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DECOMPOSITIONS OF THE MAXIMAL IDEAL SPACE OF L^∞

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ABSTRACT

DECOMPOSITIONS OF THE MAXIMAL IDEAL SPACE OF L^∞

By

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Let L^∞ be the Banach algebra of essentially bounded measurable functions on the unit circle and let $M(L^\infty)$ denote the maximal ideal space of L^∞. In this paper we prove some results about $M(L^\infty)$.

In Chapter 2 we show the existence of one point maximal antisymmetric sets for $H^\infty + C$, thus giving the first example of a maximal antisymmetric set that equals the support set of some multiplicative linear functional on $H^\infty + C$. We also show that each open set in a fiber contains a QC level set that is not a maximal antisymmetric set for $H^\infty + C$, extending a result due to D. Sarason [21].

In Chapter 3 using facts about the maximal ideal space of H^∞ we prove some results about closed subalgebras of L^∞ containing H^∞.

[21] D. Sarason, The Shilov and Bishop decompositions of $H^\infty + C$, to appear.

To my parents
Anne and David Gorkin

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CHAPTER 1

A complex Banach algebra B is a Banach space which is also a complex algebra such that the norm satisfies

$$\|fg\| \leq \|f\|\|g\|$$

for all functions f and g in B. The space of essentially bounded Lebesgue measurable functions on the unit circle, $\partial \mathbb{D}$, with normalized Lebesgue measure will be denoted by $L^\infty(\partial \mathbb{D}, \frac{d\theta}{2\pi})$ or simply L^∞. The space L^∞ is a Banach algebra when it is given pointwise multiplication and the essential supremum norm. Let $f \in L^\infty$. We define f in the unit disc by

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)P_r(\theta - t)dt$$

where $P_r(\theta) = \dfrac{1 - r^2}{1 - 2r\cos\theta + r^2}$. The extended f is a bounded harmonic function in the open unit disc and, as $r \to 1$ the functions $f_r(\theta) = f(re^{i\theta})$ converge to f in the weak-star topology on L^∞. The space of continuous complex valued functions on $\partial \mathbb{D}$ will be denoted by C or $C(\partial \mathbb{D})$. We note that C is a uniformly closed subalgebra of L^∞, hence is also a Banach algebra. The space of bounded analytic functions on the unit disc $\mathbb{D}$ will be denoted by either H^∞ or $H^\infty(\mathbb{D})$. The space H^∞ is a Banach algebra when it is given the norm $\|f\| = \sup_{z\in\mathbb{D}} |f(z)|$. By Fatou's Theorem, a bounded analytic function on $\mathbb{D}$ has radial limits almost everywhere. By identifying each H^∞ function with its boundary function, H^∞ is isometrically isomorphic to a uniformly closed subalgebra of L^∞. This space will also be denoted by H^∞. Once we have made this identification, we can describe another closed subalgebra of L^∞, the algebra $H^\infty + C = \{f + g : f \in H^\infty, g \in C\}$. Sarason [19] showed that $H^\infty + C$ is a closed subalgebra of L^∞. Finally, the largest C^*-subalgebra of $H^\infty + C$ will be denoted by QC. Thus $QC = H^\infty + C \cap \overline{H^\infty + C}$, where the bar denotes complex conjugation.

The maximal ideal space $M(B)$ of a commutative Banach algebra B with a unit 1 is the set of multiplicative linear functionals (nonzero complex algebra homomorphisms) of B. There is a one to one correspondence between maximal ideals of B and kernels of multiplicative linear functionals on B, hence this space is identified with the space of maximal ideals in B. It is not difficult to show that for $\varphi \in M(B)$ $|\varphi(f)| \leq \|f\|$ for all $f \in B$ and $\varphi(1) = 1$. Therefore $M(B)$ is contained in the dual space B^* of B. We give $M(B)$ the weak-* topology, so a net $\{\varphi_\alpha\}$ converges to φ if and only if $\varphi_\alpha(f) \to \varphi(f)$ for all $f \in B$. With this topology, $M(B)$ is a compact Hausdorff space. For $f \in B$, the Gelfand transform of f is the complex valued function $\hat{f} \in C(M(B))$ defined by $\hat{f}(\varphi) = \varphi(f)$ for all $\varphi \in M(B)$. In the cases we are interested in here, the Gelfand transform is an isometry and we will sometimes write f for $\hat{f}$, since the meaning will be clear from the context.

We begin by mentioning some facts about $M(H^\infty)$. Further information is available in [9], [10] and [11]. For each point $\zeta \in \mathbb{D}$ there exists $\varphi_\zeta \in M(H^\infty)$ such that $\varphi_\zeta(z) = \zeta$, where z denotes the function $f(z) = z$. In fact, the point $\varphi_\zeta \in M(H^\infty)$ is uniquely determined by the condition $\varphi_\zeta(z) = \zeta$. Hence $\zeta \to \varphi_\zeta$ defines an embedding of $\mathbb{D}$ into $M(H^\infty)$. This embedding is a homeomorphism. By identifying ζ with φ_ζ, we may regard $\mathbb{D}$ as an open subset of $M(H^\infty)$. The Gelfand transform of z defines a map $\hat{z} : M(H^\infty) \to \overline{\mathbb{D}}$ which is onto. Thus we write $M(H^\infty) = \mathbb{D} \cup \{\varphi \in M(H^\infty) : |\varphi(z)| = 1\}$. The Corona Theorem [5] states that $\mathbb{D}$ is dense in $M(H^\infty)$.

We shall also be interested in $M(L^\infty)$. Since L^∞ is a C^*-algebra, L^∞ is isometrically isomorphic (via the Gelfand transform) to $C(M(L^\infty))$. In [9] and [11] it is shown that $M(L^\infty)$ is a totally disconnected, in fact, extremally disconnected, compact Hausdorff space. For these and other relevant facts about the topology of $M(L^\infty)$ the reader is referred to [9] and [11].

For each $\varphi \in M(H^\infty)$, there is a unique positive Borel measure μ_φ on $M(L^\infty)$ such that

$$\varphi(f) = \int_{M(L^\infty)} f d\mu_\varphi \quad \text{for all} \quad f \in H^\infty .$$

If $\varphi \in M(H^\infty + C)$ the closed support of μ_φ is denoted supp μ_φ, or simply supp φ.

Let B denote a closed subalgebra of L^∞ containing the constant functions which separates the points of $M(L^\infty)$. A closed subset $S \subseteq M(L^\infty)$ is called a peak set for B if there is a function $f \in B$ such that $\hat{f}(\varphi) = 1$ for $\varphi \in S$ and $|\hat{f}(\psi)| < 1$ for $\psi \in M(L^\infty) \sim S$. The function f is said to be a peaking function for S. A closed subset S of $M(L^\infty)$ is called a weak peak set for B if it is the intersection of peak sets. If S is a weak peak set for B, then the restriction algebra $B \mid S$ is a Banach algebra [9, p.57].

Let B denote a closed subalgebra of L^∞ containing the function z. For $\lambda \in \partial \mathbb{D}$ we let $M_\lambda(B) = \{\varphi \in M(B) : \varphi(z) = \lambda\}$. We call $M_\lambda(B)$ the B-fiber over λ. It is not hard to show that $M(L^\infty) = \bigcup_{\lambda \in \partial \mathbb{D}} M_\lambda(L^\infty)$ and $M(H^\infty) = \mathbb{D} \cup \bigcup_{\lambda \in \partial \mathbb{D}} \{\varphi \in M(H^\infty) : \varphi(z) = \lambda\}$. Furthermore, $M(H^\infty + C) = \bigcup_{\lambda \in \partial \mathbb{D}} \{\varphi \in M(H^\infty) : \varphi(z) = \lambda\} = M(H^\infty) \sim \mathbb{D}$.

The L^∞-fiber over λ is a weak peak set for H^∞, hence for $H^\infty + C$. Note that z is constant on each fiber. Therefore each polynomial is also constant on each fiber. By the Stone-Weierstrass Theorem, we see that any continuous function f is constant on each fiber and its value on the fiber over λ is simply $f(\lambda)$. Therefore $H^\infty + C \mid M_\lambda(L^\infty) = H^\infty \mid M_\lambda(L^\infty)$.

There are several other important decompositions of $M(L^\infty)$. The first that we shall discuss here is the Shilov decomposition of $M(L^\infty)$. For $\psi \in M(L^\infty)$ we let $E_\psi = \{\varphi \in M(L^\infty) : \varphi(q) = \psi(q)$ for all $q \in QC\}$. We call E_ψ the QC level set corresponding to ψ. For $t \in M(QC)$ we may also write

$$E_t = \{\varphi \in M(L^\infty) : \varphi(q) = t(q) \text{ for all } q \in QC\} .$$

In this case we call E_t the QC level set corresponding to t. Each QC level set is contained in some L^∞ fiber. We have the following theorem of Shilov [22]:

<u>Theorem 1.1</u>. Let $f \in L^\infty$. If for each QC level set, E_ψ, there exists a function $g \in H^\infty$ such that $f \mid E_\psi = g \mid E_\psi$, then $f \in H^\infty + C$.

The second decomposition is Bishop's decomposition of $M(L^\infty)$. Before stating Bishop's theorem, we need to define the notion of an antisymmetric set. A set $S \subseteq M(L^\infty)$ is called an antisymmetric set for $H^\infty + C$ if whenever $f \in H^\infty + C$ and $f \mid S$ is real valued, then $f \mid S$ is constant. A maximal antisymmetric set for $H^\infty + C$ is a weak peak set for $H^\infty + C$.

