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RELATIONSHIPS BETWEEN THE RESTRICTED IDEALS AND INDUCED MODULES OF THE GROUP RING $\mathcal{F}G$

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ABSTRACT

RELATIONSHIPS BETWEEN THE RESTRICTED IDEALS AND INDUCED MODULES OF THE GROUP RING $\mathcal{F}G$

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Let H be a subgroup of the group G, $\mathcal{F}$ a field, and I an ideal of $\mathcal{F}G$. We wish to determine when the phenomenon $I = (I \cap \mathcal{F}H)\mathcal{F}G$ occurs. Our first result, an extension of a theorem published by D.S. Passman, shows that there exists a unique normal subgroup, W, of G such that $I = (I \cap \mathcal{F}H)\mathcal{F}G$ iff $W \leq H$. We also obtain a second characterization which states that $I = (I \cap \mathcal{F}H)\mathcal{F}G$ iff $I \subseteq \mathrm{Ann}_{\mathcal{F}G}\, N^G$, where $N = \frac{\mathcal{F}H}{I \cap \mathcal{F}H}$ and N^G denotes the tensor product $N \otimes_{\mathcal{F}H} \mathcal{F}G$.

If we restrict our attention to ideals I of the form $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M$ where S is a non-empty set of irreducible $\mathcal{F}G$-modules, then we obtain the following additional partial characterizations. If $H \trianglelefteq G$ and $[G : H] < \infty$, then $I = (I \cap \mathcal{F}H)\mathcal{F}G$ iff for each $M \in S$ and for each irreducible $\mathcal{F}H$-submodule W of M, $I \subseteq \mathrm{Ann}_{\mathcal{F}G} W^G$. If in addition to these hypotheses we assume

that $[G:H]$ is a unit in $\mathfrak{F}$, we see that $I = (I \cap \mathfrak{F}H)\mathfrak{F}G$ iff $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathfrak{F}G} M = \bigcap_{M \in S} \mathrm{Ann}_{\mathfrak{F}G}(M_H)^G$, where M_H denotes M viewed as an $\mathfrak{F}H$-module by restricting the domain of right multipliers to $\mathfrak{F}H$. Finally if we add to all previous assumptions the additional one that G be finite, we are able to conclude that $I = (I \cap \mathfrak{F}H)\mathfrak{F}G$ iff for each $M \in S$ and for each irreducible $\mathfrak{F}G$-submodule, L, of $(M_H)^G$ it is true that $L \in S$ (up to isomorphism).

We conclude the thesis with a brief chapter concerning some relationships between the semisimplicity of the group ring $\mathfrak{F}(G/H)$ and the phenomenon $\mathrm{Rad}\ \mathfrak{F}G \subseteq (\mathrm{Rad}\ \mathfrak{F}H)\mathfrak{F}G$, where Rad denotes the Jacobson radical.

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CHAPTER 1 INTRODUCTION

§1. Group Rings and Their Modules

Let G be a group and $\mathcal{F}$ a field. Then the group ring $\mathcal{F}G$ is the set of all finite sums of the form $\sum_{g \in G} a_g g$, where $a_g \in \mathcal{F}$. So $\mathcal{F}G$ is a vector space over $\mathcal{F}$ with the members of G serving as a basis. If we define addition componentwise and multiplication distributively via the multiplication in the group, then $\mathcal{F}G$ becomes an algebra over $\mathcal{F}$.

Let H be a subgroup of G and I an ideal of $\mathcal{F}G$. Then $\mathcal{F}H$ may be viewed as a subalgebra of $\mathcal{F}G$, and $I \cap \mathcal{F}H$ is an ideal in $\mathcal{F}H$. Our main goal is to determine those pairs (H,I) for which $I = (I \cap \mathcal{F}H)\mathcal{F}G$. It turns out that this phenomenon is intricately related to the behavior of certain $\mathcal{F}G$- and $\mathcal{F}H$-modules. Consequently, a large portion of this thesis will be concerned with the theory of modules. We begin by stating some elementary results that will be crucial to our later work. Throughout we will assume that all modules are right modules unless otherwise specified.

Let M be an $\mathcal{F}G$-module. By restricting the domain of right multipliers to $\mathcal{F}H$, M may be viewed as an $\mathcal{F}H$-module. This $\mathcal{F}H$-module will be denoted M_H.

We will frequently exploit the following lemma, whose proof involves an application of Zorn's lemma. (See Passman [9], p. 224.)

Lemma 1.1. Let H be a subgroup of G and let W be an irreducible $\mathcal{F}H$-module. Then there exists an irreducible $\mathcal{F}G$-module, M, such that W is a submodule of M_H.

If $[G:H] < \infty$ and if $H \trianglelefteq G$, then the study of $\mathcal{F}G$, $\mathcal{F}H$, and their respective modules is greatly facilitated by the following well-known theorem. (See Passman [9], p. 281.)

Theorem 1.2. (Clifford's Theorem)

Let H be a normal subgroup of G of finite index n, and let M be an irreducible $\mathcal{F}G$-module. Then M_H has an irreducible $\mathcal{F}H$-submodule W, and for suitable $x_1, x_2, \ldots, x_m \in G$ with $m \leq n$, we have $M_H = Wx_1 \oplus Wx_2 \oplus \cdots \oplus Wx_m$, an $\mathcal{F}H$-direct sum of irreducible $\mathcal{F}H$-modules. In particular, M_H is completely reducible.

Just as to each $\mathcal{F}G$-module M there corresponds the restricted $\mathcal{F}H$-module M_H, given an $\mathcal{F}H$-module N there corresponds the induced $\mathcal{F}G$-module $N \otimes_{\mathcal{F}H} \mathcal{F}G$, where $\otimes$ denotes tensor product. This induced module will be denoted N^G. If $\{x_i\}_{i \in \mathcal{I}}$ is a right transversal for H in G, then $N^G = \bigoplus_{i \in \mathcal{I}} N \otimes x_i$ as vector spaces. It follows that if $[G:H] < \infty$ and if $\dim_{\mathcal{F}} N$ is finite, then $\dim_{\mathcal{F}} N^G$ is also finite and in fact $\dim_{\mathcal{F}} N^G = [G:H] \dim_{\mathcal{F}} N$.

The following basic propositions concerning induced modules will be used freely without further comment. (See Curtis and Reiner [2].)

<u>Proposition 1.3</u>. If N is an $\mathfrak{F}H$-module such that $N = N_1 \oplus N_2$ as $\mathfrak{F}H$-direct sum, then $N^G = N_1^G \oplus N_2^G$ as $\mathfrak{F}G$-direct sum.

<u>Proposition 1.4.</u> If H and K are subgroups of G such that $H \leq K \leq G$, then $(N^K)^G \cong N^G$ as $\mathfrak{F}G$-modules.

If M is an $\mathfrak{F}G$-module, we denote by $\mathrm{Ann}_{\mathfrak{F}G}M$ the annihilator of $\mathfrak{F}G$ in M. Thus $\mathrm{Ann}_{\mathfrak{F}G}M$ is an ideal in $\mathfrak{F}G$. Such ideals will play an important role in our later work, and we will use the following basic lemma freely.

<u>Lemma 1.5</u>. If $M \cong L$ as $\mathfrak{F}G$-modules, then $\mathrm{Ann}_{\mathfrak{F}G}M = \mathrm{Ann}_{\mathfrak{F}G}L$.

<u>Proof</u>. Let $\varphi : M \to L$ be an $\mathfrak{F}G$-isomorphism. Let $\alpha \in \mathrm{Ann}_{\mathfrak{F}G}M$, and let $\ell \in L$. Then $\ell = \varphi(m)$ for some $m \in M$, and $\ell a = \varphi(m)\alpha = \varphi(m\alpha) = \varphi(0) = 0$. Thus $\alpha \in \mathrm{Ann}_{\mathfrak{F}G}L$, and $\mathrm{Ann}_{\mathfrak{F}G}M \subseteq \mathrm{Ann}_{\mathfrak{F}G}L$. The opposite inclusion holds by symmetry.

§2. The Jacobson Radical

The Jacobson radical of the algebra $\mathcal{F}G$ will be denoted Rad $\mathcal{F}G$. Thus Rad $\mathcal{F}G$ is the intersection of the annihilators of the irreducible $\mathcal{F}G$-modules. (There are, of course, other characterizations.) Many of our results will concern a class of ideals of which Rad $\mathcal{F}G$ is a minimal member. Thus Rad $\mathcal{F}G$ will be an important object of study for us, both in its own right and as a specific example of an ideal belonging to this more general class.

One of the most fundamental problems in group rings is that of determining when Rad $\mathcal{F}G = 0$. (We say that $\mathcal{F}G$ is semisimple in this case.) For infinite groups, conclusions exist for several important classes of groups, such as solvable groups and linear groups, but both the characteristic 0 and characteristic p cases remain unsolved in general. However, the semisimplicity problem has been solved for finite groups as indicated by the following theorem:

<u>Theorem 1.6</u> (Maschke's theorem)

Let G be a finite group and $\mathcal{F}$ a field. If $\mathcal{F}$ has characteristic 0, then $\mathcal{F}G$ is semisimple. If $\mathcal{F}$ has characteristic $p > 0$, then $\mathcal{F}G$ is semisimple iff p does not divide $|G|$.

Unfortunately, even for finite groups, it is not an easy matter to describe Rad $\mathcal{F}G$ once we know Rad $\mathcal{F}G \neq 0$. One natural approach to this problem is to seek relationships between Rad $\mathfrak{F}G$ and Rad $\mathfrak{F}H$, where H is some fixed subgroup of G. The next two theorems are basic results of this sort. The first of these is an immediate consequence of Lemma 1.1. (See Passman [9], p. 273 for an alternate proof.)

Theorem 1.7. Let H be a subgroup of G. Then $(\text{Rad } \mathfrak{F}G) \cap \mathfrak{F}H \subseteq \text{Rad } \mathfrak{F}H$.

Proof: Let $\alpha \in (\text{Rad } \mathfrak{F}G) \cap \mathfrak{F}H$, and let N be an irreducible $\mathfrak{F}H$-module. By Lemma 1.1, there exists an irreducible $\mathfrak{F}G$-module, M, such that $N \subseteq M_H$. Since $\alpha \in \text{Rad } \mathfrak{F}G$, α annihilates M. Certainly, then, α annihilates N. As N was an arbitrary irreducible $\mathfrak{F}H$-module, $\alpha \in \text{Rad } \mathfrak{F}H$.

If H is a normal subgroup of G with $[G:H] < \infty$, then we have the following stronger result, whose proof is a simple application of Clifford's Theorem. (See Passman [9], p. 282).

Theorem 1.8. Let $H \trianglelefteq G$ such that $[G:H] < \infty$. Then $(\text{Rad } \mathfrak{F}G) \cap \mathfrak{F}H = \text{Rad } \mathfrak{F}H$.

Additional background results concerning the Jacobson radical are given in the next section.

§3. Relative Projectivity, Property ρ, and the Complete Reducibility of Induced Modules.

Let H be a subgroup of G. An $\mathfrak{F}G$-module, M, is said to be H-projective, or projective relative to $\mathfrak{F}H$, if every exact sequence of $\mathfrak{F}G$-modules $0 \to L \to N \to M \to 0$

which is split over $\mathcal{F}H$ is also split over $\mathcal{F}G$. Note that if $H = \langle 1 \rangle$, then M is H-projective iff M is projective in the usual sense, for $0 \to L \to N \to M \to 0$ always splits over $\mathcal{F} = \mathcal{F}H$ in this case.

D. G. Higman has characterized the H-projective modules for those subgroups H having finite index in G. (See Higman [3]).

Theorem 1.9 (Higman's Criteria)

Let $[G:H] < \infty$ and let M be an $\mathcal{F}G$-module. Then the following statements concerning M are equivalent:

(a) M is H-projective.

(b) M is isomorphic to a direct summand of $(M_H)^G$

(c) There exists an $\mathcal{F}H$-endomorphism η of M such that $\sum_{i=1}^{n} x_i^{-1} \eta x_i = 1_M$, where $\{x_i\}_{i=1}^{n}$ is a right transversal for H in G.

If $[G:H] = n$ is a unit in $\mathcal{F}$, then $\eta = n^{-1} 1_M$ satisfies (c) above. So as an immediate consequence of Theorem 1.9 we have the following result:

Corollary 1.10 (Higman)

If $[G:H] = n$ is a unit in $\mathcal{F}$, then every $\mathcal{F}G$-module is H-projective.

Subgroups H for which every $\mathcal{F}G$-module is H-projective are of special interest. So, following Khatri and Sinha [6], we state

Definition 1.11. Let H be a subgroup of G. Then $(\mathcal{F}G,\mathcal{F}H)$ is said to be a projective pairing iff every $\mathcal{F}G$-module is H-projective.

So Corollary 1.10 says that if $[G:H] = n$ is a unit in $\mathcal{F}$, then $(\mathcal{F}G,\mathcal{F}H)$ is a projective pairing. In fact, the converse holds as well. Khatri and Sinha [6] established it for finite groups, and Gloria Potter [10] extended their result to include infinite groups. In summary, we state

Theorem 1.12. Let H be a subgroup of G. Then $(\mathcal{F}G,\mathcal{F}H)$ is a projective pairing iff $[G:H] = n$ is a unit in $\mathcal{F}$.

It turns out that if $H \underline{\Delta} G$, then the concept of projective pairing is strongly related to the following concept.

Definition 1.13. (Sinha [12])

Let H be a subgroup of G. Then the pair $(\mathcal{F}G,\mathcal{F}H)$ is said to have property ρ iff $\mathrm{Rad}\ \mathcal{F}G \subseteq (\mathrm{Rad}\ \mathcal{F}H)\mathcal{F}G$.

The connection between property ρ and projective pairing for normal subgroups H is made evident by the following theorem. (See Passman [9], p. 278).

Theorem 1.14. (Villamayor)

Let H be a normal subgroup of finite index such that $[G:H]$ is a unit in $\mathcal{F}$. Then $\mathrm{Rad}\ \mathcal{F}G = (\mathrm{Rad}\ \mathcal{F}H)\mathcal{F}G$. In particular, $(\mathcal{F}G,\mathcal{F}H)$ has property ρ.

As an immediate corollary we have

Corollary 1.15. (Potter [10])

If $H \triangle G$ and if $(\mathcal{F}G, \mathcal{F}H)$ is a projective pairing, then $(\mathcal{F}G, \mathcal{F}H)$ has property ρ.

For finite groups we have the following result which further relates the concepts of project pairing and property ρ.

Theorem 1.16. (Motose and Ninomiya [7])

Let G be a finite group and H a subgroup of G such that $\operatorname{Rad} \mathcal{F}G \subseteq (\operatorname{Rad} \mathcal{F}H)\mathcal{F}G$. Then $(\mathcal{F}G, \mathcal{F}H)$ is a projective pairing.

In general, the concepts of property ρ and projective pairing are independent of each other. (That is, there exist pairs $(\mathcal{F}G, \mathcal{F}H)$ having property ρ but not projective pairing, and vice versa. See Potter [10].) However, both of these are consequences of a third more stringent condition, as described in the next theorem.

Theorem 1.17. Let H be a subgroup of G, and consider the following statements:

(i) $(\mathcal{F}G, \mathcal{F}H)$ is a projective pairing.

(ii) $(\mathcal{F}G, \mathcal{F}H)$ has property ρ.

(iii) For each irreducible $\mathcal{F}H$-module N, the corresponding induced module, N^G, is a completely reducible $\mathcal{F}G$-module.

Then (iii) $\Rightarrow$ (i) (Potter [10]), and

(iii) $\Rightarrow$ (ii) (Sinha and Srivastava [13])

If $H \trianglelefteq G$, then (iii) $\Leftrightarrow$ (i). (Potter [10]).

