

EQUIVALENCE OF TUBULAR NEIGHBORHOODS

Thesis for the Degree of Ph. D. MICHIGAN STATE UNIVERSITY JOAN ELIZABETH QUINN 1970



THESIS



This is to certify that the

thesis entitled EQUIVALENCE OF TUBULAR NEIGHBORHOODS

presented by

Joan Elizabeth Quinn

has been accepted towards fulfillment of the requirements for

). degree in Mathematics

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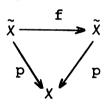
ABSTRACT

EQUIVALENCE OF TUBULAR NEIGHBORHOODS

By

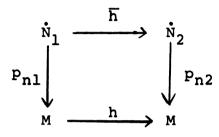
Joan Elizabeth Quinn

Let p: \tilde{X} +X be a connected covering projection. We say that p is almost regular (AR) if and only if for every f: \tilde{X} +X, the commutativity of



implies f is a homeomorphism. X is absolutely almost regular (AAR) if and only if every p is AR. In general, an AR covering projection may not be a regular covering projection. Theorem: Among closed surfaces, the 2-sphere, projective plane, torus, and Klein bottle are the only (AAR) spaces.

Let M be an (n-1)-manifold locally flatly embedded in an n-manifold. Then there exists a tubular neighborhood N (a topological 1-disk bundle over M) of M. The pair (N,M) is equivalent to another such pair (N',M) if and only if there exists a homeomorphism between the two pairs. There exists a 2-sheeted covering $p_n: N \rightarrow M$ and (N,M) is homeomorphic to (M_{pn}, M) (where M_{pn} is the mapping cylinder of P_n). Theorem: (N_1,M) and (N_2, M) are equivalent if and only if there exists homeomorphisms h and \overline{h} such that



is commutative. Let M be a closed surface. Let T be the set of equivalence classes of pairs (N, M), where N is obtained by considering all possible locally flat embeddings of M into all possible 3-manifolds, except for the case N = MxI. Let K_i be the subgroups of index 2 of $\Pi_1(M)$ and say K₁ and K₂ are equivalent if and only if there exists an automorphism of $\Pi_1(M)$ that maps K₁ onto K₂. Let N be the set of equivalence classes of all K_i. Theorem 3: There exists a natural 1-1 correspondence between N and T. (In particular, if $\Pi_1(M) = [C_1, C_2, \dots C_n | C_1^2 C_2^2 \dots C_n^2 = 1]$, then $T(M) \le n$, and T(projective plane) = 1 and T(Klein bottle) = 2.

Let (N,M) be defined as above. Theorem: Let h_1 and h_2 be involutions of N with M as fixed point set. Then h_1 and h_2 are equivalent, that is, there exists a homeomorphism t of (N,M) onto itself with $t|M = l_M$ and $h_1 = t^{-1}h_2t$.

EQUIVALENCE OF TUBULAR NEIGHBORHOODS

Ву

Joan Elizabeth Quinn

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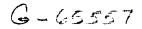
A THESIS

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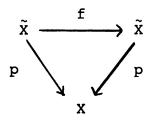
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INTRODUCTION

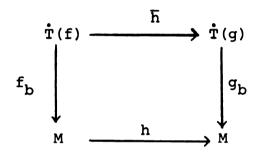
Call a covering projection, p, almost regular if and only if the commutativity of



implies f is a homeomorphism. The first section is concerned with what types of spaces have covering projections that are not almost regular. Theorem 1.1 says that if a topological space X has a fundamental group, $\pi_1(X, \mathbf{x}_0)$, which has a subgroup H and element t with tHt⁻¹ properly contained in H and an open covering \mathcal{U} with $\pi_1(\mathcal{U}, \mathbf{x}_0)$ contained in H, then X has a covering projection that is not almost regular. Then all surfaces are classified as to whether or not they have non-almost regular covering projections or not: clearly, a surface with one handle or crosscap does not, Theorem 1.2 proves all surfaces with 2 or more handles or 3 or more crosscaps do, and Theorem 1.3 proves that the Klein bottle does not. The section is concluded with an example of a covering projection of the Klein bottle that is almost regular but not regular.

