THE STABILITY OF BOUNDARY LAYER FLOW SUBJECT TO ROTATION

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This is to certify that the

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ABSTRACT

THE STABILITY OF BOUNDARY LAYER FLOW
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The stability of three dimensional Tollmien-Schlichting waves moving in a boundary layer flow on a flat plate which is rotating about its leading edge is investigated. Primary interest is in the effect of rotation on the critical point of the neutral stability curve. The problem is formulated in terms of a sixth-order differential equation, analogous to the Orr-Sommerfeld equation, with homogeneous boundary conditions. The eigen-value problem thus formed is then transformed into an initial value problem which is solved numerically.

The numerical technique developed first solves for the primary velocity profile and its derivatives. Then using a Runge-Kutta fourth order method and a special filtering technique the eigen-values are determined by iteration.

The results indicate that as the flat plate rotates into the flow it is destabilizing and as the plate rotates away from the flow it is stabilizing. There is also a span-wise wave number influence which amplifies this effect. The results compare favourably with known experimental results.

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By

Mangal Dass Chawla

A THESIS

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A DEDICATION

This thesis is dedicated to the author's loving wife Santosh who supplied encouragement and displayed great patience and tolerance during the entire course of the graduate study without which the author would not have successfully completed this work.

This is also dedicated to the author's elder son Rajeev and the newly born twins Rakesh and Renu.

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NOMENCLATURE

$C = C_r + iC_i = c$	Complex propagation speed of wave
C_r	Wave speed
C_i	Amplification factor
$D = \frac{d}{dy}$	Differential operator
$E = \exp[i(\alpha x + \beta z) - i\,\alpha ct]$	
F_i	Components of body forces
$f = f(\eta) = \frac{\Psi}{\sqrt{\nu \bar{x} U_\infty}}$	Dimensionless stream function
i,j,k	Indices
L_i	Differential operators (i = 1 to 9)
P	Pressure of undisturbed flow (Non-dimensional)
$\bar{p}$	Pressure of undisturbed flow (Dimensional)
$p = P + p'$	Pressure after perturbation
p'	Perturbation pressure
$R = \frac{U_\infty \delta}{\nu}$	Reynolds number based on boundary layer thickness δ
$R_x = \frac{U_\infty X}{\nu}$	Reynolds number based on the distance from the leading edge
ΔR	Change in R
r	Radial distance from the origin
r, r_1, r_2, r_3	Constants
S_1, S_2	Constants

T	Test function based on combined $D\Phi$ at the plate
T_1, T_2, T_3	Values of T during iteration
$t = \frac{\bar{\tau} U_\infty}{\delta}$	Time (Non-dimensional)
$U = \frac{\bar{u}}{U_\infty}$	x - Component of velocity of primary flow
U_∞	Free stream velocity
$u = U + u'$	x - component of velocity after perturbation
u'	Perturbation in x - component of velocity
$\hat{u}$	Amplitude of u' perturbation of velocity
$\bar{u}_i$	Dimensional components of velocity along coordinate axes $(i = 1,2,3)$
u_i	Dimensionless components of velocity along coordinate axes $(i = 1,2,3)$
V	y - component of velocity of primary flow
$v = V + v'$	y - component of velocity after perturbation
v'	Perturbation in y - component of velocity
$\hat{v}$	Amplitude of v' perturbation of velocity
W	z - component of velocity of primary flow

$w = W + w'$	z - component of velocity after perturbation
w'	Perturbation in z - component of velocity
$\hat{w}$	Amplitude of w' perturbation of velocity
$\bar{\omega} = \frac{\bar{p}}{\rho} - \frac{1}{2}\lvert\bar{\Omega} \times \bar{r}\rvert^2$	Pressure term including the effect of centrifugal forces (Dimensional)
$\underline{\omega} = \frac{\bar{\omega}}{U_\infty^2}$	Pressure term including the effect of centrifugal forces (Non-dimensional)
$\omega = \underline{\omega} + \omega'$	Pressure term ω after perturbation
ω'	Perturbation in pressure term ω
$\hat{\omega}$	Amplitude of ω' perturbation in pressure term
$W_1 = \frac{dz}{d\eta}$	
X	Coordinate parallel to plate in the direction of free stream
X (1), X(2)	Multiplying constants
$\bar{x}_i$	Dimensional coordinates X, Y and Z respectively $(i = 1,2,3)$
x_i	Non-dimensional coordinates x, y and z respectively $(i = 1,2,3)$
Y	Coordinate prependicular to plate
Z	Coordinate perpendicular to the planes of X and Y
$Z_1, Z_2, Z_3, Z_4, Z_5, Z_6$	New variables equal to $\hat{u}$, $\hat{v}$, $\hat{w}$, $D\hat{u}$, $R\hat{\omega}$ and $D\hat{w}$ respectively

$z = \frac{df}{d\eta}$	
α	Wave number
$\Delta\alpha$	Change in α
β	Spanwise wave number
δ	Boundary layer thickness
δ^*	Displacement thickness
$\eta = \frac{\bar{y}}{\sqrt{\frac{\nu\bar{x}}{U_\infty}}}$	Transformation variable
$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$	Laplace operator
$\zeta = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$	Vorticity
ρ	Density of fluid
ν	Kinematic viscosity of fluid
$\bar{\tau}$	Time (Dimensional)
$K^2 = \alpha^2 + \beta^2$	Square of the combined wave number
$\bar{\Omega}$	Angular rotation (Dimensional)
$\Omega = \frac{\bar{\Omega}\,\delta}{U_\infty}$	Angular rotation (Non-dimensional)
$\phi = \phi_r + i\,\phi_i = \hat{v}$	Complex eigenfunction - y dependent factor in perturbation stream function v'
$\phi_{W1}, \phi_{W2}, \phi_{W3}$	Individual eigenfunctions at the plate
Φ_W	Combined eigenfunction at the plate
$\Psi = \Psi_r + i\,\Psi_i = \hat{u}$	Complex stream function - y dependent factor in perturbation stream function u'
$\Psi_{W1}, \Psi_{W2}, \Psi_{W3}$	Individual stream-functions at the plate

Ψ_W	Combined stream function at the plate

CHAPTER I

INTRODUCTION

1.1 Review of Literature

The study of transition from laminar to turbulent flow for parallel shear flow started at the end of the ninteenth century. This was first experimentally studied by Reynolds (1883) by injecting dye in flow through pipes. On the basis of these experiments Reynolds (1895) stated that the motion of the water was direct or sinuous according as the mean velocity

$$Um \lessgtr k \frac{\mu}{D\rho}$$

where D = Diameter of the pipe

ρ = Density of water

μ = Absolute viscosity of water

k = Dimensionless number

or

$$\frac{\rho\ Um\ D}{\mu} \lessgtr k.$$

The number k was found to lie between 1900 and 2000 for a circular tube. Thus Reynolds found that steady direct motion in round tubes was stable if

$$\frac{\rho\ Um\ D}{\mu} < 1900$$

and unstable if

$$\frac{\rho\ Um\ D}{\mu} > 2000.$$

Later his theoretical analysis gave the value of k = 517 for flow between parallel plates. Then with the result that the critical velocity was inversely proportional to the hydraulic mean depth, he inferred that for a circular pipe $k = 1034$. This was below the experimental value. Reynolds stated that this discrepancy was due to not meeting all the kinematical conditions. The corresponding values for round and flat tubes were later found to be 470 and 167 respectively by Sharp (1905).

In the meantime Lord Rayleigh (1880, 1887) studied the problem of stability in perfect fluids. Lord Kelvin (1887), objecting to the neglect of viscosity by Rayleigh, found that the mathematical difficulties were enormous when viscosity was considered. Later Orr (1907) and Sommerfeld (1908) each independently made a complete theoretical analysis of the entire stability problem. The stability equation thus derived is now known as the Orr-Sommerfeld equation. Orr discussed the stability of both perfect and viscous fluids. Sommerfeld studied couette flow and found no instability. Both Orr and Sommerfeld assumed sinusoidal, periodic, infinitesimal disturbances to study stability. Rayleigh (1914) took the Orr-Sommerfeld equation for the two-dimensional frictionless case, given by

$$(U - c)(\phi'' - \alpha^2\phi) - U''\phi = 0 \qquad (1\text{-}1\text{-}1)$$

where U = mean velocity

c = complex wave velocity

α = wave number

ϕ = eigen-function,

written in the form

$$\phi'' = \frac{U''\phi}{U-c} + \alpha^2\phi$$

and found that at a distance from the boundary where $U = c$, U'' must be zero, that is there must be an inflexion point if the flow is to be unstable.He called the layer at this point the inner friction layer. Rayleigh thus found that the steady motion of an inviscid fluid in two dimensions between fixed parallel planes was stable provided the primary velocity U, parallel to the walls and a function of y only, was such that U'' does not change sign anywhere. In other words, Rayleigh proved that the existence of a point of inflexion constituted a necessary condition for the occurrence of instability. Later Tollmien (1935) showed that this was also a sufficient condition for the disturbances to grow and create instability. A similar result, found by Tietjens (1925), shows that if the basic velocity profile is approximated by multiple straight line segments stability results if the corners are convex and instability results if the corners are concave.

Prandtl (1914) defined transition, separation and drag coefficients on bodies. He first tripped a boundary layer with a wire. The discovery of a boundary layer opened new fields for the study of transition to turbulence. Prandtl (1914, 1921) started the analysis of the stability equation retaining only the larger terms involving viscous effects close to the wall and found that with this hypothesis all profiles made up of line segments were unstable for all wave lengths. This result was later confirmed by Tietjens (1925). Heisenberg (1924) applied this assumption to the basic profile, taking it as parabolic, and concluded instability. Tollmien (1929) arrived at the conclusion that viscous effects must be taken into account near the wall and at the level where $U = c$. He later found the critical Reynolds number to be 420 for flow over a flat plate at zero incidence. Tollmien's method, also known as the Asymptotic Theory, was later applied successfully by Schlichting (1933, 1935) to the velocity profiles in the boundary layer, when approximated by a straight line or a parabola. Lin (1945, 1946, 1966) gave the asymptotic theory a firm mathematical footing. He explained the behaviour of the functions near the singularities. Later, Shen (1954) working with the asymptotic method improved the accuracy of the stability calculations.

Synge (1938) surveyed the whole subject of Hydrodynamic Stability and explained in detail the physical problem and its mathematical formulation. He described the application of the method of the exponential time-factor to the couette flow with general and plane disturbances and to the Poiseuille motion in a tube of general section or in a circular tube with disturbances having rotational or axial symmetry.

The asymptotic method of solution has been successfully applied to many stability problems. Recently, the advent of large digital computers, have made numerical schemes for solving the Orr-Sommerfeld equations very popular. Numerical methods are usually preferred to the asymptotic methods as they are more accurate and relatively less time consuming, however, asymptotic methods are still employed, especially in the zones of large Reynolds number where numerical solutions show poor convergence and when general properties of the solutions are to be studied.

Gill (1950) discussed the integration of the differential equations on the computers and also developed a form that gave the highest accuracy with minimum machine instructions. His method later became very popular and has been successfully used by many authors.

Thomas (1953) investigated the problem of the stability of plane poiseuille flow and verified the work of Pekris (1948) who had used the method of asymptotic expansions. Kaplan (1964) employed a numerical scheme to study the stability of laminar incompressible boundary layers with compliant boundaries. He presented the numerical results for a variety of compliant boundaries. He forms an initial value problem from the boundary value problem at some distance from the compliant boundary before starting numerical integration as did Nachtsheim (1964). Kaplan studied the problem in detail when some roots grow very large and thus due to the limitations of the machine computations, start affecting the other roots which should otherwise stay independent. In Kaplan's terminology, the roots were called "the growing roots" and "the well behaved roots" respectively. He outlined a method of filtering the contributions of the growing roots from the well behaved roots as the integration proceeds and later repaired the well behaved roots. This method as advocated by Kaplan was later advantageously used by Lee and Reynolds (1964) and Reynolds and Potter (1967). Lee and Reynolds discussed other numerical methods and compared them to the variational solution of the Orr-Sommerfeld equations. They, of course, discussed other numerical integration methods where one may or may not use the filtering scheme used by Kaplan.

Potter (1965) studied the stability of plane Couette-Poiseuille flow by asymptotic expansions and later in 1967 of symmetrical parabolic flows numerically. Radbill and Van Driest (1966) investigated the stability of laminar boundary layers by quasilinearization and numerically.

On the experimental side work was done by Nikuradse (1933). He excited the boundary layer with sinusoidal oscillations near the leading edge of a flat plate in water. Dryden (1936) measured the velocity fluctuations in a laminar boundary layer, but these fluctuations were said to be imposed by the mean flow because they were irregular. Schubauer and Skramstad (1947) experimentally proved that induced oscillations exist in the boundary layer. Dryden (1947) and Schubauer and Skramstad (1947) experimented with very low free stream turbulence obtaining very large transition Reynolds numbers.

It is generally believed that there are three stages in the transition process before the turbulence sets in. At the first stage infinitesimal Tollmien-Schlichting waves become unstable. At the second stage two-dimensional Tollmien-Schlichting waves become three-dimensional. At the third stage bursts from the boundary start. The flow will be termed laminar up to the start of the third stage. The fourth stage is characterized by the constant burst rate from the wall. The critical Reynolds number R_{crit} given by the stability theory is the lowest Reynolds number at which a shear layer can bear and amplify an infinitesimal Tollmien-Schlichting wave. This is the start of the first stage as mentioned above. Morkovin (1958) writes that transition to turbulence is generally found to occur at Reynolds numbers much greater than R_{crit}, which indicates that certain investigators may refer to the third stage of the transition process as the beginning of transition. Emmons (1951) believes that sometimes disturbances that trigger transition may be "local in time". But once initiated, a small turbulent spot moves downstream and grows steadily in all directions, sweeping out a tongue like wedge with time. Morkovin (1958) writes that in the laminar layer the unsteadiness and vorticity of a lump of fluid decays with time during its travel, that is, the unsteadiness does not feed on the energy of the mean shear motion. Morkovin further states that many investigators feel that three-dimensionality, however small, must lead to small loops in originally straight vortex lines of the steady motion and/or the unsteady regular two-dimensional velocity distributions. These loops would inevitably stretch into longer and longer loops by the differences in mean velocity at different heights of shear layer, thus increasing ω' fluctuations.

In the area of present study Conard of Princeton University (1962) investigated the effects of rotation on the stability of laminar boundary layers on curved walls. He assumed these walls to be rotating about an axis perpendicular to both the flow direction and the radii of curvature. He considered Taylor-Görtler waves and arrived at the conclusion that positive rotations (rotations for which the absolute velocity of the fluid exceeds its relative velocity) were destabilizing and negative rotations were stabilizing. Regarding Tollmien-Schlichting waves Conard stated that when he considered two-dimensional Tollmien-Schlichting waves, the curvature and rotation terms dropped out of the equations for the order of his approximations. Had he considered three-dimensional disturbances the rotation influence would have been retained.

Experimentally, the only work available is due to that of Halleen and Johnston of Stanford (1967). Their results are similar to those obtained by Conard and now numerically verified by this study.

1.2 Purpose of the Present Investigations

The purpose of the present study is to investigate the stability of boundary layer flow subject to rotation with the axis of rotation perpendicular to the flow direction and parallel to the flat plate (see figure 1). The flow is subjected to three-dimensional Tollmien-Schlichting waves including wave number α and spanwise wave number β. Primary emphasis has been placed on neutral stability, that is, the conditions for which these infinitesimal disturbances neither decay nor grow with time. The result establishes the beginning of the first stage of transition. New information which this study furnishes is the influence of angular rotation Ω on this neutral stability of the boundary layer. Various spanwise wave numbers β are included for each Ω.

1.3 Conditions of this Study

In the numerical computations for the present investigation, the boundary value problem is transformed into an initial value problem with calculations starting at $Y = 1.5\ \delta$. This is the point which will be used as "infinity". It is dictated by the step size and the storage limitations of the machine. Another limiting condition for the present study is that convergence to the true eigenvalues is assumed to be complete when the test function T based on ϕ' at the plate divided by ϕ' at one step behind is less than 10^{-3}.

Under these conditions, this study determines the critical Reynolds number R_{crit} for different values of Ω and β at which instability due to three-dimensional, infinitesimal disturbances first sets in. The values of Ω and β along with the results are discussed in Chapter IV.

CHAPTER II

INFINITESIMAL DISTURBANCES ON A BOUNDARY LAYER FLOW

2.1 Governing Equations

Consider laminar flow of an incompressible fluid over a flat plate which is rotating with an angular velocity $\vec{\bar{\Omega}}$ about its leading edge in a counterclockwise sense. The coordinate system, shown in figure 1, is attached to the rotating plate so that the flow can be considered steady when measured in this rotating system. This requires the fluid at infinity to be rotating with the same angular velocity.

The Navier-Stokes equations, governing the motion, can be written as

$$\frac{\partial \bar{u}_i}{\partial \bar{\tau}} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial \bar{x}_j} = \bar{F}_i - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{x}_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial \bar{x}_j \partial \bar{x}_j} + \frac{\partial}{\partial \bar{x}_i}\left(\frac{1}{2}|\vec{\bar{\Omega}} \times \vec{r}|^2\right) + 2\,\epsilon_{ijk}\bar{u}_j\bar{\Omega}_k \qquad (2.1.1)$$

where the bars denote dimensional quantities and

$$\bar{\Omega}_i = (0,0,\bar{\Omega}). \qquad (2.1.2)$$

The last two terms in equation (2.1.1) account for the non-inertial coordinate system chosen. We may combine the pressure term and the centrifugal acceleration term to obtain

$$-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{x}_i} + \frac{\partial}{\partial \bar{x}_i}\left(\frac{1}{2}|\vec{\bar{\Omega}} \times \bar{r}|^2\right) = -\frac{\partial \bar{\omega}}{\partial \bar{x}_i} \qquad (2.1.3)$$

where $$\bar{\omega} = \frac{\bar{p}}{\rho} - \frac{1}{2}|\vec{\bar{\Omega}} \times \bar{r}|^2. \tag{2.1.4}$$

Neglecting body forces and introducing (2.1.3) in (2.1.1), there results

$$\frac{\partial \bar{u}_i}{\partial \bar{\tau}} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial \bar{x}_j} = - \frac{\partial \bar{\omega}}{\partial \bar{x}_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial \bar{x}_j \partial \bar{x}_j} + 2\, \epsilon_{ijk} \bar{u}_j\, \bar{\Omega}_k \tag{2.1.5}$$

The continuity equation for incompressible flow is

$$\frac{\partial \bar{u}_i}{\partial \bar{x}_i} = 0. \tag{2.1.6}$$

Equations (2.1.5) and (2.1.6) are the equations which govern the flow.

2.2 Non-Dimensional Variables

To non-dimensionalize choose u_∞ as the reference velocity and δ as a characteristic length, then the dimensionless variables will be

$$\left.\begin{aligned} x_i &= \frac{\bar{x}_i}{\delta} \\ u_i &= \frac{\bar{u}_i}{U_\infty} \\ \Omega &= \frac{\delta \bar{\Omega}}{U_\infty} \\ t &= \frac{U_\infty \bar{\tau}}{\delta} \\ \omega &= \frac{\bar{\omega}}{U_\infty^2} \end{aligned}\right\} \tag{2.2.1}$$

Introducing (2.2.1) in (2.1.5) and (2.1.6) we obtain the Navier Stokes and continuity equations in dimensionless form as

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = - \frac{\partial \omega}{\partial x_i} + \frac{1}{R} \frac{\partial^2 u_i}{\partial x_j \partial x_j} + 2\, \epsilon_{ijk} u_j \Omega_k \tag{2.2.2}$$

and

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2.2.3}$$

where the Reynolds number is

$$R = \frac{U_\infty \delta}{\nu} . \tag{2.2.4}$$

The Reynolds number can be changed by varying either U_∞, δ, ν, or $U_\infty \delta$, keeping the other variables constant. For flow over a flat plate, as in the present case, both U_∞ and ν will be kept fixed. The boundary layer thickness δ varies as the square root of the distance from the leading edge, that is

$$\delta \propto \sqrt{\frac{\nu X}{U_\infty}} \tag{2.2.5}$$

or

$$\delta \propto \frac{X}{\sqrt{R_x}} \tag{2.2.6}$$

where $R_x = \frac{U_\infty X}{\nu}$. Taking δ as a point at which the velocity is 99.9% of the free stream velocity U_∞, the constant of proportionality in (2.2.6) is approximately equal to 5.0. Using this relationship for δ in our definition of R we obtain

$$R = 5 \sqrt{\frac{U_\infty X}{\nu}} . \tag{2.2.7}$$

Knowing the Reynolds number R at which the flow becomes unstable one can then determine the distance from the leading edge along the plate at which the infinitesimal waves become unstable.

2.3 The Primary Flow

In order to observe the influence of Ω on the primary flow, let us write equations (2.2.2) in cartesian coordinates as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial \omega}{\partial x} + \frac{1}{R}\Delta u + 2v\,\Omega \tag{2.3.1}$$

and

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial \omega}{\partial y} + \frac{1}{R}\Delta v - 2u\,\Omega \tag{2.3.2}$$

The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2.3.3}$$

Differentiating (2.3.1) with respect to y, (2.3.2) with respect to x and subtracting gives

$$\begin{aligned}&\left(\frac{\partial^2 u}{\partial t\partial y} - \frac{\partial^2 v}{\partial t\partial x}\right) + u\frac{\partial^2 u}{\partial x\partial y} - u\frac{\partial^2 v}{\partial x^2} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\\ &\quad + v\frac{\partial^2 u}{\partial y^2} - v\frac{\partial^2 v}{\partial y\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} =\\ &\quad \frac{1}{R}\Delta\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + 2\,\Omega\left(\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right)\end{aligned} \tag{2.3.4}$$

The vorticity is

$$\zeta = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}. \tag{2.3.5}$$

Inserting this in (2.3.4) gives

$$\frac{D\zeta}{Dt} + \zeta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = \frac{1}{R}\Delta\zeta + 2\Omega\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{2.3.6}$$

Continuity then allows us to write

$$\frac{D\zeta}{Dt} = \frac{1}{R}\Delta\zeta\ . \tag{2.3.7}$$

Equation (2.3.7) shows that the coriolis term does not enter the vorticity equation. By an examination of equation (2.3.1) the effect of Ω is small and thus neglected because of the presence of the small velocity component v. The effect of Ω in equation (2.3.2) would be to influence the $\partial\omega/\partial y$ term. Hence Ω does not affect the primary velocity profile and the usual Blasius boundary layer results.

2.4 Three Dimensional Disturbances

Consider infinitesimal perturbations in the variables u, v, w and ω, so that after perturbing, these variables change to

$$\begin{aligned} u &= U + u' \\ v &= V + v' \\ w &= w' \\ \omega &= \underline{\omega} + \omega' \end{aligned} \tag{2.4.1}$$

where U, V and $\underline{\omega}$ pertain to the primary flow. The component of primary velocity parallel to z-axis is zero and the one parallel to y-axis is comparatively small. It is not unreasonable to assume V to be of order δ. Insert equations (2.4.1) in (2.2.2) and (2.2.3), neglect non-linear terms and write the resulting equations in cartesian coordinates to yield

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + u'\frac{\partial U}{\partial x} + v'\frac{\partial U}{\partial y} + V\frac{\partial u'}{\partial y} + w'\frac{\partial U}{\partial z} = -\frac{\partial \omega'}{\partial x} + \frac{1}{R}\Delta u' + 2v'\Omega \tag{2.4.2}$$

$$\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} + V\frac{\partial v'}{\partial y} + u'\frac{\partial V}{\partial x} + v'\frac{\partial V}{\partial y} + w'\frac{\partial V}{\partial z} = -\frac{\partial \omega'}{\partial y} + \frac{1}{R}\Delta v' - 2u'\Omega \tag{2.4.3}$$

$$\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} + V\frac{\partial w'}{\partial y} = -\frac{\partial \omega'}{\partial z} + \frac{1}{R}\Delta w' \tag{2.4.4}$$

where the Laplacian Δ is

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{2.4.5}$$

and

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \tag{2.4.6}$$

Lin (1966) has stated that, "It can be shown, by careful examination of all the terms in the first order disturbance equations of continuity, motion and energy, that the basic flow may be treated as essentially parallel. The analysis is fairly complicated, but the conclusion has been justified". The resulting equations are then

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + v'\frac{\partial U}{\partial y} = -\frac{\partial \omega'}{\partial x} + \frac{1}{R}\Delta u' + 2v'\Omega \tag{2.4.8}$$

$$\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} = -\frac{\partial \omega'}{\partial y} + \frac{1}{R}\Delta v' - 2u'\Omega \tag{2.4.9}$$

$$\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} = -\frac{\partial \omega'}{\partial z} + \frac{1}{R}\Delta w' \tag{2.4.10}$$

and

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0. \tag{2.4.11}$$

Consider further that the perturbations can be expanded in the form of three-dimensional disturbances, given by

$$\left.\begin{aligned} u' &= \hat{u}(y)\,.\,E \\ v' &= \hat{v}(y)\,.\,E \\ w' &= \hat{w}(y)\,.\,E \\ \omega' &= \hat{\omega}(y)\,.\,E \end{aligned}\right\} \tag{2.4.12}$$

where

$$E = \exp[i(\alpha x + \beta z) - i\,\alpha c t]. \tag{2.4.13}$$

The above implies a spatially periodic wave with complex amplitudes, $\hat{u}$, $\hat{v}$ and $\hat{w}$ in the x, y and z directions respectively. α and β are wave numbers which will be real. C is a complex wave velocity, given by

$$C = C_r + i\,C_i \tag{2.4.14}$$

where C_i is the amplification rate. C_i positive means growth and C_i negative implies decay. C_i equal to zero yields neutral stability, that is, neither growth nor decay of the perturbations. Actually, only the real parts of equations (2.4.12) are used to give the physical velocity and pressure perturbations.

2.5 Mathematical Formulation of the Three-Dimensional Linear Stability Problem

Introduce equations (2.4.12) in (2.4.8) to (2.4.11), cancel E throughout and rearrange. The following system of second order linear differential equations results:

$$\left.\begin{aligned}
&[D^2 - (\alpha^2 + \beta^2) - i\alpha R(U - c)]\hat{u} = R(DU - 2\Omega)\hat{v} + i\alpha R\hat{\omega} \\
&[D^2 - (\alpha^2 + \beta^2) - i\alpha R(U - c)]\hat{v} = 2R\Omega\hat{u} + RD\hat{\omega} \\
&[D^2 - (\alpha^2 + \beta^2) - i\alpha R(U - c)]\hat{w} = i\beta R\hat{\omega} \\
&\quad i(\alpha\hat{u} + \beta\hat{w}) + D\hat{v} = 0
\end{aligned}\right\} \qquad (2.5.1\text{ a to d})$$

where D is a differential operator given by

$$D \equiv \frac{d}{dy} \qquad (2.5.2)$$

In carrying out the above operations the set of partial differential equations, (2.4.8) to (2.4.11), resulted in a set of ordinary differential equations, (2.5.1). The above equations can be expressed as six, first order, linear differential equations by introducing $(Z_1,Z_2,Z_3,Z_4,Z_5,Z_6)$ in place of $(\hat{u},\hat{v},\hat{w},D\hat{u},R\hat{\omega},D\hat{w})$. These six equations are

$$\left.\begin{aligned}
DZ_1 &= Z_4 \\
DZ_2 &= -i(\alpha Z_1 + \beta Z_3) \\
DZ_3 &= Z_6 \\
DZ_4 &= [(\alpha^2 + \beta^2) + i\alpha R(U-c)]Z_1 + R(DU-2\Omega)Z_2 + i\alpha Z_5 \\
DZ_5 &= -[(\alpha^2 + \beta^2) + i\alpha R(U-c)]Z_2 - i(\alpha Z_4 + \beta Z_6) - 2R\Omega Z_1 \\
DZ_6 &= [(\alpha^2 + \beta^2) + i\alpha R(U-c)]Z_3 + i\beta Z_5
\end{aligned}\right\} \qquad (2.5.3)$$

These can be conveniently written in the matrix form by

$$D[Z] = [A][Z] \qquad (2.5.4)$$

where matrix

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \\ Z_6 \end{bmatrix} \tag{2.5.5}$$

and matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ -i\alpha & 0 & -i\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ b & R(DU-2\Omega) & 0 & 0 & i\alpha & 0 \\ -2R\Omega & -b & 0 & -i\alpha & 0 & -i\beta \\ 0 & 0 & b & 0 & i\beta & 0 \end{bmatrix} \tag{2.5.6}$$

Here

$$b = (\alpha^2 + \beta^2) + i\alpha R(U - c) \tag{2.5.7}$$

Matrix A is not a matrix of constants but it contains U and DU, which are functions of y.

The system of equations (2.5.1 a to d) can also be expressed as two differential equations, one fourth order and one second order. Multiply (2.5.1a) by α and (2.5.1b) by β and add; we get

$$[D^2 - (\alpha^2 + \beta^2) - i\alpha R(U-c)](\alpha\hat{u} + \beta\hat{w}) = \alpha R(DU-2\Omega)\hat{v} + iR(\alpha^2 + \beta^2)\hat{\omega} \tag{2.5.8}$$

Differentiate (2.5.8) with respect to y, multiply (2.5.1b) by $-i(\alpha^2 + \beta^2)$, add the two and use (2.5.1d). This yields

$$[\{D^2-K^2-i\alpha R(U-c)\}(D^2-K^2) + i\alpha R(D^2U-2\Omega D)]\hat{v} + 2K^2R\Omega\hat{u} = 0 \qquad (2.5.9)$$

where

$$K^2 = \alpha^2 + \beta^2. \qquad (2.5.10)$$

Again eliminating the pressure term $\hat{\omega}$ from (2.5.1a and c) by cross-differentiation, we obtain

$$[D^2 - K^2 - i\alpha R(U-c)](\beta\hat{u} - \alpha\hat{w}) = \beta R(DU - 2\Omega)\hat{v} \qquad (2.5.11)$$

Eliminate $\hat{w}$ using (2.5.1d) in (2.5.11) to obtain

$$[D^2 - K^2 - i\alpha R(U-c)](K^2\hat{u} - i\alpha D\hat{v}) = \beta^2R(DU - 2\Omega)\hat{v} \qquad (2.5.12)$$

Writing (2.5.9) and (2.5.12) in terms of the new variables Z_1 to Z_6 and rearranging

$$[(D^2-K^2)^2 - i\alpha R(U-c)(D^2-K^2) + i\alpha R(D^2U - 2\Omega D)]Z_2 + 2K^2R\Omega Z_1 = 0$$

and

$$[\{(D^2-K^2) - i\alpha R(U-c)\}D - i\frac{\beta^2}{\alpha}R(DU - 2\Omega)]Z_2 + i\frac{K^2}{\alpha}[D^2-K^2-i\alpha R(U-c)]Z_1 = 0 \qquad \text{(2.5.13a and b)}$$

We can also express the equations as a sixth order differential equation by eliminating Z_1 between (2.5.13 a and b) resulting in

$$[F_7D^6 + F_6D^5 + F_5D^4 + F_4D^3 + F_3D^2 + F_2D + F_1]Z_2 = 0 \qquad (2.5.14)$$

where

$$\left.\begin{aligned} F_7 &= 1.0 \\ F_6 &= 0.0 \\ F_5 &= -[3K^2 + 2i\alpha R(U-c)] \\ F_4 &= -2i\alpha R\ DU \\ F_3 &= [3K^2 + i\alpha R(U-c)][K^2 + i\alpha R(U-c)] \\ F_2 &= 2i\alpha R(K^2DU + D^3U) \end{aligned}\right\} \qquad \text{(2.5.15a to f)}$$

$$F_1 = i\alpha R\, D^4U - K^6 - 2i\alpha R\, K^4(U-c) + \alpha^2R^2K^2(U-c)^2 + \alpha^2R^2(U-c)D^2U + 2\beta^2R^2\Omega(DU - 2\Omega) \qquad (2.5.15g)$$

The systems of equations given by (2.5.1), (2.5.3) or (2.5.4), (2.5.13) and (2.5.14) are equivalent. All of these represent the three-dimensional linear stability problem with the boundary conditions discussed in the next section.

2.6 Boundary Conditions

The problem outlined in the previous section will be fully defined if the boundary conditions are specified. The boundary conditions are such that part of them are known at one boundary and part at infinity. The problem, therefore, falls in the category of a two point boundary value problem. In the study under consideration, velocity perturbations, u', v', w' should vanish at the flat plate, because the particles stick to the plate. Also u', v', w' should tend to zero at large distances from the plate. Equations (2.4.12) then imply that

$$\left.\begin{array}{l} \hat{u}, \hat{v}, \hat{w} = 0 \quad \text{at} \quad y = 0 \\ \text{and} \\ \hat{u}, \hat{v}, \hat{w} \Rightarrow 0 \quad \text{as} \quad y \Rightarrow \infty. \end{array}\right] \qquad (2.6.1)$$

Thus the problem of three-dimensional, linear stability is completely defined by equations (2.5.1 a to d) with boundary conditions (2.6.1). Or, in terms of the Z's the problem is defined by equations (2.5.3) or (2.5.4) with boundary conditions

$$\left.\begin{array}{l} Z_1, Z_2, Z_3 = 0 \quad \text{at} \quad y = 0 \\ \text{and} \\ Z_1, Z_2, Z_3 \Rightarrow 0 \quad \text{as} \quad y \Rightarrow \infty. \end{array}\right] \qquad (2.6.2)$$

The boundary conditions to go with the form given by (2.5.13 a and b) are

$$\left.\begin{aligned} &Z_1,\ Z_2,\ DZ_2 = 0 \quad \text{at} \quad y = 0 \\ \text{and} \quad & \\ &Z_1,\ Z_2,\ DZ_2 \Rightarrow 0 \quad \text{as} \quad y \Rightarrow \infty \end{aligned}\right] \tag{2.6.3}$$

where boundary condition on DZ_2 results from continuity.

Finally the boundary conditions that go with equation (2.5.14) are

$$Z_2,\ DZ_2 = 0 \quad \text{at} \quad y = 0$$

and

$$Z_2,\ DZ_2 \Rightarrow 0 \quad \text{as} \quad y \Rightarrow \infty$$

along with two remaining boundary conditions, one at the plate and one at infinity, which are quite complicated. They can be found by using Z_1, Z_2 and $DZ_2 = 0$ (or Z_1, Z_2 and $DZ_2 \Rightarrow 0$) in equation (2.5.13a).

