MANIFOLDS WHICH ARE HOMOLOGY DOUBLES

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This is to certify that the

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ABSTRACT

MANIFOLDS WHICH ARE HOMOLOGY DOUBLES

By

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The basic question considered is under what conditions a manifold contains a subset which is in some sense equivalent to its complement. A closed (compact, connected, and without boundary) n-manifold M is a double if it is the adjunction space $N_1 \cup_h N_2$, where N_1 and N_2 are two copies of the same manifold with boundary and h is the identity between their boundaries. Generalizing this concept, M is a t-double (for twisted) if M contains a submanifold N which is homeomorphic to the closure of its complement. If R is a principal ideal domain, M is called an R-homology double, or simply an R-double, if there exists a compact subset A in M for which there is an isomorphism $H_*(A;R) \cong H_*(M-A;R)$ for singular homology. If A can be taken as a PL subspace of the combinatorial manifold M, then M is a PL R-double. It is shown that if M is a PL R-double then it contains a PL submanifold N of dimension n satisfying $H_*(N;R) \cong H_*(\overline{M-N};R)$. The subset A or N above is called a t-half, PL R-half, etc., for M.

In chapters II and III necessary and sufficient conditions for a manifold to be a generalized double are studied. It is seen that a closed manifold is a t-double or a PL R-double only if its Euler characteristic is even. Conversely, a closed 2-manifold is a

double if its Euler characteristic is even, and every 3-manifold is a PL t-double. In higher dimensions, every closed, combinatorial manifold of odd dimension is shown to be a PL R-double for any principal ideal domain R. For a combinatorial manifold M of even dimension a similar but weaker result is proved. If R is a field, this states that if M is orientable over R and the Euler characteristic of M is even, then M is a PL R-double. Two miscellaneous results on homology doubles are that the product of a manifold with an R-double is an R-double, and that the connected sum of two PL R-doubles is a PL R-double.

There are numerous examples showing that a closed n-manifold may have many different R-halves. However, a PL R-half is shown to be homologically unique if its homology modules are those of a space of low dimension (i.e. $< [n/2]$), and in this case the modules of the half are determined by those of M.

In the last chapter compact, combinatorial manifolds with boundary are considered. Such a manifold M is a PL t-double if it contains a PL submanifold N such that $N \cong \overline{M-N}$, or a PL R-double if N satisfies $H_*(N;R) \cong H_*(\overline{M-N};R)$. Every compact 2-manifold with boundary is seen to be a PL t-double where N can be taken as a disc, and it is proved that every compact 3-manifold with boundary is a PL t-double. For higher dimensions, certain compact, combinatorial manifolds (in odd dimensions, all those whose boundary components are spheres; in even dimensions a subset of these) are shown to be PL R-doubles.

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CHAPTER I

INTRODUCTION AND PRELIMINARIES

An important method of constructing examples of manifolds has been to paste two homeomorphic manifolds together by a homeomorphism between their boundaries. Manifolds with boundary can be obtained by identifying certain subsets of the boundary of one manifold with homeomorphic subsets in the boundary of another. In this thesis I consider the converse idea. More specifically, the question is asked: Under what conditions does a manifold decompose into two pieces which are homeomorphic, or which at least have isomorphic homology groups?

The basic definitions are introduced in chapter II, and a model for the type of theorem I am concerned with is proved: roughly, that a closed, connected, 2-manifold separates into equivalent pieces if and only if its Euler characteristic is even. Chapter III deals primarily with the same type of result for closed, combinatorial manifolds of higher dimensions, and in chapter IV compact manifolds with boundary are investigated. The remainder of this chapter is devoted to basic notation and known facts which I will be using.

An <u>n-manifold</u> M is a paracompact Hausdorff space with the property that each point has a neighborhood which is homeomorphic to Euclidean n-space E^n or to $\frac{1}{2}E^n = \{(x_1,\ldots,x_n) \in E^n\colon x_n \geq 0\}$. Also, all manifolds in this paper will be connected unless otherwise indicated. The set of points which have E^n neighborhoods is called the interior of M, intM, and the set M-intM is the boundary, bdM. M is with or without boundary depending on whether bdM is not empty or empty. A closed manifold is one that is connected, compact, and without boundary.

If a manifold N is a subspace of a manifold M we avoid any confusion between the interior and boundary of N as a manifold and as a subspace by letting $\mathrm{int}_M N$ and $\mathrm{bd}_M N$ represent the latter ideas. If $A \subseteq M$ then $\bar{A}$ denotes the closure of A in M.

Most of my results are for combinatorial manifolds, the main reference here being the mimeographed notes of E.C. Zeeman [13]. In the following paragraphs some basic definitions and facts from combinatorial topology are summarized.

An n-simplex A is the convex hull of n+1 linearly independent points $a_0,\ldots,a_n$, called vertices, in some E^p. It will be convenient to denote the set of vertices by a Greek letter, say α, and we write $A = |\alpha|$ for A spans α. A simplex B spanning a subset of α is called a face of A, written $B < A$. The geometric center

$$b(A) = (a_0+\ldots+a_n)/(n + 1)$$

is called the barycenter of A. If the vertices of two simplexes A and B are linearly independent in E^p we call the simplex spanning them the join $A*B$.

A simplicial complex K is a collection of simplexes satisfying:

a. If $A \in K$ then all faces of A are in K.

b. If $A,B \in K$ then $A \cap B$ is a common face or empty.

The dimension of a complex is that of its largest simplex if this exists, and the r-skeleton K^r of a complex K is the subcomplex consisting of all simplexes of dimension $\leq r$. If $A \in K$ is an arbitrary simplex then the subcomplex $st(A,K) = \{B \in K : A < B\}$ is called the star of A in K.

A topological space X is called a polyhedron if it is homeomorphic to a subspace of some E^p which is the union of a collection of simplexes forming a simplicial complex K. For convenience it is assumed that X is actually the subspace of E^p, and we say that X is triangulated by K, written $X = |K|$. A simplicial complex L is called a subdivision of the triangulation K if $|L| = |K|$ and each simplex of L is contained in some simplex of K. For the definition of two special types of subdivision, the r-th derived and the barycentric, the reader is referred to [13, chap. I, p. 4]. I note here only that the vertices of each simplex of the barycentric first derived subdivision of K are barycenters $b(A_1),\ldots,b(A_n)$ of simplexes of K which can be ordered so that $A_1 < A_2 < \ldots < A_n$.

If X is a polyhedron, a subspace Y is a PL subspace (for piecewise linear) if Y is triangulated by a sub-complex of some triangulation of X. A continuous map $f : X \to Y$ from one polyhedron to another is called a PL map if there exist triangulations K and L of X and Y respectively with respect to which f is simplicial; that is, f maps vertices to vertices and simplexes linearly to simplexes.

A PL n-ball is a polyhedron which is PL homeomorphic to an n-simplex, and a PL n-sphere is a polyhedron PL homeomorphic to the boundary of an (n+1)-simplex. A manifold is said to be a combinatorial n-manifold if it is a polyhedron $|K|$ with the property that $|st(v,K)|$ is an n-ball for each vertex $v \in K$. If $X \subset M$ is a compact PL subspace of a combinatorial n-manifold, then a regular neighborhood of X in M is a neighborhood of X which is a compact n-manifold (PL subspace) and collapses to X (see [13, chap. III]). Some important facts about a combinatorial n-manifold $M = |K|$ which I will need are:

1.1 The star $|st(A,K)|$ is an n-ball for each simplex A in K. [13, III, p. 2].

1.2 Any derived neighborhood [13, III, p. 14] of a compact PL subspace $X \subset M$ is regular, and in particular every such subspace has a regular neighborhood.

1.3 Because a collapse preserves homotopy type, a regular neighborhood of X in M has the same homotopy type as X.

Standard references for algebraic concepts I will be using are [12] or [3]. Some of those ideas used in this paper are stated next.

Throughout this thesis R will represent a principal ideal domain, and the symbols Z and Z_n are reserved for the integers and the integers modulo n respectively. If X is a topological space then $H_i(X;R)$ and $H^i(X;R)$ represent the i-th singular homology and cohomology modules respectively of X with coefficients in R, $H_*(X;R)$ and $H^*(X;R)$ representing the corresponding graded R-modules. We write $H_*(X;R) \cong H_*(Y;R)$ to indicate that there is an R-isomorphism $H_i(X;R) \cong H_i(Y;R)$ for each integer i, and similarly for the graded cohomology modules. Similar notation is used for the relative homology and cohomology modules $H_*(X,Y;R)$ and $H^*(X,Y;R)$ of a topological pair (X,Y). It will often be convenient to abbreviate $H_*(X;R)$ by $H_*(X)$, and the coefficient module will then be clarified in the text.

If $H_i(X;R)$ is a finitely generated R-module, then it is known that $H_i(X;R) \cong F_i(X;R) \oplus T_i(X;R)$, where F_i and T_i represent the free and torsion submodules and $\oplus$ indicates the direct sum over R. From the Universal Coefficient Theorem for cohomology we have [3, p. 136]:

<u>1.4</u> If $H_i(X;R)$ is a finitely generated R-module for each i, then $H^i(X;R) \cong F_i(X;R) \oplus T_{i-1}(X;R)$.

If K is a simplicial complex there are simplicial homology modules $H_i(K;R)$ defined, and $H_i(K;R) \cong H_i(K^r;R)$ if $i < r$. Since the simplicial homology is isomorphic to the singular homology of the polyhedron $X = |K|$ [12, p. 191], the notation $H_*(X;R) = H_*(K;R)$ should cause no confusion.

