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HYPONORMAL TOEPLITZ OPERATORS AND WEIGHTED SHIFTS

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ABSTRACT

HYPONORMAL TOEPLITZ OPERATORS AND WEIGHTED SHIFTS

By

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Let L^2 denote the set of all Lebesgue-measurable, square integrable functions on the unit circle, ∂D, and let H^2 be the usual Hardy space on ∂D. For f in $L^\infty(\partial D)$, the Toeplitz operator T_f, mapping H^2 to H^2, is defined by $T_f h = P(fh)$, where P is the orthogonal projection of L^2 onto H^2.

In 1970 Paul Halmos [2] raised the question, "Is every subnormal Toeplitz operator either normal or analytic?" A subnormal operator is one which has a normal extension. An analytic Toeplitz operator is one whose symbol is a bounded analytic function on the unit disk. Chapter One contains a more detailed discussion of this problem.

In Theorem 1, we prove that the answer to Halmos' question is no. More precisely, we show that for $0 < a < 1$, if v is a Riemann mapping of the unit disk onto the ellipse with vertices $\pm 1/(1-a)$ and $\pm i/(1+a)$, and $b = v + a\bar{v}$, then T_b is a subnormal unilateral weighted shift which is neither normal nor analytic. This result was originally proven by Carl Cowen; a different proof is given here. Theorem 2 proves that for

any inner function u, the Toeplitz operator $T_{b \circ u}$ is a direct sum of copies of T_b. This is a special case of a more general theorem due to Cowen [1]. These results are in Chapter Two.

In Chapter Three the work of Sun Shunhua [3] is extended. The only finite direct sums of hyponormal weighted shifts unitarily equivalent to a Toeplitz operator are those given in Theorem 2. This result extends to infinite direct sums if some additional hypotheses are assumed for the weight sequences of the shifts.

[1] C. C. Cowen, Equivalence of Toeplitz operators, J. Operator Theory, 7(1982), 167-172.

[2] P. R. Halmos, Ten problems in Hilbert space, Bull. Amer. Math. Soc. 76(1970), 887-933.

[3] Sun Shunhua, Bergman shift is not unitarily equivalent to a Toeplitz operator, Kexue Tongbao, 28(1983), 1027-1030.

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CHAPTER ONE

For H a Hilbert space, let B(H) be the set of all bounded linear operators from H to H. The unit disk in the complex plane will be denoted by D. Let L^2 denote the set of complex-valued Lebesgue measurable functions on the unit circle, D, which are square integrable with respect to normalized Lebesgue measure. The Hilbert space L^2 has the standard orthonormal basis $\{z^n : n \text{ is an integer}\}$. Each g in L^2 can be written as

$$g = \sum_{n=-\infty}^{\infty} \hat{g}(n) z^n,$$

where $\hat{g}(n)$ is the nth Fourier coefficient of g, and the infinite sum converges in the L^2-norm. Let H^2 be the closed subspace of L^2 consisting of those L^2 functions whose negative Fourier coefficients vanish. Each g in H^2 can be written as

$$g = \sum_{n=0}^{\infty} \hat{g}(n) z^n$$

and hence g can be thought of as an analytic function on the unit disk.

Let L denote the set of essentially bounded, (with respect to normalized Lebesgue measure) complex-valued, Lebesgue measurable functions on D. For any f in L , the essential range of f will be denoted by ess ran(f). This is the set of all complex numbers w such that the measure of

$$\{z : |f(z) - w| < c\}$$

is positive for all $c > 0$. Let H^∞ be the intersection of L^∞ and H^2. Each f in H^∞ extends to an analytic function on D which is bounded, and by Fatou's theorem each bounded analytic function on D has radial limits almost everywhere. With this identification, H^∞ is isometrically isomorphic to the set of bounded analytic functions on D. All of these spaces are discussed in [11].

The continuous functions on ∂D will be denoted by $C(\partial D)$ or simply C. The algebra $H^\infty + C$ is defined to be the set

$$\{g+h : h \text{ in } H^\infty, g \text{ in } C\}$$

This is a closed subalgebra of L^∞ [14], p. 191.