2.7 Eigenvalue Problem

The linear stability problem involving rotation is thus given by:

1.) the set of four equations (2.5.1) with boundary conditions (2.6.1), in the four unknowns $\hat{u}$, $\hat{v}$, $\hat{w}$ and $\hat{\omega}$, or by

2.) a system of six ordinary linear differential equations (2.5.3) with six boundary conditions (2.6.2), in six unknowns Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, or by

3.) two higher order but linear differential equations (2.5.13) with boundary conditions (2.6.3), in two unknowns Z_1 and Z_2, or by

4.) a sixth order, ordinary linear differential equation (2.5.14), in one unknown Z_2 with four well defined and two involved boundary conditions.

Every one of the above mentioned systems with the corresponding boundary conditions, represent an eigenvalue problem with the characteristic values, c, α, β, R and Ω. The relation among these parameters would result from the secular equation. Symbolically, we will have

$$F(c,\alpha,\beta,R,\Omega) = 0 \tag{2.7.1}$$

Specifying R and Ω, c would depend on the real dimensionless wave numbers α and β. Defining c as in (2.4.14) if C_i is positive, one will observe from (2.4.12) that velocity perturbations u', v', w' would grow exponentially with time and the motion would become unstable. On the other hand for negative C_i, u', v', w' would decay exponentially with time and the motion would be stable. With C_i equal to zero neutral stability would result.

From (2.7.1), for a given Ω

$$C = f(\alpha,\beta,R) \tag{2.7.2}$$

Since this is a complex equation,

$$C_i = f_i(\alpha,\beta,R) \tag{2.7.3}$$

and

$$C_r = f_r(\alpha,\beta,R) \tag{2.7.4}$$

For no growth or decay,

$$f_i(\alpha,\beta,R) = 0 \tag{2.7.5}$$

For a specified β one could then determine neutral stability curves. These are found by picking one of the eigenvalues, say C_r, and solving for the other two from equations (2.7.4) and (2.7.5). The neutral stability curve will have a minimum R, called the critical Reynolds number R_{crit}, for which a disturbance is neutrally stable. Reynolds numbers greater than R_{crit} result in growth for that particular disturbance and Reynolds numbers smaller than R_{crit} result in decay. Hence one can find an X_{crit} which separates a region of growth from a region of decay. Associated with R_{crit} we also have a $(C_r)_{crit}$ and an α_{crit} for each β and each Ω.

2.8 Closure

If we had considered a two-dimensional disturbance ($\beta = 0$), the equations (2.5.14 & 2.5.15) governing the stability of the infinitesimal disturbances (Tollmien-Schlichting waves) would reduce to those governing the stability of flow over a flat plate for which $\Omega = 0$. Thus we see that rotation influences the linear stability problem by interacting with the spanwise component of the three-dimensional disturbances.

CHAPTER III

NUMERICAL SOLUTION OF THE PROBLEM

3.1 Solution for the Primary Velocity Profile

In order to solve the stability problem, presented in Chapter 2, and given by equations (2.5.1), (2.5.3), (2.5.13) or (2.5.14), it is necessary to know the primary velocity profile which is, as shown in Chapter 2, independent of Ω.

Consider the dimensional boundary layer equations, which are

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \nu \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \tag{3.1.1}$$

and

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{3.1.2}$$

with boundary conditions

$$\left.\begin{aligned} &\bar{u} = \bar{v} = 0 \quad \text{at} \quad \bar{y} = 0 \\ \text{and} \quad &\bar{u} \Rightarrow U_\infty \quad \text{as} \quad \bar{y} \Rightarrow \infty. \end{aligned}\right] \tag{3.1.3}$$

Introducing a transformed stream function and coordinates as

$$f(\eta) = \frac{\Psi}{\sqrt{\nu \bar{x} U_\infty}} \tag{3.1.4}$$

and

$$\eta = \bar{y}\Big/\sqrt{\frac{\nu \bar{x}}{U_\infty}} \tag{3.1.5}$$

with velocity components

$$\bar{u} = \frac{\partial \Psi}{\partial \bar{y}} = \frac{\partial \Psi}{\partial \eta}\frac{\partial \eta}{\partial \bar{y}} = U_\infty f'(\eta) \qquad (3.1.6)$$

and

$$\bar{v} = -\frac{\partial \Psi}{\partial \bar{x}} = \frac{1}{2}\sqrt{\frac{\nu U_\infty}{\bar{x}}}\,(yf' - f) \qquad (3.1.7)$$

we obtain

$$f\,f'' + 2f''' = 0 \qquad (3.1.8)$$

where a prime denotes differentiation with respect to η. The new boundary conditions are

$$\left.\begin{array}{lll} f = f' = 0 & \text{at} & \eta = 0 \\ \text{and} & & \\ f' \Rightarrow 1 & \text{as} & \eta \Rightarrow \infty \end{array}\right] \qquad (3.1.9)$$

The numerical technique used to solve equation (3.1.8) with boundary conditions (3.1.9) is outlined below.

We can express equation (3.1.8) as three first order differential equations as follows:

$$f' = z \qquad (3.1.10)$$

$$z' = W_1 \qquad (3.1.11)$$

$$W_1' = -\frac{1}{2} f\, W_1 \qquad (3.1.12)$$

The starting conditions are

$$f = z = 0 \quad \text{at} \quad \eta = 0$$

Also some guess must be made about the value of W_1 at $\eta = 0$. With this information, and starting from the plate, the computations are carried out in four steps and then the predictions about the values of the functions f, z and W_1 are made at the next step. The scheme used was the fourth order Runge-Kutta. It is outlined in a reference by Weeg and Read. That value of W was accepted, which gave $U = 1.0$ at the farthest distance from the plate. The value of W_1 came out to be 0.332 which gave $U = 1.0$ at $y = 3.29$. This corresponds to $\eta \simeq 16.5$. Further details about the computer program are given in appendix C with a list of the program in appendix C-3.

In the same program the derivatives of U which appear in the governing equations were computed.

3.2 Approximate Solutions of the Orr-Sommerfeld Equation for Zero Angular Rotation

The systems of equations (2.5.1), (2.5.3), (2.5.13) or (2.5.14) are the different forms of the Orr-Sommerfeld equation. For $\Omega = 0$, consider the set (2.5.13). The first equation yields

$$D^4Z_2 - [2K^2 + i\alpha R(U-c)]D^2Z_2 + [K^4 + i\alpha R\, K^2(U-c) + i\alpha R\, D^2U]Z_2 = 0 \qquad (3.2.1)$$

This is the same equation that Radbill and Van Driest (1966) developed, the only difference being that they had $\beta = 0$, so that K reduced to α. At large distances from the boundary $(y > \delta)$ we can let $U = 1.0$ and $D^2U = 0.0$, thereby giving an equation with constant coefficients as

$$D^4Z_2 - [2\alpha^2 + i\alpha R(1-c)]D^2Z_2 + [\alpha^4 + i\alpha^3R(1-c)]Z_2 = 0 \qquad (3.2.2)$$

This equation, valid only for large y, permits solutions of the form e^{ry}. These approximate solutions then enables us to start our forward integration procedure as we move from the free-stream to the wall. To find these approximate solutions substitute e^{ry} into equation (3.2.2), and simplifying, we are left with a polynomial of the fourth degree which is

$$r^4 - [2\alpha^2 + i\alpha R(1-c)]r^2 + [\alpha^4 + i\alpha^3 R(1-c)] = 0 \qquad (3.2.3)$$

Letting

$$r^2 = S$$

equation (3.2.3) reduces to a quadratic equation in S with roots S_1 and S_2. The four roots of equation (3.2.3) are then $\pm\sqrt{S_1}$ and $\pm\sqrt{S_2}$ which would be complex. However, those roots which are admissible in the present case are bounded at infinity. There are only two such roots, $r_1 = -\sqrt{S_1}$ and $r_2 = -\sqrt{S_2}$.

The approximate solution of (3.2.1), therefore, is the linear combination

$$Z_2 = \exp(r_1 y) + B \exp(r_2 y) \qquad (3.2.4)$$

where B is a constant.

To solve equation (3.2.1) with variable $U(y)$ and $D^2U(y)$ a numerical scheme, which uses the solution (3.2.4) at large y, outlined in section (3.4) is used. The solution yields the eigenvalues α, R and C for specified β and Ω.

3.3 Approximate Solutions of the Orr-Sommerfeld Equation for Non-Zero Angular Rotation

For Ω other than zero consider the single equation given by equation (2.5.14)

$$[F_7D^6 + F_6D^5 + F_5D^4 + F_4D^3 + F_3D^2 + F_2D + F_1]Z_2 = 0 \qquad (3.3.1)$$

where F_i, $i = 1$ to 7, are given by equation (2.5.15). This is a linear, ordinary differential equation of the sixth order with coefficients depending on the known primary velocity $U(y)$ and its derivatives.

Equation (3.3.1) again admits solutions of the form e^{ry} for large y. Substituting this in (3.3.1) and simplifying yields the polynomial

$$F_7r^6 + F_6r^5 + F_5r^4 + F_4r^3 + F_3r^2 + F_2r + F_1 = 0 \qquad (3.3.2)$$

where $U = 1.0$ and all its derivatives are zero. This gives six complex roots, three of which have negative real parts. These are selected because they are the only roots which would be bounded at infinity. Let those roots be represented by r_1, r_2 and r_3. Then the approximate solution of (3.3.1) becomes

$$Z_2 = \exp(r_1y) + B \exp(r_2y) + C_1\exp(r_3y) \qquad (3.3.3)$$

where B and C_1 are constants.

Eigenvalues α, R and C with β and Ω specified are again determined from the numerical solution of equation (3.3.1) where the coefficients are functions of y. The solution (3.3.3) is used in the numerical scheme for large y.

3.4 Numerical Integration of the Orr-Sommerfeld Equations

(a) Initialization

The numerical integration is started from the free stream and carried out step by step until the wall is reached. The initial values of the independent solutions generated, which are combined when the wall is reached, are found from the approximate solutions of sections (3.2) and (3.3). These solutions are

$$\phi_i = \exp(r_i y) \tag{3.4.1}$$

where $i = 1,2$ if $\Omega = 0$ and $i = 1,2,3$ if $\Omega \neq 0$. The higher order derivatives are found simply by differentiating equation (3.4.1). A distance of $y = 1.5$ was used as the starting point for the forward stepping integration scheme.

Thus, in the case of zero angular rotation, the boundary value problem given by the Orr-Sommerfeld equation (3.2.1) and boundary conditions

$$\left.\begin{aligned} &\phi,\ D\phi = 0 \quad \text{at} \quad y = 0 \\ \text{and} \qquad &\\ &\phi,\ D\phi \Rightarrow 0 \quad \text{for large} \quad y \end{aligned}\right\} \tag{3.4.2}$$

changes to an initial value problem where we integrate from the free stream to the wall.

For $\Omega \neq 0$, having found ϕ_i and all its derivatives, Ψ_i is computed by (2.5.13a), $D\Psi_i$ is calculated by differentiating (2.5.13a) and then $D^2\Psi_i$ is obtained from (2.5.13b). These equations are given by

$$\Psi = [D^4\phi - \{2K^2 + i\alpha R(U-c)\}D^2\phi - 2i\alpha R\,\Omega\, D\phi$$

$$+ \{K^4 + i\alpha R\, K^2(U-c) + i\alpha R\, D^2U\}\phi]/(2K^2R\,\Omega) \tag{3.4.3}$$

$$D\Psi = [D^5\phi - \{2K^2 + i\alpha R(U-c)\}D^3\phi - i\alpha R(DU + 2\Omega)D^2\phi$$
$$+ \{K^4 + i\alpha R\ K^2(U-c) + i\alpha R\ D^2U\}D\phi +$$
$$i\alpha R\{K^2DU + D^3U\}\phi]/(2K^2R\ \Omega) \tag{3.4.4}$$

$$D^2\Psi = i\ \frac{\alpha}{K^2}[D^3\phi - \{K^2 + i\alpha R(U-c)\}D\phi - i\beta^2R(DU - 2\Omega)\phi/\alpha]$$
$$+ \{K^2 + i\alpha R(U-c)\}\Psi. \tag{3.4.5}$$

Using the derivatives of ϕ_i from (3.4.1) in (3.4.3) to (3.4.5) we can initialize Ψ_i, $D\Psi_i$ and $D^2\Psi_i$.

Hence, in the case of $\Omega \neq 0$, the boundary value problem given by Orr-Sommerfeld equations (2.5.1), (2.5.3), (2.5.13), or (2.5.14) with boundary conditions as discussed in section (2.6) is transformed to an initial value problem.

The numerical integration of the equations, started from $y = 1.5$ towards the plate, utilizes a Runge-Kutta fourth-order method. This is explained in detail in appendix C. The integration proceeds to the wall where the independent solutions are combined to form the eigen-functions. When the correct eigenvalues are used the boundary conditions can all be satisfied.

(b) Growth of Eigen-functions

During the numerical integration it is observed that for $\Omega = 0$, one of the two independent solutions grows more rapidly than the other. In the case of rotation, two of the three solutions grow faster than the third one. The functions with the fastest growth are called the "Growing solutions" and the other solutions are called the "Well-behaved solutions". The growing solution stems from the inviscid part of the Orr-Sommerfeld equation and the well-behaved solution results from the contribution of viscosity.

Mathematically, if ϕ_1 and ϕ_2 are the growing and the well behaved solutions respectively, then

$$\frac{D\phi_1}{\phi_1} \gg \frac{D\phi_2}{\phi_2}$$

The Orr-Sommerfeld equation can be written as, for $\Omega = 0$, (from equation (2.5.13a))

$$[(U-c)D^2 - K^2(U-c) - D^2U]\phi = -\frac{i}{\alpha R}[(D^2 - K^2)^2]\phi \qquad (3.4.6)$$

and, for $\Omega \neq 0$ (from equations (2.5.13a and b)), we have

$$\left.\begin{aligned}
&[\{(U-c)(D^2-K^2) - (D^2U-2\Omega\, D)\}\phi + 2\, i\,\frac{K^2}{\alpha}\,\Omega\Psi] = \frac{i}{\alpha R}\,[(D^2-K^2)^2]\phi \\
&\text{and} \\
&[\{(U-c)D + \frac{\beta^2}{\alpha^2}(DU-2\Omega)\}\phi + i(U-c)\,\frac{K^2}{\alpha}\,\Psi] = -\frac{i}{\alpha R}[(D^3-K^2D)\phi + i\,\frac{K^2}{\alpha}(D^2-K^2)\Psi].
\end{aligned}\right] \qquad (3.4.7)$$

For flows at large Reynolds numbers, which are of interest in this study, in equations (3.4.6) and (3.4.7), terms in the square brackets on the left hand side represent the inviscid part and those in the square brackets on the right hand side, the viscous effects.

At the start of the integration i.e. at $y = 1.5$, all the eigen-functions are linearly independent, but as the integration proceeds, the inviscid solutions grow faster than the well behaved solutions and since the computations are carried out up to a certain number of significant places on the computer, the growing solutions start affecting the viscous solutions. The functions then do not remain independent. To reduce this effect, all the calculations in the present study were made in double precision. Additional steps to make the functions linearly independent all the way to the plate, are explained in the next section.

(c) Purification Scheme-Filtering the Contribution of the Growing Solution

The problem in the numerical solution is in the numerical integration of the governing equations. For $\Omega = 0$ equation (3.4.6) can be written as

$$(L_1 + \frac{1}{R} L_2)\phi = 0 \tag{3.4.8}$$

Similarly for $\Omega \neq 0$ equations (3.4.7) can be written as

$$\left.\begin{aligned} &(L_3\phi + L_4\Psi) + \frac{1}{R} L_5\phi = 0 \\ \text{and} \quad &(L_6\phi + L_7\Psi) + \frac{1}{R}(L_8\phi + L_9\Psi) = 0 \end{aligned}\right] \tag{3.4.9}$$

where $L_1,\ldots,L_9$ are differential operators which can be determined by comparison with equation (3.4.6) or (3.4.7). When R is of the order of 10^4, the equations (3.4.8) and (3.4.9) become very singular. In any particular case, and in the absence of any rotation, there will be two linearly independent solutions of equation (3.4.8) satisfying the boundary conditions and an appropriate linear combination of these solutions. This general solution must satisfy the remaining two boundary conditions at the plate. Similarly in the presence of a rotation the general solution is an appropriate linear combination of three linearly independent solutions of equation (3.4.9) satisfying the three boundary conditions in the free stream. Here this general solution must satisfy the remaining three boundary conditions at the plate. It is well known that for $\Omega = 0$ one of the two solutions grows very rapidly as the integration proceeds from the free stream to the plate. Similarly for $\Omega \neq 0$ the growth of two out of the three solutions is very rapid. It is difficult to generate the remaining solution which would be independent because any small round off throws in some small multiple of the growing solution. This grows as the integration proceeds and may later dominate and thus pollute the well behaved solution. Reynolds and Potter (1967) did find this to be so in a solution of Poiseuille flow where they started at the centre with inviscid conditions and proceeded towards the walls. On reaching the walls they observed that the well behaved solution was, to eight digits, a multiple of the growing solution and was of the order of 10^{10} whereas initially it was present to the order of one part in 10^{8}. Since the final form of the solution is of the same order as the well behaved solution meaning thereby that a very small amount of the growing solutions is required, the problem is to numerically generate a well behaved solution which does not contain any multiple of the growing solution.

In order to generate the linearly independent solutions the scheme employed is to purify the well behaved roots of the contribution of the growing roots. Such a scheme was first used by Kaplan (1964) and later by Lee and Reynolds (1964). In the main program given in appendix C, solutions are arranged in order of their growth. For example the solutions associated with root r_1 has the largest growth.

After initializing the independent functions and calculating their derivatives, the values of $D^4\phi$ and $D^2\Psi$ are computed by solving simultaneously equations (3.4.7) for each root for $\Omega \neq 0$. But for $\Omega = 0$ $D^4\phi$ was computed from equation (3.4.6). Then proceeding forward by the Runge-Kutta fourth order method, these values are found at the next step. Then the effects of the growing solutions are filtered from the behaved solution which are later modified to repair the extractions.

The above is schematically given in the flow chart in appendix B. The computer program that does this forms a part of the main program listed in appendix C-3.

(d) Iteration Scheme for the Eigenvalues

After generating the individual independent functions they should be suitably combined so that for the correct eigenvalues, the boundary conditions at the plate are satisfied. If the guesses on the eigenvalues are off, the correct values are determined by iteration. This is explained below for $\Omega \neq 0$.

The combined Φ, $D\Phi$ and Ψ at the wall are

$$\phi_{W1} + X(1)\phi_{W2} + X(2)\phi_{W3} = \Phi_W \qquad (3.4.10)$$

$$D\phi_{W1} + X(1)D\phi_{W2} + X(2)D\phi_{W3} = D\Phi_W \qquad (3.4.11)$$

$$\Psi_{W1} + X(1)\Psi_{W2} + X(2)\Psi_{W3} = \Psi_W \qquad (3.4.12)$$

For the right choice of eigenvalues α, R and C_r with fixed Ω, β and $C_i = 0$, the functions Φ_W, $D\Phi_W$ and Ψ_W should be zeros, in order that the no-slip condition and continuity be satisfied. To set up the iteration scheme set Φ_W and Ψ_W equal to zeros. Solve for $X(1)$ and $X(2)$ from (3.4.10) and (3.4.12). Using these values in (3.4.11) the test function T results, which is

$$T_r = \frac{[D\Phi_W]_r}{[(D\Phi_W)_{\text{one step from wall}}]_r}, \quad T_i = \frac{[D\Phi_W]_i}{[(D\Phi_W)_{\text{one step from wall}}]_i}$$

giving
$$T = \sqrt{T_r^2 + T_i^2} \tag{3.4.13}$$

If $T \le 10^{-3}$, the eigenvalues are assumed correct. If $T > 10^{-3}$, let $T_1 = T$. Increase α by 1% and let $T_2 = T$. Then increase R by 1% and let $T_3 = T$. Then

$$\frac{\partial T}{\partial \alpha} = \frac{T_2 - T_1}{\Delta\alpha} \tag{3.4.14}$$

and

$$\frac{\partial T}{\partial R} = \frac{T_3 - T_1}{\Delta R} \tag{3.4.15}$$

$\Delta\alpha$ and ΔR are calculated from the complex equation

$$\frac{\partial T}{\partial \alpha}\Delta\alpha + \frac{\partial T}{\partial R}\Delta R + T_1 = 0. \tag{3.4.16}$$

The new values of α and R are

$$\left.\begin{aligned} \alpha_{new} &= \alpha_{old} + \Delta\alpha \\ \text{and} \qquad R_{new} &= R_{old} + \Delta R \end{aligned}\right] \tag{3.4.17}$$

with these values of α and R, a new T is computed. If $T < T_1$, then this forms a converging sequence, and the values of α and R would hopefully converge towards the true values. If the initial guess is close, the true values would be reached in about three iterations. Sometimes this might oscillate and then converge. But if the first guess is bad, T may fail to converge. This would be observed by a very slow convergence of T with comparatively large changes in the values of α or R.

For $\Omega = 0$, combined Φ and $D\Phi$ at the wall are

$$\phi_{W1} + X(1) \cdot \phi_{W2} = \Phi_W \tag{3.4.18}$$

and

$$D\phi_{W1} + X(1) \cdot D\phi_{W2} = D\Phi_W \tag{3.4.19}$$

To satisfy one boundary condition at the plate, set $\Phi_W = 0$, solve for $X(1)$ from equation (3.4.18) and using this in (3.4.19) gives the test T. The iteration procedure is the same as discussed for the $\Omega \neq 0$ case.

(e) Criterion for Guessing the Eigenvalues

To start the solution, the eigenvalues are first found for the $\Omega = 0$ case. As first guesses, some values of Radbill and Van Driest (1966) were used. Once some eigenvalues are determined, other guesses are obtained by extrapolation. This is done until a complete neutral-stability curve results. Actually, we are only interested in the critical eigenvalues so when these are found for $\Omega = 0$, β is assumed equal to 0.2, then taking $\Omega = 0.01$ and -0.01 and taking some of the guesses equal to the zeroed in values corresponding to $\Omega = 0$, new values are obtained. Then by careful guessing the complete stability curves for these values of Ω are completed. Similarly for the other values of Ω, the true values of α, R, and C_r are found. For $\beta = 0.1, 0.3, 0.4$, etc., values are guessed from the corresponding $\beta = 0.2$ and so on. With these, the various curves are completed.

CHAPTER IV

CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

4.1 The Primary Velocity Profile

In order to solve the equations governing the growth of the infinitesimal disturbances, it was necessary to have the values of the primary velocity U and its derivatives at all the steps of integration. For this the third order differential equation (3.1.8) was solved. The results for U, U' and U" are given in table 1. The third and fourth derivatives were also computed at the first few points. The values of U checked very closely with those of Howarth (1938).

4.2 Eigenvalues for the Case of Neutral Stability for Zero Angular Rotation

The numerical technique used here is the same as with angular rotation. Zero rotation is studied to allow a check on the accuracy of the present technique and to provide information necessary for iterations for other neutral stability curves. The eigenvalues are given in table 4 and plotted in figure 3. In the same figure mean curves due to Schlichting (1935), Lin (1945) and Kaplan (1964) are also shown. There is good agreement in the lower limit of the neutral stability curves except that Schlichting's curve is a little lower and Kaplan's a little higher than the present calculations show.

For the upper branch the present curve falls midway between Schlichting's and Lin's curves and slightly below Kaplan's. The difference with Kaplan's curve may be due to not taking $U = 1.0$ and $D^2U = 0$ at the start of integration but using the actual values computed from the primary velocity profile described in section 4.1, and also starting at a larger distance from the plate than Kaplan did. Another difference was a larger step size used in the present case. The present calculations were also checked by superimposing the curves of Shen (1954), Kurtz (1962), and the data points of Radbill and Van Driest (1966) (see figure 3). The experimental points due to Schubauer and Skramstad (1947) are also plotted in figure 3. In this connection one should remember the different characteristic length used by each author to non-dimensionlize various quantities. Table 2 gives the relation between the eigenvalues used by different authors.

The critical eigenvalues for zero rotation as computed by different authors by various techniques are given in table 3 for immediate comparison.

In order to compare the eigen-functions $\phi = \phi_r + i\,\phi_i$ and ϕ' the same values of C_r (0.33052 for the Upper Branch and 0.33156 for the Lower Branch) as taken by Radbill and Van Driest (1966) corresponding to the experimental data of Schubauer and Skramstad (1947) were taken to compute the values of α and R. Curves of ϕ and ϕ' with the corresponding curves of Radbill and Van Driest are shown in figures 4 and 5 for comparison.

At this point, for Ω and $C_i = 0.0$, Squire's theorem (1933) was verified; it requires that:

$$(\alpha R)_{\text{Two Dim.}} = (\alpha R)_{\text{Three Dim.}}$$

$$(\alpha^2)_{\text{Two Dim.}} = (\alpha^2 + \beta^2)_{\text{Three Dim.}}$$

$$(C_r)_{\text{Two Dim.}} = (C_r)_{\text{Three Dim.}}$$

The verification of this theorem provided a partial check on the numerical calculations.

4.3 Effect of the Angular Rotation Ω and Wave Number β

The numerical scheme used was employed to investigate the effect on the neutral stability curve of angular rotation Ω and disturbance angle β. Eigenvalues for various Ω and β are given in tables 4 through 17. Neutral stability curves are shown in figures 6, 9, 11 for α verses R and in figures 7, 10, and 12 for C_r verses R. Critical eigenvalues of the neutral stability curves, R_{crit}, $(C_r)_{crit}$ and α_{crit}, are plotted against Ω for constant β in figure 8. The results show (see figure 8) that increasing Ω has a destabilizing effect on the boundary layer and as Ω decreases it stabilizes the boundary layer. The same result was found by Conard (1962) in a study of the effect of rotation on Görtler waves. The results also agree with the experimental data presented by Halleen and Johnston (1967). Although they did not specifically state it, it is clear from their mean velocity profiles for small and large Reynolds numbers that positive Ω destabilizes whereas negative Ω stabilizes the flow.

It is also observed that $(C_r)_{crit}$ increases gradually with the increase in Ω at first and later decreases and α_{crit} initially decreases and then increases. These effects are less pronounced for $\beta = 0.1$, and much larger for $\beta = 0.3$; so it appears that increasing β increases these changes although it may be that there is a critical β for which the smallest critical Reynolds number occurs.

The various spanwise wave numbers chosen were $\beta = 0.0, 0.1, 0.2$ and 0.3 for $\Omega = -0.10$, $\beta = 0.1, 0.2, 0.3, 0.4, 0.5$ and 0.6 for $\Omega = 0.10$ and $\beta = 0.2$ for $\Omega = 0.50$. It can be observed from α verses R and C_r verses R curves (see figures 16 and 17) that for negative Ω increasing β stabilizes the boundary layer, and from C_r verses R curve (see figure 17) one can also notice that as β increases, C_r decreases. For positive Ω the increase in β destabilizes the boundary layer (see figures 18 and 19). From the corresponding C_r verses R curve (see figure 19) increasing β increases C_r. Figure 20 shows the effect of spanwise wave number β on the critical Reynolds number for angular rotation $\Omega = 0.10$ and -0.10. The eigen-functions near the critical point on the lower branch of the neutral stability curve for $\Omega = -0.10$ and $\beta = 0.2$ are shown in figures 13, 14, and 15, and are also presented in table 18.

4.4 Summary of Conclusions

The results obtained in this study can be summarized as follows:

1. In the presence of angular rotation Ω, spanwise wave number β must be non zero, if the effect of Ω is to be observed.
2. For a constant β, increasing Ω is destabilizing, whereas decreasing Ω has a stabilizing influence on the boundary layer. In particular, for $\beta = 0.2$, with $\Omega = -0.1$ $R_{crit} = 1897.91$, for $\Omega = 0$ $R_{crit} = 1506.87$, for

$\Omega = 0.1$ $R_{crit} = 1347.76$ and with $\Omega = 0.5$ $R_{crit} = 1166.69$.

3. For a constant Ω, increasing β has a destabilizing effect for positive values of Ω and stabilizing effect for negative values of Ω.

4.5 Recommendations for Further Research

The present study was conducted to investigate the influence of rotation on the stability of an incompressible fluid undergoing the parallel shear flow, the laminar boundary layer. The angular rotation rate Ω was varied between -0.1 to 0.5 for different values of spanwise wave number β. It is proposed that future research in this area can be extended along the following lines:

1. To continue the present work to include a greater range of Ω say from -1 to 5 for β varying from 0 to 1.
2. To extend the present work for $C_i \neq 0$ in order to study the growth and the decay of the three-dimensional disturbances.
3. To restudy the present work and also as suggested in paragraphs 1 and 2 above for a compressible fluid.
4. To conduct research as done in this study and suggested in paragraphs 1 and 2 above for a non-Newtonian fluid.
5. To take into account the temperature variations and thus to study the flow stability when there is a temperature boundary layer superimposed on a velocity boundary layer.
6. To verify the results obtained in the present analytical work by the use of experiments on a flat plate.

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ILLUSTRATIONS

(Tables and Figures)

Table 1. Values for the Primary Velocity Profile

y	U	U'	U"
1.500	0.999981858	0.000276314	-0.003992197
1.490	0.999978886	0.000319064	-0.004569974
1.480	0.999975456	0.000367969	-0.005224437
1.470	0.999971503	0.000423839	-0.005964703
1.460	0.999966953	0.000487582	-0.006800817
1.450	0.999961722	0.000560211	-0.007743826
1.440	0.999955716	0.000642854	-0.008805861
1.430	0.999948828	0.000736768	-0.010000208
1.420	0.999940938	0.000843347	-0.011341399
1.410	0.999931913	0.000964138	-0.012845291
1.400	0.999921602	0.001100852	-0.014529150
1.390	0.999909837	0.001255382	-0.016411745
1.380	0.999896429	0.001429816	-0.018513429
1.370	0.999881167	0.001626453	-0.020856227
1.360	0.999863817	0.001847822	-0.023463924
1.350	0.999844119	0.002096699	-0.026362145
1.340	0.999821781	0.002376124	-0.029578440
1.330	0.999796483	0.002689425	-0.033142352
1.320	0.999767868	0.003040234	-0.037085497
1.310	0.999735541	0.003432510	-0.041441616
1.300	0.999699065	0.003870560	-0.046246638
1.290	0.999657961	0.004359064	-0.051538719
1.280	0.999611699	0.004903091	-0.057358276
1.270	0.999559697	0.005508128	-0.063748003
1.260	0.999501314	0.006180099	-0.070752875
1.250	0.999435850	0.006925391	-0.078420134
1.240	0.999362539	0.007750872	-0.086799250
1.230	0.999280541	0.008663918	-0.095941871
1.220	0.999188943	0.009672431	-0.105901736
1.210	0.999086747	0.010784860	-0.116734578
1.200	0.998972869	0.012010221	-0.128497983
1.190	0.998846134	0.013358116	-0.141251231
1.180	0.998705265	0.014838744	-0.155055105
1.170	0.998548881	0.016462923	-0.169971660
1.160	0.998375490	0.018242093	-0.186063966
1.150	0.998183483	0.020188331	-0.203395809
1.140	0.997971125	0.022314351	-0.222031361
1.130	0.997736552	0.024633514	-0.242034810
1.120	0.997477764	0.027159816	-0.263469950
1.110	0.997192617	0.029907891	-0.286399741
1.100	0.996878817	0.032892994	-0.310885823
1.090	0.996533915	0.036130990	-0.336988001
1.080	0.996155300	0.039638328	-0.364763695
1.070	0.995740194	0.043432019	-0.394267350
1.060	0.995285647	0.047529600	-0.425549825
1.050	0.994788530	0.051949096	-0.458657749
1.040	0.994245531	0.056708975	-0.493632858
1.030	0.993653153	0.061828094	-0.530511310
1.020	0.993007708	0.067325641	-0.569322986

Cont'd.

