ABELIAN VARIETIES ASSOCIATED TO CLIFFORD ALGEBRAS

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ABSTRACT

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The Kuga-Satake construction is a construction in algebraic geometry which associates an abelian variety to a polarized K3-surface X. This abelian variety, A, is created from the Clifford algebra arising from the quadratic space $H^2(X, \mathbb{Z})/torsion$ with its natural cohomology pairing. Furthermore, there is an inclusion of Hodge structures $H^2(X, \mathbb{Q}) \hookrightarrow H^1(A, \mathbb{Q}) \otimes H^1(A, \mathbb{Q})$ relating the cohomology of the original $K3$-surface with that of the abelian variety. We investigate when this construction can be generalized to both arbitrary quadratic forms as well as higher degree forms. Specifically, we associate an abelian variety to the Clifford algebra of an arbitrary quadratic form in a way which generalizes the Kuga-Satake construction. When the quadratic form arises as the intersection pairing on the middle-dimensional cohomology of an algebraic variety Y, we investigate when the cohomology of the abelian variety can be related to that of Y. Additionally, we explore when families of algebraic varieties give rise to families of abelian varieties via this construction. We use these techniques to build an analogous method for constructing an abelian variety from the generalized Clifford algebra of a higher degree form. We find certain families of complex projective 3-folds and 4-folds for which an abelian variety can be constructed from the respective cubic and quartic forms on H^2. The relations between the cohomology of the abelian variety and the original variety are also discussed.

To my family.

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Chapter 1

Introduction

The construction of the Jacobian variety associated to a complex projective curve is classical in algebraic geometry. Geometric questions about projective algebraic curves can often be translated to Jacobian varieties. This is attractive because various tools developed for abelian varieties can be used in addressing these questions. In fact, Torelli's Theorem tells us that two curves are isomorphic if and only if their Jacobian varieties are isomorphic. Rephrased in terms of Hodge structures, we have that two curves C_1 and C_2 are isomorphic if and only if there is an isomorphism of integral Hodge structures $H^1(C_1, \mathbb{Z}) \cong H^1(C_2, \mathbb{Z})$ which respects the intersection pairing. For higher dimensional algebraic varieties, it is thus desirable to have naturally associated abelian varieties which retain some geometric information of the initial variety. The Albanese variety associated to a surface is such an example. However, very often the Albanese variety is a point, and not too useful. Other constructions of this flavor are the intermediate Jacobian for cubic 3-folds [CG72], and the Kuga-Satake variety associated to a K3 surface [KS67]. In this paper, we study the extensions of the Kuga-Satake construction to arbitrary quadratic forms and higher degree forms.

Let V be an integral, weight 2 polarized Hodge structure with $\dim V^{2,0} = 1$. The Kuga-Satake construction associates an abelian variety A (which we call the Kuga-Satake variety) to such a V. Furthermore, there is an inclusion of weight 2 Hodge structures

$$V \otimes \mathbb{Q} \hookrightarrow H^1(A, \mathbb{Q}) \otimes H^1(A, \mathbb{Q}). \qquad (*)$$

This was applied to $V = H^2(X, \mathbb{Z})/torsion$ for a polarized K3 surface X. However, it works in general for complex projective surfaces with $h^{2,0} = 1$.

We briefly explain this construction, see Chapter 2 for more detail. Recall that there is a Clifford algebra associated to the intersection form q on $V = H^2(X, \mathbb{Z})/torsion$. The even Clifford algebra $C_q^+(V_{\mathbb{R}})$ over $\mathbb{R}$ can be given a complex structure, and this has a full lattice given by the inclusion of the even Clifford algebra $C_q^+(V)$ over $\mathbb{Z}$ into $C_q^+(V_{\mathbb{R}})$. Thus the quotient is a complex torus. This complex torus admits a very ample line bundle, and hence an embedding into projective space, making it an abelian variety. This construction relies heavily upon the fact that $\dim V^{2,0} = 1$ (see [KS67] or [vG00, Proposition 5.9] for a modern treatment).

The Kuga-Satake construction has helped a great deal in studying questions related to K3 surfaces. Deligne used this construction to prove the Weil conjectures for K3 surfaces [Del72]. Furthermore, there is a Torelli-type theorem for K3 surfaces due to Pyatetskii-Shapiro and Shafarevich [IIPS71]. It says that two polarized K3 surfaces X_1 and X_2 are isomorphic if and only if there is an isomorphism of Hodge structures $H^2(X_1, \mathbb{Z}) \cong H^2(X_2, \mathbb{Z})$ which is compatible with the intersection pairing. In other words, the Kuga-Satake variety completely determines the K3 surface.

This construction has been generalized beyond the $K3$-surface case. In [Mor85], Morrison notes that the middle dimensional cohomology of abelian surfaces satisfies the hypothesis of the Kuga-Satake construction, and computes the abelian variety associated to an abelian surface. Voison [Voi05] provides an alternative viewpoint of the Kuga-Satake construction by noting that the Kuga-Satake variety can be obtained from the weight 2 Hodge structure on the exterior algebra $\bigwedge^* V$.

The basic ingredient of the Kuga-Satake construction is a quadratic form on a free $\mathbb{Z}$-

module of finite rank V. When V is as in the Kuga-Satake construction, the signature of q is $(\dim V - 2, 2)$. Noting this, we consider the following questions: can we create an abelian variety A from the Clifford algebra of an arbitrary quadratic form in a meaningful way? If so, can we recover the Kuga-Satake variety when q has signature $(\dim V - 2, 2)$? Finally, are there other polarized Hodge structures on V for which we can construct an abelian variety with an inclusion of Hodge structures of the form $(*)$ above?

Chapter 3 provides an affirmative answer to these questions. Namely, in Theorem 3.2.8 we show that for a quadratic form in two or more variables with signature other than $(1, 1)$, we can create an abelian variety from the Clifford algebra. Furthermore, Proposition 3.4.4 discusses an inclusion of Hodge structures as in $(*)$. Finally, Proposition 3.4.3 shows that we do in fact recover the Kuga-Satake variety when the quadratic form is as in the Kuga-Satake construction.

In Chapter 4, we show that our construction works in families. We begin with a continuously varying family of polarized Hodge structures (variation of Hodge structure), instead of just a single polarized Hodge structure. In Theorem 4.2.2 we prove, under some hypotheses, that the corresponding abelian varieties obtained from our construction also vary continuously in a family. Theorem 4.2.8 provides a situation in which the hypotheses are satisfied, and we conclude with geometric examples, including a proof that the Kuga-Satake construction works in families.

The remainder of the paper applies our techniques to higher degree forms. The Lefshetz Hyperplane Theorem suggests that for a complex projective variety of dimension n, the most interesting cohomology resides in H^n. For K3 surfaces this is certainly the case, as only H^2 contains nontrivial information. On the other hand, intermediate Jacobians can be constructed from any H^k for k odd, so interesting information can be found outside the middle-dimensional cohomology groups. Our idea is to consider higher degree forms associated to a variety from the intersection form on H^2. For example, consider a complex, projective 3-fold Y. Then the (trilinear) intersection form

$$H^2(Y,\mathbb{Q}) \otimes H^2(Y,\mathbb{Q}) \otimes H^2(Y,\mathbb{Q}) \to \mathbb{Q}$$

yields a cubic form on the vector space $H^2(Y,\mathbb{Q})$. In general, the intersection form on H^2 of a d-dimensional complex variety yields a degree d form. The question we ask is: can we construct an abelian variety from a degree d form in a manner similar to that of the Kuga-Satake construction? And can we say anything meaningful in regards to the Hodge structure?

There is a notion of a "generalized" Clifford algebra C_f for an arbitrary degree d form f which has been studied by various authors including Roby [Rob69], Revoy [Rev77], and Childs [Chi78]. It is the $\mathbb{Z}/d\mathbb{Z}$-graded associative algebra

$$C_f(K) = K\langle x_1, \dots x_n\rangle / \big((a_1x_1 + \cdots + a_nx_n)^d - f(a_1,\dots,a_n) : a_i \in K\big)$$

A similar scenario to the Kuga-Satake construction would be to say that $C_f^0(\mathbb{Z})$ is a full lattice in $C_f^0(\mathbb{R})$, put a complex structure on $C_f^0(\mathbb{R})$ so that the quotient is a complex torus, and show that the complex torus is in fact an abelian variety. Unfortunately, when $d \geq 3$ and the form has two or more variables, the generalized Clifford algebra is infinite dimensional (this is well-known, see for example [BHS88, Theorem 1.8]). This idea needs modification since we don't want an infinite dimensional torus.

To remedy this, we will take a finite dimensional, graded representation of $C_f(\mathbb{C})$ into $M_N(\mathbb{C})$ for some N, and look at the induced representations on $C_f(\mathbb{R})$ and $C_f(\mathbb{Z})$. We denote the images by $c_f(\mathbb{R})$ and $c_f(\mathbb{Z})$, respectively. Since we have chosen a graded representation, $c_f^0(\mathbb{R})$ and $c_f^0(\mathbb{Z})$ will play the roles of the vector space and lattice, respectively.

The idea of considering finite-dimensional representations of C_f is not new. Van den Bergh [VdB87] shows that there is a one-to-one correspondence between finite dimensional representations of C_f and certain vector bundles (called Ulrich bundles) on the hypersurface $X_f = Z(w^d - f)$. There is an automorphism σ of X_f defined by $w \mapsto \zeta w$ for a primitive d^{th} root of unity ζ. Under Van den Bergh's correspondence, graded representations correspond to vector bundles which are invariant under the automorphism σ of X_f. Representations of C_f and their properties are also discussed in [HT88, CKM12].

Chapter 5 first discusses background material and existence of graded representations of C_f. The initial question is whether or not $c_f^0(\mathbb{R})$ can be given a complex structure for which the quotient $c_f^0(\mathbb{R})/c_f^0(\mathbb{Z})$ is an abelian variety. Section 5.2 puts conditions on the representation which allow for the construction of an abelian variety. Under a finite-dimensional, graded representation of the Clifford algebra, there is no guarantee that $c_f^0(\mathbb{Z})$ is a full lattice inside of $c_f^0(\mathbb{R})$. We discuss this in Section 5.3. This is the reason we restrict to cubic and quartic forms. Section 5.4 provides several examples of 3-folds and 4-folds for which an abelian variety can be constructed from the intersection form on H^2. Section 5.5 specializes to 4-folds with are products of surfaces. We discuss how the abelian variety associated to H^2 of a surface relates to the abelian variety associated to H^2 of the 4-fold. In particular, we show that the Kuga-Satake variety of a K3 surface is a subquotient of the abelian variety associated to the product of K3 surfaces. We conclude with a discussion of when this construction works in families.

Chapter 2

Background Material

This chapter contains the background information which will be used throughout paper. We discuss Hodge structures and the way in which they arise in geometry. We then treat abelian varieties and Clifford algebras. The chapter concludes with the Kuga-Satake construction.

2.1 Hodge Structures

Definition 2.1.1. *Let V be a free $\mathbb{Z}$-module of finite rank. A (pure)* Hodge structure *of weight k on V is a decomposition*

$$V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}$$

of $V_{\mathbb{C}} := V \otimes_{\mathbb{Z}} \mathbb{C}$ into subspaces $V^{p,q}$ with $\overline{V^{p,q}} = V^{q,p}$, where $\overline{v \otimes z} := v \otimes \bar{z}$.

Given such a decomposition, we define the Hodge filtration

$$V_{\mathbb{C}} \supset \cdots F^p \supset F^{p+1} \supset \cdots \supset 0$$

by $F^p = \bigoplus_{r \geq p} V^{r,r-p}$. Note that $V^{p,q} = F^p \cap \overline{F^{k-p+1}}$, so that equivalent data is obtained from either a Hodge structure or a Hodge filtration.

In the case where V is a finite dimensional $\mathbb{Q}$-vector space, the definition is similar. To distinguish between the two cases, we will say *integral* Hodge structure when V is a free $\mathbb{Z}$-module of finite rank, and *rational* Hodge structure when V is a $\mathbb{Q}$-vector space. We say a Hodge structure is of *type* $T = \{(p_1, q_1), \ldots, (p_n, q_n)\}$ if $V^{r,s} = 0$ unless $(r, s) \in T$. When we don't specify the type T of a Hodge structure, we assume that the $p_i, q_j \geq 0$. This assumption is due to the following example.

Example 2.1.2. *Let X be a smooth, complex projective variety. Thus X can be considered as a nonsingular complex manifold. The singular cohomology group $V_{\mathbb{Q}} = H^k(X, \mathbb{Q})$ has a natural Hodge structure of weight k with $V^{p,q} = H^q(X, \Omega^p)$ where Ω^p be the sheaf of differential p-forms. This is the example of a Hodge structure that we will have in mind throughout the paper. Similarly, $V_{\mathbb{Z}} = H^k(X, \mathbb{Z})/torsion$ has an integral Hodge structure of weight k.*

We now list several important notions which pertain to Hodge structures.

(i) The *Tate structure* $\mathbb{Z}(k)$ is defined to be the free $\mathbb{Z}$-module $V = (2\pi i)^k\mathbb{Z} \subset \mathbb{C}$ with $V^{-k,-k} = V_{\mathbb{C}}$. This is a Hodge structure of type $\{(-k,-k)\}$ and weight $-2k$. We define $\mathbb{Q}(k)$ analogously, by replacing $\mathbb{Z}$ with $\mathbb{Q}$. For our purposes, we will not need the factor of $(2\pi i)^k$ in the definition of $\mathbb{Z}(k)$, so we will drop it to simplify notation.

(ii) If V has an integral Hodge structure of weight k, we define a weight $-k$ integral Hodge structure on $V^\vee = \mathrm{Hom}_{\mathbb{Z}}(V, \mathbb{Z})$ by setting $(V^\vee)^{-p,-q} = \mathrm{Hom}_{\mathbb{C}}(V^{p,q}, \mathbb{C})$.

(iii) If V and W have Hodge structures of weight n and m, respectively. Then $V \otimes W$ has a Hodge structure of weight $n + m$ given by

$$(V \otimes W)^{p,q} = \oplus (V^{p_1,q_1} \otimes W^{p_2,q_2}) \text{ where } p_1 + p_2 = p \text{ and } q_1 + q_2 = q.$$

Since $\mathrm{Hom}(V, W) \cong V^\vee \otimes W$, we see that $\mathrm{Hom}(V, W)$ has a natural Hodge structure of weight $m - n$. We write $V(k)$ for $V \otimes \mathbb{Z}(k)$, which has weight $n - 2k$ for any k. In particular, $V(0) = V$.

(iv) We define a *morphism of Hodge structures* to be a linear map $f : V \to W$ such that the $\mathbb{C}$-linear extension satisfies $f_{\mathbb{C}}(V^{p,q}) \subset W^{p,q}$. Such a morphism is necessarily 0, unless V and W have the same weight. If V has weight n, W has weight $n+2k$, and we have a linear map $f : V \to W$ satisfying $f_{\mathbb{C}}(V^{p,q}) \subset W^{p+k,q+k}$, then we obtain a morphism of Hodge structures $V \to W(k)$ given by $v \mapsto f(v)$.

Definition 2.1.3. *An* algebraic representation *of $\mathbb{C}^*$ on V is defined to be a morphism of real algebraic groups $\rho : \mathbb{C}^* \to GL(V_{\mathbb{R}})$.*

Proposition 2.1.4. *There is a bijection between rational Hodge structures of weight k on V and algebraic representations $\rho : \mathbb{C}^* \to GL(V_{\mathbb{R}})$ with $\rho(t) = t^k$ for $t \in \mathbb{R}^*$. The Hodge structure defined by ρ is the decomposition $V^{p,q} := \{v \in V_{\mathbb{C}} : \ \rho(z)v = z^p \bar{z}^q v\}$.*

Proof. See [vG00, Proposition 1.4]. □

Using this correspondence, we now translate (i)-(iv) above:

(i) The Tate structure $\mathbb{Z}(k)$ corresponds to $V = \mathbb{Z} \subset \mathbb{C}$ with $\rho(z)(v) = (z\bar{z})^{-k}v$, for $v \in \mathbb{Z}(k)$.

(ii) If V has representation $\rho : \mathbb{C}^* \to GL(V_{\mathbb{R}})$, $V^\vee$ has representation $\rho^* : \mathbb{C}^* \to GL(V^\vee)$ defined by $\rho^*(z)(\varphi) : v \mapsto \varphi(\rho^{-1}(z)(v))$, for $\varphi \in V^\vee$ and $v \in V$.

(iii) If V and W have ρ_V and ρ_W, respectively, then $V \otimes W$ has representation $\rho_V \otimes \rho_W$.

(iv) A morphism of Hodge structures (of same weight) $f : V \to W$ means

$$f(\rho_V(z)(v)) = \rho_W(z)(f(v))$$

Definition 2.1.5. *Let V be an integral Hodge structure of weight k with corresponding representation $\rho : \mathbb{C}^* \to GL(V_{\mathbb{R}})$. A* polarization *is a morphism of Hodge structures*

$$Q : V \otimes V \to \mathbb{Z}(-k)$$

such that $Q(v, \rho(i)w)$ is a symmetric, positive definite form on $V_{\mathbb{R}}$. The pair (V, Q) is called a polarized Hodge structure.

Note that $Q(\rho(z)v, \rho(z)w) = (z\bar{z})^k Q(v, w)$ for all $v, w \in V_{\mathbb{R}}$ by definition of a morphism of Hodge structures (see (iv) above). When $k = 1$, the definition of polarization translates to the Riemann bilinear relations. Furthermore, a polarization gives an isomorphism of the weight $-k$ rational Hodge structures $V(k) \cong V^{\vee}$ via $v \otimes 1 \mapsto Q(v, -)$.