Note that each antisymmetric set is contained in some QC level set and it is easy to see that if $\{S_\alpha\}$ denotes the set of maximal antisymmetric sets for $H^\infty + C$, then $M(L^\infty) = \bigcup_\alpha S_\alpha$. Bishop's theorem [3] says the following:

Theorem 1.2. Let $\{S_\alpha\}$ denote the maximal antisymmetric sets for $H^\infty + C$. If $f \in L^\infty$ is such that for each maximal antisymmetric set S_α there exists $g \in H^\infty$ with $f \mid S_\alpha = g \mid S_\alpha$, then $f \in H^\infty + C$.

Sarason [21] has given an example of a QC level set that is not an antisymmetric set for $H^\infty + C$. Thus Bishop's decomposition for $M(L^\infty)$ is strictly finer than Shilov's decomposition for $M(L^\infty)$.

The third theorem along these lines is due to Sarason [20].

Theorem 1.3. Let $f \in L^\infty$. If for each $\varphi \in M(H^\infty + C)$ there exists $g \in H^\infty$ such that $f \mid \text{supp}\ \varphi = g \mid \text{supp}\ \varphi$, then $f \in H^\infty + C$.

The relationship of Sarason's theorem to the others is not clear. However, one can say something about how Sarason's theorem relates to Bishop's theorem. Each support set is an antisymmetric set for $H^\infty + C$. Thus Sarason's theorem is a refinement of Bishop's theorem. In [20] and [21] Sarason asked for the precise relation between support sets and sets of antisymmetry for $H^\infty + C$. Is every maximal antisymmetric set for $H^\infty + C$ the support set of a multiplicative linear functional on $H^\infty + C$? This question is still open. In fact it was unknown whether any maximal antisymmetric set equals the support set of a multiplicative linear functional on $H^\infty + C$. We shall show the existence of a maximal antisymmetric set consisting of a single point. The relation of this result to Sarason's question is indicated below.

Let $\varphi \in M(L^\infty)$. Then point evaluation at φ is a positive measure and is the unique positive measure satisfying $\varphi(f) = \int_{M(L^\infty)} f d\mu_\varphi$ for all $f \in H^\infty$. The closed support of this measure is $\{\varphi\}$. Thus a maximal anti-symmetric set for $H^\infty + C$ consisting of a single point must be a support set. We will give many examples of one point maximal anitsymmetric sets and will show that many of these are contained in QC level sets consisting of more than one point.

In Chapter 3 we use some of these results to obtain information about closed subalgebras of L^∞ containing H^∞.

CHAPTER 2

The following theorem is the main result of this chapter .

Theorem 2.1. Let $\{\lambda_n\}$ be a sequence of distinct points of $\partial \mathbb{D} \sim \{1\}$ with $\lambda_n \to 1$. Let $\psi_n \in M_{\lambda_n}(L^\infty)$ and $\psi \in \overline{\{\psi_n\}}^{M(L^\infty)} \cap M_1(L^\infty)$. Then $\{\psi\}$ is a maximal anti-symmetric set for $H^\infty + C$.

An unpublished result of K. Hoffman shows that any point of $M(L^\infty)$ in the closure of a sequence of points from distinct L^∞ fibers is a maximal support set. Our proof is independent of this fact, although Hoffman's result follows easily from Theorem 2.1.

In order to prove Theorem 2.1, we need the result given below.

Theorem 2.2. Let $\{\lambda_n\}$ be a sequence of distinct points of $\partial \mathbb{D} \sim \{1\}$ such that $\lambda_n \to 1$. Let $\{I_n\}$ be a sequence of intervals of $\partial \mathbb{D}$ with $\overline{I}_n \cap \overline{\bigcup_{m \neq n} I_m} = \phi$ and $\lambda_n \in I_n$. Then there exists $q \in QC$ satisfying :

(1) q is continuous except at $\lambda = 1$;

(2) $|\arg q(\lambda_n) - \pi| < \frac{1}{4}$ for all n ;

(3) $|\arg q(\lambda)| < \frac{1}{4}$ for $\lambda \in \partial \mathbb{D} \sim \overline{\cup I_n}$.

The proof will be in two parts. The first part is Lemma 2.3 given below. In what follows we let $\tilde{u}$ denote the harmonic conjugate of u. The space of continuous real valued functions with continuous first derivatives will be denoted by $C^1_{\mathbb{R}}$. If $u \in C^1_{\mathbb{R}}$, then $\tilde{u}$ is a real valued continuous function [11, p.79].

Lemma 2.3. Let I be an open interval contained in $\partial \mathbb{D}$ and let $w \in I$. Then given $\varepsilon > 0$ and $\lambda_0 \in \mathbb{R}^+$, there exists $u \in C^1_{\mathbb{R}}$ with $\|u\|_\infty < \varepsilon$, $|\tilde{u}(z)| < \varepsilon$ for $z \in \partial \mathbb{D} \sim I$ and $\tilde{u}(w) = \lambda_0$.

Proof: By choosing $\delta > 0$ sufficiently small and rotating, we may assume that $w = 1$ and $I = \{e^{i\theta} : -2\delta < \theta < 2\delta\}$. It is enough to show that there exists $v \in C^1_{\mathbb{R}}$ with $\|v\|_\infty < 1$, $|\tilde{v}(z)| \leq 1$ for $z \in \partial \mathbb{D} \sim I$ and $\tilde{v}(1) = \frac{\lambda_0}{\varepsilon}$, for then $u = \frac{\lambda_0 v}{\tilde{v}(1)}$ satisfies $\|u\|_\infty < \varepsilon$, $|\tilde{u}(z)| < \varepsilon$ for $z \in \partial \mathbb{D} \sim I$ and $\tilde{u}(1) = \lambda_0$.

It is not hard to show that $\lim\limits_{x \to \infty} \dfrac{1 - (\frac{1}{k})^{\frac{1}{x}}}{\frac{1}{x}} = \ln k$ for $k > 0$. We use this fact below. To find v, let $\varepsilon > 0$ and $\lambda_0 \in \mathbb{R}^+$ be given. Choose k so that $\ln k > \dfrac{2\pi\lambda_0}{\varepsilon \tan\frac{\delta}{2}}$.

Choose an odd integer m satisfying

(i) $m[1-(\frac{1}{k})^{\frac{1}{m}}] > \frac{2\pi\lambda_0}{\epsilon \tan\frac{\delta}{2}}$, (ii) $\frac{1}{m} < \delta$ and (iii) $\cos(\frac{1}{2m}) > \frac{1}{2}$.

Let $$v(z) = \begin{cases} 0 & \text{if } z \in \partial \mathbb{D} \sim \frac{I}{2} \\ (-\frac{1}{2}\tan\frac{\delta}{2})(mt)^{\frac{1}{m}} & \text{if } z = e^{it} \in \{e^{is}: \frac{1}{km} < s < \frac{1}{m}\} \end{cases}$$

and extend v so that $v \in C^1_{\mathbb{R}}$, $v(1) = 0$, $v(e^{is}) \leq 0$ for $0 \leq s \leq \pi$, $v(e^{-is}) = -v(e^{is})$, and $\|v\|_\infty < \tan\frac{\delta}{2} \leq 1$.

Writing $v(\theta)$ for $v(e^{i\theta})$ we have [11, p.79]

$$\tilde{v}(0) = \int_{-\pi}^{\pi} \frac{v(-t)-v(t)}{2\tan\frac{t}{2}} \frac{dt}{2\pi} = 2\int_0^{\pi} \frac{v(-t)}{\tan\frac{t}{2}} \frac{dt}{2\pi}$$

$$\geq 2\int_{\frac{1}{km}}^{\frac{1}{m}} \frac{v(-t)}{\tan\frac{t}{2}} \frac{dt}{2\pi} = 2\int_{\frac{1}{km}}^{\frac{1}{m}} \frac{(\frac{1}{2}\tan\frac{\delta}{2})(mt)^{\frac{1}{m}}}{\sin\frac{t}{2}} \cos\frac{t}{2} \frac{dt}{2\pi}$$

$$\geq \frac{1}{2}\int_{\frac{1}{km}}^{\frac{1}{m}} \frac{\tan\frac{\delta}{2}\, m^{\frac{1}{m}}\, t^{\frac{1}{m}}}{\frac{t}{2}} \frac{dt}{2\pi} = (\tan\frac{\delta}{2}) m^{\frac{1}{m}} \int_{\frac{1}{km}}^{\frac{1}{m}} t^{\frac{1}{m}-1} \frac{dt}{2\pi}$$

$$= \frac{(\tan\frac{\delta}{2}) m^{\frac{1}{m}+1}}{2\pi} [(\frac{1}{m})^{\frac{1}{m}} - (\frac{1}{k})^{\frac{1}{m}} (\frac{1}{m})^{\frac{1}{m}}]$$

$$= \frac{(\tan\frac{\delta}{2})}{2\pi} m[1-(\frac{1}{k})^{\frac{1}{m}}] > \frac{\lambda_0}{\epsilon}.$$

Hence $\tilde{v}(0) > \frac{\lambda_0}{\epsilon}$.

Suppose $z = e^{i\theta} \notin I$. Since the (closed) support of v is contained in $\frac{I}{2}$ we have

$$|\tilde{v}(\theta)| \le \int_{-\pi}^{\pi} \left| \frac{v(\theta+t) - v(\theta-t)}{2 \tan \frac{t}{2}} \right| \frac{dt}{2\pi}$$

$$= \int_{|t|\le\delta} \left| \frac{v(\theta+t) - v(\theta-t)}{2 \tan \frac{t}{2}} \right| \frac{dt}{2\pi} + \int_{\delta<|t|\le\pi} \left| \frac{v(\theta+t) - v(\theta-t)}{2 \tan \frac{t}{2}} \right| \frac{dt}{2\pi}$$

$$= \int_{\delta\le|t|\le\pi} \left| \frac{v(\theta+t) - v(\theta-t)}{2 \tan \frac{t}{2}} \right| \frac{dt}{2\pi} \le \frac{\|v\|_\infty}{\tan \frac{\delta}{2}} \le 1 .$$

Therefore $|\tilde{v}(z)| \le 1$ if $z \in \partial \mathbb{D} \sim I$, as desired.