1.010	0.992305315	0.073221069	-0.610090784
1.000	0.991541896	0.079534023	-0.652829911
0.990	0.990713177	0.086284257	-0.697547175
0.980	0.989814687	0.093491549	-0.744240295
0.970	0.988841757	0.101175605	-0.792897225
0.960	0.987789522	0.109355960	-0.843495509
0.950	0.986652920	0.118051871	-0.896001667
0.940	0.985426703	0.127282199	-0.950370632
0.930	0.984105434	0.137065298	-1.006545236
0.920	0.982683496	0.147418886	-1.064455757
0.910	0.981155099	0.158359921	-1.124019544
0.900	0.979514287	0.169904465	-1.185140713
0.890	0.977754948	0.182067558	-1.247709941
0.880	0.975870828	0.194863078	-1.311604349
0.870	0.973855537	0.208303604	-1.376687490
0.860	0.971702569	0.222400287	-1.442809454
0.850	0.969405312	0.237162709	-1.509807079
0.840	0.966957070	0.252598758	-1.577504291
0.830	0.964351072	0.268714496	-1.645712566
0.820	0.961580501	0.285514042	-1.714231518
0.810	0.958638505	0.302999454	-1.782849617
0.800	0.955518226	0.321170622	-1.851345033
0.790	0.952212815	0.340025172	-1.919486606
0.780	0.948715460	0.359558374	-1.987034940
0.770	0.945019409	0.379763069	-2.053743608
0.760	0.941117993	0.400629604	-2.119360475
0.750	0.937004652	0.422145782	-2.183629115
0.740	0.932672961	0.444296826	-2.246290326
0.730	0.928116657	0.467065360	-2.307083713
0.720	0.923329662	0.490431405	-2.365749353
0.710	0.918306113	0.514372392	-2.422029503
0.700	0.913040382	0.538863194	-2.475670342
0.690	0.907527108	0.563876171	-2.526423741
0.680	0.901761218	0.589381239	-2.574049033
0.670	0.895737951	0.615345950	-2.618314766
0.660	0.889452883	0.641735598	-2.659000430
0.650	0.882901946	0.668513329	-2.695898125
0.640	0.876081452	0.695640285	-2.728814168
0.630	0.868988112	0.723075744	-2.757570612
0.620	0.861619050	0.750777292	-2.782006660
0.610	0.853971826	0.778700998	-2.801979968
0.600	0.846044441	0.806801603	-2.817367808
0.590	0.837835359	0.835032725	-2.828068090
0.580	0.829343509	0.863347068	-2.834000231
0.570	0.820568300	0.891696638	-2.835105845
0.560	0.811509621	0.920032974	-2.831349279
0.550	0.802167847	0.948307371	-2.822717949
0.540	0.792543842	0.976471116	-2.809222509
0.530	0.782638956	1.004475717	-2.790896827
0.520	0.772455019	1.032273138	-2.767797785
0.510	0.761994342	1.059816023	-2.740004892
0.500	0.751259702	1.087057925	-2.707619727
0.490	0.740254338	1.113953517	-2.670765217
0.480	0.728981933	1.140458807	-2.629584752

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0.470	0.717446605	1.166531332	-2.584241153
0.460	0.705652885	1.192130354	-2.534915513
0.450	0.693605706	1.217217027	-2.481805906
0.440	0.681310375	1.241754569	-2.425126001
0.430	0.668772561	1.265708403	-2.365103575
0.420	0.655998261	1.289046298	-2.301978964
0.410	0.642993787	1.311738478	-2.236003444
0.400	0.629765735	1.333757731	-2.167437585
0.390	0.616320958	1.355079488	-2.096549572
0.380	0.602666544	1.375681894	-2.023613526
0.370	0.588809782	1.395545859	-1.948907829
0.360	0.574758143	1.414655088	-1.872713480
0.350	0.560519242	1.432996106	-1.795312487
0.340	0.546100817	1.450558254	-1.716986311
0.330	0.531510700	1.467333679	-1.638014382
0.320	0.516756784	1.483317309	-1.558672681
0.310	0.501847001	1.498506804	-1.479232417
0.300	0.486789294	1.512902513	-1.399958797
0.290	0.471591587	1.526507397	-1.321109894
0.280	0.456261764	1.539326960	-1.242935624
0.270	0.440807640	1.551369161	-1.165676837
0.260	0.425236938	1.562644319	-1.089564518
0.250	0.409557269	1.573165009	-1.014819103
0.240	0.393776104	1.582945956	-0.941649912
0.230	0.377900760	1.592003921	-0.870254700
0.220	0.361938374	1.600357584	-0.800819314
0.210	0.345895888	1.608027420	-0.733517461
0.200	0.329780031	1.615035584	-0.668510583
0.190	0.313597303	1.621405780	-0.605947835
0.180	0.297353958	1.627163145	-0.545966151
0.170	0.281055995	1.632334124	-0.488690409
0.160	0.264709139	1.636946351	-0.434233671
0.150	0.248318835	1.641028533	-0.382697510
0.140	0.231890236	1.644610339	-0.334172393
0.130	0.215428194	1.647722285	-0.288738149
0.120	0.198937252	1.650395638	-0.246464470
0.110	0.182421637	1.652662311	-0.207411485
0.100	0.165885254	1.654554774	-0.171630363
0.090	0.149331679	1.656105968	-0.139163964
0.080	0.132764161	1.657349221	-0.110047512
0.070	0.116185609	1.658318180	-0.084309313
0.060	0.099598598	1.659046742	-0.061971471
0.050	0.083005362	1.659568999	-0.043050646
0.040	0.066407792	1.659919185	-0.027558809
0.030	0.049807438	1.660131633	-0.015504013
0.020	0.033205504	1.660240739	-0.006891171
0.010	0.016602852	1.660280938	-0.001722839
0.000	0.000000000	1.660286681	0.000000000

Table 2. Definition of Eigenvalues used by Different Authors

Author	Characteristic Length	Eigenvalues equal to those used in the present study		
		α	R	C_r
Present Calculations	$\delta = 3\,\delta^*$	α	R	C_r
Kaplan	$\delta = 3.5\,\delta^*$	$\frac{3}{3.5}\,\alpha_k$	$\frac{3}{3.5}\,R_k$	$(C_r)_k$
Lin	$\delta = \frac{\delta^*}{0.28673}$	$\frac{3}{3.4876}\,\alpha_L$	$\frac{3}{3.4876}\,R_L$	$(C_r)_L$
Radbill and Van Driest	δ^*	$3\,\alpha_r$	$3\,R_r$	$(C_r)_r$
Schlichting	δ^*	$3\,\alpha_s$	$3\,R_s$	$(C_r)_s$
Schubauer and Skramstad	δ^*	$3\,\alpha_{ss}$	$3\,R_{ss}$	$(C_r)_{ss}$

Here δ = Boundary Layer Thickness

δ^* = Displacement Thickness

The subscripts k, L, r, s, ss refer to the corresponding author's value.

Table 3. Comparison of Critical Eigenvalues for Zero Rotation

Author	Critical Eigenvalue equivalent to the one derived in present study for neutral stability for $\Omega = 0.0$		
	R	α	C_r
Present study	1506.87	0.87553	0.396
Kaplan	1546.28	0.94285	0.40
Lin	1264.48	1.11549	0.411
Radbill and Van Driest	1548.75	0.89307	0.395
Schlichting	1725	0.834	0.42
Tollmien	1260	1.101	0.425

Table 4. Eigenvalues for the Orr-Sommerfeld Equation for the Case of Neutral Stability (C_i = 0.0) Ω = 0.00, β = 0.00

C_r	R	α	Remarks
0.31000	8822.88	0.83457	Upper Branch
0.33000	6099.27	0.90570	
* 0.33052	6042.47	0.90743	
0.35000	4269.23	0.96918	
0.37000	3004.11	1.01801	
0.39000	2082.86	1.03359	
0.39500	1876.92	1.02365	
0.39800	1745.12	1.00871	
0.39900	1694.89	0.99753	
0.40000	1632.00	0.98411	
0.39900	1516.99	0.91674	
0.39600	1506.87	0.87553	Critical Point
0.39300	1517.60	0.84520	
0.39000	1537.65	0.81961	
0.38000	1636.50	0.74971	
0.36000	1936.01	0.64408	
0.34000	2368.45	0.56093	
* 0.33156	2599.29	0.53015	
0.33000	2645.57	0.52468	
0.32000	2973.92	0.49131	
0.30000	3826.21	0.43135	
0.27000	5848.15	0.35514	Lower Branch

*These C_r values correspond to the data of Schubauer and Skramstad and also discussed by Radbill and Van Driest.

Table 5. Eigenvalues for the Orr-Sommerfeld Equation for the Case of Neutral Stability (C_i = 0.0) Ω = -0.01, β = 0.20

C_r	R	α	Remarks
0.380	2547.46	1.00998	Upper Branch
0.395	1894.79	0.99681	
0.399	1602.89	0.91967	
0.397	1573.87	0.87877	
0.395	1573.80	0.85301	Critical Point
0.392	1588.38	0.82187	
0.390	1603.66	0.80397	
0.380	1714.79	0.73055	
0.360	2056.52	0.62051	
0.340	2561.35	0.53372	
0.320	3290.80	0.46055	
0.300	4360.18	0.39721	Lower Branch

Table 6. Eigenvalues for the Orr-Sommerfeld Equation for the Case of Neutral Stability (C_i = 0.0) Ω = -0.10, β = 0.00

C_r	R	α	Remarks
0.360	3584.44	0.99753	Upper Branch
0.380	2517.68	1.03414	
0.385	2297.35	1.03720	
0.399	1509.12	0.91266	
0.396	1502.38	0.87287	Critical Point
0.394	1509.57	0.85276	
0.390	1534.40	0.81763	
0.385	1579.23	0.78084	
0.382	1611.08	0.76093	
0.380	1634.22	0.74843	
0.360	1933.97	0.64310	
0.340	2366.12	0.56011	Lower Branch

Table 7. Eigenvalues for the Orr-Sommerfeld Equation for the Case of Neutral Stability ($C_i = 0.0$) $\Omega = -0.10$, $\beta = 0.10$

C_r	R	α	Remarks
0.360	3546.84	0.98813	Upper Branch
0.380	2487.38	1.02115	
0.385	2265.57	1.02229	
0.397	1605.28	0.92642	
0.396	1589.35	0.90659	
0.395	1583.74	0.89174	Critical Point
0.393	1584.21	0.86767	
0.390	1598.88	0.83834	
0.385	1640.87	0.79870	
0.382	1673.20	0.77765	
0.380	1696.52	0.76465	
0.370	1837.18	0.70702	
0.360	2012.64	0.65812	
0.340	2475.87	0.57647	Lower Branch

Table 8. Eigenvalues for the Orr-Sommerfeld Equation for the Case of Neutral Stability ($C_i = 0.0$) $\Omega = -0.10$, $\beta = 0.20$

C_r	R	α	Remarks
0.360	3414.36	0.95491	Upper Branch
0.380	2355.89	0.96814	
0.385	2091.83	0.94831	
0.386	1917.42	0.89783	
0.384	1897.91	0.86574	Critical Point
0.383	1899.82	0.85411	
0.380	1921.51	0.82578	
0.375	1983.35	0.78857	
0.360	2263.17	0.70519	
0.340	2814.62	0.62342	Lower Branch

Table 9. Eigenvalues for the Orr-Sommerfeld Equation for the Case of Neutral Stability (C_i = 0.0) Ω = -0.10, β = 0.3

C_r	R	α	Remarks
0.340	4321.70	0.83378	Upper Branch
0.350	3585.03	0.84277	
0.353	3352.32	0.83768	
0.354	3258.25	0.83201	
0.354	3136.17	0.80835	
0.353	3132.25	0.79574	Critical Point
0.350	3189.73	0.77222	
0.340	3529.45	0.72037	Lower Branch

Table 10. Eigenvalues for the Orr-Sommerfeld Equation for the Case of Neutral Stability (C_i = 0.0) Ω = 0.01, β = 0.20

C_r	R	α	Remarks
0.380	2575.67	1.01568	Upper Branch
0.395	1931.37	1.01037	
0.400	1528.68	0.89431	
0.397	1522.26	0.85194	Critical Point
0.395	1529.97	0.83026	
0.392	1549.64	0.80212	
0.390	1566.49	0.78532	
0.380	1678.94	0.71390	
0.360	2012.36	0.60334	
0.340	2501.23	0.51361	
0.320	3208.42	0.43576	Lower Branch

Table 11. Eigenvalues for the Orr-Sommerfeld Equation for the Case of Neutral Stability ($C_i = 0.0$) $\Omega = 0.10$, $\beta = 0.10$

C_r	R	α	Remarks
0.360	3637.51	0.99358	Upper Branch
0.380	2550.08	1.03241	
0.395	1915.97	1.02993	
0.400	1470.11	0.89652	
0.398	1467.13	0.87058	Critical Point
0.390	1509.61	0.79669	
0.385	1555.36	0.76031	
0.380	1610.19	0.72806	
0.360	1904.46	0.62120	
0.340	2325.14	0.53542	Lower Branch

Table 12. Eigenvalues for the Orr-Sommerfeld Equation for the Case of Neutral Stability ($C_i = 0.0$) $\Omega = 0.10$, $\beta = 0.20$

C_r	R	α	Remarks
0.360	3815.29	0.98934	Upper Branch
0.380	2669.23	1.03518	
0.395	2027.86	1.04605	
0.408	1514.59	0.99625	
0.406	1349.66	0.87664	
0.405	1347.76	0.86270	Critical Point
0.400	1359.86	0.80694	
0.398	1370.38	0.78813	
0.390	1429.58	0.72321	
0.380	1529.96	0.65490	
0.360	1799.36	0.53817	
0.340	2157.41	0.43102	
0.320	2552.18	0.30861	
0.310	2610.45	0.24343	Lower Branch

Table 13. Eigenvalues for the Orr-Sommerfeld Equation for the Case of Neutral Stability ($C_i = 0.0$) $\Omega = 0.10$, $\beta = 0.30$

C_r	R	α	Remarks
0.360	4130.97	0.97924	Upper Branch
0.380	2863.98	1.03502	
0.395	2185.14	1.05851	
0.415	1174.49	0.82865	
0.413	1172.76	0.79993	Critical Point
0.410	1176.75	0.76225	
0.400	1215.53	0.65587	
0.398	1225.59	0.63611	
0.390	1267.62	0.55607	
0.380	1278.87	0.41023	Lower Branch

Table 14. Eigenvalues for the Orr-Sommerfeld Equation for the Case of Neutral Stability ($C_i = 0.0$) $\Omega = 0.10$, $\beta = 0.4$

C_r	R	α	Remarks
0.360	4651.27	0.95944	Upper Branch
0.380	3147.21	1.02830	
0.395	2388.88	1.06347	
0.405	1992.76	1.07599	
0.410	1818.64	1.07766	
0.420	1505.08	1.06666	
0.430	1204.89	1.01011	
0.432	998.49	0.84961	
0.430	970.33	0.78864	
0.428	953.66	0.73934	
0.427	946.79	0.71588	
0.425	932.32	0.66718	
0.423	911.27	0.60709	
0.4225	900.88	0.58400	
0.422	877.11	0.54589	

Table 15. Eigenvalues for the Orr-Sommerfeld Equation for the Case of Neutral Stability (C_i = 0.0) Ω = 0.10, β = 0.5

C_r	R	α	Remarks
0.430	1397.06	1.08694	Upper Branch
0.440	1145.52	1.04910	
0.445	1019.47	1.00449	
0.450	840.74	0.86321	
0.453	680.46	0.62868	
0.455	649.15	0.58823	
0.457	625.03	0.55958	
0.460	595.32	0.52685	
0.462	578.07	0.50902	
0.463	569.99	0.50091	
0.465	554.66	0.48590	
0.470	519.77	0.45346	
0.475	487.52	0.42516	

Table 16. Eigenvalues for the Orr-Sommerfeld Equation for the Case of Neutral Stability (C_i = 0.0) Ω = 0.10, β = 0.6

C_r	R	α	Remarks
0.440	1304.46	1.10778	Upper Branch
0.450	1083.42	1.08000	
0.460	882.96	1.01217	
0.470	680.09	0.84493	
0.473	626.43	0.77933	
0.475	597.04	0.74155	
0.477	572.17	0.70953	
0.480	541.15	0.67013	
0.483	515.33	0.63799	
0.485	500.19	0.61955	
0.490	467.40	0.58049	
0.495	439.59	0.54843	
0.497	429.45	0.53700	
0.500	415.06	0.52093	
0.510	371.87	0.47394	

Table 17. Eigenvalues for the Orr-Sommerfeld Equation for the Case of Neutral Stability (C_i = 0.0) Ω = 0.50, β = 0.20

C_r	R	α	Remarks
0.360	3789.20	1.03303	Upper Branch
0.380	2640.32	1.08096	
0.400	1819.80	1.09706	
0.410	1184.75	0.94655	
0.400	1166.69	0.85660	Critical Point
0.395	1177.47	0.82488	
0.390	1193.60	0.79741	
0.380	1236.30	0.75090	
0.370	1289.69	0.71261	
0.360	1351.71	0.68037	Lower Branch

Table 18. Eigen-functions near a Critical Point on the Lower Branch of the Neutral Stability Curve ($C_i = 0.0$), for $\Omega = -0.10$

$C_r = 0.38$, $\beta = 0.20$, $R = 1921.51$, $\alpha = 0.82578$

y	ϕ_r	ϕ_i	ϕ'_r	ϕ'_i	Ψ_r	Ψ_i
1.5000	1.000+000	1.175-007	-8.443-001	-2.660-003	7.360-003	-9.423-001
1.4800	1.017+000	1.602-005	-8.565-001	-7.187-004	4.811-003	-9.571-001
1.4600	1.034+000	3.102-005	-8.710-001	-7.826-004	3.383-003	-9.746-001
1.4400	1.052+000	4.735-005	-8.857-001	-8.509-004	2.220-003	-9.914-001
1.4200	1.070+000	6.509-005	-9.005-001	-9.244-004	1.484-003	-1.008+000
1.4000	1.088+000	8.438-005	-9.155-001	-1.006-003	1.141-003	-1.024+000
1.3800	1.106+000	1.054-004	-9.307-001	-1.101-003	1.082-003	-1.041+000
1.3600	1.125+000	1.285-004	-9.459-001	-1.214-003	1.201-003	-1.058+000
1.3400	1.144+000	1.542-004	-9.612-001	-1.352-003	1.418-003	-1.074+000
1.3200	1.164+000	1.828-004	-9.765-001	-1.520-003	1.687-003	-1.091+000
1.3000	1.183+000	2.152-004	-9.917-001	-1.725-003	1.991-003	-1.108+000
1.2800	1.203+000	2.521-004	-1.007+000	-1.971-003	2.328-003	-1.125+000
1.2600	1.223+000	2.944-004	-1.022+000	-2.266-003	2.706-003	-1.141+000
1.2400	1.244+000	3.431-004	-1.036+000	-2.616-003	3.140-003	-1.157+000
1.2200	1.265+000	3.994-004	-1.050+000	-3.027-003	3.640-003	-1.173+000
1.2000	1.286+000	4.646-004	-1.064+000	-3.506-003	4.221-003	-1.187+000
1.1800	1.307+000	5.402-004	-1.076+000	-4.061-003	4.892-003	-1.201+000
1.1600	1.329+000	6.276-004	-1.088+000	-4.697-003	5.663-003	-1.213+000
1.1400	1.351+000	7.286-004	-1.098+000	-5.420-003	6.540-003	-1.224+000
1.1200	1.373+000	8.451-004	-1.107+000	-6.236-003	7.531-003	-1.233+000
1.1000	1.395+000	9.787-004	-1.114+000	-7.148-003	8.639-003	-1.239+000
1.0800	1.418+000	1.132-003	-1.119+000	-8.157-003	9.866-003	-1.243+000
1.0600	1.440+000	1.306-003	-1.121+000	-9.262-003	1.121-002	-1.243+000
1.0400	1.462+000	1.503-003	-1.120+000	-1.046-002	1.267-002	-1.240+000
1.0200	1.485+000	1.725-003	-1.115+000	-1.175-002	1.423-002	-1.233+000
1.0000	1.507+000	1.973-003	-1.107+000	-1.311-002	1.589-002	-1.220+000
0.9800	1.529+000	2.249-003	-1.094+000	-1.453-002	1.762-002	-1.202+000
0.9600	1.551+000	2.555-003	-1.076+000	-1.600-002	1.941-002	-1.177+000
0.9400	1.572+000	2.889-003	-1.052+000	-1.748-002	2.122-002	-1.146+000

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0.9200	1.593+000	3.254-003	-1.021+000	-1.896-002	2.303-002	-1.107+000
0.9000	1.613+000	3.648-003	-9.839-001	-2.040-002	2.479-002	-1.059+000
0.8800	1.632+000	4.069-003	-9.391-001	-2.176-002	2.647-002	-1.002+000
0.8600	1.650+000	4.517-003	-8.861-001	-2.300-002	2.801-002	-9.345-001
0.8400	1.667+000	4.988-003	-8.244-001	-2.408-002	2.937-002	-8.568-001
0.8200	1.683+000	5.479-003	-7.535-001	-2.495-002	3.048-002	-7.676-001
0.8000	1.698+000	5.984-003	-6.730-001	-2.556-002	3.128-002	-6.666-001
0.7800	1.710+000	6.499-003	-5.825-001	-2.586-002	3.169-002	-5.532-001
0.7600	1.721+000	7.016-003	-4.816-001	-2.579-002	3.162-002	-4.271-001
0.7400	1.729+000	7.528-003	-3.703-001	-2.530-002	3.098-002	-2.880-001
0.7200	1.735+000	8.025-001	-2.485-001	-2.435-002	2.969-002	-1.359-001
0.7000	1.739+000	8.498-003	-1.161-001	-2.288-002	2.769-002	2.913-002
0.6800	1.740+000	8.937-003	2.643-002	-2.090-002	2.503-002	2.067-001
0.6600	1.738+000	9.331-003	1.789-001	-1.843-002	2.187-002	3.964-001
0.6400	1.733+000	9.672-003	3.409-001	-1.554-002	1.855-002	5.975-001
0.6200	1.724+000	9.951-003	5.118-001	-1.238-002	1.565-002	8.096-001
0.6000	1.712+000	1.017-002	6.910-001	-9.148-003	1.397-002	1.032+000
0.5800	1.697+000	1.032-002	8.781-001	-6.099-003	1.441-002	1.265+000
0.5600	1.677+000	1.041-002	1.072+000	-3.466-003	1.774-002	1.509+000
0.5400	1.654+000	1.046-002	1.274+000	-1.381-003	2.424-002	1.766+000
0.5200	1.626+000	1.047-002	1.482+000	2.580-004	3.318-002	2.037+000
0.5000	1.594+000	1.045-002	1.698+000	1.936-003	4.225-002	2.325+000
0.4800	1.558+000	1.039-002	1.920+000	4.662-003	4.714-002	2.632+000
0.4600	1.518+000	1.025-002	2.148+000	1.005-002	4.128-002	2.958+000
0.4400	1.472+000	9.953-003	2.382+000	2.031-002	1.626-002	3.303+000
0.4200	1.422+000	9.384-003	2.621+000	3.801-002	-3.730-002	3.660+000
0.4000	1.367+000	8.365-003	2.860+000	6.576-002	-1.281-001	4.020+000
0.3800	1.308+000	6.673-003	3.096+000	1.056-001	-2.619-001	4.369+000
0.3600	1.244+000	4.052-003	3.232+000	1.586-001	-4.397-001	4.689+000
0.3400	1.175+000	2.479-004	3.536+000	2.237-001	-6.555-001	4.961+000
0.3200	1.102+000	-4.955-003	3.728+000	2.978-001	-8.954-001	5.166+000
0.3000	1.026+000	-1.169-002	3.892+000	3.752-001	-1.139+000	5.293+000
0.2800	9.469-001	-1.993-002	4.024+000	4.481-001	-1.359+000	5.336+000
0.2600	8.653-001	-2.952-002	4.123+000	5.069-001	-1.530+000	5.297+000
0.2400	7.822-001	-4.004-002	4.186+000	5.410-001	-1.624+000	5.188+000

Cont'd.

0.2200	6.981-001	-5.093-002	4.215+000	5.405-001	-1.624+000	5.030+000
0.2000	6.138-001	-6.137-002	4.211+000	4.965-001	-1.520+000	4.841+000
0.1800	5.299-001	-7.045-002	4.176+000	4.026-001	-1.310+000	4.642+000
0.1600	4.470-001	-7.712-002	4.105+000	2.557-001	-1.006+000	4.442+000
0.1400	3.660-001	-8.034-002	3.992+000	5.745-002	-6.258-001	4.238+000
0.1200	2.877-001	-7.913-002	3.819+000	-1.844-001	-1.976-001	4.013+000
0.1000	2.137-001	-7.279-002	3.563+000	-4.522-001	2.438-001	3.734+000
0.0800	1.460-001	-6.108-002	3.190+000	-7.135-001	6.506-001	3.353+000
0.0600	8.719-002	-4.465-002	2.662+000	-9.126-001	9.548-001	2.816+000
0.0400	4.076-002	-2.558-002	1.948+000	-9.597-001	1.056+000	2.077+000
0.0200	1.055-002	-8.174-003	1.042+000	-7.189-001	8.062-001	1.120+000
0.0000	0.000+000	1.041-017	-2.780-004	-5.839-005	0.000+000	-2.220-016

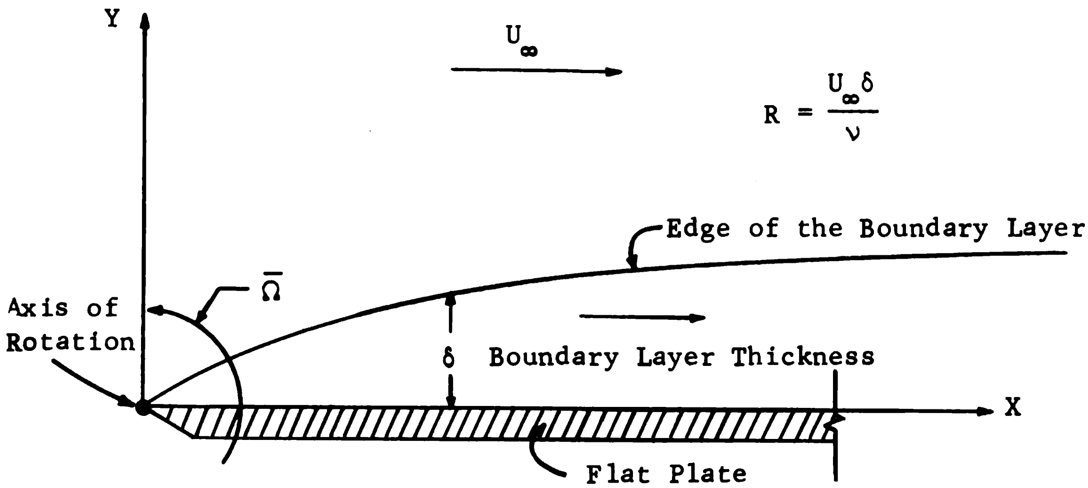


Figure 1. Primary Flow

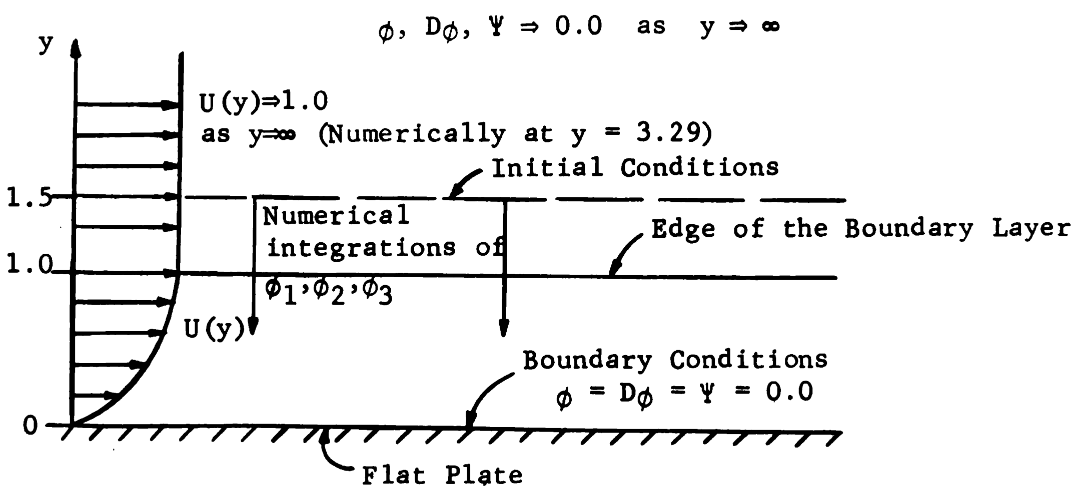


Figure 2. Numerical Integration of the Orr-Sommerfeld Equation.

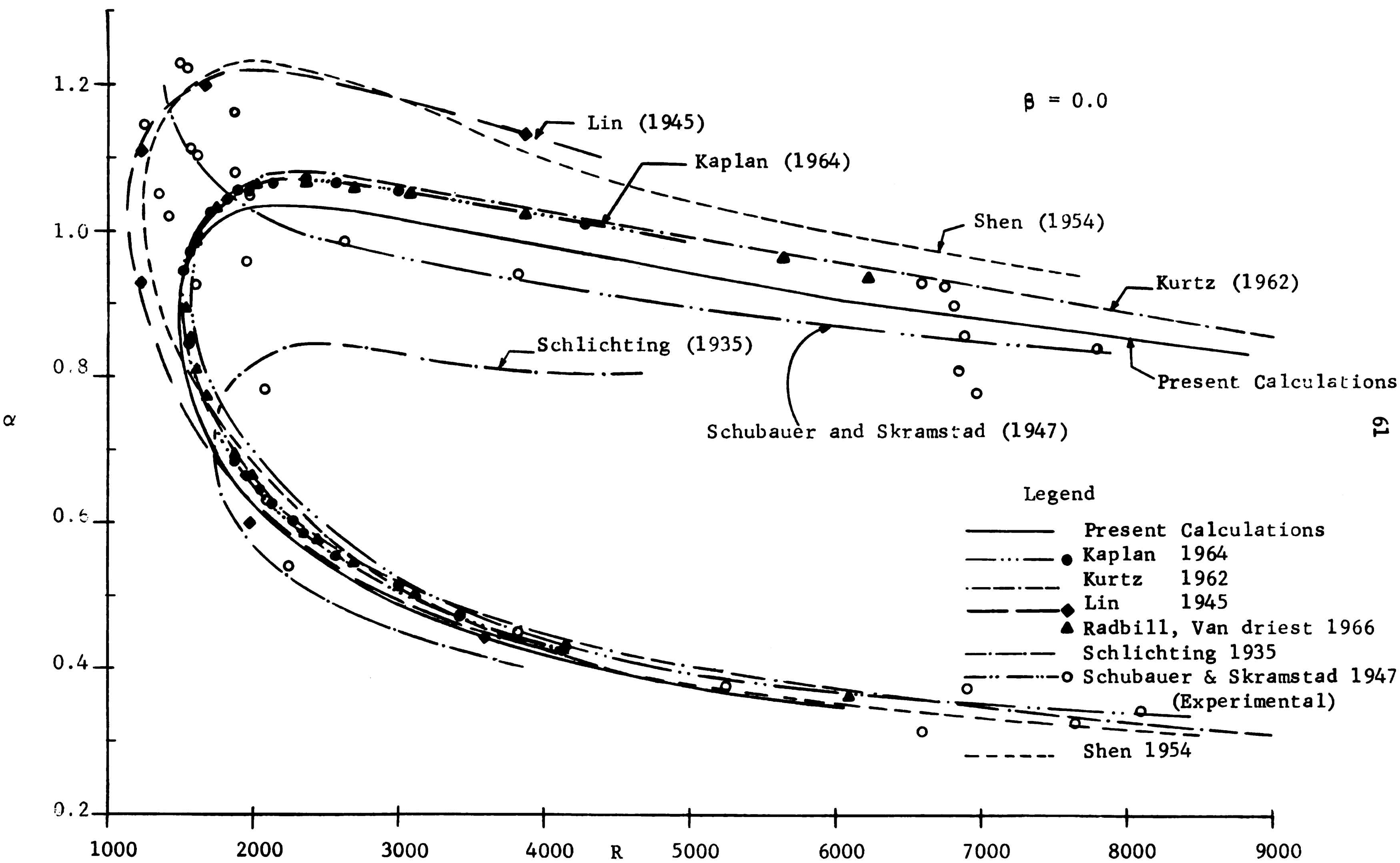


Figure 3. Comparison of the Neutral Stability Curves for the Blasius Boundary Layer over a Flat Plate for $\Omega = 0.0$

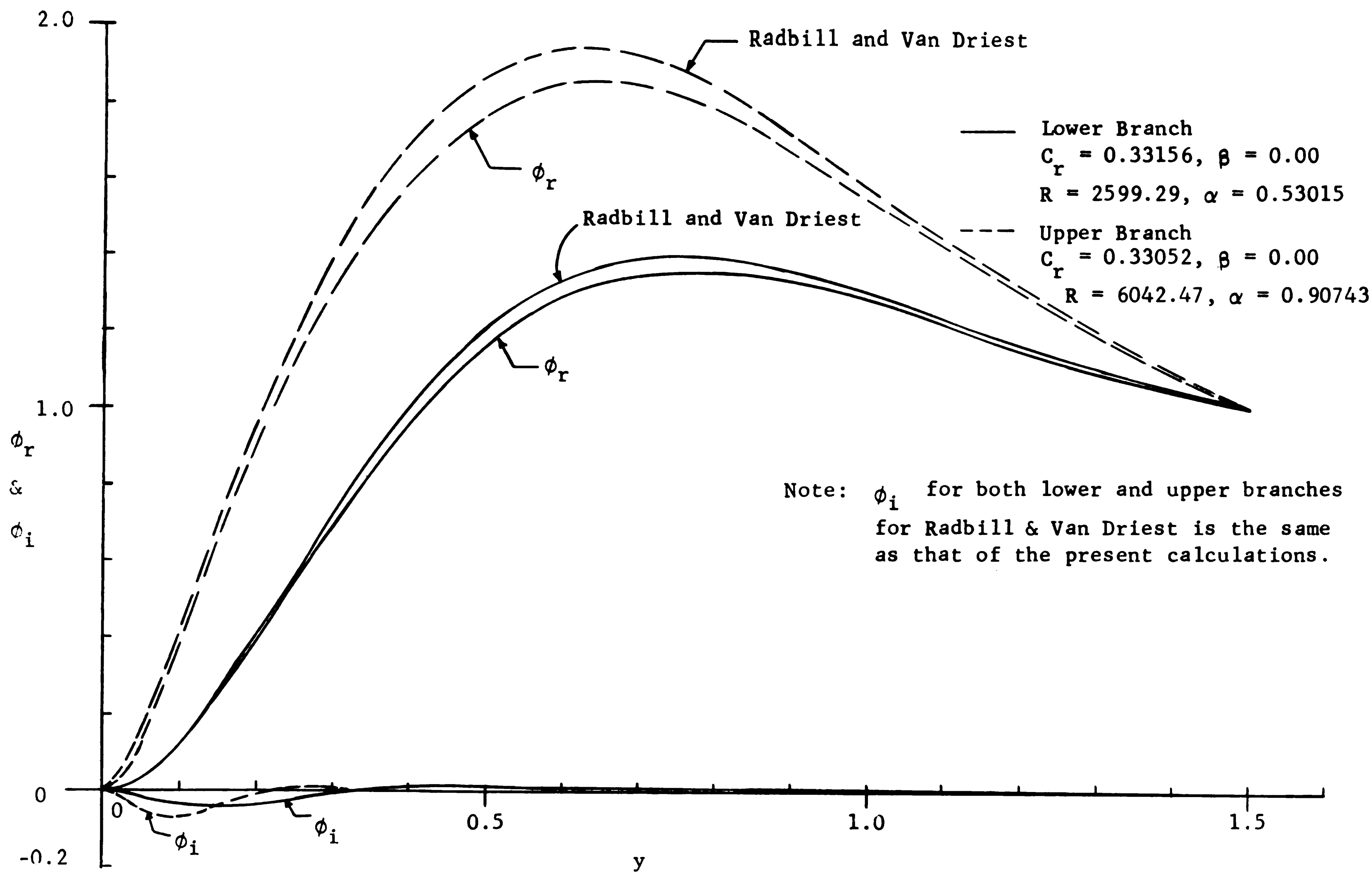


Figure 4. Comparison of Eigen-functions $\phi = \phi_r + i\,\phi_i$ with those of Radbill and Van Driest, Corresponding to the Data of Schubauer and Skramstad for the Upper and Lower Branches of the Neutral Stability Curve (C_i = 0.0) for Ω = 0.00