Example 2.1.6. *Let X be a complex, projective variety of dimension n as in Example 2.1.2. Denote by ω the class in $H^2(X, \mathbb{Q})$ corresponding to an ample divisor on X. For $\alpha, \beta \in H^k(X, \mathbb{Q})$ with $k \leq n$, define*

$$Q(\alpha, \beta) = \int_X \omega^{n-k} \wedge \alpha \wedge \beta.$$

This is a bilinear form on H^k. Then the primitive cohomology groups $P^k(X, \mathbb{Q})$, together with Q, form a polarized Hodge structure of weight k (this is well-known, see for example [VS02, Theorem 6.32]). As before, identical statements hold with $H^k(X, \mathbb{Q})$ replaced by $H^k(X, \mathbb{Z})/torsion$.

When X is a projective surface, let ω be the divisor corresponding to $\mathcal{O}_X(1)$ relative to some projective embedding. Then $P^2(X, \mathbb{Q})$ is the orthogonal complement of ω in $H^2(X, \mathbb{Q})$ and the polarization is the intersection pairing on H^2.

Lemma 2.1.7. *Let (V, Q) be a polarized Hodge structure of even weight $2k$. Then Q is $(-1)^{k-p}$-definite on the subspace $V_{\mathbb{R}} \cap (V^{p,q} \oplus V^{q,p})$.*

Proof. Let ρ denote the representation corresponding to the Hodge structure on V. Recall that $\rho(i)$ acts as i^{p-q} on the subspace $V^{p,q}$ by Proposition 2.1.4. Since V has weight $2k$,

$$i^{p-q} = i^{p-(2k-p)} = i^{2p-2k} = (-1)^{p-k} = (-1)^{k-p}.$$

Similarly, $i^{q-p} = (-1)^{k-p}$. Since Q is a polarization, we know that $Q(v, \rho(i)v) > 0$ for all $v \in V_{\mathbb{R}}$. Hence $Q(v, \rho(i)v) = (-1)^{k-p}Q(v, v) > 0$ on the subspace $V_{\mathbb{R}} \cap (V^{p,q} \oplus V^{q,p})$. □

If (V, Q) is a polarized Hodge structure of weight $2k$, we can apply the following lemma to the polarized weight 0 Hodge structure on $V(k)$.

Lemma 2.1.8. *Let (V, Q) be a polarized Hodge structure of weight $k = 0$. Let $SO(V)$ denote the special orthogonal group with respect to Q. Then the image of the corresponding representation $\rho : \mathbb{C}^* \to GL(V_{\mathbb{R}})$ lies in $SO(V_{\mathbb{R}})$.*

Proof. We have that $Q(\rho(z)v, \rho(z)w) = (z\bar{z})^k Q(v, w) = Q(v, w)$ since $k = 0$. Therefore, the image lies in $O(V_{\mathbb{R}})$. Since $\mathbb{C}^*$ is connected, the image lands in $SO(V_{\mathbb{R}})$. □

2.2 Abelian Varieties

We recall some basic facts about complex abelian varieties. All the statements here can be found in [BL04].

Let V be a complex vector space of dimension n and $\Lambda \subset V$ a full lattice. This means Λ is a free $\mathbb{Z}$-submodule of V of rank $2n$ and $\Lambda \otimes \mathbb{R} = V$. The lattice Λ acts on V by addition and we define a *complex torus* to be the quotient

$$X := V/\Lambda$$

Addition in V defines an abelian group structure on X. We note that

$$\Lambda \cong \pi_1(X) \cong H_1(X, \mathbb{Z}),$$

since $\pi_1(X)$ is abelian. The Universal Coefficient Theorem gives

$$H^1(X,\mathbb{Z}) \cong \mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{Z}) \cong \mathrm{Hom}(\Lambda,\mathbb{Z}).$$

Furthermore, the Kunneth formula tells us that

$$H^n(X,\mathbb{Z}) \cong \wedge^n H^1(X,\mathbb{Z}).$$

Definition 2.2.1. *For two complex tori X and X', a* homomorphism *is a holomorphic map $f : X \to X'$ which is also a group homomorphism. We say two complex tori X and X' are* isogenous *if there is a surjective homomorphism $X \to X'$ with finite kernel. Such a map is called an* isogeny*. Isogenies induce an equivalence relation on the set of complex tori.*

We recall the definition of the first Chern class of a line bundle. Associated to the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{e^{2\pi i \cdot}} \mathcal{O}_X^* \to 0$$

is the long exact sequence in cohomology

$$\cdots \to H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \to \cdots$$

Identifying $H^1(X,\mathcal{O}_X^*)$ with $Pic(X)$, we define the first Chern class of a line bundle to be its image in $H^2(X,\mathbb{Z})$ under the map c_1.

Since $H^2(X,\mathbb{Z}) \cong \mathrm{Hom}(\Lambda,\mathbb{Z}) \wedge \mathrm{Hom}(\Lambda,\mathbb{Z})$ from above, we can identify the first Chern class of a line bundle on X with a $\mathbb{Z}$-valued alternating form on the lattice Λ.

Proposition 2.2.2. *Let $E : V \times V \to \mathbb{R}$ be an $\mathbb{R}$-linear alternating form. Then E represents the first Chern class of a line bundle if and only if*

- *$E(\Lambda,\Lambda) \subset \mathbb{Z}$, and*
- *$E(iv,iw) = E(v,w)$ for all $v,w \in V$.*

We call a line bundle on X *positive definite* if $E(v, iv) > 0$ for all $v \in V$, where E is the alternating form associated to the line bundle as in Proposition 2.2.2.

Definition 2.2.3. *A* polarization *on a complex torus X is the first Chern class of a positive definite line bundle on X. In particular, a polarization is an alternating form $E : \Lambda \times \Lambda \to \mathbb{Z}$ whose $\mathbb{R}$-linear extension $E : V \times V \to \mathbb{R}$ satisfies*

- $E(iv, iw) = E(v, w)$
- $E(v, iv) > 0$

for all $v, w \in V$. These are called the Riemann bilinear relations*.*

Proposition 2.2.4. *On a complex torus X, a positive definite line bundle is equivalent to an ample line bundle. The third power of such a line bundle is very ample, and hence defines an embedding of X into projective space. Therefore, a polarization on X has a corresponding line bundle whose third power is very ample.*

Definition 2.2.5. *An* abelian variety *is a complex torus with a positive definite line bundle. The above discussion shows that an abelian variety is a complex, projective variety which is also an algebraic group.*

Lemma 2.2.6. *There is a bijection between the set of isomorphism classes of complex tori and the set of isomorphism classes of integral Hodge structures of weight 1.*

$$\left\{ \begin{array}{c} \textit{Complex} \\ \textit{Tori} \end{array} \right\} \xleftrightarrow{\textit{1-1}} \left\{ \begin{array}{c} \textit{Integral Hodge structures} \\ \textit{of weight 1} \end{array} \right\}$$

Furthermore, under this correspondence, abelian varieties correspond to polarized, integral Hodge structures of weight 1.

$$\left\{\begin{array}{c}\textit{Abelian}\\ \\ \textit{Varieties}\end{array}\right\}\overset{\textit{1-1}}{\longleftrightarrow}\left\{\begin{array}{c}\textit{Polarized, integral Hodge structures}\\ \\ \textit{of weight 1}\end{array}\right\}$$

Proof. Let $X = V/\Lambda$ be a complex torus. Then $\Lambda \otimes \mathbb{R} = V$ has a complex structure J. Let $\Lambda^{1,0}$ and $\Lambda^{0,1}$ be the subspaces of $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = V \otimes_{\mathbb{R}} \mathbb{C}$ on which J acts as i and $-i$, respectively. This defines an integral Hodge structure of weight 1 on Λ. Convsersely, suppose $V_{\mathbb{Z}}$ has a weight 1 Hodge structure with $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$. Then we can project $V_{\mathbb{Z}}$ isomorphically into $V^{1,0}$ and $V^{1,0}/V_{\mathbb{Z}}$ is a complex torus.

Note that the definition of a polarization on a complex torus and that of a polarization on a weight 1 integral Hodge structure are identical. The correspondence between abelian varieties and polarized, integral Hodge structures of weight 1 follows immediately. □

Remark 2.2.7. *We note that if we change "integral" to "rational" in the previous lemma, we recover complex tori (and abelian varieties) up to isogeny.*

2.3 Clifford Algebras

Definition 2.3.1. *Let $q(X_1, \ldots, X_n)$ be a quadratic form with coefficients in $\mathbb{Z}$. We define the* Clifford algebra *associated to q to be the associative algebra*

$$C_q(\mathbb{Z}) = \mathbb{Z}\langle x_1, \ldots x_n\rangle / \big((a_1x_1 + \cdots + a_nx_n)^2 - q(a_1, \ldots, a_n) : a_i \in \mathbb{Z}\big)$$

where $\mathbb{Z}\langle x_1, \ldots x_n\rangle$ denotes the tensor algebra in n variables.

For notational convenience, we use capital letters for the variables of the form q and lowercase letters for the corresponding generators of the Clifford algebra. We will often consider $C_q(K) \cong C_q(\mathbb{Z}) \otimes K$ where $K \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$. The tensor algebra is $\mathbb{Z}/2\mathbb{Z}$-graded (into even and odd degree pieces), and since the two-sided ideal defining the Clifford algebra is generated by even degree elements, the grading descends to a $\mathbb{Z}/2\mathbb{Z}$-grading on the Clifford algebra:

$$C_q(\mathbb{Z}) = C_q^+(\mathbb{Z}) \oplus C_q^-(\mathbb{Z})$$

These summands are called the even and odd Clifford algebras, respectively.

For some situations in which we are interested, we will have a quadratic form q on a free $\mathbb{Z}$-module V of finite rank. In this case, the Clifford algebra is

$$C_q(V) = T(V)/\big(v \otimes v - q(v) : v \in V\big)$$

where $T(V)$ is the tensor algebra on V. The original definition of $C_q(\mathbb{Z})$ is recovered by choosing a basis of V. We have $C_q(V_K) \cong C_q(V) \otimes K$ where $V_K = V \otimes K$. We also deal with V a finite dimensional vector space over $\mathbb{Q}$.

For a quadratic space (V, q), we have an anti-involution of the tensor algebra $\iota : T(V) \to T(V)$ which swaps components: $v_1 \otimes \cdots \otimes v_n \mapsto v_n \otimes \cdots \otimes v_1$. Since ι preserves the ideal defining the Clifford algebra, it descends to an anti-automorphism, still denoted ι, of the Clifford algebra (and also the even Clifford algebra).

The *Clifford group* is the algebraic group $CSpin(V_\mathbb{R}) = \{g \in C_q^+(V_\mathbb{R}) : gV_\mathbb{R}g^{-1} = V_\mathbb{R}\}$. In this definition, we identify $V_\mathbb{R}$ with its image in $C_q(V_\mathbb{R})$. Note that $CSpin(V_\mathbb{R})$ naturally acts on $V_\mathbb{R}$ by conjugation. This defines a map $CSpin(V_\mathbb{R}) \to O(V_\mathbb{R})$ into the orthogonal group on $V_\mathbb{R}$, and the image lies in $SO(V_\mathbb{R})$.

2.4 The Kuga-Satake Construction

We now outline the classical Kuga-Satake construction from [Sat66, KS67]. This construction was created with $K3$-surfaces in mind, since the middle dimensional (primitive) cohomology of a $K3$-surface satisfies the assumptions listed below.

Let V be a free $\mathbb{Z}$-module of finite rank n. Suppose V has a weight 2 Hodge structure such that $\dim V^{2,0} = 1$. Such a Hodge structure is said to be of *K3-type*. Furthermore, suppose $-q$ polarizes this Hodge structure. We have that the signature of q is $(n-2,2)$. Then there is a basis of $V_{\mathbb{R}}$ such that $q = -X_1^2 - X_2^2 + X_3^2 + \cdots + X_n^2$. Let x_i represent the generators of the Clifford algebra corresponding to X_i. Then $J := x_1x_2$ has $J^2 = -1$, and so left-multiplication by J induces a complex structure on $C_q^+(\mathbb{R})$. It is easily verified that J (up to sign) is independent of the choice of orthonormal basis for $V_{\mathbb{R}}$. The quotient $C_q^+(\mathbb{R})/C_q^+(\mathbb{Z})$ is then a complex torus. Equivalently, $C_q^+(\mathbb{Z})$ has a weight 1 Hodge structure determined by J.

We are now left with showing that this complex torus is an abelian variety. In other words, the weight 1 Hodge structure on $C_q^+(\mathbb{Z})$ defined above is polarized. By diagonalizing q over $\mathbb{Q}$, one sees that there exists a $t > 0$ such that $tJ \in C_q^+(\mathbb{Z})$. Let $\alpha = tJ$ for the smallest such t. Then the map

$$C_q^+(\mathbb{Z}) \otimes C_q^+(\mathbb{Z}) \to \mathbb{Z}(-1) \quad \text{defined by} \quad x \otimes y \mapsto tr(\alpha\iota(x)y)$$

defines a polarization of the weight 1 Hodge structure on $C_q^+(\mathbb{Z})$. Here $tr(L)$ means trace of the endomorphism of $C_q^+(\mathbb{Z})$ which is left-multiplication by L. We summarize these results in the following theorem

Theorem 2.4.1. *Let (V,q) be an integral, polarized Hodge structure of K3-type. Then $C_q^+(\mathbb{R})$ can be given a complex structure for which the quotient $C_q^+(\mathbb{R})/C_q^+(\mathbb{Z})$ is an abelian variety. We call this quotient the Kuga-Satake variety of (V,q).*

One may wonder how the weight 1 Hodge structure on $C_q^+(\mathbb{Z})$ is related to the original weight 2 Hodge structure on V. We have the result mentioned in the introduction:

Proposition 2.4.2. *Let V be a rational, polarized Hodge structure of K3-type with polarization q. There is an inclusion of weight 2 Hodge structures*

$$V \hookrightarrow C_q^+(V) \otimes C_q^+(V).$$

Furthermore, the Hodge structure on V can be recovered from that of $C_q^+(V)$.

Proof. For a constructive proof, see [Huy15, Proposition 4.2.4]. □

There is an equivalent viewpoint of the Kuga-Satake construction taken by Deligne [Del72] which we outline now. By an abuse of notation, we say a Hodge structure is of $K3$-type if some Tate twist of it is a weight 2 Hodge structure of $K3$-type. For (V, q) a rational, polarized weight 0 Hodge structure of $K3$-type, there is a commutative diagram

$$\begin{tikzcd} & CSpin(V_{\mathbb{R}}) \arrow[d] \\ \mathbb{C}^* \arrow[ur, "\tilde{h}"] \arrow[r, "h"'] & SO(V_{\mathbb{R}}), \end{tikzcd}$$

where h is the map defining the Hodge structure on V from Lemma 2.1.8, $\tilde{h}(a+bi) = a+bJ$ where $J = x_1x_2$ was defined above, and the vertical map is the orthogonal representation defined by conjugation of $CSpin(V_{\mathbb{R}})$ on $V_{\mathbb{R}}$.

Since $CSpin(V_{\mathbb{R}})$ acts on $C_q^+(V_{\mathbb{R}})$ by left-multiplication, this gives rise to a map $\sigma : CSpin(V_{\mathbb{R}}) \to GL(C_q^+(V_{\mathbb{R}}))$. The composition $\sigma \circ \tilde{h}$ defines a Hodge structure on $C_q^+(V)$ by Proposition 2.1.4. This Hodge structure on $C_q^+(V)$ is both polarized and of weight 1 [Del72, Proposition 4.5]. By the equivalence in Lemma 2.2.6, the result is an abelian variety. Furthermore, Deligne shows that this construction works in families (see Definition 4.1.4 for the definition of a variation of Hodge structure):

Theorem 2.4.3. *Let $(\mathcal{V}, \psi)$ be a polarized variation of Hodge structure of $K3$-type over a smooth and connected scheme S of finite type over $\mathbb{C}$. Then there is a finite étale extension*

$\pi : S' \to S$ and an abelian scheme A over S' satisfying: for $s \in S'$, the fiber A_s is the Kuga-Satake variety associated to the polarized Hodge structure on $\mathcal{V}_{\pi(s)}$.

Chapter 3

Abelian Variety from the Clifford Algebra

In this chapter, we show that we can construct an abelian variety from the Clifford algebra of an arbitrary quadratic form in a way that generalizes the Kuga-Satake construction, provided the signature of the form is other than $(1,0),(0,1)$, or $(1,1)$. Our construction involves viewing the Clifford algebra as a subalgebra of a matrix algebra. This approach allows us to build an analogous theory for cubic and quartic forms in the final chapter of this paper.

3.1 Representations of Clifford Algebras

We now discuss representations of Clifford algebras starting with the diagonal form $q = \sum_{i=1}^{n} X_i^2$. The construction is as follows: set

$$\epsilon_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \epsilon_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \epsilon_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Notice that $\epsilon_2 = \epsilon_1 \cdot \epsilon_3^{-1}$ and that $\epsilon_3 \cdot \epsilon_1 = -\epsilon_1 \cdot \epsilon_3$. When n is even, we define a map

$$C_q(\mathbb{C}) \to \overbrace{M_2(\mathbb{C}) \otimes \cdots \otimes M_2(\mathbb{C})}^{n/2} \cong M_{2^{n/2}}(\mathbb{C})$$

by

$$
\begin{aligned}
x_1 &\mapsto \epsilon_3 \\
x_2 &\mapsto i \cdot \epsilon_1 \\
x_3 &\mapsto \epsilon_2 \otimes \epsilon_3 \\
x_4 &\mapsto i \cdot \epsilon_2 \otimes \epsilon_1 \\
&\vdots \\
x_{n-1} &\mapsto \epsilon_2 \otimes \cdots \otimes \epsilon_2 \otimes \epsilon_3 \\
x_n &\mapsto i \cdot \epsilon_2 \otimes \cdots \otimes \epsilon_2 \otimes \epsilon_1,
\end{aligned}
$$

where ϵ_3 means $\epsilon_3 \otimes 1 \otimes \cdots \otimes 1$ inside of the $(n/2)$-fold tensor product. For simplicity, let $E_1, \ldots, E_n$ denote the images of $x_1, \ldots, x_n$, respectively. It is easily verified that $E_j E_k = -E_k E_j$ for any $j < k$ and that $E_i^2 = I$. Hence

$$(c_1 E_1 + \cdots + c_n E_n)^2 = f(c_1, \ldots, c_n) \cdot I$$

so the map is well defined. Since ϵ_1 and ϵ_3 generate all of $M_2(\mathbb{C})$, this map is an isomorphism for dimension reasons.

