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thesis entitled

ON THE PROBLEM OF A PLANE, FINITE, LINEAR-ELASTIC

REGION CONTAINING A HOLE OF ARBITRARY SHAPE:

A BOUNDRY INTEGRAL APPROACH

presented by

Ali Reza Mir Mohamad Sadegh

has been accepted towards fulfillment of the requirements for

Ph.D. degree in Metallurgy, Mechanics and Material Science

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Major professor

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ON THE PROBLEM OF A PLANE, FINITE, LINEAR-ELASTIC REGION CONTAINING A HOLE OF ARBITRARY SHAPE: A BOUNDARY INTEGRAL APPROACH

By

Ali Reza Mir Mohamad Sadegh

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
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Department of Metallurgy, Mechanics and Material Science

1978

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ABSTRACT

ON THE PROBLEM OF A PLANE, FINITE, LINEAR-ELASTIC REGION CONTAINING A HOLE OF ARBITRARY SHAPE: A BOUNDARY INTEGRAL APPROACH

By

Ali Reza Mir Mohamad Sadegh

Previous boundary integral equation methods have been developed for problems of two-dimensional elastostatics which yield excellent results everywhere except near the boundary. This presents a major disadvantage for problems in which a hole, slot or sharp crack is present, since such an opening must be considered as boundary. Thus, results in the vicinity of the hole are not reliable. In this dissertation a new formulation of the boundary integral method is presented which eliminates this inaccuracy on and near the opening. This is done by replacing the kernel of the integrand (the influence function) with one which includes the effect of the opening. This influence function is determined in terms of the complex potential functions for an infinite elastic plane containing the opening and subjected to a concentrated line load at an arbitrary point. This is accomplished using the Muskhelishvili method of plane elasticity. Potential functions are found for the cases of a circular hole, an

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elliptical hole and a sharp crack. The determination of these functions for other opening shapes is also discussed. The boundary integral equation method is then applied to some finite regions containing either a circular hole, an elliptical hole, or a sharp crack. The results are presented and compared with exact solutions and experimental results where available.

To my brother, my mother

and my father

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LIST OF SYMBOLS

a	diameter of the circular hole or semi-major axis of the elliptical hole
a_k	coefficient of the power series expansion of $\phi_1^*(\zeta)$
A_{ij}	sub-matrix of matrix RM
b	semi-minor axis of the elliptical hole
b_k	coefficient of the power series expansion of $\Psi_1^*(\zeta)$
B	boundary of region
B_{ij}	sub-matrix of matrix RM
BV_{xi}	x component of the boundary value at sub-division i
BV_{yi}	y component of the boundary value of sub-division i
C	contour of the hole
C_{ij}	sub-matrix of matrix RM
dn	coefficient of power series in the equation of opening
D_{ij}	sub-matrix of matrix RM
F	field point
g_k	coefficient used in the transformation function
h_n	coefficient of power series in the equation of opening
$H_{ij;q}(Z,Z_0)$	ijth stress component at the point Z due to a unit load acting in the q direction at the point Z_0

α_k	coefficient for the expansion series of transformation function
β_k	coefficient for the expansion series of inverse transformation function
γ	circumference of unit circle
Γ	a function of Ω and W
δ	coefficient used in the transformation function
ε	constant coefficient used in the equation of opening
ζ	point in the transformed plane
η	coefficient used in the transformation function
θ	angle (measured counterclockwise)
λ	coefficient used in the transformation function
μ	shear modulus of elasticity (MPa)
ν	Poisson's ratio
ξ	coefficient used in the transformation function
ρ_i	coefficient of expansion of inverse transformation
σ_{ij}	ijth stress component at field point
$\sigma, \overline{\sigma}$	boundary point of the unit circle
ϕ	harmonic function
$\phi(Z), \phi_1(\zeta)$	complex potential functions
$\Psi(Z), \Psi_1(\zeta)$	and transformed complex potentials
$\chi(Z)$	an analytic function
$\omega(\zeta)$	the mapping function

$I_{i;q}(Z,Z_0)$	ith displacement component at the point Z due to a unit load acting in the q direction at the point Z_0
ℓ	constant used in the transformation function
m	constant used in the equation of ellipse
n_i	outward direction cosine to boundary
P_{xi}, P_{yi}	real line load components in the ith interval
P^*_{xi}, P^*_{yi}	fictitious line load components in the ith interval
PD	function of ζ
PDD	derivative of PD
r_i, r_o	roots of $A(\sigma)=0$ which fall inside and outside the unit circle, respectively
r_x, r_y	x, y distances from field point to boundary point
s	co-ordinate along boundary
t	point on hole boundary
t_i, t_o	roots of $B(\sigma)=0$ which fall inside and outside the unit circle, respectively
U_i	ith component of displacement at field point
U	biharmonic function
X_0	X component of boundary point Z_0
$X(x,y)$	harmonic function
Y_0	Y component of boundary point Z_0
$Y(x,y)$	harmonic function
Z	field point
Z_0	point on contour B where line load is applied
Z_1	point on contour B where boundary conditions are to be satisfied
α	function of Poisson's ratio

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INTRODUCTION

One of the most fundamental problem areas in elasticity is the effect of an opening in an elastic region. The literature is filled with analyses, analytical, numerical and experimental, of problems involving holes in finite bodies subjected to prescribed load. A renewed, and intense, interest in this type of problem has evolved with the advent of fracture mechanics, in which the "hole" takes on the shape of a sharp crack. Although specific problems (i.e., specific shaped regions containing a specific shape of opening and subjected to specific boundary conditions) have been defined and solved, there still exist a large number of fundamentally important problems which do not lend themselves to analytic solution. These analytic approaches can be accomplished only when the geometry and loading are simple. Numerical analyses (such as finite elements, finite differences and boundary integral methods) have played a big role in the solution of problems of arbitrary geometry and loading.

In this dissertation, a boundary integral equation method has been used to solve problems of this type. This method has definite advantages over finite differences and finite elements for various types of problems. In bou

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boundary integral methods, the stresses are obtained at a point and, for locations greater than one boundary subdivision from the boundary, they are extremely accurate. These methods do not require discretization of the domain as with finite elements or finite differences, but merely discretization of the boundary. This leads to a coefficient matrix which is of lower order than would be obtained by finite elements and finite differences. Indeed, within the past ten years, boundary integral equation methods have been applied to three dimensional isotropic [1] and anisotropic [2] elasticity, plasticity [3], plate theory [4], fracture mechanics [5] and a broad range of other applications [6].

The main limitation of the boundary integral equation method is the inaccuracy of the results near the boundary. This is due to the fact that the boundary is discretized and clearly the error worsens near the discretization. This is not usually a bothersome limitation but, in the case of problems involving an opening, multiply connected regions, the opening must also be considered as a part of the boundary and therefore the largest error occurs in the most important region (i.e., near the opening).

In this dissertation, a new formulation of the boundary integral equation method is presented in which the opening is no longer considered as part of the boundary since its effect on the stress and displacement fields is incorporated into the kernel of the integral equations. This is done using the theory of complex vari

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variables in elasticity, namely the Muskhelishvili method, and the Cauchy integral theorems. The results are highly accurate along and near the opening.

In Chapter I, the boundary integral equation method, the complex variable analysis of elasticity (the Muskhelishvili method) and the Cauchy integral theorems are presented. A mapping technique and a general solution for finding the kernel of the integrands of the boundary integral equations is introduced in Chapter II.

In Chapter III, the boundary integral equations presented in Chapter II are used to solve the problem of a plane finite linear elastic region containing a circular hole. The kernel of these equations is replaced by a kernel which incorporates the effects of the hole on the elastic field. This kernel is derived using the mapping technique, the Muskhelishvili method and complex variable theory. Some examples are presented and compared to some known solutions.

In Chapter IV, this solution technique is extended to the problem of a finite plane linear elastic region containing an elliptical hole or a sharp crack. Again, a kernel which incorporates the effect of the elliptical hole or the sharp crack is derived and replaces the kernel of the equations of Chapter II. Some example problems are presented and, in the case of the sharp crack, solutions are compared to some recently obtained experimental results.

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In Chapter V, the extension of the solution technique to the problem of a finite plane linear elastic region containing other types of opening is discussed.

Finally, the closure and conclusions are presented in Chapter VI. The computer programs used for computations in Chapters III and IV are included in the Appendices.

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CHAPTER I

BACKGROUND AND PRELIMINARIES

I.1 AN INTEGRAL EQUATION METHOD

Introduction

The first application of the methods of potential theory to classical elasticity theory was introduced by E. Betti [7] in 1872. Later this work was expanded by Somigliana [8], Lauricella [9] and others. In particular, Betti's contribution, i.e., the general method of integrating the equations of elasticity, was simply a development of the potential methods of Green and Poisson. Thus, some fundamental results from potential theory should first be discussed. Let a function ϕ be the solution of Laplace's equation throughout a region R:

$$\nabla^2\phi = 0 \qquad \text{in } R \tag{1.1}$$

subjected to the boundary conditions

$$\phi = f \qquad \text{on } \partial R_1$$

$$\frac{\partial\phi}{\partial n} = g \qquad \text{on } \partial R_2 \tag{1.2}$$

where $\partial R = \partial R_1 + \partial R_2$ is the boundary of R. Note that $1/r$, where r is the distance between two points in R, is a sin

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a singular solution to Laplace's equation. Combining 1/r with the solution ϕ in the classical Green's theorem of integral calculus [10] results in the identity,

$$\phi(Z) = \frac{1}{4\pi}\int_{\partial R}\left[g(Z_0)\frac{1}{r(Z,Z_0)} - f(Z_0)\,\frac{\partial}{\partial n}\left(\frac{1}{r(Z,Z_0)}\right)\right]\cdot\, ds(Z_0) \tag{1.3}$$

where Z is any point in R and Z_0 any point on ∂R. Since f and g are both needed everywhere on ∂R, only one of them is known at each point, then the other has to be found. To accomplish this [11-13], consider taking the limit of equation (1.3) as Z approaches a boundary point Z_1, on ∂R. The result is:

$$f(Z_1) - \frac{1}{2\pi}\int_{\partial R}\left[g(Z_0)\;\frac{1}{r(Z_1,Z_0)} - f(Z_0)\;\frac{\partial}{\partial n_{Z_0}}\left(\frac{1}{r(Z_1,Z_0)}\right)\right] ds(Z_0)=0 \tag{1.4}$$

where the limit as $Z \to Z_1$, of $\phi(Z)$ is, by definition, $f(Z_1)$ and the integral in equation (1.3) has a jump of $f(Z_1)/2$ in the limit [11]. Thus, this integral is to be interpreted in the Cauchy principal value sense. Equation (1.4) is the "Boundary-Integral Equation" which relates g and f. Solving the integral equation (1.4) for either f, if g is given, or g, if f is given, leads to the solution ϕ.

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In spite of this classical foundation of the boundary integral equation method, the literature contains at least two seemingly distinct formulations for the treatment of elasticity problems. One of these, due to Rizzo et al. [14-16] and Cruse [17], follows directly from Somigliana's identity of elasticity [18]. The other formulation due to Massonnet [19] and extended by Altiero and Sikarskie [20] attacks the problem by embedding the region in an infinite plane and distributing a layer of body force on the proposed boundary in such a way that the desired solution is produced within the region of interest. Both approaches will be discussed in this chapter and the latter will be employed in the subsequent analysis.

The formulation of the boundary integral equation method due to Rizzo [14] is based on Somigliana's identity:

$$U_q(Z) = \int_B I_{i;q}(Z,Z_0)\, t_i(Z_0)\, ds(Z_0) - \int_B H_{ij;q}(Z,Z_0)\, U_i(Z_0)\, n_j(Z_0)\, ds(Z_0) \tag{1.5}$$

where $\vec{U}$ is the displacement vector, $I_{i;q}(Z,Z_0)$ is the ith component of the displacement at Z produced by a unit force applied in the q direction at Z_0 in an infinite medium, and t_i and $H_{ij;q}(Z,Z_0)\, n_j(Z_0)$ are the components of the boundary traction corresponding to the displacements U_i and $I_{i;q}$, respectively. In three dimensions, ds represents an element of area, in two dimensions, an element of arc length.

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Taking the limit as Z approaches a point Z_1 on B from the inside leads to

$$\frac{1}{2} U_q(Z_1) + \int_B H_{ij;q}(Z_1,Z_0)\ U_i(Z_0)\ n_j(Z_0)\ ds(Z_0)$$

$$= \int_B I_{i;q}(Z_1,Z_0)\ t_i(Z_0)\ ds(Z_0) \qquad (1.6)$$

where the integral on the left-hand side is to be interpreted in the Cauchy principal value sense. If the traction is prescribed everywhere on B, then the right-hand side of equation (1.6) is known and a system of singular integral equations can be solved for the boundary displacement $\vec{U}$. The interior displacements can then be found from equation (1.5). If the displacements are prescribed everywhere on B, then the left-hand side of equation (1.6) is known but the resulting set of integral equations are not singular. For the mixed boundary conditions, some of the equations are singular and some are not.

The second formulation of the boundary integral method, i.e., that of Altiero and Sikarskie [20], is an extension of the work of Massonnet [19]. Massonnet introduced a method for solution of traction boundary value problems in which the real body is embedded in a series of "fictitious" half planes which are sequentially tangent to the real boundary. To demonstrate the idea, consider a finite two dimensional region with a prescribed traction all around the boundary, Figure 1.1. Choose the

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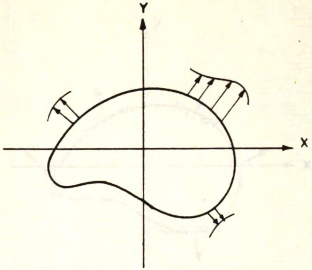

Figure 1.1. Finite two-dimensional region with prescribed traction all around the boundary.

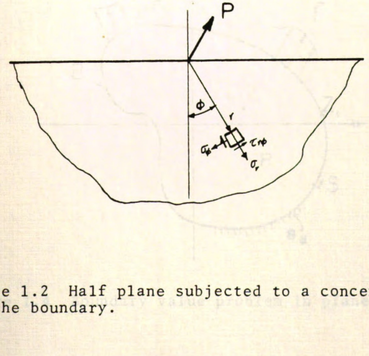

Figure 1.2 Half plane subjected to a concentrated force on the boundary.

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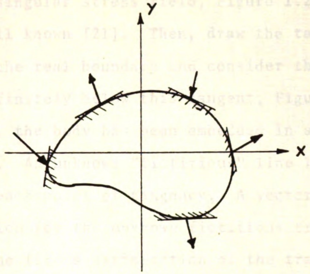

Figure 1.3 Region of Figure 1.1 embedded successively in half planes.

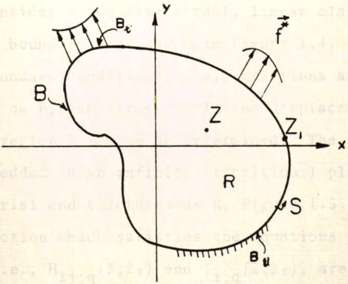

Figure 1.4 Boundary value problem in plane elasticity.

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simple radial stress distribution, i.e., a half plane subjected to a concentrated line load on the boundary, as a fundamental singular stress field, Figure 1.2. This solution is well known [21]. Then, draw the tangent to a point, Z_0, of the real boundary and consider the half plane extending indefinitely below this tangent, Figure 1.3. In other words, the body has been embedded in a succession of half planes. An unknown "fictitious" line load is introduced at each point of tangency. A vector boundary integral equation for the unknown fictitious tractions results when one forces satisfaction of the traction boundary conditions of the original problem.

An approach somewhat similar to Massonnet's has been developed [22] for anisotropic regions subject to traction boundary conditions. This approach, later, was extended by Altiero and Sikarskie [20] to mixed boundary value problems. Consider a two dimensional, linear elastic region R with boundary B as shown in Figure 1.4. For prescribed boundary conditions, i.e., tractions and/or displacements on B, the stress field and displacement field in the region R are to be determined. The region R will be embedded in an infinite (fictitious) plane of the same material and thickness as R, Figure 1.5. The influence function which satisfies the equations of elasticity, i.e., $H_{ij;q}(Z,Z_0)$ and $I_{i;q}(Z,Z_0)$, are known [23], where $H_{ij;q}(Z,Z_0)$ is the ijth stress component at a field point Z due to a unit line load in the q direction

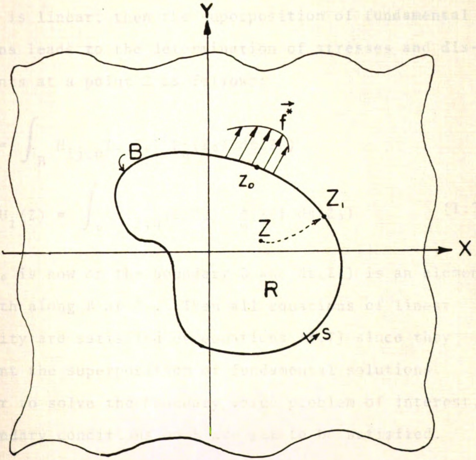

Figure 1.5 Region R embedded in an infinite plane.

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at a source point Z_0 and $I_{i;q}(Z,Z_0)$ is the displacement in the i direction at Z due to the unit line load at Z_0. Consider now a fictitious layer of body force $\vec{P}^*$ (unknown) acting along the contour B, see Figure 1.5. Since the problem is linear, then the superposition of fundamental solutions leads to the determination of stresses and displacements at a point Z as follows:

$$\sigma_{ij}(Z) = \int_B H_{ij;q}(Z,Z_0)\ P_q^*(Z_0)\ ds(Z_0)$$

$$U_i(Z) = \int_B I_{i;q}(Z,Z_0)\ P_q^*(Z_0)\ ds(Z_0) \qquad (1.7)$$

where Z_0 is now on the boundary B and $ds(Z_0)$ is an element of length along B at Z_0. Then all equations of linear elasticity are satisfied by equations (1.7) since they represent the superposition of fundamental solutions. In order to solve the boundary value problem of interest, the boundary conditions on B are yet to be satisfied. These conditions are:

$$\sigma_{ij}\ n_j = P_i^s \qquad \text{on } B_t$$

$$U_i = U_i^s \qquad \text{on } B_u \qquad (1.8)$$

where n_j is the j-component of the outward unit normal to a point on B and P_i^s, U_i^s are the specified traction and displacement components, respectively. Note that one may also specify one traction component and one displacement compo

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component at a particular boundary point, provided they are mutually orthogonal. Let the interior point Z approach a boundary point Z_1, on B, Figure 1.5. Then the stresses and displacements, equations (1.7), must satisfy the boundary conditions of equations (1.8). Thus, substitution of equations (1.7) into equations (1.8) leads to

$$\frac{1}{2} P_i^*(Z_1) + \oint\!\!\!\!-_B H_{ij;q}(Z_1,Z_0)\ P_q^*(Z_0)\ n_j(Z_1)\ ds(Z_0)$$

$$= P_i^S(Z_1)$$

$$Z_1 \text{ on } B_t \tag{1.9}$$

$$\oint_B I_{i;q}(Z_1,Z_0)\ P_q^*(Z_0)\ ds(Z_0) = U_i^S(Z_1)$$

$$Z_1 \text{ on } B_u \tag{1.10}$$

Note that the subscript i refers to a co-ordinate direction at a boundary point Z_1. Equations (1.9) and (1.10) represent coupled integral equations in the unknown fictitious traction $\vec{P}^*$. Note that the singularity has been extracted from equation (1.9) and the integral of this equation is to be interpreted in the Cauchy principal value sense. Equations (1.9) and (1.10) contain several types of problem. For the first fundamental problem of plane elasticity, i.e., traction boundary conditions only, the vector equation (1.9) is to be used. For the mixed boundary value problem, both equations appear but not in the same direction at the same point, i.e., if a traction is specified in the i direction at Z_1, then equation (1.9) holds; and if a displacement is specified, then equation (1.10) holds.

Like the Rizzo formulation, equations (1.9), i.e., the traction boundary value problem, are singular. However, for displacement boundary value problems, equations (1.10) are used and these are not singular. For mixed boundary value problems, some of the equations will be singular and some not.

It is felt that the formulation of Altiero and Sikarskie is preferable for the following reason. In the method of Rizzo, one must first perform integration around the boundary before the required integral equations are defined. This is clearly not necessary in the Altiero and Sikarskie formulation, where one merely needs to specify the tractions and displacements themselves and the right-hand sides of the required integral equation are immediately known. Therefore, the Altiero and Sikarskie formulation will be used here.

Note the fact that the Altiero and Sikarskie formulation is not restricted to embedment in an infinite plane. Massonnet, as discussed earlier, used embedment in a succession of half planes. However, to obtain singular equations for the traction problem, and therefore more numerically efficient equations, this approach requires tangency of the half plane to the embedded body successively around the boundary.

The Massonnet approach is therefore somewhat cumbersome and particularly inconvenient, particularly for the solution of anisotropic elasticity problems [22]. Also, it is very difficult to apply this method to multiply connected regions. Whichever formulation is used, the fundamental solutions for the fictitious region should be simple. This is best satisfied by the infinite plane.

Once $\vec{P}^*$ has been determined, the stresses and displacements at any field point can be determined by substituting $\vec{P}^*$ into equations (1.7). These stresses and displacements represent the solution to the boundary value problem of interest within R. The influence functions, i.e., the stress and displacement fields in component form, due to a concentrated line load $\vec{P}^*ds$ in an infinite plane are given by Love [23]. These are:

$$H_{xx;q}P^*_q = -\frac{1}{4\pi r^4}\left[P^*_x r_x(a_1 r_x^2 + a_2 r_y^2) + P^*_y r_y(a_3 r_x^2 - a_2 r_y^2)\right]$$

$$H_{yy;q}P^*_q = -\frac{1}{4\pi r^4}\left[P^*_x r_x(a_3 r_y^2 - a_2 r_x^2) + P^*_y r_y(a_1 r_y^2 + a_2 r_x^2)\right]$$

$$H_{xy;q}P^*_q = -\frac{1}{4\pi r^4}\left[P^*_x r_y(a_1 r_x^2 + a_2 r_y^2) + P^*_y r_x(a_2 r_x^2 + a_1 r_y^2)\right] \tag{1.11}$$

$$I_{x;q}P^*_q = -\frac{1}{4\pi r^2}\left[P^*_x(a_4 r^2 \log r + a_5 r_y^2) - P^*_y a_5 r_x r_y\right]$$

$$I_{y;q}P^*_q = -\frac{1}{4\pi r^2}\left[-P^*_x a_5 r_x r_y + P^*_y(a_4 r^2 \log r + a_5 r_x^2)\right] \tag{1.12}$$

where r_x and r_y are the x,y components of the radius vector from Z to Z_0 and the constants a_1 through a_5 for the problem of plane strain are

$$a_1 = (3-2\nu)/(1-\nu)$$

$$a_2 = (1-2\nu)/(1-\nu)$$

$$a_3 = (1+2\nu)/(1-\nu)$$

$$a_4 = (3-4\nu)(1+\nu)/(1-\nu)$$

$$a_5 = (1+\nu)/(1-\nu) \tag{1.13}$$

and, for plane stress, ν is replaced by ν^* in all the coefficients of equation (1.13) where:

$$\nu^* = \frac{\nu}{1+\nu}$$

The influence functions found in equations (1.11) and (1.12) can be obtained using the complex potential functions associated with the concentrated line load in an infinite plane. This will be discussed further in the next section.

The solution to any boundary value problem of plane elasticity is contained in equations (1.9), (1.10), and (1.7). For tractions specified everywhere on B, equations (1.9) are to be solved for $\vec{P}^*$. These values of $\vec{P}^*$ are then substituted into equations (1.7) to find the stresses and displacements at any field point. Equations (1.7) may give the displacement field to within a rigid body displacement. The rigid body displacement, however, can be el

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be eliminated by suitably prescribing sufficient boundary displacement information. For displacements specified everywhere on B, equations (1.10) are solved for $\vec{P}^*$ and equations (1.7) again used to find the stress and displacement fields. For mixed conditions at a point either the x component equation (1.9) and the y component of equations (1.10) or the converse must be satisfied.

To obtain a numerical solution to equations (1.9) and (1.10), the boundary is first replaced by an N-sided polygon with sides of arbitrary length ΔS_i. The resultant boundary data over the interval ΔS_i is now defined at the midpoint of each interval as follows:

$$P^*_{xi} = \int_{\Delta S_i} P^*_x ds \quad , \qquad P^*_{yi} = \int_{\Delta S_i} P^*_y ds$$

$$P_{xi} = \int_{\Delta S_i} P^s_x ds \quad , \qquad P_{yi} = \int_{\Delta S_i} P^s_y ds$$

$$U_{xi} = \int_{\Delta S_i} U^s_x ds \quad , \qquad U_{yi} = \int_{\Delta S_i} U^s_y ds \tag{1.14}$$

Note that the superscript * implies the fictitious component. Multiplying equations (1.9) and (1.10) by $ds(Z_1)$ leads to

$$\frac{1}{2} P^*_i(Z_1)\, ds(Z_1) + \oint_B H_{ij;q}(Z_1,Z_0)\, P^*_q(Z_0)\, ds(Z_0) \cdot n_j(Z_1) ds(Z_1)$$

$$= P^s_i(Z_1)\, ds(Z_1)$$

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$$\oint_B I_{i;q}(Z_1,Z_0)\ P_q^*(Z_0)\ ds(Z_0)\ \cdot ds(Z_1)$$

$$= U_i^s(Z_1)\ ds(Z_1) \qquad (1.15)$$

Integrating equations (1.15) over boundary interval ΔS_i and assuming that the influence functions are independent of Z_0 over a particular interval yields:

$$\frac{1}{2} P_i^*(Z_1) + \oint_B H_{ij;q}(Z_1,Z_0)\ P_q^*(Z_0)\ \cdot n_j(Z_1)\ ds(Z_1)$$

$$= P_i(Z_1)$$

$$\oint_B I_{i;q}(Z_1,Z_0)\ P_q^*(Z_0)\ ds(Z_1) = U_i(Z_1) \qquad (1.16)$$

A simple integration procedure can be employed here, i.e., simply multiplying the value of the integrand at the mid-point of an interval by the interval length. This is sufficient for all intervals except for the second equation in which the kernel of $I_{i;q}(Z,Z_0)$ is undefined when $Z_1 = Z_0$. Thus, this one interval must be integrated analytically and the integrals can be written as:

$$\frac{1}{2} P_i^*(Z_1) + \sum_{\substack{Z_0=1 \\ Z_0 \neq Z_1}}^{N} H_{ij;q}(Z_1,Z_0)\ P_q^*(Z_0)\ n_j(Z_1)\ \Delta S(Z_1)$$

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$$g_i\, P_i^*(Z_1) + \sum_{\substack{Z_0=1 \\ Z_0 \neq Z_1}}^{N} I_{i;q}(Z_1,Z_0)\, P_q^*(Z_0)\, \Delta S(Z_1)$$

$$= U_i(Z_1) \tag{1.17}$$

$$\text{where } g_i = \int_{\Delta S_i} I_{i;q}(Z_1,Z_0)\, ds(Z_1) \tag{1.18}$$

over the interval which includes $Z_1 = Z_0$. Note that the boundary points Z_0 and Z_1 in the discrete terms will now represent the center-point location of the intervals, numbered counterclockwise. Separating x and y components, one obtains

$$\frac{1}{2} P_x^*(Z_1) + \sum_{\substack{Z_0=1 \\ Z_0 \neq Z_1}}^{N} \Big[H_{xx;q}(Z_1,Z_0)\, P_q^*(Z_0)\, n_x(Z_1)$$

$$+ H_{xy;q}(Z_1,Z_0)\, P_q^*(Z_0)\, n_y(Z_1) \Big] \Delta S(Z_1) = P_x(Z_1)$$

$$\frac{1}{2} P_y^*(Z_1) + \sum_{\substack{Z_0=1 \\ Z_0 \neq Z_1}}^{N} \Big[H_{xy;q}(Z_1,Z_0)\, P_q^*(Z_0)\, n_x(Z_1)$$

$$+ H_{yy;q}(Z_1,Z_0)\, P_q^*(Z_0)\, n_y(Z_1) \Big] \Delta S(Z_1) = P_y(Z_1) \tag{1.19}$$

$$g_x P_x^*(Z_1) + \sum_{Z_0=1}^{N} \Big[I_{x;q}(Z_1,Z_0)\, P_q^*(Z_0) \Big] \Delta S(Z_1) = U_x(Z_1)$$

$$g_y P_y^*(Z_1) + \sum_{Z_0=1}^{N} \Big[I_{y;q}(Z_1,Z_0)\, P_q^*(Z_0) \Big] \Delta S(Z_1) = U_y(Z_1) \tag{1.20}$$

where the influence functions can now be written as

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$$H_{ij;q}(Z_1,Z_0)\ P^*_q(Z_0) = H_{ij;x}(Z_1,Z_0)\ P^*_x(Z_0)$$

$$+ H_{ij;y}(Z_1,Z_0)\ P^*_y(Z_0)$$

$$I_{i;q}(Z_1,Z_0)\ P^*_q(Z_0) = I_{i;x}(Z_1,Z_0)\ P^*_x(Z_0)$$

$$+ I_{i;y}(Z_1,Z_0)\ P^*_y(Z_0)$$

$$i,j = x,y \qquad (1.21)$$

Substituting equations (1.21) into (1.19) and (1.20) and rearranging the equations leads to

$$\tfrac{1}{2}P^*_x(Z_1) + \sum_{\substack{Z_0=1\\Z_0\neq Z_1}}^{N} \Big\{ \Big[H_{xx;x}(Z_1,Z_0)\ n_x(Z_1)$$

$$+ H_{xy;x}(Z_1,Z_0)\ n_y(Z_1) \Big] P^*_x(Z_0) + \Big[H_{xx;y}(Z_1,Z_0)\ n_x(Z_1)$$

$$+ H_{xy;y}(Z_1,Z_0)\ n_y(Z_1) \Big] P^*_y(Z_0) \Big\} \Delta S(Z_1) = P_x(Z_1)$$

$$\tfrac{1}{2}P^*{}_y(Z_1) + \sum_{\substack{Z_0=1\\Z_0\neq Z_1}}^{N} \Big\{ \Big[H_{xy;x}(Z_1,Z_0)\ n_x(Z_1)$$

$$+ H_{yy;x}(Z_1,Z_0)\ n_y(Z_1) \Big] P^*_x(Z_0) + \Big[H_{xy;y}(Z_1,Z_0)\ n_x(Z_1)$$

$$+ H_{yy;y}(Z_1,Z_0)\ n_y(Z_1) \Big] P^*_y(Z_0) \Big\} \Delta S(Z_1) = P_x(Z_1) \qquad (1.22)$$

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$$g_x \cdot P_x^*(Z_1) + \sum_{\substack{Z_0=1 \\ Z_0 \neq Z_1}}^{N} \Bigg\{ I_{x;x}(Z_1,Z_0)\ P_x^*(Z_0)$$

$$+ I_{x;y}(Z_1,Z_0)\ P_y^*(Z_0) \Bigg\} \Delta S(Z_1) = U_x(Z_1)$$

$$g_y \cdot P_y^*(Z_1) + \sum_{\substack{Z_0=1 \\ Z_0 \neq Z_1}}^{N} \Bigg\{ I_{y;x}(Z_1,Z_0)\ P_x^*(Z_0)$$

$$+ I_{y;y}(Z_1,Z_0)\ P_y^*(Z_0) \Bigg\} S(Z_1) = U_y(Z_1) \tag{1.23}$$

Equations (1.22) and (1.23) can be written in the compact form:

$$K_{xi}\ P_{xi}^* + \sum_{\substack{j=1 \\ j \neq i}}^{N} \left(A_{ij}\ P_{xj}^* + B_{ij}\ P_{yj}^* \right) = BV_{xi}$$

$$K_{yi}\ P_{yi}^* + \sum_{\substack{j=1 \\ j \neq i}}^{N} \left(C_{ij}\ P_{xj}^* + D_{ij}\ P_{yj}^* \right) = BV_{yi}$$

$$i=1,\ldots N \tag{1.24}$$

where i and j represent Z_1 and Z_0, respectively, and where

$$A_{ij} = \left[H_{xx;x}(Z_1,Z_0)\ n_x(Z_1) + H_{xy;x}(Z_1,Z_0)\ n_y(Z_1) \right] \Delta S(Z_1)$$

$$B_{ij} = \left[H_{xx;y}(Z_1,Z_0)\ n_x(Z_1) + H_{xy;y}(Z_1,Z_0)\ n_y(Z_1) \right] \Delta S(Z_1)$$

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$$C_{ij} = \left[H_{xy;x}(Z_1,Z_0)\ n_x(Z_1) + H_{yy;x}(Z_1,Z_0)\ n_y(Z_1) \right] \Delta S(Z_1)$$

$$D_{ij} = \left[H_{xy;y}(Z_1,Z_0)\ n_x(Z_1) + H_{yy;y}(Z_1,Z_0)\ n_y(Z_1) \right] \Delta S(Z_1)$$

$$K_{xi} = K_{yi} = \frac{1}{2}$$

$$BV_{xi} = P_x(Z_1) \quad , \quad BV_{yi} = P_y(Z_1) \tag{1.25}$$

for a traction condition specified and

$$A_{ij} = I_{x;x}(Z_1,Z_0)\ \Delta S(Z_1) \qquad B_{ij} = I_{x;y}(Z_1,Z_0)\ \Delta S(Z_1)$$

$$C_{ij} = I_{y;x}(Z_1,Z_0)\ \Delta S(Z_1) \qquad D_{ij} = I_{y;y}(Z_1,Z_0)\ \Delta S(Z_1)$$

$$K_{xi} = g_x \qquad K_{yi} = g_y$$

$$BV_{xi} = U_x(Z_1) \qquad BV_{yi} = U_y(Z_1) \tag{1.26}$$

or a specified displacement condition. There are several methods for solving equation (1.24) for the fictitious traction. The first and simplest method is iteration. Iteration works particularly well for traction boundary value problems, equations (1.22). An initial choice of fictitious fractions equal to the actual tractions produces fairly rapid convergence. For the mixed problem, equation (1.23), the iteration, in general, does not converge. A second method is matrix inversion or elimination. Equation (1.24) can be written in the matrix form

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$$\begin{bmatrix} \left[[K_x] - [A] \right] & [B] \\ -[C] & \left[[K_y] - [D] \right] \end{bmatrix} \begin{Bmatrix} \{P^*_x\} \\ \{P^*_y\} \end{Bmatrix} = \begin{Bmatrix} \{BV_x\} \\ \{BY_y\} \end{Bmatrix} \tag{1.27}$$

where all sub-matrixes can be found using equations (1.25) and/or (1.26). Equation (1.27) is written more compactly as:

$$\left[RM \right] \left\{ P^* \right\} = \left\{ BV \right\} \tag{1.28}$$

Once the fictitious tractions are determined, the stresses and displacements are found from the numerical approximation of equations (1.7):

$$\sigma_{xx} = \sum_{i=1}^{N} \left[H_{xx;x}(F,i)\; P^*_{xi} + H_{xx;y}(F,i)\; P^*_{yi} \right]$$

$$\sigma_{xy} = \sum_{i=1}^{N} \left[H_{yy;x}(F,i)\; P^*_{xi} + H_{yy;y}(F,i)\; P^*_{yi} \right]$$

$$\sigma_{xy} = \sum_{i=1}^{N} \left[H_{xy;x}(F,i)\; P^*_{xi} + H_{xy;y}(F,i)\; P^*_{yi} \right]$$

$$U_x = \sum_{i=1}^{N} \left[I_{x;x}(F,i)\; P^*_{xi} + I_{x;y}(F,i)\; P^*_{yi} \right]$$

$$U_y = \sum_{i=1}^{N} \left[I_{y;x}(F,i)\; P^*_{xi} + I_{y;y}(F,i)\; P^*_{yi} \right] \tag{1.29}$$

where F is a field point and P^*_{xi}, P^*_{yi} are the components of the known fictitious traction at the interval i.

It is clear from equations (1.29) that the stresses and displacements can be found at small expense anywhere in the field by simple summation.

It is important to note that the embedment in an infinite plane can also be used for multiply connected domains, such as a region containing a hole. However, the hole would need to be treated as boundary. Discretization of the boundary would therefore cause inaccuracy of the solution near the edge of the hole where the solution is most important. The goal of this dissertation is to eliminate the contour of the hole from the boundary and to find a new influence function for the problem, which contains the effect of the hole, thus improving the accuracy of the solution along and near the hole. To accomplish this goal, the Muskhelishvili method will be employed.

I.2 THE MUSKHELISHVILI METHOD: A COMPLEX VARIABLE METHOD IN ELASTICITY

After the formulation of the linear theory of elasticity had been largely completed (by the middle of the nineteenth century), functions of a complex variable were introduced into plane elasticity problems in 1909 by Kolossoff [24] who, together with Muskhelishvili [25], developed the theory. However, nearly forty years elapsed before the theory, based on Kolossoff's idea, was brought to a successful conclusion. This was accomplished, in the main, by a group of Russian mathematicians inspired by the work of Muskhelishvili. The development has been described by Muskhelishvili in two works [26,27]. The general solution of the fundamental biharmonic boundary-value problem can be made by means of two analytic functions of a complex variable. Consider the biharmonic boundary-value problem

$$\nabla^2\nabla^2\, U(x,y) = 0 \qquad \text{in } R$$

$$U_{,\theta} = f_{\theta}(s) \qquad \text{on } \partial R \tag{1.30}$$

and let

$$\nabla^2 U = X(x,y) \tag{1.31}$$

Then, clearly, the function X is harmonic in R. Note that a harmonic function is a single-valued function of class C^2 which satisfies Laplace's equation in R, i.e., $\nabla^2 X=0$. For every harmonic function there is a conjugate harmonic function which satisfies $\nabla^2 Y=0$ where the function X + iY is an analytic function. Every analytic function is a C^∞ function because it has a series expansion. Also, an analytic function satisfies the Cauchy-Riemann equations and the Cauchy integral formulae. Thus, every analytic function is a harmonic function [28].

The complex conjugate of function X, i.e., Y(x,y), can be easily found by the Cauchy-Riemann equations to within an arbitrary constant. Thus, an analytic function of a complex variable Z = X + iy can be constructed

$$F(Z) = X + iY.$$

Let

$$\phi(Z) = \frac{1}{4} \int F(Z)dZ$$

$$= X^0 + iY^0 \tag{1.32}$$

where X^0 and Y^0 are the integrated functions of X and Y. Then, $\phi(Z)$ is analytic and its derivative is

$$\phi'(Z) = \frac{\partial X^0}{\partial x} + i\,\frac{\partial Y^0}{\partial x} = \frac{1}{4}(X+iY).$$

From the Cauchy-Riemann equations, it is clear that

$$X^0_{,x} = Y^0_{,y} = \frac{1}{4}X$$

$$X^0_{,y} = -Y^0_{,x} = \frac{1}{4}Y$$

Let

$$H(x,y) = U - X^0_{\cdot}x - Y^0_{\cdot}y \tag{1.33}$$

Then it is easy to verify that H(x,y) is a harmonic function, because

$$\nabla^2\ (U - X^0_{\cdot}x - Y^0_{\cdot}y) = \nabla^2 U - \nabla\cdot\nabla(X^0_{\cdot}x) - \nabla\cdot\nabla(Y^0_{\cdot}y)$$

and the fact that X^0, Y^0 are harmonic leads to

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$$= X - \frac{1}{2}X + \frac{1}{2}Y - \frac{1}{2}X - \frac{1}{2}Y$$

$$= 0$$

Thus, $H(x,y)$ is a harmonic function, the complex conjugate of which can be easily found. Calling this conjugate $K(x,y)$, function can be constructed so that:

$$\chi(Z) = H(x,y) + iK(x,y) \tag{1.34}$$

Solving equation (1.33) for U leads to:

$$U = X^0 \cdot x + Y^0 \cdot y + H(x,y)$$

and substituting the analytic functions of equations (1.34) and (1.32) into the above equation, the biharmonic function is obtained in terms of the two analytic functions, $\phi(Z)$ and $\chi(Z)$:

$$U = \mathrm{Re}\left[\overline{Z}\phi(Z) + \chi(Z)\right] \tag{1.35}$$

Since $\phi(Z)$ and $\chi(Z)$ are analytic functions, it follows that $U(x,y)$ is of class C^∞ in R. Denoting the complex conjugate values by bars, the equation can also be written as

$$2U = \overline{Z}\phi(Z) + Z\overline{\phi(Z)} + \chi(Z) + \overline{\chi(Z)} \tag{1.36}$$

The determination of stresses and displacements in terms of the two analytic functions will now be discussed. The stresses can be written in terms of the biharmonic function:

$$\sigma_{xx} = U_{,yy}$$

$$\sigma_{yy} = U_{,xx}$$

$$\sigma_{xy} = -U_{,xy} \tag{1.37}$$

This leads to:

$$\sigma_{xx} + i\sigma_{xy} = -i(U_{,x} + iU_{,y})_{,y}$$

$$\sigma_{yy} - i\sigma_{xy} = (U_{,x} + iU_{,y})_{,x} \tag{1.38}$$

Let

$$\Psi(Z) = \chi'(Z)$$

Then, from equation (1.36), the expression $U_{,x} + iU_{,y}$ can be written

$$U_{,x} + iU_{,y} = \phi(Z) + Z\overline{\phi'(Z)} + \overline{\Psi(Z)} \tag{1.39}$$

Calculating the derivatives of equation (1.39) with respect to x and y and substituting into equations (1.38) leads to:

$$\sigma_{xx} + i\sigma_{xy} = \phi'(Z) + \overline{\phi'(Z)} - Z\overline{\phi''(Z)} - \overline{\Psi'(Z)}$$

$$\sigma_{yy} - i\sigma_{xy} = \phi'(Z) + \overline{\phi'(Z)} + Z\overline{\phi''(Z)} + \overline{\Psi'(Z)}$$

Stresses in terms of two analytic functions, $\phi(Z)$ and $\Psi(Z)$, can now be written as:

$$\sigma_{xx} + \sigma_{yy} = 4\,\mathcal{R}e\left[\phi'(Z)\right]$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2\left[\overline{Z}\phi''(Z) + \Psi'(Z)\right] \qquad (1.40)$$

Finally, displacements in terms of the two analytic functions in the compact formula will be

$$2\mu(U_x + iU_y) = \alpha\phi(Z) - Z\overline{\phi'(Z)} - \overline{\Psi(Z)} \qquad (1.41)$$

where

$$\alpha = 3-4\nu \qquad \text{for plane-strain problem}$$

or

$$\alpha = \frac{3-\nu}{1+\nu} \qquad \text{for plane-stress problem}$$

Stresses and displacements can be found individually, using equations (1.40) and (1.41), and are:

$$\sigma_{xx} = \mathrm{Re}\left[2\phi'(Z) - \overline{Z}\phi''(Z) - \Psi'(Z)\right]$$

$$\sigma_{yy} = \mathrm{Re}\left[2\phi'(Z) + \overline{Z}\phi''(Z) + \Psi'(Z)\right]$$

$$\sigma_{xy} = \mathrm{Im}\left[\overline{Z}\phi''(Z) + \Psi'(Z)\right]$$

$$U_x = \mathrm{Im}\left[\alpha\phi(Z) - Z\overline{\phi'(Z)} - \overline{\Psi(Z)}\right]/2\mu$$

$$U_y = \mathrm{Im}\left[\alpha\phi(Z) - Z\overline{\phi'(Z)} - \overline{\Psi(Z)}\right]/2\mu \qquad (1.42)$$

Now that stresses and displacements have been formulated in terms of the two analytic functions, $\phi(Z)$ and $\Psi(Z)$, the structure and arbitrariness in the definition of the two functions is an issue to be discussed. If the state of stress in the region R is specified, from equations (1.40), one can prove that the single-valued analytic functions $\phi(Z)$ and $\Psi(Z)$ could be determined to within a linear function $Ci+\gamma$ and a constant β, respectively [29]. In addition, if the displacements are prescribed, following equation (1.41), one can find that

$$c = 0$$

and

$$\alpha\gamma - \overline{\beta} = 0$$

Hence, when the stresses are given, the three constants c, γ, β will be chosen in such a way that

$$\phi(0) = 0$$

$$\mathrm{Im}\phi'(0) = 0$$

$$\Psi(0) = 0 \tag{1.43}$$

and when the displacements are given, a suitable choice of γ will be assured by the condition

$$\phi(0) = 0 \tag{1.44}$$

Thus, using the conditions (1.43) and (1.44), the functions $\phi(Z)$ and $\Psi(Z)$ will be determined uniquely [27].

The structure of the two analytic functions for a finite and infinite simply connected regions has been discussed in [27].

Since the state of stress and the displacements can be expressed by means of the two complex functions $\phi(Z)$ and $\Psi(Z)$, the fundamental boundary-value problems of plane elasticity lead to the determination of these functions from prescribed values of certain combinations of these functions on the boundary of the region.

Beginning with the first boundary-value problem in which tractions are prescribed on the boundary, the biharmonic function in terms of applied tractions, f(s), can be written as

$$U_{,x} + iU_{,y} = f(s) \qquad \text{on } \partial R$$

The equation (1.39) leads to:

$$\phi(Z) + Z\overline{\phi'(Z)} + \overline{\Psi(Z)} = f(s) \qquad \text{on } \partial R \qquad (1.45)$$

The corresponding boundary conditions of the second boundary-value problem follow from equation (1.41):

$$\alpha\phi(Z) - Z\overline{\phi'(Z)} - \overline{\Psi(Z)} = g(s) \qquad \text{on } \partial R \qquad (1.46)$$

where g(s) is a prescribed displacement function on the boundary. From either equations (1.45) or (1.46) one can obtain the two complex functions. However, mapping the region R into the inside or outside of a unit circle makes the determination of the two functions much simpler. Suppose the mapping function

$$Z = \omega(\zeta) \tag{1.47}$$

maps point in the region R, Z plane, into a unit circle $|\zeta| \leq 1$. The mapping function for a finite region where the origin is taken in the interior can be represented as a power series

$$Z = \sum_{n=1}^{\infty} k_n \zeta^n \qquad |\zeta| \leq 1$$

whereas for an infinite region, where the origin is an exterior point, the function is given by:

$$Z = \frac{c}{\zeta} + \sum_{n=0}^{\infty} k_n \zeta^n \qquad |\zeta| \leq 1$$

The boundary conditions, equations (1.45) and (1.46), can then be written as

$$\phi_1(\zeta) + \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}} \overline{\phi_1'(\zeta)} + \overline{\Psi_1(\zeta)} = F(\zeta)$$

$$\alpha\phi_1(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}} \overline{\phi_1'(\zeta)} - \overline{\Psi_1(\zeta)} = G(\zeta) \tag{1.48}$$

where

$$\phi[\omega(\zeta)] = \phi_1(\zeta) \quad \text{and} \quad \Psi[\omega(\zeta)] = \Psi_1(\zeta)$$

Equations (1.48) can now be solved for the two functions $\phi_1(\zeta)$ and $\Psi_1(\zeta)$ by a power series expansion method or integrodifferential equations using Cauchy integral formulae [27]. Since the solution of the integrodifferential equation reduces to the solution of the standard Fredholm integral equation, then the existence of a solution of equations (1.48) would follow, almost directly, from the Fredholm theory [27].

I.3 CAUCHY INTEGRALS AND RELATED THEOREMS

Since the integrodifferential equations method will be used to determine the two complex functions, it is important to discuss Cauchy integrals and related theorems briefly. The proof of the following theorems has been presented in [30] and in [27].

Suppose R^+ is a finite open simply connected region enclosed by the contour B described in a counterclockwise sense. Denote the region exterior to $(R^+ + B)$ by R^- and the points on the boundary B by t. Let f(Z) be a complex function analytic (holomorphic) in R^+ and continuous on C. Then

$$\frac{1}{2\pi i}\int_B \frac{f(t)dt}{t-Z} = f(Z) \qquad \text{for } Z\varepsilon R^+ \tag{1.49}$$

and

$$\frac{1}{2\pi i}\int_B \frac{f(t)}{(t-Z)^{n+1}}\, dt = f^{(n)}(Z) \qquad \text{for } Z\varepsilon R^+$$

$$\frac{1}{2\pi i}\int_B \frac{f(t)}{t-Z}\, dt = 0 \qquad \text{for } Z\varepsilon R^- \tag{1.50}$$

Equation (1.50) is a necessary and sufficient condition that the continuous function f(t) defined on B can be the boundary value of a function analytic in R^+. Let f(Z) be a complex fu

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function analytic in R^- including the point at infinity and continuous on B. Then

$$\frac{1}{2\pi i}\int_B \frac{f(t)}{t-Z}\,dt = f(\infty) \qquad \text{for } Z\varepsilon R^+ \qquad (1.51)$$

$$\frac{1}{2\pi i}\int_B \frac{f(t)}{t-Z}\,dt = f(\infty) - f(Z) \qquad \text{for } Z\varepsilon R^- \qquad (1.52)$$

The condition (1.51) that the Cauchy integral have a constant value in R^+ is both necessary and sufficient for the continuous function f(t), defined on B, to be the boundary value of a function analytic in R^-.

Let $\phi(t)$ be a complex function which satisfies the Hölder condition on an arc L. Then the Cauchy integral

$$\phi_1(Z) = \frac{1}{2\pi i}\int_L \frac{\phi(t)}{t-Z}\,dt \qquad (1.53)$$

may be shown to be a sectionally analytic function in the whole plane cut along the arc L. Further, the limiting values $\phi^+(t)$, $\phi^-(t)$ may be shown to exist on L and satisfy the relations

$$\phi_1^+(t_0) - \phi_1^-(t_0) = \phi(t_0)$$

$$\phi_1^+(t_0) + \phi_1^-(t_0) = \frac{1}{\pi i}\int_L \frac{\phi(t)}{t-t_0}\,dt \qquad (1.54)$$

where t_0 is a point on L and the integral in equation (1.54) is represented as a principal value. The assumption that $\phi(t)$ satisfies the Hölder condition is sufficient for the existence of the principal value. These results are referre

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Since the unit circular region will be used in the determination of principal value of some integrals in the following chapters, the following integral form will be used frequently. Let a and b be constants where $b/a<1$ and let γ be the circumference of the unit circle. Then for the points inside the unit circle

$$\frac{1}{2\pi i}\oint_{\gamma} \frac{\ln(\frac{at-b}{t})}{t-Z}\, dt = \ln(a) \qquad (1.55)$$

This is simply because the function in the integrand

$$\ln(\frac{at-b}{t}) = \ln(at-b) - \ln(t)$$

has two essential singularity inside the unit circle, $t = b/a$ and $t=0$, whereas the limit of the function at infinity exists and is equal to $\ln(a)$. Thus, the function is analytic outside the unit circle. Then following the Cauchy theorem, equation (1.51), the result of the integral (1.55) will be $\ln(a)$. This result can also be achieved by the change of variable

$$\eta = \frac{1}{t}$$

or

$$dt = -\frac{1}{\eta^2}\, d\eta$$

Thus, the integral becomes:

$$\frac{1}{2\pi i}\oint_{\gamma} \frac{\ln(a-b\eta)}{-\eta(1-Z\eta)}\,d\eta = \frac{1}{2\pi i}\oint_{\gamma} \frac{\ln(a-b\eta)}{\eta(1-Z\eta)}\,d\eta \tag{1.56}$$

The only singular point inside the unit circle is $\eta = 0$, which is the corresponding point of $t = \infty$. Obtaining the residue at $\eta = 0$ will prove equation (1.56).

CHAPTER II

GENERAL SOLUTION AND A MAPPING TECHNIQUE

II.1 INTRODUCTORY REMARKS

A general solution which leads to the influence functions for an infinite plate containing an arbitrarily shaped cavity is discussed here. These influence functions describe the stress field and displacement field generated by an isolated concentrated point force, P, applied on the plane. It is obvious that these functions must be defined everywhere except at the point where the load is applied.

It is important to note that the load, P, is a concentrated point force, if just a very thin layer of the plane is considered. In the cases of plane stress or plane strain the load, P, is a line load along a line perpendicular to the layers of the plane, as shown in Figure 2.1.

II.2 A MAPPING TECHNIQUE

Consider the problem of an infinite plane bounded by an arbitrarily shaped cavity at the origin and having a concentrated point force, P, acting in the plane at some point Z_0, where Z_0 is a point in the region outside of the

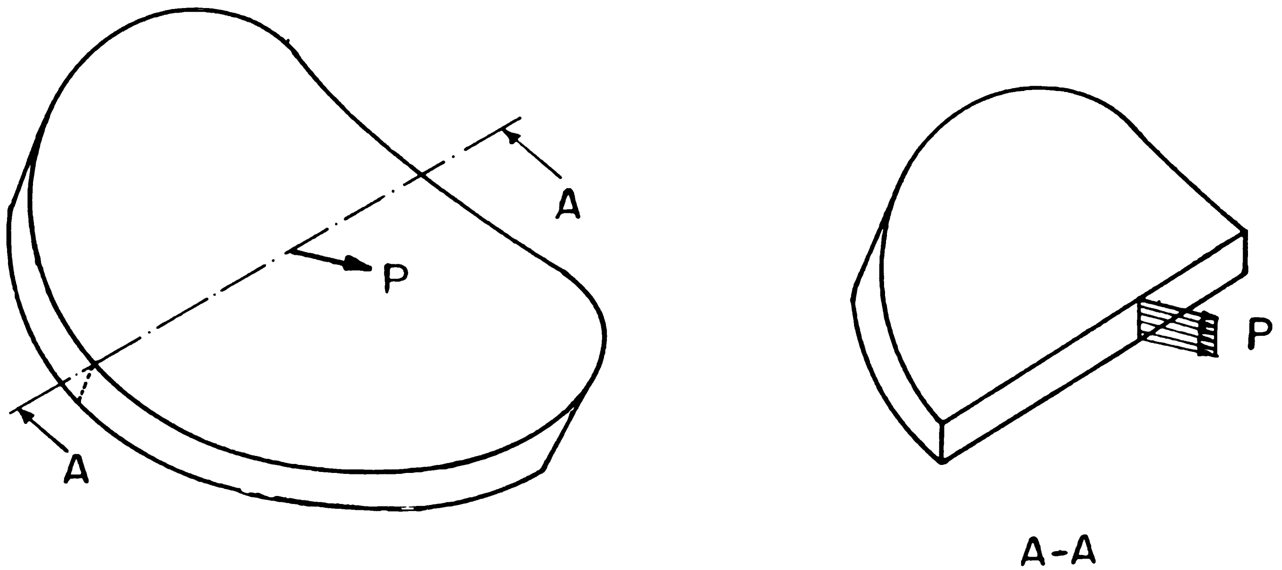

Figure 2.1 Concentrated line load.

hole. This problem can be expressed as the superposition of two problems (Figure 2.2).

Let the first problem, Figure 2.2(B) be that of a concentrated point force applied at the point, Z_0, on an infinite plane with no cavity. Let the second problem, Figure 2.2(C) be an infinite region bounded by the hole with some traction acting on the boundary C. Let this traction be equal in magnitude and opposite in direction to the traction generated on contour C in the problem of Figure 2.2(B) by the concentrated point load P.

Clearly superposition of the problems of Figure 2.2(B) and 2.2(C) gives the original problem of Figure 2.2(A), where there is no traction acting on the hole.

The complex potential functions of the problem of Figure 2.2(B) is known (Muskhelishvili [27]) and obviously since the contour C of the hole is known, then the applied traction of the problem of Figure 2.2(C) can be found.

To find the complex potential functions for the problem of Figure 2.2(C), a mapping technique is used. The problem of Figure 2.2(C) will be mapped to a unit circular disc (Figure 2.3).

If the type of contour, C, mentioned earlier in this section, is known, then the transformation function can be found. The mapping function, $\omega(\zeta)$, has to be conformal and one to one, mapping points at infinity of the Z plane to the origin of the ζ plane, and mapping points on the contour to points on the circumference of the unit disc.

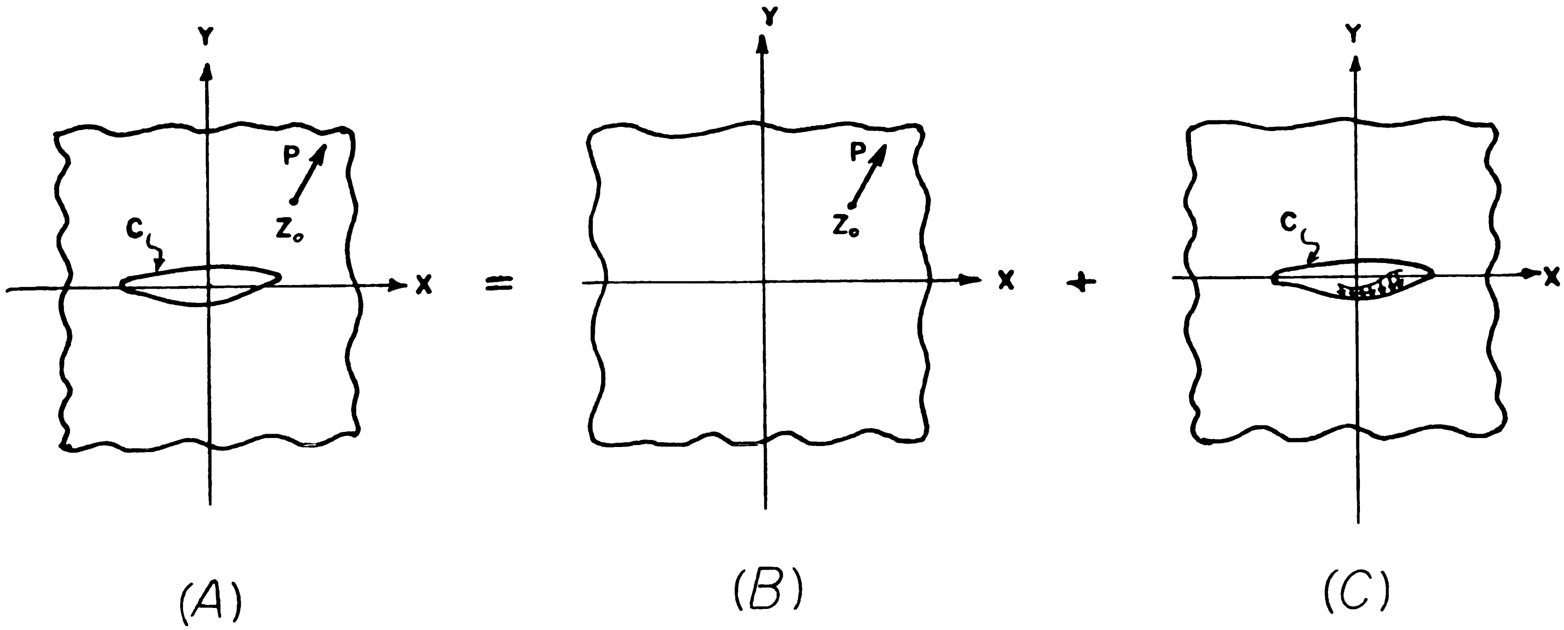

Figure 2.2 The problem of interest expressed as the superposition of two problems.

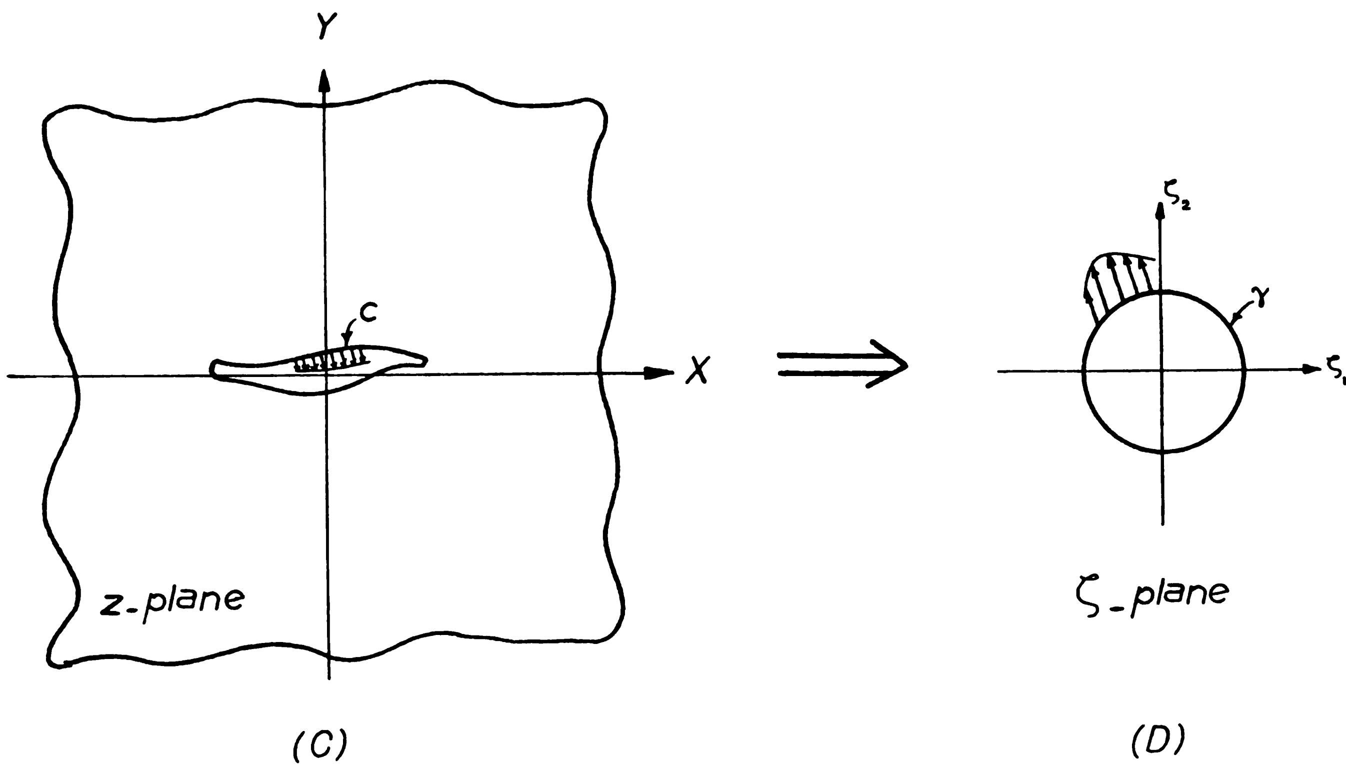

Figure 2.3 Mapping the auxiliary problem to the disc.

The traction boundary condition on the contour, C, will automatically transform to new boundary conditions acting on the disc.