If $|G| < \infty$, then (iii) $\Leftrightarrow$ (ii). (Motose and Ninomiya [7])

Because of the strong connections between conditions (i), (ii) and (iii) of the previous theorem, Khatri [5] and Potter [10] studied those groups G for which the classes of subgroups satisfying (i), (ii) and (iii), respectively, exactly coincide with each other. We make no attempt to list all their results. Suffice it to say that there are many non-trivial examples of such groups. We do, however, mention one result along these lines, since we will call upon it later.

<u>Theorem 1.18.</u> (Khatri [5])

Let p, q be distinct primes and let $\mathfrak{F}$ be a field of characteristic p. Suppose further that G is a finite group of order p^n, pq or pq^2. Then for any $H \leq G$, H satisfies (i) (of Theorem 1.17) iff H satisfies (ii) iff H satisfies (iii).

It is clear from the results described in this section that the relationship between Rad $\mathfrak{F}G$ and Rad $\mathfrak{F}H$ depends on such factors as $[G:H]$, the behavior of the induced modules N^G where N is an irreducible $\mathfrak{F}H$-module, and the behavior of modules of the form $(M_H)^G$, where M is an irreducible $\mathfrak{F}G$-module. It is reasonable, therefore, to examine these factors in our study of more general ideals. This we do in the next two chapters.

CHAPTER 2 ON IDEALS IN $\mathcal{F}G$ AND THEIR RESTRICTIONS TO $\mathcal{F}H$

§1. Statement of the Problem

Let I be a fixed ideal in $\mathcal{F}G$. In this chapter we seek conditions on $H < G$ which are necessary and/or sufficient for $I = (I \cap \mathcal{F}H)\mathcal{F}G$ to hold.

Let $H \leq G$, and $\{g_i\}_{i\in\mathcal{J}} = T$ be a right transversal of H in G such that $1 \in T$. Then $\mathcal{F}G = \sum_{i\in\mathcal{J}} (\mathcal{F}H)g_i$ exhibits $\mathcal{F}G$ as a free left $\mathcal{F}H$-module. Thus each $\alpha \in \mathcal{F}G$ has a unique representation of the form $\alpha = \sum_{i\in\mathcal{J}} \alpha_i g_i$, where $\alpha_i \in \mathcal{F}H$ and $g_i \in T$.

Given $\alpha = \sum_{g\in G} a_g g \in \mathcal{F}G$, we may project α onto $\mathcal{F}H$ via the mapping $\pi_H : \mathcal{F}G \to \mathcal{F}H$ defined by $\pi_H(\alpha) = \pi_H(\sum_{g\in G} a_g g) = \sum_{g\in H} a_g g$. If α has $\sum_{i\in\mathcal{J}} \alpha_i g_i$ as its unique representation with respect to the right transversal T of H, it then follows that $\alpha_i = \pi_H(\alpha g_i^{-1})$, $\forall i$.

We begin with a basic lemma which restates the problem in terms of π_H and the α_i's.

<u>Lemma 2.1</u>. Let I be an ideal of $\mathcal{F}G$. Then, using the above notation, the following are equivalent:

(1) $I = (I \cap \mathcal{F}H)\mathcal{F}G$

(2) $\forall \alpha = \sum_{i\in\mathcal{J}} \alpha_i g_i \in I$, $\{\alpha_i\}_{i\in\mathcal{J}} \subseteq I \cap \mathcal{F}H$

(3) $I \cap \mathcal{F}H = \pi_H(I)$

<u>Proof</u>: (1) $\Rightarrow$ (2). By assumption, $\alpha = \sum_{t=1}^{n} B_t\gamma_t$ for some positive n, where $B_t \in I \cap \mathcal{F}H$ and $\gamma_t \in \mathcal{F}G$. Write $\gamma_t = \sum_{i\in\mathcal{J}} \epsilon_{ti}g_i$, $\epsilon_{ti} \in \mathcal{F}H$, $g_i \in T$. Thus $\alpha = \sum_{t=1}^{n} B_t(\sum_{i\in\mathcal{J}} \epsilon_{ti}g_i) = \sum_{i\in\mathcal{J}}(\sum_{t=1}^{n} B_t\epsilon_{ti})g_i$. It follows that $\forall i \in \mathcal{J}$, $\alpha_i = \sum_{t=1}^{n} B_t\epsilon_{ti}$. Now $B_t \in I \trianglelefteq \mathcal{F}G \Rightarrow B_t\epsilon_{ti} \in I$. Also, $B_t\epsilon_{ti} \in \mathcal{F}H$. So $\forall t$, $B_t\epsilon_{ti} \in I \cap \mathcal{F}H \Rightarrow \alpha_i = \sum_{t=1}^{n} B_t\epsilon_{ti} \in I \cap \mathcal{F}H$, as required.

(2) $\Rightarrow$ (3). If $\alpha \in I \cap \mathcal{F}H$, then $\alpha = \pi_H(\alpha) \in \pi_H(I)$. Conversely, let $\alpha \in I$. Then $\alpha = \sum_{i\in\mathcal{J}} \pi_H(\alpha g_i^{-1})g_i = \pi_H(\alpha) + \sum_{\substack{i\in\mathcal{J}\\ 1\neq g_i}} \pi_H(\alpha g_i^{-1})g_i$. By assumption, $\pi_H(\alpha g_i^{-1}) = \alpha_i$ belongs to $I \cap \mathcal{F}H$, $\forall i$. In particular, $\pi_H(\alpha) \in I \cap \mathcal{F}H$. Since $\alpha \in I$ was arbitrary, $\pi_H(I) \subseteq I \cap \mathcal{F}H$.

(3) $\Rightarrow$ (1). Since $I \trianglelefteq \mathcal{F}G$, it is clear that $(I \cap \mathcal{F}H)\mathcal{F}G \subseteq I$. Conversely, let $\alpha \in I$. Then $\alpha = \sum_{i\in\mathcal{J}} \pi_H(\alpha g_i^{-1})g_i$. Since $\alpha \in I$, so does αg_i^{-1}. Thus $\pi_H(\alpha g_i^{-1}) \in \pi_H(I)$. But $\pi_H(I) = I \cap \mathcal{F}H$, by assumption. Thus $\alpha \in (I \cap \mathcal{F}H)\mathcal{F}G$.

We pause to give a couple examples of the phenomenon $I = (I \cap \mathcal{F}H)\mathcal{F}G$.

<u>Theorem 2.2.</u> Let $H \trianglelefteq G$ such that $[G:H] = n$ is a unit in $\mathcal{F}$. Take $I = \operatorname{Rad} \mathcal{F}G$. Then $I = (I \cap \mathcal{F}H)\mathcal{F}G$.

Proof: By Villamayor's Theorem (Thm 1.14), $\operatorname{Rad} \mathcal{F}G = (\operatorname{Rad} \mathcal{F}H)\mathcal{F}G$. But by Theorem 1.8 $\operatorname{Rad} \mathcal{F}H = (\operatorname{Rad} \mathcal{F}G) \cap \mathcal{F}H$. Hence the result.

Under the assumptions of Theorem 2.2, we note that from Lemma 2.1 it follows that $\operatorname{Rad} \mathcal{F}H = \pi_H(\operatorname{Rad} \mathcal{F}G)$ and, equivalently, that those elements α belonging to $\operatorname{Rad} \mathcal{F}G$ are precisely the ones of the form $\alpha = \sum \alpha_i g_i$, $\alpha_i \in \operatorname{Rad} \mathcal{F}H$.

Before turning to the next example, we mention a piece of convenient notation. Let $\alpha = \sum\limits_{g \in G} a_g g \in \mathcal{F}G$. Then $\operatorname{Supp} \alpha = \{g \in G \mid a_g \neq 0\}$. Recall that $\mathcal{F}G$ consists of finite sums of the form $\sum\limits_{g \in G} a_g g$. Thus $\forall \alpha \in \mathcal{F}G$, $\operatorname{Supp} \alpha$ is a finite set.

Now let $G = \langle x \rangle$ be an infinite cyclic group and let H be a subgroup of G. Let $0 \neq I \trianglelefteq \mathcal{F}G$. In this case $\mathcal{F}G$ is known to be a principal ideal domain, so $I = \alpha\mathcal{F}G$, some $\alpha \in \mathcal{F}G$. By multiplication by x^k, we may choose a generator, α, for I of the form

$$(*) \qquad \alpha = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n x^n, \quad \alpha_0 \neq 0, \quad \alpha_n \neq 0 .$$

Theorem 2.3. Let G be infinite cyclic, H a subgroup of G, and $\alpha\mathcal{F}G = I \trianglelefteq \mathcal{F}G$, where α is of the form (*). Then $I = (I \cap \mathcal{F}H)\mathcal{F}G$ iff $\operatorname{Supp} \alpha \subseteq H$.

Proof: If $\operatorname{Supp} \alpha \subseteq H$, then $I \cap \mathcal{F}H = \alpha\mathcal{F}H$ and so $(I \cap \mathcal{F}H)\mathcal{F}G = (\alpha\mathcal{F}H)\mathcal{F}G = \alpha\mathcal{F}G = I$. Conversely, suppose $I = (I \cap \mathcal{F}H)\mathcal{F}G$. By Lemma 2.1, $\pi_H(I) = I \cap \mathcal{F}H$. In particular, $\pi_H(\alpha) \in I$. SO $\pi_H(\alpha) = \alpha\gamma$, some $\gamma \in \mathcal{F}G$. Now if $x^t \in \text{Supp}\ \gamma$, then $t > 0 \Rightarrow x^{n+t} \in \text{Supp}\ \alpha\gamma = \pi_H(\alpha)$, while $t < 0 \Rightarrow x^{-t} \in \text{Supp}\ \alpha\gamma = \pi_H(\alpha)$. But each of these is impossible since $\text{Supp}\ \pi_H(\alpha) \subseteq \{1,x,\ldots,x^n\}$. Thus $t = 0$ and $\gamma \in \mathcal{F} \Rightarrow \text{Supp}\ \pi_H(\alpha) = \text{Supp}\ \alpha$. Hence $\text{Supp}\ \alpha \subseteq H$.

§2. The Controller Subgroup

If $H \trianglelefteq G$ and $I = (I \cap \mathcal{F}H)\mathcal{F}G$, then H is said to control I. The following lemma is taken from Passman's text. (See Passman [9], p. 304).

Lemma 2.4. Let I be an ideal of $\mathcal{F}G$. Then there exists a unique normal subgroup, W, called the controller of I, with the property that $H \trianglelefteq G$ controls I iff $H \supseteq W$.

It is clear that the controller, W, of I described in Lemma 2.4 is $W = \cap H$, where the intersection is over all normal subgroups H for which $I = (I \cap \mathcal{F}H)\mathcal{F}G$.

Lemma 2.4 extends to non-normal subgroups H as indicated in the following theorem.

Theorem 2.5. Let I be an ideal of $\mathcal{F}G$. Then there exists a unique normal subgroup, W, of G such that for $H \leq G$, H not necessarily normal, $I = (I \cap \mathcal{F}H)\mathcal{F}G$ iff $H \supseteq W$.

Proof: Let $S = \{H \leq G \mid I = (I \cap \mathcal{F}H)\mathcal{F}G\}$. Note that $G \in S$ and so $S \neq \emptyset$. Set $W = \bigcap_{H \in S} H$. We claim that

(1) $W \trianglelefteq G$

(2) $I = (I \cap \mathcal{F}W)\mathcal{F}G$

<u>Proof of (1)</u>: Let $x \in W$, $g \in G$. Given $H \in S$ we have $I = (I \cap \mathcal{F}H)\mathcal{F}G \Rightarrow I = g^{-1}Ig = [g^{-1}(I \cap \mathcal{F}H)g][g^{-1}\mathcal{F}Gg]$. But $g^{-1}(I \cap \mathcal{F}H)g = I \cap \mathcal{F}H^g$. Thus $I = (I \cap \mathcal{F}H^g)\mathcal{F}G$, and $H^g \in S$. Since $x \in W = \bigcap\limits_{H \in S} H$, $x \in H^g \Rightarrow gxg^{-1} \in H$. But H was an arbitrary member of S. So $gxg^{-1} \in \bigcap\limits_{H \in S} G = W$, and $W \trianglelefteq G$.

<u>Proof of (2)</u>: By Lemma 1.1, it suffices to show that $I \cap \mathcal{F}W = \pi_W(I)$. Clearly $I \cap \mathcal{F}W = \pi_W(I \cap \mathcal{F}W) \subseteq \pi_W(I)$. We prove the converse by mimicking an argument given by Passman in his proof of Lemma 2.4. Given $\alpha \in I$, we need to show that $\pi_W(\alpha) \in I$. We proceed by induction on $|\mathrm{Supp}\ \alpha|$. If $|\mathrm{Supp}\ \alpha| = 0$, then $\alpha = 0$ and certainly $\pi_W(\alpha) = 0 \in I$. Suppose $|\mathrm{Supp}\ \alpha| = n > 0$, and that the result is true for all smaller support sizes. If $\alpha \in \mathcal{F}W$, then $\pi_W(\alpha) = \alpha \in I$, and we are done. On the other hand, if $\alpha \notin \mathcal{F}W$, then, by the definition of W, there exists some $H \in S$ such that $\mathrm{Supp}\ \alpha \not\subseteq H$. Since $H \in S$, we have $I = (I \cap \mathcal{F}H)\mathcal{F}G$ and it follows from Lemma 2.1 that $\pi_H(\alpha) \in I$. Furthermore, $|\mathrm{Supp}\ \pi_H(\alpha)| < |\mathrm{Supp}\ \alpha|$, and therefore, because $H \supseteq W$, induction yields $\pi_W(\alpha) = \pi_W(\pi_H(\alpha)) \in I$.

The theorem now follows easily. For if $I = (I \cap \mathcal{F}H)\mathcal{F}G$, then by definition of W, $W \leq H$. Conversely, if $W \leq H$, then, by (2), $I = (I \cap \mathcal{F}W)\mathcal{F}G \subseteq (I \cap \mathcal{F}H)\mathcal{F}G \subseteq I$, and so $I = (I \cap \mathcal{F}H)\mathcal{F}G$.

Corollary 2.6. If G is a simple group and $0 \neq I$ is a proper ideal of $\mathcal{F}G$, then for all proper subgroups H of G, $(I \cap \mathcal{F}H)\mathcal{F}G \subsetneq I$.

Proof: If $I = (I \cap \mathcal{F}H)\mathcal{F}G$, then by the previous theorem, $I = (I \cap \mathcal{F}W)\mathcal{F}G$ for some normal subgroup W contained in H. As G is simple, the only such W is $\langle 1 \rangle$, and in this case $I = (I \cap \mathcal{F}W)\mathcal{F}G$ is clearly impossible.

Example 2.7. Let G be a finite group, $\mathcal{F}$ a field of characteristic $p > 0$, and $I = \text{Rad}\,\mathcal{F}G$. By Corollary 1.15, Theorem 1.16, and Theorem 1.12, if $H \trianglelefteq G$ then $\text{Rad}\,\mathcal{F}G \subseteq (\text{Rad}\,\mathcal{F}H)\mathcal{F}G$ iff H contains a Sylow-p-subgroup of G. By Theorem 1.8, if $H \trianglelefteq G$ then each of these is equivalent to $I = (I \cap \mathcal{F}H)\mathcal{F}G$. Hence if $H \trianglelefteq G$, then $I = (I \cap \mathcal{F}H)\mathcal{F}G$ iff H contains a Sylow-p-subgroup of G. In particular, it follows that W, the controller of I, is the unique normal subgroup of G which is minimal among all normal subgroups containing Sylow-p-subgroups.

Example 2.8. Let $G = \langle x \rangle$ be infinite cyclic, H a subgroup of G, and I a non-zero ideal of $\mathcal{F}G$. We have previously seen that $I = \alpha\mathcal{F}G$ for some generator α of the form $\alpha = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n x^n$, where $\alpha_0 \neq 0$ and $\alpha_n \neq 0$. Then it is clear from Theorem 2.3 that W, the controller of I, is the subgroup of G which is generated by $\text{Supp}\,\alpha$.

