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ABSTRACT

HOMOGENEOUS UNIVERSAL
GROUPS

By

Kenneth Keller Hickin

This thesis contains a recasting and generalization-with-application of Jonsson's Theorem on homogeneous universal structures, distinct from the notions of saturated and special structures.

Let κ be an infinite regular cardinal and let $\mathfrak{M}$ be a class of algebras or relational systems of type $< \kappa$. $\mathfrak{M}^{<\kappa}$ denotes the class of $\mathfrak{M}$ algebras generated by subsets of power $< \kappa$.

We note that homogeneous universal structures of power κ can be characterized by an "injectivity" property which we generalize to <u>κ-injectivity of complete chains of $\mathfrak{M}$ algebras of arbitrary order types</u>. In the case of a chain $1 < A < B$ with two jumps, the definition is as follows: $\{1,A,B\} < \mathfrak{M}$ is <u>κ-injective for $\mathfrak{M}$</u> iff, for all $E,X,Y \in \mathfrak{M}^{<\kappa}$ such that $X < Y$, $E < Y$ and $X \cap E \in \mathfrak{M}^{<\kappa}$ and for all embeddings $f: E \to B$ such that $f(E) \cap A = f(E \cap X)$, f extends to an embedding $\bar{f}: Y \to B$ such that $\bar{f}(Y) \cap A = \bar{f}(X)$. This definition generalizes easily to arbitrary complete chains.

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To construct κ-injective chains, an assumption called the <u>subamalgam property (s.p.)</u> is used. $\mathcal{A} = \begin{array}{ccc} B & & C \\ & \diagdown\;\diagup & \\ & A & \end{array}$ is an <u>$\mathcal{M}$ amalgam</u> iff $A, B, C \in \mathcal{M}$, $\mathcal{A} = B \cup C$, and $B \cap C = A$. $\mathcal{A}_0 = \begin{array}{ccc} B_0 & & C_0 \\ & \diagdown\;\diagup & \\ & A_0 & \end{array}$ is a <u>subamalgam of</u> $\mathcal{A}$ iff $\mathcal{A}_0 \leq \mathcal{A}$ and $B_0 \cap A = C_0 \cap A = A_0$. $\mathcal{M}$ has the <u>s.p.</u> iff for every $\mathcal{M}$ amalgam $\mathcal{A}$ there exists $M = \mathrm{alg}_M(\mathcal{A}) \in \mathcal{M}$ such that, for all $\mathcal{M}$ subamalgams $\mathcal{A}_0$ of $\mathcal{A}$, we have $\mathrm{alg}_M(\mathcal{A}_0) \cap \mathcal{A} = \mathcal{A}_0$ and $\mathrm{alg}_M(\mathcal{A}_0) \in \mathcal{M}$. To construct well-ordered κ-injective chains for any ordinal $< \kappa^+$ and to construct ω-injective chains of any countable jump-type, we require that $\mathcal{M}^{<\kappa}$ have the s.p. plus the usual set-theoretic assumptions of Jonsson's Theorem. For arbitrary jump-types of power $\leq \kappa$, we also need the <u>descendance</u> <u>condition</u>: the algebra M above can be chosen so that, for every chain $\{\mathcal{A}_\alpha | \alpha \in I\}$ of $\mathcal{M}$ subamalgams of $\mathcal{A}$, we have $\bigcap\{\mathrm{alg}_M(\mathcal{A}_\alpha) | \alpha \in I\} = \mathrm{alg}_M(\bigcap_{\alpha \in I} B_\alpha, \bigcap_{\alpha \in I} C_\alpha)$. $\mathcal{M}$ algebras of power κ possessing such chains are unique up to <u>isomorphism of chains</u>.

Classes of groups closed under free amalgamations enjoy these properties, such as the class of all groups, of torsion free groups, etc., and classes obtained from certain e.c. groups; furthermore, the class of finite groups has the s.p.

A natural generalization of Jonsson's Theorem is obtained by considering κ-injective chains with homogeneous universal jump-types.

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An application is given to the spectrum problem. In particular, it is shown that there are $2^{\aleph_1}$ non-isomorphic universal locally finite groups of power $\aleph_1$. Of course, this result is known [A. Macintyre and S. Shelah: J. of Algebra 43 (1976), 168-175], but our method of obtaining it from incompatible order types of ω-injective chains is new and quite natural. The main result needed for this application is the <u>Maximality Theorem for $\mathfrak{M}$:</u> if $\mathcal{J}$ is a κ-injective chain for $\mathfrak{M}$ and H is a subgroup of $\bigcup\mathcal{J}$ such that $J < H$ for some $1 \neq J \in \mathcal{J}$, then $H \in \mathcal{J}$.

There is great freedom to construct κ-injective chains with special properties not explored in this thesis and which will yield further applications.

HOMOGENEOUS UNIVERSAL GROUPS

By

Kenneth Keller Hickin

A DISSERTATION

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Dedicated to my father

George Keller Hickin

and

Helen Glisson; the Dimitries - Dolly, Tom, Elizabeth, Tommy, and Vicky Dimitry Williams; Deborah Platt; my brother Gregory Dale Hickin; my aunt Teressa Kaufman; my cousins Gretchen and Roger Trevino and Stuart and Paula Raines; Margaret Neely; Dr. Romeo, Mrs. Gichaud and LaPlante, and Mr. Charles Suhor of Franklin High School in New Orleans; Diane Barnes; Cindy and Helen Lack; Chana Galapak; Mark, Margaret, and William Gamble; Bob Underhill; Therese Tomasko; Mary Sonneborn; Marge Elliott; and Kathy Trebilcott.

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PREFACE OF NOTATIONS AND CONVENTIONS

§0.1 Set Theory

Inclusion of sets is denoted by $\leq$ and proper inclusion by $<$.

<u>0.1.0</u> <u>Cardinals and Ordinals</u>. A cardinal number κ is the smallest ordinal number of cardinality κ. An ordinal equals the set of its predecessors.

κ, λ, σ always denote <u>infinite</u> <u>cardinals</u>.

α, β, γ always denote ordinals in statements such as '$\alpha < \kappa$'.

The α^{th} cardinal is ω_α. The smallest infinite cardinal will always be written ω instead of ω_0.

The cardinality (power, order) of a set S is $|S|$.

The cardinal <u>successor</u> of κ is κ^+; the ordinal successor of α is $\alpha + 1$.

The <u>Generalized</u> <u>Continuum</u> <u>Hypothesis</u> (<u>G.C.H.</u>) asserts that for all κ, $\kappa^+ = 2^\kappa$, the cardinal of the power set of κ.

κ is <u>regular</u> if κ is not the union of fewer than κ sets, each of power $< \kappa$. Thus, ω is regular, and κ^+ is regular for all κ.

An ordinal γ is <u>even</u> if $\gamma = \alpha + 2n$ where α is a limit ordinal or 0 and $0 \leq n < \omega$. All other ordinals are <u>odd</u>.

<u>0.1.1</u> <u>Functions</u>. The identity function on a set S is 1_S. The notation '$f \equiv g$ on S' means that, for all $s \in S$, $f(s) = g(s)$. We write $f \equiv 1$ on S instead of $f \equiv 1_S$ on S.

<u>0.1.2</u> <u>Chains</u>. A <u>chain</u> is a set of sets which are totally ordered by set inclusion. If the members of a chain $\mathcal{C}$ are <u>indexed by ordinals</u>, as in the statement $\mathcal{C} = \{C_\alpha | \alpha < \kappa\}$, it is always assumed that, for all $\alpha < \kappa$, $C_\alpha \leq C_{\alpha+1}$. $\mathcal{C}$ is <u>continuous</u> if, for all limit ordinals $\gamma < \kappa$, $C_\gamma = \bigcup_{\alpha<\gamma} C_\alpha$.

A chain $\mathcal{C}$ is <u>complete</u> if $\mathcal{C}$ contains the union and the intersection of each of its subsets. A <u>jump</u> of the complete chain $\mathcal{C}$ is a pair (A, B), with $A, B \in \mathcal{C}$ and $A < B$, such that for all $C \in \mathcal{C}$, if $A \leq C \leq B$, then $C = A$ or $C = B$. The <u>order type of a complete chain</u> $\mathcal{C}$, denoted $^\circ\mathcal{C}$, is the order type of the set of jumps of $\mathcal{C}$ where the jumps have the order induced by set inclusion. Thus the order type of a continuous chain $\mathcal{C} = \{C_\alpha | \alpha \leq \gamma\}$ is γ.

<u>0.1.3</u> <u>Trees</u>. A <u>tree</u> is a partially ordered set $(T, <)$ such that (<u>1</u>) Every chain in T is well ordered and (<u>2</u>) For all $a, b, c \in T$, if $a < c$ and $b < c$, then either $a \leq b$ or

$b \leq a$. Each $u \in T$ is called a vertex. The level of a vertex $u \in T$ is the ordinal type of the set $T(u) = \{a \in T \mid a < u\}$ ordered by $<$.

0.1.4 Cardinal Arithmetic. Where an appeal is made to "simple cardinal arithmetic" only the following facts are generally needed where $\varkappa$ and σ are infinite cardinals. (1) $\varkappa + \sigma = \varkappa\sigma = \max(\varkappa, \sigma)$; (2) the set of finite subsets of $\varkappa$ has power $\varkappa$; and (3) the definition of regular cardinals (see 0.1.0).

§0.2 General Algebra.

Our reference for general algebra is [1; Chapt. 4], but some of our terms are different, so we will define them here.

0.2.0 General Definitions and Conventions. An algebra M is a set together with a sequence of n-ary functions and relations, for various $n < \omega$, and constant symbols which are considered elements of M. The type τ of M is a sequence specifying the arity of each function and relation on M and naming the constants of M.

A class of algebras $\mathfrak{M}$ is a (set-theoretic) class of algebras of fixed type τ which is isomorphism-closed, that is, if $M \in \mathfrak{M}$ and $N \cong M$, then $N \in \mathfrak{M}$. Isomorphism of algebras is always denoted by $\cong$.

We make the general assumption that _all algebras have countable type_, that is, there are at most ω operations, relations, and constants.

A subset $A \leq B$ is a _subalgebra_ of B if A contains all the constants and is closed under all the operations of B. If A and B are algebras, then $A \leq B$ means (_1_) A is a subset of B, (_2_) A and B have the same type, and (_3_) the operations and relations of A are identical to those induced on A in B. More loosely, $A \leq B$ means A is a subalgebra of B.

An algebra without operations is called a _relational system_. Every subset of a relational system containing the constants is a subalgebra.

_0.2.1_ _Special Notations_. Let $\mathcal{M}$ be a class of algebras and let $A, B, C, M \in \mathcal{M}$.

$\mathrm{ISO}(A,B)$ is the set of _isomorphisms of_ A _onto_ B.

Members of $\mathcal{M}$ are called $\mathcal{M}$ _algebras_.

If S is a subset of M, then $\langle S \rangle = \mathrm{alg}_M(S)$ is the smallest subalgebra of M containing S, that is, the subalgebra of M generated by S.

$\mathcal{M}^{<\kappa}$ is the class of all $\mathcal{M}$ algebras which are _generated by a subset of power_ $< \kappa$. Thus $\mathcal{M}^{<\omega}$ is the class of _finitely generated_ (_f.g._) $\mathcal{M}$ algebras. $\mathcal{M}^{\leq\kappa}$ is defined similarly. $\mathcal{M}^{\kappa}$ is the class of all $\mathcal{M}$ algebras of power κ.

Since all our algebras have countable type, simple cardinal arithmetic shows that a subset of power $\leq \kappa$ generates a subalgebra of power $\leq \kappa$, and hence $\mathfrak{M}^{\leq\kappa}$ is the class of all $\mathfrak{M}$ algebras of power $\leq \kappa$.

A function $f: A \to B$ is an <u>embedding of</u> A <u>into</u> B if $f(A)$ is a subalgebra of B and $f \in \mathrm{ISO}(A, f(A))$. A is <u>embeddable</u> in B if such an f exists.

AutM = the group of automorphisms of M.

An $\mathfrak{M}$ <u>amalgam</u>, $\mathcal{A}$, is the union of two $\mathfrak{M}$ algebras, $\mathcal{A} = B \cup C$, where B and C intersect in a common $\mathfrak{M}$ subalgebra, $B \cap C = A \in \mathfrak{M}$, and we write

$$\mathcal{A} = \begin{array}{ccc} B & & C \\ & \diagdown\ \diagup & \\ & A & \end{array} .$$

<u>0.2.2</u> <u>Trivial Algebras</u>. Let $\mathfrak{M}$ be a class of algebras of type τ and let Q be the set of constants, possibly empty, for algebras of type τ. So, Q is a subset of every $\mathfrak{M}$ algebra. We say that $\mathfrak{M}$ <u>has a trivial algebra</u> if, for all $M, N \in \mathfrak{M}$ there exists $\varphi \in \mathrm{ISO}(\mathrm{alg}_M(Q), \mathrm{alg}_N(Q))$ and $\mathrm{alg}_M(Q) = \langle Q \rangle \in \mathfrak{M}$. $\langle Q \rangle$ is called <u>the trivial algebra</u> of $\mathfrak{M}$. If $Q = \emptyset$, then $\mathfrak{M}$ has a trivial algebra $\emptyset$. Note that every isomorphism of $\mathfrak{M}$ algebras extends 1_Q.

§0.3 Group Theory.

Let G be a group.

If $x \in G$, then the order of x is $|x|$.

$A \leq G$ will mean that A is a subgroup of G unless the context demands a different interpretation. If S is a subset of G, then $gp_G(S) = \langle S \rangle$ is the subgroup of G generated by S.

$\mathrm{Inn}G$ is the group of inner automorphisms of G.

$\oplus$ denotes the internal direct sum of groups.

1 denotes both the trivial element and the trivial group.

The notation $G = (X;R)$ means that G has the presentation with generators X and relations $R = 1$. If G has generators Y and X_α and relations W and R_α for each $\alpha \in I$, we write $G = (Y, X_\alpha, \alpha \in I; W, R_\alpha, \alpha \in I)$. If H is a group and H is naturally embedded in $G = (H, X; \text{(all) relations of } H, R)$, where X is a set of formal letters and R involves $H \cup X$, then we always assume $H \leq G$.

If $G \cap H = 1$, then $G*H = (G,H; \text{relations of } G, \text{relations of } H)$ is the free product of G and H.

A group is <u>locally finite</u> (<u>ℓ.f.</u>) if all of its finitely generated subgroups are finite.

$\mathbb{Z}$ will denote the infinite cyclic group.

$\mathfrak{G}$ denotes the class of all groups.

CHAPTER I

INTRODUCTION

§1.1 Definitions of Homogeneous and Universal Algebras and Motivation for this Group-Theoretic Study.

Homogeneous universal groups are a special case of structurally unique objects which exist in many classes of algebras and relational systems, the most familiar being the dense linear ordering of the rational numbers.

<u>1.1.0</u> <u>Definitions of Homogeneous and Universal Algebras</u>. Let $\mathfrak{M}$ be a class of algebras or relational systems and let $\kappa \geq \omega$ be a cardinal. Let $M \in \mathfrak{M}$.

<u>1.1.1</u> M is κ-<u>universal for</u> $\mathfrak{M}$ if every $\mathfrak{M}^{<\kappa}$ algebra is embeddable in M.

<u>1.1.2</u> M is κ-<u>homogeneous for</u> $\mathfrak{M}$ if, for all $\mathfrak{M}^{<\kappa}$ algebras $A, B \leq M$ and for all $f \in ISO(A,B)$, f extends to an automorphism of M.

<u>1.1.3</u> $HU_{\kappa}(\mathfrak{M})$ is the class, possibly empty, of all $\mathfrak{M}$ algebras which are both κ-universal and κ-homogeneous for $\mathfrak{M}$.

<u>1.1.4</u> If $\lambda \geq \kappa$ is a cardinal, then $HU_{\kappa}^{\lambda}(\mathfrak{M}) = HU_{\kappa}(\mathfrak{M}) \cap \mathfrak{M}^{\lambda}$.

We now formalize our bias toward groups.

1.1.5 If $\mathcal{G}$ is the class of all groups, we denote $HU_{\kappa}(\mathcal{G})$ by $\mathcal{U}_{\kappa}$ and $HU_{\kappa}^{\lambda}(\mathcal{G})$ by $\mathcal{U}_{\kappa}^{\lambda}$.

If $\mathcal{O}$ is the class of all linear orderings and η is the ordering of the rationals, then $\eta \in HU_{\omega}^{\omega}(\mathcal{O})$. η is ω-universal for $\mathcal{O}$ because every finite ordered set is embedded in η, and η is ω-homogeneous for $\mathcal{O}$ because given two sequences $a_1 < \cdots < a_n$ and $b_1 < \cdots < b_n$ of rationals there is an order-automorphism Φ of the rationals such that $\Phi(a_i) = b_i$ for $1 \leq i \leq n$. Note that, since $\mathcal{O}$ has a single binary relation and no operations, every subset of the rationals is an $\mathcal{O}$ subalgebra (see 0.2.0).

Several isolated results concerning the existence of homogeneous universal systems were known before B. Jonsson in 1960 proved a very general existence and uniqueness theorem for them, [8] and [9], following a suggestion of Reinhold Baer. Concurrently, Morley and Vaught [13] gave an even more general construction in the setting of model theory. For a more complete bibliographical survey, one can refer to [1; Chapt. 10].

A full account of Jonsson's construction is given in §1.3. We will now state his result for the classes of all groups and of locally finite (ℓ.f.) groups.

1.1.6 Jonsson's Theorem for the Classes of all Groups and Locally Finite Groups.

(i) (G.C.H.) If $\kappa > \omega$ is regular, then $\mathcal{K}^{\kappa}_{\kappa}$ has exactly one member, H_{κ}, up to $\cong$. Every group of power κ is embeddable in H_{κ}.

(ii) The class $HU^{\omega}_{\omega}(\ell.f.$ groups) has exactly one member, H_{ω}, up to $\cong$. Every countable $\ell.f.$ group is embeddable in H_{ω}.

The group H_{ω} was studied by Philip Hall [2] a little before Jonsson's second paper. A group in the class $HU_{\omega}(\ell.f.$ groups) is usually called a <u>universal locally finite group</u>, and we write ULF for this class. Hall gave a concrete construction of H_{ω}. He obtained it as the union of a chain of finite symmetric groups $\ldots R_n < R_{n+1} \ldots$ where R_{n+1} is the symmetric group on R_n, and R_n is contained in R_{n+1} as its Cayley (right regular) representation.

There is no countable group which is ω-universal for the class of all groups because every countable group has at most ω finitely generated subgroups, but there are 2^{ω} non-isomorphic 2-generated groups [14].

The members of $HU^{\kappa}_{\kappa}(\mathcal{M})$ can be defined by a single property which we will use in §1.3 to simplify the account of Jonsson's theorem, and which we will generalize in §2.1.

1.1.7 Definition of a κ-Injective Algebra. An $\mathcal{M}$ algebra M is κ-<u>injective for</u> $\mathcal{M}$ if, for all $A, B \in \mathcal{M}^{<\kappa}$ with $A < B$, every embedding $f: A \to M$ extends to an embedding of B into

M. The class of κ-injective algebras for $\mathfrak{M}$ is denoted $\mathrm{INJ}_\kappa(\mathfrak{M})$ and $\mathrm{INJ}_\kappa^\lambda(\mathfrak{M}) = \mathrm{INJ}_\kappa(\mathfrak{M}) \cap \mathfrak{M}^\lambda$ for $\lambda \geq \kappa$. We define $\mathcal{J}_\kappa = \mathrm{INJ}_\kappa(\mathcal{G})$ where $\mathcal{G}$ is the class of all groups.

<u>1.1.8</u> <u>Proposition</u>. For any class $\mathfrak{M}$, $\mathrm{HU}_\kappa(\mathfrak{M}) \leq \mathrm{INJ}_\kappa(\mathfrak{M})$.

<u>Proof</u>. Suppose $M \in \mathrm{HU}_\kappa(\mathfrak{M})$, $A < B$ are $\mathfrak{M}^{<\kappa}$ algebras, and $f: A \to M$ is an embedding. Since M is κ-universal for $\mathfrak{M}$ there is an embedding $g: B \to M$. Since M is κ-homogeneous for $\mathfrak{M}$, the isomorphism $fg^{-1}: g(A) \to f(A)$ extends to some $\varphi \in \mathrm{Aut}\, M$. Now, $\varphi g: B \to M$ extends f. Hence $M \in \mathrm{INJ}_\kappa(\mathfrak{M})$.

With some weak assumptions on $\mathfrak{M}$ to be discussed in §1.3, we have $\mathrm{INJ}_\kappa^\kappa(\mathfrak{M}) = \mathrm{HU}_\kappa^\kappa(\mathfrak{M})$, but generally one cannot prove $\mathrm{INJ}_\kappa(\mathfrak{M}) \leq \mathrm{HU}_\kappa(\mathfrak{M})$ because the required automorphisms of algebras of power $> \kappa$ cannot be constructed.

However, for groups we have

<u>1.1.9</u> <u>Proposition</u>. (<u>i</u>) For all $\kappa > \omega$, $\mathcal{J}_\kappa = \mathcal{K}_\kappa$, and (<u>ii</u>) $\mathrm{INJ}_\omega(\ell.\mathrm{f.\ groups}) = \mathrm{ULF}$.

<u>Proof</u>. Let $G \in \mathcal{J}_\kappa$ (resp., $\mathrm{INJ}_\omega(\ell.\mathrm{f.})$). To show that $G \in \mathcal{K}_\kappa$ (resp., ULF), suppose A and B are subgroups of G of power $< \kappa$ $(\kappa \geq \omega)$, and $\varphi \in \mathrm{ISO}(A,B)$. We must extend φ to an automorphism of G. Let $J = \mathrm{gp}_G(A,B)$ and note that $|J| < \kappa$. J is contained in a group $H = \langle J, t\rangle$ such that for all $a \in A$, $t^{-1}at = \varphi(a)$ and such that $|H| < \kappa$. To prove this, we can use Philip Hall's construction where t is a certain element of the symmetric group on J [14; p. 537], or, for $\kappa > \omega$, we can use the HNN extension to be discussed

in §1.2. Since G is $\varkappa$-injective, there is an embedding $f\colon H \to G$ such that $f \equiv 1$ on J. Thus, for all $a \in A$, $f(t)^{-1}af(t) = \varphi(a)$, that is, φ extends to an <u>inner</u> automorphism of G.

A group $G \in \mathscr{V}_{\varkappa}$ can be thought of as an "algebraic universe for group theory for groups of power $< \varkappa$". Jonsson's theorem says that there is exactly one such universe up to $\cong$ of power $\varkappa$ (assuming the G.C.H and that $\varkappa$ is regular). Such groups have intrinsic interest because their structure reflects, in many senses, the structure of all group theory. Any general construction possible in group theory can be applied within the groups $H_{\varkappa}$ and correlated with algebraic properties of these specific groups. The simplest example of this phenomenon is the previous proof where the existence of a particular group theoretical construction implies that all the automorphisms involved in the homogeneity condition for any $G \in \mathscr{V}_{\varkappa}$ can be chosen inner. Because group theory is rich in general constructions, the homogeneous universal groups are an archetypal case of this phenomenon. We will attempt to give further evidence of this in §1.2 by proving two non-obvious structure theorems for the groups $H_{\varkappa}$, $\varkappa > \omega$.

The purpose of this study is the presentation of a general algebraic construction which gives considerable information about the structure of certain homogeneous universal algebras.

This construction involves generalizing the concept of a κ-injective algebra, which we show in §1.3 to be a natural concept in Jonsson's original proof, to the concept of a κ-<u>injective</u> <u>chain</u> <u>of</u> <u>algebras</u> (for the definition see 2.1.9). In Chapters 2 and 3 existence and uniqueness theorems are proved for κ-injective chains in a general algebraic setting divorced from group theory, and some special properties of κ-injective chains of groups are established. These special properties are developed to give an application of κ-injective chains to a question of much interest in model theory - the so-called "spectrum problem" of constructing many non-isomorphic models of theories.

We will use ω-injective chains to construct 2^{ω_1} non-isomorphic groups belonging to the class $HU_{\omega}^{\omega_1}(\mathfrak{M})$ where $\mathfrak{M}$ can be various classes of groups including the class of locally finite groups, classes obtained from algebraically closed groups (see 1.4.0 and 2.2.0), and classes admitting free amalgamations which we call "free ω-classes" (see 1.4.20). This is not intended to be a definitive application of κ-injective chains, but an illustration of their potential usefulness. These results are not new; it is the method of obtaining them which is. The existence of 2^{κ} non-isomorphic ULF groups of power κ for all $\kappa > \omega$ was obtained by Macintyre and Shelah [12] using some deep techniques of model theory, and other ways to construct ULF groups are known [3] and [22]. Our application is not a complete redundancy since these groups have a special lattice property due to their construction as transfinite ω-injective chains (2.2.1).

Chapter 2 contains our results and many of the proofs. The remaining proofs are given in Chapter 3. In §1.4 we give some concrete examples of groups to which our application applies. We feel that the concept of κ-injective chains is a direct extension of Jonsson's original idea, and that κ-injective chains may be of general use in model theory.

It should be noted that there is a well known generalization of homogeneous universal systems. This is the concept of saturated models and chains of models due to Morley, Vaught, and Keisler (see [1; Chapt. 11]). The concept of a κ-injective chain is, of course, distinct from this and offers new possibilities.[1]

---

[1] I have asked S. Shelah if he ever saw this construction. He said that the only thing vaguely resembling it is a construction he used in an unpublished proof to construct recursive automorphisms of Boolean algebras, and he expressed hope that κ-injective chains would prove helpful in certain constructions.

§1.2 Two Structure Theorems for $H_\varkappa$, $\varkappa > \omega$, Obtained From the Free Amalgamation and HNN Constructions.

We will first state the essential properties of these constructions.

1.2.0 Definition. Let $\mathcal{A} = \begin{smallmatrix} H & & K \\ & \diagdown\!\diagup & \\ & F & \end{smallmatrix}$ be a group amalgam. The free amalgamated product of $\mathcal{A}$, denoted $\mathrm{gp}_*(\mathcal{A})$, is the group with presentation ($\mathcal{A}$; all relations of H and all relations of K). A sequence $c_1,\ldots,c_n$ in $\mathcal{A}$ is called reduced if, for all i, $c_i \notin F$ and successive c_i, c_{i+1} come from distinct factors of $\mathcal{A}$. If $\mathcal{A}$ is contained in a group G, and if $1_{\mathcal{A}}$ induces an isomorphism from $\mathrm{gp}_*(\mathcal{A})$ to $\mathrm{gp}_G(\mathcal{A})$, then we will say that $\mathcal{A}$ generates $\mathrm{gp}_*(\mathcal{A})$ in G.

1.2.1 Definition. Let A and B be isomorphic subgroups of a group G and suppose $\varphi \in \mathrm{ISO}(A,B)$. The HNN extension $G_\varphi = \langle G,t\rangle$ of G relative to φ is the group with presentation (G,t; all relations of G and $t^{-1}at = \varphi(a)$ for all $a \in A$). t is called the stable letter. A sequence g_0, $t^{\epsilon_1},g_1,\ldots,t^{\epsilon_n},g_n$ (where $\epsilon_i = \pm 1$, $g_i \in G$ and $n \geq 1$) is called reduced if there is no consecutive subsequence of the form t^{-1},g_i,t with $g_i \in A$ or t,g_i,t^{-1} with $g_i \in B$. If G and t are contained in a group H, and if $1_{G\cup\{t\}}$ induces an isomorphism between G_φ and $\mathrm{gp}_H(G,t)$, then we will say that G and t generate the HNN extension G_φ in H.

<u>1.2.2</u> <u>Normal Form Theorem for Free Amalgamated Products</u> [19: Th. III]. Suppose $\mathcal{A}$ is a group amalgam, A is a group, and $\mathcal{A} \leq A$. Then $\mathcal{A}$ generates $gp_*(\mathcal{A})$ in A iff the product of every reduced sequence in $\mathcal{A}$ is non-trivial.

<u>1.2.3</u> <u>Britton's Lemma</u> (<u>Normal Form Theorem for</u> HNN <u>Extensions</u>). [19: p. 614]. Let $G_\varphi = \langle G, t \rangle$ be the HNN extension of G relative to φ. If $g_0, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n$ is a reduced sequence and $n \geq 1$, then $g_0 t^{\epsilon_1} \cdots t^{\epsilon_n} g_n \neq 1$ in $gp(G,t)$.

We will need the following two lemmas which are easy consequences of the normal form theorems.

<u>1.2.4</u> If $\mathcal{A} = \begin{array}{ccc} H & & K \\ & \diagdown\diagup & \\ & F & \end{array}$ is a group amalgam, $G = gp_*(\mathcal{A})$, and U and V are subgroups of K such that $V \cap gp_K(F,U) = 1$, then $V \cap gp_G(H,U) = 1$.