II.3 GENERAL SOLUTION

Let the complex potential functions for the problem of Figure 2.2(B) be $\phi^0(Z)$ and $\Psi^0(Z)$, and the complex potential functions for the problem of Figure 2.2(C) be $\phi^*(Z)$ and $\Psi^*(Z)$. By superposition, the complex potential functions for the problem of Figure 2.2(A), $\phi(Z)$ and $\Psi(Z)$ will be

$$\phi(Z) = \phi^0(Z) + \phi^*(Z)$$

$$\Psi(Z) = \Psi^0(Z) + \Psi^*(Z) \tag{2.1}$$

Since the problem of Figure 2.2(C) is to be transformed to the problem of Figure 2.2(D), then the transformed complex potential functions are

$$\phi_1^*(\zeta) = \phi^*[\omega(\zeta)] = \phi^*(Z)$$

$$\Psi_1^*(\zeta) = \Psi^*[\omega(\zeta)] = \Psi^*(Z)$$

$$\phi_1^0(\zeta) = \phi^0[\omega(\zeta)] = \phi^0(Z)$$

$$\Psi_1^0(\zeta) = \Psi^0[\omega(\zeta)] = \Psi^0(Z)$$

and the derivatives are

$$\overset{*}{\phi}_1{}'(\zeta) = \phi^{*}{}'(Z) \cdot \omega'(\zeta)$$

$$\overset{*}{\Psi}_1{}'(\zeta) = \Psi^{*}{}'(Z) \cdot \omega'(\zeta)$$

$$\overset{*}{\phi}_1{}''(\zeta) = \phi^{*}{}''(Z) \cdot \omega'^{2}(\zeta) + \phi^{*}{}'(Z) \cdot \omega''(\zeta) \quad (2.2)$$

To find the influence function, the derivatives of equations (2.1) are needed, so

$$\phi'(Z) = \phi^{0}{}'(Z) + \phi^{*}{}'(Z)$$

$$\Psi'(Z) = \Psi^{0}{}'(Z) + \Psi^{*}{}'(Z)$$

$$\phi''(Z) = \phi^{0}{}''(Z) + \phi^{*}{}''(Z) \quad (2.3)$$

where $\phi^{*}{}'(Z)$, $\phi^{*}{}''(Z)$ and $\Psi^{*}{}'(Z)$ can be easily found from equations (2.2):

$$\phi^{*}{}'(Z) = \frac{\overset{*}{\phi}_1{}'(\zeta)}{\omega'(\zeta)}$$

$$\phi^{*}{}''(Z) = \frac{\overset{*}{\phi}_1{}''(\zeta)}{\omega'^{2}(\zeta)} - \frac{\overset{*}{\phi}_1{}'(\zeta) \cdot \omega''(\zeta)}{\omega'^{3}(\zeta)}$$

$$\Psi^{*}{}'(Z) = \frac{\overset{*}{\Psi}_1(\zeta)}{\omega'(\zeta)} \quad (2.4)$$

Substituting equations (2.4) into equation (2.3) and reconsidering equations (2.1), the requirements for the influence function become:

$$\phi(Z) = \phi^0(Z) + \overset{*}{\phi}_1(\zeta)$$

$$\Psi(Z) = \Psi^0(Z) + \overset{*}{\Psi}_1(\zeta)$$

$$\phi'(Z) = \phi^{0\prime}(Z) + \frac{\overset{*}{\phi}_1{}'(\zeta)}{\omega'(\zeta)}$$

$$\Psi'(Z) = \Psi^{0\prime}(Z) + \frac{\Psi_1{}''(\zeta)}{\omega'(\zeta)}$$

$$\phi''(Z) = \phi^{0\prime\prime}(Z) + \frac{\overset{*}{\phi}_1{}''(\zeta)}{\omega'^2(\zeta)} - \frac{\overset{*}{\phi}_1{}'(\zeta)\cdot\omega''(\zeta)}{\omega'^3(\zeta)} \qquad (2.5)$$

The complex potential functions for an infinite plane with a concentrated force, P, at a point Z_0, $\phi^0(Z)$ and $\Psi^0(Z)$, are known (Maskhelishvili [27], Sokolnikoff [29], Green and Zerna [31]).

$$\phi^0(Z) = -\frac{P}{2\pi(\alpha+1)}\ln(Z-Z_0)$$

$$\Psi^0(Z) = \alpha\frac{\overline{P}}{2\pi(\alpha+1)}\ln(Z-Z_0) + \frac{P}{2\pi(\alpha+1)}\cdot\frac{\overline{Z}_0}{Z-Z_0} \qquad (2.6)$$

To find the complex potential functions for the problem of Figure 2.2(A), it is necessary to find the complex potential function for the problem of Figure 2.3(D), i.e., $\overset{*}{\phi}_1(\zeta)$ and $\overset{*}{\Psi}_1(\zeta)$.

Since $\overset{*}{\phi}_1(\zeta)$ and $\overset{*}{\Psi}_1(\zeta)$ have to be analytic in the domain, then as mentioned in Chapter I, just one boundary condition is necessary to find the complex potential functions, i.e., either of equations (1.45) or (1.46).

Note that in the original problem, Figure 2.2(A), the boundary of the hole is traction-free. Recall equation (1.45) for the traction boundary condition [29]:

$$f(s) = f_1 + if_2 + \text{const.} = 0$$

Then equation (1.45) becomes:

$$\phi(t) + t\,\overline{\phi'(t)} + \overline{\Psi(t)} = 0 \qquad \text{on } C \qquad (2.7)$$

where t represents the values of Z on the contour C. Substituting equations (2.1) into (2.7) leads to

$$\phi^*(t) + t\,\overline{\phi^{*\prime}(t)} + \overline{\Psi^*(t)} = -\,\phi^0(t) + t\,\overline{\phi^{0\prime}(t)} + \Psi^0(t) \qquad \text{on } C \qquad (2.8)$$

Clearly, the left-hand side of equation (2.8) is in the form of the traction boundary condition for the problem of Figure 2.2(C), since $\phi^*(t)$ and $\Psi^*(t)$ are the boundary values of $\phi^*(Z)$ and $\Psi^*(Z)$. Also, the right-hand side of equation (2.8) is known, since $\phi^0(t)$ and $\Psi^0(t)$ are the values of $\phi^0(Z)$ and $\Psi^0(Z)$ on the fictitious contour in the problem of Figure 2.2(B).

Since the boundary condition for the problem of Figure 2.3(C) is known (equation 2.8), then the boundary condition for the problem of Figure 2.3(D) can be obtained by transforming (2.8) to the ζ plane using the transformation functions

$$Z = \omega(\zeta)$$

so that the boundary transforms by:

$$t = \omega(\sigma)$$

where σ represents the values of ζ on the circumference of the disc. Hence, equation (2.8) becomes:

$$\overset{*}{\phi}_1(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\overset{*}{\phi}_1{}'(\sigma)} + \overline{\overset{*}{\Psi}_1(\sigma)} = -\left(\phi_1^0(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\phi_1^0(\sigma)} + \overline{\Psi_1^0(\sigma)}\right) \tag{2.9}$$

where $\overset{*}{\phi}_1(\sigma)$ and $\overset{*}{\Psi}_1(\sigma)$ are the boundary values of $\overset{*}{\phi}_1(\zeta)$ and $\overset{*}{\Psi}_1(\zeta)$, respectively. Also, $\phi_1^0(\sigma)$ and $\Psi_1^0(\sigma)$ are the boundary values of $\phi_1^0(\zeta)$ and $\Psi_1^0(\zeta)$, respectively.

The right-hand side of equation (2.9) is known, so let

$$F(\sigma) = -\left[\phi_1^0(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\phi_1^0{}'(\sigma)} + \overline{\Psi_1^0(\sigma)}\right] \tag{2.10}$$

then, equation (2.9) becomes:

$$\overset{*}{\phi}_1(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\overset{*}{\phi}_1{}'(\sigma)} + \overline{\overset{*}{\Psi}_1(\sigma)} = F(\sigma) \tag{2.11}$$

This is the mixed boundary condition for the problem of Figure 2.3(D) from which the two analytic functions $\overset{*}{\phi}_1(\zeta)$ and $\overset{*}{\Psi}_1(\zeta)$ will be found. It is necessary to point out some characteristics of $\overset{*}{\phi}_1(\zeta)$ and $\overset{*}{\Psi}_1(\zeta)$ before proceeding.

As mentioned in section I.2, $\overset{*}{\phi}_1(\zeta)$ and $\overset{*}{\Psi}_1(\zeta)$ must be analytic (holomorphic) inside γ, the unit circle. Also, without loss of generality, it can be assumed that $\overset{*}{\phi}_1(0) = 0$. Thus, $\overset{*}{\phi}_1(\zeta)$ and $\overset{*}{\Psi}_1(\zeta)$ may be developed for $|\zeta|<1$ in power series of the form

$$\overset{*}{\phi}_1(\zeta) = \sum_{k=1}^{\infty} a_k\zeta^k \quad , \quad \overset{*}{\Psi}_1(\zeta) = \sum_{k=0}^{\infty} b_k\zeta^k \tag{2.13}$$

where, in the first series, the constant term is absent because of the condition $\phi_1^*(0) = 0$. Furthermore,

$$\overline{\phi_1^*(\zeta)} = \sum_{k=1}^{\infty} \bar{a}_k \bar{\zeta}^k \quad , \quad \Psi_1^*(\zeta) = \sum_{k=0}^{\infty} \bar{b}_k \bar{\zeta}^k \tag{2.14}$$

Let ζ approach the boundary γ, i.e., $\zeta \to \sigma$. Note that since the radius of the disc is equal to one, then:

$$\sigma\bar{\sigma} = 1 \tag{2.15}$$

Equations (2.13) and (2.14) are valid for the boundary values $\phi_1^*(\sigma)$ and $\Psi_1^*(\sigma)$. Substituting equation (2.15) into equations (2.14) along with equations (2.13) for the boundary values, these become:

$$\phi_1^*(\sigma) = \sum_{k=1}^{\infty} a_k \sigma^k \quad , \quad \Psi_1^*(\sigma) = \sum_{k=0}^{\infty} b_k \sigma^k \tag{2.16.a}$$

$$\overline{\phi_1^*(\sigma)} = \sum_{k=1}^{\infty} \bar{a}_k \sigma^{-k} \quad , \quad \overline{\Psi_1^*(\sigma)} = \sum_{k=0}^{\infty} \bar{b}_k \sigma^{-k} \tag{2.16.b}$$

Equations (2.16.a) show that $\phi_1^*(\sigma)$ and $\Psi_1^*(\sigma)$ have poles at infinity, so they are analytic functions inside the unit circle. Also, equations (2.16.b) show that $\overline{\phi_1^*(\sigma)}$ and $\overline{\Psi_1^*(\sigma)}$ have poles at the origin, so they are analytic outside of the unit circle.

Using this analysis and employing the Cauchy integral formulas, the complex potential functions for the problem of Figure 2.3(D), $\phi_1^*(\zeta)$ and $\Psi_1^*(\zeta)$, can be computed. To find $\phi_1^*(\zeta)$, let both sides of equation (2.11) be multiplied by

$$\frac{1}{2\pi i} \cdot \frac{d\sigma}{\sigma-\zeta}$$

where ζ is a point inside γ, the unit circle.

Integrating both sides of the equation counterclockwise around the unit circle leads to:

$$\frac{1}{2\pi i}\oint_{\gamma} \frac{\phi_1^*(\sigma)}{\sigma-\zeta}\, d\sigma + \frac{1}{2\pi i}\oint_{\gamma} \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\, \frac{\overline{\phi_1^{*\prime}(\sigma)}}{\sigma-\zeta}\, d\sigma$$

$$+ \frac{1}{2\pi i}\oint_{\gamma} \frac{\overline{\Psi_1^*(\sigma)}}{\sigma-\zeta}\, d \;=\; \frac{1}{2\pi i}\oint_{\gamma} \frac{F(\sigma)}{\sigma-\zeta}\, d\sigma \qquad (2.16)$$

Since $\phi_1^*(\sigma)$ is analytic inside γ and $\Psi_1^*(\sigma)$ is analytic outside γ, then due to Cauchy integral formulas (section I.3), equation (2.16) can be written as:

$$\phi_1^*(\zeta) + \frac{1}{2\Pi i}\oint_{\gamma} \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\, \frac{\overline{\phi_1^*(\sigma)}}{\sigma-\zeta}\, d\sigma + \overline{\Psi_1^*(0)} = \frac{1}{2\pi i}\oint_{\gamma} \frac{F(\sigma)}{\sigma-\zeta}\, d\sigma$$

$$(2.17)$$

Note that the third integral of equation (2.16) becomes:

$$\frac{1}{2\pi i}\oint_{\gamma} \frac{\overline{\Psi_1^*(\sigma)}}{\sigma-\zeta}\, d\sigma = \overline{\Psi_1^*(\sigma)}\Big|_{\sigma=\infty} = \overline{\Psi_1^*(0)} = \bar{b}_0$$

where $\overline{\Psi_1^*(\sigma)}$ is a constant.

Equation (2.17) is an intergodifferential equation for $\phi_1^*(\zeta)$. It contains an unknown constant $\overline{\Psi_1^*(\sigma)}$, which can be determined by letting $\zeta = 0$ and imposing the condition $\phi_1^*(0) = 0$. Thus, if the value of $\Psi_1^*(0)$ in equation

(2.17) is chosen arbitrarily and the corresponding solution for $\phi_1^*(\zeta)$ is found, then the actual value of $\overline{\Psi_1^*(0)}$ can be computed from the condition $\phi_1^*(0) = 0$. This is due to the fact that if $\phi_1^{**}(\zeta)$ is any solution of (2.17) for a given $\Psi_1^*(0)$, and if $\phi_1^{**}(0) = a_0 \neq 0$, then $\phi_1^{**}(\zeta) - a_0$ is a solution of (2.17) with $\overline{\Psi_1^*(0)}$ replaced by $\overline{\Psi_1^*(0)} + a_0$. Thus, $\Psi_1^*(0)$ can be tentatively fixed, say $\overline{\Psi_1^*(0)} = 0$. Also, as mentioned in section I.2, in order to have a unique solution for $\phi_1^*(\zeta)$ and $\Psi_1^*(\zeta)$, the following conditions must be satisfied:

$$\phi_1^*(0) = 0 \qquad \Psi_1^*(0) = 0 \tag{2.18}$$

To find $\Psi_1^*(\zeta)$, take the conjugate of equation (2.11) and multiply both sides of the equation by

$$\frac{1}{2\pi i} \frac{d\sigma}{\sigma - \zeta}$$

where ζ is a point inside γ.

Integrating both sides of the equation counterclockwise around the unit circle leads to:

$$\frac{1}{2\pi i} \oint_\gamma \frac{\overline{\phi_1^*(\sigma)}}{\sigma - \zeta} d\sigma + \frac{1}{2\pi i} \oint_\gamma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} \frac{\phi_1^{*\prime}(\sigma)}{\sigma - \zeta} d\sigma$$
$$+ \frac{1}{2\pi i} \oint_\gamma \frac{\Psi_1^*(\sigma)}{\sigma - \zeta} d\sigma = \frac{1}{2\pi i} \oint_\gamma \frac{\overline{F(\sigma)}}{\sigma - \zeta} d\sigma \tag{2.19}$$

An argument similar to that presented for reducing equation (2.16) to equation (2.17) can also be presented here to obtain:

$$\overline{\phi_1^*(0)} + \frac{1}{2\pi i}\oint_\gamma \frac{\overline{\omega(\sigma)}}{\overline{\omega'(\sigma)}} \frac{\phi_1^{*\prime}(\sigma)}{\sigma-\zeta}\, d\sigma + \psi_1^*(\zeta) = \frac{1}{2\pi i}\oint_\gamma \frac{\overline{F(\sigma)}}{\sigma-\zeta}\, d\sigma \tag{2.20}$$

Substituting condition (2.18) into equations (2.17) and (2.20) and rearranging leads to

$$\phi_1^*(\zeta) = \frac{1}{2\pi i}\oint_\gamma \frac{F(\sigma)}{\sigma-\zeta}\, d\sigma - \frac{1}{2\pi i}\oint_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \frac{\overline{\phi_1^{*\prime}(\sigma)}}{\sigma-\zeta}\, d\sigma \tag{2.21}$$

$$\psi_1^*(\zeta) = \frac{1}{2\pi i}\oint_\gamma \frac{\overline{F(\sigma)}}{\sigma-\zeta}\, d\sigma - \frac{1}{2\pi i}\oint_\gamma \frac{\overline{\omega(\sigma)}}{\overline{\omega'(\sigma)}} \frac{\phi_1^{*\prime}(\sigma)}{\sigma-\zeta}\, d\sigma \tag{2.22}$$

It is easy to reduce the solution of the integrodifferential equation (2.21) to the solution of the standard Fredholm integral equation. The existence of a solution of equation (2.21) would then follow, almost directly, from the Fredholm theory.

The second integrals of the right-hand sides of equations (2.21) and (2.22) are left in general form since function $\omega(\zeta)$ has not yet been specified.

CHAPTER III

CIRCULAR HOLE IN A FINITE TWO-DIMENSIONAL REGION

III.1. INTRODUCTION

The effect of a circular hole on the stress distribution in an elastic region has attracted considerable attention for the past seventy years. The effect of a circular hole on an infinite plate subjected to uniaxial tension was first solved by Kirsch [32]. This work was extended to other load conditions by Bickley [33]. Howland [34] solved the problem of a long strip weakened by a circular hole subjected to uniaxial tension. Other load conditions were considered by Savin [35]. The effect of a circular hole on the stress distribution in a finite elastic region has been treated numerically and experimentally using several methods.

In this chapter, the solution of the problem of a finite plane elastic region containing a circular hole and subjected to traction boundary conditions is presented. This is the first implementation of the mapping technique and boundary integral equation method. In section 2 of this chapter, some known complex potential functions [36] are used to find the influence function for an infinite domain weakened by a circular hole. In section 3, the

Muskhishvili method is used and the mapping technique is employed to determine the influence function directly. It is shown that the two results are identical. In the last section some example problems are considered and the results are compared to some known solutions, where available. The computer program for the computation is included in Appendix B.

III.2 DERIVATION OF THE INFLUENCE FUNCTIONS USING KNOWN POTENTIAL FUNCTIONS

Consider an infinite elastic plane with a circular cavity of radius a centered at the origin. Let a force P act at a point Z_0 where $|Z_0| \geqslant a$. Bhargava and Kapoor [36] have constructed the potential functions, $\phi(Z)$ and $\Psi(Z)$, for this problem. They assume, following Green and Zerna [31], that the complex potentials are of the form:

$$\phi(Z) = \frac{P}{2\pi(\alpha+1)} \left\{ -\ln(Z-Z_0) - \alpha \ln\left(Z-\frac{a^2}{\overline{Z}_0}\right) + A\left(Z-\frac{a^2}{\overline{Z}_0}\right)^{-1} + \alpha \ln Z \right\} \tag{3.1}$$

$$(Z) = \frac{\overline{P}}{2\pi(\alpha+1)} \left\{ \alpha \ln(Z-Z_0) + \overline{Z}_0 \frac{P}{\overline{P}} (Z-Z_0)^{-1} + \ln\left(Z-\frac{a^2}{\overline{Z}_0}\right) + B\left(Z-\frac{a^2}{\overline{Z}_0}\right)^{-1} + C\left(Z-\frac{a^2}{\overline{Z}_0}\right)^{-2} - \ln Z + DZ^{-1} + E Z^{-2} \right\} \tag{3.2}$$

This choice clearly gives proper singularities at the point of action of the concentrated force Z_0. Also, it satisfies the condition of zero stresses at infinity. The unknown constants can be found from the condition that normal and tangential stresses, σ_{rr} and $\sigma_{r\theta}$, are zero at the boundary of the hole, i.e.,

$$(\sigma_{rr} - i\sigma_{r\theta})_{Z=\sigma} = 0$$

The boundary condition can be written in terms of $\phi(Z)$ and $\Psi(Z)$ following Muskhelishvili [27] as follows:

$$\sigma_{rr} - i\sigma_{r\theta} = \phi'(Z) + \overline{\phi'(Z)} - e^{2i\theta}[\overline{Z}\ \phi''(Z) + \Psi'(Z)] \qquad (3.3)$$

Substituting equations (3.1) and (3.2) into equation (3.3) leads to:

$$A = \frac{\overline{P}}{P}\,\frac{\beta^2-1}{\alpha^2\beta^6}\,Z_0$$

$$B = \frac{P}{\overline{P}}\,\alpha\,\overline{Z}_0 - \frac{\beta^2-1}{\beta^2}\,Z_0$$

$$C = \frac{\beta^2-1}{\beta^4}\,Z_0^2$$

$$D = \frac{\beta^2-1}{\beta^2}\,Z_0 - \frac{P}{\overline{P}}(\alpha-\frac{1}{\beta^2})Z_0$$

$$E = -\frac{P}{\overline{P}}\,\alpha\,a^2$$

where $\beta^2 = \frac{Z_0\overline{Z}_0}{a^2}$. Thus, for an isolated point-force P acting at the point Z_0, the complex potentials at the point Z are:

$$\phi(Z) = \frac{P}{2\pi(\alpha+1)} \left\{ -\ln(Z-Z_0) - \alpha \ln(Z-\frac{a^2}{\overline{Z}_0}) + \alpha \ln Z \right\}$$

$$+ \frac{\overline{P}}{2\pi(\alpha+1)} \left\{ \frac{Z_0\overline{Z}_0-1}{a^2\overline{Z}_0^3} (Z-\frac{a^2}{\overline{Z}_0})^{-1} \right\}$$

$$\Psi(Z) = \frac{\overline{P}}{2\pi(\alpha+1)} \left\{ \alpha \ln(Z-Z_0) + \ln(Z-\frac{\alpha^2}{\overline{Z}_0}) - \ln Z - \frac{Z_0\overline{Z}_0^{-1}}{\overline{Z}_0}(Z-\frac{\alpha^2}{\overline{Z}_0})^{-1} \right.$$

$$\left. + \frac{Z_0\overline{Z}_0-1}{\overline{Z}_0^2} (Z-\frac{\alpha^2}{\overline{Z}_0})^{-2} + \frac{Z_0\overline{Z}_0-1}{\overline{Z}_0} \cdot \frac{1}{Z} \right\} + \frac{P}{2\pi(\alpha+1)} \left\{ \overline{Z}_0(Z-Z_0)^{-1} \right.$$

$$\left. + \alpha \overline{Z}_0(Z-\frac{\alpha^2}{\overline{Z}_0})^{-1} - \alpha \overline{Z}_0 Z^{-1} - \frac{1}{\overline{Z}_0} \cdot Z^{-1} - \alpha a^2 Z^{-2} \right\}$$

Without loss of generality, assume $a = 1$. Then $\phi(Z)$ and $\Psi(Z)$ become:

$$\phi(Z) = \frac{P}{2\pi(\alpha+1)} \left\{ -\ln(Z-Z_0) - \alpha \ln(\frac{Z\overline{Z}_0-1}{Z\overline{Z}_0}) \right\}$$

$$+ \frac{\overline{P}}{2\pi(\alpha+1)} \left\{ \frac{Z_0\overline{Z}_0-1}{\overline{Z}_0^2} \cdot \frac{1}{Z\overline{Z}_0-1} \right\}$$

$$\Psi(Z) = \frac{\overline{P}}{2\pi(\alpha+1)} \left\{ \alpha \ln(Z-Z_0) + \ln(\frac{Z\overline{Z}_0-1}{Z\overline{Z}_0}) - \frac{Z_0\overline{Z}_0-1}{Z\overline{Z}_0-1} \right.$$

$$\left. + \frac{Z_0\overline{Z}_0-1}{(Z\overline{Z}_0-1)^2} + \frac{Z_0\overline{Z}_0-1}{Z\overline{Z}_0} \right\} + \frac{P}{2\pi(\alpha+1)} \left\{ \frac{\overline{Z}_0}{Z-Z_0} + \frac{\alpha\overline{Z}_0^2}{Z\overline{Z}_0-1} \right.$$

$$\left. - \frac{\alpha Z_0\overline{Z}_0+1}{Z Z_0} - \frac{\alpha}{Z^2} \right\} \qquad (3.3)$$

Let these potential functions be written as:

$$\phi(Z) = \frac{P}{2\pi(\alpha+1)}\left\{\phi_I(Z)\right\} + \frac{\overline{P}}{2\pi(\alpha+1)}\left\{\phi_{II}(Z)\right\}$$

$$\Psi(Z) = \frac{P}{2\pi(\alpha+1)}\left\{\Psi_I(Z)\right\} + \frac{\overline{P}}{2\pi(\alpha+1)}\left\{\Psi_{II}(Z)\right\}$$

where ϕ_I, ϕ_{II}, Ψ_I and Ψ_{II} can be found by comparison to equations (3.3).

Since $\phi'(Z)$, $\phi''(Z)$ and $\Psi'(Z)$ will be needed, they will be listed here:

$$\phi'(Z) = \frac{P}{2\pi(\alpha+1)}\left\{\phi_I'\right\} + \frac{\overline{P}}{2\pi(\alpha+1)}\left\{\phi_{II}'\right\}$$

$$\phi''(Z) = \frac{P}{2\pi(\alpha+1)}\left\{\phi_I''\right\} + \frac{\overline{P}}{2\pi(\alpha+1)}\left\{\phi_{II}''\right\}$$

$$\Psi'(Z) = \frac{P}{2\pi(\alpha+1)}\left\{\Psi_I'(Z)\right\} + \frac{\overline{P}}{2\pi(\alpha+1)}\left\{\Psi_{II}'(Z)\right\} \qquad (3.4)$$

where

$$\phi_I'(Z) = -\frac{1}{Z-Z_0} - \frac{\alpha}{Z(Z\overline{Z}_0-1)}$$

$$\phi_I''(Z) = +\frac{1}{(Z-Z_0)^2} + \frac{2Z\overline{Z}_0-1}{Z^2(Z\overline{Z}_0-1)^2}$$

$$\phi_{II}'(Z) = -\frac{Z_0\overline{Z}_0-1}{\overline{Z}_0^2} \cdot \frac{\overline{Z}_0}{(Z\overline{Z}_0-1)^2}$$

$$\phi''_{II}(Z) = \frac{Z_0\overline{Z}_0-1}{\overline{Z}_0^2} \cdot \frac{2\overline{Z}_0^2}{(Z\overline{Z}_0-1)^3}$$

$$\Psi'_I(Z) = -\frac{\overline{Z}_0}{(Z-Z_0)^2} - \frac{\alpha\overline{Z}_0^3}{(Z\overline{Z}_0-1)^2} + \frac{\alpha Z_0\overline{Z}_0+1}{Z_0 Z^2} + \frac{2\alpha}{Z^3}$$

$$\Psi'_{II}(Z) = \frac{\alpha}{Z-Z_0} + \frac{1}{Z(Z\overline{Z}_0-1)} + \frac{(Z_0\overline{Z}_0-1)Z_0}{(Z\overline{Z}_0-1)^2} - \frac{2(Z_0\overline{Z}_0-1)\overline{Z}_0}{(Z\overline{Z}_0-1)^3}$$

$$- \frac{Z_0\overline{Z}_0-1}{\overline{Z}_0 Z^2} \tag{3.5}$$

Hence, the influence function can be easily found as described in section I.2. They are

$$H_{xx;q}P^*_q = \text{Re}\ [2\phi'(Z) - \overline{Z}\phi''(Z) - \Psi'(Z)]$$

$$H_{yy;q}P^*_q = \text{Re}\ [2\phi'(Z) + \overline{Z}\phi''(Z) + \Psi'(Z)]$$

$$H_{xy;q}P^*_q = \text{Im}\ [\overline{Z}\phi''(Z) + \Psi'(Z)]$$

$$I_{x:q}P^*_q = \frac{1}{2\mu}\ \text{Re}\ [\alpha\phi(Z) - Z\overline{\phi'(Z)} - \overline{\Psi(Z)}]$$

$$I_{y:q}P^*_q = \frac{1}{2\mu}\ \text{Im}\ [\alpha\phi(Z) - Z\overline{\phi'(Z)} - \overline{\Psi(Z)}] \tag{3.6}$$

Substituting (3.4) into (3.6) leads to

$$H_{xx;q}(Z,Z_0)P^*_q(Z_0) = \frac{\text{Re}}{2\pi(\alpha+1)}\Big[\{2\phi'_I(Z) - \overline{Z}\phi''_I(Z) - \Psi'_I(Z)\}P^*$$

$$+ \{2\phi'_{II}(Z) - \overline{Z}\phi''_{II}(Z) - \Psi'_{II}(Z)\}\ \overline{P^*}\Big]$$

$$H_{yy;q}(Z,Z_0)P^*_q(Z_0) = \frac{\mathrm{Re}}{2\pi(\alpha+1)}\Big[\{2\phi'_I(Z) + \bar{Z}\phi''_I(Z) + \Psi'_I(Z)\}\ P^*$$

$$+ \{2\phi'_{II}(Z) + \bar{Z}\phi''_{II}(Z) + \Psi'_{II}(Z)\}\ \bar{P}^*\Big]$$

$$H_{xy;q}(Z,Z_0)P^*_q(Z_0) = \frac{\mathrm{Im}}{2\pi(\alpha+1)}\Big[\{\bar{Z}\phi''_I(Z) + \Psi'_I(Z)\}\ P^*$$

$$+ \{\bar{Z}\phi''_{II}(Z) + \Psi'_{II}(Z)\}\ \bar{P}^*\Big]$$

$$I_{x;q}(Z,Z\ )P^*_q(Z_0) = \frac{\mathrm{Re}}{4\pi\mu(\alpha+1)}\Big[\{\alpha\phi_I(Z) - Z\overline{\phi'_I(Z)} - \overline{\Psi_I(Z)}\}\ P^*$$

$$+ \{\alpha\phi_{II}(Z) + Z\overline{\phi'_{II}(Z)} - \overline{\Psi_{II}(Z)}\}\ \bar{P}^*\Big]$$

$$I_{y;q}(Z,Z_0)P^*_q(Z_0) = \frac{\mathrm{Im}}{4\pi\mu(\alpha+1)}\Big[\{\alpha\phi_I(Z) - Z\overline{\phi'_I(Z)} - \overline{\Psi_I(Z)}\}\ P^*$$

$$+ \{\alpha\phi_{II}(Z) + Z\overline{\phi'_{II}(Z)} - \overline{\Psi_{II}(Z)}\}\ \bar{P}^*\Big] \quad (3.7)$$

where $\phi_I(Z)$, $\phi'_I(Z)$, $\phi''_I(Z)$, $\phi_{II}(Z)$, $\phi'_{II}(Z)$, $\phi''_{II}(Z)$, $\Psi_I(Z)$, $\Psi'_I(Z)$, $\Psi_{II}(Z)$ and $\Psi'_{II}(Z)$ are defined by equations (3.4). These influence functions will be used to solve a simple problem by the boundary-integral method in section 4 of this chapter.

III.3 DERIVATION OF THE INFLUENCE FUNCTIONS USING A MAPPING TECHNIQUE

The mapping technique which is presented in Chapter II is now employed to obtain the influence functions. Consider the problem of an infinite plane having a circular hole of radius a at the origin and a concentrated point force P acting in the plane at some points Z_0 where $|Z_0| \geqslant a$. This problem can be expressed as the superposition of two problems, Figure 3.1.

In the problem of Figure 3.1(B), the concentrated point force P acting at the point Z_0 in an infinite plane is considered. In the problem of Figure 3.1(C), the infinite plane contains a circular hole with a prescribed traction acting on its circumference. The traction on the circular hole is equal in magnitude and opposite in direction to the generated traction on a circular contour in the problem of Figure 3.1(B). By adding the solutions to the problems of Figure 3.1(B) and 3.1(C), the zero traction on the hole of the problem of Figure 3.1(A) is obtained.

The solution of the problem of Figure 3.1(B) is well known (Muskhelishvili [27]) so that the required traction can be found. Also, the problem of Figure 3.1(C) may be handled by mapping into a unit circle (disc), Figure 3.2.

Clearly, the mapping function for this problem is

$$Z = \omega(\zeta) = \frac{a}{\zeta}$$

which is conformal and one to one [37]. Without loss of generality, let a = 1. Let the complex potential function for the problems of Figure 3.1(A), 3.1(B), and 3.1(C) be $\phi(Z)$, $\Psi(Z)$; $\phi^0(Z)$, $\Psi^0(Z)$; and $\phi^*(Z)$ and $\Psi^*(Z)$, respectively.

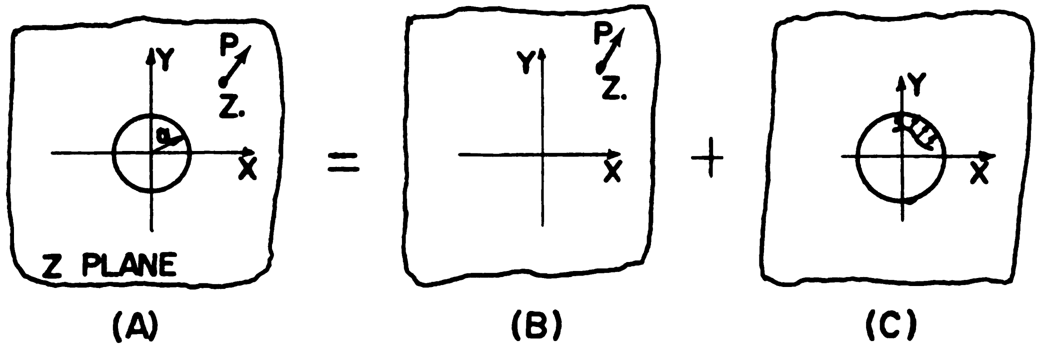

Figure 3.1 Fundamental problem expressed as superposition of two problems.

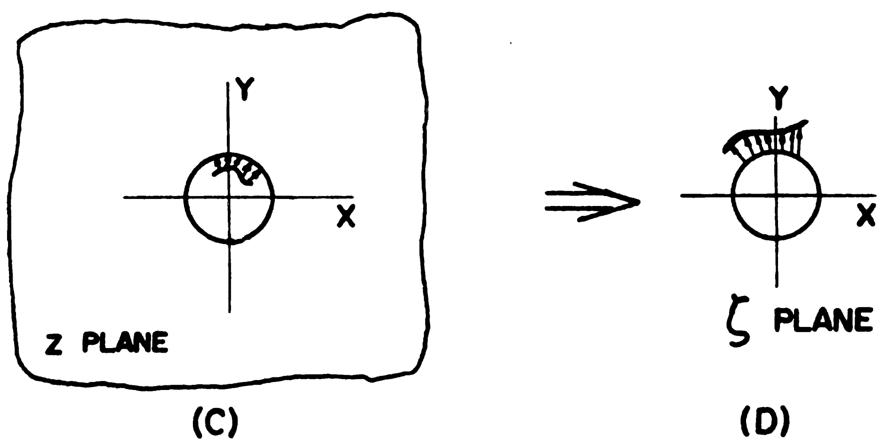

Figure 3.2 Mapping the auxiliary problem to a unit disc.

Then the complex potential functions for the problem of Figure 3.1(A) are

$$\phi(Z) = \phi^0(Z) + \phi^*(Z)$$

$$\Psi(Z) = \Psi^0(Z) + \Psi^*(Z)$$

where the derivatives are given by equations (2.2) through (2.5) in section II.3. To find $\phi^*(Z)$ and $\Psi^*(Z)$, the transformed complex potential functions which are given by equations (2.21) and (2.22) will be used. It is first necessary to calculate the integrals of equations (2.21) and (2.22).

$$I_1 = \frac{1}{2\pi i} \oint_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \frac{\overline{\phi_1^{*\prime}(\sigma)}}{\sigma - \zeta} d\sigma \qquad (3.8)$$

$$I_2 = \frac{1}{2\pi i} \oint_\gamma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} \frac{\phi_1^{*\prime}(\sigma)}{\sigma - \zeta} d\sigma \qquad (3.9)$$

Taking the derivative of equation (2.16) leads to

$$\phi_1^{*\prime}(\sigma) = \sum_{k=1}^{\infty} k a_k \sigma^k \qquad (3.10)$$

and the complex conjugate is:

$$\overline{\phi_1^{*\prime}(\sigma)} = \sum_{k=1}^{\infty} k \bar{a}_k \bar{\sigma}^{k-1}$$

Since $\sigma\bar{\sigma} = 1$, then

$$\overline{\phi_1^{*\prime}(\sigma)} = \sum_{k=1}^{\infty} k\bar{a}_k \sigma^{-k+1} \tag{3.11}$$

The mapping function and its derivative, evaluated on the boundary, are

$$\omega(\sigma) = \frac{1}{\sigma} \quad , \quad \omega'(\sigma) = -\frac{1}{\sigma^2}$$

so that

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} = -\frac{1}{\sigma^3} \tag{3.12}$$

Multiplying equations (3.11) by (3.12) gives:

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \cdot \overline{\phi_1^{*\prime}(\sigma)} = \sum_{k=1}^{\infty} -k\bar{a}_k \sigma^{-k-2} \tag{3.13}$$

From equation (3.13), it is clear that the right-hand side is an analytic function outside of γ, the unit circle. The value of the right-hand side at infinity is zero. Thus, due to the Cauchy integral formulas, the principal value of the integral of equation (3.8) leads to:

$$I_1 = \frac{1}{2\pi i} \oint_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \frac{\overline{\phi_1^{*\prime}(\sigma)}}{\sigma - \zeta} d\sigma = 0 \tag{3.14}$$

To calculate the integral of equation (3.9), consider the complex conjugate of equation (3.13), which is

$$\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} = -\sigma^3 \tag{3.15}$$

Multiplying equations (3.10) by (3.15) leads to:

$$\frac{\overline{\omega(\sigma)}}{\overline{\omega'(\sigma)}} \phi_1^{*'}(\sigma) = \sum_{k+1}^{\infty} - ka_k\sigma^{k+3} \qquad (3.16)$$

Clearly, the right-hand side of equation (3.16) is analytic inside the unit circle. Hence, following the Cauchy integral formulas, the principal value of the integral of (3.9) leads to:

$$I_2 = \frac{1}{2\pi i} \oint_{\gamma} \frac{\overline{\omega(\sigma)}}{\overline{\omega'(\sigma)}} \frac{\phi_1^{*'}(\sigma)}{\sigma-\zeta} d\sigma = -\zeta^3\phi^{*'}(\zeta) \qquad (3.17)$$

Substituting integrals (3.14) and (3.17) into the general formulation for $\phi_1^*(\zeta)$ and $\Psi_1^*(\zeta)$ (equations [2.21] and [2.22]), the complex potential functions for a circular disc with a specified boundary value, $F(\sigma)$, are obtained:

$$\phi_1^*(\zeta) = \frac{1}{2\pi i} \oint_{\gamma} \frac{F(\sigma)}{\sigma-\zeta} d\sigma \qquad (3.18)$$

$$\Psi_1^*(\zeta) = \frac{1}{2\pi i} \oint_{\gamma} \frac{\overline{F(\sigma)}}{\sigma-\zeta} d\sigma + \zeta^3\phi^{*'}(\zeta) \qquad (3.19)$$

For the case considered, $F(\sigma)$ and $\overline{F(\sigma)}$ will now be calculated. Rewriting equation (2.10) and taking the conjugate leads to:

$$F(\sigma) = -\left[\phi_1^0(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \overline{\phi_1^{0'}(\sigma)} + \overline{\Psi_1^0(\sigma)}\right] \qquad (3.20)$$

$$\overline{F(\sigma)} = -\left[\overline{\phi_1^0(\sigma)} + \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} \phi_1^{0'}(\sigma) + \Psi_1^0(\sigma)\right] \qquad (3.21)$$

Substituting equation (3.7) into (2.6), the transformed complex potential functions $\phi_1^0(\zeta)$ and $\Psi_1^0(\zeta)$ are:

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$$\phi_1^0(\zeta) = -\frac{P}{2\pi(\alpha+1)} \ln\left(\frac{1-Z_0\zeta}{\zeta}\right) \qquad (3.22)$$

$$\Psi_1^0(\zeta) = \alpha\frac{\overline{P}}{2\pi(\alpha+1)} \ln\left(\frac{1-Z_0\zeta}{\zeta}\right) + \frac{P}{2\pi(\alpha+1)}\frac{\overline{Z}_0\zeta}{1-Z_0\zeta} \qquad (3.23)$$

Taking the derivative of equation (3.22) and substituting $\zeta = \sigma$ into equations (3.22) and (3.23) leads to:

$$\phi_1^0(\sigma) = -\frac{P}{2\pi(\alpha+1)} \ln\left(\frac{1-Z_0\sigma}{\sigma}\right)$$

$$\phi_1^{0\prime}(\sigma) = -\frac{P}{2\pi(\alpha+1)}\frac{1}{(1-Z_0\sigma)}$$

$$\Psi_1^0(\sigma) = \alpha\frac{\overline{P}}{2\pi(\alpha+1)} \ln\left(\frac{1-Z_0\sigma}{\sigma}\right) + \frac{P}{2\pi(\alpha+1)}\cdot\frac{\overline{Z}_0\sigma}{1-Z_0\sigma} \qquad (3.24)$$

Taking the complex conjugates of equations (3.24) along with equations (3.12) and (3.15) will provide all the terms on the right-hand side of equations (3.20) and (3.12). Then $F(\sigma)$ and $\overline{F(\sigma)}$ will be

$$F(\sigma) = Q\left\{\ln\left(\frac{1-Z_0\sigma}{\sigma}\right) - \alpha\ln(\sigma-\overline{Z}_0)\right\} + \overline{Q}\left\{\frac{1-Z_0\sigma}{\sigma(\sigma-\overline{Z}_0)}\right\} \qquad (3.25)$$

$$\overline{F(\sigma)} = Q\left\{\frac{\sigma(\sigma-\overline{Z}_0)}{1-Z_0\sigma}\right\} + \overline{Q}\left\{\ln(\sigma-\overline{Z}_0) - \alpha\ln\frac{1-Z_0\sigma}{\sigma}\right\} \qquad (3.26)$$

where $Q = \frac{P}{2\pi(\alpha+1)}$.

Substituting equation (3.25) into (3.18) leads to

$$\phi_1^*(\zeta) = \frac{Q}{2\pi i}\oint_\gamma \frac{\ln\left(\frac{1-Z_0\sigma}{\sigma}\right)}{\sigma-\zeta}\,d\sigma - \frac{\alpha Q}{2\pi i}\oint_\gamma \frac{\ln(\sigma-\overline{Z}_0)}{\sigma-\zeta}\,d\sigma$$

$$+\frac{\overline{Q}}{2\pi i}\oint_\gamma \frac{(1-Z_0\sigma)}{\sigma(\sigma-\overline{Z}_0)(\sigma-\zeta)}\,d\sigma \qquad (3.27)$$

Recalling the discussion in section I.3, $\ln(\frac{1-Z_0\sigma}{\sigma})$ is an analytic function outside of γ, the unit circle. Because $Z_0>1$, the function has two essential singular points at σ_0 and $\sigma = 1/Z_0$. The function is defined at infinity as:

$$[\ln\ (\frac{1-Z_0\sigma}{\sigma})]_{\sigma=\infty} = \ln\ (-Z_0)$$

then

$$\frac{1}{2\pi i}\oint_\gamma \frac{\ln(\frac{1-Z_0\sigma}{\sigma})}{\sigma-\zeta}\,d\sigma = \ln\ (-Z_0) \qquad (3.28)$$

Clearly, $\ln(\sigma-Z_0)$ is an analytic function inside γ, the unit circle. Hence, the Cauchy integral formulas lead to:

$$\frac{1}{2\pi i}\oint_\gamma \frac{\ln(\sigma-\overline{Z}_0)}{\sigma-\zeta}\,d\sigma = \ln(\zeta-\overline{Z}_0) \qquad (3.29)$$

Also

$$\frac{1}{2\pi i}\oint_\gamma \frac{(1-Z_0\sigma)}{\sigma(\sigma-\overline{Z}_0)(\sigma-\zeta)}\,d\sigma = \text{Residu}\Big|_{\sigma=\zeta}\left\{\frac{1-Z_0\sigma}{\sigma(\sigma-\overline{Z}_0)(\sigma-\zeta)}\right\}$$

$$+ \text{Residu}\Big|_{\sigma=0}\left\{\frac{1-Z_0\sigma}{\sigma(\sigma-\overline{Z}_0)(\sigma-\zeta)}\right\}$$

or

$$\frac{1}{2\pi i}\oint_\gamma \frac{(1-Z_0\sigma)}{\sigma(\sigma-\overline{Z}_0)(\sigma-\zeta)} = \frac{1-Z_0\zeta}{\zeta(\zeta-\overline{Z}_0)} + \frac{1}{\overline{Z}_0\zeta} \qquad (3.30)$$

Substituting the integrals of equations (3.29) and (3.30) into equation (3.27), the complex potential function can be obtained:

$$\phi_1^*(\zeta) = Q\left\{\ln(-Z_0) - \alpha \ln(\zeta - Z_0)\right\} + \bar{Q}\left\{\frac{1-Z_0\bar{Z}_0}{\bar{Z}_0(\zeta-\bar{Z}_0)}\right\} \tag{3.31}$$

To obtain $\Psi_1^*(\zeta)$ it is necessary to find the integral of equation (3.19), i.e.,

$$I_3 = \frac{1}{2\pi i}\oint_\gamma \frac{\overline{F(\sigma)}}{\sigma-\zeta}\,d\sigma$$

Then from equation (3.26):

$$I_3 = \frac{Q}{2\pi i}\oint_\gamma \frac{\sigma(\sigma-\bar{Z}_0)d\sigma}{(1-Z_0\sigma)(\sigma-\zeta)} + \frac{\bar{Q}}{2\pi i}\oint_\gamma \frac{\ln(\sigma-\bar{Z}_0)}{\sigma-\zeta}\,d\sigma$$

$$- \frac{\bar{Q}\cdot\alpha}{2\pi i}\oint_\gamma \frac{\ln\left(\frac{1-Z_0\sigma}{\sigma}\right)}{\sigma-\zeta}\,d\sigma \tag{3.32}$$

where

$$\frac{1}{2\pi i}\oint_\gamma \frac{\sigma(\sigma-\bar{Z}_0)}{(1-Z_0\sigma)(\sigma-\zeta)}\,d\sigma = \text{Residu}\Big|_{\sigma=\zeta}\left\{\frac{\sigma(\sigma-\bar{Z}_0)}{-Z_0(\sigma-\frac{1}{Z_0})(\sigma-\zeta)}\right\}$$

$$+ \text{Residu}\Big|_{\sigma=\frac{1}{Z_0}}\left\{\frac{\sigma(\sigma-Z_0)}{-Z_0(\sigma-\frac{1}{Z_0})(\sigma-\zeta)}\right\}$$

After some simplification:

$$\frac{1}{2\pi i}\oint_\gamma \frac{\sigma(\sigma-\bar{Z}_0)}{(1-Z_0\sigma)(\sigma-\zeta)}\,d\sigma = -\frac{\zeta}{Z_0} - \frac{1-Z_0\bar{Z}_0}{Z_0{}^2} \tag{3.33}$$

Substituting equations (3.33), (3.29) and (3.28) into equation (3.32) leads to:

$$\frac{1}{2\pi i}\oint_{\gamma} \frac{\overline{F(\sigma)}}{\sigma-\zeta}\,d\sigma = Q\left\{-\frac{\zeta}{Z_0} - \frac{1-Z_0\overline{Z}_0}{Z_0{}^2}\right\} + \overline{Q}\left\{\ln(\zeta\cdot\overline{Z}_0)\right.$$

$$\left. - \alpha\ln(-Z_0)\right\} \tag{3.34}$$

Taking the derivative of equation (3.31) gives:

$$\phi_1^{*\prime}(\zeta) = Q\left\{\frac{-\alpha}{\zeta-\overline{Z}_0}\right\} + \overline{Q}\left\{\frac{1-Z_0\overline{Z}_0}{\overline{Z}_0}\cdot\frac{-1}{(\zeta-\overline{Z}_0)^2}\right\} \tag{3.35}$$

Substituting equations (3.35), (3.34) into equation (3.19) and following some simplification, the other complex potential function is:

$$\Psi_1^*(\zeta) = Q\left\{-\frac{\zeta}{Z_0} - \frac{1-Z_0\overline{Z}_0}{Z_0{}^2} - \frac{\alpha\zeta^3}{\zeta-\overline{Z}_0}\right\} + \overline{Q}\left\{\ln(\zeta-\overline{Z}_0) - \alpha\ln(-Z_0)\right.$$

$$\left. - \frac{1-Z_0\overline{Z}_0}{\overline{Z}_0}\cdot\frac{\zeta^3}{(\zeta-\overline{Z}_0)^2}\right\} \tag{3.36}$$

As is discussed in section I.2, the two complex potential functions $\phi_1^*(\zeta)$ and $\Psi_1^*(\zeta)$, expressed by equations (3.31) and (3.36), are not a unique set of functions. Since the origin of the coordinates is within γ, then, following section I.2, the uniqueness conditions for $\phi_1^*(\zeta)$ and $\Psi_1^*(\zeta)$ are:

$$\phi_1^*(0) = 0 \qquad , \qquad \Psi_1^*(0) = 0 \tag{3.37}$$

Conditions (3.37) lead to the unique complex potential functions:

$$\phi_1^*(\zeta) = Q\left\{\alpha \ln(-\overline{Z}_0) - \alpha \ln(\zeta-\overline{Z}_0)\right\} + \overline{Q}\left\{\frac{1-Z_0\overline{Z}_0}{\overline{Z}_0{}^2}\right.$$

$$\left. + \frac{1-Z_0\overline{Z}_0}{\overline{Z}_0} \cdot \frac{1}{\zeta-\overline{Z}_0}\right\} \tag{3.38}$$

$$\Psi_1^*(\zeta) = Q\left\{-\frac{\zeta}{Z_0} - \frac{\alpha\zeta^3}{\zeta-\overline{Z}_0}\right\} + \overline{Q}\left\{\ln(\zeta-\overline{Z}_0) - \ln(-\overline{Z}_0)\right.$$

$$\left. - \frac{1-Z_0\overline{Z}_0}{\overline{Z}_0} \cdot \frac{\zeta^3}{(\zeta-\overline{Z}_0)^2}\right\} \tag{3.39}$$

These can be rewritten as:

$$\phi_1^*(\zeta) = Q\left\{\phi_{I}^*(\zeta)\right\} + \overline{Q}\left\{\phi_{II}^*(\zeta)\right\}$$

$$\Psi_1^*(\zeta) = Q\left\{\Psi_{I}^*(\zeta)\right\} + \overline{Q}\left\{\Psi_{II}^*(\zeta)\right\}$$

where $\phi_I^*(\zeta)$, $\phi_{II}^*(\zeta)$, $\Psi_I^*(\zeta)$ and $\Psi_{II}^*(\zeta)$ can be obtained by comparison to equations (3.38) and (3.39) and are given in Appendix A.

Then

$$\phi_1^{*\prime}(\zeta) = Q\left\{\phi_{I}^{*\prime}(\zeta)\right\} + \overline{Q}\left\{\phi_{II}^{*\prime}(\zeta)\right\}$$

$$\phi_1^{*\prime\prime}(\zeta) = Q\left\{\phi_{I}^{*\prime\prime}(\zeta)\right\} + \overline{Q}\left\{\phi_{II}^{*\prime\prime}(\zeta)\right\}$$

$$\Psi_1^{*\prime}(\zeta) = Q\left\{\Psi_{I}^{*\prime}(\zeta)\right\} + \overline{Q}\left\{\Psi_{II}^{*\prime}(\zeta)\right\} \tag{3.40}$$

where

$$\phi_{I}^{*\prime}(\zeta) = -\frac{\alpha}{\zeta-\overline{Z}_0}$$

$$\phi_{II}^{*\prime}(\zeta) = -\frac{1-Z_0\overline{Z}_0}{\overline{Z}_0}\cdot\frac{1}{(\zeta-\overline{Z}_0)^2}$$

$$\phi_{I}^{*\prime\prime}(\zeta) = \frac{\alpha}{(\zeta-\overline{Z}_0)^2}$$

$$\phi_{II}^{*\prime\prime}(\zeta) = \frac{1-Z_0\overline{Z}_0}{\overline{Z}_0}\cdot\frac{2}{(\zeta-\overline{Z}_0)^3}$$

$$\Psi_{I}^{*\prime}(\zeta) = -\frac{1}{\overline{Z}_0} - \frac{\alpha\zeta^2(2\zeta-3\overline{Z}_0)}{(\zeta-\overline{Z}_0)^2}$$

$$\Psi_{II}^{*\prime}(\zeta) = \frac{1}{\zeta-\overline{Z}_0} - \frac{1-Z_0\overline{Z}_0}{\overline{Z}_0}\cdot\frac{\zeta^2(\zeta-3\overline{Z}_0)}{(\zeta-\overline{Z}_0)^3} \qquad (3.41)$$

Complex potential functions of equation (2.6) can be rewritten as:

$$\phi^0(Z) = -Q.\ln(Z-Z_0)$$

$$\Psi^0(Z) = Q\cdot\frac{\overline{Z}_0}{Z-\overline{Z}_0} + \overline{Q}\cdot\ln(Z-\overline{Z}_0) \qquad (3.42)$$

and the derivatives are:

$$\phi^{0\prime}(Z) = -Q\cdot\frac{1}{Z-Z_0}$$

$$\phi^{0\prime\prime}(Z) = +Q\frac{1}{(Z-Z_0)^2}$$

$$\Psi^{0\prime}(Z) = -Q\frac{\overline{Z}_0}{(Z-\overline{Z}_0)^2} + \overline{Q}\cdot\frac{\alpha}{Z-\overline{Z}_0} \qquad (3.43)$$

Substituting equations (3.40), (3.42) and (3.43) into equations (2.5) leads to:

$$\phi(Z) = Q\left\{-\ln(Z-Z_0) + \phi_I^*(\zeta)\right\} + \overline{Q}\left\{\phi_{II}^*(\zeta)\right\}$$

$$\Psi(Z) = Q\left\{\frac{\overline{Z}}{Z-\overline{Z}_0} + \Psi_I^*(\zeta)\right\} + \overline{Q}\left\{\alpha\ln(Z-\overline{Z}_0) + \Psi_{II}^*(\zeta)\right\}$$

$$\phi'(Z) = Q\left\{\frac{-1}{Z-Z_0} + \frac{\phi_I^{*\prime}(\zeta)}{\omega'(\zeta)}\right. + \overline{Q}\left\{\frac{\phi_{II}^{*\prime}(\zeta)}{\omega'(\zeta)}\right\}$$

$$\Psi'(Z) = Q\left\{\frac{-\overline{Z}_0}{(Z-\overline{Z}_0)^2} + \frac{\Psi_I^{*\prime}(\zeta)}{\omega'(\zeta)}\right\} + \overline{Q}\left\{\frac{\alpha}{Z-\overline{Z}_0} + \frac{\Psi_{II}^{*\prime}(\zeta)}{\omega'(\zeta)}\right\}$$

$$\phi''(Z) = Q\left\{\frac{1}{(Z-Z_0)^2} + \frac{\phi_I^{*\prime\prime}(\zeta)}{\omega'^2(\zeta)} - \frac{\phi_I^{*\prime}(\zeta)\omega''(\zeta)}{\omega'^3(\zeta)}\right\}$$

$$+ \overline{Q}\left\{\frac{\phi_I^{*\prime\prime}(\zeta)}{\omega'^2(\zeta)} - \frac{\phi_{II}^{*\prime}(\zeta)\omega''(\zeta)}{\omega'^3(\zeta)}\right\} \tag{3.44}$$

where $\phi_I^{*\prime}(\zeta)$, $\phi_I^{*\prime\prime}(\zeta)$, $\phi_{II}^{*\prime}(\zeta)$, $\phi_{II}^{*\prime\prime}(\zeta)$, $\Psi_I^{*\prime}(\zeta)$ and $\Psi_{II}^{*\prime}(\zeta)$ are defined by equations (3.41).

Substituting equations (3.44) into equations (3.6) along with the mapping function, $\omega(\zeta) = 1/\zeta$, and its derivatives, leads to the influence functions for a circular opening:

$$H_{xx;q}(Z,Z_0)P_q^*(Z_0) = \mathrm{Re}\left\{Q^*\left(\frac{-2}{Z-Z_0} - 2\zeta^2\phi_I^{*\prime}(\zeta) - \overline{Z}\left[\frac{1}{(Z-Z_0)} + \zeta^4\phi_I^{*\prime\prime}(\zeta) + 2\zeta^3\phi_I^{*\prime}(\zeta)\right] + \frac{\overline{Z}_0}{(Z-\overline{Z}_0)^2} + \zeta^2\Psi_I^{*\prime}(\zeta)\right) + \overline{Q}^*\left(-2\zeta^2\phi_{II}^{*\prime}(\zeta) - \overline{Z}\left[\zeta^4\phi_{II}^{*\prime\prime}(\zeta) + 2\zeta^3\phi_{II}^{*\prime}(\zeta)\right] - \frac{\alpha}{Z-\overline{Z}_0} + \zeta^2\Psi_{II}^{*\prime}(\zeta)\right)\right\}$$

$$H_{yy;q}(Z,Z_0)P_q^*(Z_0) = \mathrm{Re}\left\{Q^*\left(\frac{-2}{Z-Z_0} - 2\zeta^2\phi_I^{*\prime}(\zeta) + \overline{Z}\left[\frac{1}{(Z-Z_0)^2} + \zeta^4\phi_I^{*\prime\prime}(\zeta) + 2\zeta^3\phi_I^{*\prime}(\zeta)\right] - \frac{\overline{Z}_0}{(Z-\overline{Z}_0)^2} - \zeta^2\Psi_I^{*\prime}(\zeta)\right) + \overline{Q}^*\left(-2\zeta^2\phi_{II}^*(\zeta) + \overline{Z}\left[\zeta^4\phi_{II}^{*\prime\prime}(\zeta) + 2\phi_{II}^{*\prime}(\zeta)\zeta^3\right] + \frac{\alpha}{Z-\overline{Z}_0} - \zeta^2\Psi_{II}^{*\prime}(\zeta)\right)\right\}$$

$$H_{xy;q}(Z,Z_0)P_q^*(Z_0) = \mathrm{Im}\left\{Q^*\left(\overline{Z}\left[\frac{1}{(Z-Z_0)^2} + \zeta^4\phi_I^{*\prime\prime}(\zeta) + 2\phi_I^{*\prime}(\zeta)\cdot\zeta^3\right] - \frac{\overline{Z}_0}{(Z-\overline{Z}_0)^2} - \zeta^2\Psi_I^{*\prime}(\zeta)\right) \overline{Q}^*\left(\overline{Z}\left[\zeta^4\phi_{II}^{*\prime\prime}(\zeta) + 2\phi_{II}^{*\prime}(\zeta)\cdot\zeta^3\right] - \frac{\alpha}{Z-\overline{Z}_0} + \zeta^2\Psi_{II}^{*\prime}(\zeta)\right)\right\}$$

$$I_{x;q}(Z,Z_0)P_q^*(Z_0) = \mathrm{Re}\left\{\frac{Q^*}{2\mu}\left(-\alpha\ln(Z-Z_0) + \alpha\phi_I^*(\zeta) - Z\left[\frac{-1}{\overline{Z}-\overline{Z}_0} - \overline{\zeta}^2\overline{\phi_I^{*\prime}(\zeta)}\right] - \frac{Z_0}{\overline{Z}-Z_0} - \overline{\Psi_I^*(\zeta)}\right) + \frac{\overline{Q}^*}{2\mu}\left(\alpha\phi_{II}^*(\zeta) + Z\overline{\zeta}^2\,\overline{Q_{II}^{*\prime}(\zeta)} - \alpha\ln(\overline{Z}-Z_0) - \overline{\Psi_{II}^*(\zeta)}\right)\right\}$$

$$I_{y;q}(Z,Z_0)P_q^*(Z_0) = \mathrm{Im}\left\{\frac{Q^*}{2\mu}\left(-\alpha\ln(Z-Z_0) + \alpha\phi_I^*(\zeta) - Z\left[\frac{-1}{\overline{Z}-\overline{Z}_0} - \overline{\zeta}^2\overline{\phi_I^{*\prime}(\zeta)}\right] - \frac{Z_0}{\overline{Z}-Z_0} - \overline{\Psi_I^*(\zeta)}\right) + \frac{\overline{Q}^*}{2\mu}\left(\alpha\phi_{II}^*(\zeta) + Z\overline{\zeta}^2\overline{\phi_{II}^{*\prime}(\zeta)} - \alpha\ln(\overline{Z}-Z_0) - \overline{\Psi_{II}^*(\zeta)}\right)\right\} \quad (3.44)$$

Hence the influence functions for an infinite plate with a unit circular hole at the origin are obtained.

It has been stated that these influence functions are unique. Therefore, they must be identical to those found in section III.2. To show this, let $\phi^*(Z)$ and $\Psi^*(Z)$ be determined, using the known solutions

$$\phi^*(Z) = \phi(Z) - \phi^0(Z) = \frac{P}{2\pi(\alpha+1)}\left\{-\alpha\ln\frac{Z\bar{Z}_0-1}{Z\bar{Z}_0}\right\} + \frac{\bar{P}}{2\pi(\alpha+1)}\left\{\frac{Z_0\bar{Z}_0-1}{\bar{Z}_0^{\,2}}\cdot\frac{1}{Z\bar{Z}_0-1}\right\}$$

$$\Psi^*(Z) = \Psi(Z) - \Psi^0(Z) = \frac{P}{2\pi(\alpha+1)}\left\{\frac{\alpha\bar{Z}_0^{\,2}}{Z\bar{Z}_0-1} - \frac{\alpha Z_0\bar{Z}_0+1}{Z\bar{Z}_0} - \frac{\alpha}{Z^2}\right\}$$
$$+ \frac{\bar{P}}{2\pi(\alpha+1)}\left\{\ln\left(\frac{Z\bar{Z}_0-1}{Z\bar{Z}_0}\right) - \frac{Z_0\bar{Z}_0-1}{Z\bar{Z}_0-1} + \frac{Z_0\bar{Z}_0-1}{(Z\bar{Z}_0-1)^2} + \frac{Z_0\bar{Z}_0-1}{Z\bar{Z}_0}\right\} \qquad (3.45)$$

where $\phi(Z)$, $\Psi(Z)$ were given by equations (3.3) and $\phi^0(Z)$, $\Psi^0(Z)$ were given by equation (2.6).

Transforming the two equations (3.45) into the ζ-plane, i.e., $Z = \frac{1}{\zeta}$, leads to

$$\phi_1^*(\zeta) = Q\left\{\alpha\ln(-\bar{Z}_0) - \alpha\ln(\zeta-\bar{Z}_0)\right\} + \bar{Q}\left\{\frac{1-Z_0\bar{Z}_0}{\bar{Z}_0^{\,2}} + \frac{1-Z_0\bar{Z}_0}{\bar{Z}_0}\cdot\frac{1}{\zeta-\bar{Z}_0}\right\}$$

$$\Psi_1^*(\zeta) = Q\left\{-\frac{\zeta}{Z_0} - \frac{\alpha\zeta^3}{\zeta-\overline{Z}_0}\right\} + \overline{Q}\left\{\ln(\zeta-\overline{Z}_0) - \ln(-\overline{Z}_0)\right.$$

$$\left. - \frac{1-Z_0\overline{Z}_0}{\overline{Z}_0} \cdot \frac{\zeta^3}{(\zeta-\overline{Z}_0)^2}\right\} \qquad (3.46)$$

The two potential functions, found by equations (3.46), have also been determined by the mapping technique, equations (3.39), and as one can see, they are identical.

III.4 THE BOUNDARY INTEGRAL EQUATION METHOD APPLIED TO PLANE FINITE REGIONS WEAKENED BY A CIRCULAR HOLE

The basic idea of the Boundary Integral Equation Method has been discussed in section I.1, where the discretized form of the integral equations is given, see equation (1.19). Consider a plane finite region with boundary B subjected to specified traction boundary condition, t, and containing a unit circular hole at the origin, Figure 3.3. Divide the boundary, B, into N meshes (not necessarily equal) and embed the region R in an infinite (fictitious) plane of the same material as R containing a unit circular hole at the origin, see Figure 3.4. Note that the influence functions $H_{ij;q}(Z,Z_0)$ and $I_{i;q}(Z,Z_0)$ for this fictitious region are given by equations (3.44).

Following section I.1, the fictitious traction P* around the fictitious boundary can be found from:

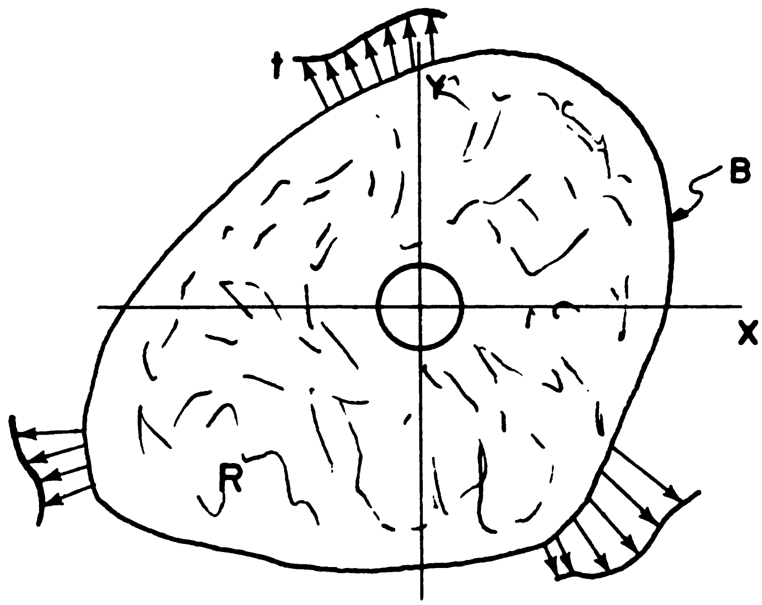


Figure 3.3 A unit circular hole in a plane finite region with prescribed traction on the boundary.

