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ABSTRACT

THE OPTIMUM ANALYTICAL DESIGN OF TRANSIENT EXPERIMENTS FOR SIMULTANEOUS DETERMINATIONS OF THERMAL CONDUCTIVITY AND SPECIFIC HEAT

by James Vere Beck

Accurate values of thermal properties of new and conventional materials are indispensable for the design of space vehicles. In order to (a) determine the thermal conductivity and specific heat for some modern materials which degrade when heated and (b) provide a more rapid method of simultaneously determining both these properties, a transient experiment and a method of analysis are needed.

A method of calculating the properties from transient temperature and heat flux measurements has been developed and includes (a) the calculation of temperatures and (b) the iteration procedure called nonlinear estimation. The temperatures can be accurately calculated using a modified Crank-Nicloson finite-difference approximation applied to the heat conduction equation. The nonlinear estimation procedure is an extension of linear regression analysis. Properties can be readily calculated with less than 0.1% error due to approximations in the numerical procedure.

To determine the optimum experiment a criterion is developed and is applied to find the optimum boundary conditions and locations of thermocouples. It is proved that both thermal properties can not be and the second s

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independently calculated from temperature measurements in a homogeneous body unless the heat flux is measured. The optimum heat flux boundary condition is one which causes a step rise in the surface temperature. The optimum locations for the thermocouples for a finite body are at the boundaries. Constant and temperature-variable thermal conductivity and specific heat are calculated from experimental data for nickel and copper.

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THE OPTIMUM ANALYTICAL DESIGN OF TRANSIENT EXPERIMENTS FOR SIMULTANEOUS DETERMINATIONS OF THERMAL CONDUCTIVITY AND SPECIFIC HEAT

Ву

James Vere Beck

A THESIS

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To his wife Barbara, the author dedicates this dissertation for her understanding and cooperation during graduate study and research.

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I. DESCRIPTION OF THE PROBLEM

1.1 Introduction

In the space program the values of thermal properties of new and conventional materials are indispensable for the design of components such as reentry vehicle heat shields, rocket nozzles and others. With the development of a host of new materials some of which degrade when heated, a rapid transient method of determining thermal properties is needed. The method should be rapid, be amenable to automation and be competitive in cost with present methods of measuring thermal properties. With the advent of the large scale digital computer it is not necessary that the computations associated with the method be simple enough to be readily performed by hand calculations. The method given herein is intended to help meet this need. The method is not restricted, however, to the determination of only thermal properties but can be utilized for finding a wide category of physical properties.

The basic objective is to investigate the design of optimum experiments for the simultaneous determination of thermal conductivity and specific heat of solids from transient temperature measurements. The thermal properties can be functions of temperature. An "optimum" experiment permits the properties to be determined more accurately than any other similar experiment with the same temperature range and duration of the experiment. The large-scale digital computer is to be used whenever needed and thus the optimum experiment

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need not be one possessing a solution convenient for hand calculations. On the other hand, the optimum experiment should have boundary conditions that can be simulated readily in the laboratory.

Literally hundreds of papers have been written on the subjects of determining thermal conductivity k, specific heat c_p and density p [1-3]. Recently a number of papers have been written about the determination of thermal diffusivity, $a = k/p c_p$ [4-10]. Thermal conductivity can be found using solely steady-state measurements while c_p and a require transient measurements. To determine k and c_p the heat flux and temperatures must be measured; while for a temperatures alone are required. Within these limitations the methods proposed for determining thermal properties are quite varied. There are a number of respects in which the present work differs substantially from the preceding, however. Some of these differences are listed below.

a) Each previous method uses a relatively simple exact solution for the temperature-distribution to facilitate the calculation of the properties (except [10]).

b) Because of a), certain boundary conditions must be maintained and heat losses must be reduced at the heated surface. These heat losses for certain transient cases can be difficult to eliminate.

c) All the solutions employ the assumption that the thermal properties do not vary with temperature.

d) Only a limited number of temperature measurements are used, usually only two or three. This is frequently true even for transient techniques for which temperature histories are obtained with one, two or more thermocouples. Not all the data is used; or if it is,

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e) Questions relating to the optimum heat flux, heating time or best location of thermocouples for the transient cases are not investigated theoretically.

f) Except for a few papers [6, 11, 12] each experiment isdesigned to determine only a single property.

The method to be described utilizes the finite-difference solution of the heat-conduction equation and thus is flexible enough to treat any given time-variable heat flux for constant or temperature-dependent properties. Boundary conditions are suggested for which heat losses are negligible. <u>All</u> the transient temperatures measured are utilized in a straight forward manner using a method which is an extension of least-squares analysis. It is called "nonlinear estimation." A criterion for determining the optimum conditions is developed and employed. Particular emphasis is given to the determination of both thermal conductivity k and specific heat c_p from a single experiment. Actually we shall find k and ρc_p from the experiment. For convenience let

$$c = \rho c_n \tag{1.1.1}$$

The density ρ is relatively easy to measure accurately from gravimetric and linear measurements. Moreover, the density is much less sensitive to temperature than either k or c_p . For this reason the effect of elongation of the body during heating upon the temperature distribution is usually negligible and is not considered in this analysis.

Parker and co-workers [6] utilized a pulse of high-intensity short-duration light impinging on the blackened surface of the specimen.

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The other surface, which was insulated, was instrumented with a thermocouple from which the temperature data was obtained. The relatively simple analytical solution was derived for the temperature at the insulated surface for a body exposed to heat pulse of known magnitude. This solution was then utilized to calculate the thermal diffusivity $\mathbf{a} = \mathbf{k/c}$, thermal conductivity k and heat capacity c. The accuracy obtained was about $\frac{1}{2}$ 5%.

A Russian, Smekalin [11], considered two semi-infinite specimens of the same material separated by an electrical heater producing a known constant heat flux. A thermocouple was located inside one of the specimens 7-10 mm. from the interface and the temperature was measured at times 2, 3, 4, 5 and 6 minutes. From temperatures at two different times the thermal diffusivity and thermal conductivity were determined. He reported accuracies of $\frac{1}{2}$ 3% for a and $\frac{1}{2}$ 5% for k.

Hsu [12] performed an experiment with two semi-infinite solids initially at two different uniform temperatures. If one of the solids (the standard) has known thermal properties, then properties can be determined for the other (the specimen). The experiment was begun by suddenly pressing the two solids together. Several thermocouples were carefully placed in the specimen at different positions and the temperatures were measured at a number of instants. When the two semi-infinite bodies were brought into contact, the temperature at the interface suddenly increased to some intermediate temperature between the initial temperature of the two solids. If the properties were independent of temperature remained constant as long as the bodies were in contact. He performed a very careful experiment with

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1.2 Problem Definition

In this dissertation a number of problem areas relative to the efficient determination of the thermal properties are investigated. They are suggested in this section and analyzed later.

Boundary conditions: For simplicity the bodies considered will have a plane, one-dimensional heat flow. The boundary conditions necessary to separately calculate k and c are determined and those that are insufficient are also indicated. Typical boundary conditions are: given temperatures, prescribed insulated surface, prescribed heat flux and convective heating.

Location of thermocouples: The optimum location and number of thermocouples are determined. The optimum thickness of a finite specimen for a given heating time is also found.

<u>Development of a general criterion</u>: In order to determine an "optimum" experiment some measure is needed of the effectiveness of an experiment to determine thermal properties. This measure would provide a criterion to be used to determine if one experiment is superior to another. A criterion is developed and applied for a number of different boundary conditions.

Effect of temperature errors: The calculation of the properties can be affected by errors from the following sources: inaccurate temperature measurement, errors in the finite-difference calculations and an imperfect model. The effect of errors due to the first two cases are examined. For most materials the heat-conduction equation (or model)

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:::::::e i it âl ت و. _{ام} ا u e e 91.7**4**11 389 an Ś . 19 met jas dec 202 **1** Rođe En sin Seeter Sectia ziele I describes the heat flow very accurately and thus errors from this source are not considered. An improved model is needed for decomposing materials such as are used in heat shields of reentry vehicles. That is beyond the present scope of this work, however.

1.3 Nonlinear Estimation Procedure

The problem of calculating parameters appearing in a differential equation which describes a physical process is called "nonlinear estimation." Supplied with experimental data obtained from the process and a method of solving the differential equation it is possible using the nonlinear estimation procedure to calculate the parameters. The method has been developed from a statistical viewpoint in the past decade by G. E. P. Box and co-workers [13-16]. The first known reference to the method was written by Gauss in 1809 [17]. Due to the large number of calculations involved it has only recently become practical with the advent of the large-scale digital computer. Box's work is mainly related to the statistical development of the method rather than the application to any specific case. He has, however, given some examples involving first-order ordinary differential equations arising in chemical engineering. He has not given an application utilizing a partial differential equation.

The governing equation describing one-dimensional heat conduction is

$$\frac{\partial}{\partial \mathbf{x}} \left(\mathbf{k} \ \frac{\partial \mathbf{T}}{\partial \mathbf{x}} \right) = \mathbf{c} \frac{\partial \mathbf{T}}{\partial \theta}$$
(1.3.1)

where T is temperature, x is position and θ is time. The thermal properties, thermal conductivity k and heat capacity c, are to be determined. For continuous transient temperature measurements

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the sum of squares function F for n thermocouples,

$$F(k,c) = \sum_{j=1}^{n} A_{j} \int_{0}^{\theta} (T_{j}(\theta) - T_{e,j}(\theta))^{2} d\theta \qquad (1.3.2)$$

is to be minimized with respect to k and c. The temperature $T_j(\theta)$ is found using a finite-difference approximation for (1.3.1) at position x_j and time θ ; it is a function of k and c. Temperature $T_{e,j}(\theta)$ is the experimental temperature at x_j and θ . The quantity A_j is a weighting factor for each position and is frequently equal to unity. The sum of squares function F is a parametric function with the properties k and c being the parameters.

The properties can, in general, be prescribed functions of temperature. The method of minimizing F can be illustrated by assuming k and c as constants. A number of procedures have been suggested for minimizing F; they include the method of steepest descent and modifications to the Taylor series approach [13, 15, 17-19]. For a well-designed experiment and with an estimate in error less than $\frac{1}{2}$ 30% of the correct properties, the Taylor series approach permits very rapid convergence. The Taylor series approach in its simplest form has proven to be adequate in our work; it is outlined below.

The calculated temperature is a nonlinear function of k and c. The Taylor series method is an iterative procedure, however, which assumes at each step that the temperature is a linear function of k and c, or

$$T(k,c) \approx T \Big|_{k_0,c_0} + T_k \Delta k + T_c \Delta c \qquad (1.3.3)$$

For convenience the subscript j and functional dependence upon θ have

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been omitted. The derivatives T_k and T_c are defined by

$$T_{k} = \frac{\partial T(k,c)}{\partial k} \Big|_{c_{0},k_{0}}$$
(1.3.4)
$$\frac{\partial T(k,c)}{\partial k} \Big|_{c_{0},k_{0}}$$

$$T_{c} = \frac{\partial T(k, c)}{\partial c} \Big|_{c_{o}, k_{o}}$$
(1.3.5)

and Δk and Δc are given by

$$\Delta k = k - k_0 \tag{1.3.6}$$

$$\Delta c = c - c_0 \qquad (1.3.7)$$

The properties k_0 and c_0 are the zeroth estimate of k and c. At the point (k*, c*) at which F is a minimum the first derivatives of F with respect to k and to c are equal to zero. These two equations can then be solved simultaneously for the first correction to the properties k and c. These corrections are given by (see Appendix A)

$$\Delta \mathbf{k} = \frac{1}{\Delta} \begin{bmatrix} \begin{pmatrix} n \\ \Sigma \\ j=1 \end{pmatrix}^{n} \mathbf{A}_{j} \int_{0}^{\theta} \mathbf{T}_{c,j} (\mathbf{T}_{j} - \mathbf{T}_{e,j}) d\theta \end{pmatrix} (\mathbf{T}_{k}, \mathbf{T}_{c})$$
$$- \begin{pmatrix} n \\ \Sigma \\ j=1 \end{pmatrix}^{n} \mathbf{A}_{j} \int_{0}^{\theta} \mathbf{T}_{k,j} (\mathbf{T}_{j} - \mathbf{T}_{e,j}) d\theta \end{pmatrix} (\mathbf{N}\mathbf{T}_{c})]; \Delta \neq 0 \qquad (1.3.8)$$

$$\Delta c = \frac{1}{\Delta} \begin{bmatrix} \prod_{j=1}^{n} A_{j} \int_{0}^{\theta_{m}} T_{k,j} (T_{j} - T_{e,j}) d\theta \end{bmatrix} (T_{k}, T_{c})$$

$$- \prod_{j=1}^{n} A_{j} \int_{0}^{\theta_{m}} T_{c,j} (T_{j} - T_{e,j}) d\theta \end{bmatrix} (NT_{k})]; \Delta \neq 0 \qquad (1.3.9)$$

where

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$$\Delta = (NT_k) (NT_c) - (T_k, T_c)^2 \qquad (1.3.10)$$

$$NT_{k} = \sum_{j=1}^{n} A_{j} \int_{0}^{0} (T_{k,j})^{2} d\theta \qquad (1.3.11)$$

$$NT_{c} = \sum_{j=1}^{n} A_{j} \int_{0}^{0} (T_{c,j})^{2} d\theta \qquad (1.3.12)$$

$$(T_k, T_c) = \sum_{j=1}^{n} A_j \int_{0}^{0} (T_{k,j}T_{c,j})d\theta$$
 (1.3.13)

The integrals NT_k and NT_c are norms of T_k and T_c and (T_k, T_c) is an inner product. In the simplest iterative procedure improved values of k and c are given by

$$k_1 = k_0 + \Delta k$$
 (1.3.14)

$$c_1 = c_0 + \Delta c \tag{1.3.15}$$

after which another smaller pair of values Δk and Δc are obtained. The iteration procedure is continued until, say,

$$\frac{\Delta k}{k} < 0.0001$$
 (1.3.16)

$$\frac{\Delta c}{c} < 0.0001$$
 (1.3.17)

For a well-designed experiment and with the initial estimate of k and c about 10% in error, only three iterations are usually necessary to satisfy (1.3.16) and 1.3.17). On the next iteration after satisfying these relations $\Delta k/k$ and $\Delta c/c$ are much smaller -- about 0.00001. In any new experiment F-values should be calculated surrounding the point which is thought to be minimum to verify that it is actually the local minimum. Cases for which a local minimum is not the true • 2 18 (1**11**2) nate th . jettes : 11 mei 1.iim 242.007 31, 1s Rerage ∺s: e]tocedo minimum have not been found for well-designed experiments, however.

If temperature measurements $T_{e,j}$ are not made continuously with time but at discrete times (which is the most common case), then the integrals in (1.3.2) and subsequent equations would be replaced by a summation over time. For example, (1.3.11) would become

$$NT_{k} = \sum_{j=1}^{n} A_{j} \sum_{\ell=1}^{m} T^{\ell}_{k,j}$$
(1.3.18)

where the l superscript is for time $l \Delta \theta$ for l = 1, 2, ... m.

A tendency of some investigators who measure thermal properties is to use only part of the data [6, 11]. The nonlinear estimation method by using all the transient temperature data reduces the errors in the calculated property if the errors in the data are random which is a common case. (If the errors are not random, then corrections for the biased errors can be made.) Others, like Hsu, use all the data by repeatedly applying a simple equation to calculate a number of values for the properties which are then averaged. (See section 6.3) It can be proved, however, that the "best" estimates of the parameters are given by the least-squares procedure incorporated in nonlinear estimation [20].
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II. GENERAL RELATIONS BETWEEN TEMPERATURE RISE AND PROPERTY DERIVATIVES

2.1 Relations for Properties k and c

Much helpful information can be obtained from general relations between the temperature, T, and the derivatives

$$T_{k} = \frac{\partial T}{\partial k} \Big|_{c}$$
; $T_{c} = \frac{\partial T}{\partial c} \Big|_{k}$ (2.1.1)

This information is useful to aid in determining the optimum boundary conditions, to help check accuracy of numerical calculations and to provide certain insights into the property-determination problem. We will restrict our attention to a one-dimensional, Cartesian system and to constant thermal properties k and c. (Temperature-variable thermal properties are considered in chapter VII.)

Let us first consider the transient heat-conduction equation

$$k \frac{\partial^2 T}{\partial x^2} = c \frac{\partial T}{\partial \theta}$$
 (2.1.2)

with the temperature boundary conditions,

$$T(0,\theta) = T_{0}(\theta) \qquad (2.1.3)$$

$$T(L,\theta) = T_{I}(\theta) \qquad (2.1.4)$$

and the initial condition,

$$T(x, 0) = T_{i}(x)$$
 (2.1.5)

The temperatures $T_{0}(\theta)$, $T_{1}(\theta)$ and $T_{1}(x)$ are known.

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12. 1. j In order to simplify the notation, the operator D is defined

$$D = k \frac{\partial^2}{\partial x^2} - c \frac{\partial}{\partial \theta}$$
 (2.1.6)

and then (2.1.2) can be written

$$DT = 0$$
 (2.1.7)

Equations for the calculation of T_c are obtained by taking the partial derivative of eqs. (2.1.2) to (2.1.5) with respect to c holding k constant; we obtain

$$DT_{c} = \frac{\partial T}{\partial \theta}$$
 (2.1.8)

$$T_{c}(0,\theta) = T_{c}(L,\theta) = T_{c}(x,0) = 0$$
 (2.1.9)

In a similar manner taking the partial derivative with respect to k yields

$$DT_{k} = -\frac{\partial^{2}T}{\partial x^{2}} = -\frac{1}{a} \frac{\partial T}{\partial \theta}$$
(2.1.10)

$$T_{k}(0,\theta) = T_{k}(L,\theta) = T_{k}(x,0) = 0$$
 (2.1.11)

By comparing eqs. (2.1.8) and (2.1.9) with (2.1.10) and (2.1.11) after multiplying the latter two by -a, we observe that

$$T_c = -a T_k$$

or

$$cT_{c} + kT_{k} = 0$$
 (2.1.12)

If the insulation boundary condition, viz.,

$$0 = \frac{T6}{x6}$$

were to replace one or both of the temperature boundary conditions (2.1.3) or (2.1.4), relation (2.1.12) would still be obtained. Further-

as

more, this relation can be shown to be valid also for these boundary conditions if the body is semi-infinite.

Other very important boundary conditions are in terms of prescribed heat fluxes at the boundaries, viz.,

$$-k \frac{\partial T(0, \theta)}{\partial x} = q_0(\theta) \qquad (2.1.13)$$

$$-k \frac{\partial T(L, \theta)}{\partial x} = q_{L}(\theta) \qquad (2.1.14)$$

The initial temperature distribution considered here is simply

$$T(x, 0) = T_{i} = constant$$
 (2.1.15)

rather than the more complicated boundary condition (2.1.5).