<u>1.2.5</u> If $G_\varphi = \langle G, t \rangle$ is the HNN extension of G relative to $\varphi \in ISO(A,B)$, and if U is a subgroup of G such that $U \cap A = U \cap B = 1$, then U and $\langle t \rangle$ generate their free product in $gp(G,t)$ and $G \cap \langle U, t \rangle = U$.

Our first structure theorem is

<u>1.2.6</u> <u>Normal Basis Theorem for</u> $H_\varkappa$. (G.C.H.). Suppose $\varkappa > \omega$ is regular. Let F be the free group on $\{x,y\}$. For each ordinal $\gamma < \varkappa$, let $w_\gamma(x,y) \in F$ be a reduced word of length at least 2 such that $\langle w_\gamma \rangle$ is a maximal cyclic subgroup of F.

$H_\varkappa$ has a generating set $\{a_\gamma, b_\gamma | \gamma < \varkappa\}$ such that for all $\gamma < \varkappa$, (i) $F_\gamma = \langle a_\gamma, b_\gamma \rangle$ is free on $\{a_\gamma, b_\gamma\}$ and, putting $H_\gamma = \langle a_\beta, b_\beta | \beta < \gamma \rangle$, (ii) if π is a product in which non-trivial elements of H_γ alternate with non-trivial elements of F_γ, then $\pi = 1$ in $H_\varkappa$ only if π contains a consecutive subword of the form $w_\gamma(a_\gamma, b_\gamma)^j$, $j \neq 0$.

1.2.7 Corollary. (a) If σ is a non-trivial reduced word in the free group with free basis $\{a_\gamma, b_\gamma | \gamma < \varkappa\}$, then $\sigma = 1$ in $H_\varkappa$ only if σ has a consecutive subword of the form $w_\gamma(a_\gamma, b_\gamma)^j$, $j \neq 0$, and for some $\beta < \gamma$ either a_β or b_β occurs in σ. (b) Let $A = \langle a_\gamma | \gamma < \varkappa \rangle$ and $B = \langle b_\gamma | \gamma < \varkappa \rangle$. Then A and B are free subgroups of $H_\varkappa = \langle A, B \rangle$, and if no word w_γ has the form $(x^i y^j)^{\pm 1}$, then for all $1 \neq u \in A$ and $1 \neq v \in B$, $\langle u \rangle$ and $\langle v \rangle$ generate their free product in $H_\varkappa$.

Discussion. The theorem has been stated in the manner most convenient for proof. Part (a) of the Corollary is actually a sharper formulation of the theorem describing the "freeness" with which the generators $\{a_\gamma, b_\gamma | \gamma < \varkappa\}$ generate $H_\varkappa$. Part (b) is the most quotable result, saying "$H_\varkappa$ is generated by two disjoint free subgroups which interact pairwise freely".

Proof of (a). Suppose $\sigma = 1$ in $H_\varkappa$. Let γ be the largest ordinal for which a_γ or b_γ occurs in σ. Then $\sigma = h_0 f_1 \cdots f_n h_n$, $n \geq 1$, where $1 \neq f_i \in \langle a_\gamma, b_\gamma \rangle = F_\gamma$, h_i is a reduced word in the free group G on $\{a_\beta, b_\beta | \beta < \gamma\}$ and

$h_1,\ldots,h_{n-1}$ are non-trivial in G. Since $\sigma = 1$ in $H_\varkappa$, part (ii) of the Theorem (with the above decomposition of σ used as π) implies that either some h_i which is non-trivial in G is trivial in $H_\varkappa$ or some $f_i = w_\gamma^j$. We can rule out the former case by induction, and assume $f_i = w_\gamma^j$. Since $f_i \neq 1$ in $H_\varkappa$ by part (i), some $h_j \neq 1$ in G and hence some a_β or b_β, $\beta < \gamma$, occurs in h_j and hence in σ.

Proof of (b). That A and B are free groups follows from part (a) and the assumption that each word w_γ has length at least 2. If $1 \neq u \in A$ and $1 \neq v \in B$ and π is a product where powers of u alternate with powers of v, then the only consecutive subwords of π belonging to $\langle a_\gamma, b_\gamma \rangle$ for any γ can be of the form $(a_\gamma^i b_\gamma^j)^{\pm 1}$ unless $u \in \langle a_\gamma \rangle$ and $v \in \langle b_\gamma \rangle$. So $\pi \neq 1$ in $H_\varkappa$ by part (a).

Proof of the Normal Basis Theorem for $H_\varkappa$. Let $\{z_\gamma \mid 0 < \gamma < \varkappa,$ γ even$\}$ be a list of all the elements of $H_\varkappa$. We will construct the sets $\{a_\gamma, b_\gamma\}$ inductively in such a way that (i) and (ii) of the Theorem hold and so that, if γ is even, then $z_\mu \in H_\gamma = \langle a_\beta, b_\beta \mid \beta < \gamma \rangle$ for all even $\mu < \gamma$. This guarantees that $\{a_\gamma, b_\gamma \mid \gamma < \varkappa\}$ does generate $H_\varkappa$.

Assume $\{a_\beta, b_\beta \mid \beta < \gamma\}$ have been chosen and γ is even. Let μ be minimal such that $z_\mu \notin H_\gamma$. Note that $|H_\gamma| < \varkappa$ and that $\mu \geq \gamma$ by the inductive assumption. Let $J = \langle u, v \rangle$ be a group such that $z_\mu \in J$, u and v have infinite order, $\mathcal{A} = \left(\begin{array}{ccc} \langle H_\gamma, z_\mu \rangle & & J \\ & \langle z_\mu \rangle & \end{array} \right)$ is an amalgam, and

<u>1.2.8</u> $\langle v\rangle \cap \langle u, z_\mu\rangle = 1 = \langle u\rangle \cap \langle z_\mu\rangle$.

(We can define $J = \langle v\rangle * \langle z_\mu\rangle$ where v is a formal letter, and let $u = v^{-1}z_\mu v$). Using κ-injectivity of H_κ, let f be an embedding of $gp_*(\mathcal{A})$ into H_κ such that $f \equiv 1$ on $\langle H_\gamma, z_\mu\rangle$. Intuitively, f reproduces the amalgam $\mathcal{A}$ in H_κ, with $\langle H_\gamma, z_\mu\rangle$ fixed, so that $f(\mathcal{A})$ generates $gp_*(f(\mathcal{A}))$ in H_κ. To simplify notation we will assume now that $\mathcal{A} < H_\kappa$, and so $\mathcal{A}$ generates $gp_*(\mathcal{A})$ in H_κ. Note that

<u>1.2.9</u> $\langle v\rangle \cap \langle u, H_\gamma\rangle = 1$ by 1.2.4 and 1.2.8.

Now choose elements $a_\gamma, b_\gamma \in H_\kappa$ so that $F_\gamma = \langle a_\gamma, b_\gamma\rangle$ is free on $\{a_\gamma, b_\gamma\}$, $u = w_\gamma(a_\gamma, b_\gamma)$, and $\mathcal{A}_1 = \begin{array}{ccc} F_\gamma & & \langle \mathcal{A}\rangle \\ & \diagdown\ \diagup & \\ & \langle u\rangle & \end{array}$ is an amalgam which generates $gp_*(\mathcal{A}_1)$ in H_κ. This can be done using κ-injectivity as before. Since $\langle u\rangle \cap \langle H_\gamma\rangle = 1$, the Normal Form Theorem 1.2.2 implies that F_γ has the property (ii) of our Normal Basis Theorem since $u = w_\gamma(a_\gamma, b_\gamma)$. Note that 1.2.4 and 1.2.9 imply

<u>1.2.10</u> $\langle v\rangle \cap \langle H_\gamma, F_\gamma\rangle = 1$.

Again using κ-injectivity, choose $a_{\gamma+1}, b_{\gamma+1} \in H_\kappa$ such that $F_{\gamma+1} = \langle a_{\gamma+1}, b_{\gamma+1}\rangle$ is free on $\{a_{\gamma+1}, b_{\gamma+1}\}$, $v = w_{\gamma+1}(a_{\gamma+1}, b_{\gamma+1})$, and $\mathcal{A}_2 = \begin{array}{ccc} F_{\gamma+1} & & \langle \mathcal{A}_1\rangle \\ & \diagdown\ \diagup & \\ & \langle v\rangle & \end{array}$ is an amalgam which generates $gp_*(\mathcal{A}_2)$ in H_κ. As above, 1.2.2 and 1.2.10 imply that (ii) holds for $H_{\gamma+1} = \langle H_\gamma, F_\gamma\rangle$ and $F_{\gamma+1}$. We also have $z_\gamma \in \langle u, v\rangle < \langle \mathcal{A}_2\rangle = \langle H_{\gamma+1}, F_{\gamma+1}\rangle = H_{\gamma+2}$, and this completes the inductive step.

Before stating our second theorem we need a definition.

<u>1.2.11</u> <u>Definition</u>. Let $(T,<)$ be a tree and for each ordinal α, let T_α be the set of vertices of T at level α. If κ is an infinite cardinal, we will call T a <u>κ-tree</u> provided

(<u>i</u>) For all $\alpha < \kappa$, $|T_\alpha| < \kappa$,

(<u>ii</u>) For all $\alpha < \kappa$, $x \in T_\alpha$, and β, $\alpha < \beta \leq \kappa$, there exists $y \in T_\beta$ such that $x < y$, and

(<u>iii</u>) For all $u \neq v \in T_\kappa$, there exists $\alpha < \kappa$ and $x \in T_\alpha$ such that $x < u$ and $x \nless v$.

Thus the members of T_κ are in correspondence maximal chains in $T_{<\kappa} = \bigcup\{T_\alpha | \alpha < \kappa\}$, and each maximal chain in $T_{<\kappa}$ has power κ. We have not specified the size of T_κ, but if branching occurs at enough vertices of $T_{<\kappa}$ and the G.C.H. holds, then $|T_\kappa| = \kappa^+$.

Our second structure theorem is

<u>1.2.12</u> <u>Maximal Subgroup-Tree Theorem for</u> H_κ. (G.C.H.) Suppose $\kappa > \omega$ is regular and $(T,<)$ is a κ-tree. For every $u \in T_\kappa$, let $T(u) = \{x \in T_{<\kappa} | x < u\}$. There is a one-to-one function $f: T_{<\kappa} \to H_\kappa$ such that

(<u>i</u>) For all $u \in T_\kappa$, $f(T(u))$ freely generates a maximal subgroup $M(u)$ of H_κ,

(<u>ii</u>) For all $u \neq v \in T_\kappa$, $M(u) \cap M(v) = \langle f(T(u) \cap T(v)) \rangle$ (in particular $|M(u) \cap M(v)| < \kappa$), and

(iii) Let $u \in T_\varkappa$ and $a,b \in H_\varkappa$ with $|a| = |b| = \omega$, and $\langle a \rangle \cap M(u) = \langle b \rangle \cap M(u) = 1$. Then there exists $x \in M(u)$ such that $x^{-1}ax = b$.

The third property gives the key to the construction which uses HNN extensions.

Proof. Let $\{z_\mu | 0 \leq \mu < \varkappa,\ \mu$ an even ordinal$\}$ be a list of $H_\varkappa$. Let $q \in H_\varkappa$ be a fixed element of infinite order.

We will construct f inductively on each level T_γ of T, $\gamma < \varkappa$. The construction will have the following properties. If $v \in T_\gamma$, $\gamma \leq \varkappa$, let $T(v) = \{u \in T | u < v\}$. Assume f is defined on $T_{<\gamma}$.

(a) For all $u \in T_\gamma$, $f(T(u))$ freely generates a subgroup $M(u)$ of $H_\varkappa$,

(b) For all $u \neq v \in T_{\leq\gamma}$, $M(u) \cap M(v) = \langle f(T(u) \cap T(v)) \rangle$,

(c) For all $u \in T_\gamma$, $\langle q \rangle \cap M(u) = 1$,

(d) Suppose γ is even, $\mu < \gamma$ is even, $v \in T_\gamma$, $v > u \in T_\mu$, and $z_\mu \notin M(u)$. Then $q \in \langle z_\mu, M(v) \rangle$.

(e) Suppose γ is even, $\mu < \gamma$ is even, $u \in T_\mu$, $u < v \in T_{\mu+1}$, $|z_\mu| = \omega$, and $\langle z_\mu \rangle \cap M(u) = 1$. Then $q = f(v)^{-1} z_\mu f(v)$.

Suppose these properties hold for all $\gamma < \varkappa$. We will check that (i) - (iii) of the Theorem hold. Condition (iii) is an easy consequence of (e) since (e) guarantees that a and b are both conjugated to q by members of $f(T(u))$. Condition (ii) is immediate from (b). Since the freeness of each $M(u)$ is asserted in (a), we need only prove that each $M(u)$ is maximal in $H_\varkappa$. For this we need a small lemma.

<u>1.2.13</u> For all $u \in T_\varkappa$ and $y \in H_\varkappa$, there exist $s,t \in H_\varkappa$ such that $|s| = |t| = \omega$, $\langle s \rangle \cap M(u) = \langle t \rangle \cap M(u) = 1$, and $y \in \langle s,t \rangle$.

<u>Proof</u>. By $\varkappa$-injectivity, let $a,b \in H_\varkappa$ such that $|a| = |b| = \omega$ and $\langle a \rangle \oplus \langle b \rangle \oplus \langle y \rangle$ exists in $H_\varkappa$. We claim that either the pair of subgroups $\{\langle a \rangle, \langle a+y \rangle\}$ or the pair $\{\langle b \rangle, \langle b+y \rangle\}$ has the property that both of its members intersect $M(u)$ trivially. For otherwise $M(u)$ would contain a free abelian subgroup of rank 2 contrary to the freeness of $M(u)$. Now take s and t to be the generators of this good pair above.

Now to show $M(u)$ is maximal in $H_\varkappa$, suppose $x \in H_\varkappa - M(u)$. We will show $H_\varkappa = \langle x, M(u) \rangle$. Since $x = z_\mu$ for some μ, applying (d) for $\gamma > \mu$, we have $q \in \langle x, M(u) \rangle$. Let $y \in H_\varkappa$ and let $s = z_\alpha$ and $t = z_\beta$ be as in 1.2.13. Applying (e) for some γ larger than α and β, there exist $a,b \in T(u)$ such that $q = f(a)^{-1}sf(a) = f(b)^{-1}tf(b)$. Hence $y \in \langle s,t \rangle < \langle q, f(a), f(b) \rangle < \langle x, M(u) \rangle$. Hence $M(u)$ is maximal, and (i) - (iii) follow from (a) - (e).

<u>The Construction of</u> f <u>on</u> T_ρ <u>and</u> $T_{\rho+1}$, ρ <u>even</u>.

Assume f is defined on $T_{<\rho}$ for some even $\rho \geq 0$, and that (a) - (e) are satisfied for $\gamma \leq \rho$. We will construct f inductively first on T_ρ, and then on $T_{\rho+1}$.

Suppose f has been defined on a subset U of T_ρ and that (a) - (c) hold where $\gamma = \rho + 1$ and $u,v \in T_{\leq\rho}$ or $u,v \in T_{\rho+1}$ are successors of elements in U. ((d) and (e) are vacuous because $\gamma = \rho + 1$ is odd). We must define f on some $s \in T_\rho - U$. Let $J = \langle f(T_{<\rho} \cup U), z_\rho, q \rangle$ and note that $|J| < \kappa$. By κ-injectivity, choose $f(s) \in H_\kappa$ so that $|f(s)| = \omega$ and J and $\langle f(s) \rangle$ generate their free product in H_κ. Properties (a) - (c) are easily checked in the new cases where u or v is a successor of t in $T_{\rho+1}$ by appeal to normal form in $J * \langle f(s) \rangle$ and the inductive assumptions. Thus f can be defined on T_ρ.

We now assume f is defined on $T_{\leq\rho}$ and on a subset U of $T_{\rho+1}$ and that (a) - (e) hold for $\gamma = \rho + 2$ and for relevant $u, v \in T_{\leq\gamma}$. We must define f on some $t \in T_{\rho+1} - U$. Let $G = \langle f(T_{\leq\rho} \cup U), z_\rho, q \rangle$.

<u>Case 1</u>. $|z_\rho| = \omega$ <u>and</u> $\langle z_\rho \rangle \cap M(s) = 1$ <u>where</u> $t > s \in T_\rho$. In this case, we must take care to satisfy (e) for $\gamma = \rho + 2$, $\mu = \rho$, $u = s$, and $v = t$. Note that (d) follows easily from this where $\gamma = \rho + 2$, $\mu = \rho$, and v is a successor of t in T_γ. Now $\langle z_\rho \rangle \cap \langle M(s), f(s) \rangle = \langle z_\rho \rangle \cap M(t) = 1$ by the previous construction, using normal form in $J * \langle f(s) \rangle$. Let $\varphi \in \mathrm{ISO}(\langle z_\rho \rangle, \langle q \rangle)$ be such that $\varphi(z_\rho) = q$. By κ-injectivity, choose $f(t) \in H_\kappa$ so that G and $f(t)$ generate the HNN extension G_φ with stable letter $f(t)$ in H_κ. Thus (e) is satisfied, and (a) - (c) in the new cases where u or v is a successor of t in T_γ are all easy consequences of the corollary 1.2.5 to Britton's Lemma and our inductive assumptions.

<u>Case 2</u>. <u>For</u> <u>some</u> $i > 1$, $z_\rho^i \in M(s)$, <u>but</u> $z_\rho \notin M(s)$. In this case (e) is vacuous, but we must take care that (d)

holds where $\mu = \rho$, $\gamma = \rho + 2$, and v is a successor of t in T_γ. Let $y = z_\rho f(s)$. By the previous construction of $f(s)$, we have $|y| = \omega$ and $\langle y \rangle \cap \langle M(s), f(s) \rangle = \langle y \rangle \cap M(t) = 1$ by normal form in $J * \langle f(s) \rangle$. Let $\varphi \in ISO(\langle y \rangle, \langle q \rangle)$ be such that $\varphi(y) = q$. By κ-injectivity choose $f(t) \in H_\kappa$ so that G and $f(t)$ generate the HNN extension G_φ with stable letter $f(t)$ in H_κ. Now (a) - (c) hold as in the previous case, and (d) holds since $q \in \langle y, f(t) \rangle < \langle z_\rho, f(s), f(t) \rangle < \langle z_\rho, M(v) \rangle$ if $s < v \in T_{\rho+2}$.

<u>Case III</u>. $z_\rho \in M(s)$. This is the trivial case. We need only satisfy (a) - (c), and this can be done by choosing $f(t)$ so that G and $\langle f(t) \rangle$ generate their free product in H_κ.

This completes the proof of the Maximal Subgroup-Tree Theorem.

§1.3 The Construction of Homogeneous Universal Algebras.

Our sketch of Jonsson's construction will be similar to [1: Chapt. 10] but we will use the idea of κ-injectivity to unify the presentation.

The construction can be motivated by asking several questions about any class of algebras $\mathfrak{M}$.

(<u>1</u>) What minimal property must $\mathfrak{M}$ have if there exists a κ-injective algebra for $\mathfrak{M}$?

(<u>2</u>) What properties must $\mathfrak{M}$ have in order that any two members of $INJ_\kappa^\kappa(\mathfrak{M})$ be isomorphic?

(3) What properties will guarantee that $INJ_{\kappa}^{\kappa}(\mathfrak{M}) = HU_{\kappa}^{\kappa}(\mathfrak{M})$?

(4) Are any further properties needed to actually construct a κ-injective algebra for $\mathfrak{M}$?

In answer to question (1), we want a relative version of κ-injectivity which does not make reference to any particular algebra. This is defined as follows.

1.3.0 The κ-injective property for $\mathfrak{M}$. $\mathfrak{M}$ is κ-injective provided given any three $\mathfrak{M}^{<\kappa}$ algebras A, $\bar{A}$, and B with $A < \bar{A}$ and given an embedding $f: A \to B$, there exists $\bar{B} \in \mathfrak{M}$ with $B \leq \bar{B}$ and an embedding $\bar{f}: \bar{A} \to \bar{B}$ such that $\bar{f}$ extends f. (Note that any $M \in INJ_{\kappa}(\mathfrak{M})$ will serve as $\bar{B}$ provided $B \leq M$). $\mathfrak{M}$ is injective if $\mathfrak{M}$ is κ-injective for all κ.

The injective property is a concrete version of what is usually called the "amalgamation property" and defined as follows.

1.3.1 The amalgamation property for $\mathfrak{M}$ [1: p. 203]. If $A, B_0, B_1 \in \mathfrak{M}$, f_0 is an embedding of A into B_0, and f_1 is an embedding of A into B_1, then there exists $C \in \mathfrak{M}$ and embeddings g_0 of B_0 into C and g_1 of B_1 into C such that the following diagram commutes:

$$\begin{tikzcd} & B_0 \arrow[dr, "g_0"] & \\ A \arrow[ur, "f_0"] \arrow[dr, "f_1"'] & & C \\ & B_1 \arrow[ur, "g_1"'] & \end{tikzcd} .$$

This is equivalent to the injective property by identifying A with a subalgebra of $B_0 = \bar{A}$ and B_1 with a subalgebra of $C = \bar{B}$.

To answer question (2), the idea of constructing an isomorphism between κ-injective algebras for $\mathfrak{M}$ is quite simple because κ-injectivity can be used to enlarge any partial isomorphism between $\mathfrak{M}^{<\kappa}$ subalgebras. Specifically, suppose $M,N \in \mathrm{INJ}_\kappa(\mathfrak{M})$, $A,B \in \mathfrak{M}^{<\kappa}$ are subalgebras of M,N respectively, $f \in \mathrm{ISO}(A,B)$, and $x \in M - A$. We can use κ-injectivity of N to extend f to an embedding defined at x provided there exists an $\mathfrak{M}^{<\kappa}$ algebra C such that $\langle A,x\rangle \leq C \leq M$. Alternatively, if $y \in N - B$ and there is an $\mathfrak{M}^{<\kappa}$ algebra D such that $\langle B,y\rangle \leq D \leq N$, we can use κ-injectivity of M to define an embedding g of D into M such that $g \equiv f^{-1}$ on $f(A)$, and then define $f_1 = g^{-1}$. Then f_1 is an extension of f to $C = g(D)$ and we have $y \in f_1(C)$. In steps like these, both the domain and the range of the partial isomorphisms can be enlarged to include arbitrary elements. But to guarantee the existence of C and D we need the "κ-local property".

<u>1.3.2</u> <u>The</u> κ-<u>local property for</u> $\mathfrak{M}$. Suppose $M \in \mathfrak{M}$ and X is a subset of M with $|X| < \kappa$. Then X is contained in some $\mathfrak{M}^{<\kappa}$ subalgebra of M.

Thus, the κ-injective and κ-local properties seem to allow us to build an isomorphism φ of M onto N, provided $|M| = |N| = \kappa$, by defining φ piecemeal in a back-and-forth

manner on $\mathcal{M}^{<\kappa}$ subalgebras of M. In fact, two more properties are needed to accomplish this. We must guarantee that, during the construction of φ, the domain of each partial φ is in $\mathcal{M}$. For this we need

<u>1.3.3</u> <u>The</u> κ-<u>inductive property</u>. $\mathcal{M}$ is κ-<u>inductive</u> if, for all chains $\mathcal{C}$ of $\mathcal{M}$ algebras with $|\mathcal{C}| \leq \kappa$, we have $\bigcup\mathcal{C} \in \mathcal{M}$. $\mathcal{M}$ is <u>inductive</u> if $\mathcal{M}$ is κ-inductive for all κ.

Finally, we need a property which guarantees that the isomorphism φ can be started, that is, that M and N have isomorphic $\mathcal{M}$ subalgebras.

<u>1.3.4</u> <u>The comparison property for</u> $\mathcal{M}$. If $A, B \in \mathcal{M}$, then there exist $\bar{A}, \bar{B} \in \mathcal{M}$ with $A \leq \bar{A}$ and $B \leq \bar{B}$ such that there exists $C \in \mathcal{M}$ which is embeddable in both $\bar{A}$ and $\bar{B}$.

The role of this property appears in the following proof.

<u>1.3.5</u> <u>Lemma</u>. If $\mathcal{M}$ has the comparison and κ-local properties, then every κ-injective algebra for $\mathcal{M}$ is κ-universal for $\mathcal{M}$.

<u>Proof</u>. Suppose $M \in INJ_\kappa(\mathcal{M})$ and $A \in \mathcal{M}^{<\kappa}$. We must embed A into M. Using the κ-local property, let B be any $\mathcal{M}^{<\kappa}$ subalgebra of M, and let $\bar{A}, \bar{B}$, and C be as in the comparison property and assume $C \leq \bar{A}$. By the κ-local property we can also assume $\bar{A}, \bar{B}, C \in \mathcal{M}^{<\kappa}$. Let h be an embedding of C into $\bar{B}$. Since M is κ-injective there is an embedding $f\colon \bar{B} \to M$ such that $f \equiv 1$ on B, and there is an embedding

$g\colon \bar{A} \to M$ which extends $fh\colon C \to M$. $g \upharpoonright A$ is the required embedding of A into M.

The theorem concerning isomorphisms between κ-injective algebras can be stated as follows. The previous comments hopefully make clear how the proof goes. This lemma is similar to [1: Isomorphism Theorem, p. 207].

<u>1.3.6</u> <u>Isomorphism Lemma</u>. Suppose $\mathcal{M}$ has the κ-inductive and κ-local properties.

(<u>a</u>) If $M, N \in \mathrm{INJ}_\kappa(\mathcal{M})$, $A < M$, $B < N$, $A, B \in M^{<\kappa}$, $f \in \mathrm{ISO}(A,B)$, and S and T are subsets of power κ of M and N respectively, then f has an extension to some $\varphi \in \mathrm{ISO}(\bar{A},\bar{B})$ where $S \leq \bar{A} \leq M$, $T \leq \bar{B} \leq N$ and $\bar{A}, \bar{B} \in \mathcal{M}^\kappa$.

(<u>b</u>) If $M \in \mathcal{M}^\kappa$, $N \in \mathrm{INJ}_\kappa(\mathcal{M})$, $A < M$, $B < N$, $A, B \in \mathcal{M}^{<\kappa}$ and $f \in \mathrm{ISO}(A,B)$, then f has an extension to an embedding of M into N.

<u>1.3.7</u> <u>Corollary</u>. Suppose $\mathcal{M}$ has the κ-inductive and κ-local properties.

(<u>a</u>) If $M, N \in \mathrm{INJ}^\kappa_\kappa(\mathcal{M})$ and $f \in \mathrm{ISO}(A,B)$ where A and B are $\mathcal{M}^{<\kappa}$ subalgebras of M and N respectively, then f has an extension to some $\varphi \in \mathrm{ISO}(M,N)$.

(<u>b</u>) If $\mathcal{M}$ also has the comparison property, then $\mathrm{INJ}^\kappa_\kappa(\mathcal{M}) = \mathrm{HU}^\kappa_\kappa(\mathcal{M})$, and this class has at most one member up to $\cong$. Furthermore, any κ-injective algebra for $\mathcal{M}$ is κ^+-universal for $\mathcal{M}$.

Proof of the Corollary. (a) follows from 1.3.6(a), taking $S = M$ and $T = N$. To prove (b), let $M \in INJ^{\kappa}_{\kappa}(\mathcal{M})$. M is κ-universal for $\mathcal{M}$ by 1.3.5, and κ-homogeneity of M follows from 0.3.6(a) with $N = M = S = T$. Now, recalling 0.1.8, we have $INJ^{\kappa}_{\kappa}(\mathcal{M}) = HU^{\kappa}_{\kappa}(\mathcal{M})$. Now suppose $M, N \in INJ^{\kappa}_{\kappa}(\mathcal{M})$. To show $N \cong M$, choose $A < M$ with $A \in \mathcal{M}^{<\kappa}$ by the κ-local property. Since N is κ-universal for $\mathcal{M}$ (1.3.5), there is an embedding f of A into N, and, by 1.3.7(a), f has an extension to some $\varphi \in ISO(M,N)$ as required. The proof that every $N \in INJ_{\kappa}(\mathcal{M})$ is κ^{+}-universal is similar to the previous proof, except we apply 1.3.6(b).

Questions (2) and (3) are answered by this corollary.

We need a special term for classes which satisfy the properties we have listed so far.

1.3.8 Definition of a κ-class. $\mathcal{M}$ is a κ-class if $\mathcal{M}$ satisfies the comparison, κ-inductive, κ-local, and κ-injective properties, and, if $\kappa = \omega$, $\mathcal{M}^{<\omega}$ has at most ω members up to $\cong$.

