PRESERVICE TEACHERS' UNDERSTANDINGS OF VOLUME AND ITS MEASUREMENT IN EVERYDAY AND SCHOOL CONTEXTS

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ABSTRACT

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To examine preservice teachers' understandings of volume and its measurement, this study explored how seven preservice teachers conceived of and measured volume in everyday and school contexts through task-based interviews. The interviews were audio- and video-recorded, and notes written during the interviews were collected as well. The collected data was analyzed according to the coding scheme inductively developed for this study. The preservice teachers in this study revealed different understandings about volume and its measurement by contexts: volume as capacity and its measurement by filling—revealed in everyday contexts—and volume as geometric volume and its measurement by calculation—revealed in school contexts. The findings of this study point to the need for enriching connections between everyday and school mathematics knowledge in volume measurement instruction.

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CHAPTER ONE INTRODUCTION

In *Principles and Standards for School Mathematics*, the National Council of Teachers of Mathematics (NCTM, 2000) asserts that "teachers must know and understand deeply the mathematics they are teaching and be able to draw on that knowledge with flexibility in their teaching tasks" (p. 17). Given the importance of mathematics content knowledge for teaching, educators have been concerned that preservice teachers have an insufficient understanding of school mathematics concepts, and are therefore inadequately prepared for teaching (Conference Board of the Mathematical Sciences, 2012). College students enter teacher education programs with both school mathematical understandings—what is learned in school—and everyday mathematical understandings—developed through everyday experience. Therefore, the challenge for teacher educators is how to build on preservice teachers' existing mathematical understandings in ways that develop and deploy their mathematics content knowledge for teaching.

This study explored preservice teachers' everyday and school understandings in one mathematical domain—volume measurement. Individuals' understandings of volume measurement are grounded in everyday experience. In addition, as school mathematics content, volume measurement is applicable to solving mathematical problems in everyday life. Considering that understandings about volume measurement span both everyday mathematics knowledge gained from everyday experience and mathematics knowledge taught in school, it is a good domain for examining preservice teachers' everyday and school mathematical understandings. Thus, I examined how preservice teachers' understandings of volume measurement are framed and shaped within the contexts of everyday and school experiences.

Research Purpose

This study explored preservice elementary teachers' knowledge of volume measurement, in terms of their mathematical understandings of volume and its measurement in everyday and school contexts. In this study, *preservice teacher* refers to a student in a college mathematics course who intends to become certified to teach grades preK-5. Drawing on Battista's (2007) conception of volume as "an attribute (e.g., length, area, volume) contained in a geometric object" (p. 891), and Bright's (1976) conceptions of measurable attribute as "a characteristic that can be quantified by comparing it to standard unit" (p. 88), and measuring as "the process of the comparing an attribute of a physical object to some unit selected to quantify that attribute" (p. 88); the term *volume* in this study is conceptualized as a measurable attribute of three-dimensional objects, and its *measurement* is viewed as a process of measuring volume in relation to the number of units used to measure the volume of the objects.

Underlying this study is the assumption that "Learning is contextual: what students learn is fundamentally connected to how they learn it" (Lappan & Briars, 1995, p. 133); according to Lappan and Briars, students' mathematical understandings are developed from their own experiences and interactions with other people around them. The study is also guided by the assumption, drawing on Barsalou's (1982) and McNeil and Alibali's (2005) views on the activation of certain properties of a concept by contextual relevance, that an individual develops understandings of volume measurement by working and thinking within different contexts (both in school and outside of school), and thus the different conceptions are activated by everyday and school contexts. The study examined what preservice teachers understand about volume and its measurement, and in what ways measurement situations and contexts influence preservice teachers' conceptions of volume and their measurement strategies.

Research Questions

Preservice teachers' content knowledge of volume measurement develops through learning both in school and out of school. In this study, I attended to preservice teachers' everyday and school understandings of volume and its measurement, as well as the relationship between those understandings. The assumption of the current study is that different contexts—everyday and school contexts—may evoke preservice teachers' different volume measurement conceptions.

The first focus of this study is how preservice teachers conceive of and measure volume in everyday contexts. Because everyday mathematical understandings are developed through everyday experiences (Lave, Murtaugh, & de la Rocha, 1984; Nunes, Schliemann, & Carraher, 1993; Saxe, 1988; Scribner, 1985), I examined preservice teachers' everyday understandings about volume and its measurement in everyday contexts—the context of everyday objects (e.g., a fish tank or a recipe).

The volume concepts and volume formulas, as well as the measurement concepts that underlie applying formulas, are treated as essential contents/concepts in school mathematics standards (e.g., *Principles and Standards for School Mathematics*, NCTM, 2000; *Common Core State Standards* for Mathematics [CCSS-M], National Governors Association Center for Best Practices and the Council of Chief State School Officers [NGA Center & CCSSO], 2010). Thus, preservice teachers' understandings of the concepts, and about volume formulas, can be considered their school knowledge of volume measurement—what is learned in school—as the second focus of the study. In considering the volume measurement concepts taught in school regarding three-dimensional geometric objects, I examined preservice teachers' school understandings of volume and its measurement in school contexts—the contexts of geometric objects (e.g., a cube, a rectangular prism made of unit cubes, a cylinder, etc.).

How preservice teachers' everyday and school understandings about volume and its measurement are related is the final focus of this study. In response to these study foci, the overall research questions guiding this study are:

1. How do preservice teachers conceive of and measure volume in everyday contexts?
2. How do preservice teachers conceptualize (understand) volume measurement in the context of school mathematics?
3. How are preservice teachers' everyday and school understandings of volume and its measurement related?

Dissertation Structure

Including the introduction, my dissertation consists of seven chapters. In Chapter 2, I provide the review of literature in the domain of mathematics education and mathematics teacher education, as well as some relevant cognitive science literature, to identify and support this study on preservice teachers' understandings of volume and its measurement in everyday and school contexts. In Chapter 3, I explain the method of data collection through structured, task-based interviews with individual preservice teachers, and the analysis method of the collected interview data. Then, I present the analysis results of the collected data in the following three chapters. In Chapter 4, I present the analysis results of preservice teachers' understandings of volume and its measurement in everyday contexts. In Chapter 5, I present the results of the analyses of preservice teachers' understandings of volume and its measurement in school contexts. In Chapter 6, I present the analysis results of the relationship between preservice teachers' everyday and school volume measurement understandings. Finally, in Chapter 7, I discuss the issues raised by the findings of the study and some issues for further study, as well as the implications of this study for mathematics teacher education, mathematics education, and educational psychology.

CHAPTER TWO LITERATURE REVIEW

This literature is organized around three themes: (a) teacher content knowledge of mathematics; (b) everyday understandings of mathematics; and (c) volume measurement knowledge. Through the literature review, I aim to position this study around preservice teachers' understandings of volume and its measurement in everyday and school contexts in the field of mathematics teacher education and mathematics education, as well as educational psychology.

Teacher Content Knowledge of Mathematics

Over the past 30 years, there has been broad interest in the "knowledge needed for teaching" (Ball, Hill, & Bass, 2005). Since Shulman (1986) proposed a three-pronged conceptual framework of teacher knowledge that he identified as subject matter content knowledge (knowledge of facts, concepts, principles and frameworks in a subject domain), pedagogical content knowledge (subject matter knowledge for teaching), and curricular knowledge (knowledge of curricular alternatives available for instruction), much has been written about teacher knowledge.

Building on the Shulman's framework, scholars studying teacher knowledge have focused on the nature of content knowledge needed for teaching mathematics in terms of "mathematical knowledge for teaching" (Ball et al., 2005). Ball et al. initially conceptualized mathematics content knowledge for teaching as the knowledge consisting of common mathematics knowledge, which refers to knowledge "any well-educated adult should have" (p. 22), and the specialized knowledge of mathematics, which refers to "mathematical knowledge that is "specialized" to the work of teaching and that only teachers need know" (p. 22).

As a refinement to the Shulman's (1986) framework, Ball, Thames, and Phelps (2008) conceptualized a domain map of mathematical knowledge for teaching, consisting of subject matter knowledge and pedagogical content knowledge. For Ball et al., subject matter knowledge consists of common content knowledge (the mathematical knowledge known in common with other professions that know and use mathematics), specialized content knowledge (the mathematical content knowledge unique to the work of teaching), and horizon content knowledge (an awareness of the relationships among different content in mathematics curriculum). There are three knowledge domains related to pedagogical content knowledge: knowledge of content and students (the knowledge that links knowing about mathematics and knowing about students), knowledge of content and teaching (the knowledge that links knowing about mathematics and knowing about teaching), and knowledge of content and curriculum (the knowledge about instructional materials that can be used for and in the curriculum). Ball et al.'s the domain map of mathematical knowledge for teaching has influenced teaching and teacher knowledge research.

Among scholars working on mathematics content knowledge for teaching, there is a shared assumption about the relationship between teachers' understandings of subject matter and their students' learning of mathematics (see Ball, 1990; Ball et al., 2005; Hill, Rowan, & Ball, 2005). Based on this view, researchers have examined the effect of teachers' content knowledge on their students' mathematics performance. For example, Hill et al. (2005) conducted a large quantitative study to explore whether and how teachers' mathematical content knowledge influences first and third graders' mathematics performance. Hill et al. collected student achievement and survey data from students (1,190 first graders and 1,773 third graders) and teachers (334 first-grade and 365 third-grade teachers). Through statistical analysis, they found a positive relationship between teacher content knowledge and student achievement. Based on the findings, Hill et al. suggest that teachers' mathematical content knowledge is a significant predictor of their students' mathematics performance.

In attending to the role of teacher content knowledge in effective mathematics teaching and student learning, one of the main goals of teacher education programs is to support the subject matter preparation of preservice teachers. To understand preservice teachers' conceptual understandings of mathematical concepts in terms of teacher content knowledge, Ball (1990) examined preservice teachers' understandings of the division of fractions, regarded as their mathematics content knowledge. Ball collected data from 217 elementary and 35 secondary preservice teachers by using a questionnaire item, asking them to choose an appropriate representation of a statement of division involving fractions from a set of four representations, and then interviewed the preservice teachers about their learning experience about the division of fractions and the strategies used to solve a problem of division involving fractions. Ball found that only 30% of the 252 participating preservice teachers chose an appropriate representation of a given division statement; although almost all of the 35 preservice teachers participated in the interview can calculate a given problem of division-by-fractions by using the strategy of inverting and multiplying, only four secondary preservice teachers of the 35 elementary and secondary preservice teachers were able to show a mathematically appropriate representation of the division-by-fractions expression in the problem. Ball thus argued that their conceptual understandings of dividing fractions were rule-bound and shallow, and that preservice teachers' subject matter knowledge for teaching mathematics is insufficient "to be able to represent it appropriately and in multiple ways—with story problems, pictures, situations, and concrete materials" (p. 458).

Other studies have reported preservice teachers' insufficient subject matter knowledge for teaching mathematics. For instance, in a study of elementary preservice teachers' subject matter content knowledge and pedagogical content knowledge of the division of fractions, Tirosh (2000) reported findings to those of Ball (1990). In the diagnostic questionnaire items, which asked participants in the study to solve problems of division-by-fractions, seven of the 30 participating preservice teachers made mistakes in solving such problems. For the items that asked the possible reasons for seventh graders' mistakes (after they talked about possible mistakes commonly made by seventh graders), 26 of the 30 participating preservice teachers responded that one reason was that the seventh graders incorrectly applied the standard algorism of dividing fractions, revealing their belief that "the steps of the algorithm are memorized and that if a step is forgotten, students will be unable to reconstruct it thorough mathematical inquiry" (pp. 10-11). The observation from Tirosh's study reflects preservice teachers' insufficient conceptual understandings of mathematical concepts. These results draw more attention to the need for subject matter preparation of elementary preservice teachers in teacher education programs.

In sum, research on teachers' mathematics content knowledge has shown the importance of teachers' mathematics content knowledge and raised concerns for preservice teachers' insufficient understanding of mathematical concepts. These issues led me to focus on preservice teachers' conceptual understandings of mathematical concepts in this study.

Everyday Understandings of Mathematics

Educators generally agree that learning occurs when an individual builds upon his or her existing knowledge (Tirosh, 2000). In considering the potential role of students' existing knowledge in teaching and learning in school, studies on cognition and learning have attended to “the activity and context in which learning takes place” (Brown, Collins, & Duguid, 1989, p. 32), especially, the knowledge developed in everyday life situations.

Leinhardt (1988) characterized such knowledge as intuitive knowledge (i.e., everyday understanding) as “applied, everyday life, circumstantial knowledge, which can be either correct or incorrect” (p. 120), arguing that intuitive knowledge is applicable to solving mathematical problems presented in familiar everyday contexts. In mathematics education research, everyday understandings of mathematics, which is often called informal mathematics knowledge, are contrasted with or distinguished from formal, school knowledge of mathematics—learned and taught in school (see Hiebert, 1988; Mack, 1990).

Ethnographic studies on mathematical practices in everyday life situations have reported that children—and adults—acquire rich everyday mathematical knowledge through engaging in different everyday activities (e.g., Lave et al., 1984; Nunes et al., 1993; Saxe, 1988; Scribner, 1985). For example, Saxe (1988) assessed 5- to 15-year-old candy sellers’ mathematical understandings compared to school children’s (i.e., nonsellers) understandings, and found that the candy sellers performed better in solving their selling practice-linked arithmetic problems than the nonsellers, and that the sellers developed a mathematical system that was different from the mathematical system of nonsellers, being more experienced in dealing with the complex problems of selling practices. Saxe thus argued that “sellers develop a mathematics that is adapted to the practice and over time, manifests mathematical operations of increasing complexity and power” (p. 20). This study suggested that children develop rich mathematical understandings when engaging in everyday arithmetic activities, which are different than the arithmetic activities presented in school mathematics. The difficulty of learning mathematics in school is often regarded as being caused by the gap between everyday experience and school mathematics (Gravemeijer, 1999).

Research on the relationship between everyday and school mathematical knowledge investigates how children's school knowledge of mathematics can be developed, building on their everyday mathematical understandings developed through everyday experiences. Mack (1990) examined the growth of eight sixth graders' mathematical knowledge of fractions symbols and algorithmic procedures through 11 to 12 one-to-one instructional sessions, which were designed to build on the students' everyday understandings of fractions. At the beginning of the study, all students showed rich everyday mathematical understandings about fractions, but insufficient mathematical understandings of fractions symbols and algorithmic procedures. When the instruction began to build on their everyday understandings, students could construct meaningful algorithms from the fraction problems, building on their everyday mathematical understandings of fractions symbols; however, this happened only when the fraction problems in everyday contexts were presented before the similar symbolic problems with the previous everyday contexts problems were presented. During the instruction, students' knowledge of rote (often incorrect) algorithmic procedures hindered their everyday understandings of fractions when solving the fraction problems represented symbolically and in everyday contexts. As the instruction progressed, students were able to develop the meaning of fraction symbols and procedures relating to their everyday understandings; however, this transfer was limited considering their conceptions of fractions as "the number of pieces," not thinking of the size of a fraction as a whole. Mack suggested that students were able to develop meaningful understandings of fraction symbols and procedures by building on their everyday understandings. Similarly, Warrington (1997) examined how fifth and sixth graders naturally think about dividing fractions, reporting that students constructed school knowledge based on their existing understandings about fractions. The research evidence supports the view that children's everyday mathematical understandings developed through everyday experiences can leverage to develop school knowledge of mathematics.

To examine children's understandings of length measurement related to their previous experience of measuring at home and in school, Irwin, Ell, and Vistro-Yu (2004) interviewed 43 New Zealand and 48 Filipino 8- and 9-year-olds and their teachers. Irwin et al. presented children with five measurement tasks designed to explore their understandings of everyday measurement—using informal units (e.g., the clip part of a mechanical pencil, eraser, their fingers, or tiles)—and school measurement—using rulers. Irwin et al. found that children from both countries have limited experience of measuring at home and in school, and the interview tasks were challenging for children in both countries, revealing their insufficient understanding of the underlying concepts in measurement. Related to the findings, Irwin et al. shared New Zealand teachers' reported concerns of spending "too short of a time in the transition from informal to formal measurement" (p. 17), arguing that there is the need for better support in mathematics instruction for the transition from everyday to school measurement experiences.

The need for connecting everyday understandings of mathematics to school mathematics is also supported by the studies reporting the inappropriate transfer and application of the school mathematical knowledge to solving mathematical problems presented in everyday contexts. For instance, Zevenbergen (2005) examined 98 Australian preservice teachers' understandings of volume concepts through their responses to a quiz problem: find the amount of concrete needed to fill a barbecue area 8.5 m long, 3.2 m wide, and 30 cm deep. Up to 50% of the 98 participating preservice teachers responded that 8.16 in cubic meters (or 8.16 m^3) of concrete was needed.

Some preservice teachers responded in cubic centimeters (i.e., 8,160,000 cm^3) or converted their responses to 8000 liters. These responses are mathematically correct but contextually inappropriate—cubic centimeters and liters are not an appropriate unit for purchasing concrete. Zevenbergen argued that the preservice teachers who determined the volume of concrete in cubic centimeters or liters had knowledge of school mathematics but failed to transfer it to the practical context. Zevenbergen's study points to the need for building better connections between knowledge of school mathematics and everyday mathematical knowledge in mathematics instruction.

By building better connections between everyday mathematics and school mathematics, everyday mathematical understandings outside of school can provide a conceptual base for making sense of school mathematics, and can facilitate applying school mathematical knowledge to everyday practices (Putnam, Lampert, & Peterson, 1990). With respect to preservice teachers, this view supports the idea that both school and everyday mathematical understandings are critical for preservice teachers' learning in teacher education programs. This view also supports the need for teacher educators and researchers studying preservice teachers' mathematics content knowledge to attend to preservice teachers' everyday mathematical understanding, as well as their school mathematical understanding, as existing knowledge.

Overall, the literature on everyday understandings of mathematics supports the need to examine preservice teachers' everyday understandings of mathematics to make sense of how their current understandings of mathematics concepts are developed. The next section of this literature review focuses on volume measurement, the content domain of this study.

Volume Measurement Knowledge

"Measurement is one of the principal real-world applications of mathematics" (Clements & Battista, 2001, p. 406), and is rooted in everyday experiences of measuring (Lehrer, 2003). As a content domain of geometrical measurement, volume measurement constitutes part of school mathematics curriculum (e.g., CCSS-M, NGA Center & CCSSO, 2010). In everyday life situations, individuals measure the volume of various substances (e.g., flour, orange juice, or gasoline). For that reason, volume measurement can be a practical and contextually situated domain of mathematics. Because of the significance of volume measurement in both school mathematics and everyday life, it is important to attend to how children—and adults—conceive of volume and develop their understandings about its measurement, as well as the difficulties they have in conceptualizing and measuring volume. The review of literature in the domain of volume measurement is thus organized around the following three themes: (a) concepts of volume and individuals' volume conceptions; (b) measuring volume as geometrical measurement; and (c) relationships between volume and capacity.

Volume Concepts and Individual Conceptions

One mathematical definition of the term volume is "a number describing the three-dimensional extent of a set" (James, James, & Alchian, 1976, p. 410). Similarly, in a mathematics textbook (written for preservice teachers), volume is described as "the size of an object (or a part of an object) that is three-dimensional" (Beckmann, 2014, p. 495); the volume of a three-dimensional object refers to the number of cubic units of space taken up by the object. These two definitions reflect the concept of volume as a mathematical idea that is accepted in the mathematics community and commonly shared by mathematics educators.

For Sfard (1991), *concept* represents a mathematical idea in its official, school form; *conception* refers to an individual's internal, subjective representations and associations evoked by the concept. As a subjective understanding towards a mathematical concept, therefore, an individual's conception (conceptualization) may not be fully consistent with the mathematical concept that has been accepted by the mathematics community as a school mathematical idea.

The early classification of children's volume conceptions is found in Piaget and his colleague's study on the conservation and measurement of volume. In defining the developmental stages of children's reasoning about the volume conservation and measurement, Piaget, Inhelder, and Szeminska (1960) examined children's volume conceptions in terms of three different kinds of volume: (a) interior volume (the number of unit cubes contained in an object, or the amount of matter defined by the boundary surfaces of an object); (b) occupied volume (the amount of space occupied by an object as a whole in a setting of other objects around it); and (c) complementary volume (the volume of the water displaced when immersing an object in water). The three volume conceptions observed in the Piaget et al.'s study raise a question: what makes children evoke different volume conceptions?

About mathematics knowledge in general, McNeil and Alibali (2005) asserted, "an individual's knowledge of a concept may depend on the context in which the knowledge is elicited" (p. 286). In his study of the nature of human conceptual processing, Barsalou (1982) stated that context-dependent properties of a concept are activated by contextual relevance; this helps to assume that an individual might evoke different conceptions of volume when thinking or working within different contexts. That is, an individual thinking about how much gasoline his car gas tank can hold may think about volume differently than when he is thinking about how many cubes will be needed to build a rectangular solid in a school context. This assumption is supported by studies on individuals' volume conceptions in different contexts.

Sáiz (2003) examined 22 elementary inservice teachers' conceptions of volume of everyday objects (e.g., chair, sheet of paper, spinning top). Teachers revealed conceptions of volume as: (a) capacity (the volume of matter filling a container as capacity); (b) enclosed volume (the space enclosed in the closed surface of an object); (c) a number (obtained from numerical information and by volume formulas); and (d) a mental object or quality related to an object's characteristic of having three dimensions.

To explore children's volume conceptions in different contexts, Potari and Spiliotopoulou (1996) asked 38 11-year-olds to compare the volume of paired objects, which had the same shape but exhibited different characteristics of materials or substances; for example, one object was filled with water and the other empty, one object was solid and the other hollow, or one object was open and the other covered. Potari and Spiliotopoulou observed that presenting children with different objects evoked different volume conceptions. That is, the presented contexts evoked different aspects of children's thinking about volume. When presented with a drawing of two stemmed glasses (one empty and one filled with water), for example, children described the volume of the glass in terms of the volume of the air inside the glass or in terms of how much the glass could hold; thus, their responses were examples of the conception of volume as capacity. Potari and Spiliotopoulou identified five different volume conceptions: (a) volume as occupied space by an object; (b) volume as capacity; (c) volume as the material substance of an object; (d) volume as the geometric properties of an object (such as its shape, size, and dimensions); and (e) volume as weight.

Geometric Volume Measurement

In school mathematics, geometric measurement is defined as "the assignment of a numerical value to an attribute of an object" (NCTM, 2000, p. 44), such as length, area, or volume of an object. According to Battista (2007), understanding geometric measurement requires (a) conceptualization of assigning numbers to quantify the amount of an attribute of a geometric object in relation to the number of attribute-units fitted in the object, and (b) implementation of measuring process, such as iterating units, using a ruler, and choosing appropriate measurement units. In geometrical measurement, thus, it is important to understand the role of units in measuring attributes and the process of using the units in its measurement in terms of the iteration of units. In measuring volume, according to Battista (2003), four mental processes underlie the meaningful iteration of units: (a) forming and using a mental model process (creating and using mental representation to visualize, understand, and reason about the encountered contexts); (b) a spatial structuring process (abstracting the composition and structure of an object by recognizing, interconnecting, and organizing its components); (c) a unit locating process (coordinating the composite of cubes along the three dimensions of an object); and (d) an organizing-by-composites process (forming the iterable composite units to generate the whole object).

In CCSS-M (NGA Center & CCSSO, 2010), measuring volume by counting unit cubes and by applying volume formulas (i.e., calculating volume by mathematical multiplication of volume formulas) are introduced as the principal strategies of volume measurement in the domain of geometric measurement. Studies of volume measurement have shown that children have difficulty in finding the number of cubes in rectangular prisms built from unit cubes (e.g., Battista & Clements, 1996; Ben-Haim, Lappan, & Houang, 1985; Hirstein, 1981).

Hirstein (1981) analyzed 13- and 17-year-olds' performance for volume measurement items from the second assessment of the National Assessment of Educational Progress, asking for the number of cubes in rectangular prisms built from unit cubes (presented in two-dimensional representations). Hirstein found that the 13- and 17-year-olds only counted visible surface squares, visible cubes (on the top layer), or all surface squares (i.e., surface area). Based on the findings, Hirstein argued that students confused volume and surface area.

During a pilot study to develop the spatial visualization assessments for examining fifth- to eighth-graders' spatial visualization abilities towards two-dimensional representations of rectangular prisms built from unit cubes, Ben-Haim et al. (1985) found that the children sometimes counted only visible surface squares (and doubled that number) or visible unit cubes of the rectangular prisms; these findings are consistent with Hirstein (1981). Ben-Haim et al. argued that students have difficulty in visualizing the hidden parts of the rectangular prisms when they are presented in two-dimensional representations.

Battista and Clements (1996) also found that children experience difficulties in finding the number of cubes in rectangular prisms built from unit cubes. They examined third- and fifth-graders' solution strategies for three volume measurement items that involved presenting two-dimensional representations of rectangular prisms built from unit cubes and asking how many cubes make up (or are in) the rectangular prisms. Battista and Clements described the students' solution strategies within five categories, involving the errors of counting all or a subset of the visible cubes of the rectangular prisms, multiplying the number of squares on a face of the rectangular prisms to the number on another face of it, or double-counting of cubes in the rectangular prisms. Battista and Clements attributed children's difficulty in finding the number of cubes in the rectangular prisms to an insufficient *spatial structuring*, defined as “the mental act of constructing an organization or form for an object or set of objects” (p. 282).

The studies reported children’s difficulty relating three-dimensional objects to two-dimensional representations of those objects, raising the issue of whether preservice teachers experience the same difficulties; thus, in this study on preservice teachers’ school understandings of volume measurement, I include some spatial visualization tasks as a component to examine.

Researchers have observed that many children solving volume measurement problems use and apply volume formulas by rote without thinking of why they work (e.g., Battista & Clements, 1996; Vasilyeva, Ganley, Casey, Dulaney, Tillinger, & Anderson, 2013). Other studies reveal similar concerns for the rote memorization and application of volume formulas by teachers (e.g., Enochs & Gabel, 1984; Sáiz, 2003). Enochs and Gabel (1984) examined 128 preservice teachers’ misconceptions of volume and surface area measurement by using six tasks designed for the study. Enochs and Gabel found that preservice teachers tended to use volume formulas by rote in inappropriate contexts, and confused volume and surface area. Enochs and Gabel argued that “the major difficulty that students appear to have in determining volume and surface area is that they rely on the use of formulas to solve the problems rather than on the conceptual meaning of volume and surface area” (pp. 676-677). Interestingly, in the task involving the volume of a rectangular prism, 77% of the 128 preservice teachers thought that its volume can be determined by length times width times height, but only 44% of the preservice teachers were sure that the volume formula of base area times height can be used in calculating its volume.

In her study of elementary inservice teachers’ conceptions of volume in everyday contexts, Sáiz (2003) found that teachers considered some objects (e.g., a sheet of paper or a handkerchief) to not have volume unless they have an apparent third dimension and that some everyday things (like a chair, or a spinning top) do not have measurable volume due to their irregular shape (i.e., volume formulas could not be applied to irregular shapes directly).

The difficulties that individuals—children and teachers—appeared to have in determining volume by using volume formulas suggest the importance of using and applying the volume formulas with sufficient understanding of the meaning of the triple multiplication. According to Piaget et al. (1960), the triple multiplication of lengths in three dimensions should be conceptualized as logical multiplication, which represents the relationship between length or area and volume. Battista and Clements (1996) argued that spatial structuring is important to the students' enumeration strategies of layers, which lead to understanding and visualizing the volume formulas in terms of layer structuring. Additionally, Battista and Clements (1998) pointed out that "only students who have already constructed such a layering conceptualization seem "ready" to begin formulating their enumeration procedure more abstractly in terms of a formula" (p. 260).

Overall, understanding of measuring volume (in terms of geometric measurement) requires conceptualizing of the meaningful unit iteration and visualizing the spatial structure of the objects to be measured. Researchers in this field have indicated that individuals'—children and teachers—difficulties in implementing school volume measurement tasks are caused by their rote memorization and application of volume formulas without sufficient understanding how volume formulas were derived. From this perspective, I attended to preservice teachers' conceptions of volume formulas, as well as their reasoning about the spatial/layer structuring of the volume formulas, as the main components of school volume measurement understandings to be examined in this study.