§3. Concerning Induced Modules

In this section we offer another characterization of those subgroups H for which $I = (I \cap \mathcal{F}H)\mathcal{F}G$. We begin with a definition.

Definition 2.9. Let I be a fixed ideal of $\mathcal{F}G$. Then

$$C(I) = \left\{ H \leq G \middle/ \begin{array}{l} I \subseteq \mathrm{Ann}_{\mathcal{F}G} N^G \text{ for each } \mathcal{F}H\text{-module } N \text{ such that} \\ \qquad\qquad I \cap \mathcal{F}H \subseteq \mathrm{Ann}_{\mathcal{F}H} N \end{array} \right\}$$

Sinha [11] observed that a connection exists between those subgroups H of G satisfying $I = (I \cap \mathcal{F}H)\mathcal{F}G$ and the subgroups of G which belong to $C(I)$. We quote his result.

Lemma 2.10. (Sinha [11]) Let $I \trianglelefteq \mathcal{F}G$. Then

(i) If $H \in C(I)$ then $I = (I \cap \mathcal{F}H)\mathcal{F}G$.

(ii) If $H \trianglelefteq G$, then $H \in C(I)$ iff $I = (I \cap \mathcal{F}H)\mathcal{F}G$.

Unfortunately, we see from the definition of $C(I)$ that any direct application of Lemma 2.10 would involve testing each $\mathcal{F}H$-module N, and the class of all $\mathcal{F}H$-modules is, at best, unwieldy. We now offer an improvement of Lemma 2.10 which simplifies the criterion for membership in $C(I)$, and which shows that in fact $C(I)$ consists precisely of those subgroups H of G for which $I = (I \cap \mathcal{F}H)\mathcal{F}G$.

Theorem 2.11. Let I be an ideal of $\mathcal{F}G$ and H a subgroup of G. Then the following are equivalent:

(i) $I = (I \cap \mathfrak{F}H)\mathfrak{F}G$

(ii) $H \in C(I)$

(iii) $I \subseteq \mathrm{Ann}_{\mathfrak{F}G} N^G$, where $N = \dfrac{\mathfrak{F}H}{I \cap \mathfrak{F}H}$.

<u>Proof</u>: (i) $\Rightarrow$ (ii) Suppose that $I = (I \cap \mathfrak{F}H)\mathfrak{F}G$. Then by Theorem 2.5, $H \supseteq W$, where W is the controller of I. Let N be any $\mathfrak{F}H$-module such that $I \cap \mathfrak{F}H \subseteq \mathrm{Ann}_{\mathfrak{F}H} N$. Certainly, then, $I \cap \mathfrak{F}W \subseteq \mathrm{Ann}_{\mathfrak{F}H} N$.

Let $\{g_i\}$ be a right transversal for H in G, and let $\{k_j\}$ be a right transversal for W in G. Finally, let $\alpha \in I$. Since W is the controller of I, $I = (I \cap \mathfrak{F}W)\mathfrak{F}G$. In particular, by Lemma 2.1, $\alpha = \sum \alpha_j k_j$, where $\alpha_j \in I \cap \mathfrak{F}W$, $\forall j$. Note that since $W \trianglelefteq G$, $g_i \alpha_j g_i^{-1} \in I \cap \mathfrak{F}W \subseteq \mathrm{Ann}_{\mathfrak{F}H} N$, $\forall i,j$. Thus $(N^G)\alpha = (\sum_i N \otimes_{\mathfrak{F}H} g_i)(\sum \alpha_j k_j) \subseteq \sum_{i,j} (N g_i \alpha_j g_i^{-1} \otimes g_i k_j) = 0$. Hence $\alpha \in \mathrm{Ann}_{\mathfrak{F}G} N^G$, and $H \in C(I)$.

(ii) $\Rightarrow$ (iii) Suppose $H \in C(I)$. Then since $(I \cap \mathfrak{F}H) \subseteq \mathrm{Ann}_{\mathfrak{F}H} N$, it follows from the definition of $C(I)$ that $I \subseteq \mathrm{Ann}_{\mathfrak{F}G} N^G$.

(iii) $\Rightarrow$ (i) Suppose $I \subseteq \mathrm{Ann}_{\mathfrak{F}G} N^G$. Let $\alpha = \sum \alpha_i g_i \in I$, where $\{g_i\}$ is a right transversal for H in G, and where $\alpha_i \in \mathfrak{F}H$, $\forall i$. By assumption, $(N^G)\alpha = 0$. In particular, $\forall n \in N$ we have $(n \otimes 1)\alpha = (n \otimes 1)(\sum \alpha_i g_i) = {}$ $= \sum_i (n\alpha_i \otimes g_i) = 0$. Thus $n\alpha_i = 0$, $\forall i$. Since $n \in N$ was arbitrary, we have $N\alpha_i = 0$, $\forall i$. So each $\alpha_i \in \mathrm{Ann}_{\mathfrak{F}H} N = {}$ $= I \cap \mathfrak{F}H$, and $I = (I \cap \mathfrak{F}H)\mathfrak{F}G$.

Let H be a subgroup of G. Now the class of irreducible $\mathfrak{F}H$-modules is always of special interest in the study of $\mathfrak{F}H$. Furthermore, in some cases this class is completely determined; for example, in the case where H is finite and $\mathfrak{F}$ is algebraically closed of characteristic 0. So it is natural to wonder how $C(I)$ compares to the following class of subgroups of G.

Definition 2.12. Let I be an ideal of $\mathfrak{F}G$. Then we denote by $A(I)$ the set of all $H \leq G$ such that for each irreducible $\mathfrak{F}H$-module N, $I \cap \mathfrak{F}H \subseteq \mathrm{Ann}_{\mathfrak{F}H}N \Rightarrow I \subseteq \mathrm{Ann}_{\mathfrak{F}G}N^G$.

From their respective definitions, it is clear that $C(I) \subseteq A(I)$. Equivalently, by Theorem 2.11, if $H \leq G$ such that $I = (I \cap \mathfrak{F}H)\mathfrak{F}G$, then $H \in A(I)$. The converse fails, as demonstrated by the next example.

Example 2.13. Let $G = \{1,a,b,ab\}$, the Klein 4-group, and let $H = \langle a \rangle$. Let $\mathfrak{F}$ be the field of two elements. Since H is a p-group where $p = \mathrm{char}\,\mathfrak{F}$, it is known that $\mathrm{Rad}\,\mathfrak{F}H = w(\mathfrak{F}H)$, the augmentation ideal of $\mathfrak{F}H$. Since $w(\mathfrak{F}H)$ is a maximal ideal in $\mathfrak{F}H$ of dimension 1, it follows that $\frac{\mathfrak{F}H}{\mathrm{Rad}\,\mathfrak{F}H}$ is a field of dimension 1 over $\mathfrak{F}$. Thus $\frac{\mathfrak{F}H}{\mathrm{Rad}\,\mathfrak{F}H} \cong \mathfrak{F}$. Since each irreducible $\mathfrak{F}H$-module is also an irreducible $\frac{\mathfrak{F}H}{\mathrm{Rad}\,\mathfrak{F}H}$ - module, we see that there is but one irreducible $\mathfrak{F}H$-module, denote it $N = \langle n \rangle$, with the action of H defined by $n \cdot 1 = n = n \cdot a$.

We compute $\mathrm{Ann}_{\mathfrak{F}G}N^G$. Let $\alpha = \alpha_1 + \alpha_2 a + \alpha_3 b + \alpha_4 ab \in \mathrm{Ann}_{\mathfrak{F}G}N^G$, where each $\alpha_i \in \mathfrak{F}$. Now an arbitrary member of N^G has the

form $(n\delta_1 \otimes_{\mathcal{F}H} 1) + (n\delta_2 \otimes_{\mathcal{F}H} b)$, where each $\delta_i \in \mathcal{F}$.
Since $\alpha \in \mathrm{Ann}_{\mathcal{F}G}N^G$, we have

$$0 = [(n\delta_1 \otimes 1) + (n\delta_2 \otimes b)](\alpha_1 + \alpha_2 a + \alpha_3 b + \alpha_4 ab)$$
$$= [n(\delta_1\alpha_1 + \delta_1\alpha_2 + \delta_2\alpha_3 + \delta_2\alpha_4) \otimes 1] + [n(\delta_1\alpha_3 + \delta_1\alpha_4 + \delta_2\alpha_1 + \delta_2\alpha_2) \otimes b].$$

Thus $\forall \delta_i \in \mathcal{F}$, $\delta_1\alpha_1 + \delta_1\alpha_2 + \delta_2\alpha_3 + \delta_2\alpha_4 = 0$, and

$$\delta_1\alpha_3 + \delta_1\alpha_4 + \delta_2\alpha_1 + \delta_2\alpha_2 = 0 .$$

If $\delta_1 = 1$ and $\delta_2 = 0$, we have $\alpha_1 + \alpha_2 = 0 = \alpha_3 + \alpha_4$.
If $\delta_1 = 0$ and $\delta_2 = 1$, we have $\alpha_3 + \alpha_4 = 0 = \alpha_1 + \alpha_2$.
If $\delta_1 = 1 = \delta_2$, we have $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$.
The only other possibility is $\delta_1 = 0 = \delta_2$, which imposes no restrictions on the α_i. Thus we conclude that
$\mathrm{Ann}_{\mathcal{F}G}N^G = \{\alpha_1 + \alpha_2 a + \alpha_3 b + \alpha_4 ab \,/\, \alpha_1 = \alpha_2 \text{ and } \alpha_3 = \alpha_4\}$.

Now let $I = \langle 1 + a + b + ab \rangle$. That is, $I = \{\alpha(1 + a + b + ab)/\alpha \in \mathcal{F}\}$. We claim that $H \in A(I)$, but $I \neq (I \cap \mathcal{F}H)\mathcal{F}G$. Recall that there is only the trivial $\mathcal{F}H$-module, N, to consider. Now $I \cap \mathcal{F}H = 0$, so certainly $I \cap \mathcal{F}H \subseteq \mathrm{Ann}_{\mathcal{F}H}N$. So we must verify that $I \subseteq \mathrm{Ann}_{\mathcal{F}G}N^G$. But this is clear since

$$\mathrm{Ann}_{\mathcal{F}G}N^G = \{\alpha_1 + \alpha_2 a + \alpha_3 b + \alpha_4 ab/\alpha_1 = \alpha_2,\ \alpha_3 = \alpha_4\} \supseteq$$
$$\{\alpha_1 + \alpha_2 a + \alpha_3 b + \alpha_4 ab/\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4\} = I.$$

Thus $H \in A(I)$. However, $(I \cap \mathcal{F}H)\mathcal{F}G = 0 \neq I$.

Although $C(I)$ and $A(I)$ do not coincide in general, there are special cases in which $H \in A(I) \Rightarrow H \in C(I)$. In fact, Theorem 2.11 gives rise to some of these, which we now state as corollaries.

Corollary 2.14. Let I be an ideal of $\mathfrak{F}G$ and suppose that $I \cap \mathfrak{F}H$ is a maximal right ideal of $\mathfrak{F}H$. Then $H \in A(I) \Rightarrow I = (I \cap \mathfrak{F}H)\mathfrak{F}G$.

Proof: In this case $N = \frac{\mathfrak{F}H}{I \cap \mathfrak{F}H}$ is an irreducible $\mathfrak{F}H$-module such that $I \cap \mathfrak{F}H \subseteq \mathrm{Ann}_{\mathfrak{F}H} N$. Thus if $H \in A(I)$, then $I \subseteq \mathrm{Ann}_{\mathfrak{F}G} N^G$ by the very definition of $A(I)$. The result follows from Theorem 2.11.

Corollary 2.15. Let I be an ideal of $\mathfrak{F}G$ and suppose that $N = \frac{\mathfrak{F}H}{I \cap \mathfrak{F}H}$ is a completely reducible $\mathfrak{F}H$-module. Then $H \in A(I) \Rightarrow I = (I \cap \mathfrak{F}H)\mathfrak{F}G$.

Proof: By assumption, $N = N_1 \oplus N_2 \oplus \cdots \oplus N_k$, where each N_i is an irreducible $\mathfrak{F}H$-module. Now $I \cap \mathfrak{F}H \subseteq \mathrm{Ann}_{\mathfrak{F}G} N \Rightarrow I \cap \mathfrak{F}H \subseteq \mathrm{Ann}_{\mathfrak{F}H} N_i$, $\forall i$. If $H \in A(I)$, then since each N_i is irreducible, it follows that $I \subseteq \mathrm{Ann}_{\mathfrak{F}G}(N_i)^G$, $\forall i$. Thus $I \subseteq \mathrm{Ann}_{\mathfrak{F}G}(\bigoplus_{i=1}^{k} (N_i)^G) = \mathrm{Ann}_{\mathfrak{F}G} N^G$. Again, the result follows from Theorem 2.11.

Corollary 2.16. Let I be an ideal of $\mathfrak{F}G$, and let H be a finite subgroup of G. Suppose that $I \cap \mathfrak{F}H \supseteq \mathrm{Rad}\, \mathfrak{F}H$. Then $H \in A(I) \Rightarrow I = (I \cap \mathfrak{F}H)\mathfrak{F}G$.

Proof: In this case $N = \frac{\mathfrak{F}H}{I \cap \mathfrak{F}H}$ may be viewed as an $\frac{\mathfrak{F}H}{\mathrm{Rad}\, \mathfrak{F}H}$ - module. Since $\frac{\mathfrak{F}H}{\mathrm{Rad}\, \mathfrak{F}H}$ is semisimple artinian, all its right modules are completely reducible. In particular, N is a completely reducible $\frac{\mathfrak{F}H}{\mathrm{Rad}\, \mathfrak{F}H}$ - module. It follows that N is also completely reducible when viewed as an $\mathfrak{F}H$-module. The result now follows from the previous corollary.

We conclude this section with a theorem which describes another instance in which $H \in A(I) \Rightarrow I = (I \cap \mathcal{F}H)\mathcal{F}G$.

Theorem 2.17. Let I be an ideal of $\mathcal{F}G$ and let H be a finite subgroup of G. Suppose that $(I \cap \mathcal{F}H)$ is a prime ideal of $\mathcal{F}H$. Then $H \in A(I) \Rightarrow I = (I \cap \mathcal{F}H)\mathcal{F}G$.

Proof: Since H is finite, there exists an irreducible right $\mathcal{F}H$-submodule L, say, of $\frac{\mathcal{F}H}{I\cap\mathcal{F}H}$. Furthermore, L is of the form $L = \frac{L^*}{I\cap\mathcal{F}H}$, where L^* is a right ideal of $\mathcal{F}H$ containing $I \cap \mathcal{F}H$. Now $I \cap \mathcal{F}H \subseteq \mathrm{Ann}_{\mathcal{F}H}L$ and so since $H \in A(I)$ we have $I \subseteq \mathrm{Ann}_{\mathcal{F}G}L^G$. Let $\alpha = \sum \alpha_i g_i \in I$, where $\{g_i\}$ is a right transversal for H in G, and where $\alpha_i \in \mathcal{F}H, \forall i$. Then $(L^G)\alpha = 0$. In particular, $\forall \ell \in L$, $(\ell \otimes 1)\alpha = (\ell \otimes 1)(\sum \alpha_i g_i) = \sum (\ell\alpha_i) \otimes g_i = 0 \Rightarrow \ell\alpha_i = 0, \forall i$. Since $\ell \in L$ was arbitrary, it follows that $\alpha_i \in \mathrm{Ann}_{\mathcal{F}H}L, \forall i$.

Let $0 \neq \ell \in L$. Then $\ell = \ell^* + I \cap \mathcal{F}H$, where $\ell^* \in L^* \subseteq \mathcal{F}H$, but $\ell^* \notin I \cap \mathcal{F}H$. Since $\alpha_i \in \mathrm{Ann}_{\mathcal{F}H}L, \forall i$, we have $\ell^*\alpha_i \in I \cap \mathcal{F}H$. Since $\ell^* \notin I \cap \mathcal{F}H$ and since $I \cap \mathcal{F}H$ is prime, it follows that $\alpha_i \in I \cap \mathcal{F}H, \forall i$. Thus $\alpha \in (I \cap \mathcal{F}H)\mathcal{F}G$, and as $\alpha \in I$ was arbitrary, we have $I = (I \cap \mathcal{F}H)\mathcal{F}G$.