Question (4) asks if any further assumptions are needed, besides that $\mathcal{M}$ is a κ-class, for the existence of κ-injective algebras for $\mathcal{M}$. The answer is that the κ-injective property of $\mathcal{M}$ permits the inductive construction of a κ-injective algebra M for $\mathcal{M}$ - only a set-theoretical assumption is needed to insure that all the required "injections" can be built into the constructed algebra M. M is built in steps indexed by ordinals $< \kappa$. At stage μ we have constructed $M_\mu \in \mathfrak{M}^{<\varkappa}$. Let $\mathcal{J}$ be a set containing one member of each isomorphism class of $\mathfrak{M}^{<\varkappa}$ algebras. Let $U \in \mathfrak{M}^{<\varkappa}$, $U < V \in \mathcal{J}$ and let f be an embedding of U into M_μ. At some future step ρ of the construction we must guarantee that f has an extension to an embedding of V into $M_{\rho+1}$. This is done by direct use of $\varkappa$-injectivity of $\mathfrak{M}$ (refer to 1.3.0), letting $U = A$, $V = \bar{A}$, $M_\rho = B$, and choosing $M_{\rho+1}$ to be $\bar{B} \in \mathfrak{M}^{<\varkappa}$ by the $\varkappa$-local property.

It suffices to assume that $\varkappa$ is regular and that the G.C.H. holds in order that $\mathcal{J}$ and the set of functions f have power $\leq \varkappa$. In this way we can prove

<u>1.3.9</u> <u>Jonsson's Universal Homogeneous Structure Theorem</u>. Suppose $\varkappa$ is regular and the G.C.H. holds. If $\mathfrak{M}$ is a $\varkappa$-class and $\mathfrak{M}^{\varkappa} \neq \emptyset$, then $HU_\varkappa^\varkappa(\mathfrak{M})$ has a member $U_\varkappa$ which is unique up to $\cong$. If $M \in \mathfrak{M}^{\varkappa}$, $N \in \mathfrak{M}^{<\varkappa}$, and $N < M$, then every embedding of N into $U_\varkappa$ has an extension to an embedding of M into $U_\varkappa$. In particular $U_\varkappa$ is $\varkappa^+$-universal for $\mathfrak{M}$.

The assumption that $\mathfrak{M}^{\varkappa} \neq \emptyset$ is needed to insure that the constructive procedure outlined above does build an algebra of power $\varkappa$. This algebra is $\varkappa$-injective, and so the other conclusions of Jonsson's theorem follow from 1.3.7. Note also that for all $N \in INJ_\varkappa(\mathfrak{M})$ and for all subsets $T < N$ with $|T| = \varkappa$, there is an embedding f of $U_\varkappa$ into N such that $T \leq f(U_\varkappa)$. This follows from 1.3.6(a) with $M = S = U_\varkappa$.

Set-theoretical details similar to those needed in Jonsson's theorem will be given in Chapter 3, so we have omitted them here. The G.C.H. is not needed in the case $\kappa = \omega$; instead, the assumption that $\mathfrak{M}^{<\omega}$ has at most ω members up to $\cong$ is used (see 1.3.8).

In many important cases an injective algebra is given, and the class for which it is injective is defined from it. This idea is used in [1; Chapt. 10] to simplify the proof of Jonsson's Theorem, via the concept of "homoiogeneity".

<u>1.3.10</u> <u>Definition</u> [1: p. 207]. Suppose $M \in \mathfrak{M}^{\kappa}$. M is <u>homoiogeneous for</u> $\mathfrak{M}$ if, given $B, C \in \mathfrak{M}^{<\kappa}$ with $B \leq C < M$ and an embedding f of B into M, then f extends to an embedding of C into M.

Because of the way we have defined a κ-class we can give a theorem which relates this concept to Jonsson's Theorem. First we need

<u>1.3.11</u> <u>Definition</u>. If $M \in \mathfrak{M}$, $\mathfrak{M}\restriction M$ = the class of all $\mathfrak{M}$ algebras embeddable in M.

It is clear that if $M \in \mathfrak{M}^{\kappa}$, then M is homoiogeneous for $\mathfrak{M}$ if and only if M is κ-injective for $\mathfrak{M}\restriction M$. The next theorem relates this to Jonsson's Theorem and shows that our discussion is not less general than [1: Chapt. 10].

<u>1.3.12</u> <u>Theorem</u>. Suppose κ is regular, $\mathfrak{M}$ has the κ-local and κ-inductive properties, and $M \in \mathfrak{M}^{\kappa}$. Then, the following conditions are equivalent.

(i) M is κ-injective for $\mathcal{M}\restriction M$,

(ii) M is κ-homogeneous for $\mathcal{M}$, and

(iii) $\mathcal{M}\restriction M$ is a κ-class and $M \in HU_{\kappa}^{\kappa}(\mathcal{M}\restriction M)$.

Proof. The proof that (ii) $\Rightarrow$ (i)is similar to 1.1.8. Since (iii) $\Rightarrow$ (ii) is obvious, we need only show that (i) $\Rightarrow$ (iii). So assume $M \in INJ_{\kappa}^{\kappa}(\mathcal{M})$. To prove (iii) we need only show that $\mathcal{M}\restriction M$ is a κ-class and then apply 1.3.7(b) replacing $\mathcal{M}$ by $\mathcal{M}\restriction M$. It is clear that $\mathcal{M}\restriction M$ has the comparison, κ-local, and κ-injective properties.

To prove that $\mathcal{M}\restriction M$ is κ-inductive, suppose $\mathcal{C}$ is any chain of $\mathcal{M}\restriction M$ algebras with $|\mathcal{C}| \leq \kappa$. Let $C = \bigcup\mathcal{C}$, and note $C \in \mathcal{M}$ since $\mathcal{M}$ is κ-inductive. We must construct an embedding φ of C into M. For this it suffices to show that C is the union of a chain $\mathcal{U}$ where, for all $A \in \mathcal{U}$, $A \in \mathcal{M}\restriction M$ and $|A| < \kappa$. The embedding φ can then be constructed by using κ-injectivity of M for $\mathcal{M}\restriction M$ in the manner outlined for 1.3.6. To construct $\mathcal{U}$ we must show (*) for every $S < C$ with $S \in \mathcal{M}^{<\kappa}$, there exists $A \in \mathcal{M}\restriction M$ with $S \leq A \leq C$. The proof of (*) requires a set-theoretic maneuver, which we will sketch.

We can assume that $\kappa > \omega$ and $\mathcal{C}$ has a subchain $\mathcal{D}$ such that $C = \bigcup\mathcal{D}$ and $|\mathcal{D}| < \kappa$, because otherwise every S of (*) is contained in some $A \in \mathcal{C}$. Now suppose $S < C$ with $|S| < \kappa$ is given. Let $D \in \mathcal{D}$. We will construct $S_1 < C$ such that $S \leq S_1 \in \mathcal{M}^{<\kappa}$ and $S_1 \cap D \in \mathcal{M}\restriction M$. Put $T_0 = S$ and assume $T_0 \leq \cdots \leq T_n < C$ have been chosen and $T_n \in \mathcal{M}^{<\kappa}$. By the κ-local property of $\mathcal{M}$ there is some $X_n \leq D$

such that $X_n \in \mathfrak{M}^{<\varkappa}$ and $T_n \cap D \leq X_n$. Again by the $\varkappa$-local property, choose $T_{n+1} \in \mathfrak{M}^{<\varkappa}$ such that $T_n \cup X_n \leq T_{n+1} \leq C$. Put $S_1 = \bigcup_{n\geq 0} T_n$. Then $|S_1| < \varkappa$ since $\varkappa$ is regular. Since $\mathfrak{M}$ is $\varkappa$-inductive we have $S_1 \cap D = \bigcup_{n\geq 0} X_n \in \mathfrak{M}\restriction M$. Now assume $\mathcal{D} = \{D_\alpha | \alpha$ is an ordinal $< \rho\}$ where $\rho < \varkappa$. We construct algebras S_μ, $\mu < \rho$, as follows. Let $S_0 = S$, and construct $S_{\alpha+1}$ from S_α in the manner above, replacing S by S_α, D by $D_{\alpha+1}$, and S_1 by $S_{\alpha+1}$. At limit α, put $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$. The regularity of $\varkappa$ insures that $|S_\mu| < \varkappa$ for all $\mu < \rho$. We have by the construction that $\overline{S_\alpha} = S_{\alpha+1} \cap D_{\alpha+1} \in \mathfrak{M}\restriction M$ for all $\alpha < \rho$, and that if $\beta < \alpha$, then $\overline{S_\beta} \leq \overline{S_\alpha}$. Now $S \leq \bigcup_{\alpha<\rho} \overline{S_\alpha} = A$, $|A| < \varkappa$ by regularity, and we can show that $A \in \mathfrak{M}\restriction M$ by constructing an embedding of A into M as φ of the second paragraph of this proof is to be constructed.

We can omit the hypothesis that $\varkappa$ is regular in the above theorem if we assume instead that $\mathfrak{M}$ is λ-local for some $\lambda < \varkappa$. It would follow that $\mathfrak{M}$ is σ-local for all $\lambda \leq \sigma \leq \varkappa$ [1: Lemma 2.4, p. 206], and in the above proof, if $|S| \leq \sigma$ and $\rho \leq \sigma < \varkappa$, then $|A| \leq \sigma$.

Theorem 1.3.12 expresses a type of duality involved in the existence of homogeneous universal structures. We can <u>either</u> begin with a $\varkappa$-class $\mathfrak{M}$ and construct, via Jonsson's Theorem, $HU^{\varkappa}_{\varkappa}(\mathfrak{M})$ from it <u>or</u>, if $\mathfrak{M}$ is $\varkappa$-local and $\varkappa$-inductive, we can begin with any $M \in \mathfrak{M}^{\varkappa}$ which is injective for $\mathfrak{M}\restriction M$ (or homogeneous for $\mathfrak{M}$, which is prima facie stronger) and obtain a κ-class $\mathfrak{M}\restriction M$ from it to which Jonsson's Theorem, if applied, would yield the original $M \in HU_{\kappa}^{\kappa}(\mathfrak{M}\restriction M)$. Of course, to apply Jonsson's Theorem we need the G.C.H. if $\kappa > \omega$, but no such assumption is needed to apply Theorem 1.3.12. Another way to look at the content of Theorem 1.3.12 is that an algebra $M \in \mathfrak{M}^{\kappa}$ which is κ-injective for $\mathfrak{M}\restriction M$ is determined up to $\cong$ by its $\mathfrak{M}^{<\kappa}$ subalgebras; that is, if M_0 and M_1 are any two such, and $M_0 \not\cong M_1$, then, for $i = 0$ or 1, some $\mathfrak{M}^{<\kappa}$ subalgebra of M_i is not embeddable in M_{1-i}.

§1.4 Some Groups of this Study; Free Algebraic Closures and Free κ-Classes.

Let $\mathfrak{G}$ be the class of all groups and $\mathfrak{U}$ the class of all groups of order ω which are ω-homogeneous for $\mathfrak{G}$; that is, by Theorem 1.3.12, $G \in \mathfrak{U}$ iff $G \in INJ_{\omega}^{\omega}(\mathfrak{G}\restriction G) = HU_{\omega}^{\omega}(\mathfrak{G}\restriction G)$ iff, given any $\varphi \in ISO(A,B)$ where A and B are f.g. subgroups of G, φ extends to an automorphism of G. For example, every divisible abelian group has this property. Theorem 1.3.12 says that any such group is determined uniquely in $\mathfrak{U}$ by the set of its f.g. subgroups. An important subclass of $\mathfrak{U}$ are those groups G such that every φ as above extends to an inner automorphism of G. We will call such a group <u>inner-homogeneous for</u> $\mathfrak{G}^{<\omega}$. Hall's ULF group H_{ω} is such a group by the proof of 1.1.8, and there is a very important subclass of inner-homogeneous groups - the algebraically closed groups.

<u>1.4.0</u> <u>Definition of an algebraically closed group</u>. Let G be a group and W a finite set of formal products involving variable letters (henceforth, variables) $x_1,\ldots,x_n$ and elements of G. We call $w \in W$ a <u>word over</u> G. W is said to be <u>consistent over</u> G if there is a group K with $G \leq K$ such that the equations $W = 1$ have a solution, $x_i = a_i \in K$ $(1 \leq i \leq n)$, in K. G is <u>algebraically closed</u> (<u>a.c.</u>) if every consistent set of words over G has a solution in G.

<u>1.4.1</u> <u>Definition of an existentially closed group</u>. Let G be a group and W and R finite sets of words over G with variables $x_1,\ldots,x_n$. The pair (W,R) is <u>consistent over</u> G if there is a group K with $G \leq K$ such that the equations $W = 1$ and inequations $r \neq 1$, $r \in R$ have a (simultaneous) solution, $x_i = a_i \in K$ $(1 \leq i \leq n)$, in K. G is <u>existentially closed</u> (<u>e.c.</u>) if every consistent pair (W,R) over G has a solution in G.

The concept of an e.c. group was first defined in 1951 by W.R. Scott [20] who called them "algebraically closed". The situation was clarified in less than a year by B.H. Neumann.

<u>1.4.2</u> <u>Theorem</u> (B.H. Neumann [17]). Every non-trivial algebraically closed group is existentially closed.

The point of Neumann's theorem is that any set of consistent inequations over a group can be forced by a larger set of equational relations. We will indicate the nature of Neumann's argument by considering the case of one inequation. Suppose G is a group, $1 \neq b \in G$ and $(W, \{r\})$ is a consistent pair over G. Let K be a group in which $W = 1$, $r \neq 1$ has a solution $x_i = a_i \in K$, $1 \leq i \leq n$. Let $\bar{r} \in K$ be the result of substituting $x_i = a_i$ in r. The idea is to add new relations to W which guarantee that b belongs to the normal closure of r. We will show that such a set of relations exist which are consistent over G. There is an extension $K_1 = \langle K, y, z \rangle$ such that y and z have infinite order and $b = yz$, and there is an extension $K_2 = K * \langle u \rangle * \langle v \rangle$ such that $c = (u^{-1}\bar{r}u)(v^{-1}\bar{r}v)$ has infinite order. K_1 can be defined by an easy free amalgamation. Let $A = K_1 *_K K_2$ and let $J = \langle A, t, s \rangle$ be a group such that $t^{-1}ct = y$ and $s^{-1}cs = z$. J can be defined by a sequence of two HNN extensions. The existence of J implies that the set of words $W \cup \{yzb^{-1}, t^{-1}cty^{-1}, s^{-1}csz^{-1}\}$ with variables $x_1, \ldots, x_n, u, v, y, z, s, t$ is consistent over G, and this set forces $r \neq 1$ since the non-trivial element $b \in G$ is in the normal closure of r.

We will now check our initial assertion.

<u>1.4.3</u> Every a.c. group is inner-homogeneous for $\mathcal{L}^{<\omega}$ ([11: Lemma 1]).

<u>Proof</u>. Suppose G is an a.c. group, $A = \langle a_1, \ldots, a_n \rangle$ and B are f.g. subgroups of G, and $\varphi \in ISO(A,B)$. The set of relations $x^{-1}a_i x = \varphi(a_i)$, $1 \leq i \leq n$, is consistent over G

since they are solvable in an HNN extension of G. Hence for some $t \in G$, $t^{-1}a_i t = \varphi(a_i)$, $1 \leq i \leq n$.

Thus a countable a.c. group is determined uniquely in the class of countable a.c. groups by its set of f.g. subgroups.

Algebraically closed groups have been and continue to be the subject of research since Scott introduced them [18], [11], [4], [22]. The main technique of this study ($\varkappa$-injective chains) is not directly applicable to arbitrary a.c. groups, but can be applied to a natural class of a.c. groups, one of which is discussed by Macintyre with the method of forcing [11: Theorem 8, p. 81]. We will now discuss some members of this class.

<u>1.4.4</u> <u>Definition of free extensions of groups</u>. Suppose $W = W(A,X)$ is a set of words over a group G which involve only elements of the subgroup $A \leq G$ and the variables X. The <u>free extension of</u> G <u>by</u> W, denoted $G^* = G^*(W)$, is the group with presentation $(G,X;$ all relations of G and $W = 1)$. We say that G^* is a <u>finitary free extension</u> (<u>f.f.e.</u>) <u>of</u> G if X and W are both finite and W is consistent over G.

The structure of G^* is somewhat clarified by the next observation.

<u>1.4.5</u> <u>Lemma</u>. Let $W = W(A,X)$ be a consistent set of words over G and put $\mathbf{E} = G^*(W)$. Then, G is a subgroup of $\mathbf{E}$,

$gp_E(A,X) = A^*(W)$, $G \cap A^*(W) = A$, and the amalgam $\mathcal{a} = \begin{smallmatrix} G & & A^*(W) \\ & \diagdown\ \diagup & \\ & A & \end{smallmatrix}$

generates $gp_*(\mathcal{a})$ in E.

Proof. To show that $G < E$ use the universal mapping property of presentations and the fact that W is consistent over G. The remaining assertions follow from the fact that $gp_*(\mathcal{a})$ has the same presentation as E.

1.4.6 Definition of the free extension property. Let $G < K$ be groups. K has the free extension property (f.e.p.) over G if, for every finite subset $S \leq K$ and for every finitary free extension J^* of $J = \langle G,S \rangle$, there is an embedding f of J^* into K such that $f \equiv 1$ on J. We say that K is an f.e.p. group if K has the f.e.p. over 1.

Notice that every f.e.p. group has the f.e.p. over each of its f.g. subgroups, and is therefore prima facie an a.c. group.

1.4.7 Lemma. If K has the f.e.p. over G, then K has the f.e.p. over every subgroup of G.

Proof. Suppose K has the f.e.p. over G and $A \leq G$. Let S be a finite subset of K, $J = \langle A,S \rangle$, and let $J^*(W)$ be a f.f.e. of J. By Lemma 1.4.5 we have $J^*(W) \leq \langle G,S \rangle^*(W) = Q$, and by 1.4.6 there is an embedding f of Q into K such that $f \equiv 1$ on $\langle G,S \rangle$. Now the restriction of f to $J^*(W)$ satisfies 1.4.6 proving that K has the f.e.p. over A.

<u>1.4.8</u> <u>Lemma</u>. If B is a f.f.e. of A and C is a f.f.e. of B, then C is a f.f.e. of A.

<u>Proof</u>. Let $B = A^*(W_1)$ and $C = B^*(W_2)$ and let X_1 and X_2 be the sets of variables involved in W_1 and W_2 respectively. Then C has the presentation $(A, X_1 \cup X_2;$ relations of A, $W_1 \cup W_2)$ showing that C is a f.f.e. of A.

<u>1.4.9</u> <u>Definition of a free algebraic closure of a group</u>. Suppose $G < K$ are groups. K is called a <u>free algebraic closure</u> (<u>f.a.c.</u>) of G provided

(<u>i</u>) K has the f.e.p. over G, and

(<u>ii</u>) For every finite subset $S < K$, there is a f.f.e. G^* of G and an embedding f of G^* into K such that $f \equiv 1$ on G and $S < f(G^*)$.

This concept has apparently never been studied. The universal algebraic closures of Jonsson [10] are much different.

<u>1.4.10</u> <u>Theorem</u>.

(<u>a</u>) Every group G has a f.a.c. K with $|K| = \max(|G|, \omega)$.

(<u>b</u>) If $G < H < K$ are groups, H is a f.a.c. of G, and K is a f.a.c. of H, then K is a f.a.c. of G.

(<u>c</u>) If $G < H < K$ are groups, H is a f.f.e. of G, and K is a f.a.c. of H, then K is a f.a.c. of G.

(<u>d</u>) If G is a f.g. subgroup of an f.e.p. group F, then there is a f.a.c., K, of G with $G < K \leq F$.

(e) If G is a countable group and K_1 and K_2 are countable f.a.c.'s of G, then there exists $\varphi \in ISO(K_1, K_2)$ such that $\varphi \equiv 1$ on G.

Proof of (a). This proof follows a suggestion of Professor Sonneborn. It is more natural than my original proof.

Let H be any group and let $\mathcal{W} = \mathcal{W}(H) = \{W_\alpha | \alpha \in I\}$ be a "complete" set of consistent finitary sets of words over H in the sense that if $H^*(W)$ is any f.f.e. of H, then there is some $\alpha \in I$ such that the sets W and W_α are identical under some one-to-one correspondence of their variables. Let X_α be the set of variables involved in W_α and put $X(\mathcal{W}) = \bigcup_{\alpha \in I} X_\alpha$. We assume that these variable sets are pairwise disjoint; that is, if $\alpha \neq \beta \in I$, then $X_\alpha \cap X_\beta = \emptyset$. We define the group $H^*(\mathcal{W})$ to have the presentation $(H, X_\alpha, \alpha \in I;$ relations of $H, W_\alpha, \alpha \in I)$, and we observe

1.4.11 If $W_1, \ldots, W_n \in \mathcal{W}$ and $X_1, \ldots, X_n$ are the respective sets of variables, then the subgroup $J = \langle H, X_1, \ldots, X_n \rangle$ of $H^*(\mathcal{W})$ has the presentation $P = (H, X_1, \ldots, X_n;$ relations of $H, W_1, \ldots, W_n)$. In particular, J is a f.f.e. of H.

The proof of 1.4.11 is that we can construct $H^*(\mathcal{W})$ in two steps: first presenting the group P as above, and then adding all the other letters X_α, $\alpha \in I$ with $X_\alpha \neq X_i$, $1 \leq i \leq n$, with relations W_α. An application of Lemma 1.4.5, using the disjointness of the sets X_α, shows that $P = J < H^*(\mathcal{W})$.

Now let G be any group. We will iterate the above construction ω times. Specifically, let $\mathscr{W}_1 = \mathscr{W}(G)$ and define $G_1 = G^*(\mathscr{W}_1)$; having defined G_n, let $\mathscr{W}_{n+1} = \mathscr{W}(G_n)$ and put $G_{n+1} = G_n^*(\mathscr{W}_{n+1})$. Finally, put $K = \bigcup_{n \geq 1} G_n$. In this construction we assume that the sets of variables $X(\mathscr{W}_n)$, $n \geq 1$, are pairwise disjoint.

For any group H, simple cardinal arithmetic shows that $|\mathscr{W}(H)| = \max(|H|,\omega) = |H^*(\mathscr{W})|$, and thus, in our construction, $|K| = \max(|G|,\omega)$.

Now we must check that K is a f.a.c. of G.

To show that K has the f.e.p. over G, let S be a finite subset of K, let $J = \langle G,S \rangle$, and let $J^*(W)$ be any f.f.e. of J. Now $J < G_n$ for some $n \geq 1$, and there exists $W_\alpha \in \mathscr{W}_{n+1}$ such that W and W_α are identical under some one-to-one correspondence of their variables. Thus there exists $f \in \mathrm{ISO}(J^*(W),J^*(W_\alpha))$ with $f \equiv 1$ on J. Since $J^*(W_\alpha) < G_n^*(W_\alpha) < G_n^*(\mathscr{W}_{n+1})$ by 1.4.5 and 1.4.11, f meets the condition of 1.4.6.

To prove (ii) of 1.4.9, assume inductively that if T is any finite subset of G_n, for some $n \geq 1$, then T is contained in some f.f.e., $G^*(U) < G_n$, of G. Note that, if $n = 1$, this is immediate from 1.4.11. Let $S < G_{n+1}$ be finite. There exist $W_1,\ldots,W_m \in \mathscr{W}(G_n)$ with respective variable sets $X_1,\ldots,X_m$ such that $S < \langle G_n,X_1,\ldots,X_m \rangle$. Now the words $W_1 \cup \cdots \cup W_m$ involve only finitely many elements T of G_n, and each W_i, $i \leq i \leq m$, is a set of words over $\langle T \rangle$. By the induction hypothesis $T < G^*(U) < G_n$, and, by 1.4.5 and 1.4.11, $C = \langle G^*(U), X_1, \ldots, X_m \rangle < \langle G_n, X_1, \ldots, X_n \rangle$ is a f.f.e. of $G^*(U)$. By Lemma 1.4.8, it follows that C is a f.f.e. of G, and since $S < C$ we have proved (ii) of 1.4.9. Hence K is a f.a.c. of G.

Proof of (b). Suppose K is a f.a.c. of H and H is a f.a.c. of G. By Lemma 1.4.7, K has the f.e.p. over G. So we need to check (ii) of 1.4.9. Let $S < K$ be finite and choose $H^*(W) = H^*$, a f.f.e. of H, and an embedding f of H^* into K such that $f \equiv 1$ on H and $S < f(H^*)$. Let X be the set of variables in W and let T be the (finite) set of elements of H which occur in words of W. Choose $G^*(U) = G^*$, a f.f.e. of G, and an embedding g of G^* into H such that $g \equiv 1$ on G and $T < g(G^*)$. Let Y be the set of variables in U. Now $J = \langle G, g(Y), f(X) \rangle < \langle H, f(X) \rangle$ is seen to be a f.f.e. of G by applying 1.4.5 and 1.4.8 (via the isomorphisms g and f). Since $S < J$, we have now proved that K is a f.a.c. of G.

Proof of (c). The proof is similar to (b), with the simplification that $g(G^*) = H$ is fixed.

Proof of (d). Let G be a f.g. subgroup of the f.e.p. group F. We will construct a chain $\cdots G_n < G_{n+1} < \cdots$ of f.g. subgroups of F with $G = G_0$ and the additional properties

1.4.12 For each $m \geq 0$ there is a f.f.e. G_m^* of G_m and $\varphi \in \mathrm{ISO}(G_m^*, G_{m+1})$ such that $\varphi \equiv 1$ on G_m, and

1.4.13 For every $n \geq 0$ and for every $W \in \mathscr{W}(G_n)$ (as in part (a)), there exists $m > n$ such that $G_m^* = G_n^*(W)$ where G_m^* is the f.f.e. of 1.4.12.

If G_m has been constructed for some m and is f.g., then, once we choose G_m^*, we can find $G_{m+1} < F$ by a direct application of the f.e.p. of F. Since each of the sets $\mathscr{W}(G_n)$ is countable, there is no difficulty in choosing the G_m^* so that 1.4.13 holds. Thus we can construct $K = \bigcup_{n \geq 1} G_n$ so that 1.4.12 and 1.4.13 hold. The proof that K is an f.a.c. of G is very similar to the proof in part (a), so we will omit it. In fact, this was our original proof of the existence of free algebraic closures.

Proof of (e). First we need

1.4.14 Lemma. Suppose $G < H < G^*$ for some f.f.e. G^* of G and $H = \langle G, S \rangle$ with S finite. Then there is a f.f.e. H^* of H and $f \in \mathrm{ISO}(H^*, G^*)$ such that $f \equiv 1$ on H.

Proof. Suppose $G^* = (G, X;$ relations of $G, W)$. For each $s \in S$ let $u(s) \in G^*$ be an expression for s as a product of elements of G and X. Let $U = \{s^{-1}u(s) \mid s \in S\}$. Let $\bar{X}$ be letters in one-to-one correspondence with X and let $\bar{W}$ and $\bar{U}$ be the sets of words obtained by substituting the $\bar{X}$ for the X. Define $H^* = (H, \bar{X};$ relations of H, $\bar{W}, \bar{U})$, and for every $\bar{z} \in H^*$ define $f(\bar{z}) = z \in G^*$. It is easy to see that $f \in \mathrm{ISO}(H^*, G^*)$ since every relation of H^* is satisfied in G^* (and vice versa) under the correspondence $\bar{x} \leftrightarrow x$, and $f \equiv 1$ on H because the relations U guarantee this. This completes the proof of 1.4.14.

Now suppose K_1 and K_2 are countable f.a.c.'s of G. We will build $\varphi \in \mathrm{ISO}(K_1,K_2)$, with $\varphi \equiv 1$ on G, inductively on a chain of subgroups of K_1 which are f.g. over G. The proof is similar to that outlined for the Isomorphism Lemma 1.3.6. We will give the essential step.

Suppose $H = \langle G,T\rangle < K_1$ with T finite and an embedding h of H into K_2 has been defined such that $h \equiv 1$ on G. (h is an approximation to φ). We must be able to extend h to an embedding of $A = \langle H,F\rangle < K_1$ into K_2 where F is an arbitrary finite subset of K_1. Using 1.4.9(ii) (with $S = T \cup F$), let G^* be a f.f.e. of G and g an embedding of G^* into K_1 such that $g \equiv 1$ on G and $A < g(G^*)$. By Lemma 1.4.14, there is a f.f.e. H^* of H and $f \in \mathrm{ISO}(H^*, g(G^*))$ with $f \equiv 1$ on H. Since K_2 has the f.e.p. over G (letting $J = h(H)$ in 1.4.6), there is an embedding e of H^* into K_2 such that $e \equiv h$ on H. Now $\bar{h} = ef^{-1}$ embeds $g(G^*)$ into K_2 and extends h. Hence $\bar{h}\upharpoonright A$ is the desired extension of h. (See the diagram below).

$$\begin{tikzcd}
 & g(G^*) = f(H^*) \arrow[r, "\bar{h}"] \arrow[d, dash] & e(H^*) \arrow[d, dash] \\
A \arrow[ur, dash] \arrow[r, dash] & H \arrow[r, "h"] \arrow[d, dash] & h(H) \arrow[dl, dash] \\
 & G &
\end{tikzcd}$$

At the next stage of the construction, we will reverse the procedure and extend $\bar{h}^{-1}$ in K_2. Since K_1 and K_2

are countable, an isomorphism φ with domain K_1 and range K_2 can be built in this way, and we have $\varphi \equiv 1$ on G since this holds at every stage.

