BIFURCATION THEORY WITH APPLICATIONS TO CHEMICAL REACTION EQUATIONS

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ABSTRACT

BIFURCATION THEORY WITH APPLICATIONS TO CHEMICAL REACTION EQUATIONS

By

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This thesis concerns families of nonlinear differential equations in a Banach space which depend on one or more parameters. At certain critical values of the parameters, non-trivial equilibrium states may bifurcate from the trivial solution. We consider two cases.

In the first case, the generalized null space of the linear part of the system is one-dimensional at the bifurcation point, and the system depends on a single parameter. We determine the number and magnitude of the bifurcating solutions and their stability properties.

The second case involves dependence on two parameters. We consider the situation where there are two "bifurcation" curves in the parameter plane which intersect transversally. The linear part of the system which corresponds to these curves has a one-dimensional generalized null space, except at the intersection where it is two-dimensional. We develop analytical methods which can be applied to study the number and magnitude of the bifurcating solutions as a function of

the parameters near the bifurcation point. We then apply these techniques to a system of partial differential equations which arises in the study of chemical reactions.

BIFURCATION THEORY WITH APPLICATIONS TO CHEMICAL REACTION EQUATIONS

Ву

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In memory of my grandmother,
Amelia Zegaren

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CHAPTER I

PRELIMINARIES

§1. Bifurcation

As a simple example of bifurcation, consider what is observed experimentally when a compressive axial thrust is applied to a thin elastic rod [12]. As the thrust T is gradually increased from zero, the rod first becomes thicker and shorter, but its center line remains straight. The classical linear theory of elasticity predicts this straight state to be the unique equilibrium state of the rod for all values of T. However, when T reaches a certain critical value To, the rod is observed to buckle into a bent state which becomes more pronounced as T The classical theory is inadequate to describe the phenomenon of buckling. When nonlinear effects are no longer neglected, one may construct a model which predicts both the straight state and the bent state as possible equilibrium states for T greater than T_O . Intuitively, the straight state "loses its stability" to the bent state at this critical value.

Let X, Y, and Λ be Banach spaces, and let F: $\Lambda \times X \to Y$ be continuous. We will say that x_O is an equilibrium

state corresponding to α_O in Λ if \mathbf{x}_O is a solution of $\mathbf{F}(\alpha_O,\mathbf{x})=0$. We will say that (α_O,\mathbf{x}_O) is a bifurcation point for T if and only if

(i)
$$F(\alpha_0, x_0) = 0$$

(ii) For every neighborhood V of (α_0, x_0) , there exist α in Λ and x_1, x_2 in X with $x_1 \neq x_2$, such that (α, x_1) and (α, x_2) are in V, and $F(\alpha, x_1) = F(\alpha, x_2) = 0$. That is, we can always find α arbitrarily close to α_0 such that there is more that one equilibrium state near x_0 that corresponds to α .

§2. Differential Equations and Stability

First we shall mention some standard results, from the theory of ordinary differential equations [8]. Let A: $\mathbb{R}^n \to \mathbb{R}^n$ be a continuous linear map, and let N: $\mathbb{R}^n \to \mathbb{R}^n$ be continuous, with N(0) = 0, DN(0) = 0 [where DN(0) denotes the Fréchet derivative of N at x = 0]. Consider the ordinary differential equation

$$\frac{dx}{dt} = Ax + N(x)$$

and the associated linear differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax.$$

Definition 2.1. The zero solution of (2.1) (or of (2.2)) is said to be uniformly asymptotically stable if and only if there exists $\epsilon > 0$ such that $|x_0| < \epsilon$

implies $|\mathbf{x}(t)| \to 0$ as $t \to \infty$ where $\mathbf{x}(t)$ is the solution of (2.1) (or of (2.2)) with $\mathbf{x}(0) = \mathbf{x}_0$. A necessary and sufficient condition that the system (2.2) be uniformly asymptotically stable is that all the eigenvalues of A have negative real parts. If this is the case, there exist positive constants, K and α , such that

$$|\mathbf{x}(t)| < |\mathbf{x}_{0}| \text{ Ke}^{-\alpha (t-t_{0})}$$

where x(t) is the solution of (2.2) with initial condition $x(t_0) = x_0$. If N(x) is $O(|x|^2)$ as x approaches zero, the asymptotic stability of (2.2) implies that the zero solution of (2.1) is asymptotically stable.

We will now consider differential equations in a Banach space X. Let A be a closed linear operator whose domain is a dense subspace D(A) contained in X. It is useful to consider D(A) as a Banach space Y with the so-called graph norm: $|z|_Y = |z|_X + |Az|_X$. The injection $j\colon Y\to X$ is continuous with dense range, and A: $Y\to X$ is then a continuous linear map. Let N: $Y\to X$ be continuous, with N(O)=0 and DN(O)=0. Consider the differential equation

(2.3)
$$\frac{dz}{dt} = Az + N(z)$$

and the associated linear differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = Az.$$

Here

$$\frac{dz}{dt} = \lim_{h \to 0} \frac{z(t+h)-z(t)}{h}$$

where the limit is taken in the norm $|\cdot|_{\mathbf{x}}$.

Motivated by the results for ordinary differential equations, we make the following definition.

Definition 2.2. We shall say that the system (2.4) is stable if there exists a $\delta < 0$ such that whenever λ is in the spectrum of A, $\text{Re}\lambda < \delta$. We shall say that the system (2.4) is unstable if part of the spectrum lies to the right of the imaginary axis.

In many cases, given that (2.4) is stable in this sense, it is possible to prove stability results analogous to those for ordinary differential equations [10], [13], [14]. That is, if the linear system (2.4) is stable in the sense of definition (2.2), then the zero solution of the nonlinear system (2.3) is stable. These results depend on the particular properties of the operator A, and we will not go into them here. Instead we make the following hypothesis.

Principle of Linearized Stability. Let $z=z_0$ be an equilibrium solution of (2.3). If the linearization of (2.3) about z_0 , given by

(2.5)
$$\frac{dz}{dt} = Az + DN(z_0)z$$

is stable in the sense of definition (2.2), then the solution $z = z_0$ of (2.3) is stable.

In what follows, we shall say that z_0 is stable, if (2.5) is stable in the sense of definition (2.2), and that z_0 is unstable, if (2.5) is unstable. Statements about stability of z_0 will actually be statements about the location of the spectrum of the operator $A + DH(z_0)$.

§3. Analytic Functions and the Implicit Function Theorem

For the sake of completeness, we mention the following
results, which may be found in Dieudonne's Foundation of

Modern Analysis [5].

Definition 3.1. Let D be an open subset of K^p , where $K = \mathbb{R}$ or \mathbb{C} . We say that a mapping f of D into a Banach space \mathbb{E} over K is analytic if, for every point $a \in D$, there is an open polycylinder $P = \{z \in K^p | |z_i - a_i| < r_i, 1 \le i \le p\}$, such that in P, f(z) is equal to the sum of an absolutely summable power series in the p variables $(z_k - a_k)$, 1 < k < p. The following are true:

- (3.2) The power series in (3.1) is unique.
- (3.3) Let $A \subset \mathbb{C}^p$ be an open connected set, f and g two analytic functions in A with values in a complex Banach space E. If there is a nonempty open

subset U of A such that f(x) = g(x) in U, then f(x) = g(x) for every x in A. Let U be an open subset of A, b a point of U, and suppose that f(x) = g(x) in the set $U \cap (b + \mathbb{R}^p)$, then f(x) = g(x) for all x in A.

- (3.4) Let E be a complex Banach space, A an open subset of \mathbb{R}^p , f an analytic mapping of A into E.

 Then there is an open set $B \subset \mathbb{C}^p$ such that $B \cap \mathbb{R}^p = A$ and an analytic mapping of B into E into which extends f.
- (3.5) A continuously (Fréchet) differentiable mapping f of an open subset of \mathbb{C}^p into a complex Banach space is analytic. [Henceforth, differentiable will mean Fréchet differentiable].

Implicit Function Theorem. Let E, F, G be three Banach spaces, f a continuously differentiable mapping of an open subset A of $\mathbf{E}_{\times}\mathbf{F}$ into G. Let $(\mathbf{x}_{0},\mathbf{y}_{0})$ be a point of A such that $f(\mathbf{x}_{0},\mathbf{y}_{0})=0$ and the partial derivative $\mathbf{D}_{2}f(\mathbf{x}_{0},\mathbf{y}_{0})$ be a linear homeomorphism of F onto G. Then there is an open neighborhood \mathbf{U}_{0} of \mathbf{x}_{0} in E such that, for every open connected neighborhood V of \mathbf{x}_{0} , contained in \mathbf{U}_{0} , there is a unique continuous mapping V into F such that $\mathbf{u}(\mathbf{x}_{0})=\mathbf{y}_{0}$, $(\mathbf{x},\mathbf{u}(\mathbf{x}))\in\mathbf{A}$ and $f(\mathbf{x},\mathbf{u}(\mathbf{x}))=0$ for any $\mathbf{x}\in\mathbf{V}$. Furthermore, \mathbf{u} is continuously differentiable in V, and its derivative is given by

(3.6)
$$u(x) = -[D_2f(x,u(x))]^{-1}[D_1f(x,u(x))].$$

If f is p times continuously differentiable in a neighborhood of (x_0, y_0) , then u is p times continuously differentiable in a neighborhood of x_0 . The following also hold

- (3.7) If E, F, G are finite dimensional and f is an-alytic in A, then u is analytic in a neighborhood of x_0 . (Here $A \subset \mathbb{C}^p$ or $A \subset \mathbb{R}^p$).
- (3.8) If $\mathbf{E} = \mathbf{C}^{\mathbf{D}}$, then u: $\mathbf{E} \to \mathbf{F}$ is continuously differentiable, hence analytic by (3.5).

§4. Remarks

In this thesis we will be interested in two special cases of the following problem which we briefly outline here. Given a family of differentiable equations

$$\frac{dz}{dt} = A(\alpha)z + N(\alpha, z) \qquad (as in §2),$$

which depend on a parameter α in C^n , describe the set of equilibrium solutions near a bifurcation point $(\alpha_0,0)$ of the operator $F(\alpha,z) = A(\alpha)z + N(\alpha,z)$. In particular, we shall be interested in the number of real equilibrium solutions which correspond to α in \mathbb{R}^n . We shall introduce the hypotheses we need and make our notions more precise in the chapters that follow.

In Chapter II, we use the Liapunov-Schmidt method [7] to reduce the problem to a finite system of "bifurcation equations" on a finite dimensional space. We shall then specialize to a case where α is in C and generalized null space of $A(\alpha_{\Omega})$ is one dimensional, and determine the set of bifurcating solutions along with their stability properties. This situation arises in fluid dynamics and has been studied by Kirchgassner and Sorger [13] in the context of the Taylor problem, and by Kirchgässner and Kielhöfer [14] in a general survey of bifurcation in fluid dynamics. Sattinger [17] has used Leray-Schauder degree to study the stability of bifurcating solutions, and has obtained results which overlap those of Chapter II. The technique which we employ is different, and shows how the sign of the critical eigenvalue is related to the leading terms in the bifurcation equations.

Chow, Hale, and Mallet-Paret [4] have studied a two parameter bifurcation problem which concerns the buckling of a rectangular plate. In one of the situations they studied, the generalized null space is two-dimensional. In Chapter III, we shall develop methods for analyzing the bifurcation set in a different general setting where the generalized null space is two-dimensional and the system depends on two complex-valued parameters. We apply these results to a system of chemical reaction equations in Chapter IV.

CHAPTER II

BIFURCATION AND STABILITY-ONE DIMENSIONAL NULL SPACE

§1. Preliminaries

This chapter is divided into two parts. The first two sections are largely introductory; we consider the problem of determining nontrivial equilibrium states which bifurcate from the trivial solution of

$$\frac{dz}{dt} = A(\alpha)z + N(\alpha,z),$$

and give basic hypotheses under which this problem can be reduced to a finite-dimensional problem. In §3, we consider the case where the generalized null space of A(O) is one-dimensional. We shall assume that for real α , A(α) has a simple real eigenvalue $\lambda^*(\alpha)$ which crosses the imaginary axis as α moves through zero. The number of bifurcating solutions and their dependence upon α will be discussed. In §4, we shall study the stability properties of these solutions and show how stability is related to the leading terms of the bifurcation equation.

We now give our basic hypotheses and discuss their consequences.

(H1) Let Λ be an open set in \mathbb{C}^n , and let X be a complex Banach space with dense subspace D(A). Let $A: \Lambda \times D(A) \to X$ be a closed linear operator with domain D(A) for each α in Λ , and let A be analytic in Λ in the sense that $A(\alpha)z$ has a Taylor expansion at each α_0 in Λ which converges in a disc $|\alpha - \alpha_0| < r$ independent of z. Assume the origin is in Λ .

It follows from (H1), that if Y is the Banach space consisting of D(A) endowed with the norm

$$|z|_{\mathbf{Y}} = |z|_{\mathbf{X}} + |\mathbf{A}(0)z|_{\mathbf{X}},$$

then for some neighborhood U of the origin in \mathbb{C}^n , we may regard A: U x Y \rightarrow X as a continuous map for each α in U.

(H2) Let N: $U \times Y \rightarrow X$ be a continuous map such that $N(\alpha,0) = 0$ and $D_2N(\alpha,0) = 0$ for all α in U; [i.e., N has zero linear part at $(\alpha,0)$]. We assume that N is continuously Fréchet differentiable in a neighborhood V of (0,0).

According to Nachbin [15], this implies that for every v_{0} in V, there is a $\,\rho\,>\,0\,$ and a power series

$$\sum_{m=0}^{\infty} \frac{1}{m!} D_m (v - v_0)^m$$

that converges to N uniformly for $|v-v_O^{}|<\rho$. $D_m^{}$ is a symmetric m-linear form on $\left[\sigma^n_{} \times Y\right]^m$. In fact, $D_m^{}$ is the

mth Frechet derivative of N at v_0 , denoted by $D^mF(v_0)$. In most applications, this series is finite.

In most physical problems, we deal with real spaces and real parameters. Thus we assume

(H3) A and N are extensions of "real" operators in the sense that $\overline{A(\alpha)z} = A(\overline{\alpha})\overline{z}$ and $\overline{N(\alpha,z)} = N(\overline{\alpha},\overline{z})$.

Now consider the differential equation

(1.1)
$$\frac{dz}{dt} = A(\alpha)z + N(\alpha, z)$$

and the steady state equation

$$(1.2) 0 = A(\alpha)z + N(\alpha,z).$$

The trivial solution is always an equilibrium state for α near $\alpha=0$. If A(0) is a linear homeomorphism of Y onto X, then the implicit function theorem guarantees that the only solution of (1.2) in a neighborhood of (0,0) is the trivial solution. Thus we may expect nontrivial equilibrium solutions to bifurcate from the trivial solution at (0,0) only if A(0) is not a linear homeomorphism; i.e., $\lambda=0$ is in the spectrum of A(0). We now assume

(H4) $\Lambda=0$ is an isolated eigenvalue of A(0) with finite dimensional generalized null space. For our purposes, we shall assume that the null space is equal to the generalized null space.

§2. Reduction to the Finite-dimensional Problem

We shall show that the problem of determining equilibrium solutions of (2.1) corresponding to α near $\alpha=0$, can be reduced to a finite system of equations on the generalized null space of A(0). In our case the generalized null space is the null space and is finite-dimensional. Let $P_{\mathbf{v}}$ denote this null space.

By standard results from theory of closed operators we have

Proposition 2.1. There is a continuous projection $P: X \rightarrow P_X$ which commutes with A(0) in the sense that A(0)Pz = PA(0)z for z in D(A) = Y.

Furthermore, the spectrum of the restriction of A(O) to $Q_{\mathbf{Y}}$ does not contain λ = O, and we have

Proposition 2.2. The restriction $A(0): Q_Y \rightarrow Q_X$ is a linear homeomorphism.

The Liapunov-Schmidt method [7] consists in using the projections P and I-P given by proposition (2.1) to decompose the equation

$$(2.1) 0 = A(\alpha)z + N(\alpha,z)$$

into an equivalent system of two equations as follows. Let z in Y be rewritten as z=w+b where w is in Q_Y and b is in P_Y ; b may be considered as a point in \mathbb{C}^m where m is the dimension of P_Y . Equation (1.2) is then equivalent to the system

$$(2.2) \begin{cases} O = (I - P) \{A(O) + [A(\alpha) - A(O)]\}(w + b) + (I - P)N(\alpha, w + b) \\ \\ O = P\{A(O) + [A(\alpha) - A(O)]\}(w + b) + PN(\alpha, w + b) \end{cases}$$

Since P and (I-P) commute with A(O), we have

$$\begin{cases} (2.3) (a) & 0 = F_1(\alpha, b, w) = A_0(w) + (I - P)[A(\alpha) - A(0)](w + b) \\ & + (I - P)N(\alpha, w + b) \end{cases}$$

$$(2.3) (b) & 0 = F_2(\alpha, b, w) = P[A(\alpha) - A(0)](w + b) + PN(\alpha, w + b)$$

where

$$F_1: U \times \mathbb{C}^m \times Q_v \rightarrow Q_v$$

and

$$F_2: U \times \mathbb{C}^m \times Q_Y \to \mathbb{C}^m$$

are continuously differentiable. Since

 $D_3F_1(0,0,0)[\alpha,b,w] = A(0)w + (I - P)D_2N(0,0)[w] = A(0)w,$

and the restriction $A(0): Q_Y \to Q_X$ is a linear homeomorphism, the implicit function theorem allows us to solve (2.3)(a) for a unique $w(\alpha,b)$ for (α,b) near (0,0), such that $F_1(\alpha,b,w(\alpha,b))=0$ and w(0,0)=0. The function $w(\alpha,b)$ is analytic in some neighborhood of (0,0) in $\mathbb{C}^n \times \mathbb{C}^m$ and may be expanded in a convergent power series in that neighborhood.

We will refer to (2.3) (a) as the <u>auxiliary equation</u> and (2.3) (b) as the <u>bifurcation equation</u>. Upon substitution of $w(\alpha,b)$ into (2.3) (b) we have

Proposition 2.3. Under the hypotheses (H1) - (H4), the problem of finding equilibrium solutions of (1.1) near the trivial solution for α close to zero, is equivalent to solving the finite dimensional problem

(2.4)
$$O = F(\alpha,b) = F_2(\alpha,b,w(\alpha,b))$$

for (α,b) near (0,0). [F(α ,b) is analytic in a neighborhood of (0,0).]

The simplest case occurs where α is in \mathbb{C} , and the subspace P_Y is one-dimensional. It is this case which we shall consider in the remainder of the chapter. In the following chapter we shall discuss a case where P_Y is two dimensional.

§3. Bifurcation [One Dimensional Null Space]

Let A and N satisfy the hypotheses (H1) - (H4), with Λ an open subset of C. A and N are to be thought of as extensions of "real" operators defined for a real parameter. Let

$$A(\alpha)z = A_0z + \alpha A_1z + \alpha^2 A_2z + \cdots$$

Recalling the example of the elastic rod in chapter I, we see that we intuitively expect bifurcation to occur when the trivial solution "loses its stability". In keeping with this intuition we make the following assumptions

(H5) λ = 0 is a simple real eigenvalue of A(0) with eigenfunction ϕ_0 .

By results from analytic perturbation theory, (H5) implies that $A(\alpha)$ has a simple isolated eigenvalue

$$\lambda^*(\alpha) = \lambda_1 \alpha + \lambda_2 \alpha^2 + \cdots$$

which is an analytic function in a neighborhood of zero. Since $\overline{A(\alpha)z} = A(\overline{\alpha})\overline{z}$, $\chi^*(\alpha)$ is real for real α . In order to assure that $\chi^*(\alpha)$ actually crosses the imaginary axis as α passes through zero, we assume

(H6)
$$\lambda_1 \neq 0$$
.

Thus in the case where the remaining spectrum of $A(\alpha)$ initially lies to the left of some line $Re\lambda = \delta$ ($\delta < 0$)

and $\lambda^*(\alpha) < 0$ for $\alpha < 0$, the trivial solution is stable for $\alpha < 0$ and becomes unstable for $\alpha > 0$.

Example (3.1). Let $A(\mu) = L + \mu B$ for μ in C, where L and B are densely defined closed linear operators on a Banach space X, with $D(L) \subset D(B)$. Suppose that L has a continuous compact inverse and that $L^{-1}B$ has a continuous compact extension to all of X. The spectrum of A then consists of isolated eigenvalues with finite multiplicities. Suppose $\lambda = 0$ is a simple eigenvalue of A at $\mu_0 \neq 0$, with eigenfunction ϕ_0 . Then we have analytic expansions for the eigenvalue

$$\lambda (\mu) = \lambda_1 (\mu - \mu_0) + \lambda_2 (\mu - \mu_0)^2 + \cdots$$

and the eigenfunction

$$\varphi(\alpha) = \varphi_0 + \varphi_1 (\mu - \mu_0) + \varphi_2 (\mu - \mu_0)^2 + \cdots$$

which are valid, in a neighborhood of $\mu = \mu_0$ [11]. We claim that (H6) is satisfied.