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Letting f: $M^{n-1} \rightarrow N^n$ be a locally flat embedding as defined by Brown in [2], a tubular neighborhood T(f) of M in N corresponding to f is a topological 1-disk bundle of f(M) that is contained in N. Section 2 is concerned with when there is a homeomorphism h: $(T(f), M) \longrightarrow (T(g), M)$. Theorem 2.1 proves that 2 tubular neighborhoods, T(f) and T(g), of M are equivalent if and only if there is a homeomorphism \overline{h} : $\dot{T}(f) \longrightarrow \dot{T}(g)$ for which

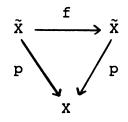


commutes (where f_b and g_b are restrictions of the bundle map). For M connected, it is shown that the number of non-equivalent tubular neighborhoods of M, T(M), $\geq 1 + number of non-equivalent subgroups of index 2 of <math>\pi_1(M)$, where H_1 and H_2 (subgroups of $\pi_1(M)$) are equivalent if and only if there is an automorphism a: $\pi_1(M) \rightarrow \pi_1(M)$ with $a(H_1) = H_2$. Theorem 2.2 shows that if M is a surface with n (2 or more) crosscaps, then $3 \leq T(M) \leq n+1$. Also, T(projective plane) = 2 and T(Klein bottle) = 3. Section 2 concludes with Theorem 2.3 on involutions of tubular neighborhoods: If h_1 and h_2 are involutions of T(f) with M as fixed point set, then h_1 is equivalent to h_2 .

Section 1. Almost Regular Covering Projections.

Let p: $\tilde{X} \rightarrow \tilde{X}$ be a covering projection with X connected and locally path-connected and \tilde{X} connected. Note that this implies \tilde{X} is locally path-connected, since p is a local homeomorphism.

<u>Definition</u> 1.1. p is almost regular if and only if the commutativity of

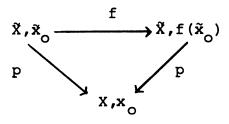


implies f is a homeomorphism.

<u>Remark</u> 1.1. One can also state Definition 1.1 in terms of a property of the fundamental group of the covering space, \tilde{X} . That is,

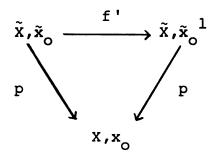
p is almost regular if and only if $p_{\#} \pi_1(\tilde{X}, \tilde{X}_0)$ is not properly contained in any of its conjugates in $\pi_1(X, X_0)$ (where X_0 is any point of X and \tilde{X}_0 is any preimage of X_0 under p, and $p_{\#}$ means the map on fundamental groups induced by p).

Proof of Remark 1.1: a. Assume the commutativity of



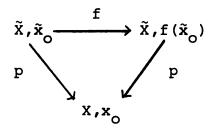
implies f is a homeomorphism.

Suppose $p_{\#}\pi_{1}(\tilde{X},\tilde{x}_{o})$ were properly contained in one of its conjugates, say $tp_{\#}\pi_{1}(\tilde{X},\tilde{x}_{o})t^{-1}$. Then since $\{p_{\#}\pi_{1}(\tilde{X},\tilde{x}_{o}) | \tilde{x}_{o} \in p^{-1}(x_{o})\}$ is a conjugacy class in $\pi_{1}(X,x_{o})$ by [11, Thm. 6, p. 73], $tp_{\#}\pi_{1}(\tilde{X},\tilde{x}_{o})t^{-1} = p_{\#}\pi_{1}(\tilde{X},\tilde{x}_{o}^{1})$ for some \tilde{x}_{o}^{-1} in \tilde{X} for which $p(\tilde{x}_{o}^{-1}) = x_{o}$. And therefore, there exists f': $\tilde{X} + \tilde{X}$ such that



commutes by [11, Thm. 5, p. 76]. Also by [11, Thm. 5, p. 76], there exists no such map from $(\tilde{X}, \tilde{x}_0^{-1})$ to (\tilde{X}, \tilde{x}_0) . So f' is not a homeomorphism which contradicts our assumption. Therefore, $p_{\#}\pi_1(\tilde{X}, \tilde{x}_0)$ is not properly contained in any of its conjugates in $\pi_1(X, x_0)$.

b. Assume $p_{\#}\pi_{1}(\tilde{X},\tilde{x}_{O})$ is not properly contained in any of its conjugates. Suppose