Let M be a compact, connected, n-manifold where bdM may or may not be empty. Then $H_n(M,bdM;R)$ is either R or 0 [12, p. 302] and we say that M is <u>orientable</u> over R if the former holds, otherwise that M is <u>non-orientable</u> over R. An orientation of M is a choice of generator in $H_n(M,bdM;R)$ and a homeomorphism h between two oriented manifolds M and N is said to be orientation preserving if the induced isomorphism $h_* : H_n(M,bdM;R) \to H_n(N,bdN;R)$ sends the chosen generator to the chosen generator. For orientable manifolds some important duality theorems hold:

1.5 (See [12, p. 296]). If M is a closed, combinatorial n-manifold which is orientable over R and if N is a closed, compact, PL submanifold of dimension n, then there are isomorphisms:

a. $H_i(M,N;R) \cong H^{n-i}(\overline{M-N};R)$,

b. $H_i(N;R) \cong H^{n-i}(M,\overline{M-N};R)$.

Note that here $\overline{M-N}$ is also an n-manifold [13, chap. III, p. 20].

1.6 (Poincare Duality [12, p. 297]). If M is a closed n-manifold orientable over R, then for each i there is an isomorphism $H_i(M;R) \cong H^{n-i}(M;R)$.

1.7 (Lefschetz Duality [3, p. 186]). Let M be a compact n-manifold with boundary which is orientable over R. Then there is a sign-commutative diagram of R-modules (all coefficients in R)

$$\begin{array}{ccccccccc} \cdots \to & H^{q-1}(M) & \to & H^{q-1}(bdM) & \to & H^q(M,bdM) & \to & H^q(M) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & H_{n-q+1}(M,bdM) & \to & H_{n-q}(bdM) & \to & H_{n-q}(M) & \to & H_{n-q}(M,bdM) & \to \cdots \end{array}$$

where the rows are the exact Mayer-Vietoris sequences and the vertical arrows are isomorphisms induced by the cap product.

Another useful result, based on the idea of relative homeomorphism (see [12, p. 202]), is:

1.8 If M is a closed, combinatorial n-manifold, $N_1 \subset M$ is a compact PL submanifold of dimension n, and $N_2 = \overline{M-N_1}$ is the complementary manifold, then for any R

$$H_*(M,N_1;R) \cong H_*(N_2,\mathrm{bd}N_2;R).$$

If A is a finitely generated R-module, then rankA denotes the number of elements in a basis for its free submodule. The following lemma [3, p. 100] will be needed.

1.9 Given an exact sequence of finitely generated R-modules

$$0 \to A_1 \to A_2 \to \dots \to A_r \to 0,$$

then

$$\mathrm{rank}A_1 - \mathrm{rank}A_2 + \dots + (-1)^{r+1}\mathrm{rank}A_r = 0.$$

For any space X such that $H_*(X;R)$ is a finitely generated graded R-module, the Euler characteristic of X is defined as the finite sum

$$\chi(X) = \sum_i (-1)^i \mathrm{rank}H_i(X;R).$$

Although the rank of the i-th homology module may depend on the coefficients R, it follows from the Universal Coefficients Theorem for homology that $\chi(X)$ is independent of R [3, p. 103]. Some useful results involving the Euler characteristic are listed:

1.10 Let M be a closed, combinatorial n-manifold and N a compact, n-dimensional PL submanifold. Then

$$\chi(M) = \chi(N) + \chi(\overline{M-N}) - \chi(\mathrm{bd}N).$$

Proof This follows by applying 1.9 above to the Mayer-Vietoris sequence of the couple $\{N,\overline{M-N}\}$.

<u>1.11</u> The Euler characteristic of a closed, odd dimensional manifold is 0.

<u>Proof</u> [11, p. 246] or: since any manifold is orientable over Z_2, apply Poincare duality (1.6) and 1.4 obtaining $H_i(M;Z_2) \cong H_{n-i}(M;Z_2)$.

I conclude this chapter with the statement of the important Newman-Gugenheim homogeneity theorem [4, p. 32].

<u>1.12</u> Let B_1 and B_2 be two n-balls which are PL subspaces of the closed, combinatorial n-manifold M which is oriented over Z, and let $P \subset M-(B_1 \cup B_2)$ be a compact polyhedron which does not disconnect M. Let $h : B_1 \to B_2$ be an orientation preserving homeomorphism, where the orientation on the B_i is induced by that of M. Then there is an orientation preserving PL homeomorphism H of M onto itself which is PL isotopic to the identity and such that $H|B_1 = h$ and $H|P$ is the identity.

CHAPTER II

DEFINITIONS AND GENERAL RESULTS

In the literature (see, for example, [10, pp. 4, 56]) a closed manifold M is a double if $M = N_1 \cup_h N_2$, the adjunction space of two copies of the same manifold with boundary, h being the identity map on their boundaries. I will call each N_i a half of the manifold M.

In this chapter the weaker concepts of a twisted double and a homology double are defined. The relationship between these ideas is discussed and some general results and examples are presented.

Definition 2.1 Let M be a closed n-manifold and R a principal ideal domain. M is said to be an R-homology double, or simple an R-double, if it contains a compact subset A such that the following isomorphism for singular homology holds:

$$H_*(A;R) \cong H_*(M-A;R).$$

The set A will be called an R-half of M.

In the special case that M is a Z-double (and hence, by the Universal Coefficient Theorem for homology, an R-double for any R) I may also refer to M as a homology double.

Definition 2.2 The combinatorial manifold M is called a PL R-double if there exists an A as above which is a PL subspace of M.

The following facts result immediately from the Universal Coefficients Theorem for cohomology [12, p. 243] and from the corresponding theorem which expresses homology in terms of cohomology [12, p. 248].

2.3 If A is an R-half for the manifold M, then

$$H^*(A;R) \cong H^*(M-A;R).$$

2.4 If M is a closed n-manifold, A is a compact subset such that $H^*(A;R) \cong H^*(M-A;R)$, and $H^*(A;R)$ is a finitely generated R-module, then M is an R-double with half A.

In the case of a PL R-double more can be said about a half.

Proposition 2.5 The closed, combinatorial n-manifold M is a PL R-double if and only if M has an R-half which is an n-dimensional PL submanifold.

Proof If the condition holds then M is a PL R-double by definition. If M has an R-half A which is a PL subspace, let N be a regular neighborhood of A in M. Then N has the same homotopy type as A (1.3) and similarly for M-N and M-A. Thus $H_*(N;R) \cong H_*(M-N;R)$.

Corollary 2.6 A closed, combinatorial n-manifold M is a PL R-double if and only if M contains an n-dimensional PL submanifold N with $H_*(N;R) \cong H_*(\overline{M-N};R)$.

Proof If N is as in the theorem, then $\overline{M-N}$ is an n-manifold with interior M-N. Because the boundary of a manifold is collared, M-N and $\overline{M-N}$ have the same homotopy type.

If a closed manifold M contains a submanifold N which is homeomorphic to $\overline{M-N}$, then N is a homology half for M in a stronger sense than that of definition 2.1. This motivates the following:

Definition 2.7 A closed manifold M will be called a t-double (for twisted) with t-half N if it contains a submanifold N such that N and $\overline{M-N}$ are homeomorphic. If N can be taken as a PL submanifold of the combinatorial manifold M then M will be called a PL t-double.

Considerable study has been made of t-doubles, especially in the case of 3-manifolds. Heegard splittings [11, p. 220], [5, pp. 40, 83] and toroidal manifolds [1, p. 99] are examples. It is clear that every double is a t-double, but the toroidal manifolds, for example, show that a t-half of a manifold may not be a half in the stronger sense. The key here is the way in which the t-halves are pasted together on their common boundary. The toroidal manifolds also show that a given manifold with boundary may be a t-half for different closed manifolds, whereas the double of a manifold with boundary is topologically unique. That the concept of a t-double is stronger than that of a homology double, even in the PL case, is shown by the following:

Example 2.8 Let M be a closed, combinatorial Poincare space; that is, M is a closed 3-manifold which has the homology groups (coefficients in Z) of a 3-sphere but which is not a sphere (See, for example, [11, p. 226]). Let $A \subset M$ be a 3-ball (PL subspace) and set $B = \overline{M-A}$. B cannot be a 3-ball for then M would be a sphere, and thus A is not a t-half of M.

On the other hand, from 1.5b and because the reduced homology $\tilde{H}^*(A) = 0$,

$$H_i(B) \cong H^{3-i}(M,A) \cong \tilde{H}^{3-i}(M)$$

for each i. Thus $H_*(A) \cong H_*(B)$ and M is a Z-double with Z-half A.

The following theorem gives a necessary condition for manifolds to be certain types of doubles.

Theorem 2.9 If the closed n-manifold M is a t-double or a PL R-double for some R, then the Euler characteristic of M is even.

Proof Since every odd dimensional manifold has Euler characteristic 0 (1.11), we consider only the case that n is even. By hypothesis there is some principal ideal domain R and some submanifold $A \subset M$ with $H_*(A;R) \cong H_*(B;R)$, where $B = \overline{M-A}$ and $A \cap B = \text{bd}A$. Applying 1.9 to the Mayer-Vietoris sequence of the couple $\{A,B\}$ (coefficients in R)

$$\cdots \to H_{i+1}(M) \to H_i(\text{bd}A) \to H_i(A) \oplus H_i(B) \to H_i(M) \to \cdots,$$

it follows that

$$\chi(M) - (\chi(A) + \chi(B)) + \chi(\text{bd}A) = 0. \qquad (1)$$

Since bdA is a closed, odd-dimensional manifold and $\chi(A) = \chi(B)$, we have $\chi(M) = 2\chi(A)$.

Corollary 2.10 The boundary of a compact n-manifold has even Euler characteristic.

Proof We need only consider the case that A is an odd-dimensional manifold with boundary. Letting M be the double of A, formula (1) still holds but now $\chi(M) = 0$ and $\chi(bdA) = 2\chi(A)$.