<u>1.4.15</u> <u>Remarks on generalizing the concepts of free extensions and free algebraic closures</u>. These concepts can be defined in any class $\mathfrak{M}$ in which algebras have presentations, that is, in which there exist free algebras with the universal mapping property. This is the case if $\mathfrak{M}$ is any variety of algebras. But, to prove Theorem 1.4.10 for a variety $\mathfrak{M}$, the following fact is needed to replace Lemma 1.4.5: if W is a consistent set of words over $M \in \mathfrak{M}$ and $M \leq N \in \mathfrak{M}$, then W is consistent over N and $M^*(W) \leq N^*(W)$. This can be easily proved if $\mathfrak{M}$ is injective (refer to 1.3.0). Letting $A = M$, $\bar{A} = N$, $B = M^*(W)$, and $f = 1_M$, the proof would be to choose $\bar{B}$ and $\bar{f}$ as in 1.3.0. Put $J = \mathrm{alg}_{\bar{B}}(M^*(W), \bar{f}(N))$. Then J is a homomorphic image of $N^*(W)$ by a homomorphism extending $\bar{f} \cup 1_X$ where X is the set of variables of W. Thus $N \leq N^*(W)$ and M and X generate $M^*(W)$ in $N^*(W)$. So Theorem 1.4.10 holds in any injective variety. For example, the variety $\mathfrak{A}$ of abelian groups is injective. The f.a.c. <u>in</u> $\mathfrak{A}$ of every $\mathfrak{A}^{\leq\omega}$ group is the divisible group whose torsion-free rank and p-rank, for every prime p, equals ω. This group is also the (unique) member of $HU^{\omega}_{\omega}(\mathfrak{N})$.

<u>1.4.16</u> <u>Definition</u>. A group K is a <u>f.a.c. group</u> if K is a f.a.c. of one of its subgroups.

We have introduced two natural classes of a.c. groups - the class of f.e.p. groups and the class of f.a.c. groups, which is contained in the former. Several observations can be made.

<u>1.4.17</u> <u>Theorem</u>. (<u>a</u>) There are 2^ω non-isomorphic countable f.a.c. groups; (<u>b</u>) Let K_0 = the countable f.a.c. of the trivial group. K_0 is embeddable in every f.e.p. group; every finitely presented group P is embeddable in K_0, and, hence, K_0 is the countable f.a.c. of P.

The proof of (a) is a simple counting argument: there are 2^ω non-isomorphic f.g. groups, but a countable f.a.c. group has only ω f.g. subgroups. That K_0 is embeddable in every f.e.p. group is immediate using 1.4.10(d). If P is finitely presented, then $P \simeq$ a f.f.e. of the trivial group; so P is embeddable in every f.e.p. group, and a simple application of 1.4.14 shows that K_0 is the countable f.a.c. of P.

The countable a.c. group G, discussed by Macintyre [11; Theorem 8], such that its f.g. subgroups are exactly the f.g. recursively presented groups, is none other than K_0. Every f.g. subgroup $H < K_0$ is contained in a finitely presented subgroup (a f.f.e. of 1), and hence H is recursively presented. On the other hand, the celebrated theorem of Graham Higman asserts that every recursively presented group is a subgroup of a finitely presented group [5], and hence is embeddable in K_0.

A countable f.e.p. group, K, exists which is not a f.a.c. of any of its f.g. subgroups. To construct K we need only arrange that, for every f.g. $G < K$, there is some f.g. $\bar{G} < K$ such that $\bar{G}$ is not embeddable in the countable f.a.c. of G. However, K might be a f.a.c. of a non-f.g. subgroup. This creates a problem we have not been able to solve.

<u>1.4.18</u> <u>Problem</u>. Construct a countable f.e.p. group which is not a f.a.c. group.

We now return to our main topic.

If K is a countable f.e.p. group, then $\mathcal{J} = \mathcal{S}\restriction K$ is an ω-class by Theorem 1.3.12. In particular, $\mathcal{J}$ has the ω-injective (i.e., amalgamation) property. In fact, $\mathcal{J}$ enjoys a much stronger amalgamation property which will serve to motivate the main concept of this section.

<u>1.4.19</u> <u>Theorem</u>. Suppose K is a f.e.p. group, $\mathcal{J} = \mathcal{S}\restriction K$, and $\mathcal{A} = \begin{smallmatrix} A & & B \\ & \bigvee & \\ & E & \end{smallmatrix}$ is an $\mathcal{J}^{<\omega}$ amalgam. Then $gp_*(\mathcal{A}) \in \mathcal{J}$.

<u>Proof</u>. Let f and g be embeddings of A and B into K respectively, $E = \langle e_1,\ldots,e_n\rangle$, and $G = \langle f(A),g(B)\rangle$. If $1 \leq i \leq n$, let $w_i = x^{-1}f(e_i)x[g(e_i)]^{-1}$, and let h be an embedding of $G^* = G^*(w_i \mid 1 \leq i \leq n) = \langle G,x\rangle$ into K such that $h \equiv 1$ on G. Now G and $h(x) = y$ generate in K the HNN extension of G relative to $\varphi \in ISO(f(E),g(E))$

with $\varphi f(e_i) = g(e_i)$, $1 \leq i \leq n$, because G^* is the HNN extension. It follows easily from Britton's Lemma (1.2.3)

that $\mathcal{a}' = $
$$\begin{tikzcd}[column sep=small] y^{-1}f(A)y \arrow[dr, dash] & & g(B) \arrow[dl, dash] \\ & g(E) & \end{tikzcd}$$
generates $gp_*(\mathcal{a}') \cong gp_*(\mathcal{a})$ in K.

The κ-classes of groups (see 1.3.8) to which we will apply the construction of κ-injective chains, and which the last theorem gives some natural examples of, are classes which admit free amalgamations.

<u>1.4.20</u> <u>Definition of a free κ-class of groups</u>. A class of groups $\mathfrak{F}$ is a <u>free</u> κ-<u>class</u> if the following axioms hold.

(<u>i</u>) $1 \in \mathfrak{F}$ and $\mathfrak{F}$ contains some non-trivial group,

(<u>ii</u>) $\mathfrak{F}$ has the κ-local property,

(<u>iii</u>) $\mathfrak{F}$ is inductive,

(<u>iv</u>) If $\mathcal{a}$ is an $\mathfrak{F}^{<\kappa}$ amalgam, then $gp_*(\mathcal{a}) \in \mathfrak{F}$, and

(<u>v</u>) If $\kappa = \omega$, then $\mathfrak{F}^{<\omega}$ has at most ω members up to isomorphism.

Many natural classes of groups are free κ-classes for all $\kappa > \omega$. If Π is any set of primes, we have as examples

(<u>1</u>) $\mathfrak{F}$ = the class of all groups in which every element of finite order is a Π-element, and

(<u>2</u>) $\mathfrak{F}$ = the class of all groups in which every periodic Π-subgroup is locally finite.

Both of these classes $\mathfrak{F}$ contain $gp_*(\mathcal{a})$ whenever $\mathcal{a}$ is an $\mathfrak{F}$ amalgam. The proof of this for (1) is quite easy

[19; Theorem II], while for (2) we can use the subgroup theorem for amalgamated free products [6; p. 228].

Every free κ-class is a κ-class. The comparison property follows from (i), while condition (iv) implies that every $\mathfrak{F}^{<\kappa}$ amalgam is contained in an $\mathfrak{F}$ group and this implies that $\mathfrak{F}$ is κ-injective because (refer to 1.3.0) we can form an $\mathfrak{F}^{<\kappa}$ amalgam $\mathcal{A} = \begin{smallmatrix} g(\bar{A}) & & B \\ & g(A) & \end{smallmatrix}$ where g is an isomorphism with $g \equiv f$ on A, and choose $\bar{B} \in \mathfrak{F}$ with $\mathcal{A} < \bar{B}$. Then $g = \bar{f}$ is the required extension of f.

Note that a free κ-class $\mathfrak{F}$ satisfies the full inductive property (iii), that is, $\mathfrak{F}$ is closed under arbitrary directed unions. This simplifies the statement of our results since we are interested in constructing many non-isomorphic members of $HU_{\kappa}^{\kappa^+}(\mathfrak{F})$ and these groups are obtained as direct unions of $HU_{\kappa}^{\kappa}(\mathfrak{F})$ groups.

Theorem 1.4.19 says that f.e.p. groups provide examples of free ω-classes via Theorem 1.3.12. The precise relation is

<u>1.4.21</u> <u>Theorem</u>. If K is a countable f.e.p. group, then there is a free ω-class $\mathfrak{F}$ such that $\mathcal{S}{\restriction}K = \mathfrak{F}^{\leq\omega}$ and $K \in HU_{\omega}^{\omega}(\mathfrak{F})$.

<u>Proof</u>. $\mathcal{K} = \mathcal{S}{\restriction}K$ is an ω-class by Theorem 1.3.12 and 1.4.3. Theorem 1.4.19 shows that $\mathcal{K}$ has property (iv) of 1.4.20, namely that $\mathcal{K}^{<\omega}$ contains $gp_*(\mathcal{A})$ for every $\mathcal{K}^{<\omega}$ amalgam $\mathcal{A}$. Let $\mathfrak{F}$ = the closure of $\mathcal{K}$ under directed unions

of its members. Thus $\mathcal{F}$ is inductive and $\mathcal{F}^{<\omega} = \mathcal{K}^{<\omega}$ since no new f.g. group occurs in a directed union. Hence $\mathcal{F}$ is a free ω-class. Every $G \in \mathcal{F}^{\omega}$ is a directed union of countably many members of $\mathcal{K}$ and so $G \in \mathcal{K}$ since $\mathcal{K}$ is ω-inductive. Thus $\mathcal{K} = \mathcal{F}^{\leq\omega}$, and the final conclusion is obvious.

The class $\mathcal{S}{\upharpoonright}K$ of the preceding theorem is always enlarged by passing to the free ω-class $\mathcal{F}$. In fact, the next three results will show that if $\mathcal{F}$ is any free κ-class, then $HU_{\kappa}^{\kappa^+}(\mathcal{F}) \neq \emptyset$.

<u>1.4.22</u> <u>Proposition</u>. Suppose (<u>a</u>) $\mathfrak{M}$ is a κ-class, (<u>b</u>) Every $\mathfrak{M}^{<\kappa}$ amalgam is contained in a member of $\mathfrak{M}$, and (<u>c</u>) There are two non-isomorphic $\mathfrak{M}$ algebras. Then $\mathfrak{M}^{\kappa} \neq \emptyset$. Thus $\mathfrak{M}$ satisfies the hypotheses of Jonsson's Theorem and, if κ is regular and the G.C.H. holds if $\kappa > \omega$, then $HU_{\kappa}^{\kappa}(\mathfrak{M})$ has a unique member up to $\cong$. In particular, every free κ-class $\mathfrak{M}$ has these properties.

<u>Proof</u>. We need only show that $\mathfrak{M}^{\kappa} \neq \emptyset$. For this it suffices to show that, for every $M \in \mathfrak{M}^{<\kappa}$, there exists $N \in \mathfrak{M}$ with $M < N$ because we can then use the κ-inductive property to obtain a member of $\mathfrak{M}^{\kappa}$. We can assume that M has a proper $\mathfrak{M}$ subalgebra E by the comparison property and assumption (c). Now the amalgam $\begin{smallmatrix} M & & M_1 \\ & \diagdown\!\diagup & \\ & E & \end{smallmatrix}$, where $M_1 \cong M$, is contained in some $N \in \mathfrak{M}$ by (b).

Jonsson remarked in [8] that assumption (b), "the strong amalgamation property", is enough to prove $\mathcal{M}^{\kappa} \neq \emptyset$.

<u>1.4.23</u> <u>Theorem</u>. Assume that κ is regular and the G.C.H. holds if $\kappa > \omega$. Let $\mathcal{M}$ satisfy the hypotheses (a) - (c) of Prop. 1.4.22 and suppose also that $\mathcal{M}$ is κ^+-inductive. Then $\mathrm{INJ}_{\kappa}^{\kappa^+}(\mathcal{M}) \neq \emptyset$. In particular, this is true for every free κ-class $\mathcal{M}$.

<u>Proof</u>. Let $U \in \mathrm{HU}_{\kappa}^{\kappa}(\mathcal{M})$ and $M \in \mathcal{M}^{\kappa}$. Lemma 1.3.6(b) guarantees that there is some embedding of M into U, but to construct a member of $\mathrm{INJ}_{\kappa}^{\kappa^+}(\mathcal{M})$ we will need to obtain proper embeddings. Assumption (b) is ideal for this purpose, but we will need a small lemma.

<u>1.4.24</u> <u>Lemma</u>. Suppose κ is regular, $\mathcal{M}$ is a κ-class and also κ^+-inductive, $\mathcal{C} = \{U_\alpha | \alpha < \kappa^+\}$ is a chain of $\mathrm{HU}_{\kappa}^{\kappa}(\mathcal{M})$ algebras, and, for all $\alpha < \kappa^+$, $U_\alpha < U_{\alpha+1}$. Then $C = \bigcup \mathcal{C} \in \mathrm{INJ}_{\kappa}^{\kappa^+}(\mathcal{M})$.

To prove this lemma, note that $C \in \mathcal{M}$ since $\mathcal{M}$ is κ^+-inductive. Suppose $A \leq B$ are $\mathcal{M}^{<\kappa}$ algebras and $f\colon A \to C$ is an embedding. Since κ is regular, $f(A) < U_\alpha$ for some $\alpha < \kappa$. Since U_α is κ-injective for $\mathcal{M}$, there is an embedding of B into U_α, and hence into C, which extends f. Hence C is κ-injective for $\mathcal{M}$, and clearly $|C| = \kappa^+$.

To prove the theorem, we must construct a chain $\mathcal{C}$ as in 1.4.24, and this is clearly possible if an arbitrary $M \in \mathcal{M}^{\kappa}$ can be properly embedded into U. Since $\mathcal{M}$ is κ-local,

M is the union of a continuous chain $\{M_\alpha \mid \alpha < \kappa\}$ of $M^{<\kappa}$ algebras. Let $f_0\colon M_0 \to U$ be an embedding and choose $N \in \mathfrak{M}^{<\kappa}$ with $f_0(M_0) = N_0 < N < U$. We can inductively construct an embedding $\varphi\colon M \to U$ with $\varphi(M) \cap N = N_0$, and hence $\varphi(M) < U$. In the inductive step we have an embedding f_α: $M_\alpha \to U$ which extends f_0 and such that $f_\alpha(M_\alpha) \cap N = N_0$. Choose $J \in \mathfrak{M}^{<\kappa}$ with $\langle f_\alpha(M_\alpha), N\rangle \leq J < U$. Form an amalgam

$$\mathcal{A} = \begin{tikzcd}[column sep=small] h(M_{\alpha+1}) & & J \\ & f_\alpha(M_\alpha) \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

where h is an isomorphism with $h \equiv f_\alpha$ on M_α. Now $\mathcal{A} \leq A \in \mathfrak{M}^{<\kappa}$ by assumption (b), and by κ-injectivity of U, there is an embedding $g\colon A \to U$ such that $g \equiv 1$ on J. It follows that $gh = f_{\alpha+1}\colon M_{\alpha+1} \to U$ extends f_α and $f_{\alpha+1}(M_{\alpha+1}) \cap N = N_0$. Thus $\bigcup_{\alpha<\kappa} f_\alpha = \varphi\colon M \to U$ is the desired proper embedding, completing the proof of 1.4.23.

We have not yet reached our goal which was to prove $HU_\kappa^{\kappa^+}(\mathfrak{F}) \neq \emptyset$ for every free κ-class $\mathfrak{F}$. To do this we will show that $INJ_\kappa(\mathfrak{F}) = HU_\kappa(\mathfrak{F})$. This will be accomplished, as it was in Prop. 1.1.9 for $\mathcal{G}$, by showing that the automorphisms involved in the homogeneity of $U \in HU_\kappa^\kappa(\mathfrak{F})$ can be chosen inner. Notice that we already know this in the special case $U = K$ is a countable f.e.p. group by 1.4.3.

<u>1.4.25</u> <u>Theorem</u>. Let $\mathfrak{F}$ be a free κ-class.

(<u>a</u>) Suppose $A, B, G \in \mathfrak{F}^{<\kappa}$, $A \leq G$, $B \leq G$, and $\varphi \in ISO(A,B)$. Then there exists $V \in \mathfrak{F}$, with $G < V$, and some $t \in V$ such that G and t generate the HNN extension G_φ in V.

(b) Every $U \in \mathrm{INJ}_{\varkappa}(\mathfrak{F})$ is inner-homogeneous for $\mathfrak{F}^{<\varkappa}$, that is, if $\varphi \in \mathrm{ISO}(A,B)$ where A and B are $\mathfrak{F}^{<\varkappa}$ subgroups of U, then φ extends to an inner automorphism of U. Hence,

(c) $\mathrm{INJ}_{\varkappa}(\mathfrak{F}) = \mathrm{HU}_{\varkappa}(\mathfrak{F})$.

Proof of (a). Axiom (iv) of a free $\varkappa$-class permits the construction of the HNN extension G_{φ} in the classical way [15; p. 536].

Let G_1 be an $\cong$ copy of G. Choose $N \in \mathfrak{F}^{<\varkappa}$ with an element $x \in N$ of infinite order, and let N_1 be an $\cong$ copy of N. Note that G and $x^{-1}Ax$ generate $G*(x^{-1}Ax)$ in $G*N$. The classical way to define the HNN extension G_{φ} is to form an amalgam

1.4.26 $$\mathcal{A} = \begin{tikzcd}[column sep=small] h(G_1 * N_1) & & G*N \\ & G*(x^{-1}Bx) \arrow[ul, no head] \arrow[ur, no head] & \end{tikzcd}$$ with $h(g_1) = g$ for all

$g \in G$ and $h(x_1^{-1}a_1x_1) = x^{-1}\varphi(a)x$ for all $a \in A$.

The subgroup of $\mathrm{gp}_*(\mathcal{A})$ generated by G and $t = h(x_1)x^{-1}$ is the HNN extension G_{φ}.

Axiom (iv) guarantees that $\mathcal{A}$ is an $\mathfrak{F}^{<\varkappa}$ amalgam and also that $G_{\varphi} \leq \mathrm{gp}_*(\mathcal{A}) \in \mathfrak{F}$, proving part (a).

Parts (b) and (c) follow easily from (a).

Free $\varkappa$-classes can easily be constructed. We will discuss this for $\varkappa = \omega$, but the construction is easily extended to all $\varkappa$.

1.4.27 Definition. Let $\mathfrak{X}$ be a class of f.g. groups containing at most ω members up to $\cong$, and such that $1 \in \mathfrak{X}$ and some non-trivial group is in $\mathfrak{X}$. Put $\mathfrak{X}_0 = \mathfrak{X}$ and inductively define $G \in \mathfrak{X}_{n+1}$ iff $G \cong gp_*(\mathcal{A})$ for some $\mathfrak{X}_n$ amalgam $\mathcal{A}$ (possibly $\mathcal{A}$ is a single $\mathfrak{X}_n$ group). Define $\mathfrak{F}(\mathfrak{X})$ to be the inductive closure of $\bigcup_{n\geq 0} \mathfrak{X}_n$, that is, the class of all directed unions of groups belonging to $\bigcup_{n\geq 0} \mathfrak{X}_n$.

1.4.28 Proposition. Let $\mathfrak{X}$ be as in 1.4.27. Then (a) $\mathfrak{F}(\mathfrak{X})$ is a free ω-class and (b) Suppose K is an f.e.p. group and every $\mathfrak{X}$ group is embeddable in K. Then $F[\mathfrak{X}] \leq K$ where $F[\mathfrak{X}] \in HU_\omega^\omega(\mathfrak{F}(\mathfrak{X}))$.

Proof of (a). From the inductive definition we see that every $\bigcup_{n\geq 0} \mathfrak{X}_n$ group is f.g. Assume inductively that $\mathfrak{X}_n$ has at most ω members up to $\cong$. Then there are at most ω $\mathfrak{X}_n$ amalgams up to $\cong$, and it follows that $\mathfrak{X}_{n+1}$ has at most ω members. Hence $\bigcup_{n\geq 0} \mathfrak{X}_n$ has at most ω members, and this shows that $\mathfrak{F}(\mathfrak{X})$ satisfies axiom (v) for a free ω-class because

1.4.29 $$\mathfrak{F}(\mathfrak{X})^{<\omega} = \bigcup_{n\geq 0} \mathfrak{X}_n$$

due to the fact that no new f.g. groups are introduced by taking directed unions. Thus $\mathfrak{F}(\mathfrak{X})$ contains $gp_*(\mathcal{A})$ for every $\mathfrak{F}(\mathfrak{X})^{<\omega}$ amalgam $\mathcal{A}$, so that axiom (iv) holds. Axioms (i), (ii), and (iii) are immediate.

Proof of (b). If follows from 1.4.19 and an easy induction that, for all $n \geq 0$, every $\mathfrak{X}_n$ group is embeddable in K. Since $\mathcal{S}\restriction K$ is an ω-class (see 1.3.12 and 1.4.3) and $F[\mathfrak{X}]$ is a countable union of groups in $\bigcup_{n\geq 0} \mathfrak{X}_n$, $F[\mathfrak{X}]$ is embeddable in K.

1.4.30 Definition. Let G be a non-trivial f.g. group. $K[G]$ will denote the countable f.a.c. of G, and $F[G]$ will denote $F[\mathfrak{X}]$ where $\mathfrak{X} = \{G,1\}$ (see 1.4.27 and 1.4.28). Thus $F[G]$ is embeddable in $K[G]$ by 1.4.28(b).

For many choices of G, it is easy to check that $K[G] \not\cong F[G]$ because $K[G]$ has f.g. subgroups, by virtue of being algebraically closed, which $F[G]$ does not have. For example, if G is cyclic of prime order p, then every element of finite order in $F[G]$ is a p-element, as can easily be shown inductively. On the other hand, every a.c. group (and $K[G]$ in particular) has subgroups isomorphic to every f.g. group with solvable word problem [18], which includes all finite groups. If $G = \mathbb{Z}$ is infinite cyclic, then $F[\mathbb{Z}]$ is torsion-free. A result of Baumslag, Karrass, and Solitar [0] shows that there is a finitely presented torsion-free group H not embeddable in $F[\mathbb{Z}]$, even if we enlarge each class $\mathfrak{X}_n$ by adding all subgroups of $\mathfrak{X}_n$ groups in the definition of $\mathfrak{F}(\mathfrak{X})$. This H is embeddable in every f.e.p. group since it is finitely presented (see 1.4.17(b)).

CHAPTER II

DISCUSSION OF κ-INJECTIVE CHAINS

§2.1 Definitions, Existence, and Uniqueness of κ-Injective Chains

Before presenting the central concept of κ-injective chains of algebras we will develop terminology for chains of algebras.

<u>2.1.0</u> <u>Definition of an $\mathcal{M}$-Chain</u>. $\mathcal{M}$ denotes a class of algebras of type τ. Suppose $\mathcal{C}$ is any chain of τ algebras. We write $1 \in \mathcal{C}$ if $\mathcal{C}$ contains the subalgebra generated by the constants of $\bigcup\mathcal{C}$. We say that $\mathcal{C}$ is an <u>$\mathcal{M}$-chain</u> iff $\mathcal{C}$ is a complete chain of τ algebras (see 0.1.2), $1 \in \mathcal{C}$, and every non-trivial member of $\mathcal{C}$ is an $\mathcal{M}$ algebra. A trivial algebra is an algebra generated by its constants (see 0.2.2).

Suppose $\mathcal{C}$ is any complete chain. If (A,B) is a jump of $\mathcal{C}$ we also call $B \in \mathcal{C}$ a <u>jump</u> of $\mathcal{C}$ and write $B^- = A$. Thus, a jump of $\mathcal{C}$ is any member of $\mathcal{C}$ possessing an immediate predecessor in $\mathcal{C}$. The set of jumps of $\mathcal{C}$ will be denoted

<u>2.1.1</u> $j\mathcal{C} = \{J_\alpha \mid \alpha \in I\}$ where $(I,\leq)$ is a totally ordered index set and the indexing is one-to-one.

$^{\circ}\mathcal{C}$ is the order type of $j\mathcal{C}$ (see 0.1.2). We denote a complete chain by $(\mathcal{C}, I)$ to indicate that I indexes the jumps of $\mathcal{C}$ as in 2.1.1, and the notation $^{\circ}\mathcal{C} = I$ has the same meaning. Note that if $^{\circ}\mathcal{C} = \gamma$ is an ordinal, then $\mathcal{C}$ is a well ordered chain with $\gamma + 1$ members.

We will note a simple fact about trivial algebras. Let Q be the set of constants for τ algebras.

<u>2.1.2</u> If $\mathfrak{M}$ has the comparison property and $A, B \in \mathfrak{M}$, then there exists $\varphi \in \mathrm{ISO}(\mathrm{alg}_A(Q), \mathrm{alg}_B(Q))$.

<u>Proof</u>. Using the notation of 1.3.4 we have $A \leq \bar{A} \in \mathfrak{M}$ and $B \leq \bar{B} \in \mathfrak{M}$ where $\bar{A}$ and $\bar{B}$ contain $\cong$ subalgebras. Since every $\cong$ of τ algebras extend 1_Q, our conclusion follows.

In view of 2.1.2, if $\mathfrak{M}$ has the comparison property we will simply write $1 \in \mathfrak{M}$ to indicate that $\langle Q \rangle \in \mathfrak{M}$, that is, that $\mathfrak{M}$ has a (unique) trivial algebra.

<u>2.1.3</u> <u>Definition of the Induced Chain</u> $\mathcal{C}_E$. Suppose $(\mathcal{C}, I)$ is a complete chain and $E \leq \bigcup\mathcal{C}$. The chain $(\mathcal{C}_E, I_E)$ <u>induced on</u> E <u>by</u> $\mathcal{C}$ is defined by $\mathcal{C}_E = \{E \cap X \mid X \in \mathcal{C}\}$ and $I_E = \{\alpha \in I \mid E \cap (J_\alpha - J_\alpha^-) \neq \emptyset\}$ (see 2.1.1). Note that $\mathcal{C}_E$ is also a complete chain.

<u>2.1.4</u> <u>Definition of a $\varkappa$-Local Chain for</u> $\mathfrak{M}$. Suppose $(\mathcal{C}, I)$ is an $\mathfrak{M}$-chain. We say that $\mathcal{C}$ is a $\varkappa$-<u>local chain for</u> $\mathfrak{M}$ if $j\mathcal{C} \neq \emptyset$ and, for every subset $S \leq \bigcup\mathcal{C}$ with $|S| < \varkappa$, we have $S \leq M \lneq \bigcup\mathcal{C}$ where $\mathcal{C}_M$ is an $\mathfrak{M}^{<\varkappa}$-chain and

$|I_M| < \varkappa$. The <u>class of $\varkappa$-local chains for</u> $\mathcal{M}$ will be denoted by $CH_\varkappa(\mathcal{M})$, and we define $\mathcal{C} \in CH_\varkappa^{\leq\varkappa}(\mathcal{M})$ iff $\mathcal{C} \in CH_\varkappa(\mathcal{M})$ and $|\bigcup\mathcal{C}| \leq \varkappa$.

The distinction between $\varkappa$-local chains for $\mathcal{M}$ and $\mathcal{M}$-chains is important if $\varkappa = \omega$, but otherwise is usually unnecessary. In fact, we have

<u>2.1.5</u> <u>Proposition</u>. If $\varkappa > \omega$ is regular and $\mathcal{M}$ is $\varkappa$-local and $\varkappa$-inductive, then every $\mathcal{M}$-chain with at least one jump is $\varkappa$-local for $\mathcal{M}$.

The proof is similar to the last part of the proof of 1.3.12.

<u>2.1.6</u> <u>Definition of</u> $\mathcal{O}(\mathcal{C}_1,\mathcal{C}_2)$. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are complete chains, then $\mathcal{O}(\mathcal{C}_1,\mathcal{C}_2)$ is the set of strictly order-preserving maps from $j\mathcal{C}_1$ into $j\mathcal{C}_2$.