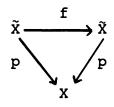


commutes. (Note that f is a covering projection by [11, Lemma 1, p. 79]). Suppose $f_{\#}: \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(\tilde{X}, f(\tilde{x}_0))$ is not a surjection. Then $p_{\#}f_{\#}: \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow p_{\#}\pi_1(\tilde{X}, f(\tilde{x}_0))$ is not a surjection, since $f_{\#}$ and $p_{\#}$ are monomorphisms by [11, Thm. 4, p. 72]. But $p_{\#}f_{\#} \pi_{1}(\tilde{X}, \tilde{x}_{o}) = p_{\#}\pi_{1}(\tilde{X}, \tilde{x}_{o})$ and by [11, Thm. 6, p. 73], $p_{\#} \pi_{1}(\tilde{X}, \tilde{x}_{o}) = p_{\#} \pi_{1}(\tilde{X}, f(\tilde{x}_{o}))$, so $p_{\#}f_{\#}\pi_{1}(\tilde{X}, \tilde{x}_{o}) = p_{\#}\pi_{1}(\tilde{X}, f(\tilde{x}_{o}))$. Therefore, $f_{\#}$ is a surjection. It follows that the multiplicity of p is 1 by [11, Thm. 9, p. 73] and thus that f is a homeomorphism.

Examples of almost regular covering projections:

 All regular covering projections are almost regular by [11, Thm. 11, p. 74].

2. All covering projections, p, with finite multiplicity are almost regular. Since p is n to 1, for any commutative diagram

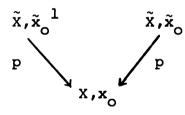


pf is n to 1. So f must be 1 to 1 and thus a homeomorphism. 3. If $\pi_1(X, x_0)$ has the ascending or descending chain condition, then, clearly, any covering projection of X is almost regular.

Definition 1.2. A group G is said to be <u>not regular</u> if there exists a subgroup H of G and an element t of G for which tHt^{-1} is properly contained in H. Otherwise, G is said to be <u>regular</u>. Any subgroup H and element t of a group G that is not regular, mentioned in the following pages, is understood to behave as in Definition 1.2.

<u>Theorem</u> 1.1. If $\pi_1(X, x_0)$ is not regular and there is an open covering U of X such that $\pi_1(U, x_0)$ is contained in H, then there is a covering projection with base space X that is not almost regular.

Proof: Let p: $(\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ be the covering projection with $p_{\#} \pi_1(\tilde{X}, \tilde{x}_0) = H$ constructed as in [11, Thm. 13, p. 82]. \tilde{X} is clearly connected. Let \tilde{x}_0^{-1} be the element of \tilde{X} which corresponds to a loop w(t) about x_0 in X which corresponds to the element t of $\pi_1(X, x_0)$. Consider the diagram:



 $p_{\#}\pi_{1}(\tilde{X},\tilde{x}_{o}) = H$. Since p(path from \tilde{x}_{o}^{-1} to \tilde{x}_{o}) is $(w(t))^{-1}$ which corresponds to t^{-1} , $p_{\#}\pi_{1}(\tilde{X},\tilde{x}_{o}^{-1}) = tHt^{-1}$. But tHt^{-1} is properly contained in H, so p is not almost regular.

<u>Remark</u> 1.2. One observes from the construction in Theorem 1.1 that if M is a connected manifold for which $\pi_1(M)$ is not regular, then there is a non-almost regular covering projection of M (because there is a covering of M by open sets {U}, for which $\pi_1(U, \mathbf{x}_0) = 0$). Clearly, all covering projections of a surface with precisely one handle or one crosscap are almost regular. The following two theorems will complete a classification of surfaces according to whether or not they have non-almost regular covering projections or not.

<u>Theorem</u> 1.2. All surfaces with 2 or more handles or 3 or more crosscaps have non-almost regular covering projections.

Proof: Fact 1. If a group G is given by the generators $a_1, a_2, \ldots a_n$ and one relation $f(a_1, a_2, \ldots a_n) = 1$, and if, further, the element a_n occurs in this relation and cannot be removed from it by transformations, then the subgroup $\{a_1, a_2, \ldots, a_{n-1}\}$ is free and $a_1, a_2, \ldots, a_{n-1}$ is free and $a_1, a_2, \ldots, a_{n-1}$ are free generators of the subgroup by [8, p. 77].

