PIECEWISE LINEAR INVOLUTIONS ON $P^2 \times S^1$

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ABSTRACT

PIECEWISE LINEAR INVOLUTIONS ON $P^2 \times S^1$

By

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This thesis is to classify the PL involutions on $P^2 \times S^1$. The main technique used is the P-equivariant surgery developed by Tollefson [10] and Tollefson and Kim [5]. If h is an involution on a 3-manifold M; we look for an appropriate surface S properly embedded in M for which $h(S) = S$ or $h(S) \cap S = \emptyset$, and then cut M along $S \cup h(S)$ to get a manifold M' and an induced involution $h' : M' \longrightarrow M'$, where h' is easier to classify than h. Pasting back what we cut help us to classify h.

In this thesis our manifold M is $P^2 \times S^1$ and the surface we are looking for is an embedded P^2 in $P^2 \times S^1$.

<u>Lemma 1</u>: Let $h : P^2 \longrightarrow P^2$ be a PL involution. Then $F \neq \emptyset$, moreover $F = \alpha \cup \{a\}$, where α is a non-separating simple closed curve in P^2.

<u>Lemma 2</u>: Let $h : P^2 \times S^1 \longrightarrow P^2 \times S^1$ be a PL involution. Then there exists a projective plane P embedded in $P^2 \times S^1$ such that $h(P) = P$ or $h(P) \cap P = \emptyset$.

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<u>Theorem 3</u>: Up to PL equivalence there are 3 PL involutions on $P^2 \times I$ with fixed point sets homeomorphic to (i) a projective plane, (ii) a disjoint union of a simple closed curve and a single point, or (iii) a disjoint union of an annulus and a simple arc.

<u>Theorem 4</u>: Up to PL equivalence there are six PL involutions on $P^2 \times S^1$ with fixed point sets homeomorphic to (i) $P^2 \cup P^2$, (ii) $P^2 \cup S^1 \cup *$, (iii) $S^1 \times S^1 \cup S^1$, (iv) $K \cup S^1$, (v) $S^1 \cup S^1 \cup S^0$ or (vi) $\emptyset$.

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ON $P^2 \times S^1$

By

Muhammad Arafat Natsheh

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To my mother and father

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INTRODUCTION

In this thesis we will classify the piecewise linear (PL) involutions on $P^2 \times S^1$. Tollefson [10] showed that up to PL equivalence there is only one free PL involution on $P^2 \times S^1$, which is the obvious one.

Since $S^2 \times S^1$ is the orientable double covering of $P^2 \times S^1$ we will make use of the PL involutions on $S^2 \times S^1$ which was classified by Tao [9], Kwun [8], Fremon [3], and Tollefson [10]. Moreover, we will use the P-equivariant surgery developed by Tollefson [10] and Tollefson and Kim [5]. The idea is if $h : M \longrightarrow M$ is an involution on a 3-manifold M, then we look for an appropriate surface S properly embedded in M for which $h(S) = S$ or $h(S) \cap S = \emptyset$ and then cut along $S \cup h(S)$ to get a manifold M' and an induced involution $h_1 : M' \longrightarrow M'$ which is easier to handle than M.

In case $h : P^2 \times S^1 \longrightarrow P^2 \times S^1$ we will be able to find a P^2 embedded in $P^2 \times S^1$ such that either $h(P^2) = P^2$ in case $F \neq \emptyset$ or $h(P^2) \cap P^2 = \emptyset$ in case $F = \emptyset$; and cutting along $P^2 \cup h(P^2)$ we get $M' \approx P^2 \times I$ and $h_1 : P^2 \times I \longrightarrow P^2 \times I$. In theorem 2.2 we classify all involutions h_1 and this leads to the classification of the involutions $h : P^2 \times S^1 \longrightarrow P^2 \times S^1$ where it turns out that there are up to PL equivalence five PL involutions with nonempty fixed point set homeomorphic to (i) $P^2 \cup P^2$, (ii) $P^2 \cup S^1 \cup *$, (iii) $S^1 \times S^1 \cup S^1$, (iv) $K \cup S^1$, or (v) $S^1 \cup S^1 \cup S^0$. This together with Tollefson's result of the free case completes the classification of all involutions on $P^2 \times S^1$. Thus up to PL equivalence there are six PL involutions on $P^2 \times S^1$.

CHAPTER I

INTRODUCTORY REMARKS AND P-EQUIVARIANT SURGERY

We work in the PL category, all manifolds are assumed to have a piecewise linear structure and all maps are to be piecewise linear maps unless otherwise stated.

S^n will denote the n-sphere, P^n the real projective n-space, K the Klein bottle, and I the closed unit interval $[0,1]$. We will use "s.c.c." for a "simple closed curve", $\chi(M)$ for the Euler characteristic of M, and if $h : M \longrightarrow N$ is a map then $\lambda(h)$ will denote the Lefschetz number of h. If M is an n-dimensional manifold, then a map $h : M \longrightarrow M$ is an involution if h is not the identity and $h \circ h$ = the identity map on M; $F(h)$ will denote the fixed point set of h.

A surface S in a 3-dimensional manifold M is properly embedded in M if $F \cap \partial M = \partial F$; two surfaces S_1 and S_2 properly embedded in M are called parallel if there is an embedding of $S_1 \times [-1,1]$ in M such that $S_1 = S_1 \times -1$ and $S_2 = S_1 \times 1$. A surface S properly embedded in M is two-sided if there is a neighborhood of S in M of the form $S \times [-1,1]$ with $S = S \times 0$ and $S \times [-1,1] \cap \partial M = \partial S \times [-1,1]$. A surface S properly embedded in M is one-sided if S does not separate any connected neighborhood of S.

<u>Definition 1.1</u>: Let S be a 2-sided surface in a 3-manifold M. The manifold M' obtained by <u>splitting</u> M at S

is the manifold whose boundary contains two copies of S S_1 and S_2 such that there is a natural projection $p : (M', S_1 \cup S_2) \longrightarrow (M, S)$ with $P|M - (S_1 \cup S_2)$ is a homeomorphism onto $M-S$ and M' is homeomorphic to $M - (S \times (-1,1))$. If S is one-sided, the manifold N obtained by splitting M at S is the manifold whose boundary contains S_1 a double cover of S and $N-S_1$ is homeomorphic to $M-S$.

Definition 1.2: Let h be an involution on a manifold M. The quotient space M/h of M which is obtained by identifying x in M with $h(x)$ is called the orbit space of h and the quotient map $q : M \text{---} M/h$ is called the orbit map.

Definition 1.3: Let $h_1, h_2 : M \longrightarrow M$ be two homeomorphisms. h_1 and h_2 are equivalent if there is a PL homeomorphism $T : M \longrightarrow M$ such that $h_1 T = T h_2$, in such a case T is called PL equivariant with respect to h_1 and h_2.

