EQUIVALENCE OF TUBULAR NEIGHBORHOODS

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ABSTRACT

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By

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Let $p\colon \tilde{X} \to X$ be a connected covering projection. We say that p is almost regular (AR) if and only if for every $f\colon \tilde{X} \to X$, the commutativity of

$$\begin{tikzcd} \tilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

implies f is a homeomorphism. X is absolutely almost regular (AAR) if and only if every p is AR. In general, an AR covering projection may not be a regular covering projection. Theorem: Among closed surfaces, the 2-sphere, projective plane, torus, and Klein bottle are the only (AAR) spaces.

Let M be an (n-1)-manifold locally flatly embedded in an n-manifold. Then there exists a tubular neighborhood N (a topological 1-disk bundle over M) of M. The pair (N,M) is equivalent to another such pair (N',M) if and only if there exists a homeomorphism between the two

pairs. There exists a 2-sheeted covering $p_n:\dot{N}\to M$ and (N,M) is homeomorphic to (M_{pn}, M) (where M_{pn} is the mapping cylinder of p_n). Theorem: (N_1,M) and (N_2, M) are equivalent if and only if there exists homeomorphisms h and $\bar{h}$ such that

$$\begin{tikzcd} \dot{N}_1 \arrow[r, "\bar{h}"] \arrow[d, "p_{n1}"'] & \dot{N}_2 \arrow[d, "p_{n2}"] \\ M \arrow[r, "h"] & M \end{tikzcd}$$

is commutative. Let M be a closed surface. Let $\mathcal{T}$ be the set of equivalence classes of pairs (N, M), where N is obtained by considering all possible locally flat embeddings of M into all possible 3-manifolds, except for the case $N = M\mathrm{x}I$. Let K_i be the subgroups of index 2 of $\Pi_1(M)$ and say K_1 and K_2 are equivalent if and only if there exists an automorphism of $\Pi_1(M)$ that maps K_1 onto K_2. Let $\mathcal{N}$ be the set of equivalence classes of all K_i. Theorem 3: There exists a natural 1-1 correspondence between $\mathcal{N}$ and $\mathcal{T}$. (In particular, if $\Pi_1(M) = [C_1, C_2,\ldots C_n | C_1{}^2 C_2{}^2 \ldots C_n{}^2 = 1]$, then $\mathcal{T}(M) \leq n$, and $\mathcal{T}$(projective plane) = 1 and $\mathcal{T}$(Klein bottle) = 2.

Let (N,M) be defined as above. Theorem: Let h_1 and h_2 be involutions of N with M as fixed point set. Then h_1 and h_2 are equivalent, that is, there exists a homeomorphism t of (N,M) onto itself with $t|M = 1_M$ and $h_1 = t^{-1}h_2t$.

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INTRODUCTION

Call a covering projection, p, almost regular if and only if the commutativity of

$$\begin{tikzcd} \tilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

implies f is a homeomorphism. The first section is concerned with what types of spaces have covering projections that are not almost regular. Theorem 1.1 says that if a topological space X has a fundamental group, $\pi_1(X,x_0)$, which has a subgroup H and element t with tHt^{-1} properly contained in H and an open covering $\mathcal{U}$ with $\pi_1(\mathcal{U},x_0)$ contained in H, then X has a covering projection that is not almost regular. Then all surfaces are classified as to whether or not they have non-almost regular covering projections or not: clearly, a surface with one handle or crosscap does not, Theorem 1.2 proves all surfaces with 2 or more handles or 3 or more crosscaps do, and Theorem 1.3 proves that the Klein bottle does not. The section is concluded with an example of a covering projection of the Klein bottle that is almost regular but not regular.

Letting f: $M^{n-1} \to N^n$ be a locally flat embedding as defined by Brown in [2], a tubular neighborhood T(f) of M in N corresponding to f is a topological 1-disk bundle of f(M) that is contained in N. Section 2 is concerned with when there is a homeomorphism h: $(T(f),M) \longrightarrow (T(g),M)$. Theorem 2.1 proves that 2 tubular neighborhoods, T(f) and T(g), of M are equivalent if and only if there is a homeomorphism $\bar{h}$: $\dot{T}(f) \longrightarrow \dot{T}(g)$ for which

$$\begin{tikzcd} \dot{T}(f) \arrow[r, "\bar{h}"] \arrow[d, "f_b"'] & \dot{T}(g) \arrow[d, "g_b"] \\ M \arrow[r, "h"] & M \end{tikzcd}$$