Proof of Theorem 2.2: Given intervals I_n with $I_n \cap \overline{\bigcup_{m\neq n} I_m} = \phi$, $\lambda_n \in I_n$ and $\lambda_n \to 1$ choose functions $u_n \in C^1_{\mathbb{R}}$ with $\|u_n\|_\infty < \frac{1}{2^{n+3}}$, $|\tilde{u}_n(z)| < \frac{1}{2^{n+3}}$ for $z \in \partial \mathbb{D} \sim I_n$ and $\tilde{u}_n(\lambda_n) = (2n+1)\pi$. Let $u = \sum_{n=1}^{\infty} u_n$. Then $u \in C_{\mathbb{R}}$ and since the map $T : L^2 \to L^2$ defined by $T(f) = \tilde{f}$ is continuous, $\tilde{u} = \sum_{n=1}^{\infty} \tilde{u}_n$ in L^2 norm. Since each $u_n \in C^1_{\mathbb{R}}$, $\tilde{u}_n \in C_{\mathbb{R}}$. It is easy to see that $\{\sum_{n=1}^{m} \tilde{u}_n\}_m$ converge uniformly to $\tilde{u}$ on compact subsets of $\partial \mathbb{D} \sim \{1\}$. Hence $\tilde{u}$ is continuous except possibly at $\lambda = 1$.

Let $q = e^{i\tilde{u}}$. Then $q = e^{u + i\tilde{u}} e^{-u} \in H^\infty + C$ and $\bar{q} = e^{-u - i\tilde{u}} e^{u} \in H^\infty + C$. Therefore $q \in QC$.

For any n we have

$$|\arg q(\lambda_n) - \pi| = |\arg e^{i\tilde{u}(\lambda_n)} - \pi| = |\arg e^{i\sum_m \tilde{u}_m(\lambda_n)} - \pi|$$

$$= |\arg e^{i[(2n+1)\pi + \sum_{m\neq n} \tilde{u}_m(\lambda_n)]} - \pi|$$

$$= |\arg(-e^{i\sum_{m\neq n} \tilde{u}_m(\lambda_n)}) - \pi| < \frac{1}{4}$$

and if $\lambda \in \partial \mathbb{D} \sim \overline{\cup I_n}$, then

$$|\arg q(\lambda)| = |\arg e^{i\sum_m \tilde{u}_m(\lambda)}| < \frac{1}{4} .$$

Before we present the proof of Theorem 2.1 we prove a corollary of Theorem 2.2 that will be used frequently.

<u>Corollary 2.4</u>. Let $t \in M_1(QC)$ and $\{\lambda_n\}$ be a sequence of distinct points of $\partial \mathbb{D} \sim \{1\}$ such that t is in the $M(QC)$ closure of a sequence of points $\{t_n\}$, where $t_n \in M_{\lambda_n}(L^\infty)$ and $\lambda_n \to 1$. Then $E_t \subseteq \overline{\bigcup_n M_{\lambda_n}(L^\infty)}$.

<u>Proof</u>: Suppose $\varphi \in M(L^\infty) \sim \overline{\bigcup_n M_{\lambda_n}(L^\infty)}$. If $\varphi \in M(L^\infty) \sim M_1(L^\infty)$, then $\varphi \in M(L^\infty) \sim E_t$. Therefore we may assume that $\varphi \in M_1(L^\infty)$. Since $M(L^\infty)$ has a basis of clopen sets (sets that are both closed and open), we can find a clopen set $F \subseteq M(L^\infty)$ with $\varphi \in F \subseteq M(L^\infty) \sim \overline{\bigcup_n M_{\lambda_n}(L^\infty)}$. For each n, $M_{\lambda_n}(L^\infty) \subseteq M(L^\infty) \sim F$ and therefore $\bigcup_{m=1}^{\infty} \{\varphi' \in M(L^\infty) : |\varphi'(z) - \lambda_n| > \frac{1}{m}\} \supsetneq F$. Since F is compact, there exists N such that $\bigcap_{m=1}^{N} \{\varphi' \in M(L^\infty) : |\varphi'(z) - \lambda_n| \leq \frac{1}{m}\}$

is contained in $M(L^\infty) \sim F$. Thus there exists an interval I_n with $\lambda_n \in I_n$ satisfying $M_\lambda(L^\infty) \subseteq M(L^\infty) \sim F$ for all $\lambda \in \overline{I_n}$. By choosing I_n sufficiently small we may assume that $\overline{I_n} \cap \overline{\bigcup_{m \neq n} I_m} = \phi$. Note that (*) $\bigcup_n \{M_\lambda(L^\infty) : \lambda \in \overline{I_n}\} \subseteq M(L^\infty) \sim F$. Thus there is a QC function q satisfying conditions (1) - (3) of Theorem 2.2.

For any n and any $\psi \in M_{\lambda_n}(L^\infty)$ we have, by (1) and (2) of Theorem 2.2, that $|\arg q(\psi) - \pi| \leq \frac{1}{4}$. Passing to $M(QC)$ we have $|\arg q(t) - \pi| \leq \frac{1}{4}$ for any $t \in M_{\lambda_n}(L^\infty)$. Therefore for any $\psi' \in E_t$ we have $|\arg q(\psi') - \pi| \leq \frac{1}{4}$.

To see that $\varphi \in M(L^\infty) \sim E_t$ we shall show that $|\arg q(\varphi)| \leq \frac{1}{4}$. Choose $\varepsilon > 0$ and let $F_\varepsilon = \{\eta \in M(L^\infty) : |\arg \varphi(q) - \arg \eta(q)| < \varepsilon\}$. Then $F_\varepsilon \cap F$ is an open set in $M(L^\infty)$ containing φ. We claim that there exists $\lambda_0 \neq 1$ such that $M_{\lambda_0}(L^\infty) \cap F \cap F_\varepsilon \neq \phi$. To establish this it is enough to show the following:

<u>Claim</u> $M_1(L^\infty)$ has no interior in $M(L^\infty)$.

<u>Proof of claim</u>: Suppose F is a clopen subset of $M(L^\infty)$ with $F \subseteq M_1(L^\infty)$. Then $\chi_F \in C(M(L^\infty))$. Hence there exists a measurable set $E \subseteq \partial \mathbb{D}$ of positive measure such that $\chi_F = \hat{\chi}_E$. If f is a nonconstant function in H^∞ such that f is continuous at 1, then f is constant on $M_1(L^\infty)$. Therefore f is constant on F. Hence f is constant on E. Since E has positive measure, f

must be a constant function. This contradiction establishes the claim.

Choose $\lambda_0 \in \partial \mathbb{D}$ satisfying $\lambda_0 \neq 1$ and $M_{\lambda_0}(L^\infty) \cap F \cap F_\varepsilon \neq \phi$. By (*), $\lambda_0 \in \partial \mathbb{D} \sim \overline{\cup I_n}$. Hence $|\arg q(\lambda_0)| \leq \frac{1}{4}$. Let $\psi_{\varepsilon,0} \in M_{\lambda_0}(L^\infty) \cap F \cap F_\varepsilon$. Then $|\arg \psi_{\varepsilon,0}(q)| \leq \frac{1}{4}$. Therefore $|\arg \varphi(q)| \leq \frac{1}{4} + \varepsilon$. Since ε was arbitrary $|\arg \varphi(q)| \leq \frac{1}{4}$. Therefore $\varphi \in M(L^\infty) \sim E_t$, so $M(L^\infty) \sim (\cup M_\lambda(L^\infty)) \subseteq M(L^\infty) \sim E_t$ which implies the result.

Proof of Theorem 2.1. Choose $\varphi \in M(L^\infty)$ with $\varphi \neq \psi$ such that φ and ψ are in the same QC level set. If no such φ exists, then $E_\psi = \{\psi\}$ and hence the maximal antisymmetric set containing ψ, S_ψ, satisfies $S_\psi = \{\psi\}$ and we are done. We assume then that such a φ exists. Since $\varphi \neq \psi$, there exists a clopen set F with $\varphi \in F$ and $\psi \in M(L^\infty) \sim F$. Thus passing to a subsequence of $\{\psi_n\}$ if necessary, we may assume that $\overline{\{\psi_n\}} \subseteq M(L^\infty) \sim F$. By a theorem of Axler [1] for each n we can find $f_n \in H^\infty + C$ with $\|f_n\|_\infty = 1$ such that $|\psi_n(f_n)| = 1$ and $\eta(f_n) = 0$ for all $\eta \in F$. Using an idea of Sarason we let G_n denote the open ellipse with major axis $[-1,1]$ and minor axis $[-i/n, i/n]$. Let T_n denote a conformal mapping of the open unit disc $\mathbb{D}$ onto G_n such that $T_n(0) = 0$ and by [18, p.309] we may assume $T_n \in C$. Choose $z_n \in \mathbb{D}$ with $|z_n| > \frac{n}{n+1}$, $T_n(z_n)$ real and $T_n(z_n) > \frac{n}{n+1}$. By multiplying f_n by a constant of modulus one, we may assume that $|z_n|\psi_n(f_n) = z_n$.