Since $0 = (L - \mu_0 B) \phi_0$, we have $0 = [\frac{1}{\mu_0} I - L^{-1} B] \phi_0$, and ϕ_0 is an eigenfunction for $L^{-1} B$ corresponding to the simple eigenvalue $\frac{1}{\mu_0}$. As in proposition 2.1, there is a projection P: $X \to \text{span}\{\phi_0\}$ that commutes with $L^{-1} B$. Then

$$\lambda(\mu)\phi(\mu) = (L - \mu B)\phi(\mu),$$

and

$$\chi(\mu) \operatorname{PL}^{-1} \varphi(\mu) \ = \ \operatorname{PL}^{-1} \left(\operatorname{L} - \mu \operatorname{B} \right) \varphi(\mu) \ .$$

Differentiating with respect to u gives

$$\lambda (O) PL^{-1} \phi_1 + \lambda_1 PL^{-1} \phi_O = PL^{-1} (L - \mu_O B) \phi_1 + PL^{-1} B \phi_O$$
or
$$\lambda_1 PL^{-1} \phi_O = L^{-1} B \phi_O.$$

Since $O = (I - L^{-1}B)\phi_O$ and $\phi_O \neq O$, $L^{-1}B\phi_O \neq O$. Thus $\lambda_1 \neq O$.

Many of the linear operators encountered in mathematical physics fall into this category.

The aim of this section and its successor is to prove the following

Theorem 3.1. Let A and N satisfy (H1) - (H6). In addition, suppose that the remaining spectrum of A satisfies $|\text{Re}\lambda| > \delta$, for some $\delta > 0$. Then one of the following occurs

- (i) There is an infinite number of real nontrivial solutions of (2.2) near the zero solution for α = 0 ["vertical bifurcation"]
- (ii) For each real α near 0, there is a real non-zero solution $z(\alpha)$ of (2.2) which may be expanded in a fractional power series about $\alpha = 0$. $z(\alpha)$ is the only nontrivial solution such that $z_1(\alpha)$ tends to zero with α .
- (iii) There are two real nonzero solutions $z_1(\alpha)$ and $z_2(\alpha)$ for (2.2) for $\alpha>0$ ($\alpha<0$) and none for $\alpha<0$ ($\alpha>0$). These solutions may also be expanded in fractional power series.

Theorem 3.2. Let $L(\alpha)$ denote the linearization of (2.2) about $z_1(\alpha)$ [or $z_2(\alpha)$] given by

$$L(\alpha) = A(\alpha) + D_2 N(\alpha, z_1(\alpha)).$$

Under the hypotheses of Theorem 3.1, the spectrum of $L(\alpha)$ remains near that of A(0). For α near zero there is a simple real eigenvalue $\chi^+(\alpha)$; $\chi^+(\alpha)$ and $\chi^*(\alpha)$ have opposite signs in a neighborhood of $\alpha = 0$.

Corollary 3.3. If the spectrum of $A(\alpha)$ (with the exclusion of λ^*) satisfies $\text{Re}\lambda < \delta < 0$ for α near zero and $\lambda^*(\alpha) < 0$ for real $\alpha < 0$, and there is no vertical bifurcation, then real nontrivial solutions are unstable for $\alpha < 0$ and stable for $\alpha > 0$.

Sattinger [17] has proven a similar result using Leray Schauder degree when the linear operator $A(\alpha)$ is given as in Example 3.1. Gavalas [6] has also applied degree theory to the stability of bifurcating solutions. In some cases the stability for particular systems which arise in fluid mechanics has been determined by perturbation methods [13], [18], [19]. The proof we give here is different and directly relates the sign of $\lambda^+(\alpha)$ to the derivative of the bifurcation equation.

We now return to the analysis of the system

$$\begin{cases} (3.1) (a) & O = F_1(\alpha,b,w) = A(O)w + (I-P)[A(\alpha)-A(O)](w+b) \\ & + (I-P_O)N(\alpha,w+b) \end{cases}$$

$$(3.1) (b) & O = F_2(\alpha,b,w) = P[A(\alpha)-A(O)](w+b) + PN(\alpha,w+b)$$

where $F_1: U \times C \times Q_Y \to Q_X$, $F_2: U \times C \times Q_X \to C$. We have already shown that for (α,b) near (0,0), there is a unique analytic $w(\alpha,b)$ such that $F_1(\alpha,b,w(\alpha,b)) = 0$. Let $w(\alpha,b) = b_{10}\alpha + b_{01}b + b_{11}\alpha b + b_{20}\alpha^2 + \cdots + b_{k}\alpha^k b^k$.

Since w=0 is a solution for $(\alpha,b)=(\alpha,0)$, and $w(\alpha,b)$ is unique in a neighborhood of (0,0), we must have $w(\alpha,0)=0$. Thus $b_{k0}=0$ for all k.

We now determine $b_{10}=\frac{\partial w}{\partial b}$ (0,0). Since $F(\alpha,b,w(\alpha,b))=0$ and $D_2N(0,0)=0$, implicit differentiation shows that $A(0)\frac{\partial w}{\partial b}$ (0,0) = 0. Since b_{01} is in Q_Y , $\frac{\partial w}{\partial b}$ (0,0) = 0, and we have

Proposition 3.1. There exists an analytic function $w(\alpha,b)$ which solves (3.1)(a) in a neighborhood of (0,0). This function has the form

$$w(\alpha,b) = b_{11}\alpha b + b_{02}b^2 + o(\alpha^2 b + b^2 \alpha + b^3).$$

Upon substitution of $w(\alpha,b)$ into (3.1)(b), the bifurcation equation becomes

(3.2)
$$O = F(\alpha,b) = \alpha PA_{1}b + \alpha PA_{1}w(\alpha,b) + P[A(\alpha) - \alpha A_{1} - A_{0}](w(\alpha,b) + b) + PN(\alpha,b+w(\alpha,b)),$$

where $PN(\alpha,b+w(\alpha,b))$ is analytic and has a power series expansion in a neighborhood of $(\alpha,b)=(0,0)$. Recall that $N(\alpha,0)\equiv 0$ and $D_2N(\alpha,0)\equiv 0$ for all α in a neighborhood of zero. Thus $D_1^kN(\alpha,0)=0$ and $D_1^kD_2N(\alpha,0)=0$ for all α near zero, and we have $PN(\alpha,b+w(\alpha,b))=b^2N_1(\alpha,b)$ where $N_1(\alpha,b)$ is analytic in a neighborhood of (0,0).

It is useful to calculate the term PA_1b . Let $\phi(\alpha)=\phi_0+\alpha\phi_1+\alpha^2\phi_2+\cdots$ be the eigenfunction corresponding to

$$\lambda^*(\alpha) = \alpha \lambda_1 + \alpha^2 \lambda_2 + \cdots$$

Then $P[\lambda^*(\alpha)I - A(\alpha)]\phi(\alpha) = 0$. Differentiating with respect to α at $\alpha = 0$ gives

$$P[\lambda_1 I - A_1] \phi_O + P[-A_O] \phi_1 = O$$

or
$$\lambda_1 \phi_0 = P A_1 \phi_0$$
.

Thus $PA_1b = \lambda_1b$. The bifurcation equation then becomes

$$0 = \lambda_{1} \alpha b + \alpha P_{0} A_{1} [b_{11} \alpha b + b_{02} \alpha b^{2} + \cdots]$$

$$+ P_{0} [\alpha^{2} A_{2} + \alpha^{3} A_{3} + \cdots] (w (\alpha, b) + b)$$

$$+ b^{2} N_{1} (\alpha, b).$$

Thus α and b are related by a power series in two variables with real coefficients:

(3.3)
$$0 = b[\lambda_1 \alpha + c_{10}b + c_{02}\alpha^2 + 0(\alpha b + b^2)].$$

The trivial solution b=0 is a solution for all α near zero. One may calculate c_{10} to be $\frac{1}{2} PD_2^2 N (0,0) < \phi_0, \phi_0 >$ where ϕ_0 is the eigenfunction corresponding to $\lambda=0$, with $|\phi_0|=1$. If c_{10} is not zero, the implicit function theorem gives a unique solution $b(\alpha)$ of (3.3) which depends on α analytically in a neighborhood of zero. $b(\alpha)$ is real for real α and b(0)=0.

A more general analysis of (3.6) may be carried out by using Newton's polygonal method [1]. Consider a power series in two complex variables that converges in a neighborhood of (0,0).

$$G(\alpha,b) = c_{10}b + c_{01}\alpha + c_{20}b^2 + \cdots + c_{ij}b^i\alpha^j + \cdots$$

Suppose the coefficients are real. We wish to find real solutions of $G(\alpha,b)=0$ corresponding to real α in a neighborhood of $\alpha=0$. Let c_{k0} be the first of the coefficients c_{i0} that do not vanish. The following two lemmas [1] apply:

Lemma 3.7. Let k be even, $c_{01} \neq 0$. Then if $c_{01}/c_{k0} < 0$ $(c_{01}/c_{k0} > 0)$, the equation $G(\alpha,b) = 0$ has two different real roots $b_1(\alpha)$, $b_2(\alpha)$ for $\alpha > 0$ $(\alpha < 0)$ which are simple real roots, and has no real roots for $\alpha < 0$ $(\alpha > 0)$.

Lemma 3.8. Let k be odd and $c_{Ol} \neq 0$. Then $G(\alpha,b)=0$ has precisely one real root for both $\alpha>0$ and $\alpha<0$ and this root is simple.

Under the above conditions, these solutions are given by fractional power series in $\sqrt[k]{\alpha}$. The leading coefficients are given by $\pm \sqrt[k]{|c_{01}/c_{k0}|}$ for k even, and by $\sqrt[k]{-c_{01}/c_{k0}}$ for k odd.

Since $\lambda_1 \neq 0$, these lemmas apply to (3.6). The number and nature of the real roots $b(\alpha)$ will be determined by the first term $c_{k0} \neq 0$, the sign of λ_1/c_{k0} and the parity of k.

We now conclude the proof of Theorem 3.1.

(i) If no c_{iO} is nonzero, then (0,b) is a solution of (3.3) for arbitrary b, and "vertical bifurcation" occurs.

Therefore, we assume that $c_{\mbox{Ok}}$ is the first such non-zero coefficient and apply the lemmas:

(ii) If k is odd,

(3.4)
$$b_{1}(\mu) = \sqrt[k]{-\lambda_{1}/c_{k0}} \mu + \cdots, \text{ where } \mu = \sqrt[k]{\alpha}$$
and
$$z_{1}(\alpha) = w(\alpha, b_{1}(\mu)) + b_{1}(\mu)$$

(iii) Let k be even. If $-\lambda_1/c_{\mathrm{kO}}^{}>0$

(3.5)
$$\begin{cases} b_1(\mu) = \sqrt[k]{-\lambda_1/c_{k0}} \ \mu + \cdots, \text{ where } \mu = + \sqrt[k]{\alpha} \\ b_2(\mu) = \sqrt[k]{-\lambda_1/c_{k0}} \ \mu + \cdots, \text{ where } \mu = - \sqrt[k]{\alpha} \end{cases}.$$

If $-\lambda_1/c_{k0}<$ 0, replace α by $-\alpha,$ and use λ_1/c_{k0} under the radical sign. In either case,

$$z_1(\alpha) = w(\alpha, b_1(\mu)) + b_1(\mu)$$

and

$$z_2(\alpha) = w(\alpha, b_2(\mu)) + b_2(\mu)$$
.

In fact, the analytic function

(3.6)
$$z(\mu) = w(\mu^{k}, b(\mu)) + b(\mu)$$

is a solution of

(3.7)
$$0 = A(\mu^{k})z + N(\mu^{k},z)$$

for all (complex) μ in a neighborhood of zero. Here $b(\mu)$ is any of the functions given by (3.4) - (3.5) and $\mu^k = \alpha$. [In the case $\mu^k = -\alpha$, we replace μ^k by $-\mu^k$ in (3.6) - (3.7)]. For each α near zero, there correspond k distinct $z_i(\alpha)$ ($i=1,\ldots,k$) corresponding to the k roots of α . These $z_i(\alpha)$ form a cyclic system of solutions of the system (1.2). If α moves along some Jordan curve about zero, the values $z_i(\alpha)$ undergo a cyclic permutation when we return to the starting point. We also note that $z(\mu)$ may be expanded in a power series with real coefficents.

§4. Stability

In this section we give the proof of Theorem 3.2. Recall the differential equation

(4.1)
$$\frac{dz}{dt} = A(\alpha)z + N(\alpha, z)$$

and the stationary problem

$$(4.2) 0 = A(\alpha)z + N(\alpha,z).$$

We have shown that real nontrivial solutions $z(\alpha)$ of (4.2) bifurcate from the trivial solution for real α near zero. These solutions may be found by substituting an appropriate real kth root of α (or $-\alpha$) into an analytic function $z(\mu) = \gamma_1 \mu + \gamma_2 \mu^2 + \cdots$ where γ_1 is given in (3.4) - (3.5). For all μ in a neighborhood of zero, $z(\mu)$ is an equilibrium solution of

(4.3)
$$\frac{dz}{dt} = A(\mu^k)z + N(\mu^k, z) \quad \text{if} \quad \mu^k = \alpha$$

or

(4.4)
$$\frac{dz}{dt} = A(-\mu^{k})z + N(-\mu^{k},z)$$
 if $\mu^{k} = -\alpha$.

We wish to discuss the stability of the equilibrium solutions $z(\mu)$ where μ is a real kth root of α (or $-\alpha$). Without loss of generality, we shall use $\mu^k = \alpha$ in our calculations.

Consider the linearization of (4.3) about $z(\mu)$ for fixed u:

(4.5)
$$\frac{dy}{dt} = A(\mu^{k})y + D_{2}N(\mu^{k}, z(u))y = L(\mu)y.$$

For each μ , $L(\mu)$: $Y \to X$ is a continuous linear map. $L(\mu)$ depends on μ analytically, and $\overline{L(\mu)}\overline{y} = L(\overline{\mu})\overline{y}$. For μ

sufficiently close to zero, $L(\mu)$ may be regarded as a closed operator with dense domain D(A) in X; thus we may refer to Kato [11] for information about the dependence of its spectrum on μ . Since L(0) = A(0), the spectrum of $L(\mu)$ satisfies the following when μ is sufficiently close to zero.

- (4.6) There is a simple eigenvalue $\lambda^+(\mu)$ such that $\lambda^+(0) = 0$ and $\lambda^+(\mu)$ is analytic in a neighborhood of zero. $\lambda^+(\mu)$ is real for real μ near zero.
- (4.7) The remainder of the spectrum satisfies $|\text{Re}\lambda| > \delta/2 > 0$.

Our aim is to relate the sign λ^+ to that of λ^* when α is real and close to zero.

Let y in Y be written as $y=x+\xi$ where $x\in Q_Y$ and $\xi\in P_Y$ and consider $N(\alpha,z)$ as $N(\alpha,x,\xi)$ where $z=x+\xi$. The eigenvalue problem for $L(\mu)$ may be written as the system

$$\begin{cases} (4.8) (a) & \lambda x = [A(0) + (I - P)[A(\mu^{k}) - A(0) + D_{x}N(\mu^{k}, z(\mu))]x \\ & + (I - P_{0})[A(\mu^{k}) - A(0) + D_{\xi}N(\mu^{k}, z(\mu))]\xi \end{cases}$$

$$(4.8) (b) & \lambda \xi = P[A(\mu^{k}) - A(0) + D_{x}N(\mu^{k}, z(\mu))]x \\ & + P[A(\mu^{k}) - A(0) + D_{\xi}N(\mu^{k}, z(\mu))]\xi$$

or more simply

$$\begin{cases} (4.9) (a) & O = [B(\mu) - \lambda]x + C(\mu)\xi \\ (4.9) (b) & O = D(\mu)x + [E(\mu) - \lambda]\xi \end{cases}$$

where $B(\mu)$, $C(\mu)$, $D(\mu)$, and $E(\mu)$ are linear maps for each μ , and depend on μ analytically in a neighborhood of the origin.

Lemma 4.1. For λ and μ sufficiently small, $B(\mu)$ and $\lambda - B(\mu)$ are invertible and

$$(B(\mu) - \lambda)^{-1} = B(\mu)^{-1} + \lambda[B(\mu)^{-1}]^{2} + \cdots + \lambda^{n}[B(\mu)^{-1}]^{n+1} + \cdots$$

converges uniformly in the operator norm for λ and μ in a neighborhood of the origin.

Proof. Since the set of linear homeomorphisms in $L(Q_Y,Q_X) \quad \text{is open [5] and } B(O) = (I-P)A(O) \quad \text{is a linear homeomorphism, by proposition 2.1, } B(\mu) \quad \text{and} \quad B(\alpha) - \lambda$ are invertible for μ and λ sufficiently small. Also $B(\mu)$ is close to A(O) in norm. The rest follows from the fact that if T_n are elements of Banach space and $\sum_{i=1}^{\infty} \|T_n\| < \infty, \text{ then } \sum_{i=1}^{\infty} T_n \quad \text{converges.} \square$

We may now solve (4.9) (a) for x in terms of ξ and substitute into (4.9) (b) to get

$$O = [E(\mu) - D(\mu) (B(\mu) - \lambda)^{-1} C(\mu) - \lambda] \xi$$

(4.10)
$$O = \mathbf{E}(\mu) - \mathbf{D}(\mu) \mathbf{B}(\mu)^{-1} \mathbf{C}(\mu) - \lambda$$

$$- \mathbf{D}(\mu) \left[\lambda \left[\mathbf{B}(\mu)^{-1} \right]^{2} + (\lambda)^{2} \left[\mathbf{B}(\mu)^{-1} \right]^{3} + \cdots \right] \mathbf{C}(\mu)$$

$$= \mathbf{f}(\mu) - \lambda - \lambda \mathbf{g}(\mu, \lambda)$$

where f is an analytic function μ in a neighborhood of zero, and g is an analytic function of μ and λ near the origin. A Newton's polygon argument shows that for real μ there is a real solution $\lambda(\mu) = d_1 \mu^p + d_2 \mu^{2p} + \cdots$ where d_1 is the coefficient of the lowest power of μ in $f(\mu) = E(\mu) - D(\mu)B(\mu)^{-1}C(\mu)$ and p is that lowest power.

We now make the following observation. Recall that $z(\mu) = w(\mu, b(\mu)) + b(\mu) \quad \text{satisfies}$

$$(4.11) \begin{cases} 0 = (I-P)[A(\mu^{k})[w+b] + N(\mu^{k}, w, b)] = F_{1}(\mu, w, b) \\ \\ 0 = P[A(\mu^{k})[w+b] + N(\mu^{k}, w, b)] = F_{2}(\mu, w, b). \end{cases}$$

Let μ near 0 be fixed. Differentiating $F_2(\mu,w,b)$ with respect to b at $b(\mu)$ gives

(4.12)
$$P[A(\mu^{k}) + D_{3}N(\mu^{k}, w(\mu^{k}, b(\mu)), b(\mu)]$$

 $+ P[A(\mu^{k})\frac{\partial w}{\partial b} + D_{2}N(\mu^{k}, w(\mu^{k}, b(\mu)), b(\mu)]\frac{\partial w}{\partial b}(\mu^{k}, b(\mu)).$

Since by I (3.6),

$$\frac{\partial w}{\partial b} = -[D_2F_1(\mu, w(\mu^k, b(\mu)), b(\mu)]^{-1}[D_3F_1(\mu, w(\mu^k, b(\mu)), b(\mu)],$$

(4.12) is exactly $E(\mu) - D(\mu)B(\mu)^{-1}C(\mu)$. We have proved Lemma 4.12. For μ sufficiently close to zero, we may determine the leading term of $\lambda^+(\mu)$ by differentiating $F_2(\mu,w,b)$ with respect to b at $b(\mu)$. The leading term of this derivative will be the leading term of $\lambda^+(\mu)$. [Note that $F_2(\mu,w,b) = 0$ is the bifurcation equation.]

Recall that the bifurcation equation may be written in the form

(4.13)
$$F(\alpha,b) = \lambda_1 \alpha b + c_{02} \alpha^2 + c_{10} b^2 + \cdots$$

where $\alpha = \mu^k$. Let $c_{Ok}b^{k+1}$ be the first term of the form $c_{iO}b^{i+1}$ that does not vanish. In this case

$$b\left(\mu\right)=\left\{ \begin{array}{ll} \frac{k}{\sqrt{\left|\lambda/c_{\mathbf{k}0}\right|}}\;\mu\;+\;\cdots\;\;\text{if }\;\;k\;\;\text{is even}\\ \\ \frac{k}{\sqrt{-\lambda/c_{\mathbf{k}0}}}\;\;\mu\;\;\text{if }\;\;k\;\;\text{is odd,}\;\;\mu^{k}\;=\;\alpha. \end{array} \right.$$

[For k even, $\mu^k=\alpha$ if $-\lambda/c_{kO}>0$ and $\mu^k=-\alpha$ if $-\lambda/c_{kO}<0.$] Thus

$$\begin{split} \frac{\partial F}{\partial b}(\alpha,b(\mu)) &= \lambda_1 \alpha + c_{02} \alpha^{2k} + \cdots + (k+1) c_{k0} [b(\mu)]^k + \cdots \\ &= \lambda_1 \alpha - (k+1) \lambda_1 \alpha + \text{ higher order terms in } \sqrt[k]{\alpha} \\ &= -k \lambda_1 \alpha + \text{higher order terms in } \sqrt[k]{\alpha} \end{split}$$

Thus $d_1=-k\lambda$, p=k, and $\lambda^+(\mu)$ is actually an analytic function of α : $\lambda^+(\alpha)=-k\lambda_1\alpha+\cdots$. Recall that

 $\lambda^*(\alpha) = \lambda_1 \alpha + \lambda_2 \alpha^2 + \cdots$. Thus for real α near zero, λ^+ and λ^* have opposite signs. This completes the proof of Theorem 3.2.