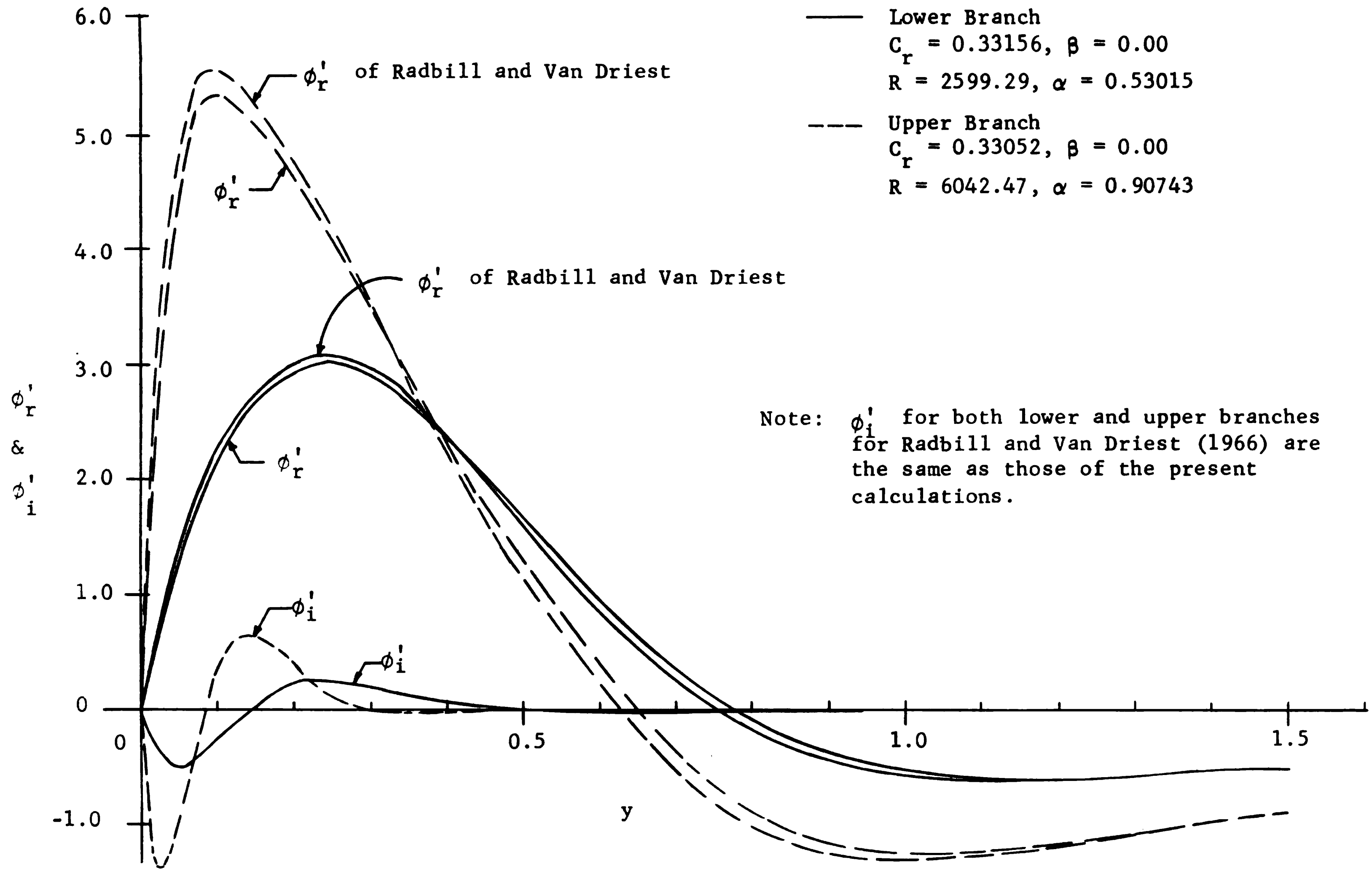


Figure 5. Comparison of the First Derivative of the Eigen-function $\phi' = \phi_r' + i\ \phi_i'$ with those of Radbill & Van Driest, Corresponding to the Data of Schubauer and Skramstad for the Upper and Lower Branches of the Neutral Stability Curve $(C_i = 0.0)$ for $\Omega = 0.0$

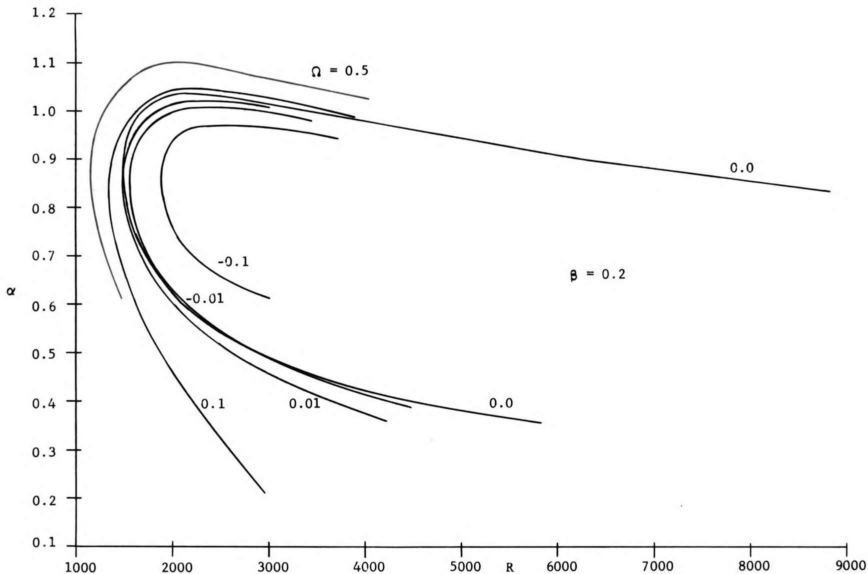


Figure 6. Plot of Wave Number α against Reynolds Number R for the Neutral Stability Curves for different values of Angular Rotation Ω

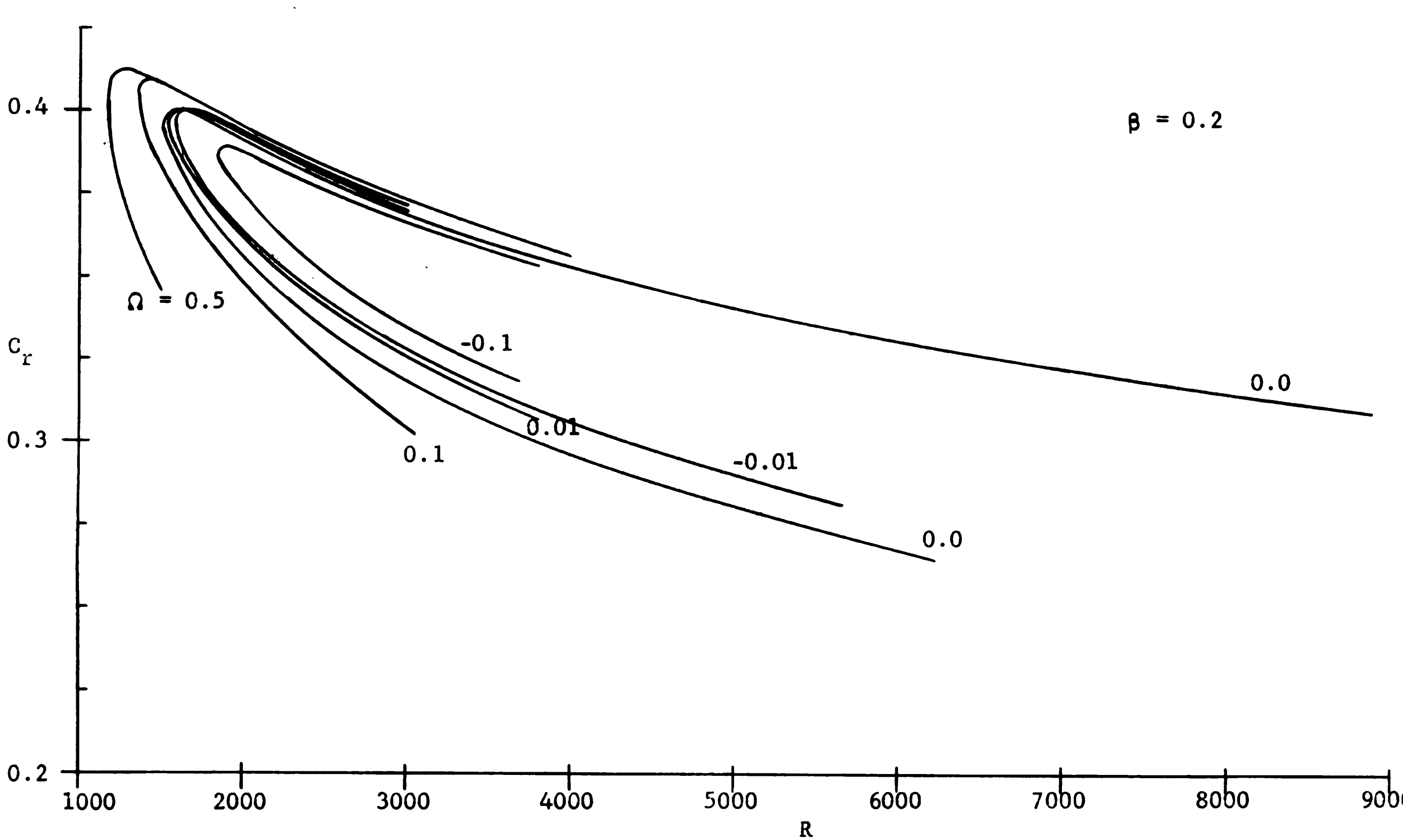


Figure 7. Plot of Wave Velocity C_r against Reynolds Number R for the Neutral Stability Curves for different values of Angular Rotation Ω

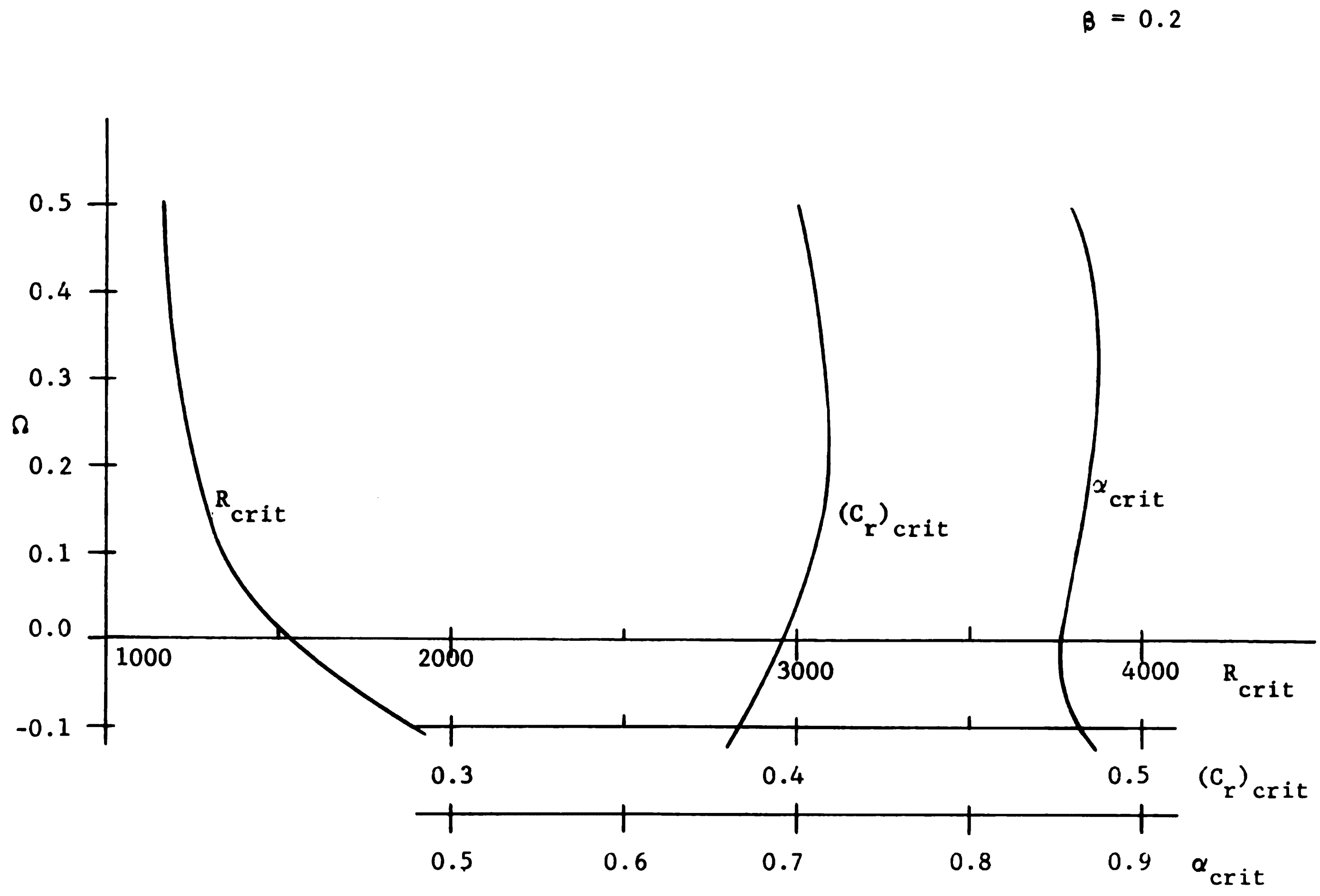


Figure 8. Effect of Angular Rotation Ω on Critical Eigenvalues R_{crit}, $(C_r)_{crit}$ and α_{crit} for $\beta = 0.2$

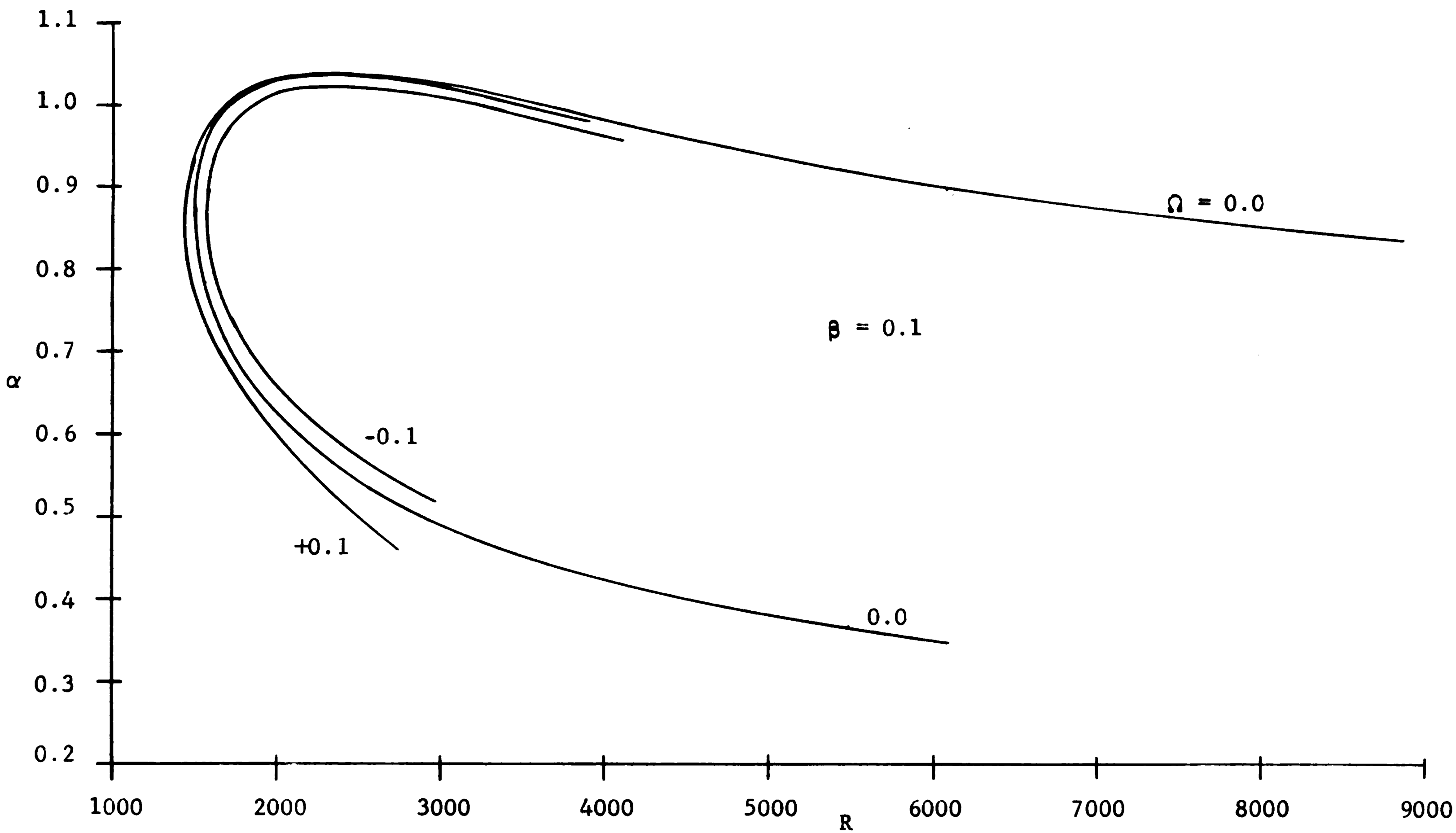


Figure 9. Effect of varying Angular Rotation Ω on the Neutral Stability Curves for $\beta = 0.1$

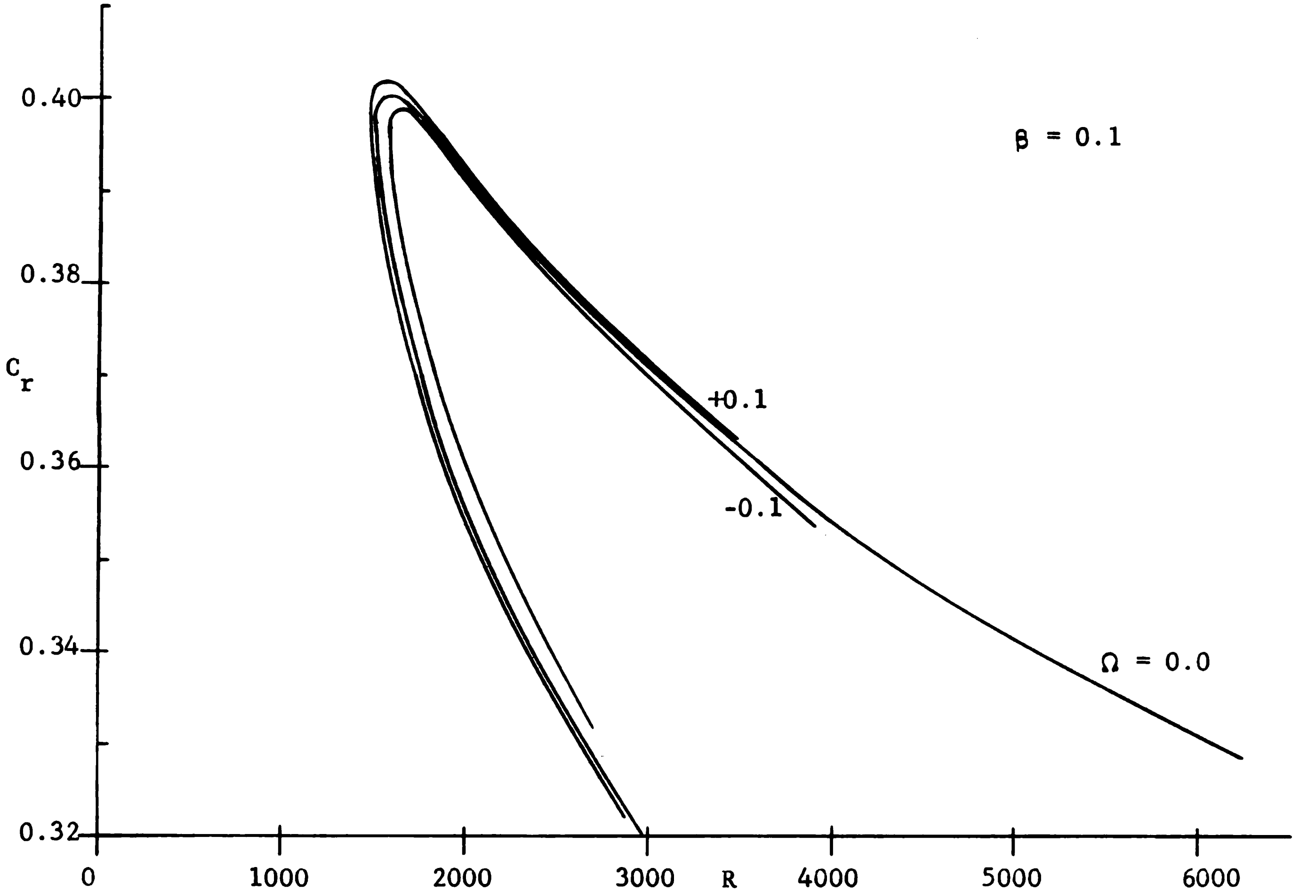


Figure 10. Effect of varying Angular Rotation Ω on the plot of Wave Velocity C_r against Reynolds Number R for the Neutral Stability for $\beta = 0.10$

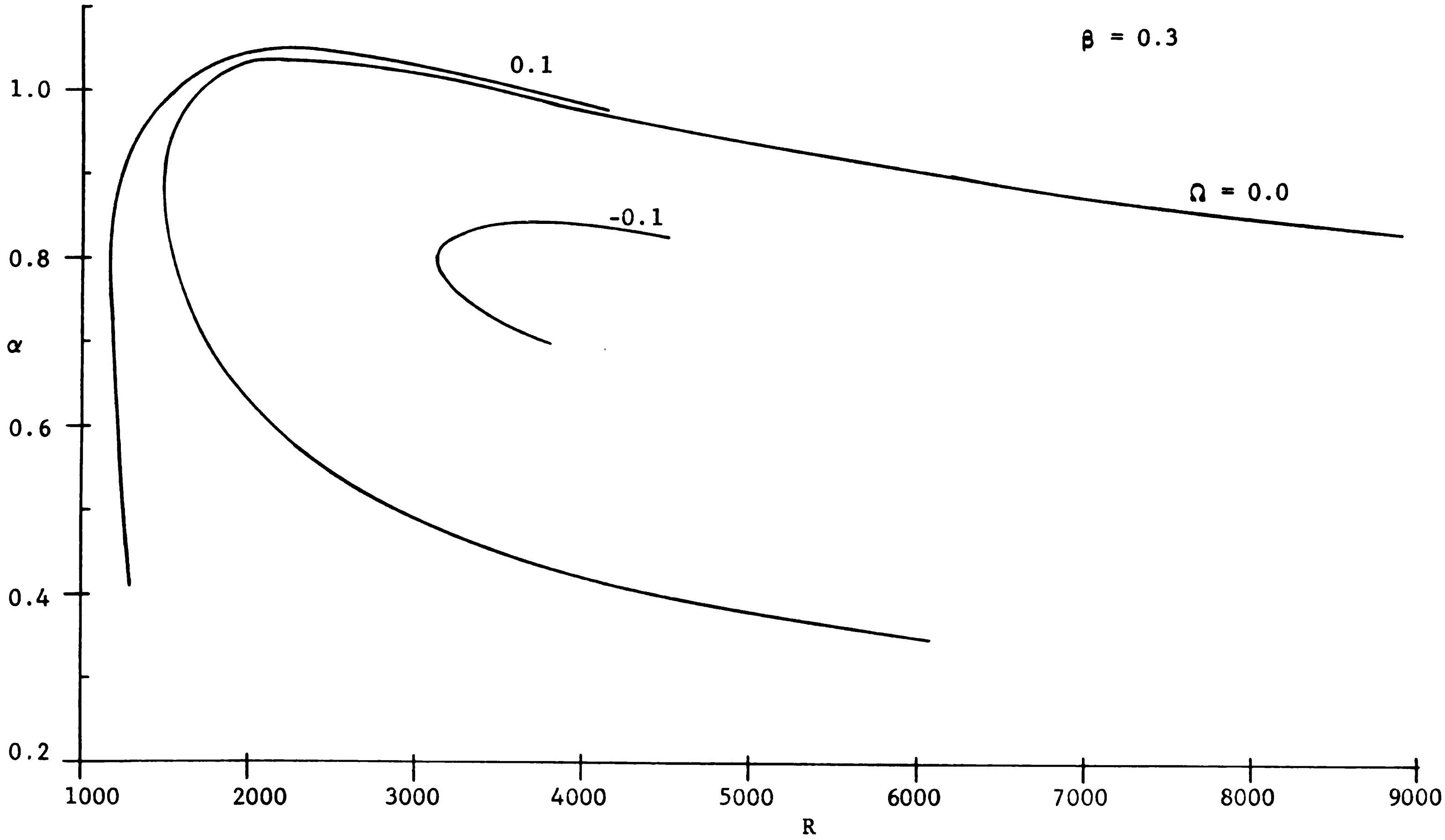


Figure 11. Effect of varying Angular Rotation Ω on the Neutral Stability Curves for $\beta = 0.30$

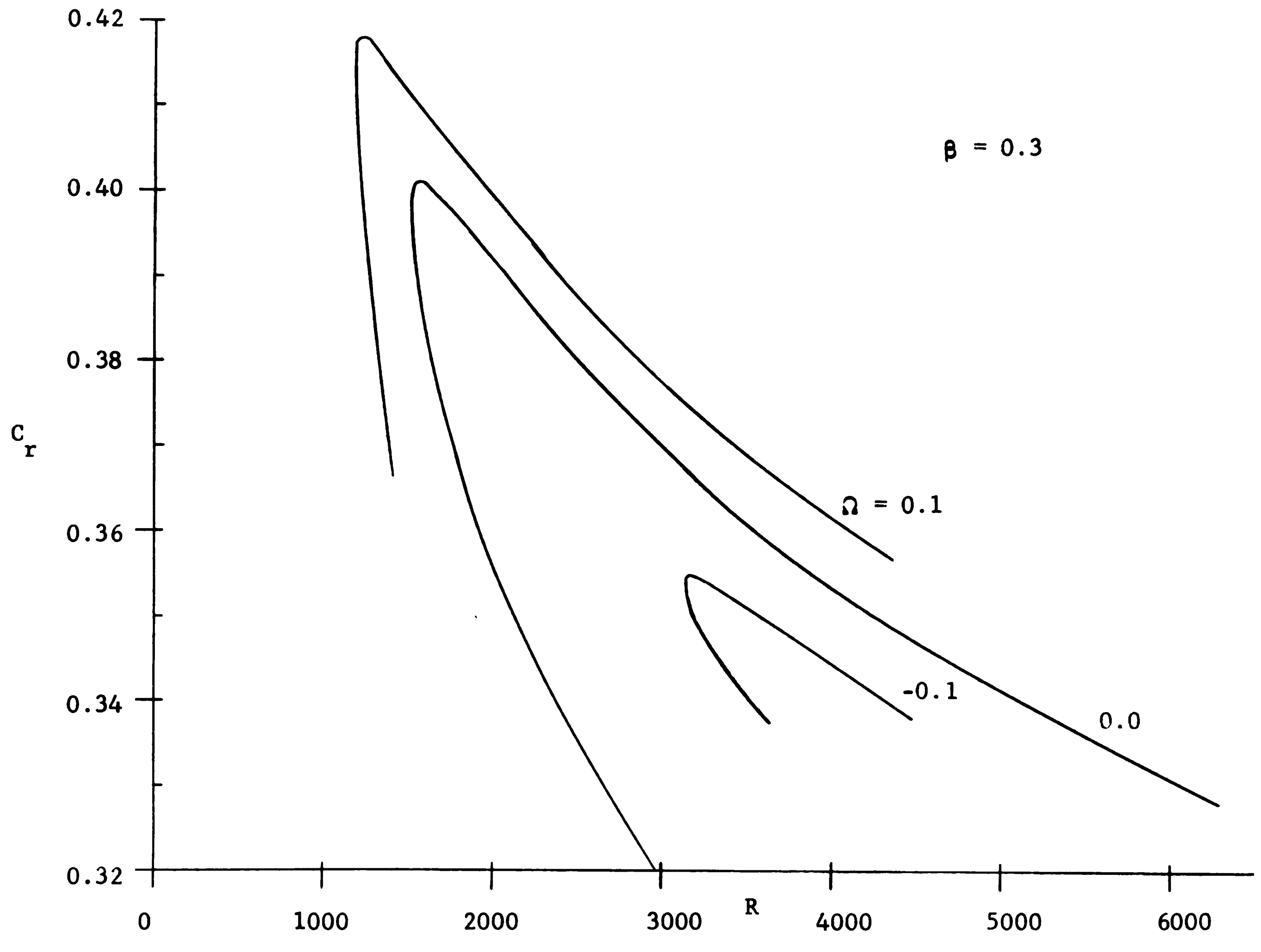


Figure 12. Effect of varying Angular Rotation Ω on the plot of Wave Velocity C_r against Reynolds Number R for the Neutral Stability for $\beta = 0.30$

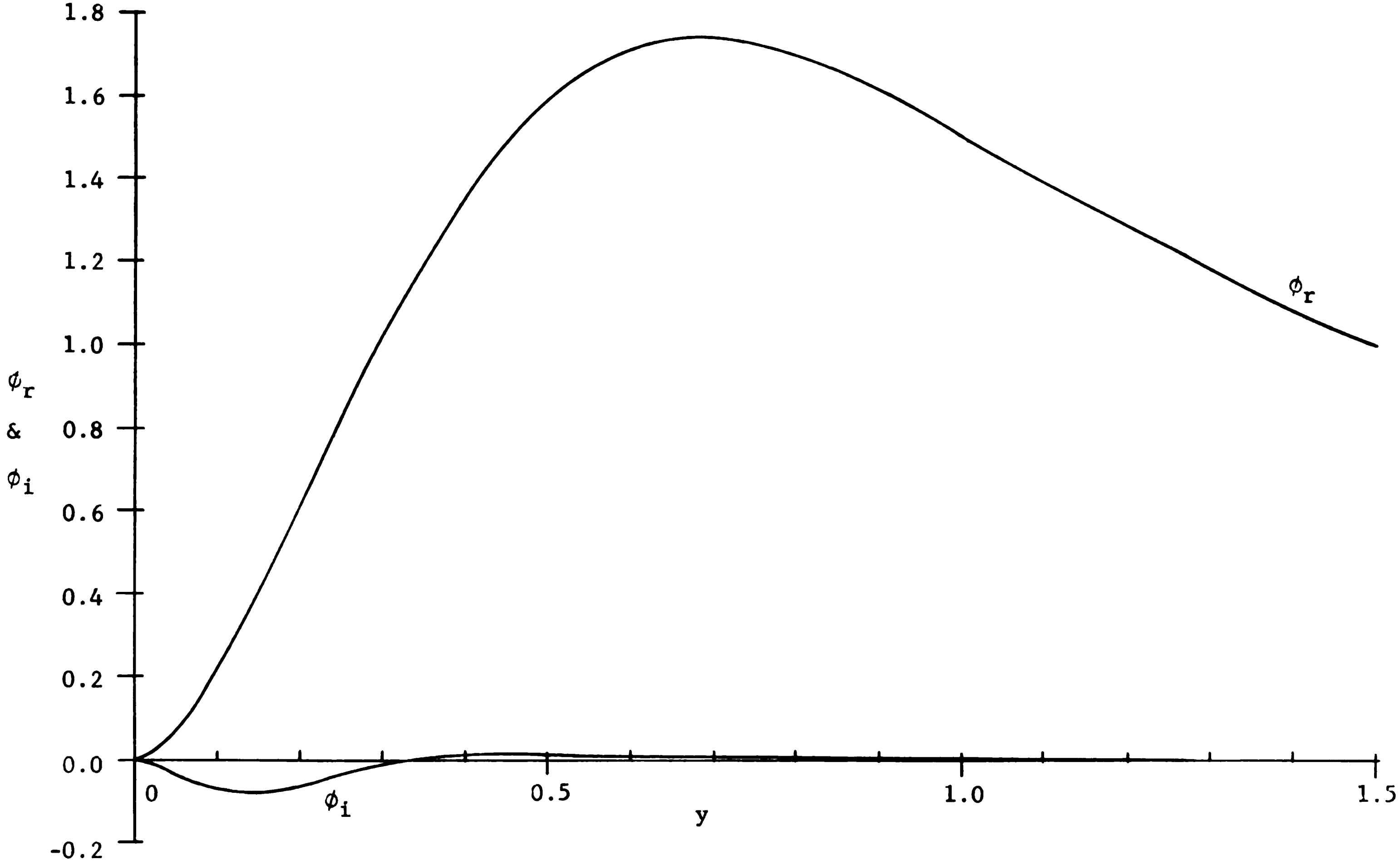


Figure 13. Plot of Eigen-functions $\phi = \phi_r + i\ \phi_i$ near a Critical Point on the Lower Branch of the Neutral Stability Curve ($C_i = 0.0$) for $\Omega = -0.10$ and $\beta = 0.20$

