, d\zeta'.$$

Naively, we can bound A_1 by

$$A_1 \le \|\phi\|_\infty \left\|\nabla\psi_{2,\epsilon}\right\|_\alpha \int_{|\zeta'|<M} |\zeta'|^{1+\alpha-n}.$$

We easily conclude the bound

$$A_1 \le C(M,\alpha) \, \|\phi\|_\infty \, C(\epsilon). \tag{4.11}$$

For A_2, we recall that $b = \delta + \psi_{2,\epsilon}(\zeta' + x') - \psi_{1,\epsilon}(x')$ and use Lemma 4.3 to conclude the bound

$$A_2 \le \|\phi\|_\infty \int_{|\zeta'|<M} \left|\nabla\psi_{2,\epsilon}(x'+\zeta')\right| \left|\psi_{2,\epsilon}(x'+\zeta')\right|^{-(1-\mu)\frac{n}{2}} \, |\zeta'|^{1-(1+\mu)\frac{n}{2}} \, d\zeta',$$

then we use lemma (4.2) to bound $\nabla\psi_{2,\epsilon}(x'+\zeta')$, that is, we obtain the following bound

$$A_2 \le C\big(\alpha, \|\psi_2\|_{1,\alpha_0}\big) \, \|\phi\|_\infty \int_{|\zeta'|<M} \left|\psi_{2,\epsilon}(x'+\zeta')\right|^{\frac{\alpha}{1+\alpha}-(1-\mu)\frac{n}{2}} \, |\zeta'|^{1-(1+\mu)\frac{n}{2}} \, d\zeta'.$$

Choosing $1 - \left(\frac{\alpha}{1+\alpha}\right)\frac{2}{n} < \mu < 1$, we clearly approach the desired bound bound, *i.e*, we get

$$A_2 \le C\big(\alpha, \|\psi_2\|_{1,\alpha_0}\big) \, \|\phi\|_\infty \left\|\psi_{2,\epsilon}\right\|_{1,\alpha}^{\frac{\alpha}{1+\alpha}-(1-\mu)\frac{n}{2}} \int_{|\zeta'|<M} |\zeta'|^{1-(1+\mu)\frac{n}{2}} \, d\zeta'.$$

Thus we have

$$A_2 \leq C\big(\alpha, n, M, \|\psi_2\|_{1,\alpha_0}\big)\, \|\phi\|_\infty\, C(\epsilon). \tag{4.12}$$

Finally, combining (4.11) and (4.12), we have the desired bound for A which is

$$A \leq C\big(\alpha, n, M, \|\psi_2\|_{1,\alpha_0}\big)\, \|\phi\|_\infty\, C(\epsilon). \tag{4.13}$$

Continuing in the same fashion to bound the term B uniformly, where

$$B = \|\phi\|_\infty \int_{|\zeta'|<M} \frac{\left|(\left|\zeta'\right|^2 + a^2)^{\frac{n}{2}} - (\left|\zeta'\right|^2 + b^2)^{\frac{n}{2}}\right| \left|a - \zeta' \cdot \nabla\psi_{1,\epsilon}(x')\right|}{(|\zeta'|^2 + b^2)^{\frac{n}{2}}(|\zeta'|^2 + a^2)^{\frac{n}{2}}}\, d\zeta'.$$

By using the mean value theorem, we have

$$\left|(\left|\zeta'\right|^2 + a^2)^{\frac{n}{2}} - (\left|\zeta'\right|^2 + b^2)^{\frac{n}{2}}\right| \leq \frac{n}{2}\left(\left|\zeta'\right|^2 + z^2\right)^{\frac{n}{2}-1} \left|b^2 - a^2\right|,$$

for some z^2 that lies in the line segment joining a^2 to b^2. Without loss of generality we may assume that $a^2 < z^2 < b^2$. On other hand, same arguments would hold if $b^2 < z^2 < a^2$. Therefor we get the following bound

$$B \leq C\,\|\phi\|_\infty \int_{|\zeta'|<M} \frac{\left|a - \zeta' \cdot \nabla\psi_{1,\epsilon}(x')\right|\ |b-a|\,|b+a|}{\left(|\zeta'|^2 + b^2\right)\ \left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}}\, d\zeta'.$$

Using Schwartz inequality to bound the quantity $\left|a - \zeta' \cdot \nabla\psi_{1,\epsilon}(x')\right| \; |b + a|$, it follows that

$$B \leq C \, \|\phi\|_\infty \int_{|\zeta'|<M} \frac{|b-a| \left(\left|a - \zeta' \cdot \nabla\psi_{1,\epsilon}(x')\right|^2 + |b+a|^2 \right)}{\left(|\zeta'|^2 + b^2\right) \; \left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}} \, d\zeta'.$$

Notice the following

$$\left|a - \zeta' \cdot \nabla\psi_{1,\epsilon}(x')\right|^2 + |b+a|^2 \leq 6\Big(b^2 + \left|\zeta'\right|^2 \left\|\nabla\psi_{1,\epsilon}\right\|_\infty^2\Big).$$

So we conclude the following

$$B \leq C \, \|\phi\|_\infty \int_{|\zeta'|<M} \frac{|b-a| \left(b^2 + \left|\zeta'\right|^2 \; \left\|\nabla\psi_{1,\epsilon}\right\|_\infty^2\right)}{\left(|\zeta'|^2 + b^2\right) \; \left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}} \, d\zeta'. \tag{4.14}$$

Since

$$\frac{\left(b^2 + \left|\zeta'\right|^2 \; \left\|\nabla\psi_{1,\epsilon}\right\|_\infty^2\right)}{|\zeta'|^2 + b^2} \leq 1 + \left\|\nabla\psi_{1,\epsilon}\right\|_\infty^2,$$

then B can be bounded by

$$B \leq C\big(1 + \left\|\nabla\psi_{1,\epsilon}\right\|_\infty^2\big) \, \|\phi\|_\infty \int_{|\zeta'|<M} \frac{|b-a|}{\left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}} \, d\zeta'.$$

Using the bound (2) form Lemma 4.4, the following bound follows

$$B \leq C\big(1 + \left\|\nabla\psi_{1,\epsilon}\right\|_\infty^2\big) \, \|\phi\|_\infty \int_{|\zeta'|<M} \frac{\left\|\nabla\psi_{2,\epsilon}\right\|_\alpha \; \left|\zeta'\right|^{1+\alpha}}{\left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}} + \frac{\left|\nabla\psi_{2,\epsilon}(x')\right| \; \left|\zeta'\right|}{\left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}} \, d\zeta'.$$

Thus easily we have the bound

$$B \leq C\big(1 + \|\nabla \psi_{1,\epsilon}\|_\infty^2\big) \|\phi\|_\infty \int_{|\zeta'|<M} \|\nabla \psi_{2,\epsilon}\|_\alpha |\zeta'|^{\alpha+1-n} + \frac{|\nabla \psi_{2,\epsilon}(x')| \, |\zeta'|}{|\zeta'|^{\frac{n}{2}(1+\mu)} a^{\frac{n}{2}(1-\mu)}} \, d\zeta',$$

where we used Lemma 4.3 for the second integrand with the choice $1 - \frac{2}{n}\left(\frac{\alpha}{1+\alpha}\right) < \mu < 1$. Again for the second integrand we use Lemma 4.2 and the trivial inequality $\psi_{2,\epsilon}(x') \leq a$ to obtain the following

$$B \leq C\big(1 + \|\nabla \psi_{1,\epsilon}\|_\infty^2\big) \|\phi\|_\infty \left(C(\epsilon)\frac{M^\alpha}{\alpha} + \int_{|\zeta'|<M} |\psi_{2,\epsilon}(x')|^{m_1} \; |\zeta'|^{m_2} \; d\zeta' \right),$$

where $m_1 = \frac{\alpha}{1+\alpha} - \frac{n}{2}(1-\mu)$ and $m_2 = 1 - \frac{n}{2}(1+\mu)$. Clearly $m_1 > 0$ and the radial function $|\zeta'|^{m_2}$ is integrable in the domain $|\zeta'| < M$. Thus we have

$$B \leq C(\alpha, M, n)\Big(1 + \|\nabla \psi_{1,\epsilon}\|_\infty^2\Big) \|\phi\|_\infty C(\epsilon). \tag{4.15}$$

Therefore the desired uniform bound for $J_2^{\epsilon,\delta}$ follows by combining (4.13) and (4.15),*that is*,

$$\left| J_2^{\epsilon,\delta} \varphi(x) \right| \leq C(\epsilon), \tag{4.16}$$

where $C(\epsilon)$ approaches 0 as ϵ approaches 0.

4.3 Hölder continuity of $J_2^{\epsilon,\delta}$, $\delta > 0$.

For $\varphi \in C^{\alpha}(\Gamma_2)$, let $x, y \in \Gamma_1 \cap B(0, R_0)$ and set $d = |x' - y'|$. To show the Hölder continuity of the operator $J_2^{\epsilon,\delta}$, we form the following

$$J_2^{\epsilon,\delta}\varphi(x) - J_2^{\epsilon,\delta}\varphi(y) = \int_{|\zeta'|<M} \mathbf{k}_2^{\epsilon,\delta}(\zeta', x')\phi(\zeta' + x') - \mathbf{k}_2^{\epsilon,\delta}(\zeta', y')\phi(\zeta' + y')\, d\zeta'. \tag{4.17}$$

We use the same decomposition for the case $n = 2$ [1], *that is*, we form the following

$$\begin{aligned} J_2^{\epsilon,\delta}\varphi(x) - J_2^{\epsilon,\delta}\varphi(y) &= \int_{|\zeta'|<M} \mathbf{k}_2^{\epsilon,\delta}(\zeta', x')\Big(\phi(\zeta' + x') - \phi(\zeta' + y')\Big)\, d\zeta' \\ &+ \int_{|\zeta'|<M} \Big(\mathbf{k}_2^{\epsilon,\delta}(\zeta', x') - \mathbf{k}_2^{\epsilon,\delta}(\zeta', y')\Big)\Big(\phi(\zeta' + y') - \phi(y')\Big)\, d\zeta' \\ &+ \int_{|\zeta'|<M} \Big(\mathbf{k}_2^{\epsilon,\delta}(\zeta', x') - \mathbf{k}_2^{\epsilon,\delta}(\zeta', y')\Big)\phi(y')\, d\zeta' \\ &:= R_1 + R_2 + R_3. \end{aligned}$$

The fact that we split up R_2 and R_3 is technical. Naively, one may try to not do the splitting and estimate directly their sum (canceling out the factors of $\phi(y')$), but this won't give the desired estimate that we want.

We easily bound R_1 as follows

$$|R_1| \le \|\phi\|_{\alpha} \int_{|\zeta'|<M} |x' - y'|^{\alpha} \left|\mathbf{k}_2^{\epsilon,\delta}(\zeta', x')\right|\, d\zeta'.$$

In the previous section 4.2 we have shown that

$$\int_{|\zeta'|<M} \left| \mathbf{k}_2^{\epsilon,\delta}(\zeta', x') \right| \; d\zeta' \leq C(\epsilon).$$

Thus we conclude that

$$|R_1| \leq C(\epsilon)\; d^{\alpha} \;\; \|\phi\|_{\alpha}. \tag{4.18}$$

The work for the terms R_2 and R_3 are tedious so we will deal with them separately in the next subsections.

4.3.1 Boundedness of R_2.

For R_2 we have

$$R_2 = \int_{|\zeta'|<M} \left(\mathbf{k}_2^{\epsilon,\delta}(\zeta', x') - \mathbf{k}_2^{\epsilon,\delta}(\zeta', y') \right) \left(\phi(\zeta' + y') - \phi(y') \right) \; d\zeta',$$

where

$$\mathbf{k}_2^{\epsilon,\delta}(\zeta', x') = \frac{b - \zeta' \cdot \nabla \psi_{1,\epsilon}(x')}{\left[|\zeta'|^2 + b^2 \right]^{\frac{n}{2}}} - \frac{a - \zeta' \cdot \nabla \psi_{1,\epsilon}(x')}{\left[|\zeta'|^2 + a^2 \right]^{\frac{n}{2}}},$$

and

$$\mathbf{k}_2^{\epsilon,\delta}(\zeta', y') = \frac{\hat{b} - \zeta' \cdot \nabla \psi_{1,\epsilon}(y')}{\left[|\zeta'|^2 + \hat{b}^2 \right]^{\frac{n}{2}}} - \frac{\hat{a} - \zeta' \cdot \nabla \psi_{1,\epsilon}(y')}{\left[|\zeta'|^2 + \hat{a}^2 \right]^{\frac{n}{2}}}.$$

We rewrite R_2 as follows

$$R_2 = S_1 + S_2 + S_3 + S_4,$$

where

$$S_1 = \int_{|\zeta'|<d} \mathbf{k}_2^{\epsilon,\delta}(\zeta', x') \left(\phi(\zeta' + y') - \phi(y')\right) \, d\zeta',$$

$$S_2 = -\int_{|\zeta'|<d} \mathbf{k}_2^{\epsilon,\delta}(\zeta', y') \left(\phi(\zeta' + y') - \phi(y')\right) \, d\zeta',$$

$$S_3 = \int_{d<|\zeta'|<M} \left(\frac{b - \zeta' \cdot \nabla\psi_{1,\epsilon}(x')}{\left[|\zeta'|^2 + b^2\right]^{\frac{n}{2}}} - \frac{\hat{b} - \zeta' \cdot \nabla\psi_{1,\epsilon}(y')}{\left[|\zeta'|^2 + \hat{b}^2\right]^{\frac{n}{2}}} \right) \left(\phi(\zeta' + y') - \phi(y')\right) \, d\zeta',$$

and

$$S_4 = -\int_{d<|\zeta'|<M} \left(\frac{a - \zeta' \cdot \nabla\psi_{1,\epsilon}(x')}{\left[|\zeta'|^2 + a^2\right]^{\frac{n}{2}}} - \frac{\hat{a} - \zeta' \cdot \nabla\psi_{1,\epsilon}(y')}{\left[|\zeta'|^2 + \hat{a}^2\right]^{\frac{n}{2}}} \right) \left(\phi(\zeta' + y') - \phi(y')\right) \, d\zeta'.$$

For S_1, we use the Hölder continuity of ϕ to obtain the following

$$|S_1| \leq \|\phi\|_\alpha \int_{|\zeta'|<d} \left|\zeta'\right|^\alpha \left|\mathbf{k}_2^{\epsilon,\delta}(\zeta', x')\right| \, d\zeta'.$$

Then we clearly have the following bound

$$|S_1| \leq \|\phi\|_\alpha \; d^\alpha \int_{|\zeta'|<M} \left|\mathbf{k}_2^{\epsilon,\delta}(\zeta', x')\right| \, d\zeta'.$$

Thus by using the L^∞-norm for $J_2^{\epsilon,\delta}$ (4.16), we obtain the desired bound for S_1. *That is,*

$$|S_1| \leq C(\epsilon) \|\phi\|_\alpha \; d^\alpha. \tag{4.19}$$

Clearly, the same bound holds for S_2. Thus we conclude that

$$|S_1| + |S_2| \leq C(\epsilon) \left\|\phi\right\|_\alpha \, d^\alpha. \tag{4.20}$$

We still have to show the uniform boundedness for S_3 and S_4. For S_3 we have

$$S_3 = \int_{d<|\zeta'|<M} \left(\frac{b - \zeta' \cdot \nabla\psi_{1,\epsilon}(x')}{\left(|\zeta'|^2 + b^2\right)^{\frac{n}{2}}} - \frac{\hat{b} - \zeta' \cdot \nabla\psi_{1,\epsilon}(y')}{\left(|\zeta'|^2 + \hat{b}^2\right)^{\frac{n}{2}}} \right) \left(\phi(\zeta' + y') - \phi(y')\right) \, d\zeta',$$

then trivially by using the least common denominator, it follows that

$$\begin{aligned} |S_3| = \int_{d<|\zeta'|<M} \Bigg\{ & \frac{\left(b - \zeta' \cdot \nabla\psi_{1,\epsilon}(x')\right) \left[\left(\left|\zeta'\right|^2 + \hat{b}^2\right)^{\frac{n}{2}} - \left(\left|\zeta'\right|^2 + b^2\right)^{\frac{n}{2}} \right]}{\left(|\zeta'|^2 + b^2\right)^{\frac{n}{2}} \left(|\zeta'|^2 + \hat{b}^2\right)^{\frac{n}{2}}} \\ & - \frac{\left(\zeta' \cdot \nabla\psi_{1,\epsilon}(y') - \zeta' \cdot \nabla\psi_{1,\epsilon}(x')\right)}{\left(|\zeta'|^2 + \hat{b}^2\right)^{\frac{n}{2}}} - \frac{(\hat{b} - b)}{\left(|\zeta'|^2 + \hat{b}^2\right)^{\frac{n}{2}}} \Bigg\} \left(\phi(\zeta' + y') - \phi(y')\right) \, d\zeta' \end{aligned}$$

Using the Hölder continuity of ϕ and $\nabla\psi_{1,\epsilon}$ and the mean value theorem for some z^2 that lies on line segment of b^2 and $\hat{b}^2$, we can form the following decomposition

$$\begin{aligned} |S_3| \leq\; & C \left\|\phi\right\|_\alpha \int_{d<|\zeta'|<M} \frac{\left|\zeta'\right|^\alpha \left|b - \zeta' \cdot \nabla\psi_{1,\epsilon}(x')\right| \left(\left|\zeta'\right|^2 + z^2\right)^{\frac{n}{2}-1} \left|b^2 - \hat{b}^2\right|}{\left(|\zeta'|^2 + b^2\right)^{\frac{n}{2}} \left(|\zeta'|^2 + \hat{b}^2\right)^{\frac{n}{2}}} \, d\zeta' := S_{3,1} \\ & + C \left\|\phi\right\|_\alpha \int_{d<|\zeta'|<M} \frac{d^\alpha \, \left|\zeta'\right|^{\alpha+1} \left\|\nabla\psi_{1,\epsilon}\right\|_\alpha}{\left(|\zeta'|^2 + \hat{b}^2\right)^{\frac{n}{2}}} \, d\zeta' := S_{3,2} \\ & + C \left\|\phi\right\|_\alpha \int_{d<|\zeta'|<M} \frac{\left|\hat{b} - b\right| \left|\zeta'\right|^\alpha}{\left(|\zeta'|^2 + \hat{b}^2\right)^{\frac{n}{2}}} \, d\zeta' := S_{3,3}. \end{aligned}$$

For $S_{3,1}$ we assume $\hat{b}^2 < z^2 < b^2$ and again the same work holds when $b^2 < z^2 < \hat{b}^2$. We obtain the following

$$\left|S_{3,1}\right| \leq C \left\|\phi\right\|_\alpha \int_{d<|\zeta'|<M} \frac{\left|b-\hat{b}\right| \left|b+\hat{b}\right| \left|b-\zeta'\cdot\nabla\psi_{1,\epsilon}(x')\right| \left|\zeta'\right|^\alpha}{\left(|\zeta'|^2+b^2\right)\left(|\zeta'|^2+\hat{b}^2\right)^{\frac{n}{2}}}\, d\zeta'.$$

Using the bound (4) from Lemma 4.4, it follows that

$$\left|S_{3,1}\right| \leq C \left\|\phi\right\|_\alpha d \left(\left\|\nabla\psi_{2,\epsilon}\right\|_\alpha + \left\|\nabla\psi_{1,\epsilon}\right\|_\alpha\right) \int_{d<|\zeta'|<M} \frac{\left|b+\hat{b}\right| \left|b-\zeta'\cdot\nabla\psi_{1,\epsilon}(x')\right| \left|\zeta'\right|^\alpha}{\left(|\zeta'|^2+b^2\right)\left(|\zeta'|^2+\hat{b}^2\right)^{\frac{n}{2}}}\, d\zeta'.$$

Since

$$\left|b+\hat{b}\right| \left|b-\zeta'\cdot\nabla\psi_{1,\epsilon}(x')\right| \leq 3\left(b^2+\left|\zeta'\right|^2\left|\nabla\psi_{1,\epsilon}(x')\right|^2\right),$$

we conclude the following

$$\left|S_{3,1}\right| \leq \left\|\phi\right\|_\alpha C(\epsilon)\, d \int_{d<|\zeta'|<M} \frac{\left(b^2+\left|\zeta'\right|^2\ \left|\nabla\psi_{1,\epsilon}(x')\right|^2\right)\left|\zeta'\right|^\alpha}{\left(|\zeta'|^2+b^2\right)\left(|\zeta'|^2+\hat{b}^2\right)^{\frac{n}{2}}}\, d\zeta'.$$

Thus we approach the desired bound, *that is*, we have

$$\left|S_{3,1}\right| \leq \left\|\phi\right\|_\alpha \left(1+\left\|\nabla\psi_{1,\epsilon}\right\|_\infty^2\right) C(\epsilon)\, d \int_{d<|\zeta'|<M} \left|\zeta'\right|^{\alpha-n}\ d\zeta'.$$

That is, we have proved that

$$\left|S_{3,1}\right| \leq C(n,\alpha) \left\|\phi\right\|_\alpha \left(1+\left\|\nabla\psi_{1,\epsilon}\right\|_\infty^2\right) C(\epsilon)\, d^\alpha. \tag{4.21}$$

We easily bound $S_{3,2}$ as follows. We have

$$S_{3,2} = C \left\|\phi\right\|_\alpha \ d^\alpha \int_{d<|\zeta'|<M} \frac{\left|\zeta'\right|^{\alpha+1} \left\|\nabla\psi_{1,\epsilon}\right\|_\alpha}{\left(|\zeta'|^2+\hat{b}^2\right)^{\frac{n}{2}}} \, d\zeta'.$$

Trivially, we have the bound

$$\left|S_{3,2}\right| \leq C \left\|\phi\right\|_\alpha \ \left\|\nabla\psi_{1,\epsilon}\right\|_\alpha \ d^\alpha \int_{d<|\zeta'|<M} \left|\zeta'\right|^{\alpha+1-n} \ d\zeta'.$$

There fore we have

$$\left|S_{3,2}\right| \leq C(M,\alpha) \left\|\phi\right\|_\alpha C(\epsilon) \ d^\alpha. \tag{4.22}$$

For the last term $S_{3,3}$ we use the bound (4) from Lemma 4.4 to get

$$\left|S_{3,3}\right| \leq C \left\|\phi\right\|_\alpha C(\epsilon) \ d \int_{d<|\zeta'|<M} \left|\zeta'\right|^{\alpha-n} \ d\zeta'.$$

Thus

$$\left|S_{3,3}\right| \leq C(M,\alpha) \left\|\phi\right\|_\alpha C(\epsilon) d^\alpha. \tag{4.23}$$

Combining (4.21), (4.22) and (4.23) , the desired estimate for S_3 follows, *that is*, we get

$$|S_3| \leq C(n,M,\alpha) \left\|\phi\right\|_\alpha C(\epsilon) \ d^\alpha. \tag{4.24}$$

Observe that S_4 has the same bound as S_3 does because the quantities $\left|b-\hat{b}\right|$ and $|a-\hat{a}|$ have the same upper bound which is $d\big(\left\|\nabla\psi_{2,\epsilon}\right\|_\alpha + \left\|\nabla\psi_{2,\epsilon}\right\|_\alpha \big)$. Therefore we gather all the above uniform bounds to end with the desired uniform bound for R_2 which is

$$|R_2| \leq C(n, M, \alpha) \, \|\phi\|_{\alpha} \, C(\epsilon) \, d^{\alpha}. \tag{4.25}$$

4.3.2 Boundedness of R_3.

Even we are working in higher dimensions $n \geq 2$, we form the same decomposition that has been presented for the dimension $n = 2$ [1] but we need some technical modification to deal with the exponent $\frac{n}{2}$. For the term

$$R_3 = \int_{|\zeta'|<M} \left(\mathbf{k}_2^{\epsilon,\delta}(\zeta', x') - \mathbf{k}_2^{\epsilon,\delta}(\zeta', y')\right) \phi(y') \, d\zeta'.$$