<u>2.1.7</u> <u>Definition of Embeddings of Chains</u>. Suppose $(\mathcal{C}_1,I_1)$ and $(\mathcal{C}_2,I_2)$ are complete chains of τ-algebras with $j\mathcal{C}_i = \{J^i_\alpha | \alpha \in I_i\}$, $\eta \in \mathcal{O}(\mathcal{C}_1,\mathcal{C}_2)$, and f is an embedding of $\bigcup\mathcal{C}_1$ into $\bigcup\mathcal{C}_2$. We say that f is an <u>η-embedding of $\mathcal{C}_1$ into</u> $\mathcal{C}_2$ iff, for all $\alpha \in I_1$, $f(J^1_\alpha - (J^1_\alpha)^-) \leq J^2_{\eta(\alpha)} - (J^2_{\eta(\alpha)})^-$, or equivalently, $f(J^1_\alpha) \leq J^2_{\eta(\alpha)}$ and $f(J^1_\alpha) \cap (J^2_{\eta(\alpha)})^- = f((J^1_\alpha)^-)$. ($\eta(\alpha) \in I_2$ is such that $\eta(J^1_\alpha) = J^2_{\eta(\alpha)}$). We say that f is an <u>embedding of $\mathcal{C}_1$ into $\mathcal{C}_2$</u> iff f is an η-embedding of $\mathcal{C}_1$ into $\mathcal{C}_2$ for some $\eta \in \mathcal{O}(\mathcal{C}_1,\mathcal{C}_2)$, and that f is an <u>isomorphism of $\mathcal{C}_1$ onto $\mathcal{C}_2$</u> iff f is an embedding of $\mathcal{C}_1$ into $\mathcal{C}_2$ and $f \in ISO(\bigcup\mathcal{C}_1,\bigcup\mathcal{C}_2)$.

2.1.8 Embeddings of Induced Chains. Using the notation above, suppose $\mathbf{E} \leq \bigcup \mathcal{C}_1$ is a subalgebra, $\eta \in \mathcal{O}((\mathcal{C}_1)_{\mathbf{E}}, \mathcal{C}_2)$, and $\bar{\eta} \in \mathcal{O}(\mathcal{C}_1, \mathcal{C}_2)$. We will say that $\bar{\eta}$ is an extension of (extends) η if, for every $\alpha \in (I_1)_{\mathbf{E}}$, $\bar{\eta}(J_1^{\alpha}) = \eta(J_1^{\alpha} \cap \mathbf{E})$. Notice that if $\mathbf{E} \in \mathcal{C}_1$, then '$\bar{\eta}$ extends η' has its usual functional meaning.

2.1.9 Definition of a $\varkappa$-Injective Chain for $\mathfrak{M}$. $\mathcal{J}$ is a $\varkappa$-injective chain for $\mathfrak{M}$ iff

(1) $\mathcal{J} \in \mathrm{CH}_{\varkappa}(\mathfrak{M})$ and

(2) Suppose $\mathcal{S}$ is an $\mathfrak{M}^{<\varkappa}$-chain with $|j\mathcal{S}| < \varkappa$, $\mathbf{E} \leq \bigcup \mathcal{S}$ is such that $\mathcal{S}_{\mathbf{E}}$ is an $\mathfrak{M}^{<\varkappa}$-chain, f is an η-embedding of $\mathcal{S}_{\mathbf{E}}$ into $\mathcal{J}$ for some $\eta \in \mathcal{O}(\mathcal{S}_{\mathbf{E}}, \mathcal{J})$, and $\bar{\eta} \in \mathcal{O}(\mathcal{S}, \mathcal{J})$ is an extension of η. Then, f has an extension to an $\bar{\eta}$-embedding, $\bar{f}$, of $\mathcal{S}$ into $\mathcal{J}$.

We denote the class of $\varkappa$-injective chains for $\mathfrak{M}$ by $\mathrm{INCH}_{\varkappa}(\mathfrak{M})$, and we define $\mathcal{J} \in \mathrm{INCH}_{\varkappa}^{\leq\varkappa}(\mathfrak{M})$ iff $\mathcal{J} \in \mathrm{INCH}_{\varkappa}(\mathfrak{M})$ and $\mathcal{J} \in \mathrm{CH}_{\varkappa}^{\leq\varkappa}(\mathfrak{M})$.

Notice that every initial segment of a $\varkappa$-injective chain for $\mathfrak{M}$ is $\varkappa$-injective for $\mathfrak{M}$, that is, if $\mathcal{J} \in \mathrm{INCH}_{\varkappa}(\mathfrak{M})$ and $1 \neq M \in \mathcal{J}$, then $\mathcal{J}_M \in \mathrm{INCH}_{\varkappa}(\mathfrak{M})$.

Our isomorphism theorem for $\varkappa$-injective chains is a direct analogue of the isomorphism lemmas 1.3.7(a) and 1.3.6(b) for $\varkappa$-injective algebras.

<u>2.1.10</u> <u>Isomorphism Theorem for</u> $\varkappa$-<u>Injective Chains</u>. Suppose $\mathcal{M}$ is $\varkappa$-local and $\varkappa$-inductive.

(<u>a</u>) Suppose $\mathcal{B}, \mathcal{J} \in \mathrm{INCH}_{\varkappa}^{\leq\varkappa}(\mathcal{M})$; $\mathbf{E} \leq \bigcup\mathcal{B}$ is such that $\mathcal{B}_{\mathbf{E}}$ is an $\mathcal{M}^{<\varkappa}$-chain; f is an η-embedding of $\mathcal{B}_{\mathbf{E}}$ into $\mathcal{J}$ for some $\eta \in \mathcal{O}(\mathcal{B},\mathcal{J})$; and $\bar{\eta} \in \mathcal{O}(\mathcal{B},\mathcal{J})$ is an order-isomorphism of $j\mathcal{B}$ onto $j\mathcal{J}$ which extends η. Then, f extends to an $\bar{\eta}$-isomorphism of $\mathcal{B}$ onto $\mathcal{J}$.

(<u>b</u>) Suppose $\mathcal{B} \in \mathrm{CH}_{\varkappa}^{\leq\varkappa}(\mathcal{M})$ and $\mathcal{J} \in \mathrm{INCH}_{\varkappa}(\mathcal{M})$; $\mathbf{E} \leq \bigcup\mathcal{B}$ is such that $\mathcal{B}_{\mathbf{E}}$ is an $\mathcal{M}^{<\varkappa}$-chain; f is an η-embedding of $\mathcal{B}_{\mathbf{E}}$ into $\mathcal{J}$ for some $\eta \in \mathcal{O}(\mathcal{B}_{\mathbf{E}},\mathcal{J})$; and $\bar{\eta} \in \mathcal{O}(\mathcal{B},\mathcal{J})$ is an extension of η. Then, f extends to an $\bar{\eta}$-embedding, $\bar{f}$, of $\mathcal{B}$ into $\mathcal{J}$.

This theorem has a corollary similar to 1.3.7(b).

<u>2.1.11</u> <u>Corollary</u>. Suppose $\mathcal{M}$ has the comparison property.

(<u>a</u>) Suppose $\mathcal{B}, \mathcal{J} \in \mathrm{INCH}_{\varkappa}^{\leq\varkappa}(\mathcal{M})$ and Ψ is an order-isomorphism of $j\mathcal{B}$ onto $j\mathcal{J}$. Then, there exists some Ψ-isomorphism of $\mathcal{B}$ onto $\mathcal{J}$.

(<u>b</u>) Suppose $\mathcal{B} \in \mathrm{CH}_{\varkappa}^{\leq\varkappa}(\mathcal{M})$, $\mathcal{J} \in \mathrm{INCH}_{\varkappa}(\mathcal{M})$, and $\Psi \in \mathcal{O}(\mathcal{B},\mathcal{J})$. Then, there exists a Ψ-embedding of $\mathcal{B}$ into $\mathcal{J}$.

This corollary is an easy consequence of the Isomorphism Theorem above and the definitions we have given. We can let $\mathbf{E}$ = the trivial subalgebra of $\bigcup\mathcal{B}$ and let f be the $\cong$ between $\mathbf{E}$ and the trivial subalgebra of $\bigcup\mathcal{J}$ which exists by 2.1.2. This f is an η-embedding of $\mathcal{B}_{\mathbf{E}}$ into $\mathcal{J}$ where $\eta = \emptyset$, and so f extends to a Ψ-embedding by the Isomorphism Theorem.

If $\mathcal{M}$ has the comparison and κ-local properties and $1 \neq M \in \mathrm{INJ}_\kappa(\mathcal{M})$, then the chain $\{1,M\}$ is κ-injective for $\mathcal{M}$. This is checked using the fact that M is κ-universal for $\mathcal{M}$ by 1.3.5 to obtain extensions of trivial maps.

If $\mathcal{M}$ has a κ-injective chain with <u>more than one jump</u>, we can show that $\mathcal{M}$ satisfies an amalgamation property stronger than κ-injectivity.

<u>2.1.12</u> <u>Definition of the Strong Amalgamation Property</u>. $\mathcal{M}$ has the κ-<u>strong amalgamation property</u> (κ-<u>s.a.p.</u>) iff, for every $\mathcal{M}^{<\kappa}$ amalgam $\mathcal{A}$, there exists $M \in \mathcal{M}$ with $\mathcal{A} \leq M$. $\mathcal{M}$ has the s.a.p. iff $\mathcal{M}$ has the κ-s.a.p. for all $\kappa \geq \omega$.

<u>2.1.13</u> <u>Proposition</u>. Suppose $\mathcal{M}$ has the comparison property. If there exists $\mathcal{J} \in \mathrm{INCH}_\kappa(\mathcal{M})$ with $|j\mathcal{J}| \geq 2$, then $\mathcal{M}$ has the κ-s.a.p..

<u>Proof</u>. Let $J_1, J_2 \in j\mathcal{J}$ with $J_1 < J_2$. Note that $J_1 \leq J_2^-$. Let $\mathcal{A} = \begin{array}{ccc} B & & D \\ & \diagdown\ \diagup & \\ & A & \end{array}$ be any $\mathcal{M}^{<\kappa}$-amalgam. The trivial map (an $\emptyset$-embedding) extends to an embedding $f\colon D \to J_1$ such that $f(D-1) \leq J_1 - J_1^-$. We can assume $A < B$. If $A = 1$, there is an embedding $g\colon B \to J_2$ such that $g(B-1) \leq J_2 - J_2^-$ and hence the amalgam $\begin{array}{ccc} g(B) & & f(D) \\ & \diagdown\ \diagup & \\ & 1 & \end{array}$ copies $\mathcal{A}$ in $J_2 \in \mathcal{M}$. If $A \neq 1$, then we have an $\mathcal{M}^{<\kappa}$-chain $\mathcal{B} = \{1,A,B\}$ with jumps A and B. $f\restriction A$ is an η-embedding of $\mathcal{B}_A$ into $\mathcal{J}$ with $\eta(A) = J_1$ and so, letting $\bar{\eta}(A) = J_1$ and $\bar{\eta}(B) = J_2$,

there is an $\bar{\eta}$-embedding $\bar{f}$ of $\mathcal{B}$ into $\mathcal{J}$. We have $\bar{f}(B-A) \leq J_2 - J_2^-$ and so the amalgam

$$\begin{tikzcd}[column sep=small] f(B) & & f(D) \\ & \bar{f}(A) = f(A) \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

copies $\mathcal{A}$ in J_2.

To construct κ-injective chains for $\mathfrak{M}$, we need to make an assumption much stronger than the s.a.p..

<u>2.1.14</u> <u>Definition of the Subamalgam Property</u>. Suppose

$$\mathcal{A} = \begin{tikzcd}[column sep=small] B & & C \\ & A \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

is an amalgam.

$$\mathcal{A}_0 = \begin{tikzcd}[column sep=small] B_0 & & C_0 \\ & A_0 \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

is a subamalgam of $\mathcal{A}$ iff $B_0 \leq B$, $C_0 \leq C$, and $B_0 \cap A = C_0 \cap A = A_0$.

$\mathfrak{M}$ has the <u>subamalgam property (s.p.)</u> if for every $\mathfrak{M}$ amalgam $\mathcal{A}$ as above, there exists $M = \mathrm{alg}_M(\mathcal{A}) \in \mathfrak{M}$ such that, for every $\mathfrak{M}$ subamalgam $\mathcal{A}_0 \leq \mathcal{A}$ as above, we have $\mathrm{alg}_M(\mathcal{A}_0) \cap B = B_0$, $\mathrm{alg}_M(\mathcal{A}_0) \cap C = C_0$, and $\mathrm{alg}_M(\mathcal{A}_0) \in \mathfrak{M}$.

$\mathfrak{M}$ has the <u>subamalgam property with the descendance condition (s.p.d.)</u> iff $\mathfrak{M}$ satisfies the definition of the s.p. above with the extra condition that, for every chain $\{\mathcal{A}_\alpha \mid \alpha \in S\}$ of $\mathfrak{M}$ subamalgams

$$\mathcal{A}_\alpha = \begin{tikzcd}[column sep=small] B_\alpha & & C_\alpha \\ & A_\alpha \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

of $\mathcal{A}$, we have $\bigcap_{\alpha \in S} \mathrm{alg}_M(\mathcal{A}_\alpha) = \mathrm{alg}_M(\bigcap_{\alpha \in S} B_\alpha, \bigcap_{\alpha \in S} C_\alpha)$.

<u>2.1.15</u> <u>Existence Theorem for κ-Injective Chains, Part I</u>. Suppose $\mathfrak{M}$ is a κ-class, $1 \in \mathfrak{M}^{<\kappa}$, and $\mathfrak{M}^{<\kappa}$ satisfies the s.p.. Then

(a) (G.C.H.) If $1 \leq \gamma < \varkappa^+$ is an ordinal, then there exists $\mathcal{J} \in \mathrm{INCH}_{\varkappa}^{\leq\varkappa}(\mathcal{M})$ with $°\mathcal{J} = \gamma$

(b) If $\varkappa = \omega$, and I is an ordered set with $1 \leq |I| \leq \omega$, then there exists $\mathcal{J} \in \mathrm{INCH}_{\omega}^{\leq\omega}(\mathcal{M})$ with $°\mathcal{J} = I$.

2.1.16 Existence Theorem, Part II. (G.C.H.) Suppose $\varkappa > \omega$ is regular, $\mathcal{M}$ is a $\varkappa$-class, $1 \in \mathcal{M}^{<\varkappa}$, and $\mathcal{M}^{<\varkappa}$ satisfies the s.p.d. If I is any ordered set with $1 \leq |I| \leq \varkappa$, then there exists $\mathcal{J} \in \mathrm{INCH}_{\varkappa}^{\leq\varkappa}(\mathcal{M})$ with $°\mathcal{J} = I$.

2.1.17 Jonsson's Theorem for $\varkappa$-Injective Chains. (G.C.H.) Suppose $\varkappa \geq \omega$ is regular, $\mathcal{M}$ is a $\varkappa$-class, $\mathcal{M}^{<\varkappa}$ has the s.p., and, if $\varkappa > \omega$, $\mathcal{M}^{<\varkappa}$ has the s.p.d.. Then, there exists $\mathcal{J} \in \mathrm{INCH}_{\varkappa}^{\leq\varkappa}(\mathcal{M})$, which is unique up to isomorphism of chains, such that if $\mathcal{C} \in \mathrm{CH}_{\varkappa}^{\leq\varkappa}(\mathcal{M})$, $\mathbf{E} \leq \bigcup\mathcal{C}$ is such that $\mathcal{C}_{\mathbf{E}}$ is an $\mathcal{M}^{<\varkappa}$-chain, and f is an embedding of $\mathcal{C}_{\mathbf{E}}$ into $\mathcal{J}$, then f extends to an embedding of $\mathcal{C}$ into $\mathcal{J}$.

Proof. By Jonsson's Theorem (1.3.9) there is a $\varkappa$-homogeneous universal order type $\nu_{\varkappa}$ of cardinal $\varkappa$ [1: p. 203]. Let $\mathcal{J} \in \mathrm{INCH}_{\varkappa}^{\leq\varkappa}(\mathcal{M})$ be such that $°\mathcal{J} = \nu_{\varkappa}$ by 2.1.16 or 2.1.15(b) if $\varkappa = \omega$. Now by 2.1.10(b) and $\varkappa$-injectivity of the order type $\nu_{\varkappa}$, $\mathcal{J}$ has the embedding property of the theorem.

To show that $\mathcal{J}$ is unique up to $\cong$ of chains, let $\mathcal{J} \in \mathrm{INCH}_{\varkappa}^{\leq\varkappa}(\mathcal{M})$ be another chain with the property of the theorem. If we show that $°\mathcal{J} = \nu_{\varkappa}$, then 2.1.11(a) will give the result. Put $j\mathcal{J} = \{J_\alpha \mid \alpha \in I\}$ and let $T < I$ be a subset with $|T| < \kappa$. We regard T as an ordered subset of I. By 2.1.16 there exists a κ-injective chain $\mathcal{C}$ for $\mathfrak{M}$ with $^\circ\mathcal{C} = T$, and since $\mathcal{C}$ is κ-local for $\mathfrak{M}$, there exists an $\mathfrak{M}^{<\kappa}$-chain $\mathcal{B}$ with $^\circ\mathcal{B} = T$. Let $\eta \in \mathcal{O}(\mathcal{B},\mathcal{J})$ and by 2.1.11(b), let f be an η-embedding of $\mathcal{B}$ into $\mathcal{J}$ and let g be any embedding of $\mathcal{B}$ into $\mathcal{g}$. Now, by our hypothesis concerning $\mathcal{J}$, fg^{-1} extends to an $\bar{\eta}$-embedding of $\mathcal{g}$ into $\mathcal{J}$ for some $\bar{\eta} \in \mathcal{O}(\mathcal{g},\mathcal{J})$ such that $\eta(j\mathcal{B}) < \bar{\eta}(j\mathcal{g})$. This shows that T is contained in a subset of I of order type $^\circ\mathcal{g} = \nu_\kappa$. It follows that the order type $^\circ\mathcal{J}$ is κ-injective, and so, by 1.3.7(b), $^\circ\mathcal{J} = \nu_\kappa$.

In order that κ-injective chains be an adequate generalization of κ-injective algebras, it should be the case that "most" members of κ-injective chains are κ-injective algebras. This is not immediate from our definitions, but we can prove it making use of the κ-s.a.p. for $\mathfrak{M}$ which holds by 2.1.13.

<u>2.1.18</u> <u>Lemma</u>. Suppose $\mathcal{J} \in \mathrm{INCH}_\kappa(\mathfrak{M})$, $J \in j\mathcal{J}$, $A < B$ are $\mathfrak{M}^{<\kappa}$ algebras, and f is an embedding of A into J. Then f extends to an embedding g of B into J such that

<u>2.1.19</u> $g(B) \cap J^- = f(A) \cap J^-$.

<u>2.1.20</u> <u>Corollary</u>. Suppose $\mathfrak{M}$ has the comparison property, $\mathcal{J} \in \mathrm{INCH}_\kappa(\mathfrak{M})$, and $M \in \mathcal{J}$. If either M is a jump of $\mathcal{J}$ or M does not equal the union of any subchain of $\mathcal{J}_M - \{M\}$ of power $< \kappa$, then $M \in \mathrm{INJ}_\kappa(\mathfrak{M})$.

<u>Proof of the Corollary</u>. Suppose $A < B$ are $\mathcal{M}^{<\kappa}$ algebras and f is an embedding of A into M. Under either asumption on M, there exists a jump J of $\mathcal{J}$ such that $f(A) \leq J \leq M$. Now we can use 2.1.18 to extend f to an embedding of B into J.

<u>Proof of 2.1.18</u>. If $\mathcal{J} = \{1, J\}$, then a direct application of κ-injectivity of $\mathcal{J}$ shows that f extends to an embedding of B into J (see the remarks following 2.1.11), and 2.1.19 is trivial since $J^- = 1$.

So we can assume $|j\mathcal{J}| \geq 2$, and hence $\mathcal{M}$ has the κ-s.a.p. by 2.1.13. Since $\mathcal{J}$ is a κ-local chain for $\mathcal{M}$ and $f(A) \in \mathcal{M}^{<\kappa}$ there exists $N \leq \bigcup\mathcal{J}$ such that $f(A) \leq N$ and $\mathcal{J}_N$ is an $\mathcal{M}^{<\kappa}$-chain. Since $N \cap J \in \mathcal{M}^{<\kappa}$ we will assume $N \leq J$, and we will enlarge N, if need be, to insure that $N \not\leq J^-$, so $N \in j\mathcal{J}_N$ and $N^- = N \cap J$. Now form an amalgam

$$\mathcal{A} = \begin{array}{ccc} h(B) & & N \\ & \diagdown \ \diagup & \\ & f(A) & \end{array} \quad \text{where } h \equiv f \text{ on } A \text{ and let } \mathcal{A} \leq M \in \mathcal{M}^{<\kappa}$$

by the κ-s.a.p.. Let $\bar{\mathcal{J}}_N$ be the chain obtained by replacing N by M in $\mathcal{J}_N$. Since N is the last jump in $\mathcal{J}_N$, $\bar{\mathcal{J}}_N$ is an $\mathcal{M}^{<\kappa}$-chain with $^\circ\bar{\mathcal{J}}_N = {}^\circ\mathcal{J}_N$. Since 1_N is an η-embedding of $\mathcal{J}_N$ into $\mathcal{J}$ where η is the induced map, by κ-injectivity of $\mathcal{J}$, 1_N has an extension $\bar{f}$ such that $\bar{f}$ is an $\bar{\eta}$-embedding of $\bar{\mathcal{J}}_N$ into $\mathcal{J}$ where $\bar{\eta}$ extends η (see 2.1.8). Thus $\bar{f}$ embeds M into J and, since $\bar{\eta}(M) = J$ and $M^- = N^- = N \cap J^-$, we have $f(M) \leq J$ and $f(M - N \cap J^-) \leq J - J^-$ (see 2.1.7). Considering the amalgam $\mathcal{A}$, we have

$\bar{f}(h(B) - f(A) \cap J^-) \leq J - J^-$, that is, $\bar{f}h(B) \cap J^- = f(A) \cap J^-$ since $\bar{f} \equiv 1$ on $f(A)$. Hence $g = \bar{f}h\colon B \to J$ extends f and satisfies 2.1.19.

In order to apply our Existence Theorems and Jonsson's Theorem for $\varkappa$-injective chains we will note that many classes of interest have the subamalgam property.

<u>2.1.21</u> <u>Proposition</u>. If $\mathfrak{M}$ is a free $\varkappa$-class, then $\mathfrak{M}^{<\varkappa}$ satisfies the s.p.d..

The proofs of all the conditions of 2.1.14 follow easily from the Normal Form Theorem for $M = gp_*(\mathcal{A})$ (1.2.2) and axiom (iv) of a free $\varkappa$-class (1.4.20).

<u>2.1.22</u> <u>Theorem</u>. (G.C.H.) Let $\mathcal{G}$ be the class of all groups and let $\varkappa > \omega$ be regular. Then, there exist $\varkappa$-injective chains for $\mathcal{G}$ of arbitrary order types of power $\leq \varkappa$. Every jump in one of these chains is isomorphic to $H_\varkappa$ (see 1.1.6).

This is immediate from 2.1.21 with $\mathfrak{M} = \mathcal{G}$, our 2nd Existence Theorem 2.1.16, and 2.1.20.

Also note the following consequence of the Isomorphism Theorem 2.1.10: If $\mathcal{J} \in \mathrm{INCH}_\varkappa^{\leq\varkappa}(\mathfrak{M})$, then every order-automorphism of $j\mathcal{J}$ is induced by some automorphism of $\bigcup\mathcal{J}$. Together with 2.1.22 and 2.1.20, this provides an easy way to see that $|\mathrm{Aut}\ \aleph_\varkappa| = 2^\varkappa$.

§2.2 An Application of ω-Injective Chains of Groups

We will apply the construction of κ-injective chains to classes of groups $\mathfrak{M}$ of the following types.

<u>2.2.0</u> (I) $\mathfrak{M}$ is a free κ-class, where $\kappa \geq \omega$ is regular, and the infinite cyclic group $\mathbb{Z} \in \mathfrak{M}$.

(II) $\mathfrak{M}$ is the smallest inductive class containing the subgroups of a fixed countable algebraically closed group G such that the class of f.g. subgroups of G satisfies the subamalgam property. Note that such a class $\mathfrak{M}$ is an ω-class, and $M \in \mathfrak{M}$ iff every f.g. subgroup of M is embeddable in G.

(III) $\mathfrak{M}$ is the class of locally finite groups (an ω-class).

Note that the free ω-class generated by the subgroups of a countable f.e.p. group (see 1.4.21) is a special case of both (I) and (II) by 1.4.19 and 2.1.21.

Our application will be the construction of 2^{ω_1} non-isomorphic $HU_{\omega}^{\omega_1}(\mathfrak{M})$ groups in each of these three cases, with $\kappa = \omega$ in case (I).

We will first discuss the hypotheses (I) and (II), and then develop the properties of ω-injective chains needed for the application.

If $\mathcal{M}$ is a free $\varkappa$-class, the condition that $\mathbf{Z} \in \mathcal{M}$, as in case (I) above, has considerable strength. It allows free amalgamations over $\mathbf{Z}$, and this permits the full use of the HNN and free amalgamation constructions. In particular, the normal basis theorem and the maximal subgroup-tree theorems of §1.2 can be proved for the unique group $U_\varkappa \in HU_\varkappa^\varkappa(\mathcal{M})$ where $\mathcal{M}$ is a free $\varkappa$-class and $\mathbf{Z} \in \mathcal{M}$. We will not give the details of this, which are mostly obvious modifications in the proofs of §1.2, making essential use of 1.4.25(a) with $A = B = \mathbf{Z}$ for the subgroup-tree theorem. Let us observe that the above groups $U_\varkappa$ are all simple. This follows from the following easy facts: (i) Every non-trivial normal subgroup of $U_\varkappa$ has an element of infinite order; (ii) Every pair of elements of infinite order in $U_\varkappa$ are conjugate in $U_\varkappa$; and (iii) $U_\varkappa$ is generated by elements of infinite order. The axioms of a free $\varkappa$-class imply that $U_\varkappa$ is the union of groups which are non-trivial free products, and the facts (i) and (iii) are easily deduced from this. For (ii), we must use 1.4.25(a) and the axiom $\mathbf{Z} \in \mathcal{M}$, as already noted.

These simple observations permit us to test the strength of the axiom $\mathbf{Z} \in \mathcal{M}$ because there is a free $\varkappa$-class $\mathcal{F}$ such that $F \in HU_\varkappa^\varkappa(\mathcal{F})$ is not simple. Using the notation of 1.4.30, let $F = F[\mathbf{Z}_2]$ where $|\mathbf{Z}_2| = 2$. Recall that $F \in HU_\omega^\omega(\mathcal{F})$ where $\mathcal{F}$ is the smallest free ω-class containing $\mathbf{Z}_2$ which is closed under free amalgamations of its f.g. members. We can show, by induction on the classes $\mathfrak{X}_n$,

$n \geq 0$, (see 1.4.27), that, for every $G \in \mathfrak{F}^{<\omega} = \bigcup_{n=0}^{\infty} \mathfrak{X}_n$, there is a unique homomorphism $\varphi: G \to Z_2$ such that ($\dagger$) $\varphi(x) \neq 1$ for all $x \in G$ with $|x| = 2$. Clearly such a homomorphism is unique because every $\mathfrak{X}_n$ group, $n \geq 0$, and so every $\mathfrak{F}$ group, is generated by elements of order 2. Suppose there is a unique φ satisfying ($\dagger$) for every $G \in \mathfrak{X}_n$. Now let $G = gp_*(\mathcal{a}) \in \mathfrak{X}_{n+1}$ where

$$\mathcal{a} = \begin{array}{ccc} B & & C \\ & \diagdown \diagup & \\ & A & \end{array}$$

is a $\mathfrak{X}_n$ amalgam. There are unique homomorphisms $\varphi_1: B \to \mathbb{Z}_2$ and $\varphi_2: C \to \mathbb{Z}_2$ satisfying ($\dagger$), and φ_1 and φ_2 have unique and hence equal restrictions to A. Hence, by the universal mapping property of $gp_*(\mathcal{a})$, φ_1 and φ_2 have a unique extension to a homomorphism $\varphi: G \to \mathbb{Z}_2$, and φ satisfies ($\dagger$) because every element of order 2 in G is conjugate to an element of B or C. It follows that there is a unique homomorphism from $F = F[\mathbb{Z}_2]$ onto $\mathbb{Z}_2$ (the uniqueness follows from the fact that any two elements of order 2 in F are conjugate in F by 1.4.25(a)). Clearly the same result holds if we close $\mathfrak{F}$ under free amalgamations of all $\mathfrak{F}$ amalgams to obtain a free $\varkappa$-class for all $\varkappa$.

The situation is more complicated if we consider $F = F[\mathbb{Z}_p]$ where $|\mathbb{Z}_p| = p$ is an odd prime because the only abelian image of F is trivial. This is because F is generated by elements of order p, any two of which are conjugate in F, and so every element of order p in F

is conjugate to its square, and hence is trivial in every abelian image of F.