Definition 1.4: [Tollefson [10] and Tollefson and Kim [5]] Let $h : M \longrightarrow M$ be a PL involution on the 3-manifold M, with fixed point set F. Let S be a surface properly embedded in M. S is said to be in h-general position modulo F if (i) both $(S, \partial S)$ and $(h(S), \partial h(S))$ are in general position with respect to F, (ii) $S-F$ and $h(S)-F$ are in general position, and (iii) all cuts among S, $h(S)$ and F are locally piercing cuts.

We observe that any properly embedded surface in M can be put into h-general position modulo F by a series of arbitrarily small isotopies, and if S meets F at a nonpiercing point or curve then S and $h(S)$ can be simultaneously pulled away from F at this place. This can be done by restriction of h to a small invariant regular neighborhood of F.

Let $h: P^2 \times S^1 \longrightarrow P^2 \times S^1$, let Σ be the set of all projective planes embedded in $P^2 \times S^1$ which are either invariant and in general position with respect to F or in h-general position modulo F. For any $P \in \Sigma$, define the <u>complexity</u> of P, $c(P) = (a, b)$, where a = the number of components of $[P \cap h(P)] - F$ and b = the number of components of $P \cap F$; we order the complexities in a lexigraphical order.

<u>Remarks 1.5</u>: Any simple closed curve in P^2 either bounds a disk and separates P^2 or does not bound and is nonseparating. A nonseparating s.c.c. in P^2 is covered by a s.c.c. in S^2 (the orientable double cover of P^2) which is invariant under the covering transformation; hence any two nonseparating simple closed curves in P^2 has a nonempty intersection.

Any P^2 embedded in $P^2 \times S^1$ does not separate, for P^2 does not bound any manifold.

Any embedded $P^2 \subset P^2 \times S^1$ is two-sided and $P^2 \times S^1 - P^2$ is homeomorphic to $P^2 \times (0, 1)$. For if $p: S^2 \times S^1 \longrightarrow P^2 \times S^1$ is the orientable double covering then $p^{-1}(P^2) = S^* \subset S^2 \times S^1$, where S^* is a two sphere which does not bound a 3-cell; and $S^* \times [-1, 1]$ is a 2-sided regular neighborhood of S^* which double cover $P^2 \times [-1, 1]$; moreover, $S^2 \times S^1 - S^*$ is homeomorphic to $S^2 \times (0,1)$ which double cover $P^2 \times S^1 - P^2$, hence $P^2 \times S^1 - P^2 \approx P^2 \times (0,1)$. Let $i: P^2 \longrightarrow P^2 \times S^1$ be an embedding then $i_*: \pi_1(P^2) \longrightarrow \pi_1(P^2 \times S^1)$ is a monomorphism, $j_*: \pi_1(P^2) \longrightarrow \pi_1(P^2 \times I)$ is a monomorphism too, where j is an embedding.

Lemma 1.6: Let $h: P^2 \longrightarrow P^2$ be a PL involution with $\mathrm{Fix}(h) = F$. Then $F \neq \emptyset$; moreover, $F = \alpha \cup \{a\}$ where α is a nonseparating s.c.c. in P^2.

Proof: Since $\lambda(h) = 1 \neq 0$, then $F \neq \emptyset$, and since F is a submanifold of P, F is a finite number of disjoint simple closed curves and points.

By Conner [1], $\chi(F) = \lambda(h)$, hence $\chi(F) = 1$; and by Floyd [2] $\sum \dim H_i(F; Z_2) \leq \sum \dim H_i(P^2; Z_2) = 3$. Hence F has to contain a single point a and may have at most one s.c.c.

If α is a s.c.c. in F then α cannot bound a disk because if so then the disk is invariant and its boundary in F hence the whole disk is contained in F, which cannot happen. So if $\alpha \subset F$, it has to be a nonseparating s.c.c.

Let J be any nonseparating s.c.c. in P^2 such that $a \notin J$ and $J, h(J)$ are in general position. $J \cap h(J) \neq \emptyset$ and either $J \subset F$ or $J \cap h(J)$ is an odd number of points, for by considering the commutative diagram:

$$\begin{tikzcd} H_1(P^2;Z_2) \otimes H_1(P^2;Z_2) \arrow[d, "\approx"] \arrow[dr] & \\ H^1(P^2;Z_2) \otimes H^1(P^2;Z_2) \arrow[r] & Z_2 \end{tikzcd}$$

if $[J]$ generates $H_1(P^2, Z_2)$, $\bar{J}$ its dual and z_2 is the generator of $H_2(P^2; Z_2)$ then $\langle [\bar{J}] \cup [h^* \bar{J}], z_2 \rangle = 1 \in Z_2$, hence the intersection number of J and $h(J) \equiv 1 \pmod 2$, i.e. an odd integer.

h acts as a permutation of order 2 on the points $J \cap h(J)$ whose number is odd, hence there is a fixed point $x \in J \cap h(J)$.

Therefore there is a nonseparating s.c.c. $\alpha \subset F$ such that $x \in \alpha$ and $F = \alpha \cup \{a\}$.

Lemma 1.7: (P-equivariant surgery) Let $h : P^2 \times S^1 \longrightarrow P^2 \times S^1$ be a PL involution. Then there exists a $P^2 \subset P^2 \times S^1$ such that $h(P^2) = P^2$ or $h(P^2) \cap P^2 = \emptyset$.

Proof: Let Σ be the set of all projective planes embedded in $P^2 \times S^1$ which are invariant and in general position with respect to F, or in h-general position modulo F. If there is a $P^2 \in \Sigma$ disjoint from F, then we choose P such that its complexity is minimal among all such P's in Σ which are disjoint from F. If every $P \in \Sigma$ meets F then choose an arbitrary $P \in \Sigma$ with minimal complexity.

We argue that $c(P) = (0,0)$, for if $c(P) > (0,0)$ we can obtain $P' \in \Sigma$ of the same type with lower complexity by performing P-equivariant surgery once on P. Hence our original choice of P must satisfy $h(P) = P$ or $h(P) \cap P = \emptyset$.

Choose $P \in \Sigma$ of minimal complexity and suppose $c(P) \searrow (0,0)$. In $P \cap h(P)$ we have the following types of intersection curves: (a) an isolated point which is in F, (b) a s.c.c. in $P-F$, (c) a s.c.c. with one point in F, (d) a s.c.c. in F, (e) a simple arc with its end points in F.

First, we rule out case (a) using Tollefson argument [10, lemma 2]. If x is isolated in F then we move P and $h(P)$ simultaneously off x. If x is a point of a one-dimensional component of F, then let N be an invariant 3-cell neighborhood of x such that $N \cap F$ is an arc. Then $h|N$ is simply a rotation about this arc, we adjust P and $h(P)$ slightly so that $P \cup h(P)$ is in general position with respect to ∂N. There are simple closed curves in $P \cap N$ and $h(P) \cap N$ that bound innermost disks (containing x) $R \subset P$ and $Q \subset h(P)$. $R \cup Q$ separates N into three components U, V, W, where $R \subset \partial U$, $h(U) = V$ and $h(W) = W$. Clearly $F \cap U \subset W$.