Let f be in L^∞. The multiplication operator M_f is defined by $M_f g = fg$, for all g in L^2. Each M_f is a bounded linear operator on L^2, in fact $\|M_f\| = \|f\|_\infty$. The adjoint of M_f is given by $M_f^* = M_{\bar{f}}$. A normal operator is one that commutes with its adjoint. Since all multiplication operators commute with each other, M_f is a normal operator.

An operator S in B(H) is called subnormal if it extends to a normal operator, that is there exists a Hilbert space K such that H is a subspace of K, and a normal operator N in B(K) such that the restriction of N to H equals S. Alternatively, subnormal operators are restrictions of normal operators to invariant subspaces. A general reference for subnormal operators is [4].

Let f be in L^∞, and let P denote the orthogonal projection of L^2 onto H^2. The Toeplitz operator with symbol f, denoted T_f, is multiplication by f compressed to H^2, so $T_f h = P(fh)$ for all h in H^2. The operator T_f is contained in $B(H^2)$ and $\|T_f\| = \|f\|_\infty$. The adjoint of T_f is $T_{\bar{f}}$. See [7], Chapter Seven for the basic properties of Toeplitz operators.

In general Toeplitz operators do not commute with each other. If $T_f T_g = T_g T_f$, then either both f and g are in H^∞, both $\bar{f}$ and $\bar{g}$ are in H^∞, or there exist constants c_1, c_2 and c_3 (not all zero) such that $c_1 f + c_2 g = c_3$. From this it can be shown that T_f is normal if and only if $f = c_1 + c_2 g$, where c_1 and c_2 are constants and g is real-valued [3], p. 98.

If f is in H^∞, the Toeplitz operator T_f is called analytic. When f is in H^∞, for all h in H^2, fh is also in H^2. Thus the Toeplitz operator T_f, with symbol in H^∞, is the restriction of M_f (defined on L^2) to H^2. Analytic Toeplitz operators are subnormal.

A unilateral weighted shift W, defined on a Hilbert space H, is a linear operator such that

$$We_n = w_n e_{n+1},$$

where $\{e_n : n \geq 0\}$ is an orthonormal basis of H and $\{w_n : n \geq 0\}$ is a sequence of complex numbers. The shift W is bounded if and only if the weight sequence $\{w_n\}$ is bounded, in fact

$$\|W\| = \sup\{|w_n| : n \geq 0\}.$$

There is no loss of generality in assuming $w_n \geq 0$, since W is unitarily equivalent to a weighted shift with weights $\{|w_n|\}$ [16], p. 52. We will thus consider only weighted shifts with nonnegative weights. The adjoint of W is given by

$$W^* e_0 = 0 \text{ and } W^* e_n = w_{n-1} e_{n-1},$$

for all $n > 0$. If w_{-1} is defined to be zero, then

$$W^* e_n = w_{n-1} e_{n-1},$$

for all $n \geq 0$.

Let z denote the identity function. Since $T_z z^n = z^{n+1}$ for all $n \geq 0$, this is an example of a weighted shift with $w_n = 1$ for all n. The operator T_z is called the (unweighted) unilateral shift. The function z is analytic, so T_z is subnormal.

Let L_a^2 be the set of all analytic functions on D which are square integrable with respect to area measure. This Hilbert space is called the Bergman space. The Bergman shift S is the restriction of multiplication by z on $L^2(D,\text{area})$ to L_a^2, so S is subnormal. Clearly S shifts the orthogonal basis $\{z^n : n \geq 0\}$ and normalizing shows that S is a weighted shift with weight sequence $\{[(n+1)/(n+2)]^{1/2}\}$.

An inner function u is an H^∞ function such that $|u| = 1$ almost everywhere on ∂D. A basic reference for inner functions is [13], Chapter Seventeen. For example, given a complex number w in D, the function

$$u(z) = (z - w)/(1 - \bar{w}z)$$

is an inner function. This is called a Blaschke factor. It can be shown that a product of Blaschke factors b, (a Blaschke product) is an inner function if and only if

$$\sum_n (1 - |z_n|) < \infty,$$

where $\{z_n\}$ is the set of zeros of b counted according to multiplicity [13], p. 333. The order of an inner function u, ord(u), is defined to be the number of zeros of u (counted according to multiplicity) if u is a finite Blaschke product, otherwise the order of u is infinite.