From theorem 2.8 we obtain manifolds which cannot be homology doubles.

Example 2.11 a. The homology groups of real projective n-space P^n are well known (See, for example, [6, p. 137]). In particular, if n is even $\chi(P^n) = 1$. Thus in every even dimension we have a non-orientable, closed, combinatorial manifold which cannot be a PL homology double.

b. If CP^n is the complex projective space of dimension 2n, then it is known [6, p. 143] that $\chi(CP^n) = n + 1$. Thus if n is even, CP^n is an orientable 2n-manifold which cannot be a PL homology double.

Much of chapter III will be devoted to the question whether the converse of theorem 2.9 is true. For closed 2-manifolds the answer is easy to obtain.

Theorem 2.12 A closed 2-manifold is a double if and only if its Euler characteristic is even.

Proof If M is a closed 2-manifold whose Euler characteristic χ is even, then by the classification theorem [7, p. 33] M is the connected sum of $g = (2-\chi)/2$ tori in the orientable case or $g = 2-\chi$ projective planes if M is non-orientable (g is the genus of the surface).

If M is orientable then it is the double of a disc with g holes punched out (Figure 1).

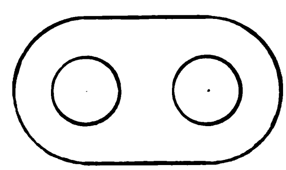
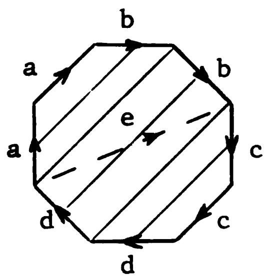


Figure 1

If M is non-orientable there is a circle which separates M into halves each of which is the connected sum of $g/2$ projective planes with one hole. For example, if $g = 4$,



has half

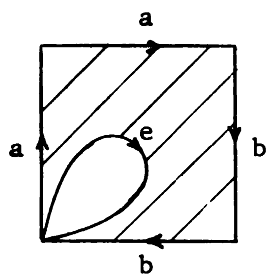


Figure 2

The decompositions above are not unique. For example, an orientable surface S of genus 2m contains a circle which separates S into two halves each of which is an orientable surface having genus m and one boundary circle (Figure 3).

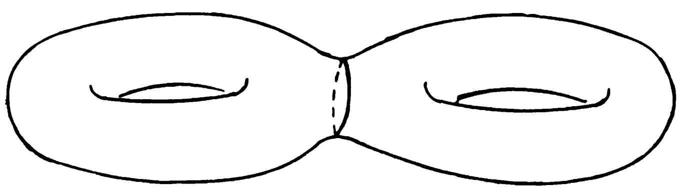


Figure 3

In chapter III all the R-halves obtained will be manifolds with connected boundary. Thus it is interesting to ask whether every closed 2-manifold with even Euler characteristic has a PL R-half which is a 2-manifold with connected boundary.

Suppose A is a compact, orientable (over Z) 2-manifold with one boundary circle. It follows from [7, chap. 1, section 10] that $H_1(A;Z)$ is a free abelian group of rank 2g, where g is the genus of A. Hence for any R, rank$H_1(A;R) = 2g$. Now suppose such an A is a PL R-half for the closed, orientable 2-manifold M. $\overline{M-A}$ is also an orientable 2-manifold with one boundary component so its genus is also g, and it follows that M is a surface of genus 2g. Thus although every closed, orientable surface has even Euler characteristic, for those with odd genus the answer to the above question is no. The best we can do in this case is to find a half with two boundary components, as indicated in Figure 4.

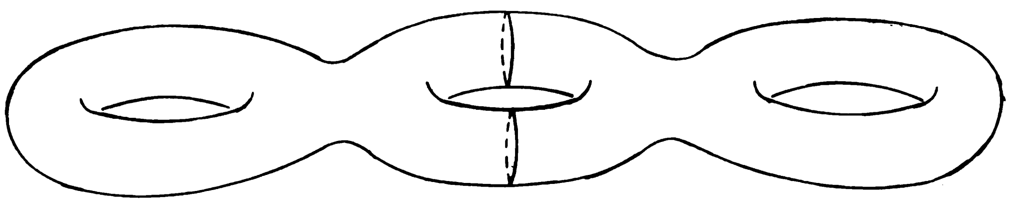
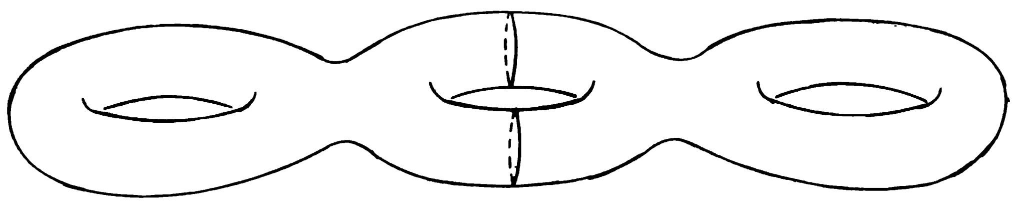


Figure 4

A compact, orientable 2-manifold is characterized by its homology groups and number of boundary components. (Again see [7, chap. I].) Applying proposition 2.5 to the orientable case, theorem 2.12 is equivalent to the statement: A closed, orientable 2-manifold M has a PL Z-half (not necessarily a manifold) if and only if $\chi(M)$ is even. The following example of P.H. Doyle shows the importance of choosing A to be a PL subspace here.

Example 2.13 Every closed 2-manifold has a decomposition as a disjoint union $M = E^2 \cup P$, where P is the one-point union of p circles [2, p. 74]. If we replace P by P', the one-point union of p closed sin(1/x) curves, it is still true that $M-P' \cong E^2$. Since we are dealing with singular homology, $\tilde{H}_*(P';Z) \cong 0 \cong \tilde{H}_*(E^2;Z)$, genus of A. Hence for any R, rank$H_1(A;R) = 2g$. Now suppose such an A is a PL R-half for the closed, orientable 2-manifold M. $\overline{M-A}$ is also an orientable 2-manifold with one boundary component so its genus is also g, and it follows that M is a surface of genus 2g. Thus although every closed, orientable surface has even Euler characteristic, for those with odd genus the answer to the above question is no. The best we can do in this case is to find a half with two boundary components, as indicated in Figure 4.



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I conclude this chapter with two miscellaneous results.

Theorem 2.14 If M and N are closed manifolds and one of them is a t-double (resp. R-double), then the manifold $M \times N$ is a t-double (R-double).

Proof a. If A is a t-half of M, then $A \times N$ is a t-half of $M \times N$.

b. If M is an R-double, then there is a compact $A \subset M$ such that $H_*(A) \cong H_*(M-A)$. (All coefficients are in R.) Then in $M \times N$, $A \times N$ is an R-half as the following application of the Künneth formula [3, p. 198] shows:

$$\begin{aligned} H_k(A \times N) &\cong \bigoplus_{i=o}^{k} (H_i(A) \otimes H_{k-i}(N)) \oplus \bigoplus_{i=o}^{k} (H_i(A) * H_{k-i-1}(N)) \\ &\cong \bigoplus_{i=o}^{k} (H_i(M-A) \otimes H_{k-i}(N)) \oplus \bigoplus_{i=o}^{k} (H_i(M-A) * H_{k-i-1}(N)) \\ &\cong H_k((M-A) \times N). \end{aligned}$$

In the following discussion all maps and subspaces are PL.

The connected sum $M_1 \# M_2$ of two closed, combinatorial, oriented (over Z) n-manifolds is obtained by removing the interior of an n-ball from each and then matching the resulting boundaries by an orientation reversing homeomorphism (See [8, p. 1]). This operation is independent of the choice of balls by the homogeneity theorem (1.12) and is uniquely determined up to orientation preserving PL homeomorphism.

A related idea for manifolds with boundary is that of the disc sum. The <u>disc sum</u> $N_1 \Delta N_2$ of two combinatorial, oriented, n-manifolds with connected boundary is obtained by taking an (n-1)-ball in the boundary of each and matching them by an orientation reversing homeomorphism. Because the boundary of a combinatorial manifold has a PL collar [13, chap. V, p. 13], the homogeneity theorem shows that the disc sum is also independent of the balls chosen. To see this, let B_1 and B_2 be two balls in bdN and $H : \text{bdN} \times I \to \text{bdN} \times I$ be the isotopy such that H_1 is the identity on bdN and $H_0(B_1) = B_2$. Since $\text{bdN} \times I$ is (homeomorphic to) the collar, H extends by the identity to an orientation preserving homeomorphism on all of M.

<u>Theorem 2.15</u> The connected sum $M_1 \# M_2$ of two PL t-doubles (resp. PL R-doubles) is a PL t-double (PL R-double).