Let D be a disk close to and parallel to R such that $\operatorname{Int} D \subset U$ and $\partial D = \partial R$. Define $P' = (P - R) \cup D$. The only difference between $P \cap h(P)$ and $P' \cap h(P')$ is that we have removed the point x, but because of our choice of P this case cannot appear.

Second: There is a s.c.c. J in $P \cap h(P)$ of type (b), (c), (d) or a simple arc α of type (e); since $\alpha \neq h(\alpha)$, $\alpha \cup h\alpha$ is a s.c.c. in $P \cap h(P)$.

<u>Case I</u>: There is a s.c.c. $J \subset P \cap h(P)$ which bounds an innermost disk E in $h(P)$; where a surface E is innermost in $h(P)$ if $E \cap P = \emptyset$ and $\partial E \subset P \cup h(P)$. There always exists an innermost disk, for if J bounds the disk E in $h(P)$ which is not innermost we can find a s.c.c. $J' \subset E$ where $J' \subset P \cap h(P)$ and bounds an innermost disk E' in $h(P)$. Hence $J \subset P \cap h(P)$ bounds an innermost disk E in $h(P)$. $J \cap F$ may be one of the following: $\emptyset$, J, a single point, or a couple of points.

Since J bounds E in $h(P)$, it bounds a disk in $P^2 \times S^1$ and hence it bounds a disk in P, for i_* and j_* are monomorphisms in:

$$\pi_1(P) \xrightarrow{i_*} \pi_1(P^2 \times S^1) \xleftarrow{j_*} \pi_1(h(P))$$

Hence J separates P into two components E_1 and E_2, let E_1 be the disk. We have either $h(J) = J$ or $h(J) \neq J$ in either case let U be a small regular neighborhood of E. In U find a disk E' parallel to E and such that:
(i) $E' \cap h(P) = F \cap J$, (ii) $\partial E' \cup J$ bounds a semi-degenerate annulus A contained in P, which is pinched along $J \cap F$, (iii) the interior of the 3-cell bounded by $E \cup A \cup E'$ is disjoint from $P \cup h(P)$. Take $P = E' \cup (E - A)$. If P is not already in general position with respect to F along the curve J then move P slightly off J where necessary to achieve this general position. Then $P' \in \Sigma$ and $c(P') < c(P)$.

Repeating the last process we can assume that there are no s.c.c. α in $P \cap h(P)$ such that α bounds a disk in $h(P)$; and hence we can assume all s.c.c.'s in $P \cap h(P)$ do not bound in $h(P)$ and intersect in a single point x, otherwise we are going to find a s.c.c. which bounds an innermost disk in $h(P)$.

Case II: Let E be an innermost surface in $h(P)$ bounded by two such nonseparating s.c.c.'s α and β. E is a pinched annulus pinched at x. Now $\alpha \cup \beta \sim 0$ in $h(P)$ and this implies that $h(\alpha) \cup h(\beta) \sim 0$ in P. So $h(\alpha) \cup h(\beta)$ bounds a pinched annulus in P.

Subcase (a): $h(E) \cap E = x$ a single point. Let U be a small regular neighborhood of E. In U find a pinched annulus E' (pinched at x) which is parallel to E and such that: (i) $E' \cap h(P) = x$, (ii) $\partial E' \cup \alpha \cup \beta$ bounds two pinched annuli (each pinched at x) A_1, A_2 both in P, (iii) the interior of the pinched solid torus (pinched at x) that is bounded by $E \cup E' \cup A_1 \cup A_2$ is disjoint from $P \cup h(P)$. Now $\alpha \cup \beta$ separate P into two components E_1 and E_2 let E_2 contain $h(\alpha) \cup h(\beta)$. Take $P' = E' \cup [E_2 - (A_1 \cup A_2)]$, in this way we get rid of α, β, $h(\alpha)$, $h(\beta)$ in $P \cap h(P)$.

<u>Subcase (b)</u>: $h(E) \cap E = \beta$, $h(\alpha) \cap \alpha = x$

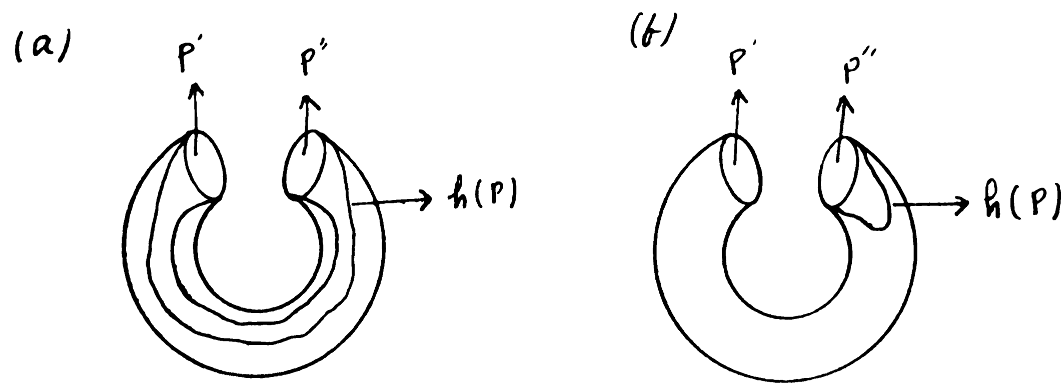
Let U be a small regular neighborhood of E. In U find a pinched annulus E' (pinched at x) parallel to E and such that (i) $E' \cap h(P) = \beta$, (ii) $\partial E' \cup \alpha$ bounds a pinched annulus $A \subset P$, (iii) the interior of the pinched solid torus which is bounded by $E \cup A \cup E'$ is disjoint from $P \cup h(P)$. $\alpha \cup \beta$ separate P into two components E_1 and E_2, let E_2 contain $h(\alpha)$. Take $P' = E' \cup (E_2 - A)$, hence we get rid of α and $h(\alpha)$ in $P \cap h(P)$.

<u>Subcase (c)</u>: $E \cap h(E) = \alpha \cup \beta$ where $h(\alpha) = \beta$ we get rid of α and β the same way as in subcase (a).

<u>Subcase (d)</u>: Now we can assume that every s.c.c. $\alpha \subset P \cap h(P)$ is nonseparating in $h(P)$ and so in P and $h(\alpha) = \alpha$. We have two cases: (i) there exists more than one α (ii) there is only one α.

(i) If E is an innermost surface in $h(P)$ which is bounded by $\alpha \cup \beta$ where $h(\alpha) = \alpha$, $h(\beta) = \beta$, let $h(E) = E_1$ in P. $E \cup E_1$ is either a projective plane which is invariant and we are done (finding invariant projective plane embedded in $P^2 \times S^1$) or $E \cup E_1$ bounds a pinched solid torus T which is invariant under h. Hence there exists a pinched annulus $E' \subset T$ such that $\partial E' = \alpha \cup \beta$ and $h(E') = E'$. If $h(P) = E_1 \cup E_2 --- \cup E_n$, $h(\alpha_i) = \alpha_i$ where every α_i is a simple closed curve in ∂E_i, then let $P' = E'_1 \cup E'_2 \cup --- \cup E'_n$ $P' \in \Sigma$ and $h(P') = P'$.