We form the following

$$\begin{aligned}
R_3 &= \phi(y') \int_{|\zeta'|<M} \left(\frac{b - \zeta' \cdot \nabla\psi_{2,\epsilon}(\zeta' + x')}{\left(|\zeta'|^2 + b^2\right)^{\frac{n}{2}}} - \frac{\hat{b} - \zeta' \cdot \nabla\psi_{2,\epsilon}(\zeta' + y')}{\left(|\zeta'|^2 + \hat{b}^2\right)^{\frac{n}{2}}} \right) d\zeta' := \phi(y') \, T_1 \\
&+ \phi(y') \int_{|\zeta'|<M} \left(\frac{\hat{a}}{\left(|\zeta'|^2 + \hat{a}^2\right)^{\frac{n}{2}}} - \frac{a}{\left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}} \right) d\zeta' := \phi(y') \, T_2 \\
&+ \phi(y') \int_{|\zeta'|<M} \left(\frac{\zeta' \cdot \nabla\psi_{2,\epsilon}(\zeta' + x') - \zeta' \cdot \nabla\psi_{1,\epsilon}(x')}{\left(|\zeta'|^2 + b^2\right)^{\frac{n}{2}}} - \frac{\zeta' \cdot \nabla\psi_{2,\epsilon}(\zeta' + y') - \zeta' \cdot \nabla\psi_{1,\epsilon}(y')}{\left(|\zeta'|^2 + \hat{b}^2\right)^{\frac{n}{2}}} \right) d\zeta' := \phi(y') T_3 \\
&+ \phi(y') \int_{|\zeta'|<M} \left(\frac{\zeta' \cdot \nabla\psi_{1,\epsilon}(x')}{\left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}} - \frac{\zeta' \cdot \nabla\psi_{1,\epsilon}(y')}{\left(|\zeta'|^2 + \hat{a}^2\right)^{\frac{n}{2}}} \right) d\zeta' := 0.
\end{aligned}$$

Notice that for the term T_1 we have used $\zeta' \cdot \nabla\psi_{2,\epsilon}(\zeta' + .)$ instead of $\zeta' \cdot \nabla\psi_{1,\epsilon}(.)$ in order to obtain an exact differential form as you will see later. Also notice that the last integral is vanishing. We start with the most technical term which is T_1, that is, we begin with

$$T_1 = \int_{|\zeta'|<M} \left(\frac{b - \zeta' \cdot \nabla\psi_{2,\epsilon}(\zeta' + x')}{\left(|\zeta'|^2 + b^2\right)^{\frac{n}{2}}} - \frac{\hat{b} - \zeta' \cdot \nabla\psi_{2,\epsilon}(\zeta' + y')}{\left(|\zeta'|^2 + \hat{b}^2\right)^{\frac{n}{2}}} \right) d\zeta'.$$

We remark here that the work for this term is depending on the dimension n. Since for the dimension $n = 2$ it has already proven in [1] that $|T_1| \leq C(\epsilon)\, d$, so we restrict our work for $n \geq 3$. Now let us bound T_1 uniformly when $n = 3$. For such case we have

$$T_1 = \int_{|\zeta'|<M} \left(\frac{b - \zeta' \cdot \nabla\psi_{2,\epsilon}(\zeta' + x')}{\left(|\zeta'|^2 + b^2\right)^{\frac{3}{2}}} - \frac{\hat{b} - \zeta' \cdot \nabla\psi_{2,\epsilon}(\zeta' + y')}{\left(|\zeta'|^2 + \hat{b}^2\right)^{\frac{3}{2}}} \right) d\zeta'.$$

Since $b = \delta + \psi_{2,\epsilon}(\zeta' + x') - \psi_{1,\epsilon}(x')$, then $\nabla_{\zeta'}\, b = \nabla\psi_{2,\epsilon}$, where $\nabla_{\zeta'}$ means that the gradient is taken with respect to the variable ζ'. By utilizing the polar coordinates $\zeta_1 = r\cos\theta$ and $\zeta_2 = r\sin\theta$, then we clearly see that $r\;\partial_r b = \zeta' \cdot \nabla_{\zeta'}\, b$. Thus it follows that

$$\begin{aligned}
\int_{|\zeta'|<M} \frac{b - \zeta' \cdot \nabla_{\zeta'}\, b}{\left(|\zeta'|^2 + b^2\right)^{\frac{3}{2}}}\, d\zeta' &= \int_0^{2\pi}\int_0^M \frac{\big(b(r,\theta,x') - r\;\partial_r\, b\big)\, r}{\left(r^2 + b^2(r,\theta,x')\right)^{\frac{3}{2}}}\, drd\theta \\
&= \int_0^{2\pi} \frac{-b(r,\theta,x')}{\sqrt{r^2 + b^2(r,\theta,x')}}\, \Big|_{r=0}^{r=M}\;\; d\theta.
\end{aligned}$$

Noticing that $b(0,\theta,x') = \delta + \psi_{2,\epsilon}(x') - \psi_{1,\epsilon}(x') > 0$, therefore we have

$$\int_{|\zeta'|<M} \frac{b - \zeta' \cdot \nabla_{\zeta'}\, b}{\left(|\zeta'|^2 + b^2\right)^{\frac{3}{2}}}\, d\zeta' = \int_0^{2\pi} \frac{-b(M,\theta,x')}{\sqrt{M^2 + b^2(M,\theta,x')}}\, d\theta + 2\pi. \tag{4.26}$$

Similarly we have

$$\int_{|\zeta'|<M} \frac{\hat{b} - \zeta' \cdot \nabla_{\zeta'} \hat{b}}{\left(|\zeta'|^2 + \hat{b}^2\right)^{\frac{3}{2}}} \, d\zeta' = \int_0^{2\pi} \frac{-b(M,\theta,y')}{\sqrt{M^2 + b^2(M,\theta,y')}} \, d\theta + 2\pi. \tag{4.27}$$

Using (4.26) and (4.27), we have the following

$$|T_1| \leq \int_0^{2\pi} \left| \frac{-b(M,\theta,x')}{\sqrt{M^2 + b^2(M,\theta,x')}} + \frac{b(M,\theta,y')}{\sqrt{M^2 + b^2(M,\theta,y')}} \right| \, d\theta.$$

We use the mean value theorem for some ξ' lies on the line segment that joining x' to y' to obtain the following

$$|T_1| \leq \int_0^{2\pi} \left| \frac{M^2 \, \nabla_{\xi'} \, b(M,\theta,\xi')}{\left(M^2 + b^2(M,\theta,\xi')\right)^{\frac{3}{2}}} \right| \, \left|x' - y'\right| \, d\theta. \tag{4.28}$$

Since

$$\nabla_{\xi'} \, b(M,\theta,\xi') = \nabla\psi_{2,\epsilon}(M,\theta,\xi') - \nabla\psi_{1,\epsilon}(\xi'),$$

thus we have

$$|T_1| \leq \frac{C}{M} \int_0^{2\pi} \left(\left\|\nabla\psi_{2,\epsilon}\right\|_\alpha + \left\|\nabla\psi_{1,\epsilon}\right\|_\alpha\right) \left|x' - y'\right| \, d\theta.$$

That is for dimension $n = 3$ we have the desired bound which is

$$|T_1| \leq C(M) \, C(\epsilon) \, d^\alpha. \tag{4.29}$$

We continue to find the uniform boundedness for T_1 in higher dimensions $n > 3$. We use spherical coordinates in dimension $n-1$ to get the following

$$T_1 = \int_{|\zeta'|<M} f(\zeta')\, d\zeta' = \int_{\theta_1=0}^{\pi} \cdots \int_{\theta_{n-3}=0}^{\pi} \int_{\theta_{n-2}=0}^{2\pi} \int_0^M f(\zeta'(r,\Theta))\, J(n,r,\Theta)\, dr\, d\Theta,$$

where

$$f(\zeta'(r,\Theta)) = \frac{\left(b - \zeta' \cdot \nabla_{\zeta'} b\right)}{\left(|\zeta'|^2 + b^2\right)^{\frac{n}{2}}} = \frac{\Big(b(r,\Theta,x') - \partial_r\, b(r,\Theta,x')\Big)}{\Big(|\zeta'|^2 + b^2(r,\Theta,x')\Big)^{\frac{n}{2}}},$$

$$J(n,r,\Theta) = r^{n-2}(\sin\theta_1)^{n-3} \ldots (\sin\theta_{n-4})^2\, (\sin\theta_{n-3}),$$

and

$$\Theta = (\theta_1, \cdots, \theta_{n-2}).$$

We use integration by parts to compute the following

$$\int_0^M \frac{(b - r\,\partial_r\, b)\, r^{n-2}}{\left(r^2+b^2\right)^{\frac{n}{2}}}\, dr = \int_0^M \frac{(b - r\,\partial_r\, b)\, r}{\left(r^2+b^2\right)^{\frac{3}{2}}}\, \frac{r^{n-3}}{\left(r^2+b^2\right)^{\frac{n-3}{2}}}\, dr. \tag{4.30}$$

Let

$$A_1(r) := A_1(r,\Theta,x') = \frac{r^{n-3}}{\left(r^2+b^2\right)^{\frac{n-3}{2}}}.$$

Notice that $b(0,\Theta,x') = \delta + \psi_{2,\epsilon}(x') - \psi_{1,\epsilon}(x') > 0$ and $A_1(0) = 0$. Thus by integration by parts, it follows that

$$\int_0^M \frac{(b - r\,\partial_r\, b)\, r}{\left(r^2+b^2\right)^{\frac{3}{2}}}\, A_1(r)\, dr = \frac{-A_1(M)\, b(M,\Theta,x')}{\sqrt{M^2 + b^2(M,\Theta,x')}} + \int_0^M \frac{b}{\sqrt{r^2+b^2}}\, A_1'(r)\, dr, \tag{4.31}$$

where

$$A_1'(r) = (n-3)\left(\frac{r^{n-4}}{\left(r^2+b^2\right)^{\frac{n-3}{2}}} - \frac{r^{n-3}\,(r + b\,\partial_r b)}{\left(r^2+b^2\right)^{\frac{n-1}{2}}}\right) \tag{4.32}$$

We treat the last integral as follows

$$\begin{aligned}
&\int_0^M \frac{b}{\sqrt{r^2+b^2}}\,A_1'(r)\,dr = (n-3)\int_0^M \frac{b\,r^{n-4}}{\left(r^2+b^2\right)^{\frac{n-2}{2}}} - \frac{b\,r^{n-2}}{\left(r^2+b^2\right)^{\frac{n}{2}}} - \frac{b^2\,\partial_r b\,r^{n-3}}{\left(r^2+b^2\right)^{\frac{n}{2}}}\,dr\\
&= (n-3)\int_0^M \frac{b\,r^{n-4}}{\left(r^2+b^2\right)^{\frac{n-2}{2}}} - \frac{(b - r\,\partial_r b)\,r^{n-2}}{\left(r^2+b^2\right)^{\frac{n}{2}}} - \frac{(b^2+r^2)\,\partial_r b\,r^{n-3}}{\left(r^2+b^2\right)^{\frac{n}{2}}}\,dr\\
&= (n-3)\int_0^M \frac{(b - r\,\partial_r b)\,r^{n-4}}{\left(r^2+b^2\right)^{\frac{n-2}{2}}} - \frac{(b - r\,\partial_r b)\,r^{n-2}}{\left(r^2+b^2\right)^{\frac{n}{2}}}\,dr.
\end{aligned} \tag{4.33}$$

Substituting (4.33) and (4.31) in (4.30), it follows that

$$\int_0^M \frac{(b - r\,\partial_r\,b)\,r^{n-2}}{\left(r^2+b^2\right)^{\frac{n}{2}}}\,dr = \left(\frac{-1}{n-2}\right)\frac{A_1(M)b(M,\Theta,x')}{\sqrt{M^2+b^2(M,\Theta,x')}} + \left(\frac{n-3}{n-2}\right)\int_0^M \frac{(b - r\,\partial_r\,b)\,r^{n-4}}{\left(r^2+b^2\right)^{\frac{n-2}{2}}}\,dr. \tag{4.34}$$

Repeating the same process $\frac{n-2}{2}$ times if n is even and $\frac{n-3}{2}$ times if n is odd, we see that

$$\int_0^M \frac{(b - r\,\partial_r\,b)\,r^{n-2}}{\left(r^2+b^2\right)^{\frac{n}{2}}}\,dr = \begin{cases} \sum_{j=1}^{\frac{n-3}{2}} C_j(n)\;B_j(M) + \tilde{C}_o(n)\int_0^M \frac{(b - r\,\partial_r\,b)\,r}{\left(r^2+b^2\right)^{\frac{3}{2}}}\,dr & \text{if } n \text{ is odd,}\\[2ex] \sum_{j=1}^{\frac{n-2}{2}} C_j(n)\;B_j(M) + \tilde{C}_e(n)\int_0^M \frac{(b - r\,\partial_r\,b)}{r^2+b^2}\,dr & \text{if } n \text{ is even,} \end{cases} \tag{4.35}$$

where

$$B_j(M) := B_j(M,\Theta,x') = \frac{A_j(M,\Theta,x')b(M,\Theta,x')}{\sqrt{M^2+b^2(M,\Theta,x')}}, \tag{4.36}$$

$$A_j(r) = \frac{r^{n-(2j+1)}}{\left(r^2+b^2\right)^{\frac{n-(2j+1)}{2}}}, \tag{4.37}$$

$$\tilde{C}_o(n) = \prod_{j=1}^{\frac{n-3}{2}} \frac{n-(2j+1)}{n-2j}, \tag{4.38}$$

$$\tilde{C}_e(n) = \prod_{j=1}^{\frac{n-2}{2}} \frac{n-(2j+1)}{n-2j}, \tag{4.39}$$

and

$$\tilde{C}_j(n) = \left(\frac{-1}{n-2j}\right) \prod_{i=1}^{j-1} \frac{n-(2i+1)}{n-2i}. \tag{4.40}$$

To estimate T_1 it is enough to bound $\left|B_j(M,\Theta,x') - B_j(M,\Theta,y')\right|$ because the term involving the integral $\int_0^M \frac{(b - r\,\partial_r\, b)\,r}{\left(r^2+b^2\right)^{\frac{3}{2}}}\,dr$ is treated in the case $n=3$ and also the term involving the integral $\int_0^M \frac{(b - r\,\partial_r\, b)}{r^2+b^2}\,dr$ is treated in the case $n=2$ [1]. Now we bound the term $\left|B_j(M,\Theta,x') - B_j(M,\Theta,y')\right|$

$$\left|B_j(x') - B_j(y')\right| = M^{n-(2j+1)} \left| \frac{b(x')}{\left(M^2+b^2(x')\right)^{(\frac{n-2j}{2})}} - \frac{b(y')}{\left(M^2+b^2(y')\right)^{(\frac{n-2j}{2})}} \right|,$$

where $b(x') := b(M,\Theta,x')$. Using the mean value theorem, it follows that

$$\left|B_j(x') - B_j(y')\right| \le M^{n-(2j+1)} \left| \frac{\nabla b(\xi')}{\left(M^2+b^2(\xi')\right)^{(\frac{n-2j}{2})}} - \frac{(n-2j)b^2(\xi')\nabla b(\xi')}{\left(M^2+b^2(\xi')\right)^{(\frac{n-2j+2}{2})}} \right| \left|x'-y'\right|,$$

for some ξ' lies on the line segment joining x' to y'. Thus we have

$$\begin{aligned}\left|B_j(x') - B_j(y')\right| &\leq C(n,j,M)\left(1 + b^2(\xi')\right)\left|\nabla b(\xi')\right|\ \left|x' - y'\right| \\ &\leq C(n,j,M)\ C(\epsilon)\ d^{\alpha}. \qquad (4.41)\end{aligned}$$

Therefore from all above estimates, we have the desired bound for T_1, *that is*,

$$|T_1| \leq C(n,M)\ C(\epsilon)\ d^{\alpha}. \qquad (4.42)$$

If we proceed in the same fashion as we did for T_1 and replace b by a, $\hat{b}$ by $\hat{a}$ and noticing that a and $\hat{a}$ are independent of ζ', we would bound T_2 by the same bound that we found for T_1. That is,

$$|T_2| \leq C(n,M)\ C(\epsilon)\ d^{\alpha}. \qquad (4.43)$$

For the last integral T_3 we have

$$T_3 = \int_{|\zeta'|<M}\left(\frac{\zeta' \cdot \nabla\psi_{2,\epsilon}(\zeta' + x') - \zeta' \cdot \nabla\psi_{1,\epsilon}(x')}{\left(|\zeta'|^2 + b^2\right)^{\frac{n}{2}}} - \frac{\zeta' \cdot \nabla\psi_{2,\epsilon}(\zeta' + y') - \zeta' \cdot \nabla\psi_{1,\epsilon}(y')}{\left(|\zeta'|^2 + \hat{b}^2\right)^{\frac{n}{2}}}\right)d\zeta'$$

We write T_3 as

$$T_3 = T_{3,1} + T_{3,2} + T_{3,3} + T_{3,4},$$

where

$$T_{3,1} = \int_{d<|\zeta'|<M}\frac{\zeta' \cdot \left(\nabla\psi_{2,\epsilon}(\zeta' + x') - \nabla\psi_{1,\epsilon}(x') - \nabla\psi_{2,\epsilon}(\zeta' + y') + \nabla\psi_{1,\epsilon}(y')\right)}{\left(|\zeta'|^2 + b^2\right)^{\frac{n}{2}}}\ d\zeta',$$

$$T_{3,2} = \int_{d<|\zeta'|<M} \zeta' \cdot \left(\nabla\psi_{2,\epsilon}(\zeta'+y') - \nabla\psi_{1,\epsilon}(y')\right)\left(\frac{1}{\left(|\zeta'|^2+b^2\right)^{\frac{n}{2}}} - \frac{1}{\left(|\zeta'|^2+\hat{b}^2\right)^{\frac{n}{2}}}\right) d\zeta',$$

$$T_{3,3} = \int_{|\zeta'|<d} \frac{\zeta' \cdot \left(\nabla\psi_{2,\epsilon}(\zeta'+x') - \nabla\psi_{1,\epsilon}(x')\right)}{\left(|\zeta'|^2+b^2\right)^{\frac{n}{2}}}\, d\zeta',$$

and

$$T_{3,4} = \int_{|\zeta'|<d} \frac{\zeta' \cdot \left(\nabla\psi_{1,\epsilon}(y') - \nabla\psi_{2,\epsilon}(\zeta'+y')\right)}{\left(|\zeta'|^2+\hat{b}^2\right)^{\frac{n}{2}}}\, d\zeta'.$$

We can easily bound $T_{3,1}$ by

$$\left|T_{3,1}\right| \le d^{\alpha}\left(\left\|\nabla\psi_{1,\epsilon}\right\|_{\alpha} + \left\|\nabla\psi_{2,\epsilon}\right\|_{\alpha}\right)\int_{d<|\zeta'|<M} \frac{|\zeta'|}{\left(|\zeta'|^2+b^2\right)^{\frac{n}{2}}}\, d\zeta'.$$

For any $\alpha' < \alpha < \alpha_0$, we would have

$$\left|T_{3,1}\right| \le d^{\alpha'} C(\epsilon)\int_{d<|\zeta'|<M} \frac{\left|\zeta'\right|^{1+(\alpha-\alpha')}}{\left(|\zeta'|^2+b^2\right)^{\frac{n}{2}}}\, d\zeta'.$$

Thus by using polar coordinates, we obtain the following

$$\left|T_{3,1}\right| \le C\ d^{\alpha'}\ C(\epsilon)\int_d^M \frac{r^{(\alpha-\alpha')+n-1}}{\left(r^2+b^2\right)^{\frac{n}{2}}}\, dr.$$

Therefore we have the bound

$$\left|T_{3,1}\right| \le C(\alpha,\alpha',M)\ d^{\alpha'}\ C(\epsilon). \tag{4.44}$$

For $T_{3,2}$ we use the mean value theorem for some z^2 that lies on the line segment joining $\hat{b}^2$

to b^2 to obtain the following

$$\left|T_{3,2}\right| \leq C \int_{d<|\zeta'|<M} \left|\nabla\psi_{2,\epsilon}(\zeta'+y') - \nabla\psi_{1,\epsilon}(y')\right| \frac{\left(\left|\zeta'\right|^2 + z^2\right)^{\frac{n}{2}-1} \left|b-\hat{b}\right| \; \left|\zeta'\right| \left|b+\hat{b}\right|}{\left(|\zeta'|^2 + b^2\right)^{\frac{n}{2}} \left(|\zeta'|^2 + \hat{b}^2\right)^{\frac{n}{2}}} \, d\zeta'.$$

Let first assume that $\hat{b}^2 < z^2 < b^2$. Then we clearly see that

$$\left|\zeta'\right| \left|b+\hat{b}\right| \leq 2\big(\left|\zeta'\right|^2 + b^2\big).$$

Therefore we have

$$\left|T_{3,2}\right| \leq C \int_{d<|\zeta'|<M} \left|\nabla\psi_{2,\epsilon}(\zeta'+y') - \nabla\psi_{1,\epsilon}(y')\right| \frac{\left|b-\hat{b}\right|}{\left(|\zeta'|^2 + \hat{b}^2\right)^{\frac{n}{2}}}.$$

Using the bound (5) from Lemma 4.4, we obtain the following

$$\begin{aligned}
\left|T_{3,2}\right| \leq C \int_{d<|\zeta'|<M} & \left|\nabla\psi_{2,\epsilon}(\zeta'+y') - \nabla\psi_{1,\epsilon}(y')\right| \frac{\left(\left\|\nabla\psi_{2,\epsilon}\right\|_\alpha + \left\|\nabla\psi_{1,\epsilon}\right\|_\alpha \right) d^{1+\alpha}}{\left(|\zeta'|^2 + \hat{b}^2\right)^{\frac{n}{2}}} \, d\zeta' := \mathcal{T}_{3,2,1} \\
+ C \int_{d<|\zeta'|<M} & \left|\nabla\psi_{2,\epsilon}(\zeta'+y') - \nabla\psi_{1,\epsilon}(y')\right| \frac{\left(\left|\nabla\psi_{2,\epsilon}(\zeta'+y')\right| + \left|\nabla\psi_{1,\epsilon}(y')\right| \right) d}{\left(|\zeta'|^2 + \hat{b}^2\right)^{\frac{n}{2}}} \, d\zeta' := \mathcal{T}_{3,2,2}.
\end{aligned}$$

The first integral $\mathcal{T}_{3,2,1}$ follows easily, *that is*,

$$\mathcal{T}_{3,2,1} \leq C(\epsilon)\, d^\alpha. \tag{4.45}$$

For the second integral $\mathcal{T}_{3,2,2}$, we have the bound

$$\mathcal{T}_{3,2,2} \leq C\ d \int_{d<|\zeta'|<M} \frac{\left|\nabla\psi_{2,\epsilon}(\zeta'+y')\right|^2 + \left|\nabla\psi_{1,\epsilon}(y')\right|^2}{\left(|\zeta'|^2+\hat{b}^2\right)^{\frac{n}{2}}}\ d\zeta'.$$

Then we apply Lemma 4.3 for some $0<\mu<1$ to bound the denominator and Lemma 4.2 to bound the numerator, it follows that

$$\mathcal{T}_{3,2,2} \leq C\ d \int_{d<|\zeta'|<M} \frac{\left|\psi_{2,\epsilon}(\zeta'+y')\right|^{\frac{2\alpha}{1+\alpha}}}{|\zeta'|^{(1+\mu)\frac{n}{2}}\ \hat{b}^{(1-\mu)\frac{n}{2}}}\ d\zeta' + C\ d \int_{d<|\zeta'|<M} \frac{\left|\psi_{1,\epsilon}(y')\right|^{\frac{2\alpha}{1+\alpha}}}{|\zeta'|^{(1+\mu)\frac{n}{2}}\ \hat{b}^{(1-\mu)\frac{n}{2}}}\ d\zeta'.$$