CHAPTER III

A TWO-DIMENSIONAL NULL SPACE

§1. Introduction

Our aim in this chapter is to develop tools which will allow us to describe the set of nontrivial equilibrium solutions which bifurcate from the trivial solution of

$$\frac{dz}{dt} = A(\alpha,\beta)z + N(\alpha,\beta,z)$$

at (0,0), where (α,β) is in \mathbb{C}^2 . Part of the problem will be to choose suitable hypotheses so that the dependence of the bifurcating solutions on the parameters may be studied in a full neighborhood of $(\alpha,\beta) = (0,0)$.

In the case which we shall study, A(0,0) has a two-dimensional null space, and $A(\alpha,\beta)$ has two eigenvalues $\lambda(\alpha,\beta)$ and $\mu(\alpha,\beta)$ which pass through zero as (α,β) passes through the origin. The curves $\lambda(\alpha,\beta)=0$ and $\mu(\alpha,\beta)=0$, defined for (α,β) in \mathbb{R}^2 , are assumed to cross transversally at the origin. This last assumption will allow us to change coordinates to $\tau=\lambda(\alpha,\beta)$, $\eta=\mu(\alpha,\beta)$. We will formalize these hypotheses and discuss their implications at the end of this section.

In §2 we discuss the bifurcation equations and the coordinate change mentioned above. In §3 we give a simple example which demonstrates that the behavior of the bifurcation set depends on the ratio $Y = \eta/\tau$ (or τ/η). Motivated by this observation, we make a change of scale which allows us to introduce the ratio Y in place of one of the parameters τ, η . Section 4-6 discuss methods for analyzing the scaled equations and the interpretation of these results in the (τ, η) -space.

We now list the basic hypotheses of this chapter.

(H1) - (H3) are from Chapter II and are repeated for convenience

(H1) Let Λ be an open set in \mathbb{C}^2 that contains the origin and let X be a complex Banach space with dense subspace D(A). Let $A: \Lambda \times D(A) \to X$ be a closed linear operator with domain D(A) for each (α,β) in Λ , and let A be analytic in Λ in the sense that $A(\alpha,\beta)z$ has a Taylor expansion at each (α_0,β_0) in Λ which converges in a disc $|(b,\alpha)-(b_0,\alpha_0)|< r$ independent of z.

It follows from (H1), that if Y is the Banach space consisting of D(A) endowed with the graph norm, then for some neighborhood U of \mathbb{C}^2 we may regard A: U X Y \rightarrow X as a continuous linear map.

(H2) Let N: $U \times Y \to X$ be a continuous map such that $N(\alpha,\beta,0) = 0$ and $D_3N(\alpha,\beta,0) = 0$ for all (α,β) in U.

We assume that N is continuously Fréchet differentiable in a neighborhood V of (0,0,0).

(H3) A and N are extensions of "real" operators in the sense that $\overline{A(\alpha,\beta)z} = A(\bar{\alpha},\bar{\beta})\bar{z}$ and $\overline{N(\alpha,\beta,z)} = N(\bar{\alpha},\bar{\beta},\bar{z})$.

In addition, we make the following additional assumptions about $A(\alpha,\beta) = A_0z + \alpha A_1z + \beta A_2z + \cdots$

(H7) Zero is an isolated eigenvalue of A_0 , and the generalized null space is two-dimensional. In addition, we assume that there are two distinct branches of eigenvalues that depend on the parameters analytically and take on the value zero at (0,0).

$$\lambda(\alpha,\beta) = \lambda_{10}\alpha + \lambda_{01}\beta + \cdots$$

and

$$\mu(\alpha,\beta) = \mu_{10}\alpha + \mu_{01}\beta + \cdots$$

are eigenvalues of $A(\alpha,\beta)$. Moreover, there are two distinct eigenfunctions ϕ_O and Ψ_O corresponding to $\lambda(0,0)=0$ = 0 = $\mu(0,0)$, and we have the expansions

$$\varphi(\alpha,\beta) = \varphi_0 + \varphi_{10}\alpha + \varphi_{01}\beta + \cdots$$

$$\Psi(\alpha, \beta) = \Psi_0 + \Psi_{10}\alpha + \Psi_{01}\beta + \cdots$$

which converge in some neighborhood of (0,0).

(H8) For real values of the parameters, λ and μ are simple real eigenvalues (except when $\mu(\alpha,\beta)=\lambda(\alpha,\beta)$), and $\overline{\Psi}(\alpha,\beta)=\Psi(\alpha,\beta)$, $\overline{\varphi}(\alpha,\beta)=\varphi(\alpha,\beta)$. Furthermore, the curves $\lambda(\alpha,\beta)=0$, $\mu(\alpha,\beta)=0$, defined for real α and β , cross transversely at the origin (see Fig. 1); i.e.,

$$\det \begin{bmatrix} \frac{\partial \lambda}{\partial \alpha} & \frac{\partial \lambda}{\partial \beta} \\ & & & \\ \frac{\partial \mu}{\partial \alpha} & \frac{\partial \mu}{\partial \beta} \end{bmatrix} \neq 0 \quad \text{at (0,0)}$$

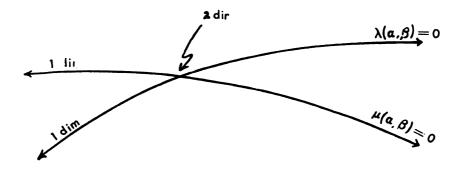


Figure 1. Transversal crossing of zero eigencurves.

This last assumption is important as it will allow us to make a local change of coordinates about the origin in \mathbb{R}^2 in which the curves $\chi(\alpha,\beta)=0$ and $\mu(\alpha,\beta)=0$ become the axes.

Example 1.2. The hypothesis (H7) is necessary as the analytic dependence of the eigenvalues and eigenfunctions upon the parameters does not necessarily follow from that of A. Let

$$A = \begin{bmatrix} \alpha & \beta \\ \\ \\ \beta & -\alpha \end{bmatrix}$$

map c^2 into c^2 . The eigenvalues are

$$\lambda = \sqrt{\alpha^2 + \beta^2}$$
 and $\mu = -\sqrt{\alpha^2 + \beta^2}$

which are not differentiable at $(\alpha,\beta)=(0,0)$. The problem is that the eigenvalues are given by different branches of the same multivalued function $\sqrt{\alpha^2+\beta^2}$ and (0,0) is the branch point.

§2. The Bifurcation Equations

Let P be the projection given by proposition II 2.1, and P_X , P_Y , Q_X , Q_Y be the subspaces described in §2 of chapter II. In our case $P_X = P_Y = \operatorname{span}\{\phi_O, \Psi_O\}$ and is homeomorphic to \mathbb{C}^2 . The projection $P: X \to P_X$ is the sum of two projections, P_1 and P_2 . $P_1: X \to \operatorname{span}\{\phi_O\} = P_X^1$ and $P_2: X \to \operatorname{span}\{\psi_O\} = P_X^2$ where $P_X = P_X^1 \oplus P_X^2$. If z is in X, we may rewrite z as z = w + x + y where $w \in Q_X$, $x \in P_X^1$ and $y \in P_X^2$. As in §2 of chapter II, the equation (1.3) is equivalent to the system

$$\begin{cases} (2.1) (a) & O = A_0 w + (I - P)[A_0 - A(\alpha, \beta)](w + x + y) \\ & + (I - P)N(\alpha, \beta, x, y, w) \end{cases}$$

$$(2.1) (b) & O = P_1[\alpha A_1 + \beta A_2 + \cdots](w + x + y) + P_1N(\alpha, \beta, x, y, w)$$

$$(2.1) (c) & O = P_2[\alpha A_1 + \beta A_2 + \cdots](w + x + y) + P_1N(\alpha, \beta, x, y, w)$$

Furthermore, we have shown in chapter II, §2, that there is a unique $w(\alpha,\beta,x,y)$ such that w(0,0,0,0)=0 and $(\alpha,\beta,x,y,w(\alpha,\beta,x,y))$ is a solution of (2.1)(a) for (α,β,x,y) near (0,0,0,0); $w(\alpha,\beta,x,y)$ can be expanded in a power series in some neighborhood of the origin. Also $\bar{w}(\alpha,\beta,x,y)=w(\bar{\alpha},\bar{\beta},\bar{x},\bar{y})$.

As before, we may show that

$$\frac{\partial x}{\partial w}(0,0,0,0) = \frac{\partial y}{\partial w}(0,0,0,0) = 0.$$

Moreover, since $w(\alpha,\beta,0,0) = 0$ for all values of (α,β) near (0,0), the uniqueness of w implies that this power series will contain no terms which contain only α or β . Thus $w(\alpha,\beta,x,y)$ is of the form

$$\begin{aligned} w(\alpha,\beta,x,y) &= c_1 \alpha x + c_2 \beta x + c_3 \alpha y + c_4 \beta y + c_5 x^2 \\ &+ c_6 x y + c_7 y^2 + \text{higher order terms in } x,y,\alpha,\beta. \end{aligned}$$

We may substitute this expression for w into the bifurcation equations (2.1)(b) and (2.1)(c), and obtain two power series in 4 variables:

(2.2)
$$\begin{cases} o = P_1[\alpha A_1 + \beta A_2]x + P_1[\alpha A_1 + \beta A_2]y + \cdots \\ o = P_2[\alpha A_1 + \beta A_2]x + P_2[\alpha A_1 + \beta A_2]y + \cdots \end{cases}$$

We now determine the terms listed in (2.2). Differentiating the expression

$$O = P_{1}[\lambda(\alpha,\beta) - A(\alpha,\beta)]\phi(\alpha,\beta)$$

with respect to α at (0,0) gives $\frac{\partial \lambda}{\partial \alpha}$ (0,0) $\phi_O = P_1 A_1 \phi_O$. A similar argument shows that

$$\frac{\partial \lambda}{\partial \beta}(0,0) \varphi_0 = P_1 A_2 \varphi_0$$

$$\frac{\partial \mu}{\partial \alpha}(0,0) \Psi_0 = P_2 A_1 \Psi_0$$
 and $\frac{\partial \mu}{\partial \beta}(0,0) \Psi_0 = P_2 A_2 \Psi_0$

Also $P_2A_1\phi_0 = P_2A_2\phi_0 = P_1A_1\psi_0 = P_1A_2\psi_0 = 0$. Thus the bifurcation equations become

$$(2.3) \begin{cases} 0 = \alpha \frac{\partial \lambda}{\partial \alpha} x + \beta \frac{\partial \lambda}{\partial \beta} x + P_{1}[\alpha A_{1} + \beta A_{1}]w \\ + P_{1}[A(\alpha, \beta) - A_{0} - \alpha A_{1} - \beta A_{2}][w + x + y] + P_{1}N(\alpha, \beta, x, y, w) \\ 0 = \alpha \frac{\partial \mu}{\partial \alpha} x + \beta \frac{\partial \mu}{\partial \beta} x + P_{2}[\alpha A_{1} + \beta A_{1}]w \\ + P_{2}[A(\alpha, \beta) - A_{0} - \alpha A_{1} - \beta A_{2}][w + x + y] + P_{2}N(\alpha, \beta, x, y, w) \end{cases}$$

Henceforth, we will seek real solutions (x,y) of (2.3) corresponding to real parameters (α,β) . We may as well assume that the norm is Euclidean.

It is convenient to make a change of variables in the parameter plane so that the curves $\lambda(\alpha,\beta)=0$ and $\mu(\alpha,\beta)=0$ become the axes. Define the transformation T: $(\alpha,\beta) \rightarrow (\tau,n)$ by $\tau=\lambda(\alpha,\beta)$, $\eta=\mu(\alpha,\beta)$. Since $\lambda(\alpha,\beta)=0$ and $\mu(\alpha,\beta)=0$

cross transversely at (0,0), T is a homeomorphism in a neighborhood of (0,0). Then

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \frac{\partial \lambda}{\partial \alpha} & \frac{\partial \lambda}{\partial \beta} \\ \frac{\partial \mu}{\partial \alpha} & \frac{\partial \mu}{\partial \beta} \end{bmatrix}^{-1} \begin{bmatrix} \tau \\ \eta \end{bmatrix} + O(\tau, \eta)^{2}.$$

Thus terms which are first order in α and β , are first order in τ and η ; terms which are second order in α and β , are second order in τ and η ; etc. The bifurcation equations take the form

(2.4)
$$\begin{cases} O = \tau x \\ O = \eta y \end{cases} + \begin{cases} 2nd \text{ order terms in } x \text{ and } y + higher \\ order \text{ terms in } x, y, \tau, \eta \end{cases}$$

There are no terms which contain only powers of τ and η . Terms which contain only powers of x and y are derived from the nonlinear term which we refer to as $N(\tau, \eta, x, y, w(x, y, \tau, \eta))$.

§3. An Appropriate Change of Scale

As motivation for what we are about to do, consider the following.

Example 3.1. Consider the system

(3.1)
$$\begin{cases} 0 = \tau x + \frac{1}{4}y^2 + x^2 \\ 0 = \eta y + \frac{1}{4}x^2 + y^2 \end{cases}$$

The numbers of nontrivial solutions of (3.1) corresponding to each (τ, η) are given schematically in Figure 2.

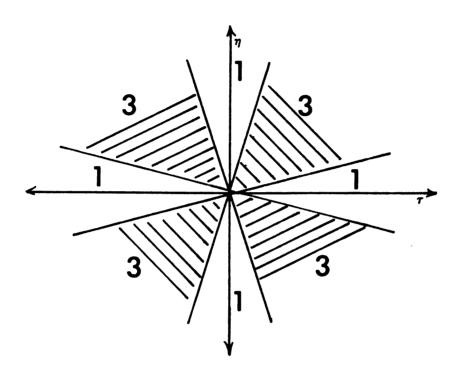


Figure 2. Dependence on (τ, η) of the number of nontrivial solutions of (3.1).

As we can see, the number of nontrivial solutions depends on the ratio τ/η or (η/τ) . Also note that it is impossible to apply the implicit function theorem to obtain solutions of (3.1) of the form $(x(\tau,\eta),y(\tau,\eta),\tau,\eta)$ in a neighborhood of $(x,y,\tau,\eta) = (0,0,0,0)$.

Now consider the following change of scale. Let $\tau = \gamma\eta$, and replace x by ηx and y by ηy in (3.1). After division by η^2 , (3.1) becomes

(3.2)
$$\begin{cases} 0 = \gamma x + \frac{1}{4} y^2 + x^2 \\ 0 = y + \frac{1}{4} x^2 + y^2 \end{cases}$$

Sectors in the (τ, η) -plane and the corresponding regions in the (η, γ) -plane are represented in Figure 3. Points on the γ -axis correspond to different slopes through the origin in the (τ, η) -plane.

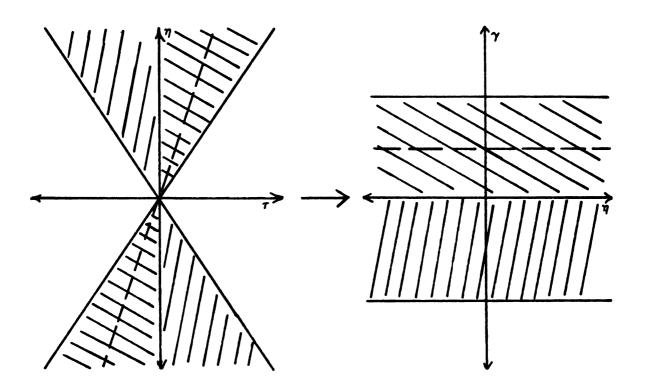


Figure 3. Correspondence between (τ, η) -plane and (η, γ) -plane.

The numbers of nontrivial solutions of (3.2) corresponding to each (η, Y) are given in Figure 4.

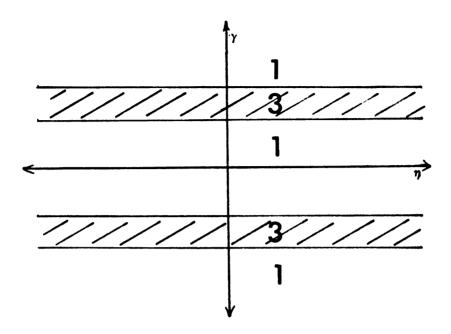


Figure 4. Dependence on (η, γ) of the number of nontrivial solutions of (3.2).

It is easy to check that at most of the solutions (x_0, y_0, γ_0) of (3.2), the implicit function theorem applies. The points at which it fails correspond to the lines in Figure 4 which indicate a change in the number of solutions.

In what follows we shall give hypotheses under which a similar scaling may be determined for the bifurcation equations (2.4). Solutions (x,y) of (2.4) which tend to (0,0) with (τ,η) will correspond to solutions of the scaled equation which remain bounded as (τ,η) approaches (0,0). In order to accomplish this, we shall require the bifurcation equations to be of a certain form so that we

can obtain an a priori estimate of the magnitude of solutions of (2.4) which tend to (0,0) with (τ,η) . This a priori estimate will determine the scaling we choose.

Let $_T = \eta = 0$, and suppose that the first terms of (2.4) in x and y which do not vanish have the same degree q in each equation. The terms of degree q form a continuous q-linear form $M_q: \mathbb{R}^{2q} \to \mathbb{R}^2$

(3.3)
$$M_{q}((x,y),...,(x,y)) = \begin{bmatrix}
a_{0}x^{n} + a_{1}x^{n-1}y + \cdots + a_{n-1}xy^{n-1} + a_{n}y^{n} \\
b_{0}x^{n} + b_{1}x^{n-1}y + \cdots + b_{n-1}xy^{n-1} + b_{n}y^{n}
\end{bmatrix}$$

We say that M_{α} is nondegenerate if

(3.4)
$$M_{q}((x,y),...,(x,y)) \neq (0,0)$$
 for all $(x,y) \neq (0,0)$ in \mathbb{R}^{2} .

Since the unit ball in ${\rm I\!R}^2$ is compact and ${\rm M}_{\rm Q}$ is continuous, (3.4) implies there are positive constants ${\rm c}_1$ and ${\rm c}_2$ such that

(3.5)
$$c_1 | (x,y) |^q \le M_q ((x,y),...,(x,y)) \le c_2 | (x,y) |^q$$

for (x,y) in \mathbb{R}^2 .

(H9) We assume the bifurcation equation has the form (3.6)

$$\begin{bmatrix} O \\ O \end{bmatrix} = \begin{bmatrix} \tau \mathbf{x} \\ \eta \mathbf{y} \end{bmatrix} + S(\tau, \eta, \mathbf{x}, \mathbf{y}) + M_{\mathbf{q}}((\mathbf{x}, \mathbf{y}), \dots, (\mathbf{x}, \mathbf{y})) + \mathbb{R}(\tau, \eta, \mathbf{x}, \mathbf{y})$$

where $S(0,0,x,y) \equiv 0$, $S(\tau,\eta,x,y)$ contains terms in x and y of order less than q and there is a positive constant c_3 such that

(3.7)
$$|S(\tau, \eta, x, y)| \le c_3 |(\tau, \eta)| \cdot |(x, y)|$$

when $(\tau, \eta, \mathbf{x}, \mathbf{y})$ is sufficiently close to (0,0,0,0). We also assume that

(3.8)
$$\frac{\left|R\left(\tau_{k}, \eta_{k}, x_{k}, y_{k}\right)\right|}{\left|\left(x_{k}, y_{k}\right)\right|^{q}} \rightarrow 0 \quad \text{for all sequences}$$

 $\{\tau_k\}, \{\eta_k\}, \{x_k\}, \{y_k\}$ that tend to zero with $|(x_k, y_k)| \neq 0$ for all k, and that M_q is a nondegenerate q-linear form.

Proposition 3.1. Let (H9) be satisfied. Then for any (τ, η, x, y) sufficiently close to (0,0,0,0), such that (x,y) is a solution of (3.6) corresponding to $(\tau, \eta) \neq (0,0)$, we have

$$|(x,y)| \le m^{q-1}\sqrt{|(\tau,\eta)|}$$

where m is a constant independent of τ , η , x, y.