§4. Concerning Chains of Subgroups

We conclude this chapter with a couple of results dealing with the situation where $K < H < G$, and where one of the subgroups K, H belongs to $A(I)$, I being an ideal of $\mathcal{F}G$.

Theorem 2.18. Let $K < H < G$ and let I be an ideal of $\mathcal{F}G$. Suppose $H \in A(I)$, $K \in A(I \cap \mathcal{F}H)$, and that for each irreducible $\mathcal{F}K$-module, N, the corresponding induced module N^H is completely reducible. Then $K \in A(I)$.

Proof: Let N be an irreducible $\mathcal{F}K$-module such that $N(I \cap \mathcal{F}K) = 0$. Since $K \in A(I \cap \mathcal{F}H)$, it follows that $(N^H)(I \cap \mathcal{F}H) = 0$. Now by assumption, N^H is completely reducible; say $N^H = L_1 \oplus \cdots \oplus L_t$, where the L_i are irreducible $\mathcal{F}H$-modules. Since $(N^H)(I \cap \mathcal{F}H) = 0$, it is certainly true that $L_i(I \cap \mathcal{F}H) = 0$, $\forall i$. But since $H \in A(I)$, this implies that $I \subseteq \bigcap_{i=1}^{L} \mathrm{Ann}_{\mathcal{F}G}(L_i)^G = \mathrm{Ann}_{\mathcal{F}G}(L_1^G \oplus L_2^G \oplus \cdots \oplus L_t^G) = \mathrm{Ann}_{\mathcal{F}G}(N^H)^G = \mathrm{Ann}_{\mathcal{F}G}N^G$, and so $K \in A(I)$.

Theorem 2.20. Suppose $K \trianglelefteq H < G$ with $[H:K] = n$ a unit in $\mathcal{F}$, and let I be an ideal of $\mathcal{F}G$. Then if K belongs to $A(I)$, so does H.

Proof: Suppose N is an irreducible $\mathcal{F}H$-module such that $N(I \cap \mathcal{F}H) = 0$. By Clifford's Theorem, $N_K = N_1 \oplus N_2 \oplus \cdots \oplus N_t$, where each N_i is an irreducible $\mathcal{F}K$-module. Since $N(I \cap \mathcal{F}H) = 0$, we certainly have $N(I \cap \mathcal{F}K) = 0$. Thus $N_i(I \cap \mathcal{F}K) = 0$, $\forall i$. Since $K \in A(I)$, it follows that $I \subseteq \mathrm{Ann}_{\mathcal{F}G}(N_i)^G$, $\forall i$. Therefore, $I \subseteq \mathrm{Ann}_{\mathcal{F}G}(N_1^G \oplus N_2^G \oplus \cdots \oplus N_t^G) = \mathrm{Ann}_{\mathcal{F}G}(N_K)^G$.

Now since $[H:K]$ is a unit in $\mathcal{F}$, $(\mathcal{F}H, \mathcal{F}K)$ is a projective pairing by Theorem 1.12. It follows from Theorem 1.9 that N is isomorphic to a direct summand of $(N_K)^H$, say $(N_K)^H \cong N \oplus L$, for some $\mathcal{F}H$-module L. So we have the following; $(N_K)^G \cong [(N_K)^H]^G \cong [N \oplus L]^G \cong N^G \oplus L^G$. Since I annihilates $(N_K)^G$, it follows that I also annihilates N^G. Thus $H \in A(I)$.

CHAPTER 3 ANNIHILATORS OF IRREDUCIBLE MODULES

§1. The General Case

Let G be a group and let $\mathrm{Irr}(\mathcal{F}G)$ denote the class of all irreducible $\mathcal{F}G$-modules. In this chapter we restrict our attention to ideals I of the form $I = \bigcap_{M\in S} \mathrm{Ann}_{\mathcal{F}G}M$. where S is a non-empty subset of $\mathrm{Irr}(\mathcal{F}G)$. We note that $\mathrm{Rad}\ \mathcal{F}G = \bigcap_{M\in \mathrm{Irr}(\mathcal{F}G)} \mathrm{Ann}_{\mathcal{F}G}M$ is an ideal of this form, as is $\mathrm{Ann}_{\mathcal{F}G}M$ for any $M \in \mathrm{Irr}(\mathcal{F}G)$. Of course, in some cases $\mathrm{Rad}\ \mathcal{F}G = 0$. However, the augmentation ideal of $\mathcal{F}G$, $\omega(\mathcal{F}G)$, is the annihilator of the trivial irreducible $\mathcal{F}G$-module and $\omega(\mathcal{F}G) \neq 0$ provided $|G| > 1$. So the class of ideals under consideration is never trivial.

We begin with a definition.

<u>Definition 3.1</u>. Let G be a group, let $\mathcal{F}$ be fixed, let $\emptyset \neq S \subseteq \mathrm{Irr}(\mathcal{F}G)$, and let $I = \bigcap_{M\in S} \mathrm{Ann}_{\mathcal{F}G}M$. Then we say $H \leq G$ has property ρ with respect to I and S, or $H \in \rho(I_S)$, if $I = \bigcap_{M\in S} \mathrm{Ann}_{\mathcal{F}G}M \subseteq (\bigcap_{\substack{W \text{irred} \subseteq M_H \\ M\in S}} \mathrm{Ann}_{\mathcal{F}H}W)\mathcal{F}G$.

We note that our definition depends on the choice of S as well as I, since it is not clear that $I = \bigcap_{M\in S} \mathrm{Ann}_{\mathcal{F}G}M$ has a unique representation of this form.

We now pause to consider an important example.

Example 3.2. Take $S = \mathrm{Irr}(\mathcal{F}G)$ and $I = \bigcap_{M \in \mathrm{Irr}(\mathcal{F}G)} \mathrm{Ann}_{\mathcal{F}G} M = \mathrm{Rad}\ \mathcal{F}G$. Then $H \in \rho(I_S)$ iff $I = \mathrm{Rad}\ \mathcal{F}G \subseteq (\mathrm{Rad}\ \mathcal{F}H)\mathcal{F}G$. This is so since by Lemma 1.1, given any irreducible $\mathcal{F}H$-module, W, there exists some $M \in \mathrm{Irr}(\mathcal{F}G)$ such that $W \subseteq M_H$. Thus $\mathrm{Rad}\ \mathcal{F}H = \bigcap_{\substack{W \text{ irred} \subseteq M_H \\ M \in S}} \mathrm{Ann}_{\mathcal{F}H} W$.

In this special case, therefore, we see that $H \in \rho(I_S)$ iff $(\mathcal{F}G, \mathcal{F}H)$ has property ρ in the sense of Definition 1.13. So Definition 3.1 may be viewed as a generalization of the concept of property ρ.

Our main interest is still the study of those pairs (I,H), $I \trianglelefteq \mathcal{F}G$, $H \leq G$, for which $I = (I \cap \mathcal{F}H)\mathcal{F}G$. If I is of the form $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M$ for some $S \subseteq \mathrm{Irr}(\mathcal{F}G)$, then it is clear that $I \cap \mathcal{F}H = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}H} M$. Thus $I = (I \cap \mathcal{F}H)\mathcal{F}G$ iff $I = (\bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}H} M)\mathcal{F}G$. Actually, $I \subseteq (\bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}H} M)\mathcal{F}G$ is sufficient for $I = (I \cap \mathcal{F}H)\mathcal{F}G$ since the opposite inclusion $[\bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}H} M]\mathcal{F}G = (I \cap \mathcal{F}H)\mathcal{F}G \subseteq I$ automatically holds.
For future reference, we summarize these comments in a lemma.

Lemma 3.3. Let G be a group, $\emptyset \neq S \subseteq \mathrm{Irr}(\mathcal{F}G)$, and $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M$. Then $I = (I \cap \mathcal{F}H)\mathcal{F}G$ iff $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M \subseteq (\bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}H} M)\mathcal{F}G$.

Now let I be of the form $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M$, where $S \subseteq \mathrm{Irr}(\mathcal{F}G)$. Then as we'd expect from the similarities between Lemma 3.3 and Definition 3.1, the class consisting of those subgroups H of G for which $I = (I \cap \mathcal{F}H)\mathcal{F}G$ is strongly related to the class $\rho(I_S)$. Indeed, as we shall later see, these classes coincide in several important special cases. In general we have

<u>Theorem 3.4</u>. Let G be a group, $\emptyset \neq S \subseteq \mathrm{Irr}(\mathcal{F}G)$, and let $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M$. If H is a subgroup of G such that $I = (I \cap \mathcal{F}H)\mathcal{F}G$, then $H \in \rho(I_S)$.

<u>Proof</u>: Suppose $I = (I \cap \mathcal{F}H)\mathcal{F}G$. Then by Lemma 3.3 we have $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M \subseteq (\bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}H} M)\mathcal{F}G$. But $(\bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}H} M)\mathcal{F}G \subseteq (\bigcap_{\substack{W \text{ irred} \subseteq M_H \\ M \in S}} \mathrm{Ann}_{\mathcal{F}H} W)\mathcal{F}G$, since any element of $\mathcal{F}H$ which annihilates some $M \in S$ must certainly annihilate all its irreducible $\mathcal{F}H$-submodules.

We note that in the case $I = \mathrm{Rad}\ \mathcal{F}G$, Theorem 3.4 simply says that if $\mathrm{Rad}\ \mathcal{F}G = [(\mathrm{Rad}\ \mathcal{F}G) \cap \mathcal{F}H]\mathcal{F}G$, then $\mathrm{Rad}\ \mathcal{F}G \subseteq (\mathrm{Rad}\ \mathcal{F}H)\mathcal{F}G$. Otherwise put, if H contains the controller of $\mathrm{Rad}\ \mathcal{F}G$, then $(\mathcal{F}G, \mathcal{F}H)$ has property ρ. This was also an immediate consequence of Theorem 1.7.

The following example shows that in general $H \in \rho(I_S)$ is not sufficient for $I = (I \cap \mathcal{F}H)\mathcal{F}G$.

<u>Example 3.5</u>. Let $G = S_3$, $\mathrm{char}\ \mathcal{F} = 2$, $S = \mathrm{Irr}(\mathcal{F}G)$, and $I = \mathrm{Rad}\ \mathcal{F}G$. Let H be a subgroup of G of order 2. Then since $[G:H]$ is a unit in $\mathcal{F}$, it follows from

Theorem 1.12 that $(\mathcal{F}G, \mathcal{F}H)$ is a projective pairing. Furthermore, since $|G| = 2 \cdot 3$, we conclude from Theorem 1.18 that $(\mathcal{F}G, \mathcal{F}H)$ has property ρ. Equivalently, $H \in \rho(I_S)$.

However, $I \neq (I \cap \mathcal{F}H)\mathcal{F}G$. For suppose equality holds. Then H contains the controller subgroup, W, of $I = \text{Rad } \mathcal{F}G$. But the only normal subgroup of G contained in H is $\langle 1 \rangle$. This means that $W = \langle 1 \rangle$ and so consequently $(\text{Rad } \mathcal{F}G) \cap \mathcal{F}W = (\text{Rad } \mathcal{F}G) \cap \mathcal{F} = 0$. Since W controls I, we have $I = \text{Rad } \mathcal{F}G = [(\text{Rad } \mathcal{F}G) \cap \mathcal{F}W]\mathcal{F}G = 0$. But by Maschke's Theorem (Thm 1.6), Rad $\mathcal{F}G \neq 0$. Thus $I = (I \cap \mathcal{F}H)\mathcal{F}G$ is impossible.

Let $I = \bigcap_{M \in S} \text{Ann}_{\mathcal{F}G} M$ for some $S \subseteq \text{Irr } \mathcal{F}G$, and let $\mathcal{C}$ denote the class of all subgroups H of G such that $I = (I \cap \mathcal{F}H)\mathcal{F}G$. Since $H \in \mathcal{C}$ iff H contains the controller of I, it is evident that the class $\mathcal{C}$ has the property that $H \leq K \leq G$, $H \in \mathcal{C} \Rightarrow K \in \mathcal{C}$. It is not clear that the same property holds for the class $\rho(I_S)$, but the next theorem provides a result somewhat along these lines.

<u>Theorem 3.6</u>. Let $I = \bigcap_{M \in S} \text{Ann}_{\mathcal{F}G} M$ for some $\emptyset \neq S \subseteq \text{Irr}(\mathcal{F}G)$. Suppose that $H \in \rho(I_S)$ and K is a subgroup of G such that $H \trianglelefteq K \leq G$ and $[K:H] < \infty$. Then $K \in \rho(I_S)$.

<u>Proof</u>: Let L be any irreducible $\mathcal{F}K$-module. Then by Clifford's Theorem L_H is completely reducible. It is therefore clear that $\text{Ann}_{\mathcal{F}H} L = \bigcap_{W \text{ irred} \subseteq L_H} \text{Ann}_{\mathcal{F}H} W$. In

particular, this is true for any $L \subseteq M_K$, where $M \in S$. Thus, since $H \in \rho(I_S)$, we have $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M \subseteq$

$$[\bigcap_{\substack{W^{\mathrm{irred}} \subseteq M_H \\ M \in S}} \mathrm{Ann}_{\mathcal{F}H} W]\,\mathcal{F}G \subseteq [\bigcap_{\substack{L^{\mathrm{irred}} \subseteq M_K \\ M \in S}} (\bigcap_{W^{\mathrm{irred}} \subseteq L_H} \mathrm{Ann}_{\mathcal{F}H} W)]\,\mathcal{F}G$$

$$= [\bigcap_{\substack{L^{\mathrm{irred}} \subseteq M_K \\ M \in S}} \mathrm{Ann}_{\mathcal{F}H} L]\,\mathcal{F}G \subseteq [\bigcap_{\substack{L^{\mathrm{irred}} \subseteq M_K \\ M \in S}} \mathrm{Ann}_{\mathcal{F}K} L]\,\mathcal{F}G .$$

So by definition, $K \in \rho(I_S)$.

<u>Corollary 3.7</u>. Let $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M$ for some $\emptyset \neq S \subseteq \mathrm{Irr}(\mathcal{F}G)$. Suppose that $H \in \rho(I_S)$ and that K is a subgroup of G such that $H < K < G$, $[K:H] < \infty$, and H is subnormal in K. Then $K \in \rho(I_S)$.

<u>Proof</u>: Repeated applications of Theorem 3.6.

The following theorem gives a sufficient condition for $H \in \rho(I_S)$.

<u>Theorem 3.8</u>. Let $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M$ for some $\emptyset \neq S \subseteq \mathrm{Irr}(\mathcal{F}G)$. Suppose $H \leq G$ has the property that for each $M \in S$ and for each irreducible $\mathcal{F}H$-module $W \subseteq M_H$, it is true that $\mathrm{Ann}_{\mathcal{F}G} W^G \supseteq I$. Then $H \in \rho(I_S)$.

<u>Proof:</u> Let $\sum a_i x_i \in I$, where $\{x_i\}$ is a right transversal for H in G and where $a_i \in \mathcal{F}H$, $\forall i$. Let $M \in S$ and let W be any irreducible $\mathcal{F}H$-submodule of M_H. By assumption $\sum a_i x_i$ annihilates $W^G = \sum_i W \otimes x_i$. In

particular, for each $w \in W$ we have $0 = (w \otimes 1)(\sum_i a_i x_i) = \sum_i (wa_i \otimes x_i)$. Thus $wa_i = 0$, $\forall i$. Since $w \in W$ was arbitrary, $a_i \in \mathrm{Ann}_{\mathcal{F}H} W$, $\forall i$. Furthermore, since M was an arbitrary member of S and W an arbitrary irreducible $\mathcal{F}H$-submodule of M_H, it follows that for each i, $a_i \in \bigcap_{\substack{W^{\text{irred}} \subseteq M_H \\ M \in S}} \mathrm{Ann}_{\mathcal{F}H} W$. Thus $\sum a_i x_i \in (\bigcap_{\substack{W^{\text{irred}} \subseteq M_H \\ M \in S}} \mathrm{Ann}_{\mathcal{F}H} W)\mathcal{F}G$, and $H \in \rho(I_S)$.