Choosing $\mu = 1-\frac{2\alpha}{n}$ in both integrals and noticing that $\left|\hat{b}\right| \geq \left|\psi_{2,\epsilon}(\zeta'+y')\right|$ as well as $\left|\hat{b}\right| \geq \left|\psi_{1,\epsilon}(y')\right|$, we get the following bound

$$\mathcal{T}_{3,2,2} \leq C\ d \int_{d<|\zeta'|<M} \frac{\left|\psi_{2,\epsilon}(\zeta'+y')\right|^{\frac{\alpha(1-\alpha)}{1+\alpha}}}{|\zeta'|^{n-\alpha}}\ d\zeta' + C\ d \int_{d<|\zeta'|<M} \frac{\left|\psi_{1,\epsilon}(y')\right|^{\frac{\alpha(1-\alpha)}{1+\alpha}}}{|\zeta'|^{n-\alpha}}\ d\zeta'.$$

Thus we obtain the bound

$$\mathcal{T}_{3,2,1} \leq C(\epsilon)\ d^{\alpha}. \tag{4.46}$$

Then the desired bound for $T_{3,2}$ follows when $\hat{b}^2 < z^2 < b^2$. That is, we have proved that

$$T_{3,2} \leq C(\epsilon)\ d^{\alpha}. \tag{4.47}$$

While for the case $b^2 < z^2 < \hat{b}^2$ we need some modification for the above work in order to get the bound (4.47). For this case we would have that

$$\left|T_{3,2}\right| \leq C \int_{d<|\zeta'|<M} \left|\nabla\psi_{2,\epsilon}(\zeta'+y') - \nabla\psi_{1,\epsilon}(y')\right| \frac{\left|b-\hat{b}\right|}{\left(|\zeta'|^2+b^2\right)^{\frac{n}{2}}},$$

where $b = \delta + \psi_{2,\epsilon}(\zeta'+x') - \psi_{1,\epsilon}(x')$. We see that the function b depends on the variables ζ' and x' while the functions $\nabla\psi_{2,\epsilon}$ and $\nabla\psi_{1,\epsilon}$ depend on the the variables ζ' and y' so we will perturb the the functions $\nabla\psi_{2,\epsilon}$ and $\nabla\psi_{1,\epsilon}$ in order to match their variables with variables of the function b. For that we offer the following decomposition

$$\left|T_{3,2}\right| \leq \tilde{\mathcal{T}}_{3,2,1} + \tilde{\mathcal{T}}_{3,2,2} + \tilde{\mathcal{T}}_{3,2,3},$$

where

$$\tilde{\mathcal{T}}_{3,2,1} = C \int_{d<|\zeta'|<M} \left|\nabla\psi_{2,\epsilon}(\zeta'+x') - \nabla\psi_{1,\epsilon}(x')\right| \frac{\left|b-\hat{b}\right|}{\left(|\zeta'|^2+b^2\right)^{\frac{n}{2}}},$$

$$\tilde{\mathcal{T}}_{3,2,2} = C \int_{d<|\zeta'|<M} \left|\nabla\psi_{2,\epsilon}(\zeta'+y') - \nabla\psi_{2,\epsilon}(\zeta'+x')\right| \frac{\left|b-\hat{b}\right|}{\left(|\zeta'|^2+b^2\right)^{\frac{n}{2}}},$$

and

$$\tilde{\mathcal{T}}_{3,2,3} = C \int_{d<|\zeta'|<M} \left|\nabla\psi_{1,\epsilon}(x') - \nabla\psi_{1,\epsilon}(y')\right| \frac{\left|b-\hat{b}\right|}{\left(|\zeta'|^2+b^2\right)^{\frac{n}{2}}}.$$

Evidently, we can bound $\tilde{\mathcal{T}}_{3,2,1}$ by $C(\epsilon)\ d^{\alpha}$ if we use the earlier work the for $T_{3,2}$ because we have the alternative bound (6) for $\left|b-\hat{b}\right|$ that involving x' instead of (5) that involving y'. For the integrals $\tilde{\mathcal{T}}_{3,2,2}$ and $\tilde{\mathcal{T}}_{3,2,3}$ we trivially get the desired bound by using the Hölder continuity of the auxiliary functions and the bound (4) for $\left|b-\hat{b}\right|$. That is, the desired bound (4.47) still valid when $b^2 < z^2 < \hat{b}^2$. After finishing $T_{3,2}$, we are ready to bound the last two integrals $T_{3,3}$ and $T_{3,4}$ in the decomposition of the T_3 term. If we look at

$$T_{3,3} = \int_{|\zeta'|<d} \frac{\zeta' \cdot \left(\nabla\psi_{2,\epsilon}(\zeta'+x') - \nabla\psi_{1,\epsilon}(x')\right)}{\left(|\zeta'|^2 + b^2\right)^{\frac{n}{2}}}\, d\zeta',$$

and

$$T_{3,4} = \int_{|\zeta'|<d} \frac{\zeta' \cdot \left(\nabla\psi_{1,\epsilon}(y') - \nabla\psi_{2,\epsilon}(\zeta'+y')\right)}{\left(|\zeta'|^2 + \hat{b}^2\right)^{\frac{n}{2}}}\, d\zeta'.$$

We see that they are both in the same style. So we work to bound $T_{3,3}$ only. On the other hand, the bound for integral $T_{3,4}$ will follow in the same manner. To begin bounding $T_{3,3}$, we add and subtract $\nabla\psi_{2,\epsilon}(x')$ and using the advantage of the integral

$$\int_{\zeta'<d} \frac{\zeta' \cdot \left(\nabla\psi_{1,\epsilon}(x') - \nabla\psi_{2,\epsilon}(x')\right)}{\left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}}\, d\zeta' = 0,$$

to offer the beneficiary decomposition

$$\begin{aligned} T_{3,3} = &\int_{|\zeta'|<d} \frac{\zeta' \cdot \left(\nabla\psi_{2,\epsilon}(\zeta'+x') - \nabla\psi_{2,\epsilon}(x')\right)}{\left(|\zeta'|^2 + b^2\right)^{\frac{n}{2}}}\, d\zeta' := \mathcal{T}_{3,3,1} \\ &+ \int_{|\zeta'|<d} \zeta' \cdot \left(\nabla\psi_{2,\epsilon}(x') - \nabla\psi_{1,\epsilon}(x')\right) \left(\frac{1}{\left(|\zeta'|^2 + b^2\right)^{\frac{n}{2}}} - \frac{1}{\left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}} \right) d\zeta' := \mathcal{T}_{3,3,2}. \end{aligned}$$

Since $\nabla\psi_{2,\epsilon}$ is Hölder continuous, we easily bound $\mathcal{T}_{3,3,1}$ by $C(\epsilon)\, d^{\alpha}$. For the second integral, we use the mean value theorem for some z^2 that lies on the line segment that joining a^2 to b^2 to obtain the following

$$|\mathcal{T}_{3,3,2}| \leq C \int_{|\zeta'|<d} |\zeta'|\; |\nabla\psi_{2,\epsilon}(x') - \nabla\psi_{1,\epsilon}(x')|\; \frac{\left(|\zeta'|^2 + z^2\right)^{\frac{n}{2}-1}\; |b-a|\; |b+a|}{\left(|\zeta'|^2 + b^2\right)^{\frac{n}{2}} \left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}}\, d\zeta'$$

First we assume that $a^2 < z^2 < b^2$. Then by using the bound

$$\left|\zeta'\right| |a+b| \leq 2\big(\left|\zeta'\right|^2 + b^2 \big),$$

it follows that

$$\left|\mathcal{T}_{3,3,2}\right| \leq C \int_{|\zeta'|<d} \left|\nabla\psi_{2,\epsilon}(x') - \nabla\psi_{1,\epsilon}(x')\right| \; \frac{|b-a|}{\Big(|\zeta'|^2 + a^2\Big)^{\frac{n}{2}}} \; d\zeta'.$$

using the bound (2) from Lemma 4.4, we get the bound

$$\left|\mathcal{T}_{3,3,2}\right| \leq C \int_{|\zeta'|<d} \left|\nabla\psi_{2,\epsilon}(x') - \nabla\psi_{1,\epsilon}(x')\right| \; \frac{\left|\nabla\psi_{2,\epsilon}(x')\right| \left|\zeta'\right| + \left\|\nabla\psi_{2,\epsilon}\right\|_\alpha \; \left|\zeta'\right|^{\alpha+1}}{\Big(|\zeta'|^2 + a^2\Big)^{\frac{n}{2}}} \; d\zeta'.$$

Then we easily have the following

$$\begin{aligned}\left|\mathcal{T}_{3,3,2}\right| \leq C(\epsilon) \int_{|\zeta'|<d} \frac{\left|\zeta'\right|^{1+\alpha}}{\Big(|\zeta'|^2 + a^2\Big)^{\frac{n}{2}}} \; d\zeta' + C \int_{|\zeta'|<d} \frac{\left|\nabla\psi_{2,\epsilon}(x')\right|^2 \left|\zeta'\right|}{\Big(|\zeta'|^2 + a^2\Big)^{\frac{n}{2}}} \; d\zeta' \\ + C \int_{|\zeta'|<d} \frac{\left|\nabla\psi_{1,\epsilon}(x')\right|^2 \left|\zeta'\right|}{\Big(|\zeta'|^2 + a^2\Big)^{\frac{n}{2}}} \; d\zeta'.\end{aligned}$$

We clearly see the first integral is bounded by $C(\epsilon)\; d^\alpha$. The second and the third integral will be treated in the same manner. To bound the second integral, we use Lemma 4.3 with the choice $\mu = 1 - \frac{2\alpha}{n}$ to bound the denominator and also we use Lemma 4.2 to bound the nominator. Then the following follows

$$\int_{|\zeta'|<d} \frac{\left|\nabla\psi_{2,\epsilon}(x')\right|^2 \left|\zeta'\right|}{\Big(|\zeta'|^2 + a^2\Big)^{\frac{n}{2}}} \; d\zeta' \leq C \int_{|\zeta'|<d} \; \left|\zeta'\right|^{1+\alpha-n} \; \left|\psi_{2,\epsilon}(x')\right|^{\frac{\alpha(1-\alpha)}{1+\alpha}} \; d\zeta'.$$

Thus the second and the third integral are bounded by $C(\epsilon)\, d^{\alpha}$. Therefore when $b^2 < z^2 < a^2$ we obtain

$$\left|T_{3,3}\right| \leq C(\epsilon)\, d^{\alpha}.$$

Now for the case $b^2 < z^2 < a^2$ which is almost the same work but with some modification. We clearly would have the bound

$$\left|\mathcal{T}_{3,3,2}\right| \leq C \int_{|\zeta'|<d} \left|\nabla\psi_{2,\epsilon}(x') - \nabla\psi_{1,\epsilon}(x')\right| \frac{|b-a|}{\left(|\zeta'|^2 + b^2\right)^{\frac{n}{2}}}\, d\zeta'.$$

using the bound (3) from Lemma 4.4, we get the bound

$$\left|\mathcal{T}_{3,3,2}\right| \leq C \int_{|\zeta'|<d} \left|\nabla\psi_{2,\epsilon}(x') - \nabla\psi_{1,\epsilon}(x')\right| \frac{\left|\nabla\psi_{2,\epsilon}(\zeta'+x')\right| \left|\zeta'\right| + \left\|\nabla\psi_{2,\epsilon}\right\|_{\alpha} \left|\zeta'\right|^{\alpha+1}}{\left(|\zeta'|^2 + b^2\right)^{\frac{n}{2}}}\, d\zeta'.$$

Clearly the term with $\left|\zeta'\right|$ follows trivially. So we just need to bound the integral

$$\tilde{\mathcal{T}}_{3,3,2} = \int_{|\zeta'|<d} \left|\nabla\psi_{2,\epsilon}(x') - \nabla\psi_{1,\epsilon}(x')\right| \frac{\left|\nabla\psi_{2,\epsilon}(\zeta'+x')\right| \left|\zeta'\right|}{\left(|\zeta'|^2 + b^2\right)^{\frac{n}{2}}}\, d\zeta'.$$

We perturb by adding and subtracting the function $\nabla\psi_{2,\epsilon}(\zeta'+x')$ to get the following

$$\begin{aligned}\tilde{\mathcal{T}}_{3,3,2} \leq &\int_{|\zeta'|<d} \left|\nabla\psi_{2,\epsilon}(\zeta'+x') - \nabla\psi_{2,\epsilon}(x')\right| \frac{\left|\nabla\psi_{2,\epsilon}(\zeta'+x')\right| \left|\zeta'\right|}{\left(|\zeta'|^2 + b^2\right)^{\frac{n}{2}}}\, d\zeta' \\ &+ \int_{|\zeta'|<d} \left|\nabla\psi_{2,\epsilon}(\zeta'+x') - \nabla\psi_{1,\epsilon}(x')\right| \frac{\left|\nabla\psi_{2,\epsilon}(\zeta'+x')\right| \left|\zeta'\right|}{\left(|\zeta'|^2 + b^2\right)^{\frac{n}{2}}}\, d\zeta',\end{aligned}$$

then the bound follows trivially. Thus the desired bound for $T_{3,3}$ follows. That is

$$\left|T_{3,3}\right| \leq C(\epsilon)\ d^{\alpha}. \tag{4.48}$$

Finally, we enclose this subsection by combining all the above bounds to get the desired bound for R_3 which is

$$|R_3| \leq C(n, \alpha, \alpha', M)\ \|\phi\|_{\alpha}\ C(\epsilon)\ d^{\alpha'}, \quad \forall\, \alpha' < \alpha < \alpha_0. \tag{4.49}$$

Therefor, at this point, Theorem (4.1) has been proven when $\delta > 0$.

Now we show the validity of Theorem (4.1) under the case $\delta = 0$. That is, we need to show

$$\left\| \frac{\eta}{\omega_n \sqrt{1 + \left|\nabla\psi_{1,\epsilon}\right|^2}}\ J_2^{\epsilon,0} \right\|_{\mathcal{L}(C^{\alpha}(\Gamma_2), C^{\alpha}(\Gamma_1))} \leq C(\epsilon), \quad \forall\, \alpha < \alpha_0, \tag{4.50}$$

where $C(\epsilon)$ approaches 0 when ϵ approaches 0. The bound (4.50) follows if we show that for $\varphi \in C^{\alpha}(\Gamma_2)$, the following limit holds

$$\lim_{\delta \to 0} J_2^{\epsilon,\delta}\varphi(x) = J_2^{\epsilon,0}\varphi(x), \quad x \in \Gamma_1 \cap \mathcal{B}(R).$$

In fact, since $J_2^{\epsilon,\delta}$ is uniformly bounded in operator norm, then it follows that

$$J_2^{\epsilon,0}\varphi(x) \in C^{\alpha}(\Gamma_1) \cap \mathcal{B}(R),$$

and

$$\left\| J_2^{\epsilon,0}\varphi \right\|_{C^{\alpha}(\Gamma_1)} \leq C(\epsilon) \left\| \varphi \right\|_{\alpha}.$$

For $\varphi \in C^{\alpha}(\Gamma_2)$, $0 \neq x \in \Gamma_1 \cap B(0, R_0)$ the operator $J_2^{\epsilon,0}$ is defined by

$$J_2^{\epsilon,0}\varphi(x) = \int_{\mathbb{R}^{n-1}} \mathbf{k}_2^{\epsilon,0}(y', x')\phi(y')\, dy', \tag{4.51}$$

where

$$\begin{aligned}\mathbf{k}_2^{\epsilon,0}(y', x') = &\frac{\big(\psi_{2,\epsilon}(y') - \psi_{1,\epsilon}(x')\big) - \nabla\psi_{1,\epsilon}(x') \cdot (y' - x')}{\Big(|x' - y'|^2 + \big(\psi_{1,\epsilon}(x') - \psi_{2,\epsilon}(y')\big)^2\Big)^{\frac{n}{2}}} \\ &- \frac{\big(\psi_{2,\epsilon}(x') - \psi_{1,\epsilon}(x')\big) - \nabla\psi_{1,\epsilon}(x') \cdot (y' - x')}{\Big(|x' - y'|^2 + \big(\psi_{1,\epsilon}(x') - \psi_{2,\epsilon}(x')\big)^2\Big)^{\frac{n}{2}}}.\end{aligned}$$

Whereas for $x = 0$, the operator $J_2^{\epsilon,0}$ is defined by

$$J_2^{\epsilon,0}\varphi(0) = \int_{\mathbb{R}^{n-1}} \frac{\psi_{2,\epsilon}(y')}{\Big(y_1^2 + \dots + y_{n-1}^2 + \psi_{2,\epsilon}^2(y')\Big)^{\frac{n}{2}}}\, \phi(y')\, dy'. \tag{4.52}$$

After using the change of variables $\zeta' = y' - x'$ in (4.51) and recalling that ϕ has compact support in $\mathcal{B}'(M)$, it follows that

$$J_2^{\epsilon,0}\varphi(x) = \int_{|\zeta'|<M} \mathbf{k}_2^{\epsilon,0}(\zeta', x')\phi(\zeta' + x')\, d\zeta',$$

where $a_0(x') = \psi_{2,\epsilon}(x') - \psi_{1,\epsilon}(x')$, $b_0(x', \zeta') = \psi_{2,\epsilon}(\zeta' + x') - \psi_{1,\epsilon}(x')$ and

$$\mathbf{k}_2^{\epsilon,0}(\zeta', x') = \frac{b_0(x', \zeta') - \zeta' \cdot \nabla\psi_{1,\epsilon}(x')}{\left(|\zeta'|^2 + b_0^2(x', \zeta')\right)^{\frac{n}{2}}} - \frac{a_0(x') - \zeta' \cdot \nabla\psi_{1,\epsilon}(x')}{\left(|\zeta'|^2 + a_0^2(x')\right)^{\frac{n}{2}}}.$$

Proposition 4.5. *Let $\varphi \in C^\alpha(\Gamma_2)$. Then for any $2\epsilon < \epsilon_0$ and $x \in \Gamma_1 \cap \mathcal{B}(R_0)$, we have*

$$\lim_{\delta\to 0} J_2^{\epsilon,\delta}\varphi(x) = J_2^{\epsilon,0}\varphi(x).$$

Proof. For $x \neq 0$, it is clear that $b_0(x') > 0$ and $a_0(x') > 0$. Thus

$$\lim_{\delta\to 0} \mathbf{k}^{\epsilon,\delta}(\zeta', x') = \frac{b_0 - \zeta' \cdot \nabla\psi_{1,\epsilon}(x')}{\left(|\zeta'|^2 + b_0^2\right)^{\frac{n}{2}}} - \frac{a_0 - \zeta' \cdot \nabla\psi_{1,\epsilon}(x')}{\left(|\zeta'|^2 + a_0^2\right)^{\frac{n}{2}}}, \quad \text{a.e} \quad \left|\zeta'\right| < M.$$

Also we notice that the kernel function $\mathbf{k}^{\epsilon,\delta}$ is uniformly bounded because the denominator is bounded away from zero. Therefore by using the dominated convergence theorem, it follows that

$$\lim_{\delta\to 0} J_2^{\epsilon,\delta}\varphi(x) = \int_{|\zeta'|<M} \left(\frac{b_0 - \zeta' \cdot \nabla\psi_{1,\epsilon}(x')}{\left(|\zeta'|^2 + b_0^2\right)^{\frac{n}{2}}} - \frac{a_0 - \zeta' \cdot \nabla\psi_{1,\epsilon}(x')}{\left(|\zeta'|^2 + a_0^2\right)^{\frac{n}{2}}} \right) \phi(\zeta' + x')\, d\zeta'.$$

That is,

$$\lim_{\delta\to 0} J_2^{\epsilon,\delta}\varphi(x) = J_2^{\epsilon,0}\varphi(x), \quad x \neq 0. \tag{4.53}$$

For $x = 0$, we have the following

$$J_2^{\epsilon,\delta}\varphi(0) = \int_{|\zeta'|<M} \left(\frac{\delta + \psi_{2,\epsilon}(\zeta')}{\left(|\zeta'|^2 + \left(\delta + \psi_{2,\epsilon}(\zeta')\right)^2\right)^{\frac{n}{2}}} - \frac{\delta}{\left(|\zeta'|^2 + \delta^2\right)^{\frac{n}{2}}} \right) \phi(\zeta')\, d\zeta'.$$

Let

$$f_\delta(\zeta') = \frac{\delta + \psi_{2,\epsilon}(\zeta')}{\left(|\zeta'|^2 + \left(\delta + \psi_{2,\epsilon}(\zeta')\right)^2\right)^{\frac{n}{2}}} - \frac{\delta}{\left(|\zeta'|^2 + \delta^2\right)^{\frac{n}{2}}},$$

and

$$f_0(\zeta') = \frac{\psi_{2,\epsilon}(\zeta')}{\left(|\zeta'|^2 + \left(\psi_{2,\epsilon}(\zeta')\right)^2\right)^{\frac{n}{2}}}.$$

By the mean value theorem, we clearly see that

$$\left|f_\delta(\zeta')\right| \leq \frac{n\left|\psi_{2,\epsilon}(\zeta')\right|}{|\zeta'|^n},$$

which is integrable on the domain $\{\left|\zeta'\right| < M\}$. Thus by the dominated convergence theorem we obtain the pointwise convergence. That is,

$$\lim_{\delta\to 0} J_2^{\epsilon,\delta}\varphi(0) = \int_{|\zeta'|<M} \frac{\psi_{2,\epsilon}(\zeta')}{\left(|\zeta'|^2 + \psi_{2,\epsilon}^2(\zeta')\right)^{\frac{n}{2}}}\, \phi(\zeta')\, d\zeta' = J_2^{\epsilon,0}\varphi(0).$$

□

4.4 Convergence of $J_2^{\epsilon,\delta}$.

In this Section we study the convergence of the operator $J_2^{\epsilon,\delta}$ as an operator from the Banach space $C^\alpha(\Gamma_2)$ to the Banach space $C^\alpha(\Gamma_1)$ for any $\alpha < \alpha_0$, where α_0 is the regularity of the boundaries Γ_j and α is the regularity of $\nabla\psi_{j,\epsilon}$, $j = 1, 2$. Before we start showing the pointwise convergence, recall the definition of the operator $J_2^{\epsilon,\delta}$, which is

$$J_2^{\epsilon,\delta}\varphi(x) = \int_{|\zeta'|<M} \mathbf{k}_2^{\epsilon,\delta}(\zeta', x')\phi(\zeta' + x')\, d\zeta',$$

where

$$\mathbf{k}^{\epsilon,\delta}(\zeta', x') = \frac{b(\zeta', x') - \zeta' \cdot \nabla\psi_{1,\epsilon}(x')}{\left(|\zeta'|^2 + b^2(\zeta', x')\right)^{\frac{n}{2}}} - \frac{a(x') - \zeta' \cdot \nabla\psi_{1,\epsilon}(x')}{\left(|\zeta'|^2 + a^2(x')\right)^{\frac{n}{2}}},$$

$$b(\zeta', x') = \delta + \psi_{2,\epsilon}(\zeta' + x') - \psi_{1,\epsilon}(x'),$$

$$b_0(\zeta', x') = \psi_{2,\epsilon}(\zeta' + x') - \psi_{1,\epsilon}(x'),$$

$$a(x') = \delta + \psi_{2,\epsilon}(x') - \psi_{1,\epsilon}(x'),$$

and

$$a_0(x') = \psi_{2,\epsilon}(x') - \psi_{1,\epsilon}(x').$$

Theorem 4.6. *[1] Fix $0 < 2\epsilon < \epsilon_0$. Then for any $0 < \alpha' < \alpha < \alpha_0$ and for all $\varphi \in C^\alpha(\Gamma_2)$, we have*

$$\eta J_2^{\epsilon,\delta}\varphi \xrightarrow{\alpha'} \eta J_2^{\epsilon,0}\varphi \ \text{ as } \delta \longrightarrow 0,$$

where $\xrightarrow{\alpha'}$ *means the convergence take place in* $C^{\alpha'}(\Gamma_1)$.