<u>Proof</u> a. For $i = 1,2$, let N_i be a PL t-half of M_i and triangulate M_i by K_i so that N_i is triangulated by a subcomplex. Since in forming the connected sum arbitrary balls can be chosen, choose an (n-1)-simplex σ_i in bdN_i and let $B_i = |st(\sigma_i, K_i)|$ be a ball (1.1) in M_i. Note that

$$N_i - \text{int}B_i \cong N_i \cong \overline{M_i - N_i} \cong (\overline{M_i - N_i}) - \text{int}B_i .$$

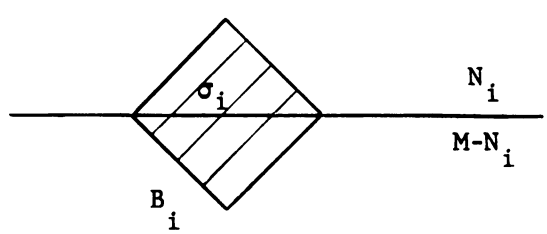

Figure 5

Now choose an orientation reversing homeomorphism $h : bdB_1 \to bdB_2$ satisfying

$$h(bdB_1 \cap N_1) = bdB_2 \cap N_2$$

and $$M_1 \# M_2 \cong (M_1 - intB_1) \cup_h (M_2 - intB_2).$$

This can always be done by first choosing an orientation reversing homeomorphism between the (n-2)-spheres $bdN_i \cap bdB_i$ and then extending to a homeomorphism between the (n-1)-balls $bdB_i \cap N_i$ and between $bdB_i \cap \overline{M-N_i}$. Then

$$(N_1 - intB_1) \cup_h (N_2 - intB_2) \cong N_1 \Delta N_2$$

$$\text{and } ((\overline{M_1 - N_1}) - intB_1) \cup_h ((M_2 - N_2) - intB_2) \cong \overline{M_1-N_1} \Delta \overline{M_2-N_2}. \quad (*)$$

Since the disc sum is well defined (Although $bdN_i = bd(\overline{M-N_i})$ may not be connected, this is no problem since the connecting discs are from the same boundary component.), these two spaces are homeomorphic and $(N_1-intB_1) \cup_h (N_2-intB_2)$ is a PL t-half for $M_1 \# M_2$.

b. Suppose for $i = 1,2$, N_i is a submanifold of M_i which is a PL R-half. Proceeding exactly as in case a, we obtain (*). From the reduced Mayer-Vietoris sequence of the couple $\{N_1, N_2\}$ in $N_1 \Delta N_2$ it follows that (with coefficients in R)

$$H_*(N_1 \Delta N_2) \cong H_*(N_1) \oplus H_*(N_2).$$

Similarly

$$H_*(\overline{M-N_1} \Delta \overline{M-N_2}) \cong H_*(\overline{M-N_1}) \oplus H_*(\overline{M-N_2}).$$

Thus $(N_1 - intB_1) \cup_h (N_2 - intB_2)$ is an R-half for $M_1 \# M_2$.

CHAPTER III

CLOSED, COMBINATORIAL n-MANIFOLDS, $n \geq 3$.

In this chapter all manifolds will be closed, combinatorial n-manifolds, where $n \geq 3$, and all maps and subspaces will be PL. In considering the converse of theorem 2.9, we will show that for odd dimensions every such manifold is a PL homology double. For even dimensional manifolds with even Euler characteristic a slightly weaker result is obtained. The last theorems concern the uniqueness of these double decompositions.

We first develop some notation and lemmas concerning combinatorial n-manifolds.

<u>Definition 3.1</u> Let K be a combinatorial triangulation of a closed manifold M, and let K' be its first barycentric subdivision. For each simplex A in K, the <u>dual cell</u> A^* is defined as the union of all simplexes of K' of the form $|\{b(A_j),\ldots,b(A_m)\}|$, where $A < A_j < \ldots < A_m$. The collection of cells $K^* = \{A^* : A \in K\}$ is called the <u>dual cell decomposition</u> of K.

Note that if A is a k-simplex of K then A^* is an (n-k)-dimensional cell which has a simplicial triangulation by a subcomplex of K'. In this triangulation the (n-k)-simplexes of A^* are precisely those (n-k)-simplexes of K' having $b(A)$ as first vertex. It is known that K^* is a cell complex [11, sect. 67] and that if K^{*r} denotes the r-skeleton of K^* (which has a simplicial triangulation by a subcomplex of K'), then for each $i < r$,

$H_i(M;R) \cong H_i(K^{*r};R)$.

Definition 3.2 Let K be a combinatorial triangulation of the n-manifold M, and let $L \subset K$ be a subcomplex. The complementary cell complex $L^* \subset K^*$ is defined as

$$L^* = \{A^* \in K^* : A \in K - L\}.$$

That is, the cells of L^* are the duals of those simplexes of K which are not in L.

Observe that every vertex of K' is in either $|L|$ or $|L^*|$. For if there is such a vertex $b \notin |L|$, then $b = b(A)$ where $A \notin L$ and thus $b \in A^* \subset |L^*|$. Also, the notation L' is appropriate for the subcomplex of K' which triangulates $|L|$ and similarly for $L^{*'} \subset K'$.

Lemma 3.3 Suppose K is a combinatorial triangulation of the manifold M, $L \subset K$ is a proper subcomplex, L^* is the complementary cell complex, and K'' is the second barycentric subdivision of K. Let $N = N(|L|,K'')$, $N^* = N(|L^*|,K'')$ be the derived neighborhoods. Then $N^* = \overline{M-N}$.

Proof (Cf. [13, chap. III]) First note that if $x \in M - (|L| \cup |L^*|)$, then there is a unique choice of $a \in |L|$, $a^* \in |L^*|$, and $0 < \lambda < 1$ such that $x = \lambda a + (1-\lambda)a^*$. (Recall that M can be considered imbedded is some Euclidean space.) To see this, let A be the unique simplex of K' such that $x \in \text{int}A$. Each vertex of A is in $|L|$ or $|L^*|$, but not all the vertices of A are in one of these, since L' and $L^{*'}$ are full subcomplexes of K' [13, III, p. 13]. Thus

A is the join $B*B^*$, where $B \in L'$, $B^* \in L^{*'}$, and $x = a + (1-\lambda)a^*$ uniquely for some $a \in B$, $a^* \in B^*$.

Now it follows that there is a unique continuous map $f : M \to [0,1]$ which is linear on simplexes of K' and such that $f^{-1}(0) = |L|$, $f^{-1}(1) = |L^*|$. From the definitions of N and N^*, $N = f^{-1}[0,\frac{1}{2}]$, $N^* = f^{-1}[\frac{1}{2},1]$. Clearly then, $N \cup N^* = M$ and $\text{bd}N = \text{bd}N^* = f^{-1}(\frac{1}{2}) = N \cap N^*$. This is equivalent to the conclusion of the lemma.

We can now prove the main result on odd dimensional manifolds.

Theorem 3.4 Let M be a closed, combinatorial manifold of odd dimension $2m + 1 \geq 3$. Then M is a PL homology double.

Proof Let K be a combinatorial triangulation of M, let $L = K^m$ be the m-skeleton, and let L^* be its complementary cell complex. Note that L^* is the m-skeleton of K^*. Let N and N^* be the derived neighborhoods of $|L|$ and $|L^*|$ respectively in K'', as in lemma 3.3. Since N collapses to $|L|$ and N^* collapses to $|L^*|$, it follows from the remarks preceeding definition 3.2 that for any R,

$$H_i(N;R) \cong H_i(M;R) \cong H_i(N^*;R), \text{ if } i < m,$$

and (1)

$$H_i(N;R) \cong 0 \cong H_i(N^*;R), \text{ if } i > m.$$

Thus to show that N is a homology half for M it remains only to check the modules in dimension m.

We first consider the case $R = Z_2$. Since all manifolds are then orientable, there is, by 1.7, a sign commutative diagram (coefficients in Z_2):

$$\begin{array}{ccccccccccc} \ldots\to & H^m(N,bdN) & \to & H^m(N) & \to & H^m(bdN) & \to & H^{m+1}(N,bdN) & \to & H^{m+1}(N) & \to\ldots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \ldots\to & H_{m+1}(N) & \to & H_{m+1}(N,bdN) & \to & H_m(bdN) & \to & H_m(N) & \to & H_m(N,bdn) & \to\ldots \end{array}$$

Because L is an m-dimensional complex, $H^{m+1}(N) \cong H_{m+1}(N) \cong 0$. Also, since all Z_2-modules are free, 1.4 gives $H^m(N) \cong H_m(N)$, and it follows that $H_m(bdN) \cong H_m(N) \oplus H_m(N)$. An identical argument shows that $H_m(bdN^*) \cong H_m(N^*) \oplus H_m(N^*)$. But $bdN = bdN^*$, implying that $H_m(N;Z_2) \cong H_m(N^*;Z_2)$. Thus N is a Z_2-half for M, and in particular, since the Euler characteristic of M is independent of the coefficient module, $\chi(N) = \chi(N^*)$.

Now considering coefficients in Z,

$$\chi(N) = \sum_{i=1}^{2m+1} (-1)^i \mathrm{rank} H_i(N;Z),$$

and similarly for $\chi(N^*)$. From equations (1) above it follows that $\mathrm{rank}H_m(N;Z) = \mathrm{rank}H_m(N^*;Z)$. But since L and L^* are m-dimensional complexes, these two Z-modules are free and hence isomorphic. This completes the proof that $H_*(N;Z) \cong H_*(N^*;Z)$.

It should be pointed out that while every closed, combinatorial manifold of odd dimension $2m + 1$ has a PL homology half, the half N exhibited in the theorem is by no means unique. Even if we insist that the half N satisfy conditions (1) for $R = Z$, $H_m(N;Z)$ may be almost any free abelian group. Examples and a discussion of uniqueness follow later in this chapter (pp. 31-33).

For 3-manifolds a result stronger than theorem 3.4 is possible - namely, that every closed 3-manifold is actually a t-double. Seifert and Threlfall have this result for orientable 3-mainfolds [11, p. 219], and it is not too difficult to extend it to the non-orientable case. Since any 3-manifold has a combinatorial triangulation [9], we will in fact show that every closed 3-manifold is a PL t-double. Recall that all maps and subspaces are PL.

Definition 3.5 ([11, p. 219]) Let B be a PL 3-ball, let $C_1,\ldots,C_n$, $D_1,\ldots,D_n$ be mutually disjoint discs in bdB, and let $h_i : C_i \to D_i$ be a PL homeomorphism, $i = 1,\ldots,n$. Then the quotient space H obtained by identifying each $x \in C_i$ with $h_i(x) \in D_i$, $i = 1,\ldots,n$, is a 3-manifold with boundary called a handle body of genus n.