Proof. Suppose not. Then there exist sequences $\{\tau_k\}$, $\{\eta_k\}$, $\{x_k\}$, $\{y_k\}$ that converge to zero, with $(x_k,y_k) \neq 0$, $(\tau_k,\eta_k) \neq (0,0)$ and

$$\frac{\left|\left(\mathbf{x}_{k},\mathbf{y}_{k}\right)\right|^{\mathbf{q}-1}}{\left|\left(\mathbf{\tau}_{k},\mathbf{\tau}_{k}\right)\right|} > k.$$

We may assume that τ_k , η_k , x_k , and y_k are small enough so that the estimates (3.5) and (3.7) hold. Consider

$$0 = \frac{1}{\left| (x_{k}, y_{k}) \right|^{q}} \left\{ \begin{bmatrix} \tau_{k} x_{k} \\ \tau_{k} y_{k} \end{bmatrix} + S(\tau_{k}, \tau_{k}, x_{k}, y_{k}) \right\} + \frac{M_{q}((x_{k}, y_{k}), \dots, (x_{k}, y_{k}))}{\left| (x_{k}, y_{k}) \right|^{q}} + \frac{R(\tau_{k}, \tau_{k}, x_{k}, y_{k})}{\left| x_{k}, y_{k} \right|^{q}}.$$

As (τ_k, η_k) and (x_k, y_k) approach the origin, so does the last term. Estimate (3.5) implies that there are positive constants m_1 and m_2 such that

$$m_{1} \leq \frac{\left| \left(\mathbf{x}_{k}, \mathbf{y}_{k} \right) \right|^{q}}{\left| \left[\mathbf{x}_{k}^{\mathsf{T}_{k}} \mathbf{x}_{k} \right] + s\left(\mathbf{x}_{k}, \mathbf{y}_{k}, \mathbf{x}_{k}, \mathbf{y}_{k} \right) \right|} \leq m_{2}$$

for k sufficiently large. Thus by (3.7),

$$m_{2} \geq \frac{\left| (\mathbf{x}_{k}, \mathbf{y}_{k}) \right|^{q}}{\left| (\mathbf{\tau}_{k}\mathbf{x}_{k}, \mathbf{\eta}_{k}\mathbf{y}_{k}) \right| + c_{3} |\mathbf{\tau}_{k}, \mathbf{\eta}_{k}) | \cdot | (\mathbf{x}_{k}, \mathbf{y}_{k}) |}$$

$$\geq \frac{\left| (\mathbf{x}_{k}, \mathbf{y}_{k}) \right|^{q}}{(1 + c_{3}) | (\mathbf{\tau}_{k}, \mathbf{\eta}_{k}) | \cdot | (\mathbf{x}_{k}, \mathbf{y}_{k}) |}$$

and

$$(1+c_3)^{m_3} \ge \frac{\left| (x_k, y_k) \right|^{q-1}}{\left| (\tau_k, \tau_k) \right|}$$

which is a contradiction.□

Using the a priori estimate of proposition (3.1), we will now exhibit appropriate scalings of the system (3.6) for different sectors of the (τ, η) -plane near (0,0).

Let $\Omega_{\rho} = \{(\tau, \eta) : |\eta| > \rho |\tau| \}$ for $\rho > 0$. Let (\mathbf{x}, \mathbf{y}) be a solution of (3.6) corresponding to (τ, η) , that tends to (0,0) with (τ, η) . Then

$$\frac{x}{q-1/|\eta|}$$
 and $\frac{y}{q-1/|\eta|}$

remain bounded as long as (τ,η) remains in Ω_{ρ} . Thus none of the bifurcating solutions corresponding to (τ,η) in Ω_{ρ} are lost if the following changes of scale are made.

Case 1. If q is even, let $\eta=\xi^{q-1},\ \tau=\gamma\xi^{q-1}$ for $|\gamma|\leq\frac{1}{\rho}$ and replace x and y by ξx and ξy respectively. Then (3.6) becomes

(3.9) (a)
$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma x \\ y \end{bmatrix} + M_{q}((x,y),...,(x,y)) + \xi \overline{R}(x,y,\gamma,\xi)$$

where \bar{R} is analytic in (x,y,γ,ξ) . [The transformation of the sector Ω_{ρ} in the parameter plane under this change of scale was given in Figure 3].

Case 2. q is odd. In this case, we must distinguish between $\eta>0$ and $\eta<0.$

To study solutions corresponding to $\eta>0$, we use the same change of scale as in case 1. Here, however, we must keep in mind that each η corresponds to $\pm\xi$, and there will be an "extra set of solutions" for $\xi<0$.

To study solutions corresponding to $\eta<0$, we let $\eta=-\xi^{\mathbf{q}-1}, \ \tau=-\gamma\xi^{\mathbf{q}-1}, \ \text{and replace} \ \mathbf{x} \ \text{by} \ \xi\mathbf{x}, \ \text{and} \ \mathbf{y} \ \text{by}$ $\xi\mathbf{y} \ \text{for} \ |\gamma|\leq \frac{1}{\rho} \,. \ \text{Then (3.6) becomes}$

(3.9) (b)
$$O = \begin{bmatrix} \gamma & O \\ O & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} - \mathbf{M}_{\mathbf{q}}(0,0,(\mathbf{x},\mathbf{y}),\ldots,(\mathbf{x},\mathbf{y})) + \xi \overline{\mathbb{R}}(\mathbf{x},\mathbf{y},\gamma,\xi)$$

where \bar{R} is analytic in (x,y,γ,ξ) . Again, we will disregard the extra branch of solutions corresponding to $\xi<0$.

To study solutions of (3.6) for (τ, η) near the τ -axes, the roles of τ and η are reversed.

In all cases, we must now determine the behavior of real solutions (x,y) corresponding to real (γ,ξ) for systems of the form

(3.10)
$$\begin{cases} 0 = \gamma x + a_n x^n + a_{n-1} x^{n-1} y + \cdots + a_0 y^n + \xi N_1(x, y, \gamma, \xi) \\ 0 = y + b_n x^n + b_{n-1} x^{n-1} y + \cdots + b_0 y^n + \xi N_2(x, y, \gamma, \xi). \end{cases}$$

§4. Solutions of Type I - Fold curves

Consider the system

$$\begin{cases} O = F(x, y, \gamma, \xi) = \gamma x + a_n x^n + a_{n-1} x^{n-1} y + \dots + a_0 y^n + \xi N_1(x, y, \gamma, \xi) \\ O = G(x, y, \gamma, \xi) = y + b_n x^n + b_{n-1} x^{n-1} y + \dots + b_0 y^n + \xi N_2(x, y, \gamma, \xi) \end{cases}$$

where

$$M_{n}((x,y),...,(x,y)) = \begin{bmatrix} a_{n}x^{n} + a_{n-1}x^{n-1}y + \cdots + a_{0}y^{n} \\ b_{n}x^{n} + b_{n-1}x^{n-1}y + \cdots + b_{0}y^{n} \end{bmatrix}$$

is a nondegenerate continuous n-linear form, $M_n: \mathbb{R}^{2n} \to \mathbb{R}^2$; and N_1 and N_2 are analytic in (x,y,γ,ξ) . Let

(4.2)
$$J(x,y,\gamma,\xi) = \det \begin{bmatrix} F_x & F_y \\ & & G_x & G_y \end{bmatrix} \text{ at } (x,y,\gamma,\xi).$$

For $\xi = 0$, (4.1) and (4.2) become

$$\begin{cases}
0 = P(x, y, y) = yx + a_n x^n + a_{n-1} x^{n-1} y + \dots + a_0 y^n \\
0 = Q(x, y, y) = y + b_n x^n + b_{n-1} x^{n-1} y + \dots + b_0 y^n
\end{cases}$$

and

(4.4)
$$O = J(x,y,\gamma,0) = P_xQ_y - Q_xP_y$$
 at $(x,y,\gamma,0)$.

Let (x_0, y_0, Y_0) be a solution of (4.3). If $J(x_0, y_0, Y_0, 0) \neq 0$, the implicit function theorem guarantees a unique solution $(x(\gamma, \xi), y(\gamma, \xi), \gamma, \xi)$ of (4.1) in a neighborhood of $(\gamma_0, 0)$ such that $x(\gamma_0, 0) = x_0$, and $y(\gamma_0, 0) = y_0$. $J(x, y, \gamma, \xi) \neq 0$ for (x, y, γ, ξ) near $(x_0, y_0, \gamma_0, 0)$. Thus the solution $(x_0, y_0, \gamma_0, 0)$ determines the behavior of nearby solutions of (4.3).

In the event that $J(x_0, y_0, \gamma_0, 0) = 0$, we wish to impose conditions on (4.1) and (4.2) so that we may still determine the local behavior of solutions near $(x_0, y_0, \gamma_0, 0)$:

(H10) We assume that for all (x_0, y_0, y_0) satisfying (4.3) with $(x_0, y_0, y_0) \neq (0,0,0)$, we have

rank
$$\begin{bmatrix} F_{\mathbf{x}} & F_{\mathbf{y}} & F_{\mathbf{y}} & F_{\mathbf{\xi}} \\ & & & \\ G_{\mathbf{x}} & G_{\mathbf{y}} & G_{\mathbf{y}} & G_{\mathbf{\xi}} \end{bmatrix} = 2 \text{ at } (\mathbf{x}_{0}, \mathbf{y}_{0}, \mathbf{y}_{0}, \mathbf{0}).$$

(HlO) guarantees that about each nontrivial solution $(x_0, y_0, \gamma_0, 0)$, there is a neighborhood of solutions to (4.1) that is homeomorphic to an open set in \mathbb{R}^2 . [The point (0,0,0,0) is a special case which corresponds to studying solutions which bifurcate from the trivial solution near the τ or η axes. This case is banished to section §6.]

(H11) We assume that for all (x_0, y_0, y_0) which satisfy both (4.3) and (4.4), the matrix

$$\begin{bmatrix} \mathbf{F}_{\mathbf{x}} & \mathbf{F}_{\mathbf{y}} & \mathbf{F}_{\mathbf{y}} & \mathbf{F}_{\mathbf{\xi}} \\ \mathbf{G}_{\mathbf{x}} & \mathbf{G}_{\mathbf{y}} & \mathbf{G}_{\mathbf{y}} & \mathbf{G}_{\mathbf{\xi}} \\ \mathbf{J}_{\mathbf{x}} & \mathbf{J}_{\mathbf{y}} & \mathbf{J}_{\mathbf{y}} & \mathbf{J}_{\mathbf{\xi}} \end{bmatrix}$$

has rank three at $(x_0, y_0, \gamma_0, 0)$.

A solution $(x_0, y_0, y_0, 0)$ of (4.1) and (4.2) for which either

(4.5) (a)
$$\det \begin{bmatrix} \mathbf{F}_{\mathbf{x}} & \mathbf{F}_{\mathbf{y}} & \mathbf{F}_{\mathbf{y}} \\ \mathbf{G}_{\mathbf{x}} & \mathbf{G}_{\mathbf{y}} & \mathbf{G}_{\mathbf{y}} \\ \mathbf{J}_{\mathbf{x}} & \mathbf{J}_{\mathbf{y}} & \mathbf{J}_{\mathbf{y}} \end{bmatrix} \neq 0$$

or

(4.5) (b)
$$\det \begin{bmatrix} \mathbf{F}_{\mathbf{x}} & \mathbf{F}_{\mathbf{y}} & \mathbf{F}_{\xi} \\ \mathbf{G}_{\mathbf{x}} & \mathbf{G}_{\mathbf{y}} & \mathbf{G}_{\xi} \\ \mathbf{J}_{\mathbf{x}} & \mathbf{J}_{\mathbf{y}} & \mathbf{J}_{\xi} \end{bmatrix} \neq 0$$

will be said to be of type I. When (4.5)(a) holds, these solutions have the pleasant feature that local behavior of nearby equations is completely determined by the polynomials (4.3) and (4.5), since (4.5)(a) is exactly the condition

(4.6)
$$\det \begin{bmatrix} \mathbf{P}_{\mathbf{x}} & \mathbf{P}_{\mathbf{y}} & \mathbf{x} \\ \mathbf{Q}_{\mathbf{x}} & \mathbf{Q}_{\mathbf{y}} & \mathbf{0} \\ \mathbf{J}_{\mathbf{x}} & \mathbf{J}_{\mathbf{y}} & \mathbf{Q}_{\mathbf{y}} \end{bmatrix} \neq \mathbf{0}$$

Note that (4.5)(a) implies

$$det \begin{bmatrix} G_{\mathbf{x}} & G_{\mathbf{y}} \\ J_{\mathbf{x}} & J_{\mathbf{y}} \end{bmatrix} \neq 0 \text{ and } \mathbf{x} \neq 0.$$

Theorem 4.7. If $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{y}_0)$ is a solution of (4.3) and (4.4), and (4.6) holds, then there is a curve $\gamma(\xi)$ and a solution curve $(\mathbf{x}(\xi), \mathbf{y}(\xi))$ such that $\gamma(0) = \gamma_0$, $(\mathbf{x}(0), \mathbf{y}(0)) = (\mathbf{x}_0, \mathbf{y}_0)$. When γ crosses $\gamma(\xi)$, the number

of solutions of (4.1) near $(x(\xi),y(\xi))$ changes by two. We will refer to $y(\xi)$ as a <u>fold curve</u>.

If (4.5) (a) fails and (4.5) (b) holds, we may prove a similar result. In this case the solution $(x_0, y_0, \gamma_0, 0)$ of (4.1) determines curves $(x(\gamma), y(\gamma)), \xi(\gamma)$ with $(x(\gamma_0), y(\gamma_0)) = (x_0, y_0)$ and $\xi(\gamma_0) = 0$. In the following we shall concentrate our attention on (4.5) (a), as results corresponding to (4.5) (b) are similar. We isolate a special case in the following lemma.

Lemma 4.8. If (x_0, y_0, γ_0) is of type I, the number of solutions (x, y, γ) of (4.3) near (x_0, y_0, γ_0) changes from zero to two as γ passes through γ_0 .

Proof. Without loss of generality, we will assume that

$$\det \begin{bmatrix} \mathbf{P}_{\mathbf{Y}} & \mathbf{x} \\ \mathbf{Q}_{\mathbf{Y}} & \mathbf{0} \end{bmatrix} \neq \mathbf{0}.$$

Thus P_{γ} and Q_{y} are nonzero at $(x_{0}, y_{0}, \gamma_{0})$, and we may use the implicit function theorem to guarantee the existence of $\gamma(x)$, $\gamma(x)$ for $\gamma(x)$, $\gamma(x)$ for $\gamma(x)$, such that $\gamma(x, \gamma(x))$ is a solution of (4.3) and $\gamma(x_{0}) = \gamma_{0}$ and $\gamma(x_{0}) = \gamma_{0}$. We shall show that γ may be written in the form

$$\gamma(x) = \gamma_0 + a_2(x - x_0)^2 + 0(x - x_0)^3$$

where $a_2 = \frac{1}{2} \gamma''(x_0)$

(i) Computation of $\gamma'(x_0)$. By the chain rule,

$$\begin{bmatrix} \mathbf{y'(x_0)} \\ \mathbf{\gamma'(x_0)} \end{bmatrix} = \frac{1}{\mathbf{P_y Q_\gamma - P_\gamma Q_y}} \begin{bmatrix} \mathbf{Q_\gamma} & -\mathbf{P_\gamma} \\ -\mathbf{Q_y} & \mathbf{P_y} \end{bmatrix} \begin{bmatrix} \mathbf{P_x} \\ \mathbf{Q_x} \end{bmatrix}$$

$$= \frac{1}{\mathbf{P_y Q_\gamma - P_\gamma Q_y}} \begin{bmatrix} \mathbf{Q_\gamma P_x} & -\mathbf{P_\gamma Q_x} \\ -\mathbf{Q_y P_x} & +\mathbf{P_y Q_x} \end{bmatrix}$$

$$= \frac{1}{\mathbf{P_y Q_\gamma - P_\gamma Q_y}} \begin{bmatrix} \mathbf{Q_\gamma P_x} & -\mathbf{P_\gamma Q_x} \\ -\mathbf{Q_y P_x} & +\mathbf{P_y Q_x} \end{bmatrix}$$

$$= \frac{1}{\mathbf{P_y Q_\gamma - P_\gamma Q_y}} \begin{bmatrix} \mathbf{Q_\gamma P_x} & -\mathbf{P_\gamma Q_x} \\ & & & \\ & & & \\ & & & \\ & & & & \\ \end{bmatrix} \quad \text{at} \quad \mathbf{x_0}.$$

Thus $\gamma'(x_0) = 0$.

(ii) Computation of $\gamma''(x_0)$. Further implicit differentiation plus part (i) gives

$$P_{xx} + 2P_{xy} \frac{dy}{dx} + P_{yy} \left(\frac{dy}{dx}\right)^2 + P_y \frac{d^2y}{dx^2} + P_y \frac{d^2y}{dx^2} = 0 \quad \text{at} \quad (x_0, y_0, y_0).$$

Also $\frac{d^2y}{dx^2} = \frac{1}{Q_y} [Q_{xx} - 2Q_{yx} \frac{dy}{dx} + Q_{yy} (\frac{dy}{dx})^2]$. Thus $\gamma''(x_0) = 0$ if and only if

$$(4.9) 0 = Q_y \left[P_{xx} + 2P_{xy} \frac{dy}{dx} + P_{yy} \left(\frac{dy}{dx} \right)^2 \right]$$

$$- P_y \left[Q_{xx} + 2Q_{xy} \frac{dy}{dx} + Q_{yy} \left(\frac{dy}{dx} \right)^2 \right].$$

By
$$(4.6)$$
,

$$\begin{aligned} &0 \neq J_{\mathbf{x}}Q_{\mathbf{y}} - J_{\mathbf{y}}Q_{\mathbf{x}} \\ &= [P_{\mathbf{x}\mathbf{x}}Q_{\mathbf{y}} + P_{\mathbf{x}}Q_{\mathbf{y}\mathbf{x}} - P_{\mathbf{y}\mathbf{x}}Q_{\mathbf{x}} - P_{\mathbf{y}}Q_{\mathbf{x}\mathbf{x}}]Q_{\mathbf{y}} \\ &- [P_{\mathbf{x}\mathbf{y}}Q_{\mathbf{y}} + P_{\mathbf{x}}Q_{\mathbf{y}\mathbf{y}} - P_{\mathbf{y}\mathbf{y}}Q_{\mathbf{x}} - P_{\mathbf{y}}Q_{\mathbf{x}\mathbf{y}}]Q_{\mathbf{x}} \\ &= P_{\mathbf{x}\mathbf{x}}Q_{\mathbf{y}}^{2} + 2Q_{\mathbf{x}\mathbf{y}}P_{\mathbf{y}}Q_{\mathbf{x}} - 2P_{\mathbf{y}\mathbf{x}}Q_{\mathbf{y}}Q_{\mathbf{y}} - Q_{\mathbf{x}\mathbf{x}}P_{\mathbf{y}}Q_{\mathbf{y}} \\ &+ P_{\mathbf{y}\mathbf{y}}Q_{\mathbf{x}}^{2} - Q_{\mathbf{y}\mathbf{y}} \frac{P_{\mathbf{y}}Q_{\mathbf{x}}^{2}}{Q_{\mathbf{y}}} \\ &= Q_{\mathbf{y}}\{Q_{\mathbf{y}}[P_{\mathbf{x}\mathbf{x}} - 2P_{\mathbf{x}\mathbf{y}}(\frac{Q_{\mathbf{x}}}{Q_{\mathbf{y}}}) + P_{\mathbf{y}\mathbf{y}}(\frac{Q_{\mathbf{x}}}{Q_{\mathbf{y}}})^{2}] \\ &- P_{\mathbf{y}}[Q_{\mathbf{x}\mathbf{x}} - 2Q_{\mathbf{x}\mathbf{y}}(\frac{Q_{\mathbf{x}}}{Q_{\mathbf{y}}}) + Q_{\mathbf{y}\mathbf{y}}(\frac{Q_{\mathbf{x}}}{Q_{\mathbf{y}}})^{2}] \} \\ &= Q_{\mathbf{y}}\{Q_{\mathbf{y}}[P_{\mathbf{x}\mathbf{x}} + 2P_{\mathbf{x}\mathbf{y}}\frac{d\mathbf{y}}{d\mathbf{x}} + P_{\mathbf{y}\mathbf{y}}(\frac{d\mathbf{y}}{d\mathbf{x}})^{2}] \} \\ &- P_{\mathbf{y}}[Q_{\mathbf{x}\mathbf{x}} + 2Q_{\mathbf{x}\mathbf{y}}\frac{d\mathbf{y}}{d\mathbf{x}} + Q_{\mathbf{y}\mathbf{y}}(\frac{d\mathbf{y}}{d\mathbf{x}})^{2}] \}. \end{aligned}$$

Thus $\gamma''(x_0) \neq 0.\square$

Proof of Theorem 4.7. Once again, we assume

$$\det \begin{bmatrix} \mathbf{P}_{\mathbf{Y}} & \mathbf{x} \\ \mathbf{Q}_{\mathbf{Y}} & \mathbf{0} \end{bmatrix} \neq \mathbf{0}.$$

Since (4.6) holds, there is a unique curve $(x(\xi), y(\xi), \gamma(\xi), \xi)$ of solutions of (4.1) and (4.2) with $x(0) = x_0$, $y(0) = y_0$, $\gamma(0) = \gamma_0$. For $\bar{\xi}$ sufficiently close to zero, (4.6) still holds. Moreover,

$$\det \begin{bmatrix} \mathbf{F}_{\mathbf{Y}} & \mathbf{F}_{\mathbf{Y}} \\ \mathbf{G}_{\mathbf{Y}} & \mathbf{G}_{\mathbf{Y}} \end{bmatrix} = \mathbf{0}$$

at $(x(\overline{\xi}), y(\overline{\xi}), \gamma(\overline{\xi}), \overline{\xi})$. Fix $\overline{\xi}$ and consider the functions

(4.10)
$$\begin{cases} \overline{P}(x,y,\gamma) = F(x,y,\gamma,\overline{\xi}) \\ \overline{Q}(x,y,\gamma) = G(x,y,\gamma,\overline{\xi}) \\ \overline{J}(x,y,\gamma) = J(x,y,\gamma,\overline{\xi}) \end{cases}$$

(4.10) satisfies

$$(4.11) \quad \det \begin{bmatrix} \overline{P}_{\mathbf{x}} & \overline{P}_{\mathbf{y}} & \overline{P}_{\mathbf{y}} \\ \overline{Q}_{\mathbf{x}} & \overline{Q}_{\mathbf{y}} & \overline{Q}_{\mathbf{y}} \end{bmatrix} \neq 0 \quad \text{at} \quad (\mathbf{x}(\overline{\xi}), \mathbf{y}(\overline{\xi}), \mathbf{y}(\overline{\xi})).$$

We now do some calculations similar to those in Lemma 4.8 and use (4.11) in place of $J_{\mathbf{x}}Q_{\mathbf{y}} - J_{\mathbf{y}}Q_{\mathbf{x}} \neq 0$ to reach the desired conclusion. Thus for fixed $\bar{\xi}$ close to zero, $\gamma(\mathbf{x},\bar{\xi})$ has the form

$$\gamma(x, \overline{\xi}) = \gamma(\overline{\xi}) + a_2(x - x(\overline{\xi}))^2 + O(x - x(\overline{\xi}))^3$$

where a_2 depends continuously on $\bar{\xi}$ and hence has constant sign for ξ near 0. Thus there are two solutions of (4.1) near $(x(\xi),y(\xi))$ on one side of $\gamma(\xi)$ and no nearby solutions on the other.