Proof. For $\varphi \in C^{\alpha}$ and $\alpha' < \alpha$, we argue the convergence of $\eta J_2^{\epsilon,\delta}$ by contradiction. If the sequence $\eta J_2^{\epsilon,\delta}\varphi$ does not converge to $\eta J_2^{\epsilon,0}\varphi$ in $C^{\alpha''}(\Gamma_1)$. Then there is a sequence $\{\eta J_2^{\epsilon,\delta n}\varphi\}$ that satisfying

$$\left\|\eta J_2^{\epsilon,\delta n}\varphi - \eta J_{\varphi}^{\epsilon,0}\right\|_{\alpha'} > C. \tag{4.54}$$

for some $C > 0$.

We have shown that $\{\eta J_2^{\epsilon,\delta n}\varphi\}$ is uniformly bounded in $C^{\alpha}(\Gamma_1)$ and since $C^{\alpha}(\Gamma_1)$ is compactly embedded in $C^{\alpha'}(\Gamma_1)$, then there exists a subsequence $\{\eta J_2^{\epsilon,\delta n_j}\varphi\}$ that converges to $\mathfrak{J}_2^{\epsilon} \in C^{\alpha'}(\Gamma_1)$. From Lemma 4.5, we have pointwise convergence for $\eta J_2^{\epsilon,\delta}\varphi_2$. That is

$$\lim_{\delta\to 0} \eta J_2^{\epsilon,\delta}\varphi(x) = \eta J_2^{\epsilon,0}\varphi(x), \quad x \in \Gamma_1 \cap \mathcal{B}(R_0).$$

From the uniqueness of a limit, we have $\mathfrak{J}_2^{\epsilon} = \eta J_2^{\epsilon,0}\varphi_2$ and this would contradict (4.54). Thus

$$\eta J_2^{\epsilon,\delta}\varphi \longrightarrow \eta J_2^{\epsilon,0}\varphi \quad \text{in } \; C^{\alpha'}, \; \alpha' < \alpha.$$

□

Chapter 5

Uniform Hölder continuity for $I_2^{\epsilon,\delta}$.

In this chapter we demonstrate the the uniform Hölder continuity of the operator $\eta I_2^{\epsilon,\delta}$ in the operator norm $\mathcal{L}(C^\alpha(\Gamma_2), C^\alpha(\Gamma_1))$. We will show that the operator $\eta I_2^{\epsilon,\delta}$ in any dimension $n \geq 2$ is uniformly bounded by the quantity $\frac{1}{2}\Big(1 + C(\epsilon)\Big)\,(1+\epsilon_0)$ which is exactly the same quantity that has been found in the earlier work for dimension $n = 2$ [1]. However, if the bound were greater than $\frac{1}{2}\Big(1 + C(\epsilon)\Big)\,(1+\epsilon_0)$, the invertibility of the operator $\Lambda_{\epsilon,\delta}$ that defined in (3.69) may fail. We can see that clearly from the definition of the operator

$$\Lambda_{\epsilon,\delta} = \begin{pmatrix} \lambda I & \frac{\eta}{\omega_n\sqrt{1+\left|\nabla\psi_{1,\epsilon}(x')\right|^2}}\left(J_2^{\epsilon,\delta} + I_2^{\epsilon,\delta}\right) \\ \frac{\eta}{\omega_n\sqrt{1+\left|\nabla\psi_{2,\epsilon}(x')\right|^2}}\left(J_1^{\epsilon,\delta} + I_1^{\epsilon,\delta}\right) & \lambda I \end{pmatrix},$$

where $\lambda = \frac{k+1}{2(k-1)}$ and k is the conductivity in the subdomains, and ϵ_0 has been chosen to satisfy $\frac{1}{2} < \frac{1+\epsilon_0}{2} < |\lambda|$. Thus for the operator $\Lambda_{\epsilon,\delta}$ to be invertible, it suffices that the off-diagonal operators be bounded by a quantity strictly less than $|\lambda|$. The invertibility and the convergence of the operator $\Lambda_{\epsilon,\delta}$ will be fully illustrated in the next chapter. The proof of the uniform Hölder continuity of $I_2^{\epsilon,\delta}$ is similar to but easier than what we have seen for $J_2^{\epsilon,\delta}$ in Chapter 4. We state and prove the main theorem of this chapter,

Theorem 5.1. *Given any $0 < \epsilon < \epsilon_0$ and any $0 \leq \delta < \delta_0$, the operator $\eta I_2^{\epsilon,\delta}$ is a continuous* *linear operator from $C^{\alpha}(\Gamma_2)$ to $C^{\alpha}(\Gamma_1)$, $\alpha < \alpha_0$. Moreover, we have*

$$\left\| \frac{\eta}{\omega_n \sqrt{1 + \left|\nabla \psi_{1,\epsilon}\right|^2}} \, I_2^{\epsilon,\delta} \right\|_{\mathcal{L}(C^{\alpha}(\Gamma_2), C^{\alpha}(\Gamma_1))} \leq \frac{1}{2}\Big(1 + C(\epsilon)\Big)\,(1 + \epsilon_0)\,,$$

where $C(\epsilon)$ converges to 0 as ϵ converges to 0, uniformly in δ.

Remark 5.2. *We use the definition 3.2 to show the uniform bound to the operator $I_2^{\epsilon,\delta}$. That is, for $\varphi \in C^{\alpha}(\Gamma_2)$ and $x, y \in \Gamma_1 \cap \mathcal{B}(\mathcal{R}_0)$, we show that*

$$\max\left(\left\| I_2^{\epsilon,\delta} \varphi \right\|_{\infty}, \sup_{|x|,|y| \leq \mathcal{R}_0} \frac{\left| I_2^{\epsilon,\delta}\varphi(x) - I_2^{\epsilon,\delta}\varphi(y) \right|}{|x - y|^{\alpha}} \right) \leq \frac{1}{2}\Big(1 + C(\epsilon)\Big)\,(1 + \epsilon_0)\,,$$

where

$$\left\| I_2^{\epsilon,\delta} \varphi \right\|_{\infty} = \sup_{|x| \leq \mathcal{R}_0} \left| I_2^{\epsilon,\delta} \varphi(x) \right|.$$

Let $\varphi \in C^{\alpha}(\Gamma_2)$, $x \in \Gamma_1 \cap \mathcal{B}(R_0)$. Recall the definition of $I_2^{\epsilon,\delta}$

$$I_2^{\epsilon,\delta}\varphi(x) = \int_{\mathbb{R}^{n-1}} \frac{\big(\delta + \psi_{2,\epsilon}(x') - \psi_{1,\epsilon}(x')\big) - \nabla\psi_{1,\epsilon}(x') \cdot (y' - x')}{\Big(|x' - y'|^2 + \big(\delta + \psi_{2,\epsilon}(x') - \psi_{1,\epsilon}(x')\big)^2\Big)^{\frac{n}{2}}} \, \phi(y') \, dy',$$

where $\phi(y') = E\varphi(y')\sqrt{1 + \left|\nabla\psi_{2,\epsilon}(y')\right|^2}$ has compact support in $\mathcal{B}'(R_0)$ and bounded by the norm

$$\|\phi\|_{\alpha} \leq \Big(1 + C(\epsilon)\Big)(1 + \epsilon_0)\, \|\varphi\|_{\alpha}\,. \tag{5.1}$$

For $x = (x', x_n) \in \Gamma_1 \cap B(0, R_0)$, we decompose the operator $I_2^{\epsilon,\delta}$ as $\mathcal{I}_2 - \mathcal{I}_1$, where

$$\mathcal{I}_1\varphi(x) = \int_{\mathbb{R}^{n-1}} \frac{\nabla\psi_{1,\epsilon}(x') \cdot (y' - x')}{\Big(|x' - y'|^2 + \big(\delta + \psi_{2,\epsilon}(x') - \psi_{1,\epsilon}(x')\big)^2\Big)^{\frac{n}{2}}} \, \phi(y') \, dy', \tag{5.2}$$

and

$$\mathcal{I}_2\varphi(x) = \int_{\mathbb{R}^{n-1}} \frac{\big(\delta + \psi_{2,\epsilon}(x') - \psi_{1,\epsilon}(x')\big)}{\Big(|x'-y'|^2 + \big(\delta + \psi_{2,\epsilon}(x') - \psi_{1,\epsilon}(x')\big)^2\Big)^{\frac{n}{2}}}\, \phi(y')\, dy'. \tag{5.3}$$

Before we start proving the Theorem , recall from Lemma 3.6 that

$$\int_{\mathbb{R}^{n-1}} \frac{dz'}{\Big(|z'|^2 + 1\Big)^{\frac{n}{2}}} = \frac{\omega_n}{2}. \tag{5.4}$$

We will use it frequently during the proof of the Theorem.

5.1 L^∞-norm for $I_2^{\epsilon,\delta}$, $\delta > 0$.

In this section we elaborate the uniform bound for $I^{\epsilon,\delta}$, when $\delta > 0$. We discuss the uniform bounds for its terms $\mathcal{I}_1$ and $\mathcal{I}_2$ separately. To begin the discussion of the uniform boundedness, we assume that $x = (x', x_n) \in \Gamma_1 \cap B(0, R_0)$.

5.1.1 L^∞-norm for $\mathcal{I}_1$, $\delta > 0$.

Since ϕ has compact support in $\mathcal{B}'(R_0)$, we see that

$$\begin{aligned}
\mathcal{I}_1\varphi(x) &= \int_{\mathbb{R}^{n-1}} \frac{\nabla\psi_{1,\epsilon}(x')\cdot(y'-x')}{\Big(|x'-y'|^2 + \big(\delta + \psi_{2,\epsilon}(x') - \psi_{1,\epsilon}(x')\big)^2\Big)^{\frac{n}{2}}}\, \phi(y')\, dy' \\
&= \int_{\mathcal{B}'(R_0)} \frac{\nabla\psi_{1,\epsilon}(x')\cdot(y'-x')}{\Big(|x'-y'|^2 + \big(\delta + \psi_{2,\epsilon}(x') - \psi_{1,\epsilon}(x')\big)^2\Big)^{\frac{n}{2}}}\, \phi(y')\, dy'.
\end{aligned}$$

Let $\zeta' = y' - x'$, then we have

$$\mathcal{I}_1\varphi(x) = \int_{|\zeta'|<M} \frac{\nabla\psi_{1,\epsilon}(x')\cdot\zeta'}{\left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}}\, \phi(\zeta' + x')\, d\zeta', \tag{5.5}$$

where

$$a := a(x') = \delta + \psi_{2,\epsilon}(x') - \psi_{1,\epsilon}(x').$$

Since

$$\int_{|\zeta'|<M} \frac{\nabla\psi_{1,\epsilon}(x')\cdot\zeta'}{\left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}}\, \phi(x')\, d\zeta' = 0,$$

then clearly we see that

$$\mathcal{I}_1\varphi(x) = \int_{|\zeta'|<M} \frac{\nabla\psi_{1,\epsilon}(x')\cdot\zeta'}{\left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}}\, \left(\phi(\zeta' + x') - \phi(x')\right)\, d\zeta'.$$

Thus we have the bound

$$|\mathcal{I}_1\varphi(x)| \le \left\|\nabla\psi_{1,\epsilon}\right\|_\infty\, \|\phi\|_\alpha \int_{|\zeta'|<M} \frac{\left|\zeta'\right|^{1+\alpha}}{\left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}}\, d\zeta'.$$

Since $a = \delta + \psi_{2,\epsilon}(x') - \psi_{1,\epsilon}(x') > 0$, we obtain

$$\begin{aligned} |\mathcal{I}_1\varphi(x)| &\le \left\|\nabla\psi_{1,\epsilon}\right\|_\alpha\, \|\phi\|_\alpha \int_{|\zeta'|<M} \left|\zeta'\right|^{1+\alpha-n}\, d\zeta' \\ &\le C(M,\alpha)\, C(\epsilon)\, \|\phi\|_\alpha . \end{aligned} \tag{5.6}$$

Therefore from (5.1) and (5.6), we have the bound

$$|\mathcal{I}_1\varphi(x)| \le C(\epsilon)\,\|\varphi\|_\alpha\,. \tag{5.7}$$

5.1.2 L^∞-norm for $\mathcal{I}_2$, $\delta > 0$.

From the definition of $\mathcal{I}_2$, we have

$$\mathcal{I}_2\varphi(x) = \int_{\mathbb{R}^{n-1}} \frac{a}{\left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}}\,\phi(\zeta' + x')\,d\zeta'.$$

By using the change of variable $\zeta' = a\,\varpi$, it follows that

$$\begin{aligned}
|\mathcal{I}_2\varphi(x)| &= \left|\int_{\mathbb{R}^{n-1}} \frac{1}{\left(1 + |\varpi|^2\right)^{\frac{n}{2}}}\,\phi(a\,\varpi + x')\,d\varpi\right| \\
&\le \|\phi\|_\infty \int_{\mathbb{R}^{n-1}} \frac{1}{\left(1 + |\varpi|^2\right)^{\frac{n}{2}}}\,d\varpi = \frac{\omega_n}{2}\,\|\phi\|_\infty\,.
\end{aligned} \tag{5.8}$$

Again from (5.1) and (5.8), we see the uniform bound

$$|\mathcal{I}_2\varphi(x)| \le \frac{\omega_n}{2}\Big(1 + C(\epsilon)\Big)(1+\epsilon_0)\,\|\varphi\|_\alpha\,. \tag{5.9}$$

Finally, we combine (5.7) and (5.9) to get the desired uniform bound for $I^{\epsilon,\delta}$

$$\left\|I^{\epsilon,\delta}\varphi\right\|_\infty \le \frac{\omega_n}{2}\Big(1 + C(\epsilon)\Big)(1+\epsilon_0)\,\|\varphi\|_\alpha\,. \tag{5.10}$$

5.2 Hölder estimate for $I_2^{\epsilon,\delta}$, $\delta > 0$.

For the Hölder continuity, we choose $x = (x', x_n)$ and $\bar{x} = (\bar{x}', x_n') \in \Gamma_1 \cap B(0, R_0)$. As before we use a for the combination of the $C^{1,\alpha}$ functions $\psi_{1,\epsilon}$ and $\psi_{2,\epsilon}$ acting at the point x' with the perturbation constant δ, *that is*,

$$a := a(x') = \delta + \psi_{2,\epsilon}(x') - \psi_{1,\epsilon}(x'),$$

and similarly for $\bar{a}$, but at the point $\bar{x}'$,*that is*,

$$\bar{a} := a(\bar{x}') = \delta + \psi_{2,\epsilon}(\bar{x}') - \psi_{1,\epsilon}(\bar{x}').$$

In order to get the desired bound for $I^{\epsilon,\delta}$, we use the same decomposition for the terms $\mathcal{I}_1$ and $\mathcal{I}_2$ that presented in dimension $n = 2$ [1] with some technical modification.

5.2.1 Hölder estimate for $\mathcal{I}_1$, $\delta > 0$.

From the definition of $\mathcal{I}_1$, we have

$$\mathcal{I}_1\varphi(x) = \int_{|\zeta'|<M} \frac{\nabla\psi_{1,\epsilon}(x') \cdot \zeta'}{\left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}} \phi(\zeta' + x')\, d\zeta',$$

and

$$\mathcal{I}_1\varphi(\bar{x}) = \int_{|\zeta'|<M} \frac{\nabla\psi_{1,\epsilon}(\bar{x}') \cdot \zeta'}{\left(|\zeta'|^2 + \bar{a}^2\right)^{\frac{n}{2}}} \phi(\zeta' + \bar{x}')\, d\zeta'. \tag{5.11}$$

Let $\hbar = x' - \bar{x}'$ and $\zeta' = \zeta_1' + \hbar$ in (5.11), it follows that

$$\mathcal{I}_1\varphi(\bar{x}) = \int_{\left|\zeta_1'+\hbar\right|<M} \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot(\zeta_1'+\hbar)}{\left(\left|\zeta_1'+\hbar\right|^2+\bar{a}^2\right)^{\frac{n}{2}}}\,\phi(\zeta_1'+\hbar+\bar{x}')\,d\zeta_1'. \tag{5.12}$$

Since ϕ has compact support, we have the following

$$\mathcal{I}_1\varphi(\bar{x}) = \int_{\mathbb{R}^{n-1}} \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot(\zeta_1'+\hbar)}{\left(\left|\zeta_1'+\hbar\right|^2+\bar{a}^2\right)^{\frac{n}{2}}}\,\phi(\zeta_1'+x')\,d\zeta_1'. \tag{5.13}$$

Letting $\zeta'=\zeta_1'$ in (5.13), it follows that

$$\mathcal{I}_1\varphi(\bar{x}) = \int_{\mathbb{R}^{n-1}} \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot(\zeta'+\hbar)}{\left(|\zeta'+\hbar|^2+\bar{a}^2\right)^{\frac{n}{2}}}\,\phi(\zeta'+x')\,d\zeta'. \tag{5.14}$$

Now we form the following decomposition

$$\begin{aligned}
\mathcal{I}_1\varphi(x)-\mathcal{I}_1\varphi(\bar{x}) = &\int_{\mathbb{R}^{n-1}} \frac{\left(\nabla\psi_{1,\epsilon}(x')-\nabla\psi_{1,\epsilon}(\bar{x}')\right)\cdot\zeta'}{\left(|\zeta'|^2+a^2\right)^{\frac{n}{2}}}\,\phi(\zeta'+x')\,d\zeta' \\
&+\int_{\mathbb{R}^{n-1}} \left(\frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot\zeta'}{\left(|\zeta'|^2+a^2\right)^{\frac{n}{2}}} - \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot(\zeta'+\hbar)}{\left(|\zeta'+\hbar|^2+\bar{a}^2\right)^{\frac{n}{2}}}\right)\,\phi(\zeta'+x')\,d\zeta' \\
&:= A+B. && (5.15)
\end{aligned}$$

For A we have

$$A = \int_{\mathbb{R}^{n-1}} \frac{\left(\nabla\psi_{1,\epsilon}(x')-\nabla\psi_{1,\epsilon}(\bar{x}')\right)\cdot\zeta'}{\left(|\zeta'|^2+a^2\right)^{\frac{n}{2}}}\,\phi(\zeta'+x')\,d\zeta'.$$

Since

$$\int_{\mathbb{R}^{n-1}} \frac{\left(\nabla\psi_{1,\epsilon}(x') - \nabla\psi_{1,\epsilon}(\bar{x}')\right)\cdot\zeta'}{\left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}}\,\phi(x')\,d\zeta' = 0,$$

then the following bound follows

$$\begin{aligned}
|A| &\leq \left\|\nabla\psi_{1,\epsilon}\right\|_\alpha\ |\hbar|^\alpha \int_{\mathbb{R}^{n-1}} \frac{|\zeta'|}{\left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}}\ \left|\phi(\zeta' + x') - \phi(x')\right|\ d\zeta' \\
&\leq \left\|\nabla\psi_{1,\epsilon}\right\|_\alpha\ |\hbar|^\alpha\ \|\phi\|_\alpha \int_{|\zeta'|<M} \frac{|\zeta'|^{1+\alpha}}{\left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}}\ d\zeta' \\
&\leq C(M,\alpha)\ C(\epsilon)\,\|\phi\|_\alpha\,|\hbar|^\alpha.
\end{aligned}$$

Using Hölder estimate (5.1) for ϕ, the desired bound for A follows

$$|A| \leq C(\epsilon)\,\|\varphi\|_\alpha\,|\hbar|^\alpha. \tag{5.16}$$

We proceed to find the bound for B, where

$$B = \int_{\mathbb{R}^{n-1}} \left(\frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot\zeta'}{\left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}} - \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot(\zeta' + \hbar)}{\left(|\zeta' + \hbar|^2 + \bar{a}^2\right)^{\frac{n}{2}}} \right)\ \phi(\zeta' + x')\,d\zeta'$$

We decompose B into four terms as follows

$$\begin{aligned}
B &= \int_{|\zeta'|<4|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot\zeta'}{\left(|\zeta'|^2+a^2\right)^{\frac{n}{2}}} \left(\phi(\zeta'+x')-\phi(x')\right) d\zeta' \\
&- \int_{|\zeta'|<4|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot(\zeta'+\hbar)}{\left(|\zeta'+\hbar|^2+\bar{a}^2\right)^{\frac{n}{2}}} \left(\phi(\zeta'+x')-\phi(\bar{x}')\right) d\zeta' \\
&- \int_{|\zeta'|<4|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot(\zeta'+\hbar)}{\left(|\zeta'+\hbar|^2+\bar{a}^2\right)^{\frac{n}{2}}} \left(\phi(\bar{x}')-\phi(x')\right) d\zeta' \\
&+ \int_{|\zeta'|>4|\hbar|} \left(\frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot\zeta'}{\left(|\zeta'|^2+a^2\right)^{\frac{n}{2}}} - \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot(\zeta'+\hbar)}{\left(|\zeta'+\hbar|^2+\bar{a}^2\right)^{\frac{n}{2}}}\right) \left(\phi(\zeta'+x')-\phi(x')\right) d\zeta'.
\end{aligned}$$

Thus we have the decomposition

$$B := B_1 + B_2 + B_3 + B_4,$$

where B_i is the corresponding integral in the above decomposition, $i = 1, 2, 3, 4$. We begin by estimating the first integral B_1, *that is*,

$$B_1 = \int_{|\zeta'|<4|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot\zeta'}{\left(|\zeta'|^2+a^2\right)^{\frac{n}{2}}} \left(\phi(\zeta'+x')-\phi(x')\right) d\zeta'.$$

Using the Hölder continuity of ϕ, we easily have the following bound

$$|B_1| \le \left\|\nabla\psi_{1,\epsilon}\right\|_\alpha \|\phi\|_\alpha \int_{|\zeta'|<4|\hbar|} \left|\zeta'\right|^{1+\alpha-n} \, d\zeta' \le C(\epsilon)\,\|\varphi\|_\alpha\,|\hbar|^\alpha. \tag{5.17}$$

Similarly we can get the appropriate bound for B_2. To show that, we have

$$B_2 = -\int_{|\zeta'|<4|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}') \cdot (\zeta' + \hbar)}{\left(|\zeta' + \hbar|^2 + \bar{a}^2\right)^{\frac{n}{2}}} \left(\phi(\zeta' + x') - \phi(\bar{x}')\right) d\zeta'.$$

Hölder continuity of ϕ allow us to have the bounds

$$\begin{aligned}
|B_2| &\leq \left\|\nabla\psi_{1,\epsilon}\right\|_\alpha \ \|\phi\|_\alpha \int_{|\zeta'|<4|\hbar|} \frac{\left|\zeta' + \hbar\right|}{\left(\left|\zeta' + \hbar\right|^2 + \bar{a}^2\right)^{\frac{n}{2}}} \ \left|\zeta' + \hbar\right|^\alpha \ d\zeta' \\
&\leq C(\epsilon) \ \|\phi\|_\alpha \int_{\left|\zeta'\right|<4|\hbar|} \left|\zeta' + \hbar\right|^{1+\alpha-n} \ d\zeta' \\
&\leq C(\epsilon) \ \|\phi\|_\alpha \int_{\left|\zeta'\right|<5|\hbar|} \left|\zeta'\right|^{1+\alpha-n} \ d\zeta'.
\end{aligned}$$

Using (5.1) and integrating $\left|\zeta'\right|^{1+\alpha-n}$ over $\left|\zeta'\right| < 5\,|\hbar|$, we obtain the desired bound

$$|B| \leq C(\epsilon) \ \|\varphi\|_\alpha \ |\hbar|^\alpha. \tag{5.18}$$

For B_3, we substitute $\zeta' = \zeta' - \hbar$ in order to write B_3 in the following form

$$B_3 = -\int_{|\zeta'-\hbar|<4|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}') \cdot \zeta'}{\left(|\zeta'|^2 + \bar{a}^2\right)^{\frac{n}{2}}} \left(\phi(\bar{x}') - \phi(x')\right) d\zeta'.$$

Since

$$\int_{|\zeta'|<\frac{1}{2}|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}') \cdot \zeta'}{\left(|\zeta'|^2 + \bar{a}^2\right)^{\frac{n}{2}}} \left(\phi(\bar{x}') - \phi(x')\right) d\zeta' = 0,$$

then we have

$$B_3 = -\int_{\{|\zeta' - \hbar| < 4|\hbar| \setminus |\zeta'| < \frac{|\hbar|}{2}\}} \frac{\nabla \psi_{1,\epsilon}(\bar{x}') \cdot \zeta'}{\left(|\zeta'|^2 + \bar{a}^2\right)^{\frac{n}{2}}} \left(\phi(\bar{x}') - \phi(x')\right) d\zeta'.$$