Relationships Between Volume and Capacity

Thinking of volume as capacity commonly appears in two empirical research studies on individuals'—children and inservice teachers—volume conceptions in different contexts (see Potari & Spiliotopoulou, 1996; Sáiz, 2003). This suggests that the concept of capacity is somehow associated with volume in individuals' minds. In her commentary on how volume and capacity should be addressed in volume measurement instruction for children, Kerslake (1976) pointed out that volume measurement in everyday life situations usually deals with capacity in terms of packing-up and filling-up activities for a container. Considering these two points highlights the need to review how scholars in this field have conceptualized the relationships between volume and capacity, and how the capacity-related volume measurement activities are addressed in the literature in the domain of volume measurement.

The concept of capacity refers to "the amount of substance that can be held in containers of different sizes" (Vasilyeva et al., 2013, p. 30). In comparing the definitions of the volume concept addressed above, it is clear that these concepts represent different properties. Volume and capacity, however, are sometimes treated as synonyms, potentially leading to confusion. Sáiz (2001) reported that "Teachers define volume as the place occupied by a body in space; nevertheless they themselves use this term, at times, as a synonym of capacity" (p. 368).

Potari and Spiliotopoulou (1996) noted a conceptual relationship between volume and capacity: in the context of a container, the interior volume is the same as the capacity of the container, and the occupied volume of the substance in the container (assuming it is full) is also the same as the capacity of the container. This reflects that both volume and capacity describe the same amount of space inside the container. Similar to Potari and Spiliotopoulou, Sáiz (2003) conjectured that thinking of "volume as 'capacity' came into view from capacity's meaning as the volume of matter that fills a container" (p. 100). That is, when thinking of capacity in terms of a (physical) quantity measured, capacity can be conceived as either equivalent to the interior volume of the container, or as the occupied volume of the material substance filling the container; namely, volume and capacity represent the same measurement value quantitatively. These relationships between volume and capacity conceptualized in the literature provide grounds not only for understanding individuals' conceptions of volume as capacity, but also for conceptualizing the relationships between volume and capacity when measuring the same quantity.

Identifying an appropriate unit for the attribute to be measured is important in measuring (Curry, Mitchelmore, & Outhred, 2006). For a given circumstance, the decision to select a unit is influenced by the contextual features perceived and reasoned by the individual in terms of contextual relevance. As a contextual feature that may influence an individual's decision of selecting a unit from either volume/cubic unit, or a capacity unit, for measuring volume, scholars have attended to the nature of the objects to be measured, such as solids and liquids (e.g., Curry & Outhred, 2005; Outhred & McPhail, 2000; Outhred, Mitchelmore, McPhail, & Gould, 2003; Owens & Outhred, 2006; Sarama & Clements, 2009). In building on the relationship between the object to be measured and a unit being able to be used for measuring volume—namely, selecting an appropriate measurement unit based on the nature of the object to be measured—scholars differentiate the characteristics of volume measurement activities as packing and filling.

Curry and Outhred (2005) addressed the distinctive characteristics between packing and filling activities in terms of measuring interior volume, such as "the space is packed with a three-dimensional array consisting of a two-dimensional array of units which is iterated in the third dimension" (p. 265) and "the space is filled by iterating a fluid unit which takes the shape of the container" (pp. 265-266), respectively; this differentiates the types of units used for each activity, namely volume/cubic units for packing and capacity (liquid) units for filling in the objects being measured. Considering that we do measure the volume of solid and liquid materials through packing and filling activities, the activities or conceptions of measuring volume as packing volume/cubic units and filling capacity units can be addressed in connection with individuals' everyday measurement experiences of volume.

According to Curry and Outhred (2005), the main difference between packing and filling volume is the *unit structure*, defined as "the pattern formed when the units fill the object to be measured" (p. 265). In their research on 96 first- to fourth-graders' conceptual understandings of length, area, and volume measurement, Curry and Outhred found that some students perceived the unit structure of filling volume as one-dimensional, similar to length measurement. In measuring the number of cups of rice needed to fill a container completely, for example, most students in the study treated the height of the rice in the container as a length unit and iteratively counted it up the side of the container. Vasilyeva et al. (2013) addressed that, in the task of filling containers with liquids, it is difficult to visualize the spatial structuring of liquids in terms of measurement units (capacity units) because of the spatial characteristic of liquids (i.e., its shape changes as it is poured from one container to another).

Overall, this review of the literature on how scholars in this field have conceptualized the relationship between volume and capacity, and characterized volume measurement activities as packing and filling, helps to understand how capacity can be conceptualized related to volume and how capacity-related measurement activities can be addressed in volume measurement, respectively. However, there remains the question of what contexts may evoke the conceptions of volume as capacity and what strategies there may be for measuring volume by packing or filling. Therefore, recognizing the features of the contexts of conceptualizing volume and its measurement related to capacity is essential in examining preservice teachers' everyday understandings of volume and its measurement.

CHAPTER THREE METHOD

In this exploratory qualitative study, I examined preservice teachers' everyday and school knowledge of volume measurement through structured, task-based interviews with individual participants (Goldin, 2000). The questions guiding the research were:

1. How do preservice teachers conceive of and measure volume in everyday contexts?
    a. What types of experiences or activities do they consider to be volume measurement?
    b. What volume measurement strategies do they use in everyday contexts?
2. How do preservice teachers conceptualize (understand) volume and its measurement in the context of school mathematics?
    a. How do they conceptualize volume of three-dimensional geometric objects?
    b. How do preservice teachers understand volume formulas?
        i. How do they explain the meaning of the volume formula, length times width times height, and the number produced by applying it?
        ii. How do they apply volume formulas to measure different three-dimensional geometric objects?
3. How are preservice teachers' everyday and school understandings of volume and its measurement related?

Participants

I interviewed seven preservice teachers who were students in a mathematics course required for their elementary (K-8) teacher preparation program at a university in the Midwestern United States. MTH1-2 is required of all elementary education majors at the university. MTH1-2 is a series of mathematics courses that focus on the mathematics content knowledge needed for K-8 teaching. These courses aim to help preservice teachers acquire solid subject matter knowledge of K-8 mathematics. While MTH1 focuses on arithmetic and algebra, MTH2 focuses mostly on geometry and measurement. Thus, I recruited volunteers for this study from MTH2 and conducted the interviews before they were taught about volume measurement in the course. During the interviews, participants were asked to share voluntarily some background information, such as what year they are in at the university and in what subjects they are majoring in the Elementary Teacher Education Program. Table 3.1 contains participants' background information that they shared voluntarily.

This study was deemed exempt from Michigan State University's Institutional Review Board (IRB) oversight, but I followed expected guidelines for obtaining informed consent. At the beginning of each interview, I explained to participants the information in the consent form. As participants signed the consent form, each interview was conducted. For participating in this study, I gave participants a $10 gift card to a dairy store at the university.

Table 3.1.

Participants Background Information

Participant[a]	Year	Major
Anne[b]	Sophomore	Language Arts/Mathematics
Becca	Sophomore	Social Studies/English
Cacie[b]	Sophomore	History/Social Study
Della[b]	Freshman	Mathematics
Eva[b]	Sophomore	Language Arts
Flora	Sophomore	Science
Grace[b]	Sophomore	Language Arts

[a]All pseudonyms [b]Former MTH1 students of the interviewer.

Data Collection

Interviews

I conducted two interviews with each participant in late February and early March, 2015. Each interview lasted around 45 minutes (ranging from 20 to 60 minutes). All the interviews were conducted before volume measurement was taught in class. The first interview was conducted in a conference room's kitchen area and the second interview was conducted in a meeting room in which participants could evoke an everyday life situation and a school setting, respectively. Note that the conceptualization of everyday and school contexts used in the development of the interview tasks are the contexts of the everyday and geometrical objects presented to participants in the interview. The interviews were audio- and video-recorded, and notes written during the interviews were collected as well.

Interview Tasks

To examine participants' understanding of volume and its measurement, I developed seven interview tasks. For the design of the tasks, certain criteria were set up, such as (a) whether they could explain what volume means; (b) whether they could use different measurement tools (e.g., a measuring cup or a ruler); (c) whether they know volume formulas; and (d) whether they could reflect on their strategies for measuring volume. (See Appendix A for the detailed descriptions of the seven interview tasks.)

Structured task-based interviews consist of a subject (preservice teacher) and an interviewer (researcher), interacting in relation to one or more tasks (questions, problems, or activities) introduced by the interviewer in a preplanned way; thus, subjects' interactions are not merely with the interviewer, but with the task environment (Goldin, 2000). According to Goldin, each task-based interview should be organized into four stages: (a) posing the question, giving sufficient time for the participant to respond, and later posing only nondirective follow-up questions (e.g., “Can you tell me more about that?”); (b) minimal heuristic suggestions, if the participant’s response is not spontaneous (e.g., “Can you show me using some of these materials?”); (c) the guided use of heuristic suggestions, if the requested description or anticipated behavior does not occur spontaneously (e.g., “Do you see a pattern in the cards?”); and (d) exploratory, metacognitive questions (e.g., “Do you think you could explain how you thought about the problem?”). Thus, I referred to Goldin’s four stages of examining participants’ mathematical understandings in designing the interview script of this study. As suggested by Goldin, the interview script was created in a sufficiently detailed manner for conducting structurally similar interviews with other participants using the same interview tasks.

Each interview task focused primarily on one of the research questions of everyday and school volume measurement conceptions and the relationship between them. The first interview consisted of four tasks that examined participants’ experiences of measuring volume in everyday life and their understandings of volume and its measurement in everyday contexts. The second interview consisted of two tasks that focused on participants’ school-based understandings of volume and its measurement and one task that required participants to reflect on the strategies for measuring volume that they used during the two interviews.

Four tasks of the first interview. First, I informed the participants that the goal of the interview was to learn how they think about volume. Then, I asked the participants to share their initial thoughts about volume (Task 1): “What is the first thought that comes to mind when you hear the term volume?”

In Task 2, I asked participants about their experiences of measuring volume in everyday life situations: “Do you ever measure volume in everyday life situations?” If participants answered that they had no experience of volume measurement, I provided them with the example of a car's gas tank: "How about gasoline in your car? How could you figure out the volume of gas in your car's tank?"

In Task 3, I presented participants with a picture of an empty rectangular prism-shaped fish tank, which was printed on 4-inch-by-6-inch photo paper (see Figure 3.1), and asked their thoughts about volume: "What if you had a fish tank like this, how would you think about the volume of the fish tank?" If participants answered with a way of figuring out volume instead of their thoughts about volume, I asked them how they would describe the volume of the fish tank. After discussing the volume of the fish tank, I asked them how they would know or figure out its volume.

Figure 3.1. Picture of a fish tank

In Task 4, I presented participants with a recipe for strawberry shortcake that included ingredients with various units (e.g., pounds, cups, and teaspoons); then, I asked them how they would figure out the amount of each ingredient (strawberries, flour, baking soda/powder, salt/sugar, milk, and butter) to use if they were going to make strawberry shortcake. (See Figure 3.2.) I then presented participants with actual strawberries, milk, flour, and butter, along

Strawberry Shortcake Recipe

Ingredients

- 1½ pounds of strawberries;
- 3 cups of all-purpose flour;
- 4 teaspoons of baking powder;
- 1 teaspoon of baking soda;
- 5 tablespoons of sugar;
- 1¼ tablespoons of salt;
- 1 cup of milk; and
- ¾ cup of butter.

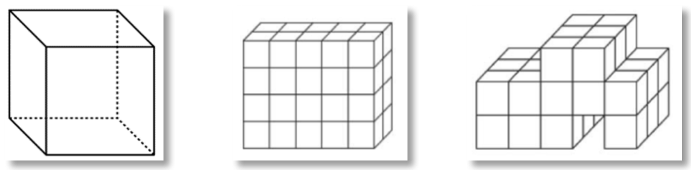
Figure 3.2. Strawberry shortcake recipe

with measurement tools, and asked them to measure the amount of strawberries, milk, flour, and butter needed for the recipe.[1] After completing these measurements, I presented them with the recipe again, asking them to identify volume measures among the listed ingredients.[2]

Three tasks of the second interview. In Task 5, I presented participants with the picture of a cube, which was printed on 8.5-inch-by-11-inch white paper (as a two-dimensional representation of a three-dimensional object, representing three-dimensional perspective by including dotted lines to represent unseen faces; see Figure 3.3(a)), and asked their thoughts

[1]In all measurement tasks of the two interviews, the same set of measurement tools were presented: a ruler, a tape measure, measuring cups (a set of dry measuring cups, including one full cup measure, one-half cup, and one-fourth cup; a set of measuring spoons, including one teaspoon, one-half teaspoon, one-fourth teaspoon and one tablespoon measure; and a 4-cup liquid measuring cup), unit cubes (one-inch-size and one-cm-size), two bowls (quart and half-quart), and an electronic scale. A calculator was also provided.

[2]For two participants (Flora and Grace), I missed the chance to ask about this question at the end of the first interview; thus, I asked about it at the beginning of the second interview.



(a) (b) (c)

Figure 3.3. Pictures of a cube (a), a rectangular prism built from unit cubes (b), and a stack of unit cubes (c)[3]

about volume: "How would you think about the volume of this object [cube]?" If the participants responded with a way of figuring out volume, rather than their thoughts about volume, I asked them to describe the volume of the object. After discussing the volume of the cube, I asked them how they would know or figure out its volume: "How would you know the volume of this object? How could you figure it out?" I asked them the same questions about the pictures of a rectangular prism built from unit cubes (as a two-dimensional representation of a three-dimensional object, representing unit cubes by including grid lines on the seen faces to form a series of squares) and a stack of unit cubes (also as a two-dimensional representation of a three-dimensional object, representing unit cubes by including grid lines on the seen faces to form a series of squares), respectively; each of them were printed on 8.5-inch-by-11-inch white paper (see Figures 3.3(b) and (c)). These pictures were presented to examine participants' understandings of volume and its measurement in a pictorial context.

In Task 6, I presented participants with a solid wooden rectangular prism, a hollow clear plastic cube, and a hollow clear plastic cylinder (see Figure 3.4), and examined their understanding of volume and its measurements in the physical context.

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[3]The figures of a rectangular prism built from unit cubes and a stack of unit cubes were adapted from "How Children Determine the Size of 3D Structures: Investigating Factors Influencing Strategy Choice," by Vasilyeva, Ganley, Casey, Dulaney, Tillinger, and Anderson, 2013, *Cognition and Instruction*, p. 33. Copyright 2013 by Taylor & Francis. doi: 10.1080/07370008.2012.742086

(a) (b) (c)

Figure 3.4. Pictures of a solid wooden rectangular prism (a), a hollow clear plastic cube (b), and a hollow clear plastic cylinder (c)

First, I presented participants with a solid wooden rectangular prism and asked how they would think about the volume of the solid wooden rectangular prism. I then presented them with the same set of measurement tools that they used before (in Task 4) and asked them to measure its volume. If participants obtained a calculated number by using a volume formula to find the volume of the solid wooden rectangular prism, I asked about the meaning of calculated number in the solid wooden rectangular prism; I also asked them about the volume formula that they used: “Where does the volume formula come from? What do the numbers in it mean?”

After completing all the questions for the first object, I presented participants with a hollow clear plastic cube, asking their thoughts about volume and then asked them to measure its volume. If participants obtained a calculated number by using a volume formula to find its volume, I asked them to estimate the number of unit cubes needed to fill the hollow clear plastic cube completely, and to justify their estimations: “How many cubes do you need to fill this container completely? If we don’t have enough cubes to check your answer, how can you convince me that your number is correct?”

For the final object, I presented participants with a hollow clear plastic cylinder, and asked their thoughts about volume. I then asked them to measure the volume of the hollow clear plastic cylinder by using the volume formula that they used for measuring the volume of the solid wooden rectangular prism and/or the hollow clear plastic cube (i.e., length times width times height).

Tasks 5 and 6 focused on the use of volume formulas to measure volume related to school volume measurement knowledge. In these tasks, the pictures of a cube, a rectangular prism built from unit cubes, and a stack of unit cubes; a solid wooden rectangular prism, a hollow clear plastic cube, and a hollow clear plastic cylinder were provided to present school contexts to the participants. The three-dimensional geometric objects in the pictorial context (i.e., the pictures of a cube, a rectangular prism built from unit cubes, and a stack of unit cubes as two-dimensional representations of three-dimensional geometric objects) were provided to examine participants' spatial visualization ability in the different measurement contexts in a school setting.

In Task 7, I asked the participants to reflect on the different strategies for measuring volume that they used thus far: "In measuring volume, you used different measurement tools like measuring cups or a ruler and you also used a volume formula. When do you measure volume by using a measuring cup? When do you measure volume by using a volume formula?" I also asked them about the everyday contexts in which they use a volume formula to measure volume and their experience of measuring volume by using a volume formula in everyday life situations.

Data Analysis

The audio- and video-recorded interviews were transcribed to capture the participants' words, actions, interactions, and gestures. After transcribing each interview, I watched the videos again to check the accuracy of the transcription and to take some screenshots of video-recorded interviews aligned with my transcription of participants' actions, interactions, and gestures.

I developed coding schemes inductively to analyze participants' understandings of volume and its measurement in everyday and school contexts. (See Appendix B for the coding scheme with sample participant responses.) All codes were developed from participants' responses to the interview questions. As I read the interview transcripts of the participants, I grouped their common responses and named each code to represent the features of each group of the responses.[4]

The method used to name the codes was: (a) if a group of responses reflected the same characteristic as a volume conception or measurement strategy already observed and named in the related literature, I adapted the term and its definition that appeared in the literature as a code name to avoid possible confusion for future readers; (b) if a group of responses referred to somewhat similar characteristics (but not the same) as the volume conception and measurement strategy in the literature, I named the code for the response group, drawing on the most similar term of the literature to reflect the connection between them, but defined the meaning of the code to reflect the difference between them; and (c) if a group of responses represented new characteristics of volume conception or measurement strategy not documented in the literature, I named and defined the code for the response group reflecting the distinctive characteristic.

Across the two interviews, I classified 10 different codes for volume conceptions: (a) substance volume (the amount of substance inside or filling a container, drawn on the meaning of volume as the material substance of an object, Potari & Spiliotopoulou, 1996), involving "liquid volume" (NGA Center & CCSSO, 2010; Vasilyeva et al., 2013); (b) volume as substance capacity (the potential amount of substance needed to fill a container, drawn on the meaning of volume as capacity, Sáiz, 2003; Potari & Spiliotopoulou, 1996), involving "liquid

[4] I used *QSR International's NVivo 10 qualitative data analysis Software* for coding.

capacity” (Joram, Subrahmanyam, & Gelman, 1998); (c) complementary volume (Piaget et al., 1960); (d) volume as enclosed liquid (the amount of liquid to fill space inside of a container); (e) volume as space occupied by liquid (the amount of space taken by liquid in a container, drawn on the meaning of volume as occupied space, Piaget et al., 1960); (f) calculated volume (the result obtained from multiplying the three length measures, drawn on the meaning of volume as number, Sáiz, 2003); (g) the number of cubic units (the number of unit cubes that can be packed in or used for building/constructing an object, drawn on the meaning of interior volume, Piaget et al., 1960; and also drawn on the meaning of packing volume, Curry & Outhred, 2005), (h) whole object (an entire object, indicating its size, drawn on the meaning of volume as the geometrical properties of an object, Potari & Spiliotopoulou, 1996), (i) occupied space (Piaget et al., 1960), and (j) enclosed space (Sáiz, 2003).

Throughout the two interviews, I coded two conceptions of measuring volume: (a) pouring/filling liquid (seeing volume measurement as the activity of pouring or filling liquid, drawn on the meaning of filling volume, Curry & Outhred, 2005); and (b) calculating volume (seeing volume measurement as a calculation with a volume formula, drawn on the meaning of volume as a number, Sáiz, 2003). I also coded three different strategies of measuring volume: (a) using containers, involving both an informal/everyday container and a formal measuring cup, (b) using formulas, involving using a ruler and cubes to measure the three dimensions of an object, and (c) reading the marks on the wrapping of butter for measuring.

In the second interview, I coded two conceptions of the volume formula of length times width times height: (a) a dimensional structure, as length-by-width-by-height structuring, and (b) layer structuring (Battista & Clements, 1996); three conceptions of the calculated number/volume obtained by length times width times height: (a) the product resulting from the triple multiplication (drawn on the meaning of volume as number, Sáiz, 2003), (b) the number of cubic units filling and comprising an object (drawn on the meaning of interior volume, Piaget et al., 1960); and (c) the amount of substance comprising an object (drawn on the meaning of volume as the material substance of an object, Potari & Spiliotopoulou, 1996); and two different reflections on the relationship between the volume formula for a rectangular prism ($l \times w \times h$) and the volume formula for a cylinder ($\pi r^2 \times h$): (a) different formulas for different shapes, and (b) the same sequence/procedure of computation. I also coded two different reflections on two different volume measurement strategies: (a) using different strategies by the nature of measurement contexts, and (b) using a measuring cup in the everyday context and using a volume formula in the school context.

After developing coding for all participants' responses, I grouped codes into the seven types of participants' understandings about volume and its measurement: (a) codes for volume conceptions revealed in everyday and school contexts, (b) codes for the conceptions of measuring volume revealed in everyday and school contexts, (c) codes for volume measurement strategies taken in everyday and school contexts, (d) codes for the conceptions of the volume formula, length times width times height, (e) codes for conceptions of the number obtained by length times width times height, (f) codes for reflections on the relationship between two volume formulas (i.e., $l \times w \times h$ and $\pi r^2 \times h$), and (g) codes for reflections on two different volume measurement strategies (i.e., using a measuring cup and a volume formula).

As a result of the process described here, I developed the coding scheme used for analyzing participants' understandings of volume and its measurement in everyday and school contexts. This coding scheme composes the preliminary findings of this study and can be used as an analytic framework for observing other preservice teachers in the future.

CHAPTER FOUR RESULTS: UNDERSTANDINGS OF VOLUME AND ITS MEASUREMENT IN EVERYDAY CONTEXTS

In this chapter, I examine how participants conceive of and measure volume in everyday contexts. I present the participants' first thoughts about volume and their experiences of measuring volume in everyday life situations; then, I present their conceptions of volume and strategies for measuring volume in everyday contexts—a fish tank and a recipe.

First Thoughts About Volume

I first asked participants, "What is the first thought that comes to mind when you hear the term volume?" Table 4.1 summarizes participants' responses to this first question. Note that participants referred to a certain context when describing volume related to liquid, space, or a formula.

Four participants—Anne, Eva, Flora, and Grace—described volume in terms of the amount of water or coffee (liquid) in/inside or the amount needed to fill a container, for example, "For some reason, water is the first thing that I think of, like how much water would fill a certain object is how I would try to explain volume," (Anne). Flora described volume as the potential, maximal amount of coffee (liquid) to fill a container: "Like if I had a cup and I wanted to know how much coffee or something I can put in it, I would need to know the volume, so how much of the coffee can actually fit inside the cup without it spilling over or anything." Flora's volume description can be distinguished from the others in considering the equivalence between the potential amount of liquid to fill a container and the capacity of the container. When these four participants described volume related to liquid, they cited everyday contextual components such

Table 4.1.

Participants' Conceptions of Volume with Contextual Components

Participant	First thought	Conception
Anne	Amount of *water* in *a swimming pool* Amount of *water* to fill *a certain object* [a container]	Liquid volume
Eva	Quantity *2-liter* Amount of substance [water] inside *a water bottle, 16.9 FL OZ*	Liquid volume
Grace	Amount of *water* inside *a cup* Amount of [abstract] substance held inside a *container*[a]	Liquid volume
Flora	Potential amount of [abstract] substance to put into a *container* Potential amount of *coffee* to fit inside *a cup*	Liquid capacity
Becca	Amount of *space* taken up by an object or inside *a bottle or a box*–related to the *problems* to find its volume as *3-D measurement* in *class*	Occupied space Enclosed space
Cacie	Amount of *space* taken up by *liquid* in *3-D space* [without referring a container to hold it]	Occupied space
Della	*Formula* in *primary, middle,* and *high schools* Rotated *diagram* in *calculus*	Formula

Note. Contextual components are italicized.
[a]Distinguished from capacity, "how big the container is" (Grace).

as a swimming pool, a cup, or coffee; thus, it is possible to infer that their volume conceptions are related to everyday contexts.

Two participants, Becca and Cacie, talked about volume connected to 3-D space. Becca described the amount of space taken by an object or inside a container: "Like how much space an object takes up or how much space, if it's a bottle or a box, how much space is inside." Cacie described volume as the amount of space taken by liquid (without referring a container to hold it): "I think of a 3-D space and the amount that it takes up or including liquid, how much space the liquid could take up in a 3-D space. … The complete amount of space it would take up, yeah." When Becca and Cacie described volume related to space, they cited school-context components such as a problem, a class, or 3-D measurement; thus, it is possible to infer that their volume conceptions are linked to school contexts.

When asked her first thoughts about volume, Della, the seventh participant, mentioned her experience using formulas in school. After an additional prompting question, she described volume as a diagram in her calculus course (i.e., a calculus-diagram): “Just rotated diagram, a curve with the *x*-axis or *y*-axis to ask for the volume is the way we think of the word volume in our mind.” Thus, Della conceptualized volume as a formula and a calculus-diagram.

In their initial thoughts about volume, participants described volume as an amount of liquid, an amount of space, or a formula. In considering the contextual components cited in participants’ volume descriptions, their conceptions of volume can be seen as everyday or as school volume conceptions. As everyday volume conceptions, participants described volume as (a) liquid volume (the amount of liquid inside or filling a container), and (b) liquid capacity (the potential amount of liquid to fill a container). As school volume conceptions, participants described volume as (a) an occupied space (the amount of space taken by an object or liquid, Piaget et al., 1960); (b) an enclosed space (the amount of space inside an object, Sáiz, 2003); or (c) a formula.

Experiences of Measuring Volume in Everyday Life Situations

The second question I asked participants was, “Do you ever measure volume in everyday life situations?” Through this question, I examined what types of experiences or activities they consider to be volume measurement. Table 4.2 summarizes responses to this question. Participants talked about their experience of measuring volume in everyday life situations as pouring or filling liquid into a container and calculation; one participant, Della, said that she had no experience measuring volume in everyday life.

Table 4.2.

Participants' Experiences of Measuring Volume and Their Conceptions of Volume Measurement

Participant	Experience	Conception
Anne	Pouring *water* into *a coffee maker* (in *cups* or *fl. Oz.)*	Pouring liquid
Flora	Measuring a *cup* of *milk* or a *tablespoon* of *cooking oil* in *cooking* Pouring *milk* into *a measuring cup*	Pouring liquid
Grace	Measuring the amount of *water* or *oil* with a *measuring cup* in *cooking*	Pouring liquid
Eva	Filling up [the gas tank to know its volume related to capacity]	Filling liquid
Becca	No experience of calculating volume	Calculating volume
Cacie	Calculating with the formula, $A \times h$	Calculating volume
Della	No experience measuring volume	None

Note. Contextual components stated by participants are italicized.

Anne, Flora, and Grace, who initially talked about volume in terms of liquid, talked about cooking and making coffee as everyday contexts of measuring volume: "I think sometimes when I'm cooking, sometimes I'll only need like a cup of milk or a tablespoon of the cooking oil. I'd measure it then," (Flora). They described measuring volume as the activity of pouring a certain amount of liquid from or into some kind of measuring cup (a coffee pot, a cup, a tablespoon, etc.); for example, "I have the measuring cups that are labeled how much it is, so I just pour it in and watch it until it gets to the line that I need it to be at, and then I use it," (Flora).

In addition to pouring liquid into a measuring cup, Flora also described measuring volume as putting a solid substance into a bowl: "I made a salad yesterday and I was trying to fit it into a container, and the first one I got was too small so I had to find [another bowl] ...." Unlike her previous description of pouring liquid into a measuring cup, this experience was a holistic, visual comparison (visual estimate of size as a whole, Piaget et al. 1960), rather than a volume measurement.

When I asked Eva, who also initially talked about volume in terms of liquid, about her experience measuring volume in everyday life situations, at first she said she had no experience measuring volume; however, with prompting questions, she described the volume of gasoline in gallons and measuring the volume of a gas tank in terms of filling.

Eun Mi: How about the gas tank in your car?

Eva: Okay, yes.

Eun Mi: So, how could you know how much gas you need for your car?

Eva: Well, I need to know how many gallons of gas fit in my car, and I don't know, kind of estimate it? […]

[*More discussion about ways to know the volume of gasoline in terms of gallons or ounces.*]

Eun Mi: How could you figure out the volume of your gas tank?