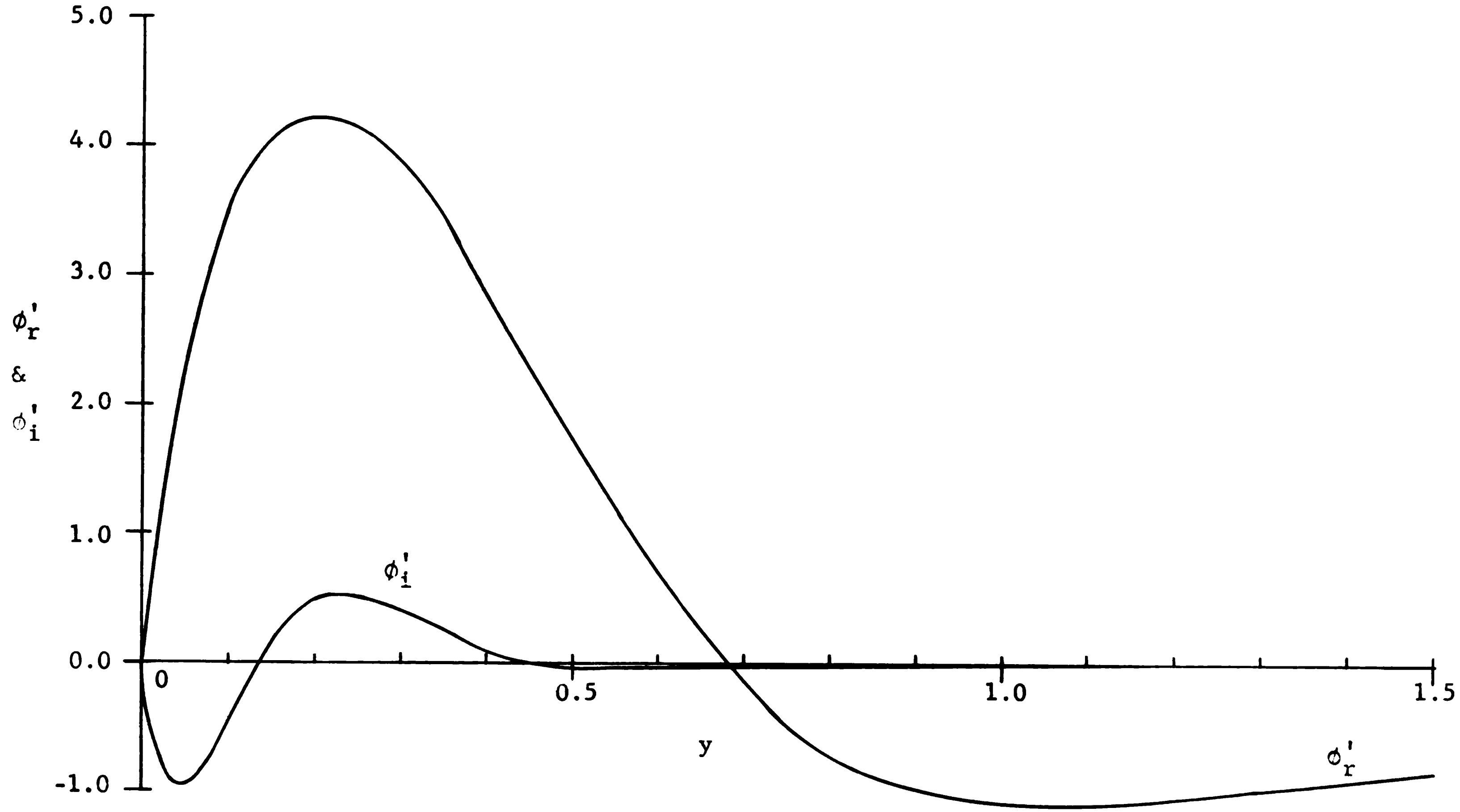


Figure 14. Plot of the First Derivative of the Eigen-function $\phi' = \phi_r' + i\ \phi_i'$ near a Critical Point on the Lower Branch of the Neutral Stability Curve (C_i = 0.0) for Ω = -0.10 and β = 0.20

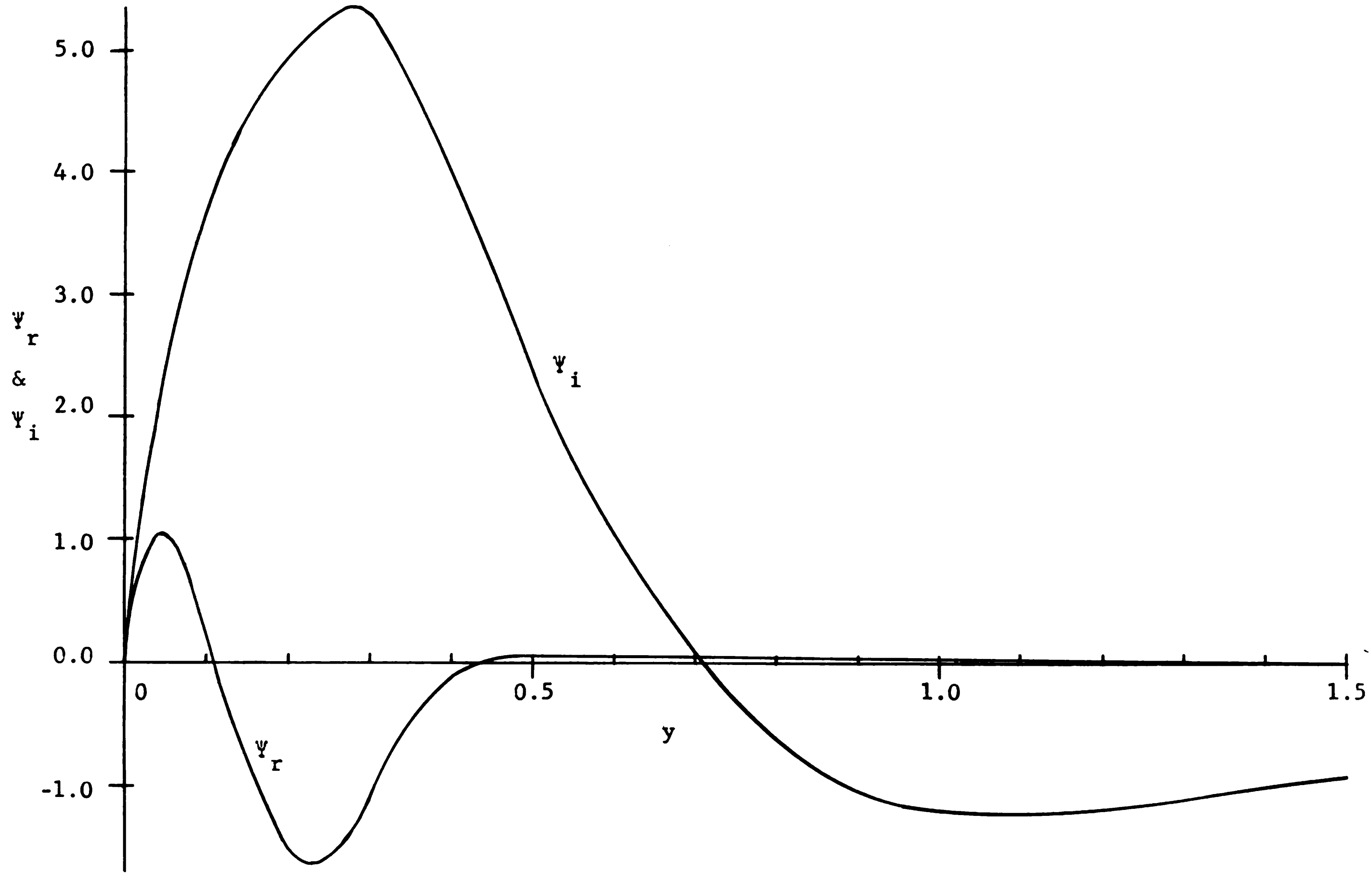


Figure 15. Plot of Eigen-functions $\Psi = \Psi_r + i\,\Psi_i$ near a Critical Point on the Lower Branch of the Neutral Stability Curve $(C_i = 0.0)$ for $\Omega = -0.10$ and $\beta = 0.20$

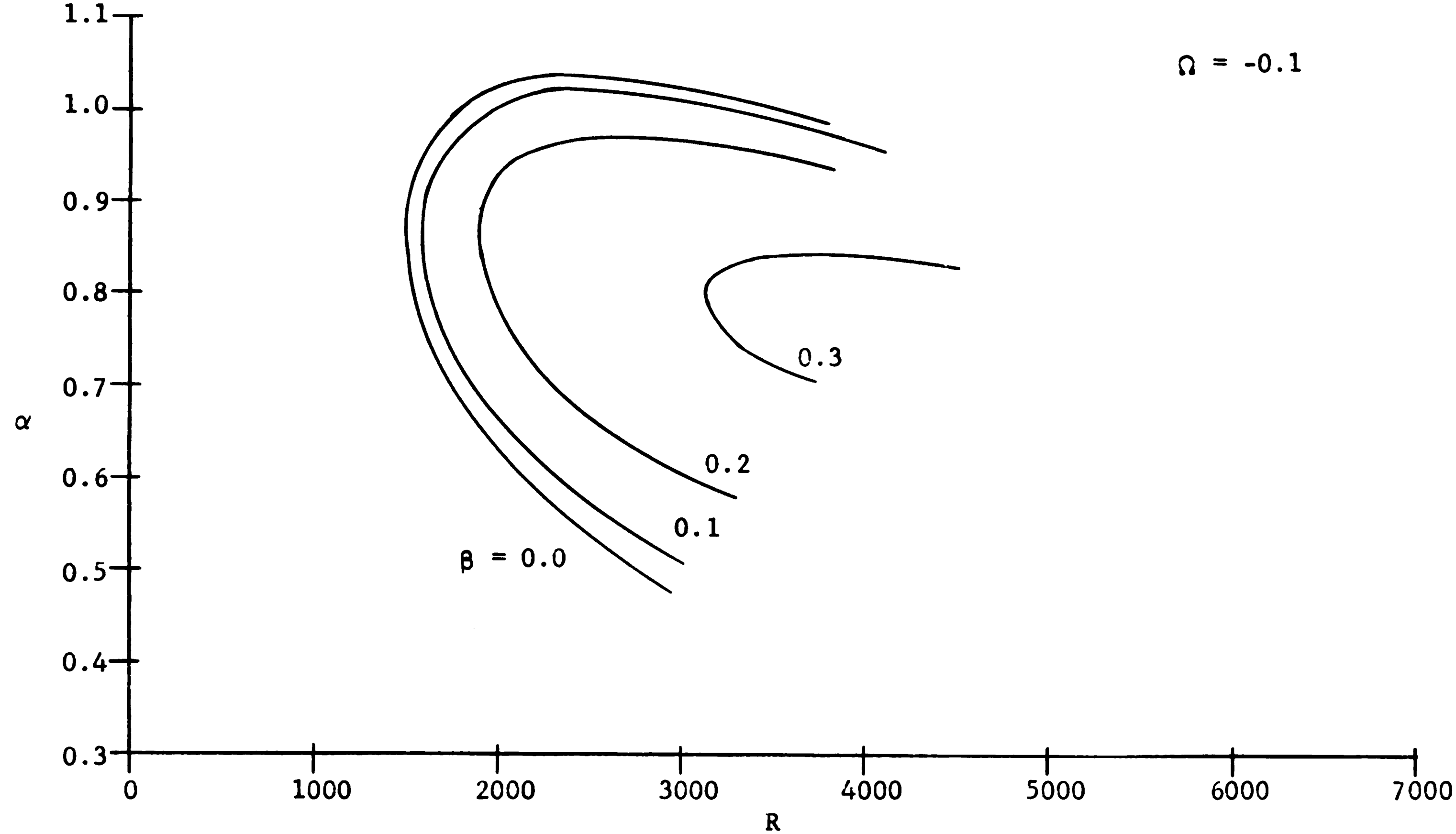


Figure 16. Effect of varying Wave Number β on the Neutral Stability Curves (C_i = 0.0) for Angular Rotation Ω = -0.10

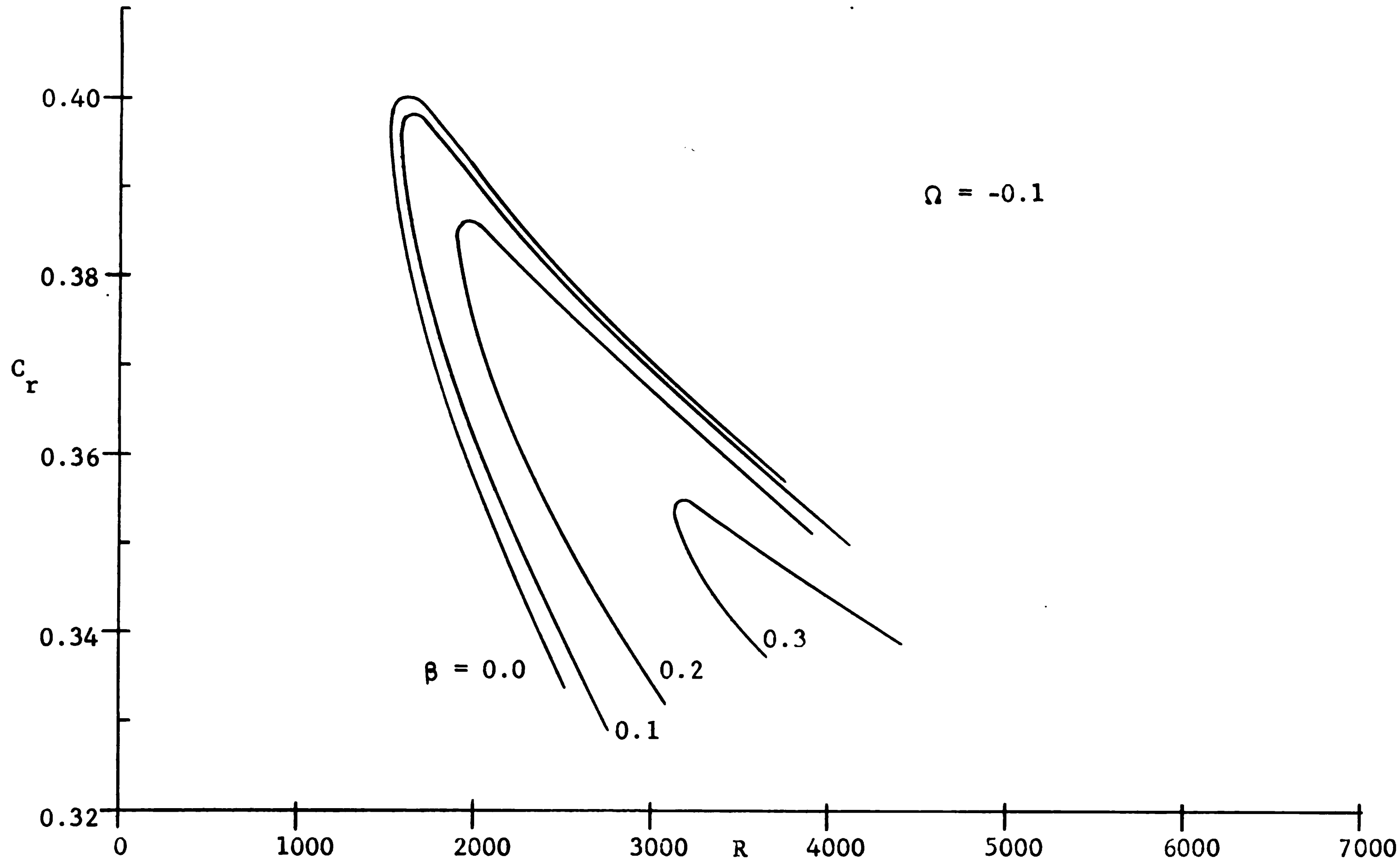


Figure 17. Effect of varying Wave Number β on the Plot of Wave Velocity C_r against Reynolds Number R for the Neutral Stability for $\Omega = -0.10$

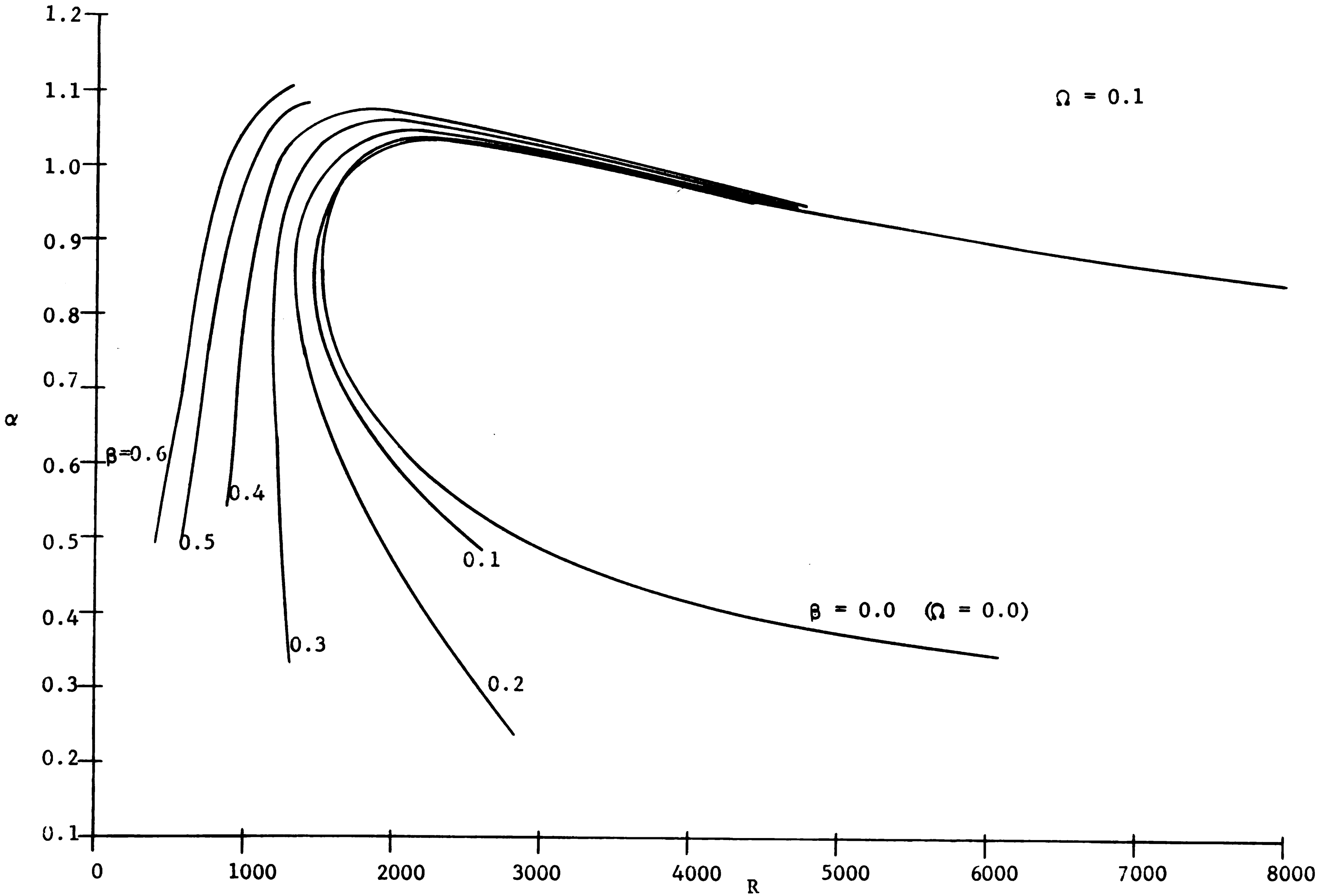


Figure 18. Effect of varying Wave Number β on the Neutral Stability Curves (C_i = 0.0) for Angular Rotation Ω = 0.10

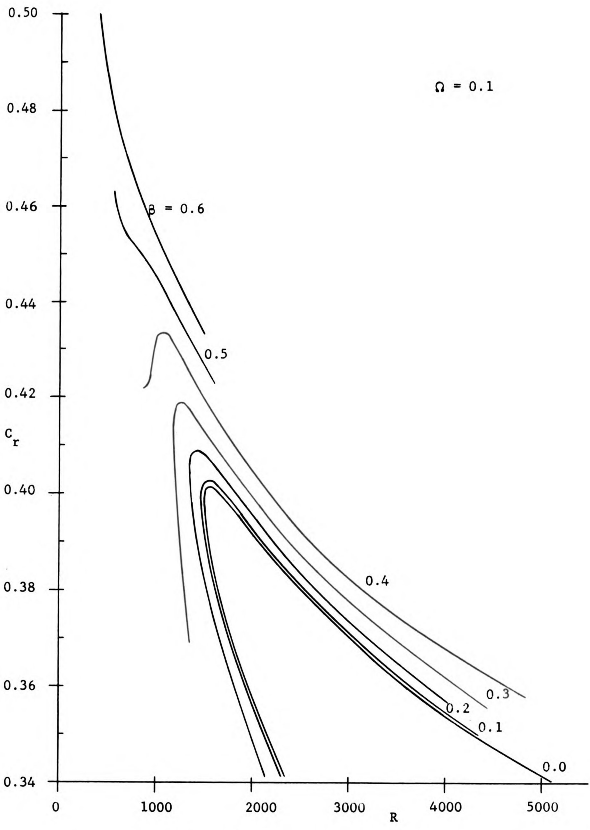


Figure 19. Effect of varying Wave Number β on the plot of Wave Velocity C_r against Reynolds Number R for the Neutral Stability for $\Omega = 0.10$

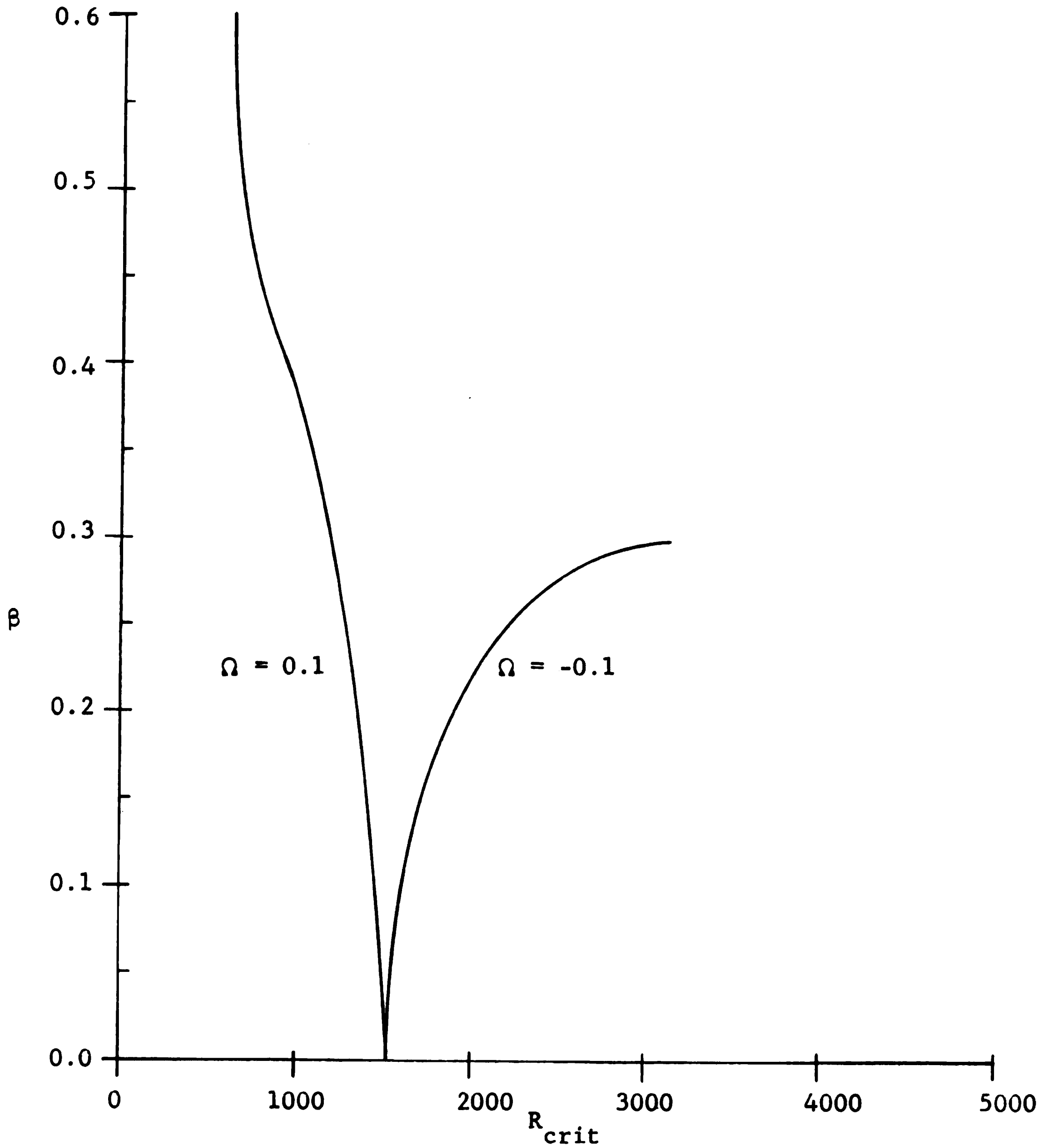


Figure 20. Plot of Wave Number β against Critical Reynolds Number R_{crit} for Angular Rotation $\Omega = 0.10$ and -0.10

APPENDICES

APPENDIX A

NUMERICAL TECHNIQUES

A-1 Numerical Technique to Solve for the Primary Velocity Profile

This numerical scheme solves the equation (3.1.8). It is a fourth order Runge-Kutta method as presented by Weeg and Reed. The accuracy of this scheme was checked by comparing the values of the primary velocity U with those given in Schlichting (1962). The agreement between the two is very good. The technique used is given in appendix C-3 and explained below:

The third order differential equation (3.1.8) is written as three first order differential equations

$$\left.\begin{aligned} F_n' &= Z_n \\ Z_n' &= W_n \\ W_n' &= -\frac{1}{2} Z_n W_n \end{aligned}\right] \qquad \text{(A-1-1)}$$

with the initial (boundary) conditions at the plate being

$$\left.\begin{aligned} F_n &= 0.0 \\ Z_n &= 0.0 \\ W_n &= 0.3 \end{aligned}\right] \qquad \text{(A-1-2)}$$

The value of W_n was guessed and was later found by iteration to be 0.332. The functions F, Z, and W at one step away from the plate are

$$F_{n-1} = F_n + \frac{1}{6}(a_1 + 2\ a_2 + 2\ a_3 + a_4)$$

$$Z_{n-1} = Z_n + \frac{1}{6}(b_1 + 2\ b_2 + 2\ b_3 + b_4)$$

$$W_{n-1} = W_n + \frac{1}{6}(c_1 + 2\ c_2 + 2\ c_3 + c_4)$$

where a_i, b_i, and c_i for $i = 1$ to 4 are computed as follows:

$$\left.\begin{aligned} a_1 &= h\ Z_n \\ b_1 &= h\ W_n \\ c_1 &= -\frac{h\ Z_n W_n}{2} \end{aligned}\right] \qquad \text{(A-1-3)}$$

$$\left.\begin{aligned} a_2 &= h(Z_n + \frac{a_1}{2}) \\ b_2 &= h(W_n + \frac{b_1}{2}) \\ c_2 &= -\frac{h}{2}(Z_n + \frac{a_1}{2})(W_n + \frac{c_1}{2}) \end{aligned}\right] \qquad \text{(A-1-4)}$$

$$\left.\begin{aligned} a_3 &= h(Z_n + \frac{a_2}{2}) \\ b_3 &= h(W_n + \frac{b_2}{2}) \\ c_3 &= -\frac{h}{2}(Z_n + \frac{a_2}{2})(W_n + \frac{c_2}{2}) \end{aligned}\right] \qquad \text{(A-1-5)}$$

and

$$\left.\begin{aligned} a_4 &= h(Z_n + \frac{a_3}{2}) \\ b_4 &= h(W_n + \frac{b_3}{2}) \\ c_4 &= -\frac{h}{2}(Z_n + \frac{a_3}{2})(W_n + \frac{c_3}{2}), \end{aligned}\right] \qquad \text{(A-1-6)}$$

where h is the step size.

Using this technique the solution is generated to free stream. That value of W_n is adopted which gives $U = 1.0$ at the farthest distance from the plate.

A-2 Numerical Technique to solve the Governing Stability Equations

The numerical integration of the Orr-Sommerfeld equation (3.2.1) for $\Omega = 0$ and equations (2.5.13) for $\Omega \neq 0$ is successfully completed by using a standard Runge-Kutta integration method for second and fourth order equations as illustrated by Collatz (1951). The $\Omega = 0$ case was solved separately because the order of the equation reduces by two and thus the numerical technique changes for the two cases. Equations (2.5.13) are solved separately but simultaneously so that the order of the equations remains less than fourth. Alternatively we can solve one sixth order ordinary linear differential equation in one variable as explained by Zurmühl (1948). The accuracy of the results is checked by comparing with the results of Kaplan (1964), Lin (1945), Schlichting (1935), Radbill and Van Driest (1966) and also by comparing with the experimental data of Schubauer and Skramstad (1947). This is further verified by varying the step size. The scheme as used in the final form for the simultaneous solution of equations (2.5.13) is outlined below:

At y

$$V_{00} = \phi(y,I)$$

$$V_{10} = -h\,\phi'(y,I)$$

$$V_{20} = \frac{h^2}{2}\,\phi''(y,I)$$

$$V_{30} = -\frac{h^3}{6}\,\phi'''(y,I)$$

$$U_{00} = \Psi(y,I)$$

$$U_{10} = -h\,\Psi'(y,I)$$

Using equations (2.5.13),

$$F_1 = \frac{h^2}{2} f(-\frac{6}{h^3} V_{30}, -\frac{1}{h} V_{10}, V_{00}, U_{00}, y)$$

$$= \frac{h^2}{2}[i \frac{\alpha}{K} \phi''' - \frac{1}{K}\{K + iR(U-c)\}\phi' - i \frac{\beta^2 R}{\alpha^2 K} (U'-2\Omega)\phi'$$

$$+ \{K + iR(U-c)\}\Psi]$$

$$G_1 = \frac{h^4}{24} f(\frac{2}{h^2} V_{20}, -\frac{1}{h} V_{10}, V_{00}, U_{00}, y)$$

$$= \frac{h^4}{24}[\{iR(U-c) + 2K\}\phi'' + 2 iR \Omega \phi' - \{K^2 + iR(K(U-c)$$

$$+ U'')\}\phi - 2K R \Omega \Psi]$$

Now at $y - h/2$

$$V_{01} = V_{00} + \frac{1}{2} V_{10} + \frac{1}{4} V_{20} + \frac{1}{8} V_{30} + \frac{1}{16} G_1$$

$$V_{11} = V_{10} + V_{20} + \frac{3}{4} V_{30} + \frac{1}{2} G_1$$

$$V_{21} = V_{20} + \frac{3}{2} V_{30} + \frac{3}{2} G_1$$

$$V_{31} = V_{30} + 2 G_1$$

$$U_{01} = U_{00} + \frac{1}{2} U_{10} + \frac{1}{4} F_1$$

Then as before

$$F_2 = \frac{h^2}{2} f(-\frac{6}{h^3} V_{31}, -\frac{1}{h} V_{11}, V_{01}, U_{01}, y - \frac{h}{2})$$

$$G_2 = \frac{h^4}{24} f(\frac{2}{h^2} V_{21}, -\frac{1}{h} V_{11}, V_{01}, U_{01}, y - \frac{h}{2})$$

Now at $y - h$,

$$V_{02} = V_{00} + V_{10} + V_{20} + V_{30} + G_2$$

$$V_{12} = V_{10} + 2V_{20} + 3V_{30} + 4G_2$$

$$V_{22} = V_{20} + 3V_{30} + 6G_2$$

$$V_{32} = V_{30} + 4G_2$$

$$U_{02} = U_{00} + U_{10} + F_2$$

Again

$$F_3 = \frac{h^2}{2} f(-\frac{6}{h^3} V_{32}, -\frac{1}{h} V_{12}, V_{02}, U_{02}, y - h)$$

$$G_3 = \frac{h^4}{24} f(\frac{2}{h^2} V_{22}, -\frac{1}{h} V_{12}, V_{02}, U_{02}, y - h)$$

Then

$$F = \frac{1}{3} (F_1 + 2F_2)$$

$$F' = \frac{1}{3} (F_1 + 4F_2 + F_3)$$

$$G = \frac{1}{15} (8G_1 + 8G_2 - G_3)$$

$$G' = \frac{1}{5} (9G_1 + 12G_2 - G_3)$$

$$G'' = 2(G_1 + 2G_2)$$

$$G''' = \frac{2}{3} (G_1 + 4G_2 + G_3)$$

Thus, the functions and their derivatives at $y - h$ are

$$\phi(y-h) = V_{00} + V_{10} + V_{20} + V_{30} + G$$

$$\phi'(y-h) = -\frac{1}{h} (V_{10} + 2V_{20} + 3V_{30} + G')$$

$$\phi''(y-h) = \frac{2}{h^2} (V_{20} + 3V_{30} + G'')$$

$$\phi'''(y-h) = -\frac{6}{h^3} (V_{30} + G''')$$

and

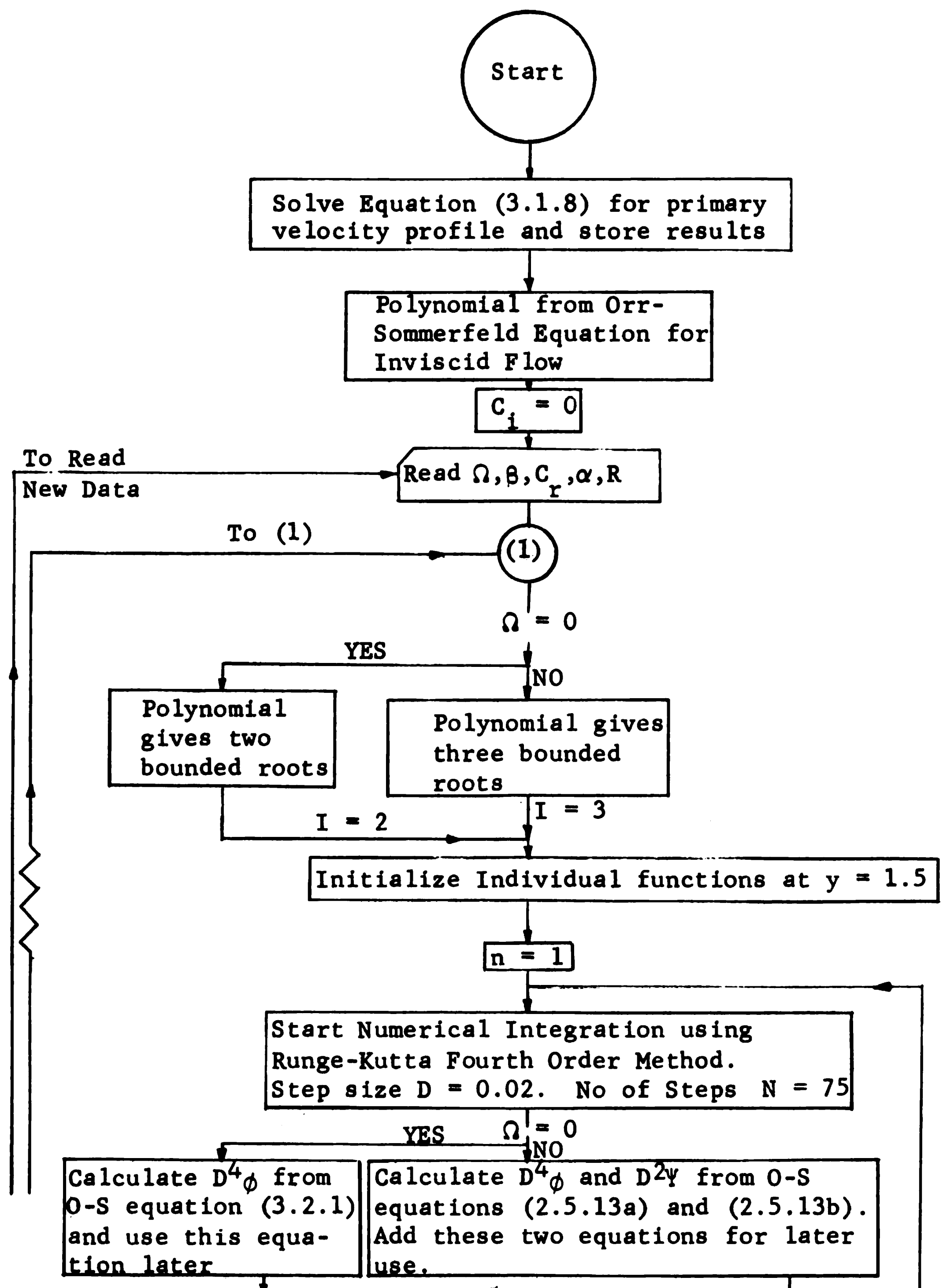
$$\Psi(y-h) = U_{00} + U_{10} + F$$

$$\Psi'(y-h) = -\frac{1}{h}(U_{10} + F')$$

This is done until the plate is reached and the combined functions are formed.