Fact 2. Any free group, G, of 2 or more generators is not regular (Example by L. M. Sonneborn - Let H be the subgroup generated by

{a, b^{-1} ab, $b^{-2}ab^2$, . . . , $b^{-n}ab^n$, . . . }

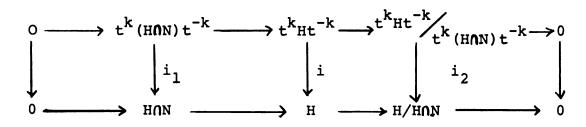
 $\pi_1(X_n)$, where X_n is a surface with n (2 or more) handles, has generators a_1 , b_1 , a_2 , b_2 , \ldots , a_n , b_n and the single relation $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}$. $a_nb_na_n^{-1}b_n^{-1} = 1$ by [11, p. 149]. Therefore, $\pi_1(X_n)$ has a free subgroup with more than one generator by Fact 1. So by Fact 2, and Remark 1.2, X_n has a non-almost regular covering projection.

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 $\pi_1(Y_n)$, where Y_n is a surface with n (3 or more) crosscaps, has generators c_1, c_2, \ldots, c_n and the single relation $c_1^2 c_2^2 \cdots c_n^2 = 1$ by [11, p. 149]. Therefore, as in the case for X_n , Y_n has a non-almost regular covering projection.

<u>Theorem</u> 1.3. All covering projections of the Klein bottle, K, are almost regular. Proof: Consider the covering projection of multiplicity 2, p: $s^1 \times s^1 \longrightarrow K$. $\pi_1(s^1 \times s^1)$ is regular, so, by the following lemma, $\pi_1(K)$ is regular. Therefore, all covering projections of K are almost regular.

Lemma 1.1. If G is not regular and N is a normal subgroup of G for which $[G:N]<\infty$, N is not regular. Proof: Since tHt⁻¹ is properly contained in H, t^kHt^{-k} is properly contained in H for each positive integer k. There is a k for which t^k is an element of N since $[G:N]<\infty$. Consider the following diagram:



 i_2 is an isomorphism since H/(HAN) is isomorphic to HN/N and [G:N]< ∞ . So, by the five lemma in [ll, p. 185], if i_1 is onto, then i is onto. But i is not onto by assumption. Therefore, i_1 is not onto and N is not regular.

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Example of closed PL manifolds of arbitrarily high dimension that have non-almost regular covering projections: Let $m \ge 4$. Embed X_n (or Y_n) in S^{m+1} . Take a regular neighborhood, N, of X_n in S^{m+1} . $\pi_1(\dot{N}) = \pi_1(X_n)$. \dot{N} is a closed PL manifold of dimension m and by Remark 1.2 has a non-almost regular covering projection.

Example of an almost regular covering projection that is not regular:

 $\pi_{1}(K) \text{ has 2 generators, } c_{1} \text{ and } c_{2}, \text{ and the relation}$ $c_{1}^{2}c_{2}^{2} = 1. Consider the covering projection p: <math>(\tilde{K}, \tilde{x}_{0}) \rightarrow (K, x_{0}) \text{ with } p_{\#}\pi_{1}(\tilde{K}, \tilde{x}_{0}) = \{c_{2}\} \text{ (where } \{a\} \text{ denotes the sub$ $group generated by a). } c_{1}^{-1}\{c_{2}\} c_{1} \text{ is not equal to } \{c_{2}\}. Consider w: (I, 0) \rightarrow (K, x_{0}) \text{ where } w(I) \text{ corresponds to } c_{1}. There is a lifting of w to w': (I, 0) \rightarrow (\tilde{K}, \tilde{x}_{0}) \text{ by [11,} Thm. 3, p. 67]. Let w'(1) = \tilde{x}_{0}^{1}. Then p(\tilde{x}_{0}^{-1}) = x_{0} \text{ and } p_{\#}\pi_{1}(\tilde{K}, \tilde{x}_{0}) = c_{1}^{-1}p_{\#}\pi_{1}(\tilde{K}, \tilde{x}_{0})c_{1} \text{ which is not equal to } p_{\#}\pi_{1}(\tilde{K}, \tilde{x}_{0}). So p \text{ is not regular by [11, Thm. 11, p. 74],} and, by Theorem 1.3, p \text{ is almost regular.}$ Section 2. Equivalence of Tubular Neighborhoods.

Let M be an n-l manifold and f: $M \rightarrow N$ be a locally flat embedding of M into any n-manifold N (where manifold and locally flat embedding are defined as in [2]).

Definition 2.1. A <u>tubular neighborhood</u>, T(f), of M in N corresponding to f is a topological 1-disk bundle of f(M) that is contained in N.