Eva: Oh, by filling it up and seeing how much it can, like the capacity that it can hold.

Eva explicitly mentioned capacity, suggesting an understanding of the equivalence between the potential amount of liquid to fill a container and the capacity of the container.

Thus, all four participants who described volume related to liquid in response to the first interview question considered measuring volume as the activity of pouring or filling liquid. When asked to describe "measuring volume," these participants cited cooking, making coffee, or the gas tank in a car as measurement contexts; thus, it is possible to infer that their conceptions of measuring volume are related to everyday contexts.

Becca and Cacie, who talked about volume related to space, described measuring volume as a calculation. By thinking of volume in this way, Becca said she had no everyday experience measuring volume: “No. I haven’t calculated, found volume. I don’t think I’ve really ever needed to do the full calculation.” With additional prompting questions, she said:

> Becca: I have a humidifier in my bedroom. I need to get enough water to get in it. I guess trying to ... I never have calculated it but I always just guess or estimate how much I need and it’s never the right amount. It’s either too much or too little. I guess if I calculated it or looked it up it would be easier to actually get the right amount so I'm not making a mess.

Although Becca talked about estimating or guessing the amount of water needed for her humidifier, this was a holistic, visual comparison similar to what Flora did; she still thought of calculation as the way to measure volume.

When I asked Cacie about her experience measuring volume in everyday life situations, she said:

> Cacie: Like if you’re passing, if you’re just walking down the street, a trash receptacle, how much it can hold. Or that fridge [as pointing the refrigerator in the corner of the room], how much that could hold, how much space is inside it.

Here, Cacie was connecting the capacity of the container to the amount of space inside of it without explicitly explaining how capacity and the space inside are related (as Eva did). When I asked her how she would know the volume of the trash can, Cacie said:

Cacie: If it's circular, it would be to find the area of the circle on the bottom times the height I think. But, I guess if the trash is completely compact and filling up all the space it'd be less, so that'd be hard to figure out probably.

Eun Mi: You also mentioned our lovely refrigerator. How would you know the volume of the refrigerator? You just mentioned the amount of space inside.

Cacie: Yeah, that would be a very complex measurement I guess, because it's not an exact square or exact shape. There are probably curves on the inside that would make it a little difficult to measure so I'm not sure.

Cacie thus described calculating volume by using the volume formula, base area times height (i.e., $A \times h$) and talked about the difficulty of calculating the volume of an everyday object because of its shape.

In sum, Becca and Cacie, who described volume related to space in response to the first interview question, considered measuring volume as a calculation (with a volume formula). Neither appeared to calculate volume in everyday life.

Della, who thought of volume as a formula, said she had no experience measuring volume in everyday life situations: "Why do I need to measure it? … Never." With additional prompting questions, Della insisted that she had no need to measure volume in everyday contexts.

In response to questions about measuring volume in everyday life situations, the four participants who initially conceptualized volume as everyday-volume conceptions conceived of measuring volume as (a) pouring liquid (pouring liquid from or into a measuring cup) and

(b) filling liquid (filling up a container with liquid). The three participants who initially conceptualized volume in terms of school volume did not appear to measure volume in everyday life. In addition, Becca and Cacie conceived of measuring volume as calculating-volume (determining volume by calculation with a volume formula). Thus, participants' conceptions of measuring volume seem to reflect how they initially conceptualized volume; the conceptions of pouring or filling liquid are linked to everyday contexts.

Conceptions of Volume in Everyday Contexts

After asking participants about their experiences of measuring volume in everyday life, I asked a series of questions about volume and its measurement in the fish tank and recipe contexts. I examined how participants' talk about the volume of the fish tank and of the recipe ingredients for evidence of their volume conceptions in these everyday contexts. Table 4.3 summarizes participants' conceptions of volume revealed in their responses to questions about the fish tank and the recipe. Note that participants often described volume in multiple ways in any given context.

Volume as the Amount of Substance

When talking about the volume of the fish tank and/or of the recipe ingredients, all participants referred to the substance in a container. In addition to liquid, participants mentioned other substances such as fish or air. Their ways of referring to substance were similar to the volume descriptions related to liquid—liquid volume and liquid capacity—observed previously in four of the participants' responses (Anne, Eva, Grace, and Flora) who cited everyday volume conceptions in answer to the first interview question. In these two contexts, two dominant conceptions of volume related to substance emerged: substance volume and substance capacity (see Figure 4.1). These two conceptions involve liquid volume and liquid capacity, respectively.

Table 4.3.

Participants' Conceptions of Volume in the Everyday Contexts

Participant	Fish tank	Cake recipe
Anne	Potential amount of substance (everything, e.g., water and fish) to put in a container Amount of liquid (water) to fill in a container Complementary volume (if putting stuff in, water level rises)	Amount of substance (milk characterized as liquid; butter, salt, sugar, baking soda, baking powder, and flour characterized as not-solid; i.e., other than strawberries) to fill a container
Eva	Potential amount of liquid (water) put to fill a container[a]	N/A[b]
Grace	Amount of substance (air) inside a container[a]	Potential amount of substance (all of them; i.e., other than strawberries) to put in/inside a container
Flora	Potential amount of substance (water, air, decorations, rocks, fish, and filters) to put in a container Complementary volume (water is displaced)	Volume [Amount] of substance (milk and melted butter … all of them other ingredients; i.e., other than strawberries) in a container
Becca	Amount of liquid (water) in a container	Amount of liquid (milk) taking the space inside a container
Cacie	Potential amount of [abstract] substance to fill a container Amount of space taken by the liquid (water) poured in a container	Amount of substance (milk characterized as liquid and thus filling space; and all, like baking powder, flour, and baking soda characterized as powder, uniform substance, and thus taking a certain amount of space; i.e., other than strawberries) in a container
Della	Calculated volume Potential amount of substance (water and fish) to fit in a container	N/A[b]

[a] Eva and Grace described volume related to capacity, but seemed to recognize the difference between volume and capacity. [b]Eva and Della did not reveal any volume conception when they were asked to identify which ingredients in the recipe were volume measures.

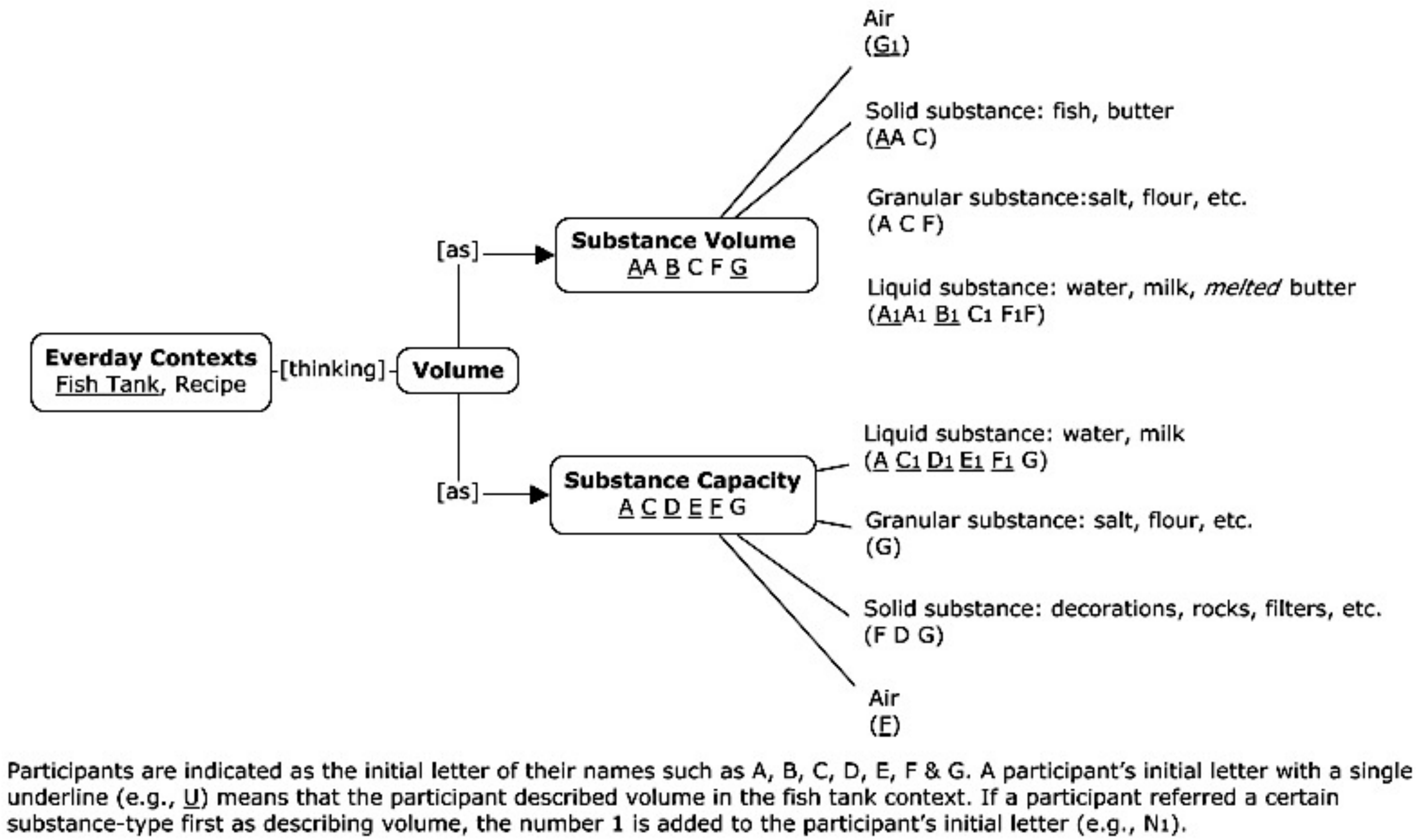

Figure 4.1. Participants' two dominant conceptions of volume revealed in the everyday contexts[5]

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[5]To draw the diagram of participants' two dominant conceptions of volume revealed in the everyday contexts, I used *Institute for Human and Machine Cognition (IHMC)'s CmapTools knowledge modeling kit Version 6.01.01*.

Substance volume. Five participants—Anne, Becca, Cacie, Flora, and Grace—exhibited their thinking of volume in terms of the amount of substance (water, fish, air, milk, melted butter, etc.) in a container. When describing the volume of the fish tank, Anne and Becca described volume as the amount of water inside or filling the container (Anne also referred to fish later): "Um, everything that goes inside like how much water would fill it. But then if you had fish in there, that would take up volume too," (Anne). When I asked Grace about the volume of the fish tank, she said:

Grace:	Zero.
Eun Mi:	Zero?
Grace:	Because it's just the case.
Eun Mi:	Looks like nothing there?
Grace:	Yeah, but if you're looking for how much inside, like how much air that's inside, then you could figure it out.

Grace thus thought of volume as the amount of substance (air) in the container.

When I presented the strawberry shortcake recipe and asked which ingredients were volume measures, Anne picked (1 cup of) milk first because it is liquid filling a cup: "I would have said milk is the volume because it is liquid and it actually fills it." She said strawberries would not fill because of "air space" between them, but flour would fill a container. Thus, Anne conceptualized volume as "anything that would easily fill it [a container]" for the reason that "it fills a space fully"; and she listed all the ingredients except strawberries and butter. In the recipe context, Cacie and Flora revealed that their thinking of volume was similar to Anne's. Thus, five participants (Anne, Becca, Cacie, Flora, and Grace) described volume as the amount of substance inside or filling a container. I classified these as a substance volume conception.

Substance capacity. Participants also described volume as the potential amount of substance (water, fish, air, decoration, etc.) needed to fill a container. For example, Eva said, "I would think about […] how much water that I could put inside of the fish tank to fill it up. Or, I mean, to not fill it up but how much, the capacity that it would hold." With prompting questions, she explained how the potential amount of water needed to fill a container is related to volume and to capacity:

Eva:	If you take the water out of the fish tank, it would still be the same quantity if it was in or not in, so the volume is the same whether the fish tank is related or just the volume is by itself. Is that what you're asking me? Sorry?
Eun Mi:	How do you think?
Eva:	Yeah, I mean that's how I feel like it's related. Because the volume ... the fish tank creates the shape or, I don't know how I want to say it, but ... the fish tank determines how much volume can be contained. But that volume can also be contained by another object that has the same capacity to hold water.

Eva thus recognized the relationships among the capacity of the tank, its (interior) volume, and the potential amount of liquid substance in it in terms of equivalent quantities.

All participants except Becca at some point described volume as the potential amount of substance—a substance capacity conception of volume. Because of its equivalence to capacity, this way of thinking of volume in terms of the maximal, potential amount of the substance to fill a container (as substance capacity) is distinguished from the volume conception of substance volume, as liquid capacity was distinguished from liquid volume.

In sum, the dominant conceptions of volume revealed in these everyday contexts were framed in terms of the substance in a container. All participants (even Becca, Cacie, and Della who revealed school volume conceptions in response to the first interview question) described volume in these ways. Conceptions involving substance fell into two categories: (a) substance volume (the amount of substance inside or filling a container, which involves liquid volume conception), and (b) substance capacity (the potential amount of substance needed to fill a container, which involves liquid capacity conception). Note that most participants tended to think of liquid (water or milk) first as the substance type when thinking of volume.

Four Additional Conceptions of Volume Revealed in Everyday Contexts

In addition to substance capacity and substance volume, four different ways of thinking of volume were revealed in participants' responses to the questions about the fish tank and the shortcake recipe (see Figure 4.2): (a) complementary volume; (b) enclosed liquid; (c) space occupied by liquid; and (d) calculated volume.

Complementary volume. In addition to describing volume related to material substance, Anne and Flora talked about their experiences of the water level in a fish tank rising when some objects (like fish, filter, or decorations) are submerged: "My brother has a fish tank and if it's full with this much water if you put stuff in it, the water level rises," (Anne). Thus, Anne and Flora conceptualized volume as complementary volume (the amount of water displaced when immersing an object in water; Piaget et al., 1960).

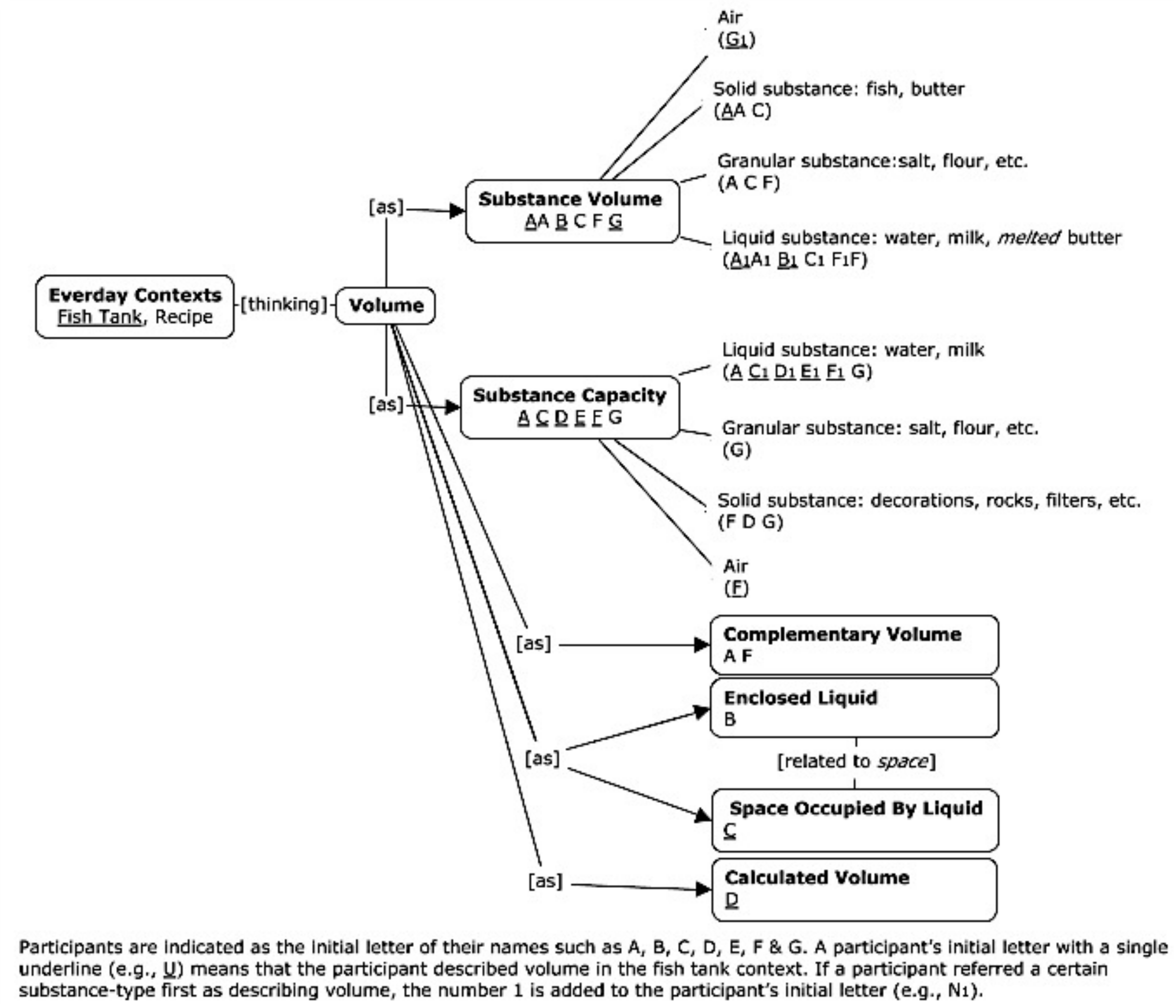

Figure 4.2. Participants' four additional conceptions of volume revealed in the everyday contexts[6]

[6]To draw the diagram of participants' four additional conceptions of volume revealed in the everyday contexts, I used *Institute for Human and Machine Cognition (IHMC)'s CmapTools knowledge modeling kit Version 6.01.01*.

Enclosed liquid and space occupied by liquid. Becca and Cacie, who initially talked about volume related to space in response to the first interview question, not only described volume as being related to substance, but also to space. In the context of a fish tank, Becca described volume as "how much water in it [fish tank]," but she described volume differently in the context of the shortcake recipe. When I presented the recipe and asked which ingredients were volume measures, she picked all of the capacity units because they measure the amount of material substance to put into a container: "I think the cups and teaspoons and tablespoons all are measuring volume because it's how much you're going to put into something, how much you're going to put in the cup." As explaining why she wouldn't think of pounds (of $1\frac{1}{2}$ pounds of strawberries) as volume measure, she said:

Becca: I guess for volume you don't necessarily have to put something in it. When I am thinking about it in terms of the recipe, I would think that the volume, you would be putting the milk into the measuring cup and then the milk would be the volume, taking up space in the cup.

Within the recipe context, thus, Becca tended to think of volume as the amount of liquid (milk) taking up the space inside a container.

In the fish tank context, Cacie described volume as "how much could fill the whole thing." With additional prompting questions, she said:

Cacie: The volume of it? Well, it'd be the amount of space that if you were to pour water in it, how much space that water is taking up, taking on like a 3-D rectangular shape. Yeah, how much it can hold.

Cacie thus appeared to think of volume as the amount of space taken by liquid in the container, as well as the potential amount of substance needed to fill a container (substance capacity).

In sum, both Becca and Cacie conceptualized volume as related to space, but in slightly different aspects; Becca attended to the amount of liquid taking up the space inside a container and Cacie to the amount of space taken by liquid in a container. I classified these as the volume conceptions of occupied space by liquid and enclosed liquid, respectively.

Calculated volume. In the context of a fish tank, Cacie and Della mentioned measuring the side lengths of the fish tank.

> Cacie: I would measure the sides, multiply it by this length here to get the area of this, and then multiply it by the height to see how much could fill the whole thing.

Here, Cacie described measuring the side lengths and multiplying them to determine volume; she perceived of volume as the potential amount of substance needed to fill the tank.

Della, who, in response to the first interview question, talked about volume being related to a formula, first described volume as the result obtained from the dimensions of the fish tank:

> Della: To measure the height, measure the base, measure the width.
>
> Eun Mi: What do you mean?
>
> Della: The dimensions of them. The height, the width, and the …. This is width, and the length and I can get the volume.

With additional prompting questions, Della described volume as the potential amount of substance (water and fish) to fill the tank; she also explained that the volume of the tank and the potential amount of water to fill it are the same: “Totally ignore the glass of the tank and then they’re exactly same.” In the context of the fish tank, thus, Della conceptualized volume as the result obtained from the calculation and the potential amount of substance; she seemed to recognize the difference between the volume of the fish tank and the potential amount of water in it.

In sum, other than the two dominant conceptions of volume (substance volume and substance capacity), participants described volume as: (a) complementary volume (the amount of water displaced by immersing an object in water, Piaget et al., 1960); (b) enclosed liquid (the amount of liquid to fill space inside a container); (c) space occupied by liquid (the amount of space taken by liquid in a container); and (d) calculated volume (the result obtained from multiplying the three length-measures of a container). Interestingly, the last three conceptions were exhibited by the three participants—Becca, Cacie, and Della—who revealed school volume conceptions in response to the first interview question. These conceptions share similar characteristics with the school volume conceptions in terms of enclosed space, occupied space, and formula, respectively.

In the everyday contexts, two dominant conceptions of volume were revealed—substance volume and substance capacity—along with four additional volume conceptions—complementary volume, enclosed liquid, space occupied by liquid, and calculated volume. Based on the two dominant conceptions, it appears that everyday contexts evoke thinking of volume as the amount of substance (mainly liquid) that fills a container. Participants ascribed to volume’s physical characteristics (especially, related to the ability of liquid to fill and the ability of the container to hold matter) in the sense of interior volume (Piaget et al., 1960).

Measuring Volume in Everyday Contexts

I examined participants' strategies for measuring volume in everyday contexts through their responses to the interview questions of how they would know or figure out the volume of the fish tank and the amount of the ingredients needed for the recipe. Tables 4.4 and 4.5 summarize participants' strategies for measuring the volume of the fish tank and of the recipe ingredients, respectively. Note that participants often described multiple strategies in any given context.

Across the contexts of the fish tank and the recipe ingredients, participants talked about different strategies to determine volume. I classified their responses as two strategies for measuring volume by (a) using containers and (b) using formulas. The first strategy was similar to the participants' experiences of measuring volume in everyday life situations related to liquid, such as pouring water into a coffee maker. In addition, when measuring butter, five participants talked about using the marks on the wrapping of butter for measuring.

Using Containers

When talking about the strategies for measuring the volume of the fish tank and/or of the recipe ingredients, all participants at some point referred to using a container as a measuring tool, such as an everyday container (a cup, a milk jug, or a water bottle), or a measuring container (a liquid/dry measuring cup, a tablespoon, or a teaspoon). Note that participants matched measurement tools with the units of ingredient measurements given in the recipe.

Everyday containers. In the context of the fish tank, Becca and Flora talked about their strategies for measuring volume as pouring water by using a water bottle or a cup, and accumulating quantities by counting: "I guess you could, the first time you fill it up, use a water bottle.… If it is 20 water bottles it would take to fill up the fish tank, then every time you know it's going to be that much," (Becca); "I took a cup and just kept going, and kind of did it like that. Usually the cups change, but for a cup that big [as indicating the height/size of a cup by using her two index fingers] it was maybe two and a half of those filled all the way up would fill it up," (Flora). The measurements of volume found by using everyday containers are the quantities of nonstandard capacity units.

Table 4.4.

Participants' Strategies for Measuring Volume in the Fish Tank Context

Participant	Fish Tank
Anne	Calculating with the formula, $l \times w \times h$ (to get a final answer) [aimed to measure substance capacity][a] Pouring a *specific* amount of water in a container [aimed to measure liquid capacity]
Eva	Putting a certain amount of water (*gallons* of water) into a container [aimed to measure liquid capacity][b] Filling up the space inside a container with salad, sand, or bounce ball (less accurate) [aimed to measure liquid capacity]
Grace	Calculating with the formula, $l \times w \times h$, to find out the volume [of a container; substance volume] Putting a certain amount of water (*gallons* of water) into a container [aimed to measure liquid volume]
Flora	[Pouring] a certain amount of water with a cup in to her own fish tank (*half* a *gallon*) [aimed to measure liquid capacity] After filled a container with water in the tank, [pouring] out water with a *gallon* milk jug [aimed to measure complementary volume]
Becca	Calculating with the formula, $l \times w \times h$ [aimed to measure liquid volume][a] Pouring a certain amount of water with a cup and water bottle into a container [aimed to measure liquid capacity]
Cacie	Calculating with the formula, $A \times h$ [aimed to measure substance capacity][a] Calculating with the formula, $l \times w \times h$ [aimed to measure space occupied by liquid]
Della	[Calculating with] the formula, $l \times w \times h$ [aimed to measure substance capacity *ideally*][c]

[a]Focusing on the shape of a container in terms of a rectangular prism, and not recognizes the gap between the quantity aiming to measure and the quantity to be measured. [b]Recognizing the relation between capacity and interior volume (the container's liquid capacity). [c]Recognizing the gap between exterior volume (the volume of a container) and interior volume (the container's liquid capacity) because of the thickness of the container.

Table 4.5.

Participants' Strategies for Measuring Volume in the Recipe Context

Participant	Strawberries	Granular	Milk	Butter
Anne	(Weighing strawberries with a scale) Converting pounds to cups	Filling flour, baking soda, salt [into a container] with three cups, a teaspoon, a tablespoon Filling a teaspoon with baking powder four times	Filling a cup	For a stick of butter, reading numbers on [the wrapping]
Eva	(Measuring strawberries with a scale)	Putting flour in a big [liquid] measuring cup up to 3-cups [scale] or 1-cup [dry] measuring cup three times Scooping a teaspoon of baking powder four times Using 5 tablespoons of sugar Using a conversion table for the relation among teaspoon, tablespoon, and cup	Pouring into a measuring-[cup]	For soft butter, putting it into a measuring cup or scooping it [by using the cup] For a stick of butter, reading measurements on [the wrapping]
Grace	(Weighing strawberries [with a scale])	Filling-up a cup with flour three times or a three-cup-sized cup with flour Filling-up a teaspoon with baking powder four times Filling-up sugar [to a tablespoon] five times	Filling-up a cup	Filling a [liquid] measuring-cup
Flora	(Using a scale) (Estimating) Converting pounds to cups	For flour, using one-cup [dry] measuring-cup or pouring it into a glass [liquid] measuring-cup For baking powder and baking soda, using a teaspoon measuring-cup four [times] and one [time] For salt, using a tablespoon and a quarter tablespoon, and do each of them For sugar, using a tablespoon and do five [times]	Filling a [dry] measuring-cup	For melted butter, pouring into a [dry] measuring - cup For butter sticks, reading numbers [on the wrapping] For the tub of butter, filling-up a [dry] measuring-cup

Table 4.5. (cont'd)

Participant	Strawberries	Granular	Milk	Butter
Becca	(Using a scale) Converting pounds to cups	Filling-up a dry measuring cup with flour three times Filling a teaspoon with baking powder four times; same for baking soda Measuring-out one tablespoon of sugar five times	Using a liquid/glass measuring-cup	Using a dry measuring-cup For a stick of butter, reading numbers on [the wrapping] (in tablespoons; 8 tbsp. = 1 cup)
Cacie	(Guessing [estimating] $1\frac{1}{2}$ pounds of Sb from 2 lbs. of Sb)	Measuring-out three cups of flour by using a cup Pouring a teaspoon of baking powder [into a container] four times Filling a tablespoon with sugar and measure out five of them	Filling-up a cup	For a stick of butter, reading label [numbers on the wrapping] in ounces and convert oz. to cups
Della	([Measuring by feeling])	Using a glass [liquid measuring] cup None for baking powder and sugar	Putting one cup of milk	None for butter

Note. Participants' strategies of figuring out the amount of strawberries needed for the recipe are added with round brackets, considering they are not volume measurement strategies.

Measuring containers. In the context of the recipe ingredients, all participants talked about using a measuring container (e.g., a measuring cup or tablespoon) to measure the volume of liquid and pourable substances (flour, baking powder/soda, salt/sugar, milk, and melted butter): "If you use a cup measurement and then fill it [flour] all the way and do it three times," (Becca); "The three-fourths cup of butter, if it's melted butter then I can pour it into a three-fourths cup," (Flora). Compared to using an everyday container, using a measuring container gives a standard measurement of volume (i.e., the quantities of standard capacity units).