Since $H^\infty + C|M_{\lambda_n}(L^\infty) = H^\infty|M_{\lambda_n}(L^\infty)$, there exists an H^∞ function whose restriction to $M_{\lambda_n}(L^\infty)$ is $f_n|M_{\lambda_n}(L^\infty)$. Multiplying that function by a suitable peaking function for $M_{\lambda_n}(L^\infty)$ we obtain a function $g_n \in H^\infty$ such that $\|g_n\|_\infty < \frac{1}{|z_n|}$ and $g_n \mid M_{\lambda_n}(L^\infty) = f_n \mid M_{\lambda_n}(L^\infty)$. Thus $T_n \circ (|z_n|g_n) \in H^\infty$. Let $\eta \in M(L^\infty)$. We claim that $\eta(T_n \circ |z_n|g_n) = T_n(|z_n|\eta(g_n))$. To see this, note that T_n is a uniform limit of polynomials $p_{m,n}$. If $f \in H^\infty$ with $\|f\|_\infty < 1$, then $\eta(T_n \circ f) = \eta(\lim_m p_{m,n}(f)) = \lim_m p_{m,n}(\eta(f)) = T_n(\eta(f))$. Therefore for each n we have

$$\psi_n(T_n \circ |z_n|g_n) = T_n(|z_n|\psi_n(g_n))$$

$$= T_n(|z_n|\psi_n(f_n)) = T_n(z_n) > \frac{n}{n+1} .$$

If $\tau \in F \cap M_{\lambda_n}(L^\infty)$ for some n, then

$$\tau(T_n \circ |z_n|g_n) = T_n(|z_n|\tau(f_n)) = T_n(0) = 0 .$$

For each λ_n, choose intervals I_n centered at λ_n with $I_n \cap \overline{\bigcup_{m\neq n} I_m} = \phi$ where the Lebesgue measure of I_n, $|I_n|$, satisfies $|I_n| < \frac{1}{2^{n+4}}$ and $1 \in \partial \mathbb{D} \sim (\bigcup_n I_n)$. Let $\mathcal{O}(I_n) = \{z \in \overline{\mathbb{D}} : |z - \lambda_n| < \frac{|I_n|}{2}\}$ and let h_n be a peaking function for $M_{\lambda_n}(L^\infty)$. By raising h_n to a sufficiently large power we may assume that $\|h_n \mid \overline{\mathbb{D}} \sim \mathcal{O}(I_n)\|_\infty < \frac{1}{2^{n+4}}$.

Let K_n be a Möbius transformation such that $K_n(1) = 0$ and $K_n(\lambda_n) = 1$. Let $\ell_n = h_n(T_n \circ |z_n|g_n)K_n$. Then

(1) $\ell_n \mid M_{\lambda_n}(L^\infty) = (T_n \circ |z_n|g_n) \mid M_{\lambda_n}(L^\infty)$ for all n

and

(2) $\|\ell_n \mid \overline{\mathbb{D}} \sim \mathcal{O}(I_n)\|_\infty < \frac{1}{2^{n+4}}$.

Let $L_m = \sum_{n=1}^{m} \ell_n$ and $L = \sum_{n=1}^{\infty} \ell_n$. It is easy to see that L_m converges to L uniformly on compact subsets of $\overline{\mathbb{D}} - \{1\}$. Furthermore, $\|L_m\| \leq 2$ and thus $L \in H^\infty(\mathbb{D})$.

To see that $L \mid E_t$ is real valued, let $\varepsilon > 0$ be given. Choose N such that $\sum_{n=N}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{3}$. Let I be an open interval of $\partial \mathbb{D}$ containing 1 such that $\max_{1\leq j\leq N} \|K_j \mid I\|_\infty < \frac{\varepsilon}{3N}$. Then $\|\ell_j \mid I\|_\infty < \frac{\varepsilon}{3N}$ $j = 1,2,\ldots,N$. Choose ψ_0 in the QC level set corresponding to ψ, E_ψ. Let $V = \{\eta \in M(L^\infty) : |\eta(L) - \psi_0(L)| < \frac{\varepsilon}{3}\} \cap \bigcup_{\lambda\in I} M_\lambda(L^\infty)$. Then V is an open set about ψ_0. By Corollary 2.4, there exists an integer m satisfying $m > \max(N, \frac{3}{\varepsilon})$ and such that $V \cap M_{\lambda_m}(L^\infty) \neq \phi$. Let $\varphi_0 \in V \cap M_{\lambda_m}(L^\infty)$. Since $\sum_n \ell_n$ converges uniformly on $\overline{I}_m$, we have

$$|\mathrm{Im}\, \varphi_0(L)| = |\mathrm{Im} \sum_n \varphi_0(\ell_n)| = |\sum_{n=1}^{N} \mathrm{Im}\, \varphi_0(\ell_n) + \mathrm{Im}\, \varphi_0(\ell_m) + \sum_{\substack{n=N+1\\ n\neq m}}^{\infty} \mathrm{Im}\, \varphi_0(\ell_n)|$$

$$\leq \sum_{n=1}^{N} |\mathrm{Im}\, \varphi_0(\ell_n)| + |\mathrm{Im}\, \varphi_0(T_m \circ (|z_m|g_m))| + \sum_{\substack{n=N+1\\ n\neq m}}^{\infty} |\mathrm{Im}\, \varphi_0(\ell_n)|$$

$$\leq \sum_{n=1}^{N} \frac{\varepsilon}{3N} + |\mathrm{Im}(T_m \circ (\varphi_0(|z_m|g_m))| + \\ + \sum_{\substack{n=N+1 \\ n\neq m}} \frac{1}{2^{n+4}}$$

$$< \frac{\varepsilon}{3} + \frac{1}{m} + \frac{\varepsilon}{3} < \varepsilon .$$

Therefore $|\mathrm{Im}\ \psi_0(L)| < \frac{4\varepsilon}{3}$. Since ε was arbitrary, $\psi_0(L)$ must be real valued.

Recall that we chose φ to be a point in $M(L^\infty)$ with $\varphi \neq \psi$ such that φ and ψ are in the same QC level set and F was a clopen subset of $M(L^\infty)$ with $\varphi \in F$ and $\psi \in \overline{\{\psi_n\}}^{M(L^\infty)} \subseteq M(L^\infty) \sim F$. Since $\{\eta \in M(L^\infty) : |\eta(L) - \psi(L)| < \frac{1}{8}\}$ is an open subset containing ψ, there exists n with $n \geq 7$ and $\psi_n \in M_{\lambda_n}(L^\infty)$ such that $|\psi_n(L) - \psi(L)| < \frac{1}{8}$. Thus

$$|\psi_n(L)| = |\psi_n(\ell_n) + \sum_{m\neq n} \psi_n(\ell_m)| \geq \frac{n}{n+1} - \frac{1}{4} \geq \frac{7}{8} - \frac{1}{4} = \frac{5}{8}$$

Therefore $|\psi(L)| \geq \frac{1}{2}$.

To determine $\varphi(L)$, note that $U = \{\eta \in M(L^\infty) : |\eta(L) - \varphi(L)| < \frac{1}{8}\} \cap F$ is an open set in $M(L^\infty)$ containing φ. By Corollary 2.4 there exists m such that $M_{\lambda_m}(L^\infty) \cap U \neq \phi$. Let $\varphi_m \in M_{\lambda_m}(L^\infty) \cap U$. Then we have

$$|\varphi_m(L)| = |\varphi_m(\ell_m) + \sum_{n\neq m} \varphi_m(\ell_n)| \leq |\varphi_m(\ell_m)| + \sum_{n\neq m} |\varphi_m(\ell_n)| \leq 0 + \frac{1}{4}$$

Therefore $|\varphi(L)| \leq \frac{3}{8}$ and $\psi(L) \neq \varphi(L)$.

The maximal antisymmetric set S_ψ containing ψ is contained in E_ψ, so $L|S$ is real valued. Thus $L|S$ is constant. Therefore $\varphi \notin S$. Since φ was an arbitrary point of E_ψ distinct from ψ, $S = \{\psi\}$ and the proof is complete.

We will show that many of the points that are in the $M(L^\infty)$ closure of a sequence of points from distinct L^∞ fibers are contained in QC level sets consisting of more than one point. We will also show that not every QC level set contains such a point. Before proceeding to the proofs of these results, some related results must be presented.

We call a sequence $\{z_j\}_{j=1}^\infty$ of distinct points in $\mathbb{D}$ an interpolating sequence if whenever $\{w_j\}_{j=1}^\infty$ is a bounded sequence of complex numbers, there exists a function $f \in H^\infty$ with $f(z_j) = w_j$ for all j. It is well known [4] that a sequence $\{z_j\}_{j=1}^\infty$ is an interpolating sequence if and only if there exists a constant $\delta > 0$ such that $\prod_{j \neq k} \left| \frac{z_k - z_j}{1 - \bar{z}_j z_k} \right| \geq \delta > 0$ for $k = 1,2,3,\ldots$. A Blaschke product with zeroes $\{z_j\}_{j=1}^\infty \subseteq \mathbb{D}$ is a function $b \in H^\infty(\mathbb{D})$ of the form

$$b(z) = \lambda \prod_j \frac{-\bar{z}_j}{|z_j|} \frac{z - z_j}{1 - \bar{z}_j z} \quad \text{for} \quad z \in \mathbb{D}$$

where $|\lambda| = 1$ and $\sum_n (1 - |z_n|) < \infty$. If the zeroes of

b form an interpolating sequence, b is called an interpolating Blaschke product. A Blaschke product is an inner function, so that $|b(e^{i\theta})| = 1$ for almost all θ. A great deal of information has been obtained about interpolating Blaschke products. In [14] P. Jones showed that the interpolating Blaschke products separate the points of $M(H^\infty)$. We will use the theorem stated below. The proof of this theorem is given in [11, p.205].

<u>Theorem 2.5</u>. Let $\{z_j\}_{j=1}^{\infty}$ be an interpolating sequence and let b be the Blaschke product with zeroes $\{z_j\}_{j=1}^{\infty}$. Then $\overline{\{z_j\}_{j=1}^{\infty}}^{M(H^\infty)}$ is homeomorphic to the Stone-Cech compactification of $\{z_j\}_{j=1}^{\infty}$ and every zero of b in $M(H^\infty) \sim \mathbb{D}$ is in the $M(H^\infty)$ closure of $\{z_j\}_{j=1}^{\infty}$.