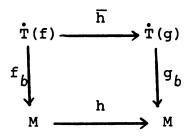
Remark 2.1. Given a locally flat embedding f: M→N, one can exhibit a tubular neighborhood corresponding to f in the style of [5] as follows: Let $\{U_{\alpha}\}$ be a basis for the topology on N such that if $U_{\alpha} \cap f(M) \neq \phi$ then there is a homeomorphism $(E^{n}, E^{n-}) \longrightarrow (U_{\alpha}, U_{\alpha} \cap f(M))$. E^{n-1} separates E^{n} , say into h: E_{+}^{n} and E_{-}^{n} . Denote $h(E_{+}^{n} \cup E^{n-1})$ by U_{α}^{+} and $h(E_{-}^{n} \cup E^{n-1})$ by U_{α}^{-} . Let $\{U_{\beta}\} = \{U \mid U \text{ is an element of } \{U_{\alpha}\} \text{ and } U \cap f(M) = \phi$ or $U = U_{\alpha}^{+}$ or U_{α}^{-} for some α }. Let $N_1 = \{(x, U_\beta) | x \text{ is an element of } U_\beta\}$. Define an equivalence relation, R, on N_1 by (x, U_R) is equivalent to (x^{1}, U_{β}^{1}) if and only if $x = x^{1}$ and $U_{\beta} \cap U_{\beta}^{1} \cap C(f(M)) \neq \phi$. Take N' = N_1/R and denote an equivalence class by $[x, U_\beta]$. Let $\{U_{\beta}^{*}\}$ be a basis for N' where $[x, U_{\gamma}]$ is an element of U_{R}^{*} if and only if x is an element of U_{R}^{*} . Define n: N'+N $n[x,U_{\beta}] = x$. n is 2 to 1 restricted to $n^{-1}(f(M))$ and by restricted to $Cn^{-1}(f(M))$ is 1 to 1. N' is an n-manifold with boundary $n^{-1}(f(M))$. $n^{-1}(f(M))$ is collared by [2], and its image is a 1-disk bundle of f(M).

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<u>Remark</u> 2.2. Let $f_b: T(f) \rightarrow M$ be the bundle map. Then $f_b | \mathring{T}(f)$ is a covering projection of multiplicity 2 and, denoting the mapping cylinder of $f_b | \mathring{T}(f)$ by M_f , (M_f, M) is homeomorphic to (T(f), M).

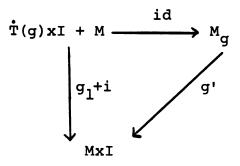
<u>Definition</u> 2.2. T(f) is equivalent to T(g) if and only if there is a homeomorphism h: $(T(f),M) \longrightarrow (T(g),M)$.

<u>Theorem</u> 2.1. T(f) is equivalent to T(g) if and only if there is a homeomorphism \overline{h} : $\mathring{T}(f) \longrightarrow \mathring{T}(g)$ for which



commutes.

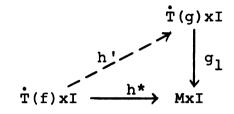
Proof: Assume T(f) is equivalent to T(g). Then there is a homeomorphism h: $(M_f, M) \longrightarrow (M_g, M)$. Let g_1 be the covering projection, g_1 : $\mathring{T}(g) \times I \longrightarrow M \times I$, induced by g_b and let i: $M \longrightarrow M \times I \longrightarrow M \times I$ be the natural inclusion. Then, by the diagram:



g' is continuous by [4, Thm. 3.2, p. 123]. (id will always indicate the obvious identification map.) Case 1. Assume $\dot{T}(f)$, $\dot{T}(g)$, and M are connected. We have

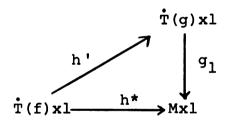
h*:
$$\dot{T}(f) \times I \xrightarrow{id} M_f \xrightarrow{h} M_g \xrightarrow{g'} M \times I$$

where h*(x,1) is in Mx1. Consider the diagram:



There exists h' with h' $|\mathring{T}(f)x0 = h$ by [11, Thm. 3, p. 67]. Note that h' $(\mathring{T}(f)x1)$ is contained in $\mathring{T}(g)x1$. Also $h^*|\mathring{T}(f)x1$ is a covering projection.