Remark 3.1.1. *This matrix representation comes from the unique (in the sense of [BHS88, Theorem 3.9]) indecomposable matrix factorization of $X_0^2 + q$ over $\mathbb{C}$ constructed in [BHS88, Example 3.12].*

Now when n is odd, $C_q(\mathbb{C})$ is not isomorphic to a matrix algebra. However, by considering the representation of $C_g(\mathbb{C})$ where $g = q + X_{n+1}^2$ constructed above, we realize $C_q(\mathbb{C})$ as a subalgebra of the matrix algebra $M_{2(n+1)/2}(\mathbb{C})$. We summarize the above discussion in the following proposition:

Proposition 3.1.2. *Let $n > 0$ be arbitrary, set $N = \lfloor (n+1)/2 \rfloor$, and suppose $q = \sum_{i=1}^{n} a_i X_i^2$ is a quadratic form with a_i nonzero integers. Let α_i be a root of $X^2 - a_i$. The map*

$$C_q(\mathbb{C}) \to M_{2^N}(\mathbb{C}) \quad \text{defined by} \quad x_i \mapsto F_i := \alpha_i E_i \tag{3.1}$$

is an isomorphism of algebras when n is even and is a faithful algebra representation when n is odd. Under this representation, the image of $C_q^+(K)$ is

$$K\left\langle \prod_{i=1}^{n} F_i^{m_i} \;\middle|\; m_i \in \{0,1\},\ \sum_{i=1}^{n} m_i \text{ is even} \right\rangle \tag{3.2}$$

where $K \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}\}$.

3.2 Construction of the Abelian Variety

In this section, we show that we can obtain an abelian variety from the Clifford algebra of any quadratic form q in ≥ 2 variables, as long as the signature of q is not $(1,1)$. The restriction on the signature comes from the following lemma.

Lemma 3.2.1. *Let a, b both be positive integers. Consider the binary quadratic forms*

$$q_1 = aX^2 + bY^2 \qquad q_2 = -aX^2 - bY^2 \qquad q_3 = aX^2 - bY^2$$

Then

1. *Up to sign, there is a unique element $J \in C_{q_1}^+(\mathbb{R})$ with $J^2 = -1$. The same result holds for $C_{q_2}^+(\mathbb{R})$.*

2. *There is no element of $C_{q_3}^+(\mathbb{R})$ with square -1.*

Proof. Part 1: To prove the first part, we note that

$$C_{q_1}(\mathbb{R}) = \mathbb{R}[x, y : x^2 = a, y^2 = b, yx = -xy].$$

so that $C^+_{q_1}(\mathbb{R}) = \langle 1, xy\rangle_{\mathbb{R}}$. Suppose we have $J = c + dxy \in C^+_{q_1}(\mathbb{R})$ with $c, d \in \mathbb{R}$ such that $J^2 = -1$. Computing, we get

$$J^2 = c^2 + d^2(xy)^2 + 2cd(xy) = (c^2 - abd^2) + 2cd(xy) = -1$$

since $(xy)^2 = -x^2y^2 = -ab$. Equating both sides, we get $c^2 - abd^2 = -1$ and $2cd = 0$. This can only happen if $c = 0$ and $d = \pm\frac{1}{\sqrt{ab}}$. Thus, we get up to sign

$$J = \frac{1}{\sqrt{ab}}xy.$$

Now when we look at $C^+_{q_2}(\mathbb{R})$, we again let $J = c + dxy$ and compute $J^2 = (c^2 - abd^2) + 2cd(xy)$ because $(xy)^2 = -ab$ as before. In order to have $J^2 = -1$, we must have the same conditions on c and d, so again $J = \frac{1}{\sqrt{ab}}xy$.

Part 2: This time, however, we will show that there is no element $J \in C^+_{q_3}(\mathbb{R})$ with $J^2 = -I$. Suppose we have $J = cI + dAB \in C^+_{q_3}(\mathbb{R})$. Then

$$J^2 = (c^2 + abd^2) + 2cd(xy)$$

since $(xy)^2 = -x^2y^2 = ab$. If $J^2 = -I$, we are to solve the equations $c^2 + abd^2 = -1$ and $2cd = 0$. This is impossible since a, b are positive. Therefore, there is no such J. □

Remark 3.2.2. *Consider case (1) from the lemma. Left multiplication by J on $C^+_q(\mathbb{R})$ defines a complex structure; furthermore, $C^+_q(\mathbb{Z})$ is a full lattice inside the complex vector space $C^+_q(\mathbb{R})$, and so the quotient $C^+_q(\mathbb{R})/C^+_q(\mathbb{Z})$ is a complex torus.*

Recall from Lemma 2.2.6 that abelian varieties (respectively, abelian varieties up to isogeny) are equivalent to polarizable, integral (resp. rational) Hodge structures of weight 1. As we proceed to show this complex torus is an abelian variety, our assumption that f is *diagonal is a very minor hypothesis. We work with a diagonal quadratic form with coefficients in $\mathbb{Z}$ and construct an abelian variety. However, any quadratic form with coefficients in $\mathbb{Z}$ can be transformed, via a $\mathbb{Q}$-linear change of variables, into a diagonal form with coefficients in $\mathbb{Q}$. This amounts to passing from an integral Hodge structure to a rational one, which means instead of creating an abelian variety on the nose, we are creating an isogeny class of abelian varieties. In what follows, all statements remain true if we replace $\mathbb{Z}$ by $\mathbb{Q}$.*

Lemma 3.2.3. *Let n be arbitrary, $N = \lfloor (n+1)/2 \rfloor$, and $q = \sum_{i=1}^{n} a_i X_i^2$ with $a_i \in \mathbb{Z}$. Let*

$$v := \prod_{i=1}^{n} F_i^{m_i} \in M_{2^N}(\mathbb{C}) \quad \text{and} \quad w := \prod_{i=1}^{n} F_i^{l_i} \in M_{2^N}(\mathbb{C})$$

for $m_i, l_i \in \{0,1\}$ with $\sum_{i=1}^{n} m_i$ and $\sum_{i=1}^{n} l_i$ even, as in (3.2) of Proposition 3.1.2. Let Tr denote the trace of a matrix and let $*$ denote conjugate transpose of a matrix.

(a) $Tr(v) = \begin{cases} 2^N & \text{if } m_i = 0 \text{ for all } i \text{ (i.e. } v = I) \\ 0 & \text{else.} \end{cases}$

(b) $Tr(x^*) = Tr(x)$ for all x in the image of $C_q^+(\mathbb{R})$ under the map in (3.1).

(c) $vv^* = \prod_{i=1}^{n} |a_i|^{m_i} \cdot I$.

(d) $Tr(vw^*) = \begin{cases} 2^N \prod_{i=1}^{n} |a_i|^{m_i} & \text{if } v = w \\ 0 & \text{otherwise.} \end{cases}$

Proof. (a) Recall from Section 3.1 that

$$Tr(\epsilon_1) = Tr(\epsilon_2) = Tr(\epsilon_3) = 0.$$

If $v \neq I$, then choose the largest i such that $m_i = 1$. Then the $\lfloor (i+1)/2 \rfloor$-th element of the tensor product that makes up v is a constant multiple of

$$\begin{cases} \epsilon_1 & \text{if } i \text{ is even and } m_{i-1} = 0 \\ \epsilon_2 & \text{if } i \text{ is even and } m_{i-1} = 1 \\ \epsilon_3 & \text{if } i \text{ is odd} \end{cases}$$

from the construction in Section 3.1. Since the trace of a tensor product of matrices is the product of the traces of the matrices that make up the tensor product, this makes $Tr(v) = 0$. In the case $v = I$, $Tr(v) = 2^N$, since I is a $2^N \times 2^N$ matrix.

For (b), note that conjugate transpose and trace are both $\mathbb{R}$-linear, so it suffices to prove the claim for basis elements of the form in (3.2) of Proposition 3.1.2. So we will show $Tr(v^*) = Tr(v)$. But

$$F_i^* = \begin{cases} F_i & \text{if } a_i \text{ is positive} \\ -F_i & \text{if } a_i \text{ is negative.} \end{cases}$$

Since $F_iF_j = -F_jF_i$ for $i \neq j$ and $(AB)^* = B^*A^*$, we find that v^* is either v or $-v$. If $v^* = v$, then $Tr(v^*) = Tr(v)$. If $v^* = -v$, then part (a) implies that $Tr(v) = 0$, and hence $Tr(v^*) = Tr(v)$.

For (c), note that $E_kE_k^* = I$ for all k, since $\epsilon_i\epsilon_i^* = I$ for $i = 1, 2, 3$. So $F_kF_k^* = (\alpha_i\bar{\alpha}_i)E_kE_k^* = |a_i| \cdot I$. Therefore,

$$vv^* = \left(F_1^{m_1} \cdots F_n^{m_n}\right)\left((F_n^*)^{m_n} \cdots (F_1^*)^{m_1}\right) = \prod_{i=1}^n (\alpha_i\bar{\alpha}_i)^{m_i} \cdot I = \prod_{i=1}^n |a_i|^{m_i} \cdot I.$$

Finally, (d) is just a combination of (a) and (c). □

Remark 3.2.4. *It is preferable to have a basis-independent anti-involution defined on the Clifford algebra which, under the representation (3.1), agrees with the conjugate transpose of a matrix. For the diagonal form $q = \sum_{i=1}^n a_iX_i^2$ with a_i nonzero integers, let x_i denote* *the generators of C_q corresponding to X_i. Under (3.1), $x_i \mapsto F_i$. As noted in the proof of Lemma 3.2.3 part (b),*

$$F_i^* = \begin{cases} F_i & \text{if } a_i \text{ is positive} \\ -F_i & \text{if } a_i \text{ is negative.} \end{cases}$$

Define $\varphi : C_q \to C_q$ by $\varphi(x_i) = \frac{a_i}{|a_i|}x_i$. Then $\varphi \circ \iota$ is an anti-involution of C_q, where ι is the anti-involution of the Clifford algebra from Section 2.3. This clearly agrees with $*$ under the representation (3.1). Unfortunately, by assuming the form is diagonal, we are implicitly choosing a basis. The following lemma gives a situation in which this issue is resolved.

Lemma 3.2.5. *Let (V, Q) be an integral, polarized Hodge structure of even weight $2k$. Then there is a basis-independent anti-involution on $C_Q(V)$ which, under the representation (3.1), agrees with the conjugate transpose of a matrix.*

Proof. We have that Q is $(-1)^{k-p}$-definite on the subspace $V \cap (V^{p,q} \oplus V^{q,p})$ by Lemma 2.1.7. Define $\varphi : V \to V$ by

$$v \mapsto (-1)^{k-p}v \quad \text{for } v \in V \cap (V^{p,q} \oplus V^{q,p}).$$

Then φ extends to a map of the Clifford algebra $C_Q(V)$. To get a diagonal form, we diagonalize $Q_{\mathbb{Q}}$ on the subspaces $V_{\mathbb{Q}} \cap (V^{p,q} \oplus V^{q,p})$. Once we have a diagonal form, we apply Remark 3.2.4, which shows that $\varphi \circ \iota$ agrees with $*$ on $C_Q(V)$, where ι is the anti-involution of the Clifford algebra from Section 2.3. $\square$

Remark 3.2.6. *It is also preferable to discuss "trace" of an element of C_q^+ without referring to the trace of a matrix under the representation (3.1). For $x \in C_q^+$, we could consider the trace of the endomorphism L_x of C_q^+ defined by left-multiplication by x. In the case that q is* *as in the Kuga-Satake construction, Lemma 3.2.3 agrees with the Kuga-Satake construction (see [vG00, Lemma 5.8]) upon replacing 2^N with 2^{n-1}.*

More generally, let (V, Q) be an integral, polarized Hodge structure of even weight $2k$, and suppose the rank of V is n. Let $*$ denote the anti-involution of $C_Q(V)$ guaranteed by Lemma 3.2.5. Then the results of Lemma 3.2.3 hold for trace of a matrix replaced by the trace of the endomorphism L_x (with 2^N replaced by 2^{n-1}).

We now arrive at the main result of this section:

Theorem 3.2.7. *Let $q = \sum_{i=1}^{n} a_i X_i^2$ be a quadratic form with coefficients in $\mathbb{Z}$. Suppose there is an element $J \in C_q^+(\mathbb{R})$ satisfying the following properties:*

1. *Left-multiplication by J induces a complex structure on $C_q^+(\mathbb{R})$, i.e. $J^2 = -I$.*
2. *$J^* = -J$ (once we identify the Clifford algebra with matrices via (3.1)).*
3. *$\alpha := tJ \in C_q^+(\mathbb{Z})$ for some $t > 0$.*

Then the bilinear form

$$Q : C_q^+(\mathbb{R}) \times C_q^+(\mathbb{R}) \to \mathbb{R}, \qquad Q(v, w) := Tr(\alpha v w^*)$$

satisfies the Riemann bilinear relations (2.2.3), and hence $C_q^+(\mathbb{R})/C_q^+(\mathbb{Z})$ is an abelian variety with polarization Q.

Proof. First, we note that Q is $\mathbb{R}$-bilinear since the trace map and taking conjugate transpose of a matrix both are. By Lemma 3.2.3 part (a), the restriction of Q to $C_q^+(\mathbb{Z}) \times C_q^+(\mathbb{Z})$ takes values in $\mathbb{Z}$. Note that $Q(Jv, Jw) = Q(v, w)$:

$$Q(Jv, Jw) = Tr(\alpha(Jv)(Jw)^*)$$

$$
\begin{aligned}
&= Tr(\alpha J v w^* J^*) \\
&= -Tr(J\alpha J v w^*) && J^* = -J \text{ and } Tr(AB) = Tr(BA) \\
&= Tr(\alpha v w^*) && J \text{ commutes with } \alpha \text{ and } J^2 = -I \\
&= Q(v, w).
\end{aligned}
$$

The bilinear form $(v, w) \mapsto Q(v, Jw)$ is also symmetric:

$$
\begin{aligned}
Q(v, Jw) &= Tr(\alpha v (Jw)^*) \\
&= Tr((\alpha v w^* J^*)^*) && \text{Lemma 3.2.3 part (b)} \\
&= Tr(J w v^* \alpha^*) \\
&= -Tr(\alpha w v^* J) && \alpha^* = -\alpha \text{ and } J \text{ commutes with } \alpha \\
&= Tr(\alpha w v^* J^*) && J^* = -J \\
&= Tr(\alpha w (Jv)^*) \\
&= Q(w, Jv).
\end{aligned}
$$

It remains to check that $Q(v, Jv) > 0$ for all $v \in C_q^+(\mathbb{R})$, $v \neq 0$. We first check this for the basis element $v = \prod_{i=1}^n F_i^{m_i}$ with $F_i^2 = a_i$, $m_i \in \{0, 1\}$ and $\sum_{i=1}^n m_i$ even. We have

$$
\begin{aligned}
Q(v, Jv) &= Tr(\alpha v (Jv)^*) \\
&= -Tr(J\alpha v v^*) \\
&= t\, Tr(v v^*) && \text{since } J\alpha = -t, \text{ recall } t > 0 \\
&= 2^N t \prod_{i=1}^n |a_i|^{m_i} > 0
\end{aligned}
$$

by Lemma 3.2.3 part (d), where $N = \lfloor (n+1)/2 \rfloor$. Since $Q(x, Jx) = t\ Tr(xx^*)$, by the $\mathbb{R}$-linearity of Tr and Lemma 3.2.3 part (d), we have positive definiteness for all $x \in C_q^+(\mathbb{R})$.

□

Theorem 3.2.8. *Let $q = \sum_{i=1}^{n} a_i X_i^2$ be a quadratic form in $n \geq 2$ variables with signature not $(1,1)$. For a choice of l and m with $a_l a_m > 0$, there is a unique (up to sign) element J in $C_q^+(\mathbb{R})$ with square -1. Furthermore, the complex structure defined by J makes $C_q^+(\mathbb{R})/C_q^+(\mathbb{Z})$ an abelian variety of complex dimension 2^{n-2}.*

Proof. As long as the signature of q is not $(1,1)$, we are able to find $l, m \in \{1, \ldots, n\}$ with $a_l a_m > 0$. We may then apply Lemma 3.2.1 to get $J = \frac{1}{\sqrt{a_l a_m}} F_l F_m$. Verifying the assumptions of Theorem 3.2.7 is straight-forward: $J^2 = -I$, $J^* = -J$, and $t = \sqrt{a_l a_m}$. □

3.3 Maps Between Abelian Varieties

In this section we discuss how maps between Clifford algebras preserving the complex structure give rise to maps between abelian varieties. We begin with a map between vector spaces (or free $\mathbb{Z}$-modules of finite rank) preserving the quadratic form, and show how it defines a map of Clifford algebras.

Suppose we have quadratic forms q_V and q_W on free $\mathbb{Z}$-modules of finite rank, V and W, respectively. Furthermore, suppose we have a linear map $L : V \to W$ which satisfies $q_V(x) = q_W(Lx)$ for all $x \in V$. Let i_V and i_W denote the inclusions of V and W into their respective Clifford algebras. Then in $C_{q_W}(W)$ we have

$$(i_W(Lx))^2 = q_W(Lx) = q_V(x)$$

so by the universal property of the Clifford algebra, there is a unique map, which we also denote by L, from $C_{q_V}(V)$ to $C_{q_W}(W)$ which makes the following diagram commute

$$\begin{tikzcd} V \arrow[r, hook, "i_V"] \arrow[d, "L"] & C_{q_V}(V) \arrow[d, "L"] \\ W \arrow[r, hook, "i_W"] & C_{q_W}(W) \end{tikzcd}$$

More concretely, $C_{q_V}(V)$ is generated by products $v_1 \cdots v_k$ with $v_i \in V$, and the map takes $v_1 \cdots v_k \mapsto L(v_1) \cdots L(v_k)$. This makes it clear that L preserves the grading, so that the map restricts to $L : C^+_{q_V}(V) \to C^+_{q_W}(W)$.