The class of algebraically closed groups described in 2.2.0(II), whose f.g. subgroups have the subamalgam property, is a rather unnatural restriction on an a.c. group, although it is surely a very large class. This restriction is necessary to secure only the existence of ω-injective chains - the actual properties of these chains will follow from results provable for an arbitrary countable a.c. group, such as the construction of maximal subgroups. I don't know of any type of extension with the uniqueness property of ω-injective chains which can be defined for the ω-class of subgroups of an arbitrary countable a.c. group. There are ways to define chains of a.c. groups with properties similar to ω-injective chains by relativizing the injection property to finite sets, but there are complications in this which would require careful discussion beyond the ken of this paper, and since an isomorphism theorem would be lacking, the present application of ω-injective chains to the spectrum problem would not be possible.

We will now describe the special properties of κ-injective chains which permit the construction of non-isomorphic homogeneous universal groups and give most of the proof.

The single most important property of κ-injective chains in the three cases of 2.2.0 is the following.

<u>2.2.1</u> <u>Maximality Theorem</u>. Suppose $\mathcal{M}$ is a κ-class of one of the three types of 2.2.0, $\mathcal{J} \in \mathrm{INCH}_{\kappa}(\mathcal{M})$, and H is a subgroup of $\bigcup\mathcal{J}$ with $J \leq H$ for some $J \in j\mathcal{J}$. Then $H \in \mathcal{J}$.

Theorem 2.2.1 asserts that κ-injective chains $\mathcal{J}$ of the types considered have a strong maximality property - not only is J^- maximal in J for every jump $J \in j\mathcal{J}$ with $1 \neq J^-$, but the only subgroups of $\bigcup\mathcal{J}$ containing J^- are the members of $\mathcal{J}$. This result requires different arguments, but with similar themes, in each of the three cases of 2.2.0.

We first give the lemma which applies if $\mathcal{M}$ is a free κ-class.

<u>2.2.2</u> <u>Lemma</u>. Suppose $\mathcal{M}$ is a free κ-class with $\mathbf{Z} \in \mathcal{M}$, $\mathcal{J} \in \mathrm{INCH}_{\kappa}(\mathcal{M})$, and $J < J_1$ are jumps of $\mathcal{J}$. Put $X = \{x \in J_1 - J_1^- \mid |x| = \omega \text{ and } \langle x\rangle \cap J_1^- = 1\}$ and, for all $x \in X$, put $X(x) = \{y \in X \mid \langle x\rangle * \langle y\rangle$ exists in J_1 and $\langle x,y\rangle \cap J_1^- = 1\}$. Then,

(<u>a</u>) For all $x \in X$, $J_1 = \langle X(x)\rangle$.

(<u>b</u>) For all $x,y \in X$ there exists $t \in J$ such that $t^{-1}xt = y$.

(<u>c</u>) For all $a \in J_1 - J_1^-$, we have $\langle a,J\rangle \cap X \neq \emptyset$.

<u>2.2.3</u> <u>Corollary</u>. The Maximality Theorem holds if $\mathcal{M}$ is a free κ-class with $\mathbf{Z} \in \mathcal{M}$.

Proof of the Corollary. (Refer to 2.2.1) For every jump J_1 of $\mathcal{J}$ with $J < J_1$ and $H \cap (J_1 - J_1^-) \neq \emptyset$, the three parts of 2.2.2 imply $J_1 \leq H$ since $J \leq H$. Hence H is the union of members of $\mathcal{J}$ and so $H \in \mathcal{J}$.

Proof of 2.2.2(a). Suppose $x \in X$. Let $u \in J_1$ be any element of infinite order. Since $\mathcal{M}$ is a free κ-class we can use axiom (iv) of 1.4.20 to find $G \in \mathcal{M}^{<\kappa}$ such that $u \in G$ and there exist $c,d \in G$ such that $|c| = |d| = \omega$, $u = cd$, and c and d generate $\langle c \rangle * \langle d \rangle$ in G. Let $A \in \mathcal{M}^{<\kappa}$ be a subgroup of J_1 such that $\langle u,x \rangle \leq A$. Since $\mathbb{Z} \in \mathcal{M}$, we can use 1.4.20(iv) to obtain

$gp_*(\mathcal{A}) \in \mathcal{M}^{<\kappa}$ where $\mathcal{A} = \begin{array}{ccc} G & & A \\ & \diagdown\ \diagup & \\ & \langle u \rangle & \end{array}$. Note that in $gp_*(\mathcal{A})$ we have

2.2.4 $\langle c,x \rangle \cap A = \langle d,x \rangle \cap A = \langle x \rangle$

by the Normal Form Theorem 1.2.2. By Lemma 2.1.18, there is an embedding $g: gp_*(\mathcal{A}) \to J_1$ with $g \equiv 1$ on A and $g(B) \cap J_1^- = A \cap J_1^-$. It follows from 2.2.4 that $\langle g(c),x \rangle \cap J_1^- \leq \langle g(c),x \rangle \cap A \cap J_1^- \leq \langle x \rangle \cap J_1^- = 1$ since $x \in X$. Since c and x generate $\langle c \rangle * \langle x \rangle$ in $gp_*(\mathcal{A})$, this proves that $g(c) \in X(x)$. Similarly, we have $g(d) \in X(x)$. This implies $u = g(c)g(d) \in \langle X(x) \rangle$. Since $J_1 \in INJ_\kappa(\mathcal{M})$ by 2.1.20, J_1 is generated by its elements u of infinite order. This proves that $J_1 = \langle X(x) \rangle$.

<u>Proof of 2.2.2(b)</u>. Let $x, y \in X$ and put $H = \langle x,y \rangle$. Since $\mathcal{J}$ is a κ-local chain for $\mathfrak{M}$, there is a subgroup $G < J_1$ such that $H \leq G$, $G \cap (J - J^-) \neq \emptyset$, and $\mathcal{J}_G$ is an $\mathfrak{M}^{<\kappa}$-chain with $|j\mathcal{J}_G| < \kappa$. Let $\varphi \in \mathrm{ISO}(\langle x \rangle, \langle y \rangle)$ be such that $\varphi(x) = y$. By 1.4.25(a), the HNN extension $G_\varphi = \langle G, t \rangle \leq M$ for some $M \in \mathfrak{M}^{<\kappa}$. Note that $\mathcal{J}_G$ has a maximum jump G and $G^- = G \cap J_1^-$. Britton's Lemma (1.2.3) implies that $\langle t \rangle$ and G^- generate $\langle t \rangle * G^-$ in G_φ since $\langle x \rangle \cap J_1^- = \langle y \rangle \cap J_1^- = 1$. Put $j\mathcal{J}_G = \{G_\alpha | \alpha \in I_G\}$ where $j\mathcal{J} = \{J_\alpha | \alpha \in I\}$ and $G_\alpha = G \cap J_\alpha$ for all $\alpha \in I_G$ as in 2.1.3. For all $\alpha \in I_G$, if $J \leq J_\alpha < J_1$, define $M_\alpha = \langle t, G_\alpha \rangle \cong \langle t \rangle * G_\alpha \in \mathfrak{M}^{<\kappa}$ since $\mathbb{Z} \in \mathfrak{M}$; and, if $J_\alpha < J$, define $M_\alpha = G_\alpha$. Define $\mathcal{C}$ to be the chain whose jumps are $j\mathcal{C} = \{M_\alpha | G_\alpha \neq G\} \cup \{M\}$. Thus, M is the maximum jump of $\mathcal{C}$ and $M^- = \bigcup\{M_\alpha | G_\alpha \neq G\} = \langle t, G^- \rangle$. Note that $\mathcal{J}_G = \mathcal{C}_G$ since, for all $\alpha \in I_G$ for which $G_\alpha \neq G$, we have $\langle t, G_\alpha \rangle \cap G = G_\alpha$ again using Britton's Lemma. Let η map the jumps of $\mathcal{J}_G$ to the jumps of $\mathcal{J}$ which induce them, that is, for all $\alpha \in I_G$, $\eta(G_\alpha) = J_\alpha$ where $G_\alpha = G \cap J_\alpha$. Let $\bar{\eta}$ be the extension of η to $j\mathcal{C}$, so that $\bar{\eta}(M_\alpha) = \eta(G_\alpha)$ for all $\alpha \in I_G$ with $G_\alpha \neq G$, and $\bar{\eta}(M) = J_1$. Since 1_G is an η-embedding and $\mathcal{J}$ is a κ-injective chain for $\mathfrak{M}$, 1_G extends to an $\bar{\eta}$-embedding $f\colon M \to \bigcup\mathcal{J}$. Since $G \cap (J - J^-) \neq \emptyset$, there is some $\alpha \in I_G$ such that $\bar{\eta}(M_\alpha) = J$; hence $f(M_\alpha) \leq J$ and since $t \in M_\alpha$, we have $f(t) \in J$ and $f(t)^{-1} x f(t) = y$ since $f \equiv 1$ on G. This proves part (b).

<u>Proof of 2.2.2(c)</u>. Let $a \in J_1 - J_1^-$. Since $\mathcal{J}$ is a κ-local chain for $\mathfrak{M}$ there is a subgroup $G < J_1$ such that $a \in G$, $G \cap (J - J^-) \neq \emptyset$, and $\mathcal{J}_G$ is an $\mathfrak{M}^{<\kappa}$-chain with $|j\mathcal{J}_G| < \kappa$. Let $M = \langle t \rangle * G$ where $|t| = \omega$. Thus $M \in \mathfrak{M}$. We define $\mathcal{C}, \eta$, and $\bar{\eta}$ in a manner identical to the proof of part (b), and obtain an $\bar{\eta}$-embedding $f : M \to \bigcup\mathcal{J}$. Since $\bar{\eta}(M) = J_1$, we have $f(M) < J_1$ and $f(M) \cap J_1^- = f(M^-) = f(\langle t, G \cap J_1^- \rangle) = \langle f(t), G \cap J_1^- \rangle$ since $f \equiv 1$ on G. Since $a \in J_1 - J_1^-$ and $\langle t, G \rangle = \langle t \rangle * G$, we have $\langle at \rangle \cap \langle t, G \cap J_1^- \rangle = 1$, and, applying f, we have $\langle af(t) \rangle \cap J_1^- = \langle af(t) \rangle \cap (f(M) \cap J_1^-) = \langle af(t) \rangle \cap \langle f(t), G \cap J_1^- \rangle = 1$. Hence $af(t) \in X$. Since $f(t) \in J$ (as in part (b)), we have $af(t) \in \langle a, J \rangle$ and so $\langle a, J \rangle \cap X \neq \emptyset$, proving (c).

We next give the lemma pertinent to the proof of 2.2.1 in the case 2.2.0(II).

<u>2.2.5</u> <u>Lemma</u>. Suppose G is a countable a.c. group and $\mathfrak{M}$ is the inductive closure of the class of subgroups of G, so that $M \in \mathfrak{M}$ iff every f.g. subgroup of M is embeddable in G. Suppose that $\mathcal{J} \in \mathrm{INCH}_\omega(\mathfrak{M})$ and $J < J_1$ are jumps of $\mathcal{J}$. Put $X_2 = \{x \in J_1 - J_1^- \mid |x| = 2\}$. Then, the following conditions are satisfied.

(<u>a</u>) $J_1 = \langle X_2 \rangle$,

(<u>b</u>) For all $x, y \in X_2$, there exists $t \in J$ such that $t^{-1}xt = y$, and

(<u>c</u>) For all $a \in J_1 - J_1^-$, $\langle a, J \rangle \cap X_2 \neq \emptyset$.

<u>2.2.6</u> <u>Corollary</u>. The Maximality Theorem holds if $\mathfrak{M}$ is the ω-class generated by the subgroups of a countable a.c. group whose f.g. subgroups satisfy the s.p..

The proof of this corollary is similar to 2.2.3.. Lemma 2.2.5 will be proved in §3.3.

The next lemma proves the Maximality Theorem in the third case of 2.2.0.

<u>2.2.7</u> <u>Lemma</u>. Let $\mathfrak{M}$ be the class of locally finite groups. Suppose $\mathcal{J} \in \mathrm{INCH}_{\omega}(\mathfrak{M})$ and $J < J_1$ are jumps of $\mathcal{J}$. For every natural number $n \geq 2$, let $X_n = \{x \in J_1 - J_1^- \mid |x| = n$ and $\langle x \rangle \cap J_1^- = 1\}$ and, for all $x \in X_n$, let $X_n(x) = \{y \in X_n \mid \langle y,x \rangle \cap J_1^- = 1\}$. Then, for all $n \geq 2$,

(<u>a</u>) For all $x \in X_n$, $J_1 = \langle X_n(x) \rangle$,

(<u>b</u>) For all $x \in X_n$ and $y \in X_n(x)$, there exists $t \in J$ such that $t^{-1}xt = y$, and

(<u>c</u>) For all $a \in J_1 - J_1^-$, $\langle a,J \rangle \cap X_n \neq \emptyset$.

<u>2.2.8</u> <u>Corollary</u>. The Maximality Theorem holds if $\mathfrak{M}$ is the ω-class of locally finite groups.

The proof of this corollary is similar to 2.2.3.

We will note here that the class of finite groups has the s.p.. This can be proved using B.H. Neumann's permutational product construction [16]. We will give the details of this in §3.2. Hence, our 1st Existence Theorem (2.1.15(b)), together with 2.1.20, imply

<u>2.2.9</u> <u>Proposition</u>. For every countable ordered set I, there exists $\mathcal{J} \in \mathrm{INCH}_{\omega}^{\leq\omega}(\ell.\mathrm{f.\ groups})$ with $''\mathcal{J} = I$. Every non-trivial member of $\mathcal{J}$ is isomorphic to the (unique) countable ULF group $H_{\omega} \in \mathrm{INJ}_{\omega}(\ell.\mathrm{f.\ groups})$ (see 1.1.6(ii)).

The proofs of parts (a) and (c) of 2.2.7 will be deferred until §3.2 because they involve special amalgamations of finite groups.

<u>Proof of 2.2.7(b)</u>. Let $x \in X_n$ and $y \in X_n(x)$. Since $|x| = |y| = n$, there is a finite group $G = \langle x,y,t \rangle$ such that $t^{-1}xt = y$ (see the proof of 1.1.9) and we can assume WLOG that $\langle t \rangle \cap \langle x,y \rangle = 1$, otherwise we can form a holomorph to accomplish this. Let $\mathcal{S} = \{1,\langle t \rangle,G\}$, $E = \langle x,y \rangle$, and $\eta(E) = J_1$. Thus, 1_E is an η-embedding of $\mathcal{S}_E = \{1,E\}$ into $\mathcal{J}$ since $E < J_1$ and $E \cap J_1^- = 1$. Define $\bar{\eta}(\langle t \rangle) = J$ and $\bar{\eta}(G) = J_1$. By κ-injectivity of $\mathcal{J}$, 1_E extends to an $\bar{\eta}$-embedding f of $\mathcal{S}$ into $\mathcal{J}$. Hence $f(t) \in J$ and $f(t)^{-1}xf(t) = y$ as required.

In order to use the Maximality and Isomorphism Theorems for κ-injective chains to construct non-isomorphic $\mathrm{HU}_{\omega}^{\omega_1}(\mathfrak{M})$ groups, one more fact is needed.

<u>2.2.10</u> <u>Inductive Property for</u> $\mathrm{INCH}_{\omega}(\mathfrak{M})$. Suppose $\mathcal{J}$ is an $\mathfrak{M}$-chain with $j\mathcal{J} \neq \emptyset$ and, for all $J \in j\mathcal{J}$, $\mathcal{J}_J \in \mathrm{INCH}_{\omega}(\mathfrak{M})$. Then $\mathcal{J} \in \mathrm{INCH}_{\omega}(\mathfrak{M})$.

<u>Proof</u>. Refer to 2.1.9. Since $\kappa = \omega$, the chain $\mathcal{B}$, which we are required to embed into $\mathcal{J}$, has only finitely many jumps. Hence, $\bar{\eta}(j\mathcal{B}) < \mathcal{J}_J$ for some $J \in j\mathcal{J}$ and the required embedding $\bar{f}$ exists because $\mathcal{J}_J$ is ω-injective for $\mathcal{M}$.

It is the failure of this inductive property if $\kappa > \omega$ that prevents the direct use of the following method to construct non-isomorphic $HU_{\kappa}^{\kappa^+}(\mathcal{M})$ groups if $\mathcal{M}$ is a free κ-class with $\mathbf{Z} \in \mathcal{M}$.

<u>2.2.11</u> <u>Definition of κ-Initial Chains</u>. $\mathcal{J}$ is a κ-<u>initial chain for</u> $\mathcal{M}$ iff $\mathcal{J} \in INCH_{\kappa}(\mathcal{M})$, $|j\mathcal{J}| = \kappa^+$, and, for all $J \in j\mathcal{J}$, $\mathcal{J}_J \in INCH_{\kappa}^{\leq\kappa}(\mathcal{M})$. We denote the class of κ-initial chains for $\mathcal{M}$ by $INTL_{\kappa}(\mathcal{M})$. If $\mathcal{J} \in INTL_{\kappa}(\mathcal{M})$, then $^{\circ}\mathcal{J}$ is an order type of power κ^+ such that every initial segment of $^{\circ}\mathcal{J}$ has power $\leq \kappa$. Such an order type will be called a κ-<u>initial order type</u>. The most familiar example of a κ-initial order type is the ordinal κ^+.

<u>2.2.12</u> <u>Theorem</u>. Suppose $\mathcal{M}$ is an ω-class, $\mathcal{M}^{<\omega}$ has the s.p., and I is an ω-initial order type. Then there exists $\mathcal{J} \in INTL_{\omega}(\mathcal{M})$ with $^{\circ}\mathcal{J} = I$.

<u>Proof</u>. This is a direct consequence of the Existence and Isomorphism Theorems 2.1.15(b) and 2.1.11(a), and the inductive property 2.2.10. If $\alpha \in I$, let $I(\alpha) = \{\gamma \in I \mid \gamma \leq \alpha\}$. Suppose $S < I$ is a countable union of sets of the form

$I(\alpha)$, $\alpha \in I$, and we have constructed $\mathcal{J}(S) \in \mathrm{INCH}_{\omega}^{\leq\omega}(\mathcal{M})$ with $j\mathcal{J}(S) = \{J_\gamma | \gamma \in S\}$. Let $\beta \in I - S$. We claim that

<u>2.2.13</u> The chain $\mathcal{J}(S)$ is contained in a chain $\mathcal{J}(\beta) \in \mathrm{INCH}_{\omega}^{\leq\omega}(\mathcal{M})$ with $j\mathcal{J}(\beta) = \{J_\gamma | \gamma \leq \beta\}$.

<u>Proof</u>. By 2.1.15(b), since $|I(\beta)| \leq \omega$, there exists $\mathcal{P} \in \mathrm{INCH}_{\omega}^{\leq\omega}(\mathcal{M})$ with $°\mathcal{P} = I(\beta)$, say $j\mathcal{P} = \{P_\gamma | \gamma \leq \beta\}$. Let $\mathcal{P}(S)$ be the complete subchain of $\mathcal{P}$ with $j\mathcal{P}(S) = \{P_\gamma | \gamma \in S\}$. Thus, $\mathcal{P}(S) \in \mathrm{INCH}_{\omega}^{\leq\omega}(\mathcal{M})$ and $°\mathcal{P}(S) = S$. By 2.1.11(a), letting $\Psi(P_\gamma) = J_\gamma$ for all $\gamma \in S$, there is an isomorphism f of $\mathcal{P}(S)$ onto $\mathcal{J}(S)$ such that $f(P_\gamma) = J_\gamma$ for all $\gamma \in S$. Clearly f can be extended to an isomorphism $\bar{f}$ with domain $\bigcup\mathcal{P}$. For all $\alpha \in I(\beta) - S$, we define $J_\alpha = \bar{f}(P_\alpha)$ and 2.2.13 follows.

If $\cdots\beta_n < \beta_{n+1}\cdots$ are members of I and chains $\mathcal{J}(\beta_n) \in \mathrm{INCH}_{\omega}^{\leq\omega}(\mathcal{M})$ with $j\mathcal{J}(\beta_n) = \{J_\gamma | \gamma \leq \beta_n\}$ have been constructed for all $n \geq 1$, then the inductive property 2.2.10 guarantees that $\mathcal{J}(S) = \bigcup_{n\geq 1}\mathcal{J}(\beta_n) \in \mathrm{INCH}_{\omega}^{\leq\omega}(\mathcal{M})$. Note that $j\mathcal{J}(S) = \{J_\gamma | \gamma \in S = \bigcup_{n\geq 1} I(\beta_n)\}$. Hence the construction can be continued to obtain chains $\mathcal{J}(\beta) \in \mathrm{INCH}_{\omega}^{\leq\omega}(\mathcal{M})$ for all $\beta \in I$ with $j\mathcal{J}(\beta) = \{J_\gamma | \gamma \leq \beta\}$ since $|I| = \omega_1$. Now $\mathcal{J} = \bigcup\{\mathcal{J}(\beta) | \beta \in I\} \in \mathrm{INTL}_{\omega}(\mathcal{M})$ with $°\mathcal{J} = I$ since $j\mathcal{J} = \{J_\beta | \beta \in I\}$, and this completes the proof of 2.2.12.

We next consider the conditions under which two groups possessing ω-initial chains can be isomorphic.

<u>2.2.14</u> <u>Lemma</u>. Suppose $\mathcal{M}$ is an ω-class which satisfies the Maximality Theorem 2.2.1 and $\mathcal{J}, \mathcal{G} \in \mathrm{INTL}_\omega(\mathcal{M})$ with $\bigcup\mathcal{J} \cong \bigcup\mathcal{G}$. Then, the order types $^\circ\mathcal{J}$ and $^\circ\mathcal{G}$ are "eventually isomorphic", that is, there exist $\alpha \in {^\circ\mathcal{J}}$ and $\beta \in {^\circ\mathcal{G}}$ such that the ordered sets $\{\gamma \in {^\circ\mathcal{J}} \mid \gamma \geq \alpha\}$ and $\{\gamma \in {^\circ\mathcal{G}} \mid \gamma \geq \beta\}$ are order-isomorphic.

<u>Proof</u>. Let $f \in \mathrm{ISO}(\bigcup\mathcal{J}, \bigcup\mathcal{G})$. We will first show that, for some $1 \neq \bar{J} \in \mathcal{J}$, we have $f(\bar{J}) \in \mathcal{G}$. For all $J \in j\mathcal{J}$, there exists $g(J) \in j\mathcal{G}$ such that $f(J) \leq g(J)$. Likewise, for all $J \in j\mathcal{G}$, there exists $h(J) \in j\mathcal{J}$ such that $J \leq f(h(J))$. This follows from the facts that all jumps of $\mathcal{J}$ and $\mathcal{G}$ are countable and $|j\mathcal{J}| = |j\mathcal{G}| = \omega_1$. Let $J \in j\mathcal{J}$ and put $J_n = (hg)^n(J)$ and $\bar{J} = \bigcup_{n\geq 1} J_n$. Thus $\bar{J} \in \mathcal{J}$ since $\mathcal{J}$ is a complete chain, and $\bigcup_{n\geq 0} (gh)^n g(J) \leq \bigcup_{n\geq 0} f(hg)^{n+1}(J)) \leq \bigcup_{n\geq 0} (gh)^{n+1} g(J)$, which implies $f(\bar{J}) \in \mathcal{G}$.

Since $\mathcal{M}$ satisfies the Maximality Theorem 2.2.1, f must induce a one-to-one correspondence between the chains $\{H \in \mathcal{J} \mid \bar{J} \leq H\}$ and $\{K \in \mathcal{G} \mid f(\bar{J}) \leq K\}$ since these chains are the entire subgroup lattices above $\bar{J}$ and $f(\bar{J})$. This correspondence clearly induces a correspondence between the jumps also; so if $j\mathcal{J} = \{J_\gamma \mid \gamma \in {^\circ\mathcal{J}}\}$ and $j\mathcal{G} = \{P_\gamma \mid \gamma \in {^\circ\mathcal{G}}\}$ and we choose $\alpha \in {^\circ\mathcal{J}}$ with $\bar{J} \leq J_\alpha$ and let $\beta \in {^\circ\mathcal{G}}$ be such that $f(J_\alpha) = P_\beta$, the conclusion of the lemma is immediate.

<u>2.2.15</u> <u>Lemma</u>. There is a set Ω of power 2^{ω_1} of ω-initial order types such that no two distinct members of Ω are eventually isomorphic.

<u>Proof</u>. Recall that ω_1 equals the set of countable ordinals. Let $\mathscr{F}$ be the set of functions f with domain ω_1 such that, for every even $\alpha < \omega_1$, $f(\alpha)$ = the order type of the rationals and, for every odd $\alpha < \omega_1$, $f(\alpha)$ is a countably infinite ordinal. If $f \in \mathscr{F}$, let $\mathrm{Ord}(f)$ be the set of ordinals in the range of f.

Let $f \in \mathscr{F}$. We define the order type $\eta(f) = \sum_{\alpha<\omega_1} f(\alpha)$. $\eta(f)$ is the order type obtained as the well-ordered sum of the order types in the range of f. It is immediate that $\eta(f)$ is an ω-initial order type and that the set $\mathrm{Ord}(f)$ is uniquely determined by $\eta(f)$.

Now there is a subset Γ of the power set of $\omega_1 - \omega$ such that (i) $|\Gamma| = 2^{\omega_1}$, (ii) For all $S \in \Gamma$, $|S| = \omega_1$, and (iii) For all $S \neq T \in \Gamma$, $S \cap T$ is countable. The existence of Γ follows from the useful theorem of Sierpinski [23: p. 451]. For each $S \in \Gamma$, let $f_S \in \mathscr{F}$ be such that $\mathrm{Ord}(f_S) = S$, and define $\Omega = \{\eta(f_S) | S \in \Gamma\}$. Suppose $S \neq T \in \Gamma$ and $\eta(f_S)$ and $\eta(f_T)$ are eventually isomorphic. Since the ordinal segments of these order types are uniquely determined, it would follow that S and T differ only by a countable set of ordinals, contrary to (iii).

<u>2.2.16</u> <u>Corollary</u>. If $\mathfrak{M}$ is an ω-class, $\mathfrak{M}^{<\kappa}$ has the s.p., and $\mathfrak{M}$ satisfies the Maximality Theorem 2.2.1, then

$HU_{\omega}^{\omega_1}(\mathfrak{M})$ contains 2^{ω_1} mutually non-isomorphic members, each possessing an ω-initial chain for $\mathfrak{M}$. In particular $\mathfrak{M}$ can be an ω-class of any of the three types of 2.2.0.

The proof is immediate from the last three results and the Maximality Theorem 2.2.1, 2.1.21, and 2.2.9.

We will next consider the effect that a κ-injective chain $\mathcal{J}$ has on the possible automorphisms of $\bigcup\mathcal{J}$. We have seen already, in the proof of 2.2.14, that the Maximality Theorem places a restriction on the automorphisms of $\bigcup\mathcal{J}$ since, whenever $\alpha \in \mathrm{Aut}(\bigcup\mathcal{J})$ and $\alpha(J) = J$ for some $J \in j\mathcal{J}$, then every member of $\mathcal{J}$ containing J is also invariant under α. In the three cases of 2.2.0, there is a much stronger condition which limits the automorphisms of $\bigcup\mathcal{J}$.

<u>2.2.17</u> <u>Theorem on Uniqueness of Automorphisms</u>. Suppose $\mathfrak{M}$ is a κ-class of one of the three types of 2.2.0, $\mathcal{J} \in \mathrm{INCH}_{\kappa}(\mathfrak{M})$, and $J \in j\mathcal{J}$. Then, for all $\alpha \in \mathrm{Aut}(\bigcup\mathcal{J})$, if $\alpha \equiv 1$ on J, then $\alpha = 1_{\bigcup\mathcal{J}}$. Thus, every automorphism of $\bigcup\mathcal{J}$ is determined uniquely by its action on any jump of $\mathcal{J}$.

We will prove 2.2.17 here in the case that $\mathfrak{M}$ is a free κ-class with $\mathbb{Z} \in \mathfrak{M}$, postponing the other cases for §3.2 and §3.3.

<u>Proof</u>. Suppose that $\alpha \in \mathrm{Aut}(\bigcup\mathcal{J})$ and, for some $J \in j\mathcal{J}$, $\alpha \equiv 1$ on J. Suppose, further, that $J_1 \in j\mathcal{J}$ with $J < J_1$. We will prove

<u>2.2.18</u> For all $x \in J_1 - J_1^-$ with $|x| = \omega$ and $\langle x\rangle \cap J_1^- = 1$, we have $\alpha(x) \in \langle x\rangle$.