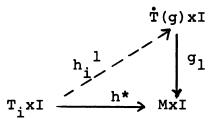
Now consider the diagram:



h' $|\dot{\mathbf{T}}(f)\mathbf{x}|$ is a covering projection, since $\dot{\mathbf{T}}(f)$, $\dot{\mathbf{T}}(g)$, and M are connected and locally path-conntected by [11, Lemma 1, p. 79] and therefore h' $|\dot{\mathbf{T}}(f)\mathbf{x}|$ is open and onto. h' $|\dot{\mathbf{T}}(f)\mathbf{x}|$ must clearly then be 1 to 1, so h' $|\dot{\mathbf{T}}(f)\mathbf{x}|$ induces the desired $\overline{\mathbf{h}}$.

Case 2. Assume $\mathring{T}(f)$ and $\mathring{T}(g)$ are not connected and that M is connected.

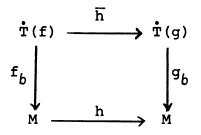
Then there are T_1 and T_2 (each connected) for which $f_b | T_i: T_i \longrightarrow M$ is a homeomorphism. As in Case 1, h*: $\dot{T}(f) \times I \longrightarrow M \times I$. For this case, consider the diagram:



There exists $h_i^{\ l}$ that makes the diagram commute and $h_i^{\ l}(T_i xI)$ is connected. So, we get a 1 to 1, onto map $h': \dot{T}(f) \longrightarrow \dot{T}(g)$ that clearly must be open.

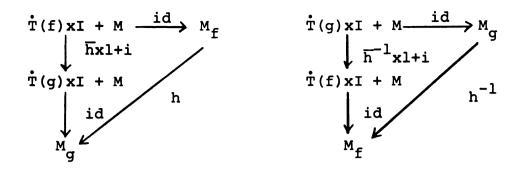
Case 3. Assume M is not connected. The argument is the same as the preceding ones applied to the components of M and their preimages.

Assume there is \overline{h} : $\overset{\bullet}{T}(f) \longrightarrow \overset{\bullet}{T}(g)$ for which



commutes for some homeomorphism h: $M \longrightarrow M$.

Consideration of the following diagrams completes the proof:

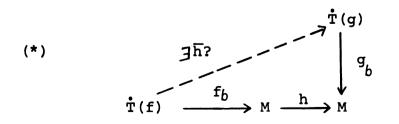


<u>Remark</u> 2.3. Given a commutative diagram as in Theorem 2.1, one can alter h to be base-point preserving by an isotopy and \overline{h} can be altered accordingly. So we will assume h is base-point preserving.

<u>Remark</u> 2.4. Up to equivalence, there is one tubular neighborhood, T(f), of M with $\dot{T}(f)$ not connected, namely, MxI.

<u>Remark</u> 2.5. Given any subgroup H of $\pi_1(M)$ of index 2, there is a corresponding tubular neighborhood, T(p'), of M. Let p: $E \longrightarrow M$ be a connected covering projection of M with $p_{\#}\pi_1(E) = H$. And take N to be the mapping cylinder of p and p': $M \longrightarrow N$ to be the natural map.

In this paragraph, assume $\mathbf{T}(f)$, $\mathbf{T}(g)$, and M are connected. Consider the diagram:



By Theorem 2.1, if T(f) is equivalent to T(g), then there exists \overline{h} that makes the diagram commute, and therefore $h_{\sharp}: \pi_1(M) \longrightarrow \pi_1(M)$ with $h_{\sharp}f_{b\sharp}\pi_1 \dot{T}(f) = g_{b\sharp}\pi_1 \dot{T}(g)$ by [11, Thm. 5, p. 76]. Suppose there is no automorphism a: $\pi_1(M) \longrightarrow \pi_1(M)$ with $af_{b\sharp}\pi_1 \dot{T}(f) = g_{b\sharp}\pi_1 \dot{T}(g)$. Then T(f) is not equivalent to T(g). One concludes that the number of non-equivalent tubular neighborhoods, T(f), of M with $\dot{T}(f)$ connected \geq the number of non-equivalent subgroups of $\pi_1(M)$ of index 2, where if H_1 and H_2 are subgroups of G, H_1 is equivalent to H_2 if and only if there is an automorphism a: $G \longrightarrow G$ with $a(H_1) = H_2$.

When M is a surface, every automorphism of $\pi_1(M)$ is induced by a homeomorphism of M by [10, Thm. 2, p. 542]. Therefore, if M is a surface, T(M) = the number of nonequivalent subgroups of $\pi_1(M)+1$. Let $\pi_1(M) =$ G. Define a subgroup of G of index 2 by a homeomorphism from G to Z_2 and let H_1 and H_2 by 2 such subgroups.