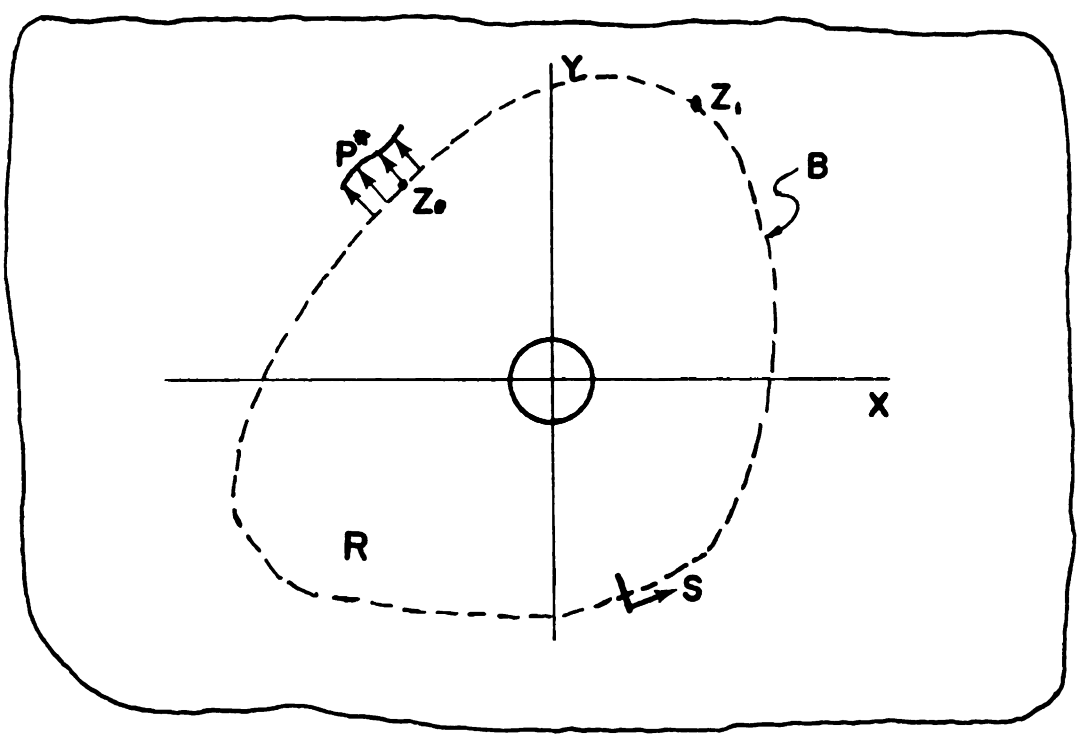


Figure 3.4 Region R embedded in an infinite plane containing a circular hole at the origin.

$$\frac{P^*_{xi}}{2} + \sum_{\substack{j=1 \\ j \neq i}}^{N} \left\{ H_{xx;q}(Z,Z_0)P^*_q(Z_0)n_{xi} + H_{xy;q}(Z,Z_0)P^*_q(Z_0)n_{yi} \right\} \Delta S_i$$

$$= P_{xi} \qquad\qquad (i=1,..N)$$

$$\frac{P^*_{yi}}{2} + \sum_{\substack{j=1 \\ j \neq i}}^{N} \left\{ H_{xy;q}(Z,Z_0)P^*_q(Z_0)n_{xi} + H_{yy;q}(Z,Z_0)P^*_q(Z_0)n_{yi} \right\} \Delta S_i$$

$$= P_{yi} \qquad\qquad (i=1,..N) \qquad (3.47)$$

where the influence functions $H_{ij;q}(Z,Z_0)P^*_q(Z_0)$ given by equation (3.44) and the resultant fictitious traction on a given interval is represented by:

$$P^*_{qi} = P^*_{xi} + iP^*_{yi} \qquad (i=1,..N)$$

In equation (3.47) n_{xi} and n_{yi} are the components of the unit normal to the interval i. Also, P_{xi} and P_{yi} are the x and y component of the real resultant traction applied to the mesh i.

Considering the influence functions, equation (3.44) and splitting each equation into two components of P*, i.e., P^*_x and P^*_y, leads to:

$$H_{xx;q}(Z,Z_0)P^*_q(Z_0) = H_{xx;x} \cdot P^*_x + H_{xx;y} \cdot P^*_y$$

$$H_{yy;q}(Z,Z_0)P^*_q(Z_0) = H_{yy;x} \cdot P^*_x + H_{yy;y} \cdot P^*_y$$

$$H_{xy;q}(Z,Z_0)P^*_q(Z_0) = H_{xy;x} \cdot P^*_x + H_{xy;x} \cdot P^*_y$$

$$I_{x;q}(Z,Z_0)P_q^*(Z_0) = I_{x;x} \cdot P_x^* + I_{x;y} \cdot P_y^*$$

$$I_{y;q}(Z,Z_0)P_q^*(Z_0) = I_{y;x} \cdot P_x^* + I_{y;y} \cdot P_y^* \qquad (3.48)$$

where $H_{ij;q}$ and $I_{i;q}$ can be easily found by comparing equations (3.48) with (3.44), see Appendix A.

Substituting equations (3.48) into equations (3.47) leads to:

$$\frac{P_{xi}^*}{2} + \sum_{\substack{j=1 \\ j\neq i}}^{N} \Big\{ (H_{xx;x} \cdot P_x^* + H_{xx;y} \cdot P_y^*) \cdot n_{xi} + (H_{xy;x} \cdot P_x^*$$

$$+ H_{xy;y} \cdot P_y^*) \cdot n_{yi} \Big\} \Delta S_i = P_{xi} \qquad (i=1,..N)$$

$$\frac{P_{yi}^*}{2} + \sum_{\substack{j=1 \\ j\neq i}}^{N} \Big\{ (H_{xy;x} \cdot P_x^* + H_{xy;y} \cdot P^*{}_y) \cdot n_{xi} + (H_{yy;x} \cdot P_x^*$$

$$+ H_{yy;y} \cdot P_y^*) \cdot n_{yi} \Big\} \Delta S_i = P_{xi} \qquad (i=1,..N) \qquad (3.49)$$

Rearranging equations (3.49):

$$\frac{P_{xi}^*}{2} + \sum_{\substack{j=1 \\ j\neq i}}^{N} \Big\{ [H_{xx;x} \cdot n_{xi} + H_{xy;x} \cdot n_{yi}] \cdot P_{xi}^* + [H_{xx;y} \cdot n_{xi}$$

$$+ H_{xy;y} \cdot n_{yi}] \cdot P_{yi}^* \Big\} \Delta S_i = P_{xi}$$

$$\frac{P^*_{yi}}{2} + \sum_{\substack{j=1\\j=i}}^{N} \Big\{ [H_{xy;x} \cdot n_{xi} + H_{yy;x} \cdot n_{yi}] \cdot P^*_{xi} + [H_{xy;y} \cdot n_{xi}$$

$$+ H_{yy;y} \cdot n_{yi}] \cdot P^*_{yi} \Big\} \Delta S_i = P_{yi} \qquad (3.50)$$

or writing equation (3.50) in the form of equation (1.24):

$$\frac{1}{2} P^*_{xi} + \sum_{\substack{j=1\\j\neq i}}^{N} \left(A_{ij} \cdot P^*_{xi} + B_{ij} \cdot P^*_{yi} \right) = BV_{xi} \quad (i=1,2..N)$$

$$\frac{1}{2} P^*_{yi} + \sum_{\substack{j=1\\j\neq i}}^{N} \left(C_{ij} \cdot P^*_{xi} + D_{ij} \cdot P^*_{yi} \right) = BV_{yi} \quad (i=1,2..N) \, (3.51)$$

where:

$$A_{ij} = H_{xx;x} \cdot n_{xi} + H_{xy;x} \cdot n_{yi}$$

$$B_{ij} = H_{xx;y} \cdot n_{xi} + H_{xy;y} \cdot n_{yi}$$

$$C_{ij} = H_{xy;x} \cdot n_{xi} + H_{yy;x} \cdot n_{yi}$$

$$D_{ij} = H_{xy;y} \cdot n_{xi} + H_{yy;y} \cdot n_{yi}$$

$$(i,j=1,...N)$$

Euqations (3.51) represent a set of 2N equations with 2N unknowns, i.e., P^*_{xi} and P^*_{yi} for i=1,...N. Methods for obtaining the solution have been discussed in section I.1.

Writing equation (3.51) in matrix form:

$$\begin{bmatrix} A_{ij} & B_{ij} \\ C_{ij} & D_{ij} \end{bmatrix} \cdot \begin{Bmatrix} P^*_{xi} \\ P^*_{yi} \end{Bmatrix} = \begin{Bmatrix} BV_{xi} \\ BV_{yi} \end{Bmatrix} \tag{3.52}$$

Note that the diagonals of submatrices $[A_{ij}]$ and $[D_{ij}]$ are 1/2 and the diagonals of submatrices $[B_{ij}]$ and $[C_{ij}]$ are zero

Equation (3.52) can be solved by matrix inversion, iteration, or elimination (Faddeeva [38]). Once the fictitious tractions are found, then the stress and displacement at the point F can be easily found following section I.2. These are:

$$\sigma_{xx} = \sum_{i=1}^{N} [H_{xx;x}(F,Z_0) \cdot P^*_{xi} + H_{xx;y}(F,Z_0) \cdot P^*_{yi}]$$

$$\sigma_{yy} = \sum_{i=1}^{N} [H_{yy;x}(F,Z_0) \cdot P^*_{xi} - H_{yy;y}(F,Z_0) \cdot P^*_{yi}]$$

$$\sigma_{xy} = \sum_{i=1}^{N} [H_{xy;x}(F,Z_0) \cdot P^*_{xi} + H_{xy;y}(F,Z_0) \cdot P^*_{yi}]$$

$$U_x = \sum_{i=1}^{N} [I_{x;x}(F,Z_0) \cdot P^*_{xi} + I_{x;y}(F,Z_0) \cdot P^*_{yi}]$$

$$U_y = \sum_{i=1}^{N} [I_{y;x}(F,Z\ ) \cdot P^*_{xi} + I_{y;y}(F,Z_0) \cdot P^*_{yi}] \tag{3.53}$$

EXAMPLE III.1

A Rectangular Plane Weakened by a Circular Hole

Consider the rectangular region (10cm x 20cm) of unit thickness (h = 1cm) which is weakened by a circular hole of radius r = 1cm at the origin, see Figure 3.5. A uniformly distributed traction (ω = 1.0 MPa) is applied to the top and the bottom of the rectangular region as shown. The boundary has been subdivided into sixty equally-spaced meshes, each of length 1.0cm, i.e., 10 meshes are defined on each of the top and bottom edges and 20 meshes on each vertical edge.

The field points, the points where the stress and displacement are calculated, are chosen along the x,y axis and include points on the edge of the hole. These are also shown in Figure 3.6.

The data, i.e., the coordinates of the nodal points, X(I) and Y(I), the resultant of the traction on each subdivision (calculated by the trapezoidal rule), BVX(I) and BVY(I), and the coordinates of the field points, XF(I) and YF(I), are read into the program (Appendix B). The results are presented in Table 3.1.

The results are compared to the theoretical solution of a long strip weakened by a circular hole subjected to uniaxial tension (Howland [34] and Savin [35]). The program required 35 seconds of CPU time on a CDC 6500 computer.

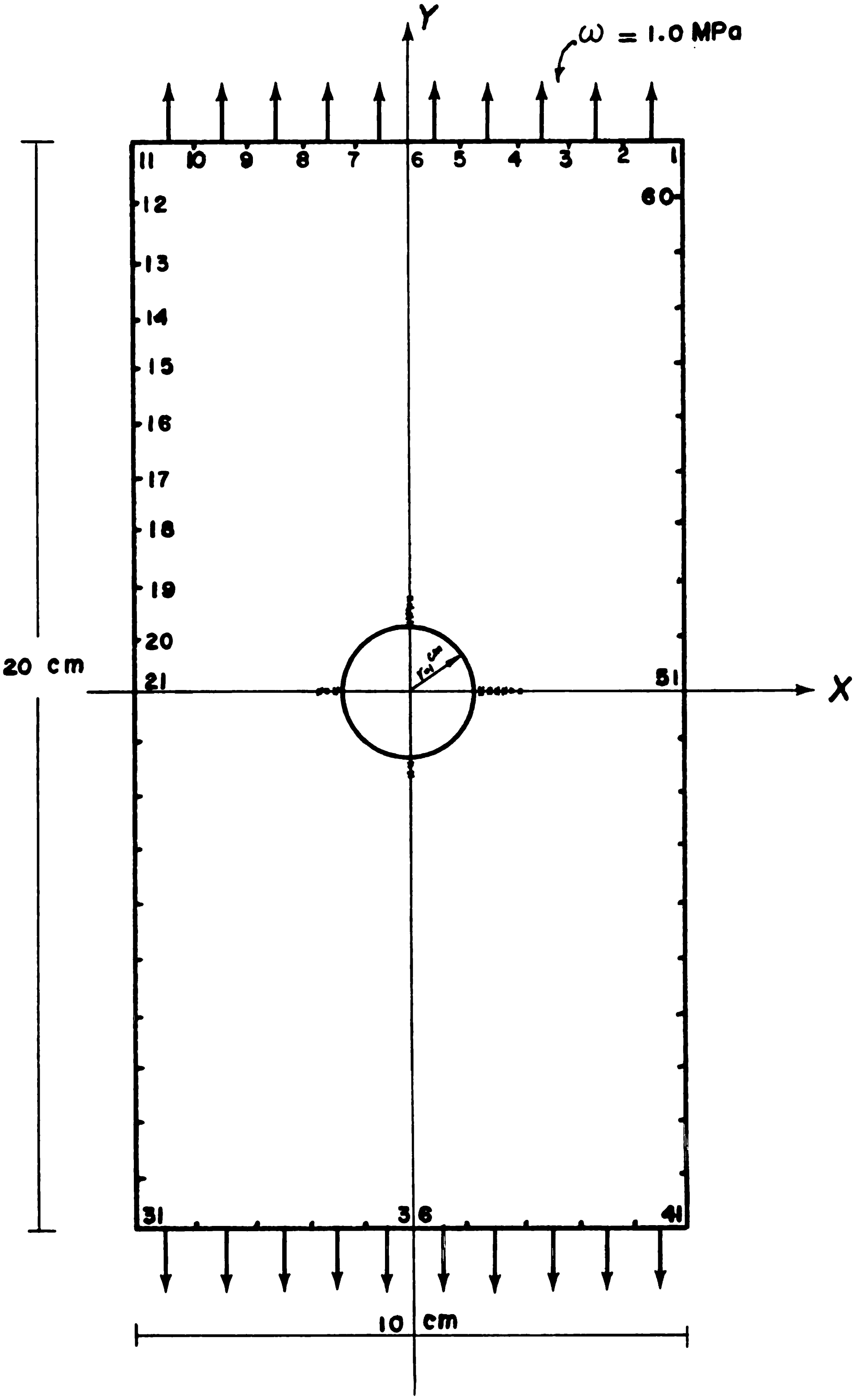


Figure 3.5 Circular hole symmetrically placed in a finite rectangular plate under uniaxial tension.

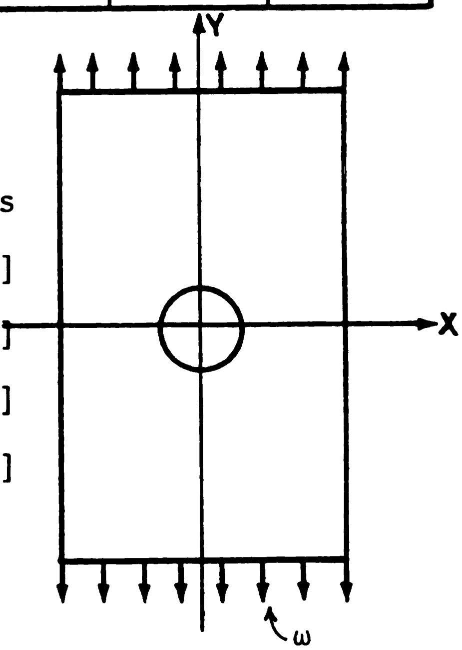
Table 3.1. Stresses and displacements in a rectangular region containing a circular hole at the origin, Case 1

Geometry: rectangular plane (10 x 20cm^2) (1 cm thickness)
Load: ω = 1.0 MPa
Eccentricity: X_0 = 0.0 Y_0 = 0.0
E = 70000 MPa, μ = 26315.79 MPa, ν = 0.33

Field Point No.	Coordinates		σ_{xx} (MPa)	σ_{yy} (MPa)	σ_{xy} (MPa)	U_x microns	U_y microns
	X cm	Y cm					
1	1.0	0.0	0.0	3.13128	0.0	-0.0014	0.0
2	1.2	0.0	0.32793	2.1486	0.0	-0.00152	0.0
3	1.4	0.0	0.38043	1.70146	0.0	-0.00154	0.0
4	1.8	0.0	0.31226	1.33672	0.0	-0.00155	0.0
5	0.0	1.0	-1.1198	0.0	0.0	0.0	0.0028
6	0.0	1.2	-0.44908	-0.0235	0.0	0.0	0.00291
7	0.0	1.4	-0.18639	0.10502	0.0	0.0	0.00298
8	0.0	1.8	-0.02031	0.36858	0.0	0.0	0.00314
9	-1.0	0.0	0.0	3.13128	0.0	0.0014	0.0
10	0.0	-1.0	-1.1198	0.0	0.0	0.0	-0.00275

Available Solution:

Field Point No.		References
1	σ_{yy}=3.14 MPa	[34],[35]
5	σ_{xx}= -1.11 MPa	[34],[35]
9	σ_{yy}=3.14 MPa	[34],[35]
10	σ_{xx}= -1.11 MPa	[34],[35]



To see the effect of the size of the plane on the stress and displacement solutions, smaller rectangular planes, 8cm x 16cm and 6cm x 12cm, weakened by the circular hole of radius r = 1cm at the origin were considered. The results are presented in Tables 3.2 and 3.4.

The program has been written in such a way that, if different dimensions of the rectangular plane are needed, only one character, WR, is to be changed. Note that the proportionality of the long side to the small side remains constant and equal to 2.0. Also, for different locations of the hole, the new coordinates of the center of the hole XO,YO must be read into the program. Finally, the example of the problem of a rectangular plane (9cm x 19cm) weakened by an unsymmetrically located circular hole is solved and the results are presented in Table 3.4. Again, the CPU time was 35 seconds for each run on a CDC 6500 computer.

EXAMPLE III.2

A Circular Plane Weakened by a Circular Hole

Let a circular plane of radius R = 6cm and unit thickness (h = 1cm), which is weakened by a circular hole of radius r = 1cm at the origin, be considered, see Figure 3.6. A radially uniform distributed load (ω = 1.0 MPa) is partially applied to the top and the bottom of the outer circumference, as shown. The boundary has been subdivided into sixty equally spaced meshes each of which covers 6 degrees of angle (0.6283cm) numbered from the top and

Table 3.2 Rectangular region containing a circular hole at the origin, Case 2

Geometry: rectangular plane ($8 \times 16 cm^2$) (1 cm thickness)
Load: $\omega = 1.0$ MPa
Eccentricity: $X_0 = 0$ $Y_0 = 0$
$E = 70000$ MPa, $\mu = 26315.79$ MPa, $\nu = 0.33$

Field Point No.	Coordinates X cm	Y cm	σ_{xx} (MPa)	σ_{yy} (MPa)	σ_{xy} (MPa)	U_x microns	U_y microns
1	1.0	0.0	0.0	3.2212	0.0	-0.218	0.0
2	1.2	0.0	0.3339	2.2013	0.0	-0.229	0.0
3	1.4	0.0	0.382	1.738	0.0	-0.232	0.0
4	1.8	0.0	0.3040	1.3619	0.0	-0.234	0.0
5	0.0	1.0	-1.1821	0.0	0.0	0.0	0.404
6	0.0	1.2	-0.4851	-0.0294	0.0	0.0	0.418
7	0.0	1.4	-0.2090	0.0992	0.0	0.0	0.427
8	0.0	1.8	-0.0293	0.3662	0.0	0.0	0.450
9	-1.0	0.0	0.0	3.2212	0.0	0.218	0.0
10	0.0	-1.0	-1.1821	0.0	0.0	0.0	-0.392

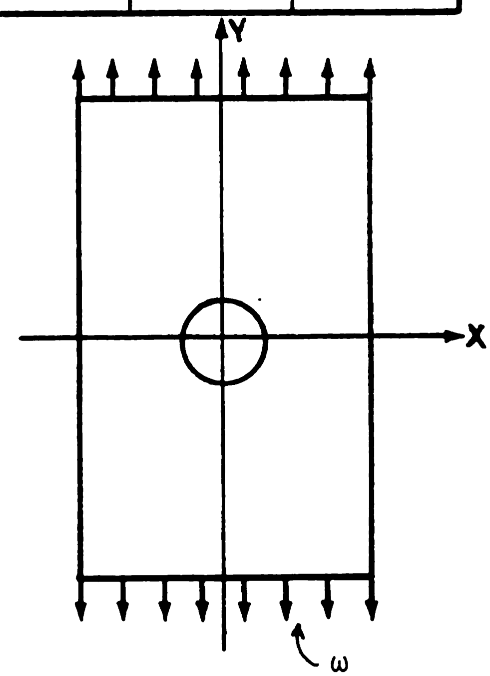


Table 3.3 Rectangular region containing a circular hole at the origin, Case 3

Geometry: rectangular plane (6 x 12cm^2) (1 cm thickness)
Load: ω = 1.0 MPa
Eccentricity: X_0 = 0.0 Y_0 = 0.0
E = 70000 MPa, μ = 26315.79 MPa, ν = 0.33

Field Point No.	Coordinates		σ_{xx} (MPa)	σ_{yy} (MPa)	σ_{xy} (MPa)	U_x microns	U_y microns
	X cm	Y cm					
1	1.0	0.0	0.0	3.4341	0.0	-0.244	0.0
2	1.2	0.0	0.3456	2.3206	0.0	-0.257	0.0
3	1.4	0.0	0.3827	1.8198	0.0	-0.260	0.0
4	1.8	0.0	0.2724	1.4061	0.0	0.265	0.0
5	0.0	1.0	-1.306	0.0	0.0	0.0	0.406
6	0.0	1.2	-0.555	-0.0425	0.0	0.0	0.429
7	0.0	1.4	-0.2486	0.0852	0.0	0.0	0.439
8	0.0	1.8	-0.0361	0.3580	0.0	0.0	0.460
9	-1.0	0.0	0.0	3.4341	0.0	0.244	0.0
10	0.0	-1.0	-1.306	0.0	0.0	0.0	-0.390

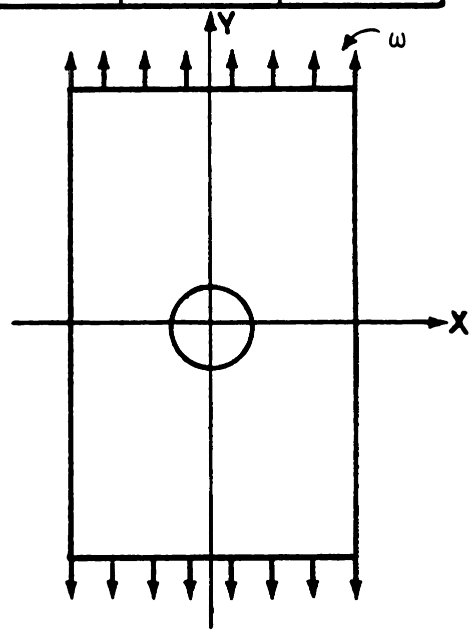
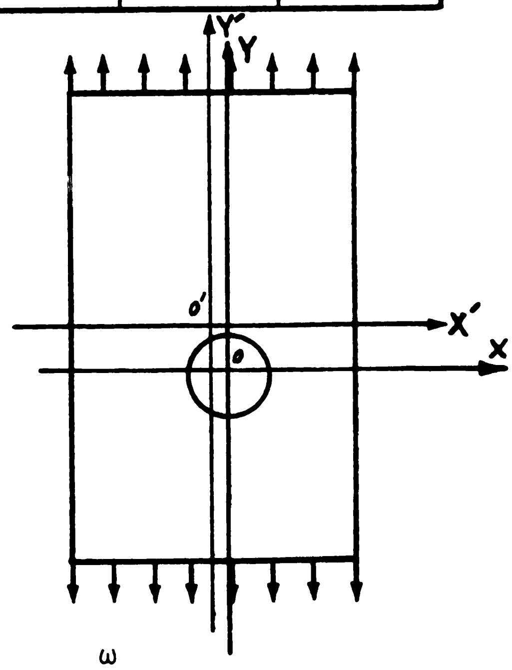


Table 3.4 Stresses and displacements in a rectangular region containing a nonsymmetrically located circular hole

Geometry: rectangular plane (9 x 18cm^2) (1 cm thickness)
Load: ω = 1.0 MPa
Eccentricity: X_0 = -0.5 cm Y_0 = 1.5 cm
E = 70000 MPa, μ = 26315.79 MPa, ν = 0.33

Field Point No.	Coordinates X cm	Y cm	σ_{xx} (MPa)	σ_{yy} (MPa)	σ_{xy} (MPa)	U_x microns	U_y microns
1	1.0	0.0	0.0	3.16445	0.0	-0.3267	0.6244
2	1.2	0.0	0.3313	2.1690	-0.0010	-0.3376	0.6248
3	1.4	0.0	0.3841	1.71526	-0.0013	-0.340	0.6251
4	1.8	0.0	0.31466	1.3436	-0.0015	-0.3415	0.6253
5	0.0	1.0	-1.1624	0.0	0.0	-0.1193	1.033
6	0.0	1.2	-0.4736	-0.0269	-0.0038	-0.1180	1.044
7	0.0	1.4	-0.2027	0.10251	-0.0047	-0.1184	1.053
8	0.0	1.8	-0.0294	0.3697	-0.0046	-0.1192	1.0763
9	-1.0	0.0	0.0	3.19990	0.0	0.0989	0.626
10	0.0	-1.0	-1.15149	0.0	0.0	-0.11812	0.2181



counterclockwise. The field points are chosen along the x,y axis and include points on the edge of the hole. These are also shown in Figure 3.6.

The data, i.e., the coordinates of the nodal points, X(I) and Y(I), the resultant of the traction on each subdivision (calculated by the trapezoidal rule), BVX(I) and BVY(I), and the coordinates of the field points, XF(I) and YF(I), are read into the program (Appendix B). The results are presented in Table 3.5. The program required 36 seconds of CPU time on a CDC 6500 computer.

The effect of the radius on the stress and displacement solution has also been considered by solving the problem for R = 4.8cm and 3.6cm and r = 1cm. The results are presented in Tables 3.6 and 3.7.

To obtain the solution for different radii of the plane, one has to change the character WR which is the ratio of the desired radius to the R = 6cm. Also, for a different location of the hole, the new coordinates of the center of the hole, XO,YO must be read into the program. To see the effect of eccentric placement of the circular hole on the stress and displacement field, the example of a circular plane (R = 5.4cm) weakened by an unsymmetrically located circular hole of radius r = 1cm is solved and the results are presented in Table 3.8. Again, the CPU time was 36 seconds for each run on a CDC 6500 computer.

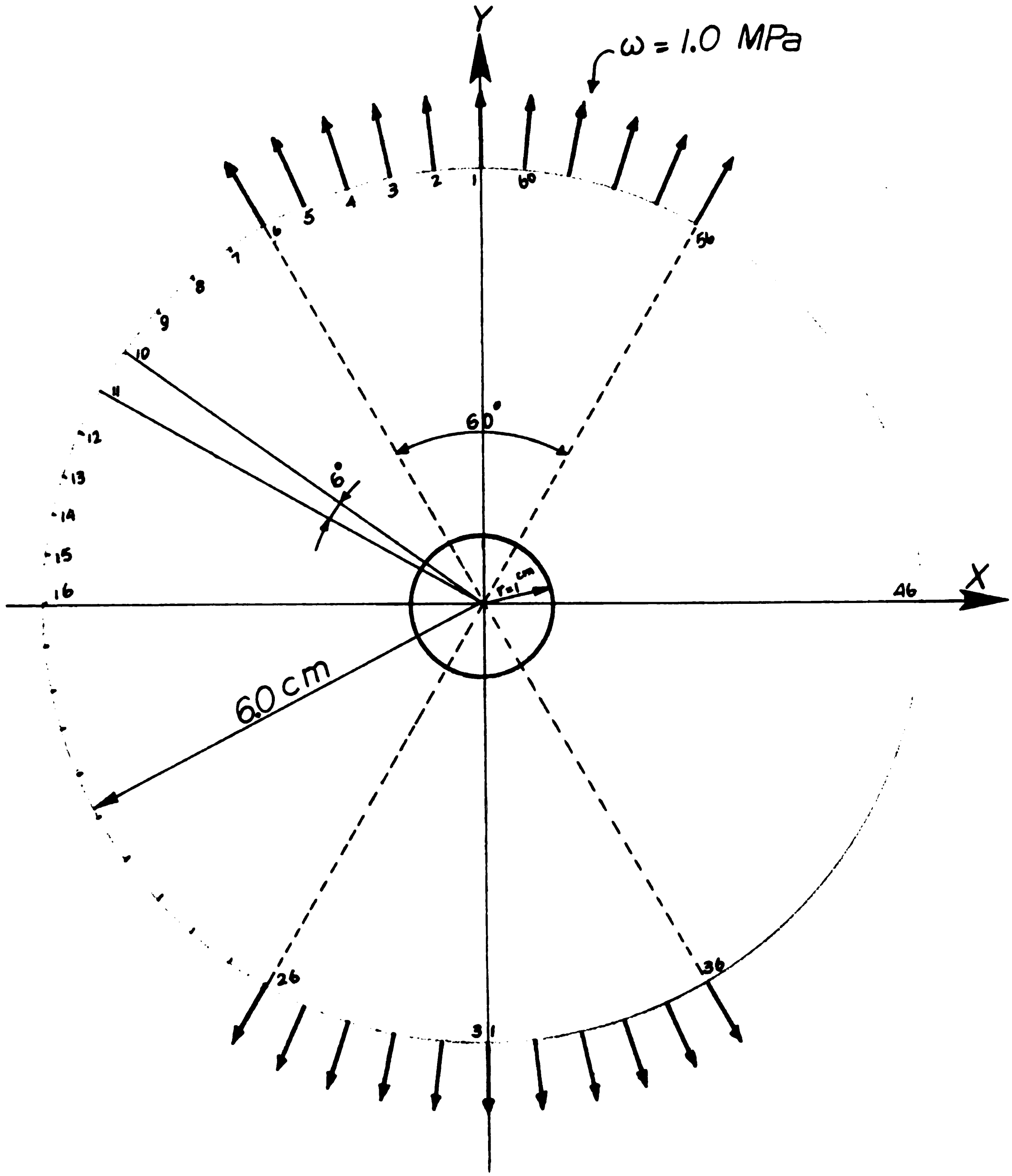


Figure 3.6 Circular plane, containing a circular hole, subjected to radially uniform tension over a portion of the boundary.

Table 3.5 Stress and displacement in a circular plane containing a circular hole at the origin, Case 1

Geometry: circular plane R = 6 cm (1 cm thickness)
Load: ω = 1.0 MPa
Eccentricity: X_0 = 0.0 Y_0 = 0.0
E = 70000 MPa, μ = 26315.79 MPa, ν = 0.33

Field Point No.	Coordinates X cm	Y cm	σ_{xx} (MPa)	σ_{yy} (MPa)	σ_{xy} (MPa)	U_x microns	U_y microns
1	1.0	0.0	0.0	2.948	0.0	-0.272	0.0
2	1.2	0.0	0.2975	1.9438	0.0	-0.284	0.0
3	1.4	0.0	0.32771	1.4742	0.0	-0.287	0.0
4	1.8	0.0	0.2345	1.06002	0.0	-0.290	0.0
5	0.0	1.0	-1.70816	0.0	0.0	0.0	0.323
6	0.0	1.2	-0.8751	-0.0846	0.0	0.0	0.380
7	0.0	1.4	-0.5090	0.0332	0.0	0.0	0.388
8	0.0	1.8	-0.2547	0.31029	0.0	0.0	0.409
9	-1.0	0.0	0.0	2.948	0.0	0.267	0.0
10	0.0	-1.0	-1.7081	0.0	0.0	0.0	-0.318

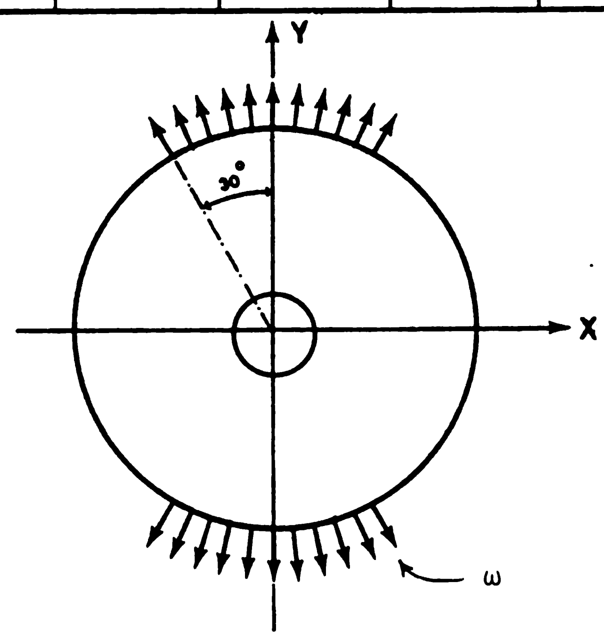


Table 3.6 Circular plane containing a circular hole at the origin, Case 2

Geometry: circular plane R = 4.8 cm (1 cm thickness)
Load: ω = 1.0 MPa
Eccentricity: X_0 = 0.0 Y_0 = 0.0
E = 70000 MPa, μ = 26315.79 MPa, ν = 0.33

Field Point No.	Coordinates X cm	Coordinates Y cm	σ_{xx} (MPa)	σ_{yy} (MPa)	σ_{xy} (MPa)	U_x microns	U_y microns
1	1.0	0.0	0.0	3.0268	0.0	-0.288	0.0
2	1.2	0.0	0.3077	1.9808	0.0	-0.3011	0.0
3	1.4	0.0	0.3414	1.4806	0.0	-0.304	0.0
4	1.8	0.0	0.2502	1.0181	0.0	-0.306	0.0
5	0.0	1.0	-1.8281	0.0	0.0	0.0	0.347
6	0.0	1.2	-0.9085	-0.0886	0.0	0.0	0.362
7	0.0	1.4	-0.528	0.0397	0.0	0.0	0.412
8	0.0	1.8	-0.232	0.339	0.0	0.0	0.434
9	-1.0	0.0	0.0	3.0268	0.0	0.282	0.0
10	0.0	-1.0	-1.8281	0.0	0.0	0.0	-0.340

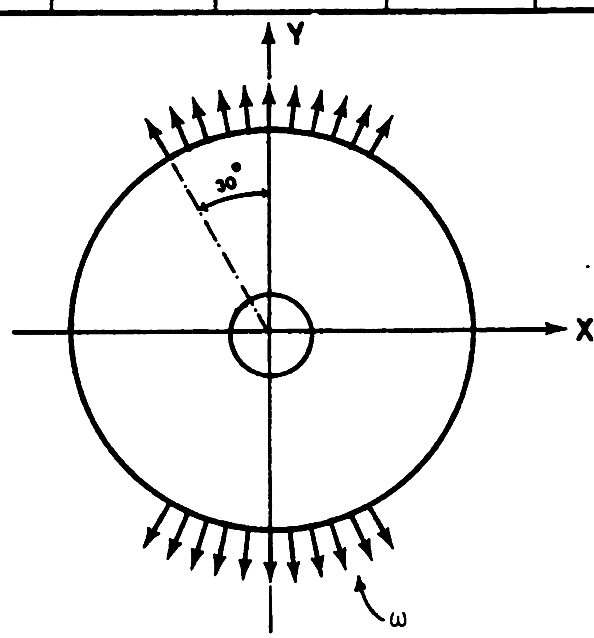


Table 3.7 Circular plane containing a circular hole at the origin, Case 3

Geometry: circular plane R = 3.6 cm (1 cm thickness) Load: ω = 1.0 MPa Eccentricity: X_0 = 0.0 Y_0 = 0.0 E = 70000 MPa, μ = 26315.79 MPa, ν = 0.33							
Field Point No.	Coordinates		σ_{xx} (MPa)	σ_{yy} (MPa)	σ_{xy} (MPa)	U_x microns	U_y microns
	X cm	Y cm					
1	1.0	0.0	0.0	3.2127	0.0	-0.327	0.0
2	1.2	0.0	0.3298	2.0625	0.0	-0.341	0.0
3	1.4	0.0	0.3686	1.4878	0.0	-0.344	0.0
4	1.6	0.0	0.2748	0.9105	0.0	-0.344	0.0
5	0.0	1.0	-2.093	0.0	0.0	0.046	0.352
6	0.0	1.2	-1.0129	-0.0980	0.0	0.0	0.407
7	0.0	1.4	-0.550	0.0526	0.0	0.016	0.417
8	0.0	1.8	-0.136	0.3982	0.0	0.0	0.485
9	-1.0	0.0	0.0	3.2127	0.0	0.319	0.0
10	0.0	-1.0	-2.0937	0.0	0.0	0.042	-0.343

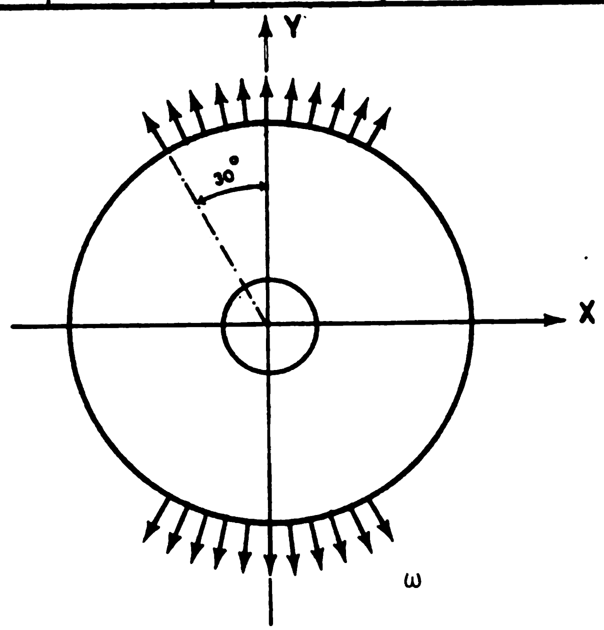
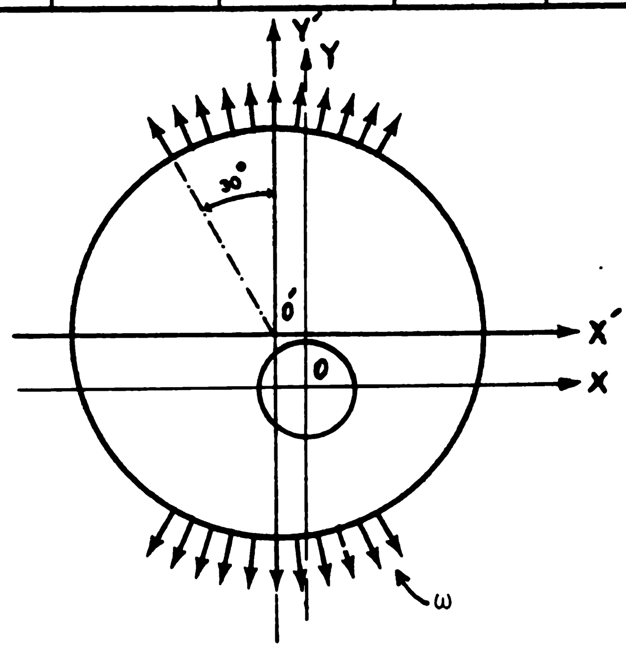


Table 3.8 Stress and displacement of a circular plane containing a nonsymmetrically located circular hole

Geometry: circular plane R = 5.4 cm (1 cm thickness)
Load: ω = 1.0 MPa
Eccentricity: X_0 = -1.0 Y_0 = 2.0
E = 70000 MPa, μ = 26315.79 MPa, ν = 0.33

Field Point No.	Coordinates X cm	Coordinates Y cm	σ_{xx} (MPa)	σ_{yy} (MPa)	σ_{xy} (MPa)	U_x microns	U_y microns
1	1.0	0.0	0.0	3.239	0.0	0.0	0.919
2	1.2	0.0	0.325	2.161	-0.019	-0.0116	0.9163
3	1.4	0.0	0.358	1.667	-0.0346	-0.0166	0.9118
4	1.6	0.0	0.259	1.239	-0.0629	-0.0232	0.898
5	0.0	1.0	-1.532	0.0	0.0	0.358	1.226
6	0.0	1.2	-0.6160	-0.0417	0.05507	0.344	1.239
7	0.0	1.4	-0.2224	0.11601	0.0629	0.332	1.247
8	0.0	1.8	0.1352	0.4443	0.04002	0.2523	1.309
9	-1.0	0.0	0.0	2.779	0.0	0.509	0.782
10	0.0	-1.0	-1.5716	0.0	0.0	0.1903	0.5704



CHAPTER IV

ELLIPTICAL HOLE OR SHARP CRACK IN A FINITE TWO-DIMENSIONAL REGION

IV.1 INTRODUCTION

Problems associated with stress concentration around holes in structures have motivated the effort to solve problems of plane elastic regions weakened by elliptical holes or sharp cracks. The solution for the stress near an elliptical hole in an infinite plane subjected to a uniform load was first obtained by Inglis [39] using complex potentials. Later this problem was examined experimentally by Durelli and Murray [40]. A method for the determination of stresses and displacements near the tip of a sharp crack in an infinite plane subjected to in plane load was developed in an infinite series form by Westergaard [41]. The effect of holes of more general shape on infinite planes has received considerable attention, most notably by Muskhelishvili [27]. The problem of an elliptical hole or a sharp crack in a long strip subjected to uniform tension and compression has been treated experimentally and numerically using several methods and techniques. Yet, no solution for an arbitrary plane region weakened by an ellipse or crack is available.

In this chapter the solution for the problem of an arbitrary, finite, two dimensional elastic region, weakened by an arbitrarily located and oriented ellipse or sharp crack, is presented. This is the extension of the implementation of the mapping technique and boundary integral equation method. In section 2 of this chapter, the Muskhelishvili method is used and the mapping technique is employed to determine the influence functions for an elliptical hole. In section 3 this influence function is extended to a sharp crack. Finally in the last section, some example problems are solved for an elliptical hole and a sharp crack at different orientations. These solutions are compared to some available experimental [42,43] and analytic [44] results. The computer programs are included in Appendices C and D.

IV.2 DERIVATION OF THE INFLUENCE FUNCTION USING THE MAPPING TECHNIQUE: THE ELLIPTICAL HOLE PROBLEM

In this section, the influence function for an infinite plane region containing an elliptical hole is derived. Consider an infinite plane containing an elliptical hole at the origin and a concentrated point force P acting in the plane at some point Z_0, where Z_0 lies on or outside of the ellipse, i.e.,

$$\frac{X_0}{a^2} + \frac{Y_0}{b^2} \geqslant 1 \tag{4.1}$$

where $Z_0 = X_0 + iY_0$ and a,b are the semi-major and semi-minor axes of the elliptic hole. The problem can be expressed as the superposition of two problems, see Figure 4.1. The problem of Figure 4.1(B) is simply that of a concentrated point force P applied at Z_0 in an infinite plane and the problem of Figure 4.1(C) is that of prescribed traction acting on an elliptic hole in an infinite region.

This applied traction on the elliptic hole is equal in magnitude and opposite in direction to the traction generated on an elliptic contour in the problem of Figure 4.1(B), by the concentrated point force P.

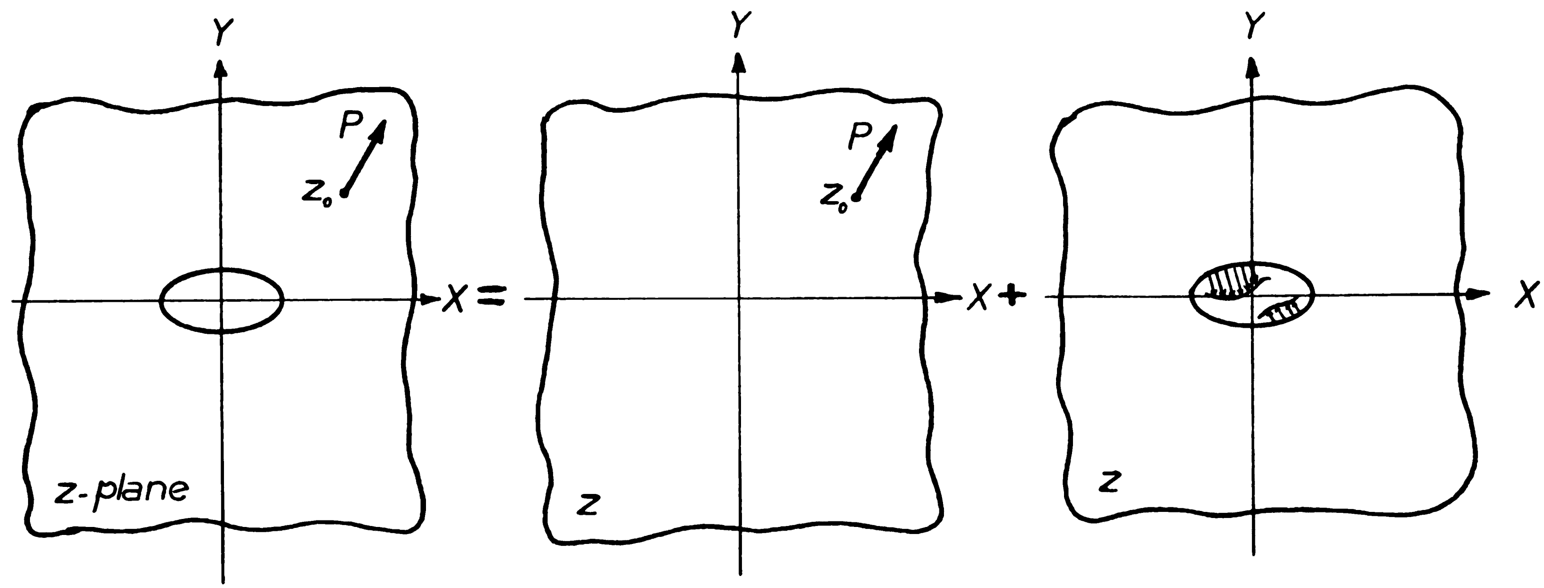
Adding the solutions of Figures 4.1(B) and 4.1(C), the zero traction condition on the hole of the problem of Figure 4.1(A) is obtained. The solution to the problem of Figure 4.1(B) is known (Muskhelishvili [27]). Thus, the required traction can be found.

To solve the problem of Figure 4.1(C), it is necessary to map this problem into a unit circle (disc), see Figure 4.2. It is easy to verify that the mapping function

$$Z = R\left(\frac{1}{\zeta} + m\zeta\right)$$

$$\text{for } R > 0 \text{ and } 0 \leqslant m \leqslant 1 \qquad (4.2)$$

transforms the region exterior to the ellipse into a unit circle $|\zeta| \leqslant 1$ (Churchill [37]), provided R and M are taken as:



(A) (B) (C)

Figure 4.1 Superposition of the problem of elliptical hole.

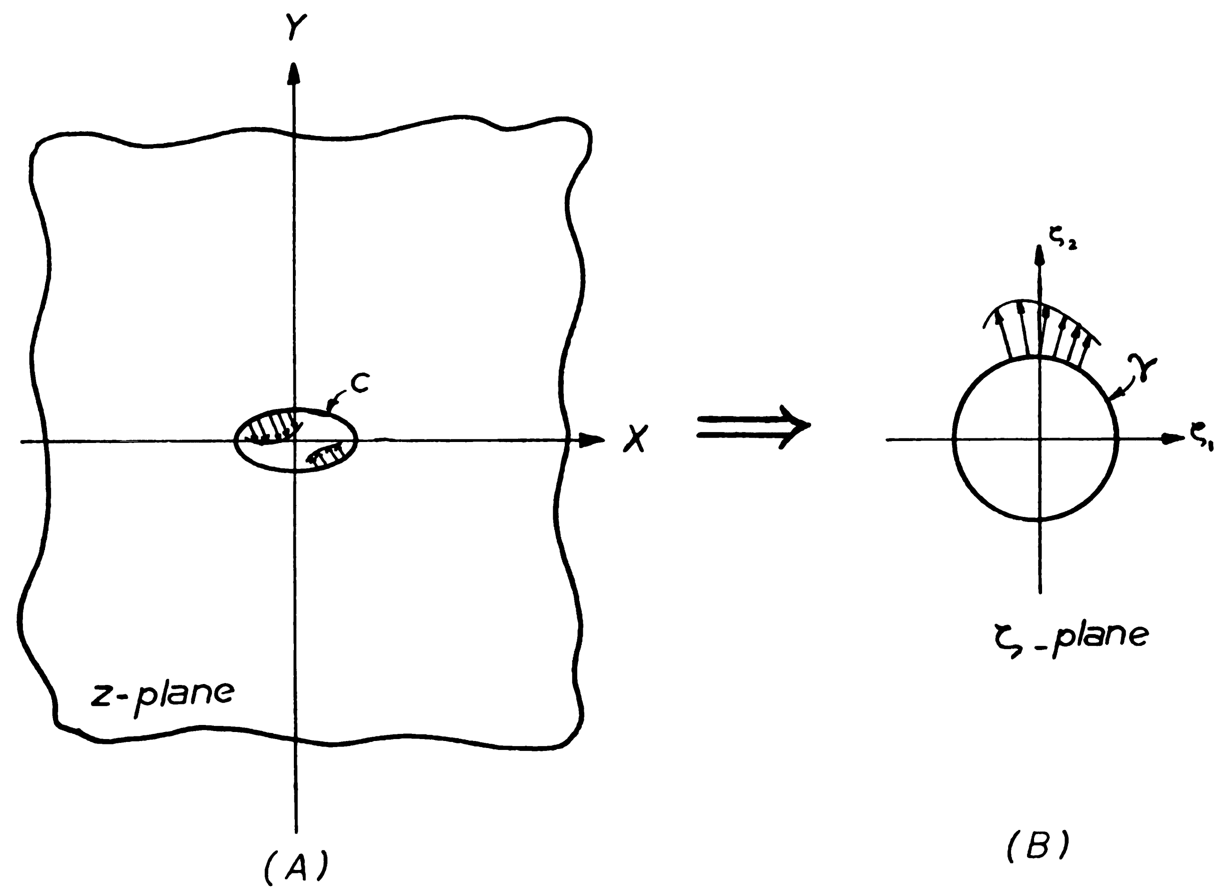


Figure 4.2 Mapping the auxiliary problem to a unit disc.

$$R = \frac{a+b}{2} \qquad \text{and} \qquad m = \frac{a-b}{a+b} \tag{4.3}$$

where a and b are the semi-major and semi-minor axes of the ellipse, respectively, and are equal to:

$$a = R(1+m) \qquad , \qquad b = R(1-m)$$

The mapping function is conformal for, if $\omega'(\zeta)$ is considered, i.e.,

$$\omega'(\zeta) = -\frac{1}{\zeta^2} + m \qquad o \leq m \leq 1$$

It is obvious that $\omega'(\zeta)$ has two roots, $\zeta = \sqrt{1/m}$, outside γ, the unit circle. Thus, $\omega'(\zeta)$ is not equal to zero inside γ, the unit circle, and following the conformal mapping theorems [37] it can be concluded that the mapping function of equation (4.2) is conformal.

It is important to note that, as the point $\zeta = e^{i\theta}$ describes the circle $|\zeta| = 1$ in the positive, counter-clockwise direction, the corresponding point traces out the ellipse in the clockwise direction. Clearly, the parametric equations of the ellipse must be taken in the form:

$$X = R(1+m) \operatorname{Cos} \theta$$

$$Y = R(1-m) \operatorname{Sin} \theta \tag{4.4}$$

If m = o, the ellipse becomes a circle and the transformation function equation (4.2) becomes $\omega(\zeta) = R/\zeta$. However, it will be seen that several expressions derived in this chapter will be singular when m = o and therefore the analysis is invalid for the case of the circular hole. Since the case of the circular hole has already been treated, the following restrictions will be placed on m:

$$o < m \leq 1$$

When m = 1, the point in the Z-plane traces out the segment of the x-axis between X = +2R and X = -2R twice as the point ζ describes the boundary of the unit circle, $|\zeta| = 1$. Thus, in this case the mapping function of equation (4.2) maps a sharp crack along the line joining the points (2R,o) and (-2R,o) to γ, the circumference of the unit circle, and thus maps the Z-plane, excluding the crack, onto the unit circle $|\zeta|<1$.

Without loss of generality, let R = 1. Then the mapping function and its derivatives are:

$$\begin{aligned} Z = \omega(\zeta) &= \frac{1}{\zeta} + m\zeta && \text{(a)} \\ \omega'(\zeta) &= -\frac{1}{\zeta^2} + m && \text{(b)} \\ \omega''(\zeta) &= \frac{2}{\zeta^3} && \text{(c)} \end{aligned} \qquad (4.5)$$

Let $\phi^0(Z)$ and $\Psi^0(Z)$ be the complex potential functions for the problem of Figure 4.1(B) and $\phi^*(Z)$ and $\Psi^*(Z)$ be the complex potential functions for the problem of Figure 4.1(C). Then the potential functions for the problem of Figure 4.1(A) are

$$\phi(Z) = \phi^0(Z) + \phi^*(Z)$$

$$\Psi(Z) = \Psi^0(Z) + \Psi^*(Z)$$

where the derivatives are given by equations (2.2) to (2.5) in section II.3. To find $\phi^*(Z)$ and $\Psi^*(Z)$, the transformed complex potential functions are

$$\phi_1^*(\zeta) = \frac{1}{2\pi i}\oint_\gamma \frac{F(\sigma)}{\sigma-\zeta}\,d\sigma - \frac{1}{2\pi i}\oint_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\frac{\overline{\phi_1^{*\prime}(\sigma)}}{\sigma-\zeta}\,d\sigma \qquad (4.6)$$

$$\Psi_1^*(\zeta) = \frac{1}{2\pi i}\oint_\gamma \frac{\overline{F(\sigma)}}{\sigma-\zeta}\,d\sigma - \frac{1}{2\pi i}\oint_\gamma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\,\frac{\phi_1^{*\prime}(\sigma)}{\sigma-\zeta}\,d\sigma \qquad (4.7)$$

It is first necessary to calculate the following integrals:

$$I_1 = \frac{1}{2\pi i}\oint_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\frac{\overline{\phi_1^{*\prime}(\sigma)}}{\sigma-\zeta}\,d\sigma \qquad (4.8)$$

$$I_2 = \frac{1}{2\pi i}\oint_\gamma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\,\frac{\phi_1^{*\prime}(\sigma)}{\sigma-\zeta}\,d\sigma \qquad (4.9)$$

To construct the arguments of the integrals of (4.8) and (4.9), it is necessary to substitute σ into equation (4.5) and note that $\sigma\bar{\sigma} = 1$. The mapping function and its complex conjugates become:

$$\omega(\sigma) = \frac{1}{\sigma} + m\sigma$$

$$\overline{\omega(\sigma)} = \sigma + \frac{m}{\sigma}$$

$$\omega'(\sigma) = \frac{1}{\sigma^2} + m$$

$$\overline{\omega'(\sigma)} = \sigma^2 + m \qquad (4.10)$$

and, since $\phi^*(\sigma)$ and $\phi^{*\prime}(\sigma)$ are analytic inside γ and $\overline{\phi^{*\prime}(\sigma)}$ is analytic outside γ, equations (3.10) and (3.11) lead to:

$$\phi_1^{*\prime}(\sigma) = \sum_{k=1}^{\infty} k a_k \sigma^k \qquad \text{and} \qquad \overline{\phi_1^{*\prime}(\sigma)} = \sum_{k=1}^{\infty} k \bar{a}_k \sigma^{-k+1}$$

Thus, the arguments of the integrals can be constructed as follows:

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \cdot \overline{\phi_1^{*\prime}(\sigma)} = \frac{1+m\sigma^2}{\sigma(m-\sigma^2)} \cdot \sum_{k=1}^{\infty} k\bar{a}_k \sigma^{-k+1} \tag{4.11}$$

$$\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} \phi_1^{*\prime}(\sigma) = -\frac{\sigma(\sigma^2+m)}{1-m\sigma^2} \cdot \sum_{k=1}^{\infty} k a_k \sigma^k \tag{4.12}$$

Substituting equation (4.11) into the integral (4.8) leads to:

$$I_1 = \frac{1}{2\pi i} \oint_\gamma \frac{[(1+m\sigma^2)/(m-\sigma^2)] \displaystyle\sum_{k=1}^{\infty} k\bar{a}_k \sigma^{-k}}{\sigma-\zeta} \, d\sigma$$

It is clear that the numerator of the argument is an analytic function outside γ, the unit circle. Hence, following the Cauchy integral formula, presented in section I.3, the principal value of the integral of equation (4.8) becomes:

$$I_1 = \frac{1}{2\pi i} \oint_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \frac{\overline{\phi_1^{*\prime}(\sigma)}}{\sigma-\zeta} \, d\sigma = 0 \tag{4.13}$$

Also, substituting equation (4.12) into the integral (4.9) leads to:

$$I_2 = \frac{1}{2\pi i} \oint_\gamma - \frac{[(\sigma^2+m)/(1-m\sigma^2)] \cdot \sum_{k=1}^{\infty} k a_k \sigma^{k+1}}{\sigma-\zeta} d\sigma$$

Obviously the numerator of the argument is an analytic function inside γ, the unit circle. Thus, due to the Cauchy integral formulae,the principal value of the integral becomes:

$$I_2 = \frac{1}{2\pi i} \oint_\gamma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} \cdot \frac{\phi^{*\prime}(\sigma)}{\sigma-\zeta} d\sigma = - \frac{\zeta(\zeta^2+m)}{1-m\zeta^2} \cdot \phi_1^{*\prime}(\zeta) \qquad (4.14)$$

Substituting the expressions (4.13) and (4.14) into equations (4.6) and (4.7) leads to:

$$\phi_1^*(\zeta) = \frac{1}{2\pi i} \oint_\gamma \frac{F(\sigma)}{\sigma-\zeta} d\sigma \qquad (4.15)$$

$$\Psi_1^*(\zeta) = \frac{1}{2\pi i} \oint_\gamma \frac{\overline{F(\sigma)}}{\sigma-\zeta} d\sigma + \frac{\zeta(\zeta^2+m)}{1-m\zeta^2} \phi_1^{*\prime}(\zeta) \qquad (4.16)$$

Recall equations (3.20) and (3.21):

$$F(\sigma) = -[\phi_1^0(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \overline{\phi_1^{0\prime}(\sigma)} + \overline{\Psi_1^0(\sigma)}] \qquad (4.17)$$

$$\overline{F(\sigma)} = -[\overline{\phi_1^0(\sigma)} + \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} \phi_1^{0\prime}(\sigma) + \Psi_1^0(\sigma)] \qquad (4.18)$$

where $\phi^0(Z)$ and $\Psi^0(Z)$ are given by equations (2.6). To find the transformed complex potential functions $\phi_1^0(\zeta)$ and $\Psi_1^0(\zeta)$, substitute the mapping function, equation (4.5.a), into equation (2.6):

$$\phi_1^0(\zeta) = -Q \ln \frac{m\zeta^2 - Z_0\zeta + 1}{\zeta} \tag{4.19}$$

$$\Psi_1^0(\zeta) = \alpha \bar{Q} \ln \frac{m\zeta^2 - Z_0\zeta + 1}{\zeta} + Q \cdot \frac{\zeta \bar{Z}_0}{m\zeta^2 - Z_0\zeta + 1} \tag{4.20}$$

where $Q = \frac{P}{2\pi(\alpha+1)}$ and $\bar{Q}$ is the complex conjugate of Q. Taking the derivative of equation (4.19) and evaluating the potential functions, equations (4.19) and (4.20) at $\zeta = \sigma$, leads to:

$$\phi_1^0(\sigma) = -Q \ln \frac{m\sigma^2 - Z_0\sigma + 1}{\sigma}$$

$$\phi_1^{0\prime}(\sigma) = -Q \cdot \frac{m\sigma^2 - 1}{\sigma(m\sigma^2 - Z_0\sigma + 1)}$$

$$\Psi_1^0(\sigma) = \bar{Q} \cdot \alpha \ln \frac{m\sigma^2 - Z_0\sigma + 1}{\sigma} + Q \cdot \frac{\sigma \bar{Z}_0}{m\sigma^2 - Z_0\sigma + 1} \tag{4.21}$$

Following equations (4.10), it is clear that:

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} = \frac{1 + m\sigma^2}{\sigma(m - \sigma^2)} \tag{4.22}$$

$$\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} = -\frac{\sigma(\sigma^2 + m)}{1 - m\sigma^2} \tag{4.23}$$

Taking the complex conjugate of equations (4.21) along with the equations (4.23) will provide all the terms needed to calculate $F(\sigma)$ and $\overline{F(\sigma)}$. Thus, equations (4.17) and (4.18) become:

$$F(\sigma) = Q\left\{\ln\frac{m\sigma^2 - Z_0\sigma + 1}{\sigma} - \alpha \ln\frac{\sigma^2 - \overline{Z}_0\sigma + m}{\sigma}\right\}$$

$$+ \overline{Q}\left\{\frac{m\sigma^2 - Z_0\sigma + 1}{\sigma^2 - \overline{Z}_0\sigma + m}\right\} \tag{4.24}$$

$$\overline{F(\sigma)} = Q\left\{\frac{\sigma^2 - \overline{Z}_0\sigma + m}{m\sigma^2 - Z_0\sigma + 1}\right\} + \overline{Q}\left\{\ln\frac{\sigma^2 - \overline{Z}_0\sigma + m}{\sigma}\right.$$

$$\left. - \alpha \ln\frac{m\sigma^2 - Z_0\sigma + 1}{\sigma}\right\} \tag{4.25}$$

Before any further calculation, it is necessary to examine the terms in equations (4.24) and (4.25). There are two distinct quadratic terms in the equations:

$$A = m\sigma^2 - Z_0\sigma + 1 \tag{4.26}$$

$$B = \sigma^2 - \overline{Z}_0\sigma + m \tag{4.27}$$

Solving equation (4.5[a]), the mapping function, for ζ, yields:

$$m\zeta^2 - Z\zeta + 1 = 0 \tag{4.28}$$

As discussed earlier, the mapping function, equation (4.5[a]), represents a conformal mapping, i.e., for every point Z exterior to the ellipse, there exists only one corresponding point in the ζ plane interior to the circle. Since equation (4.28) is of the quadratic form and has two complex roots, then one root has to fall inside ζ and the other root has to fall outside γ, the unit circle. The two quardratics, equations (4.26) and (4.28), have the same coefficients. Thus, equation (4.26) has two roots, one inside and one outside γ. Denote the root inside γ by r_i and the root outside γ by r_o. Then equation (4.26) can be written as

$$A = m\sigma^2 - Z_0\sigma + 1 = m(\sigma - r_i)(\sigma - r_o) \qquad (4.29)$$

where

$$r_{i,o} = \frac{Z_0 \pm \sqrt{Z_0{}^2 - 4m}}{2m}$$

For examination of equation (4.27), consider the following mapping function:

$$Z = \omega(\zeta) = \zeta + \frac{m}{\zeta} \qquad (4.30)$$

This function maps points in the plane exterior to the ellipse onto points in the plane exterior to the unit circle. This mapping function is also conformal. For, if $\omega'(\zeta)$ is considered

$$\omega'(\zeta) = 1 - \frac{m}{\zeta^2} \qquad o<m\leqslant 1$$

it is clear that $\omega'(\zeta)$ has two roots, $\zeta = \pm \sqrt{m}$, inside γ, the unit circle. Thus, $\omega'(\zeta)$ is not equal to zero outside γ and, following the conformal mapping theorems [37], it is thus concluded that the mapping function, equation (4.30), is also conformal.

Solving equation (4.30) for ζ leads to:

$$\zeta^2 - Z\zeta + m = 0 \tag{4.31}$$

where the roots are

$$\zeta_{1,2} = Z/2 \pm \sqrt{Z^2/4-m} \tag{4.32}$$

Since the mapping function is conformal, then for every point in the Z plane exterior to the ellipse, there exists only one corresponding point in the ζ plane exterior to the circle. Hence, one of the roots of equation (4.32) has to be inside γ and the other root has to be outside γ. Note that, if the roots of a polynomial of degree N with the complex coefficients, $f(Z) = 0$, are $\alpha_i (i=1,...N)$, then roots of $\bar{f}(Z) = 0$ are $\overline{\alpha_i} (i=1,...N)$. Hence, the roots of the following equation

$$\zeta^2 - \bar{Z}\zeta + m = 0 \tag{4.33}$$

are the complex conjugate of the roots of equation (4.31), i.e., complex conjugate of equation (4.32), and since the equation (4.31) has one root inside and the other root outside γ, then equation (4.33) has one root inside and one root outside γ, the unit circle. Comparing equations (4.33) and (4.27) leads to the fact that equation (4.27) also has two roots, one inside and the other outside γ.