In the notation of Lemma 2.2.2, this implies that $\alpha(x) \in \langle x\rangle$ for all $x \in X$. Thus, by 2.2.2(a), we have $\alpha(z) \in \langle z\rangle$ for all z lying in the free subgroup $\langle x\rangle * \langle y\rangle$ for all $x \in X$ and $y \in X(x)$, and this implies that $\alpha(x) = x$ because if $\alpha(x) = x^i$, then we have both $\alpha(xy) = x^i\alpha(y) = x^i y^j$ and $\alpha(xy) = (xy)^k$ for some $j,k \in \mathbb{Z}$, which is impossible unless $i = j = k = 1$. Thus, $\alpha(x) = x$ for all $x \in X$. Since $J_1 = \langle X\rangle$ we have $\alpha \equiv 1$ on J_1, and, since J_1 is arbitrary, $\alpha = 1_{\cup\mathcal{J}}$.

To prove 2.2.18, we first apply the $\varkappa$-local chain property of $\mathcal{J}$ to obtain $G \in \mathfrak{M}^{<\varkappa}$ such that $\langle x,\alpha(x)\rangle \leq G < J_1$ and $\mathcal{J}_G$ is an $\mathfrak{M}^{<\varkappa}$-chain with $|j\mathcal{J}_G| < \varkappa$ and $G \cap (J - J^-) \neq \emptyset$. There exists $H \in \mathfrak{M}^{<\varkappa}$ such that $\langle t\rangle \oplus \langle x\rangle \leq H$ where $|t| = \omega$ because the group $\langle t\rangle \oplus \langle x\rangle$ is the HNN extension of $\langle x\rangle$ with $\varphi(x) = x$ (here we have used the hypothesis that $\mathbb{Z} \in \mathfrak{M}$ and 1.4.25(a)).

Let $\mathcal{A} = \begin{array}{ccc} H & & G \\ & \diagdown\ \diagup & \\ & \langle x\rangle & \end{array}$. Thus, $gp_*(\mathcal{A}) = M \in \mathfrak{M}^{<\varkappa}$.

Since $\langle x\rangle \cap J_1^- = 1$, the Normal Form Theorem (1.2.2) implies that $\langle t\rangle * (G \cap J_1^-)$ exists in M. This allows us to define the chain $\mathcal{C}$, the embeddings η and $\bar{\eta}$, and the $\bar{\eta}$-embedding $f\colon M \to J_1$ which extends 1_G in a manner identical to the proof of 2.2.2(b) (p. 66, lines 11-24), and we conclude, as there, that $f(t) \in J$. Thus,

<u>2.2.19</u> $\alpha(\langle x, f(t)\rangle) = \langle \alpha(x), \alpha f(t)\rangle = \langle \alpha(x), f(t)\rangle$.

Since $f \equiv 1$ on G, we have $f(x) = x$, and so $\langle x, f(t)\rangle \cong \langle x,t\rangle = \langle x\rangle \oplus \langle t\rangle$. On the other hand, if $\alpha(x) \notin \langle x\rangle$, then t does not commute with $\alpha(x)$ in $gp_*(\mathcal{a})$ and 2.2.19 is impossible. This proves 2.2.18.

We can use 2.2.17 to obtain some information about the automorphism groups of the groups of 2.2.16, which possess ω-initial chains.

<u>2.2.20</u> <u>Corollary</u>. Suppose $\mathcal{m}$ is an ω-class of one of the three types of 2.2.0 and $\mathcal{J} \in \mathrm{INTL}_\omega(\mathcal{m})$. Put $G = \bigcup\mathcal{J}$ and, for each $J \in j\mathcal{J}$, put $A(J) = \{\alpha \in \mathrm{Aut}G \mid \alpha(J) = J\}$. Then, (<u>i</u>) For all $J \in j\mathcal{J}$, $A(J)$ is isomorphic to a subgroup of AutJ, (<u>ii</u>) For all $J < Q \in j\mathcal{J}$, $A(J) \leq A(Q)$, (<u>iii</u>) $\mathrm{Aut}G = \bigcup\{A(J) \mid J \in j\mathcal{J}\}$, and (<u>iv</u>) $|\mathrm{Aut}G| \leq 2^\omega$; so, if the Continuum Hypothesis holds, we have $|\mathrm{Aut}G| = \omega_1 = |G|$.

<u>Proof</u>. Conclusion (i) holds because, by 2.2.17, for all $\alpha \in A(J)$, α is the unique automorphism β of G such that $\beta \equiv \alpha$ on J; conclusion (ii) is a consequence of the Maximality Theorem; (iii) holds because, for all $\alpha \in \mathrm{Aut}G$, we have $\alpha(J) = J$ for some $J \in j\mathcal{J}$ (see the proof of 2.2.14); and (iv) follows because, for all $J \in j\mathcal{J}$, we have $J \in HU_\omega^\omega(\mathcal{m})$, $|J| = \omega$, $|\mathrm{Aut}J| = 2^\omega$, and, hence, $|\mathrm{Aut}G| \leq \omega_1 2^\omega = 2^\omega$.

CHAPTER III

EXISTENCE AND SPECIAL PROPERTIES OF $\varkappa$-INJECTIVE CHAINS

§3.1 Proofs of the Existence and Isomorphism Theorems.

<u>Proof of the Isomorphism Theorem 2.1.10, p. 53</u>.

We will give only the proof of part (a) since the proof of (b) is an easier application of the same idea.

The proof of part (a) is a standard back-and-forth argument, similar in outline to the sketch on p. 19 of the proof of the Isomorphism Lemma 1.3.6. But, since there is more structure in the present context, we will give most of the details.

Referring to the hypotheses of part (a), let $B = \bigcup \mathcal{B}$, $C = \bigcup \mathcal{J}$, and $j\mathcal{B} = \{J_\alpha | \alpha \in I\}$. Note that $\mathcal{B}$ and $\mathcal{J}$ are order-isomorphic since $j\mathcal{B}$ and $j\mathcal{J}$ are. We will first prove

<u>3.1.0</u> If $|j\mathcal{B}| \geq 2$, then $|B| = \varkappa = |C|$.

<u>Proof</u>. We have $|B|, |C| \leq \varkappa$ by simple cardinal arithmetic since $\mathcal{B}, \mathcal{J} \in \mathrm{INCH}_{\varkappa}^{\leq\varkappa}(\mathcal{M})$ (see 2.1.9). To obtain the reverse inequality, note that the class $\mathcal{M}\restriction B$ (see 1.3.11) has the

comparison property and $\mathcal{B} \in \mathrm{INCH}_{\varkappa}(\mathcal{M}\restriction B)$. Hence, by 2.1.13, if $|j\mathcal{B}| \geq 2$, $\mathcal{M}\restriction B$ has the $\varkappa$-s.a.p.. We can therefore use an argument similar to the proof of 1.4.22 to show that $\mathcal{M}^{\varkappa}\restriction B \neq \emptyset$, which proves 3.1.0.

If $|j\mathcal{B}| = 1$, that is, $\mathcal{B} = \{1, B\}$, then $B, C \in \mathrm{INJ}_{\varkappa}^{\leq\varkappa}(\mathcal{M})$ and, in this case, the Isomorphism Theorem follows from the Isomorphism Lemma 1.3.6. So, we can assume that $|j\mathcal{B}| \geq 2$ and, by 3.1.0,

<u>3.1.1</u> $|B| = \varkappa = |C|$.

Let $\mathrm{cf}(\varkappa)$ = the cofinality of $\varkappa$ = the smallest cardinal σ such that $\varkappa$ is the union of σ sets, each of power $< \varkappa$.

Beginning with $\mathbf{E} = \mathbf{E}_0$ and $\eta = \eta_0$ (see the hypotheses of 2.1.10(a)) we will construct a chain $\{\mathbf{E}_\gamma | \gamma < \mathrm{cf}(\varkappa)\}$ of $\mathcal{M}^{<\varkappa}$ subalgebras of B and a chain of embeddings f_γ: $\mathbf{E}_\gamma \to C$ such that each $\mathcal{B}_{\mathbf{E}_\gamma} = \mathcal{B}_\gamma$ is an $\mathcal{M}^{<\varkappa}$-chain and each f_γ is an η_γ-embedding of $\mathcal{B}_\gamma$ into $\mathcal{J}$ where $\eta_\gamma \in \mathcal{O}(\mathcal{B}_\gamma, \mathcal{J})$ is the (unique) restriction of $\bar{\eta}$, that is, putting $j\mathcal{B}_\gamma = \{\mathbf{E}_\gamma \cap J_\alpha | \alpha \in I_\gamma\}$ where $I_\gamma = \{\alpha \in I | \mathbf{E}_\gamma \cap (J_\alpha - J_\alpha^-) \neq \emptyset\}$ (see 2.1.3 and 2.1.8),

<u>3.1.2</u> for all $\alpha \in I_\gamma$, $\eta_\gamma(\mathbf{E}_\gamma \cap J_\alpha) = \bar{\eta}(J_\alpha)$.

The pairs $(\mathbf{E}_\gamma, f_\gamma)$, $\gamma < \mathrm{cf}(\varkappa)$, are defined inductively. Assume they are defined for all $\beta < \gamma$.

If γ is a limit ordinal, we put $\mathbf{E}_\gamma = \bigcup\{\mathbf{E}_\beta | \beta < \gamma\}$ and $f_\gamma = \bigcup\{f_\beta | \beta < \gamma\}$, and we check easily that

$I_\gamma = \bigcup\{I_\beta \mid \beta < \gamma\}$ and that f_γ is an η_γ-embedding of $\mathcal{B}_\gamma$ into $\mathcal{J}$. This last fact asserts that $f_\gamma(E_\gamma \cap J_\alpha - E_\gamma \cap J_\alpha^-) \leq \bar{\eta}(J_\alpha) - \bar{\eta}(J_\alpha)^-$ for all $\alpha \in I_\gamma$, and this follows from the similar inclusions which hold for all (E_β, f_β), $\beta < \gamma$. $\mathcal{B}_\gamma$ is an $\mathcal{M}^{<\kappa}$-chain by the κ-inductive property of $\mathcal{M}$ and the fact that $\gamma < cf(\kappa)$.

If $\gamma = \beta + 1$, then we must enlarge E_β to E_γ by adding certain elements of B. This can be done in any way we choose; namely, if S is any subset of B with $|S| < \kappa$, then, by the κ-local chain property of $\mathcal{B}$ for $\mathcal{M}$, there exists $E_\gamma < B$ such that $E_\beta \cup S \leq E_\gamma$ and $\mathcal{B}_\gamma$ is an $\mathcal{M}^{<\kappa}$-chain. Since η_γ is an extension of η_β (they are both restrictions of $\bar{\eta}$), the κ-injectivity of the chain $\mathcal{J}$ for $\mathcal{M}$ guarantees that f_β has an extension to an η_γ-embedding, f_γ, of $\mathcal{B}_\gamma$ into $\mathcal{J}$. During the construction we must be sure that two conditions will be met, namely,

<u>3.1.3</u> $B = \bigcup\{E_\gamma \mid \gamma < cf(\kappa)\}$ and

<u>3.1.4</u> $C = \bigcup\{f_\gamma(E_\gamma) \mid \gamma < cf(\kappa)\}$.

If these hold, we can define $\bar{f} = \bigcup\{f_\gamma \mid \gamma < cf(\kappa)\}$ and check easily that $\bar{f}$ is an $\bar{\eta}$-isomorphism of $\mathcal{B}$ onto $\mathcal{J}$.

We can choose the "enlargement sets" S mentioned above so that 3.1.3 is met precisely because $|B| = \kappa$ (see 3.1.1) and, so, B can be exhausted in exactly $cf(\kappa)$ steps. Condition 3.1.4 is met in a similar manner by reversing the roles of $\mathcal{B}$ and $\mathcal{J}$ at alternate steps of the construction so that f_β^{-1} is extended to f_γ^{-1} and $f_\gamma(E_\gamma)$ is an arbitrary enlargement of $f_\beta(E_\beta)$. This completes our proof of 2.1.10(a).

Proof of the Existence Theorems for κ-Injective Chains 2.1.15 and 2.1.16 (pp. 55 and 56).

The construction of an $\mathfrak{M}$ algebra possessing a κ-injective chain for $\mathfrak{M}$ is completely analogous to the construction of a κ-injective algebra for $\mathfrak{M}$ outlined on pp. 22-23. There are two essential components. The algebraic component asserts that a partial chain obtained at some stage of the construction can be extended to permit a given injection, and the set-theoretic component asserts that the number of required injections does not exceed the cardinal of the algebra under construction (which equals the number of available steps). We will present these components in the next two lemmas.

In the case of Jonsson's Theorem, the algebraic component is an immediate consequence of κ-injectivity, but, in the case of κ-injective chains, we need a special amalgamation procedure for chains which depends on the subamalgam property.

3.1.5 Lemma (Amalgamation of Chains). Suppose

(i) $\mathfrak{B}, \mathfrak{C} \in CH_\kappa(\mathfrak{M}^{<\kappa})$;

(ii) $j\mathfrak{B} = \{B_\alpha | \alpha \in X_1\}$ and $j\mathfrak{C} = \{C_\alpha | \alpha \in X_2\}$ where $X = X_1 \cup X_2$ is a totally ordered set;

(iii) $E \leq \bigcup\mathfrak{B}$ is such that $\mathfrak{B}_E$ is an $\mathfrak{M}^{<\kappa}$-chain and, putting $E_\alpha = E \cap B_\alpha$ and $E_\alpha^- = E \cap B_\alpha^-$, we have $X_E = \{\alpha \in X_1 | E_\alpha - E_\alpha^- \neq \emptyset\} \leq X_2$; and

(iv) f is an η-embedding of $\mathfrak{B}_E$ into $\mathfrak{C}$ where $\eta \in \mathcal{O}(\mathfrak{B}_E, \mathfrak{C})$ is induced by the identity on X_E, that is, for all $\alpha \in X_E$, $f(E_\alpha - E_\alpha^-) \leq C_\alpha - C_\alpha^-$.

Further suppose that either

(v) X is well ordered and $\mathcal{M}^{<\kappa}$ has the subamalgam property or

(vi) $\mathcal{M}^{<\kappa}$ has the subamalgam property with the descendance condition.

Then, there exists $\mathfrak{A} \in CH_\kappa(\mathcal{M}^{<\kappa})$ such that

(1) $j\mathfrak{A} = \{A_\alpha | \alpha \in X\}$,

(2) $\bigcup\mathfrak{C} \leq \bigcup\mathfrak{A}$ and, for all $\alpha \in X_2$, $C_\alpha - C_\alpha^- \leq A_\alpha - A_\alpha^-$,

(3) f has an extension to an $\bar{\eta}$-embedding, $\bar{f}$, of $\mathfrak{B}$ into $\mathfrak{A}$ where $\bar{\eta} \in \mathcal{O}(\mathfrak{B}, \mathfrak{A})$ is induced by the identity on X_1, that is, for all $\alpha \in X_1$, $\bar{f}(B_\alpha - B_\alpha^-) \leq A_\alpha - A_\alpha^-$.

Proof of 3.1.5. We can assume WLOG that $\mathfrak{B}$ and $\mathfrak{C}$ form an amalgam $\mathcal{A} = \begin{array}{ccc} \bigcup\mathfrak{B} & & \bigcup\mathfrak{C} \\ & \diagdown\ \diagup & \\ & E & \end{array}$ and that $f = 1_E$. This implies

$$\text{3.1.6} \quad \begin{cases} \text{For all } \alpha \in X_E,\ E \cap (B_\alpha - B_\alpha^-) = E \cap (C_\alpha - C_\alpha^-); \\ \text{For all } \alpha \in X_1,\ B_\alpha \cap (\bigcup\mathfrak{C}) = B_\alpha \cap E; \text{ and} \\ \text{For all } \alpha \in X_2,\ C_\alpha \cap (\bigcup\mathfrak{B}) = C_\alpha \cap E. \end{cases}$$

The chain $\mathfrak{A}$ is obtained from the amalgam $\mathcal{A}$ by direct use of the subamalgam property: Let $M \in \mathcal{M}^{<\kappa}$ be such that $M = \langle \mathcal{A} \rangle$ and M satisfies the definition 2.1.14 of either the s.p. or the s.p.d. depending on our assumptions (v) and (vi) respectively. We define the subgroups A_α and A_α^- of M and the chain $\mathfrak{A}$ as follows. For each $\alpha \in X$, let

3.1.7 $\hat{B}_\alpha = \bigcup\{B_\beta \mid \alpha \geq \beta \in X_1\}$ and
$\hat{C}_\alpha = \bigcup\{C_\beta \mid \alpha \geq \beta \in X_2\}$,

and note that

3.1.8 If $\alpha \in X_1$, then $\hat{B}_\alpha = B_\alpha$;
If $\alpha \in X_2$, then $\hat{C}_\alpha = C_\alpha$; and
For all $\alpha \in X$, $\hat{B}_\alpha \in \mathfrak{B}$ and $\hat{C}_\alpha \in \mathfrak{C}$.

We define, for all $\alpha \in X$,

3.1.9 $A_\alpha = \mathrm{alg}_M(\hat{B}_\alpha, \hat{C}_\alpha)$,

3.1.10
$$A_\alpha^- = \begin{cases} \mathrm{alg}_M(B_\alpha^-, \hat{C}_\alpha) & \text{if } \alpha \in X_1 - X_2 \\ \mathrm{alg}_M(B_\alpha^-, C_\alpha^-) & \text{if } \alpha \in X_1 \cap X_2 \\ \mathrm{alg}_M(\hat{B}_\alpha, C_\alpha^-) & \text{if } \alpha \in X_2 - X_1, \text{ and} \end{cases}$$

3.1.11 $\mathfrak{A}_0 = \{A_\alpha \mid \alpha \in X\} \cup \{A_\alpha^- \mid \alpha \in X\}$ and
$\mathfrak{A}$ = the complete chain consisting of all unions and intersections of subchains of $\mathfrak{A}$.

From 3.1.8 and the s.p., we have, for all $\alpha \in X$, $A_\alpha, A_\alpha^- \in \mathfrak{M}^{<\kappa}$. Note that $\alpha < \beta \in I$ implies $A_\alpha \leq A_\beta^-$.

We will note the facts which permit the use of the subamalgam property.

3.1.12 For all $\alpha \in X$, $\mathcal{A}_\alpha = \begin{tikzcd} \hat{B}_\alpha \arrow[dr, dash] & & \hat{C}_\alpha \arrow[dl, dash] \\ & \hat{E}_\alpha & \end{tikzcd}$ is a subamalgam of $\mathcal{A}$ where $\hat{E}_\alpha = \bigcup\{E_\beta \mid \alpha \geq \beta \in X_E\} \in \mathcal{M}^{<\kappa}$.

This is immediate from 3.1.6, 3.1.7, and 3.1.8.

3.1.13

(a) If $\alpha \in X_1 - X_2$, then $\begin{tikzcd} B_\alpha^- \arrow[dr, dash] & & \hat{C}_\alpha \arrow[dl, dash] \\ & \hat{E}_\alpha & \end{tikzcd}$

(b) If $\alpha \in X_2 - X_1$, then $\begin{tikzcd} \hat{B}_\alpha \arrow[dr, dash] & & C_\alpha^- \arrow[dl, dash] \\ & \hat{E}_\alpha & \end{tikzcd}$

(c) If $\alpha \in (X_1 \cap X_2) - X_E$, then $\begin{tikzcd} B_\alpha^- \arrow[dr, dash] & & C_\alpha^- \arrow[dl, dash] \\ & \hat{E}_\alpha & \end{tikzcd}$

(d) If $\alpha \in X_E$, then $\begin{tikzcd} B_\alpha^- \arrow[dr, dash] & & C_\alpha^- \arrow[dl, dash] \\ & E_\alpha^- & \end{tikzcd}$

is a sub-amalgam of $\mathcal{A}$.

The proof in the three cases (a), (b), and (c) is identical because, in these cases, $\alpha \notin X_E$ implies that either $B_\alpha \cap E = B_\alpha^- \cap E$ or $C_\alpha \cap E = C_\alpha^- \cap E$ (or both in case (c)) and the conclusions hold in view of 3.1.12 and 3.1.8. In case (d), we have directly that $E_\alpha^- = E \cap B_\alpha^- = E \cap C_\alpha^-$ (by 3.1.6); in this case, note that $E_\alpha^- < E_\alpha = E \cap B_\alpha = \hat{E}_\alpha$.

We can conclude immediately from the s.p., 3.1.9, and 3.1.12 that

<u>3.1.14</u> For all $\alpha \in X_1$, $A_\alpha \cap (\bigcup\mathfrak{B}) = B_\alpha$; and
For all $\alpha \in X_2$, $A_\alpha \cap (\bigcup\mathfrak{C}) = C_\alpha$.

From 3.1.10 and 3.1.13, we likewise have

<u>3.1.15</u> For all $\alpha \in X_1$, $A_\alpha^- \cap (\bigcup\mathfrak{B}) = B_\alpha^-$; and
For all $\alpha \in X_2$, $A_\alpha^- \cap (\bigcup\mathfrak{C}) = C_\alpha^-$.

Taken together, 3.1.14 and 3.1.15 imply

<u>3.1.16</u> For all $\alpha \in X_1$, $B_\alpha - B_\alpha^- \leq A_\alpha - A_\alpha^-$; and
For all $\alpha \in X_2$, $C_\alpha - C_\alpha^- \leq A_\alpha - A_\alpha^-$.

Thus, the conclusion (2) of 3.1.5 holds, and so does conclusion (3) if we define $\bar{f} = 1_{\bigcup\mathfrak{B}}$. Also note that 3.1.14 and 3.1.15 imply that, for all $\alpha \in X$, (A_α^-, A_α) is a jump of $\mathfrak{A}$.

We must yet show that conclusion (1) holds; namely, that every jump of $\mathfrak{A}$ is of the form (A_α^-, A_α) for some $\alpha \in X$. First note that

<u>3.1.17</u> For all $\alpha \in X$, $A_\alpha^- = \bigcup\{A_\beta \mid \alpha > \beta \in X\}$,
which follows from the facts that, for all $\alpha \in X_1$, $B_\alpha^- = \bigcup\{B_\beta \mid \alpha > \beta \in X_1\}$ and the analogous fact for C_α^-. Also note that $M = \bigcup\{A_\alpha \mid \alpha \in X\}$ since $M = \langle \mathfrak{a} \rangle$. Now, if X is well ordered, conclusion (1) is an immediate consequence of 3.1.17.

So, we must attend to the case in which assumption (vi) holds (the descendance condition (d.c.) - see 2.1.14) and X is not necessarily well ordered. From the d.c. we have

3.1.18 For all $\alpha \in X$, $A_\alpha = \bigcap\{A_\beta^- | \alpha < \beta \in X\}$, which is the dual of 3.1.17.

Suppose S is a subset of X and $\mathcal{T} = \{A_\alpha | \alpha \in S\}$. We will prove

3.1.19 If $\mathcal{T}$ does not have a largest member, then either $\bigcup\mathcal{T} = A_\tau^-$ for some $\tau \in X$ or $\bigcup\mathcal{T} = \bigcap\{A_\gamma | \gamma \in S^+\}$ where $S^+ = \{\gamma \in X | \gamma > \alpha$ for all $\alpha \in S\}$.

This will show that $\mathfrak{A}$ has no jumps besides (A_α^-, A_α), $\alpha \in X$: Suppose (J^-, J) is a jump of $\mathfrak{A}$ such that $J^- \neq A_\alpha^-$ for all $\alpha \in X$; then we also have $J^- \neq A_\alpha$ for all $\alpha \in X$ for, otherwise, 3.1.18 is contradicted. Hence $J^- \notin \mathfrak{A}_0$ (see 3.1.11) and so

3.1.20 $J^- = \bigcup\mathcal{T}$ where $\mathcal{T} = \{A_\alpha | \alpha \in X$ and $A_\alpha < J^-\}$ because $\mathfrak{A}$ is the completion of $\mathfrak{A}_0$. Since 3.1.20 contradicts 3.1.19, it will indeed suffice to prove 3.1.19.

To prove 3.1.19, assume $\mathcal{T} = \{A_\alpha | \alpha \in S\}$ has no largest member and put $S_1 = X_1 \cap S$ and $S_2 = X_2 \cap S$. We can assume WLOG that

3.1.21 $\bigcup\mathcal{T} = \bigcup\{A_\gamma | \gamma \in S_1\}$ and, hence, S_1 has no largest member.

We define

$$U = \bigcup\{B_\gamma | \gamma \in S_1\},$$

$$\check{X}_2 = \{\mu \in X_2 | \mu \leq \gamma \text{ for some } \gamma \in S_1\},$$

$V = \bigcup\{C_\mu \mid \mu \in \check{X}_2\}$,

$S_1^+ = \{\alpha \in X_1 \mid \alpha > \gamma$ for all $\gamma \in S_1\}$,

$\check{X}_2^+ = \{\beta \in X_2 \mid \beta > \gamma$ for all $\gamma \in \check{X}_2\}$

$= \{\beta \in X_2 \mid \beta > \gamma$ for all $\gamma \in S_1\}$, and

$\Delta_2 = \{\beta \in X_2 \mid \alpha < \beta < \gamma$ for all $\alpha \in S_1$ and $\gamma \in S_1^+\}$;

and we note that

<u>3.1.22</u> $\bigcup\mathcal{T} = \langle U, V \rangle$.

We will check that 3.1.19 holds by considering the following three sets of exhaustive cases.

<u>3.1.23</u> (1) $\Delta_2 = \emptyset$ and (2) $\Delta_2 \neq \emptyset$.

<u>3.1.24</u> (i) $U = B_\alpha^-$ for some $\alpha \in X_1$.

In this case, α is the smallest member of S_1^+.

(ii) Otherwise.

In this case, since U cannot be an upper member of a jump of $\mathcal{B}$ by 3.1.21, we have $U = \bigcap\{B_\gamma \mid \gamma \in S_1^+\}$.

<u>3.1.25</u> (a) $V = C_\beta$ for some $\beta \in X_2$.

This is the case iff β is the largest member of $\check{X}_2$.

(b) $V = C_\beta^-$ for some $\beta \in X_2$.

In this case, β is the smallest member of $\check{X}_2^+$.

(c) Otherwise.

In this case, we have $V = \bigcap\{C_\mu \mid \mu \in \check{X}_2^+\}$.

The following table shows the status of $\bigcup\mathcal{T}$ in every possible case, and 3.1.19 is readily checked from it.

<table>
<tr><th rowspan="2"></th><th rowspan="2">(1):
$\Delta_2 = \emptyset$</th><th colspan="3">(2): $\Delta_2 \neq \emptyset$</th></tr>
<tr><th>(a):
$V = C_\beta$</th><th>(b):
$V = C_\beta^-$</th><th>(c):
Otherwise</th></tr>
<tr><td>(i):
$U = B_\alpha^-$</td><td>$\bigcup\mathcal{T} = A_\alpha^-$</td><td rowspan="2">$\bigcup\mathcal{T}$
=
$\bigcap\{A_\gamma^- \mid \gamma \in \Delta_2\}$</td><td rowspan="2">$\bigcup\mathcal{T}$
=
A_β^-</td><td rowspan="2">$\bigcup\mathcal{T}$
=
$\bigcap\{A_\gamma \mid \gamma \in \Delta_2\}$</td></tr>
<tr><td>(ii):
Otherwise</td><td>$\bigcup\mathcal{T} =$
$\bigcap\{A_\gamma \mid \gamma \in S_1^+\}$</td></tr>
</table>

Note that, because of 3.1.21, we have $S_1^+ \leq S^+$ and $\Delta_2 \leq \overset{\vee}{X}{}_2^+ \leq S^+$, and so 3.1.19 does follow in all cases.

We will discuss only one case here since they are all quite similar. In the case (2)(a) we have $V = C_\beta$, $\beta =$ largest member of $\overset{\vee}{X}_2$, and $\Delta_2 \neq \emptyset$. For all $\gamma \in \Delta_2$, note that $A_\gamma = \langle U, C_\gamma \rangle$ (see 3.1.7-9). We consider two cases: (I) β has an immediate successor, τ, in X_2; evidently $\tau \in \Delta_2$, $C_\beta = C_\tau^-$, and hence $\bigcup\mathcal{T} = \langle U, V \rangle = \langle U, C_\beta \rangle = A_\tau^-$ (see 3.1.10), and (II) Δ_2 has no smallest member; in this case, $C_\beta = \bigcap\{C_\gamma \mid \gamma \in \Delta_2\} = \bigcap\{C_\gamma^- \mid \gamma \in \Delta_2\}$ since C_β is not the lower member of any jump of $\mathfrak{C}$ as in the previous case, and so $\bigcup\mathcal{T} = \langle U, V\rangle = \langle U, C_\beta\rangle = \bigcap\{A_\gamma^- \mid \gamma \in \Delta_2\}$ by the d.c..

This completes our proof of Lemma 3.1.5.