We have the following local picture when (γ, ξ) is near $(\gamma_0, 0)$ [Figures 5 and 6].

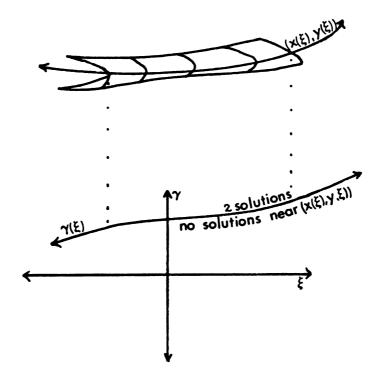


Figure 5. Behavior near a fold curve in the (ξ, γ) -plane.

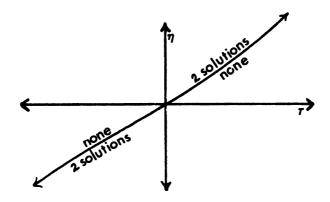


Figure 6. Behavior near a fold curve $\text{in the } (\tau,\eta)\text{-plane.}$

These pictures are local representations. It is entirely possible to have other solutions (x_1,y_1) that correspond to γ_0 . However, the solutions near (x_1,y_1) will correspond to a different branch which does not intersect the branch determined by (x_0,y_0) in some neighborhood of $(\gamma_0,0)$.

§5. Solutions of Type II

Let (H10) - (H11) hold. Solutions $(x_0, y_0, \gamma_0, 0)$ of (4.1) which satisfy neither (4.5)(a) nor (4.5)(b) will be called solutions of type II. In general, the local behavior near solutions of type II is not as easy to determine as that near a solution of type I. However, we can give a complete description of this behavior when the original bifurcation equations contain a nondegenerate bilinear form when $\tau = \eta = 0$. We shall determine some general properties of solutions of type II.

If $(x_0, y_0, \gamma_0, 0)$ is a solution of type II, we must have

(5.1)
$$\det \begin{bmatrix} P_{\mathbf{x}} & P_{\mathbf{y}} \\ Q_{\mathbf{x}} & Q_{\mathbf{y}} \end{bmatrix} = 0 \text{ at } (\mathbf{x}_{0}, \mathbf{y}_{0}, \mathbf{y}_{0}).$$

Proposition 5.1. If $P_x = P_y = Q_x = Q_y = 0$ at a solution (x_0, y_0, y_0) of (4.3), then either $(x_0, y_0, y_0) = (0,0,0)$ or the n-linear form is degenerate.

Proof. At (x_0, y_0, γ_0) , we have

$$0 = x_0^p + y_0^p = y_0^n + n[a_n^n x_0^n + a_{n-1}^n x_0^{n-1} y_0 + \dots + a_0^n y_0^n]$$

$$0 = x_0^p + y_0^p = y_0^n + n[b_n^n x_0^n + b_{n-1}^n x_0^{n-1} y_0^n + \dots + b_0^n y_0^n]$$

Since $P(\mathbf{x}_0, \mathbf{y}_0, \mathbf{y}_0) = Q(\mathbf{x}_0, \mathbf{y}_0, \mathbf{y}_0) = 0$, we have $(n-1)\mathbf{y}_0 = 0$ and $(n-1)\mathbf{y}_0\mathbf{x}_0 = 0$. Thus $\mathbf{y}_0 = 0$. If $\mathbf{y}_0 = 0$ and $\mathbf{x}_0 \neq 0$, then $\mathbf{a}_n = \mathbf{b}_n = 0$ and a degeneracy occurs in the n-linear form. If $\mathbf{x}_0 = 0$, $\mathbf{p}_{\mathbf{x}} = 0$ implies $\mathbf{y}_0 = 0$.

Recall that (0,0,0,0) is not considered in (H10) - (H11), and therefore cannot be a solution of type II. Before we go on, we shall consider two special cases.

Case 1. $x_0 = 0$. If $a_0 \neq 0$ in (4.1), then y = 0 and $J(0,0,\gamma_0,0) = \gamma_0$ cannot be zero unless $\gamma_0 = 0$. If $a_0 = 0$, then

$$y_0 = (-1/b_0)^{\frac{1}{n-1}}$$
.

[If b_0 is also zero, the n-linear form M_n is degenerate]. Suppose y_0 is real. Then the matrix in (Hll) becomes

$$\begin{bmatrix}
\gamma_{0} - a_{1}/b_{0} & 0 & 0 & N_{1}(0, y_{0}, \gamma_{0}, 0) \\
-b_{1}/b_{0} & -(n-1) & 0 & N_{2}(0, y_{0}, \gamma_{0}, 0) \\
J_{x} & J_{y} & -(n-1) & J_{\xi}
\end{bmatrix}$$

Now $J(0, y_0, y_0, 0) = 0$ if and only if $y_0 = a_1/b_0$. Case 1 can only occur if $a_0 = 0$ and $b_0 \neq 0$. In this case

(HlO) - (Hll) are satisfied if and only if $N_1(0, y_0, y_0, 0) \neq 0$. Then

$$\det \begin{bmatrix}
0 & 0 & N_{1}(0, Y_{0}, Y_{0}, 0) \\
(1-n) & 0 & N_{2}(0, Y_{0}, Y_{0}, 0)
\end{bmatrix} \neq 0$$

$$J_{y} \qquad (1-n) \qquad J_{\xi}$$

and

det
$$\begin{bmatrix} 0 & N_1(0, y_0, \gamma_0, 0) \\ (1-n) & N_2(0, y_0, \gamma_0, 0) \end{bmatrix} \neq 0.$$

Case 2. $Q_x = Q_y = 0$ at $(x_0, y_0, \gamma_0) \neq (0,0,0)$. As in the proof of proposition 5.1, we have that $y_0 = 0$. If $b_n \neq 0$, then x = 0 and $J(0,0,\gamma_0,0) = \gamma_0$. Thus as before, we assume $b_n = 0$ and $a_n \neq 0$. Then $(x_0, y_0, \gamma_0) \neq (0,0,0)$ implies

$$x_0 = (-\gamma_0/a_n)^{\frac{1}{n-1}}$$
 with $\gamma_0 \neq 0$.

The matrix of (Hll) becomes

(5.3)

$$\begin{bmatrix} \gamma_0^{-n}\gamma_0 & -a_{n-1}\gamma_0/a_n & x_0 & N_1(x_0,0,\gamma_0,0) \\ 0 & 1-\gamma_0b_{n-1}/a_n & 0 & N_2(x_0,0,\gamma_0,0) \\ -\gamma_0(n-1)^2b_{n-1}x_0^{n-2} & J_y & 1-\gamma_0b_{n-1}/a_n & J_\xi \end{bmatrix}$$

 $Q_y = 0$ implies $y_0 = a_n/b_{n-1}$ and $b_{n-1} \neq 0$. Thus Case 2 occurs only when $b_n = 0$, $a_n \neq 0$ and $b_n - 1 \neq 0$. In order

for (H10) - (H11) to hold, we must have $N_2(x_0, 0, \gamma_0, 0) \neq 0$, $x_0 \neq 0$, and $\gamma_0 \neq 0$.

Example 5.1. Case 2 may be reduced to Case 1 by a change of variables. We shall demonstrate this in the case where M_n is a nondegenerate bilinear form. If we recall the form of the nonlinear term before the scaling of §3, we may write

(5.4)
$$\begin{cases} 0 = \gamma x + a_2 x^2 + a_1 xy + a_0 y^2 + N_1 (x, y, \gamma \xi, \xi) \\ 0 = y + b_1 xy + b_0 y^2 + N_2 (x, y, \gamma \xi, \xi) \end{cases}$$

Then

(5.5)
$$x = -\frac{1}{b_1}$$
 $y = a_2/b_1$, $y = 0$ is a solution.

Now consider

(5.6)
$$\begin{cases} o = \rho u + b_0 u^2 + b_1 vu + N_1 (v, u, \sigma, \rho \sigma) \\ o = v + a_0 v^2 + a_1 uv + a_2 u^2 + N_2 (v, u, \sigma, \rho \sigma) \end{cases}$$

If (x_0, y_0, y_0, ξ_0) is a solution of (5.4) with $\gamma \neq 0$, then $u = y/\gamma$, $v = x/\gamma$, $\rho = 1/\gamma$, $\sigma = \gamma \xi$ is a solution of (5.6). The solution (5.7) becomes

$$u = 0$$
, $v = -1/a_2$, $\rho = b_1/a_2$.

This correspondence arises from interchanging the roles of τ and η in the scalings discussed in § 3.

Proposition 5.2. Let $(x_0, y_0, y_0, 0)$ be a solution of type II with $x_0 \neq 0$. If $Q_x = 0$ and $Q_y \neq 0$, then $P_x = J_x = 0$. If $Q_x \neq 0$ and $Q_y = 0$, then $P_y = J_y = 0$.

Proof. $P_xQ_y - Q_xP_y = 0$ and (4.5)(a) fails if and only if $J_xQ_y - J_yQ_x = 0$.

Since (4.5)(a) and (4.5)(b) both fail for a solution of type II, we have

(5.7)
$$\det \begin{bmatrix} \mathbf{P}_{\mathbf{x}} & \mathbf{x} & \mathbf{F}_{\xi} \\ \mathbf{Q}_{\mathbf{x}} & \mathbf{0} & \mathbf{G}_{\xi} \\ \mathbf{J}_{\mathbf{x}} & \mathbf{J}_{\gamma} & \mathbf{J}_{\xi} \end{bmatrix} \neq \mathbf{0}$$

or

(5.8)
$$\det \begin{bmatrix} \mathbf{P}_{\mathbf{Y}} & \mathbf{x} & \mathbf{F}_{\xi} \\ \mathbf{Q}_{\mathbf{Y}} & \mathbf{0} & \mathbf{G}_{\xi} \\ \mathbf{J}_{\mathbf{Y}} & \mathbf{J}_{\mathbf{Y}} & \mathbf{J}_{\xi} \end{bmatrix} \neq \mathbf{0}.$$

Proposition 5.4. Let $(x_0, y_0, \gamma_0, 0)$ be a solution of type II. Then one of the following occurs.

(a) (5.7) holds and

$$\det \begin{bmatrix} \mathbf{P}_{\mathbf{X}} & \mathbf{F}_{\xi} \\ \mathbf{Q}_{\mathbf{X}} & \mathbf{G}_{\xi} \end{bmatrix} \neq \mathbf{0} \quad \text{or} \quad \det \begin{bmatrix} \mathbf{P}_{\mathbf{X}} & \mathbf{x} \\ \mathbf{Q}_{\mathbf{X}} & \mathbf{0} \end{bmatrix} \neq \mathbf{0}$$

(b) (5.8) holds and

$$\det \begin{bmatrix} P_{y} & F_{\xi} \\ Q_{y} & G_{\xi} \end{bmatrix} \neq 0 \quad \text{or} \quad \det \begin{bmatrix} P_{y} & x \\ Q_{y} & 0 \end{bmatrix} \neq 0.$$

Proof. This follows from Case 1, Case 2 and proposition (5.2).□

Definition 5.1. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $(u,v) \to (f(u,v),v)$. We say that (u_O,v_O) is a cusp point for T if

(i)
$$\frac{\partial f}{\partial u}(u_0, v_0) = 0$$
 and $\frac{\partial^2 f}{\partial u^2}(u_0, v_0) = 0$

(ii)
$$\frac{\partial^2 f}{\partial u \partial v}(u_0, v_0) \neq 0$$

(iii)
$$\frac{\partial^3 f}{\partial u^3}(u_0, v_0) \neq 0.$$

Proposition 5.4. Let $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{y}_0, \mathbf{0})$ be a solution of type II. Then

- (a) The implicit function theorem enables us to solve (4.1) for at least one of the following in a neighborhood of $(x_0, y_0, y_0, 0)$: $\gamma(x, \xi)$, $\gamma(y, \xi)$, $\xi(x, \gamma)$, $\xi(y, \gamma)$.
- (b) At least one of the following maps exists and also satisfies (i) and (ii) of definition (5.1) at $(x_0, y_0, y_0, y_0, 0)$

$$(x, \xi) \rightarrow (\gamma(x, \xi), \xi)$$

$$(y,\xi) \rightarrow (\gamma(y,\xi),\xi)$$

$$(x, \gamma) \rightarrow (\xi(x, \gamma), \gamma)$$

$$(y, \gamma) \rightarrow (\xi(y, \gamma), \gamma).$$

Proof. We will only give the proof in the case that

(5.9)
$$\det \begin{bmatrix} P_{y} & x & F_{\xi} \\ Q_{y} & O & G_{\xi} \\ J_{y} & J_{\gamma} & J_{\xi} \end{bmatrix} \neq 0$$

$$\det \begin{bmatrix} Py & x \\ Q_y & 0 \end{bmatrix} \neq 0 \text{ at } (x_0, y_0, y_0, 0).$$

All other cases are similar.

Since $x_0Q_y \neq 0$, there exist unique real analytic functions $\gamma(x,\xi)$ and $\gamma(x,\xi)$ such that $(x,y(x,\xi),\gamma(x,\xi),\xi)$ satisfies (4.1) near $(x_0,0)$ and $\gamma(x_0,0)=0$, $\gamma(x_0,0)=0$. Recall that $x_0 \neq 0$ implies $Q_xJ_y-Q_yJ_x=0$ for a solution of type II. As in Lemma 4.8, we have

(a)
$$\frac{\partial Y}{\partial x}(x_0, 0) = 0$$
 and $\frac{\partial Y}{\partial x}(x_0, 0) = -Q_x/Q_y$

(b)
$$\frac{\partial^2 y}{\partial x^2}(x_0, 0) = 0$$
 if and only if $Q_x J_y - Q_y J_x = 0$

$$\begin{split} (c) \quad & \frac{\gamma^2 \mathbf{x}}{\partial \mathbf{x} \partial \xi}(\mathbf{x}_0, 0) \ = \ & \frac{\mathbf{J}_{\mathbf{y}} \frac{\partial \mathbf{y}}{\partial \xi} + \mathbf{J}_{\mathbf{y}} \frac{\partial \mathbf{y}}{\partial \xi} + \mathbf{J}_{\xi}}{-\mathbf{x}_0 Q_{\mathbf{y}}} \\ \\ & = \frac{\mathbf{J}_{\mathbf{y}}[\mathbf{x}_0 G_{\xi}] - \mathbf{J}_{\mathbf{y}}[\mathbf{P}_{\mathbf{y}} G_{\xi} - \mathbf{F}_{\xi} Q_{\mathbf{y}}] + \mathbf{J}_{\xi}[\mathbf{x}_0 Q_{\mathbf{y}}]}{\left(\mathbf{x}_0 Q_{\mathbf{y}}\right)^2} \ . \end{split}$$

Thus (5.9) implies $\frac{\partial^2 y}{\partial x \partial \xi}(x_0, 0) \neq 0.\square$

Definition 5.2. If $(x_0, y_0, \gamma_0, 0)$ is a solution of type I, we will call $(x_0, y_0, \gamma_0, 0)$ a <u>fold point</u>. If $(x_0, y_0, \gamma_0, 0)$ is a solution of type II, and all conditions of definition (5.1) are satisfied for one of the maps guaranteed by proposition 5.4, we will call $(x_0, y_0, v_0, 0)$ a <u>cusp point</u>.

In general, we shall not be able to verify the third condition of definition (5.1), as it depends on the order of contact between the two curves $P(x,y,\gamma_0)=0$ and $Q(x,y,\gamma_0)=0$ at (x_0,y_0) . On the intuition that two conics cannot "touch too much" without some form of degeneracy, we now specialize to the case of a nondegenerate bilinear form

$$M_{2}[(x,y),(x,y)] = \begin{cases} Ax^{2} + Bxy + Cy \\ ax^{2} + bxy + cy \end{cases}$$

where A, B, and C are not all zero. Consider the system,

$$(5.10)\begin{cases} 0 = F(x, y, \gamma, \xi) = \gamma x + Ax^{2} + Bxy + Cy^{2} + \xi N_{1}(x, y, \gamma, \xi) \\ \\ 0 = G(x, y, \gamma, \xi) = y + ax^{2} + bxy + cy^{2} + \xi N_{2}(x, y, \gamma, \xi) \end{cases}$$

As an example of what to expect, we will look at Case 1. In this case C = 0, $c \neq 0$ and $(x_0, y_0, y_0) = (0, -1/c, B/c)$. The matrix (5.2) becomes

$$\begin{bmatrix} 0 & 0 & 0 & N_{1}(0,-1/c,B/c,0) \\ -b/c & -1 & 0 & N_{2}(0,-1/c,B/c,0) \\ \\ -2A + Bb/c & -B & -1 & J_{\xi} \end{bmatrix}$$

For a type II solution, we must have -A + Bb/c = 0.

By the implicit function we may solve (5.10) for $\xi(x,\gamma)$ and $y(x,\gamma)$. If we expand $\xi(x,\gamma)$, $y(x,\gamma)$ and the equations in (5.10) in power series and then equate coefficients, we find that

$$(5.11) \qquad \frac{\partial \mathbf{x}}{\partial \mathbf{x}}(\mathbf{x}_0, \mathbf{y}_0) = - \mathbf{Q}_{\mathbf{x}}/\mathbf{Q}_{\mathbf{y}}$$

$$(5.12) \quad \frac{\partial^{2} y}{\partial x^{2}}(x_{0}, y_{0}) = -Q_{y}^{-1}[Q_{xx} + Q_{xy} \frac{\partial y}{\partial x} + Q_{yy}(\frac{\partial y}{\partial x})^{2}]$$
$$= -2Q_{y}[a + b \frac{\partial y}{\partial x} + c(\frac{\partial y}{\partial x})^{2}]$$

(5.13)
$$\frac{\partial^2 \xi}{\partial x^2}(x_0, y_0) = 0 \quad \text{if and only if} \quad P_x - P_y Q_x/Q_y = 0$$

$$(5.14) \quad \frac{\partial^{2} \xi}{\partial x^{2}}(x_{O}, \gamma_{O}) = 0 \quad \text{if and only if}$$

$$0 = \left[P_{xx} + P_{xy} \frac{\partial y}{\partial x} + P_{yy} \left(\frac{\partial y}{\partial x}\right)^{2}\right]$$

$$- P_{y}Q_{y}^{-1}\left[Q_{xx} + Q_{xy} \frac{\partial y}{\partial x} + Q_{yy} \left(\frac{\partial y}{\partial x}\right)^{2}\right]$$

$$= 2\left[A + B \frac{\partial y}{\partial x} + C \left(\frac{\partial y}{\partial x}\right)^{2}\right] - 2P_{y}Q_{y}^{-1}\left[a + b \frac{\partial y}{\partial x} + c \left(\frac{\partial y}{\partial x}\right)^{2}\right]$$

$$(5.15) \quad \frac{\partial^{3} \xi}{\partial x^{3}}(x_{Q}, \gamma_{Q}) = 0 \quad \text{if and only if}$$

$$O = \left[P_{xy} + P_{yy}\left(\frac{-Q_{x}}{Q_{y}}\right)\right] \frac{\partial^{2} y}{\partial x^{2}} - P_{y}Q_{y}^{-1}\left[Q_{xy} + Q_{yy}\left(\frac{-Q_{x}}{Q_{y}}\right)\right] \frac{\partial y^{2}}{\partial x^{2}}$$

if and only if, $J_y = 0$ or $\frac{\partial^2 y}{\partial x^2} = 0$. However, $\frac{\partial^2 y}{\partial x^2}(x_0, y_0) = 0$ implies that $(x, y) = (1, \frac{dy}{dx})$ is a nontrivial solution of

		ĺ

$$\begin{cases} 0 = ax^2 + bxy + cy^2 \\ 0 = Ax^2 + Bxy + Cy^2 \end{cases}$$

Since the bilinear form M_2 is nondegenerate, this implies $J_y = 0$. Since $Q_y \neq 0$, we must have $J_x = 0$.