Case 1. Suppose G has generators a_1 , b_1 , \dots , a_n , b_n and the relation $a_1b_1a_1^{-1}b_1^{-1} \cdots a_nb_na_n^{-1}b_n^{-1} = 1$. A. If for both H_1 and H_2 , one and only one generator (a_j for H_1 and a_i for H_2) is mapped to 1 in Z_2 , then there is clearly an automorphism a of G with $a(H_1) = H_2$.

B. If H_1 is the subgroup of G defined by sending a_1 and only a_1 to 1 and H_2 is the subgroup of G defined by sending b_1 to 1, then there is an automorphism a of G for which $a(H_1) = H_2$. Namely, define $a(a_1) = a_1b_1$ and $a(b_1) = a_1^{-1}$. Otherwise, $a(a_1) = a_1$ and $a(b_1) = b_1$. C. If H_1 is the subgroup of G defined by

sending a_i and only a_i to 1 and H_2 is the subgroup of G defined by sending a_i and b_i to 1, then there is an automorphism a of G with $a(H_1) = H_2$. Namely, $a(a_j) = a_j$ and $a(b_j) = b_j$ for j not equal to i and $a(a_i) = a_ib_i$ and $a(b_i) = a_i^{-1}$. One concludes when M is the torus, T(M) = 2.

Case 2. Suppose G has generators c_1, c_2, \ldots, c_n and the single relation $c_1^2 c_2^2 \cdots c_n^2 = 1$.

A. Let H_s denote the subgroup of index 2 defined by mapping c_s to 1, then there is an automorphism s of G with $s(H_s)=H_t$ (where without loss of generality assume s<t).

Define:
$$s(c_i) = c_i$$
 for it
 $s(c_s) = c_s^2 c_t c_s^{-2}$
 $s(c_{s+1}) = c_s^2 c_t^{-2} c_{s+1} c_t^2 c_s^{-2}$
 $s(c_{s+2}) = c_s^2 c_t^{-2} c_{s+2} c_t^2 c_s^{-2}$

 $s(c_{t-1}) = c_s^2 c_t^{-2} c_{t-1} c_t^2 c_s^{-2}$ $s(c_t) = c_s$

B. Let H_i denote the subgroup of G of index 2 defined by mapping c_1, c_2, \ldots, c_i to 1. Then there is an automorphism a of G with $a(H_i) = H_i$. Namely, ł 1 define $a = h_1 h_2 \cdot \cdot \cdot h_n$ where $h_k = i_k$ if $i_k < j_k$ and $h_k = j_k$ if $j_k < i_k$ and h_k = identity if $i_k = j_k$ (where i_k and j_k are defined as above).

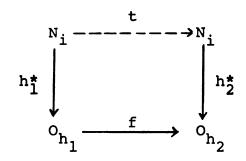
C. Suppose H_i is defined by sending c_i and only c_i to 1 and H_j is defined by sending $c_{j_1}, c_{j_2}, \ldots, c_{j_s}$ to 1 where s is even. Then there does not exist an automorphism a of G for which $a(H_i) = H_j$. Abelianize G giving H'_i and H'_j , corresponding to H_i and H_j . In H_j' , there is a non-identity element, namely $c_1c_2 \ldots c_n$, whose square is equal to 1. So H'_j is not free abelian, but H'_i is clearly free abelian. So H'_i is not isomorphic to H'_i .

One concludes:

<u>Theorem 2.2.</u> T(surface with n crosscaps) is ≥ 3 and \leq n+1 (where n ≥ 2). T(projective plane) = 2 and T(Klein bottle) = 3.

Let M be any surface and f: $M \rightarrow N$ a locally flat embedding with corresponding tubular neighborhood T(f). Let h_1 and h_2 be involutions on (T(f), f(M)) with f(M) as fixed point set. As is standard, h_1 is equivalent to h_2 if there is a t: T(f) \rightarrow T(f) with $th_1t^{-1} = h_2$.