We clearly can bound B_3 as follows

$$\begin{aligned} |B_3| &\leq \left\|\nabla \psi_{1,\epsilon}\right\|_\alpha \, \|\phi\|_\alpha \, |\hbar|^\alpha \int_{\{|\zeta' - \hbar| < 4|\hbar| \setminus |\zeta'| < \frac{|\hbar|}{2}\}} \left|\zeta'\right|^{1-n} \, d\zeta' \\ &\leq \left\|\nabla \psi_{1,\epsilon}\right\|_\alpha \, \|\phi\|_\alpha \, |\hbar|^\alpha \int_{\left(\frac{|\hbar|}{2} < |\zeta'| < 5|\hbar|\right)} \left|\zeta'\right|^{1-n} \, d\zeta'. \end{aligned}$$

Therefore,

$$|B_3| \leq C(\epsilon) \, \|\varphi\|_\alpha \, |\hbar|^\alpha . \tag{5.19}$$

For the last integral B_4, we have

$$B_4 = \int_{|\zeta'| > 4|\hbar|} \left(\frac{\nabla \psi_{1,\epsilon}(\bar{x}') \cdot \zeta'}{\left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}} - \frac{\nabla \psi_{1,\epsilon}(\bar{x}') \cdot (\zeta' + \hbar)}{\left(|\zeta' + \hbar|^2 + \bar{a}^2\right)^{\frac{n}{2}}} \right) \left(\phi(\zeta' + x') - \phi(x')\right) d\zeta'.$$

We work with some manipulations in order to have the desired bound. We start by assuming

$\zeta_1' = \zeta' + \hbar$ in the second integrand to decompose B_4 as follows

$$
\begin{aligned}
B_4 = &- \int_{3|\hbar|<|\zeta'|<4|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot\zeta'}{\left(|\zeta'|^2+a^2\right)^{\frac{n}{2}}} \left(\phi(\zeta'+x')-\phi(x')\right) d\zeta' \\
&+ \int_{|\zeta'|>3|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot\zeta'}{\left(|\zeta'|^2+a^2\right)^{\frac{n}{2}}} \left(\phi(\zeta'+x')-\phi(x')\right) d\zeta' \\
&- \int_{\left|\zeta_1'-\hbar\right|>4|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot\zeta_1'}{\left(\left|\zeta_1'\right|^2+\bar{a}^2\right)^{\frac{n}{2}}} \left(\phi(\zeta_1'-\hbar+x')-\phi(x')\right) d\zeta_1' \\
:= &\, C_1 + C_2, \qquad (5.20)
\end{aligned}
$$

where

$$C_1 = -\int_{3|\hbar|<|\zeta'|<4|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot\zeta'}{\left(|\zeta'|^2+a^2\right)^{\frac{n}{2}}} \left(\phi(\zeta'+x')-\phi(x')\right) d\zeta'$$

and

$$
\begin{aligned}
C_2 = &\int_{|\zeta'|>3|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot\zeta'}{\left(|\zeta'|^2+a^2\right)^{\frac{n}{2}}} \left(\phi(\zeta'+x')-\phi(x')\right) d\zeta' \\
&- \int_{\left|\zeta_1'-\hbar\right|>4|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot\zeta_1'}{\left(\left|\zeta_1'\right|^2+\bar{a}^2\right)^{\frac{n}{2}}} \left(\phi(\zeta_1'-\hbar+x')-\phi(x')\right) d\zeta_1'.
\end{aligned}
$$

We easily can bound C_1 by

$$
\begin{aligned}
C_1 &\leq \left\|\nabla\psi_{1,\epsilon}\right\|_\alpha \, \|\phi\|_\alpha \int_{3|\hbar|<|\zeta'|<4|\hbar|} \left|\zeta'\right|^{1-n+\alpha} \, d\zeta' \\
&\leq C(\epsilon) \, \|\varphi\|_\alpha \, |\hbar|^\alpha . \qquad (5.21)
\end{aligned}
$$

For C_2, recall that $\hbar = x' - \bar{x}'$ then we see that

$$
\begin{aligned}
C_2 = &\int_{|\zeta'|>3|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot\zeta'}{\left(|\zeta'|^2+a^2\right)^{\frac{n}{2}}} \left(\phi(\zeta'+x')-\phi(x')\right) d\zeta' \\
&- \int_{\left|\zeta_1'-\hbar\right|>4|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot\zeta_1'}{\left(\left|\zeta_1'\right|^2+\bar{a}^2\right)^{\frac{n}{2}}} \left(\phi(\zeta_1'+\bar{x}')-\phi(x')\right) d\zeta_1'.
\end{aligned}
$$

We rewrite C_2 in the following form

$$
\begin{aligned}
C_2 = &\int_{|\zeta'|>3|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot\zeta'}{\left(|\zeta'|^2+a^2\right)^{\frac{n}{2}}} \left(\phi(\zeta'+x')-\phi(x')\right) d\zeta' \\
&- \int_{\left|\zeta_1'-\hbar\right|>4|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot\zeta_1'}{\left(\left|\zeta_1'\right|^2+\bar{a}^2\right)^{\frac{n}{2}}} \left(\phi(\zeta_1'+x')-\phi(x')\right) d\zeta_1' \\
&+ \int_{\left|\zeta_1'-\hbar\right|>4|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot\zeta_1'}{\left(\left|\zeta_1'\right|^2+\bar{a}^2\right)^{\frac{n}{2}}} \left(\phi(\zeta_1'+x')-\phi(\zeta_1'+\bar{x}')\right) d\zeta_1' \\
&:= D_1 + D_2,
\end{aligned}
\tag{5.22}
$$

where

$$
\begin{aligned}
D_1 = &\int_{|\zeta'|>3|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot\zeta'}{\left(|\zeta'|^2+a^2\right)^{\frac{n}{2}}} \left(\phi(\zeta'+x')-\phi(x')\right) d\zeta' \\
&- \int_{\left|\zeta_1'-\hbar\right|>4|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot\zeta_1'}{\left(\left|\zeta_1'\right|^2+\bar{a}^2\right)^{\frac{n}{2}}} \left(\phi(\zeta_1'+x')-\phi(x')\right) d\zeta_1'.
\end{aligned}
$$

and

$$
D_2 = \int_{\left|\zeta_1'-\hbar\right|>4|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot\zeta_1'}{\left(\left|\zeta_1'\right|^2+\bar{a}^2\right)^{\frac{n}{2}}} \left(\phi(\zeta_1'+x')-\phi(\zeta_1'+\bar{x}')\right) d\zeta_1',
$$

We start with the easiest term which is D_2 and after utilizing that ϕ is compactly supported in $\mathcal{B}'(M)$, clearly we can see the bounds

$$
\begin{aligned}
|D_2| &\leq \int_{3|\hbar|<\left|\zeta_1'\right|<M} \frac{\left|\nabla\psi_{1,\epsilon}(\bar{x}')\cdot\zeta_1'\right|}{\left(\left|\zeta_1'\right|^2+\bar{a}^2\right)^{\frac{n}{2}}} \left|\phi(\zeta_1'+x')-\phi(\zeta_1'+\bar{x}')\right| \, d\zeta_1' \\
&\leq \|\phi\|_\alpha \; |\hbar|^\alpha \int_{3|\hbar|<\left|\zeta_1'\right|<M} \frac{\left|\psi_{1,\epsilon}(\bar{x}')\right|^{\frac{\alpha}{1+\alpha}} \left|\zeta_1'\right|}{\left(\left|\zeta_1'\right|^2+\bar{a}^2\right)^{\frac{n}{2}}} \, d\zeta_1' \;\; \text{(by using lemma 4.2)} \\
&\leq \|\phi\|_\alpha \; |\hbar|^\alpha \int_{3|\hbar|<\left|\zeta_1'\right|<M} \frac{\left|\psi_{1,\epsilon}(\bar{x}')\right|^{\frac{\alpha}{1+\alpha}} \left|\zeta_1'\right|}{\left|\zeta_1'\right|^{(1+\mu)\frac{n}{2}} \; \bar{a}^{(1-\mu)\frac{n}{2}}} \, d\zeta_1' \;\; \text{(by using lemma 4.3)} \\
&\leq \|\phi\|_\alpha \; |\hbar|^\alpha \int_{3|\hbar|<\left|\zeta_1'\right|<M} \left|\psi_{1,\epsilon}(\bar{x}')\right|^{\frac{\alpha(1-\alpha)}{2(1+\alpha)}} \left|\zeta_1'\right|^{1-n+\frac{\alpha}{2}} \, d\zeta_1' \; \left(\text{by choosing } \mu = 1-\frac{\alpha}{n}\right) \\
&\leq C(\epsilon) \; \|\phi\|_\alpha \; |\hbar|^\alpha \int_{3|\hbar|<\left|\zeta_1'\right|<M} \left|\zeta_1'\right|^{1+\frac{\alpha}{2}-n} \, d\zeta_1' \\
&\leq C(\epsilon) \; \|\phi\|_\alpha \; |\hbar|^\alpha .
\end{aligned}
$$

Therefore we end with the bound

$$
|D_2| \leq C(\epsilon) \, \|\varphi\|_\alpha \, |\hbar|^\alpha . \tag{5.23}
$$

To estimate D_1 we rewrite $\zeta' = \zeta_1'$ in the first integral for D_1 to get the following

$$
\begin{aligned}
D_1 = &\int_{\left|\zeta_1'\right|>3|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot\zeta_1'}{\left(\left|\zeta_1'\right|^2+a^2\right)^{\frac{n}{2}}} \left(\phi(\zeta_1'+x')-\phi(x')\right) d\zeta_1' \\
&- \int_{\left|\zeta_1'-\hbar\right|>4|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot\zeta_1'}{\left(\left|\zeta_1'\right|^2+\bar{a}^2\right)^{\frac{n}{2}}} \left(\phi(\zeta_1'+x')-\phi(x')\right) d\zeta_1'.
\end{aligned}
$$

Now we rewrite D_1 in the following form

$$D_1 = \int_{\left|\zeta_1'\right|>3|\hbar|} \left(\frac{\nabla\psi_{1,\epsilon}(\bar{x}') \cdot \zeta_1'}{\left(\left|\zeta_1'\right|^2 + a^2\right)^{\frac{n}{2}}} - \frac{\nabla\psi_{1,\epsilon}(\bar{x}') \cdot \zeta_1'}{\left(\left|\zeta_1'\right|^2 + \bar{a}^2\right)^{\frac{n}{2}}} \right) \left(\phi(\zeta_1' + x') - \phi(x')\right) d\zeta_1'$$
$$+ \int_{\left|\zeta_1'\right|>3|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}') \cdot \zeta_1'}{\left(\left|\zeta_1'\right|^2 + \bar{a}^2\right)^{\frac{n}{2}}} \left(\phi(\zeta_1' + x') - \phi(x')\right) d\zeta_1'$$
$$- \int_{\left|\zeta_1'-\hbar\right|>4|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}') \cdot \zeta_1'}{\left(\left|\zeta_1'\right|^2 + \bar{a}^2\right)^{\frac{n}{2}}} \left(\phi(\zeta_1' + x') - \phi(x')\right) d\zeta_1'.$$

Let us find out the bound for the last two integrals in D_1 and denote them by $D_{1,2}$. *That is,*

$$D_{1,2} = \int_{\left|\zeta_1'\right|>3|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}') \cdot \zeta_1'}{\left(\left|\zeta_1'\right|^2 + \bar{a}^2\right)^{\frac{n}{2}}} \left(\phi(\zeta_1' + x') - \phi(x')\right) d\zeta_1'$$
$$- \int_{\left|\zeta_1'-\hbar\right|>4|\hbar|} \frac{\nabla\psi_{1,\epsilon}(\bar{x}') \cdot \zeta_1'}{\left(\left|\zeta_1'\right|^2 + \bar{a}^2\right)^{\frac{n}{2}}} \left(\phi(\zeta_1' + x') - \phi(x')\right) d\zeta_1'.$$

Then we easily see that

$$\left|D_{1,2}\right| \leq \int_{3|\hbar|<\left|\zeta_1'\right|<5|\hbar|} \frac{\left|\nabla\psi_{1,\epsilon}(\bar{x}') \cdot \zeta_1'\right|}{\left(\left|\zeta_1'\right|^2 + \bar{a}^2\right)^{\frac{n}{2}}} \left|\phi(\zeta_1' + x') - \phi(x')\right| \, d\zeta_1'$$
$$\leq \left\|\nabla\psi_{1,\epsilon}\right\|_\alpha \, \|\phi\|_\alpha \int_{3|\hbar|<\left|\zeta_1'\right|<5|\hbar|} \left|\zeta_1'\right|^{1+\alpha-n} \, d\zeta_1'.$$

Thus we have the following bound

$$\left|D_{1,2}\right| \leq C(\epsilon) \, \|\phi\|_\alpha \, |\hbar|^\alpha . \tag{5.24}$$

For the first integral in D_1 we need to use the mean value theorem in order to get the desired bound. Let us denote it by $D_{1,1}$ and use ζ' instead of ζ_1' to simplify the writing, *that is*

$$D_{1,1} = \int_{|\zeta'|>3|\hbar|} \left(\frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot\zeta'}{\left(|\zeta'|^2+a^2\right)^{\frac{n}{2}}} - \frac{\nabla\psi_{1,\epsilon}(\bar{x}')\cdot\zeta'}{\left(|\zeta'|^2+\bar{a}^2\right)^{\frac{n}{2}}} \right) \Big(\phi(\zeta'+x')-\phi(x')\Big)\, d\zeta'.$$

In a naive way, we can have the bound

$$\left|D_{1,1}\right| \leq \left\|\nabla\psi_{1,\epsilon}\right\|_\alpha \; \|\phi\|_\alpha \int_{|\zeta'|>3|\hbar|} \left|\zeta'\right|^{1+\alpha} \; \frac{\left|\left(|\zeta'|^2+\bar{a}^2\right)^{\frac{n}{2}} - \left(|\zeta'|^2+a^2\right)^{\frac{n}{2}}\right|}{\left(|\zeta'|^2+a^2\right)^{\frac{n}{2}} \; \left(|\zeta'|^2+\bar{a}^2\right)^{\frac{n}{2}}}. \tag{5.25}$$

Using the mean value theorem for some z^2 that lies between a^2 and $\bar{a}^2$, we have the following

$$\left|D_{1,1}\right| \leq C(n) \; \left\|\nabla\psi_{1,\epsilon}\right\|_\alpha \; \|\phi\|_\alpha \int_{|\zeta'|>3|\hbar|} \left|\zeta'\right|^{1+\alpha} \; \frac{\left(|\zeta'|^2+z^2\right)^{\frac{n}{2}-1} \; |a^2-\bar{a}^2|}{\left(|\zeta'|^2+a^2\right)^{\frac{n}{2}} \; \left(|\zeta'|^2+\bar{a}^2\right)^{\frac{n}{2}}}. \tag{5.26}$$

Without loss of generality we may assume $z^2 \leq a^2$ and same work will hold when $z^2 < \bar{a}^2$. Therefore, we have the bound

$$\left|D_{1,1}\right| \leq C(n)\; C(\epsilon) \; \|\phi\|_\alpha \int_{|\zeta'|>3|\hbar|} \left|\zeta'\right|^{1+\alpha} \; \frac{|a-\bar{a}| \; |a+\bar{a}|}{\left(|\zeta'|^2+a^2\right) \; \left(|\zeta'|^2+\bar{a}^2\right)^{\frac{n}{2}}}. \tag{5.27}$$

Since $\bar{a}^2 < a^2$, we can easily have the inequality

$$\frac{|a+\bar{a}|}{\left(|\zeta'|^2+a^2\right) \; \left(|\zeta'|^2+\bar{a}^2\right)^{\frac{n}{2}}} \leq \frac{1}{|\zeta'|^{1+n}}. \tag{5.28}$$

By substituting inequality (5.28) in (5.27), we have the bound

$$\left|D_{1,1}\right| \leq C(\epsilon)\ \|\phi\|_{\alpha} \int_{\left|\zeta'\right|>3|\hbar|} \left|\zeta'\right|^{\alpha-n}\ |a-\bar{a}|\ d\zeta'. \tag{5.29}$$

Since $a=\delta+\psi_{2,\epsilon}(x')-\psi_{1,\epsilon}(x')$ and $\bar{a}=\delta+\psi_{2,\epsilon}(\bar{x}')-\psi_{1,\epsilon}(\bar{x}')$. Then we easily have the bound

$$|a-\bar{a}| \leq\ |\hbar|\ \left\|\nabla\psi_{2,\epsilon}\right\|_{\alpha}+|\hbar|\ \left\|\nabla\psi_{1,\epsilon}\right\|_{\alpha}$$

Thus we can see that

$$\begin{aligned} \left|D_{1,1}\right| &\leq C(\epsilon)\ \|\phi\|_{\alpha}\,|\hbar| \int_{\left|\zeta'\right|>3|\hbar|} \left|\zeta'\right|^{\alpha-n}\ d\zeta' \\ &\leq\ C(\epsilon)\ \|\phi\|_{\alpha}\,|\hbar| \int_{r=3|\hbar|}^{\infty} r^{\alpha-2}\ dr \\ &\leq\ C(\epsilon)\ \|\varphi\|_{\alpha}\,|\hbar|^{\alpha}\,. \end{aligned} \tag{5.30}$$

Combining (5.24) and (5.30), the bound for D_1 follows, *that is*,

$$|D_1| \leq C(\epsilon)\,\|\varphi\|_{\alpha}\,|\hbar|^{\alpha}\,. \tag{5.31}$$

While combining (5.23) and (5.31) gives the bound for C_2,*that is*,

$$|C_2| \leq C(\epsilon)\,\|\varphi\|_{\alpha}\,|\hbar|^{\alpha}\,. \tag{5.32}$$

Therefore B_4 is bounded by (5.21) and (5.32),*that is,*

$$|B_4| \leq C(\epsilon) \left\|\varphi\right\|_{\alpha} |\hbar|^{\alpha}. \tag{5.33}$$

For B term we combine (5.17), (5.18), (5.19) and (5.33) to get the bound

$$|B| \leq C(\epsilon) \left\|\varphi\right\|_{\alpha} |\hbar|^{\alpha}. \tag{5.34}$$

Finally, to let this end we combine (5.16) and (5.34) to get

$$|\mathcal{I}_1\varphi(x) - \mathcal{I}_1\varphi(\bar{x})| \leq C(\epsilon) \left\|\varphi\right\|_{\alpha} |x-\bar{x}|^{\alpha}. \tag{5.35}$$

5.2.2 Hölder estimate for $\mathcal{I}_2$, $\delta > 0$.

We follow the same idea that presented for dimension $n = 2$ in [1] to find the Hölder continuity for $\mathcal{I}_2$. From the definition of $\mathcal{I}_2$ we have

$$\mathcal{I}_2\varphi(x) = \int_{\mathbb{R}^{n-1}} \frac{a}{\left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}} \, \phi(\zeta' + x') \, d\zeta'.$$

We use change of variable $\zeta' = a\,\varpi$ to get

$$\mathcal{I}_2\varphi(x) = \int_{\mathbb{R}^{n-1}} \frac{1}{\left(1 + |\varpi|^2\right)^{\frac{n}{2}}} \, \phi(a\,\varpi + x') \, d\varpi.$$

We form

$$\mathcal{I}_2\varphi(x) - \mathcal{I}_2\varphi(\bar{x}) = \int_{\mathbb{R}^{n-1}} \frac{1}{\left(1+|\varpi|^2\right)^{\frac{n}{2}}} \left(\phi(a\,\varpi + x') - \phi(\bar{a}\,\varpi + \bar{x}')\right) d\varpi.$$

By using the Hölder continuity of ϕ, we have the following bound

$$\begin{aligned}
|\mathcal{I}_2\varphi(x) - \mathcal{I}_2\varphi(\bar{x})| &\leq \|\phi\|_\alpha \int_{\mathbb{R}^{n-1}} \frac{1}{\left(1+|\varpi|^2\right)^{\frac{n}{2}}} \ \left|a\,\varpi + x' - \bar{a}\,\varpi - \bar{x}'\right|^\alpha \ d\varpi \\
&\leq \|\phi\|_\alpha \int_{\mathbb{R}^{n-1}} \frac{1}{\left(1+|\varpi|^2\right)^{\frac{n}{2}}} \ \left|x' - \bar{x}'\right|^\alpha \ \left|\frac{x'-\bar{x}'}{|x'-\bar{x}'|} + \frac{(a-\bar{a})\,\varpi}{|x'-\bar{x}'|}\right|^\alpha \ d\varpi \\
&\leq \|\phi\|_\alpha \ \left|x' - \bar{x}'\right|^\alpha \int_{\mathbb{R}^{n-1}} \frac{1}{\left(1+|\varpi|^2\right)^{\frac{n}{2}}} \ \left(1 + |\varpi| \ \frac{|a-\bar{a}|}{|x'-\bar{x}'|}\right)^\alpha \ d\varpi.
\end{aligned}$$

We bound

$$\begin{aligned}
\frac{|a-\bar{a}|}{|x'-\bar{x}'|} &\leq C\left\|\psi_{2,\epsilon} - \psi_{1,\epsilon}\right\|_{1,\alpha} \\
&\leq C\ \epsilon^\nu \left(\|\psi_2\|_{1,\alpha_0} + \|\psi_1\|_{1,\alpha_0}\right).
\end{aligned} \tag{5.36}$$

Substituting (5.36) in the above bound for $\mathcal{I}_2$, it follows that

$$|\mathcal{I}_2\varphi(x) - \mathcal{I}_2\varphi(\bar{x})| \leq \|\phi\|_\alpha \ \left|x' - \bar{x}'\right|^\alpha \ \int_{\mathbb{R}^{n-1}} \frac{1}{\left(1+|\varpi|^2\right)^{\frac{n}{2}}} \left(1 + C(\epsilon)\,|\varpi|\,\right)^\alpha d\varpi. \tag{5.37}$$

Notice that

$$\frac{\left(1 + C(\epsilon)\,|\varpi|\,\right)^\alpha}{\left(1+|\varpi|^2\right)^{\frac{n}{2}}} \longrightarrow \frac{1}{\left(1+|\varpi|^2\right)^{\frac{n}{2}}} \quad \text{as } \ \epsilon \longrightarrow 0.$$

Thus by using the dominated convergence theorem, we see that

$$\int_{\mathbb{R}^{n-1}} \frac{\Big(1+C(\epsilon)\,|\varpi|\Big)^{\alpha}}{\Big(1+|\varpi|^2\Big)^{\frac{n}{2}}}\ d\varpi \longrightarrow \int_{\mathbb{R}^{n-1}} \frac{d\varpi}{\Big(1+|\varpi|^2\Big)^{\frac{n}{2}}} = \frac{\omega_n}{2} \quad \text{as } \ \epsilon \longrightarrow 0.$$

Therefore we have

$$|\mathcal{I}_2\varphi(x) - \mathcal{I}_2\varphi(\bar{x})| \leq \Big(\frac{\omega_n}{2} + C(\epsilon)\Big)\ \|\phi\|_{\alpha}\ \left|x' - \bar{x}'\right|^{\alpha}. \tag{5.38}$$

Involving the bound (5.1), we get the desired Hölder bound for $\mathcal{I}_2$

$$|\mathcal{I}_2\varphi(x) - \mathcal{I}_2\varphi(\bar{x})| \leq \frac{1}{2}\Big(\omega_n + C(\epsilon)\Big)(1+\epsilon_0)\,\|\varphi\|_{\alpha}\ \left|x' - \bar{x}'\right|^{\alpha}. \tag{5.39}$$

Combining (5.35) and (5.39), we have

$$\left|I_2^{\epsilon,\delta}\varphi(x') - I_2^{\epsilon,\delta}\varphi(\bar{x}')\right| \leq \frac{1}{2}\Big(\omega_n + C(\epsilon)\Big)(1+\epsilon_0)\,\|\varphi\|_{\alpha}\ \left|x' - \bar{x}'\right|^{\alpha}. \tag{5.40}$$

Thus the Hölder estimate for $I_2^{\epsilon,\delta}$ follows ,*that is,*

$$\left\|I_2^{\epsilon,\delta}\varphi\right\|_{C^{\alpha}(\Gamma_1)} \leq \frac{1}{2}\Big(\omega_n + C(\epsilon)\Big)(1+\epsilon_0)\,\|\varphi\|_{\alpha}, \tag{5.41}$$

and the theorem is proved when $\delta > 0$. For the case $\delta = 0$, it follows if we prove that for $\varphi \in C^{\alpha}(\Gamma_2)$, the following limit holds

$$\lim_{\delta\to 0} I_2^{\epsilon,\delta}\varphi(x) = I_2^{\epsilon,0}\varphi(x), \quad x \in \Gamma_1 \cap \mathcal{B}(R),$$

and this will be the aim of the next Section.