Taking the partial derivative with respect to c of (2.1.2), (2.1.13), (2.1.14) and (2.1.15) gives

$$D(-cT_{c}) = -c \frac{\partial T}{\partial \theta} \qquad (2.1.16)$$

$$\frac{\partial (-cT_c(0,\theta))}{\partial x} = \frac{\partial (-cT_c(L,\theta))}{\partial x} = -cT_c(x,0) = 0 \qquad (2.1.17)$$

In a similar manner taking the partial derivative with respect to k yields

$$D(-kT_{k}) = c \frac{\partial T}{\partial \theta}$$
(2.1.18)
$$-k \frac{\partial T_{k}(0,\theta)}{\partial x} - \frac{\partial T(0,\theta)}{\partial x} = 0$$

or using (2.1.13),

$$k \frac{\partial T_k(0,\theta)}{\partial x} = \frac{q_0(\theta)}{k}$$

and multiplying by -k gives

$$-k \frac{\partial (-kT_k(0,\theta))}{\partial x} = q_0(\theta) \qquad (2.1.19)$$

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and similarly

$$-k \frac{\partial (-kT_k(L,\theta))}{\partial k} = q_L(\theta)$$
 (2.1.20)

The initial condition can be written

$$-kT_{k}(x,0) = 0$$
 (2.1.21)

We note that the partial differential equations (2.1.16) and (2.1.18) contain the same (except for sign) non-homogeneous term which can be evaluated from the solution of (2.1.2), (2.1.13), (2.1.14) and (2.1.15). In a standard method of solution of linear, non-homogenous partial differential equations [21] the general problem is split into three distinct problems and then the results added to obtain the desired solution. Since these equations are linear, superposition is valid. Then the temperature rise $T-T_i$ with heat flux boundary conditions can be shown to be related to cT_c and kT_k by

$$T - T_i = -cT_c - kT_k$$
 (2.1.22)

This result can be obtained readily from Table 2.1.1 since the sum of the non-homogeneous terms for $-cT_c$ and $-kT_k$ gives the partial differential equation and boundary conditions for $T - T_i$. If the body is insulated at one boundary only or if the body is semi-infinite $(L \rightarrow \infty)$, (2.1.22) still applies. It is not valid for the heat flux given on one boundary and a prescribed temperature history on the other, however. (Incidentally, (2.1.22) applies if T varies with the radius in the cylindrical or spherical coordinate systems.)

It is proved in section 2.3 that the relation derived for the temperature boundary condition, (2.1.12), indicates that the properties k and c can not be separately determined for this boundary condition.

Table 2.1.1 Partial differential equations and boundary conditions for $T-T_i$, $-cT_c$ and $-kT_k$

Function	P.D.E.	Boundary Conditions
T - T _i	$D(T-T_i) = 0$	$-k \frac{\partial (T (0, \theta) - T_{i})}{\partial x} = q_{0} (\theta)$
	(T _i = constant)	$-k \frac{\partial (T (L, \theta) - T_{i})}{\partial x} = q_{L}(\theta)$
		$T(x, 0) - T_{i} = 0$
-cT _c	$D(-cT_c) = -c\frac{\partial T}{\partial \theta}$	$\frac{\partial (-cT_c(0,\theta))}{\partial x} = 0$
		$\frac{\partial(-cT_{c}(L,\theta))}{\partial x} = 0$
		$-cT_{c}(\mathbf{x},0) = 0$
-kTk	$D(-kT_k) = c \frac{\partial T}{\partial \theta}$	$-k \frac{\partial (-kT_k(0,\theta)}{\partial x} = q_0(\theta)$
		$-k \frac{\partial (-kT_k(L, \theta))}{\partial x} = q_L(\theta)$
		$-kT_{k}(x,0) = 0$
Note:	2	
	$D = k \frac{\partial^2}{\partial x^2} - c \frac{\partial}{\partial \theta}$	

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For the heat flux boundary the important relation (2.1.22) applies; this relation has a number of uses as will be suggested later.

2.2 Surface Temperature of a Semi-Infinite Body with Heat Flux Boundary Condition

The surface temperature rise of a semi-infinite body which is heated with a constant heat flux is (see section 4.1)

$$T - T_{i} = 2q_{0}(\theta / kc \pi)^{\frac{1}{2}}$$
 (2.2.1)

where T_i is the uniform initial temperature and q_0 is the constant heat flux beginning at time $\theta = 0$ (for $\theta < 0$, $q_0 = 0$). For this case kT_k and cT_c are equal and are given by

$$kT_{k} = -q_{0}(\theta / kc_{\pi})^{\frac{1}{2}}$$
 (2.2.2)

$$cT_{c} = -q_{0}(\theta / kc_{\pi})^{\frac{1}{2}}$$
 (2.2.3)

and thus

$$kT_{k} = cT_{c}$$
 (for x = 0 only) (2.2.4)

or

$$kT_k - cT_c = 0$$

From (2.2.1), (2.2.2) and (2.2.3) we find

$$T - T_i = -2kT_k = -2cT_c$$
 (for x = 0) (2.2.5)

which is consistent with the relation

$$T - T_i = -kT_k - cT_c$$
 (2.2.6)

which was derived for a homogeneous, constant property, finite or semi-finite body with heat flux boundary conditions. Eqs. (2.2.4)and (2.2.5) are given specifically for the special case of the surface temperature of a semi-infinite body with q = C.

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We now demonstrate the validity of (2.2.4) for an arbitrary heat flux. The superposition integral [22] for a given heat flux at a boundary and uniform initial temperature can be given by

$$T(x,\theta) - T_{i} = \int_{0}^{\theta} q(\lambda) \frac{\partial A(x,\theta-\lambda)}{\partial \theta} d\lambda \qquad (2.2.7)$$

where $q(\theta)$ is a known, but arbitrary time-variable heat flux and A is the temperature response for a unit constant heat flux. Let us consider only the heated surface, x = 0, for which A can be calculated from (2.2.1),

$$A = (1/kc\theta \pi)^{\frac{1}{2}}$$
 (2.2.8)

Then (2.2.7) becomes

$$T(0,\theta)-T_{i} = (\pi kc)^{-\frac{1}{2}} \int_{0}^{\theta} q(\lambda)(\theta - \lambda)^{-\frac{1}{2}} d\lambda \qquad (2.2.9)$$

By taking the derivative of (2.2.9) with respect to k and c it can be shown that (2.2.4) is valid at the surface of a semi-infinite body initially at a uniform temperature and exposed to an <u>arbitrary</u> timevariable heat flux. Since (2.2.6) is true for still more general conditions, (2.2.5) also is true.

Another position for which (2.2.4) applies is for the interface between a finite body of known thermal properties and a semi-infinite body of unknown thermal properties. At the free face of the finite body the surface temperature can be given or the heat flux can be given. The initial temperature distribution is uniform. This case is equivalent to prescribing the heat flux at the interface between the materials. The problem of calculating the heat flux at a surface or interface when the transient temperature history near the heated surface is prescribed has been considered in some detail by the author [23].

2.3 Cases for Which k and c Can Not Be Independently Determined

The thermal conductivity k and specific heat c can not be independently determined when Δ given by (1.3.10) is equal to zero. In these cases it may be possible to determine only one property or perhaps just the ratio or product of the properties. It is well-known that Δ is equal to zero if and only if the functions T_k and T_c are linearly dependent [24]. Two functions are said to be linearly dependent if and only if the relation

$$aT_{c} + bT_{b} = 0$$
 (2.3.1)

is satisfied and one of the constants (a or b) does not equal zero. Several boundary conditions have been found for a homogeneous body with temperature-independent properties for which (2.3.1) applies. These cases are listed in Table 2.3.1. One would anticipate difficulty in finding k and c for any boundary conditions which approach those listed. For example, the given surface temperature case (case 1) is approached by the convective boundary condition when hL/k is large compared to unity. (The quantity h is the heat transfer coefficient and L is the thickness of the specimen.)

At any <u>boundary</u> at which the temperature is prescribed, we have

$$T_{c} = T_{k} = 0$$
 (2.3.2)

for a finite or semi-infinite body. This relation also applies if the

body is homogeneous or composite. Hence any temperature measurement near a temperature boundary condition yields very little information about the thermal properties.

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Table 2.3.1Cases for which k and c can not be independently
determined.

Case	Location of Measured Temp. Histories	Description of Body and Boundary Conditions	Initial Temp. Distribution
1	Any no. of measurements and at any loca- tion.	Finite or semi-infinite 1-D body with a) given, arbi- trary time-variable T at the boundaries or b) insu- lated boundaries. See (2.1.12).	Arbitrary
2	Any no. of measurements and at any loca- tion.	Finite 1-D body with given, arbitrary time-variable T at one boundary and insu- lated at the other boundary. See (2.1.12).	Arbitrary
3	Heated surface only	Semi-infinite 1-D body with given, arbitrary time- variable heat flux at surface. See (2.2.4). (This also applies approxi- mately for finite body for $\tau < 0.2$. At the unheated end, the boundary condi- tion can be either $\partial T / \partial x = 0$ or T = const.)	Uniform
4	Heated surface only	Semi-finite 1-D body with the convective boundary condition, $q =$ $h(T_{\infty}-T(0, \theta))$ where h and T_{∞} are constants. See (2.2.4).	Uniform
5	Interface between bodies	Composite of finite length of material A and semi- infinite length of material B. Material A has the one free surface at which a time- variable T (or q) is given. Material A has known k and c while those of Material B are to be determined. See (2. 2. 4).	Uniform

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III. BASIC CRITERIA

3.1 Criterion for Determining Optimum Experiments

In order to efficiently determine the optimum experiment for determining the thermal properties a single criterion is needed. This criterion, if it exists, would enable us to select the <u>boundary conditions</u> for which a random distribution of errors in the temperature measurements would cause the least inaccuracies in k and c. Box [14] found a single criterion for a closely related problem. He considered, for his first example, the case of a transient, chemical reaction governed by two ordinary, first-order differential equations. Hence he had only an initial condition to consider for each variable. Instead of seeking to determine the optimum initial condition, however, he sought to find the optimum two times at which to make measurements for calculating his two parameters. Since the problem at hand is not identical to Box's, a criterion is derived in this section.

The following analysis is subject to these conditions:

 The values of the thermal conductivity k and densityspecific heat product c at the minimum of F(k, c) are known and are designated k* and c*.

2) The minimum value of F(k, c) need not be zero; however, the errors in the temperature measurements are assumed to be small.

3) While applying the criterion we fix the (a) number of temperature histories, (b) maximum temperature rise and (c) maximum duration

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of the experiment.

4) The sum of squares function F(k, c) is examined in the local region near $F(k^*, c^*)$.

We do not explicitly require that the criterion select an experiment for which k and c are of equal accuracy; we shall find, however, that for a particular optimum experiment the properties can be determined to the same accuracy.

The sum of squares function is

$$F(k,c) = \sum_{j=1}^{n} A_{j} \int_{0}^{0} (T_{j}(\theta) - T_{e,j}(\theta))^{2} d\theta \qquad (3.1.1)$$

and near the minimum a Taylor series expansion gives

$$F(k*+\Delta k, c*+\Delta c)_{\approx}F*+\frac{\partial F*}{\partial k}\Delta k+\frac{\partial F*}{\partial c}\Delta c+\frac{1}{2}\frac{\partial^{2}F*}{\partial k^{2}}(\Delta k)^{2}+\frac{1}{2}\frac{\partial^{2}F*}{\partial c^{2}}(\Delta c)^{2}$$
$$+\frac{\partial^{2}F*}{\partial k\partial c}\Delta k\Delta c \qquad (3.1.2)$$

where the starred terms are evaluated at (k^*, c^*) . At the minimum of F,

$$\frac{\partial F^*}{\partial k} = \frac{\partial F^*}{\partial c} = 0 \qquad (3.1.3)$$

The higher derivatives of F are found from (3.1.1) to be

$$\frac{\partial^{2} \mathbf{F}^{*}}{\partial \mathbf{k}^{2}} = 2\Sigma \mathbf{A}_{j} \int (\mathbf{T}^{*}_{j} - \mathbf{T}_{e,j}) \frac{\partial^{2} \mathbf{T}^{*}}{\partial \mathbf{k}^{2}} d\theta + 2\Sigma \mathbf{A}_{j} \int (\mathbf{T}^{*}_{k})^{2} d\theta$$

$$(3.1.4)$$

$$\frac{\partial^{2} \mathbf{F}^{*}}{\partial c^{2}} = 2\Sigma \mathbf{A}_{j} \int (\mathbf{T}^{*}_{j} - \mathbf{T}_{e,j}) \frac{\partial^{2} \mathbf{T}^{*}}{\partial c^{2}} d\theta + 2\Sigma \mathbf{A}_{j} \int (\mathbf{T}^{*}_{c})^{2} d\theta$$

$$(3.1.5)$$

$$\frac{\partial^{2} \mathbf{F}^{*}}{\partial k \partial c} = 2 \Sigma \mathbf{A}_{j} \int (\mathbf{T}^{*}_{j} - \mathbf{T}_{e,j}) \frac{\partial^{2} \mathbf{T}}{\partial k \partial c} d\theta + 2 \Sigma \mathbf{A}_{j} \int \mathbf{T}^{*}_{k} \mathbf{T}^{*}_{c} d\theta$$
(3.1.6)

For the properties k* and c* when $T_j^* = T_{e,j}$ (zero error in the Tmeasurements) the first term on the right hand side in each of the preceeding three equations is equal to zero. The second terms in (3.1.4) and (3.1.5) are not equal to zero unless T_k or T_c is equal to zero (which does not occur in any case in which k and c can be found). For sufficiently small values of $T_j - T_{e,j}$ (condition 2) the first terms on the right hand side of (3.1.4), (3.1.5) and (3.1.6) are negligible and hence the sum of the squares function is approximated by

$$\mathbf{F}(k*+\Delta k, c*+\Delta c) = \mathbf{F}*+(NT_{k}*)(\frac{\Delta k}{k*})^{2}+(NT_{c}*)(\frac{\Delta c}{c*})^{2}+2(T_{k}*, T_{c}*)(\frac{\Delta k}{k*})(\frac{\Delta c}{c*})$$
(3.1.7)

^

where

$$NT *_{k} = \sum_{j=1}^{n} A_{j} \int_{0}^{0} (k * T_{k}^{*})^{2} d\theta \qquad (3.1.8)$$

$$NT*_{c} = \sum_{j=1}^{n} A_{j} \int_{0}^{\theta} (c*T_{c}^{*})^{2} d\theta \qquad (3.1.9)$$

$$(T_{k}^{*}, T_{c}^{*}) = \sum_{j=1}^{n} A_{j} \int_{0}^{\theta} (c * T_{c,j}^{*} k * T_{k,j}^{*}) d\theta$$
 (3.1.10)

We now desire to find a new coordinate system for the Fcontour. This new coordinate system should be one in which the length and angle are preserved. This is readily done using the standard procedures for treating quadratic forms. In the procedure we solve for the values of λ in the determinantal equation

$$\begin{array}{c|cccc} (NT *_{k}) - \lambda & (T *_{k}, T *_{c}) \\ & & & \\ (T *_{k}, T_{c}) & (NT *_{c}) - \lambda \end{array} = 0 \qquad (3.1.11)$$

to obtain for the two roots

$$\lambda_{1} = [NT_{k}^{*} + NT_{c}^{*} + \sqrt{(NT_{k}^{*} - NT_{c}^{*})^{2} + 4(T_{k}^{*}, T_{c}^{*})^{2}}] / 2$$
(3.1.12)

$$\lambda_{2} = [NT_{k}^{*} + NT_{c}^{*} - \sqrt{(NT_{k}^{*} - NT_{c}^{*})^{2} + 4(T_{k}^{*}, T_{c}^{*})^{2}}]/2$$
(3.1.13)

Clearly λ_1 and λ_2 are real and thus the F-contour in the region of k* and c* is an ellipse and is given by

$$F(p_1, p_2) - F^* = \lambda_1 p_1^2 + \lambda_2 p_2^2$$
 (3.1.14)

where

$$P_{1} = \frac{(NT_{c}^{*} - \lambda_{1})(\Delta k/k^{*}) - (T_{k}^{*}, T_{c}^{*})(\Delta c/c^{*})}{\sqrt{(NT_{c}^{*} - \lambda_{1})^{2} + (T_{k}^{*}, T_{c}^{*})^{2}}}$$
(3.1.15)
$$P_{2} = \frac{-(T_{k}^{*}, T_{c}^{*})(\Delta k/k^{*}) + (NT_{k}^{*} - \lambda_{2})(\Delta c/c^{*})}{\sqrt{(NT_{k}^{*} - \lambda_{2})^{2} + (T_{k}^{*}, T_{c}^{*})^{2}}}$$
(3.1.16)

The angle β between the \textbf{p}_1 and k axes is

$$\beta = \tan^{-1} \left[(NT_c^* - \lambda_1) / (T_k^*, T_c^*) \right] \qquad (3.1.17)$$

The angle β for four interesting cases is shown by Figures 3.1.1 to 3.1.4. If β is near zero degrees, Figure 3.1.1, then c can be determined more accurately on the average than k. For $\beta = \frac{+}{-}90^{\circ}$ (Figure 3.1.2), k is determined more accurately. For $\beta = 45^{\circ}$ and -45°, the values k/c = a and kc are respectively determined more accurately.

The length of the major axis of the ellipse is $[(F-F*)/\lambda_2]^{\frac{1}{2}}$ and of the minor axis is $[(F-F*)/\lambda_1]^{\frac{1}{2}}$. Thus the ratio R of the major axis divided by the minor axis is

$$R = (\lambda_1 / \lambda_2)^{\frac{1}{2}}$$
 (3.1.18)

which is always greater than unity.

The area of the ellipse A_{cr} is given by

$$A_{cr} = \pi (F - F^*) (\lambda_1 \lambda_2)^{-\frac{1}{2}}$$
(3.1.19)

If the errors in the measured temperatures $T_{e,j}$ are stochastically independent, then the area A_{cr} is that of the confidence region when F-F* is calculated using standard statistical procedures. Unfortunately, the errors are probably not independent in most cases. See references 25 and 26. Let us, however, consider two different experiments with the same (a) material, (b) number of temperature measurements, (c) maximum temperature rise, (d) duration of the experiment and (e) distribution of errors in the temperature measurements. For these two experiments the area A_{cr} is smaller for fixed F - F*(from (e)) for the experiment having the largest value of

$$\lambda_{1}\lambda_{2} = (NT_{k}^{*})(NT_{c}^{*}) - (T_{k}^{*},T_{c}^{*})^{2} = \Delta(k^{*},c^{*})$$
(3.1.20)

We have noted in section 2.3 that the properties k and c can not be determined separately if $\Delta = 0$. The criterion Δ given by (3.1.20) must be modified to include the constraints of maximum temperature rise $T_m - T_i$ and maximum duration of the experiment, θ_m . The optimum dimensionless time is given by Deter more trately Tig. E = 0 Deter more ratel : İ -Ā - 1g.



$$\tau_{\rm m} = a^* \theta_{\rm m} / L^2$$
; $a^* = k^* / c^*$ (3.1.21)

for fixed values of θ_{m} and a^{*} ; thus determining the optimum value of τ_{m} is equivalent to finding the optimum specimen thickness L. For any prescribed boundary conditions and given specimen (for Tindependent thermal properties) the derivatives T_{k}^{*} and T_{c}^{*} can both be shown to be directly proportional to the temperature rise $T-T_{i}$ for the same time and position. Due to this linear behavior of T_{k}^{*} , T_{c}^{*} , and $T-T_{i}$, the criterion Δ can be made independent of the magnitude of $T-T_{i}$ by replacing T_{k}^{*} and T_{c}^{*} by $T_{k}^{*}/(T_{m}-T_{i})$ and $T_{c}^{*}/(T_{m}-T_{i})$. The maximum time θ_{m} can be introduced simply in the following manner. The first term on the right hand side of (3.1.20) can be written

$$NT_{k}^{*} = \sum A_{j} \int_{0}^{\theta} (k^{*}T_{kj}^{*})^{2} d\theta = \frac{L^{2}}{a^{*}} \sum A_{j} \int_{0}^{\tau} (k^{*}T_{kj}^{*})^{2} d\tau$$
(3.1.22)

Dividing (3.1.22) now by the fixed maximum real time θ_m causes the right side of (3.1.22) to be a function of the maximum dimensionless time τ_m (for a given heating condition). The other integrals are treated in the same manner. Hence the optimum experiment is one for which

$$\bar{\Delta} = (N\bar{T}_k) (N\bar{T}_c) - (\bar{T}_k, \bar{T}_c)^2$$
 (3.1.23)

is a maximum for fixed $T_m - T_i$, θ_m , number of temperature measurements and thermal properties. The terms on the right hand side of (3.1.23) are defined by

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$$N\bar{T}_{k} = \sum_{j=1}^{n} A_{j} \int_{0}^{\tau} \frac{(k^{*}T_{k,j}^{*})^{2}}{(T_{m}^{-}T_{i}^{-})^{2}} \frac{d\tau}{\tau_{m}}$$
(3.1.24)

$$N\bar{T}_{c} = \sum_{j=1}^{n} A_{j} \int_{0}^{1} \frac{(c * T_{c,j} *)^{2}}{(T_{m} - T_{i})^{2}} \frac{d\tau}{\tau_{m}}$$
(3.1.25)

$$(\bar{T}_{k}, \bar{T}_{c}) = \sum_{j=1}^{n} A_{j} \cdot \frac{(k*T_{k,j}*c*T_{c,j}*)}{(T_{m}-T_{i})^{2}} \frac{d\tau}{\tau_{m}}$$
 (3.1.26)

The important criterion $\overline{\Delta}$ is made a maximum by varying both the boundary conditions and dimensionless duration of the experiment; it is examined for both finite and semi-infinite bodies for a variety of boundary conditions in chapters IV and V.