commutes (where f_b and g_b are restrictions of the bundle map). For M connected, it is shown that the number of non-equivalent tubular neighborhoods of M, $\mathcal{T}(M)$, $\geq$ 1 + number of non-equivalent subgroups of index 2 of $\pi_1(M)$, where H_1 and H_2 (subgroups of $\pi_1(M)$) are equivalent if and only if there is an automorphism a: $\pi_1(M) \longrightarrow \pi_1(M)$ with $a(H_1) = H_2$. Theorem 2.2 shows that if M is a surface with n (2 or more) crosscaps, then $3 \leq \mathcal{T}(M) \leq n+1$. Also, $\mathcal{T}$(projective plane) = 2 and $\mathcal{T}$(Klein bottle) = 3. Section 2 concludes with Theorem 2.3 on involutions of tubular neighborhoods: If h_1 and h_2 are involutions of T(f) with M as fixed point set, then h_1 is equivalent to h_2.

<u>Section 1</u>. Almost Regular Covering Projections.

Let $p\colon \tilde{X} \to X$ be a covering projection with X connected and locally path-connected and $\tilde{X}$ connected. Note that this implies $\tilde{X}$ is locally path-connected, since p is a local homeomorphism.

<u>Definition</u> 1.1. p is almost regular if and only if the commutativity of

$$\begin{tikzcd} \tilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

implies f is a homeomorphism.

<u>Remark</u> 1.1. One can also state Definition 1.1 in terms of a property of the fundamental group of the covering space, $\tilde{X}$. That is,

p is almost regular if and only if $p_{\#}\, \pi_1(\tilde{X},\tilde{x}_o)$ is not properly contained in any of its conjugates in $\pi_1(X,x_o)$ (where x_o is any point of X and $\tilde{x}_o$ is any preimage of x_o under p, and $p_{\#}$ means the map on fundamental groups induced by p).

Proof of Remark 1.1: a. Assume the commutativity of

$$\begin{tikzcd} \tilde{X},\tilde{x}_o \arrow[rr, "f"] \arrow[dr, "p"'] & & \tilde{X},f(\tilde{x}_o) \arrow[dl, "p"] \\ & X,x_o & \end{tikzcd}$$

implies f is a homeomorphism.

Suppose $p_{\#}\pi_1(\tilde{X},\tilde{x}_o)$ were properly contained in one of its conjugates, say $tp_{\#}\pi_1(\tilde{X},\tilde{x}_o)t^{-1}$. Then since $\{p_{\#}\pi_1(\tilde{X},\tilde{x}_o) \mid \tilde{x}_o \in p^{-1}(x_o)\}$ is a conjugacy class in $\pi_1(X,x_o)$ by [11, Thm. 6, p. 73], $tp_{\#}\pi_1(\tilde{X},\tilde{x}_o)t^{-1} = p_{\#}\pi_1(\tilde{X},\tilde{x}_o^1)$ for some $\tilde{x}_o{}^1$ in $\tilde{X}$ for which $p(\tilde{x}_o{}^1) = x_o$. And therefore, there exists $f'\colon \tilde{X} \to \tilde{X}$ such that

$$\begin{tikzcd}
\tilde{X},\tilde{x}_o \arrow[rr, "f'"] \arrow[dr, "p"'] & & \tilde{X},\tilde{x}_o{}^1 \arrow[dl, "p"] \\
& X,x_o &
\end{tikzcd}$$

commutes by [11, Thm. 5, p. 76]. Also by [11, Thm. 5, p. 76], there exists no such map from $(\tilde{X},\tilde{x}_o{}^1)$ to $(\tilde{X},\tilde{x}_o)$. So f' is not a homeomorphism which contradicts our assumption. Therefore, $p_{\#}\pi_1(\tilde{X},\tilde{x}_o)$ is not properly contained in any of its conjugates in $\pi_1(X,x_o)$.

b. Assume $p_{\#}\pi_1(\tilde{X},\tilde{x}_o)$ is not properly contained in any of its conjugates. Suppose

$$\begin{tikzcd}
\tilde{X},\tilde{x}_o \arrow[rr, "f"] \arrow[dr, "p"'] & & \tilde{X},f(\tilde{x}_o) \arrow[dl, "p"] \\
& X,x_o &
\end{tikzcd}$$

commutes. (Note that f is a covering projection by [11, Lemma 1, p. 79]). Suppose $f_{\#}\colon \pi_1(\tilde{X},\tilde{x}_o) \longrightarrow \pi_1(\tilde{X},f(\tilde{x}_o))$ is not a surjection. Then $p_{\#}f_{\#}\colon \pi_1(\tilde{X},\tilde{x}_o) \to p_{\#}\pi_1(\tilde{X},f(\tilde{x}_o))$ is not a surjection, since $f_\#$ and $p_\#$ are monomorphisms by [11, Thm. 4, p. 72]. But $p_\# f_\# \pi_1(\tilde{X},\tilde{x}_o) = p_\#\pi_1(\tilde{X},\tilde{x}_o)$ and by [11, Thm. 6, p. 73], $p_\# \pi_1(\tilde{X},\tilde{x}_o) = p_\# \pi_1(\tilde{X},f(\tilde{x}_o))$, so $p_\# f_\#\pi_1(\tilde{X},\tilde{x}_o) = p_\#\pi_1(\tilde{X},f(\tilde{x}_o))$. Therefore, $f_\#$ is a surjection. It follows that the multiplicity of p is 1 by [11, Thm. 9, p. 73] and thus that f is a homeomorphism.