If B is given an orientation then this induces an orientation on each C_i and D_i. The homeomorphism h_i is said to be of type 1 if it is orientation reversing and of type 2 if it is orientation preserving. If n_1 of the h_i are of type 1 and n_2 are of type 2, then H is said to have n_1 handles of type 1 and n_2 handles of type 2.

As in the case of the disc sum (p. 17), the homogeneity theorem shows that the construction of H is independent of the choice of the discs C_i and D_i. On the other hand, bdH is a closed surface of genus n which is orientable if and only if all the h_i are of type 1. Thus H is an orientable 3-manifold with boundary if and only if all the h_i are of type 1. Hence an orientable handle body is determined up to PL homeomorphism by its genus, and this is determined if bdH is known. The following lemma establishes the same result for a non-orientable handle body.

<u>Lemma 3.6</u> A non-orientable handle body H of genus n can be represented as a handle body with n handles of type 2.

<u>Proof</u> Let $h_i : C_i \to D_i$, $i = 1,\ldots,n$, be the homeomorphisms between discs in bdB defining H. Since H is non-orientable, at least one of the h_i, say h_1, is of type 2. Suppose that $k \geq 1$ of the h_i, and in particular h_2, are of type 1. The lemma will follow in a finite number of steps if we can show that H can be represented as a handle body with k-1 handles of type 1.

If B is represented as the suspension of a 2-simplex A, with suspension points p_1 and p_2, then B is the disc sum of the balls $B_1 = p_1 * A$ and $B_2 = p_2 * A$, with the identification taking place on A. If $A \subset B_1$ is denoted by C_0 and $A \subset B_2$ is denoted by D_0, with $h_0: C_0 \to D_0$ the identity map, then $B = B_1 \cup_{h_o} B_2$. We can assume that

$$D_1, D_2 \subset bdB_2 - D_0,$$

and that $C_i, D_j \subset bdB_1 - C_0$ for $i = 1,\ldots,n$, $j = 3,\ldots,n$. (See figure 6)

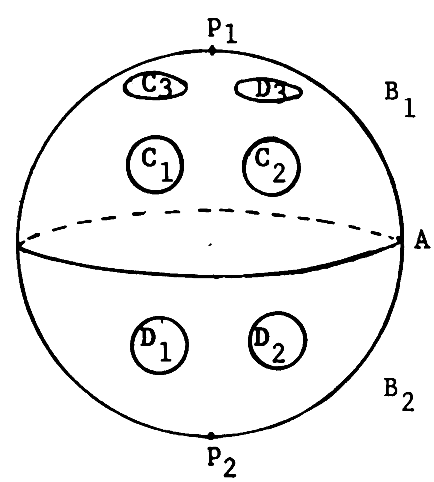

Figure 6

Now let $B^* = B_1 \cup_{h_1} B_2$, which is formed by cutting B through A and repasting by $h_1 : C_1 \to D_1$. We wish to orient B^*, and since h_1 is orientation preserving we must change the orientation on one of the pieces, say on B_2. Now for $i = 0,2,3,\ldots,n$, h_i induces a homeomorphism g_i between discs in bdB^*. Because the orientation on B_2 is reversed, g_0 and g_2 are now of type 2. On the other hand, since all the remaining discs are in B_1, g_i is of the same type as h_i for $i = 3,\dots,n$. Thus H results from the ball B^* where the number of identifications of type 1 has been reduced by one, completing the lemma.

Theorem 3.7 Every closed 3-manifold is a PL t-double.

Proof Let K be a combinatorial triangulation of M, and let N be the second derived neighborhood in M of the 1-skeleton of K as in theorem 3.4. In this case N and $N^* = \overline{M-N}$ are handle bodies with the same boundary [11, p. 219]. By lemma 3.6 and the remarks preceeding it, a handle body is determined by its boundary, and it follows that N and N^* are PL homeomorphic.

For even dimensional manifolds I have not been able to obtain a result having the full strength of theorem 3.4. In order to obtain a slightly weaker result the concept of a weak R-double is introduced.

Definition 3.8 A closed, combinatorial n-manifold M will be called a weak R-double if it contains an n-manifold with boundary N (PL subspace) such that for each i, $\text{rank}H_i(N;R) = \text{rank}H_i(\overline{M-N};R)$.

Note that if R is a field, then a weak R-double is in fact a PL R-double.

The goal is to show that if an even dimensional manifold has even Euler characteristic (Cf. theorem 2.9) and is orientable over R, then it is a weak R-double. Some notation and lemmas are needed. In the following discussion all homology modules have coefficients in

R, and rank refers to rank as an R-module.

Suppose M is a polyhedron of dimension $\geq m$, with a cell decomposition having m-cells $C_1,\ldots,C_t$. Let L_0 be the (m-1)-skeleton of the cell decomposition, and let $L_i = L_{i-1} \cup \{C_i\}$, $i = 1,\ldots,t$. Thus L_t is the m-skeleton of M and $|L_i|$ is obtained from $|L_{i-1}|$ by attaching the m-cell C_i via the identity map on its boundary. The following facts are known (See [3, pp. 89, 101]):

A. $H_m(L_i)$ is a free R-module, for $i = 0,1,\ldots,t$.

B. $H_j(L_i) = H_j(M)$ for $j < m - 1$, $i = 0,1,\ldots,t$.

C. $H_j(L_i) = 0$ for $j > m$, $i = 0,1,\ldots,t$.

D. For $i = 0,\ldots,t-1$, <u>precisely one</u> of the following holds:

1. $H_m(L_{i+1}) \cong H_m(L_i)$ and $\operatorname{rank} H_{m-1}(L_{i+1}) = \operatorname{rank} H_{m-1}(L_i) - 1$.
2. $H_m(L_{i+1}) \cong H_m(L_i) \oplus R$ and $\operatorname{rank} H_{m-1}(L_{i+1}) = \operatorname{rank} H_{m-1}(L_i)$.

<u>Lemma 3.9</u> Let M be as above. Then the m-cells of the cell decomposition can be ordered so that the following hold:

E. $H_m(L_i) = 0$ for $i = 0,\ldots,r$.

F. $H_m(L_i) = H_m(L_{i-1}) \oplus R$ for $i = r+1,\ldots,t$.

G. $\operatorname{rank} H_{m-1}(L_i) = \operatorname{rank} H_{m-1}(M)$ for $i = r,\ldots,t$.

Moreover, having chosen $C_1,\ldots,C_r$, the ordering of $C_{r+1},\ldots,C_t$ is arbitrary.

<u>Proof</u> Let $S = \{C_1,\ldots,C_r\}$ be a maximal set of m-cells having the property that $H_m(L_r) = 0$; that is, for any additional m-cell C, $H_m(L_r \cup \{C\}) \cong R$. Order the remaining m-cells arbitrarily.

Now suppose for some $i > r$, $H_m(L_i) \cong H_m(L_{i-1})$. By [3, p. 89], $\operatorname{kernal} H_{m-1}(f) = 0$, where $f : \operatorname{bd}C_i \to |L_{i-1}|$ is the identity map and

$H_{m-1}(f)$ is the induced homeomorphism on homology. But f factors as

$$\mathrm{bd}C_i \xrightarrow{f'} |L_r| \xrightarrow{j} |L_{i-1}|,$$

where f' is also the identity and j is inclusion. Since $H_{m-1}(f) = H_{m-1}(j)H_{m-1}(f')$, $\mathrm{kernal}H_{m-1}(f') = 0$. But then $H_m(L_r \cup \{C_i\}) = 0$, contradicting the maximality of S.

Therefore conditions E and F hold, and condition G holds because of D above.

Lemma 3.10 Let K be a combinatorial triangulation of the closed $2m$-manifold M, $2m \geq 4$, which is orientable over R. Suppose $L \subset K$ is a subcomplex of dimension $\leq m$ satisfying $H_m(L;R) = 0$. Then

$$\mathrm{rank}H_{m-1}(L^*;R) = \mathrm{rank}H_{m-1}(M;R).$$

Proof Let N and N^* be the derived neighborhoods of $|L|$ and $|L^*|$ respectively in K'', as in lemma 3.3. From 1.5 and 1.4,

$$H_m(M,N^*) \cong H^m(N) \cong F_m(N) \oplus T_{m-1}(N) \cong F_m(L) \oplus T_{m-1}(L) \cong T_{m-1}(L).$$

Similarly, $H_{m-1}(M,N^*) \cong F_{m+1}(L) \oplus T_m(L) = 0$. Then in the long exact sequence of the pair (M,N^*) we have

$$\ldots \to T_{m-1}(L) \to H_{m-1}(N^*) \to H_{m-1}(M) \to 0 \to \ldots$$

It follows that the torsion-free submodules of $H_{m-1}(N^*)$ and $H_{m-1}(M)$ are isomorphic, completing the lemma.

Theorem 3.11 Let M be a closed, combinatorial manifold of even dimension $2m$ and orientable over R. Then M is a weak R-double if and only if its Euler characteristic is even.

<u>Proof</u> Necessity and the case $2m = 2$ were proved in theorems 2.9 and 2.12.

Suppose that $\chi(M)$ is even and $2m \geq 4$. Let K be a combinatorial triangulation of M and K^* the dual cell decomposition. Denote by L_0 the $(m-1)$-skeleton of K and by $C_1,\ldots,C_t$ the m-simplexes in K. Using the notation established above and lemma 3.9, we can order these m-simplexes so that conditions A through G hold.