Another quantity of interest for a given experiment is the correlation coefficient $\boldsymbol{\rho}$

$$\rho = (\bar{T}_{k}, \bar{T}_{c}) / (N\bar{T}_{k}N\bar{T}_{c})^{\frac{1}{2}}$$
(3.1.27)

The correlation coefficient ρ can have values equal to and between -1 and +1. For ρ equal to either -1 or +1, $\overline{\Delta}$ is equal to zero and the properties k and c can not both be determined. It does not follow that $\overline{\Delta}$ is a maximum when $\rho = 0$, however.

3.2 Error Analysis

It is important to investigate the effect of errors upon the calculated thermal properties. There are three major sources of errors. First, there are experimental inaccuracies in measuring the temperatures. Next, in calculating the temperatures in the body an approximate finite-difference method is utilized. This second source of error

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as shown in section 6.2 can be made as small as desired (0.1% is readily obtained). Finally, the model used to describe the temperatures in the body can be imperfect. For example, the thermal properties actually vary with temperature but the thermal properties may be found as temperature-independent thermal properties. This error can be made, in general, as small as desired by reducing the temperature range. The main source of error is consequently the experimental error in the temperature measurements.

Let us then investigate the errors in k and c due to some continuous distribution of small errors δ_i in the measurements,

$$\delta_{j}(\theta) = T_{e,j}(\theta) - T_{e,j}^{*}(\theta)$$
 (3.2.1)

where $T_{e,j}(\theta)$ is an experimentally measured temperature distribution which contains errors and $T_{e,j}^{*}$ is the "true" temperature. Using (3.2.1) in the sum of squares function F(k,c) given by (3.1.1) yields

$$F(k,c) = \sum_{j=1}^{n} A_{j} \int_{0}^{0} (T_{j}(\theta) - T_{e,j}^{*}(\theta) - \delta_{j}(\theta))^{2} d\theta \quad (3.2.2)$$

The temperature $T_j(\theta)$ is the calculated temperature for the assumed model which need not be perfect. If $\delta_j(\theta) = 0$, then the correct properties k* and c* are obtained by differentiating F(k,c) with respect to k and c and setting both expressions equal to zero,

$$\frac{\partial \mathbf{F}(\mathbf{k},\mathbf{c})}{\partial \mathbf{k}} = 2 \Sigma \mathbf{A}_{j} \int (\mathbf{T}_{j} \ast - \mathbf{T}_{e,j} \ast) \frac{\partial \mathbf{T}_{j} \ast}{\partial \mathbf{k}} d\theta = 0 \quad (3.2.3)$$

$$\frac{\partial F(k,c)}{\partial c} = 2 \Sigma A_{j} \int (T_{j} * -T_{e,j} *) \frac{\partial T_{j} *}{\partial c} d\theta = 0 \quad (3.2.4)$$

where T_j^* is the calculated temperature for $\delta_j = 0$.

The quantities ${\Delta k}_e$ and ${\Delta c}_e$ are the errors introduced due to the $\delta_j{}{}^i s$,

$$\Delta k_{e} = k - k *$$
 (3.2.5)

$$\Delta c_e = c - c * \qquad (3.2.6)$$

Using the standard approximation of linearity of ${\tt T}_j$ for small $\boldsymbol{\delta}_j$ gives

$$T_{j} \approx T_{j}^{*+T}_{k,j}^{*\Delta k} e^{+T}_{c,j}^{*\Delta c} e \qquad (3.2.7)$$

where typically $T_{k,j}^{*} = \frac{\partial T_{j}}{\partial k} \Big|_{k}^{*}, c^{*}$

Introducing (3. 2. 7) in (3. 2. 2) and differentiating with respect to k and c, setting the resulting expressions equal to zero and utilizing (3. 2. 3) and (3. 2. 4) yields

$$\sum A_{j} \left(T_{k,j} * \Delta k_{e} + T_{c,j} * \Delta c_{e} - \delta_{j} \right) T_{k,j} * d\theta = 0 \qquad (3.2.8)$$

$$\sum \mathbf{A}_{j} \left(\mathbf{T}_{\mathbf{k},j} \ast \Delta \mathbf{k}_{\mathbf{e}} + \mathbf{T}_{\mathbf{c},j} \ast \Delta \mathbf{c}_{\mathbf{e}} - \delta_{j} \right) \mathbf{T}_{\mathbf{c},j} \ast d\theta = 0 \qquad (3.2.9)$$

Since the δ_j are small, iteration for Δk_e and Δc_e is not necessary. Solving for Δk_e and Δc_e and rearranging slightly gives

$$\frac{\Delta k_{e}}{k*} = \frac{(N\bar{T}_{c})I_{k} - (\bar{T}_{k},\bar{T}_{c})I_{c}}{\bar{\Delta}}$$
(3.2.10)

$$\frac{\Delta c_{e}}{c^{*}} = \frac{(N\bar{T}_{k})I_{c} - (\bar{T}_{k},\bar{T}_{c})I_{k}}{\bar{\Delta}}$$
(3.2.11)

where
$$I_{k} = \sum_{j=1}^{n} A_{j} \int_{0}^{\tau} \frac{m k T_{k,j} \delta_{j}}{(T_{m} - T_{i})^{2}} \frac{d\tau}{\tau}$$
 (3.2.12)

$$I_{c} = \sum_{j=1}^{n} A_{j} \int_{0}^{\tau} \frac{c * T_{c,j} * \delta_{j}}{(T_{m} - T_{i})^{2}} \frac{d\tau}{\tau}$$
(3.2.13)

and the other terms are defined by (3.1.24) to (3.1.26). Though it is not obvious from (3.2.10) and (3.2.11) that Δk and Δc are reduced if $\overline{\Delta}$ is maximized, this fact is demonstrated in chapters IV and V. We also note that the errors Δk_e and Δc_e are directly proportional (for small errors) to the error distribution, δ_j . Generally the derivatives T_k and T_c are continuous, slowly-varying functions and thus the integrals in the above equations involving δ_j are larger for biased onesided errors than for random (or even sinusoidal) errors which are both positive and negative. The biased, consistent errors, however, can be usually computed and then the measured temperatures corrected. The author has analyzed some problems with biased errors for thermocouples embedded in solids [26, 27].

Equations for Δk_e and Δc_e for discrete measurements in time can be obtained from (3. 2. 10) and (3. 2. 11) simply by replacing the time integral by a summation over time and the dt by $\Delta \tau$.

From the linearity of (3.2.10) and (3.2.11) with respect to δ_j , the errors in k and c for two different error distributions can be added to obtain the error due to both sources. More generally we can examine the errors Δk_e and Δc_e due to a <u>single</u> error δ_j at a time τ . In order to compare the effect of a single error for different experiments, Δk and Δc are normalized by multiplying by $(T_m - T_i)\tau_m / (\delta_j \Delta \tau)$ to obtain

$$\Delta \bar{k}_{j} = \frac{\Delta k_{e}(\tau, x_{j})}{k^{*}} \left(\frac{T_{m} - T_{i}}{\delta_{j}} \right) \left(\frac{\tau}{\Delta \tau} \right) = \frac{(N \bar{T}_{c}) \frac{k^{*} T_{k,j}^{*}(\tau)}{T_{m} - T_{i}} - (\bar{T}_{k}, \bar{T}_{c}) \frac{c^{*} T_{c,j}^{*}(\tau)}{T_{m} - T_{i}}}{\bar{\Delta}}$$

$$(3.2.14)$$

$$\Delta \bar{c}_{j} = \frac{\Delta c_{e}(\tau, x_{j})}{c^{*}} \left(\frac{T_{m} T_{i}}{\delta_{j}} \right) \left(\frac{\tau_{m}}{\Delta \tau} \right) = \frac{(N \bar{T}_{k}) \frac{c^{*} T_{c,j} (\tau)}{T_{m} T_{i}} - (T_{k}, T_{c}) \frac{k^{*} T_{k,j} (\tau)}{T_{m} T_{i}}}{\bar{\Delta}}$$
(3.1.15)

The errors $\Delta \bar{k}_j$ and $\Delta \bar{c}_j$ are caused by a single error in the temperature, δ_j , at time τ . By plotting $\Delta \bar{k}_j$ and $\Delta \bar{c}_j$ versus time τ and positions x_j , the information is given for determining the error in k and c for any distribution of small errors for the experiment considered. For a number of different boundary conditions, $\Delta \bar{k}_j$ and $\Delta \bar{c}_j$ are given usually for the τ_m - value of the maximum $\overline{\Delta}$ for each case and for two values of x_j , usually 0 and L.

3.3 Errors in Properties for Certain Cases

A biased error instructive to consider is an error in each temperature rise which is proportional to the temperature rise, or

$$\delta_{j} = \in (T_{j} - T_{i})$$
 (3.3.1)

where \in is a constant much less than unity. Now for a finite or a semi-infinite body with specified heat flux boundary conditions we have

$$T_j - T_i = -cT_{c,j} - kT_{k,j}$$
 (3.3.2)

where T_i is the uniform initial temperature. The terms involving δ_i in (3.2.10) and (3.2.11) can then be written

$$I_k = - \in (N\bar{T}_k + (\bar{T}_k, \bar{T}_c))$$
 (3.3.3)

$$I_{c} = - \in (N\bar{T}_{c} + (\bar{T}_{k}, \bar{T}_{c}))$$
 (3.3.4)

then with these latter expressions (3.2.10) and (3.2.11) readily yield

$$\Delta k_{\rho}/k^{*} = -\epsilon \qquad (3.3.5)$$

$$\Delta c_{\rho}/c^{*} = -\epsilon \qquad (3.3.6)$$

Thus if all the temperature rises were measured a constant one percent too large, then both k and c would be calculated one percent too small; the thermal diffusivity $\mathbf{a} = k/c$ would not be in error, however. In the analysis of section 3.2 the error was assumed to be only in $\mathbf{T}_{e,j}$ and not in \mathbf{T}_j . If the temperature rise $\mathbf{T}_j - \mathbf{T}_i$ were calculated a constant fraction \in too large due to the same fraction \in error in the prescribed heat flux, then the error in the properties would be the same as given by a negative constant error in $\mathbf{T}_{e,j}$. Thus for a constant fraction error in q the errors in k and c are

$$\Delta k_{e}/k^{*} = \epsilon \qquad (3.3.7)$$

$$\Delta c_{\rho}/c^{*} = \epsilon \qquad (3.3.8)$$

or a constant one percent error in the heat flux curve causes a positive one percent error in k and c.

Another error which introduces a bias is the result of incorrect measurement of the location of a thermocouple. Suppose a thermocouple junction is thought to be located at the surface heated with a known q but the junction is in reality located a small distance Δx inside. The temperature at $x = \Delta x$ can be approximately given by

$$T(0, x) \approx T(\theta, 0) + \frac{\partial T(\theta, 0)}{\partial x} \Delta x$$
 (3.3.9)

Hence, δ for this thermocouple is given by

and

$$\delta = T_e - T_e^* \approx T(\theta, 0) + \frac{\partial T(\theta, 0)}{\partial x} \Delta x - T(\theta, 0)$$
$$= \frac{\partial T(\theta, 0)}{\partial x} \Delta x = -\frac{q(\theta, 0)}{k} \Delta x \qquad (3.3.10)$$

Then the error in the surface temperature is proportional to q and Δx . At an insulated surface q = 0 and thus δ = 0 at such a boundary. Evidently the thermocouples should be located much more precisely at heated surfaces than at insulated ones.

3.4 Determination of a Criterion for the Optimum Heat Flux Input

The determination of the optimum heat flux input for finite or semi-infinite bodies is pertinent to the efficient calculation of thermal properties. Anticipating results given in chapters IV and V, the case of two thermocouples is examined; one is located at the heated surface and the other is in the interior or at the other surface. The heat flux which maximizes $\overline{\Delta}$ produces a step rise in the heated surface temperature. For conciseness this heat flux alone is discussed. Though the problem of maximizing $\overline{\Delta}$ is not directly amenable to solution using the classical methods of the calculus of variations, the method of determining a necessary condition for $\overline{\Delta}$ to be a maximum is similar in some respects to the proof of the necessary conditions of Weierstrass and Legendre given by Bliss [29].

For two temperature histories $\overline{\Delta}$ is given by

$$\bar{\Delta} = [(N\bar{T}_{k_1}) + (N\bar{T}_{k_2})][N\bar{T}_{c_1}) + (N\bar{T}_{c_2})] - [(\bar{T}_{k_1}, \bar{T}_{c_1}) + (\bar{T}_{k_1}, \bar{T}_{c_1})]^2$$
(3.4.1)

where typically

$$N\bar{T}_{k_{1}} = \int_{0}^{\tau} \left(\frac{kT_{k_{1}}}{T_{m}^{-T_{i}}} \right)^{2} \frac{d\tau}{\tau_{m}}$$
 (3.4.2)

and

$$(\bar{T}_{k}, \bar{T}_{c})_{l} = \int_{0}^{\tau} \begin{pmatrix} \tau_{k} \\ \bar{T}_{m} \\ \bar{T}_{m} \\ \bar{T}_{m} \end{pmatrix} \begin{pmatrix} cT_{c} \\ \bar{T}_{m} \\ \bar{T}_{m} \\ \bar{T}_{m} \end{pmatrix} \frac{d\tau}{\tau_{m}}$$
(3.4.3)

The subscripts 1 and 2 refer respectively to the measurements at x = 0 and x = L. The heat flux q, surface temperature T_{mo} , criterion $\overline{\Delta}$ and the derivatives kT_{k_1} and cT_{c_1} are shown on page 26 by Figures 3.4.1 to 3.4.4. The value of $\overline{\Delta}$ for the constant surface temperature is designated $\overline{\Delta}_0$. The effect upon $\overline{\Delta}$ for a small step increase in the surface temperature, δT , is now examined. It has a short duration $\Delta \tau$ and is shown by Figure 3.4.2; δT is caused by an appropriate δq . The cases that we shall consider are those for which at x = 0

$$kT_{k_1} = cT_{c_1} = -\frac{1}{2}(T - T_i)$$
 (3.4.4)

Due to the linearity of kT_k , cT_c and T, superposition is valid (see Table 2.1.1) and thus we have

$$\delta(kT_{k_1}) = \delta(cT_{c_1}) = -\frac{1}{2}\delta T_1$$
 (3.4.5)

The terms with the "l" subscript in (3.4.1) can be written for small values of δT as,

$$(N\bar{T}_{k_1}) \approx [(N\bar{T}_{k_1})_0 + 2E_{k_1}] \frac{(T_{m0} - T_i)^2}{(T_m - T_i)^2}$$
 (3.4.6)

$$(N\bar{T}_{c_1}) \approx [(N\bar{T}_{c_1})_0 + 2E_{c_1}] \frac{(T_{m0} - T_i)^2}{(T_m - T_i)^2}$$
 (3.4.7)

$$(\bar{T}_{k}, \bar{T}_{c})_{1} \approx [(\bar{T}_{k}, \bar{T}_{c})_{1,0} + E_{k,c_{1}}] \frac{(T_{mo} - T_{i})^{2}}{(T_{m} - T_{i})^{2}}$$
 (3.4.8)

where

$$E_{k_{1}} = \int_{\tau_{0}}^{\tau_{m}} \frac{(kT_{k_{1}})_{0} - \delta(kT_{k_{1}})}{(T_{m0} - T_{1})^{2}} \frac{d\tau}{\tau_{m}} =$$

$$\int_{\tau_{0}}^{\tau_{m}} \frac{\left[-\frac{1}{2}(T_{m0} - T_{1})\right]\left[-\frac{1}{2}\delta T\right] d\tau}{(T_{m0} - T_{1})^{2} - \tau_{m}} = \frac{1}{4} \frac{\delta T}{(T_{m0} - T_{1})} \frac{\Delta \tau}{\tau_{m}}$$
(3.4.9)

The value of $(N\bar{T}_k)$ with $\delta T = 0$ and $T_m = T_{mo}$ is designated $(N\bar{T}_{k_1})_0$. In deriving (3.4.9), equations (3.4.4) and (3.4.5) are used and it is noted that δT has a non-zero value only between τ_0 and $\tau_0 + \Delta \tau$. Similar expressions can be found for E_{c_1} and E_{k,c_1} ; these are related to E_{k_1} by

$$E_{c_1} = E_{k_1}$$
 (3.4.10)

$$E_{k,c_1} = 2 E_{k_1}$$
 (3.4.11)

If instead of a single increase in the surface temperature, δT , a finite number of increases (or decreases) in the surface temperature is considered, E_{k_1} is readily modified. Here we have a distribution of n values δT_i each of which represents an increase in the surface temperature for the time duration $\Delta \tau_i$. The only restriction is that δT_i is small compared to T_{mo} - T_i . Then E_{k_1} becomes
$$E_{k_{1}} = \frac{1}{4} \frac{1}{(T_{m0} - T_{i})} \sum_{i=1}^{n} \delta T_{i} \frac{\Delta \tau_{i}}{\tau_{m}}$$
(3.4.12)

and (3.4.10) and (3.4.11) also apply.

The effect of δT upon the integrals in (3.4.1) with the "2" subscript is not as readily given and depends upon the boundary conditions at the location of thermocouple 2. The values E_{k_2} , and E_{c_2} , and E_{k,c_2} are evaluated for the particular cases examined.

Let us return to the consideration of a single δT . The temperature rise for negative δT remains $T_{mo} - T_i$; but for a positive δT we can use the approximation

$$(T_{m}-T_{i})^{-2} \approx (T_{m0}-T_{i})^{-2} \left[1 - \frac{2\delta T}{T_{m0}-T_{i}}\right]$$
 (3.4.13)

Then for a positive δT (3.4.1) can be expanded to

$$\overline{\Delta} - \overline{\Delta}_{o} \approx 4 \left[B_{1}(\tau_{o}) - \overline{\Delta}_{o} \frac{\delta T}{T_{mo}^{-T} i} \right] \qquad (3.4.14)$$

and for a negative δT (and δq)

$$\overline{\Delta} - \overline{\Delta}_{0} \approx 4B_{1}(\tau_{0}) \qquad (3.4.15)$$

where

$$B_{1}(\tau_{o}) = [E_{k}(N\bar{T}_{c}) + E_{c}(N\bar{T}_{k}) - E_{k,c}(\bar{T}_{k},\bar{T}_{c})]/2 \qquad (3.4.16)$$

and typically

$$E_{k} = E_{k_{1}} + E_{k_{2}}(\tau_{0})$$
 (3.4.17)

$$E_{k,c} = E_{k,c_1} + E_{k,c_2}(\tau_0)$$
 (3.4.18)

$$N\bar{T}_{k} = (N\bar{T}_{k_{1}})_{0} + (N\bar{T}_{k_{2}})_{0}$$
 (3.4.19)

The integrals ($N\bar{T}_k$), ($N\bar{T}_c$) and (\bar{T}_k, \bar{T}_c) are constants for a given experiment duration τ_m . The value E_{k_1} is independent of τ_o as shown by (3.4.9); however, E_{k_2} , E_{c_2} and E_{k,c_2} are functions of τ_0 . It is shown by (3.4.9) that E_{k_1} is directly proportional to δT and $\Delta \tau$ and this proportionality can be also demonstrated for $E_{k_2}(\tau_0)$, $E_{c_2}(\tau_0)$ and $E_{k,c_2}(\tau_0)$. Then $B_1(\tau_0)$ can be written

$$B_{1}(\tau_{o}) = \frac{\delta T}{T_{mo}^{-}T_{i}} \left(\frac{\Delta \tau}{\tau_{m}}\right) B(\tau_{o}) \qquad (3.4.20)$$

and hence (3.4.14) is given by

$$\overline{\Delta} - \overline{\Delta}_{0} \approx 4 \frac{\delta T}{T_{m0} - T_{i}} \left[\frac{\Delta \tau}{\tau_{m}} B(\tau_{0}) - \overline{\Delta}_{0} \right] \qquad (3.4.21)$$

which is valid only for positive δT . Note that $B(\tau_0)$ is independent of δT and $\Delta \tau$.