Examples of almost regular covering projections:

1. All regular covering projections are almost regular by [11, Thm. 11, p. 74].

2. All covering projections, p, with finite multiplicity are almost regular. Since p is n to 1, for any commutative diagram

$$\begin{tikzcd} \tilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

pf is n to 1. So f must be 1 to 1 and thus a homeomorphism.

3. If $\pi_1(X,x_o)$ has the ascending or descending chain condition, then, clearly, any covering projection of X is almost regular.

<u>Definition</u> 1.2. A group G is said to be <u>not regular</u> if there exists a subgroup H of G and an element t of G for which tHt^{-1} is properly contained in H. Otherwise, G is said to be <u>regular</u>.

Any subgroup H and element t of a group G that is not regular, mentioned in the following pages, is understood to behave as in Definition 1.2.

Theorem 1.1. If $\pi_1(X,x_o)$ is not regular and there is an open covering $\mathcal{U}$ of X such that $\pi_1(\mathcal{U},x_o)$ is contained in H, then there is a covering projection with base space X that is not almost regular.

Proof: Let $p\colon (\tilde{X},\tilde{x}_o) \longrightarrow (X,x_o)$ be the covering projection with $p_\# \pi_1(\tilde{X},\tilde{x}_o) = H$ constructed as in [11, Thm. 13, p. 82]. $\tilde{X}$ is clearly connected. Let $\tilde{x}_o{}^1$ be the element of $\tilde{X}$ which corresponds to a loop w(t) about x_o in X which corresponds to the element t of $\pi_1(X,x_o)$. Consider the diagram:

$$\begin{tikzcd}
\tilde{X},\tilde{x}_o{}^1 \arrow[dr, "p"'] & & \tilde{X},\tilde{x}_o \arrow[dl, "p"] \\
& X,x_o &
\end{tikzcd}$$

$p_\#\pi_1(\tilde{X},\tilde{x}_o) = H$. Since p(path from $\tilde{x}_o{}^1$ to $\tilde{x}_o$) is $(w(t))^{-1}$ which corresponds to t^{-1}, $p_\#\pi_1(\tilde{X},\tilde{x}_o{}^1) = tHt^{-1}$. But tHt^{-1} is properly contained in H, so p is not almost regular.

Remark 1.2. One observes from the construction in Theorem 1.1 that if M is a connected manifold for which $\pi_1(M)$ is not regular, then there is a non-almost regular covering projection of M (because there is a covering of M by open sets $\{\mathcal{U}\}$, for which $\pi_1(\mathcal{U},x_o) = 0$).

Clearly, all covering projections of a surface with precisely one handle or one crosscap are almost regular. The following two theorems will complete a classification of surfaces according to whether or not they have non-almost regular covering projections or not.

<u>Theorem</u> 1.2. All surfaces with 2 or more handles or 3 or more crosscaps have non-almost regular covering projections.

Proof: Fact 1. If a group G is given by the generators $a_1, a_2, \ldots a_n$ and one relation $f(a_1, a_2, \ldots a_n) = 1$, and if, further, the element a_n occurs in this relation and cannot be removed from it by transformations, then the subgroup $\{a_1, a_2, \ldots, a_{n-1}\}$ is free and $a_1, a_2, \ldots, a_{n-1}$ are free generators of the subgroup by [8, p. 77].

Fact 2. Any free group, G, of 2 or more generators is not regular (Example by L. M. Sonneborn - Let H be the subgroup generated by

$$\{a, b^{-1}ab, b^{-2}ab^2, \ldots, b^{-n}ab^n, \ldots\}$$

$\pi_1(X_n)$, where X_n is a surface with n (2 or more) handles, has generators $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ and the single relation $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} \cdots a_nb_na_n^{-1}b_n^{-1} = 1$ by [11, p. 149]. Therefore, $\pi_1(X_n)$ has a free subgroup with more than one generator by Fact 1. So by Fact 2, and Remark 1.2, X_n has a non-almost regular covering projection.