The following lemma was proven by K. Clancey and J.A. Gosselin [7].

<u>Lemma 2.6</u>. Let u be an inner function. If $t \in M(QC)$ with $u \mid E_t$ invertible in $H^\infty \mid E_t$, then $u \mid E_t$ is constant. In fact $\{t \in M(QC) : u \mid E_t \text{ is constant}\} = \{t \in M(QC) : u \mid E_t \text{ is invertible in } H^\infty \mid E_t\}$ is an open set in $M(QC)$.

This was also proven by R.G. Douglas in [8]. Using Lemma 2.6 together with the following result of D.E. Marshall [16, p.15], we prove a similar result about characteristic functions. In what follows $H^\infty[f]$ denotes the closed subalgebra of L^∞ generated by H^∞ and f.

Theorem 2.7. Let χ_E be a nonconstant characteristic function in L^∞. Then there is an inner function u such that

$$H^\infty[\chi_E] = H^\infty[\bar{u}] .$$

Theorem 2.8. Let χ_E be a nonconstant characteristic function in L^∞. If $\chi_E \mid E_{t_0} \in H^\infty \mid E_{t_0}$ for some QC level set E_{t_0}, then $\chi_E \mid E_{t_0}$ is constant.

Sarason has given a proof of Theorem 2.8. Since his proof is unpublished we include a proof below. Our proof is different from Sarason's; his did not use Theorem 2.7.

Proof: By Theorem 2.7 there exists an inner function u such that $H^\infty[\chi_E] = H^\infty[\bar{u}]$. Therefore $M(H^\infty[\chi_E]) = M(H^\infty[\bar{u}])$. Thus $M(H^\infty \mid E_{t_0}) \subseteq M(H^\infty[\chi_E]) = M(H^\infty[\bar{u}])$. Hence $|\varphi(u)| = 1$ for all $\varphi \in M(H^\infty \mid E_t)$. Therefore $u \mid E_t$ is invertible in $H^\infty \mid E_t$. By Lemma 2.6 there exists $\mathcal{O}$ open in $M(QC)$ containing t_0 with $\bar{u} \mid E_t \in H^\infty \mid E_t$ for all $t \in \mathcal{O}$. Let $q \in QC$ with $q(t_0) = 1$, $q(s) = 0$ for $s \in M(QC) \sim \mathcal{O}$ and $0 \le q \le 1$. Choose $\psi \in M(H^\infty + C)$. If $\operatorname{supp} \psi \subseteq E_s$ and $s \in M(QC) \sim \mathcal{O}$, then $q \mid \operatorname{supp} \psi = 0$ and therefore $q\chi_E \mid \operatorname{supp} \psi = 0$. If $\operatorname{supp} \psi \subseteq E_t$ and $t \in \mathcal{O}$, then $\bar{u} \mid \operatorname{supp} \psi \in H^\infty \mid \operatorname{supp} \psi$ and hence $\chi_E \mid \operatorname{supp} \psi \in H^\infty \mid \operatorname{supp} \psi$. By Theorem 1.3

$q\chi_E \in H^\infty + C$. Since $q\chi_E$ is real valued, $q\chi_E \in QC$. Thus $q\chi_E \mid E_{t_0}$ is constant. Since $\hat{q}(\varphi) = 1$ for all $\varphi \in E_{t_0}$ we must have $\chi_E \mid E_{t_0}$ constant, as desired.

By the Shilov Idempotent Theorem we obtain the corollary below, answering a question of R.G. Douglas in [8].

Corollary 2.9. If $t \in M(QC)$, then $M(H^\infty | E_t)$ is connected.

It is a consequence of the following result of T. Wolff [25] that any function $f \in L^\infty$ is constant on some QC level set.

Theorem 2.10. Let $f \in L^\infty$. There exists an outer function $q \in QC \cap H^\infty$ such that $qf \in QC$.

We will show that for any $\lambda \in \partial \mathbb{D}$ and any clopen set F contained in $M_\lambda(L^\infty)$ there exists a QC level set contained in F.

We will make frequent use of the following [21] and [24].

Theorem 2.11. Let f and g be functions in L^∞. If for each $\varphi \in M(H^\infty + C)$ either $f \mid \text{supp}\, \varphi \in H^\infty \mid \text{supp}\, \varphi$ or $g \mid \text{supp}\, \varphi \in H^\infty \mid \text{supp}\, \varphi$, then for each QC level set E_t either $f \mid E_t \in H^\infty \mid E_t$ or $g \mid E_t \in H^\infty \mid E_t$.

This theorem together with the following unpublished result of K. Hoffman provides us with much more information about the points in $M(L^\infty)$ that are in the closure of a sequence of points from distinct L^∞ fibers.

<u>Theorem 2.12</u>. Let $\{z_j\}_{j=1}^\infty$ be an interpolating sequence such that

$$\lim_{n\to\infty} \prod_{m\neq n} \left| \frac{z_n - z_m}{1 - \bar{z}_m z_n} \right| = 1 .$$

If $\varphi \in \overline{\{z_n\}}^{M(H^\infty)}$ and $\varphi \in M(H^\infty + C)$, then $\operatorname{supp} \varphi$ is a maximal support.

<u>Theorem 2.13</u>. Let E be a nonempty clopen subset of $M_1(L^\infty)$. Then E contains a QC level set that is not a maximal antisymmetric set for $H^\infty + C$.

We remark that Sarason [21] has shown that each L^∞ fiber contains a QC level set that is not a maximal antisymmetric set for $H^\infty + C$. Some of the ideas used to prove Theorem 2.13 are similar to techniques communicated by T. Wolff (private communication).

<u>Proof</u>: Let F be a clopen subset of $M(L^\infty)$ such that $E = F \cap M_1(L^\infty)$. Then there exists a measurable subset G of $\partial \mathbb{D}$ of positive measure such that $\chi_F = \hat{\chi}_G$. Let $\{\lambda_n\}$ be a sequence of distinct points of $\partial \mathbb{D}$ with $\lambda_n = e^{i\theta_n} \to 1$ and $\lim_{r\to 1} \chi_G(re^{i\theta_n}) = 1$. We claim that there exists an interpolating sequence $\{z_m\}$ with the following properties:

$$(1) \quad \lim_{n\to\infty} \prod_{m\neq n} \left| \frac{z_n - z_m}{1 - \bar{z}_m z_n} \right| = 1$$

(2) $\{z_m\}$ is the disjoint union of interpolating sequences $\{z_{m,n}\}_{n=1}^{\infty}$ such that $z_{m,n} = r_{m,n}e^{i\theta_n}$ for suitable choices of $r_{m,n}$

and

(3) $\chi_G(z_m) \to 1$ as $m \to \infty$.

We construct such a sequence as follows: Let $z_1 = z_{1,1} = r_{1,1}e^{i\theta_1}$ where $0 < r_{1,1} < 1$. Choose $z_2 = z_{2,1} = r_{2,1}e^{i\theta_2}$ such that $\chi_G(z_2) > \frac{1}{2}$ and $\left|\frac{z_2 - z_1}{1 - \bar{z}_2 z_1}\right| > e^{-\frac{1}{2}}$. Choose $z_3 = z_{1,2} = r_{1,2}e^{i\theta_1}$ such that $\chi_G(z_3) > 1 - \frac{1}{3}$ and $\left|\frac{z_3 - z_j}{1 - \bar{z}_j z_3}\right| > e^{-\frac{1}{2^{3+j}}}$ $j = 1,2$. We choose $z_4 = z_{2,2} = r_{2,2}e^{i\theta_2}$ satisfying $\chi_G(z_4) > 1 - \frac{1}{4}$ and $\left|\frac{z_4 - z_j}{1 - \bar{z}_j z_4}\right| \geq e^{-\frac{1}{2^{4+j}}}$ $j = 1,2,3.$

We continue to choose z_n satisfying (2) such that

$\chi_G(z_n) > 1 - \frac{1}{n}$ and $\left|\frac{z_n - z_j}{1 - \bar{z}_j z_n}\right| > e^{-\frac{1}{2^{n+j}}}$ for $j < n$. It is not hard to see that (1) and (3) also hold.

Let $\varphi_n \in \overline{\{z_{m,n}\}_{m=1}^{\infty}}^{M(H^\infty + C)}$. Then $\varphi_n \in M_{\lambda_n}(H^\infty)$. Let $\varphi_0 \in \overline{\{\varphi_n\}}^{M(H^\infty + C)} \cap M_1(H^\infty)$. By Theorem 2.12 we have that for each n supp φ_n is a maximal support set. Let b be the interpolating Blaschke product with zeroes $\{z_m\}$. Choose $\psi \in M(H^\infty + C)$. If $\bar{b} \mid \text{supp}\ \psi \notin H^\infty \mid \text{supp}\ \psi$, then $|\psi(b)| < 1$. Since $b \mid \text{supp}\ \psi$ is not invertible in

$H^\infty \mid \text{supp}\ \psi$, there exists $\tau \in M(H^\infty \mid \text{supp}\ \psi) = \{\eta \in M(H^\infty) : \text{supp}\ \eta \subseteq \text{supp}\ \psi\}$ with $\tau(b) = 0$. By Theorem 2.5, $\tau \in \overline{\{z_m\}}$ and hence by Theorem 2.12, supp τ is a maximal support set. Therefore supp τ = supp ψ. Since $\chi_G(z_m) \to 1$ as $m \to \infty$, we have $\tau(\chi_G) = 1$. Thus supp $\tau \subseteq G$, so supp $\psi \subseteq G$. Thus for any $\psi \in M(H^\infty + C)$ either $\bar{b} \mid \text{supp}\ \psi \in H^\infty \mid \text{supp}\ \psi$ or $\chi_F \mid \text{supp}\ \psi \in H^\infty \mid \text{supp}\ \psi$. By Theorem 2.11 on each QC level set E_t we have $\bar{b} \mid E_t \in H^\infty \mid E_t$ or $\chi_F \mid E_t \in H^\infty \mid E_t$. Consider the QC level set E_{t_0} containing supp φ_0. Since $\bar{b} \mid \text{supp}\ \varphi_0 \notin H^\infty \mid \text{supp}\ \varphi_0$ we must have $\chi_F \mid E_{t_0} \in H^\infty \mid E_{t_0}$. By Theorem 2.8 $\chi_F \mid E_{t_0}$ is constant. By (3) above, we must have $E_{t_0} \subseteq F$.