The criterion B can aid in the determination of the optimum heat flux. We shall show in two cases of interest that $B(\tau_0)$ is positive for any value of τ_0 . If δT is negative, then from (3.4.20) and (3.4.15) a small δT at any time during the experiment makes $\overline{\Delta}$ less than $\overline{\Delta}_0$. For positive δT , (3.4.2!) is used. From the Schwarz inequality $\overline{\Delta}_0$ must be equal to or greater than zero [24]; for an optimum experiment it can not be equal to zero. Usually $B(\tau_0)$ is about as large as $\overline{\Delta}_0$; regardless of its value, however, for a sufficiently small $\Delta \tau$ the right hand side of (3.2.21) is negative. Thus if $B(\tau_0)$ can be shown to be positive, a value of δT either positive or negative makes $\overline{\Delta}$ less than $\overline{\Delta}_0$. Hence the optimum heat flux boundary condition appears to be one which produces a step rise in the temperature of the heated surface (provided $B(\tau_0)$ is positive).

IV. SEMI-INFINITE BODIES

4.1 Semi-Infinite Body with Constant Heat Flux

The temperature at a point x in a semi-infinite body which is subjected to a constant heat flux q_0 is [30]

$$T - T_{i} = 2 q_{0} (\theta/kc)^{\frac{1}{2}} \operatorname{ierfc}(0.5 \tau_{x}^{-\frac{1}{2}})$$
 (4.1.1)

where

$$\tau_{x} = a \theta / x^{2} = k \theta / c x^{2} \qquad (4.1.2)$$

Taking the partial derivative of (4.1.1) with respect to k and c gives

$$kT_{k} = k \left(\frac{\partial T}{\partial k}\right)_{c} = \frac{q_{o}x}{k} \begin{bmatrix} -\frac{\tau}{x} & \frac{1}{2} \\ -\frac{\tau}{x} & \frac{1}{2} \\ \pi^{\frac{1}{2}} & \exp\left(-\frac{1}{4\tau_{x}}\right) + \operatorname{erfc}\left(-\frac{1}{2\tau_{x}}\right) \\ & 2\tau_{x}^{\frac{1}{2}} \end{bmatrix}$$

$$(4.1.3)$$

$$cT_{c} = c\left(\frac{\partial T}{\partial c}\right)_{k} = -\frac{q_{o}x}{k} - \frac{\tau_{x}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} \exp\left(-\frac{1}{4\tau_{x}}\right)$$
(4.1.4)

1

Using the identity [30]

ierfc(z) =
$$\pi^{-\frac{1}{2}} \exp(-z^2) - z \operatorname{erfc}(z)$$
 (4.1.5)

it can be verified that

$$T - T_i = -kT_k - cT_c$$

as derived in chapter II for general q boundary conditions. It is instructive to evaluate (4.1.1), (4.1.3) and (4.1.4) at the heated surface,

$$-40 - T(0, \theta) - T_{i} = 2 q_{0} (\theta / kc \pi)^{\frac{1}{2}}$$
(4.1.6)

$$kT_{k}(0,\theta) = -q_{0}(\theta/kc_{\pi})^{\frac{1}{2}}$$
 (4.1.7)

$$cT_{c}(0,\theta) = -q_{0}(\theta / kc \pi)^{\frac{1}{2}}$$
 (4.1.8)

and we note as derived in chapter II

$$kT_k(0,\theta) = cT_c(0,\theta) = -(T(0,\theta) - T_i)/2$$
 (4.1.9)

We readily find at x = 0

$$N\bar{T}_{k} = (1/\tau_{m}) \int_{0}^{\tau_{m}} (kT_{k}/(T_{m}-T_{i}))^{2} d\tau_{x} = 1/8$$
 (4.1.10)

$$N\bar{T}_{c} = 1/8$$
 (4.1.11)

$$(\bar{T}_{k}, \bar{T}_{c}) = (1/\tau_{m}) \int_{0}^{\tau_{m}} (kT_{k} cT_{c}/(T_{m}-T_{i}))^{2} d\tau_{x} = 1/8$$

(4.1.12)

Then the criterion $\overline{\Delta}$ for x = 0,

$$\bar{\Delta} = (N\bar{T}_{c})(N\bar{T}_{k}) - (\bar{T}_{k}, \bar{T}_{c})^{2} = (1/8)(1/8) - (1/8)^{2} = 0$$
(4.1.13)

is equal to zero.

With $\overline{\Delta}$ equal to zero, it is impossible to determine both properties k and c. The surface temperature history can be used, however, to calculate the product kc.

Results for the various quantities of interest have been calculated and are tabulated for one interior measurement at x = x in Table 4.1.1. The dimensionless derivatives

$$\bar{T}_{k} = \frac{kT_{k}}{q_{o}x/k}$$
; $\bar{T}_{c} = \frac{cT_{c}}{q_{o}x/k}$ (4.1.14)

т _х	Ť(0)	Τ _c	Τ _k	Ŧ(x)	NТ _с	(Ŧ _k ,Ŧ _c)	^{NŦ} k	Δ	ρ	R	β
.05	.2523	0009	.0007	.0001	. הסווס	0000	.0000	.000000	-1.00)0	209.21	41.30
.10	. 3568	7146	.0107	.01.30	. 0 3 9 5	0005	.001		9997	81.36	34.97
• 15	. 4370	0413	.0266	.0147	. 1015	0010	• ¶ ¶ ¶ 7	.0000000	9991	50.05	34.17
• 20	. 5046	0723	.0416	.0307	. 1141	9925	.0015	.0.00000	9987	34.74	31.75
• 25	. 5642	1038	.0535	.0503	. 1075	i nn43	. 1174	.000010	- 9969	29.43	20.62
.30	.6180	1343	.0624	. (710	.0115	0960	.032	•000000	9952	24.83	27.71
• 35	.6674	1634	.0686	.0449	. 1156	1976	.0037	.000001	9932	21.70	25,94
. 4 !)	.7135	1910	.0726	.1194	. 1196	1189	.0041	.000001	9907	10.45	24.34
. 45	. 7569	2171	.0747	1425	. 1236	1199	. 1943	.000002		17.74	22.85
• 50	. 7979	2420	.0753	.1064	. ^ 274	010A	.0044	.000004	- 9443	16.42	21.40
· 6n	. P 7 4 h	2881	.1/32	.2140	1.544	0118	. 1042		9/55	14.50	14.44
.70	.9441	5 59.5	.0578	. 2025	.0407	0122	. 1 3 49	. 100011	- 9634	10.19	10.74
• B U	1,0093	3692	.0500	. 3192		6114	. 99.35	.000010	- 9443	17.23	12.0
. 91	1.9795	4034					.00.51		- 9/17	10 07	1
1.00	1.1244	4394		.3773 E660		- 010 1		000043	- 7915	10.04	7.61
1.27	1.2010	- 5940	. 0212	. 50 54	.00%0	- 0054	0012	000057	- 5090	0 4 8	A 56
1.70	1.1020	- 6470	- 0540	7010	0767		0010	000071	- 3003	0 1 7	2 0 0
2 00	1.442/	- 7041	- 0971	7012	0.012	0003	0.010	000003	0351	8 00	2.01
2.10	1.777*	- 7573	- 1104	8772	0840	0032	0012	000004	3127	8.76	-2 14
2 . ? 2	1.0720	- 3073	- 1576	0504	0880	0059	0016	000104	50.16	8 47	-7 91
2.70	1.7041	- 3543	- 1845	1 0388	A () O ()		.0020	.000112	6276	9.63	-5 44
3 00	1.0544	- 4901	- 2160	1 1151	0020	0110	.0026	.000120	7092	8.61	-6 8
3 50	2 1110	- 3827	- 2773	1 2600	1967		. 10.19	.000132	8040	8.43	-0.24
4 60	2.1110	-1 0600	- 3143	1 3964	1907	0196	.0053	.000142	8548	8.70	-11 24
4 50	2 1 2 1 7	-1 1321	3934	1 5254	1021	. 0233	.0048	.000149	8855	A. 80	-13.01
5.00	2 5231	-1.2000	- 4442	1.6482	1041	0266	.00A3	.000154	905A	A.93	-14.51
5.56	2 6 4 6 3	-1.2643	5913	1.7457	1057	0296	0098	.000159	9200	9.16	-15 82
6.00	2.7640	-1.3256	- 5527	1.6783	1071	0323	.0112	.000142	9305	9,21	-16.90
6.50	2.8765	-1.3841	A02h	1.9468	1084	0348	. 0127	.000144	9386	9,35	-18.01
7.00	2.9454	-1.4403	4511	2.0914	1094	.0371	.1141	.000145	9440	9.51	-18.94
7.50	7.0902	-1.4944		2.1924	.1104	0.392	.0154	.000146	.9501	9.66	-19.70
P. 00	3.1915	-1.5467	7441	5.550b	.1112	.0412	.0148	.000166	.9543	0.42	-20.54
8.50	3.2898	-1.5972	-,79AB	2.3860	.1119	.0431	.0181	.000147	.9579	9.97	-21.20
8.00	3.3851	-1.6462	8325	2.4787	.1126	448	.0193	.000166	.9609	10.12	-21.91
9.50	3.4779	-1.6938	^{#75} 2	2.5690	.1132	.0464	. 1215	. 101166	9635	10.24	-22.51
16.00	7.5692	-1.7401	9170	2.6571	.1137	.0479	. 7216	.000145	.9658	10.43	-23.07
12.00	3.9088	-1.9141	-1.0754	5.9900	.1155		.0259	.000162	.9724	11."3	-24.9
14.00	4.2220	-2.0736	-1.275	3.2472	.1168	1,75/5	. 9297	.000157	.9/71	11.60	-26.4
16.00	4.5135	-2.2218	-1.3621	3.5439	.11//	.0511	. 11 3 5 0	. 9991152	.940 4	12.15	- 27.5
14.00	4.7873	-2.3606	-1.4935	3.8534	.1145	.0541	. 1.340	.000147	. 9926	12.65	-28.02
20.00	5.0463	-2.4918	-1.5174	4,1092	. 1 7 9 1		. 0396	.000142	, 9845	15.18	-79.44
30.00	6.1804	-3,0545	-2.1073	7.2319	.1211	.0/62		.000120	. 9094	17.47	-34 0=
40.00	7,1365	-3,5450	-2.5350	5,1411	1920	0043	. 7740		.9974	10 44	-36.05
51.00	7.9/88	- 1, 1 1, 15	5.1492	7.0147	. 1726		014	200000	. 99.39	20 3=	-37.2/
60.00	8.7404	-1.3525	- 5 . 4 / 48	A 4744	.1757	1010	0400		.7749	22 22	-36.09
79,00	9.4407	-4.7035	-3.7749	0 1244	1275	- "6ATU	0744	000049	0041	21 60	-3n./*
00.04	10.0925		-4.17.77	5, 1 C 41	1214	1955	0742	000043	4400	24.88	-37 74
1.6.00	11 2020	-7.00/7	-4.0749	11 3120	1210	0969	0743	0000=0	0400	24.10	-3A 12
100.00		-2.26/0									·

Table 4.1.1 Quantities for q = C with thermocouple at x

are plotted in Fig. 4.1.1 for both x = 0 and x = x. (For x = 0, the value of x used in (4.1.2) and (4.1.14) in Fig. 4.1.1 is for the interior point x.) It is necessary for separate determination of k and c that \overline{T}_k and \overline{T}_c are not proportional and neither of them is equal to zero. Note that the results for the interior location satisfy both conditions. The derivative \overline{T}_c is always equal to or larger than \overline{T}_k in magnitude. Both \overline{T}_k and \overline{T}_c for x = 0 are negative for $\tau_x > 1.4$ though it is significant that \overline{T}_k is initially positive and then later decreases and becomes negative. Physically, this means for a given heat flux boundary condition the interior temperature initially rises more rapidly when the conductivity is increased slightly but later the increased conductivity reduces the temperature. An increase in k or c always decreases the surface temperature.

The integrals $N\bar{T}_k$, $N\bar{T}_c$ and (\bar{T}_k, \bar{T}_c) for the interior point x are given in Table 4.1.1. Observe that $N\bar{T}_c$ is always larger in absolute value than $N\bar{T}_k$ and (\bar{T}_k, \bar{T}_c) . One would expect that the specific heat-density product c could be obtained much more accurately than the thermal conductivity k for this experiment. This is proved by the ratio R = 9.97 and angle β = -21.3° at the maximum value of $\bar{\Delta}$ which occurs at $\tau_{\approx} 8.5$ for a single interior measurement. See Fig. 3.1.1. Hence, for a given time duration of the experiment θ_m the optimum location of the thermocouple is

$$x_1 = (k\theta_m/8.5c)^{\frac{1}{2}}$$
 (4.1.15)

for a single thermocouple.

Greatly improved accuracy can be obtained using two thermocouples, one at the heated surface and the other in the interior. A



Fig. 4.1.1 Derivatives \overline{T}_k and \overline{T}_c for semi-infinite body with q = C.



Fig. 4.1.2 Errors $\Delta \overline{k}_j$ and $\Delta \overline{c}_j$ for semi-infinite body with q = Cfor $\tau_m = 1.5$.

summary of the results for this case at the time of maximum $\overline{\Delta}$ is given as case 3 in Table 4.1.2. In order to compare the value of $\overline{\Delta}$ for two thermocouple measurements with the value of $\overline{\Delta}$ for one measurement it should be divided by 4.0 since two interior thermocouples at x increase $\overline{\Delta}$ by a factor of four. The equivalent value of $\overline{\Delta}$ for the measurements at x = 0 and x is 0.0026 compared to the value 0.00017 for a single interior measurement. Clearly the two measurements are superior.

The optimum position for the interior thermocouple located at x = x_2 is found from $\tau_m = a \theta_m / x_2^2 = 1.50$ or

$$x_2 = (k \theta_m / 1.5c)^{\frac{1}{2}}$$
 (4.1.16)

The criterion $\overline{\Delta}$ (τ_{T}) for measurements at positions

$$x = 0, y_1, y_2, \dots, y_i \dots y_n$$

at time $\tau_L = \alpha \theta / L^2$ can be readily calculated from Table 4.1.1. The length L can be any significant length dimension. The terms in $\overline{\Delta}$ are typically given by

$$(N\bar{T}_{k}) \begin{vmatrix} T_{L} & T_{L} & T_{j=1} & T_{j}(N\bar{T}_{k}) \\ T_{L} & T_{j=1} & T_{j}(N\bar{T}_{k}) \\ T_{L} & T_{j}(T_{k}) \\ T_{L} & T_{j}(T_{k}) \\ T_{k} & T_{k}(T_{k}) \\ T_{k}(T_{k}$$

where $(N\tilde{T}_k)_{\tau}$ is found from Table 4.1.1 at time $\tau_L(\frac{L}{y_j})$.

For three thermocouples $\overline{\Delta}$ is maximized by placing the third thermocouple near either x = 0 or x₂. For four thermocouples, the fourth should be located near the other position not occupied by the third. If the third and fourth thermocouples are placed somewhere between the surface and the interior thermocouple, the $\overline{\Delta}$ values are not greatly reduced from the values for the optimum

Correlation coefficient p	1.0	0.958	0.758	1.0	0.911	0.745
* * *	-45	-21.3	-36.8	-45	-25.3	-36.5
* *	8	9.97	2.78	8	5.88	2.71
*		8.5	1.5	I	10.0	1.25
Δ max	0	0.000167	0.0105	0	0.00232	0.0452
Location of Thermo- couples	x = 0	x = x	x = 0, x	x = 0	x = x	x = 0, x
Boundary Condition	q = C	а = С	а = С а	$q = a(\theta \pi)^{-1/2}$	$q = a(\theta \pi)^{-1/2}$	$q = a(\theta \pi)^{-1/2}$
Case	1	2	æ	4	ß	9

Maximum values of $\bar{\Delta}$ for semi-infinite bodies

Table 4.1.2

- * Time at which $\overline{\Delta}$ is a maximum, $\tau_{m} = \frac{\alpha}{x} \frac{\nabla_{m}}{2}$
- ** Ratio of major to minor axes of F-contour, R
- *** Angle of major axis measured from c coordinate, β^o

placement of the thermocouples.

The dimensionless errors in the thermal conductivity k and specific heat-density product c due to a single error at time τ for the optimum experiment with two thermocouples (x = 0 and x₂) are shown by Fig. 4.1.2. At a given time τ_x the values of Δk_0 and Δc_0 for x = 0 are the result of a single error at x = 0 and time τ_x . For example, the errors introduced due to a single error in the temperature measurement at the surface at time $\tau_x = 0.8$ are $\Delta k_0 = -3.69$ and $\Delta c_0 = -0.327$. This means that the fractional error in k due to this single error is

$$\Delta k/k = -3.69(\Delta \tau / \tau_{m}) (\delta / (T_{m} - T_{i})) \qquad (4.1.18)$$

where δ is the actual temperature error at x = 0 and $\Delta \tau / \tau_m$ is the reciprocal of the actual number of discrete temperature measurements at x = 0 utilized in the sum of squares function F. The actual error in k is the sum of all the errors due to each discrete temperature measurement at x = 0 and the optimum interior location x_2 .