Note that for $i = 1,\ldots,t$, $|L^*_{i-1}|$ is obtained from $|L^*_i|$ by attaching the dual cell C^*_i. In particular, L^*_r is obtained from L^*_t (which is the dual $(m-1)$-skeleton) by attaching $C^*_t, C^*_{t-1},\ldots,C^*_{r+1}$ in that order. Applying lemma 3.9 to the polyhedron $|L^*_r|$, these dual cells can be ordered so that

E^*. $H_m(L^*_i) = 0$ for $s \leq i \leq t$,

F^*. $H_m(L^*_i) \oplus R = H_m(L^*_{i+1})$ for $r \leq i < s$,

and conditions A through G still hold for this ordering. Note also that analogous conditions A^*, B^*, and C^* hold for each $|L^*_i|$. By lemma 3.10

$$\operatorname{rank} H_{m-1}(L^*_r) = \operatorname{rank} H_{m-1}(M),$$

and from F^*, D, and G it follows that

H. $\operatorname{rank} H_{m-1}(L^*_i) = \operatorname{rank} H_{m-1}(M) = \operatorname{rank} H_{m-1}(L_i)$, for $r \leq i \leq s$.

As in lemma 3.3, for $i = r, r+1,\ldots,s$, let N_i and N^*_i be the derived neighborhoods in K'' of $|L_i|$ and $|L^*_i|$ respectively. Since $N_i \cap N^*_i = \mathrm{bd}N_i$ is an odd dimensional manifold, 1.10 and 1.11 give

$$\chi(M) = \chi(N_i) + \chi(N_i^*) = \chi(L_i) + \chi(L_i^*).$$

Consider in particular the case $i = s$. From B, H, and E^* it follows that

$$\chi(M) = 2\sum_{i=o}^{m-1} (-1)^q \text{rank}H_q(M)) + (-1)^m \text{rank}H_m(L_s).$$

By Poincaré duality (1.6), this implies that

$$\text{rank}H_m(L_s) = \text{rank}H_m(M),$$

which is an even number since $\chi(M)$ is even. Now because F holds, $\text{rank}H_m(L_s) = s-r$, and thus $r+s$ is even. Let $p = (r+s)/2$. Then from F and F^*,

$$\text{rank}H_m(L_p) = (s-r)/2 = \text{rank}H_m(L_p^*),$$

and because these R-modules are free, $H_m(L_p) \cong H_m(L_p^*)$. Letting $N = N_p$, $N^* = N_p^*$, we summarize:

$H_j(N) \cong H_j(M) \cong H_j(N^*)$ for $j < m - 1$.

$\text{rank}H_{m-1}(N) = \text{rank}H_{m-1}(M) = \text{rank}H_{m-1}(N^*)$.

$H_m(N) \cong H_m(N^*)$, both free R-modules with rank $= \frac{1}{2}\text{rank}H_m(M)$.

$H_j(N) = 0 = H_j(N^*)$ for $j > m$.

Thus N is a weak R-half for M, completing the proof.

It is not difficult to construct an example of an even dimensional manifold having a weak Z-half which is not a PL Z-half.

Example 3.12 Let $L = L(p,q)$ be a lens space and $M = S^1 \times L$. Then the homology groups (coefficients in Z) of L are known [6, p. 148], and those of M are calculated from the Künneth formula:

$$H_i(L) \cong \begin{cases} Z & \text{if } i = 0,3, \\ Z_p & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases} \qquad H_i(M) \cong \begin{cases} Z & \text{if } i = 0,3,4, \\ Z \oplus Z_p & \text{if } i = 1, \\ Z_p & \text{if } i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Let B be a PL 3-ball in L and $C = \overline{L-B}$. Then the homology groups for C can be calculated using 1.4, 1.5, and the fact that $H^i(L,B) \cong \tilde{H}^i(L)$ for each i:

$$H_i(C) \cong \begin{cases} Z & \text{if } i = 0, \\ Z_p & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus B is a weak Z-half of L, and to obtain a weak half of M, let $N = S^1 \times B$, $N^* = S^1 \times C$. Then:

$$H_i(N) \cong \begin{cases} Z & \text{if } i = 0,1, \\ 0 & \text{otherwise.} \end{cases} \qquad H_i(N^*) \cong \begin{cases} Z & \text{if } i = 0, \\ Z + Z_p & \text{if } i = 1, \\ Z_p & \text{if } i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Note that this example does not show that no PL Z-half of $S^1 \times L$ exists. In fact, combining theorems 3.7 and 2.14, $S^1 \times L$ has a PL t-half. Also, the N and N^* above do not satisfy the conditions summarized at the end of the proof of theorem 3.11. I have not been able to find a counter-example to a stronger conclusion in theorem 3.11, or even find an example in which the specific construction in the proof gives a weak R-half which is not actually an R-half.

With an additional hypothesis, however, theorem 3.11 can be strengthened.

Theorem 3.13 Let M be a closed, combinatorial 2m-manifold which has even Euler characteristic and is orientable over R, and suppose $H_m(M;R) = 0$. Then M is a PL R-double.

Proof All coefficients will be in R. Let N and $N^* = \overline{M-N}$ be weak R-halves as constructed in theorem 3.11. Since $H^m(M) = 0$ (1.6), it follows from the cohomology sequence of the pair (M,N) that there is a monomorphism $H^m(N) \subset H^{m+1}(M,N)$. In this case $F_m(N) = 0$, implying (1.4) that $H^m(N) \cong T_{m-1}(N)$, and by 1.5, $H^{m+1}(M,N) \cong H_{m-1}(N^*)$. Thus there is a monomorphism $T_{m-1}(N) \subset T_{m-1}(N^*)$. Reversing the roles of N and N^* gives $T_{m-1}(N) \cong T_{m-1}(N^*)$.

In theorem 3.11, N and N^* were constructed so that their homologies agree except possibly in their (m-1)-dimensional torsion modules. Thus in this special case, $H_*(N) \cong H_*(N^*)$.

In order to discuss the uniqueness of an R-half in a closed manifold, the following concept will be useful.

Definition 3.14 A polyhedron X is said to have cohomology dimension $\leq k$ if $H^i(X;Z) = 0$ for all $i > k$. The cohomology dimension of X is the smallest integer with this property.

Observe that if X has cohomology dimension k, then for any R and any $i > k$, $H^i(X;R) = 0$ [12, p. 246]. Then it follows (1.4) that $H_i(X;R) = 0$ for $i > k$ and $T_k(X;R) = 0$.

If M is a closed, combinatorial manifold of dimension $2m + 1$, then the PL Z-half of M found in theorem 3.4 has cohomology dimension $\leq m$. Similarly, if M is a closed, combinatorial 2m-manifold with even Euler characteristic and orientable over R, then the weak R-half of M found in theorem 3.11 has cohomology dimension $\leq m$. The following examples show that the cohomology dimension of a half may be large.

Example 3.15 Let Y be the compact 2-manifold with boundary formed by cutting an open disc from the torus. Let $N = Y \times S^k$. Then $H^{k+1}(N;Z) \cong Z \oplus Z$, and $H^i(N;Z) = 0$ for $i > k + 1$. If M is the double of N, M is a closed manifold of dimension $n = k + 2$, and N is a half for M having cohomology dimension $n - 1$.

These examples suggest that the only hope for obtaining results on the uniqueness of a half is by placing restrictions on its cohomology dimension.

Theorem 3.16 Let M be a closed, combinatorial n-manifold which is orientable over R, and let m be the greatest integer in $n/2$. If M has a PL R-half N of cohomology dimension $k < m$, then $H_i(M;R) \cong H_i(N;R)$ for $i \leq m$.

Proof Assume that N is a PL submanifold of dimension n (2.5) and let $N^* = \overline{M-N}$. Then for $i \leq m$,

$$H_i(M,N;R) \cong H^{n-i}(N^*;R) = 0,$$

by 1.5. It follows from the exact sequence of the pair (M,N) that $H_i(M;R) \cong H_i(N;R)$ for $i < m$.

If n is odd, $n = 2m + 1$ and $H_{m+1}(M,N;R) \cong H^m(N^*;R) = 0$, implying $H_m(M;R) \cong H_m(N;R) = 0$.

If $n = 2m$ is even, then $\chi(N) = \frac{1}{2}\chi(M)$ as seen in the proof of theorem 2.9. From Poincaré duality it follows that $\operatorname{rank} H_m(M;R) = 0$, and also that

$$T_m(M;R) \cong T_{m-1}(M;R) \cong T_{m-1}(N;R) = 0.$$

Thus $H_m(M;R) = 0 = H_m(N;R)$, completing the proof.

Theorem 3.16 says that a closed, combinatorial n-manifold M has a PL R-half of cohomology dimension less than $m = [n/2]$ only if $H_m(M;R) = 0$, and that in this case the homology of such a half is uniquely determined by that of M. The examples below show that M may have many different halves of cohomology dimension m.

Example 3.17 a. Let X_t be the one-point union of t circles imbedded as a PL subspace in S^3. Then by Alexander duality [12, p. 296],

$$H_i(S^3 - X_t) \cong H_i(X_t) \cong \begin{cases} Z & \text{if } i = 0, \\ \overset{t}{\underset{1}{\oplus}} Z & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

b. Let Y_t be the one-point union of t projective planes imbedded as a PL subspace in S^4. Again applying Alexander duality,

$$H_i(S^4 - Y_t) \cong H_i(Y_t) \cong \begin{cases} Z & \text{if } i = 0, \\ \overset{t}{\underset{1}{\oplus}} Z_2 & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

If R is a field and M is an even dimensional manifold, note that theorem 3.13 is the converse of theorem 3.16. The last theorem in this chapter gives a similar result for odd dimensions.

Theorem 3.18 Let M be a closed, combinatorial $(2m + 1)$-manifold which is orientable over a field F. Then M has an F-half with

$$H_i(N;F) \cong \begin{cases} H_i(M;F) & \text{if } i \leq m, \\ 0 & \text{if } i > m. \end{cases}$$

Proof Let all modules have coefficients in F. The technique here is similar to that in the proof of theorem 3.11, except that now there are no torsion submodules.