<u>Corollary 3.9</u>. Let $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M$ for some $\emptyset \neq S \subseteq \mathrm{Irr}(\mathcal{F}G)$. Then $A(I) \subseteq \rho(I_S)$, where $A(I)$ is as in definition 2.12.

<u>Proof</u>: Let $H \in A(I)$ and let W be an irreducible $\mathcal{F}H$-submodule of M_H, where $M \in S$. By its definition, I annihilates M. Since $W \subseteq M_H$, it is therefore clear that $I \cap \mathcal{F}H \subseteq \mathrm{Ann}_{\mathcal{F}H} W$. But since $H \in A(I)$, this implies that $I \subseteq \mathrm{Ann}_{\mathcal{F}G} W^G$. So the hypotheses of Theorem 3.8 are satisfied and $H \in \rho(I_S)$.

§2. <u>Normal Subgroups of Finite Index</u>

If the subgroup H of G is normal in G and has finite index in G, then Clifford's Theorem may be applied to extend the results of the previous section. Indeed, in this case the converses of Theorem 3.4 and Theorem 3.8 both hold. We begin with the converse of Theorem 3.4.

<u>Theorem 3.10</u>. Let $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M$ for some $\emptyset \neq S \subseteq \mathrm{Irr}(\mathcal{F}G)$. If $H \trianglelefteq G$ and $[G:H] < \infty$, then $I = (I \cap \mathcal{F}H)\mathcal{F}G$ iff $H \in \rho(I_S)$ iff $H \in A(I)$.

<u>Proof</u>: By Clifford's Theorem, each $M \in S$ is completely reducible as an $\mathcal{F}H$-module. So an element of $\mathcal{F}H$ annihilates M_H iff it annihilates each of its irreducible $\mathcal{F}H$-submodules. Thus $\bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M =$

$$\bigcap_{\substack{W^{\mathrm{irred}} \subseteq M_H \\ M \in S}} \mathrm{Ann}_{\mathcal{F}H} W.$$

It is therefore clear from Definition 3.1 and Lemma 3.3 that $I = (I \cap \mathcal{F}H)\mathcal{F}G$ iff $H \in \rho(I_S)$.

Now $I = (I \cap \mathcal{F}H)\mathcal{F}G$ iff $H \in \mathcal{C}(I)$ (Thm 2.11) and $\mathcal{C}(I) \subseteq A(I)$ by their respective definitions. Furthermore, $A(I) \subseteq \rho(I_S)$ by Corollary 3.9. The result follows.

We note that if I is not of the form $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M$ for some $S \subseteq \mathrm{Irr}(\mathcal{F}G)$, then it is possible that $I \neq (I \cap \mathcal{F}H)\mathcal{F}G$ even though $H \trianglelefteq G$, $[G:H] < \infty$ and $H \in A(I)$. Such was the case in Example 2.13 of the previous chapter where we took I to be an ideal properly contained in Rad $\mathcal{F}G$.

We now turn out attention to the coverse of Theorem 3.8 in the case $H \trianglelefteq G$ and $[G:H] < \infty$. In view of our last result, we employ a slightly different wording than that used originally in the statement of Theorem 3.8.

<u>Theorem 3.11</u>. Let $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M$ for some $\emptyset \neq S \subseteq \mathrm{Irr}(\mathcal{F}G)$. Suppose $H \trianglelefteq G$ and $[G:H] < \infty$. Then

$I = (I \cap \mathcal{F}H)\mathcal{F}G$ iff for each $M \in S$ and for each irreducible $\mathcal{F}H$-module $W \subseteq M_H$ it is true that $I \subseteq \mathrm{Ann}_{\mathcal{F}G} W^G$.

The following simple lemma is used in the proof of Theorem 3.11.

Lemma 3.12. Let $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M$ for some $\emptyset \neq S \subseteq \mathrm{Irr}(\mathcal{F}G)$, and suppose $H \trianglelefteq G$. Then $I \cap \mathcal{F}H = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}H} M$ is a G-invariant ideal of $\mathcal{F}G$.

Proof: Let $\alpha \in I \cap \mathcal{F}H$ and $x \in G$. Then for each $M \in S$, $M\alpha^x = M(x^{-1}\alpha x) \subseteq M\alpha x$. But $I \subseteq \mathrm{Ann}_{\mathcal{F}G} M$ by its definition. Hence $M\alpha = 0$ which forces $M\alpha^x = 0$. So $\alpha^x \in \mathrm{Ann}_{\mathcal{F}G} M$ for each $M \in S$, and consequently $\alpha^x \in I$. Also, since $H \trianglelefteq G$ and $\alpha \in \mathcal{F}H$, we have $\alpha^x \in \mathcal{F}H$. Thus $\alpha^x \in I \cap \mathcal{F}H$ and $I \cap \mathcal{F}H$ is G-invariant.

Proof of Theorem 3.11: In view of Theorems 3.8 and 3.10 we need only show that if $I = (I \cap \mathcal{F}H)\mathcal{F}G$, then $I \subseteq \mathrm{Ann}_{\mathcal{F}G} W^G$ for each irreducible $\mathcal{F}H$-module W such that $W \subseteq M_H$, some $M \in S$.

So suppose that $I = (I \cap \mathcal{F}H)\mathcal{F}G$. Let $\alpha = \sum_{i=1}^{n} \alpha_i x_i \in I$, where $\{x_i\}_{i=1}^{n}$ is a right transversal for H in G and where each $\alpha_i \in \mathcal{F}H$. Then by Lemma 2.1, each $\alpha_i \in I \cap \mathcal{F}H = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}H} M$. Now $I \cap \mathcal{F}H$ is a G-invariant ideal by the previous lemma. So in particular, for each i and j and for each $M \in S$ we have that $\alpha_i^{x_j^{-1}} \in \mathrm{Ann}_{\mathcal{F}H} M$.

Now let W be an irreducible $\mathcal{F}H$-submodule of M_H, where $M \in S$. Then since $\alpha_i^{x_j^{-1}} \in \mathrm{Ann}_{\mathcal{F}H} M$ for each i and j, certainly $\alpha_i^{x_j^{-1}} \in \mathrm{Ann}_{\mathcal{F}H} W$. Hence $(W^G)\alpha =$
$W^G(\Sigma\, \alpha_i x_i) = (\sum_{j=1}^{n} W \otimes x_j)(\sum_{i=1}^{n} \alpha_i x_i) \subseteq \sum_{j=1}^{n}(\sum_{i=1}^{n} W \otimes x_j \alpha_i x_i) =$
$\sum_{j=1}^{n}(\sum_{i=1}^{n} W \otimes \alpha_i^{x_j^{-1}} x_j x_i) = \sum_{j=1}^{n}(\sum_{i=1}^{n} W\alpha_i^{x_j^{-1}} \otimes x_j x_i) = 0.$
Since $\alpha \in I$ was arbitrary, $I \subseteq \mathrm{Ann}\, W^G$, as required.

We pause to interpret Theorems 3.10 and 3.11 in the special case $I = \mathrm{Rad}\, \mathcal{F}G$.

<u>Corollary 3.13</u>. Let $I = \mathrm{Rad}\, \mathcal{F}G$, $H \trianglelefteq G$, and $[G:H] < \infty$. Then the following are equivalent.

(i) $\mathrm{Rad}\, \mathcal{F}G \subseteq (\mathrm{Rad}\, \mathcal{F}H)\mathcal{F}G$

(ii) $\mathrm{Rad}\, \mathcal{F}G = [(\mathrm{Rad}\, \mathcal{F}G) \cap \mathcal{F}H]\mathcal{F}G$

(iii) For each irreducible $\mathcal{F}H$-module W, $\mathrm{Rad}\, \mathcal{F}G \subseteq \mathrm{Ann}_{\mathcal{F}G} W^G$.

If in addition we assume that $[G:H] = n$ is a unit in $\mathcal{F}$, we obtain the following result.

<u>Theorem 3.14</u>. Let $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M$ for some $\emptyset \neq S \subseteq \mathrm{Irr}(\mathcal{F}G)$. If $H \trianglelefteq G$ and $[G:H] = n$ is a unit in $\mathcal{F}$, then $I = (I \cap \mathcal{F}H)\mathcal{F}G$ iff $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} (M_H)^G$.

<u>Proof</u>: First suppose that $I = (I \cap \mathcal{F}H)\mathcal{F}G$. Let $M \in S$. Since $[G:H] = n$ is a unit in $\mathcal{F}$, $(\mathcal{F}G, \mathcal{F}H)$ is a projective pairing by Theorem 1.12. It follows from Theorem 1.9 that M is a component of $(M_H)^G$. Consequently, $\mathrm{Ann}_{\mathcal{F}G} M \supseteq \mathrm{Ann}_{\mathcal{F}G} (M_H)^G$. Letting M range over S we have

$I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M \supseteq \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G}(M_H)^G$. On the other hand, Clifford's Theorem guarantees that for each $M \in S$, M_H is of the form $M_H = W_1 \oplus W_2 \oplus \cdots \oplus W_t$, where the W_i are irreducible $\mathcal{F}H$-modules. Since $I = (I \cap \mathcal{F}H)\mathcal{F}G$, we have by Theorem 3.11 that $I \subseteq \mathrm{Ann}_{\mathcal{F}G}(W_i)^G$, $\forall i$. Thus $I \subseteq \bigcap_{i=1}^{t} \mathrm{Ann}_{\mathcal{F}G}(W_i)^G = \mathrm{Ann}_{\mathcal{F}G}(M_H)^G$. Letting M range over S we have $I \subseteq \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G}(M_H)^G$. So $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G}(M_H)^G$, as required.

Conversely, suppose $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G}(M_H)^G$. Let $M \in S$ and W be an irreducible $\mathcal{F}H$-submodule of M_H. Then $W^G \subseteq (M_H)^G$ and so $\mathrm{Ann}_{\mathcal{F}G} W^G \supseteq \mathrm{Ann}_{\mathcal{F}G}(M_H)^G \supseteq \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G}(M_H)^G = I$. By Theorem 3.11, $I = (I \cap \mathcal{F}H)\mathcal{F}G$.

Theorem 3.14 suggests a condition sufficient for $I = (I \cap \mathcal{F}H)\mathcal{F}G$, as seen in our next result.

<u>Theorem 3.15</u>. Let $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M$ for some $\emptyset \neq S \subseteq \mathrm{Irr}(\mathcal{F}G)$, and let $H \trianglelefteq G$ such that $[G:H] = n$ is a unit in $\mathcal{F}$. Suppose that for each $M \in S$ and for each irreducible $\mathcal{F}G$-submodule, L, of $(M_H)^G$ it is true that $L \in S$ (up to isomorphism). Then $I = (I \cap \mathcal{F}H)\mathcal{F}G$.

<u>Proof</u>: Since $H \trianglelefteq G$ and $[G:H] = n$ is a unit in $\mathcal{F}$, it follows from Theorems 1.12 and 1.17 that for each irreducible $\mathcal{F}H$-module, W, the corresponding induced module, W^G, is a completely reducible $\mathcal{F}G$-module. Now by Clifford's Theorem and Proposition 1.3, for each

$M \in S$, $(M_H)^G$ is of the form $(M_H)^G = W_1^G \oplus \cdots \oplus W_t^G$, where the W_i are irreducible $\mathcal{F}H$-modules. Since each $(W_i)^G$ is completely reducible, so is $(M_H)^G$. It follows that $\bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G}(M_H)^G$ is precisely the intersection of the annihilators of the irreducible $\mathcal{F}G$-components appearing in the $(M_H)^G$. Since by assumption the only such components are among the members of S, we certainly have $\bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G}(M_H)^G \supseteq \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G}M$.

On the other hand, since by Theorem 1.12 $(\mathcal{F}G,\mathcal{F}H)$ is a projective pairing, we have by Theorem 1.9 that M is a component of $(M_H)^G$, $\forall\, M \in S$. Consequently, $\bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G}M \supseteq \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G}(M_H)^G$.

Thus $\bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G}M = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G}(M_H)^G$, and by Theorem 3.14, $I = (I \cap \mathcal{F}H)\mathcal{F}G$.

Theorem 3.15 gives rise to an interesting corollary, but before stating it we need a definition.

Definition 3.16. An $\mathcal{F}G$-module, L, is said to be homogeneous if it is a direct sum of, say, n copies of an irreducible $\mathcal{F}G$-module M.

Corollary 3.17. Let $I = \mathrm{Ann}_{\mathcal{F}G}M$ for some $M \in \mathrm{Irr}(\mathcal{F}G)$, and let $H \trianglelefteq G$ such that $[G:H] = n$ is a unit in $\mathcal{F}$. If $(M_H)^G$ is a homogeneous $\mathcal{F}G$-module, then $I = (I \cap \mathcal{F}H)\mathcal{F}G$.

Proof: Since $(\mathcal{F}G,\mathcal{F}H)$ is a projective pairing, by Theorem 1.9 M is a component of $(M_H)^G$. Since $(M_H)^G$ is homogeneous, M is the only irreducible component. Thus the hypotheses of Theorem 3.15 are satisfied, and $I = (I \cap \mathcal{F}H)\mathcal{F}G$.

We now take time out to consider an example which illustrates some of the concepts discussed so far.

Example 3.18. Let $G = D_4 = \langle a,b / a^2 = b^4 = 1,\ ab = b^3a\rangle$ and let H be the central subgroup of G generated by b^2. Let $\mathcal{F}$ be an algebraically closed field of characteristic 0. Then, in particular, the hypotheses of Theorem 3.15 are satisfied.

Now H and G have the following character tables:

H	1	b^2
ψ_1	1	1
ψ_2	1	-1

G	1	b^2	$\{b,b^3\}$	$\{a,ab^2\}$	$\{ab,ab^3\}$
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	-1	-1	1
χ_4	1	1	1	-1	-1
χ_5	2	-2	0	0	0

Let M be the irreducible $\mathcal{F}G$-module corresponding to χ_5, let $I = \mathrm{Ann}_{\mathcal{F}G}M$, and let W be the irreducible $\mathcal{F}G$-module corresponding to ψ_2. Than an easy computation shows that $(\chi_5)_H = 2\psi_2$, and $(\psi_2)^G = 2\chi_5$.

In terms of the corresponding irreducible modules, we have $(M_H)^G \cong M \oplus M \oplus M \oplus M$. Thus $(M_H)^G$ is homogeneous, and so $I = (I \cap \mathcal{F}H)\mathcal{F}G$ by Corollary 3.17. Since H is a minimal non-identity normal subgroup of G, it follows that

H is the controller of I. Consequently, if $K \leq G$, then $I = (I \cap \mathcal{F}K)\mathcal{F}G$ iff $K \geq H$.

We haven't been able to prove the converse of Theorem 3.15 as stated, but the converse does hold under the added assumption that $|G| < \infty$. The proof of this fact is the main result of the next section.

§3. Finite Groups

If G is finite, then $\mathcal{F}G$ is a finite dimensional algebra. In this case, the structure of the ideals of $\mathcal{F}G$ is more easily ascertained. Since we are only interested in ideals I of the form $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M$, and since each such ideal corresponds uniquely to some ideal of $\frac{\mathcal{F}G}{\mathrm{Rad}\ \mathcal{F}G}$, much insight can be gained by examining $\frac{\mathcal{F}G}{\mathrm{Rad}\ \mathcal{F}G}$, which is a semisimple, artinian algebra. The following facts concerning $\frac{\mathcal{F}G}{\mathrm{Rad}\ \mathcal{F}G}$ (which are also true for semisimple artinian algebras in general) will prove useful. See Isaacs [4] and Curtis and Reiner [2] for more details.