Given a collection of Hilbert spaces $\{H_j : j \text{ in } I\}$ for some index set I, the direct sum of these spaces is

$$H = \oplus\{H_j : j \text{ in } I\} = \{\sum_{j\in I} x_j : x_j \text{ in } H_j \text{ and } \sum_{j\in I} \|x_j\|^2 < \infty\}.$$

If T_j is in $B(H_j)$ for each j, then the direct sum of these operators, denoted $T = \oplus\{T_j : j \text{ in } I\}$, is defined on H by:

$$Tx = \sum_{j\in I} T_j x_j$$

for all $x = \sum x_j$ in H. It is easy to see that T is in B(H) if and only if $\|T_j\| \leq M$, for all j. In this case

$$\|T\| = \sup\{\|T_j\| : j \text{ in } I\}.$$

For example, T_{z^n} is a direct sum of unilateral shifts. For $0 \leq j \leq n-1$, let $H_j = \text{Cl span}\{z^{nk+j} : k \geq 0\}$, where Cl span indicates the closure of the linear span of the set. Then $H^2 = \oplus\{H_j : 0 \leq j \leq n-1\}$ and T_{z^n} shifts each H_j-basis. Since T_{z^n} leaves each H_j and its orthogonal complement invariant, T_{z^n} is a direct sum of unilateral shifts.

This example can be generalized. Note that z^n is an inner function and ord(z^n) is n. For any inner function u, the Toeplitz operator T_u is a direct sum of ord(u) unilateral shifts [5], Theorem 1. In Chapter Three we show that this is the only example of a direct sum of weighted shifts unitarily equivalent to an analytic Toeplitz operator.

An operator T is hyponormal if $T^*T - TT^*$ is positive. Every subnormal operator is hyponormal. Hyponormal weighted shifts are easily characterized. A weighted shift is hyponormal if and only if the weight sequence $\{w_n\}$ is increasing. Lemma 1.1 gives a method for constructing hyponormal weighted shifts which are not subnormal.

In 1970 Paul Halmos asked the question "Is every subnormal Toeplitz operator either normal or analytic?" (See [9], p. 906 and [10], p. 537.) The surprising answer is no, as was first shown by Carl Cowen. Later the writer of this paper gave another proof of Cowen's result (Theorem 1). This will be published jointly with Cowen [6].

Abrahamse [1] considered Halmos' problem and found a large (dense in L^∞) class of functions for which the answer is yes. If f or $\overline{f}$ is a function of bounded type (a ratio of two H^∞ functions) then T_f subnormal implies T_f is normal or analytic. Abrahamse also asked if this were true of Toeplitz operators unitarily equivalent to weighted shifts. In particular, is the Bergman shift unitarily equivalent to a Toeplitz operator? If so, then this would be an example of a non-normal, non-analytic, subnormal Toeplitz operator. (If S were unitarily equivalent to an analytic Toeplitz operator T_u, an analysis of the spectrum of S would imply that u was inner and all the weights would have to be one.)

But Sun Shunhua [17] proved that the Bergman shift is not unitarily equivalent to a Toeplitz operator. Moreover he showed that any weighted shift W with a strictly increasing sequence of weights that is unitarily equivalent to a Toeplitz operator T_b, must satisfy the following conditions.

i) There exists a constant a such that $0 < |a| < 1$ and $b - a\bar{b}$ is analytic.

ii) The weight sequence $\{w_n\}$ is given by $w_n = (1 - |a|^{2n+2})^{1/2}$.

Steven Power and this author independently showed that a weighted shift satisfying condition ii) is subnormal (Lemma 1.2). Theorem 1 proves the existence of a function b satisfying condition i) and shows that T_b is a weighted shift such that ii) holds.

Theorem 2 is a special case of a result due to Carl Cowen [5]. A direct sum of copies of the shift in Theorem 1 is unitarily equivalent to a Toeplitz operator. These results are in Chapter Two.