$\pi_1(Y_n)$, where Y_n is a surface with n (3 or more) crosscaps, has generators $c_1, c_2, \ldots, c_n$ and the single relation $c_1{}^2 c_2{}^2 \cdots c_n{}^2 = 1$ by [11, p. 149]. Therefore, as in the case for X_n, Y_n has a non-almost regular covering projection.

<u>Theorem</u> 1.3. All covering projections of the Klein bottle, K, are almost regular.

Proof: Consider the covering projection of multiplicity 2, p: $S^1 \times S^1 \longrightarrow K$. $\pi_1(S^1 \times S^1)$ is regular, so, by the following lemma, $\pi_1(K)$ is regular. Therefore, all covering projections of K are almost regular.

<u>Lemma</u> 1.1. If G is not regular and N is a normal subgroup of G for which $[G:N] < \infty$, N is not regular.

Proof: Since tHt^{-1} is properly contained in H, t^kHt^{-k} is properly contained in H for each positive integer k. There is a k for which t^k is an element of N since $[G:N] < \infty$. Consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & t^k(H\cap N)t^{-k} & \longrightarrow & t^kHt^{-k} & \longrightarrow & t^kHt^{-k}/t^k(H\cap N)t^{-k} & \longrightarrow & 0 \\
\downarrow & & \downarrow i_1 & & \downarrow i & & \downarrow i_2 & & \downarrow \\
0 & \longrightarrow & H\cap N & \longrightarrow & H & \longrightarrow & H/H\cap N & \longrightarrow & 0
\end{array}$$

i_2 is an isomorphism since $H/(H\cap N)$ is isomorphic to HN/N and $[G:N] < \infty$. So, by the five lemma in [11, p. 185], if i_1 is onto, then i is onto. But i is not onto by assumption. Therefore, i_1 is not onto and N is not regular.

Example of closed PL manifolds of arbitrarily high dimension that have non-almost regular covering projections: Let $m \geq 4$. Embed X_n (or Y_n) in S^{m+1}. Take a regular neighborhood, N, of X_n in S^{m+1}. $\pi_1(\dot{N}) = \pi_1(X_n)$. $\dot{N}$ is a closed PL manifold of dimension m and by Remark 1.2 has a non-almost regular covering projection.

Example of an almost regular covering projection that is not regular:

$\pi_1(K)$ has 2 generators, c_1 and c_2, and the relation $c_1^{\,2}c_2^{\,2} = 1$. Consider the covering projection $p\colon (\tilde{K},\tilde{x}_o) \longrightarrow (K,x_o)$ with $p_{\#}\pi_1(\tilde{K},\tilde{x}_o) = \{c_2\}$ (where $\{a\}$ denotes the subgroup generated by a). $c_1^{\,-1}\{c_2\}\,c_1$ is not equal to $\{c_2\}$. Consider $w\colon\ (I,0) \longrightarrow (K,x_o)$ where $w(I)$ corresponds to c_1. There is a lifting of w to $w'\colon\ (I,0) \longrightarrow (\tilde{K},\tilde{x}_o)$ by [11, Thm. 3, p. 67]. Let $w'(1) = \tilde{x}_o^1$. Then $p(\tilde{x}_o^{\,1}) = x_o$ and $p_{\#}\pi_1(\tilde{K},\tilde{x}_o^1) = c_1^{\,-1}p_{\#}\pi_1(\tilde{K},\tilde{x}_o)c_1$ which is not equal to $p_{\#}\pi_1(\tilde{K},\tilde{x}_o)$. So p is not regular by [11, Thm. 11, p. 74], and, by Theorem 1.3, p is almost regular.

<u>Section 2</u>. Equivalence of Tubular Neighborhoods.

Let M be an $n-1$ manifold and $f\colon\ M\to N$ be a locally flat embedding of M into any n-manifold N (where manifold and locally flat embedding are defined as in [2]).

<u>Definition</u> 2.1. A <u>tubular neighborhood</u>, $T(f)$, of M in N corresponding to f is a topological 1-disk bundle of $f(M)$ that is contained in N.

<u>Remark</u> 2.1. Given a locally flat embedding $f\colon M\to N$, one can exhibit a tubular neighborhood corresponding to f in the style of [5] as follows:

Let $\{\mathcal{U}_\alpha\}$ be a basis for the topology on N such that if $U_\alpha \cap f(M) \neq \phi$ then there is a homeomorphism $h\colon\ (E^n, E^{n-}) \longrightarrow (U_\alpha, U_\alpha \cap f(M))$. E^{n-1} separates E^n, say into E^n_+ and E^n_-. Denote $h(E^n_+ \cup E^{n-1})$ by U^+_α and $h(E^n_- \cup E^{n-1})$ by U^-_α. Let $\{\mathcal{U}_\beta\} = \{U \mid U$ is an element of $\{\mathcal{U}_\alpha\}$ and $U\cap f(M) = \phi$ or $U = U^+_\alpha$ or U^-_α for some $\alpha\}$.