<u>Proposition 3.19</u>. Let G be a finite group.

a. Each irreducible $\mathcal{F}G$-module may be viewed as an irreducible $\frac{\mathcal{F}G}{\mathrm{Rad}\ \mathcal{F}G}$ - module, and vice versa. Furthermore, there are only finitely many of these (up to isomorphism). Say $\mathrm{Irr}(\mathcal{F}G) = \mathrm{Irr}(\frac{\mathcal{F}G}{\mathrm{Rad}\ \mathcal{F}G}) = \{M_1, M_2, \ldots, M_t\}$.

b. Each (right) $\frac{\mathcal{F}G}{\mathrm{Rad}\ \mathcal{F}G}$ - module is completely reducible. Equivalently, each (right) $\frac{\mathcal{F}G}{\mathrm{Rad}\ \mathcal{F}G}$ - module is the sum of its irreducible $\frac{\mathcal{F}G}{\text{Rad }\mathcal{F}G}$ - modules.

c. $\frac{\mathcal{F}G}{\text{Rad }\mathcal{F}G} = \overline{I}_1 \oplus \overline{I}_2 \oplus \cdots \oplus \overline{I}_t$, where

i. Each $\overline{I}_i$ is a simple ideal

ii. Each $\overline{I}_i$ is the sum of all the right ideals of $\frac{\mathcal{F}G}{\text{Rad }\mathcal{F}G}$ which are isomorphic to M_i, where $M_i \in \text{Irr}(\frac{\mathcal{F}G}{\text{Rad }\mathcal{F}G})$ was defined in part a.

iii. $\text{Ann}_{\frac{\mathcal{F}G}{\text{Rad }\mathcal{F}G}} M_j = \bigoplus_{i \neq j} \overline{I}_i$

d. For each i, $1 \leq i \leq t$, let I_i be that unique right ideal containing $\text{Rad }\mathcal{F}G$ which corresponds to $\overline{I}_i$ under the canonical homomorphism $\mathcal{F}G \to \frac{\mathcal{F}G}{\text{Rad }\mathcal{F}G}$. Then $\text{Ann}_{\mathcal{F}G} M_j = \sum_{i \neq j} I_i$.

The considerations listed in proposition 3.19 lead us to the following theorem:

<u>Theorem 3.20</u>. Let G be a finite group. Then the ideals I of the form $I = \bigcap_{M \in S} \text{Ann}_{\mathcal{F}G} M$, $\emptyset \neq S \subseteq \text{Irr}(\mathcal{F}G)$, are precisely those proper ideals of $\mathcal{F}G$ which contain $\text{Rad }\mathcal{F}G$.

<u>Proof</u>: (We use the notation of proposition 3.19)

Clearly, if $I = \bigcap_{M \in S} \text{Ann}_{\mathcal{F}G} M$, then $I \supseteq \text{Rad }\mathcal{F}G$.

Conversely, suppose $I \supseteq \text{Rad }\mathcal{F}G$. Then $\frac{I}{\text{Rad }\mathcal{F}G}$ is an $\frac{\mathcal{F}G}{\text{Rad }\mathcal{F}G}$ - module. Consequently, by Proposition 3.19 (b), $\frac{I}{\text{Rad }\mathcal{F}G}$ is the sum of its irreducible $\frac{\mathcal{F}G}{\text{Rad }\mathcal{F}G}$ - modules, namely, those minimal right ideals of $\frac{\mathcal{F}G}{\text{Rad }\mathcal{F}G}$ which are contained in $\frac{I}{\text{Rad } \mathcal{F}G}$. Now since the $\bar{I}_i$ of proposition 3.19 (c) are simple, $\forall i$ we have that $\frac{I}{\text{Rad } \mathcal{F}G} \cap \bar{I}_i$ is either empty or equal to $\bar{I}_i$ itself. These considerations lead to the conclusion that $\frac{I}{\text{Rad } \mathcal{F}G}$ is the direct sum of some of the $\bar{I}_i$; wolog, it is the sum of the first k of them. Thus $\frac{I}{\text{Rad } \mathcal{F}G} = \bar{I}_1 \oplus \bar{I}_2 \oplus \cdots \oplus \bar{I}_k$, where $1 \leq k \leq t$. Now by Proposition 3.19 (c), $\text{Ann}_{\frac{\mathcal{F}G}{\text{Rad } \mathcal{F}G}} M_i = \bigoplus_{i \neq j} \bar{I}_i$. It follows easily that $\frac{I}{\text{Rad } \mathcal{F}G} = \bar{I}_1 \oplus \bar{I}_2 \oplus \cdots \oplus \bar{I}_k = \bigcap_{i>k} \text{Ann}_{\frac{\mathcal{F}G}{\text{Rad } \mathcal{F}G}} M_i$.

We claim that $I = \bigcap_{i>k} \text{Ann}_{\mathcal{F}G} M_i$. For let $\alpha \in I$. Then $(\alpha + \text{Rad } \mathcal{F}G) \in \frac{I}{\text{Rad } \mathcal{F}G} = \bigcap_{i>k} \text{Ann}_{\frac{\mathcal{F}G}{\text{Rad } \mathcal{F}G}} M_i$, and so for each $i > k$, $(M_i)\alpha = M_i(\alpha + \text{Rad } \mathcal{F}G) = 0$. Thus $\alpha \in \bigcap_{i>k} \text{Ann}_{\mathcal{F}G} M$. On the other hand, suppose $\alpha \in \bigcap_{i>k} \text{Ann}_{\mathcal{F}G} M$. Then $\forall i > k$, $M_i(\alpha + \text{Rad } \mathcal{F}G) = (M_i)\alpha = 0$. So $(\alpha + \text{Rad } \mathcal{F}G) \in \bigcap_{i>k} \text{Ann}_{\frac{\mathcal{F}G}{\text{Rad } \mathcal{F}G}} M_i = \frac{I}{\text{Rad } \mathcal{F}G}$. It follows that $\alpha + \text{Rad } \mathcal{F}G = B + \text{Rad } \mathcal{F}G$ for some $B \in I$. Consequently $\alpha = B + \gamma$, some $\gamma \in \text{Rad } \mathcal{F}G$. Since $I \supseteq \text{Rad } \mathcal{F}G$, $\gamma \in I$. Thus $\alpha \in I$, and $I = \bigcap_{i>k} \text{Ann}_{\mathcal{F}G} M_i$, as claimed.

Corollary 3.21. If char $\mathcal{F} = 0$, or if char $\mathcal{F} = p$ does not divide $|G|$, then every proper ideal of $\mathcal{F}G$ is of the form $I = \bigcap_{M \in S} \text{Ann}_{\mathcal{F}G} M$.

Proof: By Maschke's Theorem, $\text{Rad } \mathcal{F}G = 0$ in this case.

In the finite group case, therefore, we see that our hypothesis that I be of the form $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M$ is not terribly restrictive.

It is also clear from Proposition 3.19 that if G is finite, then each proper ideal I containing $\mathrm{Rad}\,\mathcal{F}G$ has a unique representation of the form $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M$, $\emptyset \neq S \subseteq \mathrm{Irr}(\mathcal{F}G)$. For from $\bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M = \bigcap_{M \in S'} \mathrm{Ann}_{\mathcal{F}G} M$ it follows that $\bigcap_{M \in S} \mathrm{Ann}_{\frac{\mathcal{F}G}{\mathrm{Rad}\,\mathcal{F}G}} M = \bigcap_{M \in S'} \mathrm{Ann}_{\frac{\mathcal{F}G}{\mathrm{Rad}\,\mathcal{F}G}} M$ and, due to the direct sum decomposition described in Proposition 3.19 (c), that $S = S'$. Since, therefore, I has a unique representation, Definition 3.1 no longer depends on S and we may simply say that $H \in \rho(I)$ if H is a subgroup of G satisfying Definition 3.1. With these comments in mind, we now state our next theorem.

<u>Theorem 3.22</u>. Let G be a finite group, I a proper ideal of $\mathcal{F}G$, and suppose that $\mathrm{char}\,\mathcal{F} = 0$ or $\mathrm{char}\,\mathcal{F} = p$ does not divide $|G|$. Then $\forall\, H \leq G$, $I = (I \cap \mathcal{F}H)\mathcal{F}G \Leftrightarrow H \in \rho(I)$.

<u>Proof</u>: By Corollary 3.21 and our comments above, $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M$ for some unique $S \subseteq \mathrm{Irr}\,\mathcal{F}G$, $S \neq \emptyset$. Let $H \leq G$. Then $\forall\, M \in S$, M_H is completely reducible since $\mathcal{F}H$ is semisimple and artinian. Thus $\mathrm{Ann}_{\mathcal{F}H} M = \bigcap_{W \overset{\text{irred}}{\subseteq} M_H} \mathrm{Ann}_{\mathcal{F}H} W$. By comparing Definition 3.1 and Lemma 3.3, we now see that $H \in \rho(I)$ iff $I = (I \cap \mathcal{F}H)\mathcal{F}G$.

We now prove the converse of Theorem 3.15 for finite groups.

Theorem 3.23. Let G be a finite group and H a normal subgroup such that $[G:H]$ is a unit in $\mathfrak{F}$. Let $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathfrak{F}G} M$, for some $\emptyset \neq S \subseteq \mathrm{Irr}(\mathfrak{F}G)$. Then $I = (I \cap \mathfrak{F}H)\mathfrak{F}G$ iff for each $M \in S$ and for each irreducible submodule L of $(M_H)^G$ it is true that $L \in S$ (up to isomorphism).

Proof: $\Leftarrow$ Theorem 3.15

$\Rightarrow$ Let $M \in S$ and let L be an irreducible $\mathfrak{F}G$-submodule of $(M_H)^G$. Suppose $L \notin S$. Now since $L \subseteq (M_H)^G$, we have $\mathrm{Ann}_{\mathfrak{F}G} L \supseteq \mathrm{Ann}_{\mathfrak{F}G} (M_H)^G \supseteq \bigcap_{M \in S} \mathrm{Ann}_{\mathfrak{F}G} (M_H)^G$. But by Theorem 3.14, $\bigcap_{M \in S} \mathrm{Ann}_{\mathfrak{F}G} (M_H)^G = \bigcap_{M \in S} \mathrm{Ann}_{\mathfrak{F}G} M = I$. It follows that $I = (\bigcap_{M \in S} \mathrm{Ann}_{\mathfrak{F}G} M) \cap \mathrm{Ann}_{\mathfrak{F}G} L$. But this violates the fact that I has a unique representation of the form $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathfrak{F}G} M$, as discussed before the statement of Theorem 3.22. Thus $L \in S$, and the theorem is proved.

Paraphrasing Theorem 3.23, we see that $(I \cap \mathfrak{F}H)\mathfrak{F}G$ is smaller than I precisely when for some $M \in S$, $(M_H)^G$ contains an irreducible submodule outside of S.

Of special interest is the case $I = \mathrm{Ann}_{\mathfrak{F}G} M$, some $M \in \mathrm{Irr}(\mathfrak{F}G)$. We have

Corollary 3.24. Let G be a finite group and H a normal subgroup such that $[G:H]$ is a unit in $\mathfrak{F}$. Let

$I = \mathrm{Ann}_{\mathcal{F}G}M$ for some $M \in \mathrm{Irr}\,\mathcal{F}G$. Then $I = (I \cap \mathcal{F}H)\mathcal{F}G$ iff $(M_H)^G$ is homogeneous.

Proof: We recall that as discussed in the proof of Theorem 3.15, $(M_H)^G$ is completely reducible. Since $[G:H]$ is a unit in $\mathcal{F}$, we have from Theorems 1.12 and 1.9 that M is a component of $(M_H)^G$. Thus $(M_H)^G$ is homogeneous if M is its only irreducible $\mathcal{F}G$-submodule, and the result now follows from the theorem.

We remark that if $[G:H] = n$ is not a unit in $\mathcal{F}$, then Theorem 3.23 fails. For instance, let G be a finite abelian p-group and $\mathcal{F}$ an algebraically closed field of characteristic p. Then $\mathcal{F}G$ has only one irreducible module, namely the trivial one - call it M. Take $I = \mathrm{Ann}_{\mathcal{F}G}M = \omega(\mathcal{F}G)$, the augmentation ideal of $\mathcal{F}G$. Then for $H \lneq G$, we have $I \cap \mathcal{F}H = \omega(\mathcal{F}H)$, and it is well known (see, for example, Connell [1]) that $\omega(\mathcal{F}G) \supsetneq \omega(\mathcal{F}H)\mathcal{F}G$. Thus $I = (I \cap \mathcal{F}H)\mathcal{F}G$ fails. On the other hand, since M is the only irreducible $\mathcal{F}G$-module, certainly M is the only irreducible $\mathcal{F}G$-submodule of $(M_H)G$. (Of course, it would not be accurate to say that $(M_H)^G$ is homogeneous in this case, since it fails to be completely reducible.)

We conclude this section by remarking that if G is finite and I is an ideal containing Rad $\mathcal{F}G$, then Theorem 3.23 provides a complete solution to the problem of determining those subgroups H of G for which $I = (I \cap \mathcal{F}H)\mathcal{F}G$. The procedure is to use Theorem 3.23 to test the normal subgroups of G, and to eventually locate the controller of I. (Recall that the controller is always a normal subgroup.) Admittedly, this procedure will be formidable in some cases. But if, for example, $\mathfrak{F}$ is an algebraically closed field of characteristic 0, then the computations may easily be performed in terms of characters. In section 5 of this chapter, we will illustrate these comments with some examples. First, however, we turn out attention to the special case $I = \mathrm{Ann}_{\mathfrak{F}G}M$, where $M \in \mathrm{Irr}(\mathfrak{F}G)$.

§4. The Annihilator of a Single Irreducible $\mathfrak{F}G$-Module

Throughout this section we will assume that $\mathfrak{F}$ is an algebraically closed field of characteristic 0. Our goal is to study the phenomenon $I = (I \cap \mathfrak{F}H)\mathfrak{F}G$ in the case that G is finite and $I = \mathrm{Ann}_{\mathfrak{F}G}M$ for some $M \in \mathrm{Irr}(\mathfrak{F}G)$. Many of the results in this section will depend heavily on character theory. We will sometimes use characters and modules interchangeably. For example, $\mathrm{Irr}(\mathfrak{F}G)$ will denote both the set of distinct (up to isomorphism) irreducible $\mathfrak{F}G$-modules and the set of irreducible characters of G, depending on the context.

The following version of Clifford's Theorem will facilitate a later result. Although it is stated in terms of characters, it may be interpreted in terms of modules as well. (See Isaacs [4], p. 79.)

Theorem 3.25. (Clifford) Let G be a finite group, $H \trianglelefteq G$, and $\chi \in \mathrm{Irr}(\mathcal{F}G)$. Let θ be an irreducible constituent of χ_H and suppose $\theta = \theta_1, \theta_2, \ldots, \theta_t$ are the distinct conjugates of θ in G. Then $\chi_H = e(\theta_1 + \theta_2 + \cdots + \theta_t)$, where e is the multiplicity of θ in χ_H.

Note, in particular, that if θ is an irreducible constituent of χ_H, then all the other irreducible constituents of χ_H are conjugates of θ and, conversely, that any conjugate of θ is a constituent of χ_H. Furthermore, each conjugate occurs with the same multiplicity.

The conjugates in G of irreducible $\mathcal{F}H$-modules (or characters) have a bearing on our problem. The following theorem is crucial. (See Curtis and Reiner [2], p. 329 for the proof.)

Theorem 3.26. Let G be a finite group, let $H \trianglelefteq G$, and let T be an irreducible representation of H. Then the induced representation T^G is irreducible iff $\forall x \notin H$ the representations T and $T^x : h \to T(x^{-1}hx)$ are disjoint.