Let $\{\varphi_{n_\alpha}\}$ be a subnet of $\{\varphi_n\}$ such that $\varphi_{n_\alpha} \to \varphi_0$. Choose $\psi_n \in \text{supp}\ \varphi_n$. Then some subnet of $\{\psi_{n_\alpha}\}$ converges. We may assume without loss of generality that $\psi_{n_\alpha} \to \psi_0$. Therefore $\psi_0 \mid QC = \varphi_0 \mid QC$. Hence $\psi_0 \in E_{t_0}$. By Theorem 2.1, $\{\psi_0\}$ is a maximal antisymmetric set for $H^\infty + C$. Since $\varphi_0 \in M(H^\infty) \sim M(L^\infty)$, supp φ_0 consists of more than one point. Since $\psi_0 \in E_{t_0}$ and supp $\varphi_0 \subseteq E_{t_0}$, E_{t_0} is not a maximal antisymmetric set for $H^\infty + C$.

One may ask whether every QC level set contains a point that is in the closure of a sequence of points from distinct L^∞ fibers. The answer to this question is no, as we shall see. We first state a lemma due to Sarason that will appear in [21].

<u>Lemma 2.14</u>. Let b be an inner function. If $\varphi \in M(H^\infty + C)$ and $b \mid \operatorname{supp} \varphi$ is nonconstant, then $b(\operatorname{supp} \varphi) = \partial \mathbb{D}$.

<u>Example 2.15</u>. Let $\{z_n\}_{n=1}^\infty$ be an interpolating sequence with $z_n \to 1$ and let $\varphi \in \overline{\{z_n\}}^{M(H^\infty)}$, $\varphi \in M(H^\infty + C)$. Let E_ψ denote the QC level set containing $\operatorname{supp} \varphi$. Then E_ψ does not contain a point which is in the closure of a sequence of points from distinct fibers.

<u>Proof</u>: Suppose not, that is, suppose $\varphi_0 \in E_\psi$ and $\varphi_0 \in \overline{\{\varphi_n\}}^{M(L^\infty)}$ where $\varphi_n \in M_{\lambda_n}(L^\infty)$ and $\lambda_n \neq \lambda_m$ for $n \neq m$. Let b be the interpolating Blaschke product with zeroes $\{z_n\}_{n=1}^\infty$. By assumption $z_n \to 1$. Thus b is continuous at λ for $\lambda \neq 1$. Hence $b \mid M_\lambda(L^\infty)$ is constant for $\lambda \neq 1$. Let $U = \{\psi \in M(L^\infty) : |\psi(b) - \varphi_0(b)| < \frac{1}{2}\}$. Then U is an open set in $M(L^\infty)$ containing φ_0. Therefore $\varphi_0 \in \overline{\{\varphi_n : \varphi_n \in U\}}^{M(L^\infty)}$. By Corollary 2.4, $E_\psi = E_{\varphi_0} \subseteq \overline{\cup \{M_{\lambda_n}(L^\infty) : \varphi_n \in M_{\lambda_n}(L^\infty) \cap U\}}^{M(L^\infty)}$. Hence $E_{\varphi_0} \subseteq \overline{U}$. If $b \mid E_{\varphi_0}$ were nonconstant, then by Lemma 2.14 we have $b(E_{\varphi_0}) = \partial \mathbb{D}$. Therefore there exists $\eta \in E_{\varphi_0}$ with $\eta(b) = -\varphi_0(b)$. Hence $\eta \notin \overline{U}$. Thus $b \mid E_{\varphi_0}$ is constant. Therefore $|\varphi(b)| = 1$ and hence φ cannot be in the closure of the zeroes of b. This contradiction implies the result.

CHAPTER 3

In this chapter we prove some related results about closed subalgebras of L^∞ containing H^∞. A closed subalgebra B of L^∞ containing H^∞ is called a Douglas algebra. Thus $H^\infty + C$ is a Douglas algebra; it is the smallest closed subalgebra of L^∞ properly containing H^∞ [10 p.376]. S.-Y. A. Chang and D.E. Marshall proved the following theorem about Douglas algebras [6], [15].

<u>Theorem 3.1</u>. (Chang-Marshall Theorem) Let B be a closed subalgebra of L^∞ containing H^∞. Then B is generated as an algebra by H^∞ and the set $\mathcal{B}$ where $\mathcal{B} = \{\overline{b} : b \text{ is an interpolating Blaschke product and } \overline{b} \in B\}$.

This theorem was proven in two stages. Chang showed that if B_1 is a Douglas algebra and B_2 is any other closed subalgebra of L^∞ containing H^∞ with $M(B_1) = M(B_2)$, then $B_1 = B_2$. Marshall completed the proof by showing that if B is a closed subalgebra of L^∞ containing H^∞, then there is a set $\mathcal{B}$ of interpolating Blaschke products such that $M(B) = M(H^\infty[\overline{b} : b \in \mathcal{B}])$ where $H^\infty[\overline{b} : b \in \mathcal{B}]$ denotes the algebra generated by H^∞ and $\{\overline{b} : b \in \mathcal{B}\}$. Marshall's half of the proof used the result stated below [26].

Theorem 3.2. Let u be an inner function. For each α, $0 < \alpha < 1$, there exists an interpolating Blaschke product c_α such that

(1) $|c_\alpha(z)| \leq \frac{1}{10}$ if $|u(z)| \leq \alpha$ $(z \in \mathbb{D})$

and

(2) There exists $\beta_\alpha < 1$ such that if $c_\alpha(z) = 0$, then $|u(z)| \leq \beta_\alpha$ $(z \in \mathbb{D})$.

The proof of this theorem yields more information than what was given above. In fact, the proof shows the following:

Theorem 3.3. Let $\{u_\gamma\}$ be a family of inner functions. For any α, $0 < \alpha < 1$, there is an interpolating Blaschke product c_α such that

(1) $|c_\alpha(z)| \leq \frac{1}{10}$ if $\sup\limits_\gamma |u_\gamma(z)| \leq \alpha$ $(z \in \mathbb{D})$

and

(2) There exists $\beta_\alpha < 1$ such that if $c_\alpha(z) = 0$, then $\sup\limits_\gamma |u_\gamma(z)| \leq \beta_\alpha$ $(z \in \mathbb{D})$.

Using this form of Marshall's result we prove a theorem that was proven by D. Sarason (unpublished). Our proof is different than Sarason's proof.

Theorem 3.4. Let $\{B_\gamma\}$ be a family of closed subalgebras of L^∞ containing H^∞. Then

$$M(\bigcap_\gamma B_\gamma) = \overline{\bigcup_\gamma M(B_\gamma)} .$$

If $H^\infty \subseteq B_\gamma$, then the continuous map $\Gamma : M(B_\gamma) \to M(H^\infty)$, defined by restricting each multiplicative linear functional on B_γ to H^∞, is a homeomorphism. Accordingly, we think of $M(B_\gamma)$ as a closed subset of $M(H^\infty)$. We also note that if u is an inner function, then $M(H^\infty[\bar{u}]) = \{\varphi \in M(H^\infty) : |\varphi(u)| = 1\}$. These remarks will be used in the proof of Theorem 3.4. Before we proceed with the proof of Theorem 3.4 we prove a lemma that will be used in the proof of Theorem 3.4.

<u>Lemma 3.5</u>. Let $\varphi \in M(H^\infty+C) \sim \overline{\bigcup_\gamma M(B_\gamma)}$ and let $\{z_n\}$ be a sequence of points of $\mathbb{D}$ such that $\varphi \in \overline{\{z_n\}} \subseteq M(H^\infty) \sim \overline{\bigcup_\gamma M(B_\gamma)}$. Then for each γ, there exists a Blaschke product b_γ such that $\bar{b}_\gamma \in B_\gamma$ and $\sup_n |b_\gamma(z_n)| < 1$.

<u>Proof</u>: Suppose not. Then there exists γ such that whenever b_γ is a Blaschke product invertible in B_γ, we must have $\sup_n |b_\gamma(z_n)| = 1$. For any Blaschke product b let $A_b = \{\psi \in M(H^\infty + C) : |\psi(b)| = 1\} \cap \{\psi \in M(H^\infty + C) : \psi \in \overline{\{z_n\}}\}$. Then A_b is a closed subset of $M(H^\infty + C)$ and if $\bar{b} \in B_\gamma$, then $A_b \neq \phi$. We claim that $\cap \{A_b : b$ is a Blaschke product and $\bar{b} \in B_\gamma\} \neq \phi$. By compactness it is enough to show that if $b_1, b_2, \ldots, b_n$ are Blaschke products and $\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n \in B_\gamma$, then $\bigcap_{j=1}^n A_{b_j} \neq \phi$. Since $\bar{b}_j \in B_\gamma$ for $j = 1,2,\ldots,n$, we have $\prod_{j=1}^n \bar{b}_j \in B_\gamma$. Therefore $A_{\prod_{j=1}^n b_j} \neq \phi$. Since $A_{\prod_{j=1}^n b_j} \subseteq \bigcap_{i=1}^n A_{b_j}$, we see that

$\bigcap_{j=1}^{n} A_{b_j} \neq \phi$. Let $\eta \in \cap \{A_b : b$ is a Blaschke product and $\bar{b} \in B_\gamma\}$. If $\bar{b}_\gamma \in B_\gamma$, then $|\eta(b)| = 1$. Therefore $\eta \in M(B_\gamma)$. But $\eta \in \overline{\{z_n\}} \subseteq M(H^\infty) \sim \overline{\bigcup_\gamma M(B_\gamma)}$ and this contradiction implies the result.