Denote the root inside γ by t_i and the root outside γ by t_o. Then equation (4.27) can be written as:

$$B = \sigma^2 - \overline{Z}_0\sigma + m = (\sigma - t_i)(\sigma - t_o) \tag{4.34}$$

where

$$t_{i,o} = \frac{\overline{Z}_0 \pm \sqrt{\overline{Z}_0^2 - 4m}}{2}$$

Substituting equations (4.29) and (4.34) into equations (4.24) and (4.25) leads to:

$$F(\sigma) = Q\left\{\ln\frac{m(\sigma-r_i)(\sigma-r_o)}{\sigma} - \alpha\ln\frac{(\sigma-t_i)(\sigma-t_o)}{\sigma}\right\} + \overline{Q}\left\{\frac{m(\sigma-r_i)(\sigma-r_o)}{(\sigma-t_i)(\sigma-t_o)}\right\} \tag{4.35}$$

$$\overline{F(\sigma)} = Q\left\{\frac{(\sigma-t_i)(\sigma-t_o)}{m(\sigma-r_i)(\sigma-r_o)}\right\} + \overline{Q}\left\{\ln\frac{(\sigma-t_i)(\sigma-t_o)}{\sigma} - \alpha\ln\frac{m(\sigma-r_i)(\sigma-r_o)}{\sigma}\right\} \tag{4.36}$$

Substitution of equations (4.35) and (4.36) into equations (4.16) and (4.17) leads to determination of the complex potential functions:

$$\phi_i^*(\zeta) = \frac{Q}{2\pi i}\oint_\gamma \ln\frac{m(\sigma-r_i)(\sigma-r_o)}{\sigma}\cdot\frac{d\sigma}{\sigma-\zeta} - \frac{\alpha Q}{2\pi i}\oint_\gamma \ln\frac{(\sigma-t_i)(\sigma-t_o)}{\sigma}\cdot\frac{d\sigma}{\sigma-\zeta} + \frac{\overline{Q}}{2\pi i}\oint_\gamma \frac{m(\sigma-r_i)(\sigma-r_o)}{(\sigma-t_i)(\sigma-t_o)(\sigma-\zeta)}\,d\sigma \tag{4.37}$$

$$\Psi_i^*(\zeta) = \frac{Q}{2\pi i} \oint_\gamma \frac{(\sigma-t_i)(\sigma-t_o)}{m(\sigma-r_i)(\sigma-r_o)(\sigma-\zeta)}\, d\sigma$$

$$+ \frac{\bar{Q}}{2\pi i} \oint_\gamma \ln \frac{(\sigma-t_i)(\sigma-t_o)}{\sigma} \cdot \frac{d\sigma}{\sigma-\zeta}$$

$$- \frac{\bar{Q}\alpha}{2\pi i} \oint_\gamma \ln \frac{m(\sigma-r_i)(\sigma-r_o)}{\sigma} \cdot \frac{d\sigma}{\sigma-\zeta}$$

$$+ \frac{\zeta(\zeta^2+m)}{1-m\zeta^2}\, \phi_i^{*\prime}(\zeta) \tag{4.38}$$

The following integral will now be evaluated:

$$I = \frac{1}{2\pi i} \oint_\gamma \ln \frac{m(\sigma-r_i)(\sigma-r_o)}{\sigma} \cdot \frac{d\sigma}{\sigma-\zeta} = \frac{1}{2\pi i} \quad {}_{\gamma} \quad \ln \frac{m(\sigma-r_i)}{\sigma} \cdot \frac{d\sigma}{\sigma-\zeta}$$

$$+ \frac{1}{2\pi i} \oint_\gamma \frac{\ln(\sigma-r_o)}{\sigma-\zeta}\, d\sigma$$

As discussed in section I.3, $\ln[\frac{m(\sigma-r_i)}{\sigma}]$ is an analytic function outside γ, the unit circle. Also, this function has two essential singular points (at $\sigma = 0$ and $\sigma = r_i$) inside γ and the value at infinity of:

$$\left[\ln(\frac{m(\sigma-r_i)}{\sigma})\right]_{\sigma=\infty} = \ln(m) \qquad o<m<1$$

Thus, due to the Cauchy integral formulas (section I.3):

$$\frac{1}{2\pi i} \oint_\gamma \ln \frac{m(\sigma-r_i)}{\sigma} \cdot \frac{d\sigma}{\sigma-\zeta} = \ln(m)$$

Also, $\ln(\sigma - r_o)$ is an analytic function inside γ, so that:

$$\frac{1}{2\pi i} \oint_\gamma \frac{\ln(\sigma - r_o)}{\sigma - \zeta}\, d\sigma = \ln(\zeta - r_o)$$

Thus,

$$I = \frac{1}{2\pi i} \oint_\gamma \ln \frac{m(\sigma - r_i)(\sigma - r_o)}{\sigma} \frac{d\sigma}{\sigma - \zeta} = \ln m(\zeta - r_o) \tag{4.39}$$

Following the same argument, it is clear that the second integral is:

$$II = \frac{1}{2\pi i} \oint_\gamma \ln \frac{(\sigma - t_i)(\sigma - t_o)}{\sigma} \cdot \frac{d\sigma}{\sigma - \zeta} = \ln(\zeta - t_o) \tag{4.40}$$

The third integral which needs to be calculated is:

$$III = \frac{1}{2\pi i} \oint_\gamma \frac{m(\sigma - r_i)(\sigma - r_o)}{(\sigma - t_i)(\sigma - t_o)(\sigma - \zeta)}\, d\sigma$$

Clearly, III has two poles (at $\sigma = t_i$ and $\sigma = \zeta$) inside γ; thus,

$$\mathrm{III} = \mathrm{Residue}\Big|_{\sigma=t_i} \left\{ \frac{m(\sigma-r_i)(\sigma-r_o)}{(\sigma-t_i)(\sigma-t_o)(\sigma-\zeta)} \right\}$$

$$+ \mathrm{Residue}\Big|_{\sigma=\zeta} \left\{ \frac{m(\sigma-r_i)(\sigma-r_o)}{(\sigma-t_i)(\sigma-t_o)(\sigma-\zeta)} \right\} = \frac{m(\sigma-r_i)(\sigma-r_o)}{(\zeta-t_i)(\zeta-t_o)}$$

$$+ \frac{m(t_i-r_i)(t_i-r_o)}{(t_i-t_o)(t_i-\zeta)}$$

$$\frac{1}{2\pi i} \oint_\gamma \frac{m(\sigma-r_i)(\sigma-r_o)}{(\sigma-t_i)(\sigma-t_o)(\sigma-\zeta)}\, d\sigma = \frac{m\zeta^2-Z_0\zeta+1}{\zeta^2-\overline{Z}_0\zeta+m} + \frac{mt_i^2-Z_0t_i+1}{(t_o-t_i)(\zeta-t_i)} \tag{4.41}$$

The fourth and final integral which must be calculated is:

$$\mathrm{IV} = \frac{1}{2\pi i} \oint_\gamma \frac{(\sigma-t_i)(\sigma-t_o)}{m(\sigma-r_i)(\sigma-r_o)(\sigma-\zeta)}\, d\sigma$$

Clearly, IV has two poles (at $\sigma = r_i$ and $\sigma = \zeta$) inside γ. Determination of residues at the two poles leads to:

$$\mathrm{IV} = \frac{(\zeta-t_i)(\zeta-t_o)}{m(\sigma-r_i)(\sigma-r_o)} + \frac{(r_i-t_i)(r_i-t_o)}{m(r_i-r_o)(r_i-\zeta)}$$

$$\frac{1}{2\pi i} \oint_\gamma \frac{(\sigma-t_i)(\sigma-t_o)}{m(\sigma-r_i)(\sigma-r_o)(\sigma-\zeta)}\, d\sigma = \frac{\zeta^2-\overline{Z}_0\zeta+m}{m\zeta^2-Z_0\zeta+1} + \frac{r_i^2-\overline{Z}_0r_i+m}{m(r_o-r_i)(\zeta-r_i)} \tag{4.42}$$

Substituting the evaluated integrals of equations (4.39), (4.40), and (4.41) into the equation (4.37) leads to:

$$\phi_1^*(\zeta) = Q\left(\ln m(\zeta-r_o) - \alpha \ln(\zeta-t_o)\right) + \overline{Q}\left(\frac{m\zeta^2-Z_0\zeta+1}{\zeta^2-\overline{Z}_0\zeta+m}\right.$$

$$\left. + \frac{mt_i-Z_0t_i+1}{(t_o-t_i)(\zeta-t_i)}\right) \tag{4.43}$$

Taking the derivatives:

$$\phi_1^{*\prime}(\zeta) = Q\left(\frac{1}{\zeta-r_o} - \frac{\alpha}{\zeta-t_o}\right) + \overline{Q}\left(\frac{PD}{(\zeta^2-\overline{Z}_0\zeta+m)^2} - \frac{mt_i^2-Z_0t_i+1}{(t_o-t_i)(\zeta-t_i)^2}\right)$$

(4.44)

where

$$PD = (2m\zeta-Z_0)(\zeta^2-\overline{Z}_0\zeta+m) - (2\zeta-\overline{Z}_0)(m\zeta^2-Z_0\zeta+1)$$

and

$$\phi^{*\prime\prime}(\zeta) = Q\left(\frac{-1}{(\zeta-r_o)^2} + \frac{\alpha}{(\zeta-t_o)^2}\right) + \overline{Q}\left(\frac{PDD}{(\zeta^2-\overline{Z}_0\zeta+m)^3}\right.$$

$$\left. + \frac{2(mt_i^2-Z_0t_i+1)}{(t_o-t_i)(\zeta-t_i)^3}\right) \tag{4.45}$$

where

$$PDD = [2m(m-\overline{Z}_0\zeta)-2(1-Z_0\zeta)](\zeta^2-\overline{Z}_0\zeta+m)-2(2\zeta-\overline{Z}_0)[PD]$$

Substituting the evaluated integrals of equations (4.39), (4.40), and (4.42) along with equation (4.44) into equation (4.38) leads to:

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$$\Psi_1^*(\zeta) = Q\left(\frac{\zeta^2-\overline{Z}_0\zeta+m}{m\zeta^2-Z_0\zeta+1} + \frac{r_i^2-\overline{Z}_0 r_i+m}{m(r_o-r_i)(\zeta-r_i)}\right.$$

$$\left. + \frac{\zeta(\zeta^2+m)}{1-m\zeta^2}\left\{\frac{1}{\zeta-r_o} - \frac{\alpha}{\zeta-t_o}\right\}\right) + \overline{Q}\left(\ln(\zeta-t_o) - \alpha\ln m(\zeta-r_o)\right.$$

$$\left. + \frac{\zeta(\zeta^2+m)}{1-m\zeta^2}\left\{\frac{PD}{(\zeta^2-\overline{Z}_0\zeta+m)^2} - \frac{mt_i^2-Z_0t_i+1}{(t_o-t_i)(\zeta-t_i)^2}\right\}\right) \qquad (4.46)$$

The two complex potential functions, $\phi_1^*(\zeta)$ and $\Psi_1^*(\zeta)$, expressed by equations (4.43) and (4.46), are not a unique set of functions. Since the origin of the coordinates is within γ, then, following section I.2, the uniqueness conditions for $\phi_1^*(\zeta)$ and $\psi_1^*(\zeta)$ are:

$$\phi_1^*(0) = 0 \qquad , \qquad \Psi_1^*(0) = 0$$

These conditions lead to the unique complex potential functions:

$$\phi_1^*(\zeta) = Q\left(\ln\frac{r_o-\zeta}{r_o} - \alpha\ln\frac{t_o-\zeta}{t_o}\right) + \overline{Q}\left(\frac{m\zeta^2-Z_0\zeta+1}{\zeta^2-\overline{Z}_0\zeta+m}\right.$$

$$\left. - \frac{1}{m} + \frac{(mt_i^2-Z_0t_i+1)\zeta}{t_i(t_o-t_i)(\zeta-t_i)}\right) \qquad (4.47)$$

$$\Psi_1^*(\zeta) = Q\left(\frac{\zeta^2-\overline{Z}_0\zeta+m}{m\zeta^2-Z_0\zeta+1} + \frac{(r_i^2-\overline{Z}_0r_i+m)}{mr_i(r_o-r_i)(\zeta-r_i)} + \frac{\zeta(\zeta^2+m)}{1-m\zeta^2}\cdot\phi_I^{*\prime}(\zeta) - m\right)$$

$$+ \overline{Q}\left(\ln\frac{t_o-\zeta}{t_o} - \alpha\ln\frac{r_o-\zeta}{r_o} + \frac{\zeta(\zeta^2+m)}{1-m\zeta^2}\cdot\phi_{II}^{*\prime}(\zeta)\right) \qquad (4.48)$$

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Note that equations (4.47) and (4.48) can be rewritten as:

$$\phi_1^*(\zeta) = Q\left(\phi_{I}^*(\zeta)\right) + \overline{Q}\left(\phi_{II}^*(\zeta)\right)$$

$$\Psi_1^*(\zeta) = Q\left(\Psi_{I}^*(\zeta)\right) + \overline{Q}\left(\Psi_{II}^*(\zeta)\right) \qquad (4.49)$$

where $\phi_{I}^*(\zeta)$, $\phi_{II}^*(\zeta)$, $\Psi_{I}^*(\zeta)$ and $\Psi_{II}^*(\zeta)$ can be found by comparison to equations (4.47) and (4.48). The derivatives of the complex potential functions can be written as:

$$\phi_1^{*\prime}(\zeta) = Q\left(\phi_{I}^{*\prime}(\zeta)\right) + \overline{Q}\left(\phi_{II}^{*\prime}(\zeta)\right)$$

$$\phi_1^{*\prime\prime}(\zeta) = Q\left(\phi_{I}^{*\prime\prime}(\zeta)\right) + \overline{Q}\left(\phi_{II}^{*\prime\prime}(\zeta)\right)$$

$$\Psi_1^{*\prime}(\zeta) = Q\left(\Psi_{I}^{*\prime}(\zeta)\right) + \overline{Q}\left(\Psi_{II}^{*\prime}(\zeta)\right)$$

where $\phi_{I}^{*\prime}(\zeta)$, $\phi_{I}^{*\prime\prime}(\zeta)$, $\phi_{II}^{*\prime}(\zeta)$, $\phi_{II}^{*\prime\prime}(\zeta)$, $\Psi_{I}^{*\prime}(\zeta)$ and $\Psi_{II}^{*\prime}(\zeta)$ are given in Appendix C.

The complex potential functions of Figure 4.1(B) and their derivatives are given by equations (3.42) and (3.43). Applying superposition and adding the two sets of potential functions, expressed by equations (3.42) and (4.49), leads to the potential functions of the problem of Figure 4.1(A):

$$\phi(Z) = Q\left\{-\ln(Z-Z_0) + \phi_I^*(\zeta)\right\} + \overline{Q}\left\{\phi_{II}^*(\zeta)\right\}$$

$$\Psi(Z) = Q\left\{\frac{\overline{Z}}{Z-\overline{Z}_0} + \Psi_I^*(\zeta)\right\} + \overline{Q}\left\{\alpha\ \ln(Z-\overline{Z}_0) + \Psi_{II}^*(\zeta)\right\} \qquad (4.50)$$

Since $\phi'(Z)$, $\phi''(Z)$ and $\Psi'(Z)$ are also needed for the influence functions, they are written here:

$$\phi'(Z) = Q\left\{\frac{-1}{Z-Z_0} + \frac{\phi_I^{*\prime}(\zeta)}{\omega'(\zeta)}\right\} + \overline{Q}\left\{\frac{\phi_{II}^{*\prime}(\zeta)}{\omega'(\zeta)}\right\}$$

$$\Psi'(Z) = Q\left\{-\frac{\overline{Z}_0}{(Z-\overline{Z}_0)^2} + \frac{\Psi_I^{*\prime}(\zeta)}{\omega'(\zeta)}\right\} + \overline{Q}\left\{\frac{\alpha}{Z-\overline{Z}_0} + \frac{\Psi_{II}^*(\zeta)}{\omega'(\zeta)}\right\}$$

$$\phi''(Z) = Q\left\{\frac{1}{(Z-Z_0)^2} + \frac{\phi_I^{*\prime\prime}(\zeta)}{\omega'^2(\zeta)} - \frac{\phi_I^*(\zeta)\omega''(\zeta)}{\omega'^3(\zeta)}\right\} + Q\left\{\frac{\phi_I^{*\prime\prime}(\zeta)}{\omega'^2(\zeta)}\right.$$

$$\left. - \frac{\phi_{II}^{*\prime}(\zeta)\omega''(\zeta)}{\omega'^3(\zeta)}\right\} \qquad (4.51)$$

Finally, substituting equations (4.50) and (4.51) along with the mapping function and its derivatives, equations (4.5), into equation (3.6) leads to the influence functions for an infinite region weakened by an elliptic hole:

$$H_{xx;q}(Z,Z_0)P_q^*(Z_0) = \mathrm{Re}\left\{ Q^*\left(\frac{-2}{\overline{Z-Z_0}} + \frac{2\zeta^2}{m\zeta^2-1}\,\phi_I^{*\prime}(\zeta) - \overline{Z}\left[\frac{1}{(Z-Z_0)^2}\right.\right.\right.$$

$$\left. + \frac{\zeta^4}{(m\zeta^2-1)^4}\,\phi_I^{*\prime\prime}(\zeta) - \frac{2\zeta^3}{(m\zeta^2-1)^3}\,\phi_I^{*\prime}(\zeta)\right]$$

$$\left. + \frac{\overline{Z}_0}{(Z-\overline{Z}_0)^2} - \frac{\zeta^2}{m\zeta^2-1}\,\psi_I^{*\prime}(\zeta)\right)$$

$$+ \overline{Q}^*\left(\frac{2\zeta^2}{m\zeta^2-1}\,\phi_{II}^{*\prime}(\zeta) - \overline{Z}\left[\frac{\zeta^4}{(m\zeta^2-1)^4}\,\phi_{II}^{*\prime\prime}(\zeta)\right.\right.$$

$$\left. - \frac{2\zeta^3}{(m\zeta^2-1)^3}\,\phi_{II}^{*\prime}(\zeta)\right] - \frac{\alpha}{Z-\overline{Z}_0}$$

$$\left.\left. - \frac{\zeta^2}{m\zeta^2-1}\,\psi_{II}^{*\prime}(\zeta)\right)\right\}$$

$$H_{yy;q}(Z,Z_0)P_q^*(Z_0) = \mathrm{Re}\left\{ Q^*\left(\frac{-2}{\overline{Z-Z_0}} + \frac{2\zeta^2}{m\zeta^2-1}\,\phi_I^{*\prime}(\zeta)\right.\right.$$

$$+ \overline{Z}\left[\frac{1}{(Z-Z_0)^2} + \frac{\zeta^4}{(m\zeta^2-1)^4}\,\phi_I^{*\prime\prime}(\zeta)\right.$$

$$\left.\left. - \frac{2\zeta^3}{(m\zeta^2-1)^3}\,\phi_I^{*\prime}(\zeta)\right] - \frac{\overline{Z}_0}{(Z-\overline{Z}_0)^2} + \frac{\zeta^2}{m\zeta^2-1}\,\psi_I^{*\prime}(\zeta)\right)$$

$$+ \overline{Q}^*\left(\frac{2\zeta^2}{m\zeta^2-1}\,\phi_{II}^{*\prime}(\zeta)\right.$$

$$+ \overline{Z}\left[\frac{\zeta^4}{(m\zeta^2-1)^4}\,\phi_{II}^{*\prime\prime}(\zeta) - \frac{2\zeta^3}{(m\zeta^2-1)^3}\,\phi_{II}^{*\prime}(\zeta)\right] +$$

$$\left.\left.\frac{\alpha}{Z-\overline{Z}_0} + \frac{\zeta^2}{m\zeta^2-1}\,\Psi^*_{II}(\zeta)\right)\right\}$$

$$H_{xy;q}(Z,Z_0)P^*_q(Z_0) = \mathrm{Im}\left\{Q^*\left(\overline{Z}\left[\frac{1}{(Z-Z_0)^2} + \frac{\zeta^4}{(m\zeta^2-1)^4}\,\phi^{*\prime\prime}_I(\zeta)\right.\right.\right.$$

$$\left. - \frac{2\zeta^3}{(m\zeta^2-1)^3}\,\phi^{*\prime}_I(\zeta)\right] - \frac{\overline{Z}_0}{(Z-\overline{Z}_0)^2}$$

$$\left. + \frac{\zeta^2}{m\zeta^2-1}\,\Psi^{*\prime}_I(\zeta)\right) + \overline{Q}^*\left(\overline{Z}\left[\frac{\zeta^4}{(m\zeta^2-1)^4}\cdot\phi^{*\prime\prime}_{II}(\zeta)\right.\right.$$

$$\left.\left.\left. - \frac{2\zeta^3}{(m\zeta^2-1)^3}\,\phi^{*\prime}_{II}(\zeta)\right] + \frac{\alpha}{Z-\overline{Z}_0} + \frac{\zeta^2}{m\zeta^2-1}\,\Psi^{*\prime}_I(\zeta)\right)\right\}$$

$$I_{x;q}(Z,Z_0)P^*_q(Z_0) = \mathrm{Re}\left\{\frac{Q^*}{2\mu}\left(-\alpha\,\ln(Z-Z_0) + \alpha\,\phi^*_I(\zeta)\right.\right.$$

$$- Z\left[\frac{-1}{Z-\overline{Z}_0} + \frac{\overline{\zeta}^2}{m\overline{\zeta}^2-1}\,\overline{\phi^{*\prime}_I(\zeta)}\right] - \frac{Z_0}{\overline{Z}-Z_0}$$

$$\left. - \overline{\Psi^*_I(\zeta)}\right) + \frac{\overline{Q}^*}{2\mu}\left(\alpha\,\phi^*_{II}(\zeta) - Z\left[\frac{\overline{\zeta}^2}{m\overline{\zeta}^2-1}\,\overline{\phi^{*\prime}_{II}(\zeta)}\right]\right.$$

$$\left.\left. - \alpha\,\ln(\overline{Z}-Z_0) - \overline{\Psi^*_{II}(\zeta)}\right)\right\}$$

$$I_{y;q}(Z,Z_0)P^*_q(Z_0) = \mathrm{Im}\left\{ \frac{Q^*}{2\mu}\left(-\alpha\ \ln(Z-Z_0) + \alpha\ \phi^*_I(\zeta) - Z\left[\frac{-1}{Z-\bar{Z}_0} + \frac{\bar{\zeta}^2}{m\bar{\zeta}^2-1}\ \overline{\phi^{*\prime}_I(\zeta)}\right] - \frac{Z_0}{\bar{Z}-Z_0} - \overline{\Psi^*_I(\zeta)}\right) + \frac{\bar{Q}^*}{2\mu}\left(\alpha\ \phi^*_{II}(\zeta) - Z\left[\frac{\bar{\zeta}^2}{m\bar{\zeta}^2-1}\ \overline{\phi^{*\prime}_{II}(\zeta)}\right] - \alpha\ \ln(\bar{Z}-Z_0) - \overline{\Psi^*_{II}(\zeta)}\right)\right\} \qquad (4.52)$$

Hence, the influence functions for an infinite plane with an elliptical cavity at the origin are found.

IV.3 DERIVATION OF THE INFLUENCE FUNCTION: THE SHARP CRACK PROBLEM

Considering $m = 1$, a sharp crack along the x-axis between $x = 2$ and $x = -2$ is obtained, Figure 4.3. The transformation function

$$Z = \omega(\zeta) = \frac{1}{\zeta} + \zeta \qquad (4.53)$$

transforms the whole region exterior to the crack into a unit circle $|\zeta|<1$. Substituting $m = 1$ into equations (4.52) leads to the influence functions for the crack problem:

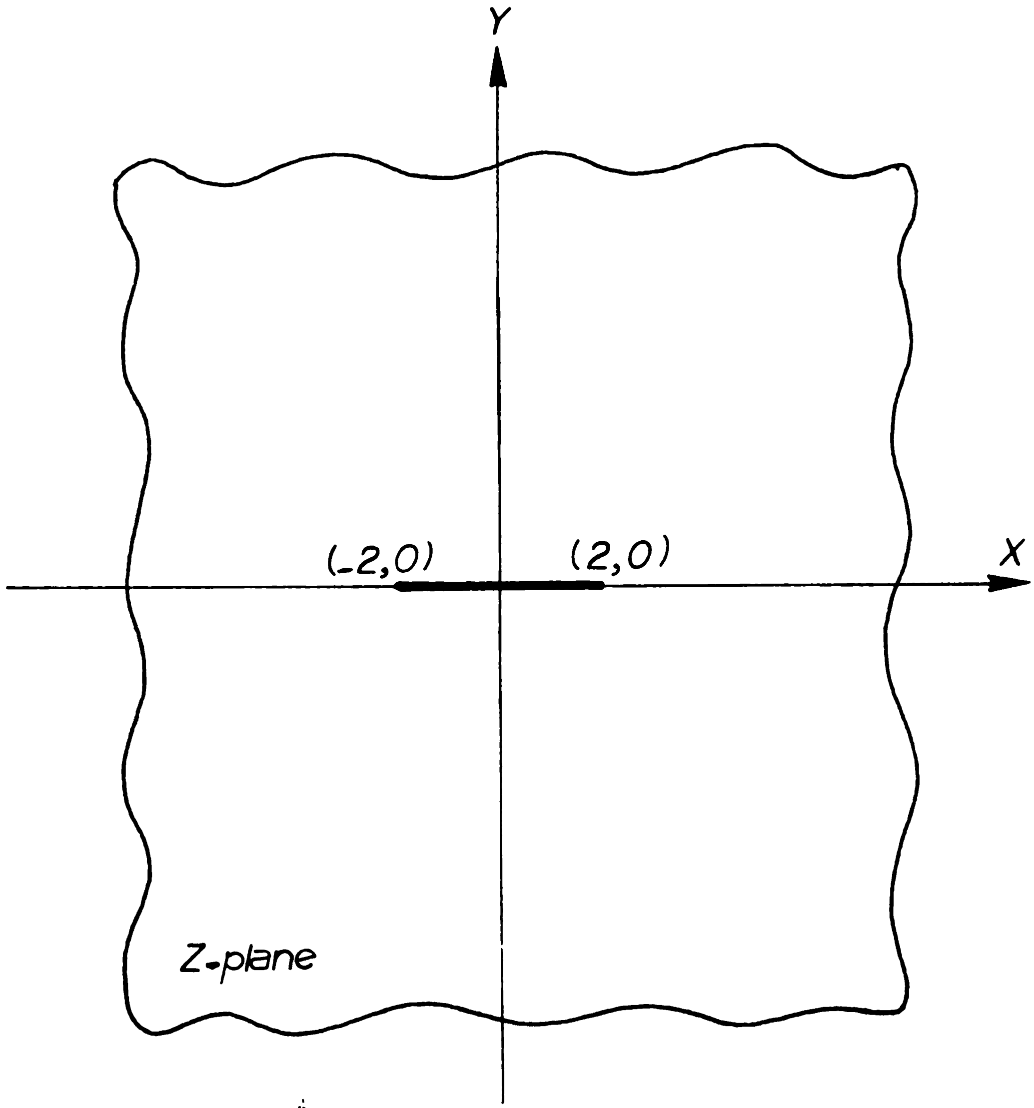

Figure 4.3 A sharp crack in an infinite plane.

$$H_{xx;q}(Z,Z_0)P_q^*(Z_0) = \mathrm{Re}\left\{Q^*\left(\frac{-2}{Z-Z_0} + \frac{2\zeta^2}{\zeta^2-1}\,\phi_I^{*\prime}(\zeta)\right.\right.$$

$$- \bar{Z}\left[\frac{1}{(Z-Z_0)^2} + \frac{\zeta^4}{(\zeta^2-1)^4}\,\phi_I^{*\prime\prime}(\zeta)\right.$$

$$\left.- \frac{2\zeta^3}{(\zeta^2-1)^3}\,\phi_I^{*\prime}(\zeta)\right] + \frac{\bar{Z}_0}{(Z-\bar{Z}_0)^2}$$

$$\left.- \frac{\zeta^2}{\zeta^2-1}\,\Psi_I^{*\prime}(\zeta)\right) + \bar{Q}^*\left(\frac{2\zeta^2}{\zeta^2-1}\,\phi_{II}^{*\prime}(\zeta)\right.$$

$$- \bar{Z}\left[\frac{\zeta^4}{(\zeta^2-1)^4}\,\phi_{II}^{*\prime\prime}(\zeta) - \frac{2\zeta^3}{(\zeta^2-1)^3}\,\phi_{II}^{*\prime}(\zeta)\right]$$

$$\left.\left.- \frac{\alpha}{Z-\bar{Z}_0} - \frac{\zeta^2}{\zeta^2-1}\,\Psi_{II}^{*\prime}(\zeta)\right)\right\}$$

$$H_{yy;q}(Z,Z_0)P_q^*(Z_0) = \mathrm{Re}\left\{Q^*\left(\frac{-2}{Z-Z_0} + \frac{2\zeta^2}{\zeta^2-1}\,\phi_I^{*\prime}(\zeta) + \bar{Z}\left[\frac{1}{(Z-Z_0)^2}\right.\right.\right.$$

$$\left.+ \frac{\zeta^4}{(\zeta^2-1)^4}\,\phi_I^{*\prime\prime}(\zeta) - \frac{2\zeta^3}{(\zeta^2-1)^3}\,\phi_I^{*\prime}(\zeta)\right]$$

$$\left.- \frac{\bar{Z}_0}{(Z-\bar{Z}_0)^2} + \frac{\zeta^2}{\zeta^2-1}\,\Psi_I^{*\prime}(\zeta)\right) + \bar{Q}^*\left(\frac{2\zeta^2}{\zeta^2-1}\,\phi_{II}^{*\prime}(\zeta)\right.$$

$$+ \bar{Z}\left[\frac{\zeta^4}{(\zeta^2-1)^4}\,\phi_{II}^{*\prime\prime}(\zeta) - \frac{2\zeta^3}{(\zeta^2-1)^3}\,\phi_{II}^{*\prime}(\zeta)\right]$$

$$\left.\left.+ \frac{\alpha}{Z-\bar{Z}_0} + \frac{\zeta^2}{\zeta^2-1}\,\Psi_{II}^{*\prime}(\zeta)\right)\right\}$$

$$H_{xy;q}(Z,Z_0)P^*_q(Z_0) = \mathrm{Im}\left\{ Q^*\left(\overline{Z}\ \frac{1}{(Z-Z_0)^2} + \frac{\zeta^4}{(\zeta^2-1)^4}\,\phi^{*\prime\prime}_{I}(\zeta)\right.\right.$$

$$- \frac{2\zeta^3}{(\zeta^2-1)^3}\,\phi^{*\prime}_{I}(\zeta) \quad - \frac{\overline{Z}_0}{(Z-\overline{Z}_0)^2} + \frac{\zeta^2}{\zeta^2-1}\,\Psi^{*\prime}_{I}(\zeta)\Bigg)$$

$$+ \overline{Q}^*\left(\overline{Z}\ \frac{\zeta^4}{(\zeta^2-1)^4}\,\phi^{*\prime\prime}_{II}(\zeta) - \frac{2\zeta^3}{(\zeta^2-1)^3}\,\phi^{*\prime}_{II}(\zeta)\right.$$

$$\left.\left. + \frac{\alpha}{Z-\overline{Z}_0} + \frac{\zeta^2}{\zeta^2-1}\,\Psi^{*\prime}_{I}(\zeta)\right)\right\}$$

$$I_{x;q}(Z,Z_0)P^*_q(Z_0) = \mathrm{Re}\left\{ \frac{Q^*}{2\mu}\left(-\ \alpha\ \ln(Z-Z_0) + \alpha\ \phi^*_I(\zeta)\right.\right.$$

$$\left. - Z\left[\frac{-1}{Z-\overline{Z}_0} + \frac{\overline{\zeta}^2}{\overline{\zeta}^2-1}\,\overline{\phi^{*\prime}_{I}(\zeta)}\right] - \frac{Z_0}{\overline{Z}-Z_0} - \overline{\Psi^*_I(\zeta)}\right)$$

$$+ \frac{\overline{Q^*}}{2\mu}\left(\alpha\ \phi^*_{II}(\zeta) - Z\cdot\frac{\overline{\zeta}^2}{\overline{\zeta}^2-1}\,\overline{\phi^{*\prime}_{II}(\zeta)}\right.$$

$$\left.\left. - \alpha\ \ln(\overline{Z}-Z_0) - \overline{\Psi^*_{II}(\zeta)}\right)\right\}$$

$$I_{y;q}(Z,Z_0)P_q^*(Z_0) = \mathrm{Im}\left\{ \frac{Q^*}{2\mu}\left(-\alpha \ln(Z-Z_0) + \alpha\, \phi_I^*(\zeta) \right.\right.$$

$$\left. - Z\left[\frac{-1}{Z-\overline{Z}_0} + \frac{\overline{\zeta}^2}{\overline{\zeta}^2-1}\,\overline{\phi_I^{*\prime}(\zeta)}\right] - \frac{Z_0}{\overline{Z}-Z_0} - \overline{\Psi_I^*(\zeta)} \right)$$

$$+ \frac{\overline{Q^*}}{2\mu}\left(\alpha \cdot \phi_{II}^*(\zeta) - Z\,\frac{\overline{\zeta}^2}{\overline{\zeta}^2-1}\,\overline{\phi_{II}^{*\prime}(\zeta)}\right.$$

$$\left.\left. - \alpha \ln(\overline{Z}-Z_0) - \overline{\Psi_{II}^*(\zeta)}\right)\right\} \tag{4.54}$$

where $\phi_I^*(\zeta)$, $\phi_{II}^*(\zeta)$, $\phi_I^{*\prime}(\zeta)$, $\phi_{II}^{*\prime}(\zeta)$, $\phi_I^{*\prime\prime}(\zeta)$, $\phi_{II}^{*\prime\prime}(\zeta)$, $\Psi_I^*(\zeta)$, $\Psi_{II}^*(\zeta)$, $\Psi_I^{*\prime}(\zeta)$ and $\Psi_{II}^{*\prime}(\zeta)$ are given in Appendix D.

Hence, the influence functions for an infinite plane with a horizontal sharp crack lying on the x-axis at the origin are found.

One must exercise some care when using these influence functions in that a singularity will occur when $Z = \overline{Z}_0$. This case will now be considered. First solve the transformation function, equation (4.53), for ζ.

$$\zeta = \frac{Z}{2} \pm \sqrt{Z^2/4 - 1} \tag{4.54}$$

Then write the two roots inside and outside γ, equations (4.29) and (4.34), for the case (m = 1):

$$r_{i,o} = \frac{Z_0}{2} \pm \sqrt{Z_0^2/4 - 1} \tag{4.55}$$

$$t_{i,o} = \overline{Z}_0/2 \pm \sqrt{\overline{Z}_0^2/4 - 1} \tag{4.56}$$

If the field point, Z, is equal to the complex conjugate of the load point, $\overline{Z}_0$, then equations (4.54) and (4.56) will be identical ($\zeta = t_i$), since $|\zeta| \leqslant 1$ and $|t_i| \leqslant 1$. Note that this makes the right-hand side of integral III, equation (4.41), infinite. Thus, integral III has to be reevaluated. Substituting $t_i = \zeta$ into the integral leads to:

$$\mathrm{III} = \frac{1}{2\pi i} \int_\gamma \frac{m(\sigma - r_i)(\sigma - r_o)}{(\sigma - t_o)(\sigma - \zeta)^2} \, d\sigma$$

Thus,

$$\mathrm{III} = \text{Residue}\Big|_{\sigma=\zeta} \left\{ \frac{m(\sigma - r_i)(\sigma - r_o)}{(\sigma - t_o)(\sigma - \zeta)^2} \right\} = \frac{d}{d\sigma} \left. \frac{m(\sigma - t_i)(\sigma - r_o)}{\sigma - t_o} \right|_{\sigma=\zeta}$$

or

$$\mathrm{III} = \frac{1}{2\pi i} \oint_\gamma \frac{m(\sigma - r_i)(\sigma - r_o)}{(\sigma - t_o)(\sigma - \zeta)^2} \, d\sigma = \frac{(2m\zeta - Z_0)(\zeta - t_o) - (m\zeta^2 - Z_0\zeta + 1)}{(\zeta - t_o)^2}$$

This change modifies the complex potential function $\phi_1^*(\zeta)$, equation (4.47), to:

$$\phi_1^*(\zeta) = Q \left\{ \ln\left(\frac{r_o - \zeta}{r_o}\right) - \alpha \ln\left(\frac{t_o - \zeta}{t_o}\right) \right\}$$

$$+ \overline{Q} \left\{ \frac{(2m\zeta - Z_0)(\zeta - t_o) - (m\zeta^2 - Z_0\zeta + 1)}{(\zeta - t_o)^2} \right.$$

$$\left. - \frac{Z_0 t_o - 1}{t_o^2} \right\} \qquad (4.57)$$

Clearly, this change affects the other complex potential function and all the derivatives. The modified functions, $\phi_I^*(\zeta)$, $\phi_{II}^*(\zeta)$, $\phi_I^{*\prime}(\zeta)$, $\phi_{II}^{*\prime}(\zeta)$, $\phi_I^{*\prime\prime}(\zeta)$, $\phi_{II}^{*\prime\prime}(\zeta)$, $\Psi_I^*(\zeta)$, $\Psi_{II}^*(\zeta)$, $\Psi_I^{*\prime}(\zeta)$ and $\Psi_{II}^{*\prime}(\zeta)$ for this special case are given in Appendix D.

Now that the influence functions have been obtained it is important to notice that at the tips of the crack where $Z = \pm 2$ the stress influence functions will become infinite as expected. These two points are the only singular points in the plane.

IV.4 THE BOUNDARY INTEGRAL EQUATION METHOD APPLIED TO A PLANE FINITE REGION WEAKENED BY AN ELLIPTICAL HOLE OR A CRACK

In this section, two classes of problems will be considered. These are: (1) a plane finite region subjected to traction boundary condition $\vec{t}$ and weakened by an elliptical hole, Figure 4.4; and (2) a plane finite region subjected to traction boundary condition $\vec{t}$ and weakened by a crack, Figure 4.5.

Solutions will be obtained by embedding the regions R_e (region with elliptical hole) and R_s (region with a sharp crack) in infinite (fictitious) planes of the same material as R_e and R_s, containing an elliptical hole, Figure 4.7, or a crack, Figure 4.8, respectively.

In the treatment of either of these problems, the boundary is divided into a finite number of divisions, N, of equal or unequal length. A concentrated line load, which is the resultant of the traction on each division, is then applied at the center of the division, i.e.,

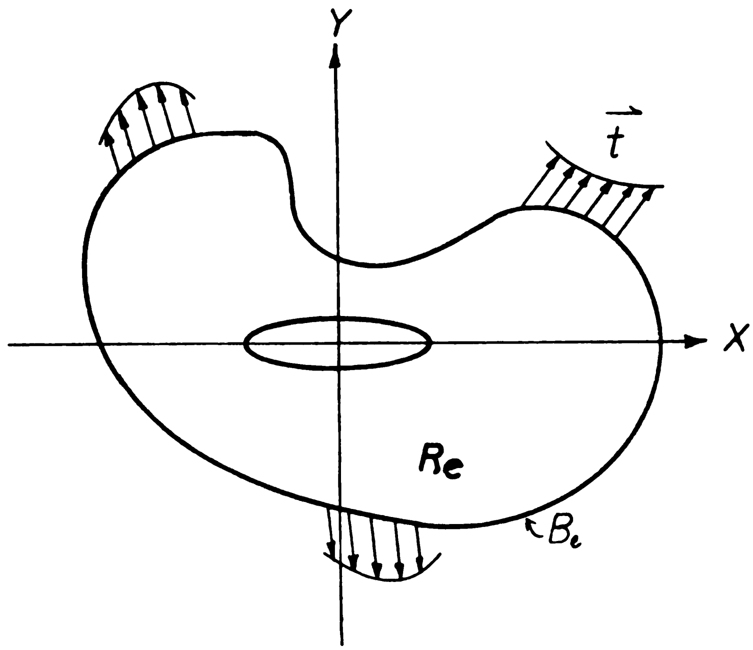

Figure 4.4 An elliptical hole in a plane finite region with prescribed traction on the boundary, B_e.

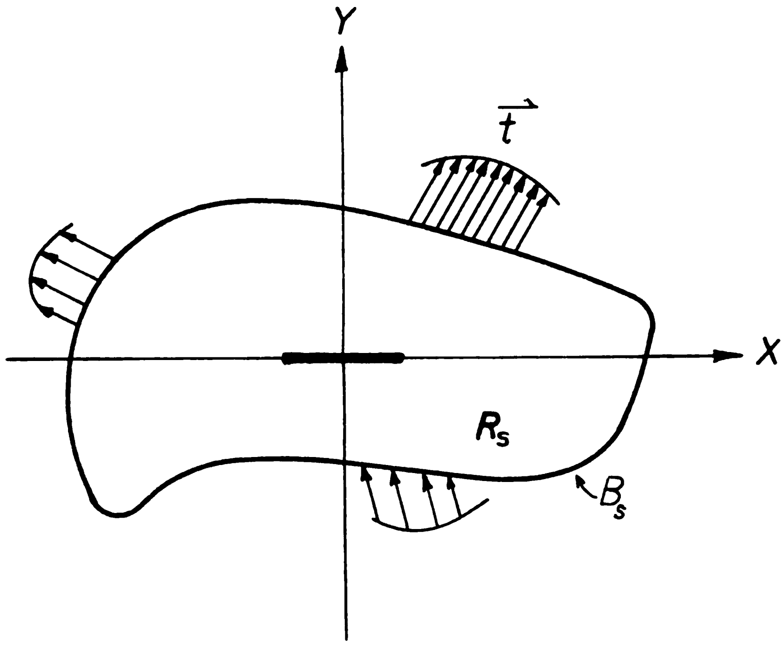

Figure 4.5 A horizontal slit in a plane finite region with prescribed traction on the boundary, B_s.

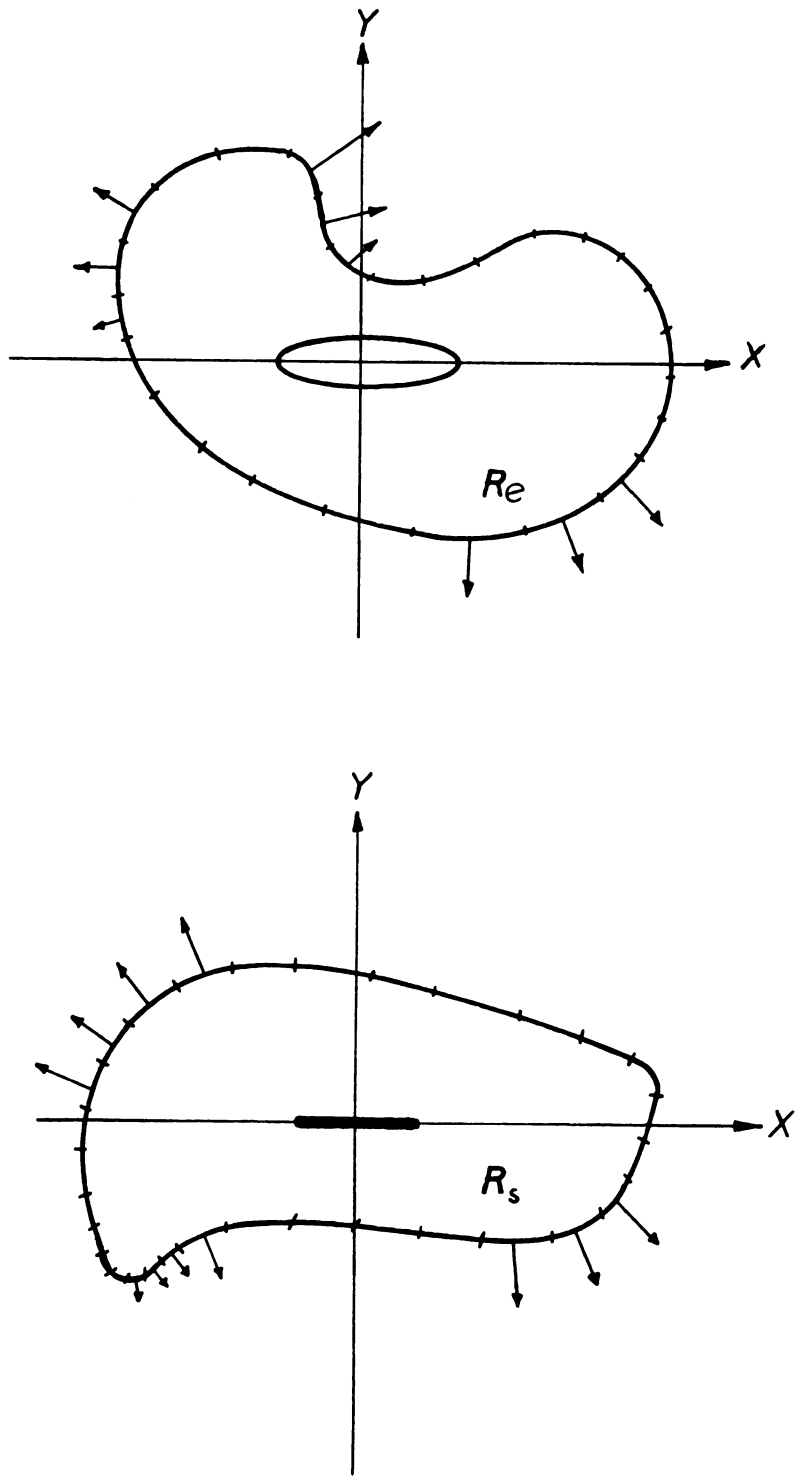

Figure 4.6 The finite regions R_e and R_s, with subdivided boundary and concentrated line loads.

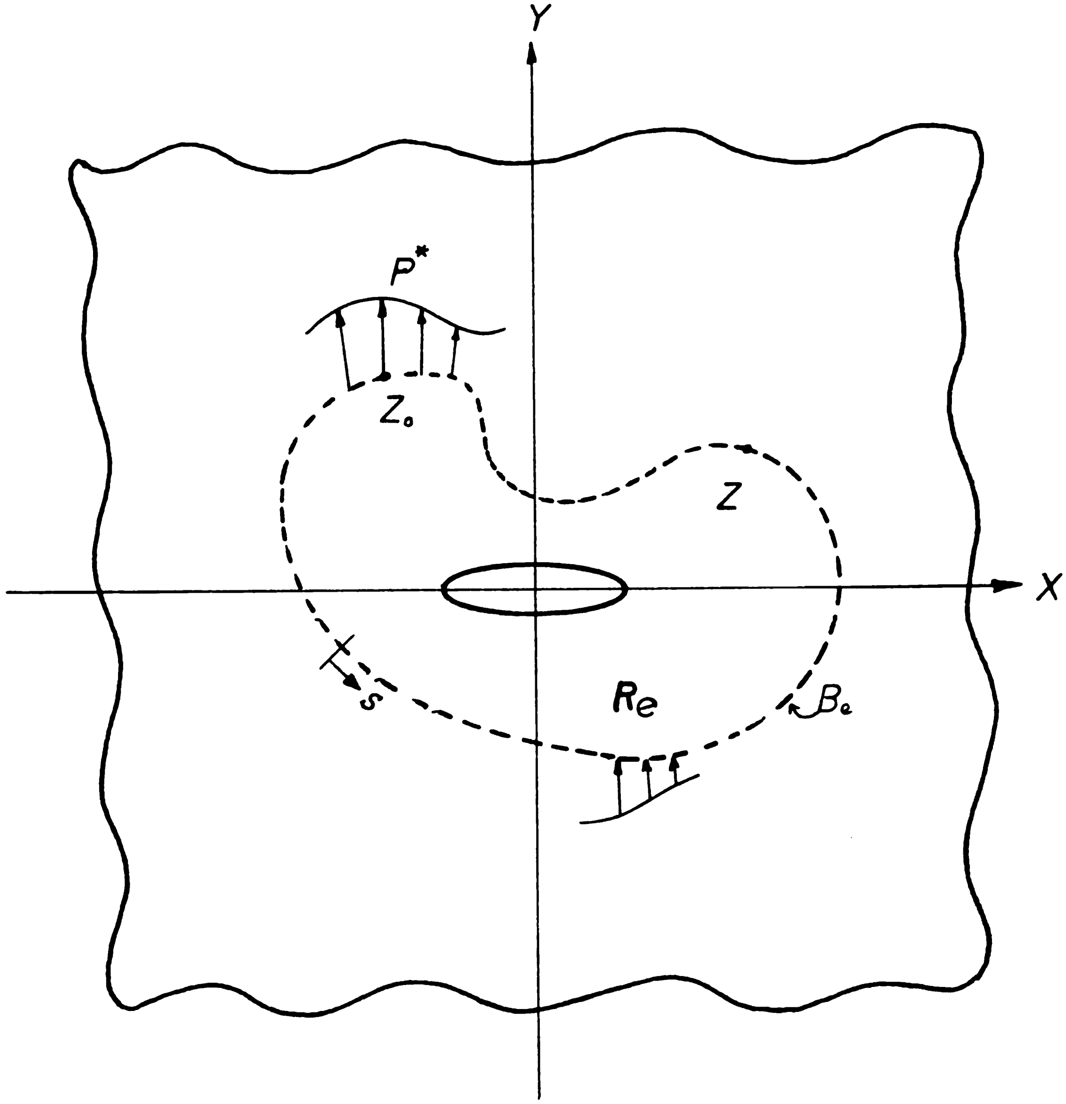

Figure 4.7 Region R_e, embedded in an infinite plane containing an elliptical hole at the origin.

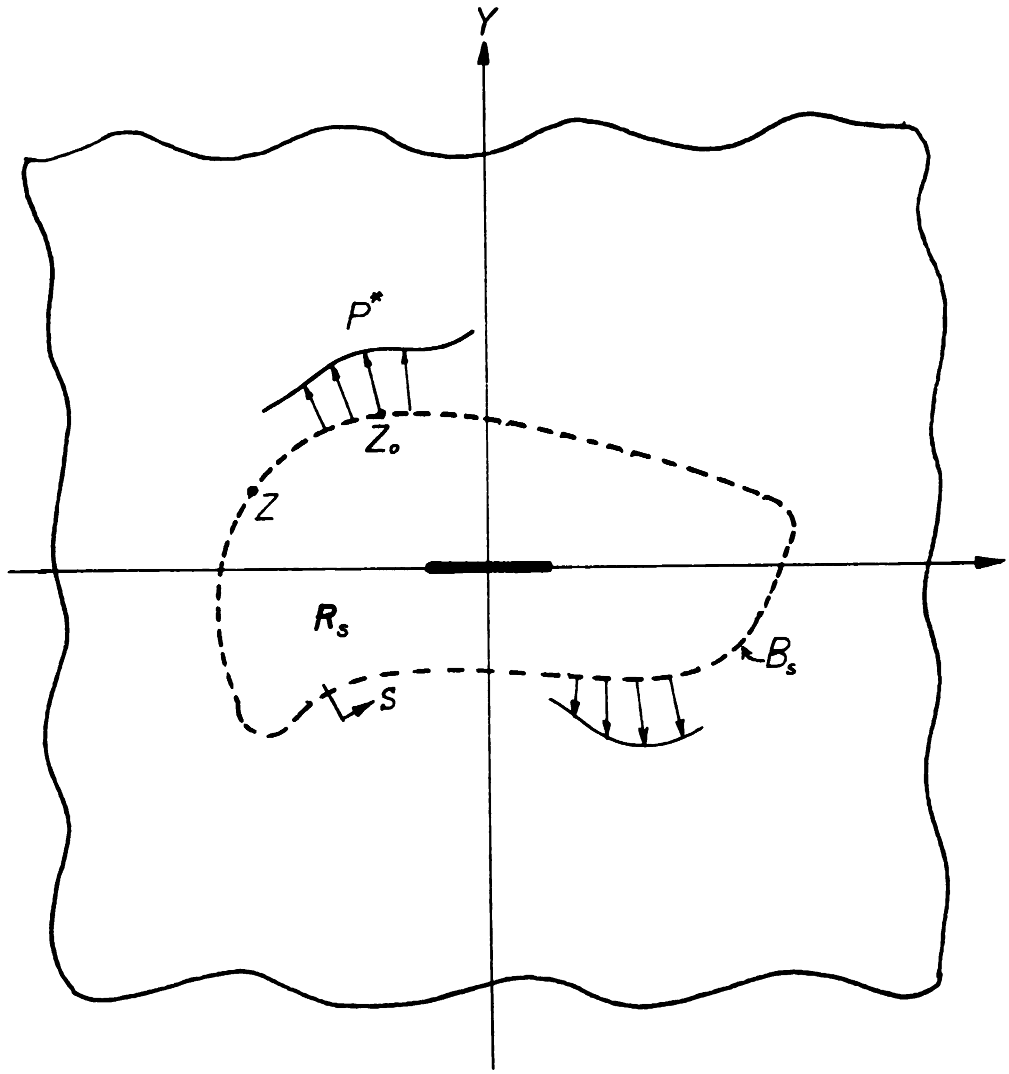

Figure 4.8 Region R_s, embedded in an infinite plane containing a horizontal slit at the origin.

$$P_{xi} = \int_{\Delta S_i} p_x \, ds$$

$$P_{yi} = \int_{\Delta S_i} p_y \, ds$$

and for the fictitious tractions

$$P^*_{xi} = \int_{\Delta S_i} p^*_x \, ds$$

$$P^*_{yi} = \int_{\Delta S_i} p_y \, ds$$

where ΔS_i is the ith interval and $i = 1, \ldots N$, see Figure (4.6). The trapezoidal rule is used to approximate these integrals. Following section I.1, the fictitious traction P^* around the fictitious boundary can be found from:

$$\frac{P^*_{xi}}{2} + \sum_{\substack{j=1 \\ j \neq i}}^{N} \left(H_{xx;q}(Z, Z_0) P^*_q \cdot n_{xi} + H_{xy;q}(Z, Z_0) P^*_q \cdot n_{yi} \right) \Delta S_i$$

$$= P_{xi}$$

$$\frac{P^*_{yi}}{2} + \sum_{\substack{j=1 \\ j \neq i}}^{N} \left(H_{xy;q}(Z, Z_0) P^*_q \, n_{xi} + H_{yy;q}(Z, Z_0) P^*_q \cdot n_{yi} \right) \Delta S_i$$

$$= P_{yi} \qquad \text{for } i = 1, \ldots N \qquad (4.58)$$

where the resultant fictitious traction on a given interval is represented by

$$P^*_{qi} = P^*_{xi} + i\,P^*_{yi} \qquad (i = 1,\ldots N) \qquad (4.59)$$

and the influence functions $H_{ij;q}(Z,Z_0)$ are given by equations (4.54), for a crack.

Note that, in equation (4.58), n_{xi} and n_{yi} are the components of the unit normal to the division i and P_{xi} and P_{yi} are the x and y component of the real resultant traction applied to the division i, i.e.,

$$P_{qi} = P_{xi} + i\,P_{yi} \qquad (i = 1,\ldots N)$$

Substituting the components of the resultant fictitious traction, equation (4.59), into the influence functions for an elliptical hole or a slit, equations (4.52) or (4.54), and rewriting them leads to:

$$H_{xx;q}(Z,Z_0)P^*_q(Z_0) = H_{xx;x} \cdot P^*_x + H_{xx;y} \cdot P^*_y$$

$$H_{yy;q}(Z,Z_0)P^*_q(Z_0) = H_{yy;x} \cdot P^*_x + H_{yy;y} \cdot P^*_y$$

$$H_{xy;q}(Z,Z_0)P^*_q(Z_0) = H_{xy;x} \cdot P^*_x + H_{xy;y} \cdot P^*_y$$

$$I_{x;q}(Z,Z_0)P^*_q(Z_0) = I_{x;x} \cdot P^*_x + I_{x;y}\,P^*_y$$

$$I_{y;q}(Z,Z_0)P^*_q(Z_0) = I_{y;x}\,P^*_x + I_{y;y}\,P^*_y \qquad (4.60)$$

where $H_{ij;q}(Z,Z_0)$, which represents the ijth stress component at a point Z due to a unit load in the q direction at a source point Z_0, and $I_{i;q}(Z,Z_0)$, which represents the ith displacement component at the point Z due to the unit load in the q direction at the source point Z_0, can be easily found by comparing equations (4.60) with either equations (4.52) for the elliptical hole (see Appendix C) or equations (4.54) for the crack (see Appendix D).

Substituting equations (4.60) into equations (4.58) and rearranging leads to:

$$\frac{P^*_{xi}}{2} + \sum_{\substack{j=1 \\ j\neq i}}^{N} \Big([H_{xx;x} \cdot n_{xi} + H_{xy;x} \cdot n_{yi}] \cdot P^*_{xi}$$

$$+ [H_{xx;y} \cdot n_{xi} + H_{xy;y} \cdot n_{yi}] \cdot P^*_{yi} \Big) \Delta S_i = P_{xi}$$

$$\frac{P^*_{yi}}{2} + \sum_{\substack{j=1 \\ j\neq i}}^{N} \Big([H_{xy;x} \cdot n_{xi} + H_{yy;x} \cdot n_{yi}] \cdot P^*_{xi}$$

$$+ [H_{xy;y} \cdot n_{xi} + H_{yy;y} \cdot n_{yi}] \cdot P^*_{yi} \Big) \Delta S_i = P_{yi} \quad (4.61)$$

or writing equation (4.61) in the form of equations (1.24) leads to:

$$\frac{1}{2}P^*_{xi} + \sum_{\substack{j=1 \\ j\neq i}}^{N} \left(A_{ij}\, P^*_{xi} + B_{ij} \cdot P^*_{yi} \right) = BV_{xi}$$

$$\frac{1}{2}P^*_{yi} + \sum_{\substack{j=1 \\ j\neq i}}^{N}\left(C_{ij}\ P^*_{xi} + D_{ij}\ P^*_{yi}\right) = BV_{yi}$$

$$\text{for } i=1,2,\ldots N \qquad (4.62)$$

where

$$A_{ij} = H_{xx;x}\ n_{xi} + H_{xy;x}\ n_{yi}$$

$$B_{ij} = H_{xx;y}\ n_{xi} + H_{xy;y}\ n_{yi}$$

$$C_{ij} = H_{xy;x}\ n_{yi} + H_{yy;x}\ n_{yi}$$

$$D_{ij} = H_{xy;y}\ n_{xi} + H_{yy;y}\ n_{yi}$$

Equations (4.62) are a set of 2N linear algebraic equations with 2N unknowns, i.e., P^*_{xi} and P^*_{yi} for i = 1,...N. The methods for obtaining the solution have been discussed in section I.1.

Clearly, from equation (4.61), one can conclude that

$$A_{ij} = \frac{1}{2} \qquad D_{ij} = \frac{1}{2}$$

$$B_{ij} = 0.0 \qquad C_{ij} = 0.0$$

for i = j. In matrix form, equations (4.62) become:

$$\begin{bmatrix} A_{ij} & B_{ij} \\ \\ C_{ij} & D_{ij} \end{bmatrix} \cdot \begin{Bmatrix} P^*_{xi} \\ \\ P^*_{yi} \end{Bmatrix} = \begin{Bmatrix} BV_{xi} \\ \\ BV_{yi} \end{Bmatrix} \quad (4.63)$$

This system of equations can be solved for P^*_{xi} and P^*_{yi} by matrix inversion as follows:

$$\begin{Bmatrix} P^*_{xi} \\ \\ P^*_{yi} \end{Bmatrix} = \begin{bmatrix} A_{ij} & B_{ij} \\ \\ C_{ij} & D_{ij} \end{bmatrix}^{-1} \cdot \begin{Bmatrix} BV_{xi} \\ \\ BV_{yi} \end{Bmatrix} \quad (4.64)$$

or by other methods for solving systems of linear equations (Faddeeva [38]).

Let F be a field point at which the stresses and displacements are to be found. Then, using Appendix C or Appendix D, the stresses and displacements at the field point due to a unit load at a boundary point such as Z_0, for the plane finite region containing either the elliptical hole or the slit, i.e., $H_{ij;q}(F,Z_0)$ and $I_{i;q}(F,Z_0)$, can be found. The known fictitious tractions, i.e., equation (4.64), will now be applied to find the real stresses and displacements at the field point. These stresses and displacements are x

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$$\sigma_{xx} = \sum_{i=1}^{N} [H_{xx;x}(F,Z_0)P^*_{xi} + H_{xx;y}(F,Z_0)P^*_{yi}]$$

$$\sigma_{yy} = \sum_{i=1}^{N} [H_{yy;x}(F,Z_0)P^*_{xi} + H_{yy;y}(F,Z_0)P^*_{yi}]$$

$$\sigma_{xy} = \sum_{i=1}^{N} [H_{xy;x}(F,Z_0) \cdot P^*_{xi} + H_{xy;y}(F,Z_0)\ P^*_{yi}]$$

$$U_x = \sum_{i=1}^{N} [I_{x;x}(F,Z_0) \cdot P^*_{xi} + I_{x;y}(F,Z_0) \cdot P^*_{yi}]$$

$$U_y = \sum_{i=1}^{N} [I_{y;x}(F,Z_0)\ P^*_{xi} + I_{y;y}(F,Z_0)\ P^*_{yi}] \tag{4.65}$$

Some example problems will now be considered. The plane stress or plane strain problem can be considered by choosing the appropriate value for α in the complex potential function. For generalized plane stress:

$$\alpha = \frac{3-\nu}{1+\nu}$$

and for generalized plane strain:

$$\alpha = 3-4\nu$$

EXAMPLE IV.1

A Rectangular Plane Weakened by an Elliptical Hole

Consider the rectangular region (10cm x 20cm) of unit thickness (h = 1cm) which is weakened by an elliptical hole described by

$$x = (1+m) \text{ Cos } \theta$$

$$y = -(1-m) \text{ Sin } \theta$$

$$o<m<1$$

with horizontal major axis 2a = 2(1+m) and minor axis 2b = 2(1-m) at the origin. A uniformly distributed traction (ω = 1.0 MPa) is applied to the top and the bottom of the rectangular region, see Figure 4.9. To obtain different ratios of major to minor axis, M can be chosen between zero and one, in this case M = 0.5, as shown. The boundary has been subdivided into sixty equally-spaced meshes, each of length 1.0cm, i.e., 10 meshes are defined on each of the top and bottom edge and 20 meshes on each vertical edge.

The points where the stress and displacement are calculated, i.e., the field points, are chosen along the major and the minor axes and include points on the edge of the hole. These are also shown in Figure 4.9.