Let $N_1 = \{(x, U_\beta) \mid x$ is an element of $U_\beta\}$. Define an equivalence relation, R, on N_1 by (x, U_β) is equivalent to $(x^1, U_\beta{}^1)$ if and only if $x = x^1$ and $U_\beta \cap U_\beta{}^1 \cap C(f(M)) \neq \phi$.

Take $N' = N_1/R$ and denote an equivalence class by $[x, U_\beta]$. Let $\{\mathcal{U}_\beta{}^*\}$ be a basis for N' where $[x, U_\gamma]$ is an element of $U_\beta{}^*$ if and only if x is an element of U_β. Define $n\colon N'\to N$ by $n[x, U_\beta] = x$. n is 2 to 1 restricted to $n^{-1}(f(M))$ and restricted to $Cn^{-1}(f(M))$ is 1 to 1. N' is an n-manifold with boundary $n^{-1}(f(M))$. $n^{-1}(f(M))$ is collared by [2], and its image is a 1-disk bundle of $f(M)$.

Remark 2.2. Let f_b: $T(f) \to M$ be the bundle map. Then $f_b | \dot{T}(f)$ is a covering projection of multiplicity 2 and, denoting the mapping cylinder of $f_b | \dot{T}(f)$ by M_f, (M_f, M) is homeomorphic to $(T(f), M)$.

Definition 2.2. $T(f)$ is equivalent to $T(g)$ if and only if there is a homeomorphism h: $(T(f), M) \longrightarrow (T(g), M)$.

Theorem 2.1. $T(f)$ is equivalent to $T(g)$ if and only if there is a homeomorphism $\bar{h}$: $\dot{T}(f) \longrightarrow \dot{T}(g)$ for which

$$\begin{tikzcd} \dot{T}(f) \arrow[r, "\bar{h}"] \arrow[d, "f_b"'] & \dot{T}(g) \arrow[d, "g_b"] \\ M \arrow[r, "h"] & M \end{tikzcd}$$

commutes.

Proof: Assume $T(f)$ is equivalent to $T(g)$. Then there is a homeomorphism h: $(M_f, M) \longrightarrow (M_g, M)$. Let g_1 be the covering projection, g_1: $\dot{T}(g) x I \longrightarrow M x I$, induced by g_b and let i: $M \longrightarrow M x 1 \longrightarrow M x I$ be the natural inclusion. Then, by the diagram:

$$\begin{tikzcd} \dot{T}(g) x I + M \arrow[r, "id"] \arrow[d, "g_1 + i"] & M_g \arrow[dl, "g'"] \\ M x I & \end{tikzcd}$$

g' is continuous by [4, Thm. 3.2, p. 123]. (id will always indicate the obvious identification map.)

Case 1. Assume $\dot{T}(f)$, $\dot{T}(g)$, and M are connected. We have

$$h^*:\quad \dot{T}(f)\times I \xrightarrow{id} M_f \xrightarrow{h} M_g \xrightarrow{g'} M\times I$$

where $h^*(x,1)$ is in $M\times 1$. Consider the diagram:

$$\begin{tikzcd} & \dot{T}(g)\times I \arrow[d, "g_1"] \\ \dot{T}(f)\times I \arrow[ur, dashed, "h'"] \arrow[r, "h^*"] & M\times I \end{tikzcd}$$

There exists h' with $h'|\dot{T}(f)\times 0 = h$ by [11, Thm. 3, p. 67]. Note that $h'(\dot{T}(f)\times 1)$ is contained in $\dot{T}(g)\times 1$. Also $h^*|\dot{T}(f)\times 1$ is a covering projection.

Now consider the diagram:

$$\begin{tikzcd} & \dot{T}(g)\times 1 \arrow[d, "g_1"] \\ \dot{T}(f)\times 1 \arrow[ur, "h'"] \arrow[r, "h^*"] & M\times 1 \end{tikzcd}$$

$h'|\dot{T}(f)\times 1$ is a covering projection, since $\dot{T}(f)$, $\dot{T}(g)$, and M are connected and locally path-conntected by [11, Lemma 1, p. 79] and therefore $h'|\dot{T}(f)\times 1$ is open and onto. $h'|\dot{T}(f)\times 1$ must clearly then be 1 to 1, so $h'|\dot{T}(f)\times 1$ induces the desired $\bar{h}$.