Proof of Theorem 3.4. Clearly $\overline{\bigcup_\gamma M(B_\gamma)} \subseteq M(\bigcap_\gamma B_\gamma)$. Suppose $\varphi \in M(H^\infty + C) \sim \overline{\bigcup_\gamma M(B_\gamma)}$. Thus $M(H^\infty) \sim \overline{\bigcup_\gamma M(B_\gamma)}$ is an open subset of $M(H^\infty)$ containing φ. The Corona Theorem [5] asserts that the open unit disk $\mathbb{D}$ is a dense subset of $M(H^\infty)$. In fact, if $\psi \in M(H^\infty + C)$, then ψ is in the $M(H^\infty)$ closure of a sequence in $\mathbb{D}$ [12]. Thus there exists a sequence of points $\{z_n\} \subseteq \mathbb{D}$ such that $\varphi \in \overline{\{z_n\}}^{M(H^\infty)}$. Without loss of generality we may assume that $\varphi \in \overline{\{z_n\}} \subseteq M(H^\infty) \sim \overline{\bigcup_\gamma M(B_\gamma)}$. By Lemma 3.5 for each γ there exists a Blaschke product b_γ such that $\sup_n |b_\gamma(z_n)| < 1$ and $\bar{b}_\gamma \in B_\gamma$. By raising b_γ to a sufficiently large power, we may assume that $\sup_n |b_\gamma(z_n)| \leq \frac{1}{2}$. By Theorem 3.3 there exists an interpolating Blaschke product c satisfying (1) and (2) of Theorem 3.3 with $\alpha = \frac{1}{2}$. Let $\{w_n\}$ denote the zero sequence of c. By (2) of Theorem 3.3, there exists $\beta < 1$ such that if $c(z) = 0$, then $\sup_\gamma |b_\gamma(z)| \leq \beta$. Hence $\sup_\gamma |b_\gamma(w_n)| \leq \beta$ for all n. By (1) of Theorem 3.3, $|c(z_n)| < \frac{1}{10}$ for all n. We claim that

[a] $\bar{c} \in \bigcap_{\gamma} B_{\gamma}$

and

[b] $|\varphi(c)| < 1$.

Since $|c(z_n)| \leq \frac{1}{10}$ for all n and $\varphi \in \overline{\{z_n\}}$, $|\varphi(c)| \leq \frac{1}{10}$. Thus [b] above holds. To establish [a], suppose [a] does not hold. Then there exists γ_0 such that $\bar{c} \notin B_{\gamma_0}$. Therefore there exists $\eta \in M(B_{\gamma_0})$ with $\eta(c) = 0$. By Theorem 2.5, $\eta \in \overline{\{w_n\}}$. By the remarks made above, $\sup_{\gamma} |b_{\gamma}(w_n)| \leq \beta$ for all n. Therefore $|\eta(b_{\gamma_0})| \leq \beta < 1$. Since $\bar{b}_{\gamma_0} \in B_{\gamma_0}$, η cannot belong to $M(B_{\gamma_0})$ which is a contradiction. Therefore [a] holds. Now [a] and [b] imply that $\varphi \in M(H^{\infty}) \sim M(\bigcap_{\gamma} B_{\gamma})$ and hence $M(\bigcap_{\gamma} B_{\gamma}) = \overline{\bigcup_{\gamma} M(B_{\gamma})}$ as desired.

Theorem 3.1 implies that if $f \in L^{\infty} \sim H^{\infty}$, then $H^{\infty}[f]$ is generated by H^{∞} and the complex conjugates of countably many interpolating Blaschke products. C. Sundberg [23] has shown that L^{∞} is not countably generated as an algebra over H^{∞}. We prove an extension of Sundberg's result below.

<u>Theorem 3.6</u>. Let B be a closed subalgebra of L^{∞} containing $H^{\infty} + C$. If there exists an open set $\mathcal{O}$ in $M(H^{\infty} + C)$ such that $\phi \neq \mathcal{O} \cap M(B) \subseteq M(L^{\infty})$, then B is not countably generated as an algebra over H^{∞}.

<u>Proof</u>: Suppose B is generated by H^{∞} together with countably many L^{∞} functions. By Theorem 3.1

and remarks made above we see that B must be generated by H^∞ together with the complex conjugates of countably many interpolating Blaschke products $\{b_j : j \in \mathbb{N}\}$. Following a method of Axler [1] we construct a single Blaschke product b such that $M(H^\infty[\overline{b}]) \subseteq M(B)$. To do this, let $\{z_{n,j}\}_{n=1}^{\infty}$ denote the zeroes of b_j. Since the zeroes of b_j form a Blaschke sequence, there exists an integer N_j such that $\sum_{n \geq N_j} (1 - |z_{n,j}|) < \frac{1}{2^j}$. Let b denote the Blaschke product with zeroes $\bigcup_{j=1}^{\infty} \{z_{n,j}\}_{n_j \geq N}$. If $\varphi \in M(H^\infty + C) \sim M(B)$, then there exists $j_0 \in \mathbb{N}$ such that $|\varphi(b_{j_0})| < 1$. Therefore $b_{j_0} | \operatorname{supp} \varphi$ is not invertible in $H^\infty | \operatorname{supp} \varphi$ and hence there exists $\tau \in M(H^\infty | \operatorname{supp} \varphi)$ such that $\tau(b_{j_0}) = 0$. By Theorem 2.5 $\tau \in \overline{\{z_{n,j_0}\}}_{n=1}^{\infty}$ and hence $\tau(b) = 0$. Thus $b | \operatorname{supp} \tau$ is not constant. Since $\operatorname{supp} \tau \subseteq \operatorname{supp} \varphi$, we see that $b | \operatorname{supp} \varphi$ is non-constant. Therefore $|\varphi(b)| < 1$ and $\varphi \in M(H^\infty + C) \sim M(H^\infty[\overline{b}])$. Thus $M(B) \supseteq M(H^\infty[\overline{b}])$.

Let $\eta \in \mathcal{O} \cap M(L^\infty)$. Since $M(L^\infty)$ has no interior in $M(H^\infty + C)$, we can choose a net $\{\eta_\alpha\} \subseteq M(H^\infty + C) \sim M(L^\infty)$ with $\eta \in \overline{\{\eta_\alpha\}}$. Since $\eta \in \mathcal{O}$ we may assume that $\{\eta_\alpha\} \subseteq \mathcal{O}$. By a result of D.J. Newman [17], there exists a Blaschke product c such that if $\varphi \in M(H^\infty + C)$ and $|\varphi(b)| < 1$, then $\varphi(c) = 0$. Since $\mathcal{O} \cap M(B) \subseteq M(L^\infty)$, and $M(H^\infty[\overline{b}]) \subseteq M(B)$, we have $\mathcal{O} \cap M(H^\infty[\overline{b}]) \subseteq M(L^\infty)$. For each α, $\eta_\alpha \in \mathcal{O}$ and $\eta_\alpha \in M(H^\infty + C) \sim M(L^\infty)$. Hence

$|\eta_\alpha(b)| < 1$. Therefore $\eta_\alpha(c) = 0$ for all α. Thus $\eta(c) = 0$. However $\eta \in M(L^\infty)$ so $|\eta(c)| = 1$, which is a contradiction. Therefore B cannot be countably generated over H^∞.

Many examples can be given of Douglas algebras that are generated by H^∞ together with one L^∞ function. For example, $H^\infty + C = H^\infty[\bar{z}]$. Donald Marshall showed in [16] that if f is a simple function, then there is an interpolating Blaschke product b such that $H^\infty[f] = H^\infty[\bar{b}]$. It is not known which algebras have the form $H^\infty[\bar{b}]$ for a single interpolating Blaschke product b. We use the following result of Sarason [21] to give two examples of Douglas algebras that are not generated by H^∞ and the complex conjugate of just one Blaschke product. The first is an example of a Douglas algebra which is generated by H^∞ and one function $f \in L^\infty$, but is not generated by H^∞ and the complex conjugate of one inner function.

<u>Theorem 3.7</u>. (Sarason [21]) There exists a function $f \in H^\infty$ such that $f \mid M_1(L^\infty)$ is real valued and f is nonconstant on some QC level set contained in $M_1(L^\infty)$.

<u>Example 3.8</u>. Let f be the function given in Theorem 3.7. Then $H^\infty[\bar{f}]$ does not equal $H^\infty[\bar{u}]$ for any inner function u.