Suppose there exists a $J_V \in C^+_{q_V}(V_{\mathbb{R}})$ so that left-multiplication by J_V defines a complex structure on $C^+_{q_V}(V_{\mathbb{R}})$. Then $J_V^2 = -1$ implies $L(J_V)^2 = -1$, so that left-multiplication by $J_W := L(J_V)$ defines a complex structure on $C^+_{q_W}(W_{\mathbb{R}})$. We have the following commutative diagram

$$\begin{tikzcd} C^+_{q_V}(V_{\mathbb{R}}) \arrow[r, "J_V"] \arrow[d, "L"] & C^+_{q_V}(V_{\mathbb{R}}) \arrow[d, "L"] \\ C^+_{q_W}(W_{\mathbb{R}}) \arrow[r, "J_W"] & C^+_{q_W}(W_{\mathbb{R}}) \end{tikzcd}$$

In such a situation, we get a map of complex tori

$$C^+_{q_V}(V_{\mathbb{R}})/C^+_{q_V}(V) \to C^+_{q_W}(W_{\mathbb{R}})/C^+_{q_W}(W).$$

When L is injective, this map is an inclusion of complex tori. If furthermore, J_W satisfies the hypotheses of Theorem 3.2.7, then $A_W := C^+_{q_W}(W_{\mathbb{R}})/C^+_{q_W}(W)$ is an abelian variety. Since a complex subtorus of an abelian variety is itself an abelian variety, $A_V := C^+_{q_V}(V_{\mathbb{R}})/C^+_{q_V}(V)$ is also an abelian variety. In this way, there is an inclusion of abelian varieties $A_V \hookrightarrow A_W$.

For the remainder of the section, we will assume that L injective and that V and W are vector spaces over $\mathbb{Q}$ or $\mathbb{R}$. Note that if we assume q_V and q_W are *nondegenerate* bilinear forms preserved by L, then L is automatically injective, for if $Lx = 0$, then $q_V(x, y) = q_W(Lx, Ly) = 0$ for all y, which means $x = 0$ since q_V is nondegenerate.

For the quadratic form q_V, we have the associated bilinear form, also denoted q_V, which is defined by

$$q_V(x,y) = \frac{1}{2}(q_V(x+y) - q_V(x) - q_V(y)),$$

and similarly for q_W. The condition $q_V(x) = q_W(Lx)$ implies $q_V(x,y) = q_W(Lx,Ly)$, which means that orthogonal elements in V are mapped to orthogonal elements in W. If we choose a basis $\{v_1,\ldots,v_n\}$ of V for which q_V is diagonal, then q_W restricted to the subspace spanned by $\{Lv_1,\ldots,Lv_n\}$ will also be diagonal. Choosing a basis of orthogonal complement of $L(V)$ in W for which q_W is diagonal, we may assume that q_V and q_W are diagonal and $q_V(x) = q_W(Lx)$ for all $x \in V$. We summarize the above discussion in the following proposition:

Proposition 3.3.1. *Let $q = \sum_{i=1}^{k} a_i X_i^2$ and $g = q + \sum_{i=k+1}^{n} a_i X_i^2$ be quadratic forms with coefficients in $\mathbb{Z}$. Suppose $J \in C_q^+(\mathbb{R}) \subset C_g^+(\mathbb{R})$ defines a complex structure on both $C_q^+(\mathbb{R})$ and $C_g^+(\mathbb{R})$. Furthermore, suppose J satisfies the hypotheses of Theorem 3.2.7 for $C_g^+(\mathbb{R})$. Then we have an inclusion of abelian varieties $C_q^+(\mathbb{R})/C_q^+(\mathbb{Z}) \hookrightarrow C_g^+(\mathbb{R})/C_g^+(\mathbb{Z})$.*

Example 3.3.2. *Let $\varphi : X \to Y$ be a map between n-dimensional complex projective varieties. Consider the intersection forms q_V and q_W on $V = H^n(Y,\mathbb{Q})$ and $W = H^n(X,\mathbb{Q})$, respectively, with $L = \varphi^* : V \to W$. Then L is injective since q_V and q_W are nondegenerate. If these intersection forms have signatures other than $(1,0),(0,1),(1,1)$, then by Theorem 3.2.8 we can find a J satisfying the hypotheses of Proposition 3.3.1. We obtain an inclusion of abelian varieties $A_Y \hookrightarrow A_X$.*

3.4 Examples and Applications

3.4.1 Examples

Example 3.4.1 (Cubic Surface). *Let X be a cubic surface in $\mathbb{P}^3$. Since X is obtained by blowing up 6 points in the plane, we know that $Pic(X) \cong \mathbb{Z}^7 = \langle H, E_1, \ldots, E_6 \rangle$ with $H^2 = 1$, $E_i^2 = -1$, $H \cdot E_i = 0$, and $E_i \cdot E_j = 0$ for $i \neq j$. Here H is the strict transform of a line not passing through any of the blown up points, and E_i are the exceptional divisors. The primitive cohomology $P^2(X, \mathbb{Z})/torsion$ is spanned by the classes of the E_i. The intersection form on $P^2(X, \mathbb{R})$ is $q = -x_1^2 - \cdots - x_6^2$, where the x_i correspond to the E_i. Let e_i be the elements of the Clifford algebra corresponding to x_i. Then $J = e_1 \cdots e_6$ satisfies the hypotheses of Theorem 3.2.7, and so makes $C_q^+(\mathbb{R})/C_q^+(\mathbb{Z})$ an abelian variety.*

In this case, J is central in $C_q^+(\mathbb{R})$. Furthermore, the weight 2 Hodge structure on $P^2(X, \mathbb{Z})/torsion$ is of type $\{(1,1)\}$. We can get a weight 0 Hodge structure V of type $\{(0,0)\}$ by taking the twist $P^2(X, \mathbb{Z})(1)/torsion$. This gives a commutative diagram as in Section 2.4:

$$\begin{tikzcd} & CSpin(V_{\mathbb{R}}) \arrow[d] \\ \mathbb{C}^* \arrow[ur, "\tilde{h}"] \arrow[r, "h"'] & SO(V_{\mathbb{R}}) \end{tikzcd}$$

where $\tilde{h}(a + bi) = a + bJ$.

In general, let V be an integral, polarized Hodge structure of weight 0 and type $\{(0,0)\}$ with $rank(V) = 2n$, n odd. Then $J = e_1 \cdots e_{2n}$ is central in $C^+(\mathbb{R})$ and satisfies the conditions of Theorem 3.2.7. Additionally, the diagram of the previous example above commutes with $\tilde{h}(a + bi) = a + bJ$.

Example 3.4.2 (K3 Cover of Enriques Surface)**.** *Let $f : X \to Y$ be the double cover of an Enriques surface by a K3 surface X. Then $H^2(Y, \mathbb{Z})/torsion$ is isometric to a sublattice inside of $H^2(X, \mathbb{Z})/torsion$ (see [BHPVdV04, p.350]). Upon choosing an appropriate J, Proposition 3.3.1 gives an injective map of abelian varieties $A_Y \to A_X$, where the complex structure on A_X is induced from the complex structure on A_Y as above. Note that the induced complex structure on A_X is not the same as the complex structure defined via the Kuga-Satake construction for K3-surfaces, because $h^{2,0}(Y) = 0$.*

3.4.2 Relation to Kuga-Satake Construction

We now describe how the construction of Section 3.2 generalizes the Kuga-Satake construction [KS67]. As an application of Theorem 3.2.8, we get

Proposition 3.4.3. *When q has signature $(n-2, 2)$, we recover the Kuga-Satake variety, up to isogeny.*

Proof. Recall Remark 3.2.2 regarding abelian varieties up to isogeny. We may write $q = \sum_{i=1}^n a_i X_i^2$ with $a_i \in \mathbb{Q}$ and $a_1, a_2 < 0$ and all other coefficients positive. Let J be the unique complex structure created from the negative coefficients as in Lemma 3.2.1. This is the same complex structure as in the Kuga-Satake construction. If n is even, the polarization $(2^{n-1}/2^{n/2})Q(v, w)$ takes the exact same values as the polarization from the Kuga-Satake construction [vG00, Proposition 5.9]. In the case that n is odd, $(2^{n-1}/2^{(n+1)/2})Q(v, w)$ does the trick. See also Remark 3.2.6. □

Now if we begin with a polarized Hodge structure V with polarization q, for which $C_q^+(V_\mathbb{R})/C_q^+(V)$ is an abelian variety, we would like to know if we can recover the Hodge structure on V from the Hodge structure on $C_q^+(V)$. We have:

Proposition 3.4.4. *Suppose V is a polarized, rational Hodge structure of type $\{(k,k)\}$ with polarization q. Also suppose* $\dim V \geq 2$*, so that there is $J \in C_q^+(V_{\mathbb{R}})$ for which the conditions of Theorem 3.2.7 are satisfied. For any such J, we have an inclusion of weight 2 Hodge structures*

$$V \otimes \mathbb{Q}(k-1) \hookrightarrow C_q^+(V) \otimes C_q^+(V)$$

Further, the Hodge structure on V can be recovered from the Hodge structure on $C_q^+(V)$.

Proof. Since V has a Hodge structure of type $\{(k,k)\}$, we know that the quadratic form defined by q is definite, this is why $\dim V \geq 2$ allows us to apply Theorem 3.2.8 to get an abelian variety $C_q^+(V_{\mathbb{R}})/C_q^+(V)$. The complex structure given to $C_q^+(V_{\mathbb{R}})$ corresponds to a weight one Hodge structure on $C_q^+(V)$ via Lemma 2.2.6.

Note that $(C_q^+(V))^* \otimes C_q^+(V) \cong End(C_q^+(V))$ and so the weight zero Hodge structure on $(C_q^+(V))^* \otimes C_q^+(V)$ gives $End(C_q^+(V))$ a Hodge structure of weight 0. The Hodge structure is determined by an action of $\mathbb{C}^*$ on $End(C_q^+(V))$. Explicitly, it is given by

$$(z \cdot \varphi)(w) := z(\varphi(z^{-1}w)) \tag{3.3}$$

where $\varphi \in End(C_q^+(V))$, $w \in C_q^+(V)$, and the action of $z = a + bi$ on the right hand side is given by left-multiplication by $a + bJ$ on $C_q^+(V)$.

We know that the polarization gives an isomorphism of weight -1 Hodge structures $C_q^+(V) \otimes \mathbb{Q}(1) \cong (C_q^+(V))^*$, as mentioned in Section 2.1. We will show that $V \otimes \mathbb{Q}(k) \hookrightarrow End(C_q^+(V))$, then

$$End(C_q^+(V)) \cong (C_q^+(V))^* \otimes C_q^+(V) \cong C_q^+(V) \otimes C_q^+(V) \otimes \mathbb{Q}(1)$$

Tensoring by $\mathbb{Q}(-1)$ will then yield the proposition. In the process, it will be clear that the Hodge structure on $C_q^+(V)$ determines the Hodge structure on V.

Fix an element $v_0 \in V$ which is invertible in $C_q(V)$. Define $V \otimes \mathbb{Q}(k) \to End(C_q^+(V))$ by $v \mapsto f_v$ where $f_v(w) = wv_0v$. This map is clearly linear. To show injectivity, note that since v_0 is invertible, $v_0^2 = q(v_0, v_0) \neq 0$. Given v, we can find a $v_1 \in V$ with $q(v_1, v) \neq 0$. Set $w = v_1v_0$, then $f_v(w) = q(v_0, v_0)v_1v \neq 0$ in $C_q^+(V)$. Thus, the map is injective.

To show that this is a morphism of Hodge structures, we just need to show that the map $v \mapsto f_v$ commutes with the action of $\mathbb{C}^*$. So we need to show

$$f_{z\cdot v} = z \cdot f_v, \tag{3.4}$$

where $z \cdot v$ is the action of z on $V \otimes \mathbb{Q}(k)$, and the action of $z \cdot f_v$ is the action described in (3.3). Since the action of $\mathbb{C}^*$ on V is given by $z \cdot v := (z\bar{z})^k v$, $\mathbb{C}^*$ acts trivially on $V \otimes \mathbb{Q}(k)$. This means $f_{z\cdot v} = f_v$. Now,

$$(z \cdot f_v)(w) = z(f_v(z^{-1}w)) = z(z^{-1}wv_0v) = wv_0v = f_v(w)$$

so $z \cdot f_v = f_v$ and equation (3.4) holds. The Hodge structure on V can be trivially obtained from the Hodge structure on $C_q^+(V)$. □

Combining Proposition 3.4.4 with the analogous one for Hodge structures of $K3$-type, we get the following:

Corollary 3.4.5. *Suppose V and W are rational, polarized weight 2 Hodge structures (with polarizations q and r, respectively) of either $K3$-type or of type $\{(1,1)\}$, and with* $\dim V, \dim W \geq 2$. *Then we have an inclusion of weight 4 Hodge structures*

$$V \otimes W \hookrightarrow C_q^+(V) \otimes C_q^+(V) \otimes C_r^+(W) \otimes C_r^+(W)$$

and furthermore, we recover the Hodge structure on $V \otimes W$ from that of $C_q^+(V)$ and $C_r^+(W)$.

Of course, there is an analogous inclusion if we instead take tensor products of three such weight 2 Hodge structures, etc. Here is the geometric example we have in mind:

Example 3.4.6. *Let X and Y be nonsingular complex, projective surfaces with Hodge structures on H^2 as in Corollary 3.4.5. Let A_X and A_Y be the abelian varieties arising from the vector spaces $H^2(X, \mathbb{Q})$ and $H^2(Y, \mathbb{Q})$, respectively. Then the corollary shows that there is an inclusion of Hodge structures*

$$\begin{aligned} H^2(X, \mathbb{Q}) \otimes H^2(Y, \mathbb{Q}) \hookrightarrow H^1(A_X, \mathbb{Q}) &\otimes H^1(A_X, \mathbb{Q}) \\ &\otimes H^1(A_Y, \mathbb{Q}) \otimes H^1(A_Y, \mathbb{Q}) \\ &\subset H^2(A_X \times A_Y, \mathbb{Q}) \otimes H^2(A_X \times A_Y, \mathbb{Q}). \end{aligned}$$

In particular, this applies to the product of two $K3$-surfaces, a $K3$-surface with an Enriques surface or a cubic surface, etc. (cf. Examples 3.4.1 and 3.4.2).

Chapter 4

Families of Abelian Varieties

The goal of this section is to show that, under certain hypotheses, our construction from the previous chapter works in families. We also provide examples of when the hypotheses are satisfied. First, we discuss some linear algebra preliminaries.

Let $V^\vee = \mathrm{Hom}_K(V, K)$ denote the dual vector space of V. Then since $\mathrm{End}(V) \cong V \otimes V^\vee$, we may define the *trace* of a linear map as follows. For $v \in V$ and $\phi \in V^\vee$, define $tr(v \times \phi) = \phi(v)$. Clearly, tr defines a map $tr : V \otimes V^\vee \to K$, and so the trace (still denoted tr) of a K-linear endomorphism of V is defined by the composition

$$\mathrm{End}(V) \xrightarrow{\cong} V \otimes V^\vee \xrightarrow{tr} K.$$

Upon choosing a basis for V, this agrees with the usual notion of trace of a matrix.

The above discussion carries over to vector bundles. For a vector bundle $V \to X$, we have the dual bundle $V^\vee \to X$ which is the same as the Hom-bundle $\mathrm{Hom}(V, K \times X) \to X$ of bundle homomorphisms (over X) of V into the trivial bundle $K \times X$. Furthermore, there is a canonical isomorphism of vector bundles $\mathrm{End}(V) \cong V \otimes V^\vee$. This allows us to define the *trace* of an endomorphism of vector bundles as the composition

$$\begin{tikzcd}
\mathrm{End}(V) \arrow[r, "\cong"] \arrow[dr] & V \otimes V^\vee \arrow[r, "tr"] \arrow[d] & K \times X \arrow[dl] \\
& X &
\end{tikzcd}$$

Note that on fibers, this agrees with the classical notion of trace of a matrix.

Recall the anti-automorphism ι of the even Clifford algebra from Section 2.3. This map also carries over to vector bundles. Specifically, if we have a vector bundle equipped with a quadratic form (V, ψ) over a base X, then we can form the tensor bundle, Clifford bundle, and even Clifford bundle as vector bundles over X. The map $\iota : C^+(V, \psi) \to C^+(V, \psi)$ is globally defined as a morphism of bundles over X and on fibers it agrees with the ι defined before.

4.1 Variations of Hodge Structure

The notion of a variation of Hodge structure is due to Griffiths [Gri68, Gri70]. This section recalls some of the necessary definitions.

Definition 4.1.1. *Let G be an abelian group and let X be a topological space. A sheaf $\mathcal{G}$ on X is called a* local system *if it is locally isomorphic to the constant sheaf with stalk G.*

For $E \to X$ a vector bundle, we denote by $\mathcal{O}(E, U)$ the set of sections over an open subset $U \subset X$. We let $\Omega^1(U)$ denote the 1-forms over U.

Definition 4.1.2. *A* connection *on a vector bundle $E \to X$ is a linear map*

$$\nabla : \mathcal{O}(E, U) \to \Omega^1(U) \otimes \mathcal{O}(E, U) \qquad U \subset X \textit{ open}$$

which satisfies the Leibniz rule

$$\nabla(f \cdot \sigma) = df \otimes \sigma + f \cdot \nabla\sigma$$

for $f \in \mathcal{O}(U)$ and $\sigma \in \mathcal{O}(E, U)$.

We say that a section $\sigma \in \mathcal{O}(E, U)$ is *flat* if $\nabla\sigma = 0$. The connection is called *flat* if there is a cover of X for which the corresponding local frames consist of flat sections. We say a vector bundle $E \to X$ is *flat* if it has a flat connection. As illustrated by the following proposition, flat vector bundles have constant transition functions.