4.2 Semi-Infinite Body with $q = a(\theta \pi)^{-\frac{1}{2}}$

Another semi-infinite case of practical interest is for the heat flux

$$q = a(\theta \pi)^{-\frac{1}{2}}$$
 (4.2.1)

where a is a known constant with appropriate units. For this case the temperature is uniform until time $\theta = 0$ when the surface temperature takes a step jump to $T_m - T_i$ and then remains constant with time. If the temperature boundary condition were prescribed rather than the heat flux, the thermal properties could not be determined independently. We obtain for $\bar{T}\text{, }\bar{T}\text{, and }\bar{T}\text{, }$

$$\bar{T} = \frac{T - T_{i}}{T_{m} - T_{i}} = (T - T_{i}) \frac{(kc)^{\frac{1}{2}}}{a} = \operatorname{erfc} \frac{1}{2\tau_{x}^{\frac{1}{2}}} = \operatorname{erfc} \frac{x}{2} \left(\frac{c}{k\theta}\right)^{\frac{1}{2}}$$
(4.2.2)

$$\bar{T}_{k} = \frac{kT_{k}}{T_{m} - T_{i}} = k \left(\frac{\partial T}{\partial k}\right)_{c} \frac{(kc)^{\frac{1}{2}}}{a} = -\frac{1}{2} \left[\text{erfc} \frac{1}{2\tau_{x}^{\frac{1}{2}}} \right]$$

$$(\pi \tau_{\rm x})^{-\frac{1}{2}} \exp(-\frac{1}{4\tau_{\rm x}})$$
] (4.2.3)

$$\bar{T}_{c} = \frac{cT_{c}}{T_{m}^{-}T_{i}} = c \left(\frac{\partial T}{\partial c}\right)_{k} \frac{(kc)^{\frac{1}{2}}}{a} = -\frac{1}{2} \left[erfc \frac{1}{2\tau_{x}^{\frac{1}{2}}} + (\pi \tau_{x})^{-\frac{1}{2}}exp(-\frac{1}{4\tau_{x}})\right]$$

$$(4.2.4)$$

For x = 0 the following results are obtained,

$$\frac{kT_{k}(0,\theta)}{T_{m}-T_{i}} = \frac{cT_{c}(0,\theta)}{T_{m}-T_{i}} = -\frac{1}{2}$$
(4.2.5)

$$N\bar{T}_{k} = N\bar{T}_{c} = (\bar{T}_{k}, \bar{T}_{c}) = \frac{1}{4}$$
 (4.2.6)

Results for the various quantities of interest are tabulated for one interior measurement in Table 4.2.1. The dimensionless derivatives \overline{T}_k and \overline{T}_c are plotted in Fig. 4.2.1 for both x = 0 and x. The maximum $\overline{\Delta}$ for the single interior measurement is 0.0023 and occurs at the dimensionless time $\tau_m \approx 10.0$ and thus the optimum position for a single thermocouple is

$$x_1 = (k\theta_m/10c)^{\frac{1}{2}}$$
 (4.2.7)

т _х	Ť(0)	Τ _c	Ŧĸ	T (x)	NŤ c	(T _k , T _c)	NT _k	۵	ρ	R	β
• 0 5	1.00	0093	.0077	.0016	.0000	0000	.0000	.000000	0999	168.64	40.13
. 15	1.00	1715	.1036	.0679	.0067	0043	.0029	.000000	99A1	35.15	33.20
. 20	1.00	2376	.1238	.1138	.0157	0092	.0055	.000001	9958	24,90	30.49
. ?5	1.00	2862	.1289	.1573	.0265	0141	.0076	.000003	9924	19.47	28,12
.30	1.00	3222	.1255	.1967	.0376	0182	.0091	.000008	9878	16.16	25.99
. 35	1.00	3494	.1174	.2320	.0484	0215	.0099	.000017	9818	13.94	24.07
. 40	1.00	3705	.1070	.2636	.0586	0239	.0102	.000031	9742	12.37	22.31
. 45	1.00	3872	.0954	.2918	.0680	0255	.0102	.00004A	9649	11.20	20.49
.50	1.00	4006	.0833	.3173	.0768	0264	.0100	.000070	9536	10.30	19.19
. 50	1 00	420/	.0794	.3013	1052	0269	.0092	.000124	-,9242	9.01	10.40
./0	1 00	- 4461	0141	4202	1148	- 0200	0002	000104		7 61	11 07
. 00	1 00	4533	0028	. 4561	.1258	- 0218	0065	.000330	- 7646	7.05	10.05
1.00	1 00	4594	0201	4795	.1341	0191	.0058	.000419	- 4829	6.69	8.30
1.25	1.00	4701	0570	.5271	.1506	0116	.0050	.000619	4241	6.09	4,55
1.50	1.00	-,4768	0869	.5637	.1629	0040	.0051	.000808	1388	5.73	1.45
1.75	1.00	4813	1116	.5930	.1724	.0034	.0058	.000982	.1081	5.50	-1.17
2.00	1.00	4846	1325	.6171	.1800	.0104	.0069	.00113A	.2938	5.35	-3.42
2.25	1.00	4870	1504	.6374	.1862	.0169	.0084	.001277	.4268	5.26	-5.37
2.50	1.00	-,4898	1659	.6547	.1914	.0229	.0101	.001400	.5221	5.19	-7.09
2.75	1.00	4902	1796	.6698	,1958	.0285	.0119	.001509	.5918	5.15	-8.61
3.00	1.00	-,4914	1917	.6831	.1996	.0337	.0137	.001005	.6441	5.13	-9.98
3.50	1.00	-,4931	- 2223	./077	.2057	.0432	.01/6	.001/00	.7104	5.12	-12.33
4 = 0	1 00	- 4052	- 2434	•/23/ 7190	2143	.0714	0217	101092	- 7034	2.14	-18 03
5 00	1 00	- 4950	- 2550	7519	2174	0.0007	0204	002972	./701	2.10 E 31	-17 75
5.80	1 00	- 496R	- 2666	.7630	.2208	0711	0327	002070		5 28	-18 89
6.00	1 00	4969	2760	.7728	.2222	.0764	.0361	.002181		5.34	-19.49
6.50	1.00	- 4972	2843	.7815	.2242	.0A12	.0393	.002220	. #650	5.41	-20.46
7.00	1.00	- 4975	2918	.7893	.2258	.0857	.0425	.002249	.8749	5.47	-21.53
7.50	1.00	4978	2985	.7963	.2273	.0897	.0454	.002272	.8832	5.54	-22.31
8.00	1.00	4980	3046	.8026	.2286	.0935	.0483	.002289	A903	5.61	-23.03
8.50	1.00	4981	3102	.8084	.2297	.0970	.0510	.002301	.8964	5.68	-23.68
9.00	1.00	4983	3154	.8137	.2307	.1003	.0536	.002309	.9018	5.74	-24.28
9.50	1.00	-,4984	3201	.8185	.2317	.1033	.0561	.002314	.9066	5.81	-24,83
10.00	1.00	-,4985	3245	.8231	,2325	.1062	.0585	.002317	.9108	5.88	-25.34
12.00	1.00	4989	3394	.8383	+2352	.1161	.00/1	.002306	.9241	0.13	-27,00
14.00	1 00	- 4007	- 3010	• 07U1	123/2	.1292	.0/90	.002276	.9337	0.30	-20.37
18 00	1 00	- 4004	- 1682	9676	.2100	1145	0868	002236	0460	10.0	-10 17
20.00	1 00	- 4905	- 1749	. 8744	.2408	1414	.0000	.002148	0503	7.04	-31.12
30.00	1.00	- 4997	3976	.8973	.2438	.1588	.1113	.001923	.9639	7.97	-33.68
40.00	1.00	- 4998	4112	.9110	.2453	.1697	.1245	.001739	.9711	8.75	-35.21
50.00	1.00	- 4999	4205	.9203	.2462	.1773	.1342	.001590	.9756	9.43	-36.24
60.00	1.00	- 4999	4274	.9273	,2468	.1831	.1418	.001469	.9788	10.04	-37.00
70.00	1.00	4999	4327	.9326	.2472	.1877	.1480	,001367	.9811	10.59	-37.59
A0.00	1.00	4999	4371	.9370	.2476	.1914	.1531	.001281	.9830	11.10	-38.07
90.00	1.00	4999	4406	.9406	.2478	.1945	.1575	.001207	,9844	11.58	-38.47
100.00	1.00	-,5000	4437	.9436	.2480	.1972	.1613	.001143	.9856	12.03	-38.AO

Table 4.2.1 Quantities for $q = a(\theta \pi)^{-1/2}$ with thermocouple at x

As noted for the constant heat flux case a much better experiment can be designed using two thermocouples, one very near the heated surface and the other in the interior as indicated by Table 4.1.2, case 6. The maximum $\overline{\Delta}$ occurs at time $\tau_m = 1.25$ and its value is 0.0452 which compares with the much smaller value of $\overline{\Delta} = 0.0105$ for the constant heat flux boundary condition, case 3. The optimum location for the interior thermocouple is

$$x_2 = (k\theta_m/1.25c)^{\frac{1}{2}}$$
 (4.2.8)

Additional thermocouples should be placed at x = 0 and x_2 or else between these two positions as previously discussed in section 4.1. The criterion $\overline{\Delta}$ for more than two thermocouples is calculated as described in section 4.1. A typical term is given by (4.1.17) except the constant 0.125 is replaced with 0.25.

The dimensionless errors $\Delta \bar{k}$ and $\Delta \bar{c}$ due to a single error at time τ_x for the optimum experiment with two thermocouples (x = 0 and x₂) are shown by Fig. 4.2.2. The interpretation of Fig. 4.2.2 is discussed in section 4.1. We note that the values of $\Delta \bar{k}$ and $\Delta \bar{c}$ for q = a($\theta \pi$)^{$-\frac{1}{2}$} are generally smaller than for the q = constant case shown by Fig. 4.1.2. This is a natural consequence of having a larger value of $\bar{\Delta}$ for the q = a($\theta \pi$)^{$-\frac{1}{2}$} case than the q = C case. Examination of Criterion for Optimum Heat Flux: In section 3.4 the parameter B(τ_0) is derived and it is shown that a necessary condition for $\bar{\Delta}$ to be a maximum is that B(τ_0) be positive. The boundary condition considered was for the heat flux which causes a step rise in the surface temperature. There were two temperature measurements considered, one at x = 0 and the other in the interior of the body. These conditions apply to the case considered in this section.



Fig. 4.2.1 Derivatives \overline{T}_k and \overline{T}_c for semi-infinite body with $q = a(\theta \pi)^{-1/2}$.



Fig. 4.2.2 Errors Δk_j and $\Delta \bar{c}_j$ for semi-infinite body with $q = a(\theta \pi)^{-1/2}$ for $\tau_m = 1.25$.

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The terms E_{k_1} , E_{c_1} and E_{k,c_1} needed in (3.4.16) are given by (3.4.9), (3.4.10) and (3.4.11). The corresponding terms for the interior thermocouple (those with "2" subscript) must be evaluated for this particular case. For the optimum time of $\tau_m = 1.25$, kT_k and δkT_k are both on the average much smaller than cT_c or δcT_c . Then it follows that

$$E_{k_2} \approx 0 \tag{4.2.9}$$

$$E_{k,c_2} \approx 0$$
 (4.2.10)

The integral E_{c_2} is not negligible, however; it is defined by

$$E_{c_{2}}(\tau_{o}) = \int_{\tau_{o}}^{\tau_{m}} \frac{(cT_{c_{2}})_{o} \delta(cT_{c_{2}})}{(T_{m}-T_{i})^{2}} \frac{d\tau}{\tau_{m}}$$
(4.2.11)

It can be proved using the linearity of T_c and T that the magnitude of δcT_c_2 is directly proportional to the pulse δT . Also for small $\Delta \tau$ it can be shown that

$$\delta cT_{c_2} \approx \frac{\partial cT_{c_2}}{\partial \tau} \Delta \tau$$
 (4.2.12)

Hence $E_{c_2}(\tau_0)$ is directly proportional to the product of δT and $\Delta \tau$, or

$$E_{c_{2}}(\tau_{0}) = \Delta \tau \ \delta T E_{c_{2}}'(\tau_{0}) \qquad (4.2.13)$$

where $E'_{c_0}(\tau_0)$ is independent of $\Delta \tau$ and δT . Then $B(\tau_0)$ as defined by (3.4.20) is

$$B(\tau_{o}) = (1/8) \left[N\bar{T}_{k} + N\bar{T}_{c} - 2(\bar{T}_{k}, \bar{T}_{c}) \right] + \left[E_{c}' N\bar{T}_{k} \right] / 2$$

$$(4.2.14)$$

Now E'_{c2} is positive since from (4.2.1!) E_{c2}(τ_0) is positive for positive δT because then both cT_{c2} and δcT_{c2} are negative. From Table 4.2.1 and (4.2.6) at time $\tau_m = 1.25$, we have $N\bar{T}_k = 0.255$, $N\bar{T}_c = 0.4006$ and $(\bar{T}_k, \bar{T}_c) = 0.2384$. Thus $B(\tau_0)$ is at least 0.0224. The positive contribution by E'_{c2} to $B(\tau_0)$ is usually less than 0.0224. At $\tau_m = 1.25$, Table 4.2.2 gives $\bar{\Delta}_0 = 0.0452$. Using these values in (3.4.21) we note that even if $\Delta \tau / \tau_m$ approaches unity, $\bar{\Delta}$ is less than $\bar{\Delta}_0$. According to the analysis it is sufficient to demonstrate that $B(\tau_0)$ is positive, however. This has been proved for this case; hence based on the $B(\tau_0)$ criterion, $q = a(\theta \pi)^{-\frac{1}{2}}$ is the optimum heat flux for the case of a semi-infinite body with one thermocouple at the surface and another located at x_2 .

V. FINITE BODIES

5.1 One Surface Insulated

A number of different heat flux conditions at x = 0 is considered in this section. In each case the surface at x = L is insulated.

The optimum thickness of the specimen can be calculated for each case in this chapter for a prescribed experiment duration θ_m using L = $(k\theta_m/\tau_m c)^{\frac{1}{2}}$ where τ_m is the optimum dimensionless duration of the experiment.

5.1.1 Constant Heat Flux

For the case of constant heat flux q_0 at x = 0 and insulated at x = L the temperature distribution and the associated property derivatives are

$$\bar{T} \equiv \frac{T - T_i}{q_0 L/k} = \tau + X$$

$$-\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}e^{-n^2}\pi^2\tau \cos n\pi(1-\frac{x}{L})$$
(5.1.1)

$$\bar{T}_{k} \equiv \frac{kT_{k}}{q_{o}L/k} = -X$$

$$+2\sum_{n=1}^{\infty}(-1)^{n}e^{-n^{2}\pi^{2}\tau}\left[\cos n\pi\left(1-\frac{x}{L}\right)\right]\left[\frac{1}{\pi^{2}n^{2}}+\tau\right] (5.1.2)$$

$$\bar{\mathbf{T}}_{c} = \frac{c\mathbf{T}_{c}}{\mathbf{q}_{o}\mathbf{L}/\mathbf{k}} = -\tau \left[1 + 2\sum_{n=1}^{\infty} (-1)^{n} e^{-n^{2} \pi^{2} \tau} \cos n \pi \left(1 - \frac{\mathbf{x}}{\mathbf{L}}\right)\right]$$
(5.1.3)

where $\tau = \mathbf{a} \theta / L^2$ and $X = +[(x/L)^2 / 2] - (x/L) + (1/3)$. These expressions are plotted versus dimensionless time τ for several positions x/L in Figures 5.1.1, 5.1.2 and 5.1.3. The derivative \overline{T}_k can be either positive or negative and approaches the quasisteady state distribution -X for times τ greater than 0.5. Consequently \overline{T}_k approaches zero at x/L = 0.422. Thus unlike the semiinfinite body it appears that an internal thermocouple located near x/L = 0.5 in ineffective in aiding the determination of k. On the other hand for $\tau > 0.3$, the derivative \overline{T}_c approaches $-\tau$.

Various parameters of interest for the maximum value of $\overline{\Delta}$ are given in Table 5.1.1 for case 1 with x = 0 and case 2 with x = L. The case of using two thermocouples is important; it is found that the optimum locations to maximize $\overline{\Delta}$ are at the boundaries: x = 0 and x = L. The results for this case are summarized in Table 5.1.1 as case 3. The ratio of the major axis of the F-contour to the minor axis, R, is 2.05 which is much smaller than for the single temperature histories at x = 0 and x = L (which have values of 7.76 and 14.5 respectively). If three thermocouples are used, $\overline{\Delta}$ is maximized by placing the third near x = 0; for four thermocouples $\overline{\Delta}$ is maximized by locating two near x = 0 and two near x = L. This is fortunate since in many cases the specimen itself need not be instrumented with thermocouples but the thermocouples can be embedded near the surfaces of the heating and insulating elements.



Fig. 5.1.1 Temperatures in a finite body with q = C at x = 0 and q = 0 at x = L.



Fig. 5.1.2 Derivative \overline{T}_k for q = C at x = 0 and q = 0 at x = L.

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Fig. 5.1.3 Derivative for q = C at x = 0 and q = 0 at x = L.



Fig. 5.1.4 Errors $\Delta \overline{k}_j$ and $\Delta \overline{c}_j$ for q = C at x = 0 and q = 0at x = L. $\tau_m = 0.64$

Case	Bour Cond x = 0	ndary ltions x = L	Location of Thermocouples	$\overline{\Delta}_{\max} \qquad \begin{array}{c} \text{Time at} \\ \text{which } \overline{\Delta} \text{ is} \\ \text{maximum,} \\ \tau_{m} = \frac{0}{L^{2}} \end{array}$		Ratio of major to minor axes of F-contour, R	Angle of major axis measured from c coordi- nate, β^0 .	Correlation coefficient, P
1	a-c	0 - 0		0 00008	1.2	7 74		0.04
	4-0 0-C	q - 0	x - 0		1.2	14 5	-22.5	0.94
2	q=0	q - 0		0.00017	0.45	2.05	14.5	-0.94
		q = 0	x = 0, L	0.0255	0.05	2.05	-16.5	0.39
	q=C τ>0.5: q=0	q = 0	x = 0, L	0.0356	0.75	2.18	-9.0	0.25
5	q=C0 ^{-1/2}	q = 0	x = 0, L	0.0981	0.48	1.855	-18.2	0.36
6	q=Cθ ^{1/2}	q = 0	x = 0, L	0.00989	0.84	2.20	-16.0	0.42
7	q≡C θ ⁻ⁿ n<-0.5	q = 0	$\begin{array}{l} \mathbf{x} = 0 \text{ and} \\ \mathbf{x} = 0, \ \mathbf{L} \end{array}$	0.0				
8	h-b.c. Bi=0.1	q = 0	x = 0	0.00083	1.16	7.41	-23.7	0.94
9	h-b.c. Bi = l	q = 0	x = 0	0.00016	0.86	7.96	-34.0	0.963
10	h-b.c. Bi = 2	q = 0	x = 0	0.00004	3.8	2.73	-51.3	0.76
11	h-b.c. Bi=0.1	q = 0	x = 0, L	0.0216	0.64	1.96	-15.8	0.36
12	h-b.c. Bi=l	q = 0	x = 0, L	0.0109	0.56	1.44	-9.0	0.11
13	h-b.c. Bi=2	q = 0	x = 0, L	0.00559	0.48	1.12	-1.8	0.01
14	q for T=Tm	q = 0	x = 0	0. 0291	1.8	4. 21	-10.3	0.57
15	q for T=T	q = 0	x = 0,L	0.1432	0.76	2.38	-8.2	0.265
16	q = C	T = 0	x = 0	0.0111	2.96	6.42	-84.3	0.528
17	0 <t< 2.2:<br="">q = C T > 2.2: q = 0</t<>	T = 0	x = 0	0.0196	2.9	3.7	-86.5	0.20
18	q for T=T _m	q for T = 0	x = 0, L	0.250	e .	1.0	-45.0	0.0

Table 5.1.1 Maximum $\overline{\Delta}$ values for different experiments with finite bodies.

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The errors in k and c for a single error at time τ for the optimum experiment using two thermocouples (case 3) with constant q are shown in Figure 5.1.4. We note that k is more sensitive to errors in the temperature measurements than c.

The effect of the weighting factor A_j upon $\overline{\Delta}$ and R was found to be insignificant in this constant-q case. In general, $\overline{\Delta}$ was reduced slightly and R was increased for A_j 's other than unity. For this reason and to reduce the scope of the problem the value of A_j for all subsequent cases is unity for each temperature history.

The case of constant q until time τ and q = 0 thereafter is of interest and is case 4 of Table 5.1.1. The optimum heating time is $\tau = 0.5$ and the soaking time is $\Delta \tau = 0.25$. This heating curve increases $\overline{\Delta}$ over that for q = C.

5.1.2 Heat Flux with $q = C \theta^n$

Results for the heat flux of the form

$$q = C \theta^n \tag{5.1.4}$$

with n = -0.5, 0.5 and n < -0.5 results are tabulated as cases 5, 6 and 7 in Table 5.1.1. In the first case the surface temperature suddenly increases to a value and remains constant until $\tau \approx 0.4$ when it begins to increase slowly. For n = 0.5 the surface temperature increases very gradually at first and then increases more rapidly at later times. For n < -0.5 the surface temperature suddenly increases at $\tau = 0^+$ to infinity and thus T_m equals infinity and $\overline{\Delta}$ equals zero.