The coordinates of the nodal points, X(I) and Y(I), the resultant of the traction on each subdivision (calculated by the trapezoidal rule), BVx(I) and BVy(I), and the

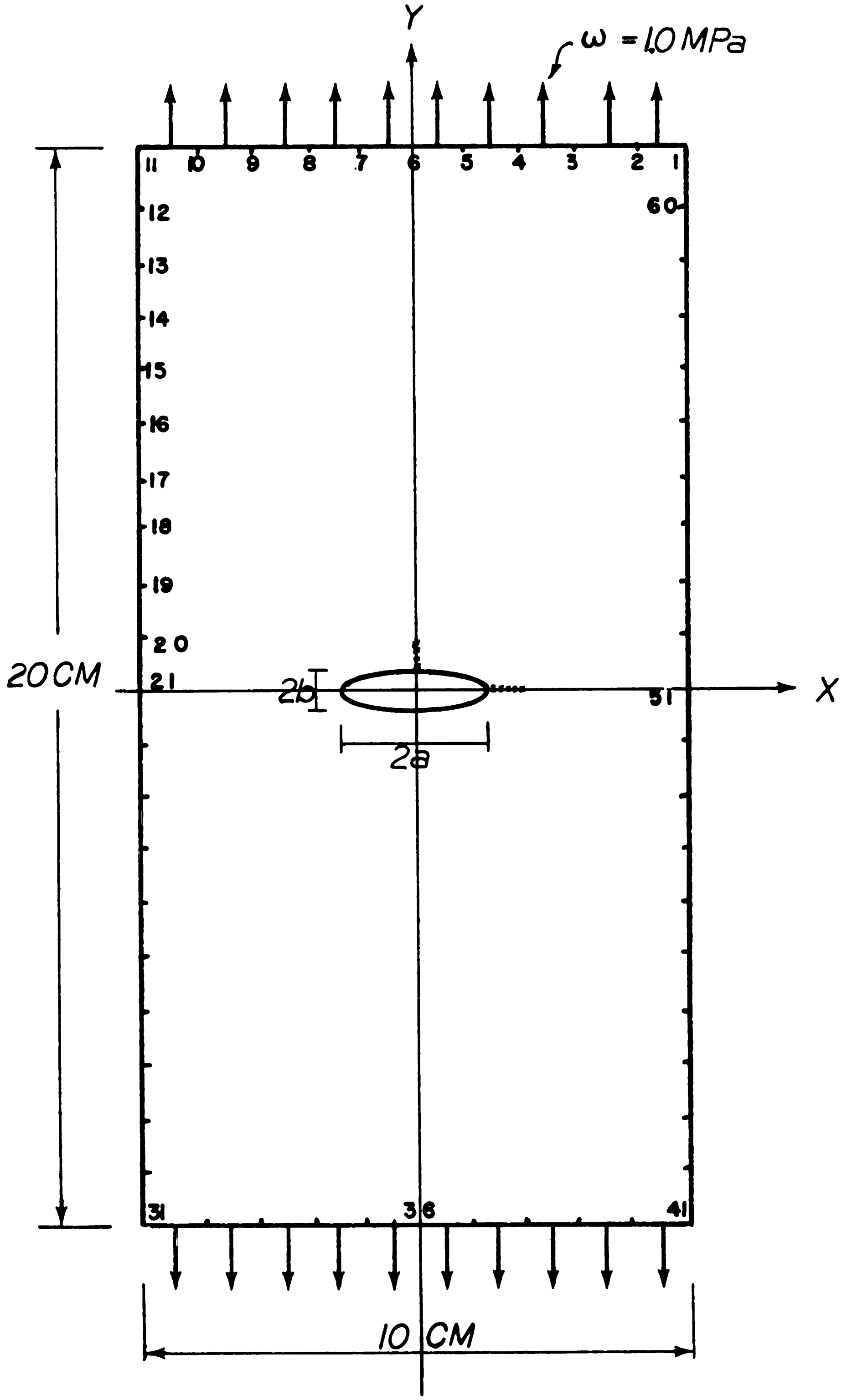


Figure 4.9 Rectangular plane weakened by an elliptical hole at the origin.

coordinates of the field points, XF(I) and YF(I), are read into the program as the data input (Appendix E). The results are presented in Table 4.1.

The results are compared to the theoretical solution of an infinite plane weakened by an elliptical hole subjected to a uniaxial tension [39] and solution of a long strip weakened by an elliptical hole subjected to uniform tension [42]. The program required 41 seconds of CPU time on a CDC 6500 computer.

Two angles of inclination of the ellipse, $\theta = 30°, 60°$, are also considered and these results are presented in Tables 4.2 and 4.3. Note that the rectangular boundary has been embedded in the infinite domain at inclination θ to the "horizontal" ellipse.

The program has been written in such a way that if different angles of inclination are desired, only one character, THETA, is to be changed. Also, different sizes of the ellipse, i.e., different a and b, can be obtained in each case by changing the character, M, in the program. For different locations of the center of the hole, the new coordinates of the center of the hole, Xo,Yo, must be read into the program. The problem of a rectangular plane subjected to uniform load and weakened by an elliptical hole with major axis 2a = 3.6cm and minor axis 2b = 0.4cm centered at Xo = 1.5cm, Yo = 2.0cm and inclined at an angle of $\theta = 30°$ is solved. The results are presented in Table 4.4. Again, the CPU time was 42 seconds for each run on a CDC 6500 computer.

Table 4.1 Stress and displacement of a rectangular plane containing an elliptical hole at the origin

Geometry: rectangle 10 x 20 cm^2 (1 cm thickness)
Load: ω = 1.0 MPa
Eccentricity: $X_0 = 0$ $Y_0 = 0$, Angle: $\theta = 0.0°$
E = 70000 MPa, μ = 26315.79 MPa, ν = 0.33, m = 0.5

Field Point No.	Coordinates X cm	Y cm	σ_{xx} (MPa)	σ_{yy} (MPa)	σ_{xy} (MPa)	U_x microns	U_y microns
1	1.5	0.0	0.0	7.4638	0.0	-0.359	0.0
2	1.7	0.0	1.0963	2.5056	0.0	-0.291	0.0
3	2.3	0.0	0.4496	1.4263	0.0	-0.251	0.0
4	2.8	0.0	0.25124	1.2630	0.0	-0.253	0.0
5	0.0	0.5	-1.1071	0.0	0.0	0.0	0.545
6	0.0	0.8	-0.5836	-0.0139	0.0	0.0	0.547
7	0.0	1.7	-0.0108	0.30356	0.0	0.0	0.564
8	0.0	2.4	0.0566	0.5381	0.0	0.0	0.605
9	-1.5	0.0	0.0	7.4638	0.0	0.359	0.0
10	0.0	-0.5	-1.1071	0.0	0.0	0.0	-0.545

Available Solution:

Field Point No.		Reference
1	σ_{yy}=7.4 MPa	[42]
1	σ_{yy}=7.0 MPa	[39] (infinite plane)
5	σ_{xx}= -1.15 MPa	[42]

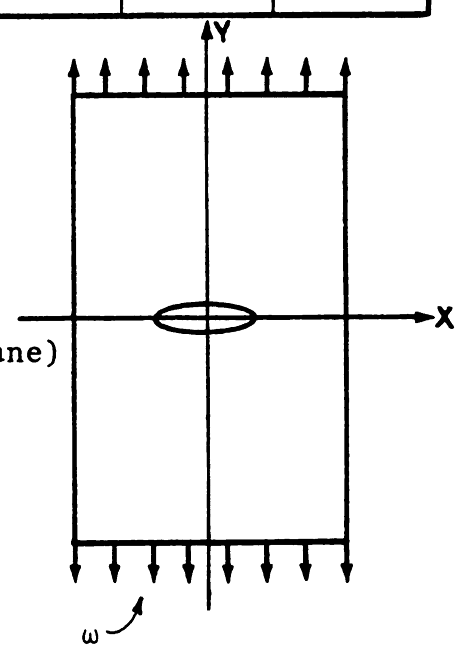


Table 4.2 Rectangular plane containing an elliptical hole (inclined major axis) at the origin, Case 1

Geometry: rectangular plane (10cm x 20cm x 1cm)
Load: ω = 1.0 MPa
Eccentricity: X_0 = 0 Y_0 = 0, Angle: θ = 30°
E = 70000 MPa, μ = 26315.79 MPa, ν = 0.33, m = 0.5

Field Point No.	Coordinates X cm	Coordinates Y cm	σ_{xx} (MPa)	σ_{yy} (MPa)	σ_{xy} (MPa)	U_x microns	U_y microns
1	1.5	0.0	0.0	5.34611	0.0	-0.1923	0.2319
2	1.7	0.0	0.8876	1.8630	0.9115	-0.1345	0.2241
3	1.9	0.0	0.7232	1.3653	0.8148	-0.1120	0.2313
4	2.8	0.0	0.4109	0.9319	0.6010	-0.075	0.295
5	0.0	0.5	-0.4309	0.0	0.0	0.2514	0.3946
6	0.0	0.8	-0.0934	0.0103	0.2562	0.214	0.3905
7	0.0	1.1	0.1047	0.0784	0.4023	0.1983	0.3878
8	0.0	2.4	0.2976	0.4357	0.5113	0.2319	0.4190
9	-1.5	0.0	0.0	5.3461	0.0	0.1923	-0.2319
10	0.0	-0.5	-0.4309	0.0	0.0	-0.2514	-0.3905

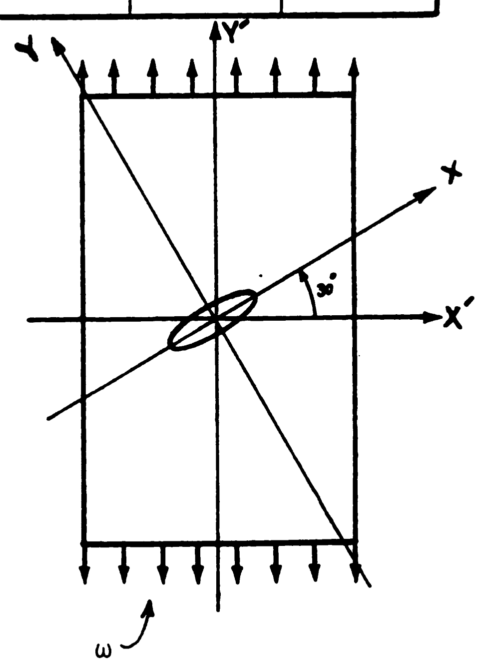


Table 4.3 Rectangular plane containing an elliptical hole (inclined major axis) at the origin, Case 2

Geometry: rectangular plane (10cm x 90cm x 1cm)
Load: ω = 1.0 MPa
Eccentricity: X_0 = 0 Y_0 = 0, Angle: θ = 60°
E = 70000 MPa, μ = 26315.79 MPa, ν = 0.33, m = 0.5

Field Point No.	Coordinates		σ_{xx} (MPa)	σ_{yy} (MPa)	σ_{xy} (MPa)	U_x microns	U_y microns
	X cm	Y cm					
1	1.5	0.0	0.0	1.0337	0.0	0.1539	0.2344
2	1.7	0.0	0.453	0.5588	0.9191	0.1903	0.2263
3	1.9	0.0	0.5660	0.4398	0.8190	0.2086	0.2334
4	2.8	0.0	0.69189	0.30451	0.5902	0.2853	0.2951
5	0.0	0.5	0.97538	0.0	0.0	0.2564	0.0818
6	0.0	0.8	0.9229	0.0593	0.2649	0.2191	0.0653
7	0.0	1.1	0.8810	0.1063	0.4156	0.2031	0.05381
8	0.0	2.4	0.7922	0.2095	0.5315	0.2385	0.0321
9	-1.5	0.0	0.0	1.0337	0.0	-0.1539	-0.2344
10	0.0	-0.5	0.9753	0.0	0.0	-0.2564	-0.0818

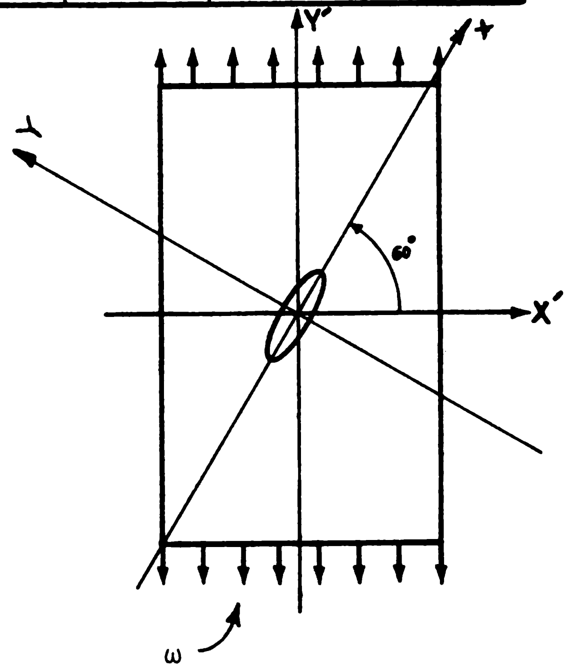
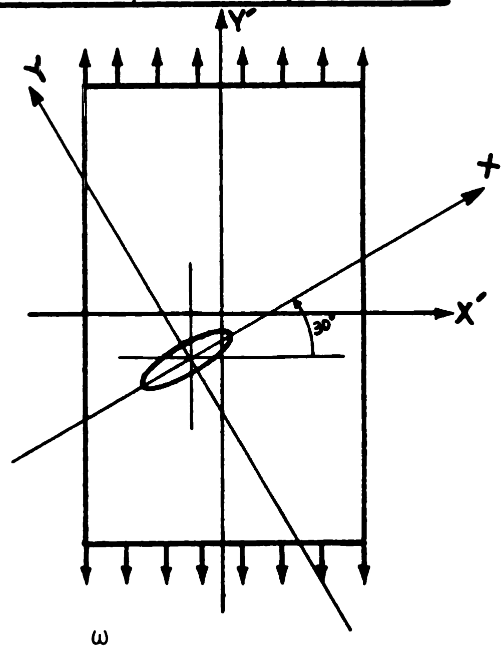


Table 4.4 Rectangular plane containing a nonsymmetrically located elliptical hole (inclined major axis)

Geometry: rectangular plane (9cm x 18cm x 1cm)
Load: ω = 1.0 MPa
Eccentricity: X_0 = 1.5 Y_0 = 2.0, Angle: θ = 30°
E = 70000 MPa, μ = 26315.79 MPa, ν = 0.33, m = 0.8

Field Point No.	Coordinates X cm	Coordinates Y cm	σ_{xx} (MPa)	σ_{yy} (MPa)	σ_{xy} (MPa)	U_x microns	U_y microns
1	1.8	0.0	0.0	17.1905	0.0	0.2434	0.3892
2	1.9	0.0	2.0736	2.9680	1.6635	0.6380	0.5744
3	2.1	0.0	1.1269	1.7636	1.1255	0.7245	0.6193
4	3.2	0.0	0.4972	0.9748	0.6480	0.8071	0.7640
5	3.8	0.0	0.2913	1.2050	0.4385	0.8116	0.8020
6	0.0	0.2	-0.5871	0.0	0.0	1.1892	0.7932
7	0.0	0.25	-0.5355	-0.0015	0.0332	1.1808	0.7930
8	0.0	0.3	-0.4849	-0.0027	0.0656	1.1728	0.7927
9	0.0	0.4	-0.3873	-0.0030	0.1277	1.1585	0.7920
10	0.0	0.7	-0.1350	0.0151	0.2847	1.1261	0.7898



EXAMPLE IV.2

A Circular Plane Weakened by an Elliptical Hole

Let a circular plane of radius R = 6cm and of unit thickness (h = 1cm) be weakened by an elliptical hole at the origin described by:

$$X = (1+m) \cos \theta$$

$$Y = -(1-m) \sin \theta$$

$$o<m<1$$

with horizontal major axis 2a = 2(1+m) and minor axis 2b = 2(1-m), see Figure 4.10. A radially uniform distributed load (ω = 1.0 MPa) is partially applied to the top and the bottom of the outer circumference as shown. Again, M can be chosen between zero and one where in the Figure M = 0.5. The boundary has been subdivided into sixty equally-spaced meshes, each of which covers 6 degrees of angle (0.6283 cm) numbered from the top and counter-clockwise. The field points are chosen along the x,y axis and include points on the edge of the hole. These are also shown in Figure 4.10.

The data, i.e., the coordinates of the nodal points, X(I) and Y(I), the resultant of the traction on each subdivision (calculated by the trapezoidal rule), BVx(I) and BVy(I), and the coordinates of the field points, XF(I) and YF(I), are read into the program (Appendix E). The

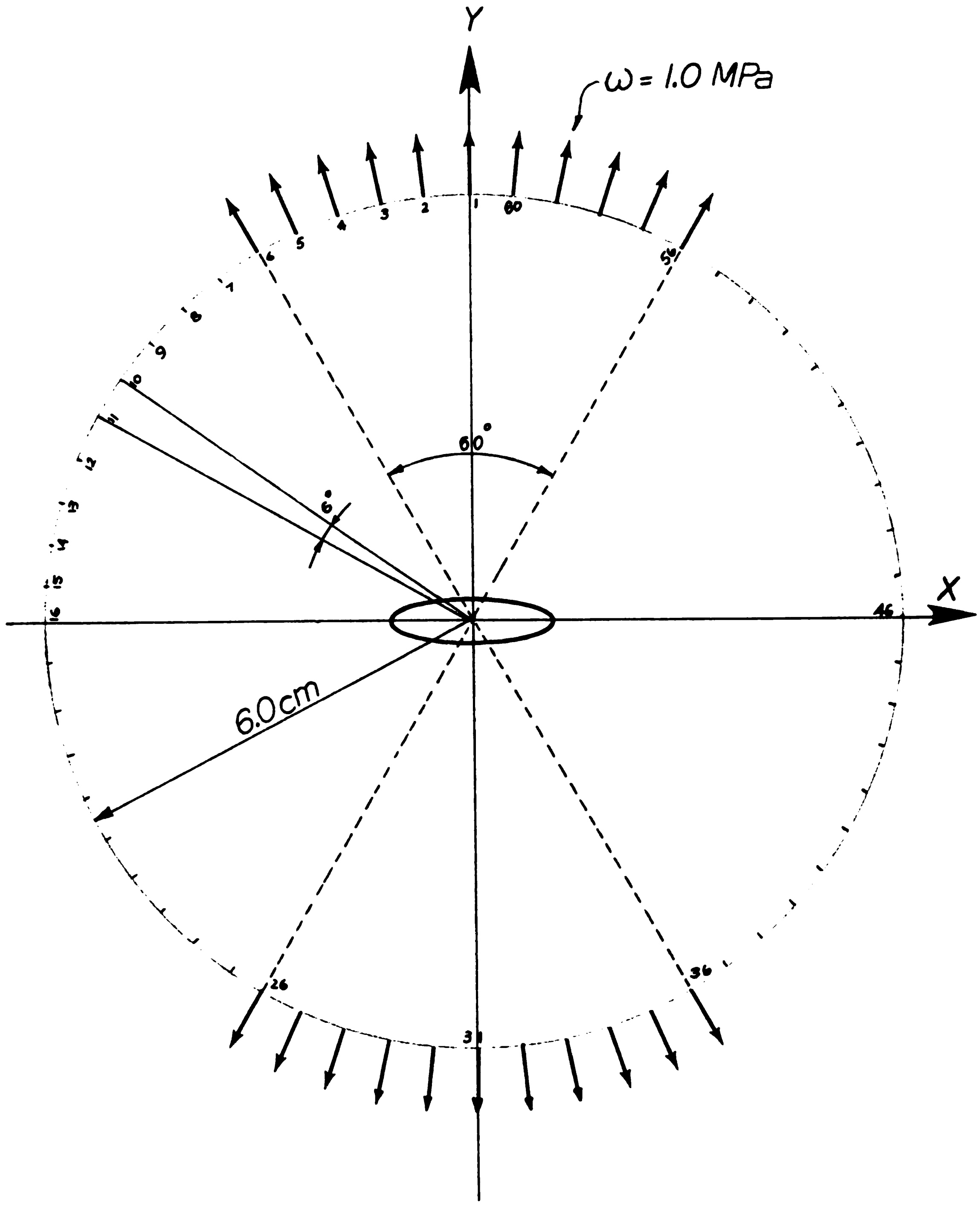


Figure 4.10 Circular plane weakened by an elliptical hole at the origin.

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results are presented in Table 4.5. The program required 42 seconds of CPU time on a CDC 6500 computer.

To see the effect of angle of inclination on the stress and displacement solution, two cases, i.e., the circular plane subjected to the load weakened by the elliptical hole at the origin but rotated around the origin counterclockwise, $\theta = 30°$ and $\theta = 60°$, were considered. The results are presented in Tables 4.6 and 4.7. Again, note that the ellipse is kept horizontal and the outer boundary is rotated clockwise.

As mentioned in Example IV.1, the solution to a problem with different angles of rotation, different hole size and different location of the hole can be obtained by reading the desired characters THETA, M, Xo and Yo into the program. The examples of a circular plane subjected to the given load and weakened by an elliptical hole with major axis 2a = 3.6cm and minor axis 2b = 0.4cm (m = 0.8), Xo = -1.0cm, Yo = 1.5cm and oriented at an angle of inclination $\theta = 30°$ is treated. The results are presented in Table 4.8. Again, the CPU time was 42 seconds for each run on a CDC 6500 computer.

Table 4.5 Stress and displacement of a circular plane containing an elliptical hole at the origin

Geometry: circular plane R = 6cm (thickness 1cm)
Load: ω = 1.0 MPa
Eccentricity: X_0 = 0 Y_0 = 0, Angle: θ = 0.0°
E = 70000 MPa, μ = 26315.79 MPa, ν = 0.33, m = 0.5

Field Point No.	Coordinates X cm	Coordinates Y cm	σ_{xx} (MPa)	σ_{yy} (MPa)	σ_{xy} (MPa)	U_x microns	U_y microns
1	1.5	0.0	0.0	6.5903	0.0	-0.4958	0.0
2	1.7	0.0	0.9213	2.1188	0.0	-0.3756	0.0
3	1.9	0.0	0.6265	1.4786	0.0	-0.3403	0.0
4	2.3	0.0	0.3084	1.0501	0.0	-0.3167	0.0
5	2.8	0.0	0.1449	0.8032	0.0	-0.3130	0.0
6	0.0	0.5	-1.4106	0.0	0.0	0.0	0.5234
7	0.0	0.8	-0.8461	-0.0236	0.0	0.0	0.5206
8	0.0	1.1	-0.5098	0.0539	0.0	0.0	0.5210
9	0.0	1.7	-0.2088	0.3062	0.0	0.0	0.5384
10	0.0	2.4	-0.0681	0.5664	0.0	0.0	0.5858

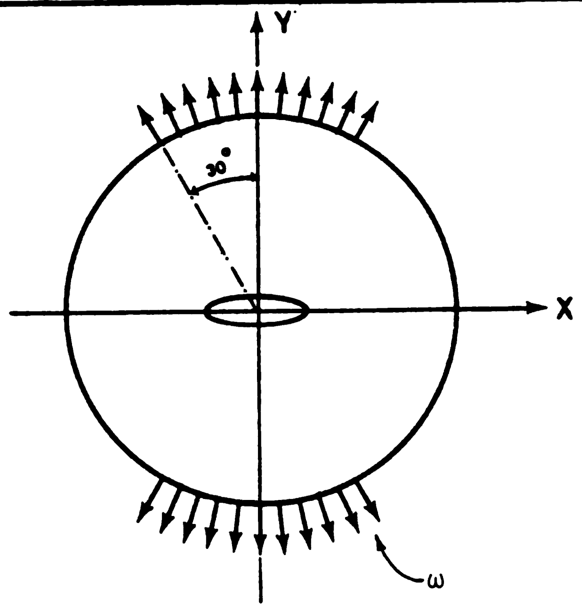

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Table 4.6 Circular plane containing an elliptical hole (inclined major axis) at the origin, Case 1

Geometry: circular plane R = 6cm (thickness 1cm)
Load: ω = 1.0 MPa
Eccentricity: X_0 = 0 Y_0 = 0, Angle: θ = 30°
E = 70000 MPa, μ = 26315.79 MPa, ν = 0.33, m = 0.5

Field Point No.	Coordinates X cm	Coordinates Y cm	σ_{xx} (MPa)	σ_{yy} (MPa)	σ_{xy} (MPa)	U_x microns	U_y microns
1	1.5	0.0	0.0	4.5582	0.0	-0.2725	0.1497
2	1.7	0.0	0.6820	1.5269	0.9271	-0.1899	0.1894
3	1.9	0.0	0.5011	1.0992	0.8071	-0.1651	0.2036
4	2.8	0.0	0.1629	0.6888	0.4720	-0.1423	0.2396
5	0.0	0.5	-0.5996	0.0	0.0	0.2734	0.3448
6	0.0	0.8	-0.2768	-0.0049	0.3027	0.2413	0.3399
7	0.0	1.1	-0.0766	0.0426	0.4676	0.2292	0.3371
8	0.0	2.4	0.2123	0.3145	0.5738	0.2648	0.3604
9	-1.5	0.0	0.0	4.5582	0.0	0.2725	-0.1497
10	0.0	-0.5	-0.5996	0.0	0.0	-0.2734	-0.3448

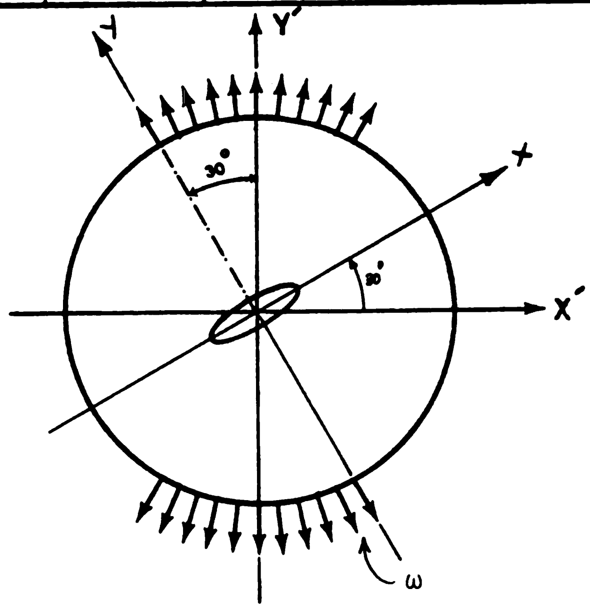


Table 4.7 Circular plane containing an elliptical hole (inclined major axis) at the origin, Case 2

Geometry: circular plane R = 6cm (thickness 1cm) Load: ω = 1.0 MPa Eccentricity: X_0 = 0.0 Y_0 = 0.0, Angle: θ = 60° E = 70000 MPa, μ = 26315.79 MPa, ν = 0.33, m = 0.5							
Field Point No.	Coordinates		σ_{xx} (MPa)	σ_{yy} (MPa)	σ_{xy} (MPa)	U_x microns	U_y microns
	X cm	Y cm					
1	1.5	0.0	0.0	-0.0207	0.0	0.2218	0.1470
2	1.7	0.0	0.222	0.1803	1.0262	0.2121	0.1963
3	1.9	0.0	0.3397	0.1848	0.9116	0.2156	0.2144
4	2.8	0.0	0.4783	0.2323	0.6416	0.2593	0.2692
5	0.0	0.5	0.9728	0.0	0.0	0.2772	-0.0174
6	0.0	0.8	0.8375	0.0466	0.2805	0.2444	-0.0266
7	0.0	1.1	0.7508	0.0657	0.4323	0.2306	-0.0340
8	0.0	2.4	0.5885	0.0416	0.4502	0.2480	-0.0620
9	-1.5	0.0	0.0	-0.0207	0.0	-0.2218	-0.1470
10	0.0	-0.5	0.9728	0.0	0.0	-0.2772	0.0174

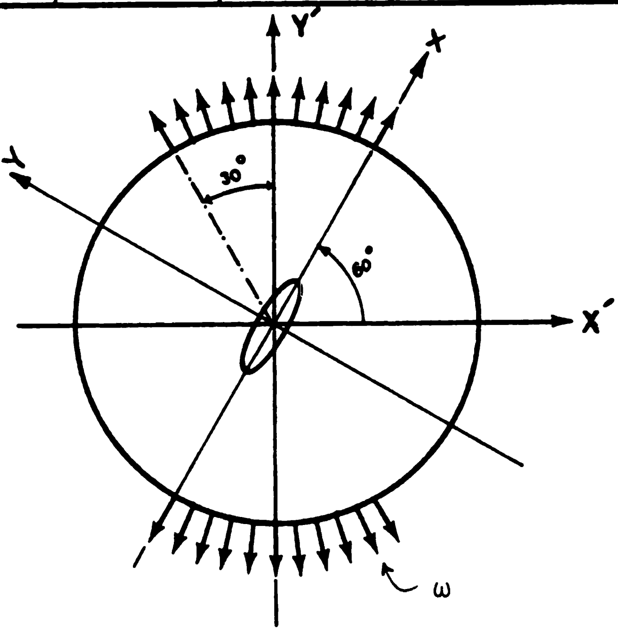
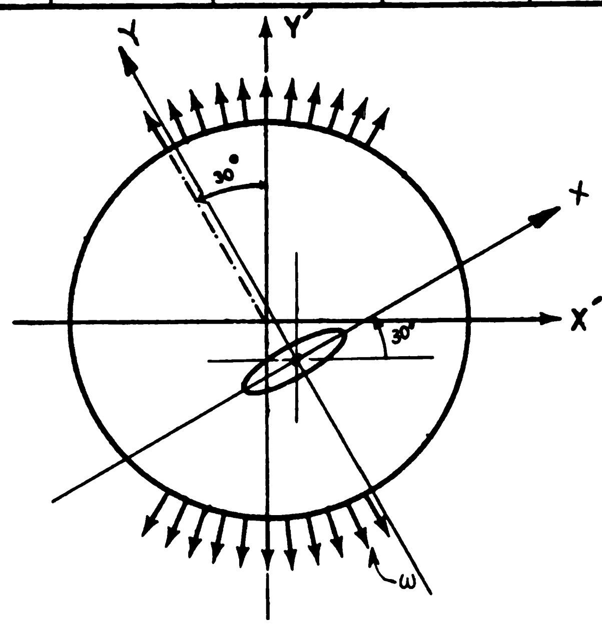


Table 4.8 Circular plane containing an elliptical hole (inclined major axis 30°) nonsymmetrically located

Geometry: circular plane R = 5.4cm (thickness 1cm)
Load: ω = 1.0 MPa
Eccentricity: X_0 = -1.0 Y_0 = 1.5, Angle: θ = 30°
E = 70000 MPa, μ = 26315.79 MPa, ν = 0.33, m = 0.8

Field Point No.	Coordinates		σ_{xx} (MPa)	σ_{yy} (MPa)	σ_{xy} (MPa)	U_x microns	U_y microns
	X cm	Y cm					
1	1.8	0.0	0.0	14.847	0.0	-0.3198	0.5865
2	1.9	0.0	1.7233	2.5786	1.4976	0.1331	0.8280
3	2.1	0.0	0.8951	1.5769	0.9539	0.2231	0.8678
4	2.8	0.0	0.3450	1.0458	0.5191	0.2767	0.8884
5	3.8	0.0	0.1156	0.7727	0.2609	0.2769	0.8414
6	0.0	0.2	-0.5221	0.0	0.0	0.7170	1.0366
7	0.0	0.25	-0.4735	-0.0014	0.0466	0.7079	1.0359
8	0.0	0.3	-0.4258	-0.0023	0.0908	0.6993	1.0353
9	0.0	0.4	-0.3337	-0.0026	0.1719	0.6838	1.0340
10	0.0	0.7	-0.0902	0.0122	0.3583	0.6486	1.0304



EXAMPLE IV.3

A Rectangular Plane Weakened by a Sharp Crack

Consider a rectangular region (10cm x 20cm) of unit thickness (h = 1cm) which is weakened by a sharp crack of length 4cm at the origin. A uniformly distributed load (ω = 1.0 MPa) is applied to the top and the bottom of the region, see Figure 4.11. The boundary has been subdivided into sixty equally-spaced meshes, each of length 1.0cm, i.e., 10 meshes are defined on each of the top and bottom edge and 20 meshes on each vertical edge. The field points are chosen along the x,y axes. Due to the singularities at the tips of the crack, the points ($x = \pm 2$, $Y = 0$) cannot be considered as field points. However, one field point is chosen very close to the tip of the crack to show the trend of the stress distribution. These are also shown in Figure 4.11.

The data, i.e., the coordinates of the nodal points X(I), Y(I), the resultant of the traction on each subdivision (calculated by the trapezoidal rule), BVx(I) and BVy(I) and the coordinates of the field points XF(I) and YF(I) are read into the program (Appendix E). The results are presented in Table 4.9.

The results are compared to the theoretical solution for the stress intensity factor of a sharp crack located in a rectangular plane subjected to a uniform load (Tada [44]). Since this solution [44] is good for points very close to the tips of the crack, just two field points,

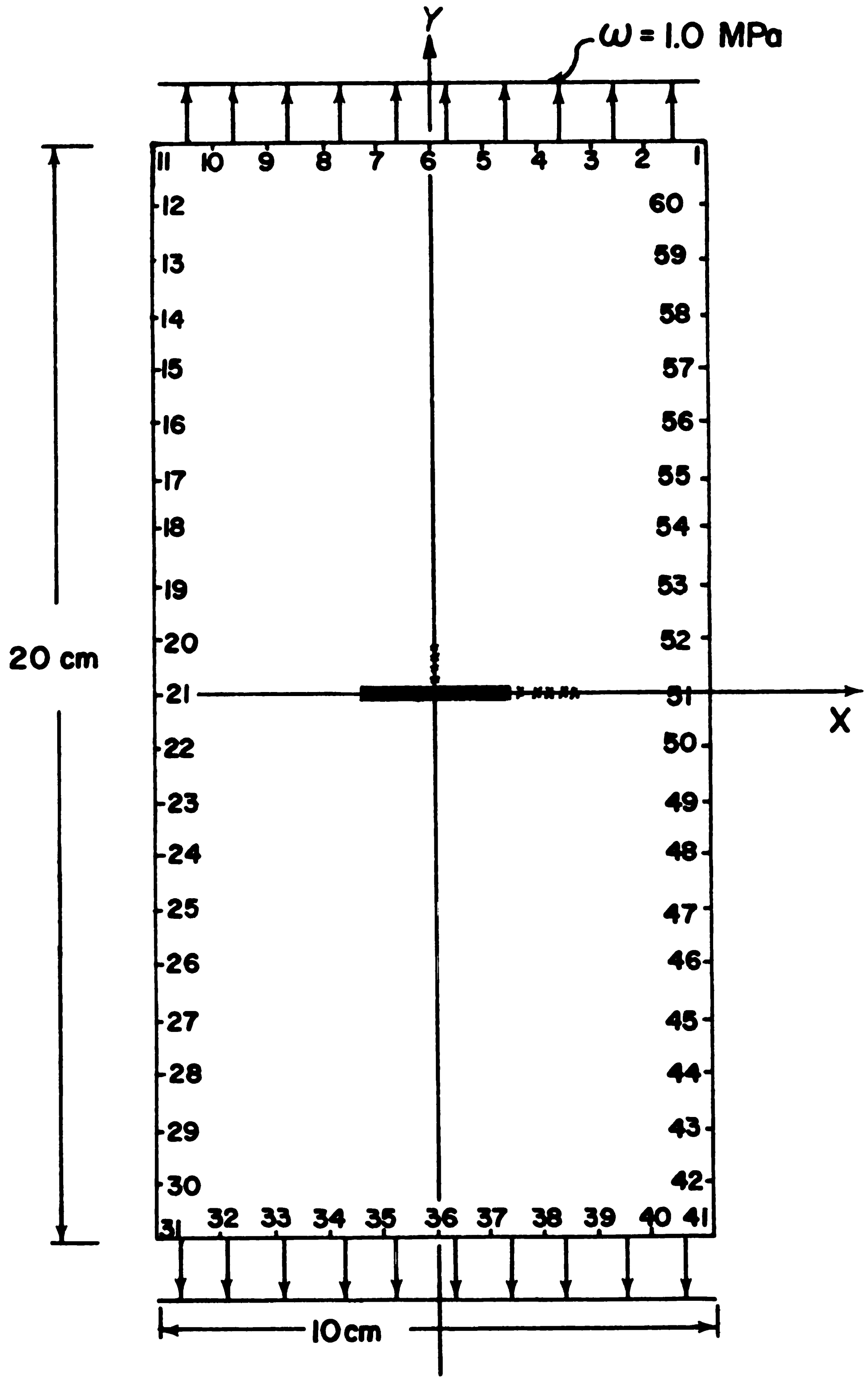


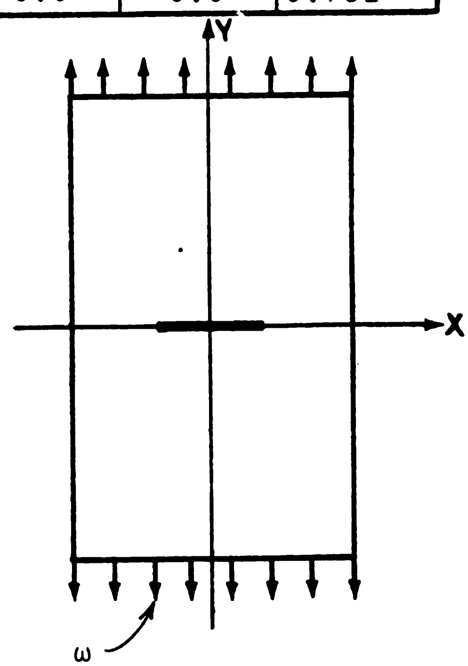
Figure 4.11 Rectangular plane weakened by a sharp crack at the origin.

Table 4.9 Stress and displacement of a rectangular plane containing a sharp crack at the origin

Geometry: rectangular plane (9.56cm x 19.12cm) (thickness 1cm), Load: ω = 1.0 MPa Eccentricity: X_0 = 0 Y_0 = 0, Angle: θ = 0.0° E = 70000 MPa, μ = 26315.79 MPa, ν = 0.33, m = 1.0(crack)							
Field Point No.	Coordinates X cm	Coordinates Y cm	σ_{xx} (MPa)	σ_{yy} (MPa)	σ_{xy} (MPa)	U_x microns	U_y microns
1	2.000000	0.0	1109.14	1110.28	0.0	-68.098	0.0
2	2.001	0.0	33.975	35.123	0.0	-2.395	0.0
3	2.1	0.0	2.492	3.643	0.0	-0.433	0.0
4	3.0	0.0	0.318	1.493	0.0	-0.285	0.0
5	4.0	0.0	0.077	1.250	0.0	-0.311	0.0
6	0.0	$1x10^{-6}$	-1.123	0.0	0.0	0.0	0.6331
7	0.0	0.001	-1.123	$6.2x10^{-9}$	0.0	0.0	0.6332
8	0.0	0.1	-1.013	$7.6x10^{-5}$	0.0	0.0	0.635
9	0.0	1.0	-0.226	0.094	0.0	0.0	0.647
10	0.0	3.0	0.117	0.613	0.0	0.0	0.751

Available Solution:

Field Point No.		Reference
1	$\sigma_{xx}=\sigma_{yy}$=1110 MPa	[44]
2	$\sigma_{xx}=\sigma_{yy}$=35.1 MPa	[44]
7	U_y=0.665 microns	[43]



1 and 2, are compared. Note that error of the solution at the two field points is less than 0.08%. The displacements of the points on and at the middle of the crack (i.e., the crack opening displacement COD) are compared to results obtained by Sharpe [43] for a slot. The program required 41 seconds of CPU time on a CDC 6500 computer.

To see the effect of the inclination of the major axis with respect to the x-axis, i.e., θ counterclockwise, on the stress and displacement solution, two cases, the rectangular plane subjected to the given load weakened by the sharp crack at the origin but rotated with respect to x-axis, $\theta = 30°$ and $\theta = 60°$ counterclockwise, were considered. The results are presented in Tables 4.10 and 4.11 and compared to the theoretical [44] and experimental [43] solutions. Again, note that the sharp crack is kept horizontal and the outer boundary has been rotated clockwise.

The program has been written in such a way that if different inclination angles are desired, only one character, THETA, is to be changed. Also, if different dimensions of the rectangular plane are needed, only one character, WR, is to be changed. Note that the proportionality of the long side to the small side and the length of the crack remain constant and equal to 2.0 and 4.0cm, respectively. For different locations of the crack, the new coordinates of the center of the crack, Xo,Yo, must be read into the program. An example, i.e., the problem of a rectangular plane (9cm x 18cm) subjected to uniform

Table 4.10 Rectangular plane containing an inclined sharp crack at the origin, Case 1

Geometry: rectangular plane (9.56cm x 19.12cm) (thickness 1cm), Load: ω = 1.0 MPa
Eccentricity: X_0 = 0.0 Y_0 = 0.0, Angle: θ = 30°
E = 70000 MPa, μ = 26315.79 MPa, ν = 0.33, m = 1.0 (crack)

Field Point No.	Coordinates X cm	Coordinates Y cm	σ_{xx} (MPa)	σ_{yy} (MPa)	σ_{xy} (MPa)	U_x microns	U_y microns
1	2.000001	0.0	842.01	842.63	450.26	-61.79	-15.49
2	2.001	0.0	26.035	26.65	14.244	-2.066	-0.194
3	2.01	0.0	7.835	8.456	4.521	-0.725	0.141
4	3.0	0.0	0.5026	1.0951	0.6268	-0.120	0.335
5	4.0	0.0	0.3513	0.8925	0.5506	-0.095	0.435
6	0.0	1×10^{-6}	-0.6327	0.0	4.3×10^{-7}	0.2541	0.4845
7	0.0	0.001	-0.6319	2.0×10^{-9}	0.00043	0.2540	0.4845
8	0.0	0.1	-0.5472	0.00012	0.0433	0.2412	0.484
9	0.0	1.0	0.0592	0.0759	0.352	0.1815	0.482
10	0.0	3.0	0.3406	0.4784	0.4721	0.239	0.541

Available Solution:

Field Point No.		Reference
6	U_y=0.448 microns	[43]
6	U_x=0.247 microns	[43]

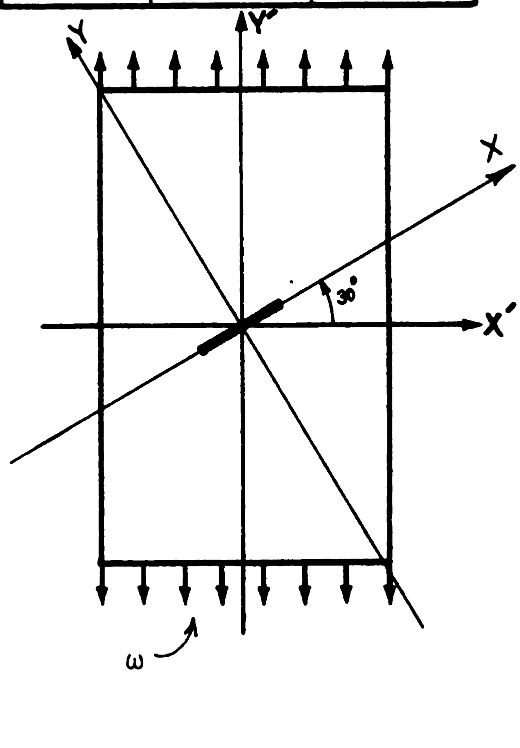


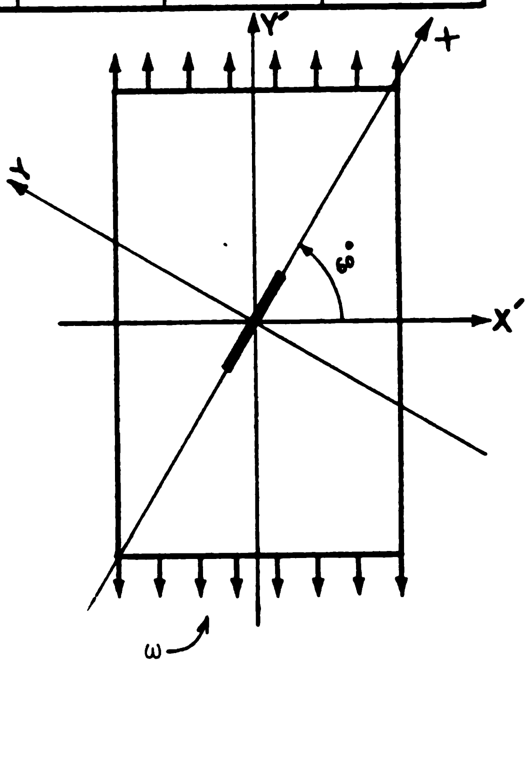
Table 4.11 Rectangular plane containing an inclined sharp crack at the origin, Case 2

Geometry: rectangular plane (9.56cm x 19.12cm) (thickness 1cm), Load: ω = 1.0 MPa
Eccentricity: X_0 = 0.0 Y_0 = 0.0, Angle: θ = 60°
E = 70000 MPa, μ = 26315.79 MPa, ν = 0.33, m = 1.0 (crack)

Field Point No.	Coordinates cm	cm	σ_{xx} (MPa)	σ_{yy} (MPa)	σ_{xy} (MPa)	U_x microns	U_y microns
1	2.000001	0.0	284.97	284.51	465.15	-45.68	-17.04
2	2.001	0.0	9.454	9.00	14.714	-1.280	-0.238
3	2.01	0.0	3.308	2.853	4.668	-0.287	0.1316
4	2.5	0.0	0.9245	0.4570	0.7719	0.1592	0.2894
5	4.0	0.0	0.7845	0.2832	0.5208	0.3365	0.4268
6	0.0	$1x10^{-6}$	0.4269	0.0	$4.6x10^{-7}$	0.2262	0.1665
7	0.0	0.001	0.4272	$7.7x10^{-9}$	$4.6x10^{-4}$	0.2661	0.1665
8	0.0	0.1	0.4573	$1.09x10^{-4}$	0.0465	0.2529	0.1662
9	0.0	1.0	0.6682	0.0303	0.3736	0.1927	0.1283
10	0.0	3.0	0.7751	0.1859	0.4931	0.2525	0.0916

Available Solution:

Field Point No.		Reference
6	U_x=0.2380 microns	[43]
6	U_y=0.1756 microns	[43]



load and weakened by a sharp crack which is inclined at an angle of $\theta = 30°$ counterclockwise and centered at $Xo = +1.0$, $Yo = 2.0$, is solved. The results are presented in Table 4.12.

The method can be applied quite simply to edge-crack problems, with any angle of inclination of the crack. An example of this extension, i.e., the problem of a rectangular plane subjected to the uniform load weakened by a crack of length 4cm at the left-hand side of the boundary, is solved by introducing the new coordinates of the crack "center" ($Xo = 3.0$, $Yo = 0$). The results are presented in Table 4.13. Again, the CPU time was 42 seconds for each run on a CDC 6500 computer.

Table 4.12 Rectangular plane containing an inclined non-symmetrically located sharp crack

Geometry: rectangular plane (9cm x 18cm) (thickness 1cm)
Load: ω = 1.0 MPa
Eccentricity: X_0 = 1.0 Y_0 = 1.5, Angle: θ = 30°
E = 70000 MPa, μ = 26315.79 MPa, ν = 0.33, m = 1.0 (crack)

Field Point No.	Coordinates X cm	Coordinates Y cm	σ_{xx} (MPa)	σ_{yy} (MPa)	σ_{xy} (MPa)	U_x microns	U_y microns
1	2.000001	0.0	964.431	905.139	485.172	-66.05	-25.73
2	2.001	0.0	27.925	28.632	15.350	-1.397	-0.2336
3	2.01	0.0	8.375	9.082	4.876	0.0539	0.328
4	2.5	0.0	0.7970	1.4687	0.8609	0.669	0.574
5	4.0	0.0	0.3550	0.9974	0.5072	0.7326	0.7586
6	0.0	$1x10^{-6}$	-0.6458	0.0	$4.6x10^{-7}$	1.1070	0.739
7	0.0	0.001	-0.6450	$-7.9x10^{-9}$	$4.6x10^{-4}$	1.1068	0.7397
8	0.0	0.1	-0.5608	$2.7x10^{-5}$	0.0467	1.0923	0.739
9	0.0	1.0	0.0498	0.0709	0.3730	1.01929	0.737
10	0.0	3.0	0.3250	0.4696	0.4880	1.0466	0.794

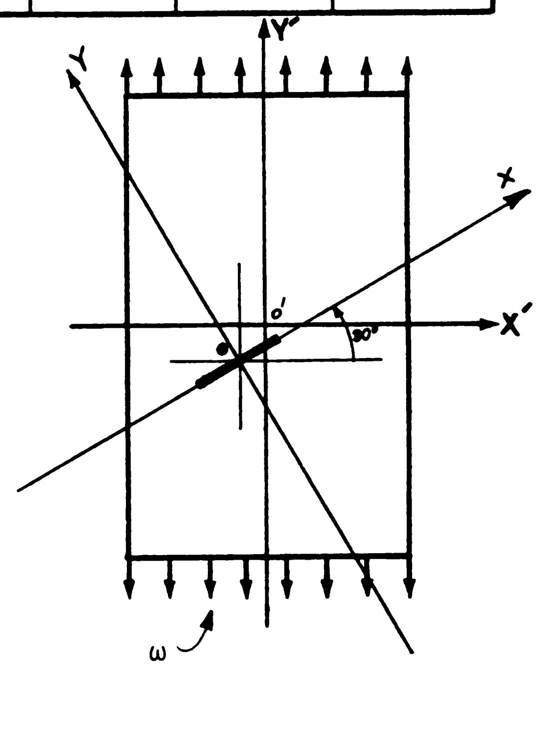
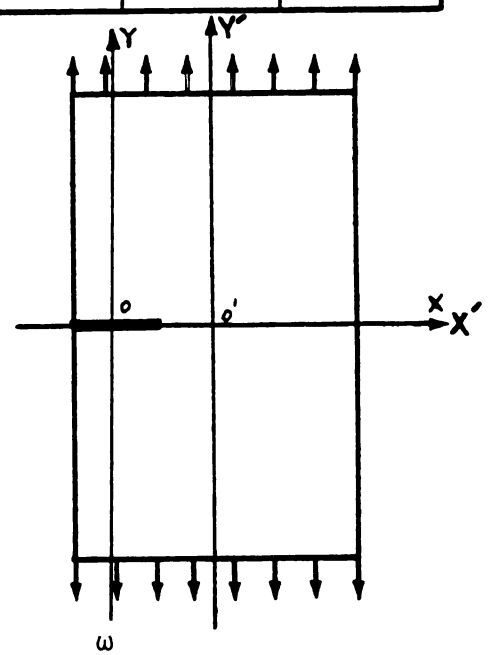


Table 4.13 Sharp crack (notch) on the side of a rectangular plane

Field Point No.	Coordinates X cm	Coordinates Y cm	σ_{xx} (MPa)	σ_{yy} (MPa)	σ_{xy} (MPa)	U_x microns	U_y microns
Geometry: rectangular plane (10cm x 20cm) (thickness 1cm) Load: ω = 1.0 MPa Eccentricity: X_0 = -4.0 Y_0 = 0.0, Angle: θ = 0.0° E = 70000 MPa, μ = 26315.79 MPa, ν = 0.33, m = 1.0 (crack)							
1	2.000001	0.0	1742.69	1743.61	0.0	-138.77	0.0
2	2.001	0.0	54.229	55.143	0.0	-27.56	0.0
3	2.01	0.0	16.540	17.454	0.0	-25.06	0.0
4	3.0	0.0	0.9894	1.8964	0.0	-23.87	0.0
5	4.0	0.0	0.5176	1.4099	0.0	-23.816	0.0
6	0.0	1×10^{-6}	-1.1361	0.0	-4.1×10^{-7}	-23.72	1.2867
7	0.0	0.001	-1.1349	-2.7×10^{-7}	-0.00041	-23.726	1.2867
8	0.0	0.1	-1.0069	-0.0023	-0.0391	23.709	1.2851
9	0.0	3.0	0.07307	0.5705	-0.1638	-23.19	1.3451
10	0.0	8.0	-0.0679	1.0586	0.0401	-22.60	2.004



CHAPTER V

ON THE PROBLEM OF AN ARBITRARILY-SHAPED HOLE IN A TWO-DIMENSIONAL REGION

V.1 INTRODUCTION

Distribution of stresses around an arbitrarily-shaped hole in an infinite elastic region was first solved by Sokolov [45] and, later, in a slightly different formulation, by Savin [46]. As an extension of the mapping technique and the integral equation method, the problem of a plane finite region weakened by an arbitrarily-shaped hole is considered in this chapter.

In section 2, the equation of the contour of any arbitrarily-shaped cavity is discussed. The mapping function is then introduced for a class of contours and the inverse transformation function is determined using the power series expansion and continued fractions methods developed by Frame [47,48]. In section 3, the influence function for openings with three axes of symmetry is discussed. In section 4, the influence function for openings with two axes of symmetry is discussed. The implementation of the boundary integral equation method is also discussed for solution of any finite two-dimensional region containing openings of this type. Finally, in the last section, the influence function for a more general class of openings is discussed.

V.2 THE CONTOUR OF AN ARBITRARILY-SHAPED HOLE AND THE MAPPING FUNCTION

A large class of smooth closed curves, e.g., triangular square or rectangular, can be written in a general Fourier series form (Lekhnitskii [50]):

$$X = R\left\{\mathrm{Cos}\ \theta + \varepsilon \sum_{n=1}^{N} (d_n\ \mathrm{Cos}\ n\ \theta + h_n\ \mathrm{Sin}\ n\ \theta)\right\}$$

$$Y = R\left\{C\ \mathrm{Sin}\ \theta + \varepsilon \sum_{n=1}^{N} (-d_n\ \mathrm{Sin}\ n\ \theta + h_n\ \mathrm{Cos}\ n\ \theta)\right\} \tag{5.1}$$

Clearly, when $\varepsilon = 0$, equation (5.1) represents an ellipse and when $C = 1$, the equation represents a circle.

An infinite plane with an opening represented by equation (5.1) can be conformally transformed to a unit circular disc, in the ζ plane. The transformation function is

$$Z = \omega(\zeta) = R\left\{\frac{1+c}{2}\,\frac{1}{\zeta} + \frac{1-C}{2}\cdot\zeta + \varepsilon\ P(\zeta)\right\} \tag{5.2}$$

where

$$P_{(\zeta)} = \sum_{n=1}^{N} (d_n - ih_n)\zeta^n$$

In order to make the transformation single-valued, one-to-one, and conformal, it is necessary that $\omega'(\zeta)=0$ for all the points inside the unit circle (Churchill [37]). Thus, all the roots of the equation

$$-\frac{1+C}{2}\frac{1}{\zeta^2}+\frac{1-c}{2}+\varepsilon\sum_{n=1}^{N} n(d_n-ih_n)\zeta^{n-1}=0 \qquad (5.3)$$

should be expressed on the planes by points located outside of the unit circle $|\zeta|=1$. Hence, the coefficients a_n, b_n and parameter ε have to be chosen such that the conformal condition of equation (5.3) is satisfied.

To present an example of smooth closed curves, equation (5.1), let a special case of the equation be considered. Consider the contour given by the equations

$$X = R\ (\text{Cos}\ \theta + \varepsilon\ \text{Cos}\ N\ \theta)$$

$$Y = R\ (c\ \text{Sin}\ \theta - \varepsilon\ \text{Sin}\ N\ \theta) \qquad (5.4)$$

where $o<c\leqslant 1$, and N is an integer. When C = 1 and N = 2 the opening has three axes of symmetry and, with an appropriate selection of parameter ε, the opening will differ little from an equilateral triangle with rounded corners, see Figure 5.1. When $c<1$ and N = 2 the opening will be a branched slot, see Figure 5.2.

When c = 1 and N = 3 there are four axes of symmetry and, at some values of ε, the opening will differ little from a square with rounded corners, see Figure 5.3. When $c<1$ and N = 3, an oval of a special type is obtained. If c and ε are taken very small, the opening will be a slot, see Figure 5.4. Also for elliptical case ($\varepsilon=0, c\neq 1$), see Figure 5.4. The computer programs for plotting Figures 5.1 to 5.4 are presented in Appendix F.

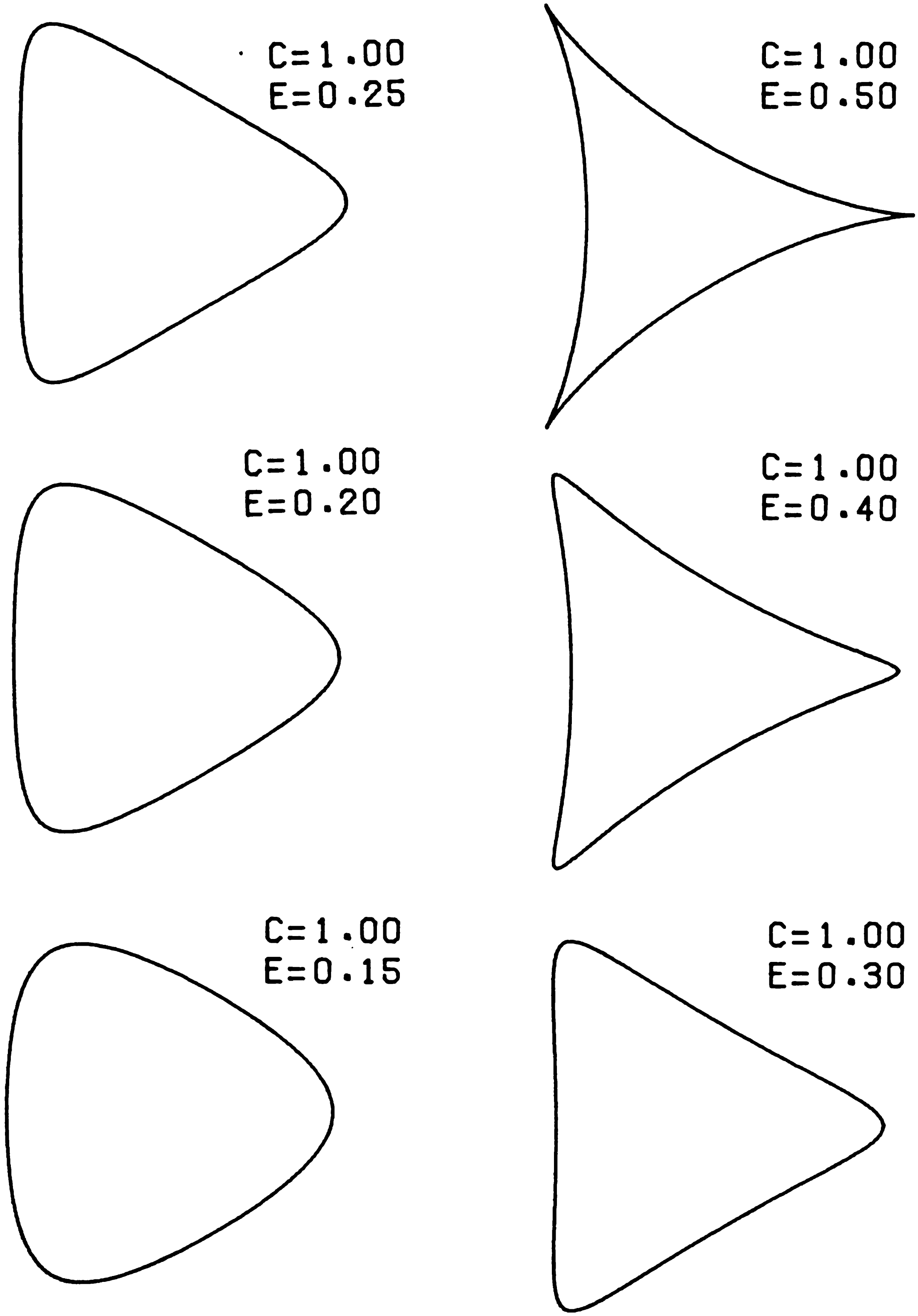


Figure 5.1 Different contours for N = 2 and c = 1.

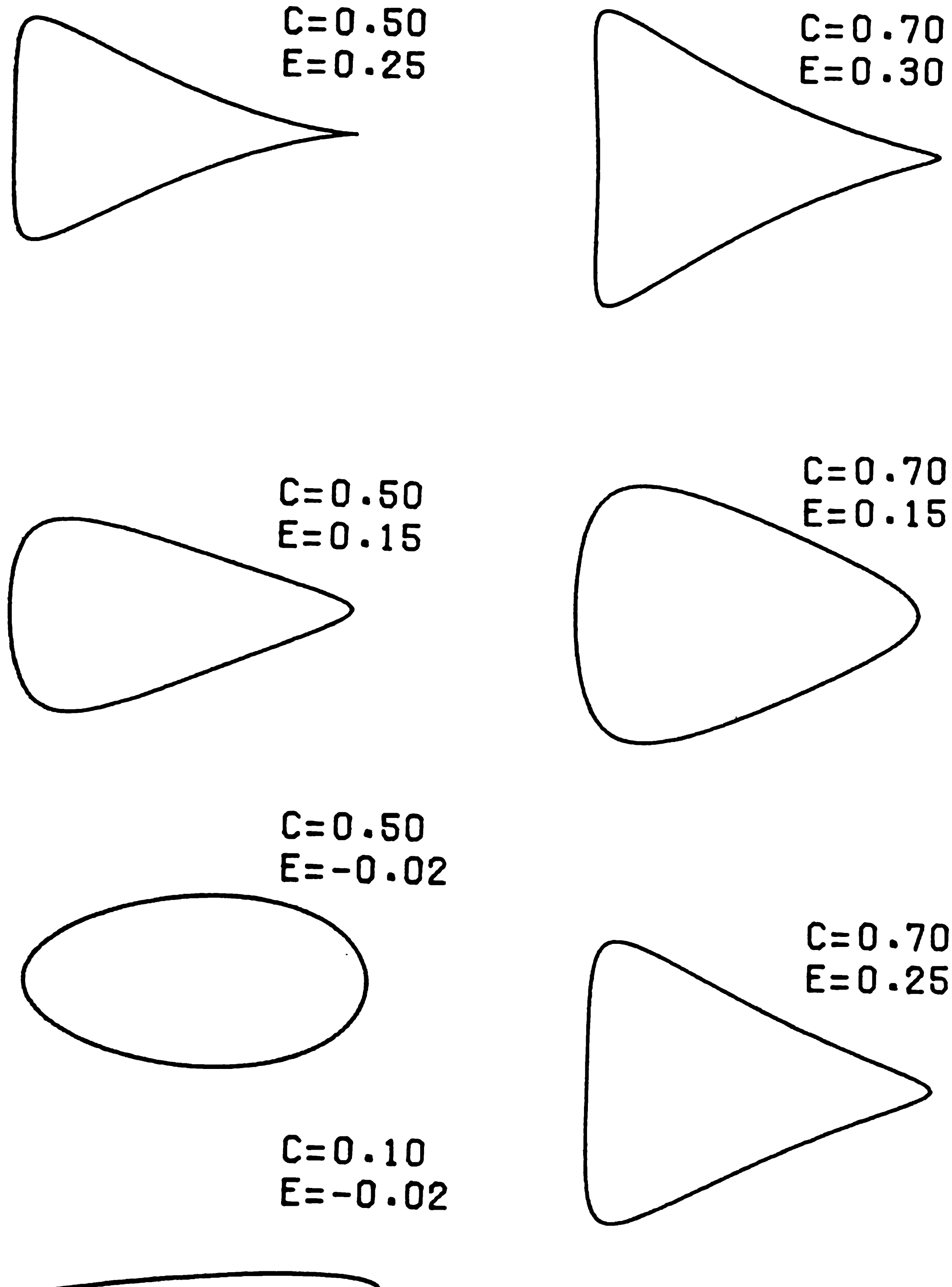


Figure 5.2 Different contours for N = 2 and $o<c<1$.

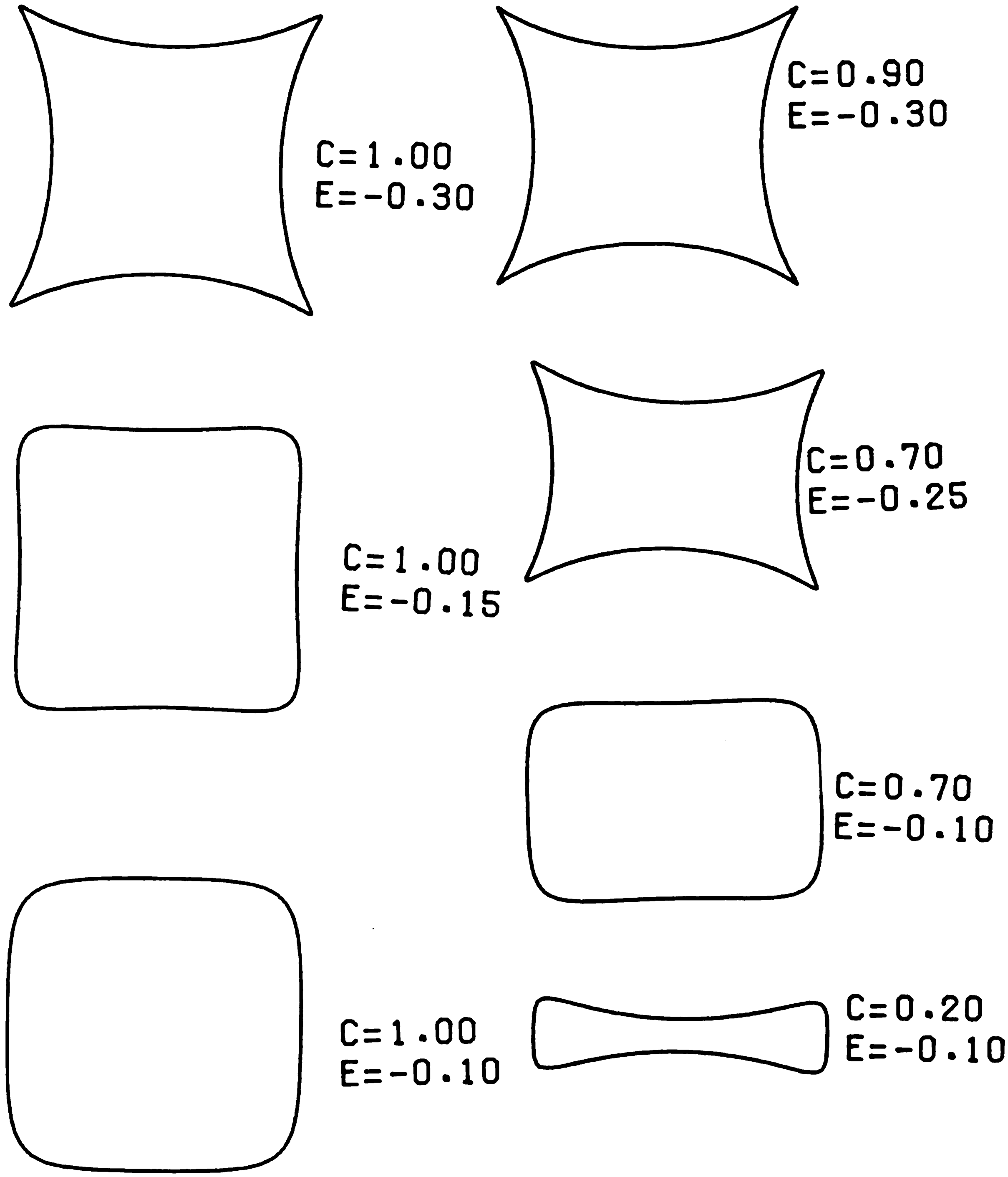


Figure 5.3 Different contours for N = 3 and c =1.