Case 2. Assume $\dot{T}(f)$ and $\dot{T}(g)$ are not connected and that M is connected.

Then there are T_1 and T_2 (each connected) for which $f_b|T_i\colon T_i \longrightarrow M$ is a homeomorphism. As in Case 1, $h^*\colon \dot{T}(f)\times I \longrightarrow M\times I$. For this case, consider the diagram:

$$\begin{tikzcd} & \dot{T}(g)\times I \arrow[d, "g_1"] \\ T_i\times I \arrow[ur, dashed, "h_i^{1}"] \arrow[r, "h^*"'] & M\times I \end{tikzcd}$$

There exists h_i^{1} that makes the diagram commute and $h_i^{1}(T_i\times I)$ is connected. So, we get a 1 to 1, onto map $h'\colon\dot{T}(f)\longrightarrow\dot{T}(g)$ that clearly must be open.

Case 3. Assume M is not connected. The argument is the same as the preceding ones applied to the components of M and their preimages.

Assume there is $\bar{h}\colon \dot{T}(f)\longrightarrow\dot{T}(g)$ for which

$$\begin{tikzcd} \dot{T}(f) \arrow[r, "\bar{h}"] \arrow[d, "f_b"'] & \dot{T}(g) \arrow[d, "g_b"] \\ M \arrow[r, "h"] & M \end{tikzcd}$$

commutes for some homeomorphism $h\colon M\longrightarrow M$.

Consideration of the following diagrams completes the proof:

$$\begin{tikzcd} \dot{T}(f)\times I + M \arrow[r, "id"] \arrow[d, "\bar{h}\times 1+i"] & M_f \arrow[dl, "h"] \\ \dot{T}(g)\times I + M \arrow[d, "id"] & \\ M_g & \end{tikzcd} \qquad \begin{tikzcd} \dot{T}(g)\times I + M \arrow[r, "id"] \arrow[d, "\bar{h}^{-1}\times 1+i"] & M_g \arrow[dl, "h^{-1}"] \\ \dot{T}(f)\times I + M \arrow[d, "id"] & \\ M_f & \end{tikzcd}$$

<u>Remark</u> 2.3. Given a commutative diagram as in Theorem 2.1, one can alter h to be base-point preserving by an isotopy and $\bar{h}$ can be altered accordingly. So we will assume h is base-point preserving.

<u>Remark</u> 2.4. Up to equivalence, there is one tubular neighborhood, T(f), of M with $\dot{T}(f)$ not connected, namely, MxI.

<u>Remark</u> 2.5. Given any subgroup H of $\pi_1(M)$ of index 2, there is a corresponding tubular neighborhood, T(p'), of M. Let p: $E \longrightarrow M$ be a connected covering projection of M with $p_{\#}\pi_1(E) = H$. And take N to be the mapping cylinder of p and p': $M \longrightarrow N$ to be the natural map.

In this paragraph, assume $\dot{T}(f)$, $\dot{T}(g)$, and M are connected. Consider the diagram:

(*)
$$\begin{tikzcd} & & \dot{T}(g) \arrow[d, "g_b"] \\ \dot{T}(f) \arrow[urr, dashed, "\exists\bar{h}?"] \arrow[r, "f_b"] & M \arrow[r, "h"] & M \end{tikzcd}$$

By Theorem 2.1, if $T(f)$ is equivalent to $T(g)$, then there exists $\overline{h}$ that makes the diagram commute, and therefore $h_{\#}\colon \pi_1(M) \longrightarrow \pi_1(M)$ with $h_{\#} f_{b\#} \pi_1 \dot{T}(f) = g_{b\#} \pi_1 \dot{T}(g)$ by [11, Thm. 5, p. 76]. Suppose there is no automorphism $a\colon \pi_1(M) \longrightarrow \pi_1(M)$ with $a f_{b\#} \pi_1 \dot{T}(f) = g_{b\#} \pi_1 \dot{T}(g)$. Then $T(f)$ is not equivalent to $T(g)$. One concludes that the number of non-equivalent tubular neighborhoods, $T(f)$, of M with $\dot{T}(f)$ connected $\geq$ the number of non-equivalent subgroups of $\pi_1(M)$ of index 2, where if H_1 and H_2 are subgroups of G, H_1 is equivalent to H_2 if and only if there is an automorphism $a\colon\ G \longrightarrow G$ with $a(H_1) = H_2$.

When M is a surface, every automorphism of $\pi_1(M)$ is induced by a homeomorphism of M by [10, Thm. 2, p. 542]. Therefore, if M is a surface, $\mathcal{T}(M)$ = the number of non-equivalent subgroups of $\pi_1(M)+1$. Let $\pi_1(M) = G$. Define a subgroup of G of index 2 by a homeomorphism from G to Z_2 and let H_1 and H_2 by 2 such subgroups.