APPENDIX B

Flow Diagram



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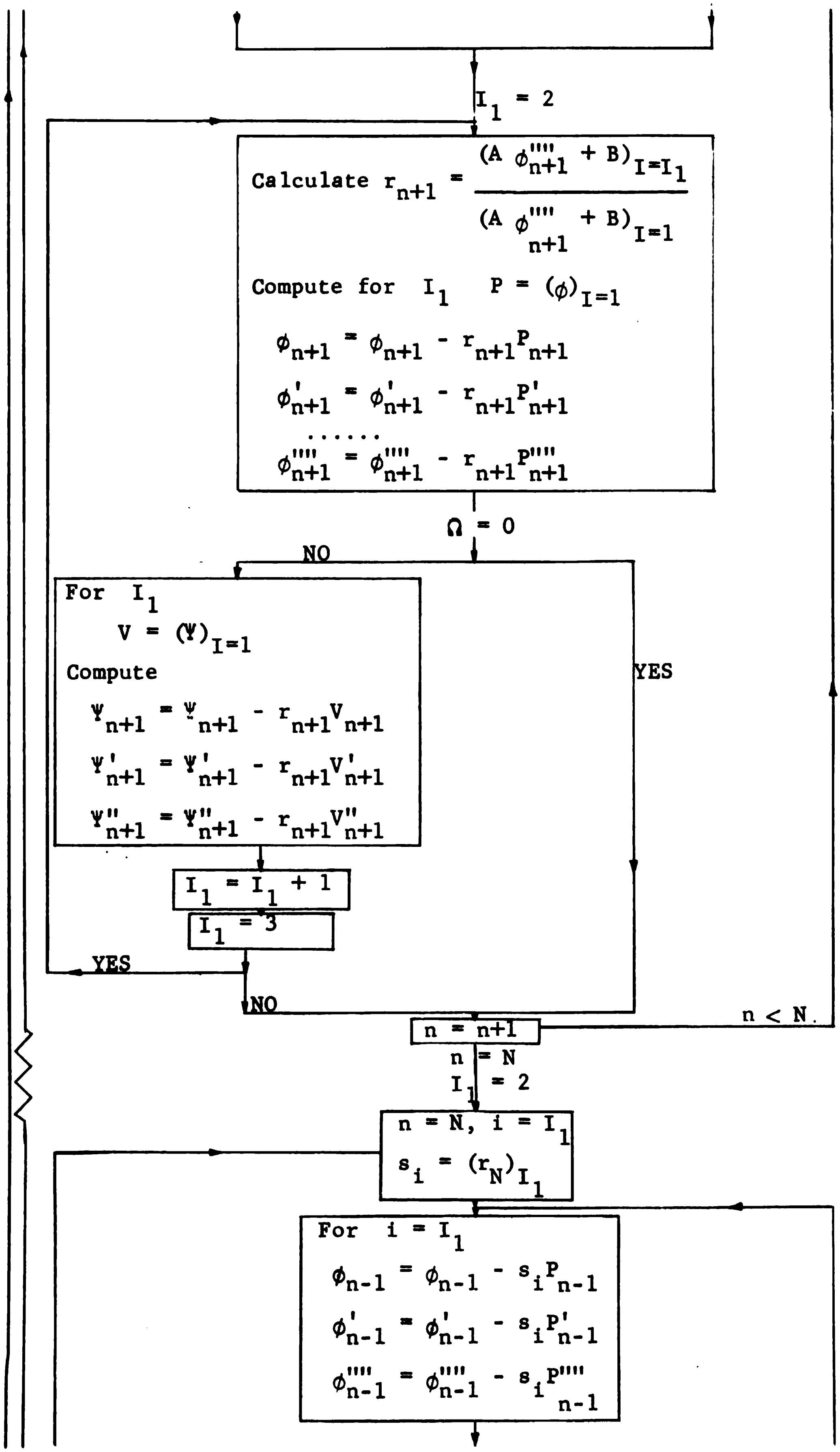
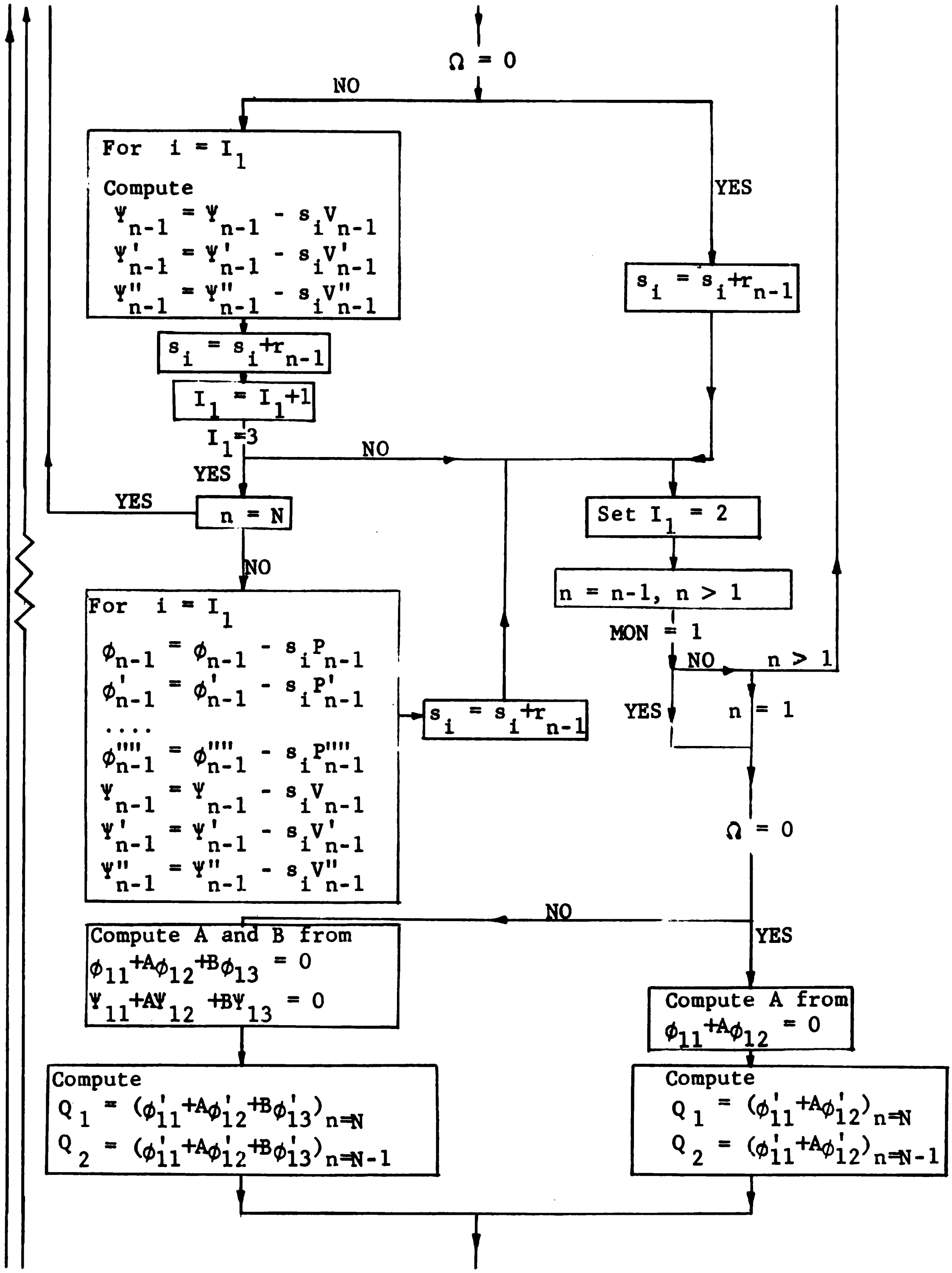
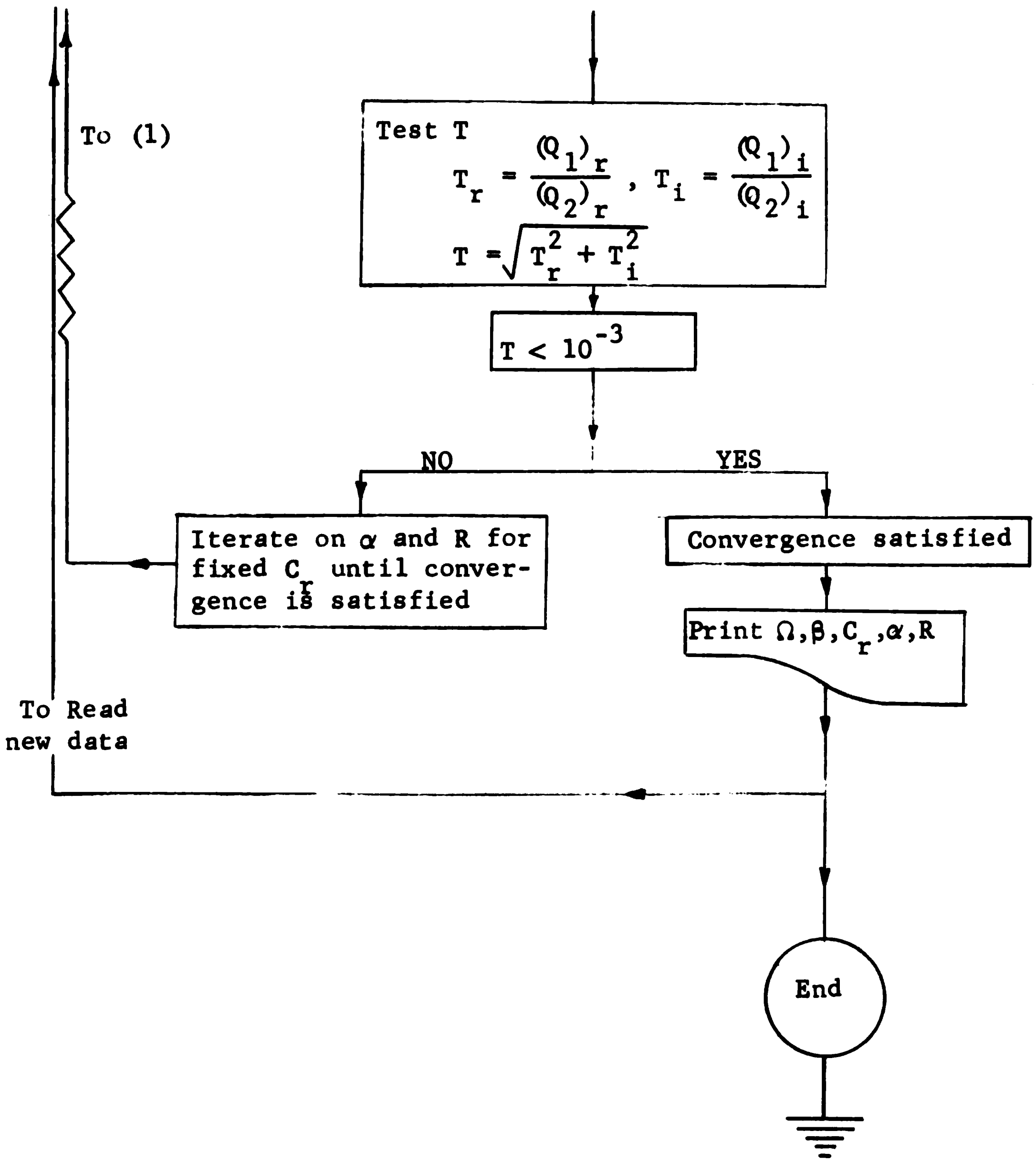

$I_1 = 2$
Calculate $r_{n+1} = \dfrac{(A\ \phi''''_{n+1} + B)_{I=I_1}}{(A\ \phi''''_{n+1} + B)_{I=1}}$
Compute for I_1 $P = (\phi)_{I=1}$
$\phi_{n+1} = \phi_{n+1} - r_{n+1}P_{n+1}$
$\phi'_{n+1} = \phi'_{n+1} - r_{n+1}P'_{n+1}$
......
$\phi''''_{n+1} = \phi''''_{n+1} - r_{n+1}P''''_{n+1}$
$\Omega = 0$
NO
YES
For I_1
$V = (\Psi)_{I=1}$
Compute
$\Psi_{n+1} = \Psi_{n+1} - r_{n+1}V_{n+1}$
$\Psi'_{n+1} = \Psi'_{n+1} - r_{n+1}V'_{n+1}$
$\Psi''_{n+1} = \Psi''_{n+1} - r_{n+1}V''_{n+1}$
$I_1 = I_1 + 1$
$I_1 = 3$
YES
NO
$n = n+1$
$n < N$
$n = N$
$I_1 = 2$
$n = N,\ i = I_1$
$s_i = (r_N)_{I_1}$
For $i = I_1$
$\phi_{n-1} = \phi_{n-1} - s_iP_{n-1}$
$\phi'_{n-1} = \phi'_{n-1} - s_iP'_{n-1}$
$\phi''''_{n-1} = \phi''''_{n-1} - s_iP''''_{n-1}$

Cont'd.


Ω = 0
NO
YES
For i = I1
Compute
Ψn-1 = Ψn-1 - siVn-1
Ψ'n-1 = Ψ'n-1 - siV'n-1
Ψ"n-1 = Ψ"n-1 - siV"n-1
si = si+rn-1
I1 = I1+1
I1=3
NO
YES
YES
n = N
NO
Set I1 = 2
n = n-1, n > 1
MON = 1
NO
n > 1
YES
n = 1
For i = I1
φn-1 = φn-1 - siPn-1
φ'n-1 = φ'n-1 - siP'n-1
....
φ""n-1 = φ""n-1 - siP""n-1
Ψn-1 = Ψn-1 - siVn-1
Ψ'n-1 = Ψ'n-1 - siV'n-1
Ψ"n-1 = Ψ"n-1 - siV"n-1
si = si+rn-1
Ω = 0
NO
YES
Compute A and B from
φ11+Aφ12+Bφ13 = 0
Ψ11+AΨ12 +BΨ13 = 0
Compute A from
φ11+Aφ12 = 0
Compute
Q1 = (φ'11+Aφ'12+Bφ'13)n=N
Q2 = (φ'11+Aφ'12+Bφ'13)n=N-1
Compute
Q1 = (φ'11+Aφ'12)n=N
Q2 = (φ'11+Aφ'12)n=N-1

Cont'd.



To (1)
Test T
$T_r = \frac{(Q_1)_r}{(Q_2)_r}$, $T_i = \frac{(Q_1)_i}{(Q_2)_i}$
$T = \sqrt{T_r^2 + T_i^2}$
$T < 10^{-3}$
NO
YES
Iterate on α and R for fixed C_r until convergence is satisfied
Convergence satisfied
Print $\Omega, \beta, C_r, \alpha, R$
To Read new data
End

APPENDIX C

COMPUTER PROGRAMS

C-1 Use of Computer Programs

Eigenvalues for the Orr-Sommerfeld equations (3.2.1) for $\Omega = 0$ or (2.5.13 a and b) for $\Omega \neq 0$ were obtained by computer programs written in fortran IV with the use of the CDC 3600 computer. A description of these programs is given in section C-2 and listings of the main program and subroutines in section C-3. Input to the main program is by two common cards and the rest by data cards. These are detailed later in this section. In these cards we specify the step sizes outside and inside the boundary layer, total number of steps in the boundary layer, number of data cards, the value of the monitor which controls output, tolerance for the roots of the polynomial, percent changes to be made in α and R, the maximum changes in these variables to be made during iteration, the desired tolerance in the convergence of the eigenvalues, specifications for the angular rotation Ω and spanwise wave number β, initial estimates for the eigenvalues C_r, α and R and the maximum no. of iterations to be performed. Output from the program is monitored. Monitor = 1 gives express runs and only prints out the eigenvalues along with the convergence obtained after every change of eigenvalue during iteration. Monitor = 2 prints the above plus the eigen-functions. Monitor = 3 prints the same as monitor = 2 plus additional intermediate results, and monitor = 4 gives the maximum print out. Monitor 3 and 4 find more use during the debugging process. Once the program is debugged, monitor = 2 shows the smoothness of the eigen-functions. Then monitor = 1 is used for finding the true eigenvalues.

Explanation of the Input Cards for the Main Program

Card Number	Column	Item	Format
1	1-10	Step size D_1 outside the boundary layer	D 10.2
	11-20	Step size D_2 inside the boundary layer	D 10.2
	21-30	No. of steps in the boundary layer	I 10
	31-40	No. of data cards	I 10
	41-50	Monitor	I 10
	51-60	Tolerance for the roots of the polynomial	D 10.2
2	1-10	Percent change in α	D 10.3
	11-20	Percent change in R	D 10.3
	21-30	Maximum change in α or R	D 10.3
	31-40	Tolerance in the convergence of eigenvalues	D 10.3
3	1-10	Angular rotation Ω	D 10.3
	11-20	Wave number β	D 10.3
	21-30	Real part of wave velocity C_r	D 10.3
	31-45	Wave number α	D 15.4
	46-60	Reynolds number R	D 15.4
	61-65	Number of iterations	I 5

Any number of data cards as shown in input card number one may be added.

C-2 Description of the Computer Programs

Main Program:

This program is labelled CHAWLA. It reads the input data, solves and stores the values of the primary velocity and its derivatives, then starting with the first estimate of the eigen-values computes the polynomial, finds out its roots by calling the subroutine polyrt, initializes the individual eigen-functions in the free stream, starts the numerical integration by the Runge-Kutta fourth order method using the filtering technique for extracting the contributions of the growing solutions from the well behaved solutions and then repairs the extractions. After this the combined functions ϕ, Ψ and ϕ' are formed. At this stage the subroutine Dsole is called to solve the linear algebraic equations formed with the boundary conditions. Then the test function T is developed to check the convergence of the eigen-values. If the test fails to converge within the specified tolerance, iteration of the eigenvalues is started, first increasing α then R by a very small amount and then the values of α and R are predicted for the next step. The whole process is repeated until the values converge to the true values. If after about four iterations the convergence is found to be slow or absent, the initial guess is revised.

Subroutines:

POLYRT.

This program was taken from the Michigan State University, Computer Library. It was modified for relative accuracy and for use in double precision arithmetic. This solves a complex poly-nomial for its roots. It makes use of other subroutines namely

Improve, Reduce, Root and Basic which are also stored in the MSU library.

V-CAL

Using Runge-Kutta fourth order method it solves for the primary velocity and its derivatives and stores these results for use in the main program at the different steps of numerical integration.

D-Sole

This subroutine solves a set of n linear algebraic equations with complex coefficients, all in double precision.

CMULT and CEXP

Subroutine CMULT multiplies two complex numbers whereas CEXP gives the exponential of a complex number, all in double precision.