> 5.1.3 Convective Heat Transfer at the Surface The convective heat transfer boundary condition is

$$q = h (T_{\omega} - T(0, \theta))$$
 (5.1.5)

Results for a thermocouple at x = 0 are given in Table 5.1.1 by cases 8, 9 and 10 and for thermocouples at x = 0 and L by cases 11, 12 and 13. Results are given for the Biot number Bi = h L/k equal to 0.1, 1 and 2.0. For Bi = 0.1 the surface temperature rises very slowly and q is almost constant for a relatively long dimensionless time. For large values of Bi the surface rapidly approaches T_{∞} and the boundary condition of a given surface temperature rather than a given surface heat flux. When the surface temperature is given, we know that the properties can not be separately determined and $\overline{\Delta} = 0$. Hence the decreasing value of $\overline{\Delta}$ as Bi increases. As Bi becomes larger the sum (-kT_k -cT_c) becomes much smaller than T - T_i while we know the equality holds for a given heat flux boundary condition.

5.2 Prescribed q at x = 0 to Produce Constant Surface Temperature and q = 0 at x = L

The heat flux at x = 0 to make $\overline{\Delta}$ a maximum for thermocouples at x = 0 and x = L (the insulated surface) is the one which makes the surface temperature at time τ = 0 increase from T_i to T_m and then remain constant. Results for a single thermocouple at x = 0 and for two thermocouples are shown by cases 14 and 15 of Table 5.1.1. The experiment is more accurate for determining c than k since β is near β = 0° and R is 2.38 for two thermocouples. The temperature T, derivatives $k\bar{T}_k$ and $c\bar{T}_c$ and the errors $\Delta \bar{k}$ and $\Delta \bar{c}$ are shown in Figures 5.2.1 to 5.2.4.

This is the optimum experiment for a finite body if the surface at x = L must be insulated. This is perhaps the best boundary condition for high conductivity materials since these are relatively easy to



Fig. 5.2.1 Temperatures in a finite body with q at x = 0 to cause $T_m = C$ and q = 0 at x = L.



Fig. 5.2.2 Derivatives \overline{T}_k for a finite body with q at x = 0 to cause $T_m = C$ and q = 0 at x = L.



Fig. 5.2.3 Derivatives T_c for a finite body with q at x = 0 to cause $T_m = C$ and q = 0 at x = L.



Fig. 5.2.4 Errors Δk_j and Δc_j for a finite body with q at x = 0 to cause $T_m = C$ and q = 0 at x = L. $\tau_m = 0.76$.

insulate. Unfortunately the $B(\tau_0)$ criterion for the optimum heating condition is not easy to apply in this case since T_k and T_c are not of simple form. Computer calculations verify, however, that $B(\tau_0)$ is positive and hence this is the optimum experiment if it is required that q = 0 at x = L.

5.3 Constant Heat Flux at x = 0 and Given T = 0 at x = L

An experiment for which k can generally be more accurately determined than c is the case of a prescribed constant heat flux at x = 0 and a constant temperature at x = L. In the standard method of determining k the temperature distribution is allowed to reach steady-state and then k alone is determined. The optimum value of $\overline{\Delta}$ for a thermocouple at x = 0 occurs at time $\tau_m = 2.96$ (case 16, Table 5.1.1). The angle β for this optimum is -84.3° and R = 6.42 indicating that k can be calculated with considerably greater accuracy than c. Since the temperature at x = L is prescribed, no information about the properties is obtained at this point. To maximize $\overline{\Delta}$ using two thermocouples the second thermocouple should also be located near x = 0. If the surface at x = 0 is insulated at time $\tau = 2.2$ and then the body is allowed to thermally soak, the value of $\overline{\Delta}$ is increased and R is decreased measurably to 3.7 (case 17).

It is worthwhile to compare the $\overline{\Delta}$, R and β values of case 16 with the case of constant q at x = 0 and insulated at x = L, case 1 of Table 5.1.1. The former is best for determining k and the latter for c. If only a single thermocouple can be used and we require q = C, the T = 0 boundary condition gives a larger $\overline{\Delta}$ than for q = 0 at x = L. Hence, in this comparison for a single thermocouple the T = 0 boundary condition at x = L is superior to the insulation condition at L. For two or more thermocouples, however, the q = 0 boundary condition at x = L is better because it causes a larger $\overline{\Delta}$ than does the T = 0boundary condition.

5.4 Heat Flux Prescribed at Both x = 0 and L

The largest possible value of $\overline{\Delta}$ that can be obtained for a homogeneous body with heat flux boundary conditions and with two or more thermocouples is found by determining the optimum heat fluxes at both x = 0 and L. The heat fluxes that will maximize $\overline{\Delta}$ are the q at x = 0to cause a step rise in the surface temperature and the q at x = L to cause the temperature at L to remain at its initial value, T = 0. Note if the <u>temperatures</u> rather than the <u>heat flux</u> boundary conditions were imposed, k and c could not be obtained. This case is unlike the other finite cases in several respects. The derivatives T_k and T_c at x = 0are related by (see Fig. 5.4.1)

$$kT_k/T_m = cT_c/T_m = -0.5$$
 (5.4.1)

and at x = L by

$$kT_{k} = -cT_{c}$$
 (5.4.2)

and thus if either temperature history at x = 0 or L is used alone to find k and c, the properties can not be separately determined. In the case of thermocouples at x = 0 and L the integrals NT_k , NT_c and (T_k, T_c) approach as $\tau \rightarrow \infty$ respectively 0.5, 0.5 and 0.0. Then the correlation coefficient goes to zero and R goes to unity indicating that both properties are found to equal accuracy. Unfortunately, $\overline{\Delta}$ reaches its maximum value of 0.25 only as τ goes to infinity. This means that



Fig. 5.4.1 Derivatives T_k and T_c for a finite body with q at x = 0 to cause T_m =C and q for T = 0 at x = L.



Fig. 5.4.2 Errors $\Delta \overline{k}_j$ and $\Delta \overline{c}_j$ for q at x = 0 to cause $T_m = C$ and q for T = 0 at x = L.

the specimen thickness must tend to zero for fixed finite duration of the experiment $\boldsymbol{\theta}_m$.

Fig. 5.4.2 gives the errors $\Delta \bar{k}_j$ and $\Delta \bar{c}_j$ due to a single error at different times at x = 0 and x = L in the measured temperatures for $\tau_m = 2.0$ for which $\bar{\Delta} = 0.222$. Note that this experiment has the smallest average errors at x = 0, $\Delta \bar{c}_0$ and $\Delta \bar{k}_0$, and at x = L, $\Delta \bar{c}_1$ and $\Delta \bar{k}_1$, of any experiment considered in this thesis, this is a result of the larger $\bar{\Delta}$ for this case than any other. The difference is most outstanding when the errors for q = C for a semi-infinite body, Fig. 4.1.2, are compared with those shown in Fig. 5.4.2; the same error in a measured temperature for the semi-infinite case as for this case can cause six times as large an error as for this case.

To experimentally introduce a heat flux at x = 0 to maintain essentially a constant surface temperature is not difficult. It can be done by utilizing another finite body of accurately known thermal properties at either a higher or lower temperature than the specimen. The two bodies are suddenly brought together at time $\theta = 0$. The heat flux into the specimen can be calculated accurately from the temperature history of the surface of the body with known properties. To accurately measure the q at x = L without the temperature increasing can be very difficult if the specimen is heated rather than cooled. If the specimen is cooled at x = 0 then an electric heater at x = L automatically adjusted could maintain the temperature constant at x = L. This is an interesting possibility since the heat capacity of the heater would not introduce any error into the calculation for the properties. **Examination of Criterion for Optimum Heat Flux**: For the heat flux at x = 0 to be an optimum boundary condition it is necessary that $B(\tau_0)$ be positive (see section 3.4). By an analogous development it can be shown that another $B(\tau_0)$ for the surface at x = L also must be positive for the optimum heat flux at x = L. Both values of $B(\tau_0)$ are positive; for conciseness $B(\tau_0)$ for x = 0 is solely considered. Due to the unique condition (5.4.2) the following relations apply,

$$E_{k_2} = E_{c_2}$$
 (5.4.3)

$$E_{k,c_2} = -2E_{c_2}$$
 (5.4.4)

and (3.4.16) becomes

$$2B_{1}(\tau_{o}) = E_{k_{1}}[(N\bar{T}_{k})+(N\bar{T}_{c})-(\bar{T}_{k},\bar{T}_{c})]+E_{c_{2}}[(N\bar{T}_{k})+(N\bar{T}_{c})+2(\bar{T}_{k},\bar{T}_{c})]$$
(5.4.5)

where E_{k_1} is given by (3.4.9) and E_{c_2} by (4.2.13). The quantities E_{k_1} and E_{c_2} are positive for positive δT and for large τ 's, $(N\bar{T}_k) = (N\bar{T}_c) = 0.5$ and $(\bar{T}_k, \bar{T}_c) = 0$; hence $B(\tau_0)$ given by (3.4.20) must be positive. Consequently the necessary condition that $B(\tau_0)$ be positive for the optimum heat flux at x = 0 is satisfied.

5.5 Summary of Observations for Heat Flux Boundary Conditions

A number of general observations can be drawn from the cases considered for the finite and semi-infinite bodies. The optimum heating boundary condition at x = 0 for at least two thermocouples appears to be the heat flux which makes the surface temperature take a step rise. This is true for both finite and semi-infinite bodies; it is indicated by the value of $B(\tau_0)$ being positive for a positive δT in each case. See section 3.4. For all the finite cases, except for case 16, two thermocouples with one at x = 0 and the other at x = L provide a much more efficient experiment than a single thermocouple at x = 0. In adding additional thermocouples the best locations to maximize $\overline{\Delta}$ are at the boundaries, x = 0 and L (except for case 16). In many cases since the optimum locations for thermocouples are at x = 0 and L the specimen itself need not be instrumented with thermocouples but the standard materials (which do not change from experiment to experiment) can be instrumented. This can mean a considerable saving in time and expense in performing experiments.

In general, the optimum finite experiments provide larger values of $\overline{\Delta}$ and smaller values of R than the optimum semi-infinite experiment. Hence the finite-body experiments are preferred. For high thermal conductivity materials the optimum experiment corresponds to case 15, Table 5.1.1. For large k's (greater than 30 Btu/hr-ft-°F) the body can readily be insulated at x = L. On the other hand, for low conductivity materials (k less than 10 Btu/hr-ft-°F) it is difficult to insulate at x = L but relatively easy to maintain a constant temperature at x = L since in this case the heat fluxes are much smaller in magnitude than for the high k case. Hence, case 18 with τ_m about 2 is recommended for low values of k. For intermediate values of k case 18 should be approximated as closely as possible.

The step change in surface temperatures for cases 15 and 18 can be experimentally obtained by utilizing a standard of accurately known thermal properties and at an initial temperature different from that of the specimen. When the standard and the specimen are brought into intimate contact a common interface temperature T_0 is suddenly attained (provided the contact resistance is negligible). Assuming temperature-independent thermal properties one can show that T_0 is

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given by

$$(T_{i,st} - T_{o}) / (T_{o} - T_{i,sp}) = (k_{sp} c_{sp} / k_{st} c_{st})^{\frac{1}{2}}$$

(5.5.1)

where T_i is initial temperature and the subscripts sp and st refer respectively to the specimen and standard materials. In order for the interface to remain constant with time it is necessary for the standard to have the same boundary condition at its other face as the specimen has: q = 0 for case 15 and T = constant for case 18. It is sufficient to also require for both cases that

$$L_{st}/L_{sp} = (a_{st}/a_{sp})^{\frac{1}{2}}$$
 (5.5.2)

Before performing the experiment one does not know accurately the thermal diffusivity of the specimen a_{sp} ; thus the ratio of the thicknesses of the standard to specimen given by (5.5.2) can not be satisfied precisely. One usually does have an estimate of a_{sp} which is sufficiently accurate to find L_{st} and approximately satisfy (5.5.2), however.

VI. CALCULATION OF CONSTANT THERMAL PROPERTIES

6.1 Finite-Difference Equations for the Heat-Conduction Equation

In order to be able to conveniently analyze transient temperature data a general digital computer program is needed that can treat a variety of boundary and initial conditions. The procedure used in the program should have the potential of being extended to consider temperature-variable thermal properties. The method of finitedifference satisfies these requirements. The only doubtful point relates to the accuracy that can be obtained with this method with a moderate expenditure of computer time. These requirements can be also satisfied since the properties can be frequently calculated to less than 0.1 percent error due to the finite difference approximations while using less than 30 seconds of CDC 3600 computer time.

In order to present concisely the finite-difference equations utilized, the interface equation between two dissimilar materials is developed. The transient heat conduction equation for constant k and c can be written

$$k \frac{\partial^2 T}{\partial x^2} = c \frac{\partial T}{\partial \theta}$$
 (6.1.1)

An energy balance written for the node n shown in Fig. 6.1.1 is



Table 6.1.1 Percent error at x = 0 for q = C at x = 0 and q = 0 at at x = L. $\eta = 0.5$ and $\lambda_1 = 0.75$ except cases 6 and 7.

				Perc	ent of er	ror at x=0 a	t times:			
Case	Δт	N	М	Δτ	4∆ т	$\tau = 0.08$	$\tau = 0.2$			
1	0.005	10	2	3.82	0.058	0.0031	0.0107			
2	0.01	10	1	14.5	-0.170	0.0347	0.0176			
3	0.02	10	0.5	20.4	-1.64	-1.64	-0.0281			
4	0.005	20	0.5	20.4	-1.64	0.0075	0.0052			
5	0.01	20	0.25	23.3	-3.51	-0.897	-0.0267			
6*	0.005	10	2	-27.6	-6.88	-1.59	-0.580			
7*	0.02	10	0.5	-23.3	-3.95	-3.95	-1.39			
* $\eta = 1.0 \text{ and } \lambda_1 = 1.0$										
· · 1										
$$\eta[k_{n-1}\frac{T_{n-1}^{m+1} - T_n^{m+1}}{\Delta x_{n-1}} + k_n \frac{T_{n+1}^{m+1} - T_n^{m+1}}{\Delta x_n}] - \omega[k_{n-1}\frac{T_{n-1}^m - T_n^m}{\Delta x_{n-1}} + k_n \frac{T_{n+1}^m - T_n^m}{\Delta x_n}]$$
$$= \frac{c_{n-1}\frac{\Delta x_{n-1}^m + c_n \Delta x_n}{2\Delta \theta}}{(T_n^{m+1} - T_n^m)}$$
(6.1.2)

which can be written in the more convenient form

$$(\eta k_{n-1})T_{n-1}^{m} - [\eta (k_{n-1} + f_n k_n) + B_n]T_n^{m+1} + (\eta k_n f_n)T_{n+1}^{m+1} =$$

$$(\omega k_{n-1})T_{n-1}^{m} - [\omega (k_{n-1} + f_n k_n) + B_n]T_n^{m} + (\omega k_n f_n)T_{n+1}^{m}$$
(6.1.3)

where

$$f_n = \Delta x_{n-1} / \Delta x_n \qquad (6.1.4)$$

$$B_n = \Delta x_{n-1} \Delta x_n (c_{n-1} f_n + c_n) / 2\Delta \theta \qquad (6.1.5)$$

$$\omega = \boldsymbol{\eta} - 1 \tag{6.1.6}$$

The temperature T_n^m is the temperature at position x_n in the body at time $m\Delta\theta$. All temperatures at time $m\Delta\theta$ are known while those at time $(m+1)\Delta\theta$ must be calculated. The quantity η can be chosen to be any value from zero to unity; for f = 1 the solution for the temperatures is stable for any size time step $\Delta\theta$ if $\eta \ge 0.5$ and the body is homogeneous (i.e., $k_{n-1} = k_n$ and $c_{n-1} = c_n$ for all n). In this case with $\eta = 0.5$, (6.1.3) reduces to the widely-used approximation first given by Crank and Nicolson in 1947 [31, 32, 33]. The most important advantage of using $\eta = 0.5$ is its excellent accuracy. Another commonly used value of η is unity; this gives the "backwarddifference" approximation.

At a heated surface an energy balance gives for node 1 shown

in Fig. 6.1.2

$$\boldsymbol{\eta} q^{m+1} - \omega q^{m} + \boldsymbol{\eta} k_{1} \frac{T_{2}^{m+1} - T_{1}^{m+1}}{\Delta x_{1}} - \omega k_{1} \frac{T_{2}^{m} - T_{1}^{m}}{\Delta x_{1}} = \frac{\Delta x_{1} c_{1}}{2} \frac{T_{0}^{m+1} - T_{0}^{m}}{\Delta \theta}$$
(6.1.7)

or with the approximation

$$T_{o}^{r} = \lambda_{1}T_{1}^{r} + \lambda_{2}T_{2}^{r}$$
; (r = m, m+1) (6.1.8)

and with

$$\lambda_2 = 1 - \lambda_1 \tag{6.1.9}$$

(6.1.7) becomes

$$-\left[2\eta k_{1}+\lambda_{1}\frac{c_{1}(\Delta x_{1})^{2}}{\Delta \theta}\right]T_{1}^{m+1} + \left[2\eta k_{1}-\lambda_{2}\frac{c_{1}(\Delta x_{1})^{2}}{\Delta \theta}\right]T_{2}^{m+1} = -\left[2\omega k_{1}+\lambda_{1}\frac{c_{1}\Delta x_{1}}{\Delta \theta}\right]T_{1}^{m} + \left[2\omega k_{1}-\lambda_{2}\frac{c_{1}(\Delta x_{1})^{2}}{\Delta \theta}\right]T_{2}^{m} + 2\Delta x_{1}(\omega q^{m}-\eta q^{m+1})$$

$$(6.1.10)$$

Experience shows that $\lambda_1 = 0.75$ gives more accurate results than the more common approximation $\lambda_1 = 1$. Eq. (6.1.10) is also used at an insulated surface by letting q = 0.

To provide a comparison of the accuracy for selected time steps and number of nodes, Table 6.1.1 is given. The percent error in the temperature rise at x = 0 is given for a finite body initially at temperature T_i and with q = C at x = 0 and q = 0 at x = L. The temperatures for this case are shown in Fig. 5.1.1. The quantities $\Delta \tau$, N and M are defined as

$$\Delta \tau = \mathbf{a} \Delta \theta / \mathbf{L}^2 \tag{6.1.11}$$

$$N = L/\Delta x \qquad (6.1.12)$$

$$M = (\Delta x)^2 / a \Delta \theta \qquad (6.1.13)$$

where $\Delta \theta$ is the time step and Δx is the spatial node spacing. We can draw several conclusions from the data of Table 6.1.1. In each case the calculation of the temperature at time $\Delta \tau$ is rather inaccurate and in some cases even at time $4\Delta\tau$ the error is still large. Since the large errors do not persist with time, however, the initial few temperatures calculated need not be used to calculate the properties. For economy in performing the experiments the time step for reading the experimental transient temperature data can be considerably larger than that required for the computer to accurately calculate the temperatures. For these reasons the program developed to calculate properties permits the calculational time step to be any even divisor of the time step used to read the experiment data. Another conclusion is that the quantity M significantly affects the accuracy. The temperatures are not necessarily calculated with greater precision for a fixed time step if N is doubled as can be observed by comparing cases 1 and 4. Omitting the first few times steps for N = 10, a value of M of either 1 or 2 yields temperatures which are well within 0.1% accuracy. For N = 20, M = 0.5 is satisfactory to provide 0.01% accuracy after several times steps. Cases 6 and 7 are for $\eta = \lambda_1 =$ 1.0 while the previous cases are for $\eta = 0.5$ and $\lambda_1 = 0.75$; a comparison of cases 1 and 6 (or 3 and 7) demonstrates that the latter values of η and λ_1 yield much better accuracy. Temperatures at interior locations have comparable accuracy with the given values.