Proposition 4.1.3. *The following three categories are equivalent*

- *Local systems of stalk $\mathbb{C}^n$ over a connected complex manifold X.*
- *Representations of the fundamental group $\pi_1(X, x) \to GL_n(\mathbb{C})$.*
- *Flat, holomorphic vector bundles $E \to X$ of rank n.*

Under this correspondence, locally constant sections, $\pi_1(X)$-invariant sections, and flat sections are all equivalent.

Proof. This is well-known, see for example [Cat14, §1.3]. $\square$

Definition 4.1.4. *Let X be a connected complex manifold. A* variation of Hodge structure (VHS) *of weight k on X consists of the following ingredients:*

- *A local system $\mathcal{V}_{\mathbb{Z}}$ of free $\mathbb{Z}$-modules on X.*
- *A decreasing filtration of the associated holomorphic vector bundle*

$$\mathcal{V}_{\mathbb{C}} \supset \cdots \supset \mathcal{F}^p \supset \mathcal{F}^{p+1} \supset \cdots \supset 0$$

  by holomorphic subbundles which satisfy:

  (i) For each point $x \in X$, the fibers $\mathcal{F}^p_x$ form a Hodge filtration of the weight k Hodge structure on the fiber $\mathcal{V}_{\mathbb{C},x}$.

  (ii) $\nabla \mathcal{O}(\mathcal{F}^p) \subset \Omega^1_X \otimes \mathcal{O}(\mathcal{F}^{p-1})$ where ∇ is the flat connection on $\mathcal{V}_{\mathbb{C}}$ guaranteed by Proposition 4.1.3.

Note that when $k = 1$ and the filtration is $\mathcal{V}_{\mathbb{C}} \supset \mathcal{F}^1 \supset 0$, condition (ii) in the definition of VHS is automatically satisfied.

Definition 4.1.5. *The* rank *of a VHS is the rank of the corresponding vector bundle. The* type *of a VHS is defined to be the type of the Hodge structure of a fiber (see Section 2.1). We assume X is connected so that these definitions are independent of the choice of fiber.*

Definition 4.1.6. *We say that the VHS is* polarized *if there is a flat, non-degenerate bilinear form ψ on $\mathcal{V}_{\mathbb{Z}}$ satisfying the following properties. First, this form is required to be symmetric or skew-symmetric (if k is even or odd, respectively). Second, we require that ψ_x polarizes the Hodge structure of the fiber $\mathcal{V}_{\mathbb{C},x}$ for each $x \in X$.*

By definition, a bilinear form on a vector bundle $E \to X$ is a section of $E^\vee \otimes E^\vee$. To say a bilinear form is flat means the corresponding section of $E^\vee \otimes E^\vee$ is flat.

Example 4.1.7. *Let $f : Y \to X$ be a proper, smooth morphism of smooth and connected schemes of finite type over $\mathbb{C}$. Then $R^k f_* \mathbb{Z}$ is a local system on X by Ehresmann's Theorem. Furthermore, the local systems $R^k f_* \mathbb{R}$ and $R^k f_* \mathbb{C}$ can also be viewed as C^∞ and holomorphic vector bundles on X, respectively. In the holomorphic case, the sheaf of holomorphic sections is simply $R^k f_* \mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_X$. The flat connection on the vector bundle $R^k f_* \mathbb{C}$ is called the* Gauss-Manin connection*. The stalks of these sheaves over any $x \in X$ are the cohomology groups of the fiber $Y_x = f^{-1}(x)$: $H^k(Y_x, \mathbb{Z})$, $H^k(Y_x, \mathbb{R})$, and $H^k(Y_x, \mathbb{C})$, respectively. Each $H^k(Y_x, \mathbb{C})$ has a Hodge filtration, and these glue together to give holomorphic subbundles $\mathcal{F}^p \subset R^k f_* \mathbb{C}$ which satisfy the conditions of the definition above. Hence we have a VHS of weight k.*

Furthermore, suppose f is projective, i.e. f factors through a closed immersion followed by projection as in the diagram

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[dr, "f"'] & \mathbb{P}^N_X \arrow[d] \\ & X \end{tikzcd}$$

Then the fibers of f come equipped with projective embeddings. This allows us to define primitive cohomology on each fiber, which gives rise to the local systems $P^k f_* \mathbb{Z}$ and similarly with $\mathbb{R}$ and $\mathbb{C}$. The polarizations on each fiber define a polarization on $P^k f_* \mathbb{Z}$. This is the geometric setting that yields a polarized VHS of weight k.

4.2 Families of Abelian Varieties

From a polarized VHS $(\mathcal{V}, \psi)$, we can construct a local system $C(\mathcal{V}, \psi)$, whose complex, real, and integral bundles are bundles of Clifford algebras. Additionally, we have the local system $C^+(\mathcal{V}, \psi)$ whose bundles correspond to the even Clifford algebras. We outline the construction below.

Since $(\mathcal{V}, \psi)$ is a VHS on X, $\mathcal{V}_{\mathbb{Z}}$ is a local system of free $\mathbb{Z}$-modules on X, and hence comes equipped with a flat connection ∇. The tensor bundle $T(\mathcal{V}_{\mathbb{Z}})$ is then a local system of free $\mathbb{Z}$-modules on X, and ∇ extends to a flat connection on $T(\mathcal{V}_{\mathbb{Z}})$. Since we are assuming $(\mathcal{V}_{\mathbb{Z}}, \psi_{\mathbb{Z}})$ is polarized by ψ, we have that ψ is flat with respect to ∇. This compatibility guarantees us that the Clifford bundle $C(\mathcal{V}_{\mathbb{Z}}, \psi_{\mathbb{Z}})$ is flat as well (the same holds for the even Clifford bundle). Thus, $C^+(\mathcal{V}_{\mathbb{Z}}, \psi_{\mathbb{Z}})$ is a local system of free $\mathbb{Z}$-modules on X. This also applies for $\mathbb{R}$ and $\mathbb{C}$.

Proposition 4.2.1. *Let $(\mathcal{V}, \psi)$ be a polarized variation of Hodge structure of weight k over a smooth scheme X of finite type over $\mathbb{C}$. Let $C^+(\mathcal{V}, \psi)$ be the associated even Clifford bundle over X. Suppose there exists a locally-constant section $\mathcal{J}$ of $C^+(\mathcal{V}_{\mathbb{R}}, \psi_{\mathbb{R}})$ such that for every*

$x \in X$, left multiplication by $\mathcal{J}_x$ defines a complex structure on the fiber $C^+(\mathcal{V}_{\mathbb{R}}, \psi_{\mathbb{R}})_x$. Then $C^+(\mathcal{V}, \psi)$ defines a weight 1 variation of Hodge structure on X.

Proof. The fact that $C^+(\mathcal{V}_{\mathbb{Z}}, \psi_{\mathbb{Z}})$ is a local system of free $\mathbb{Z}$-modules on X has been discussed above. For $x \in X$, we have a decomposition

$$C^+(\mathcal{V}_{\mathbb{R}}, \psi_{\mathbb{R}})_x \otimes \mathbb{C} = \left(C^+(\mathcal{V}_{\mathbb{R}}, \psi_{\mathbb{R}})_x\right)^{1,0} \oplus \left(C^+(\mathcal{V}_{\mathbb{R}}, \psi_{\mathbb{R}})_x\right)^{0,1}$$

into the eigenspaces on which $\mathcal{J}_x$ acts by i and $-i$, respectively. Since $\mathcal{J}$ is a global section of the even Clifford bundle, this decomposition defines

$$C^+(\mathcal{V}_{\mathbb{R}}, \psi_{\mathbb{R}})^{1,0} \qquad \text{and} \qquad C^+(\mathcal{V}_{\mathbb{R}}, \psi_{\mathbb{R}})^{0,1}$$

as holomorphic (since $\mathcal{J}_{\mathbb{C}}$ is holomorphic) subbundles of $C^+(\mathcal{V}_{\mathbb{C}}, \psi_{\mathbb{C}})$.

Set $\mathcal{F}^1 = C^+(\mathcal{V}_{\mathbb{R}}, \psi_{\mathbb{R}})^{1,0}$. Then we have a decreasing filtration

$$C^+(\mathcal{V}_{\mathbb{C}}, \psi_{\mathbb{C}}) \supset \mathcal{F}^1 \supset 0$$

and the fibers of $\mathcal{F}^1_x$ define a Hodge filtration of the weight 1 Hodge structure on the fiber $C^+(\mathcal{V}_{\mathbb{C}}, \psi_{\mathbb{C}})$. This last part is by construction, as $\mathcal{J}_x$ defines the weight 1 Hodge structure on the fiber over $x \in X$. As noted above, condition (ii) in the definition of VHS is automatic for weight 1. $\square$

Let $(\mathcal{V}, \psi)$ be a polarized VHS of weight k over a base X . We define C^∞-subbundles of $\mathcal{V}_{\mathbb{C}}$ by $\mathcal{V}^{p,q} = \mathcal{F}^p \cap \overline{\mathcal{F}^q}$, where complex conjugation is taken relative to $\mathcal{V}_{\mathbb{R}}$. Then there is a decomposition

$$\mathcal{V}_{\mathbb{C}} = \bigoplus_{p+q=k} \mathcal{V}^{p,q}$$

as C^∞-vector bundles [Sch73, 2.15]. Set

$$\mathcal{V}^p_{\mathbb{R}} = \begin{cases} \left(\mathcal{V}^{p,q} \oplus \mathcal{V}^{q,p}\right) \cap \mathcal{V}_{\mathbb{R}} & \text{if } p \neq q \\ \\ \mathcal{V}^{p,p} \cap \mathcal{V}_{\mathbb{R}} & \text{if } p = q \end{cases}$$

Then we have $\mathcal{V}_{\mathbb{R}} = \bigoplus_p \mathcal{V}^p_{\mathbb{R}}$. We can analogously define $\mathcal{V}^p_{\mathbb{Z}}$.

We define an automorphism of C^∞-vector bundles

$$\begin{tikzcd} \mathcal{V}_{\mathbb{C}} \arrow[rr, "\varphi"] \arrow[dr] & & \mathcal{V}_{\mathbb{C}} \arrow[dl] \\ & X & \end{tikzcd}$$

which acts on $\mathcal{V}^{p,q}$ as $i^k(-1)^{-p}$. When k is even, we can define $\varphi : \mathcal{V}_{\mathbb{R}} \to \mathcal{V}_{\mathbb{R}}$ by $(-1)^{k/2-p}$ on $\mathcal{V}^p_{\mathbb{R}}$, and this agrees with the above φ map. This map can also be defined on $\mathcal{V}_{\mathbb{Z}}$ in an analogous fashion.

From here on, we assume k is even, so that we can freely mention φ without specifying $\mathbb{C}, \mathbb{R}$, or $\mathbb{Z}$. This map extends to an automorphism, which we still denote by φ, of the tensor bundle $T(\mathcal{V})$. The polarization ψ is orthogonal with respect to $(\mathcal{V}^{p,q} \oplus \mathcal{V}^{q,p})$ and therefore φ descends to an automorphism of the even Clifford bundle

$$\begin{tikzcd} C^+(\mathcal{V}, \psi) \arrow[rr, "\varphi"] \arrow[dr] & & C^+(\mathcal{V}, \psi) \arrow[dl] \\ & X & \end{tikzcd}$$

We define an anti-automorphism of the even Clifford bundle as the composition $* := \varphi \circ \iota$. When $(\mathcal{V}, \psi)$ is a VHS of even weight k, this is just the global version of the anti-involution from Lemma 3.2.5.

Theorem 4.2.2. *Let $(\mathcal{V}, \psi)$ be a polarized variation of Hodge structure of even weight k over a smooth scheme X of finite type over $\mathbb{C}$. Suppose there is a locally-constant section $\mathcal{J}$ of $C^+(\mathcal{V}_{\mathbb{R}}, \psi_{\mathbb{R}})$ which satisfies the following properties: $\mathcal{J}^2 = -1$, $\mathcal{J}^* = -\mathcal{J}$, and $t\mathcal{J}$ is a section of $C^+(\mathcal{V}_{\mathbb{Z}}, \psi_{\mathbb{Z}})$ for some $t > 0$. Then $C^+(\mathcal{V}, \psi)$ defines a weight 1 polarized variation of Hodge structure on X.*

Proof. We may think of the section $\mathcal{J}$ as an endomorphism of $C^+(\mathcal{V}, \psi)$ because of the algebra structure. Thus, Proposition 4.2.1 already tells us that $C^+(\mathcal{V}, \psi)$ defines a weight 1 VHS on X. The new part here is that it is polarized.

We define the polarization as a section σ of the bundle

$$\begin{tikzcd} C^+(\mathcal{V}_{\mathbb{Z}}, \psi_{\mathbb{Z}})^\vee \otimes C^+(\mathcal{V}_{\mathbb{Z}}, \psi_{\mathbb{Z}})^\vee \arrow[d, shift left, "\pi"] \\ X \arrow[u, shift left, "\sigma"] \end{tikzcd}$$

by

$$\sigma(x) := \big(\alpha \otimes \beta \mapsto tr(L_{t\mathcal{J}_x \alpha \beta^*})\big)$$

where $\alpha, \beta \in C^+(\mathcal{V}_{\mathbb{Z}}, \psi_{\mathbb{Z}})_x$, and the trace occurs in the fiber $C^+(\mathcal{V}_{\mathbb{Z}}, \psi_{\mathbb{Z}})_x$. Recall that $tr(L_f)$ means the trace of the endomorphism which is left-multiplication by f. Since σ is continuous, it is automatically locally-constant, since we are considering it as a section of a bundle of $\mathbb{Z}$-modules.

We would like to make this map more explicit. Identify sections α and β of $C^+(\mathcal{V}_{\mathbb{Z}}, \psi_{\mathbb{Z}})$ with their images $\alpha(X)$ and $\beta(X)$, respectively. We have the map

$$C^+(\mathcal{V}_{\mathbb{Z}}, \psi_{\mathbb{Z}}) \otimes C^+(\mathcal{V}_{\mathbb{Z}}, \psi_{\mathbb{Z}}) \to \mathrm{End}(C^+(\mathcal{V}_{\mathbb{Z}}, \psi_{\mathbb{Z}}))$$

$$\alpha \otimes \beta \mapsto L_{t\mathcal{J}\alpha\beta^*}$$

Then the map described above is the composition

$$\begin{tikzcd} C^+(\mathcal{V}_{\mathbb{Z}}, \psi_{\mathbb{Z}}) \otimes C^+(\mathcal{V}_{\mathbb{Z}}, \psi_{\mathbb{Z}}) \arrow[r] \arrow[dr] & \mathrm{End}(C^+(\mathcal{V}_{\mathbb{Z}}, \psi_{\mathbb{Z}})) \arrow[r, "tr"] \arrow[d] & \mathbb{Z} \times X \arrow[dl] \\ & X & \end{tikzcd}$$

which on fibers polarizes the Hodge structure by Lemma 3.2.5, Remark 3.2.6, and Theorem 3.2.7. We see that σ is both alternating and nondegenerate by the same theorems. □

Lemma 4.2.3. *A weight 1 polarized VHS over a smooth scheme X over $\mathbb{C}$ is equivalent to an abelian scheme over X.*

Proof. See [Del72]. One direction is easy: if $A \xrightarrow{f} X$ is an abelian scheme, then $R^1 f_* \mathbb{Z}$ is a weight 1 polarized VHS. In the other direction, we can see this by looking at the fibers. The fiber over $x \in X$ has the structure of an integral weight 1 Hodge structure which is polarized. The category of such Hodge structures is equivalent to the category of abelian varieties over $\mathbb{C}$. The fact that the Hodge structure, polarization, etc. are all globally defined means that these abelian varieties glue together to give a projective family of abelian varieties over X. Furthermore, an application of Bailey-Borel shows that this is an abelian scheme over X. □

4.2.1 Examples

Theorem 4.2.4 (Riemann Existence Theorem)**.** *Let X be a scheme which is locally of finite type over $\mathbb{C}$ and let X^{an} denote the corresponding complex analytic space. The functor $Y \mapsto Y^{an}$ gives an equivalence of categories between the category of finite étale coverings Y/X and finite covering spaces Y^{an}/X^{an}.*

Let X be a smooth scheme which is locally of finite type over $\mathbb{C}$. Let $(\mathcal{V}, \psi)$ be a polarized variation of Hodge structure on X. Choose a base-point $x \in X$. The topological fundamental group $\pi_1(X, x)$ acts on the fiber $\mathcal{V}_{\mathbb{Z},x}$ of $\mathcal{V}_{\mathbb{Z}}$ above x. This action extends to an action on $\mathcal{V}_{\mathbb{R},x}$ and $\mathcal{V}_{\mathbb{C},x}$ by scalar extension. Since the action preserves the polarization, we get a map

$$\pi_1(X, x) \xrightarrow{\rho} O(\mathcal{V}_{\mathbb{Z},x}),$$

where $O(\mathcal{V}_{\mathbb{Z},x}) = \{g \in Aut(\mathcal{V}_{\mathbb{Z},x}) \mid \psi_{\mathbb{Z},x}(gv, gw) = \psi_{\mathbb{Z},x}(v, w) \text{ for all } v, w \in \mathcal{V}_{\mathbb{Z},x}\}$.

Lemma 4.2.5. *After a possible finite étale extension of X, we may assume that the image of ρ lies in $SO(\mathcal{V}_{\mathbb{Z},x})$.*

Proof. Indeed, if the image does not lie in $SO(\mathcal{V}_{\mathbb{Z},x})$, then the kernel of the composition

$$\pi_1(X,x) \xrightarrow{\rho} O(\mathcal{V}_{\mathbb{Z},x}) \xrightarrow{\det} \mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$$

defines an index 2 subgroup $H \trianglelefteq \pi_1(X,x)$. Let Y be the double cover of X corresponding to H and let $\mathcal{W}$ denote the variation of Hodge structure on Y obtained by pulling back $\mathcal{V}$ under the map $f : Y \to X$. Then f is a finite étale morphism and we have a map

$$\pi_1(Y,y) \to SO(\mathcal{W}_{\mathbb{Z},y})$$

by construction. Furthermore, Y is a smooth scheme which is locally of finite type over $\mathbb{C}$ by the Riemann Existence Theorem. $\square$

We will need the following to prove Theorem 4.2.8 below.