C-3 Listing of the Computer Programs

The programs developed to compute the eigenvalues for any given case are listed in this section. One run on the CDC 3600 with one set of estimates of the eigenvalues took approximately 10 seconds. In the programs that follow dollar sign ($) is often used. This is a legal statement separator for the CDC 3600 computer.

```
      PROGRAM CHAWLA
C     THIS PROGRAM SOLVES THE FOLLOWING DIFFERENTIAL EQUATIONS
C     (1) ONE EQUATION OF FOURTH ORDER.
C     (2) TWO EQUATIONS IN 2 VAR. OR ONE IN ONE VAR. OF 6TH ORDER
C     THREE DIMENSIONAL LINEAR STABILITY PROBLEM.
      DIMENSION PR(153,3),DPR(153,3),D2PR(153,3),
     1D3PR(153,3),D4PR(153,3),D5PR(1,3),VR(153,3),DVR(153,3),
     2D2VR(153,3),FR(7),RR(4),QR(6),TR(6),UOR(3),RR1(76),RR2(76),
     3TBR(6),TCR(6),AR(6,6),BR(6,6),XR(6),WR(6),UB(153),DUB(153),
     4D2UB(153),D3UB(3),D4UB(3),Z(153),T(6),Y(76)
      DIMENSION PI(153,3),DPI(153,3),D2PI(153,3),
     1D3PI(153,3),D4PI(153,3),D5PI(1,3),VI(153,3),DVI(153,3),
     2D2VI(153,3),FI(7),RI(4),QI(6),TI(6),UOI(3),RI1(76),RI2(76),
     3TBI(6),TCI(6),AI(6,6),BI(6,6),XI(6),WI(6)
      DOUBLE PRECISION PR,DPR,D2PR,D3PR,D4PR,D5PR,VR,DVR,D2VR,
     1FR,RR,QR,TR,UOR,RR1,RR2,TBR,TCR,AR,BR,XR,WR,RSR1,RSR2,B1R,
     2B2R,B3R,C1R,C2R,C3R,P0R,P1R,ER,DER,D2ER,D3ER,D4ER,UR,DUR,
     3DTR1,DTR2,V00R,V10R,V20R,V30R,V01R,V11R,V21R,V31R,V02R,
     4V12R,V22R,V32R,GK1R,GK2R,GK3R,GKR,GKPR,GK2PR,GK3PR,U00R
      DOUBLE PRECISION PI,DPI,D2PI,D3PI,D4PI,D5PI,VI,DVI,D2VI,
     1FI,RI,QI,TI,UOI,RI1,RI2,TBI,TCI,AI,BI,XI,WI,RSI1,RSI2,B1I,
     2B2I,B3I,C1I,C2I,C3I,P0I,P1I,EI,DEI,D2EI,D3EI,D4EI,UI,DUI,
     3DTI1,DTI2,V00I,V10I,V20I,V30I,V01I,V11I,V21I,V31I,V02I,
     4V12I,V22I,V32I,GK1I,GK2I,GK3I,GKI,GKPI,GK2PI,GK3PI,U00I
      DOUBLE PRECISION U10R,U01R,U11R,U02R,U12R,U03R,U13R,FK1R,
     1FK2R,FK3R,FKR,FKPR,A1R,FK2I,FK3I,FKI,FKPI,A1I,U10I,U01I,
     2U11I,U02I,U12I,U03I,U13I,FK1I,UB,DUB,D2UB,D3UB,D4UB,UP,DUP,
     3D2UP,D3UP,Z,T,Y,D1,D2,D,C,PPI,RAL,RBE,RCE,K,R,AL,OM,CR,CI,
     4BE,PAL,PCR,DAL,RD,PCH,TIP,DELTA2,WRT,DEN,ALN
      COMMON /2/ UB,DUB,D2UB,Z,D3UB,D4UB
C     STEP SIZES AND NO. OF STEPS.
      READ 425,D1,D2,M3,KP,MON,DELTA2
      N=0.5/D1+1.0/D2+1 $ IN=N
      READ 420,PAL,PCR,PCH,TIP
  425 FORMAT (2D10.2,3I10,2D10.2)
C     CALCULATING  UB,DUB,D2UB,D3UB,D4UB
      D1=D1/2.0D0 $ D2=D2/2.0D0 $ N=2*N-1
C     VCAL SOLVES FOR THE PRIMARY VELOCITY DISTRIBUTION.
      CALL VCAL(D1,D2,N)
      GO TO (465,465,465,22) MON
   22 PRINT 460
C     PRIMARY VEL. U OF TEXT = UB OF COMPUTER PROGRAM AND SO ON.
  460 FORMAT (10X,*UB(J)*,15X,*DUB(J)*,15X,*D2UB(J)*,9X,*Z(J)*,
     18X,*J*)
      PRINT 461,(UB(J),DUB(J),D2UB(J),Z(J),J,J=1,N)
  461 FORMAT (3(5X,F15.9),5X,F6.3,5X,I5)
      PRINT 11
   11 FORMAT (10X,*D3UB(J)*,32X,*D4UB(J)*,27X,*J*)
      PRINT 12,((D3UB(J),D4UB(J),J),J=1,3)
```

```
   12 FORMAT (2(5X,D32.25),5X,I5)
      PRINT 425,D1,D2,M3,KP,MON,DELTA2
      PRINT 420,PAL,PCR,PCH,TIP
  465 CONTINUE
      D1=2.0D0*D1 $ D2=2.0D0*D2 $ N=0.5/D1+1.0/D2+1
  500 FORMAT (6D10.3,2I5)
      CI=0.0D $ PPI=3.14159265358979323846264 3D
      KK=0
      PRINT 21,IN
   21 FORMAT (*1*,*EIGEN VALUE ITERATION   NO OF STEPS N=* I3)
C     KK IS A COUNTER FOR DATA CARDS.
      IF (MON-2) 490,400,400
  490 PRINT 135
  400 KK=KK+1
      IP=0   $   MD=0  $ ITE=1
      IF (KK-KP) 405,405,410
  405 CONTINUE
C     INITIAL GUESS FOR EIGEN-VALUES.
      READ 464,OM,BE,CR,AL,R,IT
  464 FORMAT (3D10.3,2D15.4,I5)
  420 FORMAT (4D10.3,3I5)
      PRINT 886
  886 FORMAT (30X,*R*,35X,*AL*,35X,*CR*)
      IF  (OM .EQ. 0.0D0) 661,662
  661 IR=2 $ IL=IR*2 $ GO TO 421
  662 IR=3 $ IL=IR*2
C     IT IS AN ITERATION COUNTER.
  421 MD=MD+1
      PRINT 887,R,AL,CR
  887 FORMAT (24X,3D35.25)
      IF (MD-IT) 422,422,400
  422 CONTINUE
C     K*K OF TEXT IS EQUAL TO K OF COMPUTER PROGRAM.
      K=AL*AL+BE*BE
      IF (MON-3) 471,65,65
   65 PRINT 66
   66 FORMAT (3X,*OMEGA*,3X,*ALPHA*,6X,*BE*,11X,*CR*,10X,*REY NO*)
      PRINT 67,OM,AL,BE,CR,R
   67 FORMAT (1X,D9.2,1X,D12.5,2(1X,D12.5),1X,D12.5)
  471 M=1  $ UP=UB(M) $ DUP=DUB(M) $ D2UP=D2UB(M) $ D3UP=D3UB(M)
      D4UP=D4UB(M)
      IF  (OM .EQ. 0.0D0) 505,510
  510 CONTINUE
C     FUNCTIONS -COEFFICIENTS OF THE 6TH DEGREE POLYNOMIAL.
      FR(1)=2.0D*BE*BE*R*R*OM*(DUP-2.0D*OM)+AL*AL*R*R*(UP-CR)*D2UP
     1+K*AL*AL*R*R*((UP-CR)**2-CI*CI)-2.0D*AL*R*K*K*CI-K**3
      FI(1)=-AL*AL*R*R*CI*D2UP-2.0D*(K*AL*AL*R*R)*(UP-CR)*CI-2.0D
     1*AL*R*K*K*(UP-CR)-AL*R*D4UP
      FR(2)=0.0D $ FI(2)=2.0D*AL*R*(K*DUP+D3UP)
```

```
      FR(3)=3.0D*K*K+4.0D*K*AL*R*CI-(AL*R*(UP-CR))**2+(AL*R*CI)**2
      FI(3)=4.0D*K*AL*R*(UP-CR)+2.0D*AL*AL*R*R*CI*(UP-CR)
      FR(4)=0.0D $ FI(4)=-2.0D*AL*R*DUP
      FR(5)=-3.0D*K-2.0D*AL*R*CI $ FI(5)=-2.0D*AL*R*(UP-CR)
      FR(6)=0.0D $ FI(6)=0.0D
      FR(7)=1.0D $ FI(7)=0.0D
      GO TO (10,10,10,201) MON
  201 PRINT 202
  202 FORMAT (9X,*FR(I)*,25X,*FI(I)*)
      PRINT 203,(FR(I),FI(I),I=1,7)
  203 FORMAT (2(4X,D25.15))
C     POLYRT GIVES THE ROOTS OF THE POLYNOMIAL.
   10 CALL POLYRT (FR,FI,6,QR,QI,DELTA2)
      J=1 $ DO 451 I=1,6 $ IF (QR(I) .LT. 0.0D) 450,451
  450 RR(J)=QR(I) $ RI(J)=QI(I) $ J=J+1
  451 CONTINUE $ EQAL=0 $ IF (RR(1) .EQ. RR(2)) 454,452
  452 IF (RR(1) .EQ. RR(3)) 455,453
  453 IF (RR(2) .EQ. RR(3)) 455,456
  454 RR(2)=RR(3) $ RI(2)=RI(3) $ RR(3)=RR(1) $ RI(3)=RI(1)
  455 EQAL=2 $ GO TO 456
  505 ALN=DSQRT(K) $ R=AL*R/ALN $ AL=ALN
C     FUNCTIONS -COEFFICIENTS OF THE 4TH DEGREE POLYNOMIAL.
      FR(1)=K*K+AL*R*CI*K $ FI(1)=AL*R*(K*(UP-CR)+D2UP)
      FR(2)=0.0D $ FI(2)=0.0D $ FR(3)=-((2.0D)*K+AL*R*CI)
      FI(3)=-AL*R*(UP-CR)
      FR(4)=0.0D $ FI(4)=0.0D
      FR(5)=1.0D  $  FI(5)=0.0D
      CALL POLYRT(FR,FI,4,QR,QI,DELTA2)
      J=1 $ DO 515 I=1,4 $ IF(QR(I) .LT. 0.0D) 520,515
  520 RR(J)=QR(I) $ RI(J)=QI(I) $ J=J+1
  515 CONTINUE $ EQAL=0
  537 IF (MON-3) 207,207,205
  456 GO TO (207,207,207,205) MON
  205 PRINT 470
  470 FORMAT (9X,*QR(I)*,22X,*QI(I)*,22X,*ROOTS OF THE POLYNOMIAL*)
      PRINT 203,((QR(I),QI(I)),I=1,IL)
  545 PRINT 206
  206 FORMAT (9X,*RR(I)*,22X,*RI(I)*,22X,*NEG. ROOTS OF THE
     1POLYNOMIAL*)
      PRINT 203,((RR(I),RI(I)),I=1,IR)
C     P IS EQUAL TO THE VELOCITY V-HAT, AND DP=D(V-HAT).
C     V IS EQUAL TO THE VELOCITY U-HAT.
C     BOUNDARY CONDITIONS - P=DP=V=0 AT Y=0, AND THESE APPROACH
C     ZERO AS Y APPROACHES INFINITY. CONDITION ON DP RESULTS FROM
C     CONTINUITY.
  207 CONTINUE
C     FINDING OUT THE GROWING SOLUTION I.E. WITH LARGEST DP/P.
      RR(4)=RR(1) $ RI(4)=RI(1)
      RAL=DSQRT(RR(1)*RR(1)+RI(1)*RI(1))
```

```
      RBE=DSQRT(RR(2)*RR(2)+RI(2)*RI(2))
      IF (OM .EQ. 0.0D0) 1119,1109
 1109 RCE=DSQRT(RR(3)*RR(3)+RI(3)*RI(3))
      IF (RAL-RBE) 1110,1110,1111
 1110 IF (RBE-RCE) 1115,1115,1116
 1111 IF (RAL-RCE) 1112,1112,1113
 1112 RR(1)=RR(3) $ RI(1)=RI(3) $ RR(3)=RR(2) $ RI(3)=RI(2)
      RR(2)=RR(4) $ RI(2)=RI(4) $ GO TO 1120
 1113 IF(RBE-RCE) 1114,1114,1120
 1114 RR(4)=RR(2) $ RI(4)=RI(2) $ RR(2)=RR(3) $ RI(2)=RI(3)
      RR(3)=RR(4) $ RI(3)=RI(4) $ GO TO 1120
 1115 RR(1)=RR(3)$RI(1)=RI(3) $ RR(3)=RR(4) $ RI(3)=RI(4)
      GO TO 1120
 1116 RR(1)=RR(2) $ RI(1)=RI(2) $ IF (RAL-RCE)1118,1118,1117
 1117 RR(2)=RR(4) $ RI(2)=RI(4) $ GO TO 1120
 1118 RR(2)=RR(3) $ RI(2)=RI(3) $ RR(3)=RR(4) $ RI(3)=RI(4)
      GO TO 1120
 1119 IF (RAL-RBE) 1121,1120,1120
 1121 RR(1)=RR(2) $ RI(1)=RI(2) $ RR(2)=RR(4) $ RI(2)=RI(4)
 1120 CONTINUE
      IF (MON-2) 1125,1122,1122
 1122 PRINT 203,((RR(I),RI(I)),I=1,IR)
 1125 CONTINUE
C     ASYMPTOTIC SOLUTION  AND  INITIALIZING.
      DO 6 I=1,IR $ J=1 $ M=1
      A1R=0.0D0 $ A1I=0.0D0
      CALL CXPE(A1R,A1I,PR(J,I),PI(J,I))
      CALL CMULT(RR(I),RI(I),PR(J,I),PI(J,I),DPR(J,I),DPI(J,I))
      CALL CMULT(RR(I),RI(I),DPR(J,I),DPI(J,I),D2PR(J,I),D2PI(J,I))
      CALL CMULT(RR(I),RI(I),D2PR(J,I),D2PI(J,I),D3PR(J,I),
     1D3PI(JTI))
      CALL CMULT(RR(I),RI(I),D3PR(J,I),D3PI(J,I),D4PR(J,I),
     1D4PI(J,I))
      IF  (OM .EQ. 0.0D0) GO TO 6
      CALL CMULT(RR(I),RI(I),D4PR(J,I),D4PI(J,I),D5PR(J,I),
     1D5PI(J,I))
    9 UP=UB(M) $ DUP=DUB(M) $ D2UP=D2UB(M) $ D3UP=D3UB(M)
      VR(J,I)=(D4PR(J,I)-(AL*R*CI+2.0D*K)*D2PR(J,I)+AL*R*(UP-CR)*
     1D2PI(J,I)+2.0D*AL*R*OM*DPI(J,I)+(K*K+AL*R*K*CI)*PR(J,I)-AL*
     2R*(K*(UP-CR)+D2UP)*PI(J,I))/(-2.0D*R*K*OM)
      VI(J,I)=(D4PI(J,I)-(AL*R*CI+2.0D*K)*D2PI(J,I)-AL*R*(UP-CR)*
     1D2PR(J,I)-2.0D*AL*R*OM*DPR(J,I)+(K*K+AL*R*K*CI)*PI(J,I)+AL*
     2R*(K*(UP-CR)+D2UP)*PR(J,I))/(-2.0D*R*K*OM)
      DVR(J,I)=(D5PR(J,I)-(AL*R*CI+2.0D*K)*D3PR(J,I)+AL*R*(UP-CR)
     1*D3PI(J,I)+2.0D*AL*R*OM*D2PI(J,I)+(K*K+AL*R*K*CI)*DPR(J,I)-
     2AL*R*(K*(UP-CR)+D2UP)*DPI(J,I)+AL*R*DUP*D2PI(J,I)-AL*R*(K*
     3DUP+D3UP)*PI(J,I))/(-2.0D*R*K*OM)
      DVI(J,I)=(D5PI(J,I)-(AL*R*CI+2.0D*K)*D3PI(J,I)-AL*R*(UP-CR)
     1*D3PR(J,I)-2.0D*AL*R*OM*D2PR(J,I)+(K*K+AL*R*K*CI)*DPI(J,I)+
```

```
     2AL*R*(K*(UP-CR)+D2UP)*DPR(J,I)-AL*R*DUP*D2PR(J,I)+AL*R*(K*
     3DUP+D3UP)*PR(J,I))/(-2.0D*R*K*OM)
      D2VR(J,I)=-(D3PI(J,I)-(K+AL*R*CI)*DPI(J,I)-AL*R*(UP-CR)*DPR
     1(J,I)-BE*BE*R*(DUP-2.0D*OM)*PR(J,I)/AL)*AL/K+(K+AL*R*CI)*VR
     2(J,I)-AL*R*(UP-CR)*VI(J,I)
      D2VI(J,I)=( D3PR(J,I)-(K+AL*R*CI)*DPR(J,I)+AL*R*(UP-CR)*DPI
     1(J,I)+BE*BE*R*(DUP-2.0D*OM)*PI(J,I)/AL)*AL/K+(K+AL*R*CI)*VI
     2(J,I)+AL*R*(UP-CR)*VR(J,I)
    6 CONTINUE
      DO 31 J=1,IN      $  N=2*J-1
   31 Y(J)=Z(N) $ C=D1 $ NN=IN-1
C     RUNGE KUTTA METHOD FOR THE SIMULTANEOUS SOLN. OF TWO DIFF.
C     EQUATIONS. REFERENCE COLLATZ.
      DO 55 J=1,NN             $ DO 54 I=1,IR   $ M=2*J-1
      UP=UB(M) $ DUP=DUB(M) $ D2UP=D2UB(M)
C     VALUES AT Y.
      V00R=PR(J,I) $ V00I=PI(J,I) $ V10R=-C*DPR(J,I)
      V10I=-C*DPI(J,I)
      V20R=C*C*D2PR(J,I)/2.0D0 $ V20I=C*C*D2PI(J,I)/2.0D0
      V30R=-C*C*C*D3PR(J,I)/6.0D0 $ V30I=-C*C*C*D3PI(J,I)/6.0D0
      IF  (OM .EQ. 0.0D0) GO TO 1200
      U00R=VR(J,I) $ U00I=VI(J,I) $ U10R=-C*DVR(J,I)
      U10I=-C*DVI(J,I)
      FK1R=((6.0D0*V30I/(C**3)-(K+AL*R*CI)*V10I/C-AL*R*(UP-CR)*
     1V10R/C+BE*BE*R*(DUP-2.0D0*OM)*V00R/AL)*AL/K+(K+AL*R*CI)*
     2U00R-AL*R*(UP-CR)*U00I)*C*C/2.0D0
      FK1I=((-6.0D0*V30R/(C**3)+(K+AL*R*CI)*V10R/C-AL*R*(UP-CR)*
     1V10I/C+BE*BE*R*(DUP-2.0D0*OM)*V00I/AL)*AL/K+(K+AL*R*CI)*
     2U00I+AL*R*(UP-CR)*U00R)*C*C/2.0D0
 1200 GK1R=(C**4)*((AL*R*CI+2.0D0*K)*V20R*2.0D0/(C*C)-AL*R*(UP-CR
     1)*V20I*2.0D0/(C*C)-(K*K+AL*R*K*CI)*V00R+AL*R*(K*(UP-CR)+
     2D2UP)*V00I)/24.0D
      GK1I=(C**4)*((AL*R*CI+2.0D0*K)*V20I*2.0D0/(C*C)+AL*R*(UP-CR
     1)*V20R*2.0D0/(C*C)-(K*K+AL*R*K*CI)*V00I-AL*R*(K*(UP-CR)+
     2D2UP)*V00R)/24.0D
      IF  (OM .EQ. 0.0D0) GO TO 1205
      GK1R=GK1R+(2.0D0*AL*R*OM*V10I/C-2.0D0*K*R*OM*U00R)*(C**4)/
     124.0D0
      GK1I=GK1I-(2.0D0*AL*R*OM*V10R/C+2.0D0*K*R*OM*U00I)*(C**4)/
     124.0D0
C     VALUES AT Y-C/2
 1205 M=M+1
      UP=UB(M) $ DUP=DUB(M) $ D2UP=D2UB(M)
      V01R=V00R+0.5D0*V10R+0.25D0*V20R+0.125D0*V30R+0.0625D0*GK1R
      V01I=V00I+0.5D0*V10I+0.25D0*V20I+0.125D0*V30I+0.0625D0*GK1I
      V11R=V10R+V20R+0.75D0*V30R+0.5D0*GK1R
      V11I=V10I+V20I+0.75D0*V30I+0.5D0*GK1I
      V21R=V20R+1.50D0*V30R+1.5D0*GK1R
      V21I=V20I+1.5D0*V30I+1.5D0*GK1I
```

```
      V31R=V30R+2.0D0*GK1R $ V31I=V30I+2.0D0*GK1I
      IF  (OM .EQ. 0.0D0) GO TO 1210
      U01R=U00R+0.5D0*U10R+0.25D0*FK1R
      U01I=U00I+0.5D0*U10I+0.25D0*FK1I
      U11R=U10R+FK1R $ U11I=U10I+FK1I
      FK2R=((6.0D0*V31I/(C**3)-(K+AL*R*CI)*V11I/C-AL*R*(UP-CR)*
     1V11R/C+BE*BE*R*(DUP-2.0D0*OM)*V01R/AL)*AL/K+(K+AL*R*CI)*
     2U01R-AL*R*(UP-CR)*U01I)*C*C/2.0D0
      FK2I=((-6.0D0*V31R/(C**3)+(K+AL*R*CI)*V11R/C-AL*R*(UP-CR)*
     1V11I/C+BE*BE*R*(DUP-2.0D0*OM)*V01I/AL)*AL/K+(K+AL*R*CI)*
     2U01I+AL*R*(UP-CR)*U01R)*C*C/2.0D0
 1210 GK2R=(C**4)*((AL*R*CI+2.0D0*K)*V21R*2.0D0/(C*C)-AL*R*(UP-CR
     1)*V21I*2.0D0/(C*C)-(K*K+AL*R*K*CI)*V01R+AL*R*(K*(UP-CR)+
     2D2UP)*V01I)/24.0D
      GK2I=(C**4)*((AL*R*CI+2.0D0*K)*V21I*2.0D0/(C*C)+AL*R*(UP-CR
     1)*V21R*2.0D0/(C*C)-(K*K+AL*R*K*CI)*V01I-AL*R*(K*(UP-CR)+
     2D2UP)*V01R)/24.0D
      IF  (OM .EQ. 0.0D0) GO TO 1215
      GK2R=GK2R+(2.0D0*AL*R*OM*V11I/C-2.0D0*K*R*OM*U01R)*(C**4)/
     124.0D0
      GK2I=GK2I-(2.0D0*AL*R*OM*V11R/C+2.0D0*K*R*OM*U01I)*(C**4)/
     124.0D0
C     VALUES AT Y-C.
 1215 M=M+1
      UP=UB(M) $ DUP=DUB(M) $ D2UP=D2UB(M)
      V02R=V00R+V10R+V20R+V30R+GK2R
      V02I=V00I+V10I+V20I+V30I+GK2I
      V12R=V10R+2.0D0*V20R+3.0D0*V30R+4.0D0*GK2R
      V12I=V10I+2.0D0*V20I+3.040*V30I+4.0D0*GK2I
      V22R=V20R+3.0D0*V30R+6.0D0*GK2R
      V22I=V20I+3.0D0*V30I+6.0D0*GK2I
      V32R=V30R+4.0D0*GK2R $ V32I=V30I+4.0D0*GK2I
      IF  (OM .EQ. 0.0D0) GO TO 1220
      U02R=U00R+U10R+FK2R $ U02I=U00I+U10I+FK2I
      U12R=U10R+2.0D0*FK2R $ U12I=U10I+2.0D0*FK2I
      FK3R=((6.0D0*V32I/(C**3)-(K+AL*R*CI)*V12I/C-AL*R*(UP-CR)*
     1V12R/C+BE*BE*R*(DUP-2.0D0*OM)*V02R/AL)*AL/K+(K+AL*R*CI)*
     2U02R-AL*R*(UP-CR)*U02I)*C*C/2.0D0
      FK3I=((-6.0D0*V32R/(C**3)+(K+AL*R*CI)*V12R/C-AL*R*(UP-CR)*
     1V12I/C+BE*BE*R*(DUP-2.0D0*OM)*V02I/AL)*AL/K+(K+AL*R*CI)*
     2U02I+AL*R*(UP-CR)*U02R)*C*C/2.0D0
      FKR=(FK1R+2.0D0*FK2R)/3.0D0 $ FKI=(FK1I+2.0D0*FK2I)/3.0D0
      FKPR=(FK1R+4.0D0*FK2R+FK3R)/3.0D0
      FKPI=(FK1I+4.0D0*FK2I+FK3I)/3.0D0
 1220 GK3R=(C**4)*((AL*R*CI+2.0D0*K)*V22R*2.0D0/(C*C)-AL*R*(UP-CR
     1)*V22I*2.0D0/(C*C)-(K*K+AL*R*K*CI)*V02R+AL*R*(K*(UP-CR)+
     2D2UP)*V02I)/24.0D0
      GK3I=(C**4)*((AL*R*CI+2.0D0*K)*V22I*2.0D0/(C*C)+AL*R*(UP-CR
     1)*V22R*2.0D0/(C*C)-(K*K+AL*R*K*CI)*V02I-AL*R*(K*(UP-CR)+
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      2D2UP)*V02R)/24.0D0
       IF  (OM .EQ. 0.0D0) GO TO 1225
       GK3R=GK3R+(2.0D0*AL*R*OM*V12I/C-2.0D0*K*R*OM*U02R)*(C**4)/
      124.0D0
       GK3I=GK3I-(2.0D0*AL*R*OM*V12R/C+2.0D0*K*R*OM*U02I)*(C**4)/
      124.0D0
 1225 GKR=(8.0D0*GK1R+8.0D0*GK2R-GK3R)/15.0D0
      GKI=(8.0D0*GK1I+8.0D0*GK2I-GK3I)/15.0D0
      GKPR=(9.0D0*GK1R+12.0D0*GK2R-GK3R)/5.0D0
      GKPI=(9.0D0*GK1I+12.0D0*GK2I-GK3I)/5.0D0
      GK2PR=2.0D0*GK1R+4.0D0*GK2R $ GK2PI=2.0D0*GK1I+4.0D0*GK2I
      GK3PR=(GK1R+4.0D0*GK2R+GK3R)*2.0D0/3.0D0
      GK3PI=(GK1I+4.0D0*GK2I+GK3I)*2.0D0/3.0D0
C     VALUES OF FUNCTION AND ITS DERIVATIVES AT Y-C.
      PR(J+1,I)=V00R+V10R+V20R+V30R+GKR
      PI(J+1,I)=V00I+V10I+V20I+V30I+GKI
      DPR(J+1,I)=-(V10R+2.0D0*V20R+3.0D0*V30R+GKPR)/C
      DPI(J+1,I)=-(V10I+2.0D0*V20I+3.0D0*V30I+GKPI)/C
      D2PR(J+1,I)=(V20R+3.0D0*V30R+GK2PR)*2.0D0/(C*C)
      D2PI(J+1,I)=(V20I+3.0D0*V30I+GK2PI)*2.0D0/(C*C)
      D3PR(J+1,I)=-(V30R+GK3PR)*6.0D0/(C**3)
      D3PI(J+1,I)=-(V30I+GK3PI)*6.0D0/(C**3)
      IF  (OM .EQ. 0.0D0) GO TO 1230
      VR(J+1,I)=U00R+U10R+FKR $ VI(J+1,I)=U00I+U10I+FKI
      DVR(J+1,I)=-(U10R+FKPR)/C $ DVI(J+1,I)=-(U10I+FKPI)/C
      J=J+1
      D2VR(J,I)=-(D3PI(J,I)-(K+AL*R*CI)*DPI(J,I)-AL*R*(UP-CR)*DPR
     1(J,I)-BE*BE*R*(DUP-2.0D*OM)*PR(J,I)/AL)*AL/K+(K+AL*R*CI)*VR
     2(J,I)-AL*R*(UP-CR)*VI(J,I)
      D2VI(J,I)=( D3PR(J,I)-(K+AL*R*CI)*DPR(J,I)+AL*R*(UP-CR)*DPI
     1(J,I)+BE*BE*R*(DUP-2.0D*OM)*PI(J,I)/AL)*AL/K+(K+AL*R*CI)*VI
     2(J,I)+AL*R*(UP-CR)*VR(J,I)
       J=J-1
 1230 D4PR(J+1,I)=(AL*R*CI+2.0D0*K)*D2PR(J+1,I)-AL*R*(UP-CR)*D2PI
     1(J+1,I)-(K*K+AL*R*K*CI)*PR(J+1,I)+AL*R*(K*(UP-CR)+D2UP)*
     2PI(J+1,I)
      D4PI(J+1,I)=(AL*R*CI+2.0D0*K)*D2PI(J+1,I)+AL*R*(UP-CR)*D2PR
     1(J+1,I)-(K*K+AL*R*K*CI)*PI(J+1,I)-AL*R*(K*(UP-CR)+D2UP)*
     2PR(J+1,I)
       IF  (OM .EQ. 0.0D0) GO TO 1100
      D4PR(J+1,I)=D4PR(J+1,I)-2.0D0*AL*R*OM*DPI(J+1,I)-2.0D0*K*R*
     1OM*VR(J+I,I)
      D4PI(J+1,I)=D4PI(J+1,I)+2.0D0*AL*R*OM*DPR(J+1,I)-2.0D0*K*R*
     1OM*VI(J+I,I)    $ GO TO 1101
 1100 UOR(I)=AL*CI*D2PR(J+1,I)-AL*(UP-CR)*D2PI(J+1,I)-
     1AL*K*CI*PR(J+1,I)+AL*(K*(UP-CR)+D2UP)*PI(J+1,I)
      UOI(I)=AL*(UP-CR)*D2PR(J+1,I)+AL*CI*D2PI(J+1,I)-
     1AL*(K*(UP-CR)+D2UP)*PR(J+1,I)-AL*K*CI*PI(J+1,I)
      GO TO 54
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1101 U1R=AL*CI*D2PR(J+1,I)-AL*(UP-CR)*D2PI(J+1,I)-2.0D0*AL*OM*
    1DPI(J+1,I)-AL*K*CI*PR(J+1,I)+AL*(K*(UP-CR)+D2UP)*PI(J+1,I)
    2-2.0D0*K*OM*VR(J+1,I)
     U1I=AL*(UP-CR)*D2PR(J+1,I)+AL*CI*D2PI(J+1,I)+2.0D0*AL*OM*
    1DPR(J+1,I)-AL*(K*(UP-CR)+D2UP)*PR(J+1,I)-AL*K*CI*PI(J+1,I)
    2-2.0D0*K*OM*VI(J+1,I)
     U2R=AL*CI*DPR(J+1,I)-AL*(UP-CR)*DPI(J+1,I)-BE*BE*(DUP-2.0D0
    1*OM)*PI(J+1,I)/AL-K*(UP-CR)*VR(J+1,I)-K*CI*VI(J+1,I)
     U2I=AL*(UP-CR)*DPR(J+1,I)+AL*CI*DPI(J+1,I)+BE*BE*(DUP-2.0D0
    1*OM)*PR(J+1,I)/AL-K*(UP-CR)*VI(J+1,I)+K*CI*VR(J+1,I)
     UOR(I)=U1R+U2R $ UOI(I)=U1I+U2I
  54 CONTINUE
C    EXTRACT THE GROWING SOLUTION.
     L=J $ DEN=(UOR(1))**2+(UOI(1))**2
     CALL CMULT(UOR(2),UOI(2),UOR(1),-UOI(1),POR,POI)
     RR1(L+1)=POR/DEN $ RI1(L+1)=POI/DEN
       PR(J+1,2)=  PR(J+1,2)-RR1(L+1)*  PR(J+1,1)+RI1(L+1)*  PI
    1(J+1,1)
       PI(J+1,2)=  PI(J+1,2)-RR1(L+1)*  PI(J+1,1)-RI1(L+1)*  PR
    1(J+1,1)
     D PR(J+1,2)=D PR(J+1,2)-RR1(L+1)*D PR(J+1,1)+RI1(L+1)*D PI
    1(J+1,1)
     D PI(J+1,2)=D PI(J+1,2)-RR1(L+1)*D PI(J+1,1)-RI1(L+1)*D PR
    1(J+1,1)
     D2PR(J+1,2)=D2PR(J+1,2)-RR1(L+1)*D2PR(J+1,1)+RI1(L+1)*D2PI
    1(J+1,1)
     D2PI(J+1,2)=D2PI(J+1,2)-RR1(L+1)*D2PI(J+1,1)-RI1(L+1)*D2PR
    1(J+1,1)
     D3PR(J+1,2)=D3PR(J+1,2)-RR1(L+1)*D3PR(J+1,1)+RI1(L+1)*D3PI
    1(J+1,1)
     D3PI(J+1,2)=D3PI(J+1,2)-RR1(L+1)*D3PI(J+1,1)-RI1(L+1)*D3PR
    1(J+1,1)
     D4PR(J+1,2)=D4PR(J+1,2)-RR1(L+1)*D4PR(J+1,1)+RI1(L+1)*D4PI
    1(J+1,1)
     D4PI(J+1,2)=D4PI(J+1,2)-RR1(L+1)*D4PI(J+1,1)-RI1(L+1)*D4PR
    1(J+1,1)
     IF  (OM .EQ. 0.0D0) 1102,1103
1103   VR(J+1,2)=  VR(J+1,2)-RR1(L+1)*  VR(J+1,1)+RI1(L+1)*  VI
    1(J+1,1)
       VI(J+1,2)=  VI(J+1,2)-RR1(L+1)*  VI(J+1,1)-RI1(L+1)*  VR
    1(J+1,1)
     D VR(J+1,2)=D VR(J+1,2)-RR1(L+1)*D VR(J+1,1)+RI1(L+1)*D VI
    1(J+1,1)
     D VI(J+1,2)=D VI(J+1,2)-RR1(L+1)*D VI(J+1,1)-RI1(L+1)*D VR
    1(J+1,1)
     D2VR(J+1,2)=D2VR(J+1,2)-RR1(L+1)*D2VR(J+1,1)+RI1(L+1)*D2VI
    1(J+1,1)
     D2VI(J+1,2)=D2VI(J+1,2)-RR1(L+1)*D2VI(J+1,1)-RI1(L+1)*D2VR
    1(J+1,1)
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      CALL CMULT(UOR(3),UOI(3),UOR(1),-UOI(1),P1R,P1I)
      RR2(L+1)=P1R/DEN $ RI2(L+1)=P1I/DEN
        PR(J+1,3)=  PR(J+1,3)-RR2(L+1)*  PR(J+1,1)+RI2(L+1)*  PI
     1(J+1,1)
        PI(J+1,3)=  PI(J+1,3)-RR2(L+1)*  PI(J+1,1)-RI2(L+1)*  PR
     1(J+1,1)
      D PR(J+1,3)=D PR(J+1,3)-RR2(L+1)*D PR(J+1,1)+RI2(L+1)*D PI
     1(J+1,1)
      D PI(J+1,3)=D PI(J+1,3)-RR2(L+1)*D PI(J+1,1)-RI2(L+1)*D PR
     1(J+1,1)
      D2PR(J+1,3)=D2PR(J+1,3)-RR2(L+1)*D2PR(J+1,1)+RI2(L+1)*D2PI
     1(J+1,1)
      D2PI(J+1,3)=D2PI(J+1,3)-RR2(L+1)*D2PI(J+1,1)-RI2(L+1)*D2PR
     1(J+1,1)
      D3PR(J+1,3)=D3PR(J+1,3)-RR2(L+1)*D3PR(J+1,1)+RI2(L+1)*D3PI
     1(J+1,1)
      D3PI(J+1,3)=D3PI(J+1,3)-RR2(L+1)*D3PI(J+1,1)-RI2(L+1)*D3PR
     1(J+1,1)
      D4PR(J+1,3)=D4PR(J+1,3)-RR2(L+1)*D4PR(J+1,1)+RI2(L+1)*D4PI
     1(J+1,1)
      D4PI(J+1,3)=D4PI(J+1,3)-RR2(L+1)*D4PI(J+1,1)-RI2(L+1)*D4PR
     1(J+1,1)
        VR(J+1,3)=  VR(J+1,3)-RR2(L+1)*  VR(J+1,1)+RI2(L+1)*  VI
     1(J+1,1)
        VI(J+1,3)=  VI(J+1,3)-RR2(L+1)*  VI(J+1,1)-RI2(L+1)*  VR
     1(J+1,1)
      D VR(J+1,3)=D VR(J+1,3)-RR2(L+1)*D VR(J+1,1)+RI2(L+1)*D VI
     1(J+1,1)
      D VI(J+1,3)=D VI(J+1,3)-RR2(L+1)*D VI(J+1,1)-RI2(L+1)*D VR
     1(J+1,1)
      D2VR(J+1,3)=D2VR(J+1,3)-RR2(L+1)*D2VR(J+1,1)+RI2(L+1)*D2VI
     1(J+1,1)
      D2VI(J+1,3)=D2VI(J+1,3)-RR2(L+1)*D2VI(J+1,1)-RI2(L+1)*D2VR
     1(J+1,1)
 1102 IF (J-2) 55,1105,1105
 1105 D4PR(J  ,2)=D4PR(J  ,2)-RR1(L+1)*D4PR(J  ,1)+RI1(L+1)*D4PI
     1(J  ,1)
      D4PI(J  ,2)=D4PI(J  ,2)-RR1(L+1)*D4PI(J  ,1)-RI1(L+1)*D4PR
     1(J  ,1)
      IF  (OM .EQ. 0.0D0) GO TO 1235
      D2VR(J  ,2)=D2VR(J  ,2)-RR1(L+1)*D2VR(J  ,1)+RI1(L+1)*D2VI
     1(J  ,1)
      D2VI(J  ,2)=D2VI(J  ,2)-RR1(L+1)*D2VI(J  ,1)-RI1(L+1)*D2VR
     1(J  ,1)
      D2VR(J  ,3)=D2VR(J  ,3)-RR2(L+1)*D2VR(J  ,1)+RI2(L+1)*D2VI
     1(J  ,1)
      D2VI(J  ,3)=D2VI(J  ,3)-RR2(L+1)*D2VI(J  ,1)-RI2(L+1)*D2VR
     1(J  ,1)
      D4PR(J  ,3)=D4PR(J  ,3)-RR2(L+1)*D4PR(J  ,1)+RI2(L+1)*D4PI
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     1(J   ,1)
      D4PI(J   ,3)=D4PI(J   ,3)-RR2(L+1)*D4PI(J   ,1)-RI2(L+1)*D4PR
     1(J   ,1)
1235 IF (J-3) 55,32,32
  32 D4PR(J-1,2)=D4PR(J-1,2)-RR1(L+1)*D4PR(J-1,1)+RI1(L+1)*D4PI
     1(J-1,1)
      D4PI(J-1,2)=D4PI(J-1,2)-RR1(L+1)*D4PI(J-1,1)-RI1(L+1)*D4PR
     1(J-1,1)
      IF  (OM .EQ. 0.0D0) GO TO 1240
      D2VR(J-1,2)=D2VR(J-1,2)-RR1(L+1)*D2VR(J-1,1)+RI1(L+1)*D2VI
     1(J-1,1)
      D2VI(J-1,2)=D2VI(J-1,2)-RR1(L+1)*D2VI(J-1,1)-RI1(L+1)*D2VR
     1(J-1,1)
      D2VR(J-1,3)=D2VR(J-1,3)-RR2(L+1)*D2VR(J-1,1)+RI2(L+1)*D2VI
     1(J-1,1)
      D2VI(J-1,3)=D2VI(J-1,3)-RR2(L+1)*D2VI(J-1,1)-RI2(L+1)*D2VR
     1(J-1,1)
      D4PR(J-1,3)=D4PR(J-1,3)-RR2(L+1)*D4PR(J-1,1)+RI2(L+1)*D4PI
     1(J-1,1)
      D4PI(J-1,3)=D4PI(J-1,3)-RR2(L+1)*D4PI(J-1,1)-RI2(L+1)*D4PR
     1(J-1,1)
1240 IF (J .EQ. 3) GO TO 55
      D4PR(J-2,2)=D4PR(J-2,2)-RR1(L+1)*D4PR(J-2,1)+RI1(L+1)*D4PI
     1(J-2,1)
      D4PI(J-2,2)=D4PI(J-2,2)-RR1(L+1)*D4PI(J-2,1)-RI1(L+1)*D4PR
     1(J-2,1)
      IF  (OM .EQ. 0.0D0) GO TO 55
      D2VR(J-2,2)=D2VR(J-2,2)-RR1(L+1)*D2VR(J-2,1)+RI1(L+1)*D2VI
     1(J-2,1)
      D2VI(J-2,2)=D2VI(J-2,2)-RR1(L+1)*D2VI(J-2,1)-RI1(L+1)*D2VR
     1(J-2,1)
      D2VR(J-2,3)=D2VR(J-2,3)-RR2(L+1)*D2VR(J-2,1)+RI2(L+1)*D2VI
     1(J-2,1)
      D2VI(J-2,3)=D2VI(J-2,3)-RR2(L+1)*D2VI(J-2,1)-RI2(L+1)*D2VR
     1(J-2,1)
      D4PR(J-2,3)=D4PR(J-2,3)-RR2(L+1)*D4PR(J-2,1)+RI2(L+1)*D4PI
     1(J-2,1)
      D4PI(J-2,3)=D4PI(J-2,3)-RR2(L+1)*D4PI(J-2,1)-RI2(L+1)*D4PR
     1(J-2,1)
  55 CONTINUE
      GO TO (19,19,19,16) MON
  16 IF  (OM .EQ. 0.0D0) 1041,1044
1041 PRINT 1042
1042 FORMAT ( 3X,*PR(J,I)*,5X,*PI(J,I)*,5X,*DPR(J,I)*,4X,*DPI(J,
     1I)*,4X,*D2PR(J,I)*,2X,*D2PI(J,I)*,2X,*D3PR(J,I)*,2X,*D3PI(J
     2,I)*,2X,*D4PR(J,I)*,2X,*D4PI(J,I)*,4X,*Z(M)*,9X,*I*)
      GO TO 19
1044 PRINT 17
  17 FORMAT ( 3X,*PR(J,I)*,5X,*PI(J,I)*,5X,*DPR(J,I)*,4X,*DPI(J,
```

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     1I)*,4X,*D2PR(J,I)*,2X,*D2PI(J,I)*,4X,*VR(J,I)*,5X,*VI(J,I)*
     2,5X,*DVR(J,I)*,4X,*DVI(J,I)*,3X,*Z(M)*,9X,*I*)
   19 CONTINUE
   18 FORMAT (10(1X,D11.3),1X,F7.4,1X,I7)
      GO TO (1055,1055,1055,1051) MON
 1051 PRINT 1052
 1052 FORMAT (* EIGEN FUNCTIONS LESS CONTRIBUTIONS OF GROWING
     1SOLUTIONS*)
      IF (OM .EQ. 0.0D0) 1053,1054
 1053 PRINT 18  , ((( PR(J,I),PI(J,I),DPR(J,I),DPI(J,I),D2PR(J,I)
     1,D2PI(J,I),D3PR(J,I),D3PI(J,I),D4PR(J,I),D4PI(J,I),Y(J),I),
     2J=1,IN),I=1,IR)
      GO TO 1055
 1054 PRINT 18  , ((( PR(J,I),PI(J,I),DPR(J,I),DPI(J,I),D2PR(J,I)
     1,D2PI(J,I),VR(J,I),VI(J,I),DVR(J,I),DVI(J,I),Y(J),I),J=1,IN
     2),I=1,IR)
 1055 MMM=0  $  KJ=1
C     REPAIR THE EXTRACTIONS.
 1025 NI=IN $ NN=0.5/D1+1.0/D2+1
      RSR1=RR1(NN) $ RSI1=RI1(NN)
      IF (OM .EQ. 0.0D0) GO TO 1027 $ RSR2=RR2(NN) $ RSI2=RI2(NN)
 1027 MMM=MMM+1 $ IF (MMM .EQ. 1) 1066,1067
 1067 IF (MON-2) 1030,1066,1066
 1066 CONTINUE
      IF (MON-3) 34,34,33
   33 PRINT 35,RR1(NN),RSR1,RI1(NN),RSI1,RR2(NN),RSR2,RI2(NN),
     1RSI2,NN
   35 FORMAT (8(1X,D15.5),2X,I5)
   34 NI=NI-KJ $ NN=NN-1 $ I=2
       P R(NI,I)= P R(NI,I)-RSR1* P R(NI,1)+RSI1* P I(NI,1)
       P I(NI,I)= P I(NI,I)-RSR1* P I(NI,1)-RSI1* P R(NI,1)
      DP R(NI,I)=DP R(NI,I)-RSR1*DP R(NI,1)+RSI1*DP I(NI,1)
      DP I(NI,I)=DP I(NI,I)-RSR1*DP I(NI,1)-RSI1*DP R(NI,1)
      D2PR(NI,I)=D2PR(NI,I)-RSR1*D2PR(NI,1)+RSI1*D2PI(NI,1)
      D2PI(NI,I)=D2PI(NI,I)-RSR1*D2PI(NI,1)-RSI1*D2PR(NI,1)
      D3PR(NI,I)=D3PR(NI,I)-RSR1*D3PR(NI,1)+RSI1*D3PI(NI,1)
      D3PI(NI,I)=D3PI(NI,I)-RSR1*D3PI(NI,1)-RSI1*D3PR(NI,1)
      D4PR(NI,I)=D4PR(NI,I)-RSR1*D4PR(NI,1)+RSI1*D4PI(NI,1)
      D4PI(NI,I)=D4PI(NI,I)-RSR1*D4PI(NI,1)-RSI1*D4PR(NI,1)
      IF (OM .EQ. 0.0D0) GO TO 1029
       V R(NI,I)= V R(NI,I)-RSR1* V R(NI,1)+RSI1* V I(NI,1)
       V I(NI,I)= V I(NI,I)-RSR1* V I(NI,1)-RSI1* V R(NI,1)
      DV R(NI,I)=DV R(NI,I)-RSR1*DV R(NI,1)+RSI1*DV I(NI,1)
      DV I(NI,I)=DV I(NI,I)-RSR1*DV I(NI,1)-RSI1*DV R(NI,1)
      D2VR(NI,I)=D2VR(NI,I)-RSR1*D2VR(NI,1)+RSI1*D2VI(NI,1)
      D2VI(NI,I)=D2VI(NI,I)-RSR1*D2VI(NI,1)-RSI1*D2VR(NI,1)
      I=3
       P R(NI,I)= P R(NI,I)-RSR2* P R(NI,1)+RSI2* P I(NI,1)
       P I(NI,I)= P I(NI,I)-RSR2* P I(NI,1)-RSI2* P R(NI,1)
```