5.3 Convergence of $I_2^{\epsilon,\delta}$.

The operator $I_2^{\epsilon,\delta}$ that we deal with in this chapter is nicer than the operator $J_2^{\epsilon,\delta}$ because it is produced from the normal derivative of the single layer potential that defined on the constant hypersurface. Whereas the operator $J_2^{\epsilon,\delta}$ is the difference between the normal derivative of the single layer potential that defined on the approximate surface $\Gamma_{2,\epsilon}$ and the operator $I_2^{\epsilon,\delta}$. We have shown in the previous sections that $I_2^{\epsilon,\delta}$ is continuous linear operator from $C^\alpha(\Gamma_2)$ to $C^\alpha(\Gamma_1)$ for any $\alpha < \alpha_0$, as well as its limit operator $I_2^{\epsilon,0}$ is again continuous linear operator from $C^\alpha(\Gamma_2)$ to $C^\alpha(\Gamma_1)$ for any $\alpha < \alpha_0$. Before proceeding to study the convergence, it is appropriate to write down the definition of the operator $I_2^{\epsilon,\delta}$ in order to visualize the singularity of its kernel function. For $\varphi \in C^\alpha(\Gamma_2)$ and $x \in \Gamma_1 \cap \mathcal{B}(R_0)$ we rewrote the operator $I_2^{\epsilon,\delta}$ in the following form

$$I_2^{\epsilon,\delta}\varphi(x) = \mathcal{I}_2^{\epsilon,\delta}\varphi(x) - \mathcal{I}_1^{\epsilon,\delta}\varphi(x),$$

where for $\delta > 0$ or $x \neq 0$ the operators $\mathcal{I}_1^{\epsilon,\delta}$ and $\mathcal{I}_1^{\epsilon,\delta}$ are defined by

$$\mathcal{I}_1^{\epsilon,\delta}\varphi(x) = \int_{|\zeta'|<M} \frac{\nabla\psi_{1,\epsilon}(x')\cdot\zeta'}{\left(|\zeta'|^2 + a^2(x')\right)^{\frac{n}{2}}}\, \phi(\zeta' + x')\, d\zeta',$$

$$\mathcal{I}_2^{\epsilon,\delta}\varphi(x) = \int_{\mathbb{R}^{n-1}} \frac{a(x')}{\left(|\zeta'|^2 + a^2(x')\right)^{\frac{n}{2}}}\, \phi(\zeta' + x')\, d\zeta',$$

where

$$a := a(x') = \delta + \psi_{2,\epsilon}(x') - \psi_{1,\epsilon}(x').$$

While for $x = 0$ and $\delta = 0$, we have that

$$\mathcal{I}_1^{\epsilon,0}\varphi(0) = 0,$$

and

$$\mathcal{I}_2^{\epsilon,0}\varphi(0) = \phi(0)\frac{\omega_n}{2}.$$

Proposition 5.3. *Let $\varphi \in C^{\alpha}(\Gamma_2)$. Then for any $\epsilon < \epsilon_0$ we have the uniform convergence*

$$\lim_{\delta \to 0} \left\| \eta I_2^{\epsilon,\delta}\varphi - \eta I_2^{\epsilon,0}\varphi \right\|_{\infty} = 0,$$

where η is the smooth cut-off function defined in (3.11) which satisfies the following properties $0 \leq \eta \leq 1$, η equals to 1 in the ball $\mathcal{B}(\epsilon_0)$, supported in the ball $\mathcal{B}(R_0)$ and $\|\nabla \eta\|_{\infty} \leq \epsilon_0$.

Proof. For $\varphi \in C^{\alpha}(\Gamma_2)$ and $0 \neq x \in \Gamma_1 \cap \mathcal{B}(R_0)$, we form the following

$$\mathcal{I}_1^{\epsilon,\delta}\varphi(x) - \mathcal{I}_1^{\epsilon,0}\varphi(x) = \int_{|\zeta'|<M} \left(\frac{\zeta' \cdot \nabla\psi_{1,\epsilon}(x')}{\left(|\zeta'|^2 + a^2(x')\right)^{\frac{n}{2}}} - \frac{\zeta' \cdot \nabla\psi_{1,\epsilon}(x')}{\left(|\zeta'|^2 + a_0^2(x')\right)^{\frac{n}{2}}} \right) \phi(\zeta' + x')\, d\zeta'.$$

We take advantage of the integral

$$\int_{|\zeta'|<M} \left(\frac{\zeta' \cdot \nabla\psi_{1,\epsilon}(x')}{\left(|\zeta'|^2 + a^2(x')\right)^{\frac{n}{2}}} - \frac{\zeta' \cdot \nabla\psi_{1,\epsilon}(x')}{\left(|\zeta'|^2 + a_0^2(x')\right)^{\frac{n}{2}}} \right) \phi(x')\, d\zeta' = 0,$$

to gain some regularity from the Hölder continuity of the function ϕ. That is, we form the following

$$\mathcal{I}_1^{\epsilon,\delta}\varphi(x) - \mathcal{I}_1^{\epsilon,0}\varphi(x) = \int_{|\zeta'|<M} \left(\frac{\zeta' \cdot \nabla\psi_{1,\epsilon}(x')}{\left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}} - \frac{\zeta' \cdot \nabla\psi_{1,\epsilon}(x')}{\left(|\zeta'|^2 + a_0^2\right)^{\frac{n}{2}}} \right) \left(\phi(\zeta' + x') - \phi(x')\right) \, d\zeta'.$$

It is a worthwhile to split the domain of the integral $\{|\zeta'| < M\}$ into two subdomains $\{|\zeta'| < \delta\}$ and $\{\delta < |\zeta'| < M\}$ and then we proceed to argue the uniform convergence on each subdomain separately. For that, we rewrite

$$\mathcal{I}_1^{\epsilon,\delta}\varphi(x) - \mathcal{I}_1^{\epsilon,0}\varphi(x) = \mathcal{I}_{1,1}^{\epsilon,\delta} + \mathcal{I}_{1,2}^{\epsilon,\delta},$$

where

$$\mathcal{I}_{1,1}^{\epsilon,\delta} = \int_{|\zeta'|<\delta} \left(\frac{\zeta' \cdot \nabla\psi_{1,\epsilon}(x')}{\left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}} - \frac{\zeta' \cdot \nabla\psi_{1,\epsilon}(x')}{\left(|\zeta'|^2 + a_0^2\right)^{\frac{n}{2}}} \right) \left(\phi(\zeta' + x') - \phi(x')\right) \, d\zeta',$$

and

$$\mathcal{I}_{1,2}^{\epsilon,\delta} = \int_{\delta<|\zeta'|<M} \left(\frac{\zeta' \cdot \nabla\psi_{1,\epsilon}(x')}{\left(|\zeta'|^2 + a^2\right)^{\frac{n}{2}}} - \frac{\zeta' \cdot \nabla\psi_{1,\epsilon}(x')}{\left(|\zeta'|^2 + a_0^2\right)^{\frac{n}{2}}} \right) \left(\phi(\zeta' + x') - \phi(x')\right) \, d\zeta'.$$

We easily can bound $\mathcal{I}_{1,1}^{\epsilon,\delta}$ by the following

$$\left|\mathcal{I}_{1,1}^{\epsilon,\delta}\right| \leq \left\|\nabla\psi_{1,\epsilon}\right\|_\infty \, \|\phi\|_\alpha \int_{|\zeta'|<\delta} \left|\zeta'\right|^{1+\alpha-n} \, d\zeta'.$$

Thus we have

$$\left|\mathcal{I}_{1,1}^{\epsilon,\delta}\right| \leq C \left\|\nabla\psi_{1,\epsilon}\right\|_{\infty} \ \|\phi\|_{\alpha} \ \delta^{\alpha}. \tag{5.42}$$

Therefore we clearly see the uniform convergence from the bound (5.42) over the subdomain $\{|\zeta'| < \delta\}$. Again for the second integral $\mathcal{I}_{1,2}^{\epsilon,\delta}$ we trivially obtain the following bound

$$\left|\mathcal{I}_{1,2}^{\epsilon,\delta}\right| \leq \|\phi\|_{\alpha} \int_{\delta<|\zeta'|<M} \left|\nabla\psi_{1,\epsilon}(x')\right| \left|\zeta'\right|^{1+\alpha} \left|\frac{1}{\left(|\zeta'|^2+a^2\right)^{\frac{n}{2}}} - \frac{1}{\left(|\zeta'|^2+a_0^2\right)^{\frac{n}{2}}}\right| d\zeta'.$$

Noticing that, $0 \leq a_0(x') < a(x')$ and $a(x') - a_0(x') = \delta$. Then by apply the mean value theorem for some z^2 that satisfying $a_0^2(x') < z^2 < a^2(x')$, we have the following

$$\left|\mathcal{I}_{1,2}^{\epsilon,\delta}\right| \leq C \|\phi\|_{\alpha} \int_{\delta<|\zeta'|<M} \left|\nabla\psi_{1,\epsilon}(x')\right| \left|\zeta'\right|^{\alpha} \frac{\delta \ \left|\zeta'\right| \left|a(x')+a_0(x')\right|}{\left(|\zeta'|^2+a_0^2(x')\right)^{\frac{n}{2}} \left(|\zeta'|^2+a^2(x')\right)} \, d\zeta'$$

since

$$\left|\zeta'\right| \left|a(x')+a_0(x')\right| \leq 2\left(\left|\zeta'\right|^2 + a^2(x')\right),$$

we conclude that

$$\left|\mathcal{I}_{1,2}^{\epsilon,\delta}\right| \leq C \|\phi\|_{\alpha} \, \delta \int_{\delta<|\zeta'|<M} \frac{\left|\nabla\psi_{1,\epsilon}(x')\right| \left|\zeta'\right|^{\alpha}}{\left(|\zeta'|^2+a_0^2(x')\right)^{\frac{n}{2}}} \, d\zeta'.$$

By using Lemma 4.2 to bound the function $\nabla\psi_{1,\epsilon}(x')$, we have

$$\left|\mathcal{I}_{1,2}^{\epsilon,\delta}\right| \leq C \|\phi\|_{\alpha} \, \delta \int_{\delta<|\zeta'|<M} \frac{\left|\psi_{1,\epsilon}(x')\right|^{\frac{\alpha}{1+\alpha}} \left|\zeta'\right|^{\alpha}}{\left(|\zeta'|^2+\left(\psi_{2,\epsilon}(x')-\psi_{1,\epsilon}(x')\right)^2\right)^{\frac{n}{2}}} \, d\zeta'.$$

Also by using Lemma 4.3 to bound the denominator, we get the following

$$\left|\mathcal{I}_{1,2}^{\epsilon,\delta}\right| \leq C\,\delta\,\|\phi\|_\alpha \int_{\delta<|\zeta'|<M} \frac{\left|\psi_{1,\epsilon}(x')\right|^{\frac{\alpha}{1+\alpha}}\left|\zeta'\right|^{\alpha}}{|\zeta'|^{\frac{n}{2}(1+\mu)}\left|\psi_{1,\epsilon}(x')\right|^{\frac{n}{2}(1-\mu)}}\,d\zeta'.$$

By choosing $\mu = 1 - \frac{2\alpha}{n(1+\alpha)}$, it follows that

$$\left|\mathcal{I}_{1,2}^{\epsilon,\delta}\right| \leq C\ \|\phi\|_\alpha\ \delta^{\alpha}(\delta^{1-\alpha}) \int_{\delta<|\zeta'|<M} \ \left|\zeta'\right|^{\alpha-n+\frac{\alpha}{1+\alpha}}\ d\zeta'.$$

Then we clearly see that

$$\left|\mathcal{I}_{1,2}^{\epsilon,\delta}\right| \leq C\ \|\phi\|_\alpha\ \delta^{\alpha} \int_{\delta<|\zeta'|<M} \ \left|\zeta'\right|^{1-n+\frac{\alpha}{1+\alpha}}\ d\zeta'.$$

Therefore we obtain the following bound

$$\left|\mathcal{I}_{1,2}^{\epsilon,\delta}\right| \leq C\ \|\phi\|_\alpha\ \delta^{\alpha}. \tag{5.43}$$

Thus the uniform convergence follows on the subdomain $\{\delta < \left|\zeta'\right| < M\}$. After finishing the convergence of $\mathcal{I}_1^{\epsilon,\delta}$. We continue to show the uniform convergence of the operator $\mathcal{I}_2^{\epsilon,\delta}$. For $0 \neq x \in \Gamma_1 \cap B(0, R_0)$, we have

$$\mathcal{I}_2^{\epsilon,\delta}\varphi(x) - \mathcal{I}_2^{\epsilon,0}\varphi(x) = \int_{\mathbb{R}^{n-1}} \left(\frac{a(x')\phi(\zeta'+x')}{\left(|\zeta'|^2 + a^2(x')\right)^{\frac{n}{2}}} - \frac{a_0(x')\phi(\zeta'+x')}{\left(|\zeta'|^2 + a_0^2(x')\right)^{\frac{n}{2}}} \right) d\zeta'.$$

For the first integrand we use the change of variables $\zeta' = a(x')\varpi$, while for the second integrand we use the change of variables $\zeta' = a_0(x')\varpi$, then it follows that

$$\mathcal{I}_2^{\epsilon,\delta}\varphi(x) - \mathcal{I}_2^{\epsilon,0}\varphi(x) = \int_{\mathbb{R}^{n-1}} \frac{1}{\left(1+|\varpi|^2\right)^{\frac{n}{2}}} \Big(\phi(a(x')\,\varpi + x') - \phi(a_0(x')\,\varpi + x')\Big)\, d\varpi$$

Since ϕ is Hölder continuous, then we easily obtain the following bound

$$\left|\mathcal{I}_2^{\epsilon,\delta}\varphi(x) - \mathcal{I}_2^{\epsilon,0}\varphi(x)\right| \le \|\phi\|_\alpha \int_{\mathbb{R}^{n-1}} \frac{\delta^\alpha\,|\varpi|^\alpha}{\left(1+|\varpi|^2\right)^{\frac{n}{2}}}\, d\varpi.$$

Observe that

$$\int_{\mathbb{R}^{n-1}} \frac{|\varpi|^\alpha}{\left(1+|\varpi|^2\right)^{\frac{n}{2}}}\, d\varpi = \int_{|\varpi|\le 1} \frac{|\varpi|^\alpha}{\left(1+|\varpi|^2\right)^{\frac{n}{2}}}\, d\varpi + \int_{|\varpi|>1} \frac{|\varpi|^\alpha}{\left(1+|\varpi|^2\right)^{\frac{n}{2}}}\, d\varpi,$$

$$\int_{|\varpi|\le 1} \frac{|\varpi|^\alpha}{\left(1+|\varpi|^2\right)^{\frac{n}{2}}}\, d\varpi \le \int_{\mathbb{R}^{n-1}} \frac{1}{\left(1+|\varpi|^2\right)^{\frac{n}{2}}}\, d\varpi = \frac{\omega_n}{2}.$$

and

$$\int_{|\varpi|>1} \frac{|\varpi|^\alpha}{\left(1+|\varpi|^2\right)^{\frac{n}{2}}}\, d\varpi \le \int_{|\varpi|>1} |\varpi|^{\alpha-n}\, d\varpi \le \frac{C}{1-\alpha}.$$

Therefore we have

$$\left|\mathcal{I}_2^{\epsilon,\delta}\varphi(x) - \mathcal{I}_2^{\epsilon,0}\varphi(x)\right| \le C\,\|\phi\|_\alpha\,\delta^\alpha.$$

That is, $\mathcal{I}_2^{\epsilon,\delta}$ converges uniformly to $\mathcal{I}_2^{\epsilon,0}$ when $x \neq 0$. Finally, we show the uniform convergence when $x = 0$. In this case we have

$$I_2^{\epsilon,\delta}\varphi(0) - I_2^{\epsilon,0}\varphi(0) = \int_{\mathbb{R}^{n-1}} \left(\frac{\delta\ \phi(\zeta')}{\left(|\zeta'|^2 + \delta^2\right)^{\frac{n}{2}}} - \frac{\phi(0)}{\left(1 + |\zeta'|^2\right)^{\frac{n}{2}}} \right) d\zeta'.$$

After changing the variables we obtain the following

$$I_2^{\epsilon,\delta}\varphi(0) - I_2^{\epsilon,0}\varphi(0) = \int_{\mathbb{R}^{n-1}} \frac{\left(\phi(\delta\ \varpi) - \phi(0)\right)}{\left(|\varpi|^2 + 1\right)^{\frac{n}{2}}}\, d\varpi$$

Then we clearly see the bound

$$\left|I_2^{\epsilon,\delta}\varphi(0) - I_2^{\epsilon,0}\varphi(0)\right| \leq C\, \|\phi\|_\alpha\ \delta^\alpha \int_{\mathbb{R}^{n-1}} \frac{|\varpi|^\alpha}{\left(1 + |\varpi|^2\right)^{\frac{n}{2}}} \leq C\, \|\phi\|_\alpha\ \delta^\alpha.$$

Therefore the theorem follows. □

The following Theorem shows the convergence of $I_2^{\epsilon,\delta}\varphi$ in $C^{\alpha'}(\Gamma_1)$ for any $\alpha' < \alpha$ where $\varphi \in C^\alpha(\Gamma_2)$ and it can be proved in the same manner as Theorem 4.6.

Theorem 5.4. *[1] Fix $0 < 2\epsilon < \epsilon_0$. Then for any $0 < \alpha' < \alpha < \alpha_0$ and for all $\varphi \in C^\alpha(\Gamma_2)$, we have*

$$\eta I_2^{\epsilon,\delta}\varphi \xrightarrow{\alpha'} \eta I_2^{\epsilon,0}\varphi \ \ \text{as } \delta \longrightarrow 0.$$

Chapter 6

The convergence and the main results.

In this Chapter we will show the operators $\left(T^{\delta}\right)^{-1}$ exist for any $0 \le \delta \le \delta_0$ and are uniformly bounded in the operator norm. We have shown in Section 3.1 the operators T^{δ} are invertible for $\delta > 0$, but it has proven in [1] that the operators T^{δ} do not converge in norm to their limiting operator T^{0}. From elementary functional analysis we know that **If the operators T^{δ} were to converge in norm to T^{0} and if $(T^{0})^{-1}$ exists, then we immediately would get the uniform boundedness for $(T^{\delta})^{-1}$ in operator norm**. Therefore, according to the absence of the norm convergence, we need to study further properties for the operators T^{δ} in order to have the uniform boundedness for their inverses. In [1] for the dimension $n = 2$, they have used the notion of *collectively compact operators* to get the invertibility of T^{0} and the uniform boundedness of the operators $\left(T^{\delta}\right)^{-1}$. We will mimic the idea that presented in [1] to obtain the uniform boundedness for $(T^{\delta})^{-1}$ in any dimension $n \ge 2$.

6.1 Preliminaries

Let X, Y and Z be real or complex Banach spaces and let $\mathcal{B}$ be the closed unit ball in X, *that is*,

$$\mathcal{B} = \{x \in X : \|x\| \le 1\}.$$

Denote by $\mathcal{L}(X, Y)$ the Banach space of bounded linear operators $T : X \longrightarrow Y$ with the usual operator norm,

$$\|T\| = \sup_{x \in \mathcal{B}} \|Tx\| \,.$$

It is known from elementary functional analysis that a subset $X' \subset X$ is relatively compact iff X' is sequentially compact iff X' is totally bounded. Recall that a set is totally bounded if for any $\epsilon > 0$, there exists a finite cover by ϵ-balls.

In this Chapter, we will use

$$\|T_n - T\| \longrightarrow 0$$

to denote for convergence in norm and

$$T_n \longrightarrow T$$

to denote for pointwise convergence (strong convergence), that is

$$T_n x \longrightarrow Tx \;\; \forall x \in X.$$

An operator $K \in \mathcal{L}(X, Y)$ is called compact iff the set $K\mathcal{B}$ is relatively compact. The concept of compact operators plays an important role in solving equations of the form

$$(I - K)x = y. \tag{6.1}$$

In applied mathematics, equation (6.1) is known as *Fredholm alternative*, that is, (6.1) has unique solution iff the homogeneous equation

$$(I - K)x = 0, \tag{6.2}$$

has only the trivial solution $x = 0$. In such a case, the *Fredholm operator* $I - K : X \longrightarrow Y$ has a bounded inverse $(I - K)^{-1}$. The practical solution of equation (6.1) often depends on the approximation operators. That is, to find a solution of (6.1), it often requires to find a solution of the corresponding approximate equation

$$(I - K_n)x_n = y, \tag{6.3}$$

where K_n are compact operators and $\|K_n - K\| \longrightarrow 0$. Then $(I - K)^{-1}$ exists iff for some N and all $n \geq N$ there exist uniformly bounded $(I - K_n)^{-1}$. In such a case,

$$\left\|(I - K_n)^{-1} - (I - K)^{-1}\right\| \longrightarrow 0.$$

We notice this scheme requires convergence in norm. In some cases, convergence in norm can not hold. Indeed, it is possible to have pointwise convergence only. Therefore, we seek an alternative scheme that requires pointwise convergence instead of the norm convergence. Such a scheme is indeed available: ***For a sequence of collectively compact operators*** K_n***, such that*** $K_n \longrightarrow K$ ***(where*** K ***is a compact operator). Then,*** $(I - K)^{-1}$ ***exists if and only if for some*** $n \geq N$ ***the operators*** $(I - K_n)^{-1}$ ***exist and are bounded uniformly, in such a case***

$$(I - K_n)^{-1} \longrightarrow (I - K)^{-1}.$$

Let us introduce the notion of ***collectively compact*** operators which is a generalization of the notion of compact operators. Collectively compact operators are very important tools for solving integral equations of the second kind and have been studied in several papers [4, 5, 6]. We use [3, 5] to introduce the definition of collectively compact and some basic results that are needed for our work.

Definition 6.1. *[3] A set $\mathcal{K} \subset \mathcal{L}(X, Y)$ is called collectively compact if the set*

$$\mathcal{K}\mathcal{B} = \{Kx : K \in \mathcal{K}, \ \ x \in \mathcal{B}\}$$

is relatively compact.

We notice from the definition of collectively compactness, every operator in collectively compact set is compact.

Example 6.2. *[3] Let $X = \ell^2$. Define $K_n \in \mathcal{L}(X), n = 1, 2, \cdots,$ by*

$$K_n x = (x_n, 0, 0, \cdots).$$

Let $\mathcal{K} = \{K_n\}$. Then $\mathcal{K}\mathcal{B}$ is bounded and dim$(\mathcal{K}X) = 1$. Thus $\mathcal{K}$ is collectively compact.

The following Lemma shows that pointwise limit of collectively compact operators is compact.