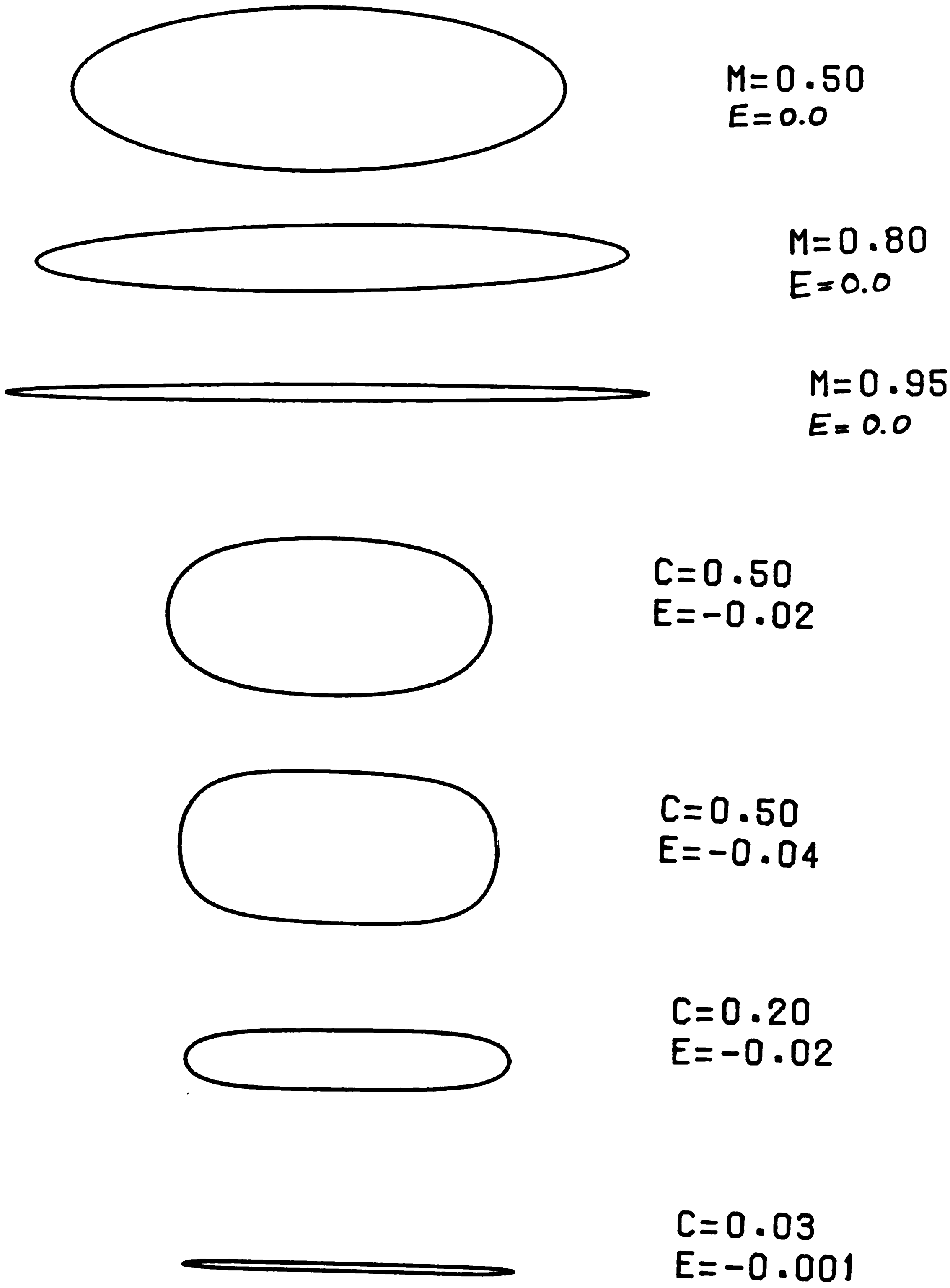


Figure 5.4 Different contours for N = 3, $c \neq 1$ and $\varepsilon = 0$.

The transformation function which includes these special cases of equation (5.4) is

$$Z = \omega(\zeta) = R\left\{ \frac{1+c}{2} \cdot \frac{1}{\zeta} + \frac{1-c}{2}\,\zeta + \varepsilon\zeta^N \right\} \qquad (5.5)$$

In order to find points in the ζ plane corresponding to points in the Z plane, the inverse of the transformation function, i.e., $\omega^{-1}(\zeta)$, is needed. In most cases, e.g., $N>2$ in equation (5.5) or $n>2$ in equation (5.2), it is very cumbersome to find the inverse of the transformation function even though some numerical technique could be employed, but the following two methods appear to be the most powerful methods for determining the inverse transformation function of equation (5.2).

Method 1: Power Series Expansions for the Inverse Transformation Function

The transformation function of equation (5.2) is written in the following form:

$$Z = \frac{R(1+c)}{2} \cdot \zeta^{-1}\left\{ 1 + \frac{1-c}{1+c}\,\zeta^2 + \frac{2\varepsilon}{1+c} \sum_{n=1}^{N} (d_n - ih_n)\zeta^{n+1} \right\} \qquad (5.6)$$

and letting

$$W = \frac{R(1+c)}{2Z}$$

equation (5.6) becomes

$$W^{-1} = \zeta^{-1}\left\{ 1 + \sum_{k=1}^{\infty} \alpha_k \zeta^k \right\} \qquad (5.7)$$

where

$$\alpha_1 = 0$$

$$\alpha_2 = \frac{2\varepsilon}{1+c}(d_1 - ih_1) + \frac{1-c}{1+c}$$

$$\alpha_k = \frac{2\varepsilon}{1+c}(d_{k-1} - ih_{k-1}) \quad \text{for } k=1,3,4....N$$

$$\alpha_k = 0 \quad \text{for } k>N$$

Equation (5.7) is a special case of:

$$W^P = \zeta^P \left\{ 1 + \sum_{k=1}^{\infty} \alpha_k Z^{kv} \right\}$$

for which the inverse function is given by Frame [47]:

$$\zeta^q = W^q \left\{ 1 + \sum_{k=1}^{\infty} \beta_k W^{kv} \right\}$$

where

$$\beta_k = \frac{q}{kv+q} \sum_{r=1}^{k} \binom{-(kv+q)/P}{n} \alpha_k^{(n)} \tag{5.8}$$

Note that $\binom{-(kv+q)/P}{n}$ is the binomial coefficient. The $\alpha_k^{(n)}$ are homogeneous polynomials of degree n in $a_1.....a_k$ defined implicitly by:

$$\left[\sum_{k=1}^{\infty} \alpha_k \zeta^k \right]^n = \sum_{k=1}^{\infty} \alpha_k^{(n)} \zeta^{kv}$$

and defined explicitly as the sum of all ordered products of r factors in which the sum of subscripts is k. For example, for k = 6, r = 3 the coefficient $\alpha_k^{(r)}$ is

$$\alpha_6^{(3)} = 3\alpha_1^2\alpha_4 + 6\alpha_1\alpha_2\alpha_3 + \alpha_2^3$$

Thus, in this case, the inverse function of equation (5.7) will be

$$\zeta = W\left\{1 + \sum_{k=1}^{\infty} \beta_k W^{kv}\right\}$$

where p = -1, q = 1 and v = 1. Hence, β_k can be found by equation (5.8). An example using this method is presented in the next section of this chapter.

Method 2: Continued Fractions

Writing equation (5.2) as

$$Z\zeta = R\left\{\frac{1+c}{2} + \frac{1-c}{2}\zeta^2 + \varepsilon\sum_{n=1}^{N}(d_n - ih_n)\zeta^{n+1}\right\}$$

or

$$\varepsilon(d_N - ih_N)\zeta^{N+1} + \varepsilon(d_{N-1} - ih_{N-1})\zeta^N + \ldots\ldots\ldots$$

$$+ \frac{1-c}{2} + \varepsilon(d_1 - ih_1)\zeta^2 \quad - Z\zeta + \frac{R(1+c)}{2} = 0$$

then this equation can be written in the following form:

$$f(\zeta) = \zeta^m + \alpha_{m-1}\zeta^{m-1} + \dots\dots\dots$$

$$+ \alpha_1\zeta + \alpha_0 = 0 \tag{5.9}$$

where the α_i's can be easily found by comparing the coefficients.

Since the parameters and coefficients in equation (5.2) have been chosen in such a way that the transformation function is conformal, then the polynomial $f(\zeta)$, equation (5.9), has one root inside the unit circle and m-1 roots outside the unit circle, provided that $f(\zeta)$ has many continuous derivatives in a neighborhood of the root inside the unit circle.

$$\eta = -\frac{f(\rho)}{f'(\rho)}\ ,\ \xi = \frac{f''(\rho)}{2f'(\rho)}\ ,\ \tau = \frac{f'''(\rho)}{3f'(\rho)}\ ,\ \delta = \frac{f''''(\rho)}{4f'''(\rho)}\ \dots\dots$$

Then, upon expanding the difference between the required root, ζ, and the estimated root, ρ, with partial numerators d_k, denominators 1, and remainder g_k, the required root can be written as [48]:

$$\zeta = \rho + \cfrac{d_1}{1+\cfrac{d_2}{1+\cfrac{\dots}{1+\cfrac{d_k}{1+g_{k+1}}}}} = \frac{P_k + g_{k+1}P_{k-1}}{Q_k + g_{k+1}Q_{k-1}}$$

where

$$g_k = \frac{d_k}{1+g_{k+1}}$$

The partial numerators d_k in the continued fraction may be determined by means of the series expansion of the difference as a power series in η

$$\zeta - \rho = \sum_{k=0}^{\infty} C_k \eta^k$$

in which C_k is a certain rational function of the first k-1 of the quantities ξ, τ, δ, ... evaluated at the chosen first estimate ρ. The first few partial numerators may be computed as follows:

$$d_1 = \eta \quad , \qquad d_2 = \eta\xi \quad ,$$

$$d_3 = \eta\xi - \gamma\eta \quad , \quad d_4 = \eta\xi - \frac{\eta\tau(\tau-\delta)}{\xi-\gamma}$$

Thus, an explicit form for $\zeta = P(Z)$ will be obtained. An example using this method is presented in the next section of this chapter.

V.3 ON THE INFLUENCE FUNCTION OF A PARTICULAR CLASS OF OPENING, CASE 1

The general formulation of the transformation function for any smooth closed contour is presented in the previous section. The influence functions for an infinite plane containing such an arbitrarily-shaped hole will be obtained by expressing the problem as the superposition of two problems, Figure 2.2. Transforming the second problem into a unit circle, Figure 2.3, and following the general solution presented in Chapter II, the influence function can be obtained.

Thus, to find the influence function, one has to have a specific opening, i.e., a specific equation and transformation function. For example, consider an infinite plane bounded by a contour which is given by the equation:

$$X = R(\text{Cos } \theta + \varepsilon \text{ Cos } 2\theta)$$

$$Y = R(c \text{ Sin } \theta - \varepsilon \text{ Sin } 2\theta) \qquad (5.10)$$

where $0<c\leq 1$. The equation represents a contour with three axes of symmetry which will differ little from an equilateral triangle with rounded corners. By choosing the right $c<1$ and small ε one can obtain a branched slot. The transformation function which transfers the region into the unit circle is

$$Z = \omega(\zeta) = R\left(\frac{1+c}{2} \cdot \frac{1}{\zeta} + \frac{1-c}{2}\zeta + \varepsilon\zeta^2\right) \qquad (5.11)$$

where c and ε are fixed constants. Then letting

$$m = \frac{R(1+c)}{2}$$

$$\ell = \frac{R(1-c)}{2} \qquad \text{and } r = R\cdot\varepsilon$$

the transformation function will be:

$$Z = \omega(\zeta) = \frac{m}{\zeta} + \ell\zeta + r\zeta^2 \tag{5.12}$$

The inverse transformation $\omega^{-1}(\zeta)$ can be easily found by solving the cubic equation [49]:

$$r\zeta^3 + \ell\zeta^2 - Z\zeta + m = 0 \tag{5.13}$$

Clearly, since the transformation is conformal, then one of the roots of equation (5.13) has to be inside the unit circle and the other two have to be outside the unit circle. A concentrated point force P is acting in the plane at some point Z_0 where Z_0 lies on or outside of the opening, i.e.,

$$X_0 \geqslant R(\text{Cos } \theta + \quad \text{Cos } 2\theta)$$

$$Y_0 \geqslant R(c \text{ Sin } \theta - \varepsilon \text{ Sin } 2\theta)$$

where

$$Z_0 = X_0 + iY_0$$

The problem can now be expressed as the superposition of two problems, see Figure 5.5. The problem of Figure 5.5(B) is that of a concentrated point force P applied at Z_0 in an infinite region and the problem of Figure 5.5(C) is that of an infinite region containing the hole with specified traction on the hole. This applied traction can be found following section II.2.

The solution of the problem of Figure 5.5(C) can be obtained using the mapping technique, i.e., transforming

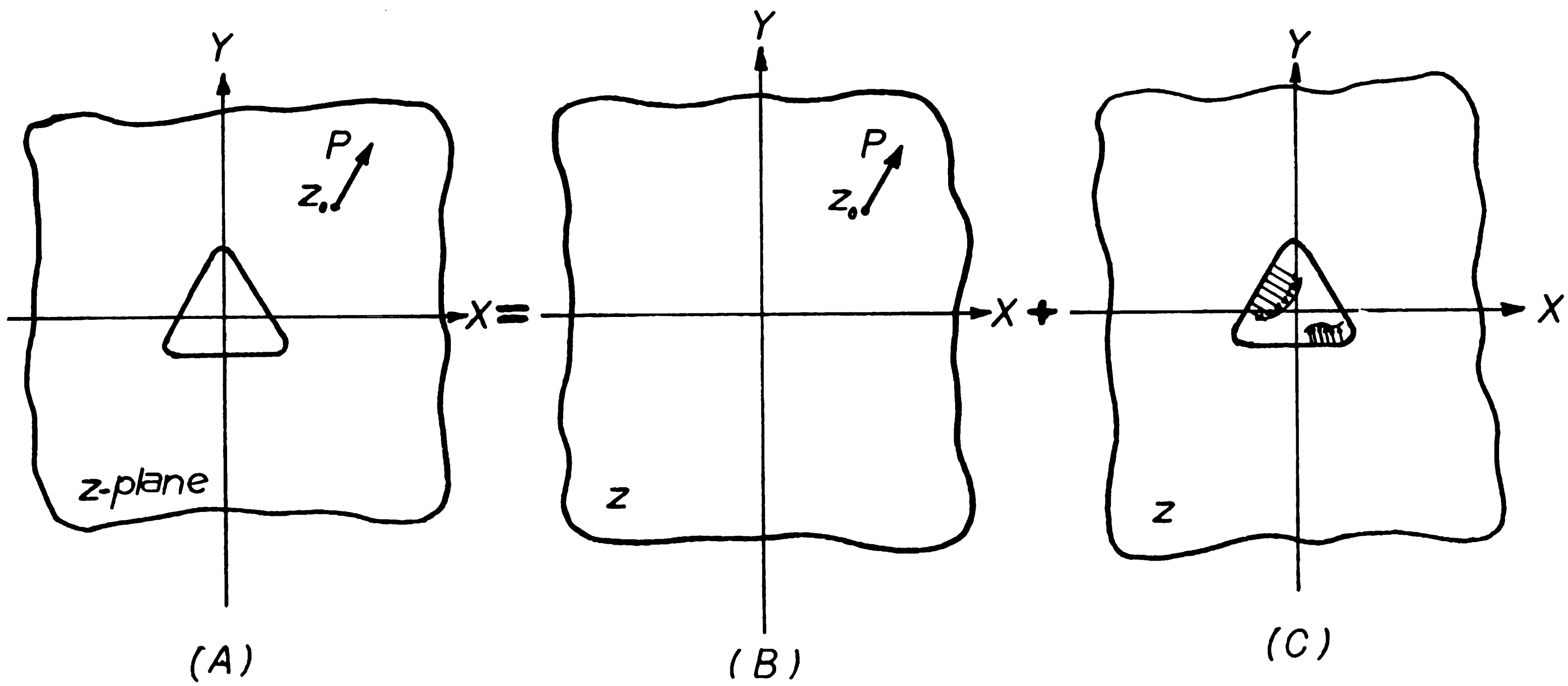


Figure 5.5 The problem of infinite plane containing a triangular hole expressed as the superposition of the two problems.

the problem to the unit circular disc, see Figure 5.6. The mapping function is given by equation (5.11).

The complex potential functions for the problem of Figure 5.5(A) will now be obtained following the general method presented in section II.3. Let $\phi^0(Z)$ and $\Psi^0(Z)$ be the complex potential functions for the problem of Figure 5.5(B), and let $\phi^*(Z)$ and $\Psi^*(Z)$ be the complex potential functions for the problem of Figure 5.5(C). Then the potential functions for the problem of Figure 5.5(A) are

$$\phi(Z) = \phi^0(Z) + \Psi^*(Z)$$

$$\Psi(Z) = \Psi^0(Z) + \Psi^*(Z) \tag{5.14}$$

where $\phi^0(Z)$ and $\Psi^0(Z)$ are known [27] and, following section II.3, the transformed complex potential functions of $\phi^*(Z)$ and $\Psi^*(Z)$ are given by

$$\phi_1^*(\zeta) = \frac{1}{2\pi i}\oint_\gamma \frac{F(\sigma)}{\sigma-\zeta}\,d\sigma - \frac{1}{2\pi i}\oint_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \cdot \frac{\overline{\phi^{*\prime}(\sigma)}}{\sigma-\zeta}\,d\sigma \tag{5.15}$$

$$\Psi_1^*(\zeta) = \frac{1}{2\pi i}\oint_\gamma \frac{\overline{F(\sigma)}}{\sigma-\zeta}\,d\sigma - \frac{1}{2\pi i}\oint_\gamma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} \cdot \frac{\phi^{*\prime}(\sigma)}{\sigma-\zeta}\,d\sigma \tag{5.16}$$

where $F(\sigma)$ and $\overline{F(\sigma)}$ are given by equations (3.20) and (3.21). Substituting σ into the mapping function, equation (5.12), and taking the complex conjugate leads to:

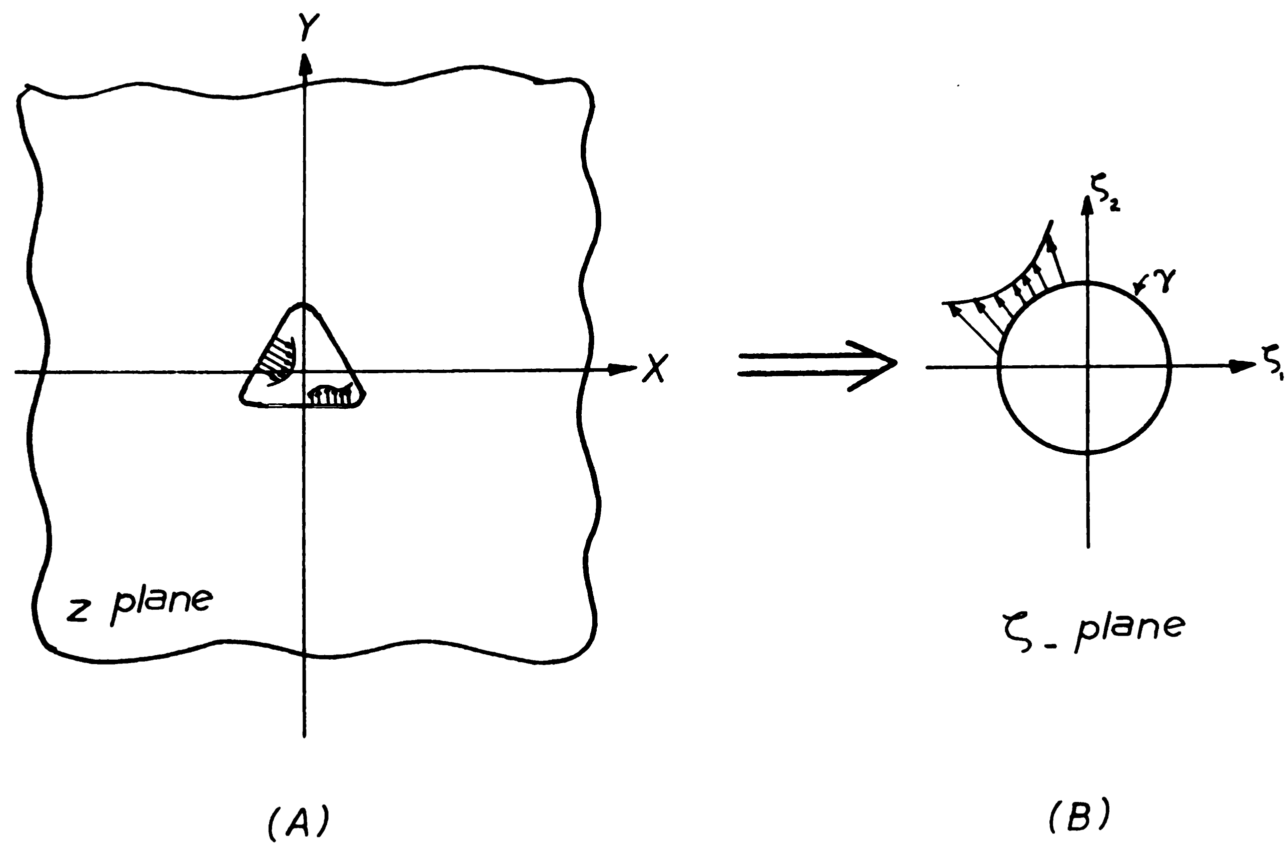


Figure 5.6 Mapping the problem of the hole and the applied traction into a unit circular disc.

$$\omega(\sigma) = \frac{m}{\sigma} + \ell\sigma + r\sigma^2$$

$$\overline{\omega(\sigma)} = m + \frac{\ell}{\sigma} + \frac{r}{\sigma^2}$$

for which the derivatives are

$$\omega'(\sigma) = \frac{-m}{\sigma^2} + \ell + 2r\sigma$$

$$\overline{\omega'(\sigma)} = -m\sigma^2 + \ell + \frac{2r}{\sigma}$$

Note that $\sigma\bar{\sigma} = 1$. Then clearly:

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} = \frac{m+\ell\sigma^2+r\sigma^3}{2r+\ell\sigma-m\sigma^3} = -\sum_{n=0}^{\infty} \alpha_n\sigma^{-n} \tag{5.17}$$

and

$$\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} = \frac{m\sigma^3+\ell\sigma+r}{2r\sigma^3+\ell\sigma^2-m} \tag{5.18}$$

$\phi_1^*(\sigma)$ and $\phi_1^{*\prime}(\sigma)$ are analytic inside γ, the unit circle and following section II.3, $\overline{\phi_1^{*\prime}(\sigma)}$ is analytic outside γ. Thus,

$$\phi_1^{*\prime}(\sigma) = \sum_{k=1}^{\infty} ka_k\sigma^k \tag{5.19}$$

$$\overline{\phi_1^{*\prime}(\sigma)} = \sum_{k=1}^{\infty} k\bar{a}_k\sigma^{-k+1} \tag{5.20}$$

Multiplying equation (5.20) by (5.17) leads to:

$$\frac{\omega(\sigma)}{\omega'(\sigma)}\overline{\phi_1^{*\prime}(\sigma)} = -\sum_{n=0}^{\infty}\alpha_n\sigma^{-n}\cdot\sum_{k=1}^{\infty}k\bar{a}_k\sigma^{-k+1} = -\sum_{n=0}^{\infty}e_n\sigma^{-n} \quad (5.21)$$

The right-hand side of equation (5.21) is analytic outside γ, the unit circle. Hence:

$$\frac{1}{2\pi i}\oint_{\gamma}\frac{\omega(\sigma)}{\omega'(\sigma)}\frac{\overline{\phi_1^{*\prime}(\sigma)}}{\sigma-\zeta}\,d\sigma = e_o$$

where e_o is a constant. Thus, equation (5.15) becomes:

$$\phi_1^{*}(\zeta) = \frac{1}{2\pi i}\oint_{\gamma}\frac{F(\sigma)}{\sigma-\zeta}\,d\sigma \quad (5.22)$$

Multiplying equation (5.19) by equation (5.18) leads to:

$$\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\phi_1^{*\prime}(\sigma) = \frac{m\sigma^3+\ell\sigma+r}{2r\sigma^3+\ell\sigma^2-m}\cdot\sum_{k=1}^{\infty}ka_k\sigma^{k-1} \quad (5.23)$$

The numerator of equation (5.23) is analytic inside γ. Then

$$\frac{1}{2\pi i}\oint_{\gamma}\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\cdot\frac{\phi_1^{*\prime}(\sigma)}{\sigma-\zeta}\,d\sigma = \frac{m\zeta^3+\ell\zeta+r}{2r\zeta^3+\ell\zeta^2-m}\cdot\phi_1^{*\prime}(\zeta) + f(\zeta)$$

where $f(\zeta)$ contains the residues of

$$\frac{(m\sigma^3+\ell\sigma+r)\phi_1^{*\prime}(\sigma)}{(2r\sigma^3+\ell\sigma^2-m)(\sigma-\zeta)}$$

at the roots which are inside γ. Thus, equation (5.16) becomes

$$\Psi_1^*(\zeta) = \frac{1}{2\pi i} \oint_\gamma \frac{\overline{F(\sigma)}}{\sigma-\zeta}\, d\sigma - \frac{m\zeta^3+\ell\zeta+r}{2r\zeta^3+\ell\zeta^2-m}\, \phi_1^{*\prime}(\zeta) - f(\zeta) \quad (5.24)$$

Rewriting equations (3.20) and (3.21)

$$F(\sigma) = -\left[\phi_1^0(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\, \overline{\phi_1^{0\prime}(\sigma)} + \overline{\Psi_1^0(\sigma)}\right] \quad (5.25)$$

$$\overline{F(\sigma)} = -\left[\overline{\phi_1^0(\sigma)} + \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\, \phi_1^{0\prime}(\sigma) + \Psi_1^0(\sigma)\right] \quad (5.26)$$

where $\phi^0(Z)$ and $\Psi^0(Z)$ are given by equation (2.6). To find the transformed complex potential functions $\phi_1^0(\zeta)$ and $\Psi_1^0(\zeta)$, substitute the mapping function, equation (5.12), into equations (2.6). Thus,

$$\phi_1^0(\sigma) = -Q \cdot \ln\left(\frac{m-Z_0\sigma+\ell\sigma^2+r\sigma^3}{\sigma}\right) \quad (5.27)$$

$$\Psi_1^0(\sigma) = Q \cdot \frac{\overline{Z}_0\sigma}{m-Z_0\sigma+\ell\sigma^2+r\sigma^3} + \overline{Q} \cdot \alpha \cdot \ln\left(\frac{m-Z_0\sigma+\ell\sigma^2+r\sigma^3}{\sigma}\right) \quad (5.28)$$

Let

$$A(\sigma) = m - Z_0\sigma + \ell\sigma^2 + r\sigma^3 \quad (5.29)$$

$$B(\sigma) = r + \ell\sigma - \overline{Z}_0\sigma^2 + m\sigma^3 \quad (5.30)$$

Then the complex conjugates of equations (5.27) and (5.28) are:

$$\overline{\phi_1^o(\sigma)} = - \overline{Q} \ln \left[\frac{B(\sigma)}{\sigma^2}\right]$$

$$\overline{\Psi_1^o(\sigma)} = Q \cdot \alpha \cdot \ln \left[\frac{B(\sigma)}{\sigma^2}\right] + \overline{Q} \cdot \frac{\overline{Z}_0 \sigma^2}{B(\sigma)} \qquad (5.31)$$

Clearly:

$$\phi_1^{0\prime}(\sigma) = Q \cdot \frac{2r\sigma^3 + \ell\sigma^2 - m}{\sigma \cdot A(\sigma)}$$

$$\overline{\phi_1^{0\prime}(\sigma)} = \frac{(2r + \ell\sigma - m\sigma^3)\sigma}{B(\sigma)} \qquad (5.32)$$

Substituting equations (5.17), (5.18) and (5.27) through (5.32) into equations (5.25) and (5.26) leads to:

$$F(\sigma) = Q \left\{ \ln \frac{A(\sigma)}{\sigma} - \alpha \ln \frac{B(\sigma)}{\sigma^2} \right\} + \overline{Q} \left\{ \sigma \cdot \frac{A(\sigma)}{B(\sigma)} \right\} \qquad (5.33)$$

$$\overline{F(\sigma)} = Q \left\{ \frac{B(\sigma)}{\sigma A(\sigma)} \right\} + \overline{Q} \left\{ \ln \frac{B(\sigma)}{\sigma^2} - \alpha \ln \frac{A(\sigma)}{\sigma} \right\} \qquad (5.34)$$

where $A(\sigma)$ and $B(\sigma)$ are defined by equations (5.29) and (5.30).

Conformity of the transformation function allows that the inverse transformation function, equation (5.13), has one root inside γ, the unit circle, and two roots outside γ. Let r_i, r_{o_1} and r_{o_2} be the roots, inside and outside, respectively. Then the similarity of equations (5.13) and (5.29) leads to:

$$A(\sigma) = m - Z_0\sigma + \ell\sigma^2 + r\sigma^3 = r(\sigma - r_i)(\sigma - r_{o_1})(\sigma - r_{o_2}) \quad (5.35)$$

Let the function $\omega(1/\zeta)$ be considered. Clearly, the function transforms the problem onto an infinite plane bounded by the unit circle.

$$Z = \omega(1/\zeta) = m\zeta + \ell \cdot \frac{1}{\zeta} + \frac{r}{\zeta^2}$$

and the inverse transformation function $\omega^{-1}(1/\zeta)$ can be obtained by solving the following cubic equation [49]:

$$m\zeta^3 - Z\zeta^2 + \ell\zeta + r = 0 \quad (5.36)$$

The conformality of the transformation function, $\omega(1/\zeta)$, allows that the implicit form of the inverse transformation function, equation (5.36), has two roots inside γ and one outside. Comparing equations (5.30) and (5.36) and noting that all coefficients of the two equations are real except Z and $\bar{Z}_0$, one can take the conjugate of these coefficients, thus yielding the conjugates of the roots of equation (5.36), one outside and two inside the unit circle.

Let t_{i_1} and t_{i_2} be the two roots inside and t_o be the root outside the unit circle. Then equation (5.30) becomes

$$B(\sigma) = m\sigma^3 - \bar{Z}_0\sigma^2 + \ell\sigma + r = m(\sigma - t_o)(\sigma - t_{i_1})(\sigma - t_{i_2}) \quad (5.37)$$

To find $\frac{1}{2\pi i}\oint_\gamma \frac{F(\sigma)}{\sigma-\zeta}\,d\sigma$ and $\frac{1}{2\pi i}\oint_\gamma \frac{\overline{F(\sigma)}}{\sigma-\zeta}\,d\sigma$, let the following integrals be calculated. The first one is

$$I = \frac{1}{2\pi i} \oint_\gamma \ln\left[\frac{A(\sigma)}{\sigma}\right] \cdot \frac{d\sigma}{\sigma-\zeta}$$

Substituting equation (5.35) into this integral and following section I.3 leads to

$$I = \ln r(\zeta - r_{o_1})(\zeta - r_{o_2}) \tag{5.38}$$

Note that equation (5.38) is evaluated in the same manner as equation (4.39).

The second integral is

$$II = \frac{1}{2\pi i} \oint_\gamma \ln\left[\frac{B(\sigma)}{\sigma^2}\right] \cdot \frac{d\sigma}{\sigma-\zeta}$$

Substituting equation (5.37) into this integral and following section I.3 leads to

$$II = \ln m(\zeta - t_o) \tag{5.39}$$

The third integral is

$$III = \frac{1}{2\pi i} \oint_\gamma \frac{\sigma A(\sigma)}{B(\sigma)} \cdot \frac{d\sigma}{\sigma-\zeta}$$

Substituting equations (5.35) and (5.37) into this integral and following the Cauchy integral theorem leads to

$$III = \frac{1}{2\pi i}\oint_{\gamma} \frac{\sigma A(\sigma)}{m(\sigma-t_o)} \cdot \frac{d\sigma}{(\sigma-t_{i_1})(\sigma-t_{i_2})(\sigma-\zeta)} = \frac{\zeta A(\zeta)}{B(\zeta)}$$

$$+ \frac{t_{i_1}A(t_{i_1})}{m(t_{i_1}-t_o)(t_{i_1}-t_{i_2})(t_{i_1}-\zeta)} + \frac{t_{i_2}A(t_{i_2})}{m(t_{i_2}-t_o)(t_{i_2}-t_{i_1})(t_{i_2}-\zeta)} \tag{5.40}$$

The last integral is

$$IV = \frac{1}{2\pi i}\oint_{\gamma} \frac{B(\sigma)}{\sigma A(\sigma)} \frac{d\sigma}{\sigma-\zeta}$$

Clearly, this integral leads to

$$IV = \frac{-r}{m\zeta} + \frac{B(\zeta)}{\zeta A(\zeta)} + \frac{B(r_i)}{r_i(r_i-r_{o_1})(r_i-r_{o_2})(r_i-\zeta)} \tag{5.41}$$

Using the four evaluated integrals, equations (5.38) to (5.41), it follows from equation (5.38) that

$$\frac{1}{2\pi i}\oint_{\gamma} \frac{F(\sigma)}{\sigma-\zeta}\, d\sigma = Q\left\{I - \alpha \cdot II\right\} + \overline{Q}\left\{III\right\} \tag{5.42}$$

Substituting equation (5.42) into equation (5.22) leads to:

$$\phi_1^*(\zeta) = Q\left\{\ln r(\zeta-r_{o_1})(\zeta-r_{o_2}) - \alpha \ln m(\zeta-t_o)\right\} + \bar{Q}\left\{\frac{\zeta A(\zeta)}{B(\zeta)}\right.$$

$$\left. + \frac{t_{i_1}A(t_{i_1})}{m(t_{i_1}-t_o)(t_{i_1}-t_{i_2})(t_{i_1}-\zeta)} + \frac{t_{i_2}A(t_{i_2})}{m(t_{i_2}-t_o)(t_{i_2}-t_{i_1})(t_{i_2}-\zeta)}\right\} \tag{5.43}$$

The following integral can also be obtained by recalling equation (5.34) and using the evaluated integrals of equations (5.38) to (5.41):

$$\frac{1}{2\pi i}\int_\gamma \frac{\overline{F(\sigma)}}{\sigma-\zeta}\,d\sigma = Q\left\{IV\right\} + \bar{Q}\left\{II - \alpha\ I\right\} \tag{5.44}$$

Substituting equation (5.44) into equation (5.24) leads to:

$$\psi_1^*(\zeta) = Q\left\{\frac{-r}{m\zeta} + \frac{B(\zeta)}{\zeta A(\zeta)} + \frac{B(r_i)}{r_i(r_i-r_{o_1})(r_i-r_{o_2})(r_i-\zeta)}\right\}$$

$$+ \bar{Q}\left\{\ln m(\zeta-t_o) - \alpha \ln r(\zeta-r_{o_1})(\zeta-r_{o_2})\right.$$

$$\left. - \frac{m\zeta^3+\ell\zeta+r}{2r\zeta^3+\ell\zeta^2-m}\,\phi_1^{*\prime}(\zeta) - f(\zeta)\right\} \tag{5.45}$$

Hence, the complex potential functions for the opening described by equation (5.10) are obtained, and given by equations (5.43) and (5.45).

The influence functions can now be found by taking the derivatives of equations (5.43) and (5.45) and substituting into equations (3.6). Since the procedure is very straightforward, the details of this calculation are omitted.

V.4 ON THE INFLUENCE FUNCTION OF A PARTICULAR CLASS OF OPENING, CASE 2

Consider an infinite plane bounded by the contour described by:

$$X = R(\cos\theta + \varepsilon \cos 3\theta)$$

$$Y = R(c \sin\theta - \varepsilon \sin 3\theta) \qquad (5.46)$$

where $o<c\leqslant 1$. Equation (5.46) represents an elongated contour symmetric about the x and y axes. Changing C and ε will produce contours which vary from a square with rounded corners to an oval or a slot. For example, when $C = .36$ and $\varepsilon = -0.04$, a contour is obtained which will differ little from a rectangle with semi-circular short sides and straight long sides. When $C = 0.537$ and $\varepsilon = -0.038$, the opening is an oval and when $C = 0.026$ and $\varepsilon = -0.004$, the opening is a slot.

A concentrated point force P is acting in the plane at some point Z where Z lies on or outside of the opening, i.e.,

$$X_0 \geqslant R(\cos\theta + \varepsilon \cos 3\theta)$$

$$Y_0 \geqslant R(c \sin\theta - \varepsilon \sin 3\theta)$$

where

$$Z_0 = X_0 + iY_0$$

Again, as presented in the previous section, the problem will be expressed as the superposition of two problems,

Figure 5.7. The problem of Figure 5.7(B) is just the concentrated point force P applied at the same point in an infinite plane and the problem of Figure 5.7(C) is the infinite plane bounded by the opening on which the appropriate specified traction is applied.

The problem of Figure 5.7(C) can be solved using the mapping technique, i.e., transferring the problem to the unit circular disc, Figure 5.8. The mapping function is given by the following equation:

$$Z = \omega(\zeta) = R\left(\frac{1+c}{2} \cdot \frac{1}{\zeta} + \frac{1-c}{2}\zeta + \varepsilon\zeta^3\right) \qquad (5.47)$$

where R, c and ε are constants. Let

$$m = \frac{R(1+c)}{2}$$

$$\ell = \frac{R(1-c)}{2}$$

$$r = R \cdot \varepsilon$$

Thus, the transformation function will be

$$Z = \omega(\zeta) = \frac{m}{\zeta} + \ell\zeta + r\zeta^3 \qquad (5.48)$$

The inverse transformation function $\omega^{-1}(\zeta)$ can be found by employing the two methods presented in the first section of this chapter.

Following the first method, power series expansion for the inverse transformation, the transformation function is to be written in the form of equation (5.7). For simplicity,

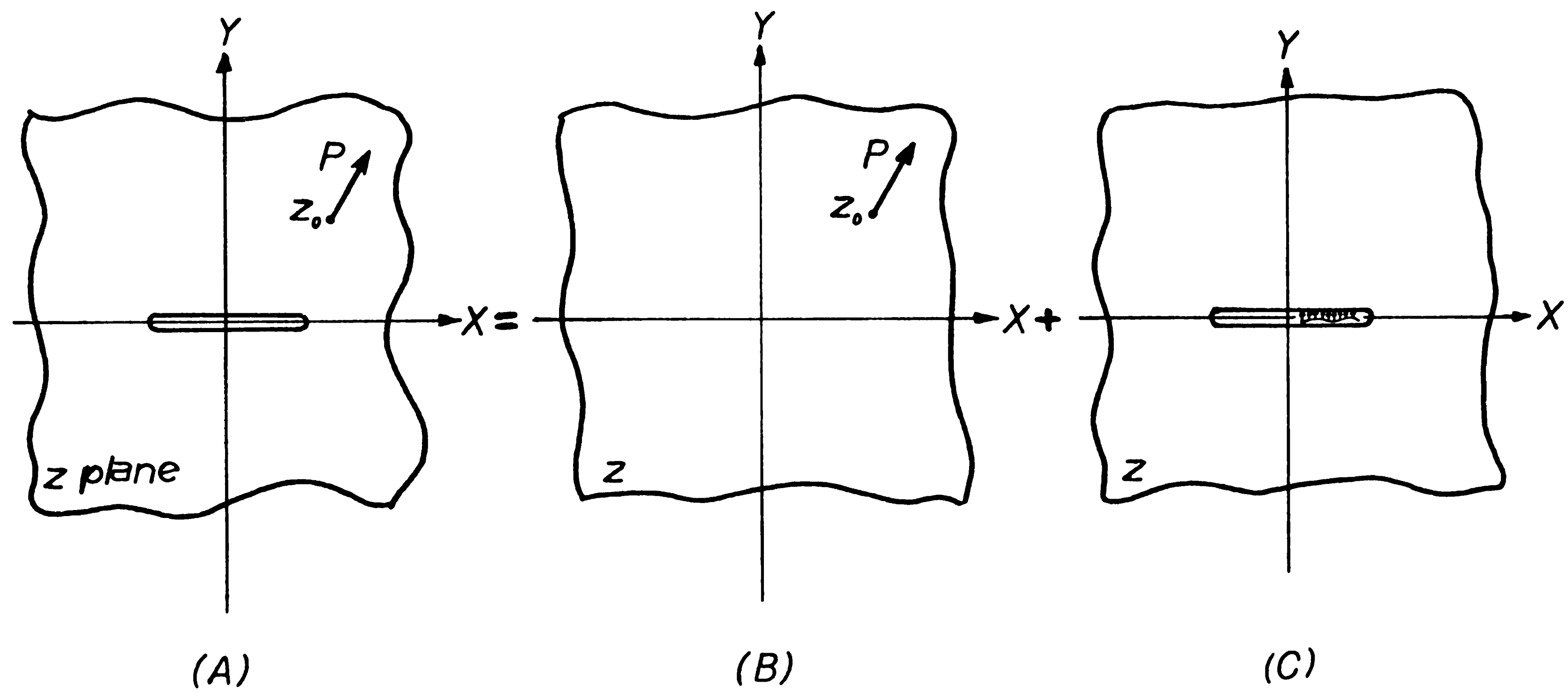


Figure 5.7 The problem of infinite plane containing an arbitrarily-shaped hole expressed as the superposition of the two problems.

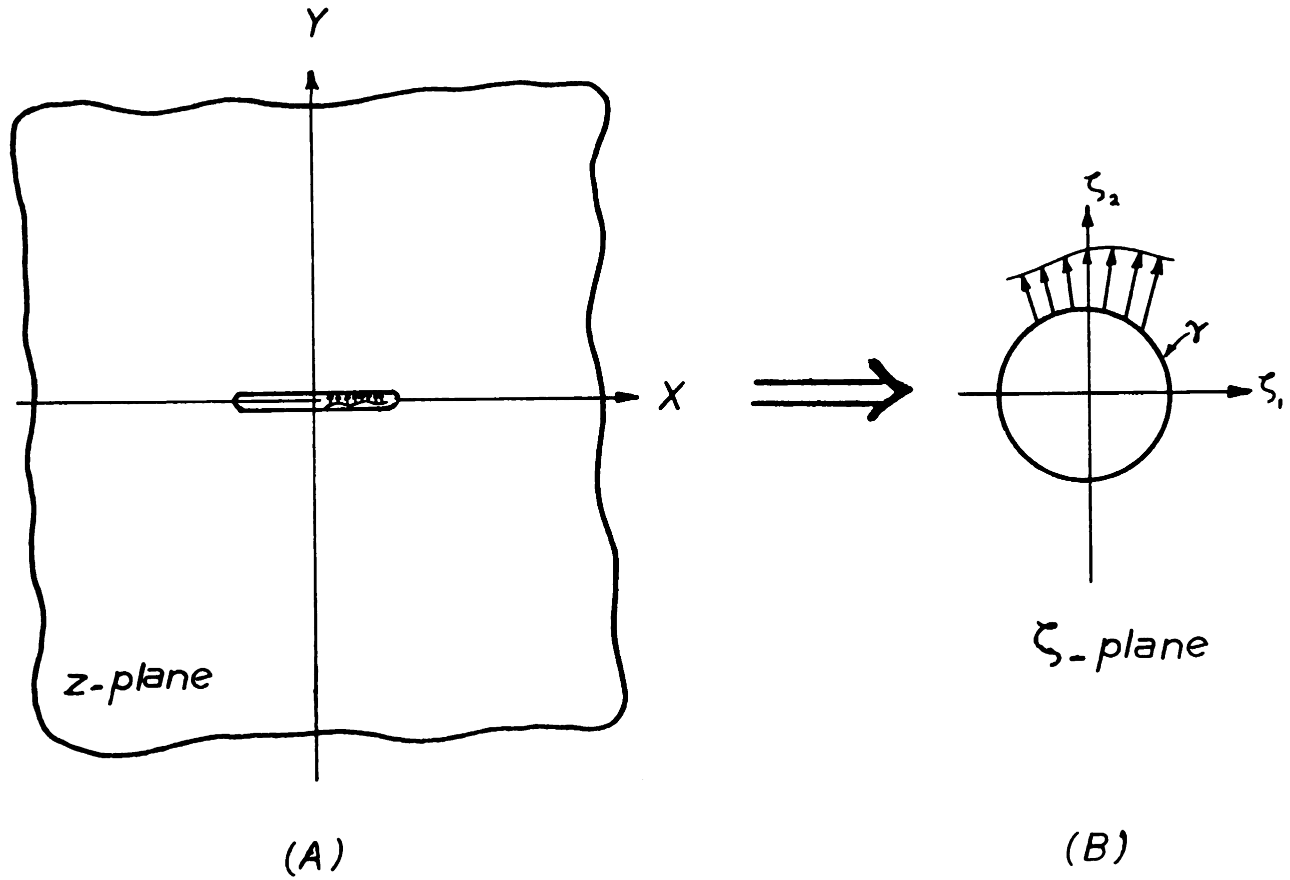


Figure 5.8 Mapping the problem of the hole subjected to applied traction into a unit circular disc.

let $\Omega = (\sqrt{m/\ell})/\zeta$. Then equation (5.48) can be written as follows:

$$Z = \frac{\Omega}{\sqrt{m/\ell}} + \frac{1\sqrt{m/\ell}}{\Omega} + \frac{r(\sqrt{m/\ell})^3}{\Omega^3}$$

or

$$W = \Omega(1 + \Omega^{-2} + \lambda\Omega^{-4}) \qquad (5.49)$$

where

$$W = \frac{Z}{m} \cdot \sqrt{\frac{m}{\ell}}$$

and

$$\lambda = (m \cdot r)/\ell^2$$

Finally, equation (5.49) can be written in the following form:

$$W = \Omega\left\{1 + \sum_{k=1}^{\infty} \alpha_k \Omega^{-2k}\right\} \qquad (5.50)$$

where $\alpha_1 = 1$, $\alpha_2 = \lambda$ and $\alpha_k = 0$ for $k \geqslant 3$.

The inverse power series of equation (5.50) can now be obtained following equation (5.8) where, in this case, $P = q = 1$ and $V = -2$. Thus,

$$\Omega = W\left\{1 + \sum_{k=1}^{\infty} \beta_k W^{-2k}\right\} \qquad (5.51)$$

where

$$\beta_k = \frac{1}{1-2k} \sum_{r=1}^{k} \binom{2k-1}{n} \alpha_k^{(n)}$$

and the $\alpha_k^{(n)}$'s can be formulated as

$$\alpha_k^{(n)} = \alpha_1^{2n-k} \, \alpha_2^{k-n} \binom{n}{k-n}$$

Since $\alpha_k = 0$ for $k>3$ and $\alpha = 1$, then

$$\alpha_k^{(n)} = \binom{n}{k-n} \lambda^{k-n}$$

so that

$$\beta_k = \frac{1}{1-2k} \sum_{n=1}^{k} \binom{2k-1}{n} \binom{n}{k-n} \lambda^{k-n} \tag{5.52}$$

which leads to:

$$\beta_1 = -1$$

$$\beta_2 = -1 - \lambda$$

$$\beta_3 = -2 - 4\lambda$$

$$\beta_4 = -5 - 15\lambda - 3\lambda^2$$

$$\beta_5 = -14 - 56\lambda - 28\lambda^2$$

It is clear that the β_k's are not converging very rapidly, a deficiency of this method. However, the β_k's are all functions of powers of λ, which indicates that there would be some general closed form for equation (5.51):

$$\frac{\Omega}{W} = 1 - \rho_0 - \rho_1\lambda - \rho_2\lambda^2 - \rho_3\lambda^3 \ldots\ldots\ldots \tag{5.53}$$

The coefficient ρ_i's will now be found. To find ρ_0, let $\lambda = 0$ in equation (5.53), then

$$\frac{\Omega}{W} = 1 - \rho_0 \tag{5.54}$$

Also, when $\lambda = 0$, the transformation function, equation (5.49), becomes

$$\Omega^2 - W\Omega + 1 = 0$$

or

$$\frac{\Omega^2}{W^2} - \frac{\Omega}{W} + \frac{1}{W^2} = 0$$

Solving for $\frac{\Omega}{W}$ leads to:

$$\frac{\Omega}{W} = \frac{1 \pm \sqrt{1-4/W^2}}{2} \tag{5.55}$$

The minus sign is not valid since the limit of the solution, equation (5.55), must approach unity to satisfy conformality of the transformation function, i.e., infinity is transformed to infinity. Equating the right-hand sides of equations (5.55) and (5.54) leads to:

$$\rho_0 = \frac{1 - \sqrt{1-4/W^2}}{2} \tag{5.56}$$

Now let n = k in equation (5.52). Then

$$\beta_k = \frac{1}{1-2k} \binom{2k-1}{k} \binom{k}{o} \lambda^0 = -\frac{(2k-2)!}{k!(k-1)!}$$

or

$$\beta_{P+1} = - \binom{2P}{P} \frac{1}{P+1}$$

Substituting into the inverse transformation function, equation (5.51) leads to:

$$\frac{\Omega}{W} = 1 - \sum_{P=0}^{\infty} \binom{2P}{P} \frac{W^{-2P-2}}{P+1} \qquad (5.57)$$

Comparing (5.57) and (5.54), it is found that

$$\rho_0 = \sum_{p=0}^{\infty} \binom{2P}{P} \frac{W^{-2P-2}}{P+1} \qquad (5.58)$$

Now let k - n = 1 in equation (5.52). Then

$$\beta_k = \frac{1}{1-2k} \binom{2k-1}{k-1} \binom{k-1}{1} \lambda + \frac{1}{1-2k} \binom{2k-1}{k} \binom{k}{o} \lambda^0$$

Substituting β_k into the inverse transformation function, equation (5.51), yields:

$$\frac{\Omega}{W} = 1 - \sum_{p=0}^{\infty} \binom{2P}{P} \frac{W^{-2P-2}}{P+1} - \lambda \sum_{k=1}^{\infty} \binom{2k-2}{k-2} W^{-2k}$$

Comparing this equation with equation (5.53) leads to:

$$\rho_1 = \sum_{k=1}^{\infty} \binom{2k-2}{k-2} W^{-2k} \qquad (5.59)$$

To find a relation between ρ_1 and ρ_0, i.e., a closed form formulae for ρ_1, equations (5.58) and (5.59) can now be used.

Since $\binom{2k-2}{k-2} = 0$ for $k = 0,1$, then letting $k = P + 1$

$$\binom{2k-2}{k-2} = \binom{2P}{P-1}$$

But

$$\binom{2P}{P-1} = \frac{P}{P+1}\binom{2P}{P} = \binom{2P}{P} - \frac{1}{P+1}\binom{2P}{P}$$

Substituting $k = P + 1$ and the binomial coefficient above into equation (5.59) leads to:

$$\rho_1 = - \sum_{P=0}^{\infty} \binom{2P}{P} \frac{W^{-2P-2}}{P+1} + \sum_{P=0}^{\infty} \binom{2P}{P} W^{-2P-2} \tag{5.60}$$

Taking the derivative of ρ_0, equation (5.58), and constructing $\frac{1}{2}\rho_0'W$

$$\frac{1}{2}\rho_0'W = - \sum_{P=0}^{\infty} \binom{2P}{P} W^{-2P-2} \tag{5.61}$$

Substituting equations (5.61) and (5.58) into the right-hand side of equation (5.60) leads to the expression:

$$\rho_1 = - \rho_0 - \frac{1}{2}\rho_0'W \tag{5.62}$$

Taking the derivative of ρ_0 in its closed form, equation (5.56), and calculating the right-hand side of equation (5.62) leads to:

$$\rho_1 = \frac{\rho_0^2}{1-2\rho_0} \tag{5.63.a}$$

Similarly, one can find ρ_2 in terms of ρ_0, which will lead to better convergence of equation (5.53). Thus, the inverse transformation function becomes

$$\Omega = W\ (1 - \rho_0 - \frac{\rho_0^2}{1-2\rho_0}\ \lambda - \ \ldots) \tag{5.63}$$

where

$$\rho_0 = \frac{1 - \sqrt{1-4/W^2}}{2}$$

This was an example of using the power series expansion to find the inverse transformation function, even though the method did not prove efficient. The second method, continued fractions, will also give the coefficient of λ^2, i.e., ρ_2. The second method will now be applied. Rewrite equation (5.49) in the following form:

$$\Omega^4 - W\ \Omega^3 + \Omega^2 + \lambda = 0$$

Dividing the equation by W^4 leads to

$$(\frac{\Omega}{W})^4 - (\frac{\Omega}{W})^3 + (\frac{\Omega}{W})^2 \cdot \frac{1}{W^2} + \frac{\lambda}{W^4} = 0$$

or

$$(\frac{\Omega}{W})^3\ (\frac{\Omega}{W} - 1) + (\frac{\Omega}{W})^2\ \frac{1}{W^2} + \frac{\lambda}{W^4} = 0 \tag{5.64}$$

In order to check the accuracy and efficiency of this method with the previous method, let the following

assumption be made:

$$\Gamma = 1 - \frac{\Omega}{W} \tag{5.65}$$

and

$$\rho_0 = \frac{1 - \sqrt{1-4/W^2}}{2} \tag{5.66}$$

Constructing $(\rho_0 - \rho_0^2)$ using equation (5.66) leads to:

$$\rho_0 - \rho_0^2 = \frac{1}{W^2}$$

Substituting these results into equation (5.64) leads to:

$$f(\Gamma) = \Gamma(\Gamma-1)^3 + (\Gamma-1)^2 (\rho_0-\rho_0^2) + \lambda(\rho_0-\rho_0^2)^2 = 0 \tag{5.67}$$

Equation (5.67) is a polynomial in Γ of the form of equation (5.9), which can now be solved by the continued fractions method.

Let the first estimate to the required root Γ of the equation $f(\Gamma) = 0$ be ρ_0. Then, after some simplification, the function and the derivatives of the function at the estimated root are:

$$f(\rho_0) = (\rho_0-\rho_0^2)^2$$

$$f'(\rho_0) = (\rho_0-1)^2 (2\rho_0-1)$$

$$f''(\rho_0) = 2(5\rho_0-3) (\rho_0-1)$$

$$f'''(\rho_0) = 6(4 \rho_0-3)$$

$$f''''(\rho_0) = 24$$

$$f^{(n)}(\rho_0) = 0 \qquad \text{for } n \geqslant 4$$

Thus

$$\eta = -\frac{f(\rho_0)}{f'(\rho_0)} = \frac{\lambda\rho_0^2}{1-2}$$

$$\xi = \frac{f''(\rho_0)}{2f'(\rho_0)} = \frac{5\rho_0-3}{-(1-\rho_0)(2\rho_0-1)}$$

$$\tau = \frac{f'''(\rho_0)}{3f''(\rho_0)} = \frac{4\rho_0-3}{(5\rho_0-3)(\rho_0-1)}$$

$$\delta = \frac{f''''(\rho_0)}{4f'''(\rho_0)} = \frac{1}{4\rho_0-3}$$

Then $d_1 = +\eta$, $d_2 = \eta\xi$ and $d_3 = \eta(\xi-\gamma)$

$$d_4 = \eta(\xi - \frac{\tau(\tau-\delta)}{\xi-\gamma}$$

Substitution into the continued fractions leads to:

$$\Gamma = \rho_0 + \cfrac{d_1}{1+\cfrac{d_2}{\ddots}}$$

or

$$\Gamma = \rho_0 - \cfrac{\eta}{1-\cfrac{(-\eta\xi)}{1-\eta(\gamma-\xi)}}$$

For simplicity, consider the first numerator. Then expanding as a binomial series and just choosing the first two terms, the equation leads to:

$$\Gamma = \rho_0 - \eta - \eta^2\xi + \ldots\ldots$$

Substituting equation (5.65), η and ξ into the above equation leads to

$$\Omega = W(1 - \rho_0 - \frac{\rho_0^2}{1-2\rho_0}\lambda - \frac{\rho_0^4(5\rho_0-3)}{(1-2\rho_0)^3(1-\rho_0)} \cdot \lambda^2 \ldots\ldots) \quad (5.68)$$

where

$$\rho_0 = \frac{1 - \sqrt{1-4/W^2}}{2}$$

The inverse transformation function has now been obtained by the two methods. Comparison of the two equations (5.63) and (5.68) shows that the first three terms of both equations are exactly the same except that the second method, continued fractions, provides one more term. It is also clear that the second method was more efficient. The first method, however, can be more useful in some special cases.

Returning to the solution of the problem of Figure 5.7(A) and equation (5.46), the complex potential functions will now be obtained following the general procedure presented in section II.3.

Let $\phi^0(Z)$ and $\Psi^0(Z)$ be the complex potential functions for the problem of Figure 5.7(B) and $\phi^*(Z)$ and $\Psi^*(Z)$ be the complex potential functions for the problem of Figure 5.7(C). Hence, the potential functions for othe problem of Figure 5.7(A) are

$$\phi(Z) = \phi^0(Z) + \phi^*(Z)$$

$$\Psi(Z) = \Psi^0(Z) + \Psi^*(Z) \qquad (5.69)$$

where $\phi^0(Z)$ and $\Psi^0(Z)$ are known [27]. The transformed complex potential functions of $\phi^*(Z)$ and $\Psi^*(Z)$ can be obtained in a manner similar to that presented in section II.3. These are

$$\phi_1^*(\zeta) = \frac{1}{2\pi i}\oint_\gamma \frac{F(\sigma)}{\sigma-\zeta}\,d\sigma - \frac{1}{2\pi i}\oint_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\frac{\overline{\phi_1^{*\prime}(\sigma)}}{\sigma-\zeta}\,d\sigma \qquad (5.70)$$

$$\Psi_1^*(\zeta) = \frac{1}{2\pi i}\oint_\gamma \frac{\overline{F(\sigma)}}{\sigma-\zeta}\,d\sigma - \frac{1}{2\pi i}\oint_\gamma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\,\frac{\phi^{*\prime}(\sigma)}{\sigma-\zeta}\,d\sigma \qquad (5.71)$$

where $F(\sigma)$ and $\overline{F(\sigma)}$ are given by equations (3.20) and (3.21). Equation (5.70) is an integral equation in which the second integral can be evaluated. Let the transformation function of equation (5.48) and its complex conjugate be evaluated at $\zeta = \sigma$:

$$\omega(\sigma) = \frac{m}{\sigma} + \ell\sigma + r\sigma^3$$

$$\overline{\omega(\sigma)} = m\sigma + \frac{\ell}{\sigma} + \frac{r}{\sigma^3}$$

Then the derivative and conjugate derivatives are

$$\omega'(\sigma) = -\frac{m}{\sigma^2} + \ell + 3r\sigma^2$$

$$\overline{\omega'(\sigma)} = -m\sigma^2 + \ell + \frac{3r}{\sigma^2}$$

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} = \frac{\sigma(m+\ell\sigma^2+r\sigma^4)}{3r+\ell\sigma^2-m\sigma^4}$$

$$= -\sigma \quad \frac{r}{m} + \frac{\ell(m+r)}{m^2} \cdot \frac{1}{\sigma^2} + \sum_{k=2}^{\infty} \alpha_k\sigma^{-2k}$$

$$= -\frac{r}{m}\sigma - \sum_{k=1}^{\infty} \alpha_k\sigma^{-2k+1} \qquad (5.72)$$

where the α_k's are the coefficient of the expansion. Similarly,

$$\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} = \frac{m\sigma^4+\ell\sigma^2+r}{(3r\sigma^4+\ell\sigma^2-m)}$$

$$= \sum_{k=0}^{\infty} \beta_k\sigma^{-2k-1} \qquad (5.73)$$

where the β_k's are the coefficient of the expansion. Note that $\phi_1^*(\sigma)$ and $\phi_1^{*\prime}(\sigma)$ are analytic inside γ and $\overline{\phi_1^{*\prime}(\sigma)}$ is analytic outside γ. Thus, following section III.3:

$$\phi_1^{*\prime}(\sigma) = \sum_{k=1}^{\infty} k a_k \sigma^{k-1} \tag{5.74}$$

$$\overline{\phi_1^{*\prime}(\sigma)} = \sum_{k=1}^{\infty} k \bar{a}_k \sigma^{-k+1} \tag{5.75}$$

Multiplying equation (5.72) by (5.75) leads to:

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \overline{\phi_1^{*\prime}(\sigma)} = - \frac{r}{m} \cdot \sum_{k=1}^{\infty} k \bar{a}_k \sigma^{-k+2} - \sum_{k=1}^{\infty} k \bar{a}_k \sigma^{-k+1} \cdot \sum_{k=1}^{\infty} \alpha_k \sigma^{-2k+1}$$

$$= - \frac{r}{m} \bar{a}_1 \sigma - \frac{2r}{m} \bar{a}_2 - \sum_{n=1}^{\infty} e_n \sigma^{-n}$$

where e_n is a coefficient of a power series expressed in terms of $\bar{a}_k$ and α_k. Since the summation $\sum_{e=1}^{\infty} e_n \sigma^{-n}$ is an analytic function outside γ, and the value of the summation at infinity is zero, then, following section I.3, the value of the second integral in equation (5.70) is

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \frac{\overline{\phi_1^{*\prime}(\sigma)}}{\sigma - \zeta} d\sigma = - \frac{r}{m} \bar{a}_1 \zeta - \frac{2r}{m} \bar{a}_2 \tag{5.76}$$

Note that the $\overline{a_i}$'s are the complex conjugates of the coefficients of the expansion series of $\phi_1^*(\zeta)$ which are unknown. Since $-(2r/m)\bar{a}_2$ is a constant term and does not have any effect in obtaining $\phi_1^*(\zeta)$, it will be omitted. Substituting equation (5.76) into (5.70) leads to:

$$\phi_1^*(\zeta) = \frac{1}{2\pi i} \oint_{\gamma} \frac{F(\sigma)}{\sigma - \zeta} d\sigma + \frac{r}{m} \bar{a}_1 \zeta - \frac{2r}{m} \bar{a}_2 \tag{5.77}$$

In order to obtain a closed form for $\phi_1^*(\zeta)$, $\bar{a}_1$ has to be evaluated. This can be easily determined. Since

$$\frac{1}{\sigma-\zeta} = \frac{1}{\sigma(1-\frac{\zeta}{\sigma})} = \frac{1}{\sigma} + \frac{\zeta}{\sigma^2} + \ldots\ldots + \frac{\zeta^n}{\sigma^{n+1}} + \ldots$$

then

$$\frac{1}{2\pi i}\oint_\gamma \frac{F(\sigma)}{\sigma-\zeta}\,d\sigma = \sum_{k=0}^{\infty} c_k\zeta^k \tag{5.78}$$

where the c_k's can be found by substituting the expansion of $1/(\sigma-\zeta)$ into (5.78):

$$c_k = \frac{1}{2\pi i}\oint_\gamma \frac{F(\sigma)}{\sigma^{k+1}}\,d\sigma$$

The c_k's could also be obtained by taking the kth derivative of equation (5.78) and setting $\zeta = 0$. The function $\phi_1^*(\zeta)$, however, is analytic inside γ and has the series form given by equation (2.13). Hence, substitution of equations (5.78) and (2.13) into (5.77) leads to:

$$\sum_{k=1}^{\infty} a_k\zeta^k = \sum_{k=0}^{\infty} c_k\zeta^k + \frac{r}{m}\,\bar{a}_1\zeta$$

Equating the coefficients of ζ leads to:

$$a_1 = c_1 + \frac{r}{m}\,\bar{a}_1$$

where

$$c_1 = \frac{1}{2\pi i}\oint_\gamma \frac{F(\sigma)}{\sigma^2}\,d\sigma$$

Finally, equating the real parts and the imaginary parts leads to evaluation of $\bar{a}_1$:

$$\bar{a}_1 = m\left\{\frac{Re(c_1)}{m-r} - i \cdot \frac{Im(c_1)}{m+r}\right\} \tag{5.79}$$

Substituting equation (5.79) into (5.77) leads to the closed form expression for the transformed complex potential function:

$$\phi_1^*(\zeta) = \frac{1}{2\pi i}\oint_\gamma \frac{F(\sigma)}{\sigma-\zeta}\, d\sigma + r\left\{\frac{Re(c_1)}{m-r} - i\,\frac{Im(c_1)}{m+r}\right\}\zeta \tag{5.80}$$

In order to find $\Psi_1^*(\zeta)$, the second integral of equation (5.71) must first be calculated. Multiplying equation (5.73) by equation (5.75)

$$\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\,\phi_1^{*\prime}(\sigma) = \frac{m\sigma^4+\ell\sigma^2+r}{(3r\sigma^4+\ell\sigma^2-m)} \cdot \sum_{k=1}^{\infty} k a_k \sigma^{k-1}$$

It is clear that the numerator of the equation above is analytic inside γ and the denominator has five roots, one of which is zero and four of which can be found by solving the biquadratic

$$3r\sigma^4 + \ell\sigma^2 - m = 0$$

Some of these roots are inside γ and some outside. Thus, the following integral can be evaluated by the Cauchy integral theorem presented in section I.3:

$$\frac{1}{2\pi i} \oint_\gamma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} \frac{\phi^{*\prime}(\sigma)}{\sigma-\zeta} d\sigma = \frac{m\zeta^4+\ell\zeta^2+r}{\zeta(3r\zeta^4+\ell\zeta^2-m)} \phi_1^{*\prime}(\zeta) + g(\zeta)$$

where $g(\zeta)$ represents the sum of the residues of

$$\frac{(m\sigma^4+\ell\sigma^2+r) \ \phi_1^{*\prime}(\sigma)}{\sigma(3r\sigma^4+\ell\sigma^2-m)(\sigma-\zeta)}$$

at the roots which are inside γ. Thus, the other transformed complex potential function, i.e., equation (5.71), becomes:

$$\Psi_1^*(\zeta) = \frac{1}{2\pi i} \oint_\gamma \frac{\overline{F(\sigma)}}{\sigma-\zeta} d\sigma - \frac{(m\zeta^4+\ell\zeta^2+r)}{\zeta(3r\zeta^4+\ell\zeta^2-m)} \cdot \phi_1^{*\prime}(\zeta) - g(\zeta) \tag{5.81}$$

Recall that $F(\sigma)$ and $\overline{F(\sigma)}$ were given by equations (5.25) and (5.26). The transformed complex potential functions $\phi_1^0(\sigma)$ and $\Psi_1^0(\sigma)$ can be easily found by substituting the transformation function, equation (5.48), into equations (2.6). The results are

$$\phi_1^0(\sigma) = -Q \ln \frac{m-Z_0\sigma+\ell\sigma^2+r\sigma^4}{\sigma} \tag{5.82}$$

$$\Psi_1^0(\sigma) = Q \frac{\overline{Z}_0\sigma}{m-Z_0\sigma+\ell\sigma^2+r\sigma^4} + \overline{Q} \cdot \alpha \ln \frac{m-Z_0\sigma+\ell\sigma^2+r\sigma^4}{\sigma} \tag{5.83}$$

For simplicity, let

$$A(\sigma) = m - Z_0\sigma + \ell\sigma^2 + r\sigma^4 \qquad (5.84)$$

$$B(\sigma) = r + \ell\sigma^2 - \overline{Z}_0\sigma^3 + m\sigma^4 \qquad (5.85)$$

Taking the derivative of equation (5.82), obtaining the complex conjugate of equations (5.82) and (5.83), and substituting into equations (5.25) and (5.26) leads to the expressions for $F(\sigma)$ and $\overline{F(\sigma)}$. The calculation is omitted and the results are

$$F(\sigma) = Q\left\{\ln \frac{A(\sigma)}{\sigma} - \alpha \ln \frac{B(\sigma)}{\sigma^3}\right\} + \overline{Q}\left\{\sigma^2 \cdot \frac{A(\sigma)}{B(\sigma)}\right\} \qquad (5.86)$$

$$\overline{F(\sigma)} = Q\left\{\frac{B(\sigma)}{\sigma^2 A(\sigma)}\right\} + \overline{Q}\left\{\ln \frac{B(\sigma)}{\sigma^3} - \alpha \ln \frac{A(\sigma)}{\sigma}\right\} \qquad (5.87)$$

where $A(\sigma)$ and $B(\sigma)$ are given by equations (5.84) and (5.85).