In the fish tank context, three participants (Eva, Grace and Flora) talked of filling up the tank with gallons of water to measure volume, but without referring to the use of a measuring container: "You could put it [water] in, you could put one gallon at a time in. So, say you fill it up, you fill up 2 gallons and you put that in, and then it doesn't take another full gallon but maybe a half-gallon?" (Eva). Considering that a gallon is a standard capacity unit in the U.S., I classified these responses to the strategy for measuring volume by using a measuring container. This strategy also suggested that the participants conceived gallons as the units for measuring water; it may represent their contextual knowledge of measuring volume—choosing appropriate units for different substances or in a certain context.

Choosing different measurement tools by units. In the context of the shortcake recipe, participants matched measurement tools with measurement units listed in the recipe. They talked about using a scale to measure $1\frac{1}{2}$ pounds of strawberries and about using different measuring containers to measure the volume of (granular and liquid) ingredients in terms of capacity/volume units (e.g., using a measuring cup for 3 cups of flour or using a tablespoon for 4 tablespoon of baking powder). Participants seemed to understand that different units are used in measuring different attributes—pounds for weight and capacity units for volume. For example, Becca said, "I guess you could use a scale and measure it. Pounds is hard because it's not cups or they're not the same measurement as the other things [as pointing other measurements of the recipe ingredients]."

In sum, participants chose different containers for measuring volume by contexts—everyday containers in the fish tank context and measuring containers in the recipe context. Participants particularly chose a measuring container for measuring the volume of an ingredient, which corresponds with the capacity unit of the ingredient measurement. The participants who mentioned pouring gallons of water into the fish tank seemed to perceive the relationship between the unit and the nature of substance to be measured. Participants' strategies for measuring volume by using containers demonstrated their conceptions of measuring volume as the activity of filling a container with liquid/pourable substances in order to quantify its volume in relation to the capacity of the container. The measurements of volume determined by using containers represent the quantities of capacity units (i.e., nonstandard/idiosyncratic units from using everyday containers or standard units from using measuring containers).

Using Formulas

In the context of the fish tank, all participants (other than Della, who talked only of using the volume formula of length times width times height as her strategy of measuring volume) described more than one strategy for measuring its volume. Like Della, four participants (Anne, Becca, Cacie, and Grace) talked of using the volume formulas—length times width times height and/or base area times height—to find/compute volume measurements: "Since it's a rectangle, I could find the length and multiply it by the height and the width to figure out how much water would go in it [pointing each dimension]," (Becca). In considering Becca's response, the shape of the fish tank—a rectangular prism—seemed to trigger the use of the volume formulas for measuring volume. By thinking this way, there might be a gap between the measurement to be found/computed and the quantity aiming to be measured; namely the difference between the exterior volume of a container in cubic units (the resultant product obtained by multiplying the three exterior length-measures) and the interior volume of the container (as the potential amount of water to fill the container; usually represented as the quantity of capacity units). Like Becca, the other three other participants (Anne, Cacie, and Grace) did not think of the difference; only Della showed that distinction.

In sum, the participants' strategies for measuring volume by using formulas demonstrated their conceptions of measuring volume as a calculation. None of the four participants (except

Della) appeared to notice the potential gap between the quantity aiming to measure and the quantity to be measured.

Reading the Marks On the Wrapping of Butter for Measuring

In addition to the two strategies for measuring volume, when measuring $\frac{3}{4}$ cup of butter, five participants (Anne, Becca, Cacie, Eva and Flora) talked about reading the marks printed on the wrapping for measuring a stick of butter and cutting on the $\frac{3}{4}$ cup mark of: “Butter? I know. Um ... Butters, they usually, when it’s a stick of butter, it has the numbers on it so you could do three fourths of a cup. Just cut it,” (Anne). This strategy for measuring volume by reading the marks on the wrapping of butter for measuring is a context-specific strategy, because it only works for a stick of butter.

In sum, in the contexts of the fish tank and of the recipe ingredients, participants described three strategies for measuring volume: (a) using containers to determine volume measurements related to the capacities of the containers; (b) using formulas to compute volume measurements; and (c) reading the marks on the wrapping of butter for measuring its volume. Participants used different strategies for each context and they chose appropriate measurement tools and units for each context. In contrast to calculation, the strategies for measuring volume by using containers reflect perceptual, visual measurements (as the quantities of capacity units).

In response to the questions about the strategies for measuring volume in everyday contexts, participants conceived of measuring volume as (a) filling volume (measuring out the volume of liquid/pourable substances by using a container in terms of capacity units, Curry & Outhred, 2005), and (b) calculation (calculating volume with volume formulas). The concept of measuring volume as filling volume is dominant within these everyday contexts; it involves the pouring liquid and filling liquid conceptions that were observed from the previous interview questions that asked about participants' experiences of measuring volume in everyday life situations.

Chapter Summary

In everyday contexts, participants described volume as the amount of liquid that fills a container and conceptualized the measurement of volume as quantities measured in capacity units. They described strategies for measuring volume by using containers and conceptualized measuring volume as the activity of filling a container with liquid to quantify volume. This is evidence that within everyday contexts, participants conceptualize volume as *capacity*—the volume of matter filling a container as capacity (Sáiz, 2003)—and its measurement as *filling volume* (Curry & Outred, 2005).

CHAPTER FIVE RESULTS: UNDERSTANDINGS OF VOLUME AND ITS MEASUREMENT IN SCHOOL CONTEXTS

In this chapter, I examine how participants understand volume measurement in the context of school mathematics in terms of their volume conceptions and understandings of volume formulas. I present participants' volume conceptions and strategies for measuring volume in school contexts—a three-dimensional geometric object in the pictorial or physical context, such as two-dimensional representations of a cube, a rectangular prism built from unit cubes, and a stack of unit cubes; a solid wooden rectangular prism, a hollow clear plastic cube, and a hollow clear plastic cylinder (as geometric manipulatives; hereafter referred to as *a cube*, *a rectangular prism*, *stacked cubes, a wooden block, a hollow cube,* and *a hollow cylinder*, respectively).

Conceptions of Volume in School Contexts

In the second interview, I presented participants with each of the six geometrical objects in the pictorial and physical contexts, asking how they would think of the volume of the presented object. Tables 5.1 and 5.2 summarize participants' conceptions of volume revealed in their responses to questions about the six geometric objects. Across the school contexts, five different ways of thinking of volume were revealed in participants' responses to the questions about the presented geometric objects, such as thinking of volume related to (a) volume formulas (calculated volume); (b) space (enclosed and occupied space); (c) an object presented (a whole object); (d) unit cubes (the number of cubic units); and (e) substance (substance volume and capacity). Note that participants often described volume in multiple ways in any given context (see Tables 5.1 and 5.2). By the contexts, certain volume conceptions were more noticeable than others (see Figure 5.1).

Table 5.1.

Participants' Conceptions of Volume in the Pictorial Contexts

Participant	Cube	Rectangular prism	Stacked cubes
Anne	Calculated volume with $l \times w \times h$ ($A \times h$) Everything—all the space—*contained*	Number of cubes to fill *and* the amount of space filled in each cube (3-D object)	Number of cubes to fill *and* the amount of space filled in each cube
Eva	Everything (abstract) inside and the outside (like its overall mass) as the *whole* object	[Calculated] number of squares [cubes] by counting them for each side and with $l \times w \times h$ (or adding the number of cubes by the layers) 40 *cubes* (because "they [cubes] are 3-D").	Calculated number of cubes by counting cubes for each [layer] ($l \times w$) and adding them all
Grace	Calculated volume (three x [x^3]) with $l \times w \times h$ Amount of [abstract] substance inside	Number of squares [cubes] inside (which can be figured out by counting squares [cubes] for each side and with $l \times w \times h$)	Calculated number of cubes, 33 units^3 [by counting cubes for each side of each part and with $l \times w \times h$ and adding or subtracting them] Number of squares [cubes] in
Flora	Three lengths ("the leng … the height and then the width and the length") [calculated volume with $l \times w \times h$] All the space inside of the box	Three lengths ("to do length, width, and height") [calculated volume with $l \times w \times h$] Number of blocks [cubes] to fill up completely 40 units *squared* calculated with $l \times w \times h$.	Calculated number of cubes (of each part) with $l \times w \times h$ and adding them 33 units/*ounces* of water to fill as the potential amount of water to fill (when filled up with water)

Table 5.1. (cont'd)

Participant	Cube	Rectangular prism	Stacked cubes
Becca	Amount of space inside (cf. occupied space, "if I put a box on a counter")	Calculated number of cubes by multiplying the counted cube number of the front face to its depth (A×h) 40 [unit not mentioned] as the result of calculation and the number of cubes to make up	Same way [calculated volume with l×w×h] Calculated number of cubes by counting cubes for each [layer] (l×w) and adding them all 33 units *cubed* (because volume is 3-D)
Cacie	3-D space within Amount of space in	Calculated number of cubes by counting cubes for each side and with l×w×h (as a certain amount of units) Amount of space *contained* within	Calculated number of cubes by counting cubes for each side and with l×w×h
Della	Cubic units as the calculated volume with l×w×h in cubic units ($V=a^3\text{unit}^3$)	Calculated number of cubic units by counting cubes for each side and with l×w×h	Calculated number of cubes by counting [cubes for each side and with l×w×h]

Table 5.2.

Participants' Conceptions of Volume in the Physical Contexts

Participant	Wooden block	Hollow cube	Hollow cylinder
Anne	Whole wood Wood [materials] contained within/inside	The space within/*contained*	Amount of substance (air) to fill a *container*
Eva	Entire object taking a certain amount of space (Physical object has volume which can be calculated with $l \times w \times h$)	Potential amount of [abstract] substance to put/fit inside (because "it's hollow, so there's space to fill")[a] Capacity Calculated number of cubes to fit (when cubes are filled by filling them for the first layer and multiplied it by number of layers, $A \times h$)	Amount of space taken Potential amount of [abstract] substance to fill
Grace	Amount of wood in/made-of	Potential number of cubes to fit inside	Potential amount of substance (cubes, but "hard to use physical cubes") to fill inside Potential number of cubes (not physical) to fit inside (Calculated volume with [the volume formula for a cylinder,] $A \times h$)
Flora	Potential amount of substance (liquid or anything) to put inside	Potential amount of liquid to put/fit in (when filled up with liquid) Potential number of blocks [cubes] to fit in (no space)	Calculated volume with [the volume formula for a cylinder,] $A \times h$

Table 5.2. (cont'd)

Participant	Wooden block	Hollow cube	Hollow cylinder
Becca	Three lengths ("the height and the length and the width") [calculated volume with $l \times w \times h$]	Amount of [abstract] substance to fill Potential number of U-cubes to fit in (20 *square* one-inch-size blocks) [Potential] amount of substance (air) inside	Potential amount of substance [air] to fit into
Cacie	Amount of space taken by (3-D object having three lengths, "a width and a height and a length")	[Amount of] the space inside Amount of water to fit in (when water is filled) Number of cubes to fill (when cubes are filled)	Potential amount of space to fit inside [determined by formulas]
Della	Calculated volume with the formula, $l \times w \times h$ For the *whole* object	Units cubed as the calculated volume with $l \times w \times h$	Calculated volume with $\pi r^2 \times h$

[a]Distinction between exterior and interior volume: "Maybe it's because it's hollow, so there's space to fill, whereas this object [wooden block], I don't have any space to fill, so it's just the whole object is the volume" (Eva).

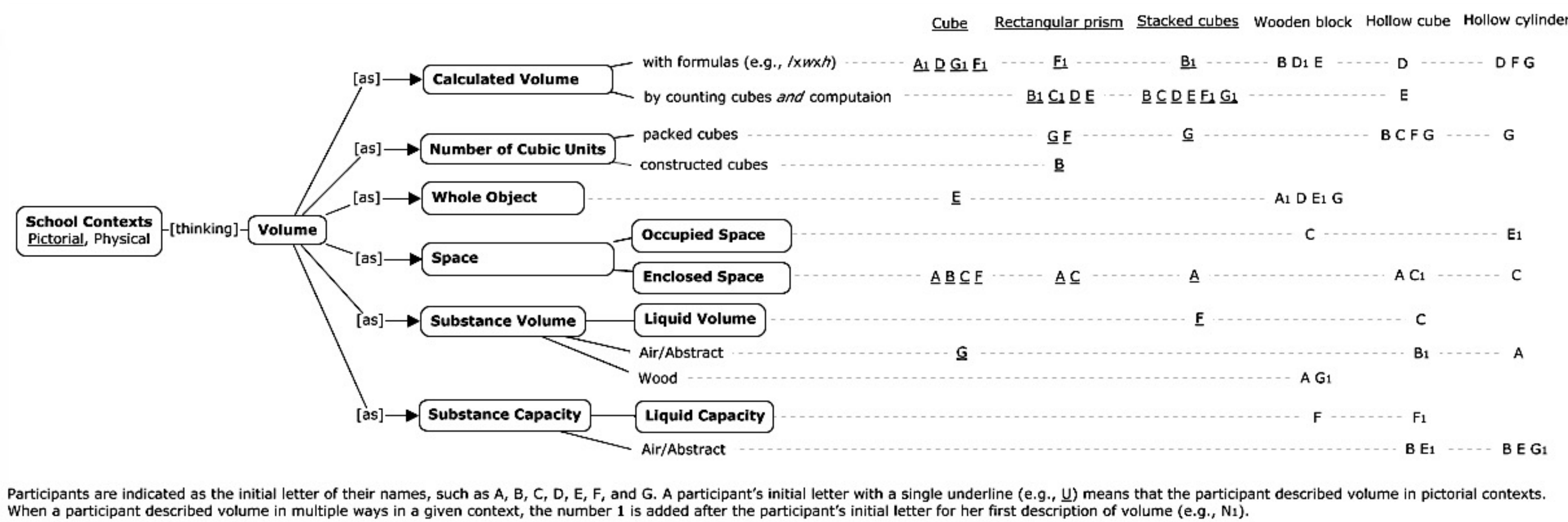

Figure 5.1. Participants' conceptions of volume revealed in the school contexts[7]

[7]To draw the diagram of participants' conceptions of volume revealed in the school contexts, I used *Institute for Human and Machine Cognition (IHMC)'s CmapTools knowledge modeling kit Version 6.01.01*.

Calculated Volume

When talking about the volume of the six geometric objects, participants attended to the dimensional aspects of the objects, for example, “a cube is a 3-D object,” (Anne) or “it’s a three-dimensional cube,” (Della). All participants at some point described volume as the result of a calculation using a volume formula—length times width times height or base area times height. For example, Grace said, “Well, all the sides would be equal. So, you just multiply this side by this side and this side and they would all be the same so. If one side was *x*, it would be three *x* [x^3], the volume of it.”

Especially in the pictorial contexts of the rectangular prism and stacked cubes, all participants except Anne described counting the squares or cubes of each dimension to know the length measures and multiplying them in terms of conceptualizing volume as a calculated result: “This one you could count the cubes, the individual cubes. There’s five across and four tall and two deep, so you could just multiply that way or count,” (Cacie).

These volume descriptions were similar to Della’s responses to the first interview question that asked her initial thoughts about volume and the question that asked how she would think about the volume of a fish tank that were classified as calculated volume. Thus, I also classified these responses as the volume conception of calculated volume; this conception emerged across the school contexts.

Enclosed Space and Occupied Space

Four participants—Anne, Becca, Cacie, and Flora—described volume as the amount of space inside an object: “the volume is everything contained within the cube like all the space,” (Anne); “All the space that’s inside of the box,” (Flora). In the context of the wooden block and/or the hollow cylinder, Cacie and Eva described volume as the amount of space taken by an object: “How much space it’s [hollow cylinder] taking up,” (Eva). As in the first interview, I classified these responses as enclosed space and occupied volume, respectively. The volume conception of enclosed space appeared in the contexts of the cube, rectangular prism, stacked cubes, hollow cube, and hollow cylinder, not for the wooden block.

Whole Object

For the cube and/or wooden block, four participants—Anne, Della, Eva, and Grace—described volume as a whole/entire object: “If I’m just thinking about volume as what volume is, I would say this entire object is a volume,” (for the wooden block, Eva); “The volume is kind of for the whole object,” (for the wooden block, Della). These volume descriptions were not observed in the previous interview questions that asked participants’ initial thoughts about volume and their thought of volume in the everyday contexts. In considering the contexts in which they were revealed (cube and/or wooden block), objects that are solid (or seem to be solid) and are not divided into individual unit cubes evoked thinking of volume as a whole object.

Number of Cubic Units

In addition to the dominant volume conception of calculated volume, four participants—Becca, Cacie, Flora, and Grace—described volume as the number of (unit) one-inch-size cubes that could be packed in or used for building/constructing an object: “Mm … how many cubes that can fit inside maybe in this box,” (for the hollow cube, Grace); “You could actually count all those and see how many cubes make up that. Then that could be the volume,” (for the stacked cubes, Becca).

These volume descriptions were not observed before in the previous interview questions that asked participants’ initial thoughts about volume and their thoughts of volume in the everyday contexts; they emerged in the contexts of the rectangular prism and stacked cubes, and in the context of the hollow cube and hollow cylinder. Thus, these contexts evoked thinking of volume as the number of cubes to be packed in or for building/constructing an object (number of cubic units). This volume conception is closely related to the view on packing aspect of volume—packing an object with cubic units (Curry & Outhred, 2005).

Substance Capacity and Substance Volume

In the context of the hollow cube or the hollow cylinder, six participants—Anne, Becca, Cacie, Eva, Grace, and Flora—described volume as the amount of substance filling an object or its potential amount needed to fill an object: "The volume of this [hollow cylinder], this is like a very similar container so it's all the air within there," (Anne); "How much it [hollow cylinder] can fill inside … maybe," (Grace). In these school contexts only two participants—Cacie and Flora—referred to liquid as the type of substance, whereas in the everyday contexts, all participants described volume related to liquid. As in the previous interview, these responses were classified as substance volume and substance capacity, respectively. In considering the contexts in which these volume descriptions appeared, it seems that the physical contexts evoke the thinking of volume as the amount of substance—either substance volume or substance capacity.

In sum, across the school contexts, all participants described the volume of geometric objects in terms of calculated volume (the products computed with volume formulas). They also described volume as (a) the number of cubic units (the number of unit cubes that could be packed in or used for building/constructing an object); (b) an enclosed space (the amount of space inside an object, Sáiz, 2003) and occupied space (the amount of space taken by an object, Piaget et al. 1960); (c) a whole object (an entire three-dimensional object, indicating its whole size); and (d) the substance capacity and substance volume (the potential amount of substance needed to fill an object and the amount of substance filling an object, respectively). By the contexts, certain volume conceptions were more dominant than others.

In the school contexts, participants conceptualized volume predominantly as calculated volume related to its quantification by calculation. According to the perceived contextual information, participants presented different volume conceptions regarding cubic units, space, solids, and substances. This reflects that participants ascribed to volume the numerical and geometric characteristics (especially, related to the dimensions of rectangular prism-shaped objects and the number of cubic units of space taken up by the objects) in the contexts.

Measuring Volume in School Contexts

I examined participants' strategies for measuring volume in school contexts and their understandings of volume formulas through: (a) their responses to the question of how they would know or figure out the volume of the cube, rectangular prism, and stacked cubes; (b) their approaches to measuring the volume of the wooden block, hollow cube, and hollow cylinder; and (c) their explanations of the meaning of the calculated result of a volume formula. Table 5.3 summarizes participants' strategies for measuring volume in the pictorial contexts; Tables 5.4, 5.5, and 5.6 summarize participants' strategies for measuring volume in the contexts of the wooden block, hollow cube, and hollow cylinder, respectively. Note that participants often described multiple strategies in any given context.

Pictorial Contexts

Cube. In the context of the cube, all participants talked of using the volume formula of length times width times height ($l \times w \times h$) to find/compute volume measurements: "Then if I wanted to calculate it [the amount of space inside the cube], I would do just the standard calculation, the length times the width times the height," (Becca). In addition to calculating

Table 5.3.

Participants' Strategies for Measuring Volume in the Pictorial Contexts

Participant	Cube	Rectangular prism	Stacked cubes
Anne	("It's a cube so",) cubes one side length (l×w×h and A×h)	("If I knew the volume of one cube") its volume times the number of cubes to fill	Figures out the volume of one cube and then multiplies it by the number of cubes
Eva	[Calculates volume with] l×w×h	Calculates the number of squares [cubes] by counting them for each side and multiplying them (l×w×h) or adding the number of cubes by 20 [vertical layer] ("20 and then 20 because it's 4 times 5 times 2")[a]	Calculates the number of cubes ("33 cubes") by moving two rows of three cubes to make/visualize two 4×3×1 and one 3×3×1 rectangular prism shapes [horizontal layers];[a] counting cubes for each part [layer] with l×w; adding them ("9 and 12 and 12")
Grace	Multiplies three side [lengths] Calculates volume with l×w×h	Figures out the number of squares [cubes] inside (in "units^3") by counting squares [cubes] for each side and multiplying them (l×w×h)	Figures out the number of squares [cubes] in ("33 units^3") by sectioning the stacked cubes into two rectangular prism shapes; counting squares [cubes] for three dimensions of each part and multiplying them (l×w×h; "5×3×2," "2×3×1"); adding them and then subtracting the missing cubes of "3×1×1" from the addition ("30 units^3+6 units^3-3units^3")
Flora	Measures the three lengths and multiplies them (l×w×h)	Calculates the number of blocks [cubes] (in "units squared") by counting cubes for each side (in the unit of "units") and multiplying them (l×w×h) or A×h ("five rows of four, and then you have that two times")[a]	Calculates the number of cubes ("33 units^2") by sectioning the stacked cubes into four rectangular prism shapes; counting squares [cubes] for three dimensions of each part and multiplying them (l×w×h; "(2)(2)(3)," "(1)(1)(3)," "(2)(2)(3)," "(2)(1)(3)"); adding them

Table 5.3. (cont'd)

Participant	Cube	Rectangular prism	Stacked cubes
Becca	Does the standard calculation, $l \times w \times h$ [Calculates volume with] $l \times w \times h$, if having the [length] measurements (related with the 3-D [aspect] of a cube/box)	Figures out the number of cubes by counting all the cubes of the front [vertical layer] and multiplying the number to its depth (20×2)[a] Calculates the number of cubes [unit not mentioned] by counting cubes for each side and doing $l \times w \times h$ [multiplying them]	Calculates the number of cubes ("33 unit cubed") by sectioning the stacked cubes into 5 rectangular prism shapes; counting cubes for each part with $l \times w$; adding them (as skip counting as 6, 12, 18, 24 and adding 9)[a]
Cacie	[Calculates volume with] $l \times w \times h$ Fills with substances (something liquid) to measure it if the cube is a real object	Counts the individual cubes Calculates the number of cubes (in "unit cubed") by counting cubes for each side and multiplying them ($l \times w \times h$) or $A \times h$ ("five times four, so 20 times two because it's two deep.")[a]	Calculates the number of cubes ("33 units cubed") by moving/visualizing the row of three cubes to make a rectangular prism of 5×2×3; counting cubes for each side and multiplying them ($l \times w \times h$); adding the product (30) to the thee left cubes ("3")
Della	Cubes one side length ($l \times w \times h$, a^3 in cubic units)	Calculates the number of cubic units (40 "unit cubed") by counting cubes for each side and multiplying them ($l \times w \times h$)	Calculates the number of cubes by moving/ visualizing three cubes to make a rectangular prism of 6[5]×3×2; counting cubes for each side and multiplying them ($l \times w \times h$); adding the produce to the three left cubes ("3×1×1")

[a]Layer as the maximal composite unit for volume and its spatial structuring (Battista, 2003)

Table 5.4.

Participants' Strategies for Measuring Volume in the Wooden Block Context

Participant	Measuring	Calculation	Meaning
Anne	Measured side lengths with a ruler in inches and substituted the measures (of decimal fraction) in $l \times w \times h$	3.1 in × 1.5 in × 1.5 in = 6.975 in^3	Resulting quantity of triple multiplication in cubic units
Eva	Measured side lengths with a ruler in inches and substituted the measures (of dyadic fraction) in $l \times w \times h$	$1\frac{1}{2} \cdot 1\frac{1}{2} \cdot 3 = [1.5 \cdot 1.5 \cdot 3 =]\ 6.75 in^{2}$[a]	Don't know for meaning of 6.75
	[With prompting question for alternative way] measured side lengths with (one-cm-size) cubes and substituted the measures in $l \times w \times h$	$4 \cdot 4 \cdot 8 = 128\ cubes$	Number of cubes to make the large object [wooden block] [After building one 4-by-8-by-1 rectangular prism shape next to the wooden block,] 4 rows/layers of 32 is same volume of the wooden block
Grace	Measured side lengths in centimeters and substituted the measures in $l \times w \times h$	3.8 cm × 3.9 cm × 8 cm = 118.56 cm^3	The 1 centimeter cubed Number of [one-cm-size] cubes inside, 118.56 cubes
Flora	Measured side lengths in centimeters and substituted the measures in $l \times w \times h$	8 cm × 4 cm × 4 cm = 128 cm^3	Potential number of cubic centimeters to put in If 128 of cubes are stacked next to the shape [wooden block], they are same [volume]
Becca	Measured side lengths with a ruler in inches and substituted the measures (of decimal fraction) in $l \times w \times h$	1.5 × 1.5 × 3.2 V = 7.2 in^3	7.2 inches of wood make up the wooden block

Table 5.4. (cont'd)

Participant	Measuring	Calculation	Meaning
Cacie	Measured side lengths in centimeters and substituted the measures in $l \times w \times h$ ("base times height times length") [also $A \times h$]	("4 times 8, and then times 4 again.") $16 \times 8 = 128$ cm^3	128 square [cubic] centimeters to fit within wooden block If stacking 128 [cubes] in the same shape as wooden block, "16 on the bottom, 8 tall," get the wooden block
Della	Measured side lengths in centimeters and substituted the measures in $l \times w \times h$	V = 8 cm × 4 cm × 4 cm = 32 × 4 = 128 cm^3	Volume; cm^3 is the unit of volume because the object is 3-D and it is obtained by centimeter times centimeter times centimeter

Note. Calculation was taken from each participant's written note.
[a]Confusion between volume/cubic unit and area/square units

Table 5.5.