Let t be the rank of $H_m(M)$ as an F-module, let L_0 be the m-skeleton of a combinatorial triangulation K of M, and let L_0^* be the complementary subcomplex in the dual cell decomposition K^*. Then $H_m(L_0) \cong H_m(L_0^*)$ (theorem 3.4), and we denote by r the rank of these modules. As in lemma 3.9, we can adjoin $s = r - t$ $(m+1)$-simplexes to L_0, obtaining L_s satisfying:

$H_i(L_s) \cong H_i(M)$ if $i < m$,

$H_m(L_s)$ has rank t, and hence $H_m(L_s) \cong H_m(M)$,

$H_i(L_s) = 0$ if $i > m$.

Let N and N^* be the derived neighborhoods of L_s and L_s^* in K'', as in lemma 3.3.

Since the m-skeleton L_0^* of K^* is obtained from L_s^* by adjoining m-cells, $H_i(N^*) \cong H_i(M)$ for $i < m - 1$ and $H_i(N^*) = 0$ for $i > m$. Because $H^i(N) = 0$ for all $i > m$ and applying 1.5 and 1.6,

$$H_{m-1}(N^*) \cong H^{m+2}(M,N) \cong H^{m+2}(M) \cong H_{m-1}(M).$$

Thus adjoining s m-cells to L_s^* gives L_0^* with no change in the (m - 1)-dimensional homology. Thus by condition D, p. 26, $H_m(L_s^*)$ has rank $r - s = t$. Therefore $H_i(N) \cong H_i(N^*)$ for all i, completing the theorem.

CHAPTER IV

COMPACT, COMBINATORIAL MANIFOLDS WITH BOUNDARY

In this chapter compact, combinatorial manifolds with non-void boundary are considered. As in chapters II and III, the question considered is: Under what circumstances does such a manifold decompose into two pieces in some sense equivalent? All compact manifolds with boundary of dimension 2 or 3 are shown to be PL t-doubles (definition below). In higher dimensions, homological results for manifolds whose boundary components are spheres are obtained by applying theorems from chapter III. As in chapter III, all subspaces and maps will be PL.

Definition 4.1 (Cf. 2.6 and 2.7) A compact, combinatorial n-manifold with boundary, M, is a PL t-double if it contains a PL submanifold N such that $N \cong \overline{M-N}$. M is said to be a PL R-double if it contains a PL submanifold N of dimension n satisfying $H_*(N;R) \cong H_*(\overline{M-N};R)$. The submanifold N is called a PL t-half or a PL R-half of M.

Theorem 4.2 Every compact 2-manifold with boundary is a PL t-double having a half which is a disc.

Proof It is known ([7, pp. 43-45]) that any compact 2-manifold with boundary can be represented by attaching strips to a disc. Moreover, as can be seen in figures 7 and 8, the strips can be attached symmetrically so that cutting the manifold on the line of symmetry separates it into two discs. Figure 7 shows an orientable 2-manifold having genus 2 and 3 boundary components, while figure 8 represents a non-orientable surface having genus 2 and 2 boundary circles.

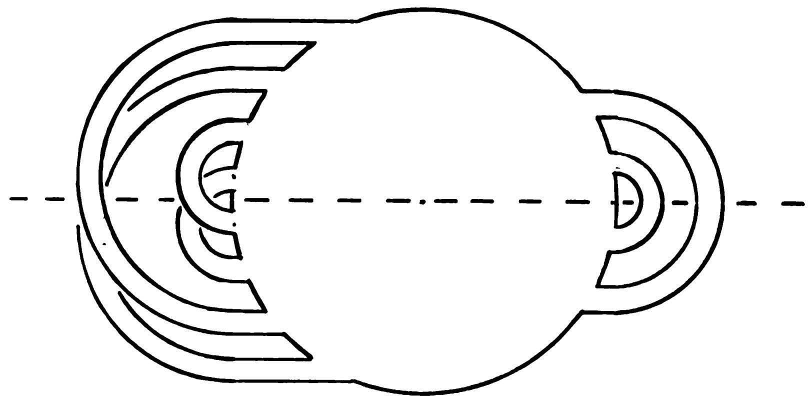

Figure 7

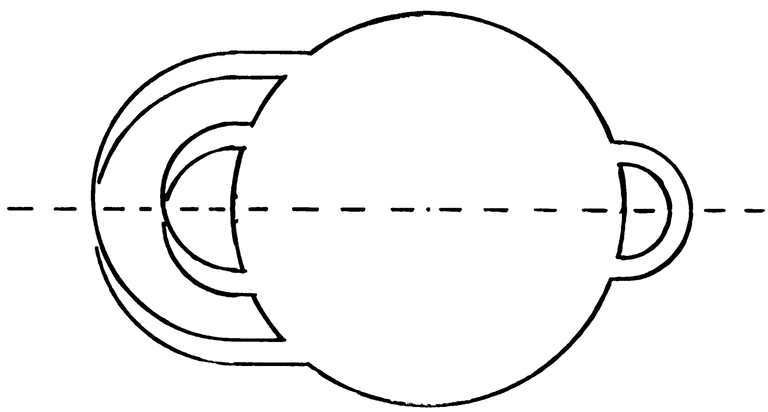

Figure 8

The following examples show that in higher dimensions there are many manifolds with boundary not having a homology ball as a PL Z-half.

<u>Example 4.3</u> If M is a compact n-manifold with boundary and $H_1(M;Z)$ has torsion, then M cannot have a PL Z-half which is a homology n-ball.

Proof If B_1 and $B_2 = \overline{M-B_1}$ are homology n-balls, then from the reduced Mayer-Vietoris sequence of the couple $\{B_1,B_2\}$ we have $\tilde{H}_1(M;Z) \cong \tilde{H}_0(B_1 \cap B_2;Z)$, which is a torsion-free module.

Example 4.4 Let K be a non-trivial polygonal knot in S^3, let N be a regular neighborhood of K, and $M = \overline{S^3 - N}$. By Alexander duality, $H_i(M;Z)$ is Z for $i = 0,1$ and 0 otherwise. On the other hand, the fundamental group of M is the knot group of K which is not cyclic.

Now suppose B_1 and $B_2 = \overline{M - B_1}$ are homology 3-balls. Then $H_1(B_1 \cap B_2;Z) \cong H_2(M;Z) = 0$, and since $B_1 \cap B_2$ is a finite collection of connected 2-manifolds with boundary it follows that $B_1 \cap B_2$ consists of disjoint discs only. Since two 3-balls meeting on a disc form a 3-ball ([13, III, p. 4]), M must be a handle body. But then $H_1(M;Z) \cong Z$ implies that the fundamental group of M is also Z. This contradiction shows that M cannot have a Z-half which is a homology 3-ball.

Example 4.4 above is interesting in view of the following result.

Proposition 4.5 Let M be a compact, connected 3-manifold with non-void boundary such that $H_i(M;Z)$ is torsion-free for $i = 1,2$. Then there is a manifold with boundary M' having a 3-ball as PL t-half and satisfying $H_*(M';Z) \cong H_*(M;Z)$.

Proof Let B be a 3-ball. In bdB embed a connected graph X such that $H_1(X;Z) \cong H_2(M;Z)$, which is possible since $H_2(M;Z)$ is free. Let A be a regular neighborhood of X in bdB. If

$\text{rank}H_1(M;Z) = k$, let $D_1,\ldots,D_k$ be disjoint discs in bdB all of which miss A. Now take B_1 and B_2 to be copies of B and h a PL homeomorphism from $A \cup D_1 \cup\ldots\cup D_k$ in B_1 to the corresponding set in B_2. Then $M' = B_1 \cup_h B_2$ is the required manifold as the following discussion shows.

Since for each i (coefficients in Z),

$$\tilde{H}_i(M') \cong \tilde{H}_{i-1}(B_1 \cap B_2) \cong \tilde{H}_{i-1}(A \cup D_1 \cup\ldots\cup D_k),$$

and the union is disjoint, it follows that $H_0(M') \cong Z$, $H_1(M')$ is free of rank k, $H_2(M') \cong H_1(A) \cong H_2(M)$, and $H_i(M') = 0$ for all other i. Thus $H_*(M) \cong H_*(M')$.

Although not all compact 3-manifolds with boundary have a half which is a ball, we will show that every such manifold is a PL t-double. The proof relies on theorem 3.7 and a technique for modifying a compact 3-manifold with boundary into a closed manifold.

In the following discussion Y will always represent a PL homeomorph of the set of all points $(x,y) \in E^2$ which satisfy

$$x \in \{0,1,\ldots,k\} \text{ and } -1 \le y \le 1$$

or

$$|y| = 1 \text{ and } 0 \le x \le k.$$

Also, X will represent a homeomorph of $\{(x,y) \in Y : y \ge 0\}$, with $x_0,x_1,\ldots,x_k$ being the end vertices of this tree. Note in particular that $Y = X_1 \cup X_2$, two copies of X attached at the points $x_0,x_1,\ldots,x_k$, as illustrated in figure 9.

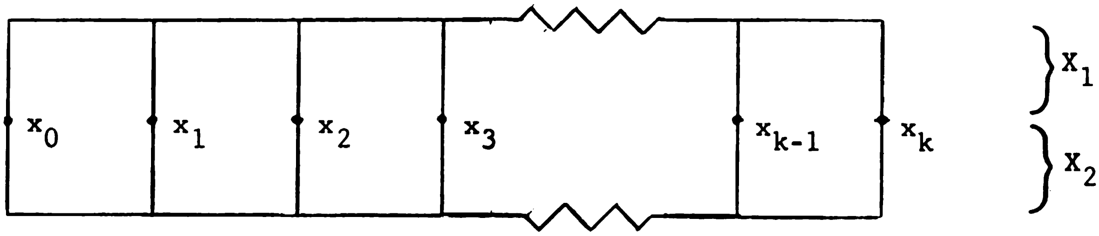

Figure 9

<u>Definition 4.6</u> Let H be a handle body with j handles and let X as above be embedded as a PL subspace of H such that $X \cap \mathrm{bd}H = \{x_0, x_1, \ldots, x_k\}$. Let N be a regular neighborhood of X in H. X is said to be unknotted in H if $H - \mathrm{int}_H N$ is a handle body with $j + k$ handles.