Lemma 6.3. *[3] Let $K, K_n \in \mathcal{L}(X), n = 1, 2, \cdots,$ be such that $\mathcal{K} = \{K_n, n = 1, 2, \cdots\}$ is a collectively compact set and K_n converges pointwise to the operator K. Then K is compact.*

Proof.

$$K\mathcal{B} \subset \overline{\{K_n x : n \geq 1, \;\; x \in \mathcal{B}\}} = \overline{\{K_n\}\mathcal{B}}.$$

Thus K is compact. $\square$

Lemma 6.4. *[5] Let $\mathcal{K} \subset \mathcal{L}(X,Y)$ be a collectively compact set and $\mathcal{M} \subset \mathcal{L}(Z,X)$ be a bounded set. Then the set $\mathcal{K}\mathcal{M}$ is collectively compact.*

Proof. Let $\mathcal{B}$ be the closed unit ball in X and $\mathcal{B}'$ be the closed unit ball in Z. Since $\mathcal{M}$ is bounded then there exist $r > 0$ be such that $\|M\| < r$ for all $M \in \mathcal{M}$. Thus

$$\mathcal{K}\mathcal{M}\mathcal{B} \subset r\, \mathcal{K}\mathcal{B}'.$$

Since $\mathcal{K}$ is collectively compact, then $\overline{\mathcal{K}\mathcal{M}\mathcal{B}}$ is compact. Therefore we see that $\mathcal{K}\mathcal{M}$ is collectively compact. $\square$

Remark 6.5. *[3] A collectively compact set is a bounded set of compact operators, but the converse is false as we will see in the following example.*

Example 6.6. *[3] Let $X = \ell^2$ with the orthonormal basis $\{\varphi_\alpha : \alpha \in I\}$ and let P_α be the orthogonal projection onto the one-dimensional subspace spanned by φ_α. Then*

$$P_\alpha x = (x, \varphi_\alpha)\varphi_\alpha, \quad \|P_\alpha\| = 1,$$

and P_α is compact since $dim P_\alpha \ell^2 = 1$. Thus $\mathcal{K} = \{P_\alpha,\ \alpha \in I\}$ is a bounded set of compact *operators in $\mathcal{L}(\ell^2)$. However, $\mathcal{K}$ is not collectively compact because*

$$\varphi_\alpha = P_\alpha \varphi_\alpha \in \mathcal{K}\mathcal{B}, \quad \forall\, \alpha \in I,$$

and

$$\left\|\varphi_\alpha - \varphi_\beta\right\| = \sqrt{2} \ \textit{for} \ \alpha \neq \beta.$$

Therefore $\mathcal{K}\mathcal{B}$ is not totally bounded.

The following Lemma is very useful and it plays a fundamental role in our work because it just requires pointwise convergence.

Lemma 6.7. *[3] Let X be a Banach space and K, K_n be bounded linear operators on X for $n = 1, 2, \ldots$. Assume that $\{K_n\}$ is collectively compact and K is compact and $K_n \longrightarrow K$. Then, $(I - K)^{-1}$ exists if and only if for some $n \geq N$ the operators $(I - K_n)^{-1}$ exist and are bounded uniformly, in which case $(I - K_n)^{-1} \longrightarrow (I - K)^{-1}$.*

6.2 Collective Compactness and convergence of $K_2^{\epsilon,\delta}$.

After introducing the concept of collectively compactness in the previous Section, we use the concept to show the set $\mathcal{K} = \{K_2^{\epsilon,\delta},\ 0 < \delta < \delta_0\}$ is collectively compact, where the operators $K_2^{\epsilon,\delta}$ are defined in Section 3.3.

Theorem 6.8. *Let ϵ be fixed with $0 < 2\epsilon < \epsilon_0$. The operators $K_2^{\epsilon,\delta} : C^\alpha(\Gamma_2) \longrightarrow C^\alpha(\Gamma_1)$, $0 < \delta < \delta_0$, form a collectively compact family of operators.*

Proof. For $\varphi \in C^\alpha(\Gamma_2)$ and $x \in \Gamma_1 \cap \mathcal{B}(R_0)$, recall the definition of the operator $K_2^{\epsilon,\delta}$ from

Section 3 which is

$$K_2^{\epsilon,\delta}\varphi(x) = \frac{-1}{\omega_n}\int_{\Gamma_2} \frac{(x-y-\delta\mathbf{e_n})\cdot\nu(x)}{|x-y-\delta\mathbf{e_n}|^n}\,\varphi(y)\,d\sigma(y)$$
$$+\frac{1}{\omega_n}\int_{\Gamma_{2,\epsilon}} \frac{(x_\epsilon-y_\epsilon-\delta\mathbf{e_n})\cdot\nu_\epsilon(x_\epsilon)}{|x_\epsilon-y_\epsilon-\delta\mathbf{e_n}|^n}\,E\varphi\circ P(y_\epsilon)\,d\sigma(y_\epsilon),$$

where

$$\nu_\epsilon(x_\epsilon) = \frac{1}{\sqrt{1+\left|\nabla\psi_{1,\epsilon}(x')\right|^2}}\begin{pmatrix}-\nabla\psi_{1,\epsilon}(x')\\ 1\end{pmatrix},$$

P is the projection map from $\Gamma_{2,\epsilon}$ onto $\mathbb{R}^{n-1}$ and E is the extension operator on $C^\alpha(\Gamma_2)$ into $C^\alpha(\mathbb{R}^{n-1})$ that defined in (3.13). Now, when $|x| < \frac{\epsilon}{2}$ we clearly see that for $\left|y'\right| < \epsilon$ the two integrands in the above expression coincide. Thus we have the following

$$K_2^{\epsilon,\delta}\varphi(x) = \frac{-1}{\omega_n}\int_{\{\Gamma_2\cap|y'|\geq\epsilon\}} \frac{(x-y-\delta\mathbf{e_n})\cdot\nu(x)}{|x-y-\delta\mathbf{e_n}|^n}\,\varphi(y)\,d\sigma(y)$$
$$+\frac{1}{\omega_n}\int_{\{\Gamma_{2,\epsilon}\cap|y'|\geq\epsilon\}} \frac{(x_\epsilon-y_\epsilon-\delta\mathbf{e_n})\cdot\nu_\epsilon(x_\epsilon)}{|x_\epsilon-y_\epsilon-\delta\mathbf{e_n}|^n}\,E\varphi\circ P(y_\epsilon)\,d\sigma(y_\epsilon),$$

and more explicitly with the help of the auxiliary functions we rewrite the operator $K_2^{\epsilon,\delta}$ in the following form

$$K_2^{\epsilon,\delta}\varphi(x) = \frac{-1}{\omega_n}\int_{\{\Gamma_2\cap|y'|\geq\epsilon\}} \frac{(x-y-\delta\mathbf{e_n})\cdot\nu(x)}{|x-y-\delta\mathbf{e_n}|^n}\,\varphi(y)\,d\sigma(y) := \mathcal{K}_{2,1}^{\epsilon,\delta}\varphi(x)$$
$$+\frac{1}{\omega_n\sqrt{1+\left|\nabla\psi_{1,\epsilon}(x')\right|^2}}\int_{|y'|\geq\epsilon} \frac{(y'-x')\cdot\nabla\psi_{1,\epsilon}(x')+\Big(\psi_{1,\epsilon}(x')-\psi_{2,\epsilon}(y')-\delta\Big)}{\Big(|x'-y'|^2+(\psi_{1,\epsilon}(x')-\psi_{2,\epsilon}(y')-\delta)^2\Big)^{n/2}}\,\phi(y')\,dy' := \mathcal{K}_{2,2}^{\epsilon,\delta}\varphi(x).$$

Then we proceed to discuss the regularity and the compactness of each term that involved in the above expression separately. First, let us assume that $\mathcal{H}_1^\delta$ and $\mathcal{H}_2^{\epsilon,\delta}$ be the kernel functions that associated with the operators $\mathcal{K}_{2,1}^{\epsilon,\delta}$ and $\mathcal{K}_{2,2}^{\epsilon,\delta}$ respectively. That is,

$$\mathcal{H}_1^{\delta}(x,y) := \frac{(x-y-\delta\mathbf{e_n})\cdot\nu(x)}{|x-y-\delta\mathbf{e_n}|^n}, \tag{6.4}$$

and

$$\mathcal{H}_2^{\epsilon,\delta}(x,y) := \frac{1}{\sqrt{1+\left|\nabla\psi_{1,\epsilon}(x')\right|^2}}\frac{(y'-x')\cdot\nabla\psi_{1,\epsilon}(x')+\Big(\psi_{1,\epsilon}(x')-\psi_{2,\epsilon}(y')-\delta\Big)}{\Big(|x'-y'|^2+(\psi_{1,\epsilon}(x')-\psi_{2,\epsilon}(y')-\delta)^2\Big)^{n/2}}. \tag{6.5}$$

For the operators

$$\mathcal{K}_{2,1}^{\epsilon,\delta}\varphi(x) = \frac{-1}{\omega_n}\int_{\{\Gamma_2\cap|y'|\geq\epsilon\}}\mathcal{H}_1^{\delta}(x,y)\varphi(y)\,d\sigma(y), \quad |x|<\frac{\epsilon}{2}.$$

We obviously see the kernel function $\mathcal{H}_1^{\delta}(x,y)$ is α_0-Hölder continuous for any $0<\alpha_0\leq 1$ with respect to x and smooth with respect to y as well as $\mathcal{H}_1^{\delta}$ is uniformly bounded independently of δ. Thus the compactness of the operators $\mathcal{K}_{2,1}^{\epsilon,\delta}$ follows. Similarly, for the operators

$$\mathcal{K}_{2,2}^{\epsilon,\delta}\varphi(x) = \frac{1}{\omega_n}\int_{|y'|\geq\epsilon}\mathcal{H}_2^{\epsilon,\delta}(x,y)\phi(y)\,dy', \quad |x|<\frac{\epsilon}{2}.$$

The kernel function $\mathcal{H}_2^{\epsilon,\delta}(x,y)$ is α_0-Hölder continuous for any $0<\alpha_0\leq 1$ with respect to x and y also $\mathcal{H}_2^{\epsilon,\delta}$ is uniformly bounded independently of δ. Therefore, the operators $\mathcal{K}_{2,2}^{\epsilon,\delta}$ are compact. Thus the compactness and the regularity of the operators $K_2^{\epsilon,\delta}$ follow when $|x|<\frac{\epsilon}{2}$. Then we continue to discuss the properties of the operators when $|x|\geq\frac{\epsilon}{2}$. In this case we have

$$K_2^{\epsilon,\delta}\varphi(x) = \frac{-1}{\omega_n}\int_{\Gamma_2} \mathcal{H}_1^{\delta}(x,y)\,\varphi(y)\,d\sigma(y) + \frac{1}{\omega_n}\int_{\mathbb{R}^{n-1}} \mathcal{H}_2^{\epsilon,\delta}(x,y)\,\phi(y')\,dy', \qquad (6.6)$$

where $\mathcal{H}_1^{\delta}(x,y)$ and $\mathcal{H}_2^{\epsilon,\delta}(x,y)$ are defined in (6.4) and (6.5) respectively. Again, clearly we see that the kernel function $\mathcal{H}_1^{\delta}(x,y)$ is α_0-Hölder continuous for any $x \in \Gamma_1 \cap \mathcal{B}(R_0) \setminus \mathcal{B}(\frac{\epsilon}{2})$ and $y \in \Gamma_2$ even though $\delta = 0$ as well as $\mathcal{H}_1^{\delta}(x,y)$ is uniformly bounded independently of δ. Thus the compactness and the regularity of the first integral in (6.6) follow. For the second integral in (6.6), we see that the denominator of the kernel function $\mathcal{H}_2^{\epsilon,\delta}(x,y)$ never vanishes even though $\delta = 0$ because the auxiliary function $\psi_{1,\epsilon}$ always negative when $|x'| > 0$ therefore the regularity of $\mathcal{H}_2^{\epsilon,\delta}$ and the uniform boundedness follow. Therefore we obtain the regularity and the uniform boundedness as well as the compactness of the second integral in (6.6). From the above observation we conclude that the operators $K_2^{\epsilon,\delta}$ are compact operators from $C^{\alpha}(\Gamma_2)$ into $C^{\alpha_0}(\Gamma_1)$ as well as they are uniformly bounded independently of δ. Thus the family

$$\mathcal{F} = \{K_2^{\epsilon,\delta},\ 0 < \delta < \delta_0\}$$

is uniformly bounded in operator norm from $C^{\alpha}(\Gamma_2)$ into $C^{\alpha_0}(\Gamma_1)$. Since, the embedding

$$C^{\alpha_0}(\Gamma_1) \hookrightarrow C^{\alpha}(\Gamma_1), \quad \alpha < \alpha_0,$$

is compact, then we see that the family $\mathcal{F}$ is collectively compact. $\square$

As a result from the above theorem, we have the pointwise convergence of the operators

$K_2^{\epsilon,\delta}$. That is,

Corollary 6.9. *Let* $0 < \alpha < \alpha_0$*, and fix* $0 < 2\epsilon < \epsilon_0$*. Then for all* $\varphi \in C^\alpha(\Gamma_2)$*, we have*

$$K_2^{\epsilon,\delta}\varphi \longrightarrow K_2^{\epsilon,0}\varphi, \quad \textit{in } C^\alpha(\Gamma_1),$$

as δ *approaches* 0.

Proof. The denominators of the kernel functions associated to the operators $K_2^{\epsilon,\delta}$ are bounded away from 0 even though $\delta = 0$ as well as the kernel function has C^α regularity. Thus by using the dominated convergence theorem we can pass the limit inside the integral sign. □

6.3 Pointwise Convergence of T^δ.

This Section deals with the pointwise convergence of the operator T^δ that defined in Section 3.4 by the following, for $\delta > 0$, T^δ defined as an operator in $\mathcal{L}\big(C^\alpha(\Gamma_1) \times C^\alpha(\Gamma_2)\big)$ and has the form

$$T^\delta = \Lambda_{\epsilon,\delta} + C_{\epsilon,\delta},$$

where

$$\Lambda_{\epsilon,\delta} = \begin{pmatrix} \lambda I & \frac{\eta}{\omega n\sqrt{1+\left|\nabla\psi_{1,\epsilon}\right|^2}}\left(J_2^{\epsilon,\delta} + I_2^{\epsilon,\delta}\right) \\ \frac{\eta}{\omega n\sqrt{1+\left|\nabla\psi_{2,\epsilon}\right|^2}}\left(J_1^{\epsilon,\delta} + I_1^{\epsilon,\delta}\right) & \lambda I \end{pmatrix},$$

and

$$C_{\epsilon,\delta} = \begin{pmatrix} -K_1^* & \eta K_2^{\epsilon,\delta} \\ \eta K_1^{\epsilon,\delta} & -K_2^* \end{pmatrix} + (1-\eta)\begin{pmatrix} 0 & L_2^\delta \\ L_1^\delta & 0 \end{pmatrix}.$$

For $\delta = 0$, T^{δ} is defined by

$$T^0 = \Lambda_{\epsilon,0} + C_{\epsilon,0},$$

where

$$\Lambda_{\epsilon,0} = \begin{pmatrix} \lambda I & \frac{\eta}{\omega_n\sqrt{1+\left|\nabla\psi_{1,\epsilon}\right|^2}}\left(J_2^{\epsilon,0} + I_2^{\epsilon,0}\right) \\ \frac{\eta}{\omega_n\sqrt{1+\left|\nabla\psi_{2,\epsilon}\right|^2}}\left(J_1^{\epsilon,0} + I_1^{\epsilon,0}\right) & \lambda I \end{pmatrix},$$

and

$$C_{\epsilon,0} = \begin{pmatrix} -K_1^* & \eta K_2^{\epsilon,0} \\ \eta K_1^{\epsilon,0} & -K_2^* \end{pmatrix} + (1-\eta)\begin{pmatrix} 0 & \mathbf{L}_2^0 \\ \mathbf{L}_1^0 & 0 \end{pmatrix}.$$

As a consequence of Theorems 4.6, 5.4 and Corollary 6.9, we obtain the following,

Corollary 6.10. *Fix $0 < 2\epsilon < \epsilon_0$. Then for any $0 < \alpha' < \alpha$ and for all $(\varphi_1, \varphi_2) \in C^{\alpha}(\Gamma_1) \times C^{\alpha}(\Gamma_2)$, we have*

$$\Lambda_{\epsilon,\delta}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \longrightarrow \Lambda_{\epsilon,0}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad \textit{in } C^{\alpha'}(\Gamma_1) \times C^{\alpha'}(\Gamma_2),$$

$$C_{\epsilon,\delta}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \longrightarrow C_{\epsilon,0}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad \textit{in } C^{\alpha}(\Gamma_1) \times C^{\alpha}(\Gamma_2),$$

and

$$T^{\delta}\begin{pmatrix}\varphi_1\\ \\ \varphi_2\end{pmatrix}\longrightarrow T^{0}\begin{pmatrix}\varphi_1\\ \\ \varphi_2\end{pmatrix},\quad \textit{in } C^{\alpha'}(\Gamma_1)\times C^{\alpha'}(\Gamma_2).$$

Furthermore, the operators $C_{\epsilon,\delta}$ are uniformly bounded in $\mathcal{L}\big(C^{\alpha}(\Gamma_1)\times C^{\alpha}(\Gamma_2)\big)$.

Proof. By the bounded principle theorem the uniform boundedness of the operators $C_{\epsilon,\delta}$ follows. □

Theorem 6.11. *For $\lambda = \frac{k+1}{2(k-1)}$ and $0 < \alpha' < \alpha$. There exists $\delta_0 > 0$ such that the operators $\Lambda_{\epsilon,\delta}$, $0 \le \delta \le \delta_0$, are invertible with inverses that satisfying*

$$\forall\ 0\le\delta\le\delta_0,\ \left\|\Lambda_{\epsilon,\delta}^{-1}\right\| \le \frac{C}{|\lambda| - \frac{1}{2}\Big(1+C(\epsilon)\Big)(1+\epsilon_0)}, \tag{6.7}$$

uniformly with respect to δ in the operator norm $\mathcal{L}\big(C^{\alpha}(\Gamma_1)\times C^{\alpha}(\Gamma_2)\big)$ and $C(\epsilon)$ approaches 0 as ϵ approaches 0. Furthermore, for any $(\varphi_1,\varphi_2)\in C^{\alpha}(\Gamma_1)\times C^{\alpha}(\Gamma_2)$, we have

$$\Lambda_{\epsilon,\delta}^{-1}\begin{pmatrix}\varphi_1\\ \\ \varphi_2\end{pmatrix}\longrightarrow \Lambda_{\epsilon,0}^{-1}\begin{pmatrix}\varphi_1\\ \\ \varphi_2\end{pmatrix}\quad \textit{in } C^{\alpha'}(\Gamma_1)\times C^{\alpha'}(\Gamma_2),\quad \alpha'<\alpha.$$

Proof. From Theorems (4.1) and (5.1), we have shown that for any $0 \le \delta \le \delta_0$ and $\alpha' < \alpha$ the following bound holds

$$\left\|\frac{\eta}{\omega_n\sqrt{1+\left|\nabla\psi_{1,\epsilon}\right|^2}}\left(J_2^{\epsilon,\delta}+I_2^{\epsilon,\delta}\right)\right\|_{\mathcal{L}(C^{\alpha}(\Gamma_2),C^{\alpha}(\Gamma_1))} \le \frac{1}{2}\Big(1+C(\epsilon)\Big)\left(1+\epsilon_0\right),$$

and in the similar manner we have the bound

$$\left\| \frac{\eta}{\omega_n \sqrt{1+\left|\nabla\psi_{1,\epsilon}\right|^2}} \left(J_1^{\epsilon,\delta} + I_1^{\epsilon,\delta} \right) \right\|_{\mathcal{L}(C^\alpha(\Gamma_1), C^\alpha(\Gamma_2))} \leq \frac{1}{2}\Big(1 + C(\epsilon)\Big)\,(1+\epsilon_0)\,.$$

Recall that from equation (3.10), ϵ_0 has been chosen to satisfy the inequality

$$\frac{1+\epsilon_0}{2} < |\lambda|\,.$$

Thus we may choose $\epsilon > 0$ sufficiently small such that

$$\frac{1}{2}\Big(1 + C(\epsilon)\Big)\,(1+\epsilon_0) < |\lambda|\,.$$

Therefore $\Lambda_{\epsilon,\delta}$ are invertible and satisfy the bound (6.7). From Corollary (6.10), it follows that for $(\varphi_1, \varphi_2) \in C^\alpha(\Gamma_1) \times C^\alpha(\Gamma_2)$ we have the pointwise convergence

$$\Lambda_{\epsilon,\delta}^{-1} \begin{pmatrix} \varphi_1 \\ \\ \varphi_2 \end{pmatrix} \longrightarrow \Lambda_{\epsilon,0}^{-1} \begin{pmatrix} \varphi_1 \\ \\ \varphi_2 \end{pmatrix} \quad \text{in } C^{\alpha'}(\Gamma_1) \times C^{\alpha'}(\Gamma_2), \quad \alpha' < \alpha.$$

□

In the following Theorem we will use the notion of collectively compact operators that introduced in Section 6.1.

Theorem 6.12. *For $0 < \alpha < 1$, the following hold*

1. *The operators $\{C_{\epsilon,\delta}\, \Lambda_{\epsilon,\delta}^{-1}\}$ are collectively compact on $C^\alpha(\Gamma_1) \times C^\alpha(\Gamma_2)$,*

2. *$C_{\epsilon,\delta}\, \Lambda_{\epsilon,\delta}^{-1} \longrightarrow C_{\epsilon,0}\, \Lambda_{\epsilon,0}^{-1}$ pointwise in $\mathcal{L}\big(C^\alpha(\Gamma_1) \times C^\alpha(\Gamma_2)\big)$ as δ approaches 0.*

Consequently, $\{C_{\epsilon,0}\,\Lambda_{\epsilon,0}^{-1}\}$ *is compact operator on* $C^{\alpha}(\Gamma_1)\times C^{\alpha}(\Gamma_2)$.

Proof. The collectively compactness of the set $\{C_{\epsilon,\delta}\,\Lambda_{\epsilon,\delta}^{-1}\}$ follows easily from the uniform bound (6.7), Theorem 6.8, the compactness of the operators $K_i^*, i=1,2$ and then Lemma 6.4.

For the pointwise convergence, let $\varphi\in C^{\alpha}(\Gamma_1)\times C^{\alpha}(\Gamma_2)$. Since the operators $C_{\epsilon,\delta}$ are collectively compact on $C^{\alpha}(\Gamma_1)\times C^{\alpha}(\Gamma_2)$ and $(\Lambda_{\epsilon,\delta}^{-1}-\Lambda_{\epsilon,0}^{-1})\varphi$ are uniformly bounded in $C^{\alpha}(\Gamma_1)\times C^{\alpha}(\Gamma_2)$, then for any subsequence $C_{\epsilon,\delta_n}(\Lambda_{\epsilon,\delta_n}^{-1}-\Lambda_{\epsilon,0}^{-1})\varphi$ there exists further subsequence $C_{\epsilon,\delta_{n_j}}(\Lambda_{\epsilon,\delta_{n_j}}^{-1}-\Lambda_{\epsilon,0}^{-1})\varphi$ which converges to some function $\mathfrak{L}_\epsilon\in C^{\alpha}(\Gamma_1)\times C^{\alpha}(\Gamma_2)$. From Theorem 6.11, we have

$$\Lambda_{\epsilon,\delta}^{-1}\varphi\longrightarrow\Lambda_{\epsilon,0}^{-1}\varphi\quad\text{in }\; C^{\alpha'}(\Gamma_1)\times C^{\alpha'}(\Gamma_2),\quad \alpha'<\alpha,$$

and also we have from Corollary 6.10 that $C_{\epsilon,\delta}$ are uniformly bounded in $\mathcal{L}\left(C^{\alpha}(\Gamma_2)\times C^{\alpha}(\Gamma_1)\right)$. Therefore,

$$C_{\epsilon,\delta_{n_j}}(\Lambda_{\epsilon,\delta_{n_j}}^{-1}-\Lambda_{\epsilon,0}^{-1})\varphi\longrightarrow 0\ \text{ in }\ C^{\alpha'}(\Gamma_1)\times C^{\alpha'}(\Gamma_2).$$

By uniqueness of the limit, we obtain that $\mathfrak{L}_\epsilon\equiv 0$. That is,

$$C_{\epsilon,\delta_{n_j}}(\Lambda_{\epsilon,\delta_{n_j}}^{-1}-\Lambda_{\epsilon,0}^{-1})\varphi\longrightarrow 0\ \text{ in }\ C^{\alpha}(\Gamma_1)\times C^{\alpha}(\Gamma_2).$$

Therefore the sequence $C_{\epsilon,\delta}(\Lambda_{\epsilon,\delta}^{-1}-\Lambda_{\epsilon,0}^{-1})\varphi$ converges to 0 in $C^{\alpha}(\Gamma_1)\times C^{\alpha}(\Gamma_2)$. Now, we form the following

$$\left(C_{\epsilon,\delta}\,\Lambda_{\epsilon,\delta}^{-1}-C_{\epsilon,0}\,\Lambda_{\epsilon,0}^{-1}\right)\varphi=C_{\epsilon,\delta}\left(\Lambda_{\epsilon,\delta}^{-1}-\Lambda_{\epsilon,0}^{-1}\right)\varphi+\left(C_{\epsilon,\delta}-C_{\epsilon,0}\right)\Lambda_{\epsilon,0}^{-1}\,\varphi.$$

Thus we obtain that $C_{\epsilon,\delta}\,\Lambda_{\epsilon,\delta}^{-1}$ converges pointwise to $C_{\epsilon,0}\,\Lambda_{\epsilon,0}^{-1}$ in $C^{\alpha}(\Gamma_1)\times C^{\alpha}(\Gamma_2)$. $\square$

Since $C_{\epsilon,0}$ is a compact operator and $\Lambda_{\epsilon,0}$ is invertible, then $T^0=\Lambda_{\epsilon,0}+C_{\epsilon,0}$ is a Fredholm operator. Thus by the help of Theorem 6.12 we have all ingredients to show the invertibility of the limiting operator T^0.