Participants' Strategies for Measuring Volume in the Hollow Cube Context

Participant	Measuring	Calculation	Estimating	Checking
Anne	Measured side lengths with a ruler in inches, and substituted the measures (of decimal fraction) in $l \times w \times h$	4 in × 4 in × 4 in = 64 in^3	1000 *cubes* by putting 10 [one-cm-size] cubes inside only for one side (10th one was not aligned with others); multiplied 10 by 10 by 10	Put 10 [one-cm-size] cubes for another side [height] to get length measure in [one-cm-size] cubes; assumed/ visualized 10 for all the three dimensions
Eva	Put one-cm-size cubes for two dimensions of the base, but stopped because the last cube didn't fit in for both Measured side lengths with a ruler in inches and substituted the measures in $l \times w \times h$	$4 \cdot 4 \cdot 4 = 64$	64 by putting one-inch-size cubes inside for one side, but the last cube didn't fit; and then by measuring two outside lengths with (aligning 4 one-inch-size cubes along the *outside* of the hollow cube and put 3 more for its second side [height]; and ended to 64 *cubes* (number of cube to fit or same volume with the hollow cube)	Aligned one-inch-size cubes along the *outside* of the hollow cube in terms of the three dimensions

Table 5.5. (cont'd)

Participant	Measuring	Calculation	Estimating	Checking
Grace	Put one-cm-size cubes for three dimensions as 10 by 10 by 10 (the last cubes for base were aligned off because they didn't fit in); so visualized it as 10 by 10 by 10 and substituted 10s in $l \times w \times h$ [After done measuring with one-cm-size cubes, with prompting question for alternative way] filled up one-inch-size cubes as 3×3×3 (because the last cube didn't fit in for three dimensions); visualized it as 4×4×4	$10 \times 10 \times 10 = 1000\ cm^3$ ("There's 1000 of these cubes inside") $4 \times 4 \times 4$ 16×4 4 groups of 16 $= 64\ in^3$	1000 cubes 64	Computed the actual number of cubes for the base with $l \times h$ ("a hundred") and explained the multiplication of the product ("10×10") with the another length ("10") as "10 groups of a hundred" "4 groups of 16"
Flora	Put one-inch-size cubes for one side; stopped because the last cube didn't fit in for the side and decided putting water to fill instead Filled the hollow cube up with water by using a liquid measuring cup three times [With prompting question asking to use formula] measured side lengths with a ruler in inches and substituted the measures (of dyadic fraction) in $l \times w \times h$	$v =$ 3 cups + 1 cup $+\frac{3}{8} = 4\frac{3}{8}$ cup $v = (4\text{ in})(4\frac{2}{16}\text{ in})(4\frac{2}{16}\text{ in})$	64 by filling one-inch-size cubes as 3×3×3; visualized it as 4×4×4 and computed as $v = (4)(4)(4) = 64$ blocks^3 (as saying "the $\frac{2}{16}$ came from this little plastic piece here")	Use 5 one-inch-size cubes to measure two lengths; use 10 one-inch-size cubes for three dimensions in explaining the area of 4×4 and adding them up 4 times related to $(l \times w) \times h$, so $A \times h$

Table 5.5. (cont'd)

Participant	Measuring	Calculation	Estimating	Checking
Becca	Measured side lengths with a ruler in inches and substituted the measures in $l \times w \times h$	$l = 4$ in $w = 4$ in $h = 4$ in $v = 64$ in^3	"64 one-inch[-size] blocks"	Put one-inch-size cubes inside for one side first, but stopped because the 4th cube couldn't fit in; saying if put 64 cubes next to the hollow cube, they look same but "can't put them inside"
Cacie	Thought of using blocks or water ("faster and easier") Measured side lengths with a ruler in centimeters and substituted the measures in $l \times w \times h$ [With prompting question asking to use water] poured water to fill with a half-cup-size measuring cup eight times and then stopped although the hollow cube was not filled up ("It's like a fish tank.")	$10 \times 10 \times 10 = 1000$ cm^3 "Little more than 4 cups" (saying "they take up the same amount of space".)	1000 for one-cm-size cubes	Described the process of putting 10 cubes for two dimensions [length and height], computing the number of cubes for the whole front face with $l \times h$ ("10 times 10") and visualizing to fill the product all the way back to 10
Della	Measured side lengths with a ruler in inches, and substitute measures (of decimal fraction) in $l \times w \times h$	V = 64 in^3 [$= 4 \times 4 \times 4$ $= 4^3$ in^3]	64	Described the process of putting one-inch-size cubes for three dimensions to find the number of cubes to fill

Note. Calculation was taken from each participant's written note.

Table 5.6.

Participants' Strategies for Measuring Volume in the Hollow Cylinder Context

Participant	Measuring	Calculation	Reflection on formulas
Anne	Referred to using the volume formula of $\pi r^2 \times h$	[Got πr^2 and multiplied it by the height]	Similarity in computing procedure of determining the area of the center shape [base] first then multiplying it by the height
Eva	Referred to calculation, but not specified the process for calculating Filled up the hollow cylinder with water first and poured water to a one-cup-size measuring cup three times and to a third-cup-size dry measuring cup one time	$3\frac{1}{3}$ cups	Difference because of the base (circle)
Grace	Referred to using the volume formula for a cylinder, but not remembered the exact formula of $\pi r^2 \times h$ Poured water with a one-cup-size measuring cup three times and with a one-fourth-cup-size dry measuring cup two times	$3\frac{1}{2}$ cups	Same in figuring out the [area of] shape [base] and then multiplying it by the height
Flora	Referred to compute the area for a circle and multiplied the product by its height, but not remembered the area formula for a circle Because not knowing the exact formula, filled the hollow cylinder with water by using a liquid measuring cup	28 oz.	Difference because of the base (circle)

Table 5.6. (cont'd)

Participant	Measuring	Calculation	Reflection on formulas
Becca	Couldn't use the formula of $l \times w \times h$ because of the base (circle) [and not specified the volume formula for a cylinder as $\pi r^2 \times h$] Couldn't put [unit cubes] because they are not "aligned" Described pouring water into the hollow cylinder by one cup until filled	[Said "8 cups" as an example volume measure][a]	Difference because of the base (circle)
Cacie	Measured the radius and the height in centimeters and substituted the measures in $\pi r^2 \times h$	$25 \times 10 = 250 \times \pi$ cm^3	Similarity in accounting for three measures to make a 3-D shape
Della	Measured the radius and the height in inches and substituted the measures in $\pi r^2 \times h$	V= $2^2\, \pi \cdot (3.7)$ $= 14.8\, \pi$ in^3	Similarity in computing procedure of determining [the area of] the base first then multiplying it by the height

Note. Calculation was taken from each participant's written note.
[a]As she described the strategy of pouring water, Becca said: "Then, say this uses 8 cups. That's how much the volume is, how much it holds."

volume with $l \times w \times h$, Cacie talked of filling the cube with liquid: "If it was a real object, we could fill it with something or a liquid or something to measure that." Thus, the cube context evoked thinking of measuring volume as a standard calculation with $l \times w \times h$.

Rectangular prism. For the rectangular prism, all participants except Anne also described using the volume formulas—$l \times w \times h$ and/or base area times height ($A \times h$)—to figure out its volume. In contrast to the cube context, before using $l \times w \times h$, participants talked of the number of cubes/squares for the three dimensions in terms of the length measures: "This one you could count the cubes, the individual cubes. There's five across and four tall and two deep, so you could just multiply that way or count," (for $l \times w \times h$, Cacie). Four participants (Becca, Cacie, Eva, and Flora, who described doing $A \times h$ as well as $l \times w \times h$), counted or calculated cubes/squares on the front of the building and multiplied it by its depth: "You count all these [cubes on the front of the building], and you multiply by two, then to figure out how many cubes are in there," (for $A \times h$, Becca). Thus, these four participants visualized the spatial structure of the rectangular prism in terms of vertical layers (2 layers of 20 cubes).

Stacked cubes. In the context of the stacked cubes (not a rectangular prism shape), all participants except Anne carried out additional steps to compose or decompose the stacked cubes before counting the number of cubes for each dimension and multiplying them. The activity of rearranging cubes showed participants' visualization of a spatial structure of unit cubes.

To compose/visualize the stacked cubes into rectangular prism shapes, Cacie, Della, and Eva described moving cubes around: "I would probably move these squares up here and these squares over one. Then I could say ... I could just count it," (Eva, see Figure 5.2). As she mentally moved unit cubes, Eva inferred the hidden cubes of the object properly: "I know that it's 3 deep, and I know that I have 1, 2, 3, 4 [pointing each cube at the bottom of the stacked

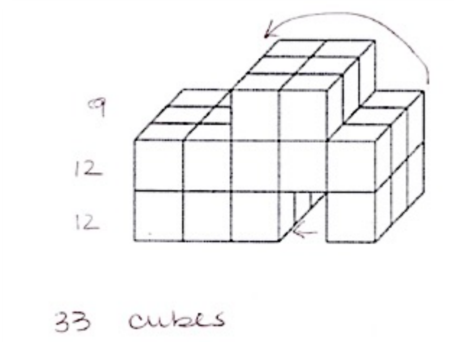

Figure 5.2. Eva’s note

cubes with her pen]. Then I have 12 there, and then I could say that’s [the stack of cubes in the middle] the same as this one.” Cacie and Della similarly showed awareness of the spatial structure of unit cubes. Moving cubes to visualize rectangular prism shapes reflects the conception of conservation (an object’s size does not change when it is moved or changes position, Stephan & Clements, 2003). For example, Eva said, “Because no matter how you move these around, they’re still going to be the same volume.” Thus, this strategy of measuring volume represents the three participants’ understandings of volume conservation.

Instead of mentally moving cubes, Becca, Flora, and Grace talked of decomposing the given stacked cubes into smaller rectangular prism shapes: “Maybe do this block here and kind of break it off there so … You have for “Block 1” if you called it that, it would be two times two times three,” (Flora, see Figure 5.3).

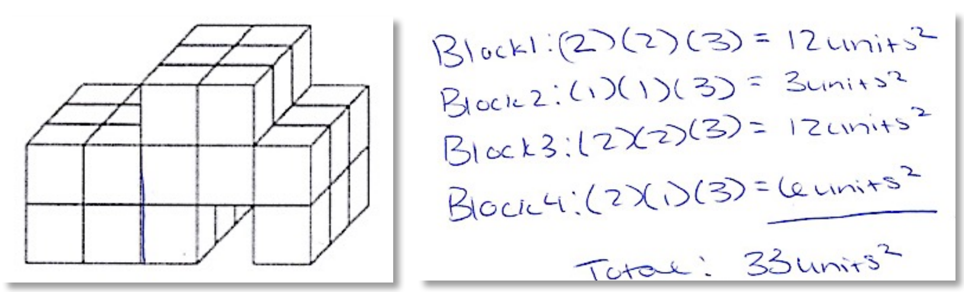

Figure 5.3. Flora’s note

All three participants (Becca, Grace, and Flora) properly visualized the hidden part of each section in terms of the spatial structure of unit cubes: "This is 2. It's 3 deep, so that would be 6," (for the first left column of the stacked cubes, Becca). The conception of accumulation (the result of parted units that indicates the whole size, Stephan & Clements, 2003) may underlie these participants' visualizations of the given stacked cubes made up of smaller rectangular prism shapes. Thus, the activity of rearranging—composing or decomposing—the stacked cubes showed the participants' visualization of a spatial structure of cubes on the invisible parts of the given object and their understandings of measurement properties, like conservation or accumulation.

In contrast to other participants, for the both contexts of the rectangular prism and stacked cubes, Anne said she would figure out the volume of one cube and then multiply it by the number of cubes to fill:

> Anne: I would figure out the volume of one cube again and then just multiply it by how many cubes there are. Because there is space missing right there so I wouldn't do a general multiplication. I would make sure I know how many cubes there are.

She thus described measuring volume as counting all the cubes composing the object. Without additional prompting, Anne did not show an awareness of the spatial structuring of unit cubes in the pictorial contexts of the rectangular prism and stacked cubes.

In sum, across the pictorial contexts (presenting a two-dimensional representation of a three-dimensional object), using $l \times w \times h$ was the predominant strategy for all participants to figure out volume. In contrast to using $l \times w \times h$ in the cube context, participants' strategies for measuring volume of the rectangular prism and stacked cubes were more complex: (a) figuring out the volume of an object related to the volume of a unit cube (multiplying the volume of one unit cube by the number of the cubes comprising the objects, the rectangular prism and stacked cubes); (b) thinking of ways to measure the lengths (counting cubes for each dimension of the rectangular prism); or (c) mentally rearranging cubes to visualize the rectangular prism shapes (composing or decomposing of the stacked cubes). The strategy of mentally rearranging cubes is consistent with Vasilyeva et al.'s (2013) view on an appropriate strategy for determining the volume of an irregular-shaped object, which is not available to use $l \times w \times h$ as a whole, that "one could try to break it into smaller components and estimate the volume of each component by mentally filling the small spaces with unit cubes" (p. 32). Thus, participants' strategies varied by context characteristics—presence of unit cubes (comprising objects) and object shape.

Physical Contexts

Participants were presented with a wooden block, a hollow cube, and a hollow cylinder, along with several measurement tools (the same set of measurement tools were used in the first interview), and for each, were asked to measure volume. When participants used volume formulas to determine volume, I asked questions about the volume formulas: (a) the meaning of calculated volume in the wooden block; (b) the number of unit cubes needed to fill in the hollow cube; and (c) the relationship between the volume formula for a rectangular prism ($l \times w \times h$) and the volume formula for a cylinder ($\pi r^2 \times h$) in the hollow cylinder.

Wooden block. All participant measured the side lengths with a ruler, using the resulting measurements in the volume formula $l \times w \times h$. For instance, Cacie used the centimeter side on the ruler: "I would probably take a ruler and measure each side. This side is about 4 times … It's about 8. 4 times 8, and then times 4 again." (See Figure 5.4.)

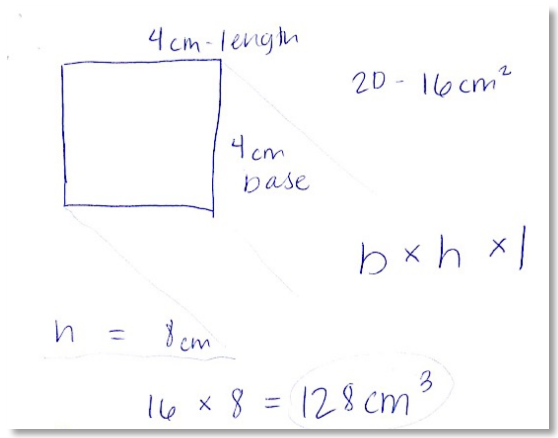

Figure 5.4. Cacie's note

Three participants—Anne, Becca, and Eva—chose the inch side on the ruler and others used its centimeter side. As the reason to choose one unit over the other, participants explained their unit choice in terms of (a) familiarity with unit system (e.g., Eva said, "I'm used to using inches over centimeters."); (b) efficiency related to the size of an object to be measured and the accuracy of measurement (e.g., Flora said, "I would probably get a bit more accurate if the units were smaller so I used the centimeters."); and (c) complication to determine measurements (e.g., Cacie said, "Centimeters didn't have to deal with fractions of inches."). Participants' decisions about unit reflect their understandings of the relationship unit choice for measuring and resulting measurements, as well as their perception about given context (object size or ruler scale). (See Table 5.7.)

In addition to using a ruler, Eva also used one-cm-size cubes to measure lengths of the wooden block and substituted the measures in the volume formula $l{\times}w{\times}h$ (see Figures 5.5 and 5.6.):

Eva: Yeah, I want to try these cubes [one-cm-size cubes]. This one's too big to do [one-inch-size cubes]. … [She puts eight cubes along one side of the wooden block.] It's like these were made for this [as putting four cubes along another side of the wooden block]. I know my length and my width are going to be the same. I mean my ... yeah, the length and width, right? Yes. That's going to be 4 times 4 times. 2, 3, 4, 5, 6, 7, 8. 4 times 4 is 16, but I don't know 16 times 8. [She uses a calculator] 128?

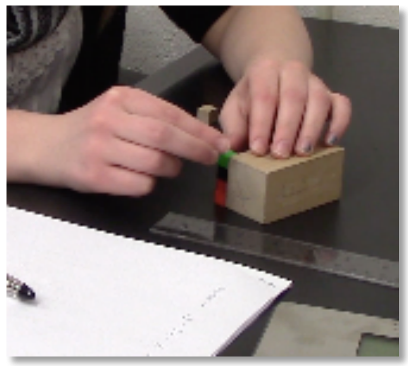

Figure 5.5. Eva's activity to measure one side of a wooden block with one-cm-size cubes

Table 5.7.

Participants' Unit Choice for Measuring Side Lengths with a Ruler

Participant	Unit	Reason
Anne	Inch	Wooden block is long enough to measure in inches and centimeter has bigger number [of units]
Eva	Inch	Being used to using inches over centimeters
Grace	Centimeter	"The size [of an object] is smaller, so I use the smaller units"; centimeter is more accurate and easier (because of decimal) than inch (because of fraction)
Flora	Centimeter	For the wooden block, it "get a bit more accurate if the units were smaller, so I used the centimeters"
Becca	Inch	Inch (customary units) is more comfortable than centimeter (a metric unit) For the wooden block, this object is big enough to use inch; for a smaller object, use centimeters
Cacie	Centimeter	For the wooden block, centimeter is an easier and more efficient (because of whole number [measure]) than inch (because of fraction)
Della	Centimeter	Chooses to use centimeter: More familiar with centimeters than inches

Figure 5.6. Eva's note

Eva thus used the cubes as a length measurement tool; her strategy of measuring volume is using the formula $l \times w \times h$.

All participants obtained a number representing the volume of the wooden block through the calculation $l \times w \times h$. I asked each what that number means. Anne and Della explained the obtained number as the quantity/product in cubic units resulting from the triple multiplication of length measures: "Since I measured in inches, volume is always the measurement, like the unit ends up being cubed because you multiplied the three lengths," (Anne). Cacie, Flora, and Grace explained the obtained number as the number of cubic units filling the object or composing an object with the same shape: "128, it means that that's the number of cubic centimeters that can be put in here," (Flora); "those many cubes in the inside of it," (Grace; number of cubes); "If you were to stack 128 in the same shape as this, so 16 on the bottom, 8 tall, then you'd get this shape," (Cacie). Eva did not provide any meaningful explanations for 6.75 (the product of computation with ruler measures), but she explained 128 (the product of computation with unit cube measures) as the number of cubes comprising the object. Becca explained 7.2 as the amount of material substance comprising the object: "7.2 inches of wood make up this block."

In sum, in the wooden block context, all participants used a ruler to measure length measures (Eva used both a ruler and one-cm-size cubes in measuring the three dimensions of the wooden block), using the resulting measures in the volume formula $l \times w \times h$. Participants chose to use inches or centimeters based on their perception of contexts (object size to be measured or scales of units to be used in measuring). Relative to the conceptions of scale, Lehrer, Jaslow, and Curtis (2003) stated that "The choice of units in relation to the object determines the relative precision of the measure" (p. 102).

For the meaning of the calculated volume, three different explanations were given: (a) the product resulting from triple multiplication (Anne and Della); (b) the number of cubic units filling and comprising an object (Cacie, Eva, Flora, and Grace); and (c) the amount of substance comprising an object (Becca).

Hollow cube. Five participants—Anne, Becca, Cacie, Della, and Eva—measured side lengths with a ruler and substituted the measures in the formula $l \times w \times h$: "It ended up being 64 inches cubed. It is just 4 times 4 times 4," (Becca, see Figure 5.7).

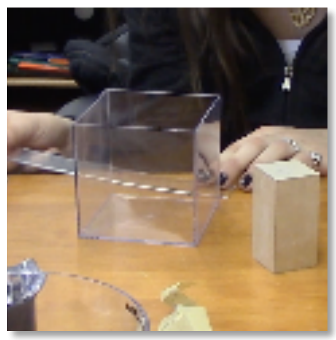

Figure 5.7. Becca's activity to measure one side of a hollow cube with a ruler

As a way of measuring volume, four participants—Cacie, Eva, Flora, and Grace—thought of using cubes/blocks. Whereas Cacie changed her mind (using cubes as opposed to using a ruler), Eva, Flora, and Grace started to put cubes along one or two dimensions of the hollow cube (for Eva, see Figure 5.8).

Eva: I feel like this ... I could use the ruler, but I like the cubes better because I feel like my answer could be better. I'm not going to fill the whole thing. I'm just going to do it the same way I did that one [wooden block] at the beginning and find the width [as aligning one-cm-size cubes along one edge of the hollow cube]. No, no [because she couldn't put in the last cube].

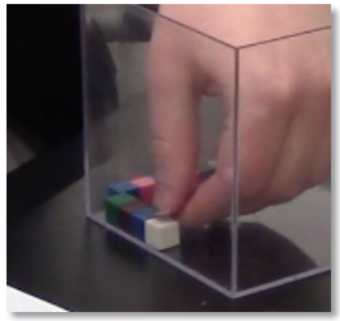

Figure 5.8. Eva's activity putting one-cm-size cubes along inside lengths

When the last whole size cube (i.e., the tenth one-cm-size cube) did not fit for two dimensions of the base, Eva and Flora stopped using cubes and decided to use another way of measuring—using a ruler (Eva), or pouring water with a measuring cup (Flora). In contrast to Eva and Flora, Grace put the tenth one-cm-size cube alongside the ninth one-cm-size cube in the row (see Figure 5.9).

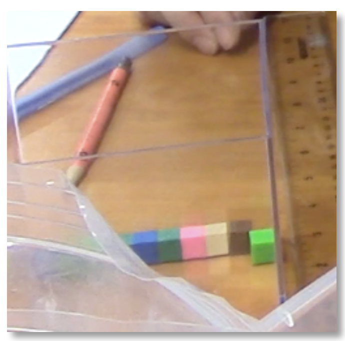

Figure 5.9. Grace's activity putting the tenth cube alongside the ninth cube

She did this for the two dimensions of the base (in the form of a 10-by-10 stacked cubes) and then stacked ten cubes for the height (see Figures 5.10 and 5.11).

Grace: There's 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 [for one side of the base]. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 … 4, 5, 6 … [for another side of the base and the height of the hollow cube]. I'm pretty sure this could fit but never mind ... anyways. So, then … like I explained there's like 10 here and then 10 here and 10 here. So you just do 10 times 10 times 10.

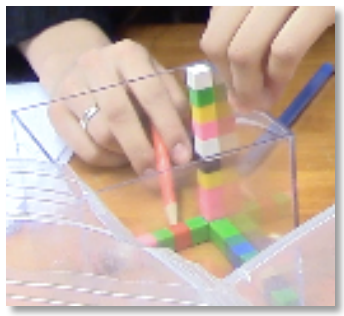

Figure 5.10. Grace's activity putting one-cm-size cubes along inside lengths

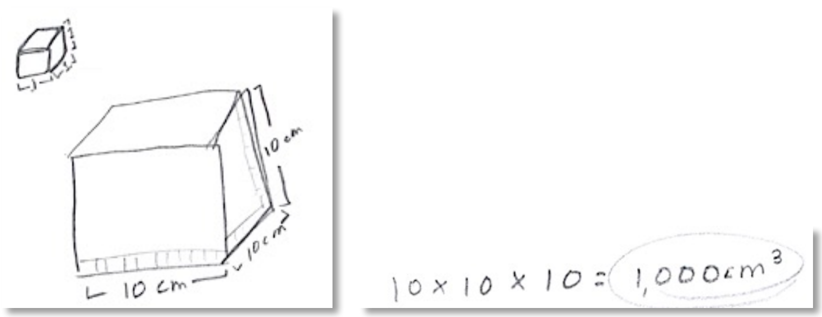

Figure 5.11. Grace's note

Grace thus visualized the stacked cubes in terms of a 10 by 10 by 10 (dimensional) structure and computed its volume using $l \times w \times h$; she did not attend explicitly to the difference between outside and inside length measures.

Because four one-inch-size cubes did not fit in the hollow cube, Flora decided to fill the container with water: "They're [one-inch-size cubes] not going to fit. No. All right. So, since that won't fit we can probably put water in it to see how much goes in there." She poured three cups of water by using a liquid measuring cup first, then added one more cup, recording how much of water she poured each time (3 cups and 1 cup). (See Figure 5.12.)

Figure 5.12. Flora's activity filling a hollow cube with water by using a measuring cup

Because the container was not completely filled using the four cups of water, she filled water to the half-cup line of the measuring cup and poured it into the container.

Flora: Is that half? That's [hollow cube] the top. Though, it looks like we got a little less than a fourth of a cup. So, if we rounded it, we could say four and half cups, but I think that might be like one eighth of a cup probably. So, this is … What did I do? I did a half of a cup. You say one half minus the one eighth that's in there [liquid measuring cup].

She thus determined the volume of the hollow cube as the total amount of water poured in the object/container, subtracting the amount of water left in the cup from the one half cup of water ($\frac{1}{2}$ - $\frac{1}{8}$) and then adding the product to the amount of water poured before (3 cups + 1 cup, see Figure 5.13).

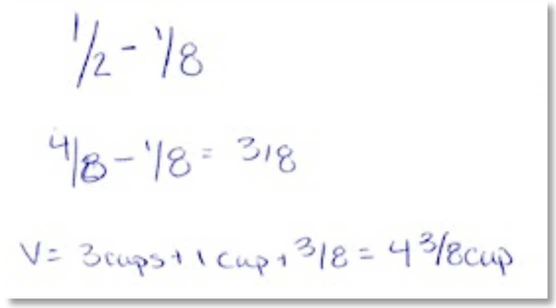

Figure 5.13. Flora's note

To answer the question that asked why she decided to use water instead of a formula, Flora pointed to the characteristics of the object, "It was hollow and I could put something in there to see how much it is exactly." She added, "And like I could do that with the ruler, too, but we really wouldn't know what inches cubed like … how much that meant if we wanted to put something in it." This reflects that, at first, she perceived the hollow cube as a geometric object that can be measured with unit cubes; however, then she started to see the object as a container by filling it with water.

I then asked Flora if she could measure the volume of the hollow cube with a formula. Flora said "yes" and proceeded to measure lengths with a ruler in inches and substituted the measures in $l \times w \times h$, namely v = (4 in)($4\frac{2}{16}$ in)($4\frac{2}{16}$ in); she emphasized its unit as "inches cubed." (This prompting question was next asked to raise the question about the meaning of calculated volume.)

After all participants computed the volume of the hollow cube using $l \times w \times h$, I asked them to estimate the number of cubes needed to fill the container completely. Once they had given an estimate, I asked them to explain or justify their estimate.

When asked to estimate the number of cubes, four participants—Becca, Cacie, Della, and Grace—took the calculated volume (in the previous measurement task) as their estimate: "It would be 64 one-inch-size blocks could fit in here," (Becca); "If they were this size [one-cm-size cube], you'd need a thousand," (Cacie). This reflects their conception of the calculated volume (the number obtained by the volume formula) as the number of cubes needed to fill the container.

Anne, Eva, and Flora estimated the number of cubes by estimating the dimensions of the container and visualizing the dimensional structure of cubes in terms of length by width by height. Anne put 10 one-cm-size cubes inside for one dimension (the 10th cube was not aligned with others) and substituted three 10s in $l \times w \times h$ (like Grace did when measuring the volume of the hollow cube); she obtained 1000 as her estimate. (See Figure 5.14.)

Anne: [Taking one-cm-size cubes] Do this. … [Aligning the cubes to the width of the front face.] That's pretty close. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. So 10 by 10 by 10. It would be [one] thousand cubes.

Figure 5.14. Anne's activity to estimate the number of cubes to fill a hollow cube

Rather than using cubes for one dimension, Flora filled the container with one-inch-size cubes (three layers of 9 one-inch-size cubes, namely a 3-by-3-by-3 rectangular prism shape) to show "how many big cubes can fit in there" and estimated inside lengths as "almost four cubes"; she substituted three 4s in $l \times w \times h$ and got 64 as her estimate. (See Figure 5.15.)

Figure 5.15. Flora’s gesture estimating inside lengths as four one-inch-size cubes

Because four one-inch-size cubes didn’t fit inside, Eva aligned the four cubes along the outside lengths of the front of the hollow cube (four one-inch-size cubes by four one-inch-size cubes). (See Figures 5.16 and 5.17.)

Eva: If I measure it on the outside, I'm measuring the entire object. So, I still know that it’s 4 times 4 times 4 [as pointing the stacked cubes] because the sides are still the same, which I know is the same as … [as pointing the number, “64,” on her note.]

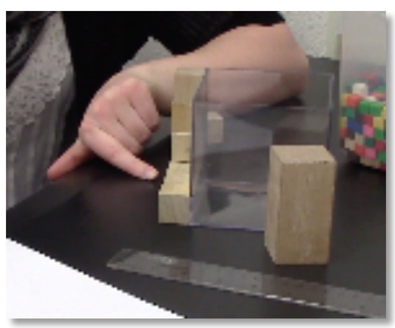

Figure 5.16. Eva’s gesture pointing outside lengths as four one-inch-size cubes

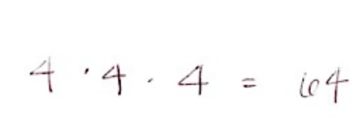

Figure 5.17. Eva’s note

She thus recognized that the outside length measured by using one-inch-size cubes is the same as the measure in inches obtained by using a ruler; thus, she calculated the volume 64 as her estimate. These participants' strategies of estimating the number of cubes reflects their internalized unit structure as a mental model of volume measurement in terms of a dimensional structure—length by width by height.

When asked to check their estimates, Anne, Della, and Eva described putting cubes along three dimensions (length by width by height): "We can put them [one-inch-size cubes] in length by length, side by side. We find that. We can put 4 of them in each one," (Della).

Cacie, Flora, and Grace described putting cubes for two dimensions of the base and multiplying the number of cubes for the base with the third dimension (for Grace, see Figure 5.18).

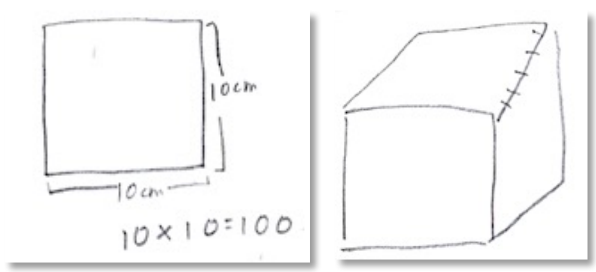

Figure 5.18. Grace's work to check her estimate

Grace: Yeah. If you … if you just were looking at one side [as drawing a square on her note] there would be 10 blocks here … as you can see. And then, 10 blocks here. But if you fill the whole thing up so … I fill the whole thing up … No, I don't want to fill it up. I can explain it … Then, it would be 10 times 10, like length times height, which is a hundred. You will know that … If you know that one side is a hundred and there are 10 of these hundreds, then you can more like see there would be a thousand because you have 10 groups of a hundred.