Proposition 4.2.6. *Let $n \equiv 2 \mod 4$. Let V be a real vector space of dimension n with positive definite quadratic form q. Then in $C_q^+(V)$ there is a unique (up to sign) central element J with $J^2 = -1$. Furthermore, for any $T \in SO(V,q)$, $T(J) = J$.*

Proof. Choose a basis $\{e_1, \ldots, e_n\}$ of V so that $q = x_1^2 + \cdots + x_n^2$, where e_i corresponds to x_i. Then a basis for $C_q^+(V)$ is given by $\{e_{i_1} \cdots e_{i_{2j}} \mid 0 \leq j \leq n/2\}$. We will show that $J = e_1 \cdots e_n$ (or $J = -e_1 \cdots e_n$) is the desired element. First, J is central since

$$(e_i e_j)J = (-1)^{n-1} e_i J e_j = (-1)^{n-1}(-1)^{n-1} J e_i e_j = J(e_i e_j)$$

and similarly for the other basis elements of $C_q^+(V)$. We also have

$$J^2 = (e_1 \cdots e_n)(e_1 \cdots e_n) = (-1)^{\frac{n(n-1)}{2}} e_1^2 \cdots e_n^2 = -1$$

since n is even and $n/2$ is odd.

Now for uniqueness: the structure theorem for Clifford algebras shows that $C_q^+(V)$ is a central simple algebra over $\mathbb{C}$, hence there is a unique (up to sign) central element which squares to -1. As we have shown, J is that element.

Finally, we show that $T(J) = J$. For any $T \in SO(V, q)$, T defines a map on $C_q^+(V)$ by the universal property of Clifford algebras. In particular, $T(J) = T(e_1) \cdots T(e_n)$. Since $e_i e_j = -e_j e_i$ for $i \neq j$, we have that

$$T(J) = T(e_1) \cdots T(e_n) = (\det T) e_1 \cdots e_n = J.$$

□

Proposition 4.2.7. *Let $n \equiv 2 \mod 4$. Let $(\mathcal{V}, \psi)$ be a polarized variation of Hodge structure of even weight k, of type $\{(\frac{k}{2}, \frac{k}{2})\}$, and rank n over a smooth scheme X of finite type over $\mathbb{C}$. Then (after possibly replacing X by a finite étale extension) there is a locally-constant section $\mathcal{J}$ of $C^+(\mathcal{V}_{\mathbb{R}}, \psi_{\mathbb{R}})$ which defines a complex structure on each fiber.*

Proof. Upon applying Lemma 4.2.5 above, we may assume that the action of $\pi_1(X, x)$ on $(\mathcal{V}_{\mathbb{Z}}, \psi_{\mathbb{Z}})$ lands in $SO(\mathcal{V}_{\mathbb{Z}}, \psi_{\mathbb{Z}})$. By the universal property of the Clifford algebra, the action of $\pi_1(X, x)$ on the fiber $\mathcal{V}_{\mathbb{Z},x}$ extends to an action of $\pi_1(X, x)$ on $C^+(\mathcal{V}_{\mathbb{Z}}, \psi_{\mathbb{Z}})_x$ and similarly for $\mathbb{R}$ and $\mathbb{C}$.

Choose a trivializing cover $\{U_\alpha\}$ of the local system $C^+(\mathcal{V}_{\mathbb{R}}, \psi_{\mathbb{R}})$ over X. Over U_α, the local system is trivial, and we write this as

$$U_\alpha \times C^+(\mathcal{V}_{\mathbb{R}}, \psi_{\mathbb{R}})_\alpha.$$

There is a unique (up to sign) $J_\alpha \in C^+(\mathcal{V}_{\mathbb{R}}, \psi_{\mathbb{R}})_\alpha$ guaranteed by Proposition 4.2.6. Note that the type of the VHS guarantees that ψ is positive definite. When $U_\alpha \cap U_\beta \neq \emptyset$, we obtain an automorphism

$$\varphi_{\alpha\beta} : C^+(\mathcal{V}_{\mathbb{R}}, \psi_{\mathbb{R}})_\alpha \to C^+(\mathcal{V}_{\mathbb{R}}, \psi_{\mathbb{R}})_\beta.$$

This automorphism is constant (doesn't depend on points in $U_\alpha \cap U_\beta$), since we are dealing with a local system. This automorphism preserves the center and the algebra structure, and hence $J_\beta = \varphi_{\alpha\beta}(J_\alpha)$ is the unique (up to sign) element of $C^+(\mathcal{V}_{\mathbb{R}}, \psi_{\mathbb{R}})_\beta$ guaranteed by Proposition 4.2.6.

For γ with $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, we have two elements of $C^+(\mathcal{V}_{\mathbb{R}}, \psi_{\mathbb{R}})_\gamma$ which are central and square to -1:

$$J_\gamma = \varphi_{\alpha\gamma}(J_\alpha) \quad \text{and} \quad J'_\gamma = \varphi_{\beta\gamma}(\varphi_{\alpha\beta}(J_\alpha)),$$

and hence $J_\gamma = J'_\gamma$ or $J_\gamma = -J'_\gamma$. Checking that these two elements are the same is equivalent to checking that for $x \in U_\alpha \cap U_\beta \cap U_\gamma$, the action of $\pi_1(X, x)$ on a fiber $C^+(\mathcal{V}_{\mathbb{R}}, \psi_{\mathbb{R}})_x$ preserves J_x. Since we are assuming the action of $\pi_1(X, x)$ lands in $SO(\mathcal{V}_{\mathbb{R},x}, \psi_{\mathbb{R},x})$, Proposition 4.2.6 assures us that $J_\gamma = J'_\gamma$. Hence there is a consistent choice of J_α which yields a locally-constant section $\mathcal{J}$ of $C^+(\mathcal{V}_{\mathbb{R}}, \psi_{\mathbb{R}})$. By construction, $\mathcal{J}$ defines a complex structure on each fiber. $\square$

Theorem 4.2.8. *Let $n \equiv 2 \mod 4$. Let $(\mathcal{V}, \psi)$ be a polarized variation of Hodge structure of even weight k, of type $\{(\frac{k}{2}, \frac{k}{2})\}$, and rank n over a smooth scheme X of finite type over $\mathbb{C}$. Then (after possibly replacing X by a finite étale extension) we can define a weight 1 polarized VHS on $C^+(\mathcal{V}, \psi)$.*

Proof. We need to show the assumptions of Theorem 4.2.2 are satisfied. Proposition 4.2.7 gives us the section $\mathcal{J}$ which satisfies $\mathcal{J}^2 = -\mathcal{J}$. Choose a basis for $\mathcal{V}_{\mathbb{R},x}$ for which $\psi_{\mathbb{R},x}$ is diagonal. Using the notation of the proof of Proposition 4.2.6, we have that $\mathcal{J}_x = \pm e_1 \cdots e_n$. Now $(e_1 \cdots e_n)^* = (-1)^{k/2 - k/2}(e_n \cdots e_1) = -e_1 \cdots e_n$. Hence $J_x^* = -J_x$. Therefore, $\mathcal{J}^* = -\mathcal{J}$ since it holds on fibers.

It remains to show that $t\mathcal{J}$ is a section of $C^+(\mathcal{V}_\mathbb{Z}, \psi_\mathbb{Z})$ for some $t > 0$. Using the notation from the proof of Proposition 4.2.7, we know that $tJ_\alpha \in C^+(\mathcal{V}_\mathbb{Z}, \psi_\mathbb{Z})_\alpha$ for some $t > 0$ as discussed in Section 2.4. For $x \in U_\alpha$, the action of $\pi_1(X, x)$ on the fibers lands in $SO(\mathcal{V}_\mathbb{Z}, \psi_\mathbb{Z})$, and hence the same argument as in the proof of Proposition 4.2.7 shows that the choice of t is consistent and yields a global section $t\mathcal{J}$ of $C^+(\mathcal{V}_\mathbb{Z}, \psi_\mathbb{Z})$. $\square$

Example 4.2.9. *A smooth, projective family of cubic surfaces* $Y \xrightarrow{f} X$ *satisfies the assumptions of Theorem 4.2.8 with* $\mathcal{V}_\mathbb{Z} = P^2 f_*(\mathbb{Z})$ *and* ψ *the intersection pairing, as in Example 4.1.7 (upon possibly replacing* X *by a finite étale extension* X' *and* Y *by the fibered product* $Y \times_X X'$*). Therefore, we obtain an abelian scheme* $A \to X$ *whose fibers are the abelian varieties corresponding to the fibers of* f *(which are cubic surfaces, see Example 3.4.1).*

Example 4.2.10 (Kuga-Satake Construction in Families)**.** *Theorem 2.4.3 shows that the Kuga-Satake construction works in families.*

Chapter 5

Abelian Varieties Associated to Higher Degree Forms

In this chapter, we apply the techniques of Chapter 3 to construct abelian varieties from the Clifford algebras of higher degree forms. Due to the fact that these generalized Clifford algebras are infinite dimensional, we consider their finite-dimensional representations. This is the reason we took the representation-theoretic viewpoint when investigating the Clifford algebras of quadratic forms. We give criteria for which a representation of a generalized Clifford algebra yields an abelian variety. This is done in a manner which parallels the Kuga-Satake construction. It turns out that cubic and quartic forms are the best candidates for this construction. We provide geometric examples of 3-folds and 4-folds for which we can create an abelian variety from the cubic and quartic forms, respectively, on H^2. This includes 4-folds which arise as the product of two surfaces. Finally, we study when this construction is natural enough to work in families as in Chapter 4.

5.1 Generalized Clifford Algebras

We begin by discussing graded algebras over $\mathbb{C}$ (the discussion works just as well for arbitrary fields). We say an algebra A is $\mathbb{Z}/d\mathbb{Z}$-graded if there is a decomposition into subspaces $A = A_0 \oplus \cdots \oplus A_{d-1}$ with $A_j A_k \subset A_{j+k}$, where the subscript is taken mod d. An element

$a \in A_k$ is called homogeneous of degree k. A map $\varphi : A \to B$ of $\mathbb{Z}/d\mathbb{Z}$-graded algebras is called a *graded homomorphism* if it is a homomorphism with $\varphi(A_j) \subset B_j$.

Example 5.1.1. *The matrix algebra $M_d(\mathbb{C})$ can be given a $\mathbb{Z}/d\mathbb{Z}$-grading for which the homogeneous elements of degree k are those matrices whose (i,j) entry is 0 unless $j - i \equiv k$* mod d. *When $d = 4$, we have*

$$\text{grade 0:} \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \qquad \text{grade 1:} \begin{pmatrix} 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \\ * & 0 & 0 & 0 \end{pmatrix}$$
$$\text{grade 2:} \begin{pmatrix} 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \\ * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \end{pmatrix} \qquad \text{grade 3:} \begin{pmatrix} 0 & 0 & 0 & * \\ * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \end{pmatrix}$$

When $d \mid N$, we can give $M_N(\mathbb{C})$ a $\mathbb{Z}/d\mathbb{Z}$-grading as follows. Since $M_N(\mathbb{C}) \cong M_d(\mathbb{C}) \otimes \cdots \otimes M_d(\mathbb{C})$, declare an element of $M_N(\mathbb{C})$ to be homogeneous if its tensor components under this decomposition are homogeneous. The degree of a homogeneous element of $M_N(\mathbb{C})$ is the sum (taken mod d) of the degrees of its tensor components.

Let f be a degree d homogeneous polynomial in n variables over K, where K is either $\mathbb{Z}$ or is a field. We define the *generalized Clifford algebra* of f to be

$$C_f(K) = K\langle x_1, \dots x_n\rangle / \big((a_1x_1 + \cdots + a_nx_n)^d - f(a_1, \dots, a_n) : a_i \in K\big),$$

where $K\langle x_1, \dots x_n\rangle$ denotes the tensor algebra in n variables. If we instead have a degree d form on a vector space V of dimension n, the definition is:

$$C_f(V) = T(V)/\langle v^d - f(v) : v \in V\rangle,$$

where $T(V)$ is the tensor algebra on V and v^d means $v \otimes \cdots \otimes v$ (d-times). Choosing a basis for V recovers the first definition. The tensor algebra is $\mathbb{Z}/d\mathbb{Z}$-graded by degree, and since the ideal defining the Clifford algebra preserves this grading, we find that $C_f(K)$ is $\mathbb{Z}/d\mathbb{Z}$-graded:

$$C_f(K) = C_f^0(K) \oplus C_f^1(K) \oplus \cdots \oplus C_f^{d-1}(K).$$

It is well-known that when $d \geq 3$ and $n \geq 2$, this is an infinite-dimensional algebra.

5.1.1 Graded Representations

We now discuss graded representations of Clifford algebras. For a representation $C_f(\mathbb{C}) \to M_N(\mathbb{C})$, where f is a form of degree d, we necessarily have that $d \mid N$ [HT88]. A graded representation is just a graded homomorphism between a graded algebra and a matrix algebra (which we will take to be graded as in Example 5.1.1). We first consider diagonal forms, then discuss the general case.

Throughout, we let $\rho \in \mathbb{C}$ be a primitive d^{th} root of unity. Suppose we have a diagonal form of degree d

$$f(X_1, \ldots, X_n) = a_1 X_1^d + \cdots + a_n X_n^d.$$

Let x_i denote the generators of the Clifford algebra corresponding to X_i. For each integer $n \geq 1$, we will construct a representation of the Clifford algebra $C_f(K)$. Set $N = d^n$ and write

$$M_N(\mathbb{C}) = \overbrace{M_d(\mathbb{C}) \otimes \cdots \otimes M_d(\mathbb{C})}^{n\text{-times}}.$$

Define a representation $\varphi_f : C_f(K) \to M_N(\mathbb{C})$ by setting

$$\varphi_f(x_j) = A_j := \overbrace{\Omega \otimes \cdots \otimes \Omega}^{j-1} \otimes A \otimes I \cdots \otimes I$$

where $\Omega \in M_d(\mathbb{C})$ is

$$\Omega = diag(1, \rho, \ldots, \rho^{d-1}),$$

and

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ a_i & 0 & \cdots & 0 \end{pmatrix}. \tag{5.1}$$

It is easily verified that $\rho A_k A_j = A_j A_k$ for $j < k$. The fact that this map is a representation of C_f follows from repeated use of the next lemma. This representation is graded since each A_j has grade 1 (see Example 5.1.1 for grading on matrix algebras) and because the relation $\rho A_k A_j = A_j A_k$ for $j < k$ preserves the grading.

Lemma 5.1.2. *Let $\mathcal{A}$ be an associative $\mathbb{C}$-algebra and let ρ be a primitive d^{th} root of unity. Suppose $x, y \in \mathcal{A}$ satisfy $\rho yx = xy$. Then $(x+y)^d = x^d + y^d$.*

Proof. We give the proof for $d = 4$, and sketch the general case. By expanding, we get

$$\begin{aligned}(x+y)^4 = x^4 &+ (x^3y + x^2yx + xyx^2 + yx^3) \\ &+ (x^2y^2 + xy^2x + xyxy + yxyx + yx^2y + y^2x^2) \\ &+ (y^3x + y^2xy + yxy^2 + xy^3) + y^4 \\ = x^4 &+ y^4,\end{aligned}$$

where we have noted that, because of the assumptions on x and y, the sum of the three middle terms is 0.

For the general case, upon expanding $(x+y)^d$, we find elements $x^k * y^{d-k}$ (monomials with k x-terms and $(d-k)$ y-terms in any order) for each k, $1 \leq k < d$. The monomials of $(x+y)^d$ can be partitioned into sets whose elements are obtained by cyclically permuting the variables. For example, in the $d = 4$ case, we have that $x^2 * y^2$ splits up into the two sets

$S_1 = \{x^2y^2, xy^2x, y^2x^2, yx^2y\}$ and $S_2 = \{xyxy, yxyx\}$. Cyclically permuting the variables amounts to multiplying by a power of ρ. Choosing a representative of each partition set, we write each other element of the set as ρ to some power times the representative. Now we sum each partitioned set and find that the coefficient is just $1 + \omega + \cdots + \omega^{m-1}$ for some primitive m^{th} root of unity ω where $m|d$. Thus the sum of each partitioned set is 0, and hence $x^k * y^{d-k} = 0$ when $1 \leq k < d$. $\square$

Now we construct a representation of the Clifford algebra of an arbitrary degree d form using the method of Chapman [Cha13]. Suppose we have a form

$$f(X_1, \ldots, X_n) = \textstyle\sum_{i_j \geq 0} c_{i_1 \cdots i_n} X_1^{i_1} \cdots X_n^{i_n}$$

of degree $d = \sum_j i_j$. Let I be the set of indices $i = i_1 \cdots i_n$ corresponding to the nonzero $c_{i_1 \ldots i_n}$'s of the form f. Fix an ordering of the set I. Define

$$M_M(\mathbb{C}) = \textstyle\bigotimes_{i \in I} M_i$$

where M_i is a copy of $M_d(\mathbb{C})$. For $i = i_1 \cdots i_n \in I$ and fixed $k \in \{1, \ldots, n\}$, define the matrices $B, \tilde{B} \in M_d(\mathbb{C})$ as follows:

- B has $c_{i_1 \cdots i_n}$ in the $(d, 1)$-entry, 1 in the entries $(1, 2), \ldots, (i_k - 1, i_k)$, and 0 elsewhere.

- $\tilde{B}$ has 1 in the entries $(i_1 + \cdots + i_{k-1}, i_1 + \cdots + i_{k-1} + 1), \ldots, (i_1 + \cdots + i_k - 1, i_1 + \cdots + i_k)$ and 0 elsewhere.