(ii) $P \cap h(P) = \alpha$ a single nonseparating s.c.c. in P (and so in $h(P)$). Split $P^2 \times S^1$ along P we get a space homeomorphic to $P^2 \times I$ and the following two cases:



Since $P - \alpha \approx \mathrm{Int}\, D^2$, an open disk (a) cannot happen because $h(P) - \alpha$ is an annulus, (b) only may happen and in this case P and $h(P)$ does not cross each other. Now $\alpha \cap F = \emptyset$, α or 2 points. $\alpha \cap F = \emptyset$ cannot happen since P is in h-general position modulo F. In case $\alpha \subset F$ we move P and $h(P)$ simultaneously off α, to get $P \cap h(P) = \emptyset$. So if $\alpha \cap F = \{x,y\}$ let U be a small regular neighborhood of α pinched along x and y. Let A be the semi-degenerate annulus contained in ∂U such that $\partial A = \partial U \cap P$ and $A \cap h(P) = \emptyset$. Let $A' = U \cap P$ and put $P' = A \cup (P - A')$.

If P' is not in general position with respect to F along α, move P' slightly off $\alpha \cap F$ to achieve the general position. $P' \in \Sigma$ and $c(P') < c(P)$. Hence in all cases we can achieve the conclusion of the theorem, i.e. there is a $P \in \Sigma$ such that $h(P) = P$ or $h(P) \cap P = \emptyset$.

CHAPTER II

PL INVOLUTIONS ON $P^2 \times S^1$

In this chapter we will prove the main theorem of the classification of PL involutions on $P^2 \times S^1$. First we classify the PL involutions on $P^2 \times I$, then using this we prove our final theorem. We will use for P^2 the more convenient one of the two notations: (i) $P^2 = S^2/\sim$ where $x \sim -x \ \forall \ x \in S^2$. or (ii) $P^2 = D^2/\sim$ where $D^2 = \{\rho z \in C \mid |z| = 1, o \leq \rho \leq 1\}$ and $z \sim -z$.

<u>Lemma 2.1</u>: Let $h: P^2 \times I \longrightarrow P^2 \times I$ be a PL involution. Then there is an annulus $A \subset P^2 \times I$, whose boundary components are non-separating simple closed curves in $P^2 \times o$ and $P^2 \times 1$ and such that $h(A) = A$.

<u>Proof</u>: Let α be a non-separating s.c.c. in P and let $S = \alpha \times I$. $(\alpha \times I, \alpha \times o \cup \alpha \times 1)$ can be deformed in $(P^2 \times I, P^2 \times o \cup P^2 \times 1)$ so that $S \approx \alpha \times I$ is either invarient and in general position with respect to F (and hence we are done) or in h-general position modulo F.

Let Σ be the set of all annuli $S \subset P^2 \times I$. which are in h-general position modulo F and such that the boundary components of every S are non-separating simple colsed curves in $P^2 \times o$ and $P^2 \times 1$. Define the complexity $c(S)$ as before and choose $S \in \Sigma$ of minimal complexity.

Again as in lemma 1.7 we choose E an innermost surface in $h(S)$. We have the following cases:

Case I: E is a disk in $\mathrm{Int}(h(S))$. Let $J = \partial E$. $J \cap F$ may be one of the following : $\emptyset$, J, a simple arc, a point, two or more components each is a point or a simple arc.

J separates A into two components E_1 and E_2 and since $J \subset \mathrm{int}(h(S))$, $J \subset \mathrm{int}\, S$. If E_1 and E_2 are annuli then J is homotopic in $P^2 \times I$ to one of the boundary components of S, but each of the boundary components of S is not null homotopic in $P^2 \times I$, hence J is not null homotopic in $P^2 \times I$, a contradiction since $J = \partial E$, E is a disk in $h(S)$ and so in $P^2 \times I$. Hence one of E_1, E_2 has to be a disk, let E_1 be the disk. We handle this case the same way as in Lemma 1.7 Case I.

Case II: E is a disk in $h(S)$ which meets one boundary component of $h(S)$. Let $J = S \cap E$, $B = E \cap \partial h(S)$. $J \cap F$ is the same as in case I. Let U be a small regular neighborhood of E. In U find a disk E' parallel to E and such that (i) $E' \cap h(S) = J \cap F$, (ii) $\partial E' \cup J \cup B$ bounds a semi-degenerate annulus $A \subset S$ pinched along $J \cap F$, (iii) the interior of the 3-cell bounded by $E \cup A \cup E'$ is disjoint from $S \cup h(S)$. Now J separates S into two components a disk E_1 and an annulus E_2. Let $S' = E' \cup (E_2 - A)$. If S' is not in general position with respect to F along J, move S' slightly off J to achieve this general position.

Case III: Now we can assume that $S \cap h(S)$ is a finite number of disjoint simple arcs each one of them starts at a point in $P^2 \times o$ and ends at a point in $P^2 \times 1$. The number of these simple arcs is odd because the number of components of $S \cap h(S)$ equals 1(mod 2), (same reasoning as in lemma 1.6).

If the number of these simple closed curves is greater than one, let E be an innermost disk in $h(S)$ bounded by two arcs J_1 and J_2, $J_1 \cup J_2 = E \cap S$.

Let $\beta_o = h(S) \cap P^2 \times o = B_o \cup S_o$, where $B_o = \beta_o \cap E$, and let $\alpha_o = S \cap P^2 \times o = C_o \cup \gamma_o$, where $\gamma_o \cup B_o$ is a non-separating simple closed curve in $P^2 \times o$ (this is possible since both α_o and β_o are non-separating simple closed curves in $P^2 \times o$).

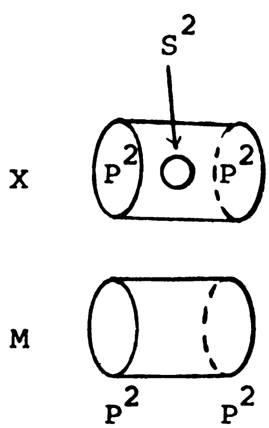
Let U be a small regular neighborhood of E. In U choose a disk E' parallel to E such that (i) $E' \cup h(S) = (J_1 \cup J_2) \cap F$, (ii) $\partial E \cup \partial E'$ bounds a semi-degenerate annulus A pinched along $J \cap F$, (iii) the interior of the 3-cell bounded by $E \cup E' \cup A$ is disjoint from $S \cup h(S)$. $J_1 \cup J_2$ separates S into two disks E_1 and E_2. let $\gamma_o \subset E_1$ then take $S' = E' \cup (E_1 - A)$. If S' is not in general position with respect to F along $J_1 \cup J_2$, move S' slightly off $F \cap (J_1 \cup J_2)$ to achieve this general position.