In Chapter Three the following question is considered. Can a Toeplitz operator be unitarily equivalent to a direct sum of distinct hyponormal weighted shifts? In many cases, the answer is no. The only hyponormal weighted shift unitarily equivalent to a non-analytic Toeplitz operator is the one given in Theorem 1. The only example of a finite direct sum of hyponormal weighted shifts that is unitarily equivalent to a Toeplitz operator is the example in Theorem 2. This result extends to infinite direct sums of hyponormal weighted shifts, but some additional hypotheses are required for the weight sequences.

Before proving these results, we list some of the facts that will be used in the proofs.

The kernel of an operator T in B(H) will be denoted by ker T. A useful fact about the kernels of Toeplitz operators is Coburn's proposition [7], p. 185, which states that for any nonzero Toeplitz operator T_g, either ker $T_g = \{0\}$, or ker $T_{\bar{g}} = \{0\}$.

Consider a weighted shift W with nonnegative weights $\{w_n\}$ and basis $\{e_n\}$. Then

$$\ker W = \mathrm{Cl\ span}\{e_n : w_n = 0\}.$$

Suppose W is unitarily equivalent to a Toeplitz operator. Since e_0 is in ker W^*, Coburn's Proposition implies that ker W is trivial. Thus W is injective and all of the weights $\{w_n\}$ are positive. This result also holds for direct sums of weighted shifts unitarily equivalent to Toeplitz operators.

For all the weighted shifts under consideration we may assume that all of the weights are positive.

The spectrum of an operator T in B(H) is

$$\{w : (w-T) \text{ is invertible}\}$$

and will be denoted by sp(T). Let K(H) be the set of compact operators in B(H). This set is a closed ideal in B(H) and the quotient space B(H)/K(H) is called the Calkin algebra; see [7], Chapter Five. The essential spectrum of T, denoted $sp_e(T)$, is the spectrum of T as an element of the Calkin algebra. If T is invertible in the Calkin algebra, then T is called a Fredholm operator. Atkinson's Theorem states that an operator T in B(H) is Fredholm if and only if the range of T is closed, the dimension of ker T is finite, and the dimension of ker T^* is finite [7], p. 129. The index of a Fredholm operator is

$$i(T) = \dim \ker T - \dim \ker T^*$$

The index is a continuous mapping from the set of Fredholm operators to the integers which satisfies

$$i(ST + K) = i(S) + i(T),$$

where S and T are Fredholm operators and K is compact [7], p. 138.

The spectrum of an operator T in B(H) can be divided into two parts, the left spectrum and the right spectrum. The left spectrum of T, denoted $sp_l(T)$, is the set of points w such that w-T is not left invertible; the right spectrum of T, denoted $sp_r(T)$, is the set of points w such that w-T is not right invertible. It is not hard to show that the complement of $sp_l(T)$ consists of those numbers w such that w-T is one to one and has closed range. This last condition is equivalent to w-T being bounded away from zero, that is there exists a constant $c > 0$ such that

$$||(w-T)x|| \geq c||x||,$$

for all x in H. An operator T is right invertible if and only if T is onto.

The left and right essential spectra of an operator T in B(H), denoted $sp_{le}(T)$ and $sp_{re}(T)$ respectively, are defined to be the left and right spectra of T as an element of the Calkin algebra. Left and right invertible elements in the Calkin algebra are called left and right Fredholm operators respectively. If an operator T does not have closed range, then T is neither left nor right Fredholm [4] p. 40. Note that $sp_{le}(T)$ is a subset of $sp_l(T)$ and $sp_{re}(T)$ is a subset of $sp_r(T)$.

For a Toeplitz operator T_g, the essential range of g is contained in both the left and right essential spectra of T_g. First suppose that T_g is left invertible. Then T_g is bounded away from zero. The proof of Proposition 7.6 in Douglas' book [7], shows that T_g bounded away from zero implies that g is bounded away from zero. Thus ess ran(g) lies in $sp_l(T_g)$. If T_g is right invertible, apply the above reasoning to $T_{\bar{g}}$, which is left invertible. Thus ess ran(g) is also contained in the right spectrum of T_g. Finally, suppose the range of T_g is closed. If T_g is also one to one, then T_g is left invertible and we are done. If T_g is not one to one, then by Coburn's Proposition, $T_{\bar{g}}$ is one to one, which implies that $T_{\bar{g}}$ is left invertible and hence g is bounded away from zero. Thus the essential range of g lies in the intersection of $sp_{le}(T_g)$ and $sp_{re}(T_g)$.