For $i = i_1 \cdots i_n \in I$ and fixed $k \in \{1, \ldots, n\}$, we define matrices $B_{k;i} \in M_M(\mathbb{C})$ by:

- If $i_k = 0$, set $B_{k,i} = 0$, the zero matrix.

- If $i_k \neq 0$ and $i_1 = \cdots = i_{k-1} = 0$,

$$B_{k,i} = \Omega \otimes \cdots \otimes \Omega \otimes B \otimes I \cdots \otimes I,$$

where B is in the i^{th} component of the tensor product.

- Otherwise,

$$B_{k,i} = \Omega \otimes \cdots \otimes \Omega \otimes \tilde{B} \otimes I \cdots \otimes I,$$

where $\tilde{B}$ is in the i^{th} component of the tensor product.

Define a map $\varphi_f : C_f \to M_M(\mathbb{C})$ by

$$\varphi_f(x_k) = \sum_{i \in I} B_{k;i}. \tag{5.2}$$

Note that when f is diagonal, this agrees with the previous definition of φ_f from the beginning of this section. To see what is happening, we give a non-diagonal example:

Example 5.1.3. *Let $f(X,Y) = X^4 + 2X^3Y + 5XY^3 + Y^4$. Then the representation is*

$$\varphi_f(x) = \begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0 \end{pmatrix} + \Omega \otimes \begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&0\\ 2&0&0&0 \end{pmatrix} + \Omega \otimes \Omega \otimes \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 5&0&0&0 \end{pmatrix}$$

and

$$\varphi_f(y) = \Omega \otimes \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} + \Omega \otimes \Omega \otimes \begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} + \Omega \otimes \Omega \otimes \Omega \otimes \begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0 \end{pmatrix}$$

where we leave out the trailing $\otimes I$, for example

$$\begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0 \end{pmatrix} \qquad \textit{means} \qquad \begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0 \end{pmatrix} \otimes I \otimes I \otimes I$$

Since $(ax+by)^4 = f(a,b)$ in the Clifford algebra, that relation better hold in the image for this to be a representation. To check this, notice that $ax+by$ maps to

$$\begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & 0 & 0 \end{pmatrix} + \Omega \otimes \begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ 2a & 0 & 0 & 0 \end{pmatrix} + \Omega \otimes \Omega \otimes \begin{pmatrix} 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \\ 5a & 0 & 0 & 0 \end{pmatrix} + \Omega \otimes \Omega \otimes \Omega \otimes \begin{pmatrix} 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \\ b & 0 & 0 & 0 \end{pmatrix}$$

Now using Lemma 5.1.2, it is easy to see that the fourth power of this matrix is $f(a, b) \cdot I$.

We now prove in general, that the map (5.2) is well-defined, and is a graded representation of the Clifford algebra.

Proposition 5.1.4. *The map $\varphi_f : C_f \to M_M(\mathbb{C})$ defined above is a graded representation.*

Proof. In order for φ_f to be a representation, we need to show that

$$\big(a_1\varphi_f(x_1) + \cdots + a_n\varphi_f(x_n)\big)^d = f(a_1, \ldots, a_n) \cdot I.$$

By definition,

$$\big(a_1\varphi_f(x_1) + \cdots + a_n\varphi_f(x_n)\big)^d = \Big(\textstyle\sum_{i\in I} a_1B_{1;i} + \cdots + a_nB_{n;i}\Big)^d$$

But by Lemma 5.1.2 and the above construction,

$$\begin{aligned}\Big(\sum_{i\in I} a_1B_{1;i} + \cdots + a_nB_{n;i}\Big)^d &= \sum_{i\in I} \Big(a_1B_{1;i} + \cdots + a_nB_{n;i}\Big)^d \\ &= \sum_{c_{i_1\cdots i_n}\neq 0} c_{i_1\cdots i_n} a_1^{i_1} \cdots a_n^{i_n} \cdot I \\ &= f(a_1, \ldots, a_n) \cdot I\end{aligned}$$

We are now left with showing that this representation is graded. By construction, the nonzero $B_{k;i}$ are homogeneous of degree 1. For each $x_k \in C_f$ corresponding to X_k of the form f, $\varphi_f(x_k)$ contains at least one nonzero $B_{k,i}$ and hence is homogeneous of degree 1. By construction, the relations among the $B_{k,i}$ are such that the grading is preserved, hence the representation φ_f is graded. □

Although this representation is graded, one may find representations which are not graded. Since we are interested in graded representations of the Clifford algebra, we call upon a result from Childs [Chi78, Lemma 6, p. 274]. It says the following

Lemma 5.1.5. *If $\varphi_f : C_f \to M_M(\mathbb{C})$ is a representation sending x_i to A_i, then*

$$C_f \mapsto M_M(\mathbb{C}) \otimes M_d(\mathbb{C}) \qquad \textit{defined by} \qquad x_i \mapsto A_i \otimes \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ & & & 1 \\ 1 & & \cdots & 0 \end{pmatrix}$$

is a graded representation.

5.2 Abelian Varieties

One of the things that makes the Kuga-Satake construction work is the existence of the anti-involution ι of the (quadratic) Clifford algebra defined by flipping basis elements: $x_0 \cdots x_n \mapsto x_n \cdots x_0$. Thus, for $x = x_0 \cdots x_n$, $\iota(x)x$ is in the center of the Clifford algebra since x_i^2 is central. This plays a key role in showing that the Kuga-Satake complex torus is an abelian variety (see [vG00, Lemma 5.8 and Proposition 5.9] and Lemma 3.2.3). However, for the Clifford algebras of higher degree forms, we do not have that x_i^2 is central. For degree $d \geq 3$ forms, the "flipping" anti-involution seems to be of little use, since for example $\iota(x_0x_1)x_0x_1 = x_1x_0^2x_1$, which is unlikely to be central in the Clifford algebra. The purpose of Theorem 3.2.7 was to show that once we view the Clifford algebra of a quadratic form as a matrix algebra, the conjugate transpose $*$ is a terrific substitute for ι. We will show that the same idea carries over to higher degree forms.

We first fix some notation. Fix a degree d form f. Suppose we have a $\mathbb{Z}/d\mathbb{Z}$-graded representation $\varphi_f : C_f(\mathbb{C}) \to M_N(\mathbb{C})$. We may consider the induced representations on $C_f(K)$ for $K \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$ via the natural inclusion $C_f(K) \subset C_f(\mathbb{C})$. Let $c_f(K)$ denote the image of $C_f(K)$ under this representation. Since the representation is graded, we let $c_f^0(K)$ denote the degree 0 part of the image. Let $*$ denote the conjugate transpose of a matrix in $M_N(\mathbb{C})$. We make the following assumption:

1. There is a $J \in c_f^0(\mathbb{R})$ so that left multiplication by J on $c_f^0(\mathbb{R})$ defines a complex structure, and $\alpha := tJ \in c_f^0(\mathbb{Z})$ for some $t > 0$. Further $J^* = -J$.

As a start toward constructing an abelian variety, we define

$$Q : c_f^0(\mathbb{R}) \times c_f^0(\mathbb{R}) \to \mathbb{R}$$

by

$$Q(v, w) = Re(tr(\alpha v w^*))$$

where α is a positive multiple of J which is in $c_f^0(\mathbb{Z})$ as in assumption (1). Note that Q is $\mathbb{R}$-bilinear since taking Re, tr, and $*$ all are. We now prove some facts about Q.

Proposition 5.2.1. *$Q(Jv, Jw) = Q(v, w)$ for all $v, w \in c_f^0(\mathbb{R})$.*

Proof.

$$\begin{aligned}
Q(Jv, Jw) :&= Re(tr(\alpha(Jv)(Jw)^*)) \\
&= Re(tr(\alpha J v w^* J^*)) \\
&= Re(tr((-J)\alpha J v w^*)) && J^* = -J \text{ and } tr(AB) = tr(BA) \\
&= Re(tr(\alpha v w^*)) && J \text{ commutes with } \alpha, \text{ and } J^2 = -I \\
&= Q(v, w)
\end{aligned}$$

$\square$

Proposition 5.2.2. *$Q(v, Jw)$ is a positive definite, symmetric form.*

Proof. First note that for any square, complex matrix $A = (a_{i,j})$, we have $Re(tr(A)) = Re(tr(A^*))$ since

$$Re(tr(A)) = Re\Big(\sum_i a_{i,i}\Big) = Re\Big(\sum_i \overline{a_{i,i}}\Big) = Re(tr(A^*)).$$

Now we compute

$$\begin{aligned}
Q(v, Jw) &= Re(tr(\alpha v(Jw)^*)) \\
&= Re(tr((\alpha v w^* J^*)^*)) && \text{since } Re(tr(A)) = Re(tr(A^*)) \\
&= Re(tr(Jwv^*\alpha^*)) \\
&= -Re(tr(\alpha w v^* J)) && \alpha^* = -\alpha \text{ and } J \text{ commutes with } \alpha \\
&= Re(tr(\alpha w v^* J^*)) && J^* = -J \\
&= Re(tr(\alpha w(Jv)^*)) \\
&= Q(w, Jv)
\end{aligned}$$

so we get the symmetric part.

Finally, we need to check that $Q(v, Jv) > 0$ for all nonzero $v \in c_f^0(\mathbb{R})$. We have

$$\begin{aligned}
Q(v, Jv) &= Re(tr(\alpha v(Jv)^*)) \\
&= -Re(tr(J\alpha v v^*)) \\
&= t\ Re(tr(vv^*)) && \text{since } J\alpha = -t, \text{ for some } t > 0
\end{aligned}$$

But $tr(AA^*) > 0$ for any complex matrix $A = (a_{i,j})$, since $tr(AA^*) = \sum_{i,j} |a_{i,j}|^2$. $\square$

Theorem 5.2.3. *Let f be a degree d form such that $c_f^0(\mathbb{Z}) \subset c_f^0(\mathbb{R})$ is a full lattice and assume the existence of a J as in (1). Furthermore, suppose that Q restricted to the lattice $c_f^0(\mathbb{Z}) \times c_f^0(\mathbb{Z})$ takes values in $\mathbb{Z}$. Then $c_f^0(\mathbb{R})/c_f^0(\mathbb{Z})$ is an abelian variety.*

Proof. The assumption that Q restricted to the lattice takes values in $\mathbb{Z}$ combined with Propositions 5.2.1 and 5.2.2 show that Q satisfies the Riemann bilinear relations. Hence the complex torus $c_f^0(\mathbb{R})/c_f^0(\mathbb{Z})$, with complex structure defined by J, is an abelian variety. $\square$

Although there are several assumptions for Theorem 5.2.3, the remainder of the paper shows that there are cases in which the assumptions are satisfied.

5.3 Lattices

We now justify the restriction to the case of cubic and quartic forms made in the introduction to this chapter. Proposition 5.1.4 gives a graded representation of the Clifford algebra C_f (over $\mathbb{C}, \mathbb{R}, \mathbb{Q}$, and $\mathbb{Z}$) into matrices over $\mathbb{C}$. As before, let $c_f^0(K)$ denote the degree 0 part of the image of the representation of $C_f(K)$ for $K \in \{\mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}\}$. If we have any hope of getting a complex torus out of this, we will need $c_f^0(\mathbb{Z})$ to be a full lattice inside of $c_f^0(\mathbb{R})$. Of course, this means that $c_f^0(\mathbb{Z})$ is a free $\mathbb{Z}$-module of finite rank equal to $\dim_{\mathbb{R}} c_f^0(\mathbb{R})$ and that $c_f^0(\mathbb{Z}) \otimes \mathbb{R} = c_f^0(\mathbb{R})$. We show the following:

Proposition 5.3.1. *If f is a cubic or quartic form, then $c_f^0(\mathbb{Z})$ is a full lattice in $c_f^0(\mathbb{R})$ under the representation of Section 5.1.1. This fails if* $\deg f > 4$.

Proof. For a cubic form, the elements of $c_f^0(\mathbb{Z})$ are matrices with entries in $\mathbb{Z}[\omega]$ where $\omega = \frac{1+i\sqrt{3}}{2}$ is a primitive cube root of unity. Hence $c_f^0(\mathbb{Z})$ lives in $M_N(\mathbb{Z}[\omega])$ for some N, which is a free $\mathbb{Z}$-module. Since submodules of free $\mathbb{Z}$-modules are themselves free, we have that $c_f^0(\mathbb{Z})$ is a free $\mathbb{Z}$-module. We must show that the rank of $c_f^0(\mathbb{Z})$ is the same as the dimension of $c_f^0(\mathbb{R})$. But clearly the $\mathbb{R}$-span of $c_f^0(\mathbb{Z})$ is $c_f^0(\mathbb{R})$, which shows rank $c_f^0(\mathbb{Z}) \geq \dim c_f^0(\mathbb{R})$. Conversely, since $\{1, \omega\}$ is a linearly independent set over $\mathbb{R}$, any $\mathbb{Z}$-linearly independent set of matrices in $M_N(\mathbb{Z}[\omega])$ will be $\mathbb{R}$-linearly independent as well. This shows rank $c_f^0(\mathbb{Z}) \leq \dim c_f^0(\mathbb{R})$, and hence $c_f^0(\mathbb{Z})$ is a full lattice in $c_f^0(\mathbb{R})$ when f is a cubic form.

When f is a quartic form, the argument is identical, just replace ω with i everywhere. For higher degree forms, the last part of the argument breaks down. As an example, with f a quintic form and ξ a primitive 5^{th} root of unity, the elements of $c_f^0(\mathbb{Z})$ are inside of $M_N(\mathbb{Z}[\xi])$ just as before. However, the set $\{1, \xi, \xi^2\}$ is $\mathbb{Z}$-linearly independent, but not $\mathbb{R}$-linearly independent. $\square$

Corollary 5.3.2. *Let f be a cubic or quartic form. Suppose that under the representation of Section 5.1.1, there is an element $J \in c_f^0(\mathbb{R})$ satisfying the properties in (1). Then $c_f^0(\mathbb{R})/c_f^0(\mathbb{Z})$ is an abelian variety.*

Proof. By the construction of the representation in Section 5.1.1, $c_f^0(\mathbb{Z})$ for f quartic (respectively cubic) consists of matrices with entries in $\mathbb{Z}[i]$ (resp. $\mathbb{Z}[\omega]$). This tells us that Q (resp. $2Q$) restricted to the lattice $c_f^0(\mathbb{Z}) \times c_f^0(\mathbb{Z})$ takes values in $\mathbb{Z}$. Proposition 5.3.1 shows that $c_f^0(\mathbb{Z}) \subset c_f^0(\mathbb{R})$ is a full lattice. Thus, $c_f^0(\mathbb{R})/c_f^0(\mathbb{Z})$ is an abelian variety by Theorem 5.2.3. $\square$

5.4 Applications

This section deals with situations in which we can find an element J satisfying the condition (1). Suppose $f(X_1, \ldots, X_n)$ is a cubic form with coefficients in $\mathbb{Z}$ of the shape

$$a_1 X_1^3 + a_2 X_2^3 + g(X_3, \ldots, X_n)$$

Let $x_1, \ldots, x_n$ be the corresponding generators of C_f. Then under the representation constructed above with $x_i \mapsto A_i$, we have

$$J := \frac{1}{\sqrt{3}} \left(\frac{2}{a_1 a_2} A_1 A_2 A_1^2 A_2^2 - I \right) = \frac{2\omega - 1}{\sqrt{3}} I = iI$$

With $t = |a_1 a_2| \sqrt{3}$, J clearly satisfies the requirements for condition (1) above. Proposition 5.3.1 and Theorem 5.2.3 tell us $c_f^0(\mathbb{R})/c_f^0(\mathbb{Z})$ is an abelian variety. This works in particular with diagonal cubic forms (in ≥ 2 variables).

Similarly, suppose $f(X_1, \ldots, X_n)$ is a quartic form with coefficients in $\mathbb{Z}$ of the shape

$$a_1 X_1^4 + a_2 X_2^4 + g(X_3, \ldots, X_n)$$

Let $x_1, \ldots, x_n$ be the corresponding generators of C_f. Then under the representation constructed above with $x_i \mapsto A_i$, we have

$$J := \frac{1}{a_1 a_2} A_1 A_2 A_1^3 A_2^3 = iI$$

Then J (with $t = |a_1 a_2|$) clearly satisfies the requirements for Theorem 5.2.3 above, and hence $c_f^0(\mathbb{R})/c_f^0(\mathbb{Z})$ is an abelian variety. This works in particular with diagonal quartic forms (in ≥ 2 variables).

Remark 5.4.1. *In the case that $f(X, Y) = aX^4 + bY^4$ is a binary diagonal from of degree 4, the graded representation $C_f \to M_{16}(\mathbb{C})$ constructed above is not surjective. However, the image is the cyclic algebra*

$$(a, b)_{4,\mathbb{C}} = \mathbb{C}\langle x, y \ : \ x^4 = a, y^4 = b, iyx = xy \rangle.$$

Since we have $(a, b)_{4,\mathbb{C}} \cong M_4(\mathbb{C})$, this gives a minimal-dimensional, surjective, graded representation of C_f. According to an unpublished result of Rajesh Kulkarni, such a representation is unique (up to conjugation). Hence, the abelian variety we construct is unique in this sense.

The above discussion holds as well for binary diagonal forms of degree 3.

5.4.1 Example

Consider the blowup of $\mathbb{P}^5$ at a point. Note $4H - E$ is very ample (where H is the strict transform of a hyperplane in $\mathbb{P}^5$ and E is the exceptional divisor). Let X be a hyperplane section under this embedding. Then $H^2(X, \mathbb{Z})/torsion$ is two dimensional with intersection form $f = 4x^4 - y^4$, and we may apply the above construction to get an abelian variety.