This gives the following

$$-B = 0$$

$$-2A + B(b/c) = 0$$
and
$$-A + Bb/c = 0$$

Hence $A = O = B.\square$

Because of example 5.1, we may neglect Case 2. It remains to examine the situation where $x_0 \neq 0$, and one of Q_x or Q_y is nonzero. By proposition 5.3, one of the following must hold

$$(5.16) \quad \det \begin{bmatrix} \mathbf{P}_{\mathbf{Y}} & \mathbf{x} & \mathbf{F}_{\mathbf{\xi}} \\ \mathbf{Q}_{\mathbf{Y}} & \mathbf{O} & \mathbf{G}_{\mathbf{\xi}} \\ \mathbf{J}_{\mathbf{Y}} & \mathbf{J}_{\mathbf{Y}} & \mathbf{J}_{\mathbf{\xi}} \end{bmatrix} \neq 0 \quad \text{and} \quad \det \begin{bmatrix} \mathbf{P}_{\mathbf{Y}} & \mathbf{x} \\ \mathbf{Q}_{\mathbf{Y}} & \mathbf{O} \end{bmatrix} \neq 0$$

or

$$(5.17) \quad \det \begin{bmatrix} P_{\mathbf{x}} & \mathbf{x} & F_{\xi} \\ Q_{\mathbf{x}} & O & G_{\xi} \\ J_{\mathbf{x}} & J_{\gamma} & J_{\xi} \end{bmatrix} \neq 0 \quad \text{and} \quad \det \begin{bmatrix} P_{\mathbf{x}} & \mathbf{x} \\ Q_{\mathbf{x}} & O \end{bmatrix} \neq 0$$

at
$$(x_0, y_0, y_0, 0)$$
.

We shall only consider (5.16), as essentially the same result holds for (5.17). We use the implicit function theorem to obtain $\gamma(\mathbf{x}, \xi)$ and $\gamma(\mathbf{x}, \xi)$ such that $(\mathbf{x}, \gamma(\mathbf{x}, \xi), \gamma(\mathbf{x}, \xi), \xi)$ is a solution of (5.10) in a neighborhood of $(\mathbf{x}_0, \gamma_0, \gamma_0, 0)$, and $\gamma(\mathbf{x}_0, 0) = \gamma_0$, $\gamma(\mathbf{x}_0, 0) = \gamma_0$. The analogues of (5.11) - (5.15) now hold, and if $\frac{\partial^2 \gamma}{\partial \mathbf{x}}(\mathbf{x}_0, 0) = 0$, the system

(5.18)
$$\begin{cases} 0 = P(x, y, \gamma) = \gamma x + Ax^{2} + Bxy + Cy^{2} \\ 0 = Q(x, y, \gamma) = y + ax^{2} + bxy + cy^{2} \end{cases}$$

must satisfy $J(x_0, y_0, y_0) = P_xQ_y - Q_xP_y = 0$, $J_x = J_y = 0$ at (x_0, y_0, y_0) . We will take care of this situation with the following lemma.

Lemma 5.5. Given A, B, C not all zero, there does not exist $(x,y,\gamma) \neq (0,0,0)$, $x \neq 0$ and a, b, c such that M_2 is nondegenerate and the system in (5.18) satisfies $J(x_0,y_0,\gamma_0) = J_y = J_x = 0$.

Proof. We must satisfy the equations

(i)
$$O = \gamma x + Ax^2 + Bxy + Cy^2$$

(ii)
$$0 = y + ax^2 + bxy + cy^2$$

(iii)
$$O = J(x,y,Y) = \gamma + \gamma bx + 2\gamma cy + 2Ax + 2Ax^2 + 4Acxy + By + 2Bcy^2 - 2aBx^2 - 4aCxy - 2bCy^2$$

(iv)
$$O = J_x = \gamma b + 2A + 4Abx + 4Acy - 4aBx - 4aCy$$

(v)
$$O = J_y = 2\gamma c + 4Acx + B + 4Bcy - 4aCx - 4bCy$$

We may use the fact that $xJ_x + yJ_y = 0$ to replace (iii) by

(vi)
$$O = \gamma - 2Abx^2 - 4Acxy - 2Bcy^2 + 2aBx^2 + 4aCxy + 2bCy^2$$
.

Since $x \neq 0$, we may multiply equations (iv), (v) and (vi) by x and use (i) to eliminate γ . We now have four equations which are linear in a, b, and c,

$$\begin{cases}
0 = ax^{2} + bxy + cy^{2} + y \\
0 = a(2Bx^{3} + 4Cx^{2}y) + b(-2Ax^{3} + 2Cy^{2}x) + c(4Ax^{2}y - 2By^{2}x) \\
+ (-Ax^{2} - Bxy - Cy^{2})
\end{cases}$$

$$0 = a(-4Bx^{2} - 4Cxy) + b(-Ax^{2} - Bxy - Cy^{2} + 4Ax^{2}) + c(4Axy) + 2Ax$$

$$0 = a(-4Cx^{2}) + b(-4Cxy) + c(2Ax^{2} + 2Bxy - 2Cy^{2}) + Bx.$$

This system may be reduced by the usual row operations to the following

$$\begin{cases}
0 = ax^{2} + bxy + cy^{2} + y \\
0 = b(Ax^{3} + Bx^{2}y + Cxy^{2}) + Ax^{2} + Bxy - Cy^{2} \\
0 = b(3Ax^{3} + 3Bx^{2}y + 3Cxy^{2}) + c(4Ax^{2}y + 4By^{2}x + 4Cy^{3}) \\
+ 2Ax^{2} + 4Byx + 4Cy^{2} \\
0 = c(2Ax^{2}y + 2Bxy + 2Cy^{2}) + Bx + 4Cy.
\end{cases}$$

An appropriate linear combination of the last three equations shows that

$$0 = 3[Ax^{2} + Bxy - Cy^{2}] + 2y[Bx + 4Cy]$$
$$- [2Ax^{2} + 4Bxy + 4Cy^{2}]$$
$$= Ax^{2} + Bxy + Cy^{2}.$$

Now (5.19) implies

$$0 = Ax^{2} + Bxy - Cy^{2}$$

$$0 = Ax^{2} + 2Bxy + 2Cy^{2}$$

$$0 = Bx + 4Cy.$$

Thus Cy = 0 = Bx = Ax. Since $x \neq 0$, we have A = B = 0. If $C \neq 0$, we must have y = 0. Then (ii) implies a = 0 and

$$M_{2}[(x,y)(x,y)] = \begin{cases} cy^{2} \\ bxy + cy^{2} \end{cases}$$

has the nontrivial solution (x,0), and therefore is degenerate.

We have just proved the following:

Theorem 5.6. If the bilinear form

$$M_{2}[(x,y),(x,y)] = \begin{cases} Ax^{2} + Bxy + Cy^{2} \\ ax^{2} + bxy + cy^{2} \end{cases}$$

is nondegenerate and A, B, and C are not all zero, then all solutions $(x_0, y_0, y_0, 0) \neq (0,0,0,0)$ of

$$\begin{cases}
0 = \gamma x + Ax^{2} + Bxy + Cy^{2} + \xi N_{1}(x, y, \gamma, \xi) \\
0 = y + ax^{2} + bxy + cy^{2} + \xi N_{2}(x, y, \gamma, \xi) \\
0 = J(x, y, \gamma, \xi)
\end{cases}$$

are either cusp points or fold points, provided that (HlO) - (Hll) are satisfied.

We shall use the following example to demonstrate behavior near a cusp point.

Example 5.2. Consider the system

$$\begin{cases} (5.20) (a) & 0 = \gamma x + xy + \tau N_1(x, y, \gamma, \tau) \\ \\ (5.20) (b) & 0 = y + x^2 + y^2 + \tau N_2(x, y, \gamma, \tau) \end{cases}$$

where $N_1(0,-1,1,0) \neq 0$ At the solution (0,-1,1,0), the matrix corresponding to (5.2) is given by

$$\begin{bmatrix} 0 & 0 & 0 & N_{1}(0,-1,1,0) \\ 0 & -1 & 0 & N_{2}(0,-1,1,0) \\ 0 & -1 & -1 & J_{\xi} \end{bmatrix}$$

and has rank three. By theorem 5.6, (0,-1,1,0) is a cusp point. If we solve (5.16) (b) for $y(x,y,\tau)$, we obtain

$$y(x, y, \tau) = -1 + x^2 + higher order terms.$$

Substitution into (5.20)(a) gives

(5.21)
$$0 = (\gamma - 1)x + x^{3} + \tau N_{1}(x, y(x, \gamma, \tau), \gamma, \tau).$$

We make the change of variables $\bar{\gamma} = -(\gamma - 1)$ $\bar{\tau} = -\tau N_1(x, y(x, \gamma, \tau), \gamma, \tau)$ in a neighborhood of (0, -1, 1, 0). Since $N_1(0, -1, 1, 0)$ is nonzero, this change of variables is a homeomorphism by the inverse mapping theorem. Now (5.21) becomes

$$\bar{\tau} = y^3 - \bar{\gamma}y$$
.

The mapping $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$(5.22) \qquad \begin{bmatrix} x \\ \overline{y} \end{bmatrix} \rightarrow \begin{bmatrix} x^3 - \overline{y}x \\ \overline{y} \end{bmatrix} = \begin{bmatrix} \overline{\tau} \\ \overline{y} \end{bmatrix}$$

is the standard form for the cusp singularity [3]. We have the following local picture, [Figure 7]

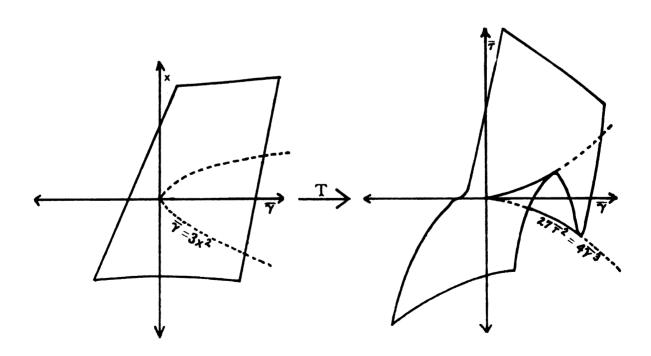


Figure 7. Behavior near a cusp.

The number of real solutions (x,y) corresponding to each (γ,τ) in a neighborhood of (1,0) is given schematically by the following bifurcation diagram [Figure 8]:

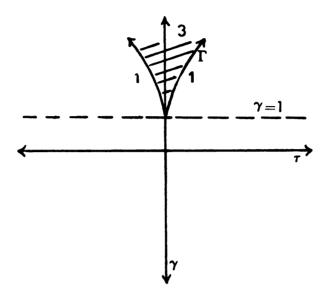


Figure 8. Bifurcation diagram in the (τ, γ) -plane.

If we change back to the (τ,η) -plane by setting $\eta=\gamma\tau$, the corresponding diagram is given by Figure 9. The curve Γ in Figure 8 has become a curve $\overline{\Gamma}$ tangent to the line $\eta=\tau$.

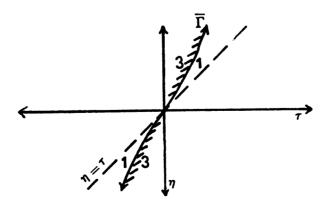


Figure 9. Bifurcation diagram in the (τ, η) -plane.

Note that this is only a local picture, and is valid only in a neighborhood of the curve Γ . The picture can be completed only when we have analyzed the behavior of solutions near every $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{y}_0, \mathbf{y}_0)$ which is a solution of

$$\begin{cases}
0 = yx + yx \\
0 = y + x^2 + y^2
\end{cases}$$

In general, if $T: \mathbb{R}^2 \to \mathbb{R}^2$ satisfies definition (5.1), there are local changes of coordinates in the domain and in the range of T such that the local picture is given by Figure 7 and T has the form (5.22) [3].

The hypotheses (H10) - (H11) were chosen to insure that the set of solutions (x,y,γ,τ) of (4.3) is a two manifold in the neighborhood of $(x_0,y_0,\gamma_0,0) \neq (0,0,0,0)$. That is, there is only one solution branch passing through $(x_0,y_0,\gamma_0,0)$. As we shall see in Chapter IV, this does not always happen in applications. If two solution branches pass through $(x_0,y_0,\gamma_0,0)$, both hypotheses (H10) and (H11) of §4 are violated. In this case, it is necessary to find a way to "factor out" a branch so that the methods of §4 and §5 apply. One is then faced with determining how the branches intersect. We shall demonstrate how this may be done in an application to chemical reaction equations discussed in Chapter IV.

§6. Solutions near $(x_0, y_0, Y_0, 0) = (0,0,0,0)$

In this section, we shall informally discuss some ways of determining the behavior of solutions of (4.1) near (0,0,0,0). At (0,0,0,0), the matrix

$$\begin{bmatrix} \mathbf{F}_{\mathbf{x}} & \mathbf{F}_{\mathbf{y}} & \mathbf{F}_{\mathbf{y}} & \mathbf{F}_{\mathbf{\xi}} \\ \mathbf{G}_{\mathbf{x}} & \mathbf{G}_{\mathbf{y}} & \mathbf{G}_{\mathbf{y}} & \mathbf{G}_{\mathbf{\xi}} \end{bmatrix} = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{1} & \mathbf{O} & \mathbf{O} \end{bmatrix}$$

and does not have rank two. Thus the hypotheses (H9) - (H10) of §4 do not apply. The system

$$\begin{cases} (6.1) (a) & 0 = F(x,y,\gamma,\xi) = \gamma x + a_n x^n + a_{n-1} x^{n-1} y + \dots + a_0 y^n \\ & + \xi N_1(x,y,\gamma,\xi) \end{cases}$$

$$(6.1) (b) & 0 = G(x,y,\gamma,\xi) = y + b_n x^n + b_{n-1} x^{n-1} y + \dots + b_0 y^n + \xi N_2(x,y,\gamma,\xi)$$

is similar to the 1-dimensional case of Chapter II as γ passes through O at ξ = O, except for the extra parameter ξ .

Example 6.1. Consider the system

(6.2)
$$\begin{cases} o = \eta y - x^{2} \\ o = \tau x + \eta x^{2} + y^{2}. \end{cases}$$

After the previous scaling, we have

(6.3)
$$\begin{cases} o = y - x^2 \\ o = \gamma x + \eta x^2 + y^2, & \tau = \gamma \eta. \end{cases}$$

Solutions near $(x,y,\gamma,\eta)=(0,0,0,0)$ correspond to solutions which bifurcate from the η -axis near (0,0). If (x,y) is a nontrivial solution, elimination of y shows that x must satisfy

$$0 = y + nx + x^3.$$

The discriminant of this cubic is given by $-4\eta^3 - 27\gamma^2$. Thus along the curve $4\eta^3 = -27\gamma^2$ in the (γ, η) -plane (or $4\eta^5 = -27\tau^2$ in the (τ, η) -plane) the number of non-trivial solutions changes by two.

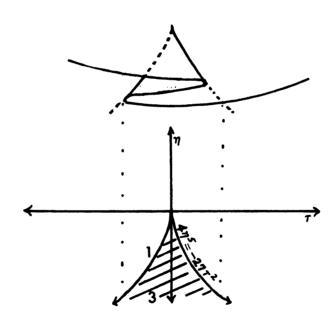


Figure 10. Behavior of nontrivial solutions of system (6.2).

Figure 10 shows the projection of the "solution space" unto the (τ, η) -plane. This example shows the advantage of considering a full neighborhood of $(\tau, \eta) = (0,0)$ in our

studies. It does not suffice to fix η and vary τ to study solutions which bifurcate from the η -axis.

This example shows that we may expect to find a cusp on curves tangent to the axes in the (τ, η) -plane. These curves and cusps play a role similar to the fold curves discussed earlier.

Since $F_y = 1$ at (0,0,0,0), we may use the implicit function theorem to solve for a unique $y(x,\gamma,\xi)$ such that $F(x,y(x,\gamma,\xi),\gamma,\xi) = 0$ and y(0,0,0) = 0. Since $(0,0,\gamma,\xi)$ is a solution for all (γ,ξ) sufficiently close to (0,0), $y(0,\gamma,\xi) = 0$ and $y(x,\gamma,\xi) = xH(x,\gamma,\xi)$ where H is analytic in x, γ and ξ . Substitution of $y(x,\gamma,\xi)$ in F_1 , and division by x gives an equation of the form

(6.4)
$$O = \overline{F}(x, y, \xi) = y + [f_1(y, \xi)]x + [f_2(y, \xi)]x^2 + \cdots$$

and we have successfully "divided out" the solution (x,y) = (0,0).

At this point we may take the following approach: solve (6.4) for $\gamma(x,\xi)$ and analyze the singularity of the map T: $(x,\xi) \rightarrow (\gamma(x,\xi),\xi)$ at $(x,\xi) = (0,0)$. We shall sketch how this may be done for the following special case

$$\begin{cases} (6.5) (a) & 0 = \gamma x + a_1 x^2 + b_1 xy + c_1 y^2 + \xi N_1 (x, y, \gamma, \xi) \\ (6.5) (b) & 0 = y + a_2 x^2 + b_2 xy + c_2 y^2 + \xi N_2 (x, y, \gamma, \xi) \end{cases}$$

We now find that

$$y(x, \gamma, \xi) = -a_2 x^2 + b_2 a_2 x^3 + xH(x, \gamma, \xi).$$

Substitution of $y(x, y, \xi)$ in (6.5)(a) and division by x yields,

$$0 = \gamma + [f_1(\gamma, \xi)]x + f_2(\gamma, \xi)x^2 + f_3(\gamma, \xi)x^3 + \cdots$$

where $f_1(0,0) = a_1$, $f_2(0,0) = -b_1 a_2$, and $f_3(0,0) = b_1 b_2 a_2 - c_1 a_2$.

If $a_1 \neq 0$, we may solve for a unique $x(\gamma, \xi)$. Since $a_1 a_2 = 0$ would make the bilinear form

$$M((x,y),(x,y)) = \begin{cases} a_1x^2 + b_1xy + c_1y^2 \\ a_2x^2 + b_2xy + c_2y^2 \end{cases}$$

degenerate, we now assume that $a_1 = 0$, $a_2 \neq 0$. As before, we may solve for $\gamma(x,\xi)$. We find that $\frac{\partial \gamma}{\partial x} = a_1 = 0$ and $\frac{\partial^2 \gamma}{\partial x^2} = -2b_1a_2$, at $(x,\xi) = (0,0)$. If $b_1 \neq 0$, there is a "fold" [3] tangent to the axis. If $b_1 = 0$, $\frac{\partial^3 \gamma}{\partial x^3} = -6c_1a_2$. If $c_1 \neq 0$, and in addition, $\frac{\partial^2 \gamma}{\partial x \partial \xi} \neq 0$ at $(x,\xi) = (0,0)$, there is a cusp [3] at (0,0). The situation is similar to Example 6.2.