Repeating this process one finally gets $h(S) \cap S = J$ a simple arc, since we cannot get rid of all of them for $h(S) \cap S \neq \emptyset$.

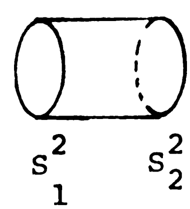
Now let $P : S^2 \times [0,1] \longrightarrow P^2 \times [0,1]$ be the covering map, $S \cap h(S) = J$ a simple arc. $P^{-1}(S) = S^*$ an annulus in $S^2 \times I$ which is invariant under the covering transformation. Same for $P^{-1}(h(S))$. $S^* \cap P^{-1}(h(S)) = \tilde{J}_1 \cup \tilde{J}_2$ two copies of J. Cut along S we get a manifold $M' \approx D^2 \times [0,1]$ and the disk $h(S)' \subset M'$ is $h(S)$ cut along J to get J_1, J_2 two copies of J in $\partial h(S)'$. Cutting again along $h(S)'$ we get two manifolds each homeomorphic to $D^2 \times I$ and $\partial D^2 \times I$ is homeomorphic in each one of these to $h(S)' \cup S'$ which are pasted along two copies of J.

Now $P^2 \times I - (S \cup h(S))$ consists of two components A and B. If $h(A) = B$ then $F \subset J$ and either $F = J$ or F is a point in $\operatorname{Int} J$. If $F = J$ then $h(P^2 \times o) = P^2 \times o$ such an involution $h|P^2 \times o$ has fixed points other then that in J (by lemma 1.6) hence $F = J$ cannot happen. If F = a point in $\operatorname{Int} J$; then we rule this out too.

Let B be a small invariant 3-cell around the fixed point such that $B \subset \operatorname{int} P^2 \times I$. Let $X = P^2 \times I - \operatorname{int} B$. h acts freely on X, let M be the orbit space and $q : X \longrightarrow M$ the quotient map, q is a 2 - 1 covering map.



$$\begin{tikzcd} \tilde{X} \arrow[d, "q_1"'] \arrow[rd, "p_1"] & \\ X \arrow[d, "q"'] & \tilde{M} \\ M \arrow[ru, "p"'] & \end{tikzcd}$$



$\tilde{X} = S^2 \times [0,1] - (\tilde{B}_1 \cup \tilde{B}_2)$ is the covering space of X, where each of $\tilde{B}_1$ and $\tilde{B}_2$ are mapped onto B by the covering $S^2 \times I \longrightarrow P^2 \times I$. Let $\tilde{M}$ be the orientable double covering of M.

Since $\tilde{X}$ is simply connected it is a universal cover of M hence there is a covering $P_1 : \tilde{X} \longrightarrow \tilde{M}$ such that the diagram commutes i.e. $PP_1 = qq_1$.

$\tilde{X} = S^3 - (\tilde{B}_1 \cup \tilde{B}_2 \cup \tilde{B}_3 \cup \tilde{B}_4)$ and P_1 is $2-1$ covering projection. $P_1(S^2 \times 0) = P_1(S^2 \times 1) = S_2^2 \subset \partial\tilde{M}$ and $P_1(\partial\tilde{B}_1) = P_1(\partial\tilde{B}_2) = S_1^2 \subset \tilde{M}$, hence P_1 can be extended to a covering projection of S^3 which implies that $\tilde{M} \approx P^3 - (B_1^o \cup B_2^o)$. The covering transformation on $\tilde{M}$ is a free involution with both S_1^2 and S_2^2 are invarient spheres. This involution can be extended to $P^3 \approx \tilde{M} \cup B_1^o \cup B_2^o$ such that $T(B_1) = B_1$ and $T(B_2) = B_2$ and T is free on $\tilde{M}$; hence the fixed point set of T (in P^3) consists of a couple of points one in $\operatorname{Int} B_1$ and the other in $\operatorname{Int} B_2$, such an involution cannot happen, see [Kwun 6, 7 and Kim 4].

Hence $h(A) = A$. Cutting along $S \cup h(S)$ we get two manifolds $\tilde{A}$ and $\tilde{B}$ each homeomorphic to $D^2 \times [0,1]$. Now $\partial\tilde{A} \supset (S-J)' \cup J_1 \cup J_2 \cup (h(S)-J)' = L$ where $(S-J)'$ comes from $S - J$ after the cutting, and the same for $(h(S)-J)'$; J_1 and J_2 are two copies of J. $h : P^2 \times I \longrightarrow P^2 \times I$ induces $h' : \tilde{A} \longrightarrow \tilde{A}$ defined as follows:

for $x \in \tilde{A} - L$ let $h'(x) = h(x)'$

for $x' \in (S-J)' \cup (h(S)-J)'$ let $h'(x') = [h(x)]'$

for $x \in J_1$ let $h'(x) = h(x)'$ in J_2

and for $x \in J_2$ let $h'(x) = h(x)'$ in J_1.
$h' : \tilde{A} \longrightarrow \tilde{A}$ is an involution with $h'(J_1) = J_2$ and $h'((S - J)') = (h(S) - J)'$, hence there exists a disk $D \subset \tilde{A}$ which is invariant under h' and $J_1 \cup J_2 \subset \partial D$. Pasting back what we cut we get $P^2 \times I$ and D goes back to an invariant annulus S.

Hence always there is an invariant annulus $S \subset P^2 \times I$ whose boundary components are non-separating simple closed curves in $P^2 \times o$ and $P^2 \times 1$, and S is in general position with respect to F.

Theorem 2.2: Let $h : P^2 \times I \longrightarrow P^2 \times I$ be a PL involution, then h is equivalent to one of the following involutions:

(i) $h_1([\rho z, t]) = [\rho z, 1 - t]$ with $F \approx P^2$

(ii) $h_2([\rho z, t]) = [-\rho z, t]$ with $F \approx S^1 \times I \cup I$

(iii) $h_3([\rho z, t]) = [-\rho z, 1 - t]$ with $F \approx S^1 \cup *$

Proof: Since $\lambda(h) = 1 \neq 0$, then $F \neq \emptyset$. By Conner [1] $\lambda(h) = \chi(F)$, so $\chi(F) = 1$. By Floyd [2] $\sum \dim H_i(F; Z_2) \leq \sum \dim H_i(P^2 \times I; Z_2) = 3$. Hence a component of F may be P^2, an annulus, a mobius band, a simple arc, a s.c.c., or a point.

Since $\chi(F) = 1$ and $\sum \dim H_i(F; Z_2) \leq 3$, so if we have a mobius band we have both of $P^2 \times o$ and $P^2 \times 1$ invariant and hence F will contain 3 components or more which violates one or more of the above conditions. Hence this case cannot happen.