Participants’ strategies of counting cubes in the base layer and enumerating layers shows their understanding of layers in terms of composite units (an iterable set of individual, single units, Battista & Clements, 1996; Battista, 2003).

To check her estimate, Becca initially thought of using cubes for one dimension: “I guess we could try and see if that’s actually accurate by looking at 4 and seeing if 4 would fit in here.” Because four one-inch-size cubes did not fit inside, Becca recognized the difference between inside and outside length measures: “My only thought is maybe the little edge, we still measure it, but it’s not inside the box.” (See Figure 5.19.)

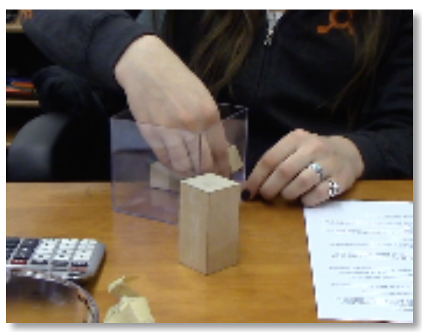

Figure 5.19. Becca’s activity to check her estimate

As an alternative way of checking her estimate of 64, Becca suggested building a stack of cubes that has the same dimensional measures as the container. She said, “I guess if you really wanted to make, put 64 cubes right next to it, and measure it out and make it look the same, but for some reason, you can’t put them inside.”

Participants’ strategies for estimating the number of cubes needed to fill in the hollow cube, and for checking and justifying their estimates, reflect not only their understandings of the calculated volume, but also their conceptualizing processes of measuring volume; both of which are consistent with Battista's (2003) four mental processes underlying the iteration of units in measuring volume: (a) forming and using a mental model process; (b) a spatial structuring process; (c) a unit locating process; and (d) an organizing-by-composites process.

On the difference between inside and outside lengths, Della, Eva, and Flora showed a reasoning process similar to Becca's.

Della: A side, it has … just like a wall. It has the same length and the same width of it. If I want to get exact volume, I need to minus that.

Eva: I tried putting the blocks on the inside, and that didn't work. I put the blocks on the outside, and then I realized that I was including the plastic part of this cube which is also part of the volume, which I'm not doing when I'm filling the water up.

Flora: I think maybe the $\frac{2}{16}$ came from this little plastic piece here. It's not exactly … like it takes up some room that would be in there, if this wasn't there then it would [be 4].

In sum, participants used three different ways to measure the volume of the hollow cube: (a) measuring outside lengths with a ruler and using $l \times w \times h$ (Anne, Becca, Cacie, Della, and Eva); (b) measuring inside lengths with unit cubes and using $l \times w \times h$ (Grace); and (c) pouring water with a measuring cup (Flora). All participants used calculated volume as their estimates of the number of cubes needed to fill the hollow cube: Becca, Cacie, Della, and Grace used the calculated volume in the previous measurement task as their estimate; Anne, Eva, and Flora estimated a number by using the numerical information obtained from the stacked cubes (the number of cubes along each sides). This shows participants' understandings of the calculated volume—that the number obtained by triple multiplication of length measures is equivalent to the total count of cubes filling up a container.

To verify their estimates, participants described the process of (a) structuring three-dimensional arrays of cubes; (b) counting the number of layers; and (c) building a stack of cubes having same dimensions as the given object. These participants' strategies of verifying their estimates reflect their conceptions of spatial structuring (of the stacked cubes) and enumerating layers (as compound units) that underlie the volume formula of $l \times w \times h$, as well the formula of $A \times h$ involving the relation between the length measure and the area measure in composing a compound unit (i.e., a layer), and the relationship between the area measure and the volume measure where the count of layers can be generated into the total count of cubes.

Hollow cylinder. To measure the volume of the hollow cylinder, three participants—Anne, Cacie, and Della—referred to using the formula $\pi r^2 \times h$: "To find this volume [hollow cylinder], you find the area of this circle and multiply by the height. So it's a little different. *Pi r* squared [πr^2] to get the area of that circle and I would multiply by the height because it fills it this many times," (Anne).

In contrast to others, Becca, Eva, Flora, and Grace said they did not know the appropriate formula for measuring the volume of the cylinder and then measured by using a measuring cup to fill the cylinder with water: "I know that there's a formula for finding the volume of cylinders, but I haven't gotten there yet so I don't know what it is," (Becca); "Since I can't remember that [the area formula for a circle], I would just fill it with water," (Flora).

After participants finished these measuring tasks, I asked additional questions to probe their understanding of the volume formula. As the last question to examine participants' understandings of the volume formula, I asked them about the relationship between the volume formula for a rectangular prism ($l \times w \times h$) and the volume formula for a cylinder ($\pi r^2 \times h$).

Becca, Eva, and Flora said that the volume formula for a rectangular prism couldn't be used for measuring the volume of a cylinder because of their different base shape (rectangle versus circle): "I don't think so. I'm assuming it has something to do with the diameter of the circle," (Eva).

In contrast to the others, Anne, Cacie, Della, and Grace saw the similarity between the two formulas in terms of computing sequence/procedure. For example, Cacie said.

> Cacie: They account for three different measurements, making it a 3-D shape. For instance, if you just were to measure one length of the base, that'd be 1-D, one-dimensional. Just like the radius is one-dimensional. Then by squaring it and multiplying it by *pi*, you're making it 2-D because you're getting the area of the circle. Same with adding like a width or a length, whatever you call it. That's 2-D. Then they both have a height going up to make it 3-D. So, they are similar.

Cacie's explanation reflects her understanding of the triple multiplication of lengths in the three dimensions of an object in terms of logical multiplication (Piaget et al., 1960); Anne, Della, and Grace showed a reasoning process similar to Cacie. Without additional prompting, none of the four participants showed an awareness of that both formulas, $l \times w \times h$ and $\pi r^2 \times h$, are special cases of the formula of $A \times h$ (i.e., $V = A \times h$; see Enochs & Gabel, 1984).

In sum, participants used two different methods to measure the volume of the hollow cylinder: (a) measuring outside lengths (radius and height) with a ruler and using the volume formula, $\pi r^2 \times h$ (Anne, Cacie, and Della), and (b) pouring water with a measuring cup (Becca, Eva, Flora, and Grace). In comparison of measuring the volume of the hollow cube, more participants used the strategy of filling the hollow cylinder with water to measure volume because they didn't know the exact formula to find the volume of a cylinder. This shows that the hollow cylinder evokes as the school context. Four participants described the similarity between the two formulas, $l \times w \times h$ and $\pi r^2 \times h$, in terms of a computing sequence/procedure; this reflects their understanding of logical multiplication underlying the volume formulas.

Summary. In the physical contexts, the participants used three different strategies for measuring volume: (a) using volume formulas to compute volume; (b) using unit cubes to measure side lengths and/or to determine the number of unit cubes needed to fill; and (c) pouring water with a measuring container to determine volume measurement (seeing volume as the amount of water needed to fill). In comparing the pictorial contexts, note that the participants used multiple strategies; it seems that the hollow cube and hollow cylinder (as a container) evoke the thinking of measuring volume as using cubes and/or filling up with water.

For the wooden block, participants conceived the calculated volume as the product resulting from triple multiplication—the number of cubic units filling and comprising an object or the amount of substance comprising an object. In the context of the hollow cube, all participants saw the calculated volume as the number of unit cubes needed to fill (related to the triple multiplication of length measures). Across the school contexts, thus, participants perceived the calculated volume by formula as the number of cubic units. They conceptualized volume formula as being related to the spatial structuring of cubic units, such as length-by-width-by-height structuring or layer structuring (Battista & Clements, 1996).

Chapter Summary

Regarding the three-dimensional geometric objects, participants described volume in terms of calculated volume—the result of triple multiplication. By contexts, they also showed different volume conceptions: number of cubic units, enclosed space (Sáiz, 2003), occupied space (Piaget et al., 1960), whole object, substance capacity, and substance volume.

Participants used volume formulas to determine the volume of geometric objects and they conceptualized the triple multiplication of $l \times w \times h$, as well as $\pi r^2 \times h$, as logical multiplication, which represents the relationship between length, or area and volume. In the context of school mathematics, thus, participants conceptualize volume as *geometric volume*—volume is obtained by a calculation of triple multiplication and the resulting number is written in cubic units—and its measurement as a quantification by formula; they visualized the spatial structure of volume formula in terms of the composition of cubic units (layers, Battista & Clements, 1996; Battista, 2003). The trade-off between the different strategies of measuring volume by given contexts/objects reflects how participants perceive the suitability of a particular strategy when using the tools (relating to the scale of a ruler and different sizes of unit cubes) at hand (see Vasilyeva et al., 2013).

CHAPTER SIX RESULTS: RELATIONSHIP BETWEEN EVERYDAY AND SCHOOL UNDERSTANDINGS OF VOLUME AND ITS MEASUREMENT

Thus far, participants' conceptions of volume and its measurement revealed in a certain context—in either an everyday or a school context—have been analyzed in terms of everyday or school conceptions of volume and its measurement strategies. As the final focus of this study, I examine how participants' everyday and school conceptions of volume measurement are related. That is, in this chapter, I address my third research question. I present participants' overall volume conceptions revealed through the two interviews; then, I present their thoughts about the different strategies for measuring volume taken in the everyday and school contexts.

Participants' Overall Volume Conceptions Revealed Through the Different Contexts

To examine how participants' everyday volume conceptions are related to their school conceptions, I drew diagrams of individual participants' overall volume conceptions that appeared throughout the two interviews.[8] First, I explain how I composed all of Anne's volume conceptions in a diagram, as well as the contextual components of each volume conception, to represent her overall volume conceptions. Then, I present my analysis of individual participants' overall volume conceptions, focusing on the relationship between everyday and school volume conceptions revealed across the two interviews.

Anne's Overall Volume Conceptions

When I asked for her first thoughts about volume, Anne described volume as the amount of water in a swimming pool. To represent this liquid-related initial volume conception connected to an everyday context, I drew a node for the liquid volume conception (light-blue

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[8]To draw the diagram of individual participants' overall volume conceptions, I used *Institute for Human and Machine Cognition (IHMC)'s CmapTools knowledge modeling kit Version 6.01.01*.

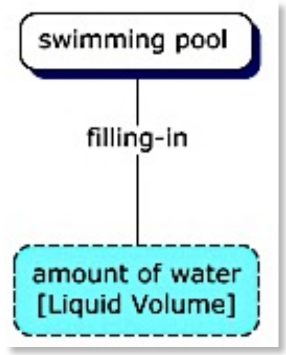


Figure 6.1. Nodes of Anne’s initial volume conception and the swimming pool context

shaded) and another node for the everyday context (dark-blue shadowed), and labeled the link between the two nodes (see Figure 6.1). Similar to the swimming pool, any other context of everyday objects node is shadowed in dark blue.

As addressed in Chapters 4 and 5, participants often described volume in multiple ways in any given context. In the context of the fish tank, Anne described volume three ways: (a) the potential amount of substances (everything, like water and fish) put in the fish tank (substance/liquid capacity); (b) the amount of water needed to fill the tank (liquid volume); and (c) the raised water level when putting stuff in the tank (complementary volume, Piaget et al., 1960). To represent the three liquid-related volume conceptions revealed in this context, I drew conception nodes for substance capacity, liquid volume, and complementary volume—all connected to the fish tank node as the evoking context. Since Anne also revealed a liquid volume conception in the recipe context, I added nodes for liquid volume and the recipe context to the diagram to represent all of Anne’s volume conceptions revealed in the everyday contexts (see Figure 6.2). Note that all liquid-related volume conceptions are shaded in blue hues; once a color was assigned for a certain volume conception node, that color was used throughout the contexts. Each conception is placed differently in consideration of its relationship to capacity (e.g., complementary volume is placed closer to the fish tank node than liquid volume and in the same

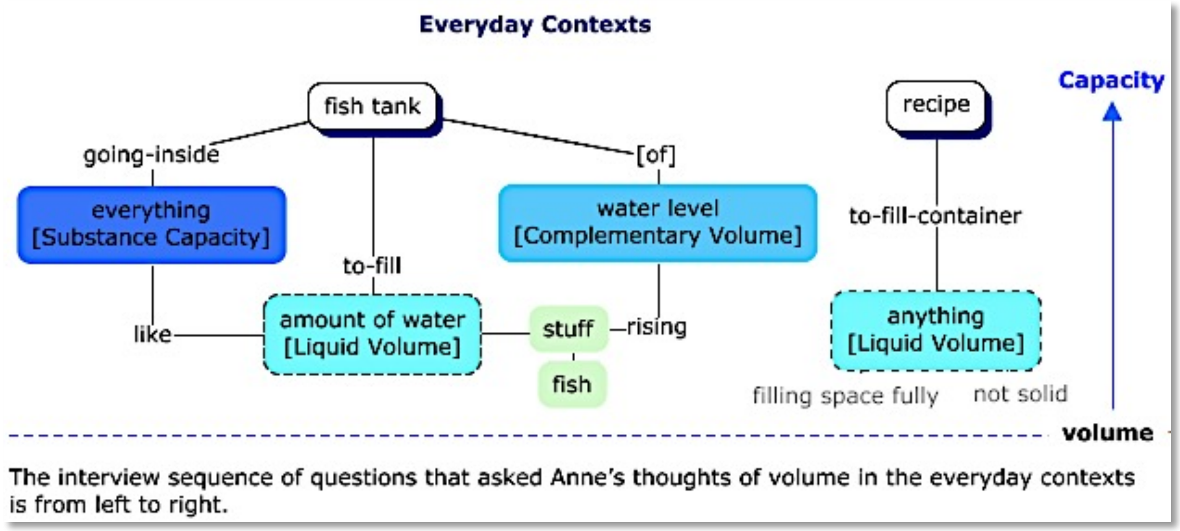


Figure 6.2. Anne's liquid-related volume conceptions revealed across the everyday contexts

level as substance/liquid capacity).

Across the contexts of the six geometric objects—a cube, a rectangular prism, stacked cubes, a wooden block, a hollow cube, and a hollow cylinder, Anne described volume in five different ways: (a) a result obtained by triple multiplication of linear dimensions (calculated volume); (b) the amount of space inside (enclosed space, Sáiz, 2003); (c) the whole wood (whole object); (d) the amount of wood contained (substance volume); and (e) the potential amount of air within (substance capacity). To represent the different volume conceptions revealed across the contexts of the geometric objects, I drew a node for each of the volume conceptions and shaded them in different colors; each conception node was connected to the context node in which the conception was revealed.[9] In contrast to the context nodes of the everyday objects, the context nodes of the geometric objects were shadowed in dark red (see Figure 6.3).

[9] Different colors were assigned for different types of volume conceptions, i.e., I shaded (a) liquid-related volume conceptions in blue hues; (b) substance (other than liquid) related conceptions in green hues; (c) space related conceptions in the yellow hues; (d) formula/calculation related conception in red; (e) cubic unit related conception in orange; and (f) whole body related conception in purple.

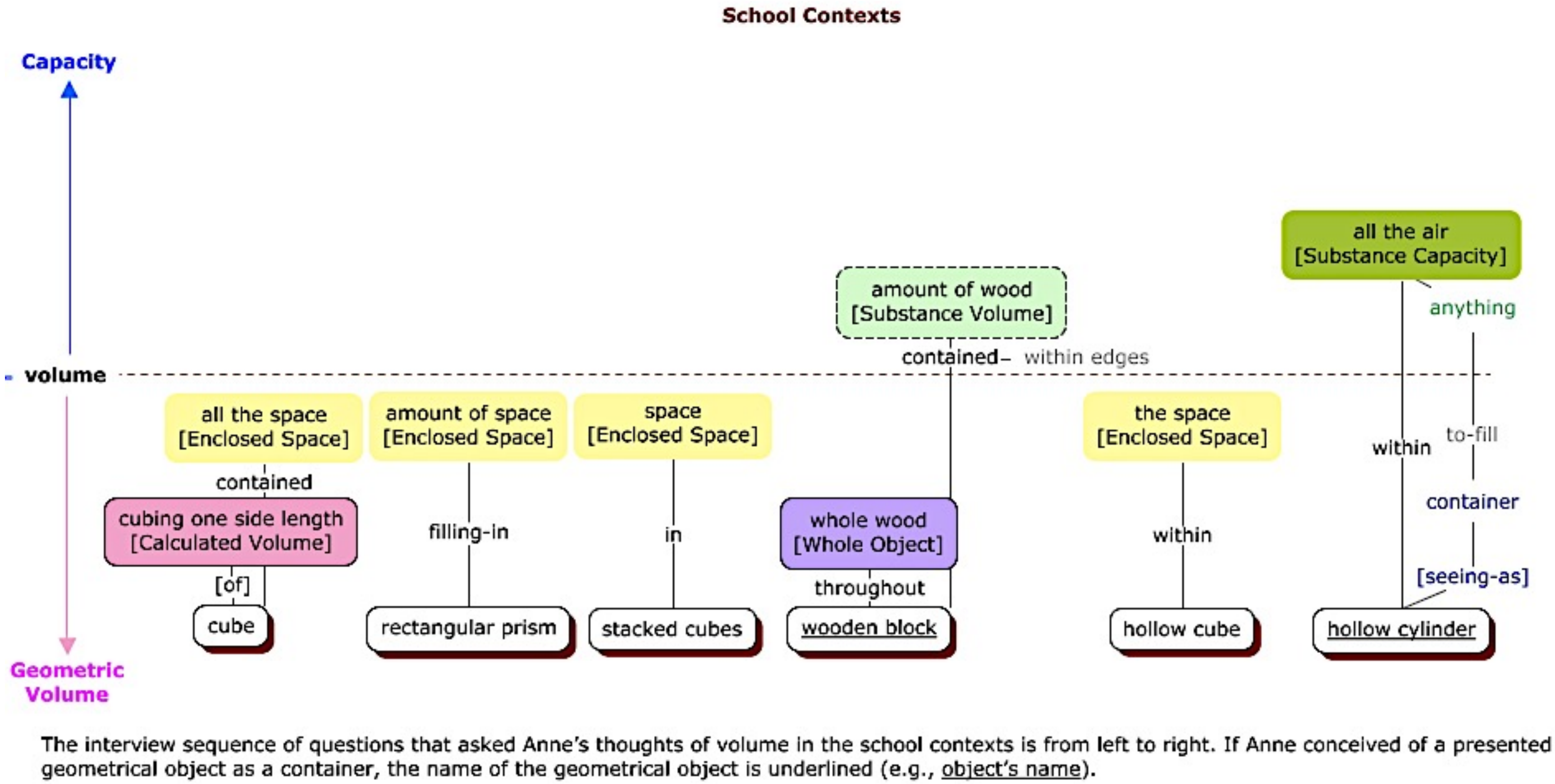


Figure 6.3. Anne's different volume conceptions revealed across the school contexts

Each volume conception node is placed in either the capacity or the geometric volume side of the diagram to reflect the proximity of each conception to the capacity and geometric volume conceptions (see the dashed line in the Figure 6.3). Among the geometric volume conceptions, each node is placed differently in consideration of the difference between exterior and interior volume. For instance, calculated volume (red shaded) is placed more closely to the context node of the cube than the enclosed space node (light-yellow shaded) and at the same distance to the whole object node (purple shaded).

Once all of the nodes of her volume conceptions and its contexts were composed in a diagram, it clearly showed how Anne's volume conceptions varied by the different contexts—everyday and school (see Figure 6.4). Anne revealed initial volume conceptions as liquid volume connected to an everyday context (swimming pool); other liquid-related conceptions (substance/liquid capacity, liquid volume, and complementary volume—all classified as a capacity volume conception in Chapter 4; for the thinking of volume as capacity, Sáiz, 2003) were consistently observed in the everyday contexts.

Anne revealed different volume conceptions in the school context: calculated volume, enclosed space, whole object, and substance volume and capacity. With the exception of two substance-related volume conceptions that appeared in the wooden block and hollow cube contexts, all of her volume conceptions revealed in the school contexts were classified as a geometric volume conception in Chapter 5. Note that Anne revealed substance-related volume conceptions when she regarded the presented geometrical object as a container: "this [hollow cylinder] is like a very similar container." That is, the context of a container evokes the thinking of volume as the (potential) amount of substance in it in terms of capacity.

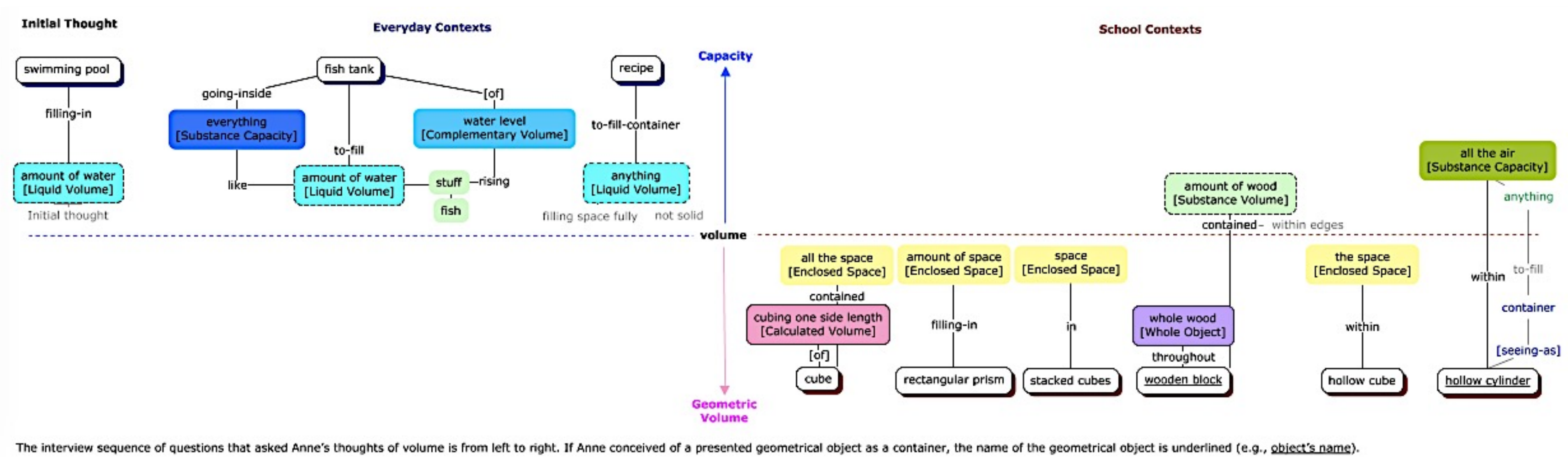


Figure 6.4. Anne's overall volume conceptions revealed across contexts

Therefore, Anne's overall volume conceptions revealed across the different contexts are strong evidence that capacity and geometric volume conceptions coexist in her mind but are activated by contextual relevance; namely, she revealed capacity volume conceptions in everyday contexts and geometric volume conceptions in school contexts. In the diagram of Anne's overall volume conceptions, the contrast between her everyday and school volume conceptions became apparent.

Patterns of Volume Conceptions Across Contexts

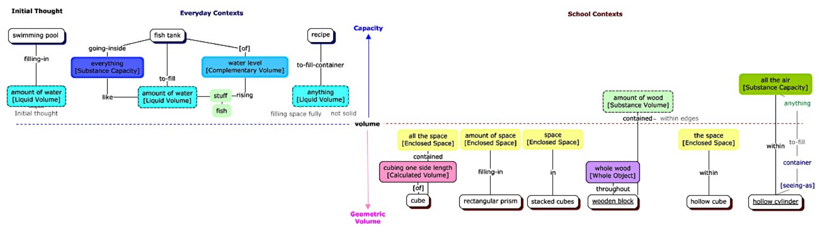
As for Anne, I drew a diagram composed of all volume conceptions revealed across the contexts for each of the other six participants to represent their overall volume conceptions. Figure 6.5 shows the diagrams of individual participants' overall volume conceptions. (See Appendix C for individual diagrams.) Two patterns of volume conceptions across the everyday and school contexts emerged.

Pattern 1. The diagrams of Anne, Eva, Grace, and Flora's overall volume conceptions represent a consistent pattern of everyday volume conceptions but multiple school volume conceptions. Like Anne, Eva, Grace, and Flora revealed liquid-related initial volume conceptions that were evoked by the everyday contexts, like a water bottle and a cup; they consistently revealed the liquid-related capacity volume conceptions across the everyday contexts.

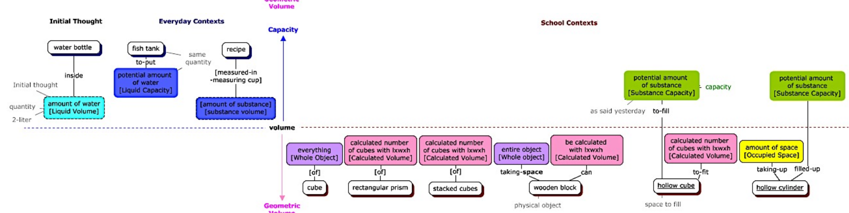
In contrast, in the school contexts, the four participants (including Anne) described volume related to space and/or the dimensional aspects of a presented geometric object, revealing geometric volume conceptions. Note that only when they perceived of the presented geometric object as a container being able to be filled, did they reveal substance-related volume conceptions: "Maybe it's because it [hollow cube] is hollow, so there's space to fill," (Eva). That is, the four participants revealed capacity volume conceptions whenever they see a presented

Pattern 1 of volume conceptions across contexts

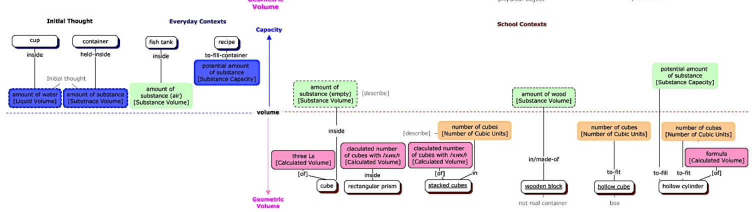
Anne:



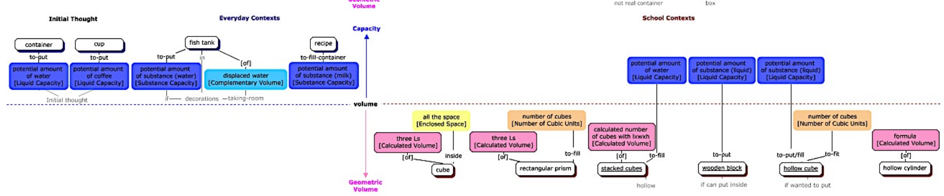
Eva:



Grace:

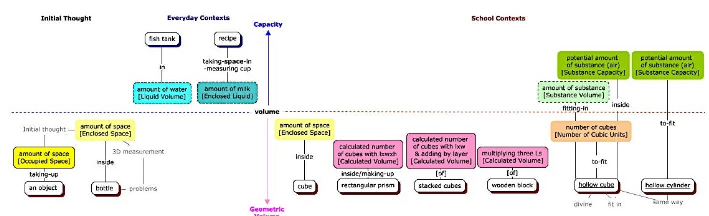


Flora:

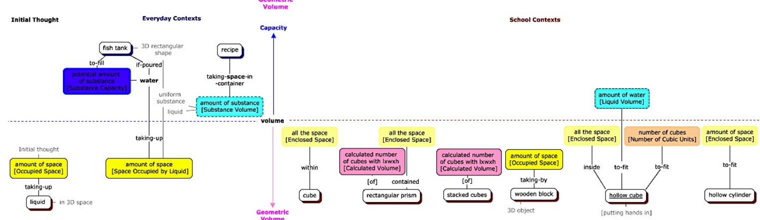


Pattern 2 of volume conceptions across contexts

Becca:



Cacie:



Della' volume conceptions across contexts

Della:

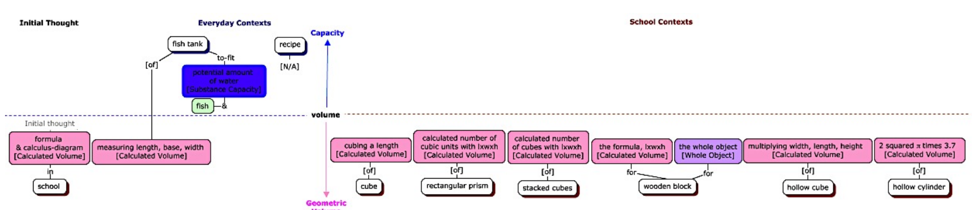


Figure 6.5. Patterns of participants' overall volume conceptions across two interviews

geometric object as a container.