We now outline an alternative approach which is more analytic in nature and does not simply entail applying classifications from singularity theory. This is useful for problems which do not fit the standard classifications. The main idea is to find curves $(x(\xi),y(\xi),\gamma(\xi),\xi)$ such that $x(0) = y(0) = \gamma(0) = 0$ and the Jacobian of system

(6.1) has zero determinant along these curves. These curves are determined by the system

$$(6.6) \begin{cases} 0 = F_1(x, y(x, \gamma, \xi), \gamma, \xi) = \gamma x + f_1(\gamma, \xi) x^2 + f_2(\gamma, \xi)^3 + \cdots \\ 0 = \frac{\partial}{\partial x} F_1(x, y(x, \gamma, \xi), \gamma, \xi) = \gamma + 2f_1(\gamma, \xi) x + 3f_2(\gamma, \xi) x^2 + \cdots \end{cases}$$

since

$$\frac{\partial}{\partial \mathbf{x}} \mathbf{F}_{1} (\mathbf{x}, \mathbf{y}(\mathbf{x}, \mathbf{y}, \mathbf{\xi}), \mathbf{\xi}) = \frac{\partial \mathbf{F}_{1}}{\partial \mathbf{x}} + \frac{\partial \mathbf{F}_{1}}{\partial \mathbf{y}} \frac{\partial \mathbf{y}}{\partial \mathbf{x}} =$$

$$= \frac{\partial \mathbf{F}_{1}}{\partial \mathbf{x}} + \frac{\partial \mathbf{F}_{1}}{\partial \mathbf{y}} \left(-\frac{\partial \mathbf{F}_{2}}{\partial \mathbf{x}} / \frac{\partial \mathbf{F}_{2}}{\partial \mathbf{y}} \right) = 0$$

at $(x,y(x,\gamma,\xi),\gamma,\xi)$ if and only if

$$\det \begin{bmatrix} \frac{\partial \mathbf{F_1}}{\partial \mathbf{x}} & \frac{\partial \mathbf{F_1}}{\partial \mathbf{y}} \\ \frac{\partial \mathbf{F_2}}{\partial \mathbf{x}} & \frac{\partial \mathbf{F_2}}{\partial \mathbf{y}} \end{bmatrix} = 0.$$

One such curve is $\mathbf{x}(\xi) = 0 = \gamma(\xi)$. This corresponds to the ξ -axis and the trivial solution $(\mathbf{x}, \mathbf{y}) = (0, 0)$. We are interested in curves where $\mathbf{x}(\xi) \neq 0$ for $\xi \neq 0$. Let $\mathbf{x}G(\mathbf{x}, \gamma, \xi) = F_1(\mathbf{x}, \mathbf{y}(\mathbf{x}, \gamma, \xi), \gamma, \xi)$. $G(\mathbf{x}, \gamma, \xi) = 0$ implies

$$\frac{\partial}{\partial x} F_1(x, y(x, \gamma, \xi), \gamma, \xi) = x \frac{\partial}{\partial x} G(x, \gamma, \xi).$$

Thus nontrivial solutions of (6.6) are found by studying

$$\begin{cases} (6.7) \text{ (a)} & 0 = G(x, y, \xi) = y + f_1(y, \xi)x + \cdots \\ (6.7) \text{ (b)} & 0 = \frac{\partial}{\partial x}G(x, y, \xi) = f_1(y, \xi) + 2f_2(y, \xi)x + \cdots \end{cases}$$

Since $\frac{\partial G}{\partial x} \neq 0$ at (0,0,0), we may use the implicit function theorem to solve for $\gamma(x,\xi)$ with $\gamma(0,0) = 0$. $\gamma(x,\xi)$ is analytic in (x,ξ) for (x,ξ) in a neighborhood of (0,0). Substitution of $\gamma(x,\xi)$ in (6.7) (b) gives a power series $\bar{G}(x,\xi)$ which converges in some neighborhood of (0,0). The curves $x(\xi)$ may be determined by Newton's polygonal method.

CHAPTER IV

APPLICATION TO A SYSTEM OF CHEMCIAL REACTION EQUATIONS

§1. Introduction

In this chapter we shall apply the method of the preceding chapter to a system of equations that arises in the study of chemical reactions [2]:

(1.1)
$$\frac{\partial x}{\partial t} = A - (B+1)x + x^2y + D \frac{\partial^2 x}{\partial r^2}$$
, $0 < r < 1$, $t > 0$
 $\frac{\partial y}{\partial t} = Bx - x^2y + \nu D \frac{\partial^2 y}{\partial r^2}$, $0 < r < 1$, $t > 0$

with boundary conditions

$$x(0,t) = y(1,t) = A,$$

 $y(0,t) = y(1,t) = B/A.$

A and D are constants, and $D = D_x$, $vD = D_y$, where D_x and D_y are diffusion coefficients for x and y respectively. B and v are taken to be the parameters.

For all values of B and ν , $x_0 = A$, $y_0 = B/A$ is an equilibrium state. Substitution of x = A + u(r,t), y = B/A + v(r,t) into (1.1) gives

(1.2)
$$\begin{cases} \frac{\partial u}{\partial t} = [B-1]u + A^2v + D \frac{\partial^2 u}{\partial r^2} + \frac{B}{A}u^2 + 2Auv + u^2v \\ \frac{\partial v}{\partial t} = -Bu - A^2v + vD \frac{\partial^2 v}{\partial r^2} - \frac{B}{A}u^2 - 2Auv - u^2v \end{cases}$$

with boundary condition

$$u(0,t) = u(1,t) = v(0,t) = v(1,t) = 0.$$

The associated linear system is

(1.3)
$$\begin{cases} \frac{\partial u}{\partial t} = [B-1]u + A^2v + D \frac{\partial^2 u}{\partial r^2} \\ \frac{\partial v}{\partial t} = -Bu - A^2v + \nu D \frac{\partial^2 v}{\partial r^2} \end{cases}$$

with the same boundary conditions.

Since we are interested in steady state solutions, we will consider u and v as functions of r, and study the stationary problem

(1.4)
$$\begin{cases} o = [B-1]u + A^{2}v + D \frac{d^{2}u}{dr^{2}} + \frac{B}{A}u^{2} + 2Auv + u^{2}v \\ o = -Bu - A^{2}v + \sqrt{D} \frac{d^{2}v}{\partial r^{2}} - \frac{B}{A}u^{2} - 2Auv - u^{2}v \end{cases}$$

with boundary conditions

(1.5)
$$u(0) = v(0) = u(1) = v(1) = 0$$

Let Y be the Banach space of twice continuously differentiable functions from [0,1] to \mathbb{R}^2 which satisfy (1.5) and

$$(1.6) u''(0) = v''(0) = u''(1) = v''(1) = 0.$$

Let $|(u,v)|_Y = ||u|| + ||v|| + ||u'|| + ||v'|| + ||u''|| + ||v''||$ where $||\cdot||$ denotes the sup norm. Note that if (u,v) is a solution of (1.4) - (1.5), then (1.6) is automatically satisfied. Let

$$A(v,B)[u,v] = \begin{bmatrix} (B-1) + D & \frac{d^2}{dr^2} & A^2 \\ -B & -A^2 + vD & \frac{d^2}{dr^2} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

and

$$N(B, u, v) = \begin{bmatrix} BA^{-1}u^{2} + 2Auv + u^{2}v \\ \\ -BA^{-1}u^{2} - 2Auv - u^{2}v \end{bmatrix}.$$

If (u,v) is in Y, then A(v,B)[u,v] and N(B,u,v) satisfy (1.5) for all v and v and v and v be the Banach space of continuous functions from v and v which satisfy (1.5). Let

$$|(u, v)|_{X} = ||u|| + ||v||.$$

Then $A(v,B): Y \to X$ is a continuous linear operator for each (v,B) in \mathbb{R}^2 , and $N: \mathbb{R}_X Y \to X$ is continuous. The stationary problem (1.4) - (1.5) is equivalent to

(1.7)
$$O = A(v, B)[u, v] + N(B, u, v).$$

If we replace \mathbb{R}^2 by \mathbb{C}^2 in the above, we see that A(v,B) and N have continuous complex extensions. A(v,B) depends on v and B analytically and N depends on B, u, and v analytically.

§2. Eigenvalues, Eigenvectors and Projections

The spectrum of the linear operator $A(\nu,B)$ consists of the eigenvalues

(2.1)
$$\sigma_{n}^{\pm} = \frac{1}{2} \{B - 1 - A^{2} - n^{2} \pi^{2} D (1 + \nu) + [[B - 1 + A^{2} + n^{2} \pi^{2} D (\nu - 1)]^{2} - 4A^{2} B]^{1/2} \}.$$

The corresponding eigenfunctions are

where M_n^{\pm} satisfies

(2.3)
$$\sigma_{n}^{+} - (B-1) + n^{2}\pi^{2}D - A^{2}M_{n}^{+} = 0.$$

For real σ_n^{\pm} , the projection unto the linear subspace spanned by ξ_n^{\pm} is given by

$$(2.4) P_n^{\frac{+}{2}} \begin{bmatrix} u \\ v \end{bmatrix} \Phi_n^{\frac{+}{2}} = \begin{bmatrix} \frac{2}{1+M_n^2 N_n^2} \int_0^1 u \sin n\pi r + N_n^{\frac{+}{2}} v \sin n\pi r dr \end{bmatrix} \Phi_n^{\frac{+}{2}}$$

where N_n^{\pm} satisfies

(2.5)
$$\sigma_{\overline{n}}^{\pm} - (B-1) + n^2 \pi^2 D + B N_{\overline{n}}^{\pm} = 0.$$

(2.6)
$$\psi_{\overline{n}}^{+} = \begin{bmatrix} \sin n\pi r \\ \frac{+}{N_{\overline{n}}} \sin n\pi r \end{bmatrix}$$

is the solution of the adjoint equation

$$(2.7) \quad \sigma_{\mathbf{n}}^{\frac{1}{2}} \begin{bmatrix} \Psi_{1} \\ \Psi_{2} \end{bmatrix} = \begin{bmatrix} (B-1) + D \frac{\partial}{\partial r^{2}} & -B \\ & & \\ & A^{2} & -A^{2} + \nu D \frac{\partial^{2}}{\partial r^{2}} \end{bmatrix} \begin{bmatrix} \Psi_{1} \\ \Psi_{2} \end{bmatrix}$$

with boundary condition $\Psi_1(0) = \Psi_2(0) = \Psi_1(1) = \Psi_2(1) = 0$. We can see that $P_n^+[\bar{\Phi}_n^+] = 1$.

We shall now determine a curve in the real (ν,B) -plane such that all the eigenvalues of $A(B,\nu)$ have negative real part when (ν,B) lies below this curve [2]. This curve will be called the <u>curve of neutral stability</u>. We can see that $\text{Re}\sigma_n^+ \geq \text{Re}\sigma_n^-$ for all n. If σ_n^+ is complex, then the curve $\text{Re}\sigma_n^+ = 0$ is given by the straight line

(2.8)
$$B = 1 + A^2 + n^2 \pi^2 D(1 + v).$$

If σ_n^+ is real, $\text{Re}\sigma_n^+ = 0$ is given by the hyperbola H_n .

(2.9)
$$B = 1 + n^2 \pi^2 D + \frac{A^2}{v} \{1 + \frac{1}{n^2 \pi^2} D \}.$$

A typical curve $\operatorname{Reo}_{n}^{+} = 0$ is given in Figure 11.

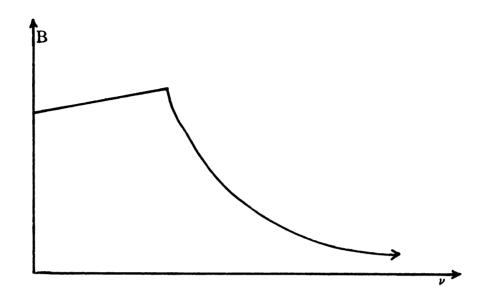


Figure 11. The curve $\operatorname{Re}\sigma_n^+ = 0$.

The curve we seek is found by joining portions of the curves $\operatorname{Re\sigma}_n^+ = 0$ which lie lowest for each (v,B). $\operatorname{Re\sigma}_1^+ = 0$ lies lowest for sufficiently small v and for sufficiently large v. For values in between, a finite number of the hyperbolas lie lowest. A typical curve is "scalloped" and is given by Figure 12.

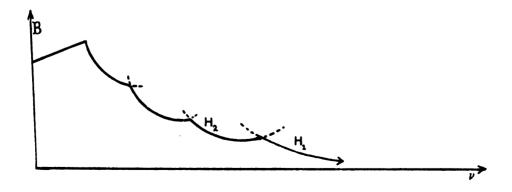


Figure 12. A typical curve of neutral stability.

The intersection of H_n and H_{n+1} is given by

$$v_{C} = \frac{A^{2}}{n^{2}(n+1)^{2}(\pi^{2}D)^{2}}$$

$$B_{C} = 1 + n^{2}\pi^{2}D + n^{2}(n+1)^{2}(\pi^{2}D)^{2}\left\{1 + \frac{1}{n^{2}\pi^{2}D}\right\}$$

$$= 1 + n^{2}\pi^{2}D + (n+1)^{2}\pi^{2}D + n^{2}(n+1)^{2}(\pi^{2}D)^{2}$$

$$= (1 + n^{2}\pi^{2}D)(1 + (n+1)^{2}\pi^{2}D)$$

The number of such intersections which lie on the curve in Figure 12 depends on the values of A and D. We will study bifurcation in the neighborhood of (v_C, B_C) for $n \ge 1$.

Proposition 2.1. If $\sigma_n^+ \neq \sigma_m^+$ for $n \neq m$ is real, then it has multiplicity 1. That is, if $[A(B, v) - \sigma_n^+ I]^k \phi = 0$, then $\phi = K \phi_n^+$.

Proof. We proceed by mathematical induction. It is true for k-1. Suppose $[A(B, v) - \sigma_n^+ I]^k \, \phi = 0$. Let $\overline{\phi} = [A(B, v) - \sigma_n^+ I]^{k-1} \, \phi$. Then $\overline{\phi} = c \, \phi_n^+$ and $P_n[c \, \phi_n^+] \, \phi_n^+ = P_n[[A(B, v) - \sigma_n^+ I]^{k-1} \, \phi] \, \phi_n^+$. Since P_n commutes with A,

$$c \Phi_{n}^{+} = [A(B, v) - \sigma_{n}^{+} I]^{k-1} [P_{n}[\Phi] \Phi_{n}^{+}] = 0.$$

Therefore c = 0 and the induction hypothesis implies $\phi = K \phi_n^+$.

Thus we have exactly the situation described in Chapter III.

§3. Calcuation of the Bifurcation Equation

Let (u,v) in X be decomposed as follows.

$$(3.1) \begin{bmatrix} u \\ v \end{bmatrix} = x \begin{bmatrix} \sin n\pi r \\ M_n \sin n\pi r \end{bmatrix} + y \begin{bmatrix} \sin (n+1)\pi r \\ M_{n+1} \sin (n+1)\pi r \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$
$$= x \phi_n + y \phi_{n+1} + w$$

where

$$x = P_n \begin{bmatrix} u \\ v \end{bmatrix}$$
, $y = P_{n+1} \begin{bmatrix} u \\ v \end{bmatrix}$, and $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ is in Q_X .

M_n, N_n, M_{n+1}, N_{n+1} correspond to $\sigma_n^+ = 0 = \sigma_{n+1}^+$ and are given by

(3.2)
$$M_{n} = -A^{-2}[B_{c} + 1 - n^{2}\pi^{2}D]$$

(3.3)
$$M_{n+1} = -A^{-2}[B_c + 1 - (n+1)^2 \pi^2 D]$$

(3.4)
$$N_n = B_c^{-1} [B_c - 1 - n^2 \pi^2 D]$$

(3.5)
$$N_{n+1} = B_c^{-1} [B_c - 1 - (n-1)^2 \pi^2 D].$$

The subspace Q_X is given by

$$\{ (w_1 w_2) \in X | P_n \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = P_{n+1} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0 \}.$$

$$Q_{\mathbf{Y}} = \mathbf{Y} \cap Q_{\mathbf{X}}$$

The auxiliary equation is given by

(3.6)
$$O = [I - P_n - P_{n+1}][A_0 w + (B - B_c)A_1(u, v) + (v - v_c)A_2(u, v)]$$

$$+ N(B, u, v)$$

where $A_0 = A(v_c, B_c)$,

$$A_1 = \begin{bmatrix} 1 & 0 \\ & \\ -1 & 0 \end{bmatrix} , \qquad A_2 = \begin{bmatrix} 0 & 0 \\ & \\ 0 & \frac{d^2}{dr^2} \end{bmatrix}$$

and
$$N(B,u,v) = N(B_C,u,v) + (B-B_C) \begin{bmatrix} A^{-1}u^2 \\ -A^{-1}u^2 \end{bmatrix}$$
.

Proposition 3.1. There exists a unique real analytic function w(x,y,v,B) such that $x_{\Phi_n} + y_{\Phi_{n+1}} + w(x,y,v,B)$ is a solution of (3.6) for all (x,y,v,B) in a neighborhood of (0,0,0,0) and w(0,0,0,0) = 0. Moreover, w(x,y,v,B) can be expressed by a convergent power series in this neighborhood. [This power series contains terms of second order or higher in x,y,v,B and all terms contain at least an x or y.]

Proof. We need only show that the restriction

$$(3.7) A_{Q}|_{Q_{\mathbf{V}}} : Q_{\mathbf{Y}} \to Q_{\mathbf{X}}$$

is a linear homeomorphism, and then apply the implicit function theorem. The rest follows from the remarks of I§3 and

equating coefficients when the power series for w(x,y,v,B) is substituted into (3.6).

Since there are no eigenvalues of H_0 in Q_Y , the map (3.7) is one-to-one. It is onto by a "Fredholm alternative" type theorem for boundary value problems [9]. Hence by the open mapping theorem it is a linear homeomorphism.

The bifurcation equations are given by

(3.8) (a)
$$O = P_n[(B-B_C)A_1(u,v) + (v-v_C)A_2(u,v) + N(B,u,v)]$$

(3.8) (b)
$$O = P_{n+1}[(B - B_c)A_1(u, v) + (v - v_c)A_2(u, v) + N(B, u, v)].$$

Substitution of w(x,y,v,B) into (3.8) gives a system of the form

$$\begin{cases}
O = F(x, y, y, B) \\
O = G(x, y, y, B)
\end{cases}$$

where F and G are power series that converge in some neighborhood of (0,0,0,0).

From this point on, we shall assume that n is odd.

F and G are actually the coefficients of

$$\Phi_{n} = \begin{bmatrix} \sin n\pi r \\ M_{n} \sin n\pi r \end{bmatrix} \quad \text{and} \quad \Phi_{n+1} = \begin{bmatrix} \sin (n+1)\pi r \\ M_{n} \sin (n+1)\pi r \end{bmatrix}$$

respectively. We shall demonstrate how symmetry properties can be used to obtain information about the bifurcation equations.

Recall the original system for the stationary problem

$$O = A(B, y)[u, v] + N(B, u, v).$$

Let θ be the transformation given by

$$\theta(u(r), v(r)) = (u(1-r), v(1-r)).$$

It is easy to see that θ is an involution and that it commutes with the operators $A(B,\nu)$ and $N(B,u,\nu)$ for all ν and ν and ν 0. We have

$$\theta (x \Phi_n + y \Phi_{n+1} + w) = x \Phi_n - y \Phi_{n+1} + \theta w$$

and

$$Aw(x,y,y,B) = w(x,-y,y,B).$$

Thus F and G must satisfy

$$F(x,-y,\nu,B) = F(x,y,\nu,B)$$
$$-G(x,-y,\nu,B) = G(x,y,\nu,B),$$

for all x,y,v,B in some ball about (0,0,0,0), and we have the following.

Proposition 3.2. If n is odd, the bifurcation equations (3.2) can be reduced to the form

(3.9) (a)
$$O = F(x,y,v,B) = Tx + ax^2 + by^2 + (higher order terms)$$

(3.9) (b)
$$O = G(x, y, v, B) = \eta y + cxy + (higher order terms)$$

where $\eta = \eta(v, B)$, $\tau = \tau(v, B)$. Here F(x, y, v, B) contains only even powers of y and G(x, y, v, B) contains only odd powers of y.

As in the previous chapter, we may show that the first terms of (3.9)(a) are

$$\frac{\partial \sigma_{\mathbf{n}}^{+}}{\partial \mathbf{B}}(\nu_{\mathbf{c}}, \mathbf{B}_{\mathbf{c}}) (\mathbf{B} - \mathbf{B}_{\mathbf{c}}) \mathbf{x} + \frac{\partial \sigma_{\mathbf{n}}^{+}}{\partial \nu} (\mathbf{v}_{\mathbf{c}}, \mathbf{B}_{\mathbf{c}}) (\nu - \nu_{\mathbf{c}}) \mathbf{x} + \cdots$$

Those of (3.9) (b) are

$$\frac{\partial \sigma_{n+1}^+}{\partial B}(v_c, B_c) (B - B_c) y + \frac{\partial \sigma_{n+1}^+}{\partial v}(v_c, B_c) (v - v_c) y + \cdots$$

Thus $\tau(\nu, B)$ and $\eta(\nu, B)$ have the same linear part at (ν_c, B_c) as that of σ_n^+ and σ_{n+1}^+ respectively.

Since bifurcation can occur at $(0,0,\nu,B)$ near $(0,0,\nu_c,B_c)$ if and only if σ_n^+ or σ_{n+1}^+ is zero, and if and only if the jacobian of (3.3) at $(0,0,\nu,B)$ vanishes, we conclude that $\tau=(\nu,B)=0$ if and only if $\sigma_n^+(\nu,B)=0$ and $\eta(\nu,B)=0$ if and only if $\sigma_{n+1}^+(\nu,B)=0$. Also $\tau(\nu,B)$ will have the same sign as $\sigma_n^+(\nu,B)$, and $\eta(\nu,B)$ will have the same sign as $\sigma_{n+1}^+(\nu,B)$ for (ν,B) near (ν_c,B_c) .