The set-theoretic component of the construction is given by

<u>3.1.26</u> <u>Lemma</u>. Suppose $\varkappa \geq \omega$ is regular and that the G.C.H. holds if $\varkappa > \omega$. Let $\mathfrak{C} \in CH_\varkappa(\mathcal{M}^{<\varkappa})$ and let χ be the class of all amalgams

$$\mathcal{A} = \begin{tikzcd}[column sep=small] \bigcup\mathfrak{B} & & \bigcup\mathfrak{C} \\ & E \arrow[ul, no head] \arrow[ur, no head] & \end{tikzcd} \quad \text{such that}$$

(1) $E \leq \bigcup\mathfrak{C}$ is such that $\mathfrak{C}_E$ is an $\mathcal{M}^{<\varkappa}$-chain and $\mathfrak{B} \in CH_\varkappa(\mathcal{M}^{<\varkappa})$ satisfies the hypotheses of Lemma 3.1.5 with $f = 1_E$; and

(2) $X_1 \cup X_2 = X \leq I$ where I is a <u>fixed</u> totally ordered set of power $\varkappa$.

Then, there are at most $\varkappa$ $\mathfrak{C}$-isomorphism classes of χ amalgams where the amalgams $\mathcal{A}_1, \mathcal{A}_2 \in \chi$ are <u>$\mathfrak{C}$-isomorphic</u> iff $E_1 = E_2, {}^\circ\mathfrak{B}_1 = {}^\circ\mathfrak{B}_2 = X_1$, and there is a function ξ of $\mathcal{A}_1$ onto $\mathcal{A}_2$ such that $\xi \equiv 1$ on $\bigcup\mathfrak{C}$ and $\xi \equiv$ some 1_{X_1}-isomorphism of $\mathfrak{B}_1$ onto $\mathfrak{B}_2$ on $\bigcup\mathfrak{B}_1$.

<u>Proof</u>. We will contrast this computation with that used in Jonsson's Theorem.

First we will give the pertinent definitions and facts from cardinal arithmetic. Suppose $\varkappa$ and σ are infinite cardinals and U is a set of power $\varkappa$.

<u>3.1.27</u> $P(U)$ = the power set of U;

$P_{<\kappa}(U)$ = the set of subsets of U which have power $< \kappa$;

$\kappa^{<\sigma} = \Sigma\{\kappa^{\lambda} \mid \lambda < \sigma\}$, and, hence,

$2^{<\kappa} = |P_{<\kappa}(U)|$.

If the G.C.H. holds when $\kappa > \omega$ and $\kappa \geq \omega$ is regular, then

<u>3.1.28</u> $\kappa = 2^{<\kappa} = \kappa^{<\kappa}$.

In proving Jonsson's Theorem we must count the number of C-isomorphism classes of $\mathfrak{M}^{<\kappa}$ amalgams $\begin{tikzcd}[column sep=small, row sep=small] B \arrow[dr, dash] & & C \arrow[dl, dash] \\ & E & \end{tikzcd}$ where C is fixed and E and B are variable. Since the elements of B can be chosen WLOG from a fixed universe U of power κ, we can count the isomorphism classes as follows.

	<u>Object Chosen</u>	<u>Bound for the # of Choices</u>
(1)	$E \in P(C)$	$2^{<\kappa} = \kappa$
(2)	$B \in P_{<\kappa}(U)$	$2^{<\kappa} = \kappa$
(3)	Ω = the set of finitary operations on B	$\|P(B^{<\omega})\|^{<\kappa} \leq \kappa^{<\kappa} = \kappa$

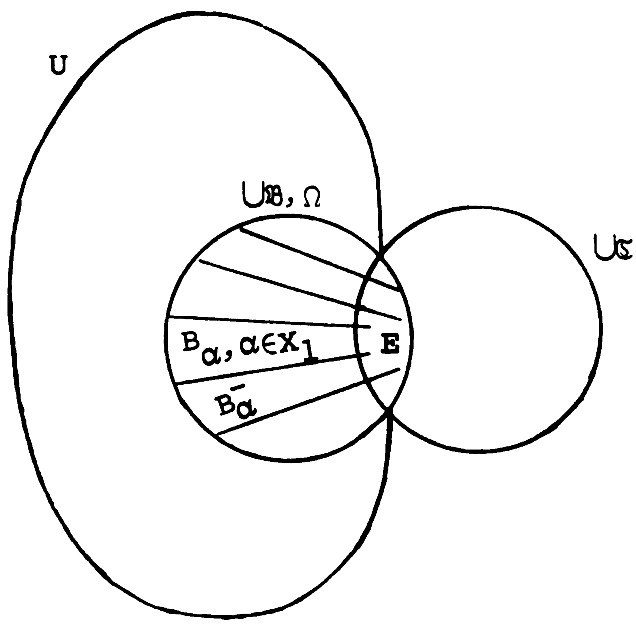
The computation (3) assumes that $\kappa > \omega$; if $\kappa = \omega$, the hypothesis that $\mathfrak{M}^{<\omega}$ has $\leq \omega$ members up to $\cong$ is used. Also, if $\kappa = \omega$, instead of choosing E in (1), we choose a finite generating set for E.

Thus, the total number of choices is at most $\varkappa^3 = \varkappa$.

In proving 3.1.26 we must also choose $X_1 \in P_{<\varkappa}(I)$ to index the jumps of $\mathfrak{B}$, and, for every $\alpha \in X_1$, we must choose B_α^- and $B_\alpha \in \mathfrak{M}^{<\varkappa}$ to be subalgebras of B. Note that if $\varkappa = \omega$, then $|j\mathfrak{B}| < \omega$ because $\bigcup\mathfrak{B} \in \mathfrak{M}^{<\omega}$ and $\mathfrak{B}$ is an ω-local chain. Thus, our count proceeds as follows.

	Object Chosen	Bound for the # of Choices
(1)	E	$2^{<\varkappa} = \varkappa$
(2)	$\bigcup\mathfrak{B} \in P_{<\varkappa}(U)$	$2^{<\varkappa} = \varkappa$
(3)	Ω	$\varkappa$ (as above)
(4)	$X_1 \in P_{<\varkappa}(I)$	$2^{<\varkappa} = \varkappa$
(5)	B_α^- and B_α, $\alpha \in X_1$	$2^{<\varkappa} = \varkappa$ for each $\alpha \in X_1$

Since $|X_1| < \varkappa$, the total number of choices is bounded by $\varkappa^{<\varkappa} = \varkappa$, as required.



<u>Completion of the Proof of the Existence Theorems</u>.

Lemmas 3.1.5 and 3.1.26 permit the construction of a tower of chains $\{\mathfrak{C}_\gamma \mid \gamma < \varkappa\} < CH_\varkappa(\mathcal{M}^{<\varkappa})$ satisfying

<u>3.1.29</u> For all $\gamma < \varkappa$, $j\mathfrak{C}_\gamma = \{J_\iota^\gamma \mid \iota \in I_\gamma\}$,

<u>3.1.30</u> For all $\alpha < \beta < \varkappa$, $I_\alpha \leq I_\beta$,

<u>3.1.31</u> For all $\alpha < \beta < \varkappa$ and $\iota \in I_\alpha$, $J_\iota^\alpha - (J_\iota^\alpha)^- \leq J_\iota^\beta - (J_\iota^\beta)^-$,

and finally,

<u>3.1.32</u> Suppose that $\alpha < \varkappa$ and $\mathfrak{B} \in CH_\varkappa(\mathcal{M}^{<\varkappa})$ are such that

$$\mathcal{A} = \begin{tikzcd} \cup\mathfrak{B} \arrow[dr, dash] & & \cup\mathfrak{C}_\alpha \arrow[dl, dash] \\ & E & \end{tikzcd}$$

is an amalgam satisfying the hypotheses of of Lemma 3.1.5 with $f = 1_E$ and

<u>3.1.33</u> $X_1 \cup I_\alpha = X \leq I$ where $X_1 = {}^\circ\mathfrak{B}$;

Then, there exists $g \colon \cup\mathfrak{B} \to \cup\mathfrak{C}_\gamma$ for some $\gamma > \alpha$ such that g is a 1_{X_1}-embedding of $\mathfrak{B}$ into $\mathfrak{C}_\gamma$ and $g \equiv 1$ on E.

<u>Proof</u>. Lemma 3.1.26 guarantees that, for each $\gamma < \varkappa$, there are at most $\varkappa$ amalgams

$$\begin{tikzcd} \cup\mathfrak{B} \arrow[dr, dash] & & \cup\mathfrak{C}_\alpha \arrow[dl, dash] \\ & E & \end{tikzcd}$$

up to $\mathfrak{C}_\alpha$-isomorphism which require the existence of an injection g as in 3.1.32 into some $\mathfrak{C}_\gamma$, $\gamma > \alpha$; and, if $\gamma > \alpha$ and

$$\mathcal{A} = \begin{tikzcd} \cup\mathfrak{B} \arrow[dr, dash] & & \cup\mathfrak{C}_\gamma \arrow[dl, dash] \\ & E & \end{tikzcd}$$

with $\mathfrak{B}$ and E as above, then we can use Lemma 3.1.5 with $f = 1_E$ to obtain the chain $\mathfrak{A} = \mathfrak{C}_{\gamma+1}$ with $I_{\gamma+1} = X_1 \cup I_\gamma \leq I$ (where $X_1 = {}^\circ\mathfrak{B}$) such that 3.1.31

is satisfied by 3.1.5(2) and such that some 1_{X_1}-injection, g, of $\mathfrak{B}$ into $\mathfrak{C}_{\gamma+1}$ exists which extends $1_{\mathbf{E}}$. Since there are $\kappa^2 = \kappa$ non-limit steps available, all of the required injections can be built into the chains.

At limit ordinals $\lambda < \kappa$, the chain $\mathfrak{C}_\lambda$ is defined as follows.

<u>3.1.34</u> $\quad I_\lambda = \bigcup\{I_\gamma \mid \gamma < \lambda\}$ and, for all $\iota \in I_\lambda$,

$$J^\lambda_\iota = \bigcup\{J^\lambda_\iota \mid \gamma < \lambda \text{ and } \iota \in I_\gamma\} \text{ and}$$

$$(J^\lambda_\iota)^- = \bigcup\{(J^\gamma_\iota)^- \mid \gamma < \lambda \text{ and } \iota \in I_\gamma\}; \text{ and}$$

$$\mathfrak{C}_\lambda = \text{the completion of the chain } \{J^\lambda_\iota, (J^\lambda_\iota)^- \mid \iota \in I_\lambda\}.$$

The property 3.1.31 implies that, for all $\gamma < \lambda$ and $\iota \in I_\gamma$, $J^\gamma_\iota - (J^\gamma_\iota)^- \leq J^\lambda_\iota - (J^\lambda_\iota)^-$. Thus, $j\mathfrak{C}_\lambda = \{J^\lambda_\iota \mid \iota \in I_\lambda\}$ and <u>all the injections</u> g <u>built into the chains at previous steps still exist into</u> $\mathfrak{C}_\lambda$.

To define $\mathcal{J} \in \mathrm{INCH}^{\leq\kappa}_\kappa(\mathfrak{M})$ with $^\circ\mathcal{J} = I$, note that $\bigcup\{I_\gamma \mid \gamma < \kappa\} = I$ since the chains $\mathfrak{B}$ have arbitrary order types $X_1 \leq I$, $|X_1| < \kappa$. Now, the definition of $\mathcal{J}$ is carried out exactly as in 3.1.34 with $\lambda = \kappa$ and $\mathcal{J} = \mathfrak{C}_\kappa$. To check that $\mathcal{J}$ is κ-injective, suppose

<u>3.1.35</u>
$$\begin{tikzcd} \bigcup\mathfrak{B} \arrow[dr, no head] & & \bigcup\mathcal{J} \arrow[dl, no head] \\ & \mathbf{E} & \end{tikzcd}$$
is an amalgam as in 3.1.32 with $\mathcal{J}$ in place of $\mathfrak{C}_\alpha$.

Since $E \in \mathfrak{M}^{<\kappa}$ and κ is regular, we have $E \leq \bigcup\mathfrak{E}_\alpha$ for some $\alpha < \kappa$ and so

$$\begin{array}{ccc} \bigcup\mathfrak{B} & & \bigcup\mathfrak{E}_\alpha \\ & \diagdown \quad \diagup & \\ & E & \end{array}$$

satisfies 3.1.32 and, hence, the chain $\mathfrak{B}$ can be injected into $\mathcal{J}$ by our construction.

This chain $\mathcal{J}$ satisfies the definition 2.1.9 (p. 52) because we can identify E with $f(E)$ and $^\circ\mathcal{B}$ with $\bar{\eta}(^\circ\mathcal{B})$ to obtain an amalgam of the form 3.1.35 with $\mathfrak{B} = \mathcal{B}$.

§3.2 ω-Injective Chains of ULF Groups.

<u>Proof That the Class of Finite Groups Has the Subamalgam Property</u>. Recall that the proof of this is required to establish Proposition 2.2.9 on p. 69 and to satisfy the second hypothesis of Corollary 2.2.16 on p. 73.

<u>3.2.0</u> <u>Lemma</u>. Suppose $\mathcal{A}$ is a finite group amalgam and, for every subamalgam $\mathcal{A}_0$ of $\mathcal{A}$, there exists a finite group $M_0 = gp_0(\mathcal{A})$ such that $gp_0(\mathcal{A}_0) \cap \mathcal{A} = \mathcal{A}_0$. Then, there exists a finite group $M = \langle \mathcal{A} \rangle$ which satisfies Definition 2.1.14 of the subamalgam property.

<u>Proof</u>. Assume the hypotheses. For every subamalgam $\mathcal{A}_0$ of $\mathcal{A}$, let $\varphi_0 \colon gp_*(\mathcal{A}) \to M_0$ be a homomorphism such that $\varphi_0 \equiv 1$ on $\mathcal{A}$. Put $K = \bigcap\{\text{kernel } (\varphi_0) \mid \mathcal{A}_0$ is a subamalgam of $\mathcal{A}\}$. Thus, $M = gp_*(\mathcal{A})/K$ is finite; and, if $\mathcal{A}$ is identified with its image mod K, an easy argument shows that $gp_M(\mathcal{A}_0) \cap \mathcal{A} = \mathcal{A}_0$ for all subamalgams $\mathcal{A}_0$ of $\mathcal{A}$.

<u>3.2.1</u> <u>Lemma</u>. Suppose $\mathcal{A}_0 = \begin{tikzcd}[column sep=small] F_0 \arrow[dr, dash] & & H_0 \arrow[dl, dash] \\ & E_0 & \end{tikzcd}$ is a subamalgam of the finite amalgam $\mathcal{A} = \begin{tikzcd}[column sep=small] F \arrow[dr, dash] & & H \arrow[dl, dash] \\ & E & \end{tikzcd}$. There is a finite group $G = \langle \mathcal{A} \rangle$ such that $gp_G(\mathcal{A}_0) \cap \mathcal{A} = \mathcal{A}_0$.

<u>Proof</u>. We will use the notation of [16: §3] with A, B, and H replaced by F, H, and E respectively, that is, G will be constructed as a permutational product of F and H with a specific choice of left transversals, S for E in F, and T for E in H. We will choose S and T so that $S_0 \leq S$ where S_0 is a transversal for E_0 in F_0 and $T_0 \leq T$ where T_0 is a transversal for E_0 in H_0. Now $G = gp(\rho F, \rho H)$ where each $\rho(a)$, $a \in F$, and $\rho(b)$, $b \in H$, is a permutation of the set $S \times T \times E$ induced by translation: $(s,t,e)\rho(a) = (s',t',e')$ where $sea = s'e'$ and $(s,t,e)\rho(b) = (s,t',e')$ where $teb = t'e'$. It follows that every permutation in $gp(\rho F_0, \rho H_0)$ permutes the triples $S_0 \times T_0 \times E_0$ among themselves, but that no element of $\rho(F - F_0)$ or $\rho(H - H_0)$ has this property. Identifying F with ρF and H with ρH, the lemma is proved.

From these two lemmas we have

<u>3.2.2</u> <u>Corollary</u> [3: §3]. The class of finite groups has the s.p..

Next, we turn to the proof of the Maximality Theorem for the class of ℓ.f. groups.

Proof of Lemma 2.2.7(a), p. 68. First we will prove

3.2.3 $J_1 = \langle X_m \rangle$ for all $m \geq 2$.

$J_1 \cong H_\omega$ (see 2.2.9, p. 69) is generated by its elements of order n [2: p. 305]; so, to prove 3.2.3, it suffices to prove

3.2.4 For every $z \in J_1$ with $|z| = m$, we have $z \in \langle X_m \rangle$.

To Prove 3.2.4, let $z \in J_1$ with $|z| = m$, and let $G = \langle a \rangle \oplus \langle b \rangle$ be such that $|a| = |b| = m$ and $z = a + b$. By 2.1.18 and ω-injectivity of $\mathcal{J}$, there is an embedding $f\colon G \to J_1$ such that $f \equiv 1$ on $\langle z \rangle$ and $f(G) \cap J_1^- = \langle z \rangle \cap J_1^-$. Hence, $f(a), f(b) \in X_m$, proving 3.2.4 and 3.2.3.

Now suppose $x \in X_n$. In view of 3.2.3, to prove 2.2.7(a), namely $J_1 = \langle X_n(x) \rangle$, we need only prove

3.2.5 $X_m < \langle X_n(x) \rangle$ for some $m \geq 2$.

Proof of 3.2.5. Let m be a prime dividing n and suppose $w \in X_m$. We wish to show that $w \in \langle X_n(x) \rangle$. Since $x \in X_n(x)$, we can assume WLOG

3.2.6 $w \notin \langle x \rangle$.

Let $H = \langle a \rangle \oplus \langle b \rangle$ be such that $|a| = |b| = n$ and

3.2.7 the amalgam $\mathcal{A} =$
$$\begin{tikzcd} \langle x,w \rangle & & H \\ & \langle w \rangle \arrow[ul] \arrow[ur] & \end{tikzcd}$$
exists, where $w = (ab)^{n/m}$.

By the s.p., let $G = \langle \mathcal{A} \rangle$ be a finite group such that $\langle \mathcal{A}_0 \rangle \cap \mathcal{A} = \mathcal{A}_0$ for every subamalgam $\mathcal{A}_0$ of $\mathcal{A}$. By Lemma 2.1.18 there is an embedding g of G into J_1 such that

<u>3.2.8</u> $g \equiv 1$ on $\langle x,w \rangle$ and $g(G) \cap J_1^- = \langle x,w \rangle \cap J_1^-$.

From 3.2.6 and 3.2.7 we see that

<u>3.2.9</u> $$\begin{tikzcd} \langle x \rangle \arrow[dr, dash] & & \langle a \rangle \arrow[dl, dash] \\ & 1 & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \langle x \rangle \arrow[dr, dash] & & \langle b \rangle \arrow[dl, dash] \\ & 1 & \end{tikzcd}$$ are subamalgams

of $\mathcal{A}$,

and, by the s.p., we have

<u>3.2.10</u> $\langle x,a \rangle \cap \langle x,w \rangle = \langle x \rangle$ and $\langle x,b \rangle \cap \langle x,w \rangle = \langle x \rangle$.

Now, 3.2.8 and 3.2.10 imply $g(\langle x,a \rangle) \cap J_1^- = g(\langle x,a \rangle) \cap \langle x,w \rangle \cap J_1^- = \langle x \rangle \cap J_1^- = 1$ since $g \equiv 1$ on $\langle x,w \rangle$ and $x \in X_n$. Hence, $g(a) \in X_n(x)$, and, similarly, we have $g(b) \in X_n(x)$. Since $w = g(w) \in g(H) = \langle g(a), g(b) \rangle$, 3.2.5 follows.

<u>Proof of 2.2.7(c), p. 68</u>. Let $a \in J_1 - J_1^-$. Since parts (a) and (b) of 2.2.7 have already been proved for all n, if we now prove

<u>3.2.11</u> $\langle a,J \rangle \cap X_m \neq \emptyset$ for some $m \geq 2$,

it will follow immediately that $\langle a,J \rangle = J_1$ proving (c).

<u>Proof of 3.2.11</u>. Put $|a| = n$ and $\langle a \rangle \cap J_1^- = \langle a^m \rangle$. Let G be the group with presentation $(a,b\colon a^n = b^n = 1,\ b^a = b,\ a^m = b^m)$. Using 2.1.18, we obtain $g\colon G \to J_1$ such that $g \equiv 1$ on $\langle a \rangle$ and $g(G) \cap J_1^- = \langle a^m \rangle$. Let us identify G with $g(G)$. Let $M = \langle G,t \rangle$ where $a^t = b$, $b^t = a$, and $|t| = 2$. Note that $\langle t \rangle \oplus \langle a^m \rangle$ exists in M. Using the

method on p. 66, we obtain $f\colon M \to J_1$ such that $f \equiv 1$ on G and $f(t) \in J$. Since $G \cap J_1^- = \langle a^m \rangle = \langle b^m \rangle$, we have $ab^{-1} \in X_m \cap \langle a, f(t) \rangle$, proving 3.2.11.

<u>Proof of the Theorem on Uniqueness of Automorphisms 2.2.17, p. 74, for the class of Locally Finite Groups</u>. Suppose $\alpha \in \mathrm{Aut}(\bigcup\mathcal{J})$ and, for some $J \in j\mathcal{J}$, $\alpha \equiv 1$ on J. We will prove

<u>3.2.12</u> For every element $x \in \bigcup\mathcal{J}$ of order 2, $\alpha(x) = x$.

This will imply that $\alpha = 1_{\bigcup\mathcal{J}}$ since $\bigcup\mathcal{J}$ is generated by its elements of order 2.

<u>Proof of 3.2.12</u>. Suppose $x \in \bigcup\mathcal{J}$ has order 2. Since we wish to show that $\alpha(x) = x$ we can assume WLOG that $x \notin J$ and that

<u>3.2.13</u> $x \in Q - Q^-$ where $J < Q \in j\mathcal{J}$.

Suppose $\alpha(x) \neq x$. Thus, by the Maximality Theorem,

<u>3.2.14</u> $\alpha(x) \in Q - Q^-$.

Let G be a finite subgroup of Q such that $x, \alpha(x) \in G$ and $G \cap (J - J^-) \neq \emptyset$, let $H = \langle t \rangle \oplus \langle x \rangle$ where $t \neq 1$ has finite order, and let

$$\mathcal{A} = \begin{tikzcd}[column sep=small] G & & H \\ & \langle x \rangle \arrow[ul, no head] \arrow[ur, no head] & \end{tikzcd}.$$

Let $M = \langle \mathcal{A} \rangle$ be a finite group such that $\langle \mathcal{A}_0 \rangle \cap \mathcal{A} = \mathcal{A}_0$ for all subamalgams $\mathcal{A}_0$ of $\mathcal{A}$. Since $\alpha(x) \notin \langle x \rangle$, t and $\alpha(x)$ do not commute in $gp_*(\mathcal{A})$, and we can assume WLOG that

<u>3.2.15</u> t and $\alpha(x)$ do not commute in M

because $gp_*(\mathcal{A})$ has a free subgroup of finite index [6: pp. 227-228] and, hence, M can be replaced by a larger finite homomorphic image of $gp_*(\mathcal{A})$ in which t and $\alpha(x)$ do not commute.

Recalling that $j\mathcal{J} = \{J_\alpha | \alpha \in I\}$, we define, as in the proof on p. 66, for each $\alpha \in I_G = \{\alpha \in I | G \cap (J_\alpha - J_\alpha^-) \neq \emptyset\}$, $G_\alpha = G \cap J_\alpha$ and $M_\alpha = \langle t, G_\alpha \rangle$. Since

$$\begin{tikzcd}[column sep=small] G \cap Q^- \arrow[dr, dash] & & \langle t \rangle \arrow[dl, dash] \\ & 1 & \end{tikzcd}$$

is a subamalgam of $\mathcal{A}$, the subamalgam property of $M = \langle \mathcal{A} \rangle$ implies $\langle G \cap Q^-, t \rangle \cap G = G \cap Q^-$ and, also,

<u>3.2.16</u> For all $\alpha \in I_G$ such that $G_\alpha \neq G$, we have

$$\langle G_\alpha, t \rangle \cap G = G_\alpha.$$

Using 3.2.16, the proof on p. 66 can be copied (lines 11-27; 3.2.16 is used in place of Britton's Lemma) to obtain the embedding $f: M \to Q$ such that $f \equiv 1$ on G and $f(t) \in J$.

Now, $x = f(x)$ commutes with $f(t)$; but, $\alpha(x) = f(\alpha(x))$ does not commute with $f(t)$ by 3.2.15. Since $f(t) \in J$, we have $\alpha(f(t)) = f(t)$ and $\alpha(\langle f(t), x \rangle) = \langle f(t), \alpha(x) \rangle$. This contradiction proves 3.2.12.

§3.3 ω-Injective Chains of Algebraically Closed Groups.

In this section we will give proofs for Lemma 2.2.5 on p. 67 which establishes the Maximality Theorem for classes obtained from a.c. groups and for the Uniqueness of

Automorphisms Theorem 2.2.17 on p. 74 for these same classes. For both proofs we will assume the hypotheses of Lemma 2.2.5, namely that

<u>3.3.0</u> G is a non-trivial countable a.c. group, $\mathcal{M}$ is the inductive closure of the class of subgroups of G, and $\mathcal{J} \in \mathrm{INCH}_{\omega}(\mathcal{M})$ with $|j\mathcal{J}| \geq 2$.

Notice that it is not at all apparent which a.c. groups G are capable of satisfying this hypothesis. The existence of $\mathcal{J}$ implies by 2.1.13 that $\mathcal{M}^{<\omega}$ has the strong amalgamation property, which we cannot expect to hold in an arbitrary a.c. group G. On the other hand, the existence of $\mathcal{J}$ of any countable order type is implied by the subamalgam property for $\mathcal{M}^{<\omega}$, which we know to hold in case G is an f.e.p. group (see 1.4.6, p. 31) because $\mathcal{M}$ is a free ω-class (see 1.4.21, p. 42). Our proofs will make use of (besides the existential assumption 3.3.0) only properties which every a.c. group possesses, namely the existence of certain subgroups given by a result of B.H. Neumann.

<u>3.3.1</u> <u>Lemma</u> [18]. Every group with a solvable word problem is embeddable in every non-trivial a.c. group.

In our proofs we will need to utilize some of the following groups, all of which have solvable word problems and, so, belong to $\mathcal{M}$.

<u>3.3.2</u> (a) finite groups;

(b) the infinite cyclic group;

(c) $gp_*(\mathcal{A})$ where $\mathcal{A} = \begin{tikzcd}[column sep=small] B \arrow[dr, no head] & & C \arrow[dl, no head] \\ & A & \end{tikzcd}$, B and C have solvable word problems, and A is finite; and

(d) the HNN extension H_φ where H has a solvable word problem and φ is an isomorphism of finite subgroups of H.

These groups have solvable word problems because, in each case, there is an obvious algorithm for reducing products to a normal form (in the case of a finite group the multiplication table is the algorithm).

Proof of 2.2.5(a). Because every non-trivial a.c. group is generated by its elements of order 2, this proof can be carried out identically to that of 3.2.3 on p. 95 because only injections of finite groups were used, and $\mathfrak{M}$ contains the class of finite groups.

Proof of 2.2.5(b). Since $H = \langle x,y \rangle$ is either a finite dihedral group or $Z_2 * Z_2$ (the infinite dihedral group), H has a solvable word problem and, hence, so does $H_\varphi = \langle H,t \rangle$, the HNN extension with $\varphi \in ISO(\langle x \rangle, \langle y \rangle)$. Hence $H_\varphi \in \mathfrak{M}$ and the proof of 2.2.2(b) on p. 66 is applicable with $M = H_\varphi$. We obtain, as there, an embedding $f\colon H_\varphi \to J_1$ such that $f \equiv 1$ on H and $f(t) \in J$ and the desired conclusion follows.

Proof of 2.2.5(c). Let $H = \langle a \rangle * \langle t \rangle$ where $|t| = 2$. Thus, H has a solvable word problem and $H \in \mathfrak{M}$. By ω-injectivity of $\mathcal{J}$, there is an embedding $f\colon H \to J_1$ such that $f \equiv 1$ on $\langle a \rangle$, $f(t) \in J$, and, for all $Q \in j\mathcal{J}$ such that $J \leq Q \leq J_1$, $f(H) \cap Q = \langle \langle a \rangle \cap Q, f(t) \rangle$. The exact details are similar to those on p. 66 and depend, of course, on the normal form theorem for H. Thus, $f(H) \cap J_1^- = \langle \langle a \rangle \cap J_1^-, f(t) \rangle$ and, since $a \notin J_1^-$, we have $a^{-1}f(t)a \notin J_1^-$. Hence $a^{-1}f(t)a \in X_2 \cap \langle a, J \rangle$, completing the proof.

<u>Proof of 2.2.17</u>. The proof is the same as that on pp. 97 and 98 of this same theorem for the class of locally finite groups with two differences: (1) we do not need to demand that $G \cap (J - J^-) \neq \emptyset$; this was done only as a matter of convenience on p. 66; hence, we can put $G = \langle x, \alpha(x) \rangle$, a group with solvable word problem; and (2) we put $M = gp_*(\mathcal{A}) \in \mathfrak{M}$ by 3.3.2(c).

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