Theorem 3.26 actually says that the representation T^G being irreducible is equivalent to T having $[G:H]$ distinct conjugate representations. Or, in terms of modules, if W is an irreducible $\mathcal{F}H$-module, then W^G is irreducible iff there are $[G:H]$ distinct isomorphism classes of $\mathcal{F}H$-modules conjugate to W. These considerations give rise to the following theorem:

Theorem 3.27. Let G be a finite group, let $H \trianglelefteq G$, and let $I = \mathrm{Ann}_{\mathcal{F}G} M$ for some $M \in \mathrm{Irr}(\mathcal{F}G)$. Suppose $M_H = W_1 \oplus \cdots \oplus W_t$, where the W_i are distinct (up to isomorphism) conjugates and where $t = [G:H]$. Then $I = (I \cap \mathcal{F}H)\mathcal{F}G$.

Proof: By Theorem 3.26, each $(W_i)^G$ is irreducible. Furthermore, by the Frobenius Reciprocity Theorem, for each i the multiplicity of M in $(W_i)^G$ is the same as the multiplicity of W_i in M_H; namely, the multiplicity is one. Thus $(M_H)^G \cong (W_1)^G \oplus (W_2)^G \oplus \cdots \oplus (W_t)^G \cong \underbrace{M \oplus M \oplus \cdots \oplus M}_{t \text{ times}}$, and so $(M_H)^G$ is homogeneous. It follows from Corollary 3.24 that $I = (I \cap \mathcal{F}H)\mathcal{F}G$.

The situation described in Theorem 3.27 can indeed occur. Consider, for example, this next result.

Theorem 3.28. (Isaacs [4], p. 86) Let G be a finite group and let $H \trianglelefteq G$ such that $[G:H] = p$ is prime. Suppose $M \in \mathrm{Irr}(\mathcal{F}G)$. Then either

(a) M_H is irreducible, or

(b) $M_H = \bigoplus_{i=1}^{p} W_i$, where the W_i are irreducible, distinct (up to isomorphism), and conjugate.

Rewording Theorem 3.28 to better suit the context of this paper, we have

Corollary 3.29. Let G be a finite group, and let $H \trianglelefteq G$ such that $[G:H] = p$ is prime. Let $M \in \mathrm{Irr}(\mathcal{F}G)$ and let $I = \mathrm{Ann}_{\mathcal{F}G} M$. If M_H is reducible, then $I = (I \cap \mathcal{F}H)\mathcal{F}G$.

Our next result is actually a restatement of Theorem 3.27. We include it because it lends itself to an interesting application.

<u>Theorem 3.30</u>. Let G be a finite group, $H \trianglelefteq G$, and suppose there exists an irreducible character, θ, of H such that $\theta^G \in \mathrm{Irr}(\mathcal{F}G)$. Then $I = (I \cap \mathcal{F}H)\mathcal{F}G$, where M is the irreducible $\mathcal{F}G$-module associated with $\chi = \theta^G$ and $I = \mathrm{Ann}_{\mathcal{F}G}M$.

<u>Proof</u>: Since $\chi = \theta^G$, it follows from the Frobenius Reciprocity Theorem that θ is a constituent of multiplicity one in χ_H. So if $\theta = \theta_1, \theta_2, \ldots, \theta_t$ are the distinct conjugates of θ in G, we have by Clifford's Theorem (Theorem 3.25) that $\chi_H = \theta_1 + \theta_2 + \cdots + \theta_t$. Furthermore, since θ^G is irreducible, by Theorem 3.26 θ has $[G:H]$ distinct conjugates. Thus $t = [G:H]$ and $I = (I \cap \mathcal{F}H)\mathcal{F}G$ by Theorem 3.27.

Before proceeding to the promised application of Theorem 3.30, we will need the following definition.

<u>Definition 3.31</u>. A finite group G is called Frobenius with kernel N and complement H if $G = NH$, $N \trianglelefteq G$, $H \cap N = 1$, and $H \cap H^x = 1$ for all $x \in G - H$.

We now quote a result from Isaac's text.

<u>Theorem 3.32.</u> (Isaac's [4], p. 94) Let G be a finite Frobenius group with kernel $N \trianglelefteq G$. Then for each character $\chi \in \mathrm{Irr}(\mathcal{F}G)$ with $N \not\subseteq \ker \chi$ we have $\chi = \varphi^G$ for some $\varphi \in \mathrm{Irr}(\mathcal{F}N)$.

As an immediate consequence of Theorems 3.30 and 3.32, we can now state

Corollary 3.33. Let G be a finite Frobenius group with kernel $N \trianglelefteq G$. Let $\chi \in \mathrm{Irr}(\mathcal{F}G)$ such that $N \not\subseteq \ker \chi$. Finally, let $I = \mathrm{Ann}_{\mathcal{F}G}M$ where M is the irreducible $\mathcal{F}G$-module corresponding to χ. Then $I = (I \cap \mathcal{F}N)\mathcal{F}G$.

The proceeding results provide some examples of the phenomenon $I = (I \cap \mathcal{F}H)\mathcal{F}G$, where $I = \mathrm{Ann}_{\mathcal{F}G}M$ for some $M \in \mathrm{Irr}(\mathcal{F}G)$. We now turn out attention to some situations in which this phenomenon cannot occur.

Theorem 3.34. Let G be a finite group, let $H \leq G$, and let M be an irreducible $\mathcal{F}G$-module with associated irreducible character χ. Finally, let $I = \mathrm{Ann}_{\mathcal{F}G}M$. If $\chi(1)^2 < [G:H]$, then $I \neq (I \cap \mathcal{F}H)\mathcal{F}G$.

Proof: Let $n = [G:H]$. Now, by assumption, $\dim(\mathrm{Hom}_{\mathcal{F}}(M,M)) = \chi(1)^2 < n$. Thus any n linear transformations on M are linearly dependent. In particular, if $\{x_i\}_{i=1}^n$ is a right transversal for H in G, then the linear transformations corresponding to the x_i via the representation associated with M are linearly dependent. So there exist $a_i \in \mathcal{F}$, not all $a_i = 0$, such that $a_1x_1 + a_2x_2 + \cdots + a_nx_n$ corresponds to the zero linear transformation. In other words, $a_1x_1 + a_2x_2 + \cdots + a_nx_n \in \mathrm{Ann}_{\mathcal{F}G}M = I$.

Suppose $a_1x_1 + a_2x_2 + \cdots + a_nx_n \in (I \cap \mathcal{F}H)\mathcal{F}G = (\mathrm{Ann}_{\mathcal{F}H}M)\mathcal{F}G$. Then by Lemma 2.1 it follows that $a_i \in \mathrm{Ann}_{\mathcal{F}H}M$, $\forall i$. But the a_i are members of $\mathcal{F}$ and so this forces $a_i = 0$ $\forall i$, a contradiction. Thus $\sum a_ix_i \notin (I \cap \mathcal{F}H)\mathcal{F}G$, and so $I \supsetneq (I \cap \mathcal{F}H)\mathcal{F}G$.

Corollary 3.35. Let G be a finite group, let H be a proper normal subgroup, and let M be an irreducible $\mathcal{F}G$-module with associated irreducible character χ. Finally, let $I = \mathrm{Ann}_{\mathcal{F}G}M$. If $H \subseteq \ker \chi$, then $I \neq (I \cap \mathcal{F}H)\mathcal{F}G$.

Proof: Suppose $H \subseteq \ker \chi$. Then χ may be viewed as an irreducible character of G/H. Now $[G:H] = \sum_{\psi \in \mathrm{Irr}(G/H)} \psi(1)^2$. Since $\chi \in \mathrm{Irr}(G/H)$ and since $|\mathrm{Irr}(G/H)| > 1$, it follows that $\chi(1)^2 < [G:H]$.

Corollary 3.36. Let G be a finite group, let M be an irreducible $\mathcal{F}G$-module with associated irreducible character χ, and let $I = \mathrm{Ann}_{\mathcal{F}G}M$. Suppose $\chi(1) = 1$. Then for $H \leq G$, $I = (I \cap \mathcal{F}H)\mathcal{F}G \Leftrightarrow H = G$.

Proof: Let W be the controller of I. Then $I = (I \cap \mathcal{F}W)\mathcal{F}G$ and so by Theorem 3.34 we must have $\chi(1) = 1 \geq [G:W]$. This forces $W = G$, and the result follows.

Corollary 3.37. Let G be a finite abelian group. Let $M \in \mathrm{Irr}(\mathcal{F}G)$ and $I = \mathrm{Ann}_{\mathcal{F}G}M$. Then for $H \leq G$, $I = (I \cap \mathcal{F}H)\mathcal{F}G \Leftrightarrow H = G$.

Proof: In this case $\chi(1) = 1$ for all irreducible characters χ of G.

§5. Examples

We conclude this chapter with a couple of examples which illustrate the use of Theorem 3.23 and the results of the previous section. Once again, we assume throughout that $\mathcal{F}$ is an algebraically closed field of characteristic 0.

Example 3.38. Let $G = S_3 = \langle (12),(123) \rangle$, and let $H = \langle (123) \rangle \trianglelefteq G$. Then the respective character tables of G and H are

G

	1	{(12),(13),(23)}	{(123,(132)}
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

H

	1	(123)	(132)
ψ_1	1	1	1
ψ_2	1	w	w^2
ψ_3	1	w^2	w

For each i, let M_i be the irreducible $\mathcal{F}G$-module corresponding to χ_i. Recall that by Corollary 3.21, each proper ideal of $\mathcal{F}G$ is of the form $I = \bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G} M$, where $\emptyset \neq S \subseteq \mathrm{Irr}(\mathcal{F}G)$. For each ideal of I, we now compute its controller subgroup.

If $I = \mathrm{Ann}_{\mathcal{F}G} M_1$ or $\mathrm{Ann}_{\mathcal{F}G} M_2$, then, by Corollary 3.36, G is the controller of I.

Suppose $I = \mathrm{Ann}_{\mathcal{F}G}M_3$. Now $(M_3)_H$ is reducible since H is abelian and so all its irreducible modules must be one-dimensional. It follows from Corollary 3.29 that $I = (I \cap \mathcal{F}H)\mathcal{F}G$. Now H is the unique non-trivial normal subgroup of G, and the controller of I must be a normal subgroup other than one. We conclude, therefore, that H is the controller of I.

Suppose $I = (\mathrm{Ann}_{\mathcal{F}G}M_1) \cap (\mathrm{Ann}_{\mathcal{F}G}M_2)$. Now $(\chi_1)_H = (\chi_2)_H = \psi_1$, and $(\psi_1)^G = \chi_1 + \chi_2$. It follows from Theorem 3.23 that $I = (I \cap \mathcal{F}H)\mathcal{F}G$. Once again we conclude that H is the controller of I.

Suppose $I = (\mathrm{Ann}_{\mathcal{F}G}M_1) \cap (\mathrm{Ann}_{\mathcal{F}G}M_3)$. Since χ_2 is a constituent of $[(\chi_1)_H]^G = (\psi_1)^G = \chi_1 + \chi_2$, it follows from Theorem 3.23 that $I \neq (I \cap \mathcal{F}H)\mathcal{F}G$. Since H is the only non-trivial candidate for the controller of I, we are forced to the conclusion that G is the controller.

An analagous computation shows that G is the controller of $I = (\mathrm{Ann}_{\mathcal{F}G}M_2) \cap (\mathrm{Ann}_{\mathcal{F}G}M_3)$.

The only remaining ideals of $\mathcal{F}G$ are 0 and $\mathcal{F}G$ itself. Clearly $\langle 1\rangle$ is the controller in each of these cases.

Example 3.39. We now do a more exhaustive study of the group $G = D_4 = \langle a,b / a^2 = b^2 = 1,\ ab = b^3a\rangle$ of example 3.18. In the previous example, we listed the ideals of $\mathcal{F}G$ and computed the controller subgroup of each. This time we will list the normal subgroups H of G, and in each case determine which ideals of $\mathfrak{F}G$ are controlled by H. The normal subgroups of G are the following:

$$\langle 1\rangle, \langle b^2\rangle, \langle b\rangle, H_1 = \langle 1, a, b^2, ab^2\rangle, H_2 = \langle 1, b^2, ab, ab^3\rangle, G$$

Before proceding with our computations, we repeat the character table of G for convenience.

G	1	b^2	$\{b, b^3\}$	$\{a, ab^2\}$	$\{ab, ab^3\}$
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	-1	-1	1
χ_4	1	1	1	-1	-1
χ_5	2	-2	0	0	0

For each i, $1 \leq i \leq 5$, let M_i be the irreducible $\mathfrak{F}G$-module associated with χ_i. We now consider the normal subgroups of G one by one.

$H = \langle 1\rangle$: It is clear that H is the controller of 0 and $\mathfrak{F}G$, and that H controls no other ideal of $\mathfrak{F}G$.

$H = \langle b^2\rangle$: Then H has character table

	1	b^2
ψ_1	1	1
ψ_2	1	-1

Furthermore, $(\chi_1)_H = (\chi_2)_H = (\chi_3)_H = (\chi_4)_H = \psi_1$

$$(\chi_5)_H = 2\psi_2$$

$$(\psi_1)^G = \chi_1 + \chi_2 + \chi_3 + \chi_4$$

$$(\psi_2)^G = 2\chi_5$$

It follows from these computations and Theorem 3.23 that H controls the non-trivial ideals $\mathrm{Ann}_{\mathfrak{F}G}M_5$ and $(\mathrm{Ann}_{\mathfrak{F}G}M_1) \cap (\mathrm{Ann}_{\mathfrak{F}G}M_2) \cap (\mathrm{Ann}_{\mathfrak{F}G}M_3) \cap (\mathrm{Ann}_{\mathfrak{F}G}M_4)$. Since H is a minimal non-identity normal subgroup of G, we see that H is the controller of these ideals.

$H = \langle b \rangle$: Then H has character table

H	1	b	b^2	b^3
ψ_1	1	1	1	1
ψ_2	1	-1	1	-1
ψ_3	1	i	-1	$-i$
ψ_4	1	$-i$	-1	i

Furthermore, $(\chi_1)_H = \psi_1$ $\qquad (\psi_1)^G = \chi_1 + \chi_4$

$(\chi_2)_H = \psi_2 \qquad (\psi_2)^G = \chi_2 + \chi_3$

$(\chi_3)_H = \psi_2 \qquad (\psi_3)^G = \chi_5$

$(\chi_4)_H = \psi_1 \qquad (\psi_4)^G = \chi_5$

$(\chi_5)_H = \psi_3 + \psi_4$

It follows from these computations and Theorem 3.23 that in addition to those ideals controlled by $\langle b^2 \rangle$,

$H = \langle b\rangle$ controls the non-trivial ideals $(\mathrm{Ann}_{\mathcal{F}G}M_1) \cap (\mathrm{Ann}_{\mathcal{F}G}M_4)$, $(\mathrm{Ann}_{\mathcal{F}G}M_2) \cap (\mathrm{Ann}_{\mathcal{F}G}M_3)$, $(\mathrm{Ann}_{\mathcal{F}G}M_1) \cap (\mathrm{Ann}_{\mathcal{F}G}M_4) \cap (\mathrm{Ann}_{\mathcal{F}G}M_5)$, and $(\mathrm{Ann}_{\mathcal{F}G}M_2) \cap (\mathrm{Ann}_{\mathcal{F}G}M_3) \cap (\mathrm{Ann}_{\mathcal{F}G}M_5)$. Furthermore, $\langle b\rangle$ is the controller of each of these additional ideals.