```
      DP R(NI,I)=DP R(NI,I)-RSR2*DP R(NI,1)+RSI2*DP I(NI,1)
      DP I(NI,I)=DP I(NI,I)-RSR2*DP I(NI,1)-RSI2*DP R(NI,1)
      D2PR(NI,I)=D2PR(NI,I)-RSR2*D2PR(NI,1)+RSI2*D2PI(NI,1)
      D2PI(NI,I)=D2PI(NI,I)-RSR2*D2PI(NI,1)-RSI2*D2PR(NI,1)
      D3PR(NI,I)=D3PR(NI,I)-RSR2*D3PR(NI,1)+RSI2*D3PI(NI,1)
      D3PI(NI,I)=D3PI(NI,I)-RSR2*D3PI(NI,1)-RSI2*D3PR(NI,1)
      D4PR(NI,I)=D4PR(NI,I)-RSR2*D4PR(NI,1)+RSI2*D4PI(NI,1)
      D4PI(NI,I)=D4PI(NI,I)-RSR2*D4PI(NI,1)-RSI2*D4PR(NI,1)
       V R(NI,I)= V R(NI,I)-RSR2* V R(NI,1)+RSI2* V I(NI,1)
       V I(NI,I)= V I(NI,I)-RSR2* V I(NI,1)-RSI2* V R(NI,1)
      DV R(NI,I)=DV R(NI,I)-RSR2*DV R(NI,1)+RSI2*DV I(NI,1)
      DV I(NI,I)=DV I(NI,I)-RSR2*DV I(NI,1)-RSI2*DV R(NI,1)
      D2VR(NI,I)=D2VR(NI,I)-RSR2*D2VR(NI,1)+RSI2*D2VI(NI,1)
      D2VI(NI,I)=D2VI(NI,I)-RSR2*D2VI(NI,1)-RSI2*D2VR(NI,1)
 1029 IF (NI-2 ) 1030,1030,1028
 1028 RSR1=RSR1+RR1(NN) $ RSI1=RSI1+RI1(NN)
      IF (OM .EQ. 0.0D0) GO TO 1027
      RSR2=RSR2+RR2(NN) $ RSI2=RSI2+RI2(NN) $ GO TO 1027
 1030 CONTINUE
      GO TO (1060,1060,1060,1056) MON
 1056 PRINT 1057
 1057 FORMAT (* REPAIRED FUNCTIONS AFTER THE EXTRACTIONS*)
      IF (OM .EQ. 0.0D0) 1058,1059
 1058 PRINT 18   , ((( PR(J,I),PI(J,I),DPR(J,I),DPI(J,I),D2PR(J,I)
     1,D2PI(J,I),D3PR(J,I),D3PI(J,I),D4PR(J,I),D4PI(J,I),Y(J),I),
     2J=1,IN),I=1,IR)
      GO TO 1060
 1059 PRINT 18   , ((( PR(J,I),PI(J,I),DPR(J,I),DPI(J,I),D2PR(J,I)
     1,D2PI(J,I),VR(J,I),VI(J,I),DVR(J,I),DVI(J,I),Y(J),I),J=1,IN
     2),I=1,IR)
 1060 CONTINUE
C     VALUES FROM ABOVE AT THE WALL.
      N=IN
      IF  (OM .EQ. 0.0D0) 555,550
  550 AR(1,1)=PR(N,1) $ AR(1,2)=PR(N,2)
      AI(1,1)=PI(N,1) $ AI(1,2)=PI(N,2)
      AR(2,1)= VR(N,1) $ AR(2,2)= VR(N,2)
      AI(2,1)= VI(N,1) $ AI(2,2)= VI(N,2)
      B1R=DPR(N,1) $ B2R=DPR(N,2) $ B3R=DPR(N,3)
      B1I=DPI(N,1) $ B2I=DPI(N,2) $ B3I=DPI(N,3)
      N=N-1
      C1R=DPR(N,1) $ C2R=DPR(N,2) $ C3R=DPR(N,3)
      C1I=DPI(N,1) $ C2I=DPI(N,2) $ C3I=DPI(N,3)
      GO TO 556
  555 AR(1,1)=PR(N,1) $ AR(1,2)=PR(N,2)
      AI(1,1)=PI(N,1) $ AI(1,2)=PI(N,2)
      B1R=DPR(N,1) $ B2R=DPR(N,2)
      B1I=DPI(N,1) $ B2I=DPI(N,2)
      N=N-1
```

```
      C1R=DPR(N,1) $ C2R=DPR(N,2)
      C1I=DPI(N,1) $ C2I=DPI(N,2)
      DEN=AR(1,1)*AR(1,1)+AI(1,1)*AI(1,1) $ AI(1,1)=-AI(1,1)
      CALL CMULT(AR(1,1),AI(1,1),AR(1,2),AI(1,2),POR,POI)
      XR(1)=-POR/DEN $ XI(1)=-POI/DEN $ AI(1,1)=-AI(1,1)
      GO TO 990
  556 N=IN
      XR(1)=-PR(N,3) $ XI(1)=-PI(N,3) $ XR(2)=-VR(N,3)
      XI(2)=-VI(N,3)
  990 IRM=IR-1
      DO 991 I=1,IRM $ DO 991 J=1,IRM $ BR(I,J)=AR(I,J)
      BI(I,J)=AI(I,J)
  991 CONTINUE
      GO TO (214,214,213,213) MON
  213 PRINT 26
   26 FORMAT (5X,*AR(I,J)*,30X,*AI(I,J)*)
      PRINT 652,(((BR(I,J),BI(I,J)),J=1,IRM),I=1,IRM)
  652 FORMAT (2(2X,D35.25))
   27 FORMAT(2(2X,D35.16))
      PRINT 58
   58 FORMAT   (5X,*B1R*,35X,*B1I*)
      PRINT 27,B1R,B1I $ PRINT 27,B2R,B2I
      IF   (OM .EQ. 0.0D0) GO TO 1245
      PRINT 27,B3R,B3I
 1245 PRINT 27,C1R,C1I $ PRINT 27,C2R,C2I
      IF   (OM .EQ. 0.0D0) GO TO 214
      PRINT 27,C3R,C3I
  214 CONTINUE
      IF   (OM .EQ. 0.0D0) GO TO 56
C     DSOLE SOLVES THE ALGEBRAIC EQUATIONS.
      CALL DSOLE(BR,BI,XR,XI,IRM,IRM)
   56 GO TO (216,216,215,215) MON
  215 PRINT 2
    2 FORMAT (5X,*XR(I)*,33X,*XI(I)*)
      PRINT 27, ((XR(I),XI(I)),I=1,IRM)
  216 CONTINUE
      IF   (OM .EQ. 0.0D0) 585,590
C     CHECK ON XR,XI.
  585 WR(1)=XR(1)*AR(1,1)-XI(1)*AI(1,1)+AR(1,2)
      WI(1)=XI(1)*AR(1,1)+XR(1)*AI(1,1)+AI(1,2)
      GO TO 586
  590 CONTINUE  $ N=IN
C     CHECKING DSOLE.
  226 WR(1)=XR(1)*AR(1,1)-XI(1)*AI(1,1)+XR(2)*AR(1,2)-XI(2)*
     1AI(1,2)+ PR(N,3)
      WI(1)=XI(1)*AR(1,1)+XR(1)*AI(1,1)+XI(2)*AR(1,2)+XR(2)*
     1AI(1,2)+ PI(N,3)
      WR(2)=XR(1)*AR(2,1)-XI(1)*AI(2,1)+XR(2)*AR(2,2)-XI(2)*
     1AI(2,2)+ VR(N,3)
```

```
      WI(2)=XR(1)*AI(2,1)+XI(1)*AR(2,1)+XR(2)*AI(2,2)+XI(2)*
     1AR(2,2)+ VI(N,3)
  586 IF (MON -2) 565,560,560
  560 PRINT38 $ PRINT 27, ((WR(I),WI(I)),I=1,IRM)
  565 WRT=0.0D $ DO 800 I=1,IRM
   38 FORMAT (5X,*WR(I)*,33X,*WI(I)* )
  800 WRT=WRT+DABS(WR(I))+DABS(WI(I))
      IF (WRT-(1.0D-6)) 830,830,805
  805 PRINT38 $ PRINT 27, ((WR(I),WI(I)),I=1,IRM)  $ GO TO 400
  830 CONTINUE  $ IP=IP+1
  100 GO TO (225,217,217,217) MON
  217 CONTINUE
      PRINT 874
  874 FORMAT (* FINAL COMBINED FUNCTIONS *)
C     E=P=Z2  AND  U=V=Z1
      IF  (OM .EQ. 0.0D0) PRINT 1030
 1030 FORMAT (3X,*PR(J)*,5X,*PI(J)*,5X,*DPR(J)*,4X,*DPI(J)*,4X,
     1*D2PR(J)*,2X,*D2PI(J)*,2X,*D3PR(J)*,2X,*D3PI(J)*,2X,*D4PR
     2(J)*,2X,*D4PI(J)*,4X,*Y(J)*,9X,*J*)
      IF (OM .EQ. 0.0D0) GO TO 1040
      PRINT 13
   13 FORMAT (3X,*PR(J)*,5X,*PI(J)*,5X,*DPR(J)*,4X,*DPI(J)*,4X,
     1*D2PR(J)*,2X,*D2PI(J)*,4X,*VR(J)*,5X,*VI(J)*,5X,*DVR(J)*
     2,4X,*DVI(J)  *,3X,*Y(J)*,9X,*J*)
 1040 DO 221 J=1 ,IN
      IF  (OM .EQ. 0.0D0) 595,605
  595 ER=XR(1)*PR(J,1)-XI(1)*PI(J,1)+PR(J,2)
      EI=XR(1)*PI(J,1)+XI(1)*PR(J,1)+PI(J,2)
      DER=XR(1)*DPR(J,1)-XI(1)*DPI(J,1)+DPR(J,2)
      DEI=XR(1)*DPI(J,1)+XI(1)*DPR(J,1)+DPI(J,2)
      D2ER=XR(1)*D2PR(J,1)-XI(1)*D2PI(J,1)+D2PR(J,2)
      D2EI=XR(1)*D2PI(J,1)+XI(1)*D2PR(J,1)+D2PI(J,2)
      D3ER=XR(1)*D3PR(J,1)-XI(1)*D3PI(J,1)+D3PR(J,2)
      D3EI=XR(1)*D3PI(J,1)+XI(1)*D3PR(J,1)+D3PI(J,2)
      D4ER=XR(1)*D4PR(J,1)-XI(1)*D4PI(J,1)+D4PR(J,2)
      D4EI=XR(1)*D4PI(J,1)+XI(1)*D4PR(J,1)+D4PI(J,2)
      PRINT 18,ER,EI,DER,DEI,D2ER,D2EI,D3ER,D3EI,D4ER,D4EI,Y(J),J
      GO TO 221
  605 ER=XR(1)*PR(J,1)-XI(1)*PI(J,1)+XR(2)*PR(J,2)-XI(2)*PI(J,2)+
     1PR(J,3)
      EI=XR(1)*PI(J,1)+XI(1)*PR(J,1)+XR(2)*PI(J,2)+XI(2)*PR(J,2)+
     1PI(J,3)
      DER=XR(1)*DPR(J,1)-XI(1)*DPI(J,1)+XR(2)*DPR(J,2)-XI(2)*DPI
     1(J,2) +DPR(J,3)
      DEI=XR(1)*DPI(J,1)+XI(1)*DPR(J,1)+XR(2)*DPI(J,2)+XI(2)*DPR
     1(J,2) +DPI(J,3)
      D2ER=XR(1)*D2PR(J,1)-XI(1)*D2PI(J,1)+XR(2)*D2PR(J,2)-XI(2)*
     1D2PI(J,2)+D2PR(J,3)
      D2EI=XR(1)*D2PI(J,1)+XI(1)*D2PR(J,1)+XR(2)*D2PI(J,2)+XI(2)*
```

```
     1D2PR(J,2)+D2PI(J,3)
      UR=XR(1)*VR(J,1)-XI(1)*VI(J,1)+XR(2)*VR(J,2)-XI(2)*VI(J,2)+
     1VR(J,3)
      UI=XR(1)*VI(J,1)+XI(1)*VR(J,1)+XR(2)*VI(J,2)+XI(2)*VR(J,2)+
     1VI(J,3)
      DUR=XR(1)*DVR(J,1)-XI(1)*DVI(J,1)+XR(2)*DVR(J,2)-XI(2)*DVI
     1(J,2)+ DVR(J,3)
      DUI=XR(1)*DVI(J,1)+XI(1)*DVR(J,1)+XR(2)*DVI(J,2)+XI(2)*DVR
     1(J,2)+ DVI(J,3)
      PRINT 18,ER,EI,DER,DEI,D2ER,D2EI,UR,UI,DUR,DUI,Y(J),J
  221 CONTINUE
C     COMPUTE TEST FUNCTION T=X(1)*B1+X(2)*B2+B3 AND CHECK HOW
C     MUCH IT DIFFERS FROM 0. USE TEST=DSQRT(T*T CONJUGATE)
  225 IF  (OM .EQ. 0.0D0) 625,630
  625 TBR(IP)=XR(1)*B1R-XI(1)*B1I+B2R
      TBI(IP)=XR(1)*B1I+XI(1)*B1R+B2I
      TCR(IP)=XR(1)*C1R-XI(1)*C1I+C2R
      TCI(IP)=XR(1)*C1I+XI(1)*C1R+C2I
      GO TO 635
  630 TBR(IP)=XR(1)*B1R-XI(1)*B1I+XR(2)*B2R-XI(2)*B2I+B3R
      TBI(IP)=XR(1)*B1I+XI(1)*B1R+XR(2)*B2I+XI(2)*B2R+B3I
      TCR(IP)=XR(1)*C1R-XI(1)*B1I+XR(2)*C2R-XI(2)*C2I+C3R
      TCI(IP)=XR(1)*C1I+XI(1)*C1R+XR(2)*C2I+XI(2)*C2R+C3I
  635 TR(IP)=TBR(IP)/TCR(IP) $ TI(IP)=TBI(IP)/TCI(IP)
      T(IP)=DSQRT(TR(IP)*TR(IP)+TI(IP)*TI(IP))
      IF (MON-2) 145,126,126
  126 IF (MD-2) 130,140,140
  130 PRINT 135
  135 FORMAT(1X,*IT*,1X,*IP*,4X,*OM*,13X,*R*,15X,*AL*,14X,*BE*,
     114X,*CR*,14X,*TR*,15X,*TI*,14X,*T*)
      GO TO 145
  140 PRINT 24
   24 FORMAT (4X,*IP*,4X,*OM*,13X,*R*,15X,*AL*,14X,*BE*,14X,*CR*,
     114X,*TR*,15X,*TI*,14X,*T*)
  145 IF (T(IP)-TIP) 102,102,101
  101 IF (IP-2) 110,115,115
  110 PRINT 28,ITE,IP,OM,R,AL,BE,CR,TR(IP),TI(IP),T(IP)
   28 FORMAT (1X,I2,1X,I2,8(1X,D15.8))
      GO TO 120
  115 PRINT 23,IP,OM,R,AL,BE,CR,TR(IP),TI(IP),T(IP)
   23 FORMAT(4X,I2,8(1X,D15.8))
C     ITERATION OF EIGEN-VALUES.
  120 GO TO (105,197,198) IP
  105 DAL=PAL*AL $ AL=AL+DAL $ GO TO 421
  197 AL=AL-DAL $ RD=PCR*R $ R=R+RD $ GO TO 421
  198 DTR1=(TR(2)-TR(1))/DAL $ DTI1=(TI(2)-TI(1))/DAL $ R=R-RD
      DTR2=(TR(3)-TR(1))/RD $ DTI2=(TI(3)-TI(1))/RD
  302 DEN=DTR1*DTI2-DTI1*DTR2 $ DAL=(TI(1)*DTR2-TR(1)*DTI2)/DEN
      RD=(TR(1)*DTI1-DTR1*TI(1))/DEN
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```
      IF(DABS(DAL) .GE. (PCH *AL)) DAL=PCH*AL*DAL/DABS(DAL)
  708 IF (DABS(RD)-(PCH*R)) 108,108,109
  109 RD=PCH*R*RD/DABS(RD)
  108 DTR1=100.0D*RD/R $ R=R+RD $ DTR2=100.0D*DAL/AL
      AL=AL+DAL $ IP=0 $ ITE=ITE+1
      PRINT 23,IP,OM,R,AL,BE,CR
      PRINT 23,IP,OM,DTR1,DTR2
      GO TO 421
  102 PRINT 211
  211 FORMAT (* CONVERGENCE TEST SATISFIED *)
      PRINT 24
      PRINT 23,IP,OM,R,AL,BE,CR,TR(IP),TI(IP),T(IP)
      GO TO 400
  410 END
C
C
C
      SUBROUTINE DSOLE(AR,AI,XR,XI,N,ND)
C     THIS SUBROUTINE WAS WRITTEN BY DR. W.C.REYNOLDS OF STANFORD
C     UNIVERSITY. IT WAS MODIFIED FOR USE IN DOUBLE PRECISION.
C     DPRE.COMPLEX SOLUTION OF N SIMULTANEOUS LINEAR ALGEBRAIC EQ.
      DIMENSION AR(6,6),AI(6,6),XR(6),XI(6)
      DOUBLE PRECISION AR,AI,XR,XI,ER,EI,C,CX,D,Y,Z
C          CONSTRUCT THE TRIANGULAR ARRAY
      NM1=N-1
      DO 50 K=1,NM1
C          NORMALIZE EACH ROW ON ITS MAXIMUM MODULUS ELEMENT
      M=N-K+1
      DO  8 I=1,M
      C=0.0D0
      DO  4 J=1,M
      CX=AR(I,J)*AR(I,J)+AI(I,1)*AI(I,J)
      IF (CX-C)  4,4,3
    3 C=CX
   21 FORMAT (10X,D25.16)
    4 CONTINUE
      C=1.0D0/DSQRT(C)
      XR(I)=XR(I)*C
      XI(I)=XI(I)*C
      DO  7 J=1,M
      AR(I,J)=AR(I,J)*C
      AI(I,J)=AI(I,J)*C
   22 FORMAT (1X,I4,1X,I4,4(5X,D25.16))
    7 CONTINUE
    8 CONTINUE
C     FIND THE OPTIMUM PIVOT ROW.
      C=0.0D0
      DO 12 I=1,M
      CX=AR(I,M)*AR(I,M)+AI(I,M)*AI(I,M)
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```
      IF (CX-C) 12,12,10
   10 C=CX
      IP=I
   23 FORMAT (5X,I4,5X,D25.16)
   12 CONTINUE
C          STORE THE PIVOT ROW
      IF (C) 70,70,13
   13 IF (IP-M) 14,20,20
   14 DO 16 J=1,M
      ER=AR(IP,J) $ EI=AI(IP,J) $ AR(IP,J)=AR(M,J)
      AI(IP,J)=AI(M,J)
      AR(M,J)=ER
   16 AI(M,J)=EI
      ER=XR(IP) $ EI=XI(IP)
      XR(IP)=XR( M) $ XI(IP)=XI( M) $ XR(M)=ER $ XI(M)=EI
   80 FORMAT (* REARRANGED MATRIX A *)
   81 FORMAT (12D11.3)
C          ELIMINATE THE MTH ELEMENTS
   20 MM1=M-1
      DO 30 I=1,MM1
      D=AR(M,M)*AR(M,M)+AI(M,M)*AI(M,M)
      ER=(AR(I,M)*AR(M,M)+AI(I,M)*AI(M,M))/D
      EI=(AI(I,M)*AR(M,M)-AR(I,M)*AI(M,M))/D
      Y=XR(I)-ER*XR(M)+EI*XI(M) $ Z=XI(I)-ER*XI(M)-EI*XR(M)
      XR(I)=Y $ XI(I)=Z
   35 FORMAT (*D=*,5(1X,D25.16))
      DO 30 J=1,MM1
      Y=AR(I,J)-ER*AR(M,J)+EI*AI(M,J)
      Z=AI(I,J)-ER*AI(M,J)-EI*AR(M,J)
      AR(I,J)=Y $ AI(I,J)=Z
   24 FORMAT (1X,I4,1X,I4,6(1X,D20.12))
   30 CONTINUE
   50 CONTINUE
C          CARRY OUT THE BACK SOLUTION
      C=AR(1,1)*AR(1,1)+AI(1,1)*AI(1,1)
      IF (C) 70,70,54
   54 Y=(XR(1)*AR(1,1)+XI(1)*AI(1,1))/C
      Z=(XI(1)*AR(1,1)-XR(1)*AI(1,1))/C
      XR(1)=Y $ XI(1)=Z
   25 FORMAT (5X,3(5X,D25.16))
      DO 60 I=2,N
      IM1=I-1
      ER=XR(I) $ EI=XI(I)
      DO 57 J=1,IM1
      ER=ER-AR(I,J)*XR(J)+AI(I,J)*XI(J)
   57 EI=EI-AR(I,J)*XI(J)-AI(I,J)*XR(J)
      D=AR(I,I)*AR(I,I)+AI(I,I)*AI(I,I)
      XR(I)=(ER*AR(I,I)+EI*AI(I,I))/D
      XI(I)=(EI*AR(I,I)-ER*AI(I,I))/D
```

```
   60 CONTINUE
      RETURN
   70 PRINT 75
   75 FORMAT (22HODSOLE SYSTEM SINGULAR)
      PRINT 81,(((AR(I,J),AI(I,J)),J=1,N),I=1,N)
      PRINT 81,((XR(I),XI(I)),I=1,N)
      RETURN
      END
C
C
C
      SUBROUTINE POLYRT(FR,FI,NORDER,RR,RI,DELTA2)
C     THIS PROGRAM ALONGWITH THE NECESSARY SUBROUTINES WAS TAKEN
C     FROM THE M.S.U. COMPUTING LIBRARY. THIS WAS WRITTEN BY
C     MR. WILLIAM HOLSTEN OF THE UNIV. OF CALIFORNIA,SAN-DIEGO.
C     IT WAS MODIFIED FOR USE IN DOUBLE PRECISION AND WITH
C     EXTENDED ACCURACY.
C     THIS SUBROUTINE ALONGWITH SUBROUTINES IMPROVE,REDUCE,ROOT,
C     AND BASIC DETERMINES A ROOT OF THE POLYNOMIAL WITH COMPLEX
C     COEFFICIENTS,WHICH IS THEN IMPROVED UNTIL THE DESIRED
C     ACCURACY DELTA2 IS REACHED. THEN THE ORDER OF THE POLYNOMIAL
C     IS REDUCED. THIS PROCESS IS REPEATED  UNTIL ALL THE ROOTS
C     ARE OBTAINED.
      DIMENSION RR(9  ),RI(9  ),FR(9  ),FI(9  ),FFR(9  ),FFI(9  )
      DOUBLE PRECISION RR,RI,FR,FI,FFR,FFI,ZD,T1,R
      K=NORDER+1
      DO 11 I=1,K
      FFR(I)=FR(I)
   11 FFI(I)=FI(I)
      NORDR=NORDER
   55 IF(FFI(K))2,33,2
   33 IF(FFR(K))3,44,3
   44 NORDR=NORDR-1
      K=K-1
      IF(NORDR-1)66,66,55
   66 STOP 66
    3 IF(FFR(K)-1.0D0)2,4,2
    2 ZD=FFR(K)*FFR(K)+FFI(K)*FFI(K)
      DO 5 I=1,K
      T1=(FFR(I)*FFR(K)+FFI(I)*FFI(K))/ZD
      FFI(I)=(FFR(K)*FFI(I)-FFR(I)*FFI(K))/ZD
    5 FFR(I)=T1
    4 IF(FFR(1))7,8,7
    8 IF(FFI(1))7,9,7
    9 DO 10 I=1,K
      FFR(I)=FFR(I+1)
   10 FFI(I)=FFI(I+1)
      RR(NORDR)=0.0D0
      RI(NORDR)=0.0D0
```

```
      NORDR=NORDR-1
      K=K-1
      GO TO 4
    7 NOR=NORDR-1
      R=1.0D0
      DO 1  I=1,NOR
      JJ=NORDR+1-I
      CALL ROOT(FFR,FFI,JJ,RR(I),RI(I),R)
      CALL IMPROVE(FFR,FFI,JJ,RR(I),RI(I),DELTA2)
      R=DSQRT(RR(I)*RR(I)+RI(I)*RI(I))
      CALL REDUCE(FFR,FFI,JJ,RR(I),RI(I))
    1 CONTINUE
      RR(NORDR)=-FFR(1)
      RI(NORDR)=-FFI(1)
      END
C
C
C
      SUBROUTINE IMPROVE(TR,TI,NORD,CR,CI,DELTA2)
      DIMENSION TR(9  ),TI(9  ),DR(9  ),DI(9  )
      DOUBLE PRECISION TR,TI,DR,DI,TTR,TTI,T1,DDR,DDI,CR,CI,
     1CRNEW,CINEW
      COMMON DR,DI
      DO 1  I=1,NORD
      ZI=I
      DR(I)=TR(I+1)*ZI
    1 DI(I)=TI(I+1)*ZI
      ICNT=-25
    7 TTR=TR(NORD+1)
      TTI=TI(NORD+1)
      DO 2  I=1,NORD
      J=NORD+1-I
      T1=TTR*CR-TTI*CI+TR(J)
      TTI=TTI*CR+TTR*CI+TI(J)
    2 TTR=T1
      MORD=NORD-1
      DDR=DR(MORD+1)
      DDI=DI(MORD+1)
      DO 3  I=1,MORD
      J=MORD+1-I
      T1=DDR*CR-DDI*CI+DR(J)
      DDI=DDI*CR+DDR*CI+DI(J)
    3 DDR=T1
      T1=DDR*DDR+DDI*DDI
      CRNEW=CR-(TTR*DDR+TTI*DDI)/T1
      CINEW=CI-(DDR*TTI-DDI*TTR)/T1
      IF(DABS((CR-CRNEW)/CRNEW)-DELTA2) 4,4,5
    4 IF(DABS((CI-CINEW)/CINEW)-DELTA2) 6,6,5
    5 CR=CRNEW
      CI=CINEW
```

```
      ICNT=ICNT+1
      IF(ICNT)7,6,6
    6 CR=CRNEW
      CI=CINEW
      RETURN
      END
C
C
C
      SUBROUTINE REDUCE(AR,AI,NORD,CR,CI)
      DIMENSION AR(9   ),AI(9   ),TR(9   ),TI(9   )
      DOUBLE PRECISION AR,AI,TR,TI,CR,CI
      COMMON TR,TI
      TR(NORD)=AR(NORD+1)
      TI(NORD)=AI(NORD+1)
      MORD=NORD-1
      DO 1 I=1,MORD
      J=MORD+1-I
      TR(J)=TR(J+1)*CR-TI(J+1)*CI+AR(J+1)
    1 TI(J)=TR(J+1)*CI+TI(J+1)*CR+AI(J+1)
      DO 2 I=1,NORD
      AR(I)=TR(I)
    2 AI(I)=TI(I)
      RETURN
      END
C
C
C
      SUBROUTINE ROOT(FR,FI,NN,CR,CI,R)
      DIMENSION FR(9   ),FI(9   )
      DOUBLE PRECISION FR,FI,CON,CR,CI,DR,DI,R,RR,C1,C2
      NUM=-8
      CON=DSQRT(.5D0)
      CR=0.0D0
      CI=0.0D0
   21 DR=CR
      DI=CI
      CALL BASIC(FR,FI,NN,R,CR,CI,NO)
      IF(NO)1,2,15
    1 STOP 4
    2 R=2.0D0*R
      CALL BASIC(FR,FI,NN,R,CR,CI,NO)
      IF(NO)1,2,3
    3 R=R/2.0D0
    4 RR=5.0D0*R/6.0D0
      C1=RR*2.0D0
      C2=C1*CON
      CR=CR+C1
      CALL BASIC(FR,FI,NN,RR,CR,CI,NO)
```

```
      IF(NO)13,5,14
    5 CR=CR-2.0D0*C1
      CALL BASIC(FR,FI,NN,RR,CR,CI,NO)
      IF(NO)13,6,14
    6 CR=CR+C1
      CI=CI+C1
      CALL BASIC(FR,FI,NN,RR,CR,CI,NO)
      IF(NO)13,7,14
    7 CI=CI-2.0D0*C1
      CALL BASIC(FR,FI,NN,RR,CR,CI,NO)
      IF(NO)13,8,14
    8 CI=CI+C1+C2
      CR=CR+C2
      CALL BASIC(FR,FI,NN,RR,CR,CI,NO)
      IF(NO)13,9,14
    9 CR=CR-2.0D0*C2
      CALL BASIC(FR,FI,NN,RR,CR,CI,NO)
      IF(NO)13,10,14
   10 CI=CI-2.0D0*C2
      CALL BASIC(FR,FI,NN,RR,CR,CI,NO)
      IF(NO)13,11,14
   11 CR=CR+2.0D0*C2
      CALL BASIC(FR,FI,NN,RR,CR,CI,NO)
      IF(NO)13,30,14
   13 STOP 3
   14 R=RR
      NUM=NUM+1
      IF(NUM)21,23,13
   30 CR=DR
      CI=DI
   23 RETURN
   15 R=R/2.0D0
      CALL BASIC(FR,FI,NN,R,CR,CI,NO)
      IF(NO)1,4,15
      END
C
C
C
      SUBROUTINE BASIC(AR,AI,NORDER,R,CR,CI,IANSWER)
      DIMENSION AR(9   ),AI(9   ),BR(9   ),BI(9   ),TR(9   ),TI(9   )
      DOUBLE PRECISION AR,AI,BR,BI,TR,TI,CR,CI,QR1,QI1,QR2,QI2,
     1T1,T2,T3,T4,T5,R
      COMMON BR,BI,TR,TI
   20 IFLAG=0
      IANSWER=0
      NN=NORDER
      N=1
      M=2
      TR(N)=CR
```

```
      TI(N)=CI
      TR(M)=R
      TI(M)=0.0D0
      BR(N)=AR(M)*CR-AI(M)*CI+AR(N)
      BI(N)=AR(M)*CI+AI(M)*CR+AI(N)
      BR(M)=AR(M)*R
      BI(M)=AI(M)*R
      DO 2 J=2,NN
      L=J+1
      QR1=TR(N)*CR-TI(N)*CI
      QI1=TR(N)*CI+TI(N)*CR
      DO 1 I=2,J
      K=I-1
      QR2=TR(I)*CR-TI(I)*CI+R*TR(K)
      QI2=TR(I)*CI+TI(I)*CR+R*TI(K)
      TR(K)=QR1
      TI(K)=QI1
      BR(K)=AR(L)*QR1-AI(L)*QI1+BR(K)
      BI(K)=AR(L)*QI1+AI(L)*QR1+BI(K)
      QR1=QR2
      QI1=QI2
    1 CONTINUE
      BR(J)=QR1*AR(L)-QI1*AI(L)+BR(J)
      BI(J)=QR1*AI(L)+QI1*AR(L)+BI(J)
      TR(L)=TR(J)*R
      TI(L)=0.0D0
      TR(J)=QR1
      TI(J)=QI1
      BR(L)=TR(L)*AR(L)
      BI(L)=TR(L)*AI(L)
    2 CONTINUE
    3 MM=NN+1
      IF(BR(1)-1.0D+100)37,37,38
   38 DO 39 I=1,MM
      BR(I)=BR(I)/1.0D+100
   39 BI(I)=BI(I)/1.0D+100
   37 J=MM
      DO 4 I=1,MM
      TR(I)=BR(J)
      TI(I)=-BI(J)
    4 J=J-1
      T2=BR(1)
      T3=BI(1)
      T4=BR(MM)
      T5=BI(MM)
      DO 5 I=1,MM
      T1=T2*BR(I)+T3*BI(I)-T4*TR(I)+T5*TI(I)
      BI(I)=T2*BI(I)-T3*BR(I)-T4*TI(I)-T5*TR(I)
    5 BR(I)=T1
```

```
      IF(BR(1))8,6,6
    6 NN=NN-1
      IF(NN+1)7,9,3
    7 STOP 1
    8 IANSWER=1
    9 RETURN
      END
C
C
C
      SUBROUTINE VCAL (D1,D2,K)
C     THIS SUBROUTINE SOLVES FOR PRIMARY VELOCITY DISTRIBUTION.
C     METHOD EMPLOYED IS RUNGE KUTTA FOURTH ORDER.
C     REFERENCE WEEG AND REED.
      DIMENSION UB(153),DUB(153),D2UB(153),Y(153),D3UB(3),D4UB(3)
      DOUBLE PRECISION UB,DUB,D2UB,Y,F,Z,W,H,D1,D2,AK1,AK2,AK3,
     1AK4,AL1,AL2,AL3,AL4,AM1,AM2,AM3,AM4,D3UB,D4UB
      COMMON /2/ UB,DUB,D2UB,Y,D3UB,D4UB
C     BOUNDARY CONDITIONS.
      N=K $ F=.0D0$Z=.0D0$W=0.332057336159082886444564 4D0
C     THE CONSTANT W WAS FOUND BY ITERATION. ITS VALUE WAS
C     CONSIDERED TO BE ONE WHICH GAVE UB=1.0 AT A DISTANCE
C     FARTHEST FROM THE FLAT PLATE.
      Y(N)=0.0D0
      N1=0.5/D1+1 $ N2=1 $ N3=3
C     DIFFERENTIAL EQUATION OF THIRD ORDER IS WRITTEN IN THE FORM
C     OF THREE FIRST ORDER EQUATIONS.
      J=3
    1 UB(N)=Z $ DUB(N)=W*.5D1$D2UB(N)=-F*W*25.0D0/2.0D0
      IF (N-N3) 3,3,4
    3 IF (N-N2) 4,6,6
    6 D3UB(J)=-62.5D0*(Z*W-0.5D0*F*F*W)
      D4UB(J)=-312.5D0*(W*W-1.5D0*F*Z*W+0.25D0*F*F*F*W) $ J=J-1
    4 IF (N-2) 20,2 ,2
    2 IF (N-N1) 11,11,12
   12 H=D2*5.0D0$GO TO 5
   11 H=D1*5.0D0
C     COMPUTATIONS IN FOUR STAGES.
    5 AK1=H*Z $ AL1=H*W $ AM1=-H*W*F/2.0D0
      AK2=H*(Z+AL1/.2D1) $ AL2=H*(W+AM1/.2D1)
      AM2=-H*(F+AK1/2.0D)*(W+AM1/2.0D)/2.0D
      AK3=H*(Z+AL2/.2D1)$AL3=H*(W+AM2/.2D1)
      AM3=-H*(F+AK2/.2D1)*(W+AM2/.2D1)/2.0D0
      AK4=H*(Z+AL3) $ AL4=H*(W+AM3)
      AM4=-H*(F+AK3)*(W+AM3)/2.0D0
C     COMPUTATIONS FOR F,Z,W AT NEXT STEP.
      F=F+(AK1+.2D1*AK2+.2D1*AK3+AK4)/6.0D0
      Z=Z+(AL1+.2D1*AL2+.2D1*AL3+AL4)/6.0D0
      W=W+(AM1+.2D1*AM2+.2D1*AM3+AM4)/6.0D0
```

```
      Y(N-1)=Y(N)+H/5.0D0$ N=N-1
      GO TO 1
   20 CONTINUE
      END
C
C
C
      SUBROUTINE CMULT(AR,AI,BR,BI,ABR,ABI)
C     THIS SUBROUTINE MULTIPLIES TWO COMPLEX NUMBERS IN DOUBLE
C     PRECISION.
      DOUBLE PRECISION AR,AI,BR,BI,ABR,ABI
      ABR=AR*BR-AI*BI $ ABI=AR*BI+AI*BR
      RETURN
      END
C
C
C
      SUBROUTINE CXPE(DR,DI,RR,RI)
C     THIS SUBROUTINE FINDS THE EXPONENTIAL OF A COMPLEX NUMBER
C     IN DOUBLE PRECISION.
      DOUBLE PRECISION DR,DI,RR,RI
      RR=DEXP(DR)*DCOS(DI) $ RI=DEXP(DR)*DSIN(DI)
      RETURN
      END
C
C
C
C     DATA CARDS
  .02D      0.02D           50         2          1  1.0D-25
 1.0D-2     1.0D-2      0.30D      1.0D-3
   .01D    0.20D      0.397D     0.880011236D    1.57937814D3      13
```