The second order terms in x and y that appear in the bifurcation equations and which do not depend on $(B-B_C)$ or $(v-v_C)$ are easy to compute. For the sake of completeness, we give them both for n-odd and n-even. In (3.8)(a) these terms are given by

$$\frac{B_{C}}{A} P_{n} \begin{cases} \left[x^{2} \sin n^{2} \pi r + 2xy \sin n \pi r \sin (n+1) \pi r \right] \\ -(x^{2} \sin^{2} n \pi r + 2xy \sin n \pi r \sin (n+1) \pi r) \right] \end{cases}$$

$$+ \begin{bmatrix} y^2 \sin^2 (n+1)\pi r \\ -y^2 \sin^2 (n+1)\pi r \end{bmatrix}$$

$$+2AP_{n} \left\{ \begin{bmatrix} x^{2}M_{n} \sin^{2} n\pi r \\ -(x^{2}M_{n} \sin^{2} n\pi r) \end{bmatrix} + \begin{bmatrix} xy(M_{n} + M_{n+1}) \sin m\pi r \sin (n+1)\pi r \\ -xy(M_{n} + M_{n+1}) \sin n\pi r \sin (n+1)\pi \end{bmatrix} \right\}$$

$$+ \begin{bmatrix} y^{2}M_{n+1} \sin^{2} (n+1)\pi r \\ -y^{2}M_{n+1} \sin^{2} (n+1)\pi r \end{bmatrix} \right\}.$$

The coefficient of x^2 is

$$\begin{cases} 0 & \text{if n is even} \\ \frac{8}{3n\pi} \left[B_C A^{-1} + 2AM_n \right] \frac{\left[1 - N_n \right]}{1 + M_n N_n} & \text{if n is odd.} \end{cases}$$

The coefficient of xy is

$$\begin{cases} \frac{2}{\pi} \left[\frac{1}{(n+1)} - \frac{1}{6n+2} - \frac{1}{2-2n} \right] \frac{[1-N_n]}{[1+M_nN_n]} \left[\frac{B_c}{A} + 2A(M_n + M_{n+1}) \right] \\ & \text{if n is even} \\ 0 & \text{if n is odd.} \end{cases}$$

The coefficient of y^2 is

$$\begin{cases} 0 & \text{if n is even} \\ \\ \frac{2}{\pi} \left[\frac{1}{n} - \frac{1}{6n+4} + \frac{1}{2n+4} \right] \frac{[1-N_n]}{[1+M_nN_n]} \left[\frac{B_C}{A} + 2AM_{n+1} \right] \\ & \text{if n is odd.} \end{cases}$$

For (3.8) (b), the coefficient of x^2 is

$$\begin{cases} \frac{2}{\pi} \left[\frac{1}{n+1} - \frac{1}{6n+2} - \frac{1}{2-2n} \right] \frac{[1-N_{n+1}]}{[1+M_{n+1}N_{n+1}]} \left[\frac{B_C}{A} + 2M_n \right] \\ & \text{if n is even} \\ 0 & \text{if n is odd.} \end{cases}$$

The coefficient of xy is

$$\begin{cases} 0 & \text{if n is even} \\ \\ \frac{2}{\pi} \left[\frac{1}{n} - \frac{1}{6n+4} + \frac{1}{2n+4} \right] \frac{[1-N_{n+1}]}{[1+M_{n+1}N_{n+1}]} \left[\frac{B_C}{A} + 2A(M_n + M_{n+1}) \right] \\ & \text{if n is odd.} \end{cases}$$

The coefficient of y^2 is

$$\begin{cases} \frac{8}{3(n+1)\pi} \frac{[1-N_{n+1}]}{[1+M_{n+1}N_{n+1}]} [\frac{B_C}{A} + 2AM_{n+1}] & \text{if n is even} \\ 0 & \text{if n is odd.} \end{cases}$$

We will carry out the analysis of the bifurcation equation for n odd. The case when n is even is similar. We

now make the change of variables from $(B-B_c)$, $(v-v_c)$ to $\tau = \sigma_n^+(v, B)$, $\eta = \sigma_{n+1}^+(v, B)$. The point (v_c, B_c) becomes the origin and (3.9) now has the form

$$\begin{cases} (3.10) (a) & 0 = \tau x + ax^2 + by^2 + \text{higher order terms in } x, y, \eta, \tau \\ \\ (3.10) (b) & 0 = \eta y + cxy + \text{higher order terms } x, y, \eta, \tau. \end{cases}$$

We will assume that A and D are such that a, b, and c are nonzero. Then the bilinear form

$$B < (x,y), (x,y) > = \begin{bmatrix} ax^2 + by^2 \\ cxy \end{bmatrix}$$

is nondegenerate, and we may use the change of scale given in III §3. Recalling the consequences of symmetry given in proposition 3.2., the bifurcation problem is equivalent to studying the two systems

$$\begin{cases} (3.11) (a) & 0 = x + ax^2 + by^2 + {}_{T}F_{1}(x, y^2, \gamma, \tau) \\ \\ (3.11) (b) & 0 = \gamma y + cxy + {}_{T}yF_{2}(x, y^2, \gamma, \tau) \end{cases}$$

and

$$\begin{cases} (3.10) (a) & 0 = \gamma x + a x^2 + b y^2 + \eta G_1(x, y^2, \gamma, \tau) \\ (3.10) (b) & 0 = y + c x y + \eta y G_2(x, y^2, \gamma, \tau) \end{cases}$$

where all terms in F_1 , F_2 , G_1 , G_2 are second order or higher terms in x and y.

If (x,y,γ,τ) is a solution of (3.11), then $(\tau x,\tau y,\tau,\gamma \tau)$ is a solution of (3.10). If (x,y,γ,η) is a solution of (3.12), then $(\eta x,\eta y,\gamma \eta,\eta)$ is a solution of (3.10). Equation (3.11) will be used to study solutions of (3.10) corresponding to (τ,η) in a set of the form $S_1=\{(\tau,\eta)\mid |\tau|\geq \rho_1|\eta|\}$ where $\rho_1>0$. Equation (3.12) will be used for sectors $S_2=\{(\tau,\eta)\mid |\eta|\geq \rho_2|\tau|\}$. S_1 and S_2 are given in Figure 13.

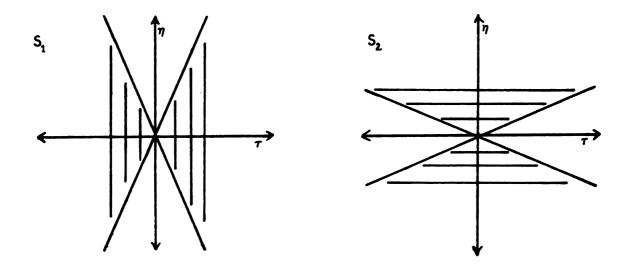


Figure 13. The sectors S_1 and S_2 .

(Of course ρ_1 and ρ_2 have been chosen so that the sectors overlap).

§4. Analysis of the Bifurcation Equations

Recall that n is odd and that we assume A and D are chosen so that a, b and c are nonzero.

When $_{T} = 0$, (3.11) becomes

(4.1)
$$\begin{cases} 0 = x + ax^2 + by^2 \\ 0 = \gamma y + cxy \end{cases}$$

which has the nontrivial solutions

(4.2) (a)
$$x = -\frac{1}{a}, y = 0$$

and

(4.2) (b)
$$x = -\frac{y}{c}, \quad y = \pm \sqrt{\frac{y(1 - \frac{a}{c}y)}{cb}}$$
.

These solutions coincide when $\gamma = c/a$.

The matrix of (HlO) at a solution $(x_0, y_0, \gamma_0, 0)$ is given by

$$\begin{bmatrix} 1 + 2ax_0 & 2by_0 & 0 & F_1(x_0, y_0, \gamma_0, 0) \\ cy_0 & \gamma + cx_0 & y_0 & F_2(x_0, y_0, \gamma_0, 0) \end{bmatrix}$$

The determinant $J(x_0, y_0, \gamma_0, 0) = (1 + 2ax_0)(\gamma + cx_0) - 2bcy_0^2$ is zero only for the solution $(-\frac{1}{a}, 0, \frac{c}{a}, 0)$ and (0,0,0,0) at $\tau = 0$. Thus if $(x_0, y_0, \gamma_0, 0)$ is any other real solution, there are unique $x(\gamma, \tau)$, $y(\gamma, \tau)$ such that $x(\gamma_0, 0) = x_0$, $y(\gamma_0, 0) = y_0$ and $(x(\gamma, \tau), y(\gamma, \tau), \gamma, \tau)$ is a solution of (3.11) in some neighborhood of $(\gamma_0, 0)$. It remains to determine the behavior of real solutions of (3.11) near $(-\frac{1}{a}, 0, \frac{c}{a}, 0)$ and (0,0,0,0).

At the solution $(-\frac{1}{a}, 0, \frac{c}{a}, 0)$, the matrix given in (H11) becomes

$$\begin{bmatrix} -1 & 0 & 0 & F_{1}(-\frac{1}{a}, 0, \frac{c}{a}, 0) \\ 0 & 0 & 0 & 0 \\ J_{x} & J_{y} & -1 & J_{\tau} \end{bmatrix}$$

and therefore does not satisfy the hypotheses (H1O) - (H11) of Chapter III. However, we shall still be able to decribe the behavior near $(-\frac{1}{a}, 0, \frac{c}{a}, 0)$. Recall that the bifurcation equations have the form

$$\begin{cases} (4.3) (a) & 0 = x + ax^2 + by^2 + {}_{T}F_{1}(x, y^2, \gamma, \tau) \\ (4.3) (b) & 0 = y(\gamma + cx + {}_{T}F_{2}(x, y^2, \gamma, \tau)) \end{cases}$$

after scaling.

We may apply the implicit function theorem to (4.3)(a) to obtain:

Proposition 4.1. There is a unique solution branch of (4.3) of the form $(\mathbf{x}(\gamma,\tau),0,\gamma,\tau)$ in a neighborhood of $(-\frac{1}{a},0,\frac{c}{a},0)$, with $\mathbf{x}(\frac{c}{a},0)=-\frac{1}{a}$. This branch is determined by (4.3)(a) and y=0.

We shall refer to this branch as branch I. We shall see that another branch is determined by

$$\begin{cases} (4.4) (a) & 0 = x + ax^2 + by^2 + {}_{T}F_{1}(x, y^2, \gamma, \tau) \\ (4.4) (b) & 0 = \gamma + cx + {}_{T}F_{2}(x, y^2, \gamma, \tau) \end{cases}$$

Let $\bar{J}(x,y,\gamma,\tau)$ denote the Jacobian of (4.4) at (x,y,γ,τ) . The matrix corresponding to (4.6) in Chapter III is given by

and has nonzero determinant. By Theorem III there is a unique fold curve $\bar{\gamma}(\tau)$, with corresponding solutions $(\bar{x}(\tau),\bar{y}(\tau))$ such that $\bar{\gamma}(0)=\frac{c}{a}$, $\bar{x}(0)=-\frac{1}{a}$, $\bar{y}(0)=0$, $0=\bar{J}(\bar{x}(\tau),\bar{y}(\tau),\bar{\gamma}(\tau),\tau)$, and (4.4) determines a solution branch near $(-\frac{1}{a},0,\frac{c}{a},0)$.

The direction of the fold is determined by the number of real solutions of (4.4) for $_{T}$ = 0 and $_{Y}$ near $\frac{c}{a}$. These solutions are given by (4.2)(b). We have the following.

Proposition 4.2. The system (4.4) determines a solution branch of (4.3) which satisfies

- (a) If bc > 0, there are two real solutions of (4.4) for $\gamma < \gamma(\tau)$ and none for $\gamma > \gamma(\tau)$.
- (b) If bc < 0, there are two real solutions of (4.4) for $\gamma > \gamma(\tau)$ and none for $\gamma < \gamma(\tau)$.

Note that this result is local and only holds in a neighborhood of $(x,y,\gamma,\tau) = (-\frac{1}{a},0,\frac{c}{a},0)$.

We shall refer to these solutions as branch II. Branch II is determined at $_{T}$ = 0 by (4.2)(b) and in this sense is distinct from branch I which is determined by (4.2)(a).

At $(-\frac{1}{a}, 0, \frac{c}{a}, 0)$, branch I and branch II coincide.

Proposition 4.3. The only intersection of branches I and II is the set of solutions $(\bar{\mathbf{x}}(\tau), \bar{\mathbf{y}}(\tau), \bar{\mathbf{y}}(\tau), \bar{\mathbf{y}}(\tau), \bar{\mathbf{y}}(\tau))$.

Proof. For any $(x,y,\gamma,0)$, we have

$$\bar{J}(x,y,\gamma,0) = \det \begin{bmatrix} 1 + 2ax + \tau & \frac{\partial F_1}{\partial x} & 2by + \tau & \frac{\partial F_1}{\partial y} \\ c + \tau & \frac{\partial F_2}{\partial x} & \tau & \frac{\partial F_2}{\partial y} \end{bmatrix} = 0$$

when y = 0, because

$$\frac{\partial F_1}{\partial Y}(x,0,\gamma,\tau) = 0 = \frac{\partial F_2}{\partial Y}(x,0,\gamma,\tau) = 0.$$

Since (4.4) determines $\mathbf{x}(\tau)$ and $\mathbf{y}(\tau)$ uniquely when y=0, the solution corresponding to the unique fold curve $\overline{\mathbf{y}}(\tau)$ has the form $(\overline{\mathbf{x}}(\tau),0)$ with $\overline{\mathbf{x}}(0)=-\frac{1}{a}$. Hence $(\overline{\mathbf{x}}(\tau),\overline{\mathbf{y}}(\tau),\overline{\mathbf{y}}(\tau),\tau)$ is part of branch I.

Secondly, the two branches can intersect only when the Jacobian $J(x,y,\gamma,\tau)$ of the system (4.3) is zero and y=0. We have

$$J(x,0,\gamma,\tau) = \det \begin{bmatrix} 1 + 2ax + \tau & \frac{\partial F_1}{\partial x} & 0 \\ & & \\ 0 & & \gamma + cx + \tau F_2(x,y^2,\gamma,\tau) \end{bmatrix}$$

We see that $J(x,0,\gamma,\tau)=0$ at a solution $(x,0,\gamma,\tau)$ of (4.3) (for τ sufficiently small), if and only if $(x,0,\gamma,\tau)$ is also a solution of (4.4). There is a unique solution of (4.4) of this form and it is the one corresponding to the fold curve.

The situation just studied is completely different from that considered in Chapter III. We may view branch II as bifurcating from the solution $(\bar{\mathbf{x}}(t),0)$ of branch I at $(\bar{\gamma}(\tau),\tau)$.

We now determine the behavior of real solutions of (4.3) near (0,0,0,0).

Proposition 4.4.(a) If cb>0, and τ and γ are sufficiently small, there are two nontrivial real solutions $(x(\gamma,\tau),y(\gamma,\tau))$ of (4.3) for $\gamma>0$ such that x and y tend to zero with γ . There are none near (0,0,0,0) for $\gamma<0$.

(b) If cb < 0, there are two nontrivial real solutions for γ < 0, and none for γ > 0. At $_{T}$ = 0, these solutions are given by (4.2)(b).

Proof. We may apply the implicit function theorem to (4.3) (a) to obtain $x(y,\gamma,\tau)$ such that $(x(y,\gamma,\tau),y,\gamma,\tau)$ is a solution of (4.3) (b) in a neighborhood of (0,0,0) and x(0,0,0) = 0. Substitution into (4.3) (b) gives an equation of the form

$$0 = yy - cby^3 + y^3q(y, y, \tau)$$

where g(0,0,0) = 0 [Terms in $g(y,\gamma,\tau)$ which do not contain τ or γ as a factor will be at least first order in y]. Division by y takes care of the trivial solution that exists for all γ . The results follow from a consideration of

(4.5)
$$0 = \gamma - cby^2 + y^2g(y, \gamma, \tau)$$

for _T and _Y sufficiently small.□

In fact, the curve $\gamma(\tau)=0$ is a fold curve for the system consisting of (4.3)(a) and (4.5), and the solution corresponding to this fold curve is $(\mathbf{x}(\tau),\mathbf{y}(\tau))=(0,0)$. The line $\gamma(\tau)=0$ corresponds to the τ -axis in the τ - η plane. Thus branch II bifurcates from the trivial solution at the τ -axis, and disappears at the curve of solutions $(\tau \bar{\mathbf{x}}(\tau),0)$ of branch I when γ reaches $\bar{\gamma}(\tau)$; i.e., (τ,η) reaches $(\tau,\bar{\gamma}(\tau)\tau)$ in the τ - η plane. In the (τ,η) -coordinates, branch II solutions have the form

$$x = -\tau \gamma/c + \cdots = -\eta/c + \cdots$$

$$y = \pm \tau \sqrt{\frac{\gamma(1 - a/c\gamma)}{cb}} + \cdots = \pm \sqrt{\frac{\eta(\tau - a/c\eta)}{cb}} + \cdots$$

All that remains is to determine the behavior of solutions for γ near zero in (3.12). [This corresponds to γ near " ω " in (3.11); or the η -axis in the τ - η plane]. When τ = 0, (3.12) becomes

$$\begin{cases}
0 = Yx + ax^2 + by^2 \\
0 = y + cxy
\end{cases}$$

Nontrivial solutions of (4.6) are given by

$$(4.7) x = -\frac{\gamma}{a}, y = 0$$

and

(4.7) (b)
$$x = -\frac{1}{c}, \quad y = \pm \sqrt{\frac{(y - c/a)}{bc}}.$$

The solutions (4.7) (b) are real for γ near 0 if and only if bc and c/a have opposite signs. The solution (4.7) (a) approaches the origin as γ tends to zero.

Proposition 4.5. There is exactly one nontrivial solution $(x(\gamma,\eta),y(\gamma,\eta))$ of (4.6) in a neighborhood of the origin, such that $x(\gamma,\eta)$ and $y(\gamma,\eta)$ tend to zero with γ . This solution is given by (4.7)(a) when $\eta=0$.

Proof. As in proposition (4.4), $y(x, \gamma, \eta)$ is determined by (3.12)(a) and the problem reduces to

(4.8)
$$0 = \gamma x + ax^{2} + x^{2}h(x, \gamma, \eta)$$

where h(0,0,0) = 0. Since $a \neq 0$, the result follows from the implicit function theorem.

Thus branch I bifurcates from the trivial solution at the η -axis. In (τ, η) -coordinates branch I solutions have

the form

$$x = - Y_{\eta}/a + \cdots = -\tau/a + \cdots$$
, $y = 0$.

We now choose the sectors S_1 and S_2 mentioned in §3. Let $S_1 = \{(\tau,\eta) \mid |\tau| \geq \rho_1 |\eta| \}$ where $0 < \rho_1 < |a/c|$. ρ_2 is chosen so that it overlaps S_1 , but does not contain the line $\eta = (c/a)\tau$.

We summarize our results by the following bifurcation diagrams in the (τ,η) -plane (Figure 14). The numerals denote the number of nontrivial solutions of (3.10) which correspond to (τ,η) in each sector. The curve $\bar{\gamma}(\tau)$ given by proposition (4.2) has become the curve $\Gamma(\tau) = (\tau,\bar{\gamma}(\tau)\tau)$ which passes through the origin with slope c/a.

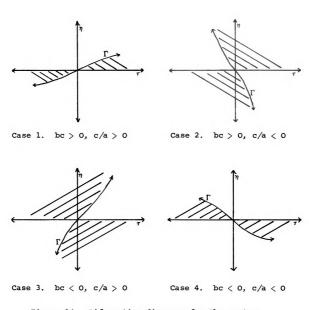


Figure 14. Bifurcation diagrams for the system $(3.10) \ \mbox{in the} \ (\tau,\eta)\mbox{-plane}.$

Figure 15 is an attempt at a more geometrical representation of the situation. We are only considering Case 4 in Figure 15.

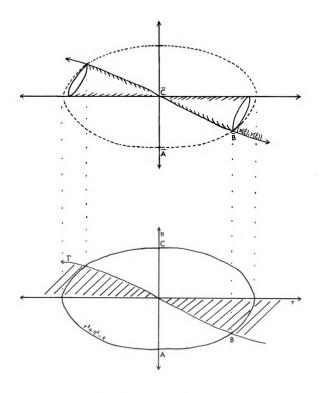


Figure 15. Intuitive representation of bifurcation $\label{eq:condition} \mbox{when } \mbox{ bc } < \mbox{ 0 } \mbox{ and } \mbox{ c/a} < \mbox{ 0.}$

Figure 15 is meant more as an aid to the intuition rather than an actual representation of the solutions (x,y,τ,η) . One must interpret the drawing in the following spirit.

Let $_{\rm T}^2 + _{\rm T}^2 = _{\rm E}$ be a sufficiently small circle in the $_{\rm T-\eta}$ plane. As we pass through point A, solutions of branch I (broken line) bifurcate from the trivial solution at the $_{\rm T-axis}$. As we move around the circle counterclockwise the corresponding solution of branch I passes through branch II at B, proceeds to bypass the trivial solution at the $_{\rm T-axis}$, and finally passes through the trivial solution when it reaches the $_{\rm T-axis}$ again at C; etc. Solutions of branch II (double valued) bifurcate from the $_{\rm T-axis}$ and continue until they disappear into branch I at the solution ($_{\rm T}\bar{x}(_{\rm T})$,0) corresponding to $_{\rm T}$.

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