5.4.2 Example

This example is similar to the varieties considered in [Lan98]. Let Y be a degree d hypersurface in $\mathbb{P}^4$ (including $d = 1$). Define

$$X = \mathbb{P}(\mathcal{E})$$

where $\mathcal{E} = \mathcal{O}_Y(n) \oplus \mathcal{O}_Y(-n)$ for $n \geq 1$. Let $\pi : X \to Y$ be the projection, h be a hyperplane section of Y, let H denote $\pi^* h$, and let ξ be the class of the divisor corresponding to $\mathcal{O}_X(1)$. We note that $h^4 = 0$ and so $H^4 = 0$. Furthermore, since $\mathcal{O}_X(1)$ restricted to a fiber over any point of Y is $\mathcal{O}_{\mathbb{P}^1}(1)$ and $h^3 = d[p]$ (where $[p]$ represents the class of a point in Y), we see that $H^3.\xi = d$.

We show that the intersection form on $H^2(X, \mathbb{Q})$ is diagonalizable. Note that $H^2(X, \mathbb{Q}) = \langle H, \xi \rangle$ and that the intersection theory on X is defined by the Leray-Hirsch formula (see [Har77, Appendix A §3] and [Hat02, Theorem 4D.1])

$$\xi^2 - (\pi^* c_1).\xi + (\pi^* c_2) = 0 \tag{$\ddagger$}$$

where c_1 and c_2 are the first and second Chern classes of the sheaf $\mathcal{E}$.

Since $\mathcal{E}$ is a direct sum of line bundles, its Chern classes are straight-forward to calculate. The Chern polynomial is

$$c_t(\mathcal{E}) = (1 + (nh)t)(1 - (nh)t) = 1 - (n^2h^2)t^2$$

Therefore, $c_1 = 0$ and $c_2 = -n^2h^2$. Equation (‡) then becomes $\xi^2 = n^2H^2$. Intersecting this with:

- H^2 gives $H^2.\xi^2 = n^2H^4 = 0$
- $H.\xi$ gives $H.\xi^3 = n^2H^3\xi = n^2d$
- ξ^2 gives $\xi^4 = n^2H^2\xi^2 = 0$

Letting U and V correspond to H and ξ, respectively, the intersection form is

$$f(U, V) = d\left[U^3V + n^2UV^3\right]$$

The linear change of variables $U \mapsto nU + nV$ and $V \mapsto U - V$ sends f to the form $2dn^3(U^4 - V^4)$, which is diagonal. Hence, the representation constructed above gives an abelian variety, A. This abelian variety is unique in the sense of Remark 5.4.1. Furthermore, we have an inclusion of Hodge structures

$$H^2(X, \mathbb{Q}) \hookrightarrow H^1(A, \mathbb{Q}) \otimes H^1(A, \mathbb{Q})$$

which is proved in the same way as Proposition 3.4.4.

5.4.3 Example

We now construct 3-folds whose intersection form on H^2 is diagonal. It parallels the previous example, so we will use analogous notation. Let Y be a degree d hypersurface in $\mathbb{P}^3$. Define

$$X = \mathbb{P}(\mathcal{E})$$

where $\mathcal{E} = \mathcal{O}_Y \oplus \mathcal{O}_Y(n)$ any $n \in \mathbb{Z}$. Then $c_t(\mathcal{E}) = 1 + nht$, so $c_1 = nh$ and $c_2 = 0$. The intersection theory on X is again defined by (‡) which now looks like

$$\xi^2 = nH.\xi$$

Intersecting this with:

- H gives $H.\xi^2 = nH^2.\xi = nd$
- ξ gives $\xi^3 = nH.\xi^2 = n^2 d$

since $H^3 = 0$ and $H^2.\xi = d$ as before. Letting U and V correspond to H and ξ, respectively, the intersection form is

$$f(U, V) = d\left[U^2 V + nUV^2 + n^2 V^3\right]$$

The linear change of variables $U \mapsto U + V$ and $V \mapsto nU - 2nV$ sends f to the form $3dn^2(U^3 + V^3)$ which is diagonal. Hence, the representation constructed above gives an abelian variety, A. This abelian variety is unique in the sense of Remark 5.4.1. Furthermore, we have an inclusion of Hodge structures

$$H^2(X, \mathbb{Q}) \hookrightarrow H^1(A, \mathbb{Q}) \otimes H^1(A, \mathbb{Q})$$

which is proved in the same way as Proposition 3.4.4.

5.5 Products of Surfaces

We now look at the case of 4-folds which are products of surfaces. The Künneth isomorphism gives

$$H^2(X \times Y, \mathbb{Q}) \cong \bigoplus_{p+q=2} H^p(X, \mathbb{Q}) \otimes H^q(Y, \mathbb{Q})$$

and the cup product is determined by

$$(a \otimes b) \cup (c \otimes d) = (-1)^{\deg b \deg c}(a \cup c) \otimes (b \cup d).$$

If we suppose that $h^1(X) = 0$, then $H^2(X \times Y) \cong H^2(X) \oplus H^2(Y)$ and the negative sign in the cup product formula is killed. Therefore, we see that the intersection (quartic) form on $H^2(X \times Y)$ is just the product of the intersection (quadratic) forms on $H^2(X)$ and $H^2(Y)$.

This leads us to consider quartic forms which are products of quadratic forms. We recall a proposition from Childs [Chi78, Theorem 8, p. 274]

Proposition 5.5.1. *Let $f = gh$ where g and h are quadratic forms. If C_g and C_h have finite-dimensional representations, then so does C_f.*

Proof. We give a sketch of the proof: we may suppose that C_g and C_h have representations of the same size (otherwise tensor the representations with appropriately sized matrix algebras), say $C_g \to M_n(\mathbb{C})$ sending generators x_i to A_i and $C_h \to M_n(\mathbb{C})$ sending generators $y_j \to B_j$. Then we obtain a representation of C_f defined by the block matrices

$$x_i \mapsto \begin{pmatrix} 0 & A_i & 0 & 0 \\ 0 & 0 & A_i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad y_j \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_j \\ B_j & 0 & 0 & 0 \end{pmatrix}$$

This is a representation of size $4n$. Note that we abusively write x_i and y_j for generators of C_f, corresponding to the generators of C_g and C_h, respectively. □

Of course this proposition is true for the product of arbitrary degree forms, we just gave the proof for the product of quadratic forms since that's all we will need. We now specialize to the case where $f = q^2$ for a quadratic form q. Geometrically, we are thinking of 4-folds of the form $X \times X$ for a surface X.

For the next theorem, we first fix some notation. Recall that if we have a graded representation of C_f, we write $c_f^0(\mathbb{R})$ to denote the degree 0 part of the image of $C_f(\mathbb{R})$. If furthermore the degree of f is even, we write c_f^+ to denote the subalgebra of the image of C_f consisting of even-graded elements. All of our previous theorems are valid with c_f^0 replaced by c_f^+.

Theorem 5.5.2. *Let X be a complex surface with $h^1 = 0, h^2 \geq 2$, and with intersection pairing of signature other than $(1,1)$. Let f denote the quartic intersection form on $H^2(X \times X)$. Then there exists a graded representation of C_f for which $c_f^+(\mathbb{R})/c_f^+(\mathbb{Z})$ is an abelian variety.*

Proof. By the discussion above, we have that $f = q^2$, where q is the intersection form on $H^2(X)$. We may suppose that $q = a_1W_1^2 + \cdots + a_nW_n^2$ is diagonal with coefficients in $\mathbb{Z}$. Therefore,

$$f = (a_1X_1^2 + \cdots + a_nX_n^2)(a_1Y_1^2 + \cdots + a_nY_n^2)$$

As usual, we will write x_i and y_i and w_i as the generators of the Clifford algebra, corresponding to X_i and Y_i and W_i, respectively. We have the representation of C_q from Section 3.1 which we briefly recall here:

$$w_1 \mapsto A_1 = \sqrt{a_1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$w_2 \mapsto A_2 = \sqrt{a_2}i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\vdots$$

$$w_n \mapsto A_n = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \sqrt{a_n} i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

for n even. When n is odd use the representation for $(n+1)$-terms and forget w_{n+1}. This map is injective. We may assume that $a_1 a_2 > 0$ (due to the restriction on the signature of q). Then in $C_q(\mathbb{R})$, the element $J = \frac{1}{\sqrt{a_1 a_2}} A_1 A_2$ satisfies $J^2 = -1$ and $J^* = -J$. Under the representation in Proposition 5.5.1, we see that

$$\frac{1}{\sqrt{a_1 a_2}}(x_1 + y_1)(x_2 + y_2) \mapsto \begin{pmatrix} 0 & 0 & J & 0 \\ 0 & 0 & 0 & J \\ J & 0 & 0 & 0 \\ 0 & J & 0 & 0 \end{pmatrix}$$

Call the image element $\mathcal{J} \in c_f^2(\mathbb{R})$. Then $\mathcal{J}^2 = -I$, $\mathcal{J}^* = -\mathcal{J}$, and $\sqrt{a_1 a_2}\mathcal{J} \in c_f^+(\mathbb{Z})$. Thus, left-multiplication by $\mathcal{J}$ defines a complex structure on $c_f^+(\mathbb{R})$ satisfying (1).

Since under the representation of C_q, $C_q(\mathbb{Z})$ is a full lattice inside $C_q(\mathbb{R})$, we immediately have that $c_f^+(\mathbb{Z}) \subset c_f^+(\mathbb{R})$ is a full lattice because it is obtained by taking blocks of the representation of C_q.

Finally, Q restricted to the lattice takes values in $\mathbb{Z}$. Note that the only elements of c_f with nonzero trace must live in c_f^0. Elements of c_f^0 are block diagonal matrices where the matrices in each block come from the representation of C_q. But the only basis elements of C_q with nonzero trace are the constant integer multiples of I (see Lemma 3.2.3). This tells us that under the representation of Proposition 5.5.1, Q restricted to the lattice takes values in $\mathbb{Z}$. Therefore, Theorem 5.2.3 says that the quotient $c_f^+(\mathbb{R})/c_f^+(\mathbb{Z})$ is an abelian variety. $\square$

Recall in Theorem 3.2.8, we created an abelian variety $C_q^+(\mathbb{R})/C_q^+(\mathbb{Z})$ from a quadratic form q in two or more variables of signature other than $(1,1)$.

Theorem 5.5.3. *Let X be a complex surface with $h^1 = 0, h^2 \geq 2$, and with intersection pairing q of signature other than $(1,1)$. Let f denote the quartic intersection form on $H^2(X \times X)$. Then the abelian variety $C_q^+(\mathbb{R})/C_q^+(\mathbb{Z})$ is a subquotient of the abelian variety $c_f^+(\mathbb{R})/c_f^+(\mathbb{Z})$ from Theorem 5.5.2.*

Proof. Let $f = q^2$ for a quadratic form $q = a_1W_1^2 + \cdots + a_nW_n^2$ with integer coefficients. We first show that the image of C_f under the representation from Proposition 5.5.1 (which we again denote by c_f) contains a subalgebra which surjects onto C_q.

Let w_i denote the generators of C_q, and let x_i and y_i denote the generators of C_f each corresponding to w_i (as in the previous theorem). The map $w_i \mapsto A_i$ discussed in the proof of the previous theorem identifies C_q with a subalgebra of a matrix algebra. The representation φ of C_f from Proposition 5.5.1 has

$$\varphi(x_i + y_i) = \begin{pmatrix} 0 & A_i & 0 & 0 \\ 0 & 0 & A_i & 0 \\ 0 & 0 & 0 & A_i \\ A_i & 0 & 0 & 0 \end{pmatrix}$$

Now the map

$$\psi : \begin{pmatrix} 0 & A_i & 0 & 0 \\ 0 & 0 & A_i & 0 \\ 0 & 0 & 0 & A_i \\ A_i & 0 & 0 & 0 \end{pmatrix} \mapsto A_i$$

defines a surjective map from the subalgebra $B = \langle I, \varphi(x_i + y_i)\rangle \subseteq c_f$ onto C_q. We don't specify the field over which the Clifford algebra is defined, since this is surjective when we consider the Clifford algebras over $\mathbb{C}, \mathbb{R}, \mathbb{Q}$, or even $\mathbb{Z}$ (we include I in the generating set of the subalgebra, since it may be necessary for surjectivity when considering the Clifford algebra over $\mathbb{Z}$). We write $B_{\mathbb{R}}$ to denote the algebra spanned by the elements of B with coefficients in $\mathbb{R}$, and similarly for $\mathbb{C}$, $\mathbb{Q}$, and $\mathbb{Z}$.

Assuming $a_1a_2 > 0$ and the complex structure on $C_q^+(\mathbb{R})$ is defined by left-multiplication by $J = \frac{1}{\sqrt{a_1a_2}}A_1A_2$, the complex structure on $c_f^+(\mathbb{R})$ is then defined by $\mathcal{J} \in c_f^2(\mathbb{R})$. We see that $\mathcal{J} \in B_{\mathbb{R}}$ and $\mathcal{J} \mapsto J$, so the map ψ is compatible with the complex structures. Therefore, we have a surjective map of complex tori $B_{\mathbb{R}}/B_{\mathbb{Z}} \to C_q^+(\mathbb{R})/C_q^+(\mathbb{Z})$. Since $B_{\mathbb{R}}/B_{\mathbb{Z}}$ is a complex subtorus of an abelian variety, it is itself an abelian variety. This finishes the theorem. $\square$

Example 5.5.4. *In particular, when X is a K3 surface, we get that the Kuga-Satake abelian variety associated to X is a subquotient of of the abelian variety associated to $H^2(X \times X)$ as in Theorem 5.5.2.*

5.6 Construction in Families

We now examine our construction for cubic and quartic forms works in families. Suppose $\mathcal{V}$ is a variation of Hodge structure of weight k on X with a flat multilinear form f, of degree d (we assume $d = 3$ or 4 throughout). Then we can form the local system of Clifford algebras $C(\mathcal{V}, f)$ as in the quadratic case, and we have its graded components: $C^0(\mathcal{V}, f), \ldots, C^{d-1}(\mathcal{V}, f)$.

The setup for performing our construction in families is the following. Suppose we have $\mathcal{W}$ a local system of complex vector spaces on X for which $\mathrm{End}(\mathcal{W})$ is $\mathbb{Z}/d\mathbb{Z}$-graded. Consider a graded morphism of local systems $C(\mathcal{V}_{\mathbb{C}}, f_{\mathbb{C}}) \to \mathrm{End}(\mathcal{W})$ and consider the induced maps on $C(\mathcal{V}_{\mathbb{R}}, f_{\mathbb{R}})$ and $C(\mathcal{V}_{\mathbb{Z}}, f_{\mathbb{Z}})$. We let $c(\mathcal{V}, f)$ denote the image of $C(\mathcal{V}, f)$. Assume that $c^0(\mathcal{V}_{\mathbb{Z}}, f_{\mathbb{Z}})$ is a full lattice inside of $c^0(\mathcal{V}_{\mathbb{R}}, f_{\mathbb{R}})$. Furthermore, suppose there is a locally-constant section $\mathcal{J}$ of $c^0(\mathcal{V}_{\mathbb{R}}, f_{\mathbb{R}})$ such that for each $x \in X$, left multiplication by $\mathcal{J}_x$ defines a complex structure on the fiber $c^0(\mathcal{V}_{\mathbb{R}}, f_{\mathbb{R}})_x$. The following is a restatement of Proposition 4.2.1.

Proposition 5.6.1. *With the above assumptions, $c^0(\mathcal{V}, f)$ is a VHS on X of weight 1.*

The major difficulty is in finding conditions for which this is a polarized VHS of weight 1. The issue is being able to globally define conjugate transpose of a matrix. If $\mathrm{End}(\mathcal{W})$ is a *unitary* bundle, we are able to globally define conjugate transpose. By unitary bundle we mean there is a cover $\{U_\alpha\}$ of X which trivializes the bundle $\mathrm{End}(\mathcal{W})$ for which the transition functions are given by conjugation by a unitary matrix. In this case, we can define the "conjugate transpose" as a map $* : \mathrm{End}(\mathcal{W}) \to \mathrm{End}(\mathcal{W})$ of C^∞-vector bundles. To do this, let U_α and U_β be two open sets in X in the trivializing cover of $\mathrm{End}(\mathcal{W})$. We write $U_\alpha \times M_N(\mathbb{C})$ and $U_\beta \times M_N(\mathbb{C})$ for the open sets above U_α and U_β, respectively, with transition function $\varphi_{\alpha\beta}$. Above $U_\alpha \cap U_\beta$, consider the following diagram

$$\begin{tikzcd}
M_N(\mathbb{C}) \arrow[r, "\varphi_{\alpha\beta}"] \arrow[d, "*"'] & M_N(\mathbb{C}) \arrow[d, "*"] \\
M_N(\mathbb{C}) \arrow[r, "\varphi_{\alpha\beta}"] & M_N(\mathbb{C})
\end{tikzcd}$$

By assumption, $\varphi_{\alpha\beta}$ is given by conjugation by a unitary matrix S (i.e. $S^* = S^{-1}$), hence we have

$$\begin{aligned}
\varphi_{\alpha\beta}(A)^* &= \left(SAS^{-1}\right)^* \\
&= (S^{-1})^* A * S^* \\
&= SA^*S^{-1} \\
&= \varphi_{\alpha\beta}(A^*)
\end{aligned}$$

so the diagram above commutes. Therefore, conjugate transpose is a globally-defined map which is $\mathbb{R}$-linear on fibers.

Proposition 5.6.2. *Assume the hypotheses of Proposition 5.6.1 and that End($\mathcal{W}$) is a unitary bundle. Suppose $t\mathcal{J}$ is a section of $c^0(\mathcal{V}_\mathbb{Z}, f_\mathbb{Z})$ for some $t > 0$ and that $\mathcal{J}$ satisfies condition (1) on fibers. Furthermore, if $Re(tr(\alpha_x v w^*)) \in \mathbb{Z}$ for any $v, w \in c^0(\mathcal{V}_{\mathbb{Z},x}, f_{\mathbb{Z},x})$, then $c^0(\mathcal{V}, f)$ is a polarized VHS of weight 1 on X.*

Proof. This is just a global version of Theorem 5.2.3. All of the assumptions we have made make the result immediate. □

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