This pattern of volume conceptions across the contexts suggests that the predominant volume conceptions of Eva, Grace, and Flora, as well as Anne, are (liquid-related) capacity volume conceptions that are evoked by the everyday contexts, even though they revealed geometric volume conceptions in school contexts.

Pattern 2. Becca and Cacie responded to the initial questions about volume by talking about volume as 3-D space, revealing enclosed space and/or occupied space volume conceptions (classified as a geometric volume conception in Chapter 5; for the thinking of volume as occupied space, Piaget et al., 1960). As with the other participants, they revealed liquid-related volume conceptions in the everyday contexts.

In the school contexts of the second interview, they again revealed their geometric volume conceptions. For the hollow cube and/or hollow cylinder, they also revealed substance-related volume conceptions. Like the others, Becca and Cacie saw the presented object as a container being able to be filled: “I can divine because it’s [hollow cube] open. I can be like, oh I can fit something in here,” (Becca); “So, if you fill [it with] water, you could measure how much water fits in it [hollow cube],” (Cacie).

The pattern of Becca and Cacie’s overall volume conceptions across the contexts demonstrates that they have relatively strong geometric volume conceptions, but also reveals capacity volume conceptions in the everyday contexts.

Della. Della did not fit either pattern. Her diagram shows a pattern dominated by the geometric volume conception, even in the everyday contexts. When asked her first thoughts about volume, Della mentioned her experience using a formula in school and she described volume as a diagram in her calculus course. In the fish tank context, she first talked of the volume of the tank as $l \times w \times h$, related to "the dimensions" of it, revealing the calculated volume conception. Then, when asked a prompting question to describe the volume of the fish tank, she talked of volume as the potential amount of water and fish (substance) to fill the tank (as a container), revealing the substance capacity conception. Thus, Della thought of the fish tank both as a geometric object and as a container, with each evoking different volume conceptions.

Across the school contexts of the second interview, she consistently talked of volume in terms of calculated volume with the volume formulas—$l \times w \times h$, $A \times h$, and $\pi r^2 \times h$. In the wooden block context, she revealed another geometric volume conception of the whole object.

The pattern of her overall volume conceptions across the contexts suggests that her predominant volume conceptions are calculated volume (as a certain type of geometric volume conceptions), and revealed her (liquid-related) capacity volume conception only when she evoked an everyday context (i.e., a container).

In sum, participants appear to hold both capacity and geometric volume conceptions, with these conceptions being activated by everyday and school contexts. Half of the participants seem to hold stronger capacity volume conceptions, with the other half holding stronger geometric volume conceptions. Containers (typically found in everyday contexts) evoke the thinking of volume as the (potential) amount of substance in the containers (as capacity volume conceptions); geometric objects trigger the thinking of volume related to dimensions, calculation with a formula, or space (as geometric volume conceptions). Sometimes, participants perceived geometric objects as containers, evoking capacity volume conceptions.

Participants' Reflections on Different Strategies of Measuring Volume

To examine how the participants thought about the different strategies for measuring volume taken in the everyday and school contexts, I analyzed their responses to the interview questions that asked about the contexts of measuring volume by using either a measuring cup or a volume formula, and the everyday contexts in which they might use/apply a volume formula to measure volume, as well as their experience of measuring volume by using a volume formula in everyday life situations. Tables 6.1 and 6.2 summarize participants' responses to these questions.

Different Strategies for Measuring Volume

When asked "When do you measure volume by using a measuring cup? And, when do you measure volume by using a volume formula?," all participants except Eva differentiated the two methods by the different characteristics of objects to be measured or in the everyday and school contexts. Eva did not provide her response to the given question.

Anne distinguished the contexts of using a measuring cup and a volume formula in terms of measuring a certain amount of substance to fill a container, and the potential amount of substance filling a rectangular container, respectively: "This [using the volume formula $l \times w \times h$] would be to figure out how much is in it [hollow cube] and that [using a measuring cup] would be to show how much you need." Like Anne, Flora described the contexts of using a measuring cup as measuring a certain amount of liquid substance needed to fill a hollow container: "Something liquid in there, so measure how many ounces or how many cups of that liquid you can put in that [hollow container]"; she specified the context of using the volume formula $l \times w \times h$ to a square- or rectangular-shaped object. Both Anne and Flora prioritized the use of a measuring cup to measure a certain amount of substance to fill, citing that it is more exact and more accurate. This view is consistent with their predominant volume conceptions of capacity that appeared in the pattern of their overall volume conceptions across contexts.

To address the contexts of using a measuring cup and using a volume formula to measure volume, Grace first gave an example of a cylinder where she used "the cup method" because she did not know its volume formula. Her example shows that she perceived the cylinder context in

Table 6.1.

Participants' Reflections on the Contexts of Using Different Strategies for Measuring Volume

Participant	Responses
Anne	Using a measuring cup to fill a container with a certain amount [of substance] and to measure out the certain amount [substance to fill]; using the measuring cup "for exactness" Using the formula to figure out the amount of substance (flour) filled up a container when the container has no numbers [scales] on it.
Eva	N/A[a]
Grace	Using cups when cooking (never measure volume "with the formula") Using a formula for a square [object] (and using "the cup method" only when she did not know the volume formula for a given object, e.g., a hollow cylinder)
Flora	Using a measuring cup to measure the amount of liquid needed to fill a hollow container (volume measure in *cups* or *ounces*; "it's more accurate") Using the formula to determine the volume of a square, rectangular, and not hollow [object]; ("I wouldn't be able to pour anything into it if it's not hollow")
Becca	Using a water bottle or a measuring cup for "daily life" ("the estimates or guessing") Calculating volume with a formula when she is "in a math situation" that needs to be "accurate"
Cacie	Using the equation "always" Not using math for baking context
Della	[Using a measuring cup] for liquid (syrup or milk) because its length dimensions cannot be measured by hands and "it [using a measuring cup] is not fit for the mathematics" that asks "to get the right answer and the accurate answer"

[a] Instead of addressing the contexts of measuring volume by using either a measuring cup or a volume formula, Eva explained how she can make sense of the resulting number obtained from by using $l \times w \times h$: "Then when I broke it down and said, oh, 32 and 32 and 32 and 32 is equal to 128, that formula made sense to me." In addition, she addressed the use of a ruler "when they [cubes] didn't fit in [hollow cube]."

Table 6.2.

Participants' Reflections on the Everyday Contexts Using Volume Formulas

Participant	Everyday experience	Everyday contexts
Anne	Nothing comes to her mind	Measuring the potential amount of water to fill a fish tank or the potential amount of substance to put into a laundry machine
Eva	Said "No."	Just using the idea of the formula $l \times w \times h$ to figure out the potential amount of water to fill her coffee pot or fish tank
Grace	Said "No." ("Not really use volume except to cook")	Fish pool [tank] (could be figured out "both ways") and building blocks, like building a tower by stacking blocks, to find volume in terms of the number of blocks
Flora	Can't think of any outside of class stuff	Measuring the amount of water needed to fill a fish tank by measuring the dimensions with a ruler; then figuring out "how much [of water] would fit inside like one cubic foot" ("water isn't measured in feet", but in "gallons or fluid ounces")
Becca	None, other than using for "math homework"	N/A[a]
Cacie	N/A[b]	Measuring (a) the sizes of rooms and the amount of space taken up [by the rooms] in the context of building a house; (b) the amount of space taken by certain things that fills a moving van; and (c) the potential amount of water that fills a swimming pool Since most moving vans are rectangular [prism shaped], doing $l \times w \times h$ to figure out a "general" amount of space (meant not exact) that has to be filled; to figure out whether a lamp and a bed or a couch fit in the van, knowing [measuring] the height and length of the van; and a table is easy to know $l \times w \times h$ and "whether you can fit it in the moving van"
Della	Said "No."	Her school experience of guessing the size of a room by putting "one-meter" cubes for each dimension; and her "lab class" using a formula [not a geometric volume formula]

[a]Becca was not asked about everyday contexts of applying a volume formula for measuring volume, since she spoke of "a humidifier" as an everyday context of measuring volume (the amount of water to fill) by calculation in the first interview. [b]Cacie was not asked for her experience of using a volume formula in her everyday life, since she described some possible applications for the volume formula $l \times w \times h$ in an everyday life situation.

the context of using a volume formula to determine its volume, and she was confident in using a measuring cup to measure volume instead of using a volume formula. This may be consistent with her predominant volume conception of capacity that was shown across contexts. In addition to the example, she addressed using a measuring cup "if you are cooking" and using a formula "if it's a square object." This reflects that she differentiates these two methods in terms of the everyday and school volume measurement strategies—using a measuring cup in everyday contexts and using a volume formula in school contexts.

In describing the contexts of using a measuring cup and using a volume formula in measuring volume, Becca, Cacie, and Della revealed their thoughts about using a volume formula as the appropriate, prioritized method to use in school: "What I've always thought is I have to sit and calculate the volume of how much flour," (Becca); "I always go to the equation just because I know that that's how you go about solving it and it's proven to work and stuff," (Cacie); "It [using a measuring cup] is not fit for the mathematics. Mathematics always asks me to get the right answer and the accurate answer," (Della). In contrast, all three participants talked of using a measuring cup for their everyday lives: "I think that the tools [a water bottle and a measuring cup] that are the estimates or guessing, at least for me is more daily life" (Becca); "For baking I wouldn't use math to figure things out," (Cacie); and "The ruler [using a volume formula] cannot count for the syrup and milk. You just need to have these cups and beakers and these containers," (Della). Thus, Becca, Cacie, and Della distinguished between using a volume formula for school from using a measuring cup for everyday life; their preference of using a volume formula to get the right, accurate answer is consistent with their predominant conceptions of geometric volume that appeared in their overall volume conceptions across contexts.

In sum, participants thought of one strategy as more appropriate than the other and is prioritized in a certain context. Most participants thought of using a measuring cup as the way of measuring volume in everyday life contexts and using a volume formula as the way that should be used in school math class.

Everyday Context of Applying a Volume Formula in Measuring Volume

When asked about the everyday experience of measuring volume by using a volume formula, no participants talked of any experience other than math homework. When asked about everyday contexts in which they might use/apply a volume formula to measure volume, participants talked of a fish tank, a swimming pool, a coffee pot, a laundry machine, a moving van, a room, a block building, or a table. Note that Becca mentioned a humidifier as an everyday context of measuring volume (as the amount of water needed to fill) by calculation to the question in the first interview that asked her experience of measuring volume in everyday life situations.

Five participants (Anne, Cacie, Eva, Flora, and Grace) spoke of a fish tank, a swimming pool, a coffee pot, and a laundry machine as the context of using a volume formula in measuring volume. Anne mentioned a fish tank and laundry machine as the everyday context of measuring volume (the potential amount of substance needed to fill) by using a formula.

Anne: I guess if you're measuring how much water to put in a fish tank you would have to figure the volume of the tank or how big your laundry machine is I guess. Like how much you are capable of putting into it, different things like that I would say.

Cacie spoke of a swimming pool: “The swimming pool idea, if someone is cleaning their swimming pool, they need to know how much water can fit in it, they would maybe measure.” In considering the shapes of a fish tank, laundry machine, and swimming pool, Anne and Cacie seemed to think of using the volume formula $l \times w \times h$ in measuring volume in terms of capacity. Eva talked of a coffee pot as the everyday context of applying the volume formula $l \times w \times h$ to estimate the potential amount of water to fill.

Eva: When I’m filling up my coffee pot, I need to know, oh, this is this deep and this wide and this ... Hmm, you know wide, length, the length times width times height I need to know that, so how much water I can put in.

In addition to the context of the coffee pot, Eva mentioned her fish tank as a possible context, but she said that she does not “actually” use the volume formula $l \times w \times h$, but uses “the idea” of the formula. Without a prompting question, Anne, Cacie, and Eva did not provide any further explanation about how the potential amount of water needed to fill a container (a fish tank, swimming pool, or coffee pot) could be measured by applying the volume formula, especially regarding the resulting quantity in cubic units determined by $l \times w \times h$.

When asked about everyday contexts of using a volume formula in measuring volume, Flora talked of filling a container (a tub or a fish tank) with water by a milk jug. When re-asked a question about the everyday contexts of using a volume “formula” to measure volume, Flora said that she can use the formula $l \times w \times h$ for the fish tank, because she can measure the three dimensions of the fish tank with a ruler.

Flora: I could take a ruler or something and measure how high it is, and how far back it goes, and how much across it is. Then I can fill it up, so I know how much water I need for a fish tank.

When asked to identify whether using a milk jug or using a volume formula was more useful in measuring a fish tank, Flora distinguished the relevant contexts of using a milk jug and using the volume formula in terms of the size of the fish tank, such as using a milk jug for filling a small fish tank with water, and using the volume formula for measuring a big fish tank.

Flora: But if I had a big one that's like one and a half feet up and that like a foot back and three feet across, and then trying to figure out how many cubic feet would be like water, how much water that would need, because water isn't measured in feet, but you can't really do that. […] It's only like ounces and gallons, or fluid ounces or something.

Flora thus recognized that the number obtained from the triple multiplication would not make sense as the amount of water needed to fill the fish tank until the quantity was converted from cubic units to the quantity in capacity units.

Grace also talked of a fish pool [tank] as the context in which volume can be figured out by using both strategies—using a measuring cup and a volume formula (no mention of substance type, like Anne, Cacie, Eva, and Flora); however, she did not provide any further explanation about the relationship between the quantities in capacity and cubic units obtained by the two methods.

Thus, the five participants (Anne, Cacie, Eva, Flora, and Grace) showed their understandings of the contextual relevance of the everyday contexts in which they can use the volume formula $l \times w \times h$, namely the application of the volume formula for measuring the volume of the rectangular prism-shaped containers; by using the volume formula $l \times w \times h$, Anne, Cacie, Eva, and Flora aimed to measure the (potential) amount of water needed to fill the containers. Without a prompting question, the three participants (Anne, Cacie, and Eva) did not show any understanding of/about what the number (in cubic units) obtained from the triple multiplication would mean in terms of the amount of water needed to fill a container; Grace also did not show her recognition of the different quantities in capacity and cubic units resulting from using a measuring cup and a volume formula, respectively. Only Flora talked of the need for unit conversion, from the quantity in cubic units obtained from the triple multiplication to the quantity of water in capacity units needed to fill a container.

In addition to the context of the swimming pool, Cacie also talked of measuring how big a room is and how much space certain things take up to fill a moving van with the volume formula $l \times w \times h$. When asked about the method of figuring out the volume of a moving van by using the formula, she said "You could just do the width of that times length times the height of it and know a general, it may not be exact, but a general amount of space that you have to fill." In addition, she addressed figuring out the volume of a table by doing $l \times w \times h$ to know "whether you can fit it in the moving van or not." This reveals that Cacie knows how to apply the volume formula $l \times w \times h$ in estimating the volumes of the moving van and a table, in terms of enclosed and occupied space, respectively. (Note that in the hollow cube context of the second interview where Cacie measured volume by calculation with the volume formula $l \times w \times h$ and filling with water using a measuring cup, she explained the quantities obtained by the two methods take up the same amount of space, although, "Unit-wise, I don't know how they convert." Like Grace, she did not think of the potential difference between the quantities obtained by the two methods in terms of exterior and interior volume.)

Della also talked about her experience of estimating how big a room is by measuring the three dimensions with a one-meter cube and using the volume formula $l \times w \times h$: "I can just guess that I put it in the edge of it, and I can guess the height of the room, the width of the room, and other. At that time, I can guess how many cubes can fit in this room." Like Cacie, Della seems to apply the volume formula $l \times w \times h$ in estimating the volume of the room in terms of enclosed space.

In addition to the context of the fish pool [tank], Grace also talked of building blocks, such as building a tower by stacking blocks, and seeing the number of blocks as an everyday context by applying a volume formula to figure out its volume (seeing volume as the number of cubic units).

Thus, Cacie, Della, and Grace showed their understandings of the contextual relevance of the everyday contexts in which they use the volume formula $l \times w \times h$, namely the application of the volume formula for measuring the amount of space inside a room, or taken up by a rectangular prism-shaped object (blocks), or estimating the amount of space taken up by an object (a table).

In sum, participants spoke of different everyday contexts in which they use or apply a volume formula to measure volume, mostly right rectangular prism-shaped objects. This reflects participants' understandings of the relevant everyday context of applying the volume formula $l \times w \times h$ in terms of the shape of the object to be measured. In describing the application of the volume formula $l \times w \times h$ in measuring the amount of water needed to fill a container, some participants (Anne, Cacie, Eva, and Grace) revealed their limited understandings in connecting their school understanding of volume measurement to the everyday measurement contexts.

Chapter Summary

Across the two interviews, participants revealed different volume conceptions and its measurement strategies by everyday and school contexts. This reflects not only the context-dependent nature of their volume conceptions but also the difference between their everyday and school volume measurement understandings. Some participants showed a partial understanding about the relationship between everyday and school understandings of volume and its measurement given a certain context, like a fish tank, a hollow cube, or a hollow cylinder. Based on the results of this analysis, it can be concluded that the everyday and school understandings of volume and its measurement coexist in the participants' mind but they exist as two sets of understandings—one set for everyday life and the other set for school—with some partial links between them (such as unit conversion between capacity and cubic units).

CHAPTER SEVEN DISCUSSION

The main finding of this study was that preservice teachers have two sets of understandings about volume and its measurement: volume as capacity and its measurement by filling—revealed in everyday contexts—and volume as geometric volume and its measurement by calculation—revealed in school contexts.

In everyday contexts, preservice teachers thought of volume related to capacity—the amount of substances (mainly liquid) that fill a container (Sáiz, 2003). I classified the volume conceptions of substance volume, substance capacity, complementary volume (Piaget et al., 1960), enclosed liquid, and space occupied liquid as capacity. Preservice teachers addressed measuring these volumes by filling a container.

In school contexts, preservice teachers thought of volume as being related to geometric aspects, such as space and dimension. The conceptions of calculated volume, number of cubic units, whole object, enclosed space (Sáiz, 2003), and occupied space (Piaget et al., 1960) appeared across the contexts; I classified these as geometric volume—volume is obtained by a calculation of triple multiplication and the resulting number is written in cubic units. Preservice teachers determined geometric volumes by calculation using a volume formula.

Overall, the difference between individual preservice teachers' everyday or school understandings of volume and its measurement was consistent across the interviews. The two sets of understandings coexist, but with limited links between them. Across the everyday and school contexts, connections between capacity and geometric volume were apparent when preservice teachers regarded a presented object as a container and/or measured the volume of a container by both filling water with a measuring cup and calculated with a volume formula.

In this chapter, I first discuss several issues raised by these findings in terms of the context-dependent nature of volume measurement understandings, followed by the issues of further research. Then, I discuss the contributions from this study, the implications for instruction, and the limitations of this study. I end this chapter with a final word about the importance of enriching school mathematics instruction by drawing on the everyday mathematics understanding developed through everyday experience.

Context-Dependent Nature of Volume Measurement Understandings

Everyday Volume Measurement Understandings Associated with Capacity

An important issue raised by the different volume measurement understandings revealed in this study concerns why preservice teachers conceive and measure volume differently in everyday and school contexts. In particular, why are their everyday volume measurement understandings associated with capacity?

Kerslake (1976) argued that volume measurement in everyday life situations usually deals with capacity, as filling or packing activities, and is often measured in capacity units, such as pints or liters. When preservice teachers' revealed their volume measurement understandings related to capacity, they referred to their own volume measurement experiences in everyday life—filling a fish tank with gallons of water or pouring cups or fluid ounces of water into a coffee maker. It may well be that the preservice teachers developed their understandings of volume and its measurement associated with capacity through their experiences of measuring volume in everyday life situations. Drawing on the capacity-related volume measurement understandings in everyday contexts is consistent with Kerslake's (1976) argument that capacity is associated with volume in everyday measurement situations.

Another answer may lie in a function of the contextual characteristics of an object presented in the interviews. For example, all preservice teachers described the volume of the fish tank in terms of capacity. They also described volume as capacity for geometric objects when they perceived the objects as containers. The context of a container may evoke preservice teachers' conception of volume as capacity in the sense of interior volume, thus they described the amount of liquid or pourable substance needed to fill the container. In addition, the context of a container also influences the preservice teachers' conception of measuring volume; most described measuring volume as filling volume—measuring the interior volume/capacity of a container by filling it with liquid poured from a measuring cup (Curry & Outhred, 2005). This suggests that the container context evokes preservice teachers' capacity-related volume conceptions and also relates to the preservice teachers' strategies for measuring volume by filling.

Literature on knowledge change (e.g., Barsalou, 1982; McNeil & Alibali, 2005) may clarify this point in that preservice teachers' volume measurement conceptions depend on the context in which the conceptions are evoked. In his study on changes in concept meanings, Barsalou (1982) distinguished two types of properties associated with concepts: context-independent properties and context-dependent properties. Context-independent properties of a concept are activated by the word of a concept on all occasions, like "unpleasant smell" for "skunk" or "gills" for "fish." In contrast, context-dependent properties of the concept are activated by contextual relevance. For example, a person might think of "floats" as a property of a basketball only when searching for something to use as a life preserver, but not when thinking about possible games to play. In general, according to Barsalou (1982), the changes in concept meanings follow as the consequence of changes in the accessibility of context-independent and context-dependent properties. This provides grounds for making an argument for the growth of preservice teachers' understandings of volume and its measurement with respect to contextual relevance. That is, it provides insight into why preservice teachers reveal a certain volume conception in one context but not in another, and how liquid or filling properties are related to the concept of volume, as well as measuring volume.

Rote Application of Volume Formulas

Volume formulas play a core role in volume measurement instruction. This is not surprising, given that most school mathematics standards introduce volume measurement as applying volume formulas (e.g., CCSS-M, NGA Center & CCSSO, 2010). However, there is an issue of applying volume formulas by rote in volume measurement tasks.

In the hollow cube example of this study, most preservice teachers determined its volume by applying the volume formula $l \times w \times \mathrm{h}$, namely measuring outside lengths with a ruler and substituting the length measures in the volume formula; they spoke of the resulting number as the number of cubic units that would fit inside the hollow cube. Until they could not put as many cubes along the inside dimensions as they could along the outside length measures, the preservice teachers did not recognize the difference between inside and outside length measures, or the difference between the interior volume and exterior volume of the hollow cube. Likewise, the strategy of measuring the side lengths of a three-dimensional object and multiplying them to determine its volume can be a rote application of volume formulas if there is no reflection on the presented contexts. The findings of this study are consistent with the prior research in which it was suggested that children and even teachers rely on rote volume formulas in volume measurement tasks (e.g., Battista & Clements, 1996; Vasilyeva et al., 2013; Enochs & Gabel, 1984; Sáiz, 2003).

Another issue related to volume measurement in applying volume formulas by rote is raised concerning the transfer and application of school mathematics to solving mathematical problems presented in everyday contexts. Most preservice teachers of this study talked of measuring the volume of a fish tank or a swimming pool by using the volume formula $l \times w \times h$, while they thought of its volume as how much water they needed to fill it. This reveals their lack of understanding of the need to choose an appropriate unit according to the nature of the object to be measured. The quantity of water in cubic units is not "an appropriately communicable form" (Zevenbergen, 2005). The inappropriate transfer and application of school volume measurement knowledge to solving everyday context problems is consistent with the findings from Zevenbergen's study.

Difference Between Everyday and School Volume Measurement Understandings

The findings of this study suggest that the preservice teachers distinguish everyday understandings of volume and its measurement from their school understandings. The difference between everyday and school understandings of volume and its measurement raises issues for mathematics education, especially if we consider that the preservice teachers in this study are a product of current schooling, and are destined to be teachers of mathematics.

Considering that the individual preservice teachers' mathematical understandings are an outcome of current schooling, the difference found in this study between everyday and school volume measurement understandings supports the need for building better connections between everyday mathematics and school mathematics (Putnam et al., 1990). Moreover, according to Gravemeijer (1999), mathematics instruction should be designed to stimulate "a process of gradual growth in which formal mathematics comes to the fore as a natural extension of the student's experiential reality" (p. 156).

The other issue concerning the difference between the preservice teachers' everyday and school volume measurement is the importance of teacher content knowledge for teaching. Many mathematics educators have argued for the effect of teachers' conceptual understandings of mathematical concepts on their teaching (e.g., Ball, 1990; Hill et al., 2005). While the preservice teachers in this study clearly revealed the difference between their everyday and school volume measurement understandings, it could be assumed that they would struggle to carry out the work of linking everyday mathematics to school mathematics in the instruction of volume measurement.

Issues for Further Research

The study as a whole raises several issues for the study on the growth of preservice teachers' understandings of volume and its measurement regarding contextual relevance. Of particular interest is the established relationship between the volume concept and its context-dependent properties, as well as its potential to change. This issue is also related to how learners' mathematical conceptions are changed as instruction proceeds. The influence of instruction on the growth of mathematical conceptions could be traced by conducting individual interviews before and after the instruction.

The association of volume and capacity is certainly an issue deserving further thought and research, particularly for the developmental progression of the levels of children's understandings and learning of volume measurement. The context of the container presented during the interviews of this study suggests that it would be appropriate to examine, in this context, how children perceive the relationships between volume and capacity.

To examine why there was confusion between the quantity measured by filling a container and the quantity determined by a volume formula, in terms of the difference between exterior and interior volume, is another issue for further research in volume measurement. The difference between exterior and interior volume could be conceived in a physical context but not in an abstract context; this may relate to the progression in mathematical thinking from the concrete to the abstract aspects of volume measurement.

A final issue for further research drawn from this study is the potential role of individual preservice teachers' preconceptions of mathematics concepts or existing mathematical understandings in their teaching and is deserving of further thought and research in teaching and learning mathematics. Namely, it would be constructive to examine how individual preservice teachers' preconceptions of mathematics concepts influence their instruction.

Contributions from This Study

The results of this study contribute to and extend the research literature in mathematics teacher education. As Senk, Park, Demir, and Crespo (2009) pointed out, there has been little research conducted into preservice teachers' content knowledge of geometry, measurement, or spatial reasoning. The research reported here is an attempt to explore elementary preservice teachers' conceptual understandings of volume measurement in terms of their everyday and school understandings of volume and its measurement. I believe that the findings of this study can be of use to teacher educators in supporting the subject matter preparation of preservice teachers in relation to their existing—everyday and school—mathematical understandings.

Another contribution of this study is to mathematics education, as well as educational psychology. As a result of my study, researchers and educators in mathematics education may understand the relationship between knowing/knowledge and the context in which the knowledge is developed in terms of the context-dependent nature of volume measurement understandings. In particular, previous research in the domain of volume measurement thus far provided evidence that context matters in how individuals conceive of volume (e.g., Sáiz, 2003; Potari & Spiliotopoulou, 1996). In addition, the findings of this study provide a clear picture of which conceptions are associated with their particular contexts. As a whole, the results of this study support the view on the influence of context on knowledge development in the field of educational psychology (e.g., Brown et al., 1989; Vasilyeva et al., 2013).

Implications for Instruction

CCSS-M (NGA Center & CCSSO, 2010) address volume as "an attribute of solid figures" (p. 37) and its measurement as "Measure volumes by counting unit cubes" (p. 37), "Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes" (p. 37), and "Apply the formulas $V = l \times w \times h$ and $V = b \times h$ for rectangular prisms" (p. 37). The concept of capacity and the filling aspect of volume measurement strategies are not addressed in the volume measurement standard of CCSS-M. Considering the absence of the capacity concept in the volume measurement standard of CCSS-M, it could be expected that any relationship between volume and capacity may not be treated in volume measurement instruction.

Given the importance of building the instruction of volume measurement on learners' capacity related everyday understandings of volume and its measurement, the findings of this study provide a number of implications for the instruction of volume measurement in school. First, considering the association of volume and capacity observed in this study, the connections between these constructs need to be addressed in volume measurement instruction in relation to contextual relevance. Volume and capacity can be conceptualized as equivalent measurement quantities in particular circumstances, such as in the context of measuring the volume of liquids or pourable substances in relation to containers (Potari & Spiliotopoulou, 1996; Sáiz, 2003). That is, contextual relevance can help to conceptualize the relationship between volume and capacity. This view of contextual relevance in thinking of volume as related to capacity could lead to identifying a contextually appropriate unit in measuring volume within the encountered everyday contexts (e.g., using gallons for gasoline and using liters for drinking water in a bottle).