<u>$H = H_1$:</u> Then H has character table

H	1	a	ab^2	b^2
ψ_1	1	1	1	1
ψ_2	1	-1	-1	1
ψ_3	1	-1	1	-1
ψ_4	1	1	-1	-1

Now $(\chi_1)_H = \psi_1$ $\qquad$ $(\psi_1)^G = \chi_1 + \chi_2$

$(\chi_2)_H = \psi_1$ $\qquad$ $(\psi_2)^G = \chi_3 + \chi_4$

$(\chi_3)_H = \psi_2$ $\qquad$ $(\psi_3)^G = \chi_5$

$(\chi_4)_H = \psi_2$ $\qquad$ $(\psi_4)^G = \chi_5$

$(\chi_5)_H = \psi_3 + \psi_4$

It follows from these computations and Theorem 3.23 that in addition to the ideals controlled by $\langle b^2\rangle$, $H = H_1$ controls the non-trivial ideals $(\mathrm{Ann}_{\mathcal{F}G}M_1) \cap (\mathrm{Ann}_{\mathcal{F}G}M_2)$, $(\mathrm{Ann}_{\mathcal{F}G}M_3) \cap (\mathrm{Ann}_{\mathcal{F}G}M_4)$, $(\mathrm{Ann}_{\mathcal{F}G}M_1) \cap (\mathrm{Ann}_{\mathcal{F}G}M_2) \cap (\mathrm{Ann}_{\mathcal{F}G}M_5)$, and $(\mathrm{Ann}_{\mathcal{F}G}M_3) \cap (\mathrm{Ann}_{\mathcal{F}G}M_4) \cap (\mathrm{Ann}_{\mathcal{F}G}M_5)$. Furthermore, $H = H_1$ is the controller of each of these additional ideals.

<u>$H = H_2$:</u> This case is completely analagous to the case $H = H_1$, so we omit the details. It turns out that $H = H_2$ is the controller of the ideals $(\mathrm{Ann}_{\mathcal{F}G}M_1) \cap (\mathrm{Ann}_{\mathcal{F}G}M_3)$, $(\mathrm{Ann}_{\mathcal{F}G}M_2) \cap (\mathrm{Ann}_{\mathcal{F}G}M_4)$, $(\mathrm{Ann}_{\mathcal{F}G}M_1) \cap (\mathrm{Ann}_{\mathcal{F}G}M_3) \cap (\mathrm{Ann}_{\mathcal{F}G}M_5)$, and $(\mathrm{Ann}_{\mathcal{F}G}M_2) \cap (\mathrm{Ann}_{\mathcal{F}G}M_4) \cap (\mathrm{Ann}_{\mathcal{F}G}M_5)$.

This now exhausts all normal subgroups except G itself. So G is the controller of all ideals not explicitly mentioned above. These are all of the form $\bigcap_{M \in S} \mathrm{Ann}_{\mathcal{F}G}M$, where $\emptyset \neq S \subseteq \mathrm{Irr}\, \mathcal{F}G$, and we will not bother listing them. We remark, however, that from the start we knew that G is the controller of $\mathrm{Ann}_{\mathcal{F}G}M_1$, $\mathrm{Ann}_{\mathcal{F}G}M_2$, $\mathrm{Ann}_{\mathcal{F}G}M_3$, and $\mathrm{Ann}_{\mathcal{F}G}M_4$ by Corollary 3.36.

CHAPTER 4 PROPERTY ρ AND SEMISIMPLE GROUPS

Wallace [14] introduced the concept of JK-groups. Their definition is as follows:

<u>Definition 4.1</u>. Let K be a field and G a group. Then G is a JK-group if for all groups T and normal subgroups S with $T/S \cong G$ the pair $(\mathfrak{F}T, \mathfrak{F}S)$ has property ρ.

It is clear that if G is a JK-group then G is semisimple. For $G/1 \cong G \Rightarrow (KG, K)$ has property $\rho \Rightarrow \text{Rad } KG = 0$. Examples of JK-groups include locally finite or abelian groups having no elements of order p where $p = \text{char } K$. (See Passman [9], p. 293).

We now generalize this concept.

<u>Definition 4.2.</u> Let $\mathfrak{F}$ be a field and let X be a class of groups which is closed under homomorphic images. Then a group $G \in X$ is a $J(X)$-group if $\forall T \in X$ and $\forall S \,\underline{\Delta}\, T$ with $T/S \cong G$, the pair $(\mathfrak{F}T, \mathfrak{F}S)$ has property ρ.

Once again, it is clear that any $J(X)$-group must be semisimple. Furthermore, any JK-group belonging to X is a $J(X)$-group.

Of particular interest to us are certain classes, X, of groups which we will call ρ-classes. We will use the notation $P \underset{G}{\leq} H$ to indicate that the subgroup H of G contains a conjugate of the subgroup P.

Definition 4.3. Let $\mathcal{F}$ be a fixed field, and let X be a fixed class of groups closed under homomorphic images such that for each $G \in X$ there exists a collection $\rho(G)$ of subgroups of G satisfying the following properties:

(a) $\forall\ P \in \rho(G)$, $(\mathcal{F}G, \mathcal{F}P)$ has property ρ.

(b) If $H \leq G$ such that $(\mathcal{F}G, \mathcal{F}H)$ has property ρ, then for each $P \in \rho(G)$, $P \underset{G}{\leq} H$.

(c) If $W \trianglelefteq G$, then $\rho(G/W) = \{\langle 1 \rangle\}$ iff for each $P \in \rho(G)$, $P \underset{G}{\leq} W$.

(d) If $G_1, G_2 \in X$ and $\varphi : G_1 \to G_2$ is an isomorphism, then $\forall\ P \in \rho(G_1)$, $\varphi(P) \in \rho(G_2)$.

Then X is called a ρ-class and for each $G \in X$, the members of $\rho(G)$ are called ρ-subgroups of G.

We pause to mention some examples of ρ-classes.

Example 4.4. Let $\mathcal{F}$ have characteristic p and let X be the class of finite groups having normal p-Sylow-subgroups. For each $G \in X$, let $\rho(G)$ be the p-Sylow-subgroup of G. We verify (a), (b) and (c) of the previous definition. The remainder is clear.

Part (a) holds by Theorem 1.12 and Corollary 1.15.

Suppose $H \leq G$ such that $(\mathfrak{F}G, \mathfrak{F}H)$ is a projective pairing. So by Theorem 1.12 $[G:H]$ is a unit in $\mathfrak{F}$ and H contains P. Thus part (b) of Definition 4.3 holds.

Now suppose $W \trianglelefteq G \ni \rho(G/W) = \{\langle 1 \rangle\}$. Then $\mathfrak{F}(G/W)$ is semisimple. By Maschke's Theorem, it follows that G/W has no elements of order p and W contains P. Conversely, if G/W contains no elements of order p, then $\langle 1 \rangle$ is the p-Sylow-subgroup of G/W and so $\rho(G/W) = \{\langle 1 \rangle\}$. Thus part (c) of Definition 4.3 holds.

Before proceeding to our next example, we need some preliminary definitions and results.

<u>Definition 4.5</u>. Let G be a locally finite group. Then a subgroup A of G is said to be locally subnormal in G if A is finite and is subnormal in all finite subgroups of G which contain it.

The characteristic subgroup $\int^p(G)$ of G was defined by Passman in [8].

<u>Definition 4.6</u>. (Passman) Let G be a locally finite group. Then

$$\int^p(G) = \left\langle A \;\middle|\; \begin{array}{l} A \text{ is locally subnormal in } G \text{ and} \\ A \text{ is generated by elements of order } p^\alpha \end{array} \right\rangle$$

Passman proved the following result concerning $\int^p(G)$.

<u>Theorem 4.7</u>. (Passman [8])

Let $\mathfrak{F}$ be a field of characteristic $p > 0$ and G a locally finite group. If $H \trianglelefteq G$ such that $\text{Rad}\, \mathfrak{F}G \subseteq (\text{Rad}\, \mathfrak{F}H)\mathfrak{F}G$, then $H \supseteq \int^p(G)$.

Finally we quote a result from Passman's text [9].

Theorem 4.8. Let $H \Delta G$ with G/H locally finite. If either char $\mathcal{F} = 0$ or char $\mathcal{F} = p$ and G/H is a p'-group, then $\text{Rad } \mathcal{F}G = (\text{Rad } \mathcal{F}H)\mathcal{F}G$.

We now make the following observations:

Theorem 4.9. Let $\mathcal{F}$ have characteristic $p > 0$. Let X be a class of locally finite groups closed under homomorphic images such that $\forall\, G \in X$, the p-Sylow-subgroups of G are all conjugate to each other. Further suppose that for each $G \in X$, $\int^{p}(G) \supseteq P$ where P is a p-Sylow-subgroup of G. Then X is a ρ-class where for each $G \in X$, $\rho(G)$ is the set of Sylow-p-subgroups of G.

Proof: Definition 4.3 (a) follows from Theorem 4.8.

For part (b), if $H \leq G$ such that $(\mathcal{F}G, \mathcal{F}H)$ has property ρ, then by Theorem 4.7 $H \supseteq \int^{p}(G) \supseteq P$, where P is a p-Sylow-subgroup of G. Since by assumption all p-Sylow-subgroups of G are conjugate, (b) holds.

For part (c), if $W \Delta G$ such that $\rho(G/W) = \{\langle 1 \rangle\}$, then G/W is a p'-group and W contains a p-Sylow-subgroup of G. Since all p-Sylow-subgroups of G are conjugate, $P \underset{G}{\leq} W \;\forall\, P \in \rho(G)$. Conversely, if $P \underset{G}{\leq} W$, then G/W is a p'-group and so $\rho(G/W) = \{\langle 1 \rangle\}$.

Part (d) is clear.

Example 4.10. As an example of a ρ-class satisfying the hypotheses of Theorem 4.9, let X be the class of locally finite nilpotent groups. Then each $G \in X$ is the direct sum of its Sylow subgroups. (See Wehrfritz and Kegel [15], p. 63). In particular, G has a unique normal p-Sylow-subgroup, P, and $\rho(G) = \{P\}$. So we need only verify that for each $G \in X$ and corresponding Sylow subgroup P, $P \subseteq \int^{P}(G)$. From the definition of $\int^{P}(G)$, it clearly suffices to show that $\forall\, x \in P$, $\langle x \rangle$ is locally subnormal in G. So let H be a finite subgroup of G containing $\langle x \rangle$. Then $P \cap H \trianglelefteq H$, and so it only remains to show that $\langle x \rangle$ is subnormal in $P \cap H$. But $P \cap H$ is a finite nilpotent group and so each of its subgroups is subnormal. In particular, $\langle x \rangle$ is subnormal in $P \cap H$, as required. [See Huppert: Chap. 3].

The following lemma further describes ρ-classes, X, as defined in Definition 4.3.

<u>Lemma 4.11</u>. Let X be a ρ-class. Then

(i) Rad $\mathfrak{F}P = 0$ for some $P \in \rho(G)$ iff $\rho(G) = \{\langle 1 \rangle\}$.

(ii) Any two members of $\rho(G)$ are conjugate and, conversely, if P belongs to $\rho(G)$ so do all its conjugates.

(iii) If $\rho(G)$ consists of a single normal subgroup P, then $\rho(G/p) = \{\langle 1 \rangle\}$.

(iv) If in addition Rad $\mathfrak{F}G = (\text{Rad } \mathfrak{F}P)\mathfrak{F}G \;\forall\, P \in \rho(G)$, then $P \underset{N_G(W)}{\leq} W \leq G \Rightarrow \text{Rad } \mathfrak{F}P \subseteq \text{Rad } \mathfrak{F}W$.

<u>Proof</u>: (i) If $\rho(G) = \{\langle 1 \rangle\}$, then certainly Rad $\mathfrak{F}P = 0$ for each $P \in \rho(G)$. Conversely, suppose Rad $\mathfrak{F}P = 0$ for each $P \in \rho(G)$. Then by Definition 4.3 (a),

Rad $\mathcal{F}G \subseteq (\text{Rad } \mathcal{F}P)\mathcal{F}G = 0$. Thus Rad $\mathcal{F}G = 0$ and $(\mathcal{F}G, \mathcal{F})$ has property ρ. By Definition 4.3 (b) $P \underset{G}{\leq} 1 \ \forall \ P \in \rho(G)$. This forces $\rho(G) = \{\langle 1 \rangle\}$.

(ii) Let $P_1, P_2 \in \rho(G)$. By Definition 4.3 (a), the pair $(\mathcal{F}G, \mathcal{F}P_1)$ has property ρ. So by Definition 4.3 (b), $P_2 \underset{G}{\leq} P_1$. Analagously, $P_1 \underset{G}{\leq} P_2$. Thus P_1 and P_2 are conjugate. On the other hand, let $P \in \rho(G)$ and $x \in G$. Now the map $\varphi : G \to G$ given by $\varphi(g) = g^x$ is an automorphism of G. By Definition 4.3 (d), $\varphi(P) = P^x \in \rho(G)$. Thus all conjugates of P belong to $\rho(G)$.

(iii) This is immediate from Definition 4.3 (c).

(iv) Suppose $P^x \leq W$, where $P \in \rho(G)$ and $x \in N_G(W)$. Then $(\text{Rad } \mathcal{F}P)^x = \text{Rad } \mathcal{F}P^x = \text{Rad } \mathcal{F}P^x \cap \mathcal{F}W =$ $(\text{Rad } \mathcal{F}P \cap \mathcal{F}W)^x \subseteq [(\text{Rad } \mathcal{F}P)\mathcal{F}G \cap \mathcal{F}W]^x = [(\text{Rad } \mathcal{F}G) \cap \mathcal{F}W]^x$ $\subseteq (\text{Rad } \mathcal{F}W)^x$. (The last inclusion follows from Theorem 1.7). Thus Rad $\mathcal{F}P \subseteq$ Rad $\mathcal{F}W$.

We close this chapter with a theorem which relates the semisimplicity of $\mathcal{F}(G/H)$ to property ρ in the case that G belongs to a ρ-class X.

<u>Theorem 4.12</u>. Let X be a ρ-class of groups such that Rad $\mathcal{F}G = (\text{Rad } \mathcal{F}P)\mathcal{F}G \ \forall \ G \in X$ and $\forall \ P \in \rho(G)$. Then for each $G \in X$

(i) $\mathcal{F}(G/H)$ is semisimple iff G/H is a $J(X)$-group.

(ii) If $H \trianglelefteq G$, then $(\mathcal{F}G, \mathcal{F}H)$ has property ρ iff G/H is a $J(X)$-group.

<u>Proof</u>: (i) It is clear that if G/H is a $J(X)$-group then $\mathfrak{F}(G/H)$ is semisimple. Conversely, suppose $\mathfrak{F}(G/H)$ is semisimple. Let $T \in X$ and $S \trianglelefteq T$ such that $T/S \cong G/H$. Thus $\mathfrak{F}(T/S)$ is semisimple. Since X is closed under homomorphic images, $T/S \in X$. Let $P \in \rho(T/S)$. Then $0 = \text{Rad}\ \mathfrak{F}(T/S) = (\text{Rad}\ \mathfrak{F}P)\mathfrak{F}(T/S) \Rightarrow \text{Rad}\ \mathfrak{F}P = 0$. By Lemma 4.11 (i), $\rho(T/S) = \langle 1 \rangle$. It follows from Definition 4.3 (c) that for each $Q \in \rho(T)$, $Q \underset{T}{\leq} S$. Lemma 4.11 (iv) now yields $\text{Rad}\ \mathfrak{F}Q \subseteq \text{Rad}\ \mathfrak{F}S$. Thus $\text{Rad}\ \mathfrak{F}T = (\text{Rad}\ \mathfrak{F}Q)\mathfrak{F}T \subseteq (\text{Rad}\ \mathfrak{F}S)\mathfrak{F}T$ and $(\mathfrak{F}T, \mathfrak{F}S)$ has property ρ. By definition, G/H is a $J(X)$-group.

(ii) Suppose $H \trianglelefteq G$ such that $(\mathfrak{F}G, \mathfrak{F}H)$ has property ρ. By Definition 4.3 (b), $P \underset{G}{\leq} H$ for each $P \in \rho(G)$. It follows from Definition 4.3 (c) that $\rho(G/H) = \{\langle 1 \rangle\}$. By Lemma 4.11 (i), $\mathfrak{F}(G/H)$ is semisimple. By part (i), G/H is a $J(X)$-group. The converse is immediate.

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