Theorem 6.13. *The operator* $T^0: C^{\alpha}(\Gamma_1)\times C^{\alpha}(\Gamma_2)\longrightarrow C^{\alpha}(\Gamma_1)\times C^{\alpha}(\Gamma_2)$ *is invertible.*

Proof. It suffices to show T^0 is injective. Let $(\varphi_1,\varphi_2)\in C^{\alpha}(\Gamma_1)\times C^{\alpha}(\Gamma_2)$ be such that

$$T^0\begin{pmatrix}\varphi_1\\ \\ \varphi_2\end{pmatrix}=0. \tag{6.8}$$

From Corollary 6.10, we have that $L_2^{\delta}\varphi_2\longrightarrow L_2^{0}\varphi_2$ in $C^{\alpha'}(\Gamma_1)$ for $\alpha'<\alpha$ as δ approaches 0. Therefore we obtain

$$\begin{aligned}\int_{\Gamma_1} L_2^0\varphi_2\,d\sigma &= \lim_{\delta\to 0}\int_{\Gamma_1} L_2^{\delta}\varphi_2\,d\sigma\\ &= -\lim_{\delta\to 0}\int_{\Gamma_1}\partial\nu\mathbf{S}_2{\varphi_2}^{\delta}(x-\delta\mathbf{e_n})\,d\sigma(x)\end{aligned}$$

Since $\mathbf{S}_2{\varphi_2}^{\delta}(x-\delta\mathbf{e_n})$ is harmonic in D_1, then we have

$$\int_{\Gamma_1} L_2^0\varphi_2\,d\sigma=0.$$

Thus we obtain the following

$$\int_{\Gamma_1}\Big((\lambda I-K_1^*)\,\varphi_1+L_2^0\varphi_2\Big)\,d\sigma(x)=\int_{\Gamma_1}(\lambda I-K_1^*)\,\varphi_1\,d\sigma(x).$$

Since K_1^* is the L^2 adjoint of K_1, then we have the following

$$\int_{\Gamma_1} (\lambda I - K_1^*)\,\varphi_1\; d\sigma(x) = \int_{\Gamma_1} \Big(\lambda\varphi_1 - K_1(1)\varphi_1\Big)\; d\sigma(x).$$

From Theorem 2.9, we have $K_1(1) = \frac{1}{2}$, thus we obtain

$$\int_{\Gamma_1} \Big((\lambda I - K_1^*)\,\varphi_1 + L_2^0\varphi_2\Big)\; d\sigma(x) = (\lambda - \frac{1}{2})\int_{\Gamma_1} \varphi_1\; d\sigma(x). \tag{6.9}$$

A similar result hold for $\big(\lambda I - K_2^*\big)\,\varphi_2 + L_1^0\varphi_1$ on Γ_2, that is

$$\int_{\Gamma_2} \Big((\lambda I - K_2^*)\,\varphi_2 + L_1^0\varphi_1\Big)\; d\sigma = (\lambda - \frac{1}{2})\int_{\Gamma_2} \varphi_2\; d\sigma. \tag{6.10}$$

From the assumption (6.8), we have

$$(\lambda I - K_1^*)\,\varphi_1 + L_2^0\varphi_2 = 0 \quad \text{on} \;\; \Gamma_1,$$

and

$$(\lambda I - K_2^*)\,\varphi_2 + L_1^0\varphi_1 = 0 \quad \text{on} \;\; \Gamma_2.$$

Therefor by using (6.9) and (6.10), it follows that

$$(\lambda - \frac{1}{2})\int_{\Gamma_1} \varphi_1\; d\sigma = (\lambda - \frac{1}{2})\int_{\Gamma_2} \varphi_2\; d\sigma = 0.$$

Since $|\lambda| > \frac{1}{2}$, we have

$$\int_{\Gamma_1} \varphi_1\; d\sigma = \int_{\Gamma_2} \varphi_2\; d\sigma = 0. \tag{6.11}$$

Now, consider the function

$$\mathbf{w} = \mathbf{S}_1\varphi_1 + \mathbf{S}_2\varphi_2 \quad \text{on } \mathbb{R}^n, \tag{6.12}$$

where $\mathbf{S}_i$ is the single layer potential on Γ_i, $i = 1, 2$. that is,

$$\mathbf{S}_i\varphi_i(x) = \int_{\Gamma_i} \Phi(x, y)\varphi_i(y)\, d\sigma(y), \quad i = 1, 2.$$

and $\Phi(x, y)$ is the fundamental solution to the Laplace equation. From the smoothness of the single layer potentials, we see that $\mathbf{w}$ is piecewise harmonic, that is, $\mathbf{w}$ is harmonic inside and outside the inclusion $D_1 \cup D_2$. By using (6.11), we easily conclude that

$$\mathbf{w}(x) = O(|x|^{1-n}) \quad \text{as} \quad |x| \longrightarrow \infty. \tag{6.13}$$

Since $\mathbf{w}$ is harmonic outside $D_1 \cup D_2$, then by using (6.13) we obtain

$$\nabla\mathbf{w}(x) = O(|x|^{-n}) \quad \text{as} \quad |x| \longrightarrow \infty. \tag{6.14}$$

Furthermore, from Lemma 2.18, we have the bounds

$$\left\|\nabla\mathbf{S}_j\varphi_j\right\|_{\alpha'\left(\overline{D_j}\right)} + \left\|\nabla\mathbf{S}_j\varphi_j\right\|_{\alpha'\left(\mathbb{R}^n\setminus D_j\right)} \leq C\left\|\varphi_j\right\|_{\alpha} \quad \alpha' < \alpha, \;\; j = 1, 2. \tag{6.15}$$

Now we show that $\mathbf{w}$ is locally a weak solution to

$$\begin{cases} \operatorname{div}(a_k\, \nabla\mathbf{w}) = 0, \quad \text{in } \mathbb{R}^n \setminus \{0\}, \\ a_k = 1 + (k-1)\chi_{(D_1 \cup D_2)}. \end{cases} \tag{6.16}$$

Let $R >> 1$ and observe that

$$\begin{aligned}\int_{\mathcal{B}(R)} a_k \nabla \mathbf{w} \nabla \eta \, dx = k &\int_{D_1} \nabla \left(\mathbf{S}_1\varphi_1(x) + \mathbf{S}_2\varphi_2(x)\right) \nabla \eta \, dx \\ + k &\int_{D_2} \nabla \left(\mathbf{S}_1\varphi_1(x) + \mathbf{S}_2\varphi_2(x)\right) \nabla \eta \, dx \\ + &\int_{\mathcal{B}(R)\setminus(D_1 \cup D_2)} \nabla \left(\mathbf{S}_1\varphi_1(x) + \mathbf{S}_2\varphi_2(x)\right) \nabla \eta \, dx.\end{aligned}$$

Using the divergence theorem, it follows

$$\begin{aligned}\int_{\mathcal{B}(R)} a_k \nabla \mathbf{w} \nabla \eta \, dx = k &\int_{\Gamma_1} \Big(\partial\nu^{-}\mathbf{S}_1\varphi_1(x) + \partial\nu\mathbf{S}_2\varphi_2(x)\Big)\eta \, d\sigma(x) \\ + k &\int_{\Gamma_2} \Big(\partial\nu\mathbf{S}_1\varphi_1(x) + \partial\nu^{-}\mathbf{S}_2\varphi_2(x)\Big)\eta \, d\sigma(x) \\ - &\int_{\Gamma_1} \Big(\partial\nu^{+}\mathbf{S}_1\varphi_1(x) + \partial\nu\mathbf{S}_2\varphi_2(x)\Big)\eta \, d\sigma(x) \\ - &\int_{\Gamma_2} \Big(\partial\nu\mathbf{S}_1\varphi_1(x) + \partial\nu^{+}\mathbf{S}_2\varphi_2(x)\Big)\eta \, d\sigma(x).\end{aligned}$$

Utilizing the jump conditions for the single layer potential (2.23), we have

$$\begin{aligned}\int_{\mathcal{B}(R)} a_k \nabla \mathbf{w} \nabla \eta \, dx = (1-k) &\int_{\Gamma_1} \Bigg(\left(\lambda I - K_1^*\right) \varphi_1(x) - \partial\nu\mathbf{S}_2\varphi_2(x) \Bigg)\eta \, d\sigma(x) \\ + (1-k) &\int_{\Gamma_2} \Bigg(\left(\lambda I - K_2^*\right) \varphi_2(x) - \partial\nu\mathbf{S}_1\varphi_1(x) \Bigg)\eta \, d\sigma(x).\end{aligned}$$

By applying (6.8), it follows that

$$\int_{\mathcal{B}(R)} a_k \nabla \mathbf{w} \nabla \eta \, dx = 0.$$

Thus $\mathbf{w}$ is a local solution to (6.16). Next, we show $\mathbf{w} \equiv 0$ on $\mathbb{R}^n$. Let $\mathbf{r} >> 1$ and $\eta = \mathbf{w}\chi$, where $\chi \in C_0^\infty(\mathbb{R}^n)$ be such that $\chi \equiv 1$ in $\mathcal{B}(\mathbf{r}-1)$ and $\chi \equiv 0$ outside $\mathcal{B}(\mathbf{r})$. Then we have

$$\int_{\mathcal{B}(\mathbf{r})} a_k \nabla \mathbf{w} \cdot \nabla\big(\mathbf{w}\chi\big)\ dx = 0.$$

That is

$$\int_{\mathcal{B}(\mathbf{r})} a_k\, |\nabla \mathbf{w}|^2\, \chi\ dx + \int_{\mathcal{B}(\mathbf{r})\setminus\mathcal{B}(\mathbf{r}-1)} a_k \big(\nabla \mathbf{w} \cdot \nabla \chi\big) \mathbf{w}\ dx = 0. \tag{6.17}$$

For the second integral in the right hand side of (6.17), we use the decay conditions (6.13) and (6.14) to obtain the following

$$\left| \int_{\mathcal{B}(\mathbf{r})\setminus\mathcal{B}(\mathbf{r}-1)} a_k \big(\nabla \mathbf{w} \cdot \nabla \chi\big) \mathbf{w}\ dx \right| \le C\ \mathbf{r}^{-n+1} \longrightarrow 0 \ \text{ as } \ \mathbf{r} \longrightarrow \infty. \tag{6.18}$$

Therefore when $\mathbf{r}$ approaches ∞ in (6.17), we conclude the following

$$\int_{\mathbb{R}^n} a_k\, |\nabla \mathbf{w}|^2\ dx = 0. \tag{6.19}$$

Thus $\mathbf{w}$ is a constant in $\mathbb{R}^n$. Then by using the decay condition (6.13), we have $\mathbf{w} \equiv 0$ in $\mathbb{R}^n$. Consequently, involving the jump conditions for the single layer potential (2.23), we have the following

$$\varphi_i(x) = \frac{\partial}{\partial \nu^+}\mathbf{w} - \frac{\partial}{\partial \nu^-}\mathbf{w} = 0, \quad x \in \Gamma_i \setminus \{0\}, \quad i = 1, 2.$$

Then by using the continuity of φ_i at 0, it follows that $\varphi_i \equiv 0$ in $\mathbb{R}^n$. Thus T^0 is injective. $\square$

Since T^0 and $\Lambda_{\epsilon,0}$ are invertible and

$$T^0 = \left(I + C_{\epsilon,0}\Lambda_{\epsilon,0}^{-1}\right)\Lambda_{\epsilon,0}.$$

Then we see that

$$\Lambda_{\epsilon,0}\, T^{0^{-1}} = \left(I + C_{\epsilon,0}\Lambda_{\epsilon,0}^{-1}\right)^{-1}.$$

That is, $\left(I + C_{\epsilon,0}\Lambda_{\epsilon,0}^{-1}\right)^{-1}$ exists. From Theorem 6.12 we have

$$I + C_{\epsilon,\delta}\Lambda_{\epsilon,\delta}^{-1} \longrightarrow I + C_{\epsilon,0}\Lambda_{\epsilon,0}^{-1} \text{ in } C^{\alpha}(\Gamma_1) \times C^{\alpha}(\Gamma_2).$$

Thus from Lemma (6.7) we have that $\left(I + C_{\epsilon,\delta}\Lambda_{\epsilon,\delta}^{-1}\right)^{-1}$ exist for δ is sufficiently small and they are uniformly bounded with respect to δ in the operator norm. Therefore, we obtain the pointwise convergence

$$\left(I + C_{\epsilon,\delta}\Lambda_{\epsilon,\delta}^{-1}\right)^{-1} \longrightarrow \left(I + C_{\epsilon,0}\Lambda_{\epsilon,0}^{-1}\right)^{-1}. \tag{6.20}$$

Since $T^\delta = \left(I + C_{\epsilon,\delta}\Lambda_{\epsilon,\delta}^{-1}\right)\Lambda_{\epsilon,\delta}$, then by using the bound (6.7) we conclude that

$$(T^\delta)^{-1} = \Lambda_{\epsilon,\delta}^{-1}\left(I + C_{\epsilon,\delta}\Lambda_{\epsilon,\delta}^{-1}\right)^{-1}$$

are uniformly norm bounded and satisfy

$$(T^\delta)^{-1} \longrightarrow (T^0)^{-1} \text{ in } C^{\alpha'}(\Gamma_1) \times C^{\alpha'}(\Gamma_2), \ \ \alpha' < \alpha,$$

as $\delta \longrightarrow 0$. Thus we have proved the following Theorem.

Theorem 6.14. *For $\lambda = \frac{k+1}{2(k-1)}$ and $\alpha < \alpha_0$. There exists $\delta_0 > 0$ such that the operators T^δ, $0 \le \delta \le \delta_0$, are invertible with inverses that are bounded independently of δ in $\mathcal{L}(C^\alpha(\Gamma_1) \times C^\alpha(\Gamma_1))$, $\alpha < \alpha_0$. Moreover, the operators $(T^\delta)^{-1}$ converge pointwise to $(T^0)^{-1}$ as δ approaches 0 in $\mathcal{L}(C^{\alpha'}(\Gamma_1) \times C^{\alpha'}(\Gamma_1))$ for any $\alpha' < \alpha < \alpha_0$.*

6.4 The main results.

The main results are similar to the case of dimension $n = 2$. The solution of problem (2.5) has the representation (2.58) in terms of the solutions $\left(\varphi_1^\delta, \varphi_2^\delta\right)$ to (2.59) and the harmonic function H_δ from (2.49). A similar relationship holds between the solution u_0 to problem (2.4) with touching inclusions and the solutions $(\varphi_1^0, \varphi_2^0)$ to

$$T^0 \begin{pmatrix} \varphi_1^0 \\ \varphi_2^0 \end{pmatrix} = \begin{pmatrix} \partial_\nu H_0 \mid_{\Gamma_1} \\ \partial_\nu H_0 \mid_{\Gamma_2} \end{pmatrix}, \tag{6.21}$$

where H_0 is the harmonic function from (2.48). This is the assertion of the following theorem.

Theorem 6.15. *[1] The solution u_0, to (2.4), may be written*

$$u_0(x) = S_{\Gamma_1}\varphi_1^0(x) + S_{\Gamma_2}\varphi_2^0(x) + H_0(x), \quad x \in \Omega, \tag{6.22}$$

where H_0 is harmonic inside Ω, and defined by (2.48), and the pair$(\varphi_1^0, \varphi_2^0) \in C^\alpha(\Gamma_1) \times C^\alpha(\Gamma_2)$ is the unique solution to (6.21).

Proof. Sine H_0 is harmonic inside Ω, and since Γ_1 and Γ_2 are $C^{1+\alpha_0}$, the right-hand side of (6.21) lies in $C^\alpha(\Gamma_1) \times C^\alpha(\Gamma_2)$ for any $\alpha \le \alpha_0$. By using theorem (6.14), the integral equation (6.21) therefor has a unique solution $(\varphi_1^0, \varphi_2^0) \in C^\alpha(\Gamma_1) \times C^\alpha(\Gamma_2)$, for any $\alpha \le \alpha_0$.

Utilizing Lemma (2.16), we see that

$$\partial_\nu H_\delta \mid_{\Gamma_i} \longrightarrow \partial_\nu H_0 \mid_{\Gamma_i}, \text{ in } C^\alpha(\Gamma_i) \quad \text{as } \delta \longrightarrow 0.$$

So we infer from theorem (6.14) that

$$\begin{aligned}
\begin{pmatrix} \varphi_1^\delta \\ \\ \varphi_1^\delta \end{pmatrix} - \begin{pmatrix} \varphi_1^0 \\ \\ \varphi_1^0 \end{pmatrix} &= \left(T^\delta\right)^{-1} \begin{pmatrix} \partial_\nu H_\delta \mid_{\Gamma_1} \\ \\ \partial_\nu H_\delta \mid_{\Gamma_2} \end{pmatrix} - \left(T^0\right)^{-1} \begin{pmatrix} \partial_\nu H_0 \mid_{\Gamma_1} \\ \\ \partial_\nu H_0 \mid_{\Gamma_2} \end{pmatrix} \\
&= \left(T^\delta\right)^{-1} \left[\begin{pmatrix} \partial_\nu H_\delta \mid_{\Gamma_1} \\ \\ \partial_\nu H_\delta \mid_{\Gamma_2} \end{pmatrix} - \begin{pmatrix} \partial_\nu H_0 \mid_{\Gamma_1} \\ \\ \partial_\nu H_0 \mid_{\Gamma_2} \end{pmatrix} \right] \\
&+ \left[\left(T^\delta\right)^{-1} - \left(T^0\right)^{-1} \right] \begin{pmatrix} \partial_\nu H_0 \mid_{\Gamma_1} \\ \\ \partial_\nu H_0 \mid_{\Gamma_2} \end{pmatrix} \\
&\longrightarrow 0 \quad \text{in } C^{\alpha'}(\Gamma_1) \times C^{\alpha'}(\Gamma_2), \ \ 0 < \alpha' < \alpha.
\end{aligned}$$

This convergence of φ_i^δ immediately implies that

$$\mathbf{S}_1\varphi_1^\delta(x + \frac{\delta}{2}\, e_n) \longrightarrow \mathbf{S}_1\varphi_1^0(x), \ \text{ and } \ \mathbf{S}_2\varphi_2^\delta(x - \frac{\delta}{2}\, e_n) \longrightarrow \mathbf{S}_2\varphi_2^0(x), \tag{6.23}$$

uniformly on compact subdomains of $\Omega \setminus (\Gamma_1 \cup \Gamma_2)$ as $\delta \longrightarrow 0$.

Consider now the solution to problem (2.5)

$$u_\delta(x) = \mathbf{S}_1\varphi_1^\delta(x + \frac{\delta}{2}\, e_n) + \ \mathbf{S}_2\varphi_2^\delta(x - \frac{\delta}{2}\, e_n) + H_\delta(x).$$

From Lemma 2.16, we know that $H_\delta \longrightarrow H_0$ uniformly on compact subdomains of Ω, and if we combine this with (6.23), it follows that

$$u_\delta(x) = \mathbf{S}_1\varphi_1^\delta(x + \frac{\delta}{2}\, e_n) + \ \mathbf{S}_2\varphi_2^\delta(x - \frac{\delta}{2}\, e_n) + H_\delta(x) \longrightarrow \mathbf{S}_1\varphi_1^0(x) + \ \mathbf{S}_2\varphi_2^0(x) + H_0(x),$$

uniformly on compact subdomains of $\Omega \setminus \Gamma_1 \cup \Gamma_2$ as $\delta \longrightarrow 0$.

Since we also know that $u_\delta \longrightarrow u_0$ in $H^1(\Omega)$, it follows from uniqueness of the limit that

$$u_0 = \mathbf{S}_1\varphi_1^0 + \ \mathbf{S}_2\varphi_2^0 + H_0, \tag{6.24}$$

on compact subdomains of $\Omega \setminus \Gamma_1 \cup \Gamma_2$. But both sides of (6.24) are continuous functions in Ω, we get that

$$u_0(x) = \mathbf{S}_1\varphi_1^0(x) + \ \mathbf{S}_2\varphi_2^0(x) + H_0(x), \quad x \in \Omega.$$

$\square$

Theorem 6.16. *[1] Let $\tilde{\epsilon} > 0$ and $0 < \alpha < \alpha_0$. The solution to (2.5) satisfies*

$$\|u_\delta\|_{C^{1,\alpha}(\Omega_{\tilde{\epsilon}} \setminus \overline{D_1^\delta \cup D_2^\delta})} + \|u_\delta\|_{C^{1,\alpha}(\overline{D_1^\delta})} + \|u_\delta\|_{C^{1,\alpha}(\overline{D_2^\delta})} \leq C\, \|g\|_{L^2(\partial\Omega)}\,, \tag{6.25}$$

where C is independent of δ and g.

Proof. Recall that u_δ has the representation

$$u_\delta(x) = \mathbf{S}_1\varphi_1^\delta(x + \frac{\delta}{2}\, e_n) + \ \mathbf{S}_2\varphi_2^\delta(x - \frac{\delta}{2}\, e_n) + H_\delta(x),$$

where $\left(\varphi_1^\delta, \varphi_2^\delta\right)$ solves (2.59) in $C^{\alpha'}(\Gamma_1) \times C^{\alpha'}(\Gamma_2)$, for any $\alpha < \alpha' < \alpha_0$. From equation

(2.62), we have

$$\left\|S_i\varphi_i^\delta\right\|_{C^{1,\alpha}\left(\overline{D_i^\delta}\right)} + \left\|S_i\varphi_i^\delta\right\|_{C^{1,\alpha}\left(\overline{\Omega_{\tilde{\epsilon}}\setminus D_i^\delta}\right)} \leq C\left\|\varphi_i^\delta\right\|_{C^{\alpha'}(\Gamma_i)}, \quad i=1,2.$$

Due to theorem (6.14) and the fact that $\left(\varphi_1^\delta, \varphi_2^\delta\right)$ solves (2.59), we have

$$\left\|\varphi_1^\delta\right\|_{C^{\alpha'}(\Gamma_1)} + \left\|\varphi_2^\delta\right\|_{C^{\alpha'}(\Gamma_2)} \leq C\ \left\|H_\delta\right\|_{C^{1,\alpha'}(\Omega_{\tilde{\epsilon}})}.$$

Applying Lemma (2.16), we have that

$$\left\|H_\delta\right\|_{C^{1,\alpha'}(\Omega_{\tilde{\epsilon}})} \leq C\left\|g\right\|_{L^2(\partial\Omega)}.$$

□

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