Second, regarding the context of measurement, given the preservice teachers reflections on the different properties of volume and its measurement, there appears to be a need to involve different physical contexts of measuring volume in mathematics instruction, such as measuring the volume of a fish tank by filling it with water using a container, or estimating the number of one-unit cubes needed to fill a hollow plastic cube. The different contexts of measuring volume may allow learners to recognize their different volume conceptions and measurement strategies in everyday and school contexts and thus, generate rich connections between their everyday and school mathematical understandings. Ideally, the opportunity for reflection on the properties of volume and its measurement in different contexts may provide grounds for helping learners to reason and operationalize measurement in the abstract.

Therefore, in the instruction of volume measurement, it would be useful to expose learners to the measurement contexts that demonstrate the contextual relevance of conceptualizing the relationship between volume and capacity, and identifying the different properties of volume and its measurement, if such learning in school is to be effectively transferred and applied into other school subjects and everyday life situations.

Limitations

Although this study contributes to research literature in mathematics education and mathematics teacher education, as well as educational psychology, there are limitations to the generalizability of the results. It is worth mentioning that the results of this study were based on the data collected from a small number of participating preservice teachers, and were acquired within the limited time frame of the interviews; thus the findings are not representative of all preservice teachers and their knowledge of volume measurement.

The conceptualization of everyday contexts considered in this study is smaller in scope than the contexts of everyday life conceptualized in the literature that investigates mathematical practices in everyday life in terms of "people-doing-in-their-usual-context" (Lave et al., 1984, p. 68), such as examining the arithmetic activities during grocery shopping in supermarkets (Lave et al., 1984), selling merchandise at a market (Nunes et al., 1993), or selling candy in the street (Saxe, 1988); or the grouping activities of dairy products related with spatial arrangement in the warehouse of a milk processing plant (Scribner, 1985). In this study, the everyday context refers to the context of an everyday object, like a fish tank or a recipe; I examined participating preservice teachers' thoughts about volume, and their measurement activities in relation with the presented everyday objects.

Considering that my first language is not English, I want to acknowledge the limitations associated with interviewing preservice teachers in English. When the interview questions were not taken by the interviewees as intended or were misinterpreted, I had difficulties in providing nondirective follow-up questions or in correcting the misinterpreted parts promptly; this caused me to miss opportunities to clarify some of the preservice teachers' responses in the interviews. Thus, the collected interview data is limited in this aspect. Additionally, the analyses of preservice teachers' volume measurement conceptions relied on my interpretation of the preservice teachers' interview data; thus, the interpretation may not be entirely representative of the preservice teachers' thoughts and knowledge about volume and its measurement.

Final Word

I began this dissertation by arguing that mathematical understandings are not only learned in school, but are also developed through everyday experiences. Through this study, I have examined differences between everyday and school mathematical understandings. I believe that school learning should build on everyday mathematical understandings. Drawing on everyday contexts in instruction may enable learners to generate rich connections between their everyday and school mathematical understandings and allow their teachers to develop and deploy those relationships for further learning.

APPENDICES

APPENDIX A

Interview Tasks

Table A.1.

Description of Seven Interview Tasks

Task	Description	Question
1		"I am trying to figure out how people think about volume, so I have a number of questions and tasks for you." "What is the first thought that comes to mind when you hear the term volume?" "Can you tell me more about that?" If the interviewee describes volume related with sound/loudness or books and the given description is not related with volume measurement, the interviewer will ask: "Can you tell me other contexts in which the term volume is used?
2		"Do you ever measure volume in everyday life situations? If the interviewee gives an example (e.g., shipping Christmas presents, fueling gas, or packing moving boxes in a truck), the interviewer will ask: "How would you know the volume of that object?" "Can you tell me more about that?" If the interviewee answers no experience of volume measurement, the interviewer will ask: "How about gasoline in your car? How could you figure out the volume of gas in your car's tank?"
3	Present a picture of an empty rectangular prism-shaped fish tank.	"What if you had a fish tank like this [pull a picture of a fish tank], how would you think about the volume of the fish tank?" If the interviewee answers the way to figure out the volume of the tank instead of describing what the volume of the fish tank is, the interviewer will ask: "OK, you tell me the way to figure the volume of the tank. And then, how would you describe the volume of the tank?" "How would you know the volume of the fish tank? I mean how could you figure it out?"

Table A.1. (cont'd)

Task	Description	Question
4	Present a strawberry shortcake recipe.	"Here is a strawberry shortcake recipe." After the interviewee reads the strawberry shortcake recipe, the interviewer will ask: "If you were going to make this, how could you figure out the amount of strawberries to use?" "How about flour?" "How about baking powder [or baking soda]? How could you figure it out?" "How about salt [or sugar]?" "How about milk?" "How could you figure out the amount of butter to use?"
	On the table, present two 1-pound-pack of strawberries, a bag of all-purpose flour, a half-gallon of milk, and a box of four quarters butter, respectively, with a set of measurement tools: a ruler, a tape measure, measuring cups (a set of dry measuring cups, a set of measuring spoons, and a 4-cup liquid measuring cup), unit cubes (one-inch-size and one-cm-size), two bowls (quart and half-quart), and an electronic scale.	Present two 1-pound-pack of strawberries. "Can you measure the amount of strawberries needed for this recipe with theses [pointing the set of measuring tools]?" "What if you don't have a scale, how can you figure it out?" Present a bag of all-purpose flour. "Can you measure the amount of flour needed for the recipe?" If the interviewee use a measuring cup, either a liquid or dry measuring cup, the interviewer will ask: "What if you don't have a measuring cup, how can you figure it out?" Pull a half-gallon of milk. "Can you measure the amount of milk needed for the recipe?" If the interviewee use a measuring cup, either a liquid or dry measuring cup, the interviewer will ask: "What if you don't have a measuring cup, how can you figure it out?" Present a box of four quarters butter. "How about this? "Can you measure the amount of butter needed for the recipe?" If the interviewee reads the marks on the wrapper of a stick of butter, the interviewer will ask: "What if you don't have the marks on the wrapper (of butter), how could you figure it out?" Pull the recipe again. "Let's look at the recipe again. Can you tell me which one is volume (measure) among the listed ingredients?" "By the way, how much experience do you have in cooking?" "Have you ever tried any cooking recipe before?"

Table A.1. (cont'd)

Task	Description	Question
5	Present a picture of a cube.	"How would you think about the volume of this object [cube]?" If the interviewee answers the way to figure out the volume of the cube instead of describing what the volume of the cube is, the interviewer will ask: "OK, you tell me the way to figure the volume of this object [cube]. And then, how would you describe what the volume of this object [cube]?" "How would you know the volume of this object [cube]? How could you figure it out?"
	Present a picture of a rectangular prism built from unit cubes [rectangular prism].	"How about this object [rectangular prism]? How would you think about the volume of this object [rectangular prism]?" "How would you know the volume of this object [rectangular prism]?"
	Present a picture of a stack of unit cubes [stacked cubes].	"How about this one [stacked cubes]? "How could you figure out the volume of this object [stacked cubes]?"
6	Present a solid wooden rectangular prism [wooden block].	"How would you think about the volume of this object [wooden block]?" "Can you measure the volume of this object with theses [pointing the set of measuring tools]?"
	On the table, present a set of measurement tools: a ruler, a tape measure, measuring cups (a set of dry measuring cups, a set of measuring spoons, and a 4-cup liquid measuring cup), unit cubes (one-inch-size and one-cm-size), two bowls (quart and half-quart), and an electronic scale. Present a calculator when needed.	If the interviewee only provides linear measures, such as length, width, height, or perimeter of the wooden block as a number with/without its measurement unit, ask again: "What is the volume of this object?" [or] "What other things (attributes) can you measure?" If the interviewee starts to calculate the volume of the wooden block based on the linear measures, a calculator will be provided. "You can use it." If the interviewee provides one calculated number with/without its measurement unit as the volume of the object, then point to the calculated number. "What is it? What does it mean?" [or] "How did you get this number?" If the interviewee mentions using a volume formula, the interviewer will ask: "Where does the volume formula come from? What do the numbers in it mean?" If the interviewee does not use or mention a volume formula, the interviewer will ask: "Why you didn't use the volume formula?" [or] "Did you (ever) consider using the volume formula?"

Table A.1. (cont'd)

Task	Description	Question
6 (cont'd)	Present a hollow clear plastic cube [hollow cube].	"How about this one [hollow cube]? How would you think about the volume of this object [hollow cube]?" "How could you measure the volume of this object [hollow cube]?" If the interviewee provides one number by applying a volume formula, the interviewer will ask: "How many cubes do you need to fill this container completely?" If the interviewee provides one number as her estimate, the interviewer will ask: "How did you figure it out?" "If we don't have enough cubes to check your answer, so how can you convince me that your number is correct?"
	Present a hollow clear plastic cylinder [hollow cylinder].	"Let's try this one [hollow cylinder]. How would you think about the volume of this object?" "Can you measure the volume of this object [hollow cylinder] by using the volume formula that you used for this object [hollow cube/wooden block]?"
7		"You described the volume of [an everyday object, e.g., a swimming pool] as [each interviewee's volume description of the everyday object given to previous interview questions] and you also described the volume of [a geometric object] as [her volume description of the geometric object given to previous interview questions]." "How do you see these volume descriptions related?" "In measuring volume, you used different measurement tools, like measuring cups or a ruler, and you also used a volume formula. When do you measure volume by using a measuring cup? When do you measure volume by using a volume formula?" "Can you tell me any everyday context of using/applying a volume formula in measuring volume?" "Do you ever use a volume formula in measuring volume in your life?"

APPENDIX B

Coding Scheme

Table B.1.

Coding Scheme

Category	Code	Coded response
Volume conceptions		
	Substance/liquid volume (drawn on Potari & Spiliotopoulou, 1996; also NGA Center & CCSSO, 2010; Vasilyeva et al., 2013)	“Um, everything that goes inside like how much water would fill it. But then if you had fish in there, that would take up volume too,” (Anne).
	Volume as substance/liquid capacity (drawn on Sáiz, 2003; Potari & Spiliotopoulou, 1996; also Joram et al., 1998)	“I would think about […] how much water that I could put inside of the fish tank to fill it up. Or, I mean, to not fill it up but how much, the capacity that it would hold,” (Eva).
	Complementary volume (Piaget et al., 1960)	“My brother has a fish tank and if it’s full with this much water if you put stuff in it, the water level rises,” (Anne).
	Volume as enclosed liquid	“When I am thinking about it [volume] in terms of the recipe, I would think that the volume, you would be putting the milk into the measuring cup and then the milk would be the volume, taking up space in the cup,” (Becca).
	Volume as space occupied by liquid (drawn on Piaget et al., 1960)	The volume of it? Well, it’d be the amount of space that if you were to pour water in it, how much space that water is taking up, taking on like a 3-D rectangular shape. Yeah, how much it can hold,” (Cacie).
	Calculated Volume (drawn on Sáiz, 2003)	“Well, all the sides would be equal. So, you just multiply this side by this side and this side and they would all be the same so. If one side was x, it would be three x [x^3], the volume of it,” (Grace).
	Number of cubic units (drawn on Piaget et al., 1960; also Curry & Outhred, 2005)	“Mm … how many cubes that can fit inside maybe in this box [hollow cube],” (Grace).
	Whole object (drawn on Potari & Spiliotopoulou, 1996)	“If I'm just thinking about volume as what volume is, I would say this entire object [wood block] is a volume,” (Eva).
	Occupied space (Piaget et al., 1960)	“How much space it’s [hollow cylinder] taking up,” (Eva).
	Enclosed space (Sáiz, 2003)	“All the space that’s inside of the box,” (Flora).

Table B.1. (cont'd)

Category	Code	Coded response
Conceptions of measuring volume		
	Pouring/filling liquid (Curry & Outhred, 2005)	"Oh, by filling it [gas tank] up and seeing how much it can, like the capacity that it can hold," (Eva).
	Calculating volume (drawn on Sáiz, 2003)	"If it's [trash can] circular, it would be to find the area of the circle on the bottom times the height I think," (Cacie).
Volume measurement strategies		
	Using containers	"I guess you could, the first time you fill it up, use a water bottle.… If it is 20 water bottles it would take to fill up the fish tank, then every time you know it's going to be that much," (Becca).
	Using formulas	"Since it's [fish tank] a rectangle, I could find the length and multiply it by the height and the width to figure out how much water would go in it [pointing each dimension]," (Becca)
	Reading the marks on the wrapping of butter for measuring	"Butter? I know. Um ... Butters, they usually, when it's a stick of butter, it has the numbers on it so you could do three fourths of a cup. Just cut it," (Anne).
Conceptions of the volume formula, $l \times w \times h$		
	Length-by-width-by-height structuring	[Taking one-cm-size cubes] Do this. … [Aligning the cubes to the width of the front face.] That's pretty close. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. So 10 by 10 by 10. It would be [one] thousand cubes," (Anne).
	Layer structuring (Battista & Clements, 1996)	"Yeah. If you … if you just were looking at one side [as drawing a square on her note] there would be 10 blocks here … as you can see. And then, 10 blocks here. But if you fill the whole thing up so … I fill the whole thing up … No, I don't want to fill it up. I can explain it … Then, it would be 10 times 10, like length times height, which is a hundred. You will know that … If you know that one side is a hundred and there are 10 of these hundreds, then you can more like see there would be a thousand because you have 10 groups of a hundred," (Grace)

Table B.1. (cont'd)

Category	Code	Coded response
Conceptions of the calculated volume by $l \times w \times h$		
	Product resulting from triple multiplication (drawn on Sáiz, 2003)	"Since I measured in inches, volume is always the measurement, like the unit ends up being cubed because you multiplied the three lengths," (Anne).
	Number of cubic units filling and comprising an object (drawn on Piaget et al., 1960)	"128, it means that that's the number of cubic centimeters that can be put in here [wooden block]," (Flora).
	Amount of substance comprising an object (drawn on Potari & Spiliotopoulou, 1996)	"7.2 inches of wood make up this [wooden] block," (Becca).
Reflections on the relationship between $l \times w \times h$ and $\pi r^2 \times h$		
	Different formulas for different shapes	"I don't think so. I'm assuming it has something to do with the diameter of the circle," (Eva).
	Same sequence/procedure of computation	"They account for three different measurements, making it a 3-D shape. For instance, if you just were to measure one length of the base, that'd be 1-D, one-dimensional. Just like the radius is one-dimensional. Then by squaring it and multiplying it by *pi*, you're making it 2-D because you're getting the area of the circle. Same with adding like a width or a length, whatever you call it. That's 2-D. Then they both have a height going up to make it 3-D. So, they are similar," (Cacie).
Reflections on two different volume measurement strategies, using a measuring cup and using a volume formula		
	Using different strategies by the nature of measurement contexts	"This [using the volume formula $l \times w \times h$] would be to figure out how much is in it [hollow cube] and that [using a measuring cup] would be to show how much you need," (Anne).
	Using a measuring cup in everyday context and using a volume formula in school	"I always go to the equation just because I know that that's how you go about solving it and it's proven to work and stuff," (Cacie); "For baking I wouldn't use math to figure things out," (Cacie).

APPENDIX C

Individual Participants’ Overall Volume Conceptions Across Contexts

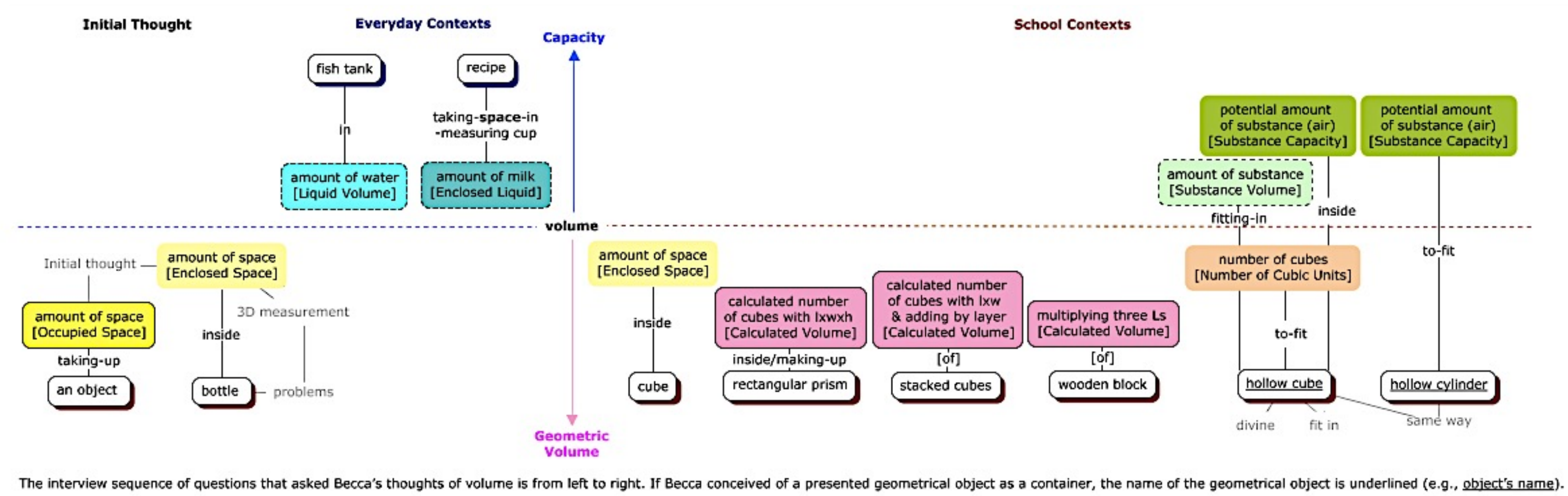


Figure C.1. Becca's overall volume conceptions revealed across contexts

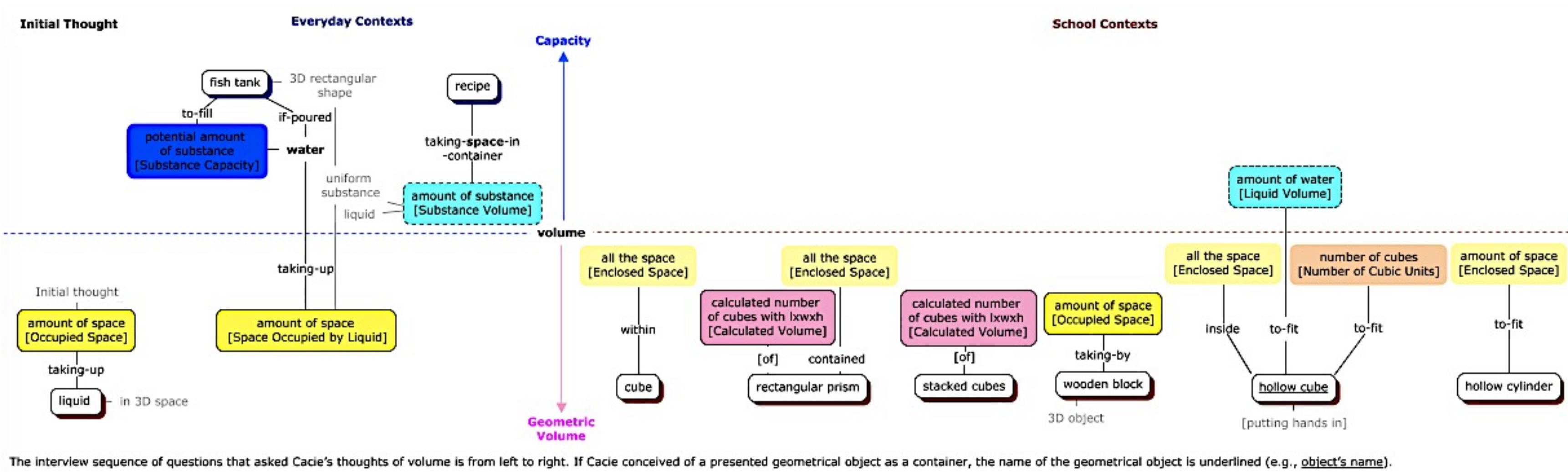


Figure C.2. Cacie's overall volume conceptions revealed across contexts

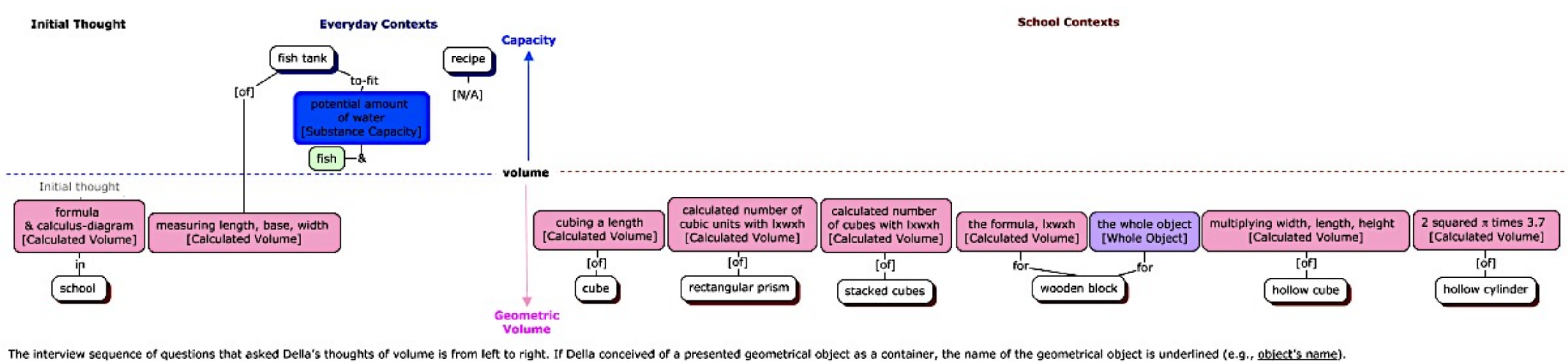
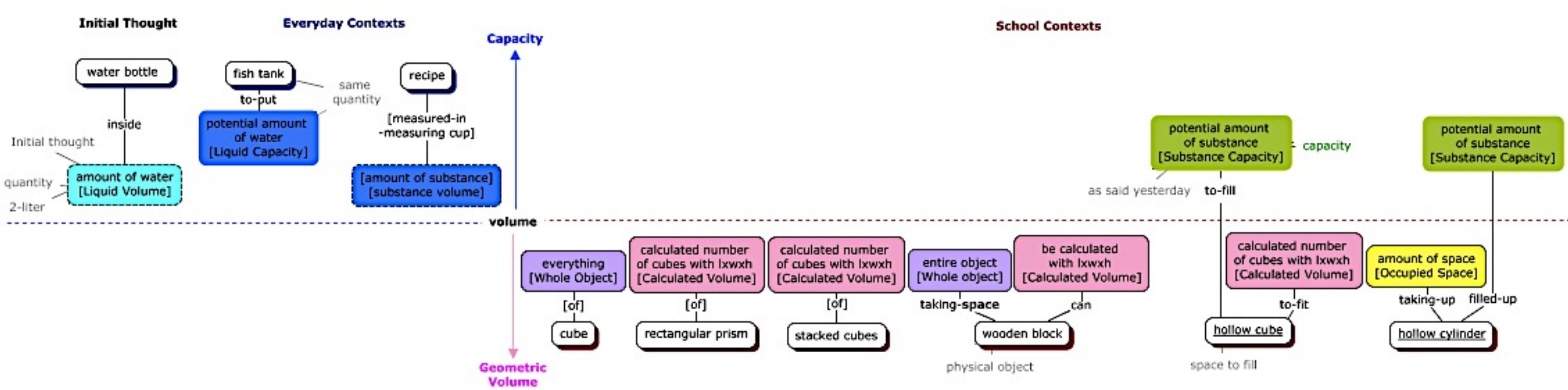


Figure C.3. Della's overall volume conceptions revealed across contexts



The interview sequence of questions that asked Eva's thoughts of volume is from left to right. If Eva conceived of a presented geometrical object as a container, the name of the geometrical object is underlined (e.g., object's name).

Figure C.4. Eva's overall volume conceptions revealed across contexts

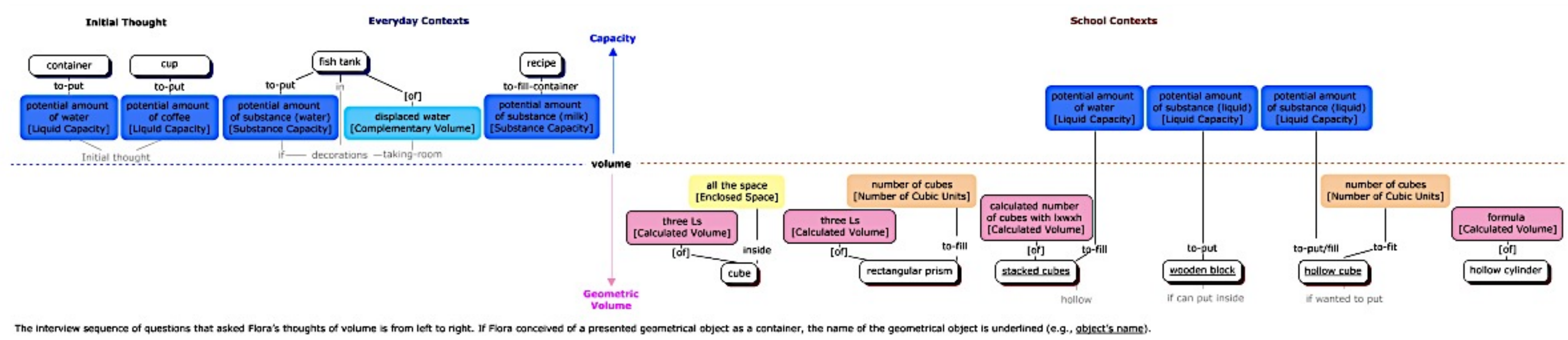


Figure C.5. Flora's overall volume conceptions revealed across contexts

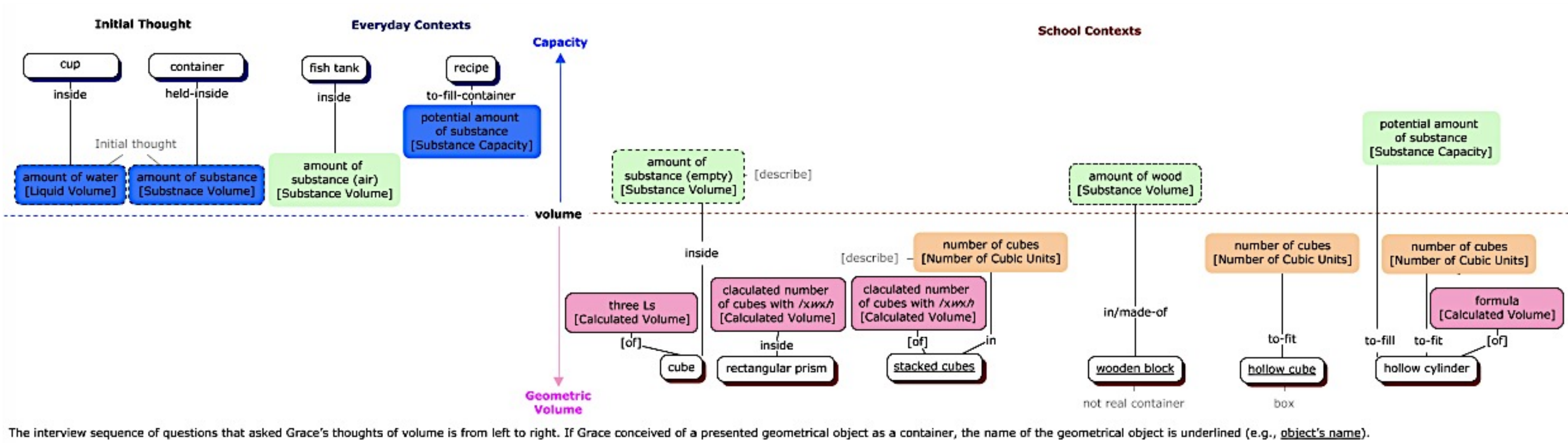


Figure C.6. Grace's overall volume conceptions revealed across contexts

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