The two remaining integrals in the potential functions of equations (5.80) and (5.81) can now be calculated since $F(\sigma)$ and $\overline{F(\sigma)}$ are known.

To evaluate these two integrals, note that the conformality of the mapping function $\omega(\zeta)$ implies that the equation $A(\sigma) = 0$ has one root inside and three roots outside γ (since the polynomial of $Z-\omega(\zeta) = 0$ and $A(\sigma) = 0$ are identical).

Denote the inside root by r_i and the three outside roots by r_{o_1}, r_{o_2}, and r_{o_3}. Root r_i has a significant role in the calculation of the two integrals whereas r_{o_1}, r_{o_2}, and r_{o_3} need not be calculated. Root r_i can be calculated by the inverse transformation function using either of the two methods presented at the beginning of this section.

Now $A(\sigma)$ may be rewritten as:

$$A(\sigma) = r(\sigma - r_i)(\sigma - r_{o_1})(\sigma - r_{o_2})(\sigma - r_{o_3})$$

Similarly, equation $B(\sigma) = 0$ is identical to $\bar{Z} - \omega(1/\zeta) = 0$, one root of which is outside γ and three of which are inside. Denote the outside root by t_o and the inside roots by t_{i_1}, t_{i_2} and t_{i_3}. Thus $B(\sigma)$ may be rewritten as:

$$B(\sigma) = m(\sigma - t_{i_1})(\sigma - t_{i_2})(\sigma - t_{i_3})(\sigma - t_o)$$

Using these forms of $A(\sigma)$ and $B(\sigma)$, the two integrations may be computed quite simply in a manner similar to that presented in the previous section. The results are:

$$\frac{1}{2\pi i}\oint_\gamma \frac{F(\sigma)}{\sigma - \zeta}\, d\sigma = Q\left\{\ln r(\zeta - r_{o_1})(\zeta - r_{o_2})(\zeta - r_{o_3}) - \alpha \ln m(\zeta - t_o)\right\}$$

$$+ \bar{Q}\left\{\frac{\zeta^2 A(\zeta)}{B(\zeta)} + \sum_{k=1}^{3} \frac{(\sigma - t_{ik})t_{ik}^2 A(t_{ik})}{B(t_{ik})(t_{ik} - \zeta)}\right\} \quad (5.88)$$

$$\frac{1}{2\pi i}\oint_\gamma \frac{\overline{F(\sigma)}}{\sigma - \zeta}\, d\sigma = Q\left\{\frac{d}{d\sigma}\left.\frac{B(\sigma)}{A(\sigma)\cdot(\sigma - \zeta)}\right|_{\sigma=0} + \frac{B(\zeta)}{\zeta^2 A(\zeta)}\right.$$

$$\left. + \frac{B(r_i)(\sigma - r_i)}{r_i^2 A(r_i)(r_i - \zeta)}\right\} + \bar{Q}\left\{\ln m(\zeta - t_o)\right.$$

$$\left. - \alpha \ln r(\zeta - r_{o_1})(\zeta - r_{o_2})(\zeta - r_{o_3})\right\} \quad (5.89)$$

Substituting equations (5.88) and (5.89) into equations (5.80) and (5.81) leads to the complex transformation functions $\phi_1^*(\zeta)$ and $\Psi_1^*(\zeta)$. The procedure for obtaining the influence functions from $\phi_1^*(\zeta)$ and $\Psi_1^*(\zeta)$ is very straightforward. This is done by substitution of these functions into equation (5.69) and subsequent substitution of this result into equation (3.6).

V.5 ON THE INFLUENCE FUNCTION OF A MORE GENERAL CLASS OF OPENING

In the previous two sections, two cases of opening were considered and the influence functions were found. In this section a general form of this special kind of opening is discussed.

Consider an infinite plane bounded by the contour given by equations

$$X = R(\text{Cos } \theta + \varepsilon \text{ Cos } N\theta)$$

$$Y = R(c \text{ Sin } \theta - \varepsilon \text{ Sin } N\theta)$$

where $o<c\leqslant 1$ and N is an integer greater than 3 (Cases N = 2 and N = 3 have been discussed in sections V.3 and V.4, respectively). A concentrated point force P is acting in the plane at some point Z_0, where Z_0 lies on or outside of the opening.

Using the mapping technique presented in Chapter II, the problem can be expressed as the superposition of two problems. Consider the second of these superposed problems. The mapping function is:

$$Z = \omega(\zeta) = \frac{m}{\zeta} + \ell\zeta + r\zeta^N \qquad (5.90)$$

where

$$m = \frac{R(1+c)}{2} \quad , \quad \ell = \frac{R(1-c)}{2} \quad \text{and } r = R\cdot\varepsilon$$

The inverse mapping function could be found by following either of the two methods presented in section V.2.

To find the complex potential functions, equations (5.70) and (5.71), the following quantities are needed:

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} = \frac{\sigma^{N-2}(m+\ell\sigma^2+r\sigma^{N+1})}{r\cdot N+\ell\sigma^{N-1}-m\sigma^{N+1}}$$

and

$$\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} = \frac{m\sigma^{N+1}+\ell\sigma^{N-1}+r}{\sigma^{N-2}(r\cdot N\sigma^{N+1}+\ell\sigma^2-m)}$$

The functions $F(\sigma)$ and $\overline{F(\sigma)}$ which appear in equations (5.70) and (5.71) can be calculated by substituting the mapping function of equation (5.90) into equations (2.6) and taking the derivatives and complex conjugates. Inserting these functions into equations (5.25) and (5.26) leads to:

$$F(\sigma) = Q\left\{\ln\frac{A(\sigma)}{\sigma} - \alpha\ln\frac{B(\sigma)}{\sigma^N}\right\} + \overline{Q}\left\{\sigma^{N-1}\cdot\frac{A(\sigma)}{B(\sigma)}\right\} \qquad (5.91)$$

$$\overline{F(\sigma)} = Q\left\{\frac{B(\sigma)}{\sigma^{N-1}A(\sigma)}\right\} + \overline{Q}\left\{\ln\frac{B(\sigma)}{\sigma^N} - \alpha\ln\frac{A(\sigma)}{\sigma}\right\} \qquad (5.92)$$

where

$$A(\sigma) = m - Z_0\sigma + \ell\sigma^2 + r\sigma^{N+1}$$

$$B(\sigma) = r + \ell\sigma^{N-1} - \overline{Z_0}\sigma^N + m\sigma^{N+1}$$

Following the discussion presented in the previous section, it is clear that $A(\sigma)$ must have one root inside γ, r_i, and N - 1 roots outside γ. Also, $B(\sigma)$ must have one root outside, t_o, and N - 1 roots inside γ. Thus,

$$A(\sigma) = r(\sigma-r_i)(\sigma-r_{o_1})(\sigma-r_{o_2})\ldots.(\sigma-r_{oN-1})$$

$$B(\sigma) = m(\sigma-t_o)(\sigma-t_{i_1})(\sigma-t_{i_2})\ldots.(\sigma-t_{iN-1})$$

The method of integration in the previous section may be used to find the following integrals:

$$\frac{1}{2\pi i}\oint_\gamma \frac{F(\sigma)}{\sigma-\zeta} = Q\Big\{\ln r(\zeta-r_{o_1})(\zeta-r_{o_2})\ldots(\zeta-r_{oN-1})$$

$$-\alpha \ln m(\zeta-t_o)\Big\}+\overline{Q}\Big\{\frac{\zeta^{N-1}A(\zeta)}{B(\zeta)}$$

$$+\sum_{k=1}^{N-1} \frac{(\sigma-t_{iK})(t_{iK}^{N-1})A(t_{iK})}{B(t_{iK})(t_{iK}-\zeta)}\Big\}$$

$$\frac{1}{2\pi i}\oint_{\gamma} \frac{\overline{F(\sigma)}}{\sigma-\zeta}\, d\sigma = Q \left\{ \frac{1}{(N-2)!} \left[\frac{d^{N-2}}{d\sigma^{N-2}} \cdot \left(\frac{B(\sigma)}{(\sigma-\zeta)A(\sigma)} \right) \right]_{\sigma=0} \right.$$

$$\left. + \frac{B(\zeta)}{\zeta^{N-1}A(\sigma)} + \frac{B(r_i)(\sigma-r_i)}{r_i^{N-1}(r_i-\zeta)A(r_i)} \right\} +$$

$$\overline{Q} \left\{ \ln m(\zeta-t_o) - \alpha \ln r(\zeta-r_{o_1})(\zeta-r_{o_2}) \right.$$

$$\left. \ldots.(\zeta-r_{oN-1}) \right\}$$

The other two terms in the complex potential functions can be found exactly in the same manner as in section V.4. Then the influence functions can be obtained. The procedure is very straightforward and the calculation is omitted.

CHAPTER VI

CLOSURE

A boundary integral equation method for the solution of finite, two-dimensional, isotropic, linear elastic regions containing arbitrarily-shaped openings has been presented. It is shown that stress and displacements at any point away from the outer boundary can be easily found by this method. Since the effect of the opening has been included in the kernels of the integral equations, solutions on or near the opening have been obtained with excellent success.

A mapping technique has been employed to find the complex potential functions which lead to the determination of the influence function. The significance of this technique is that the influence function for any shape of cavity in an infinite plane can be found. In Chapter II it was shown how, with knowledge of the transformation of the contour to a unit circle, one could employ this mapping technique to obtain the influence function.

Such an influence function was found for a circular hole in an infinite two-dimensional region in Chapter III. To show the efficiency and applicability of the method to any two-dimensional region which contains a circular hole, the boundary integral equation method presented in Chapter I was applied to two example problems. In the final example problem (i.e., a rectangular plane containing the hole and subjected to a uniaxial tension), the results were within 0.8% of available solutions. This accuracy was obtained using just 35 seconds of CPU time on a CDC 6500 computer.

To show the applicability of the method to a different geometry, a second example (i.e., circular plane containing the hole and partially loaded) was considered. No available solutions for this problem were found. Again, the CPU was 35 seconds for each run on a CDC 6500 computer.

In Chapter IV, the mapping technique was used to determine the influence function associated with an elliptical hole in an infinite region. To solve any two-dimensional problem, this influence function was used as the kernel of the integral equations and two example problems having different locations and angles of inclination of the elliptical hole were solved. In the first example problem, four different orientations and inclination angles of elliptical hole in a rectangular plane subjected to the uniaxial tension were considered. For the first case, some experimental solutions at specific points were available. The results were within 3.0%. For these cases, the computer time never exceeded 42 seconds of CPU time on a CDC 6500 computer per run. It is important to note that the major computer time consumption for this method is in determining fictitious tractions. Computation of stresses and displacements at any point uses very little

CPU time. Thus, calculation of stresses and displacements at new points has an extremely small effect on total CPU time. Although such problems can also be solved by other numerical methods (such as finite elements and finite differences), the following advantages of the BIE method are apparent. In the BIE method, different sizes and locations of the elliptical hole can be specified by changing one or two parameters in the input data, whereas with the other methods, a new discretization of the region has to be made for each case. Also, as the elliptical hole gets thinner, more difficulty will arise as one needs to discretize the region close to the ends of the hole. No such discretization of the region is required in BIE.

The second example problem (i.e., a circular plane containing the hole and partially loaded) was presented as an example of the applicability of the method to different geometries. Again, four cases were considered. No available solutions for this example problem were found. Again, computer time was 42 seconds for each run on a CDC 6500 computer.

Some modifications were done to the influence function of the elliptical opening to make it applicable for a sharp crack. Thus, the influence function for an infinite plane containing a sharp crack was determined. This influence function was then used as the kernel of the integral equations and some finite two-dimensional problems (e.g., a rectangular plane containing the crack and subjected to a uniaxial tension), with different locations and inclination angles of the crack, were considered. The tips of the crack, at which the stresses are infinite, pose extreme difficulties for other numerical methods, such as finite elements and finite differences. With this new BIE method, the stresses and displacements very close to the tips of the crack can be determined with excellent accuracy and ease of computation. Solutions for the stresses near the tips of a horizontal crack were compared to the known stress intensity factor (which is valid just very close to the tips of the crack). The differences were within 0.1%. No analytical or numerical solutions were found for the other cases (crack at inclined angles). The crack opening displacements, COD, for all the cases were compared to some recently obtained experimental measurements on rectangular specimens containing inclined slots. The differences in results were all within 11%. This difference is expected since the "slot" of the experimental study was of finite width. The trend of the stresses and the displacements can be seen in the tables. The problem of an edge crack was also considered. CPU time did not exceed 42 seconds for any run.

Finally, in the last chapter the mapping technique was extended to a larger class of problems (different shapes of opening). The complete potential functions for two of these cases (i.e., the triangular opening and square opening) were obtained and the more general case was discussed. Further development is left for future research in this area.

APPENDICES

APPENDIX A

THE POTENTIAL FUNCTIONS AND THE INFLUENCE FOR AN INFINITE PLANE REGION CONTAINING A CIRCULAR HOLE

a) The potential function used in equations (3.40) are:

$$\phi_I^*(\zeta) = \alpha \ln(-\overline{Z}_0) - \alpha \ln(\zeta - \overline{Z}_0)$$

$$\phi_{II}^*(\zeta) = \frac{1 - Z_0\overline{Z}_0}{\overline{Z}_0} + \frac{1 - Z_0\overline{Z}_0}{\overline{Z}_0^2} \cdot \frac{1}{\zeta - \overline{Z}_0}$$

$$\Psi_I^*(\zeta) = -\frac{\zeta}{Z_0} - \frac{\alpha\zeta^3}{\zeta - \overline{Z}_0}$$

$$\Psi_{II}^*(\zeta) = \ln(\zeta - \overline{Z}_0) - \ln(-\overline{Z}_0) - \frac{1 - Z_0Z_0}{\overline{Z}_0} \cdot \frac{\zeta^3}{(\zeta - \overline{Z}_0)^2}$$

b) The influence functions $H_{ij;q}$ and $I_{i;q}$ used in equations (3.48) are:

$$H_{xx;x} = \mathrm{Re}\left\{ \frac{-2}{Z - Z_0} - 2\zeta^2\phi_I^{*\prime}(\zeta) - \overline{Z}\left(\frac{1}{(Z - Z_0)^2} + \zeta^4\phi_I^{*\prime\prime}(\zeta)\right.\right.$$

$$\left.\left. + 2\zeta^3\phi_I^{*\prime}(\zeta)\right) + \frac{\overline{Z}_0}{(Z - \overline{Z}_0)^2} - \frac{\alpha}{Z - \overline{Z}_0} + \zeta^2\Psi_I^{*\prime}(\zeta)\right\} \cdot \left[\frac{1}{2\pi(\alpha + 1)}\right]$$

$$H_{xx;y} = \mathrm{Re}\left\{ i\left(\frac{-2}{Z-Z_0} - 2\zeta^2(\phi_I^{*\prime}(\zeta)-\phi_{II}^{*\prime}(\zeta)) - \bar{Z}\left[\frac{1}{(Z-Z_0)^2} \right.\right.\right.$$

$$\left. + (\phi_I^{*\prime\prime}(\zeta)-\phi_{II}^{*\prime\prime}(\zeta))\zeta^4 - (\phi_I^{*\prime}(\zeta)-\phi_{II}^{*\prime}(\zeta))\ (2\zeta^3)\right]$$

$$\left.\left. + \frac{\bar{Z}_0}{(Z-\bar{Z}_0)^2} + \frac{\alpha}{Z-\bar{Z}_0} + \zeta^2(\Psi_I^{*\prime}(\zeta)-\Psi_{II}^{*\prime}(\zeta))\right)\right\} \cdot \left[\frac{1}{2\pi(\alpha+1)}\right]$$

$$H_{yy;x} = \mathrm{Re}\left\{ \frac{-2}{Z-Z_0} - 2\zeta^2\phi_I^*(\zeta) + \bar{Z}\left[\frac{1}{(Z-Z_0)^2} + \zeta^4\phi_I^{*\prime\prime}(\zeta) \right.\right.$$

$$\left.\left. + 2\phi_I^{*\prime}(\zeta)\cdot(\zeta)^3\right] - \frac{\bar{Z}_0}{(Z-\bar{Z}_0)^2} + \frac{\alpha}{Z-\bar{Z}_0} - \zeta^2\Psi_I^{*\prime}(\zeta)\right\} \cdot \left[\frac{1}{2\pi(\alpha+1)}\right]$$

$$H_{yy;y} = \mathrm{Re}\left\{ i\left(\frac{-2}{Z-Z_0} - 2\zeta^2(\phi_I^{*\prime}(\zeta)-\phi_{II}^{*\prime}(\zeta)) + \bar{Z}\left[\frac{Z_0}{(Z-Z_0)^2} \right.\right.\right.$$

$$\left. + (\phi_I^{*\prime\prime}(\zeta)-\phi_{II}^{*\prime\prime}(\zeta))\zeta^4 - (\phi_I^{*\prime}(\zeta)-\phi_{II}^{*\prime}(\zeta))\ (2\zeta^3)\right]$$

$$\left.\left. - \frac{\bar{Z}_0}{(Z-\bar{Z}_0)^2} - \frac{\alpha}{Z-\bar{Z}_0} + \zeta^2(\Psi_I^{*\prime}(\zeta)-\Psi_{II}^{*\prime}(\zeta))\right)\right\} \cdot \left[\frac{1}{2\pi(\alpha+1)}\right]$$

$$H_{xy;x} = \mathrm{Im}\left\{ \bar{Z}\left[\frac{1}{(Z-Z_0)^2} + \zeta^4 \phi_I^{*\prime\prime}(\zeta) + 2\,\phi_I^{*\prime}(\zeta) \cdot (\zeta)^3 \right] \right.$$

$$\left. - \frac{\bar{Z}_0}{(Z-\bar{Z}_0)^2} + \frac{\alpha}{Z-\bar{Z}_0} - \zeta^2 \Psi_I^{*\prime}(\zeta) \right\} \cdot \left[\frac{1}{2\pi(\alpha+1)} \right]$$

$$H_{xy;y} = \mathrm{Im}\left\{ i\left(\bar{Z}\left[\frac{1}{(Z-Z_0)^2} + (\phi_I^{*\prime\prime}(\zeta) - \phi_{II}^{*\prime\prime}(\zeta))\,\zeta^4 - (\phi_I^{*\prime}(\zeta) \right. \right. \right.$$

$$\left. - \phi_{II}^{*\prime}(\zeta))\,(2\zeta^3) \right] - \frac{\bar{Z}_0}{(Z-Z_0)^2} - \frac{\alpha}{Z-\bar{Z}_0} + \zeta^2 (\Psi_I^{*\prime}(\zeta)$$

$$\left.\left. - \Psi_{II}^{*\prime}(\zeta)) \right) \right\} \cdot \left[\frac{1}{2\pi(\alpha+1)} \right]$$

$$I_{x;x} = \mathrm{Re}\left\{ -\alpha \ln(Z-Z_0) + \alpha\, \phi_1^*(\zeta) - Z\left[\frac{-1}{\bar{Z}-\bar{Z}_0} - \bar{\zeta}^2\, \overline{\phi_1^{*\prime}(\zeta)} \right] \right.$$

$$\left. - \frac{\bar{Z}_0}{\bar{Z}-Z_0} - \alpha \ln(\bar{Z}-Z_0) - \overline{\Psi_1^*(\zeta)} \right\} / 4\mu\pi(\alpha+1)$$

$$I_{x;y} = \mathrm{Re}\left\{ i\left(-\alpha \ln(Z-Z_0) + \alpha\,(\phi_I^*(\zeta) - \phi_{II}^*(\zeta)) - Z\left[\frac{-1}{\bar{Z}-\bar{Z}_0} \right. \right. \right.$$

$$\left. - \overline{(\phi_I^{*\prime}(\zeta) - \phi_{II}^{*\prime}(\zeta))}\;\bar{\zeta}^2 \right] - \frac{Z_0}{\bar{Z}-Z_0} + \alpha \ln(\bar{Z}-Z_0)$$

$$\left.\left. - \overline{\Psi_I^*(\zeta) + \Psi_{II}^*(\zeta)} \right) \right\} / 4\pi\mu(\alpha+1)$$

$$I_{y;x} = \mathrm{Im}\left\{ -\alpha \ln(Z-Z_0) + \alpha\, \phi_1^*(\zeta) - Z\left[-\frac{1}{\overline{Z}-\overline{Z}_0} - \overline{\zeta}^2\, \overline{\phi_1^{*\prime}(\zeta)} \right] \right.$$

$$\left. - \frac{Z_0}{\overline{Z}-Z_0} - \alpha \ln(\overline{Z}-Z_0) - \overline{\Psi_1^*(\zeta)} \right\} / 4\mu\pi(\alpha+1)$$

$$I_{y;y} = \mathrm{Im}\left\{ i\left(-\alpha \ln(Z-Z_0) + \alpha\,(\phi_I^*(\zeta) - \phi_{II}^*(\zeta)) - Z\left[\frac{-1}{\overline{Z}-Z_0} \right.\right.\right.$$

$$\left. - (\overline{\phi_I^{*\prime}(\zeta)} - \overline{\phi_{II}^{*\prime}(\zeta)})\ \overline{\zeta}^2 \right] - \frac{Z_0}{\overline{Z}-Z_0} + \alpha \ln(\overline{Z}-Z_0) - \overline{\Psi_I^*(\zeta)}$$

$$\left.\left. + \Psi_{II}^*(\zeta) \right) \right\} / 4\mu\pi(\alpha+1)$$

APPENDIX B

COMPUTER PROGRAM FOR PLANE, FINITE REGION CONTAINING A CIRCULAR HOLE

A computer program was employed for the numerical computation of the stresses and the displacements at the field points of a two-dimensional region containing a unit circular hole. A listing of that program for the rectangular region subjected to uniaxial tension (ω = 1.0 MPa) and containing a unit circular hole and for the circular plane subjected to a uniformly radially tension (ω = 1.0 MPa) and containing a unit circular hole are presented in this appendix.

A. INPUT DATA

The following information must be provided as input (this is the order of appearance in the program).

PR	- Poisson's ratio of the material
EMUD	- modulus of elasticity of the material
NML	- total number of subdivisions on the boundary
NFP	- the number of field points at which the stresses and displacements are to be computed
X(I),Y(I) I=1,NML+1	- coordinates of the outer boundary points (for circular plane case, this input has been included in the program)

WR	-	"width ratio" for the rectangular plane (WR = width/10 cm) and for the circular plane (WR = radius/6.0 cm)
X_0, Y_0	-	location of the center of the circular hole (has to be specified in the program)
$BV_x(I)$, $BV_y(I+NML)$ I=1,NML	-	boundary value (tractions) specified at each subdivision
XF(I), YF(I) I=1,NFP	-	coordinates of the field points at which the stress and displacements are to be computed

B. OUTPUT DATA

The following information is obtained as output (this is in the order of appearance in the output).

PR,EMUD	-	see input data
SHMUD	-	shear modulus of elasticity
NML,NFP	-	see input data
X(I),Y(I)	-	see input data
$BV_x(I)$, $BV_y(I+NML)$	-	see input data
PSX,PSY	-	components of the fictitious traction on the boundary, represented by the concentrated load, P^*_{xi}, P^*_{yi} at the center of each one of the subdivisions. These are computed by solving a system of linear equations (3.51) (LEQT1F, computer library)
XF(I), YF(I) I=1,NFP	-	location of the field points at which the stresses and displacements are computed

SIGMAXX, SIGMAYY, SIGMAXY, UX,UY	- components of stress and displacement tensor at each of the field points. These are computed by the use of equation (1.17).

C. COMPUTER PROGRAM FOR A RECTANGULAR PLANE CONTAINING A CIRCULAR HOLE (follows)

```
      PROGRAM  FPWCH     (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
      DIMENSION    X(61),Y(61),RM(120,120),WKAREA(120),BV(120)
      DIMENSION    XF(15),YF(15)
C     ****************************************************************
      REAL    NXI,NYI
      COMPLEX   Z(60),ZE(60),Z2(60),Z3(60),ZEC(60),F1(60),FZ(60)
      COMPLEX   ZEF(15),ZF(15)
      COMPLEX   PO1,PO2,P1,P2,H1,H2,K1,K2,S1,S2,R1,R2,R3,R4,R5,R6
      COMPLEX   POO1,POO2,ZT1,ZT2,ZO1,ZO2,WO,WO2,WO3,WOO
      COMPLEX   PZ,QM,QC,QM2,QC2
      COMPLEX   ZF2,ZF3,ZEFC,ET
      COMPLEX   ZED,ZED2,ZEFO,ZEFD2,X1,X2,QC3
C     ****************************************************************
C     ****************************************************************


100   FORMAT (2F12.8)
110   FORMAT (*0*,14X,*POISSONS RATIO*,10X,*MODULUS OF ELASTICITY*
     $ ,13X,*SHEAR MODULUS*)
120   FORMAT (*0*,18X,F10.3,10X,F20.8,10X,F20.8)
130   FORMAT (2I6)
140   FORMAT (////,10X,*NO. OF MESHES=*,I3,25X,*NO. OF FIELD POINTS=*
     $ ,I3)
150   FORMAT (6F10.5)
160   FORMAT (///,10X,*NODE NO. (I)*,22X,*X(I)*,18X,*Y(I)*/*0*/
     $ (*0*,10X,I5,25X,F8.3,15X,F8.3))
170   FORMAT (6F10.4)
180   FORMAT(////,13X,*MESH NO.*,19X,*BVX*,29X,*BVY*/*0*/(*0*,15X,I4,
     $ 16X,F10.5,25X,F10.5))
190   FORMAT  (////,*0*,38X,*FICTIOUS TRACTIONS*/*0*/*0*,4X,
     $ *MESH NO.*,19X,*PSX*,19X,*PSY*/(*0*,4X,I4,11X,2(F20.10)))
200   FORMAT (6F10.6)
210   FORMAT (////,20X,*FIELD POINT NO.*,24X,*XF(I)*,28X,*YF(I)*
     $ /*0*/(*0*,25X,I5,25X,F8.3,25X,F8.3))
220   FORMAT  (////,35X,*STRESSES  AND  DISPLACEMENTS  OF  THE  FIELD
     $ POINTS*///,2X,*NO. OF F.P.*,9X,*SIGMAXX*,18X,*SIGMAYY*,18X,
     $ *SIGMAXY*,20X,*UX*,22X,*UY*)
230   FORMAT  (///,4X,I3,5(5X,F20.10))
240   FORMAT (////,25X,*TRANSFORMED  BOUNDARY  POINTS  TO  ZETA  *,
     $ *PLANE*///,10X,*NODE NO. (I)*,22X,*REAL*,18X,*IMAG*)
250   FORMAT  (*0*,10X,I5,25X,F8.3,15X,F8.3)
260   FORMAT (////,25X,*TRANSFORMED  FIELD  POINTS  TO  ZETA PLANE*
     $ ,////,20X,*FIELD POINT NO. *,24X,*REAL*,29X,*IMAG*)
270   FORMAT (//,25X,I4,2(25X,F8.3))
C     ****************************************************************

C     ******************************
C     *     MATERIAL  PROPERTIES    *
C     ******************************
      READ (5,100) PR,EMUD
      SHMUD=EMUD/(2*(1+PR))
      WRITE (6,110)
      WRITE (6,120)   PR , EMUD , SHMUD



C     ********************************************************
C     *               GEOMETRY OF THE PROBLEM                *
C     *            TRACTION BOUNDARY CONDITION               *
C     *     RECTANGULAR,PLANE STRESS,UNIAXIAL TENSION        *
C     *  NUMBER OF MESH LENGTH=NML,NUMBER OF FIELD POINT=NFP *
C     ********************************************************

      READ  (5,130)  NML , NFP
      WRITE  (6,140)  NML , NFP
      NML1=NML+1
      NML2=NML*2
      PI=4*ATAN(1.0)
      READ  (5,150)   (X(I),Y(I),I=1,NML1)
C     **************************************************************
C     *      CALCULATION OF THE NEW BOUNDARY POINTS DUE TO          *
C     *   TRANSLATION OF THE COORDINATES AND CHANG IN WIDTH RATIO   *
C     *      WHERE  (X0,Y0)=NEW COORDINATES OF THE ORIGIN AND       *
C     *                WIDTH RATIO  WR=WIDTH/10.0                   *
C     **************************************************************
      WR=1.0
      X0=0.0
      Y0=0.0
      DO  40   I=1,NML1
```

```
      X(I)=WR*(X(I)-X0)
      Y(I)=WR*(Y(I)-Y0)
40    CONTINUE
      WRITE (6,160) (I,X(I),Y(I),I=1,NML1)
C     **************************************************************
C     *     CALCULATION  OF  COEFFICIENTS                          *
C     **************************************************************
      EI=(0.0,1.0)
      IPLANE=1
      IF(IPLANE.EQ.2)  GO TO 10
      ALPHA=(3-PR)/(1+PR)
      GO TO 11
10    ALPHA=3-4*PR
11    COEF=1/(2*PI*(1+ALPHA))
C     ******************************************************
C     *   TRANSFORMATION  OF  THE  BOUNDARY  POINTS    *
C     ******************************************************
      WRITE  (6,240)
      DO  12  I=1,NML
      ZE(I)=CMPLX((X(I)+X(I+1))/2,(Y(I)+Y(I+1))/2)
      ZEC(I)=CONJG(ZE(I))
      Z(I)=1/ZE(I)
      A=REAL(Z(I))
      B=AIMAG(Z(I))
      WRITE  (6,250)  I,A,B
      F1(I)=1-ZE(I)*ZEC(I)
      FZ(I)=F1(I)/ZEC(I)
      Z2(I)=Z(I)**2
      Z3(I)=Z2(I)*Z(I)
12    CONTINUE
C     **************************************************************
C     *  CALCULATION OF ELEMENTS OF THE RM(NML2,NML2) MATRIX  *
C     **************************************************************

C     =============================
      DO  14  I=1,NML
C     =============================
      DSI=SQRT((X(I+1)-X(I))**2+(Y(I+1)-Y(I))**2)
      NXI=(Y(I+1)-Y(I))/DSI
      NYI=(X(I)-X(I+1))/DSI
      WD=-1/Z2(I)
      WD2=WD**2
      WD3=WD2*WD
      WDD=2/Z3(I)
C     ===================
      DO  15  J=1,NML
C     ===================
      IF (I.EQ.J)  GO  TO  16
      ZED=ZE(I)-ZE(J)
      ZED2=ZED**2
      PZ=ZE(J)*Z(I)
      QM=1-PZ
      QM2=QM**2
      QC=Z(I)-ZEC(J)
      QC2=QC**2
      QC3=QC2*QC
      PD1=-ALPHA/QC
      PD2=-FZ(J)/QC2
      PDD1=ALPHA/QC2
      PDD2=FZ(J)*2/QC3
      ZD1=-ALPHA*Z2(I)*(2*Z(I)-3*ZEC(J))/QC2-1/ZE(J)
      ZD2=1/QC-FZ(J)*Z2(I)*(Z(I)-3*ZEC(J))/QC3
      H1=PD1/WD
      H2=PD2/WD
      S1=ZD1/WD
      S2=ZD2/WD
      K1=PDD1/WD2-PD1*WDD/WD3
      K2=PDD2/WD2-PD2*WDD/WD3
      H1=H1-1/ZED
      K1=K1+1/ZED2
      S1=S1-ZEC(J)/ZED2
      S2=S2+ALPHA/ZED
      R1=2*(H1+H2)
      R2=ZEC(I)*(K1+K2)+S1+S2
```

```
      R3=2*(H1-H2)
      R4=ZEC(I)*(K1-K2)+S1-S2

      RM(I,J)=(REAL(R1-R2)*NXI+AIMAG(R2)*NYI)*COEF*DSI
      RM(I,J+NML)=(REAL(EI*(R3-R4))*NXI+AIMAG(EI*R4)*NYI)*COEF*DSI
      RM(I+NML,J)=(AIMAG(R2)*NXI+REAL(R1+R2)*NYI)*COEF*DSI
     RM(I+NML,J+NML)=(AIMAG(EI*R4)*NXI+REAL(EI*(R3+R4))*NYI)*COEF*DSI

      GO  TO  15
16    RM(I,I)=1/2.0
      RM(I,I+NML)=0.0
      RM(I+NML,I)=0.0
      RM(I+NML,I+NML)=1/2.0

C     ======================================================
15    CONTINUE
C     ======================================================
14     CONTINUE
C     ======================================================

C     **********************************************************
C     *    CALCULATION OF FICTITIOUS TRACTION   P-STAR    PS(I)    *
C     **********************************************************

      READ  (5,170)   (BV(I),BV(I+NML),I=1,NML)
C     ************************************************************
C     *    CORRECTING THE B.V. DUE TO THE CHANG IN WIDTH RATIO TO    *
C     *           KEEP THE DISTRIBUTED LOAD =1.0 LB/LENGTH           *
C     ************************************************************
      DO  46  I=1,NML
      BV(I)=BV(I)*WR
      BV(I+NML)=BV(I+NML)*WR
46    CONTINUE
      WRITE  (6,180)   (I,BV(I),BV(I+NML),I=1,NML)

      CALL  LEQT1F   (RM,1,NML2,NML2,BV,8,WKAREA,IER)
      WRITE  (6,190)   (I,BV(I),BV(I+NML),I=1,NML)

C     ***************************************************
C     *   CALCULATION  OF  STRESSES  AND  DISPLACEMENTS   *
C     ***************************************************

      READ  (5,200)   (XF(I),YF(I),I=1,NFP)
      WRITE  (6,210)   (I,XF(I),YF(I),I=1,NFP)
      WRITE  (6,260)
      DO  20  I=1,NFP
      ZEF(I)=CMPLX(XF(I),YF(I))

      ZF(I)=1/ZEF(I)
24    A=REAL(ZF(I))
      B=AIMAG(ZF(I))
      WRITE  (6,270)   I,A,B
20    CONTINUE
      WRITE  (6,220)
C     ================================
      DO  21  I=1,NFP
C     ================================
      SGMXX=0.0
      SGMYY=0.0
      SGMXY=0.0
      UX=0.0
      UY=0.0
      ZF2=ZF(I)**2
      ZF3=ZF(I)*ZF2
      WO=-1/ZF2
      WO2=WO**2
      WO3=WO2*WO
      WOO=2/ZF3
      ZEFC=CONJG(ZEF(I))
C     ================================
      DO  22  J=1,NML
C     ================================
      ZEFD=ZEF(I)-ZE(J)
      ZEFD2=ZEFD**2
      PZ=ZE(J)*ZF(I)
      QM=1-PZ
      QM2=QM**2
      QC=ZF(I)-ZEC(J)
      QC2=QC**2
      QC3=QC2*QC
      X1=CLOG(QC)
```

```
      X2=CLOG(-ZEC(J))
      P1=ALPHA*X2-ALPHA*X1
      P2=FZ(J)/QC+FZ(J)/ZEC(J)
      PD1=-ALPHA/QC
      PD2=-FZ(J)/QC2
      PDD1=+ALPHA/QC2
      PDD2=FZ(J)*2/QC3
      ZT1=-ALPHA*ZF3/QC-Z(I)/ZE(J)
      ZT2=-FZ(J)*ZF3/QC2+X1-X2
      ZD1=-ALPHA*ZF2*(2*ZF(I)-3*ZEC(J))/QC2-1/ZE(J)
      ZD2=1/QC-FZ(J)*ZF2*(ZF(I)-3*ZEC(J))/QC3

      H1=PD1/WD
      H2=PD2/WD
      S1=ZD1/WD
      S2=ZD2/WD
      K1=PDD1/WD2-PD1*WDD/WD3
      K2=PDD2/WD2-PD2*WDD/WD3

      P1=P1-CLOG(ZEFC)
      H1=H1-1/ZEFD
      K1=K1+1/ZEFD2
      ZT1=ZT1+ZEC(J)/ZEFD
      ZT2=ZT2+ALPHA*CLOG(ZEFD)
      S1=S1-ZEC(J)/ZEFD2
      S2=S2+ALPHA/ZEFD

      R1=2*(H1+H2)
      R2=ZEFC*(K1+K2)+S1+S2
      R3=2*(H1-H2)
      R4=ZEFC*(K1-K2)+S1-S2
      R5=ALPHA*(P1+P2)-ZEF(I)*CONJG(H1+H2)-CONJG(ZT1+ZT2)
      R6=ALPHA*(P1-P2)+ZEF(I)*CONJG(H1-H2)+CONJG(ZT1-ZT2)

      SGMXX=SGMXX+COEF*REAL((R1-R2)*BV(J)+EI*(R3-R4)*BV(J+NML))
      SGMYY=SGMYY+COEF*REAL((R1+R2)*BV(J)+EI*(R3+R4)*BV(J+NML))
      SGMXY=SGMXY+COEF*AIMAG(R2*BV(J)+EI*R4*BV(J+NML))
      UX=UX+COEF*REAL(R5*BV(J)+EI*R6*BV(J+NML))/(2*SHMUD)
      UY=UY+COEF*AIMAG(R5*BV(J)+EI*R6*BV(J+NML))/(2*SHMUD)
C     ==================================================================
22    CONTINUE
C     ******************************************************************
      WRITE   (6,230)   I,SGMXX,SGMYY,SGMXY,UX,UY
C     ==================================================================
21    CONTINUE
C     ==================================================================
      END
```

D. COMPUTER PROGRAM FOR A CIRCULAR PLANE CONTAINING A CIRCULAR HOLE (follows)

```
      PROGRAM   FRWCH      (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
      DIMENSION     X(61),Y(61),RM(120,120),WKAREA(120),BV(120)
      DIMENSION     XF(15),YF(15)
C     ****************************************************************
      REAL    NXI,NYI
      COMPLEX   Z(60),ZE(60),Z2(60),Z3(60),ZEC(60),F1(60),F2(60)
      COMPLEX   ZEF(15),ZF(15)
      COMPLEX   PO1,PO2,P1,P2,H1,H2,K1,K2,S1,S2,R1,R2,R3,R4,R5,R6
      COMPLEX   POO1,POO2,ZT1,ZT2,ZO1,ZO2,WO,WO2,WO3,WOO
      COMPLEX   P3,QM,QC,QM2,QC2
      COMPLEX   ZF2,ZF3,ZEFC,EI
      COMPLEX   ZEO,ZEO2,ZEFO,ZEFO2,X1,X2,QC3
C     ****************************************************************
C     ****************************************************************


100   FORMAT (2F12.8)
110   FORMAT (*0*,14X,*POISSONS RATIO*,10X,*MODULUS OF ELASTICITY*
     $ ,13X,*SHEAR MODULUS*)
120   FORMAT (*0*,18X,F10.8,10X,F20.8,10X,F20.8)
130   FORMAT (2I6)
140   FORMAT (////,10X,*NO. OF MESHES=*,I3,25X,*NO. OF FIELD POINTS=*
     $ ,I3)
150   FORMAT (6F10.5)
160   FORMAT (///,10X,*NODE NO. (I)*,22X,*X(I)*,18X,*Y(I)*/*0*/
     $ (*0*,10X,I5,25X,F8.3,15X,F8.3))
170   FORMAT (6F10.9)
180   FORMAT(////,13X,*MESH NO.*,19X,*BVX*,29X,*BVY*/*0*/(*0*,15X,I4,
     $ 16X,F10.5,25X,F10.5))
190   FORMAT  (////,*0*,38X,*FICTIOUS TRACTIONS*/*0*/*0*,4X,
     $ *MESH NO.*,19X,*PSX*,19X,*PSY*/(*0*,4X,I4,11X,2(F20.10)))
200   FORMAT (6F10.6)
210   FORMAT (////,20X,*FIELD POINT NO.*,24X,*XF(I)*,28X,*YF(I)*
     $ /*0*/(*0*,25X,I5,25X,F8.3,25X,F8.3))
220   FORMAT  (////,35X,*STRESSES  AND  DISPLACEMENTS  OF  THE  FIELD
     $ POINTS*///,2X,*NO. OF F.P.*,8X,*SIGMAXX*,15X,*SIGMAYY*,15X,
     $ *SIGMAXY*,20X,*UX*,22X,*UY*)
230   FORMAT  (///,4X,I3,5(5X,F20.10))
240   FORMAT (////,25X,*TRANSFORMED  BOUNDARY  POINTS  TO  ZETA  *,
     $ *PLANE*///,10X,*NODE NO. (I)*,22X,*REAL*,15X,*IMAG*)
250   FORMAT  (*0*,10X,I5,25X,F8.3,15X,F8.3)
260   FORMAT (////,25X,*TRANSFORMED  FIELD  POINTS  TO  ZETA PLANE*
     $ ,////,20X,*FIELD POINT NO. *,24X,*REAL*,28X,*IMAG*)
270   FORMAT  (//,25X,I4,2(25X,F8.3))
C     ****************************************************************

C     ****************************
C     *    MATERIAL  PROPERTIES  *
C     ****************************
      READ (5,100) PR,EMUD
      SHMUD=EMUD/(2*(1+PR))
      WRITE (6,110)
      WRITE (6,120)   PR , EMUD , SHMUD



C     ******************************************************
C     *               GEOMETRY OF THE PROBLEM               *
C     *             TRACTION BOUNDARY CONDITION            *
C     *        RECTANGULAR,PLANE STRESS,UNIAXIAL TENSION   *
C     *  NUMBER OF MESH LENGTH=NML,NUMBER OF FIELD POINT=NFP *
C     ******************************************************

      READ  (5,130)  NML , NFP
      WRITE  (6,140)  NML , NFP
      NML1=NML+1
      NML2=NML+2
      PI=4*ATAN(1.0)
C     *********************************************************
C     *   CALCULATION OF BOUNDARY POINTS FOR THE CIRCULAR DISC  *
C     *********************************************************
      DO  50  J=1,NML1
      I=J-1
      X(J)=6*COS(PI*(1+I/15.)/2.)
      Y(J)=6*SIN(PI*(1+I/15.)/2.)
50    CONTINUE
C      ************************************************************
C     *      CALCULATION OF THE NEW BOUNDARY POINTS DUE TO          *
```

```
C     *   TRANSLATION OF THE COORDINATES AND CHANG IN WIDTH RATIO     *
C     *      WHERE  (X0,Y0)=NEW COORDINATES OF THE ORIGIN AND          *
C     *                    WIDTH RATIO  WR=WIDTH/10.0                  *
C     ******************************************************************
      WR=1.0
      X0=0.0
      Y0=0.0
      DO  40   I=1,NML1
      X(I)=WR*(X(I)-X0)
      Y(I)=WR*(Y(I)-Y0)
40    CONTINUE
      WRITE  (6,160)  (I,X(I),Y(I),I=1,NML1)
C     ***********************************************************
C     *       CALCULATION  OF  COEFFICIENTS                     *
C     ***********************************************************
      EI=(0.0,1.0)
      IPLANE=1
      IF(IPLANE.EQ.2)   GO TO 10
      ALPHA=(3-PR)/(1+PR)
      GO TO 11
10    ALPHA=3-4*PR
11    COEF=1/(2*PI*(1+ALPHA))
C     *****************************************************
C     *    TRANSFORMATION  OF  THE  BOUNDARY  POINTS      *
C     *****************************************************
      WRITE  (6,240)
      DO  12  I=1,NML
      ZE(I)=CMPLX((X(I)+X(I+1))/2,(Y(I)+Y(I+1))/2)
      ZEC(I)=CONJG(ZE(I))
      Z(I)=1/ZE(I)

      A=REAL(Z(I))
      B=AIMAG(Z(I))
      WRITE  (6,250)   I,A,B

      F1(I)=1-ZE(I)*ZEC(I)
      FZ(I)=F1(I)/ZEC(I)
      Z2(I)=Z(I)**2
      Z3(I)=Z2(I)*Z(I)
12    CONTINUE
C     ***********************************************************
C     *  CALCULATION OF ELEMENTS OF THE RM(NML2,NML2) MATRIX    *
C     ***********************************************************

C     ==============================
      DO  14   I=1,NML
C     ==============================
      DSI=SQRT((X(I+1)-X(I))**2+(Y(I+1)-Y(I))**2)
      NXI=(Y(I+1)-Y(I))/DSI
      NYI=(X(I)-X(I+1))/DSI
      W0=-1/Z2(I)
      W02=W0**2
      W03=W02*W0
      W00=2/Z3(I)
C     ====================
      DO  15   J=1,NML
C     ====================
      IF (I.EQ.J)   GO   TO   16
      ZED=ZE(I)-ZE(J)
      ZED2=ZED**2
      PZ=ZE(J)*Z(I)
      QM=1-PZ
      QM2=QM**2
      QC=Z(I)-ZEC(J)
      QC2=QC**2
      QC3=QC2*QC
      P01=-ALPHA/QC
      P02=-FZ(J)/QC2
      P001=ALPHA/QC2
      P002=FZ(J)*2/QC3
      Z01=-ALPHA*Z2(I)*(2*Z(I)-3*ZEC(J))/QC2-1/ZE(J)
      Z02=1/QC-FZ(J)*Z2(I)*(Z(I)-3*ZEC(J))/QC3
      H1=P01/W0
      H2=P02/W0
      S1=Z01/W0
      S2=Z02/W0
      K1=P001/W02-P01*W00/W03
```

```
      K2=PDD2/WD2-PC2*WDD/WD3
      H1=H1-1/ZED
      K1=K1+1/ZED2
      S1=S1-ZEC(J)/ZED2
      S2=S2+ALPHA/ZED

      R1=2*(H1+H2)
      R2=ZEC(I)*(K1+K2)+S1+S2
      R3=2*(H1-H2)
      R4=ZEC(I)*(K1-K2)+S1-S2

      RM(I,J)=(REAL(R1-R2)*NXI+AIMAG(R2)*NYI)*COEF*DSI
      RM(I,J+NML)=(REAL(EI*(R3-R4))*NXI+AIMAG(EI*R4)*NYI)*COEF*DSI
      RM(I+NML,J)=(AIMAG(R2)*NXI+REAL(R1+R2)*NYI)*COEF*DSI
     RM(I+NML,J+NML)=(AIMAG(EI*R4)*NXI+REAL(EI*(R3+R4))*NYI)*COEF*DSI

      GO TO 15
16    RM(I,I)=1/2.0
      RM(I,I+NML)=0.0
      RM(I+NML,I)=0.0
      RM(I+NML,I+NML)=1/2.0

C     ==============================================================
15    CONTINUE
C     ==============================================================
14     CONTINUE
C     ==============================================================

C     **************************************************************
C     *   CALCULATION OF FICTITIOUS TRACTION   P-STAR    PS(I)     *
C     **************************************************************

      READ  (5,170)  (BV(I),BV(I+NML),I=1,NML)
C     ***************************************************************
C     *    CORRECTING THE B.V. DUE TO THE CHANG IN WIDTH RATIO TO    *
C     *          KEEP THE DISTRIBUTED LOAD =1.0 LB/LENGTH            *
C     ***************************************************************
      DO  46  I=1,NML
      BV(I)=BV(I)*WR
      BV(I+NML)=BV(I+NML)*WR
46    CONTINUE
      WRITE  (6,180)  (I,BV(I),BV(I+NML),I=1,NML)

      CALL  LEQT1F  (RM,1,NML2,NML2,BV,3,WKAREA,IER)
      WRITE  (6,190)  (I,BV(I),BV(I+NML),I=1,NML)

C     ***********************************************************
C     *   CALCULATION  OF  STRESSES  AND  DISPLACEMENTS    *
C     ***********************************************************

      READ  (5,200)  (XF(I),YF(I),I=1,NFP)
      WRITE  (6,210)  (I,XF(I),YF(I),I=1,NFP)
      WRITE  (6,260)
      DO  20  I=1,NFP
      ZEF(I)=CMPLX(XF(I),YF(I))

      ZF(I)=1/ZEF(I)
24    A=REAL(ZF(I))
      B=AIMAG(ZF(I))
      WRITE  (6,270)  I,A,B
20    CONTINUE
      WRITE  (6,220)
C     ==================================
      DO  21  I=1,NFP
C     ==================================
      SGMXX=0.0
      SGMYY=0.0
      SGMXY=0.0
      UX=0.0
      UY=0.0
      ZF2=ZF(I)**2
      ZF3=ZF(I)*ZF2
      WD=-1/ZF2
      WD2=WD**2
      WD3=WD2*WD
      WDD=2/ZF3
      ZEFC=CONJG(ZEF(I))
C     ==================================
      DO  22  J=1,NML
C     ==================================
      ZEFD=ZEF(I)-ZE(J)
```

```
      ZEFD2=ZEFD**2
      PZ=ZE(J)*ZF(I)
      QM=1-PZ
      QM2=QM**2
      QC=ZF(I)-ZEC(J)
      QC2=QC**2
      QC3=QC2*QC
      X1=CLOG(QC)
      X2=CLOG(-ZEC(J))
      P1=ALPHA*X2-ALPHA*X1
      P2=FZ(J)/QC+FZ(J)/ZEC(J)
      PD1=-ALPHA/QC
      PD2=-FZ(J)/QC2
      PDD1=+ALPHA/QC2
      PDD2=FZ(J)*2/QC3
      ZT1=-ALPHA*ZF3/QC-Z(I)/ZE(J)
      ZT2=-FZ(J)*ZF3/QC2+X1-X2
      ZD1=-ALPHA*ZF2*(2*ZF(I)-3*ZEC(J))/QC2-1/ZE(J)
      ZD2=1/QC-FZ(J)*ZF2*(ZF(I)-3*ZEC(J))/QC3

      H1=PD1/WD
      H2=PD2/WD
      S1=ZD1/WD
      S2=ZD2/WD
      K1=PDD1/WD2-PD1*WDD/WD3
      K2=PDD2/WD2-PD2*WDD/WD3

      P1=P1-CLOG(ZEFD)
      H1=H1-1/ZEFD
      K1=K1+1/ZEFD2
      ZT1=ZT1+ZEC(J)/ZEFD
      ZT2=ZT2+ALPHA*CLOG(ZEFD)
      S1=S1-ZEC(J)/ZEFD2
      S2=S2+ALPHA/ZEFD

      R1=2*(H1+H2)
      R2=ZEFC*(K1+K2)+S1+S2
      R3=2*(H1-H2)
      R4=ZEFC*(K1-K2)+S1-S2
      R5=ALPHA*(P1+P2)-ZEF(I)*CONJG(H1+H2)-CONJG(ZT1+ZT2)
      R6=ALPHA*(P1-P2)+ZEF(I)*CONJG(H1-H2)+CONJG(ZT1-ZT2)

      SGMXX=SGMXX+COEF*REAL((R1-R2)*BV(J)+EI*(R3-R4)*BV(J+NML))
      SGMYY=SGMYY+COEF*REAL((R1+R2)*BV(J)+EI*(R3+R4)*BV(J+NML))
      SGMXY=SGMXY+COEF*AIMAG(R2*BV(J)+EI*R4*BV(J+NML))
      UX=UX+COEF*REAL(R5*BV(J)+EI*R6*BV(J+NML))/(2*SHMUD)
      UY=UY+COEF*AIMAG(R5*BV(J)+EI*R6*BV(J+NML))/(2*SHMUD)
C     ==================================================================
22    CONTINUE
C     ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      WRITE  (6,230)  I,SGMXX,SGMYY,SGMXY,UX,UY
C     ==================================================================
21    CONTINUE
C     ==================================================================
      END
```

APPENDIX C

THE POTENTIAL FUNCTIONS AND THE INFLUENCE FUNCTIONS FOR AN INFINITE PLANE REGION CONTAINING AN ELLIPTICAL HOLE

The following complex functions were used in the influence functions for an elliptical hole, equations (4.52).

$$\phi^*_I(\zeta) = \ln\left(\frac{r_o-\zeta}{r_o}\right) - \alpha\ln\left(\frac{t_o-\zeta}{t_o}\right)$$

$$\phi^*_{II}(\zeta) = \frac{m\zeta^2-Z\zeta+1}{\zeta^2-\bar{Z}\zeta+m} + \frac{(mt_i^2-Z_0t_i+1)\zeta}{t_i(t_o-t_i)(\zeta-t_i)} - \frac{1}{m}$$

$$\phi^{*\prime}_I(\zeta) = \frac{1}{\zeta-r_o} - \frac{\alpha}{\zeta-t_o}$$

$$\phi^{*\prime}_{II}(\zeta) = \frac{PD}{(\zeta^2-\bar{Z}_0\zeta+m)^2} - \frac{mt_i^2-Z_0t_i+1}{(t_o-t_i)(\zeta-t_i)^2}$$

$$\phi^{*\prime\prime}_I(\zeta) = \frac{-1}{(\zeta-r_o)^2} + \frac{\alpha}{(\zeta-t_o)^2}$$

$$\phi_{II}^{*''}(\zeta) = \frac{[2m(\zeta^2-\bar{Z}_0\zeta+m)-2(m\zeta^2-Z_0\zeta+1)](\zeta^2-\bar{Z}_0\zeta+m)-2(2\zeta-\bar{Z}_0)[PD]}{(\zeta^2-\bar{Z}_0\zeta+m)^3}$$

$$+ \frac{2(mt_i^2-Z_0t_i+1)}{(t_o-t_i)(\zeta-t_i)^3}$$

$$\Psi_I^*(\zeta) = \frac{\zeta^2-\bar{Z}_0\zeta+m}{m\zeta^2-Z_0\zeta+1} + \frac{(r_i^2-\bar{Z}_0r_i+m)\zeta}{r_im(r_o-r_i)(\zeta-r_i)} + \frac{\zeta(\zeta^2+m)}{1-m\zeta^2} \cdot \phi_I^{*'}(\zeta) - m$$

$$\Psi_{II}^*(\zeta) = \ln\left(\frac{t_o-\zeta}{t_o}\right) - \alpha\ln\left(\frac{r_o-\zeta}{r_o}\right) + \frac{\zeta(\zeta^2+m)}{1-m\zeta^2} \cdot \phi_{II}^{*'}(\zeta)$$

$$\Psi_I^{*'}(\zeta) = \frac{-PD}{(m\zeta^2-Z_0\zeta+1)^2} - \frac{r_i^2-\bar{Z}_0r_i+m}{m(r_o-r_i)(\zeta-r_i)^2} + \frac{\zeta(\zeta^2+m)}{1-m\zeta^2} \cdot \phi_I^{*''}(\zeta)$$

$$+ \frac{(3\zeta^2+m)(1-m\zeta^2)+2m\zeta^2(\zeta^2+m)}{(1-m\zeta^2)^2}\phi_I^{*'}(\zeta)$$

$$\Psi_{II}^{*'}(\zeta) = \frac{1}{\zeta-t_o} - \frac{\alpha}{\zeta-r_o} + \frac{\zeta(\zeta^2+m)}{1-m\zeta^2}\phi_{II}^{*''}(\zeta)$$

$$+ \frac{(3\zeta^2+m)(1-m\zeta^2)+2m\zeta^2(\zeta^2+m)}{(1-m\zeta^2)^2} \cdot \phi_{II}^{*'}(\zeta)$$

$$PD = (2m\zeta-Z_0)(\zeta^2-\bar{Z}_0\zeta+m) - (2\zeta-\bar{Z}_0)(m\zeta^2-Z_0\zeta+1)$$

Substituting the components of the resultant fictitious traction, equations (4.59), into the influence functions for an elliptical hole, equations (4.52),

$$H_{xx;x} = \mathrm{Re}\left\{\frac{-2}{Z-Z_0} + \frac{2\zeta^2}{m\zeta^2-1}\,\phi_I^{*\prime}(\zeta) - \bar{Z}\left[\frac{1}{(Z-Z_0)^2} + \frac{\zeta^4}{(m\zeta^2-1)^4}\,\phi_I^{*\prime\prime}(\zeta)\right.\right.$$

$$\left. - \frac{2\zeta^3}{(m\zeta^2-1)^3}\,\phi_I^{*\prime}(\zeta)\right] + \frac{\bar{Z}_0}{(Z-\bar{Z}_0)^2} - \frac{\alpha}{Z-\bar{Z}_0}$$

$$\left. - \frac{\zeta^2}{(m\zeta^2-1)}\,\Psi_I^{*\prime}(\zeta)\right\}/2\pi(\alpha+1)$$

$$H_{xx;y} = \mathrm{Re}\left\{i\left(\frac{-2}{Z-Z_0} + \frac{2\zeta^2}{m\zeta^2-1}\,(\phi_I^{*\prime}(\zeta) - \phi_{II}^{*\prime}(\zeta)) - \bar{Z}\left[\frac{1}{(Z-Z_0)^2}\right.\right.\right.$$

$$+ \frac{\zeta^4}{(m\zeta^2-1)^4}\,(\phi_I^{*\prime\prime}(\zeta) - \phi_{II}^{*\prime\prime}(\zeta)) - \frac{2\zeta^3}{(m\zeta^2-1)^3}\,(\phi_I^{*\prime}(\zeta)$$

$$\left. - \phi_{II}^{*\prime}(\zeta))\right] + \frac{\bar{Z}_0}{(Z-\bar{Z}_0)} + \frac{\alpha}{Z-\bar{Z}_0} - \frac{\zeta^2}{(m\zeta^2-1)}\,(\Psi_I^{*\prime}(\zeta)$$

$$\left.\left. - \Psi_{II}^{*\prime}(\zeta))\right)\right\}/2\pi(\alpha+1)$$

$$H_{yy;x} = \mathrm{Re}\left\{\frac{-2}{Z-Z_0} + \frac{2\zeta^2}{m\zeta^2-1}\,\phi_I^{*\prime}(\zeta) + \bar{Z}\left[\frac{1}{(Z-Z_0)^2} + \frac{\zeta^4}{(m\zeta^2-1)^4}\,\phi_I^{*\prime\prime}(\zeta)\right.\right.$$

$$\left. - \frac{2\zeta^3}{(m\zeta^2-1)^3}\,\phi_I^{*\prime}(\zeta)\right] - \frac{\bar{Z}_0}{(Z-\bar{Z}_0)^2} + \frac{\alpha}{Z-\bar{Z}_0}$$

$$\left. + \frac{\zeta^2}{m\zeta^2-1}\,\Psi_I^{*\prime}(\zeta)\right\}/(2\pi(\alpha+1))$$

$$H_{yy;y} = \mathrm{Re}\left\{ i\left(\frac{-2}{Z-Z_0} + \frac{2\zeta^2}{m\zeta^2-1}(\phi_I^{*\prime}(\zeta) - \phi_{II}^{*\prime}(\zeta)) + \overline{Z}\left[\frac{1}{(Z-Z_0)^2}\right.\right.\right.$$

$$+ \frac{\zeta^4}{(m\zeta^2-1)^4}(\phi_I^{*\prime\prime}(\zeta) - \phi_{II}^{*\prime\prime}(\zeta)) - \frac{2\zeta^3}{(m\zeta^2-1)^3}(\phi_I^{*\prime}(\zeta)$$

$$\left. - \phi_{II}^{*\prime}(\zeta))\right] - \frac{\overline{Z}_0}{(Z-\overline{Z}_0)^2} - \frac{\alpha}{Z-\overline{Z}_0} + \frac{\zeta^2}{m\zeta^2-1}(\Psi_I^{*\prime}(\zeta)$$

$$\left.\left. - \Psi_{II}^{*\prime}(\zeta))\right)\right\} / 2\pi(\alpha+1)$$

$$H_{xy;x} = \mathrm{Im}\left\{ \overline{Z}\left[\frac{1}{(Z-Z_0)^2} + \frac{\zeta^4}{(m\zeta^2-1)^4}\phi_I^{*\prime\prime}(\zeta) - \frac{2\zeta^3}{(m\zeta^2-1)^3}\phi_I^{*\prime}(\zeta)\right]\right.$$

$$\left. - \frac{\overline{Z}_0}{(Z-\overline{Z}_0)^2} + \frac{\alpha}{Z-\overline{Z}_0} + \frac{\zeta^2}{m\zeta^2-1}\Psi_I^{*\prime}(\zeta)\right\} / (2\pi(\alpha+1))$$

$$H_{xy;y} = \mathrm{Im}\left\{ i\left(\overline{Z}\left[\frac{1}{(Z-Z_0)^2} + \frac{\zeta^4}{(m\zeta^2-1)^4}(\phi_I^{*\prime\prime}(\zeta) - \phi_{II}^{*\prime\prime}(\zeta))\right.\right.\right.$$

$$\left. - \frac{2\zeta^3}{(m\zeta^2-1)^3}(\phi_I^{*\prime}(\zeta) - \phi_{II}^{*\prime}(\zeta))\right] - \frac{\overline{Z}_0}{(Z-\overline{Z}_0)^2} - \frac{\alpha}{Z-\overline{Z}_0}$$

$$\left.\left. + \frac{\zeta^2}{m\zeta^2-1}(\Psi_I^{*\prime}(\zeta) - \Psi_{II}^{*\prime}(\zeta))\right)\right\} / (2\pi(\alpha+1))$$

$$I_{x;x} = \mathrm{Re}\left\{ -\alpha\ln(Z-Z_0) + \alpha\,\phi_I^{*\prime}(\zeta) - Z\left[\frac{-1}{Z-\overline{Z}_0} + \frac{\overline{\zeta}^2}{m\overline{\zeta}^2-1}\overline{\phi_I^{*\prime}(\zeta)}\right]\right.$$

$$\left. - \frac{Z_0}{\overline{Z}-Z_0} - \alpha\ln(\overline{Z}-Z_0) - \overline{\Psi_I^{*}(\zeta)}\right\}$$

$$I_{x;y} = \mathrm{Re}\left\{ i \left(-\alpha \ln(Z-Z_0) + \alpha\,(\phi_I^{*\prime}(\zeta) - \phi_{II}^{*\prime}(\zeta)) - Z\left[\frac{-1}{Z-\bar{Z}_0} + \frac{\bar{\zeta}^2}{m\bar{\zeta}^2-1}\,(\overline{\phi_I^{*\prime}(\zeta)} - \overline{\phi_{II}^{*\prime}(\zeta)})\right] - \frac{Z_0}{\bar{Z}-Z_0} + \alpha \ln(\bar{Z}-Z_0) - (\overline{\Psi_I^{*}(\zeta)} - \overline{\Psi_{II}^{*}(\zeta)}) \right) \right\} / 4\mu\pi(\alpha+1)$$

$$I_{y;x} = \mathrm{Im}\left\{ -\alpha \ln(Z-Z_0) + \alpha\,\phi_I^{*\prime}(\zeta) - Z\left[\frac{-1}{Z-\bar{Z}_0} + \frac{\bar{\zeta}^2}{m\bar{\zeta}^2-1}\,\overline{\phi_I^{*\prime}(\zeta)}\right] + \frac{Z_0}{\bar{Z}-Z_0} - \alpha \ln(\bar{Z}-Z_0) - \overline{\Psi_I^{*}(\zeta)} \right\} / 4\mu\pi(\alpha+1)$$

$$I_{y;y} = \mathrm{Im}\left\{ i \left(-\alpha \ln(Z-Z_0) + \alpha\,(\phi_I^{*\prime}(\zeta) - \phi_{II}^{*\prime}(\zeta)) - Z\left[\frac{-1}{Z-\bar{Z}_0} + \frac{\bar{\zeta}^2}{m\bar{\zeta}^2-1}\,(\overline{\phi_I^{*\prime}(\zeta)} - \overline{\phi_{II}^{*\prime}(\zeta)})\right] - \frac{Z_0}{\bar{Z}-Z_0} + \alpha \ln(\bar{Z}-Z_0) - (\overline{\Psi_I^{*}(\zeta)} - \overline{\Psi_{II}^{*}(\zeta)}) \right) \right\} / 4\mu\pi(\alpha+1)$$