Case 1. Suppose G has generators $a_1, b_1, \ldots,$ a_n, b_n and the relation $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} = 1$.

A. If for both H_1 and H_2, one and only one generator (a_j for H_1 and a_i for H_2) is mapped to 1 in Z_2, then there is clearly an automorphism a of G with $a(H_1) = H_2$.

B. If H_1 is the subgroup of G defined by sending a_1 and only a_1 to 1 and H_2 is the subgroup of G defined by sending b_1 to 1, then there is an automorphism a of G for which $a(H_1) = H_2$. Namely, define $a(a_1) = a_1b_1$ and $a(b_1) = a_1^{-1}$. Otherwise, $a(a_i) = a_i$ and $a(b_i) = b_i$.

C. If H_1 is the subgroup of G defined by sending a_i and only a_i to 1 and H_2 is the subgroup of G defined by sending a_i and b_i to 1, then there is an automorphism a of G with $a(H_1) = H_2$. Namely, $a(a_j) = a_j$ and $a(b_j) = b_j$ for j not equal to i and $a(a_i) = a_ib_i$ and $a(b_i) = a_i^{-1}$. One concludes when M is the torus, $T(M) = 2$.

Case 2. Suppose G has generators $c_1, c_2, \ldots, c_n$ and the single relation $c_1^2c_2^2 \cdots c_n^2 = 1$.

A. Let H_s denote the subgroup of index 2 defined by mapping c_s to 1, then there is an automorphism s of G with $s(H_s)=H_t$ (where without loss of generality assume $s< t$).

Define: $s(c_i) = c_i$ for $i< s$, $i>t$

$$s(c_s) = c_s^2c_tc_s^{-2}$$
$$s(c_{s+1}) = c_s^2c_t^{-2}c_{s+1}c_t^2c_s^{-2}$$
$$s(c_{s+2}) = c_s^2c_t^{-2}c_{s+2}c_t^2c_s^{-2}$$
$$\vdots$$
$$s(c_{t-1}) = c_s^2c_t^{-2}c_{t-1}c_t^2c_s^{-2}$$
$$s(c_t) = c_s$$

B. Let H_i denote the subgroup of G of index 2 defined by mapping $c_{i_1}, c_{i_2}, \ldots, c_{i_n}$ to 1. Then there is an automorphism a of G with $a(H_i) = H_j$. Namely,

define $a = h_1 h_2 \cdot \cdot \cdot h_n$ where $h_k = i_k$ if $i_k < j_k$ and $h_k = j_k$ if $j_k < i_k$ and h_k = identity if $i_k = j_k$ (where i_k and j_k are defined as above).

C. Suppose H_i is defined by sending c_i and only c_i to 1 and H_j is defined by sending $c_{j_1}, c_{j_2}, \ldots, c_{j_s}$ to 1 where s is even. Then there does not exist an automorphism a of G for which $a(H_i) = H_j$. Abelianize G giving H_i' and H_j', corresponding to H_i and H_j. In H_j', there is a non-identity element, namely $c_1 c_2 \cdot \cdot \cdot c_n$, whose square is equal to 1. So H_j' is not free abelian, but H_i' is clearly free abelian. So H_i' is not isomorphic to H_j'.

One concludes:

<u>Theorem</u> 2.2. T(surface with n crosscaps) is ≥ 3 and $\leq n+1$ (where $n \geq 2$). T(projective plane) = 2 and T(Klein bottle) = 3.

Let M be any surface and f: $M \longrightarrow N$ a locally flat embedding with corresponding tubular neighborhood T(f). Let h_1 and h_2 be involutions on (T(f),f(M)) with f(M) as fixed point set. As is standard, h_1 is equivalent to h_2 if there is a t: $T(f) \longrightarrow T(f)$ with $th_1t^{-1} = h_2$.

<u>Theorem</u> 2.3. h_1 and h_2 are equivalent.

Proof: Case 1. f(M) separates T(f), say into N_1 and N_2.