<u>Theorem</u> 2.3. h_1 and h_2 are equivalent. Proof: Case 1. f(M) separates T(f), say into N_1 and N_2 . Claim: Either $h_1(N_1) = N_2$ and $h_1(N_2) = N_1$ or $h_1(N_j) = N_j$. Suppose not and there are n_1 and n_1^* which are elements of N_1 for which $h_1(n_1) = n'_1$ which is in N_1 and $h_1(n_1^*) = n_2$ which is in N₂. Then there is a path i in N₁ from n₁ to n_1^* and $h_1(i)$ is homeomorphic to I with $h_1(n_1) = n'_1$ and $h_1(n_1^*) = n_2$. So there must be an n in i for which $h_1(n)$ is an element of f(M). This is a contradiction so the claim holds. But we know $h_i(N_j)$ is not equal to N_j. Thus, we get $h_i(N_1) = N_2$ and $h_i(N_2) = N_1$. Therefore, the orbit spaces of h_1 and h_2 [acting on T(f)-f(M)] are clearly homeomorphic, say by f. (O_h is the orbit space of h: N → N and h* is the projection from N to O_h.) Consider the diagram:

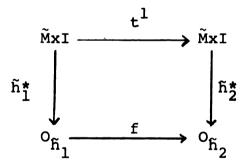


There exists t which makes the diagram commute for each of N_1 and N_2 . Now, $h_2^*th_1t^{-1}(n) = fh_1^*h_1t^{-1}(n) = fh_1^*t^{-1}(n) = h_2^*(n)$. So, $t^{-1}h_1t^{-1}(n) = n$ or $h_2(n)$. If $t^{-1}h_1t^{-1}(n) = n$, then $h_1t^{-1}(n) = t^{-1}(n)$ which is a contradiction. Extend t to T(f) by t(x) = x. Then $th_1t^{-1} = h_2$ and h_1 is equivalent to h_2 .

Case 2. f(M) does not separate T(f).

A. Assume M is not the projective plane. Let x be an element of f(M) and let U be a neighborhood of x in N such that f(M) separates U, say into U₁ and U₂. Then, as in Case 1, $h_i(U_1) = U_2$ and $h_i(U_2) = U_1$.

Let N' be defined as in Remark 2.1. Let $\tilde{M} = n^{-1}(f(M))$ and $\tilde{T} = n^{-1}(T(f))$. Let \tilde{h}_1 and \tilde{h}_2 be the involutions on \tilde{T} induced by h_1 and h_2 . Then if x is an element of \tilde{M} for which n(x) = n(x'), then $\tilde{h}_1(x) = \tilde{h}_2(x) = x'$. \tilde{T} is homeomorphic to $\tilde{M}xI$ so without loss of generality assume \tilde{T} is $\tilde{M}xI$. Let $B = \tilde{O}_{h_1}$ and $M' = O_{\tilde{h}_1} | \tilde{M}xO$ which is homeomorphic to f(M) and therefore not the projective plane. Let $h: M \longrightarrow B$ be the inclusion. $M'_1 = O_{\tilde{h}_1} | \tilde{M}x1$ and $\pi_1(M'_1) = \pi_1(M') = \pi_1(B)$. B is clearly compact, connected and a Poincare 3-manifold. Therefore, $O_{\tilde{h}_1}$ is homeomorphic to M'xI by [1, Thm. 3.1, p. 485]. Similarly for $O_{\tilde{h}_2}$. So $O_{\tilde{h}_1}$ is homeomorphic to $O_{\tilde{h}_2}$ say by f. Consider the diagram:



Since f can be chosen so that $f_{\#}\tilde{h}_{1\#}^{*}\pi_{1}(\tilde{M}xI) = \tilde{h}_{2\#}\pi_{1}(\tilde{M}xI)$ and \tilde{h}_{1}^{*} and \tilde{h}_{2}^{*} are covering projections by [11, Thm. 7, p. 87], there is a t¹ which makes the above diagram commute. As in Case 1, $t^{1}\tilde{h}_{1}t^{1-1} = \tilde{h}_{2}$. Letting t be induced by t¹, we get th₁t⁻¹ = h₂.

B. Let M be the projective plane.

By Theorem 2.2, M has 2 non-equivalent tubular neighborhoods. One is equivalent to MxI and is taken care of in Case 1. The other tubular neighborhood can be defined by the 2 to 1 covering projection p: $S^2 \rightarrow M$ as (M_p, M) . h_i: $(M_p, M) \rightarrow (M_p, M)$ induces an involution h[!] on $S^2 xI$. By [9, Thm. 1, p. 582], there is a t': $S^2 xI \rightarrow S^2 xI$ for which t'h[!]t'⁻¹ = h[!]_2. t' induces t: $(M_p, M) \rightarrow (M_p, M)$ with th₁t⁻¹ = h₂. BIBLIOGRAPHY

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