In the iteration procedure for determining k and c it is necessary to evaluate the derivatives T_k and T_c . The derivative T_k can be found by using the same program that is used to calculate the temperatures since

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$$T_{k} = \frac{\partial T}{\partial k} \Big|_{c} \approx \frac{T(k+\Delta k, c) - T(k, c)}{\Delta k}$$
(6.1.14)

The CDC 3600 computer uses 11 or 12 significant figures in its calculation so Δk can be made very small compared to k. The value of $\Delta k = 0.0001k$ gives excellent accuracy.

6.2 Accuracy of Properties Calculated from Exact Data

In evaluating the accuracy of the mathematical procedure given in this paper it is necessary to use exact temperatures rather than experimentally determined values. When the procedure is used to find values of k and c from experimental data, the error introduced into the properties due to the finite-difference approximations should be an order of magnitude less than those caused by the data itself. With reasonable care in the choice of M, N, η and λ_1 this can always be accomplished.

For the optimum heating time of $\tau = 0.64$ for case 3 of Table 5.1.1 (q = C at x = 0 and q = 0 at x = L), the percent errors in k and c are shown in Table 6.2.1. The exact temperatures at x = 0 and L at times $\tau = 0.02$, 0.04, 0.06, ..., 0.64 are used to find the properties k and c. Cases 1 through 5 which are for $\eta = 0.5$ and $\lambda_1 = 0.75$ give about 0.1 percent accuracy or better. The $\eta = \lambda_1 = 1.0$ results, cases 6 and 7, have much poorer accuracy which is consistent with Table 6.1.1.

Another boundary condition of interest is the heat flux to make the surface temperature constant at T_m and q = 0 at x = L (case 15, Table 5.1.1). The errors in k and c are shown in Table 6.2.2 for again $\Delta \tau = 0.02$ for the temperature data at x = 0 and L. For the

Table 6.2.1 Percent error in k and c for q = C at x = 0 and q = 0 at x = L with measurements at x = 0 and L and $\Delta \tau = 0.02$ for data. $\tau_m = 0.64$.

Case	Δт	N	М	η	λ 1	Percent error in k	Percent error in c
1	0.005	10	2.0	0.5	0.75	-0.0446	0.0727
2	0.01	10	1.0	0.5	0.75	-0.0543	0.0714
3	0.02	10	0.5	0.5	0.75	0.1305	0.1092
4	0.005	20	0.5	0 .5	0.75	-0.0221	0.0209
5	0.01	20	0.25	0.5	0.75	-0.1475	0.0072
6	0.005	10	2.0	1.0	1.0	-0.4878	-0.1554
7	0.02	10	0.5	1.0	1.0	-1.410	-0.0716

Table 6.2.2 Percent error in k and c for q to make T = T_m at x = 0, q = 0 at x = L with measurements at x = 0and L and $\Delta \tau = 0.02$ for data. $\tau_m = 0.8$. $\eta = 0.5$ and $\lambda_1 = 0.75$.

Case	Δτ	N	М	Percent error in k	Percent error in c
1	0.02	5	2.0	2.85	0.725
2	0.005	10	2.0	0.251	0.0559
3	0.01	10	1.0	-0.803	0.0728
4	0.02	10	0.5	4.28	0.960
5	0.00125	20	2.0	0.0299	0.0139
6	0.0025	20	1.0	0.0608	0.0164
7	0.005	20	0.5	0.659	0.0074
8	0.00125	40	0.5	0.0173	0.0051

,

same values of $\Delta \tau$ and N in Table 6.2.2 as in Table 6.2.1 the q = C case has greater accuracy. This better accuracy is due to the errors in the finite-difference approximation which are greatest at the earliest times at which time the constant temperature case is more sensitive to errors. (See Figures 5.1.7 and 5.2.4). For a random distribution of experimental errors the constant temperature case is superior as indicated by the larger $\overline{\Delta}$; the errors in the calculation for the properties can be made as small as desired simply by reducing $\Delta \tau$ and holding M fixed. Whenever a new experiment is being analyzed to determine k and c the accuracy of the calculation can be easily checked by running the same data on the computer with two or three values of $\Delta \tau$ for a fixed M value about unity. From Table 6.2.2 it appears that the error is reduced by a factor of about 10 each time $\Delta \tau$ is halved for fixed M. To run cases 1, 2 and 3 of Table 6.2.1 it took a total of 65 seconds on the CDC 3600 computer.

6.3 Analysis of Hsu Transient Temperature Data

Hsu performed in 1956 [12] an experiment using two semiinfinite sections of nickel at different initial temperatures. The nickel sections were suddenly pressed together with a pressure of about 500 psi. His data is given in Table 6.3.1. From each temperature (except at $\theta = 0$) he calculated an independent value of thermal diffusivity **a** using (4.2.2). This equation requires the interface temperature T_m which was not measured. Assuming the contact resistance between the nickel sections to be negligible, Hsu by eye drew a smooth curve through his temperature data plotted versus position. He found that the interface temperature T_m was about 46.7° C. After

		Temperature, ^o C							
	Thermocouple		Time, Seconds						
	positions, cm	0	5	10	15	20	25		
	1.495	69.9	64.6	60.85	58.65	57.3	56.15		
	0.892	69.9	59.0	55.65	54.2	53.2	52.5		
Hsu#l	0.501	69.9	53.9	51.85	50.95	50.45	50.0		
	0.209	69.9	49.85	48.9	48.45	48.25	48.1		
	-0.269	23.7	42.75	43.8	44.45	44.7	44.95		
Hsu#2	-0.996	23.7	33.7	37.1	38.65	39.7	40.4		

Table 6.3.1 Hsu temperature data [12]

Table 6.3.2 Thermal properties calculated from Hsu data. Results for Hsu #1 are given by rows 1-3 and for Hsu #2 by rows 4-6.

		a	k	с	ρ	с р	
Row	Analyst	m ² /hr	Kcal/m-hr-C	Kcal/m ³ -C	Kg/m3	Kcal/Kg-C	
		ft ² /hr	Btu/ft-hr-F	Btu/ft ³ -F	lbm/ft ³	Btu/lbm-F	
1	Hsu	0.0558	53.9	967	8870	0.109	
2	Beck	0.05510	53.55	971.8	8892	0.1093	
3*	Beck	0.5931	35.98	60.70	555.1	0.1093	
4	Hsu	0.0568	54.9	967	8870	0.109	
5	Beck	0.05925	56.30	950.3	8900	0.1068	
6*	Beck	0.6378	37.83	59.34	555.6	0.1068	
*	* English Units						

Table 6.3.3Comparison of a values calculated by present methodwith others

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T _o F	k* Btu/hr-ft-F	c _p * Btu/lbm-F	p* lbm/ft ³	a* ft ² /hr	Beck a ft ² /hr	Hsu a ft ² /hr
75	38.0	0.1055	555.9	0.649		
95	37.5	0.1062	555.6	0.636	0.6378	0.612
116	36.7	0.1072	555.4	0.618		
137	36.0	0.1082	555.1	0.600	0.5931	0.601
158	35.5	0.1090	554.8	0.588		
 From Handbook of Thermophysical Properties of Solid Materials [34] 						

obtaining the twenty values of **a** in the high temperature specimen (Hsu #1), the average **a** was calculated. Utilizing the following values, $\rho = 8870 \text{ Kg/m}^3$ and $c_p = 0.109 \text{ Kcal/KgC}$, the thermal conductivity k was calculated from

$$\mathbf{k} = \rho c_{\mathbf{p}} \mathbf{a}_{\mathbf{a} \mathbf{v} \mathbf{g}} \tag{6.3.1}$$

He repeated the same procedure with the same value of ρc_p to obtain a k-value from the average a obtained from the low temperature specimen (Hsu #2).

Hsu in his analysis of the data incorporated a number of assumptions or conditions that can be eliminated using non-linear estimation. These are as follows:

1) The contact resistance was assumed to be negligible and thus the temperature at the interface was assumed to be unique. In our calculation the surface temperature of each specimen-was assumed to be constant with time after $\theta = 0$, then the surface temperature of each specimen was calculated separately and no further assumption was made relative to the contact resistance.

2) Hsu found the interface temperature to be 46.7°C by a manual curve fit.

3) Hsu used the same value of $c = \rho c_p$ for both specimens. In our analysis it was necessary to give the q at the surface of only <u>one</u> specimen or equivalently, to give the c value of <u>one</u> specimen. Then the value of k (and c if q is given) of that specimen and the values of k and c for the other specimen were determined.

If Hsu had given the temperature history of a point near the surface of only one specimen with much finer time steps than 5 second intervals, the heat flux at the interface could have been calculated for given values of k and c. Since this information was not available, we assumed the heat flux to be

$$q = (T_m - T_i) (kc/\theta \pi)^{\frac{1}{2}}$$
 (6.3.2)

which is the heat flux for the surface temperature suddenly increasing from T_i to T_m and then remaining constant at T_m . (See section 4.2.) Utilizing the program described in sections 6.1 and 6.2 the high temperature data (Hsu #1) was analyzed first using the q calculated from (6.3.2) with the values of $T_m - T_i$ and kc that Hsu found from the low temperature data (Hsu #2). Next, the low temperature data was analyzed with the $T_m - T_i$ and kc values determined from the Hsu #1 data.

The property values calculated for Hsu #1 and #2 are shown in Table 6.3.2 along with values obtained by Hsu. English units are used in rows 3 and 6 and metric units in the other rows. The property calculated most independently is the thermal diffusivity a since neither the heat flux nor a c-value need be given to obtain a. The values of a calculated by Hsu and Beck for Hsu #1 are 0.0558 and 0.05510 and for Hsu #2, 0.0568 and 0.05925 m^2/hr . For the Hsu #1 data our values are about 1.3% lower than Hsu's and for the Hsu #2 data, about 4.6% higher than Hsu's values. The discrepancy is quite significant particularly for the Hsu #2 case. The discrepancy is due in a large measure to the interface temperature used by Hsu compared to the calculated values using nonlinear estimation. The surface temperature calculated for Hsu #1 and Hsu #2 are respectively 46.671 and 46.605°C while Hsu from his manual curve fit obtained the single value 46.7°C. Though the difference between these values appears negligible, in reality Hsu's procedure for calculating a is very sensitive

to the interface temperature particularly for the Hsu #2 data. The average a-value calculated from the Hsu #2 data and using Hsu's method with $T_m = 46.60$ °C yields about 0.05975 which is slightly higher than the "Beck" value given in Table 6.3.2 which is 0.05925 m²/hr. If average values of a are calculated separately for the two thermocouple temperature responses, (x = -0.269 and -0.996 cm), using as the surface temperature the Hsu value of 46.7 °C, one obtains 0.0553 and 0.0583 m²/hr; while using our value of 46.60 °C for the surface temperature one obtains 0.0598 and 0.0597 m²/hr. Since the latter two values are much more consistent, it is probable that surface temperatures are more accurately calculated by nonlinear estimation than the manual curve fit. Furthermore, the properties obtained by the nonlinear estimation procedure should be more accurate than those calculated by Hsu.

To obtain the k and c results given in Row 2 of Table 6.3.2 the q given by (6.3.2) was used as described above. The specific heat c_p was found from c by utilizing the accepted value of the density of nickel [34]. The density ρ is relatively easy to measure and is much more insensitive to temperature changes than either k or c_p as can be noted by comparing values given for Hsu #1 and #2 (rows 2 and 5). The values of c_p and ρ were fixed by Hsu for both specimens. In our calculation only ρ was needed and the values given in reference [34] for the average temperature were used to obtain the values in Table 6.3.2. This reference is result of a thorough literature search and analysis of published data of thermal properties for a large number of materials. The recommended values given in this reference are the generally accepted values at the present time. Before comparing the property values given in Table 6.3.2 with the recommended values the consistency of the Hsu data is examined.

Hsu does not state the accuracy of his measurements; however, from the number of significant figures used to write his measured temperatures (Table 6.3.1) evidently he thought the temperatures were measured no more accurately than about $\frac{1}{2}$ 0.05°C. The values of the rms differences between the temperatures that we calculated and those that Hsu measured are for Hsu #1 and #2 respectively 0.048 and 0.050°C. The temperatures differences are given by Figures 6.3.1 and 6.3.2. For the most part, the differences appear to be random and no one temperature or thermocouple contributes much more than the average to the rms value. This indicates that the temperatures were rather carefully measured and possess the accuracy implied by Hsu.

It is interesting to note that the temperature differences for the thermocouples nearest the interface indicate the presence of a contact resistance at the interface. The resistance would have the greatest effect at the earliest times when the heat flux is greatest. For Hsu #1 the calculated temperature is less than the measured temperature for the first thermocouple for times 5 and 10 seconds. For the Hsu #2 data the reverse is true. The effect of a contact resistance would coincide with these observations. Since these temperature differences are small, however, the assumption of constant surface temperature of each specimen is permissible in this case.

The <u>Handbook of Thermophysical Properties of Solid Materials</u> [34] gives the values for the thermal properties of nickel shown in Table 6.3.3. The values are given in English units and are to be

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Fig. 6.3.1 Differences between calculated and measured temperatures for Hsu #1.



Fig. 6.3.2 Differences between calculated and measured temperatures for Hsu #2.

compared with those values given in rows 3 and 6 of Table 6.3.2. Since the thermal conductivity varies 7% in magnitude for the temperature range 75 - 158°F (or 23.7 - 69.9°C) the property values are also given in Table 6.3.3 for Hsu #1 and #2 at the average temperatures which are 137 and 95°F. In each case the values of a calculated by us from Hsu's data are between the values for the initial temperature and the average temperature. The a-values calculated by Hsu are closer together and do not agree as well with the recommended values. A comparison of the k and c_p values calculated using the nonlinear estimation procedure (rows 3 and 6 of Table 6.3.2) with the published values shows excellent agreement. The values of the properties obtained for the Hsu #1 data using nonlinear estimation are based on the kc-value calculated by Hsu from his #2 data; the kcproduct calculated by us is only 1% greater than the Hsu product. Hence, the property values in Table 6.3.2 need be corrected only slightly.

If the specific heat-density ratio were known, for say Hsu #1, then k for Hsu #1 and k and c for Hsu #2 can be calculated. The values of $c_{p_1} = 0.1086$ and $\rho_1 = 554.9$ which are obtained from [34] and are consistent with a = 0.5931 ft²/hr were used to calculate $k_1 = 35.74$, $k_2 = 37.57$ and $c_{p_2} = 0.1061$. (The c value is obtained from c with $\rho_2 = 555.6$ lbm/ft³.) These values are very close to the recommended values; the difference is less than 0.5%.

Hsu's experiment was not designed using the concepts of optimum design developed in this research. It is interesting to note how close it compares with an optimum experimental design for a semiinfinite body. It does have the optimum heating condition represented by a step rise in the surface temperature. The surface temperature

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was not measured in either specimen but thermocouples were located near the surface (about 0.1 inch). (It is possible to place the thermocouples as close to the surface as 0.01 inch as demonstrated by Lindholm and Kirkpatrick; their experiment is discussed in section 7.4.) The maximum dimensionless times based on the maximum distances from the heated surfaces to the furthest thermocouples are 1.72 and 4.15 respectively for Hsu #1 and #2. The optimum value of τ is about 1.25; this means that the furthest thermocouple in each case should have been located still deeper in each specimen.

VII. TEMPERATURE-VARIABLE THERMAL PROPERTIES

7.1 Choice of Properties

As far as the author can determine, there is no reference in the open literature to a transient method for calculating temperaturedependent thermal properties from a single experimental run. This is probably due to the great difficulty of obtaining a convenient solution by exact methods of the heat-conduction equation with temperature-variable k and c. Temperatures can be calculated for this case using finite-difference methods almost as readily as for constant thermal properties, however. There are a number of reasons why the variable properties case is important in this analysis. One is that thermal properties do vary significantly even over small temperature ranges as noted in section 6.3 for nickel. Next, the nonlinear estimation method has the potential of being much more accurate than any method used heretofore for calculating k and c or a. This is due to the negligible effect of random errors on the temperature and the ability to eliminate the problem of heat losses at the heated surface. Hence, the temperature-variation case is more important than for less precise methods. This case is also important to make the experiments more efficient; it is desirable to perform discrete experiments covering as large temperature ranges as practical. That is, instead of possibly performing ten experiments each covering 50°F intervals, it might be possible using temperature-variable properties to perform

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one experiment covering a 500°F range.

The transient heat-conduction equation for temperature-variable k and c is

$$\frac{\partial}{\partial \mathbf{x}} \left(\mathbf{k} \; \frac{\partial \mathbf{T}}{\partial \mathbf{x}} \right) = \mathbf{c} \frac{\partial \mathbf{T}}{\partial \theta} \tag{7.1.1}$$

We shall assume that k and c are linear functions of temperature; the temperature range considered can always be made small enough to permit this assumption. The properties k and c can then be expressed as

$$k = \frac{k_1 T_2 - k_2 T_1}{T_2 - T_1} + \frac{(k_2 - k_1)T}{T_2 - T_1}$$
(7.1.2)

and

$$c = \frac{c_1 T_2 - c_2 T_1}{T_2 - T_1} + \frac{(c_2 - c_1)T}{T_2 - T_1}$$
(7.1.3)

where

$$k = k_1$$
 and $c = c_1$ at $T = T_1$
 $k = k_2$ and $c = c_2$ at $T = T_2$

The temperatures T_1 and T_2 can be chosen to be any convenient values such as the minimum and maximum temperatures obtained in the experiment.

Instead of the expressions (7.1.2) and (7.1.3) we could write

$$k = k_{a} + k \frac{T - T_{a}}{T_{a} - T_{i}}$$
 (7.1.4)

$$c = c_{a} + c \frac{T - T_{a}}{T_{a} - T_{i}}$$
 (7.1.5)

where

$$k = k_a and c = c_a at T = T_a$$

In the first two equations for k and c we need these thermal properties at two different temperatures. In the next two equations we find the properties at the temperature T_a (usually the mean temperature) and the changes of the properties with temperature. In theory it should make no difference whether we choose (7.1.2) and (7.1.3) or (7.1.4) and (7.1.5) since the same properties for k and c as functions of temperature must be found. But the convergence of the iteration procedure is affected in nonlinear estimation by the particular choice of properties functions [16].

For the special case of

$$k_1 = k_2 = k_a$$
, $c_1 = c_2 = c_a$ and $T_1 = 0$

certain relations between the property derivatives can be derived in the manner discussed in section (2.1). We find that

$$T_k = T_{k_1} + T_{k_2}$$
 (7.1.6)

$$T_{c} = T_{c_{1}} + T_{c_{2}}$$
 (7.1.7)

but from (2.1.22) for $T_i = 0$,

$$T = -kT_{k} - cT_{c} = -k_{a}(T_{k_{1}} + T_{k_{2}}) - c_{a}(T_{c_{1}} + T_{c_{2}})$$
(7.1.8)

where T_k and T_c are the derivatives for constant properties. Generally T_{k_1} and T_{k_2} have the same sign and thus both usually are smaller in absolute value than T_k . The same statement can be made for (7.1.7). For the case of $\dot{k} = \dot{c} = 0$, $T_2 = 2 T_a$ and $T_i = 0$, one can derive

$$T_{k_a} = T_k$$
(7.1.9)

$$T_{\dot{k}} = 2 T_{k_2} - T_k = T_{k_2} - T_{k_1}$$
 (7.1.10)

$$T_{c_a} = T_c$$
 (7.1.11)

$$T_{c} = 2 T_{c_{2}} - T_{c} = T_{c_{2}} - T_{c_{1}}$$
 (7.1.12)

If T_{k_2} and T_{k_1} are of approximately the same magnitude and sign (which frequently occurs), then

$$|T_{k_{m}}| > |T_{k}|$$
 (7.1.13)

and similarly for T_{c_1} and T_{c_2} ,

$$T_{c_{m}} | > > | T_{c} |$$
 (7.1.14)

Then if all the four properties are calculated simultaneously, the **F**-contour for the properties k_1 , k_2 , c_1 and c_2 can be shown to be much less attenuated than for properties k_m , \dot{k} , c_m and \dot{c} . Hence, fewer iterations are needed for the former. Even if only two properties are calculated with the other two fixed, the properties k and c as described by (7.1.2) and (7.1.3) are usually the better choice.