Claim: Either $h_i(N_1) = N_2$ and $h_i(N_2) = N_1$ or $h_i(N_j) = N_j$. Suppose not and there are n_1 and $n_1{}^*$ which are elements of N_1 for which $h_1(n_1) = n_1'$ which is in N_1 and $h_1(n_1{}^*) = n_2$

which is in N_2. Then there is a path i in N_1 from n_1 to n_1* and $h_1(i)$ is homeomorphic to I with $h_1\ (n_1) = n_1'$ and $h_1(n_1*) = n_2$. So there must be an n in i for which $h_1(n)$ is an element of f(M). This is a contradiction so the claim holds. But we know $h_i(N_j)$ is not equal to N_j. Thus, we get $h_i(N_1) = N_2$ and $h_i(N_2) = N_1$. Therefore, the orbit spaces of h_1 and h_2 [acting on T(f)-f(M)] are clearly homeomorphic, say by f. (O_h is the orbit space of $h\colon N \longrightarrow N$ and $h*$ is the projection from N to O_h.) Consider the diagram:

$$\begin{tikzcd} N_i \arrow[r, dashed, "t"] \arrow[d, "h_1^*"'] & N_i \arrow[d, "h_2^*"] \\ O_{h_1} \arrow[r, "f"] & O_{h_2} \end{tikzcd}$$

There exists t which makes the diagram commute for each of N_1 and N_2. Now,

$h_2^* t h_1 t^{-1}(n) = f h_1^* h_1 t^{-1}(n) = f h_1^* t^{-1}(n) = h_2^*(n)$.

So, $t^{-1}h_1t^{-1}(n) = n$ or $h_2(n)$.

If $t^{-1}h_1t^{-1}(n) = n$, then $h_1t^{-1}(n) = t^{-1}(n)$ which is a contradiction. Extend t to T(f) by $t(x) = x$. Then $th_1t^{-1} = h_2$ and h_1 is equivalent to h_2.

Case 2. f(M) does not separate T(f).

A. Assume M is not the projective plane.

Let x be an element of f(M) and let U be a neighborhood of x in N such that $f(M)$ separates U, say into U_1 and U_2. Then, as in Case 1, $h_i(U_1) = U_2$ and $h_i(U_2) = U_1$.

Let N' be defined as in Remark 2.1. Let $\tilde{M} = n^{-1}(f(M))$ and $\tilde{T} = n^{-1}(T(f))$. Let $\tilde{h}_1$ and $\tilde{h}_2$ be the involutions on $\tilde{T}$ induced by h_1 and h_2. Then if x is an element of $\tilde{M}$ for which $n(x) = n(x')$, then $\tilde{h}_1(x) = \tilde{h}_2(x) = x'$. $\tilde{T}$ is homeomorphic to $\tilde{M}xI$ so without loss of generality assume $\tilde{T}$ is $\tilde{M}xI$. Let $B = \tilde{O}_{h_1}$ and $M' = O_{\tilde{h}_1}|_{\tilde{M}x0}$ which is homeomorphic to $f(M)$ and therefore not the projective plane. Let $h\colon M' \hookrightarrow B$ be the inclusion. $M_1' = O_{\tilde{h}_1}|_{\tilde{M}x1}$ and $\pi_1(M_1') = \pi_1(M') = \pi_1(B)$. B is clearly compact, connected and a Poincare 3-manifold. Therefore, $O_{\tilde{h}_1}$ is homeomorphic to $M'xI$ by [1, Thm. 3.1, p. 485]. Similarly for $O_{\tilde{h}_2}$. So $O_{\tilde{h}_1}$ is homeomorphic to $O_{\tilde{h}_2}$, say by f. Consider the diagram:

$$\begin{tikzcd} \tilde{M}xI \arrow[r, "t^1"] \arrow[d, "\tilde{h}_1^*"'] & \tilde{M}xI \arrow[d, "\tilde{h}_2^*"] \\ O_{\tilde{h}_1} \arrow[r, "f"] & O_{\tilde{h}_2} \end{tikzcd}$$

Since f can be chosen so that $f_\# \tilde{h}^*_{1_\#} \pi_1(\tilde{M}xI) = \tilde{h}_{2_\#} \pi_1(\tilde{M}xI)$ and $\tilde{h}_1^*$ and $\tilde{h}_2^*$ are covering projections by [11, Thm. 7, p. 87], there is a t^1 which makes the above diagram commute. As in Case 1, $t^1 \tilde{h}_1 t^{1^{-1}} = \tilde{h}_2$. Letting t be induced by t^1, we get $th_1t^{-1} = h_2$.

B. Let M be the projective plane.

By Theorem 2.2, M has 2 non-equivalent tubular neighborhoods. One is equivalent to MxI and is taken care of in Case 1. The other tubular neighborhood can be defined by the 2 to 1 covering projection p: $S^2 \longrightarrow M$ as (M_p, M). h_i: $(M_p, M) \longrightarrow (M_p, M)$ induces an involution h_i' on $S^2 xI$. By [9, Thm. 1, p. 582], there is a t': $S^2 xI \longrightarrow S^2 xI$ for which $t'h_i't'^{-1} = h_2'$. t' induces t: $(M_p, M) \longrightarrow (M_p, M)$ with $th_1t^{-1} = h_2$.

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