Now suppose that H is a handle body in a closed 3-manifold M, and that X is embedded in bdH. We want to move X so that it is unknotted in H.

Let K be a combinatorial triangulation of M with subcomplexes K_1 and K_2 triangulating H and X respectively. Let X' be the closed subspace of X homeomorphic to X and consisting of all but the end 1-simplexes of the second barycentric subdivision K_2'', as shown in figure 10. Now let $N = N(X', K'')$ be the derived neighborhood of X' in M. Note that N is a ball and that

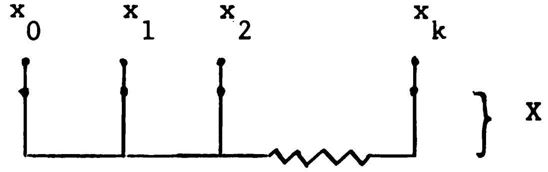

Figure 10

X is unknotted in N. Also $N \cap H$ and $N \cap \overline{M-H}$ are balls meeting in the common face $N \cap \text{bd}H$. Thus we can consider X embedded in the standard (3,2) ball pair ([13, chap. IV]) as shown in figure 11.

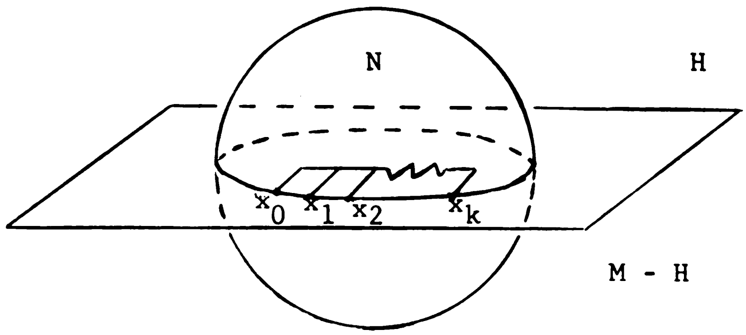

Figure 11

By lifting slightly all except the end vertices in a triangulation of X, we obtain a homeomorphism h of N onto itself which is the identity on bdN and such that h(X) is unknotted in $N \cap H$. Extending h by the identity to all of M gives a homeomorphism of M onto itself such that h(X) is unknotted in H and $h(x_i) = x_i$ for $i = 0,1,\ldots,k$.

Theorem 4.7 Any compact 3-manifold M with boundary is a PL t-double.

Proof Case 1: M has connected boundary. In this case bdM is a closed 2-manifold and there is a handle body H with $\text{bd}H \cong \text{bd}M$. Let M' be a closed 3-manifold formed by attaching H to M by a homeomorphism between their boundaries. Let Y (see figure 9) be a spine of H and K a triangulation of M' such that $Y \subset |K^1|$, the 1-skeleton. If N is a regular neighborhood of $|K^1|$ in M', it is a handle body and by theorem 3.7 is a t-half for M'. Writing

$Y = X_1 \cup X_2$ (figure 9), we need:

Lemma 4.8 There is a homeomorphism g' of M' onto itself such that $g'(X_1) \subset bdN$ and $g'(X_2)$ is unknotted in N.

Proof of lemma Since Y is contained in a 1-dimensional spine of the handle body N, we can assume that Y is embedded in a 3-ball with certain discs identified as shown in figure 12. Let N' be

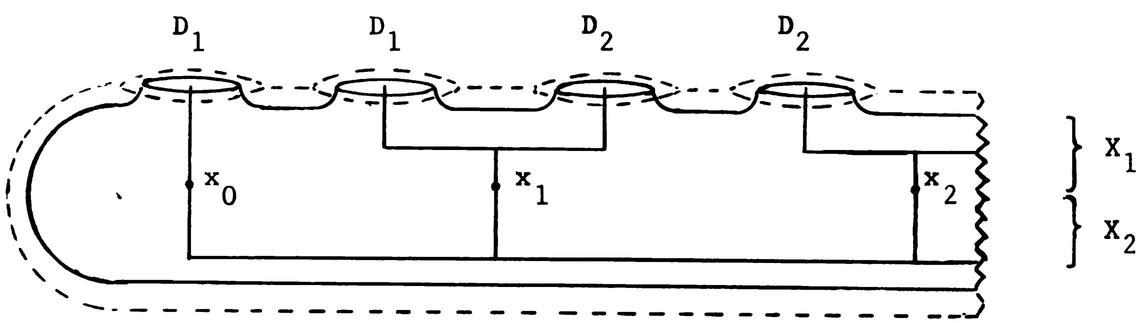

Figure 12. $Y \subset N \subset N'$.

a regular neighborhood of N in M'. By bringing X_1 straight forward (see figure 12) it is fairly easy to see a homeomorphism g' of N' onto itself which is the identity on bdN' and which satisfies the conditions $g'(X_1) \subset bdN$ and $g'(X_2)$ is unknotted in N. Extending g' to all of M' by the identity completes the lemma.

Continuing with the proof of the theorem, $\overline{M' - N}$ is a handle body and $g'(X_1)$ is contained in its boundary. By the discussion immediately preceeding the theorem, there is a homeomorphism g'' of M' onto itself such that $g''g'(X_1)$ is unknotted in $\overline{M' - N}$ and $g''|g'(X_2)$ is the identity. Thus the homeomorphism $g = g''g'$ of

M' is such that $g(X_2)$ is unknotted in N and $g(X_1)$ is unknotted in $\overline{M' - N}$.

Let J be a triangulation of M' such that $g(Y)$, N, and $\overline{M' - N}$ are all triangulated by subcomplexes. Denote $S = N(g(Y),J'')$, $S_1 = S \cap \overline{M' - N}$, $S_2 = S \cap N$. Then S_1 is a regular neighborhood of $g(X_1)$ in $\overline{M' - N}$ and S_2 is a regular neighborhood of $g(X_2)$ in N. Since N and $\overline{M' - N}$ are homeomorphic handle bodies, the unknottedness gives

$$N - \mathrm{int}_N S_1 \cong \overline{M' - N} - \mathrm{int}_{\overline{M'-N}} S_1.$$

Now $g^{-1}(S)$ is a regular neighborhood of Y in M' so there is a homeomorphism f of M' onto itself with $fg^{-1}(S) = H$. Letting $h = fg^{-1}$ we have

$$h(N - \mathrm{int}_N S_1) \cup h(\overline{M'-N} - \mathrm{int}_{\overline{M'-N}} S_2) \cong M' - \mathrm{int}_{M'} h(S) \cong M.$$

Thus $h(N - \mathrm{int}_N S_1)$ is a PL t-half for M, completing case 1.

Case 2: bdM is not connected. In this case bdM is a finite collection of connected, closed 2-manifolds. We sew a handle body H_i into each boundary component to form a closed manifold M'. Repeating the above argument for each H_i yields the homeomorphism g of M' and the rest of the argument is identical.

In higher dimensions results can be obtained for a certain set of manifolds with boundary which can be modified so that the results of chapter III are applicable. The final two results are of this nature.

Theorem 4.9 Let M be a compact, combinatorial manifold of odd dimension n, each of whose boundary components is an $(n-1)$-sphere. Then M is a PL Z-double.

Proof Let $S_1,\ldots,S_k$ be the boundary spheres. We form a closed, combinatorial n-manifold M' by attaching to each boundary component a PL n-ball B_i through a homeomorphism $bdB_i \to S_i$. For $i = 1,\ldots,k$, let p_i be a point of M' in the interior of B_i. By theorem 3.4 there exists a PL submanifold $N' \subset M'$ which is a homology half for M'. We can triangulate M' so that bdN' is triangulated by a subcomplex L. Let $A_1,\ldots,A_k$ be disjoint $(n-1)$-simplexes from L (we can always subdivide so that this is possible) and for each i choose $q_i \subset intT_i$. By homogeneity and the uniqueness of regular neighborhoods ([13, chap. III, p. 22]), there is a PL homeomorphism h of M' onto itself such that for each i, $h(p_i) = q_i$, $h(B_i) = |st(T_i,K)|$, and $(h(B_i),T_i)$ is an unknotted ball pair of codimension 1 ([13, chap. IV]).

It follows then that $h(B_i) \cap N'$ collapses across T_i to $h(bdB_i) \cap N'$ for each i. Thus

$$N' \searrow N' - (inth(B_1)\cup\ldots\cup inth(B_k)).$$

Similarly, $\overline{M' - N'} \searrow \overline{M' - N'} - (int\, h(B_1)\cup\ldots\cup inth(B_k))$. Now let $N = h^{-1}(N' - (inth(B_1)\cup\ldots\cup inth(B_k)))$. It is clear that $\overline{M - N} = h^{-1}(\overline{M' - N'} - (inth(B_1)\cup\ldots\cup inth(B_k)))$. Now N and N' have the same homotopy type, as do $\overline{M - N}$ and $\overline{M' - N'}$. Since N' was a PL Z-half for M', N is a PL Z-half for M, completing the proof.

Corollary 4.10 Suppose M is a compact, combinatorial manifold of even dimension n, each of whose boundary components is a sphere. Let M' be the closed manifold formed by sewing an n-ball into each boundary component. If $\chi(M')$ is even and M' is orientable over a field F, then M is a PL F-double.

Proof The proof is identical to that above except that we apply theorem 3.11 to obtain the F-half N' of M'.

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