7.2 Finite-Difference Equations

For temperature-variable properties (7.1.1) is nonlinear. The finite-difference equations (6.1.3) and (6.1.10) can be readily modified for this case, however. The thermal conductivity k appearing on the left hand side of (6.1.3) should be evaluated for the temperatures at time $(m+1)\Delta\theta$; k's on the right hand side should be evaluated at time $m\Delta\theta$. But the temperatures at time $(m+1)\Delta\theta$ are unknown. Rather than iterating for each time step to make both k_n^{m+1} and T_n^{m+1} consistent, sufficient accuracy can be obtained by evaluating k and c in (6.1.3) and (6.1.10) at time $m \Delta \theta$ when the temperatures are known. This method accounts for the large variation of the properties with position and the smaller variation with time (provided the time steps are made small). Another way to treat the nonlinear case is to use a predictor-corrector method [35].

The thermal conductivity k_{n-1}^{m} is evaluated at the temperature T,

$$T = (T_{n-1}^{m} + T_{n}^{m}) / 2$$

and then T is used in (7.1.2) to calculate k. For a homogeneous body with a uniform Δx , B_n given by (6.1.5) can be written as $B_n = (\Delta k)^2 c_n / \Delta \theta$, where c_n is evaluated at temperature T_n^m . The quantity c_1 used in (6.1.10) is evaluated at T_1^m . At the interface between two materials or regions of different Δx , c_{n-1}^m in (6.1.3) is evaluated at the temperature T,

$$\Gamma = 0.25T_{n-1}^{m} + 0.75T_{n}^{m}$$

7.3 Certain Cases

In analyzing the efficacy of an experiment to determine thermal properties, it is necessary to examine the derivatives of the temperature with respect to the properties. Two basic cases are considered in this section. The first case is for a heat flux producing a step rise in temperature at x = 0 and insulated at x = L. The derivatives for this case at x = 0 and L are shown in Figures 7.3.1 and 7.3.2. This case has the same boundary conditions as case 15 of Table 5.1.1. The next case has the same boundary conditions as case 18 in the same table; these boundary conditions are a heat flux at x = 0 producing a



Fig. 7.3.1 Derivatives \overline{T}_{k_1} and \overline{T}_{k_2} for a finite body with a q at x = 0 to cause $T_m = C$ and q = 0 at x = L.



Fig. 7.3.2 Derivatives \overline{T}_{c_1} and \overline{T}_{c_2} for a finite body with a q at x = 0 to cause $T_m = C$ and q = 0 at x = L.





Fig. 7.3.4 Derivatives \overline{T}_{c_1} and \overline{T}_{c_2} for a finite body with a q at x = 0 to cause $T_m = C$ and a q at x = L to cause $T = T_i = 0$.

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step rise in temperature at that surface and a heat flux at x = L to maintain the temperature at $T = T_i = 0$ at that boundary. The derivatives for this case are shown in Figures 7.3.3 and 7.3.4. For both cases T_1 and T_2 of (7.1.2) and (7.1.3) are

$$T_1 = 0$$
 (7.3.1)

$$T_2 = T_m$$
 (7.3.2)

and for simplicity the special case of

$$k_1 = k_2 = k_a$$
 (7.3.3)

$$c_1 = c_2 = c_a$$
 (7.3.4)

is treated. Then the derivatives calculated numerically can be checked with the relations (7.1.6), (7.1.7) and (7.1.8).

For any two properties p_1 and p_2 to which the derivatives are related for a particular experiment by

$$ap_1T_{p_1} = bp_2T_{p_2}$$
 (7.3.5)

for all the positions at which temperatures are measured, we know that p_1 and p_2 can not be independently determined as discussed in section 2.3. The quantities a and b are constants and one of which is not equal to zero. If (7.3.5) is not identically true but is approximated in a particular experiment, the properties p_1 and p_2 can be independently determined; the accuracy of the calculated properties in such a case of correlated properties is usually not satisfactory, however.

For both cases the derivatives T_{k_1} and T_{k_2} are highly correlated and the same also is true for T_{c_1} and T_{c_2} . These high correlations indicate that the property pairs (k_1, k_2) and (c_1, c_2) are difficult

to accurately determine from data generated by experiments similar to those mentioned. Even greater inaccuracy would result if all four properties were determined simultaneously. This is not to say that the properties k_1 , k_2 , c_1 and c_2 can not be determined simultaneously; greater accuracy would be expected, however, if the two properties k and c were obtained from a series of experiments as discussed in a later paragraph.

The property pairs (k_1, c_1) and (k_2, c_2) are much less correlated than (k_1, k_2) or (c_1, c_2) . The derivatives with respect to c_1 and c_2 are always negative. The derivatives with respect to k_1 and k_2 are both negative at x = 0 and both positive at x = L. Because the derivatives with respect to k change sign from x = 0 to L but those for c do not, the properties k_1 and c_1 (or k_2 and c_2) are less correlated than the pairs (k_1, k_2) and (c_1, c_2) . The $\overline{\Delta}$ -values for the pairs (k_1, c_1) and (k_2, c_2) are given in Table 7.3.1 along with other quantities of interest. Since the case with T = 0 at x = L has the peculiarity that $\overline{\Delta}$ is maximum at τ equal to infinity the values for this case are given at the times indicated which do not correspond to maximum $\overline{\Delta}$ for cases 3 and 4. For the two different boundary conditions considered, the lowtemperature properties k_1 and c_1 have a larger $\overline{\Delta}$ - value than for the high-temperature properties (though the difference is more marked for the last two cases than the first two). Perhaps a more precise way to refer to properties k_1 and c_1 is to call them initial-temperature properties rather than "low-temperature" properties. Hence, in general based on these cases it is better to determine the initial-temperature properties rather than the final temperature properties.

Correlation coefficient	٩	0.15	0.21	-0.22	0.38
Angle of major axis measured	from c coordi- nate. β^{0}	-3.5	-7.0	17.6	-75.7
Ratio of major to minor axes	of F-contour, K	2.75	2.20	1.46	2.15
τ m ⁼ <u>, 2</u>	L	0.43	1.34	1.0	2.0
⊼ max		0.0103	0.0094	0.0256	0.0042
Property pairs	4	kl, ^c l	k ₂ , c ₂	k ₁ , c ₁	k ₂ , c ₂
da ry itions	x = L	q = 0	q = 0	q for T = 0	q for T = 0
Boun cond	$\mathbf{x} = 0$	q for T=T	q for T=T m	q for T=T _m	q for T=T
Case		-	2	ς	4

Parameters for temperature-variable properties Table 7.3.1

Possible experiments to determine k and c as functions of temperature Table 7.3.2

4	ilibrium Temperature at which perature to evaluate properties	200 400	100 600	800
	Initial temperature Equ of specimen, ^o F tem	400	, 009	800
4	Initial temperature of standard, ^O F	0	200	400
	Experiment	1	2	ŝ

For both boundary conditions the initial temperature could be either the highest or lowest temperature in a particular experiment. To experimentally obtain the boundary condition at x = 0 most efficiently a standard material of accurately known thermal properties at initially a uniform temperature can be suddenly brought into contact with the specimen at another uniform temperature. After the first experiment is completed, the standard and specimen can be both heated (or in the case of the insulation condition at x = L, just the specimen need be heated) to higher temperatures as illustrated in Table 7.3.2. For experiment 1 the specimen is first heated to 400°F and the standard is cooled to 0°F; if the thermal properties are about equal for the standard and specimen the interface temperature during the experiment is about 200°F. If the properties k and c are known at 200°F in the specimen the properties k_1 and c_1 at the initial temperature, 400°F, can be determined. After this first experiment the temperatures can be adjusted to perform the second experiment suggested in Table 7.3.2 and properties found at 600°F. By performing a series of such experiments the properties k and c can be found as a function of temperature. The temperature range of each experiment is governed by the linearity of k and c with temperature. For many materials a 500 - 1000°F range could be covered in a single experiment instead of the 200° F used in Table 7.3.2. Since these are transient experiments which simultaneously determine both k and c, the experimental time can be a small fraction of that required for conventional measurement of these properties. If the properties k and c are not known at 200°F to start the illustrated problem, then the temperature range for solely the first experiment is reduced so that the assumption of temperature-independent properties is valid.

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7.4 Analysis of Lindholm and Kirkpatrick Transient Temperature Data

Some accurate data providing transient temperature measurements over a large temperature range was obtained by Lindholm and Kirkpatrick in 1963 [36]. In their experiment with an instrumented copper rod a very large heat flux (about 1000 cal/cm² - sec) was applied to the specimen by an arc-imaging furnace. The net heat flux to the specimen was measured using a unique scanning-type radiation sampler. The heat flux was so large that the surface temperature reached the melting temperature of about 1100°C in about one second. The experiment was designed to have an approximately uniform heat flux applied to a semi-infinite body. (See section 4.1.)

The heat flux and temperature data obtained by Lindholm and Kirkpatrick is given in Table 7.4.1. Even though the experiment was very carefully performed the accuracy of the data is not as good as is desired for determining thermal properties. Lindholm [37] estimated the accuracy of the higher temperatures to be $\frac{1}{2}$ 1% but the lower temperatures to be much less accurate and the heat flux to be within $\frac{1}{2}$ 6%. The primary purpose of this experiment, however, was not the determination of thermal properties but the investigation of the effect of very large heat fluxes upon the temperature gradients. An experiment designed using the guidelines given in this paper would be much better for determining k and c accurately.

Lindholm and Kirkpatrick calculated the temperature distribution of the copper and compared the temperatures with those measured. The properties at $T_1 = 20^{\circ}$ and $T_2 = 1020^{\circ}$ C that they used in their calculations are given in Table 7.4.2. Their values came from a handbook; more widely accepted values are given in reference 34. In Table 7.4.2

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Time	Heat flux	Temperature rise, $T - T_i$, ^o C where $T_i = 20^{\circ}C$						
seconds	cal/cm-sec ²	0.0254 cm	0.280 cm	0.660 cm	1.296 cm	1.930 cm	2.532 cm	
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8	910 918 940 967 940 922 895 842 900	0 372 541 660 757 822 868 909 972	0 174 305 414 529 595 649 698 749	0 65 128 211 282 349 399 444 489	0 25 54 84 118 152 186 222	0 4 11 23 38 55 72 83	0 0 2 3 5 9 14 20 28	

Table 7.4.1Lindholm and Kirkpatrick temperature data for
copper [36].

Table 7.4.2 Thermal properties for copper.

Temperature	^k L	^k TP	^C L ³ - ^O F	^C TP
^o C	Btu/hr-ft- ⁰ F	Btu/hr-ft- ⁰ F		Btu/ft ³ - ^o F
20	222	234	50.4	51.3
1020	192	185	68.0	62.8

Table 7.4.3Calculated thermal properties for copper fromLindholm and Kirkpatrick data

Case	Given properties	Calculated	properties	ΔT _{rms} , ^o C
1		k = 230	c = 52.6	16.7
2	Lindholm values k ₂ and c ₂	c ₁ = 47.0	k ₁ = 234	13.9
3	TP values for k_2 and c_2	c ₁ = 48.5	k ₁ = 255	16. 4
4	Lindholm values for k _l and k ₂	c ₁ = 39.5	c ₂ = 92.7	8.7

the property values used by Lindholm and coworker have an L subscript and the recommended values with subscript TP. In reference 34 the k-curve for copper was obtained from the results of fourteen experimenters one of which stated a \pm 10% accuracy, three gave \pm 5% and the others did not say. Thus the property values in Table 7..4. 2 are reasonably close considering the scatter in the data.

Since the criterion $\overline{\Delta}$ is larger for constant properties than for temperature-variable properties we expect the properties calculated for the temperature-variable case to be more sensitive to errors than for constant k and c. This is illustrated by the results for four different cases summarized in Table 7.4.3. Case 1, which is for constant properties, gives remarkably accurate values for k and c considering the accuracy of the given data. For both k and c the values are between the recommended values at 20 and 1020°C designated TP in Table 7.4.2. Further, the values are nearer the initial temperature values than the high temperature. This would be expected since many more low temperature measurements are made than high temperature measurements (see Table 7.4.1). The rms temperature difference between the calculated and measured temperatures is 16.7°C or about 1.7% of the maximum temperature rise.

In section 7.3 we noted that the initial-temperature properties could be found more accurately than the high-temperature properties for two heat flux boundary conditions. For this reason the lowtemperatures properties were calculated using the high-temperature values of k_2 and c_2 in Table 7.4.2. The results are cases 2 and 3 of Table 7.4.3. The error in the properties calculated in case 2 compared with the Lindholm values of the previous table are for k and c,

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+ 9.5% and -6.7% respectively; the error in k and c in case 3 with respect to k_{TP} and c_{TP} at 20°C are + 9.0% and -5.5%. Note that k has the largest error; this is consistent with case 1 of Table 7.3.1 for which β = 3.5°. These errors are not large considering the accuracy of the given data. (We know that a constant six percent error in q will cause about a six percent error in a calculation for constant properties.) Also note that in calculating a k_1 or c_1 value we are not calculating a <u>single</u> property value but are really calculating a property curve. Though the value given at 20°C may be 9.0% in error, the value obtained using (7.1.2) for 500°C say, usually would be much less.

Using the Lindholm values for k_1 and k_2 , the calculated values c_1 and c_2 are given in case 4, Table 7.4.3. Since c_1 and c_2 tend to be highly correlated (as noted in the previous section), these c-values are not as accurate as found for the other cases. The values of c_1 and c_2 do exhibit the correct temperature behavior, however; that is, c_1 is less than c_2 . Except for case 4 the accuracy of the calculated properties is better than might be expected from the given accuracy of the data. It is a characteristic, however, of nonlinear estimation to minimize the error introduced by random errors in the data.

VIII. CONCLUSIONS

A general criterion is developed for determining the optimum experiments for calculating simultaneously the thermal conductivity and specific heat from transient temperature and heat flux data for solids. The criterion (designated $\overline{\Delta}$) for fixed maximum temperature rise, duration of the experiment and number of thermocouples should be made a maximum for the optimum experiments.

A number of boundary conditions for finite and semi-infinite bodies are considered. The two optimum experiments found by utilizing the $\overline{\Delta}$ criterion are for finite bodies of thickness L. Each has a heat flux q at x = 0 to cause a step rise in the surface temperature; at x = L one optimum experiment has q = 0 and the other has a q to cause the temperature at L to remain constant. The former is best for high and the latter for low thermal conductivity materials. The optimum locations for thermocouples for these two experiments are at x = 0 and L.

The effect of errors in the temperature measurements is derived and general results for a number of heat flux boundary conditions are given.

Both thermal properties are calculated simultaneously from experimental data for (a) constant and (b) temperature-variable properties. The errors due to the finite-difference procedure utilized can be readily made less than 0.1%. For one set of experimental data analyzed,

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the error in the calculated values of properties is estimated to be less than 0.5%.

IX. RECOMMENDED FUTURE WORK

9.1 Design of Experimental Equipment

There are two basic experiments that are best for finding thermal properties, cases 15 and 18 of Table 5.1.1. Both have a q condition at x = 0 which causes a step rise in temperature. This condition is easily obtained experimentally as discussed in section 5.5. For the first case the finite specimen is insulated at x = L and the second has a q to make the temperature at x = L constant at the initial temperature. The first experiment is best for high conductivity materials and the second for low k materials. The insulation condition is relatively easy to obtain. For case 18 if the specimen is the high temperature body, the initial temperature at x = L can be maintained by an electric heater. On the standard material side (for case 18) the surface at x = L can be maintained near its initial temperature by simply having this surface in good contact with a high conductivity material such as copper. The heater is needed on the specimen side to obtain the heat flux boundary condition at x = L.

For transient experiments to be performed and analyzed in the most efficient manner an automatic data-reducing device is needed to read the analog thermocouple signals and reduce this data to punched cards. These cards are then used as input into the nonlinear estimation program. Before using the data, the program would direct the computer to transform the thermocouple readings to temperatures, correct the temperatures for a bias if necessary and then calculate the thermal properties.

This automated method of measuring thermal properties can determine thermal conductivity and specific heat much more rapidly, more accurately and probably less expensively than the conventional methods. It is well-adapted to meet the needs for a rapid, highvolume device for determining k and c.

9.2 Other Applications of Nonlinear Estimation

The nonlinear estimation method is very flexible and is by no means restricted to determining solely k and c. In the ablation problem, for example, transient chemical reactions and transpiration occur in certain impregnated plastic materials. Also at high temperatures some solid materials simultaneously transport energy by both radiation and conduction. Other materials, such as iron, have secondorder phase transformations which have not been thoroughly examined using a transient method to investigate the effects upon k and c. The total emissivity of solids as a function of temperature under transient conditions is of interest. In each of these problem areas (which are related to heat-conduction) properties can be determined using nonlinear estimation. In addition to determining properties, improved models to describe the physical phenomena can be developed in a rational manner by an extension of the nonlinear estimation method [38].

We need not even limit the application of the method to problems connected with heat-conduction. There are a number of problems in momentum and mass transfer that obey similar equations.

For example, the viscosity of a fluid could be calculated from

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transient velocity measurements in a fluid near a wall which is impulsively moved. The equation describing this flow is identical in form to the transient heat-conduction equation. One might also use the method for calculating eddy viscosities and diffusivities in turbulent flow. The nonlinear estimation method is particularly useful for determining parameters from experimental data for problems in which there are two dimensions or one dimension and a time-variation to be considered and for which two or more parameters are to be calculated.

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APPENDIX A

Derivation of Equations (1.3.8) and (1.3.9)

Introducing (1.3.3) into (1.3.2) and taking the derivative of F with respect to k gives

$$\frac{\partial \mathbf{F}}{\partial \mathbf{k}} = 2 \sum_{j=1}^{n} \mathbf{A}_{j} \int_{0}^{\theta} (\mathbf{T}_{j}(\mathbf{k}_{0}, \mathbf{c}_{0}) + \mathbf{T}_{k,j}(\mathbf{k}_{0}, \mathbf{c}_{0}) \Delta \mathbf{k} + \mathbf{T}_{c,j}(\mathbf{k}_{0}, \mathbf{c}_{0}) \Delta \mathbf{c} - \mathbf{T}_{e,j} \mathbf{T}_{k,j}(\mathbf{k}_{0}, \mathbf{c}_{0}) d\theta \quad (A-1)$$

or

$$\frac{\partial \mathbf{F}}{\partial k} = 2 \left[NT_{k} \Delta k + (T_{k}, T_{c}) \Delta c + \sum_{j=1}^{n} J_{j} \right] T_{k,j} (T_{j} - T_{e,j}) d\theta \right]$$
(A-2)

where (1.3.11) and (1.3.13) are used in (A-2). At the minimum v value of F it is necessary that

$$\frac{\partial \mathbf{F}}{\partial \mathbf{k}} = \frac{\partial \mathbf{F}}{\partial \mathbf{c}} = 0$$

Then also evaluating $\partial F/\partial c$, the corrections Δk and Δc are found by solving,

$$\begin{bmatrix} NT_{k} & (T_{k}, T_{c}) \\ (T_{k}, T_{c}) & NT_{c} \end{bmatrix} \cdot \begin{bmatrix} \Delta k \\ \Delta c \end{bmatrix} = -\begin{bmatrix} \Sigma A_{j} \int T_{k,j} (T_{j} - T_{e,j}) d\theta \\ \Sigma A_{j} \int T_{c,j} (T_{j} - T_{e,j}) d\theta \end{bmatrix}$$
(A-3)

The expressions for Δk and Δc are (1.3.8) and (1.3.9).

ROOM USE ONLY

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