Case I: $P^2 \subset F$, then since $\sum \dim H_i(P^2;Z_2) = 3$ we have $F = P^2 \subset \mathrm{Int}(P^2 \times I)$, for F is a properly embedded submanifold of $P^2 \times I$. Now F separates $P^2 \times I$ into two components A and B each homeomorphic to $P^2 \times I$ and $h(A) = B$. Let t be any homeomorphism from A onto $P^2 \times [0, \frac{1}{2}]$ such that $f(F) = P^2 \times \frac{1}{2}$ and let $h_1 : P^2 \times I \longrightarrow P^2 \times I$ be defined by $h_1([\rho z, t]) = [\rho z, 1-t]$. Define $T : P^2 \times I \longrightarrow P^2 \times I$ as follows: for $u \in A$, let $T(u) = t(u)$, and for $u \in B$, let $T(u) = h_1 th(u)$. Then $hT = Th_1$ and hence h is equivalent to h_1.

Case II: There exists a simple arc component $J \subset F$, then $h(P^2 \times 0) = P^2 \times 0$ and $h(P^2 \times 1) = P^2 \times 1$. Hence $\exists \alpha_i$ a non-separating curve in $P^2 \times i$, $i = 0, 1$ such that $\alpha_0 \cup \alpha_1 \subset F$. and since $\sum \dim H_i(F;Z_2) \leq 3$ there is an annulus $A \subset F$ with $\partial A = \alpha_0 \cup \alpha_1$. $A \cup J = F$ because we can not have anything else in F. Cut along A to get $D^2 \times I$ and consider the following diagram:

$$\begin{array}{ccc} D^2 \times I & \xrightarrow{\bar{h}} & D^2 \times I \\ q \downarrow & & \downarrow q \\ P^2 \times I & \xrightarrow{h} & P^2 \times I \end{array}$$

Define $\bar{h}(z,t)$ to be equal to $q^{-1}hq(z,t)$ for $(z,t) \notin \partial D^2 \times I$ and for $(z,t) \in \partial D^2 \times I$ let $\bar{h}$ be the covering transformation. Suppose $h' : P^2 \times I \longrightarrow P^2 \times I$ be any other involution with $F' = A' \cup J'$ then define $\bar{h}' : D^2 \times I \longrightarrow D^2 \times I$ as above.

Since $\bar{h} \sim \bar{h}'$ on $D^2 \times I$ there exists $t : D^2 \times I \longrightarrow D^2 \times I$ such that $\bar{h}t = t\bar{h}'$. Consider the following diagram:

$$\begin{tikzcd}
D^2 \times I \arrow[rr, "\bar{h}"] \arrow[dr, "t"] \arrow[dd, "q"'] & & D^2 \times I \arrow[dr, "t"] \arrow[dd, "q" near start] & \\
& D^2 \times I \arrow[rr, "\bar{h}'" near start] \arrow[dd, "q'" near start] & & D^2 \times I \arrow[dd, "q'"] \\
P^2 \times I \arrow[rr, "h" near start] \arrow[dr, "t'"'] & & P^2 \times I \arrow[dr, "t'"] & \\
& P^2 \times I \arrow[rr, "h'"] & & P^2 \times I
\end{tikzcd}$$

Define $t' : P^2 \times I \longrightarrow P^2 \times I$ as follows:

for $[z,t] \in P^2 \times I - A$ let $t'([\rho z,t]) = q't'q^{-1}([\rho z,t])$

for $u \in A$ let $q^{-1}(u) = \{u_1, u_2\}$ we have $\bar{h}(u_1) = u_2$ and since $\bar{h}'t(u_1) = t\bar{h}(u_1) = t(u_2)$ then $q'(t(u_1)) = q'(t(u_2))$, define $t'(u) = q'(t(u_1)) = q'(t(u_2))$

$\therefore$ the last diagram commutes and hence $h \sim h' \sim h_2$ where $h_2([\rho z,t]) = [-\rho z,t]$.

Case III: There is an isolated point $a \in F$. As in lemma 2.1 $F \neq a$ and we cannot have an annulus as another component of F because then there would be more components in F which violates $\sum \dim H_i(F;Z_2) \leq 3$. Hence the only possibility is to have α a s.c.c. in F and hence $F = \alpha \cup \{a\}$.

Let $x \in \alpha$ and $P^{-1}(x) = \{x_1, x_2\} \subset S^1 \times I$ then there exists a unique involution $\bar{h}$ such that the diagram commutes

$$\begin{array}{ccc} (S^2 \times I, x_1) & \xrightarrow{\bar{h}} & (S^2 \times I, x_1) \\ \downarrow P & & \downarrow P \\ (P^2 \times I, x) & \xrightarrow{h} & (P^2 \times I, x) \end{array}$$

If $\alpha \sim o$ in $P^2 \times I$ then $p^{-1}(\alpha) = \alpha_1 \cup \alpha_2 \subset F(\bar{h})$ and there is no such involution with α_1, α_2 as components of the fixed point set [5, lemma 6.3] and hence $\alpha \not\sim o$ in $P^2 \times I$. By Lemma 2.1 there is an invariant annulus $A \subset P^2 \times I$ whose boundary components are non-separating s.c.c.'s in $P^2 \times o$ and $P^2 \times 1$. $A \cap F \neq \emptyset$ otherwise $h|P^2 \times I - (A \cup P^2 \times \{o,1\})$ is an involution of the open 3-cell with fixed point set $\alpha \cup \{a\}$, such an involution does not occur. We have either $\alpha \subset A$, or $\alpha \cap A = x$ and $a \in A$. ($A \cap \alpha$ has an odd number of components and fix $(h|A)$ is either a circle or a couple of points.)

<u>Subcase I</u>: $\alpha \subset A$. Let $P^{-1}(a) = \{a_1, a_2\}$. There is a unique involution $\bar{h}$ that makes the following diagram commute:

$$\begin{array}{ccc} (S^2 \times I, a_1) & \xrightarrow{\bar{h}} & (S^2 \times I, a_1) \\ \downarrow P & & \downarrow P \\ (P^2 \times I, a) & \xrightarrow{h} & (P^2 \times I, a) \end{array}$$

$P^{-1}(A) = A^*$ an annulus for which $\bar{h}(A^*) = A^*$ and $T(A^*) = A^*$ where T is the covering transformation.

Let $P^{-1}(\alpha) = \alpha^*$, $\bar{h}(\alpha^*) = \alpha^*$ and $T(\alpha^*) = \alpha^*$.

$S^2 \times I - A^* = K_1 \cup K_2$, $\bar{K}_1 \approx \bar{K}_2 \approx D^2 \times I$, let $a_1 \in K_1$, $a_2 \in K_2$. $\bar{h}(K_1) = K_1$ and since $\bar{h}(a_1) = a_1$ we have $h_1(a_2) = a_2$ and $F(\bar{h}) = \{a_1, a_2\}$. Now $\bar{h}|\bar{K}_1$ is an involution with α^* invariant and $\alpha^* \subset \partial\bar{K}_1$ and has fixed point set $\{a_1\}$, hence there is an invariant disk D with $\partial D = \alpha^*$ and $a_1 \in D$.