<u>Proof</u>: Suppose $H^\infty[\overline{f}] = H^\infty[\overline{u}]$ for some inner function $\overline{u}$. By assumption $f \mid M_1(L^\infty)$ is real valued, hence f must be constant on the support set of every multiplicative linear functional $\varphi \in M_1(H^\infty)$. Thus $M_1(H^\infty) \subseteq M(H^\infty[\overline{f}]) = M(H^\infty[\overline{u}])$. Therefore for each $\varphi \in M_1(H^\infty)$, $u \mid \text{supp}\,\varphi$ is constant. We claim that there exists an open interval $I \subseteq \partial\mathbb{D}$ such that $1 \in I$ and such that for each $\lambda \in I$ and each $\varphi \in M_\lambda(H^\infty)$ we have $\overline{u} \mid \text{supp}\,\varphi \in H^\infty \mid \text{supp}\,\varphi$. If not, there exists a sequence of points $\{\lambda_n\}$ with $\lambda_n \to 1$, and $\varphi_n \in M_{\lambda_n}(H^\infty)$ with $\varphi_n(u) = 0$. By compactness there exists $\varphi \in \overline{\{\varphi_n\}}^{M(H^\infty)} \cap M_1(H^\infty)$ with $\varphi(u) = 0$. Since $u \mid \text{supp}\,\varphi$ is constant, this is a contradiction. Therefore there exists an open interval $I \subseteq \partial\mathbb{D}$ such that $1 \in I$ and $u \mid \text{supp}\,\varphi$ is constant for all $\varphi \in M_\lambda(H^\infty)$ whenever $\lambda \in I$. Let $E_t \subseteq M_1(L^\infty)$ be a QC level set such that $f \mid E_t$ is nonconstant. Choose a continuous function g such that the (closed) support of g is contained in I and $g(1) = 1$. If $\varphi \in M_\lambda(H^\infty)$ and $\lambda \in I$, then $g \mid \text{supp}\,\varphi$ is constant and $u \mid \text{supp}\,\varphi$ is constant. Thus $f \mid \text{supp}\,\varphi$ is constant. Therefore $g\overline{f} \mid \text{supp}\,\varphi \in H^\infty \mid \text{supp}\,\varphi$. If $\lambda \in \partial\mathbb{D} \sim I$, then $g\overline{f} \mid \text{supp}\,\varphi$ is identically zero. By Theorem 1.3 $g\overline{f} \in H^\infty + C$. Since $\overline{g}f \in H^\infty + C$, we have $g\overline{f} \in QC$. Therefore $g\overline{f} \mid E_t$ is constant. Hence $\overline{f} \mid E_t$ is constant, which is a contradiction. Hence $H^\infty[\overline{f}]$ does not equal $H^\infty[\overline{u}]$ for any inner function u.

The second example shows that the intersection of two algebras of the form $H^\infty[\bar{b}_1]$ and $H^\infty[\bar{b}_2]$, with b_1 and b_2 interpolating Blaschke products is not necessarily of the form $H^\infty[\bar{c}]$ where c is an inner function. We remark that if b_1 and b_2 satisfy $\lim_{|z|\to 1} \max(|b_1(z)|,|b_2(z)|) = 1$, then it has been shown [2] that $H^\infty[\bar{b}_1] \cap H^\infty[\bar{b}_2] = H^\infty + C$. Our example shows that it is possible to have $\lim_{z\to 1} \max(|b_1(z)|,|b_2(z)|) = 1$ (note that $|z| \to 1$ has been replaced by $z \to 1$) and such that $H^\infty[\bar{b}_1] \cap H^\infty[\bar{b}_2]$ is not of the form $H^\infty[\bar{u}]$ where u is an inner function.

The second example is similar to the first. We will use the result of D. Marshall [15] stated below.

<u>Theorem 3.9</u>. Let S be a set contained in $M(L^\infty)$ and containing more than one point. Then there exists an interpolating Blaschke product b which is nonconstant on S.

<u>Lemma 3.10</u>. Let E_t be a QC level set containing more than one point. Then
$$\bigcup_{\substack{S \text{ nontrivial} \\ S \text{ maximal antisymmetric}}} \{S : S \subseteq E_t\}$$
is dense in E_t.

<u>Proof</u>: If this is not the case, then there exists a clopen subset F of E_t with

$$\overline{\bigcup_{\substack{S \text{ nontrivial} \\ S \text{ maximal antisymmetric}}} \{S : S \subseteq E_t\}} \subseteq F \subseteq E_t \,.$$

Then $M(H^\infty \mid E_t) = \{\varphi \in M(H^\infty) : \operatorname{supp} \varphi \subseteq F\} \cup (E_t \sim F)$.

Therefore for each $\varphi \in M(H^\infty \mid E_t)$, $\chi_F \mid \text{supp}\,\varphi \in H^\infty \mid \text{supp}\,\varphi$. Thus $\chi_F \in H^\infty \mid E_t$. This implies $M(H^\infty \mid E_t)$ is disconnected, contradicting Corollary 2.9.

Example 3.11. There exist interpolating Blaschke products b_1 and b_2 such that for every antisymmetric set $S \subseteq M_1(L^\infty)$, either $b_1 \mid S$ is constant or $b_2 \mid S$ is constant, but on some QC level set $E_t \subseteq M_1(L^\infty)$ neither $b_1 \mid E_t$ is constant nor $b_2 \mid E_t$ is constant.

Proof: By Theorem 3.7 there exists $f \in H^\infty$ real valued on $M_1(L^\infty)$ and a QC level set $E_t \subseteq M_1(L^\infty)$ such that $f \mid E_t$ is nonconstant. Since the nontrivial maximal antisymmetric sets contained in E_t are dense in E_t, f must assume different values on two maximal antisymmetric sets S_1 and S_2 contained in E_t. Without loss of generality we may assume that $f(S_1) = 0$ and $f(S_2) = 1$. Let $\mathcal{O}_1 = \{\varphi \in M(H^\infty) : |\varphi(f) - 1| > \frac{1}{2}\}$ and $\mathcal{O}_2 = \{\varphi \in M(H^\infty) : |\varphi(f) - 1| < \frac{1}{4}\}$. Then $M(H^\infty \mid S_j) \subseteq \mathcal{O}_j$ for $j = 1,2$. By Theorem 3.9, there exist interpolating Blaschke products b_1 and b_2 such that $b_1 \mid S_1$ is nonconstant and $b_2 \mid S_2$ is nonconstant. Therefore there exists $\varphi_j \in M(H^\infty \mid S_j)$ $j = 1,2$ with $\varphi_j(b_j) = 0$ $j = 1,2$. Let $\{z_{n,1}\}_{n=1}^\infty$ and $\{z_{n,2}\}_{n=1}^\infty$ denote the zero sequences of b_1 and b_2, respectively. Without loss of generality we may assume that $\overline{\{z_{n,1}\}}^{M(H^\infty)} \subseteq \mathcal{O}_1$ and $\overline{\{z_{n,2}\}}^{M(H^\infty)} \subseteq \mathcal{O}_2$.

Let S be a maximal antisymmetric set contained in $M_1(L^\infty)$. Then either $M(H^\infty \mid S) \subseteq \mathcal{O}_1^c$, in which case

$b_1 \mid S$ is constant (since for any blaschke product either $b(S) = \partial \mathbb{D}$ or $b \mid S$ is constant) or $M(H^\infty \mid S) \cap \mathcal{O}_1 \neq \phi$. Since $f \in H^\infty$ and $f \mid M_1(L^\infty)$ is real valued, $f \mid S$ is constant. Therefore we must have $M(H^\infty \mid S) \subseteq \mathcal{O}_1$. Hence $b_2 \mid S$ is constant. Clearly neither b_1 nor b_2 is constant on E_t and we are done.

We remark that Example 3.9 shows that Theorem 2.11 does not generalize to the case where $f \mid \text{supp}\ \varphi \in H^\infty \mid \text{supp}\ \varphi$ or $g \mid \text{supp}\ \varphi \in H^\infty \mid \text{supp}\ \varphi$ for all $\varphi \in M_1(H^\infty + C)$.

<u>Example 3.12</u>. There exist interpolating Blaschke products b_1 and b_2 such that $H^\infty[\overline{b}_1] \cap H^\infty[\overline{b}_2]$ is not generated by H^∞ and the complex conjugate of a single inner function.

<u>Proof</u>: Choose b_1, b_2, E_t, φ_1 and φ_2 as in Example 3.11. Suppose there is an inner function u such that

$$H^\infty[\overline{b}_1] \cap H^\infty[\overline{b}_2] = H^\infty[\overline{u}]$$

Then by Example 3.11 $M_1(H^\infty) \subseteq M(H^\infty[\overline{b}_1]) \cup M(H^\infty[\overline{b}_2]) \subseteq M(H^\infty[\overline{u}])$. As was shown in Example 3.8, there exists an open interval $I \subseteq \partial \mathbb{D}$ containing 1, such that $M_\lambda(H^\infty) \subseteq M(H^\infty[\overline{u}]) = M(H^\infty[\overline{b}_1] \cap H^\infty[\overline{b}_2])$ for all $\lambda \in I$. By Theorem 3.4, $M_\lambda(H^\infty) \subseteq M(H^\infty[\overline{b}_1]) \cup M(H^\infty[\overline{b}_2])$ for $\lambda \in I$. Let g be a continuous function such that $\text{supp}\ g \subseteq I$ and $g(1) = 1$. Let $\varphi \in M(H^\infty + C)$. If $\varphi \in M_\lambda(H^\infty)$ and $\lambda \in \partial \mathbb{D} \sim I$, then

$g\bar{b}_1 \mid \text{supp}\ \varphi$ and $g\bar{b}_2 \mid \text{supp}\ \varphi$ are both identically zero. If $\varphi \in M_\lambda(H^\infty)$ and $\lambda \in I$ then $\varphi \in M(H^\infty[\bar{b}_1]) \cup M(H^\infty[\bar{b}_2])$. Therefore either $b_1 \mid \text{supp}\ \varphi$ is constant or $b_2 \mid \text{supp}\ \varphi$ is constant. Hence for any $\varphi \in M(H^\infty + C)$ either $g\bar{b}_1 \mid \text{supp}\ \varphi \in H^\infty \mid \text{supp}\ \varphi$ or $g\bar{b}_2 \mid \text{supp}\ \varphi \in H^\infty \mid \text{supp}\ \varphi$. By Theorem 2.11, we have either $g\bar{b}_1 \mid E_t \in H^\infty \mid E_t$ or $g\bar{b}_2 \mid E_t \in H^\infty \mid E_t$. Since $g \mid M_1(L^\infty)$ is identically 1, either $\bar{b}_1 \mid E_t \in H^\infty \mid E_t$ or $\bar{b}_2 \mid E_t \in H^\infty \mid E_t$ which is a contradiction.

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