APPENDIX D

THE POTENTIAL FUNCTIONS AND THE INFLUENCE FUNCTIONS FOR AN INFINITE PLANE REGION CONTAINING A SHARP CRACK

The following complex functions were used in the influence functions for a sharp crack, equations (4.54), m = 1:

$$\phi_I^*(\zeta) = \ln\left(\frac{r_o-\zeta}{r_o}\right) - \alpha \ln\left(\frac{t_o-\zeta}{t_o}\right)$$

$$\phi_{II}^*(\zeta) = \frac{\zeta^2-Z_0\zeta+1}{\zeta^2-\overline{Z}_0\zeta+1} + \frac{(t_i^2-Z_0t_i+1)}{t_i(t_o-t_i)(\zeta-t_i)} - 1$$

$$\phi_I^{*\prime}(\zeta) = \frac{1}{\zeta-r_o} - \frac{\alpha}{\zeta-t_o}$$

$$\phi_{II}^{*\prime}(\zeta) = \frac{PD}{(\zeta^2-\overline{Z}_0\zeta+1)^2} - \frac{t_i^2-Z_0t_i+1}{(t_o-t_i)(\zeta-t_i)^2}$$

$$\phi_I^{*\prime\prime}(\zeta) = \frac{-1}{(\zeta-r_o)^2} + \frac{\alpha}{(\zeta-t_o)^2}$$

$$\phi_{II}^{*\prime\prime}(\zeta) = \frac{[2(\zeta^2-\overline{Z}_0\zeta+1)-2(\zeta^2-Z_0\zeta+1)](\zeta^2-\overline{Z}_0\zeta+1)-2(2\zeta-\overline{Z}_0)(PD)}{(\zeta^2-\overline{Z}_0\zeta+1)^3}$$

$$+ \frac{2(t_i^2-Z_0t_i+1)}{(t_o-t_i)(\zeta-t_i)^3}$$

For the special case $Z = \bar{Z}_0$, the only changes are

$$\phi^*_{II}(\zeta) = \frac{(AN)}{(\zeta - t_o)^2} - \frac{Z_0 t_o - 1}{t_o^2}$$

$$\phi^{*\prime}_{II}(\zeta) = \frac{2}{\zeta - t_o} - \frac{2(AN)}{(\zeta - t_o)^3}$$

$$\phi^{*\prime\prime}_{II}(\zeta) = \frac{-6}{(\zeta - t_o)^2} + \frac{6(AN)}{(\zeta - t_o)^4}$$

where

$$PD = (2\zeta - Z_0)(\zeta^2 - \bar{Z}_0\zeta + 1) - (2\zeta - \bar{Z}_0)(\zeta^2 - Z_0\zeta + 1)$$

$$AN = (2\zeta - Z_0)(\zeta - t_o) - (\zeta^2 - Z_0\zeta + 1)$$

For all the cases

$$\Psi^*_I(\zeta) = \frac{\zeta^2 - \bar{Z}_0\zeta + 1}{\zeta^2 - Z_0\zeta + 1} + \frac{(r_i^2 - \bar{Z}_0 r_i + 1)}{r_i(r_o - r_i)(\zeta - r_i)} + \frac{\zeta(\zeta^2+1)}{1-\zeta^2} \cdot \phi^{*\prime}_I(\zeta) - 1$$

$$\Psi^*_{II}(\zeta) = \ln\left(\frac{t_o - \zeta}{t_o}\right) - \alpha \ln\left(\frac{r_o - \zeta}{r_o}\right) + \frac{\zeta(\zeta^2+1)}{1-\zeta^2}\, \phi^{*\prime}_{II}(\zeta)$$

$$\Psi^{*\prime}_I(\zeta) = \frac{-PD}{(\zeta^2 - Z_0\zeta + 1)^2} - \frac{r_i^2 - \bar{Z}_0 r_i + 1}{(r_o - r_i)(\zeta - r_i)^2} + \frac{\zeta(\zeta^2+1)}{1-\zeta^2} \cdot \phi^{*\prime\prime}_I(\zeta)$$

$$\frac{(3\zeta^2+1)(1-\zeta^2) + 2\zeta^2(\zeta^2+1)}{(1-\zeta^2)^2}\, \phi^{*\prime}_I(\zeta)$$

$$\Psi_{II}^{*\prime}(\zeta) = \frac{1}{\zeta - t_o} - \frac{\alpha}{\zeta - r_o} + \frac{\zeta(\zeta^2+1)}{1-\zeta^2}\,\phi_I^{*\prime\prime}(\zeta)$$

$$+ \frac{(3\zeta^2+1)(1-\zeta^2)+2\zeta^2(\zeta^2+1)}{(1-\zeta^2)^2}\,\phi_{II}^{*\prime}(\zeta)$$

Substituting the components of the resultant fictitious traction, equation (4.59) into the influence functions for a slit, equations (4.54):

$$H_{xx;x} = \mathrm{Re}\left\{\frac{-2}{Z-Z_0} + \frac{2\zeta^2}{\zeta^2-1}\,\phi_I^{*\prime}(\zeta) - \overline{Z}\left[\frac{1}{(Z-Z_0)} + \frac{\zeta^4}{(\zeta^2-1)^4}\,\phi_I^{*\prime\prime}(\zeta)\right.\right.$$

$$\left. - \frac{2\zeta^3}{(\zeta^2-1)^3}\,\phi_I^{*\prime}(\zeta)\right] + \frac{\overline{Z}_0}{(Z-\overline{Z}_0)^2} - \frac{\alpha}{Z-\overline{Z}_0}$$

$$\left. - \frac{\zeta^2}{\zeta^2-1}\,\Psi_I^{*\prime}(\zeta)\right\}/2\pi(\alpha+1)$$

$$H_{xx;y} = \mathrm{Re}\left\{ i\left(\frac{-2}{Z-Z_0} + \frac{2\zeta^2}{\zeta^2-1}\,(\phi_I^{*\prime}(\zeta)) - \phi_{II}^{*\prime}(\zeta)) - \overline{Z}\left[\frac{1}{(Z-Z_0)^2}\right.\right.\right.$$

$$\left. + \frac{\zeta^4}{(\zeta^2-1)^4}\,(\phi_I^{*\prime\prime}(\zeta) - \phi_{II}^{*\prime\prime}(\zeta)) - \frac{2\zeta^3}{(\zeta^2-1)^3}\,(\phi_I^{*\prime}(\zeta) - \phi_{II}^{*\prime}(\zeta))\right]$$

$$\left.\left. + \frac{\overline{Z}_0}{(Z-\overline{Z}_0)^2} + \frac{\alpha}{Z-\overline{Z}_0} - \frac{\zeta^2}{\zeta^2-1}\,(\Psi_I^{*\prime}(\zeta) - \Psi_{II}^{*\prime}(\zeta))\right)\right\}/2\pi(\alpha+1)$$

$$H_{yy;x} = \mathrm{Re}\left\{ \frac{-2}{Z-Z_0} + \frac{2\zeta^2}{\zeta^2-1}\phi_I^{*\prime}(\zeta) + \overline{Z}\left[\frac{1}{(Z-Z_0)^2} + \frac{\zeta^4}{(\zeta^2-1)^4}\phi_I^{*\prime\prime}(\zeta)\right.\right.$$

$$\left. - \frac{2\zeta^3}{(\zeta^2-1)^3}\phi_I^{*\prime}(\zeta)\right] - \frac{\overline{Z}_0}{(Z-\overline{Z}_0)^2} + \frac{\alpha}{Z-\overline{Z}_0}$$

$$\left. + \frac{\zeta^2}{\zeta^2-1}\Psi_I^{*\prime}(\zeta)\right\} / (2\pi(\alpha+1)$$

$$H_{yy;y} = \mathrm{Re}\left\{ i\left(\frac{-2}{Z-Z_0} + \frac{2\zeta^2}{\zeta^2-1}(\phi_I^{*\prime}(\zeta) - \phi_{II}^{*\prime}(\zeta)) + \overline{Z}\left[\frac{1}{(Z-Z_0)^2}\right.\right.\right.$$

$$+ \frac{\zeta^4}{(\zeta^2-1)^4}(\phi_I^{*\prime\prime}(\zeta) - \phi_{II}^{*\prime\prime}(\zeta)) - \left.\frac{2\zeta^3}{(\zeta^2-1)^3}(\phi_I^{*\prime}(\zeta) - \phi_{II}^{*\prime}(\zeta))\right]$$

$$\left.\left. - \frac{\overline{Z}_0}{(Z-\overline{Z}_0)^2} - \frac{\alpha}{Z-\overline{Z}_0} + \frac{\zeta^2}{\zeta^2-1}(\Psi_I^{*\prime}(\zeta) - \Psi_{II}^{*\prime}(\zeta))\right)\right\} / 2\pi(\alpha+1)$$

$$H_{xy;x} = \mathrm{Im}\left\{ \overline{Z}\left[\frac{1}{(Z-Z_0)^2} + \frac{\zeta^4}{(\zeta^2-1)^4}\phi_I^{*\prime\prime}(\zeta) - \frac{2\zeta^3}{(\zeta^2-1)^3}\phi_I^{*\prime}(\zeta)\right]\right.$$

$$\left. - \frac{\overline{Z}_0}{(Z-\overline{Z}_0)^2} + \frac{\alpha}{Z-\overline{Z}_0} + \frac{\zeta^2}{\zeta^2-1}\Psi_I^{*\prime}(\zeta)\right\} / 2\pi(\alpha+1)$$

$$H_{xy;y} = \mathrm{Im}\left\{ i\left(\overline{Z}\left[\frac{1}{(Z-Z_0)^2} + \frac{\zeta^4}{(\zeta^2-1)^4}(\phi_I^{*\prime\prime}(\zeta) - \phi_{II}^{*\prime\prime}(\zeta))\right.\right.\right.$$

$$\left. - \frac{2\zeta^3}{(\zeta^2-1)^3}(\phi_I^{*\prime}(\zeta) - \phi_{II}^{*\prime}(\zeta))\right] - \frac{\overline{Z}_0}{(Z-\overline{Z}_0)^2} - \frac{\alpha}{Z-\overline{Z}_0}$$

$$\left.\left. + \frac{\zeta^2}{\zeta^2-1}(\Psi_I^{*\prime}(\zeta) - \Psi_{II}^{*\prime}(\zeta))\right)\right\} / 2\pi(\alpha+1)$$

$$I_{x;x} = \mathrm{Re}\left\{ - \alpha \ln(Z-Z_0) + \alpha\, \phi_I^{*\prime}(\zeta) - Z\, \left[\frac{-1}{Z-\overline{Z}_0} + \frac{\overline{\zeta}^2}{\overline{\zeta}^2-1}\, \overline{\phi_I^{*\prime}(\zeta)}\right]\right.$$

$$\left. - \frac{Z_0}{\overline{Z}-Z_0} - \alpha \ln(\overline{Z}-Z_0) - \overline{\Psi_I^{*}(\zeta)} \right\} / 4\pi\mu(\alpha+1)$$

$$I_{x;y} = \mathrm{Re}\left\{ i \left(- \alpha \ln(Z-Z_0) + \alpha\, (\phi_I^{*}(\zeta) - \phi_{II}^{*\prime}(\zeta)) - Z\, \left[\frac{-1}{Z-\overline{Z}_0}\right.\right.\right.$$

$$+ \frac{\overline{\zeta}^2}{\overline{\zeta}^2-1}\, (\overline{\phi_I^{*\prime}(\zeta)} - \overline{\phi_{II}^{*\prime}(\zeta)})\Big] - \frac{Z_0}{\overline{Z}-Z_0} + \alpha \ln(\overline{Z}-Z_0) - (\overline{\Psi_I^{*}(\zeta)}$$

$$\left.\left. - \overline{\Psi_{II}^{*}(\zeta)}) \right) \right\} / 4\mu\pi(\alpha+1)$$

$$I_{y;x} = \mathrm{Im}\left\{ - \alpha \ln(Z-Z_0) + \alpha\, \phi_I^{*\prime}(\zeta) - Z\, \left[\frac{-1}{Z-\overline{Z}_0} + \frac{\overline{\zeta}^2}{\overline{\zeta}^2-1}\, \overline{\phi_I^{*\prime}(\zeta)}\right]\right.$$

$$\left. - \frac{Z_0}{\overline{Z}-Z_0} - \alpha \ln(\overline{Z}-Z_0) - \overline{\Psi_I^{*}(\zeta)} \right\} / 4\pi\mu(\alpha+1)$$

$$I_{y;y} = \mathrm{Im}\left\{ i \left(\alpha \ln(Z-Z_0) + \alpha\, (\phi_I^{*\prime}(\zeta) - \phi_{II}^{*\prime}(\zeta)) - Z\, \left[\frac{-1}{Z-\overline{Z}_0}\right.\right.\right.$$

$$+ \frac{\overline{\zeta}^2}{\overline{\zeta}^2-1}\, (\phi_I^{*\prime}(\zeta) - \phi_{II}^{*\prime}(\zeta))\Big] - \frac{Z_0}{\overline{Z}-Z_0} + \alpha \ln(\overline{Z}-Z_0)$$

$$\left.\left. - (\overline{\Psi_I^{*}(\zeta)} - \overline{\Psi_{II}^{*}(\zeta)}) \right) \right\} / 4\pi\mu(\alpha+1)$$

APPENDIX E

COMPUTER PROGRAM FOR PLANE, FINITE REGION CONTAINING AN ELLIPTICAL HOLE OR A SHARP CRACK

A computer program was employed for the numerical computation of the stresses and the displacements at the field points of a two-dimensional region containing an elliptical hole or a sharp crack. A listing of that program for the rectangular region subjected to uniaxial tension (ω = 1.0 MPa) and containing an elliptical hole or a sharp crack and for the circular plane subjected to a uniformly radially tension (ω = 1.0 MPa) and containing an elliptical hole are presented in this appendix.

A. INPUT DATA

The following information must be provided as input (this is the order of appearance in the program).

PR	- Poisson's ratio of the material
EMUD	- modulus of elasticity of the material
NML	- total number of subdivisions on the boundary
NFP	- the number of field points at which the stresses and displacements are to be computed
M	- parameter used to describe semi-major axis (a = 1+M) and semi-minor axis (b = 1-M) of elliptical hole

THETA	- angle of inclination of the elliptical hole or the sharp crack with respect to the x-axis
X_0, Y_0	- location of the center of the rectangular or circular plane with respect to the center of the ellipse or crack
WR	- "width ratio" for the rectangular plane (WR = width/10 cm) and for the circular plane (WR = radius/6.0 cm)
X(I),Y(I) I=1,NML	- coordinates of the outer boundary points, specified counterclockwise (for circular plane case, this input has been included in the program)
IPLANE	- this character specifies the type of the plane problem, i.e., for plane stress, IPLANE=1 and for plane strain, IPLANE=2
$BV_x(I)$, $BV_y(I+NML)$ I=1,NML	- x and y components of boundary tractions specified at each subdivision
XF(I), YF(I) I=1,NFP	- coordinates of the field points at which the stresses and displacements are to be computed

B. OUTPUT DATA

The following information is obtained as output (this is in the order of the appearance in the output).

PR,EMUD	- Poisson's ratio and modulus of elasticity
SHMUD	- shear modulus of elasticity
NML,NFP	- total number of subdivisions on the boundary and number of field points at which the stresses and displacements are computed

THETA - angle of inclination of the elliptical hole or the sharp crack with respect to the x-axis

X_0, Y_0 - location of the center of the rectangular or circular plane

WR - magnification of the size of the plane for the rectangular plane (WR = width/10 cm) and for the circular plane (WR = radius/6.0 cm)

X(I),Y(I) I=1,NML - coordinates of the outer boundary points, listed counterclockwise

$BV_x(I)$ $BV_y(I+NML)$ I=1,NML - x and y components of boundary tractions specified at each subdivision

PSX(I) PSY(I) I=1,NML - components of the fictitious traction on the boundary, represented by the concentrated loads P^*_{xi}, P^*_{yi} at the center of each one of the subdivisions. These are computed by solving a system of linear equations (4.61) (LEQT1F, computer library).

XF(I), YF(I) I=1,NFP - location of the field point at which the stresses and displacements are computed

SIGMAXX, SIGMAYY, SIGMAXY, UX,UY - components of stress and displacement at each of the field points. These are computed using equation (4.65).

C. THE COMPUTER PROGRAM FOR A RECTANGULAR PLANE CONTAINING AN ELLIPTICAL HOLE OR A SHARP CRACK (follows)

```
      PROGRAM  FPWEH     (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
      DIMENSION   X(61),Y(61),RM(120,120),WKAREA(120),BV(120)
      DIMENSION   XF(15),YF(15)
C     ************************************************************************
      REAL    NXI,NYI,M
      COMPLEX   Z(60),ZE(60),Z2(60),Z3(60),ZP(60),ZZ2(60),ZEC(60)
      COMPLEX   RI(60),RO(60),TI(60),TO(60)
      COMPLEX   G(60),K(60),ZEF(15),ZF(15),QMT,QCR
      COMPLEX   PD1,PC2,P1,P2,H1,H2,K1,K2,S1,S2,R1,R2,R3,R4,R5,R6
      COMPLEX   PDD1,PDD2,ZT1,ZT2,ZD1,ZD2,WO,WD2,WD3,WDD
      COMPLEX   ZE2,CS,CSC,QM,QC,QM2,QC2
      COMPLEX   PD,ZZ2F,ZF2,ZF3,ZEFC,EI,ZTI,ZTO,ZRI,ZRO,ZT02,ZR02
      COMPLEX   ZED,ZED2,ZEFD,ZEFD2,ZZ,DR,DT,F,FD
      COMPLEX   X1,X2,Y1,Y2,Y3,Y4,AN,ZT03,ZT04
C     ************************************************************************
C     ************************************************************************

100   FORMAT (2F12.8)
110   FORMAT (*0*,14X,*POISSONS RATIO*,10X,*MODULUS CF ELASTICITY*
     $,13X,*SHEAR MODULUS*)
120   FORMAT (*0*,18X,F10.8,10X,F20.8,10X,F28.8)
130   FORMAT (2I6)
140   FORMAT (////,10X,*NO. OF MESHES=*,I3,25X,*NO. CF FIELD POINTS=*
     $,I3)
150   FORMAT (6F10.5)
160   FORMAT (///,10X,*NODE NO. (I)*,22X,*X(I)*,18X,*Y(I)*/*0*/
     $(*0*,10X,I5,25X,F8.3,15X,F8.3))
170   FORMAT (6F10.8)
180   FORMAT(////,13X,*MESH NO.*,19X,*BVX*,29X,*BVY*/*0*/(*0*,15X,I4,
     $16X,F10.5,25X,F10.5))
190   FORMAT  (////,*0*,38X,*FICTIOUS TRACTIGNS*/*0*/*0*,4X,
     $*MESH NO.*,19X,*PSX*,19X,*PSY*/(*0*,4X,I4,11X,2(F20.10)))
200   FORMAT (6F10.6)
210   FORMAT (////,20X,*FIELD POINT NO.*,24X,*XF(I)*,28X,*YF(I)*
     $/*0*/(*0*,25X,I5,25X,F10.7,25X,F10.7))
220   FORMAT  (*1*,35X,*STRESSES  AND  DISPLACEMENTS  OF  THE  FIELD
     $POINTS*///,2X,*NO. OF F.P.*,8X,*SIGMAXX*,19X,*SIGMAYY*,18X,
     $*SIGMAXY*,20X,*UX*,22X,*UY*)
230   FORMAT  (///,4X,I3,5(5X,F20.10))
240   FORMAT (////,25X,*TRANSFORMED  BOUNDARY  POINTS  TO  ZETA  *,
     $*PLANE*///,10X,*NODE NO. (I)*,22X,*REAL*,18X,*IMAG*)
250   FORMAT  (*0*,10X,I5,25X,F8.3,15X,F8.3)
260   FORMAT (////,25X,*TRANSFORMED  FIELD  POINTS  TO  ZETA PLANE*
     $,////,20X,*FIELD POINT NO. *,24X,*REAL*,28X,*IMAG*)
270   FORMAT (//,25X,I4,2(25X,F8.3))
280   FORMAT  (////,25X,*TRANSLATED AND ROTATED BOUNDARY POINTS
     $DUE  TO  THETA   X0 AND Y0 *)
290   FORMAT  (////,20X,* MAJOR AXIS =*,F6.3,15X,* MINOR AXIS =*,
     $F6.3,///,20X,* ANGLE OF ROTATION THETA=*,F6.2,/*0*/,20X,
     $* TRANSLATION OF THE COORDINATES  X0 =*,F6.3,10X,*Y0 =*,F6.3,
     $///,20X, * MAGNIFICATION OF THE  SIZE CF THE PLANE  WR  =*,
     $F15.10)
300   FORMAT  (////,30X,* SHARP    CRACK          PROBLEM*)
C     ************************************************************************
C     *****************************
C     *    MATERIAL  PROPERTIES   *
C     *****************************
      READ (5,100) PR,EMUD
      SHMUD=EMUD/(2*(1+PR))
      WRITE (6,110)
      WRITE (6,120)  PR , EMUD , SHMUD
C     ***********************************************************
C     *              GEOMETRY OF THE PROBLEM                    *
C     *            TRACTION BOUNDARY CONDITION                  *
C     *      RECTANGULAR,PLANE STRESS,UNIAXIAL TENSION          *
C     *  NUMBER OF MESH LENGTH=NML,NUMBER OF FIELD PCINT=NFP    *
C     ***********************************************************
      READ  (5,130)  NML , NFP
      WRITE  (6,140)  NML , NFP
      NML1=NML+1
      NML2=NML*2
      PI=4*ATAN(1.0)
      EI=(0.0,1.0)
      M=1.0
      THETA=0.0
      X0=4.5
      Y0=0.0
```

```
      WR=1.0
      IF(M.EQ.1.0)   WRITE(6,300)
      AX=(1+M)*2.0
      AY=(1-M)*2.0
      WRITE  (6,290)  AX,AY,THETA,X0,Y0,WR
      READ  (5,150)  (X(I),Y(I),I=1,NML1)
      WRITE  (6,160)  (I,X(I),Y(I),I=1,NML1)
C     ************************************************************
C     *             CALCULATION OF THE NEW BOUNDARY POINTS        *
C     *  DUE TO THE ROTATION AND TANSLATION OF THE COORDINATES AND *
C     *        DIFFERENT WIDTH RATIO FOR ELLIPTICAL HOLE OR       *
C     *               BIMODAL  FRACTURE  OF A CRACK               *
C     *  WHERE THETA=ROTATION AND (X0,Y0)=NEW COORDINATES OF ORIGIN *
C     *          AND WIDTH RATIO  WR=WIDTH/CRACK LENGTH           *
C     ************************************************************
      DO  48  I=1,NML1
      X(I)=WR*X(I)
      Y(I)=WR*Y(I)
48    CONTINUE
      IF(X0.NE.0.0)  GO  TO  42
      IF(Y0.NE.0.0)  GO  TO  42
      IF(THETA.EQ.0.0)  GO  TO  43
42    T=PI*THETA/180
      DO  40  I=1,NML1
      XI=X(I)
      YI=Y(I)
      X(I)=(XI-X0)*COS(T)+(YI-Y0)*SIN(T)
      Y(I)=(YI-Y0)*COS(T)-(XI-X0)*SIN(T)
40    CONTINUE
43    WRITE  (6,280)
      WRITE  (6,160)  (I,X(I),Y(I),I=1,NML1)

C     ************************************************************
C     *       CALCULATION  OF  COEFFICIENTS                      *
C     ************************************************************

      IPLANE=1
      IF(IPLANE.EQ.2)  GO TO 10
      ALPHA=(3-PR)/(1+PR)
      GO TO 11
10    ALPHA=3-4*PR
11    COEF=1/(2*PI*(1+ALPHA))
C     ************************************************************
C     *     TRANSFORMATION  OF  THE  BOUNDARY  POINTS     *
C     ************************************************************
      WRITE  (6,240)
      DO  12  I=1,NML
      ZE(I)=CMPLX((X(I)+X(I+1))/2,(Y(I)+Y(I+1))/2)
      ZEC(I)=CONJG(ZE(I))
      ZE2=ZE(I)**2
      CS=CSQRT(ZE2-4*M)
      CSC=CSQRT(ZEC(I)**2-4*M)
      Y1=(ZE(I)+CS)/(2*M)
      Y2=(ZE(I)-CS)/(2*M)
      Y3=(ZEC(I)-CSC)/2.0
      Y4=(ZEC(I)+CSC)/2.0
      IF((X(I)+X(I+1))/2.LE.0.0)  GO  TO  13
      Z(I)=Y2
      RO(I)=Y1
      RI(I)=Y2
      GO  TO  17
13    Z(I)=Y1
      RO(I)=Y2
      RI(I)=Y1
17    A=REAL(Z(I))
      B=AIMAG(Z(I))
      WRITE  (6,250)  I,A,B

      IF(CABS(Y3).LT.1.0)  GO  TO  18
      TO(I)=Y3
      TI(I)=Y4
      GO  TO  19
18    TO(I)=Y4
      TI(I)=Y3

19    Z2(I)=Z(I)**2
      Z3(I)=Z2(I)*Z(I)
      ZP(I)=M*Z(I)
      ZZ2(I)=M*Z2(I)
      QMT=M*TI(I)**2-ZE(I)*TI(I)+1
      QCR=RI(I)**2-ZEC(I)*RI(I)+M
```

```
      G(I)=QMT/(TO(I)-TI(I))
      K(I)=QCR/((RC(I)-RI(I))*M)
12    CONTINUE
C     ***************************************************************
C     *  CALCULATION OF ELEMENTS OF THE RM(NML2,NML2) MATRIX  *
C     ***************************************************************
C

C     ================================
      DO  14   I=1,NML
C     ================================
      DSI=SQRT((X(I+1)-X(I))**2+(Y(I+1)-Y(I))**2)
      NXI=(Y(I+1)-Y(I))/DSI
      NYI=(X(I)-X(I+1))/DSI
      WD=-1/Z2(I)+M
      WD2=WD**2
      WD3=WD2*WD
      WDD=2/Z3(I)
C     ======================
      DO  15   J=1,NML
C     ======================
      IF (I.EQ.J)  GO  TO  16
      ZED=ZE(I)-ZE(J)
      ZED2=ZED**2

      QM=ZZ2(I)-ZE(J)*Z(I)+1
      QM2=QM**2
      QC=Z2(I)-ZEC(J)*Z(I)+M
      QC2=QC**2
      DR=2*ZP(I)-ZE(J)
      DT=2*Z(I)-ZEC(J)
      PO=DR*QC-DT*QM
      ZZ=1-ZZ2(I)
      F=Z(I)*(Z2(I)+M)/ZZ
      FD=((3*Z2(I)+M)*ZZ+2*ZZ2(I)*(Z2(I)+M))/ZZ**2
      ZTI=Z(I)-TI(J)
      ZRI=Z(I)-RI(J)
      ZTO=1/(Z(I)-TO(J))
      ZTO2=ZTO**2
      ZRO=1/(Z(I)-RO(J))
      ZRO2=ZRO**2

      PD1=ZRO-ALPHA*ZTO
      PDD1=-ZRO2+ALPHA*ZTO2
      IF (M.NE.1.0)  GO  TO  32
      IF(ZTI.NE.0.0)  GO  TO  32
      AN=DR/ZTO-QM
      ZTO3=ZTO*ZTO2
      ZTO4=ZTO3*ZTO
      PD2=2*ZTO-2*AN*ZTO3
      PDD2=-6*ZTO2+6*AN*ZTO4
      GO  TO  31
32    PD2=PO/QC2-G(J)/ZTI**2
      PDD2=((2*M*QC-2*QM)*QC-2*DT*PO)/(QC2*QC)+2*G(J)/ZTI**3
31    ZD1=(DT*QM-DR*QC)/QM2-K(J)/ZRI**2+F*PDD1+FD*PD1
      ZD2=ZTO-ALPHA*ZRO+F*PDD2+FD*PD2

      H1=PD1/WD
      H2=PD2/WD
      S1=ZD1/WD
      S2=ZD2/WD
      K1=PDD1/WD2-PD1*WDD/WD3
      K2=PDD2/WD2-PD2*WDD/WD3
      H1=H1-1/ZED
      K1=K1+1/ZED2
      S1=S1-ZEC(J)/ZED2
      S2=S2+ALPHA/ZED

      R1=2*(H1+H2)
      R2=ZEC(I)*(K1+K2)+S1+S2
      R3=2*(H1-H2)
      R4=ZEC(I)*(K1-K2)+S1-S2

      RM(I,J)=(REAL(R1-R2)*NXI+AIMAG(R2)*NYI)*COEF*DSI
      RM(I,J+NML)=(REAL(EI*(R3-R4))*NXI+AIMAG(EI*R4)*NYI)*COEF*DSI
      RM(I+NML,J)=(AIMAG(R2)*NXI+REAL(R1+R2)*NYI)*COEF*DSI
     RM(I+NML,J+NML)=(AIMAG(EI*R4)*NXI+REAL(EI*(R3+R4))*NYI)*COEF*DSI

      GO  TO  15
16    RM(I,I)=1/2.0
      RM(I,I+NML)=0.0
```

```
      RM(I+NML,I)=0.0
      RM(I+NML,I+NML)=1/2.0
C     ==============================================================
15    CONTINUE
C     ==============================================================
14     CONTINUE
C     ==============================================================
C     **************************************************************
C     *    CALCULATION OF FICTITIOUS TRACTION   P-STAR    PS(I)    *
C     **************************************************************
      READ  (5,170)  (BV(I),BV(I+NML),I=1,NML)
C     ****************************************************************
C     *    CORRECTING THE B.V. DUE TO THE CHANG IN WIDTH RATIO TO    *
C     *         KEEP THE DISTRIBUTED LOAD =1.0 LB/LENGTH             *
C     ****************************************************************
      DO  46  I=1,NML
      BV(I)=BV(I)*WR
      BV(I+NML)=BV(I+NML)*WR
46    CONTINUE
C     **************************************************
C     *   CALCULATION OF THE ROTATED BOUNDARY VALUES   *
C     **************************************************
      IF(THETA.EQ.0.0)  GO  TO  49
      DO  41  I=1,NML
      BI=BV(I)
      BJ=BV(I+NML)
      BV(I)=BI*COS(T)+BJ*SIN(T)
      BV(I+NML)=BJ*COS(T)-BI*SIN(T)
41    CONTINUE
49    WRITE  (6,180)   (I,BV(I),BV(I+NML),I=1,NML)
      CALL  LEQT1F  (RM,1,NML2,NML2,BV,8,WKAREA,IER)
      WRITE  (6,190)   (I,BV(I),BV(I+NML),I=1,NML)
C     *******************************************************
C     *   CALCULATION  OF  STRESSES  AND  DISPLACEMENTS     *
C     *******************************************************
      READ  (5,200)   (XF(I),YF(I),I=1,NFP)
      WRITE  (6,210)   (I,XF(I),YF(I),I=1,NFP)
      WRITE  (6,260)
      DO  20  I=1,NFP
      ZEF(I)=CMPLX(XF(I),YF(I))
      ZF(I)=(ZEF(I)-CSQRT(ZEF(I)**2-4*M))/(2*M)
      IF(XF(I).GT.0.0)  GO  TO  24
      IF(YF(I).LE.0.0) ZF(I)=(ZEF(I)+CSQRT(ZEF(I)**2-4*M))/(2*M)
24    A=REAL(ZF(I))
      B=AIMAG(ZF(I))
      WRITE  (6,270)   I,A,B
20    CONTINUE
      WRITE  (6,220)
C     ================================
      DO  21  I=1,NFP
C     ================================
      SGMXX=0.0
      SGMYY=0.0
      SGMXY=0.0
      UX=0.0
      UY=0.0
      ZF2=ZF(I)**2
      ZF3=ZF(I)*ZF2
      WO=-1/ZF2+M
      WO2=WO**2
      WO3=WO2*WO
      WOO=2/ZF3
      ZEFC=CONJG(ZEF(I))
      ZZ2F=M*ZF2
      ZP(I)=M*ZF(I)
C     ================================
      DO  22  J=1,NML
C     ================================
      ZEFD=ZEF(I)-ZE(J)
      ZEFD2=ZEFD**2
      QM=ZZ2F-ZE(J)*ZF(I)+1
      QM2=QM**2
      QC=ZF2-ZEC(J)*ZF(I)+M
      QC2=QC**2
      DR=2*ZP(I)-ZE(J)
```

```
      DT=2*ZF(I)-ZEC(J)
      PO=OR*QC-DT*QM
      ZZ=1-ZZ2F
      F=ZF(I)*(ZF2+M)/ZZ
      FD=((3*ZF2+M)*ZZ+2*ZZ2F*(ZF2+M))/ZZ**2
      ZTI=ZF(I)-TI(J)
      ZRI=ZF(I)-RI(J)
      ZTO=1/(ZF(I)-TO(J))
      ZTO2=ZTO**2
      ZRO=1/(ZF(I)-RO(J))
      ZRO2=ZRO**2

      X1=CLOG(-1/(ZRO*RO(J)))
      X2=CLOG(-1/(ZTO*TO(J)))
      P1=X1-ALPHA*X2
      P2=QM/QC+G(J)*(1/ZTI+1/TI(J))
      PD1=ZRO-ALPHA*ZTO
      PD2=PO/QC2-G(J)/ZTI**2
      PDD1=-ZRO2+ALPHA*ZTO2
      PDD2=((2*M*QC-2*QM)*QC-2*DT*PO)/(QC2*QC)+2*G(J)/ZTI**3
      ZT1=QC/QM+K(J)*(1/ZRI+1/RI(J))-M+F*PD1
      ZT2=+X2-ALPHA*X1
      ZD1=(DT*QM-OR*QC)/QM2-K(J)/ZRI**2+F*PDD1+FD*PD1
      ZD2=ZTO-ALPHA*ZRO+F*PDD2+FD*PD2
      H1=PD1/WD
      H2=PD2/WD
      S1=ZD1/WD
      S2=ZD2/WD
      K1=PDD1/WD2-PD1*WDD/WD3
      K2=PDD2/WD2-PD2*WDD/WD3

      P1=P1-CLOG(ZEFD)
      H1=H1-1/ZEFD
      K1=K1+1/ZEFD2
      ZT1=ZT1+ZEC(J)/ZEFD
      ZT2=ZT2+ALPHA*CLOG(ZEFD)
      S1=S1-ZEC(J)/ZEFD2
      S2=S2+ALPHA/ZEFD

      R1=2*(H1+H2)
      R2=ZEFC*(K1+K2)+S1+S2
      R3=2*(H1-H2)
      R4=ZEFC*(K1-K2)+S1-S2
      R5=ALPHA*(P1+P2)-ZEF(I)*CONJG(H1+H2)-CONJG(ZT1+ZT2)
      R6=ALPHA*(P1-P2)+ZEF(I)*CONJG(H1-H2)+CONJG(ZT1-ZT2)

      SGMXX=SGMXX+COEF*REAL((R1-R2)*BV(J)+EI*(R3-R4)*BV(J+NML))
      SGMYY=SGMYY+COEF*REAL((R1+R2)*BV(J)+EI*(R3+R4)*BV(J+NML))
      SGMXY=SGMXY+COEF*AIMAG(R2*BV(J)+EI*R4*BV(J+NML))
      UX=UX+COEF*REAL(R5*BV(J)+EI*R6*BV(J+NML))/(2*SHMUD)
      UY=UY+COEF*AIMAG(R5*BV(J)+EI*R6*BV(J+NML))/(2*SHMUD)
C     ==============================================================
22    CONTINUE
C     ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      WRITE  (6,230)  I,SGMXX,SGMYY,SGMXY,UX,UY
C     ==============================================================
21    CONTINUE
C     ==============================================================
      END
```

D. THE COMPUTER PROGRAM FOR A CIRCULAR PLANE CONTAINING AN ELLIPTICAL HOLE OR A SHARP CRACK (follows)

```
      PROGRAM  FPMEH    (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
      DIMENSION   X(61),Y(61),RM(120,120),WKAREA(120),BV(120)
      DIMENSION   XF(15),YF(15)
C     ****************************************************************
      REAL    NXI,NYI,M
      COMPLEX  Z(60),ZE(60),Z2(60),Z3(60),ZP(60),ZZ2(60),ZEC(60)
      COMPLEX  RI(60),RO(60),TI(60),TO(60)
      COMPLEX  G(60),K(60),ZEF(15),ZF(15),QMT,QCR
      COMPLEX  PD1,PD2,P1,P2,H1,H2,K1,K2,S1,S2,R1,R2,R3,R4,R5,R6
      COMPLEX  PDD1,PDD2,ZT1,ZT2,ZD1,ZD2,WD,WD2,WD3,WDD
      COMPLEX  ZE2,CS,CSC,QM,QC,QM2,QC2
      COMPLEX  PD,ZZ2F,ZF2,ZF3,ZEFC,EI,ZTI,ZTO,ZRI,ZRO,ZTO2,ZRO2
      COMPLEX  ZED,ZED2,ZEFD,ZEFD2,ZZ,DR,DT,F,FD
      COMPLEX  X1,X2,Y1,Y2,Y3,Y4,AN,ZTO3,ZTO4
C     ****************************************************************
C     ****************************************************************

100   FORMAT (2F12.8)
110   FORMAT (*0*,14X,*POISSONS RATIO*,10X,*MODULUS CF ELASTICITY*
     $ ,13X,*SHEAR MODULUS*)
120   FORMAT (*0*,18X,F10.8,10X,F20.8,10X,F20.8)
130   FORMAT (2I6)
140   FORMAT (////,10X,*NO. OF MESHES=*,I3,25X,*NO. OF FIELD POINTS=*
     $ ,I3)
150   FORMAT (6F10.5)
160   FORMAT (///,10X,*NODE NO. (I)*,22X,*X(I)*,18X,*Y(I)*/*0*/
     $ (*0*,10X,I5,25X,F8.3,15X,F8.3))
170   FORMAT (6F10.8)
180   FORMAT(////,13X,*MESH NO.*,19X,*BVX*,29X,*BVY*/*0*/(*0*,15X,I4,
     $ 16X,F10.5,25X,F10.5))
190   FORMAT  (////,*0*,33X,*FICTIOUS TRACTIONS*/*0*/*0*,4X,
     $ *MESH NO.*,19X,*PSX*,19X,*PSY*/(*0*,4X,I4,11X,2(F20.10)))
200   FORMAT (6F10.6)
210   FORMAT (////,20X,*FIELD POINT NO.*,24X,*XF(I)*,28X,*YF(I)*
     $ /*0*/(*0*,25X,I5,25X,F10.7,25X,F10.7))
220   FORMAT  (*1*,35X,*STRESSES  AND  DISPLACEMENTS  OF  THE  FIELD
     $ POINTS*///,2X,*NO. OF F.P.*,8X,*SIGMAXX*,18X,*SIGMAYY*,18X,
     $ *SIGMAXY*,20X,*UX*,22X,*UY*)
230   FORMAT  (///,4X,I3,5(5X,F20.10))
240   FORMAT (////,25X,*TRANSFORMED  BOUNDARY  POINTS  TO  ZETA  *,
     $ *PLANE*///,10X,*NODE NO. (I)*,22X,*REAL*,18X,*IMAG*)
250   FORMAT  (*0*,10X,I5,25X,F8.3,15X,F8.3)
260   FORMAT (////,25X,*TRANSFORMED  FIELD  POINTS  TO  ZETA PLANE*
     $ ,////,20X,*FIELD POINT NO. *,24X,*REAL*,28X,*IMAG*)
270   FORMAT (//,25X,I4,2(25X,F9.3))
280   FORMAT  (////,25X,*TRANSLATED AND ROTATED BOUNDARY POINTS
     $ DUE  TO  THETA   XO AND YO *)
290   FORMAT  (////,20X,* MAJOR AXIS =*,F6.3,15X,* MINOR AXIS =*,
     $ F6.3,///,20X,* ANGLE CF ROTATION THETA=*,F6.2,/*0*/,20X,
     $ * TRANSLATION OF THE COORDINATES  XO =*,F6.3,10X,*YO =*,F6.3,
     $ ///,20X, * MAGNIFICATION OF THE  SIZE GF THE PLANE  WR  =*,
     $ F15.10)
300   FORMAT  (////,30X,* SHARP    CRACK        PROBLEM*)
C     ****************************************************************

C     *******************************
C     *    MATERIAL  PROPERTIES     *
C     *******************************
      READ (5,100) PR,EMUD
      SHMUD=EMUD/(2*(1+PR))
      WRITE (6,110)
      WRITE (6,120)  PR , EMUD , SHMUD

C     ************************************************************
C     *              GEOMETRY OF THE PROBLEM                     *
C     *           TRACTION BOUNDARY CONDITION                    *
C     *      RECTANGULAR,PLANE STRESS,UNIAXIAL TENSION           *
C     *  NUMBER OF MESH LENGTH=NML,NUMBER OF FIELD PCINT=NFP     *
C     ************************************************************

      READ  (5,130)  NML , NFP
      WRITE  (6,140)  NML , NFP
      NML1=NML+1
      NML2=NML*2
      PI=4*ATAN(1.0)
      EI=(0.0,1.0)
      M=1.0
      THETA=0.0
      X0=4.5
      Y0=0.0
```

```
      WR=1.0
      IF(M.EQ.1.0)   WRITE(6,300)
      AX=(1+M)*2.0
      AY=(1-M)*2.0
      WRITE  (6,290)   AX,AY,THETA,X0,Y0,WR
C     ************************************************************
C     *   CALCULATION OF BOUNDARY POINTS FOR THE CIRCULAR DISC   *
C     ************************************************************
      DO  50   J=1,NML1
      I=J-1
      X(J)=6*COS(PI*(1+I/15.)/2.)
      Y(J)=6*SIN(PI*(1+I/15.)/2.)
50    CONTINUE
      WRITE   (6,160)  (I,X(I),Y(I),I=1,NML1)
C     *****************************************************************
C     *              CALCULATION OF THE NEW BOUNDARY POINTS           *
C     *  DUE TO THE ROTATION AND TANSLATION OF THE COORDINATES AND    *
C     *         DIFFERENT WIDTH RATIO FOR ELLIPTICAL HOLE CR          *
C     *                BIMODAL  FRACTURE  OF A CRACK                  *
C     *  WHERE THETA=ROTATION AND (X0,Y0)=NEW COORDINATES OF ORIGIN   *
C     *           AND WIDTH RATIO  WR=WIDTH/CRACK LENGTH              *
C     *****************************************************************
      DO  48   I=1,NML1
      X(I)=WR*X(I)
      Y(I)=WR*Y(I)
48    CONTINUE
      IF(X0.NE.0.0)   GO  TO  42
      IF(Y0.NE.0.0)   GO  TO  42
      IF(THETA.EQ.0.0)   GO   TO   43
42    T=PI*THETA/180
      DO  40   I=1,NML1
      XI=X(I)
      YI=Y(I)
      X(I)=(XI-X0)*COS(T)+(YI-Y0)*SIN(T)
      Y(I)=(YI-Y0)*COS(T)-(XI-X0)*SIN(T)
40    CONTINUE
43    WRITE    (6,280)
      WRITE  (6,160)  (I,X(I),Y(I),I=1,NML1)

C     ***************************************************
C     *       CALCULATION   OF   COEFFICIENTS           *
C     ***************************************************

      IPLANE=1
      IF(IPLANE.EQ.2)   GO TO 10
      ALPHA=(3-PR)/(1+PR)
      GO TO 11
10    ALPHA=3-4*PR
11    COEF=1/(2*PI*(1+ALPHA))
C     ***************************************************
C     *    TRANSFORMATION   OF   THE   BOUNDARY   POINTS    *
C     ***************************************************
      WRITE  (6,240)
      DO  12   I=1,NML
      ZE(I)=CMPLX((X(I)+X(I+1))/2,(Y(I)+Y(I+1))/2)
      ZEC(I)=CONJG(ZE(I))
      ZE2=ZE(I)**2
      CS=CSQRT(ZE2-4*M)
      CSC=CSQRT(ZEC(I)**2-4*M)
      Y1=(ZE(I)+CS)/(2*M)
      Y2=(ZE(I)-CS)/(2*M)
      Y3=(ZEC(I)-CSC)/2.0
      Y4=(ZEC(I)+CSC)/2.0
      IF((X(I)+X(I+1))/2.LE.0.0)   GO   TO   13
      Z(I)=Y2
      RO(I)=Y1
      RI(I)=Y2
      GO   TO   17
13    Z(I)=Y1
      RO(I)=Y2
      RI(I)=Y1
17    A=REAL(Z(I))
      B=AIMAG(Z(I))
      WRITE   (6,250)   I,A,B

      IF(CABS(Y3).LT.1.0)   GO   TO   18
      TO(I)=Y3
      TI(I)=Y4
      GO   TO   19
18    TO(I)=Y4
```

```
      TI(I)=Y3
19    Z2(I)=Z(I)**2
      Z3(I)=Z2(I)*Z(I)
      ZP(I)=M*Z(I)
      ZZ2(I)=M*Z2(I)
      QMT=M*TI(I)**2-ZE(I)*TI(I)+1
      QCR=RI(I)**2-ZEC(I)*RI(I)+M
      G(I)=QMT/(TO(I)-TI(I))
      K(I)=QCR/((RO(I)-RI(I))*M)
12    CONTINUE
C     ****************************************************************
C     *  CALCULATION OF ELEMENTS OF THE RM(NML2,NML2) MATRIX  *
C     ****************************************************************

C     ===============================
      DO  14  I=1,NML
C     ===============================
      DSI=SQRT((X(I+1)-X(I))**2+(Y(I+1)-Y(I))**2)
      NXI=(Y(I+1)-Y(I))/DSI
      NYI=(X(I)-X(I+1))/DSI
      WD=-1/Z2(I)+M
      WD2=WD**2
      WD3=WD2*WD
      WDD=2/Z3(I)
C     ====================
      DO  15  J=1,NML
C     ====================
      IF (I.EQ.J)  GO  TO  16
      ZED=ZE(I)-ZE(J)
      ZED2=ZED**2

      QM=ZZ2(I)-ZE(J)*Z(I)+1
      QM2=QM**2
      QC=Z2(I)-ZEC(J)*Z(I)+M
      QC2=QC**2
      DR=2*ZP(I)-ZE(J)
      DT=2*Z(I)-ZEC(J)
      PD=DR*QC-DT*QM
      ZZ=1-ZZ2(I)
      F=Z(I)*(Z2(I)+M)/ZZ
      FD=((3*Z2(I)+M)*ZZ+2*ZZ2(I)*(Z2(I)+M))/ZZ**2
      ZTI=Z(I)-TI(J)
      ZRI=Z(I)-RI(J)
      ZTO=1/(Z(I)-TO(J))
      ZTO2=ZTO**2
      ZRO=1/(Z(I)-RO(J))
      ZRO2=ZRO**2

      PD1=ZRO-ALPHA*ZTO
      PDD1=-ZRO2+ALPHA*ZTO2
      IF (M.NE.1.0)  GO  TO  32
      IF(ZTI.NE.0.0)  GO  TO  32
      AN=DR/ZTO-QM
      ZTO3=ZTO*ZTO2
      ZTO4=ZTO3*ZTO
      PD2=2*ZTO-2*AN*ZTO3
      PDD2=-6*ZTO2+6*AN*ZTO4
      GO  TO  31
32    PD2=PD/QC2-G(J)/ZTI**2
      PDD2=((2*M*QC-2*QM)*QC-2*DT*PD)/(QC2*QC)+2*G(J)/ZTI**3
31    ZD1=(DT*QM-DR*QC)/QM2-K(J)/ZRI**2+F*PDD1+FD*PD1
      ZD2=ZTO-ALPHA*ZRO+F*PDD2+FD*PD2

      H1=PD1/WD
      H2=PD2/WD
      S1=ZD1/WD
      S2=ZD2/WD
      K1=PDD1/WD2-PD1*WDD/WD3
      K2=PDD2/WD2-PD2*WDD/WD3
      H1=H1-1/ZED
      K1=K1+1/ZED2
      S1=S1-ZEC(J)/ZED2
      S2=S2+ALPHA/ZED

      R1=2*(H1+H2)
      R2=ZEC(I)*(K1+K2)+S1+S2
      R3=2*(H1-H2)
      R4=ZEC(I)*(K1-K2)+S1-S2
```

```
      RM(I,J)=(REAL(R1-R2)*NXI+AIMAG(R2)*NYI)*COEF*DSI
      RM(I,J+NML)=(REAL(EI*(R3-R4))*NXI+AIMAG(EI*R4)*NYI)*COEF*DSI
      RM(I+NML,J)=(AIMAG(R2)*NXI+REAL(R1+R2)*NYI)*COEF*DSI
     RM(I+NML,J+NML)=(AIMAG(EI*R4)*NXI+REAL(EI*(R3+R4))*NYI)*COEF*DSI
      GO  TO  15
16    RM(I,I)=1/2.0
      RM(I,I+NML)=0.0
      RM(I+NML,I)=0.0
      RM(I+NML,I+NML)=1/2.0
C     ==================================================
15    CONTINUE
C     ==================================================
14     CONTINUE
C     ==================================================
C     **************************************************
C     *    CALCULATION OF FICTITIOUS TRACTION   P-STAR    PS(I)    *
C     **************************************************
      READ  (5,170)   (BV(I),BV(I+NML),I=1,NML)
C     **************************************************
C     *    CORRECTING THE B.V. DUE TO THE CHANG IN WIDTH RATIO TO   *
C     *          KEEP THE DISTRIBUTED LOAD =1.0 LB/LENGTH         *
C     **************************************************
      DO  46  I=1,NML
      BV(I)=BV(I)*WR
      BV(I+NML)=BV(I+NML)*WR
46    CONTINUE
C     ****************************************
C     *   CALCULATION OF THE ROTATED BOUNDARY VALUES   *
C     ****************************************
      IF(THETA.EQ.0.0)   GO  TO  49
      DO  41   I=1,NML
      BI=BV(I)
      BJ=BV(I+NML)
      BV(I)=BI*COS(T)+BJ*SIN(T)
      BV(I+NML)=BJ*COS(T)-BI*SIN(T)
41    CONTINUE
49    WRITE  (6,180)   (I,BV(I),BV(I+NML),I=1,NML)
      CALL  LEQT1F  (RM,1,NML2,NML2,BV,8,WKAREA,IER)
      WRITE  (6,190)   (I,BV(I),BV(I+NML),I=1,NML)
C     ****************************************
C     *   CALCULATION  OF  STRESSES  AND  DISPLACEMENTS   *
C     ****************************************
      READ  (5,200)   (XF(I),YF(I),I=1,NFP)
      WRITE  (6,210)   (I,XF(I),YF(I),I=1,NFP)
      WRITE  (6,260)
      DO  20  I=1,NFP
      ZEF(I)=CMPLX(XF(I),YF(I))
      ZF(I)=(ZEF(I)-CSQRT(ZEF(I)**2-4*M))/(2*M)
      IF(XF(I).GT.0.0)   GO  TO  24
      IF(YF(I).LE.0.0)  ZF(I)=(ZEF(I)+CSQRT(ZEF(I)**2-4*M))/(2*M)
24    A=REAL(ZF(I))
      B=AIMAG(ZF(I))
      WRITE  (6,270)   I,A,B
20    CONTINUE
      WRITE  (6,220)
C     ================================
      DO  21  I=1,NFP
C     ================================
      SGMXX=0.0
      SGMYY=0.0
      SGMXY=0.0
      UX=0.0
      UY=0.0
      ZF2=ZF(I)**2
      ZF3=ZF(I)*ZF2
      WD=-1/ZF2+M
      WD2=WD**2
      WD3=WD2*WD
      WDD=2/ZF3
      ZEFC=CONJG(ZEF(I))
      ZZ2F=M*ZF2
      ZP(I)=M*ZF(I)
C     ================================
      DO  22  J=1,NML
```

```
C          ================================
           ZEFD=ZEF(I)-ZE(J)
           ZEFD2=ZEFD**2
           QM=ZZ2F-ZE(J)*ZF(I)+1
           QM2=QM**2
           QC=ZF2-ZEC(J)*ZF(I)+M
           QC2=QC**2
           DR=2*ZP(I)-ZE(J)
           DT=2*ZF(I)-ZEC(J)
           PD=DR*QC-DT*QM
           ZZ=1-ZZ2F
           F=ZF(I)*(ZF2+M)/ZZ
           FD=((3*ZF2+M)*ZZ+2*ZZ2F*(ZF2+M))/ZZ**2
           ZTI=ZF(I)-TI(J)
           ZRI=ZF(I)-RI(J)
           ZTO=1/(ZF(I)-TO(J))
           ZTO2=ZTO**2
           ZRO=1/(ZF(I)-RO(J))
           ZRO2=ZRO**2

           X1=CLOG(-1/(ZRO*RO(J)))
           X2=CLOG(-1/(ZTO*TO(J)))
           P1=X1-ALPHA*X2
           P2=QM/QC+G(J)*(1/ZTI+1/TI(J))
           PD1=ZRO-ALPHA*ZTO
           PD2=PD/QC2-G(J)/ZTI**2
           PDD1=-ZRO2+ALPHA*ZTO2
           PDD2=((2*M*QC-2*QM)*QC-2*DT*PD)/(QC2*QC)+2*G(J)/ZTI**3
           ZT1=QC/QM+K(J)*(1/ZRI+1/RI(J))-M+F*PD1
           ZT2=+X2-ALPHA*X1
           ZD1=(DT*QM-DR*QC)/QM2-K(J)/ZRI**2+F*PDD1+FD*PD1
           ZD2=ZTO-ALPHA*ZRO+F*PDD2+FD*PD2
           H1=PD1/WD
           H2=PD2/WD
           S1=ZD1/WD
           S2=ZD2/WD
           K1=PDD1/WD2-PD1*WDD/WD3
           K2=PDD2/WD2-PD2*WDD/WD3

           P1=P1-CLOG(ZEFC)
           H1=H1-1/ZEFD
           K1=K1+1/ZEFD2
           ZT1=ZT1+ZEC(J)/ZEFD
           ZT2=ZT2+ALPHA*CLOG(ZEFD)
           S1=S1-ZEC(J)/ZEFD2
           S2=S2+ALPHA/ZEFD

           R1=2*(H1+H2)
           R2=ZEFC*(K1+K2)+S1+S2
           R3=2*(H1-H2)
           R4=ZEFC*(K1-K2)+S1-S2
           R5=ALPHA*(P1+P2)-ZEF(I)*CONJG(H1+H2)-CONJG(ZT1+ZT2)
           R6=ALPHA*(P1-P2)+ZEF(I)*CONJG(H1-H2)+CONJG(ZT1-ZT2)

           SGMXX=SGMXX+COEF*REAL((R1-R2)*BV(J)+EI*(R3-R4)*BV(J+NML))
           SGMYY=SGMYY+COEF*REAL((R1+R2)*BV(J)+EI*(R3+R4)*BV(J+NML))
           SGMXY=SGMXY+COEF*AIMAG(R2*BV(J)+EI*R4*BV(J+NML))
           UX=UX+COEF*REAL(R5*BV(J)+EI*R6*BV(J+NML))/(2*SHMUD)
           UY=UY+COEF*AIMAG(R5*BV(J)+EI*R6*BV(J+NML))/(2*SHMUD)
C          ===================================================
22         CONTINUE
C          +++++++++++++++++++++++++++++++++++++++++++++++++++
           WRITE  (6,230)  I,SGMXX,SGMYY,SGMXY,UX,UY
C          ===================================================
21         CONTINUE
C          ===================================================
           END
```

APPENDIX F

COMPUTER PROGRAM FOR PLOTTING THE CONTOUR OF THE OPENING WITH TWO OR THREE AXES OF SYMMETRY

A computer program was employed for plotting the contour of an opening with two or three axes of symmetry (e.g., triangular hole, square hole or elliptical hole). A listing of that program for a triangular hole with rounded corners, see Figures 5.1 and 5.2, for a square hole with rounded corners, see Figures 5.3 and 5.4, and for elliptical hole are presented in this appendix.

A. INPUT DATA

The following information must be provided as input.

AM(I) I=1,6	- This character represents 6 different coefficients, ε, in equations (5.1) and (5.4)
c	- coefficient in equations (5.1) and (5.4) (has to be specified in the program)

The type of the paper, pen and ink has to be specified in the statement "CALL PLOTS."

B. OUTPUT

Six different plots associated with the six different ε's (i.e., AM(I) in the program) and specified c.

C. THE COMPUTER PROGRAM FOR PLOTTING A CONTOUR WHICH HAS THREE AXES OF SYMMETRY (TRIANGULAR HOLE WITH ROUNDED CORNERS)

```
      PROGRAM  TRPLTR (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
      DIMENSION AM(6)
      DIMENSION  IBUF(257)
      CALL  PLIMIT  (40.0)
100   FORMAT (6E10.3)
      CALL PLOTS(IBUF,257,5)
      CALL  SYMBOL  (1.5,6.5,0.28,12HFINITE PLATE,0.0,12)
      CALL  SYMBOL  (2.0,6.0,0.28,10HTRIANGULAR,0.0,10)
      CALL  SYMBOL  (2.5,5.5,0.25,4HHOLE,0.0,4)
      CALL  SYMBOL  (1.5,4.0,0.14,18HX=COS(T)+E.COS(2T),0.0,18)
      CALL  SYMBOL  (1.5,3.5,0.14,20HY=C.SIN(T)-E.SIN(2T),0.0,20)
      CALL PLOT(5.0,1.5,-3)
      READ  (5,100)  (AM(I),I=1,6)
      C=1.0
      DO  20  J=1,6
      XX=0.0
      YY=2.8
      DO 10 I=1,361
      IPEN=2
      THETA=(I-1)/360.*2*3.14159265358 97
      X=COS(THETA)+AM(J)*COS(2*THETA)
      Y=C*SIN(THETA)-AM(J)*SIN(2*THETA)
      IF (I.NE.1)  GO  TO  10
      IPEN=3
      U=X+1.0
      V=Y
10    CALL PLOT(X,Y,IPEN)
      CALL  SYMBOL (U,V,0.14,2HE=,0.0,2)
      U=U+0.3
      CALL  NUMBER (U,V,0.14,AM(J),0.0,2)
      R=U-0.3
      Z=V+0.25
      CALL  SYMBOL (R,Z,0.14,2HC=,0.0,2)
      R=R+0.3
      CALL  NUMBER (R,Z,0.14,C,0.0,2)
      IF((J/3).NE.(FLOAT(J))/3.0)  GO TO 30
      XX=XX+7.0
      YY=-2*YY
30    CALL  PLOT  (XX,YY,-3)
20    CONTINUE
      CALL PLOT (0,0,999)
      END
```

D. THE COMPUTER PROGRAM FOR PLOTTING A CONTOUR WHICH HAS TWO AXES OF SYMMETRY (E.G., SQUARE HOLE WITH ROUNDED CORNERS, OVAL HOLE AND ELLIPTICAL HOLE WHEN $\varepsilon = 0$)

```
      PROGRAM  SQPLTR (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=CUTPUT)

      DIMENSION AM(6)
      DIMENSION  IBUF(257)
      CALL  PLIMIT  (40.0)

100   FORMAT (6E10.3)
      CALL PLOTS(IBUF,257,5)

      CALL  SYMBOL  (1.5,6.5,0.28,12HFINITE PLATE,0.0,12)
      CALL  SYMBOL  (2.5,6.0,0.28,5HSQUAR,0.0,10)
      CALL  SYMBOL  (2.5,5.5,0.28,4HHOLE,0.0,4)
      CALL  SYMBOL  (1.5,4.0,0.14,19HX=COS(T)+E.COS(3T),0.0,18)
      CALL  SYMBOL  (1,5,3.5,0.14,20HY=C.SIN(T)-E.SIN(3T),0.0,20)
      CALL PLOT(9.0,1.5,-3)

      READ  (5,100)  (AM(I),I=1,6)
      C=1.0

      DO  20  J=1,6
      XX=0.0
      YY=2.8
      DO 10 I=1,361
      IPEN=2
      THETA=(I-1)/360.*2*3.141592653589 7
      X=COS(THETA)+AM(J)*COS(3*THETA)
      Y=C*SIN(THETA)-AM(J)*SIN(3*THETA)
      IF (I.NE.1)  GO  TO  10
      IPEN=3
      U=X+1.0
      V=Y
10    CALL PLOT(X,Y,IPEN)
      CALL  SYMBOL (U,V,0.14,2HE=,0.0,2)
      U=U+0.3
      CALL  NUMBER (U,V,0.14,AM(J),0.0,2)
      R=U-0.3
      Z=V+0.25
      CALL  SYMBOL (R,Z,0.14,2HC=,0.0,2)
      R=R+0.3
      CALL  NUMBER (R,Z,0.14,C,0.0,2)
      IF((J/3).NE.(FLOAT(J))/3.0)  GO TO 30
      XX=XX+7.0
      YY=-2*YY
30    CALL  PLOT  (XX,YY,-3)
20    CONTINUE
      CALL PLOT (0,0,999)
      END
```

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