Let $S = D \cup T(D)$ then $T(S) = S$ and $\bar{h}(S) = S$ for $\bar{h}T = T\bar{h}$. $p(S) = P^2$ invariant in $P^2 \times S^1$ and $\alpha \cup \{a\} \subset P^2$.

<u>Subcase II</u>: Let $\alpha \cap A = x$. Choose $y \in \alpha - A$, and left h to $\bar{h}$, as before the diagram commutes:

$$\begin{array}{ccc}
(S^2 \times I, y_1) & \xrightarrow{\bar{h}} & (S^2 \times I, y_1) \\
P \downarrow & & \downarrow P \\
(P^2 \times I, y) & \xrightarrow{h} & (P^2 \times I, y)
\end{array}$$

$\bar{h}(A) = A^*$ and $T(A^*) = A^*$. Let $P^{-1}(\alpha) = \alpha^*$ and let $S^2 \times I - A^* = K_1 \cup K_2$, so $\bar{K}_1 \approx \bar{K}_2 \approx D^2 \times I$ and $\bar{h}(K_1) = K_1$ $T(K_1) = K_2$ hence $\alpha^* = \mathrm{Fix}\,\bar{h}$. Let $\alpha^* \cap \bar{K}_1 = \alpha_1$ and $P^{-1}(a) = \{a_1; a_2\} \subset A^*$. $\bar{h}|\bar{K}_1$ is an involution on $\bar{K}_1 \approx D_2 \times I$ with fixed point set $= \alpha_1$ a simple arc. Since $\bar{h}(A^*) = A^*$ $\exists$ a s.c.c. β such that $(\alpha_1 \cap A^*) \cup \{a_1, a_2\} \subset \beta$ and $T(\beta) = \beta$. β consists of two simple arcs β_1 and β_2 where $T(\beta_1) = \beta_2 = \bar{h}(\beta_1)$ where the end points of β_1, β_2 are $\alpha_1 \cap A^*$. Let E be any disk in $\bar{K}$ with $\partial E = \alpha_1 \cup \beta_1$. $h(E)$ is a disk with $\partial h(E) = \alpha_1 \cup \beta_2$ and let $E, h(E)$ be in general position. $E \cap h(E) = \alpha_1 \cup$ a finite number of simple closed curves each bounds a disk in $h(E)$ we can pull

E and $h(E)$ apart along these s.c.c.'s by performing P-equivariant surgery once on each s.c.c. to get E such that $h(E) \cap E = \alpha_1$. Let $D = E \cup h(E)$ and $S = D \cup T(D)$ then $T(S) = S = \bar{h}(S)$ and $P(S) = P^2 \subset P^2 \times I$ an invariant projective plane such that $\alpha \cup \{a\} \subset P^2$. Hence in any case there exists $P^2 \subset P^2 \times I$ such that $\alpha \cup \{a\} \subset P^2$ and $h(P^2) = P^2$. We can define an equivariant homeomorphism $t : P^2 \times I \longrightarrow P^2 \times I$ as we did in case II to get $th = h_3 t$, where $h_3([\rho z, s]) = [-\rho z, 1-s]$.

Corollary 2.3: If $h : P^2 \times S^1 \longrightarrow P^2 \times S^1$ is a PL involution with fixed point set $F \neq \emptyset$, then there exists a projective plane $P \subset P^2 \times S^1$ such that $h(P) = P$.

Proof: By lemma 1.6 there exists a $P \subset P^2 \times S^1$ such that $h(P) = P$ or $h(P) \cap P = \emptyset$. If $h(P) \cap P = \emptyset$, cut along $P \cup h(P)$ to get two manifolds A and B each homeomorphic to $P^2 \times I$. $h(A) = A$, otherwise if $h(A) = B$ we have $F = \emptyset$. By theorem 2.2 there is an invariant projective plane $P^* \subset \operatorname{int} A$. Paste back A and B we get $P^2 \times S^1$ and $P^* \subset P^2 \times S^1$ is the invariant projective plane.

Remark 2.4: If $g : P^2 \longrightarrow P^2$ is a homeomorphism then we can get $P^2 \times S^1$ from $P^2 \times I$ by the identification $[\rho z, o] \sim [g(\rho z), 1]$. In theorem 2.5 we will use the following types of identifications (i) $P^2 \times S^1 = P^2 \times I/[\rho z, o] \sim [\rho z, 1]$, (ii) $P^2 \times S^1 = P^2 \times I/[\rho z, o] \sim [-\rho z, 1]$, (iii) $P^2 \times S^1 = P^2 \times I/[\rho z, o] \sim [\rho \bar{z}, 1]$.

Theorem 2.5: Let $h : P^2 \times S^1 \longrightarrow P^2 \times S^1$ be a PL involution with fixed point set F. Then h is equivalent to one of the following six involutions:

(i) $h_1([\rho z_1, z_2]) = [\rho z_1, -z_2]$ with $F \approx \emptyset$,

(ii) $h_2([\rho z_1, z_2]) = [\rho z_1, \bar{z}_2]$ with $F \approx P^2 \cup P^2$,

(iii) $h_3([\rho z, t]) = [\rho z, 1-t]$ with $F \approx P^2 \cup S^1 \cup *$,

(iv) $h_4([\rho z_1, z_2]) = [-\rho z_1, \bar{z}_2]$ with $F \approx S^1 \cup S^1 \cup S^o$,

(v) $h_5([\rho z_1, z_2]) = [\rho z_1, z_2]$ with $F \approx S^1 \times S^1 \cup S^1$,

(vi) $h_6([\rho z_1, t]) = [-\rho z_1, t]$ with $F \approx K \cup S^1$.

Where in (iii) $P^2 \times I/[\rho z, o] \sim [-\rho z, 1]$ and in (vi) $P^2 \times S^1 = P^2 \times I/[\rho z, o] \sim [\rho \bar{z}, 1]$.

Proof: By Tollefson [10], there is only one free involution on $P^2 \times S^1$, the obvious one, and case (i) is settled. By Cor. 2.3 there exists a projective plane $P \subset P^2 \times S^1$ such that $h(P) = P$. Split $P^2 \times S^1$ along P to get a manifold homeomorphic to $P^2 \times I$ and an induced involution $h' : P^2 \times I \longrightarrow P^2 \times I$.

By Lemma 2.2 h' is equivalent to one of the three involutions: (i) $h_1'([\rho z, t]) = [\rho z, 1-t]$, (ii) $h_2'([\rho z, t) = [-\rho z, 1-t]$, (iii) $h_3'([\rho z, t]) = [-\rho z, t]$. In $P^2 \times S^1$ let $M_+ = \{[\rho z_1, z_2] \in P^2 \times S^1 | \mathrm{Re}\, z_2 \geq o\}$, and let $M_- = Cl(P^2 \times S^1 - M_+)$, then $M_+ \approx M_- \approx P^2 \times I$, also let $P_1 = P^2 \times 1 \subset P^2 \times S^1$ and $P_2 = P^2 \times \{-1\} \subset P^2 \times S^1$. Now we have the following cases:

Case (I): h' leaves $P^2 \times \frac{1}{2}$ invariant and interchange $P^2 \times o$ and $P^2 \times 1$. In this case after pasting back we get two invariant projective planes P' and P'' and $P^2 \times S^1 - (P' \cup P'') = N_1 \cup N_2$ where $\bar{N}_1 \approx \bar{N}_2 \approx P^2 \times I$ and $h(\bar{N}_1) = \bar{N}_2$.

<u>Subcase (a)</u>: $P' \cup P'' = F$. Let t be any homeomorphism from $\bar{N}_1$ onto M_+. Define $T : P^2 \times S^1 \longrightarrow P^2 \times S^1$ as follows: $T(x) = x$ for all $x \in \bar{N}_1$ and $T(x) = h_2 th(x)$ for $x \in \bar{N}_2$. Then $hT = Th_2$ and $h \sim h_2$.

<u>Subcase (b)</u>: $F = P' \cup \alpha \cup \{a\}$, where $\alpha \cup \{a\} \subset P''$. Let f be any homeomorphism from $\bar{N}_1$ onto $P^2 \times [o, \frac{1}{2}]$ which takes P' onto $P^2 \times \frac{1}{2}$ and α onto $\{[z,o] | z \in P^2, |z| = 1\}$ and a to $[o,o]$. Define
$T : P^2 \times S^1 \longrightarrow P^2 \times S^1 = P^2 \times I/[\rho z, o] \sim [-\rho z, 1]$ as follows:
$T(x) = t(x)$ for $x \in \bar{N}_1$ and $T(x) = h_3 th(x)$ for $x \in \bar{N}_2$.
Then $hT = Th_3$ and $h \sim h_3$.

<u>Subcase (c)</u>: $F = \alpha \cup \{a\} \cup \beta \cup \{b\}$ where $\alpha \cup \{a\} \subset P'$ and $\beta \cup \{b\} \subset P''$. Let t be any homeomorphism from $\bar{N}_1$ onto M_+, which takes P' onto P_1 and P'' onto P_2 and F onto $\mathrm{Fix}(h_4)$. Define T as in the last subcase and conclude $hT = Th_4$ and hence $h \sim h_4$.

<u>Case (II)</u>: $h'([\rho z, t]) = [-\rho z, t]$. Let
$\mathrm{Fix}(h') = F' \approx S^1 \times I \cup I$ and let
$E = \{[\rho z] \in P^2 | o < \rho < 1\} \approx S^1 \times I$. Let $g : P^2 \longrightarrow P^2$ be a homeomorphism such that $g([-\rho z]) = -g([\rho z])$. $g|E$ is either orientation preserving, or orientation reversing. Let
$q : P^2 \times I \longrightarrow P^2 \times S^1$ be the quotient map where
$q([\rho z, o]) = q([g[\rho z], 1])$. In case g is orientation preserving on E we have $q(F') \approx S^1 \times S^1 \cup S^1$ and in case g is orientation reversing we get $q(F') \approx K \cup S^1$.

<u>Subcase (a)</u>: Let $h : P^2 \times S^1 \longrightarrow P^2 \times S^1$ be a PL involution with $F = T \cup \alpha \approx S^1 \times S^1 \cup S^1$.

Let $q : (P^2 \times S^1, \alpha) \longrightarrow (D^2 \times S^1, \alpha)$ be the projection map onto the orbit space. Let $h_5 : P^2 \times S^1 \longrightarrow P^2 \times S^1$ be defined $h_5([\rho z_1, z_2[) = [-\rho z_1, z_2[$; $\mathrm{Fix}(h_5) = S^1 \times S^1 \cup S^1$, let $q_1 : (P^2 \times S^1, S^1) \longrightarrow (D^2 \times S^1, S^1)$ be the projection map onto the orbit space.

Now let $t : S^1 \longrightarrow \alpha$ be any homeomorphism. Extend t to a homeomorphism $t : (D^2 \times S^1, S^1) \longrightarrow (D^2 \times S^1, \alpha)$ and define $\bar{t} : P^2 \times S^1 \longrightarrow P^2 \times S^1$ as follows:

Choose $a \notin S^1 \times S^1 \cup S^1$, and let $q^{-1}tq_1(a) = \{u, v\}$. Let $\bar{t}$ be the unique lefting of $tq_1 : ((P^2 \times S^1 - (S^1 \times S^1 \cup S^1)), a) \longrightarrow (D^2 \times S^1 - (q(T) \cup \alpha).\ tq_1(a))$ which takes a to u. For $y \in S^1 \times S^1 \cup S^1$ let $\bar{t}(y) = q^{-1}tq_1(y)$. Then $\bar{t}$ is well defined and $q\bar{t} = tq_1$. and hence $h\bar{t} = \bar{t}h_5$, by the commutativity of the diagram:

$$
\begin{tikzcd}
P^2 \times S^1 \arrow[rr, "h"] \arrow[dr, dashed, "\bar{t}"] \arrow[dd, "q_1"] & & P^2 \times S^1 \arrow[dr, dashed, "\bar{t}"] \arrow[dd, "q_1"] & \\
& P^2 \times S^1 \arrow[rr, "h_5"] \arrow[dd, "q"] & & P^2 \times S^1 \arrow[dd, "q"] \\
D^2 \times S^1 \arrow[rr, "id"] \arrow[dr, "t"] & & D^2 \times S^1 \arrow[dr, "t"] & \\
& D^2 \times S^1 \arrow[rr, "id"] & & D^2 \times S^1
\end{tikzcd}
$$

So $h \sim h_5$.

<u>Subcase (b)</u>: Let $h : P^2 \times S^1 \longrightarrow P^2 \times S^1$ be a PL involution with fixed point set $K_1 \cup \alpha$, K_1 is a Klein bottle. Then from what we discussed before the orbit space is N the non-orientable disk bundle over S^1. Let $q : (P^2 \times S^1, \alpha) \longrightarrow (N,\alpha)$ be the projection onto the orbit space. Let $h_6 : P^2 \times S^1 \longrightarrow P^2 \times S^1$ be defined as in the theorem, $\mathrm{Fix}(h_6) = K \cup S^1$ and let $q_1 : P^2 \times S^1 \longrightarrow (N,S^1)$ be the projection map onto the orbit space. Now let $t : S^1 \longrightarrow \alpha$ be a homeomorphism and extend t to a homeomorphism $t : (N,S^1) \longrightarrow (N,\alpha)$. Define $\bar{t} : P^2 \times S^1 \longrightarrow P^2 \times S^1$ exactly as we did in subcase (a) and using a similar argument we get $h \sim h_6$.

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