If f is continuous, the spectrum of T_f is easily described [7], Theorem 7.26. The essential spectrum of T_f is the range of the function f. For complex numbers w not in the range of f, $w-T_f$ is Fredholm with index equal to minus the winding number of f about w. By Coburn's proposition, $w-T_f$ is invertible if its index is zero. Thus the spectrum of T_f is the union of the range of f, together with those components of the complement of the range where the winding number of f about that component is not zero. For example, let u be a finite Blaschke product. Then u is continuous and $sp(T_u)$ is the unit disk. Furthermore, the essential spectrum of T_u is the unit circle and $i(w-T_u) = -ord(u)$, for all w in D.

This result extends to functions g in $H^\infty + C$ [7], Theorem 7.36. For g in $H^\infty + C$, let $\tilde{g}$ be the Poisson extension of g to D. Suppose there exists a $c > 0$ and $r_0 < 1$, such that for $|z| = 1$, we have $|\tilde{g}(rz)| > c$, for all r, $r_0 < r < 1$. Define the winding number of g about zero to be that of $\tilde{g}(rz)$ for any $r > r_0$. Then T_g is Fredholm, and the index of T_g is the negative of the winding number of g about zero.

For any bounded measurable function f, the spectrum of T_f is still related to the function f. As was noted earlier, the essential range of f is contained in the essential spectrum of T_f. The spectrum of T_f is contained in the closed convex hull of the essential range of f. A deep theorem due to Widom and Douglas states that both the spectrum and the essential spectrum of a Toeplitz operator are connected. See [7], Theorem 7.45 and Corollary 7.46.

Let W be an injective hyponormal weighted shift with weights $\{w_n\}$ and $\|W\| = 1$. Then the spectrum of W is the unit disk; the essential spectrum of W is the unit circle [16], p. 77. Moreover $i(z-W) = -1$, for all z in D.

Suppose that $W = \oplus\{W_j : 1 \leq j \leq N\}$, where each W_j is an injective hyponormal weighted shift and N is finite. Assume

$$\|W\| = \max\{\|W_j\| : 1 \leq j \leq N\} = 1.$$

In this case sp(W) is also the unit disk and $sp_e(W)$ is the union of the circles

$$\{z : |z| = \|W_j\|\}$$

for $1 \leq j \leq N$.

Now let W be an infinite direct sum of hyponormal weighted shifts W_j, for $j \geq 1$. Let

$$\|W\| = \sup\{\|W_j\| : j \geq 1\} = 1.$$

Then sp(W) = D, but the essential spectrum is much larger than in the previous cases. Let $\limsup\{\|W_j\|\} = a$.

<u>Proposition 1.</u> The essential spectrum of W is the union of the disk $\{z : |z| \leq a\}$ and the circles $\{z : |z| = \|W_j\|\}$ for $j \geq 1$.

<u>Proof:</u> If $|z| = \|W_j\|$ for some j, then z is in $sp_e(W_j)$, which is contained in $sp_e(W)$. If $|z| < a$, then dim ker $(z-W_j) = -1$ for infinitely many j's, thus z is in $sp_e(W)$. Since essential spectra are closed sets, $\{z : |z| \leq a\}$ is contained in $sp_e(W)$.

Suppose then that $|z| > a$ and $|z| \neq \|W_j\|$ for all j. Write $W = S_1 \oplus S_2$, where S_2 is the (finite) direct sum of $\{W_j : \|W_j\| > |z|\}$ and S_1 is $W \ominus S_2$. Since $|z| > a$, we have $\|S_1\| < |z|$ and thus $(z-S_1)$ is invertible. Clearly, $(z-S_2)$ is Fredholm, and therefore z is not contained in the essential spectrum of W.

Some additional algebraic properties of Toeplitz operators will be needed in the proofs that follow. Let g be in H^∞ and let f be in L^∞. Then for all h in H^2,

$$T_f T_g h = T_f P(gh) = T_f gh = P(fgh) = T_{fg} h.$$

Thus $T_f T_g = T_{fg}$ whenever g is analytic. Taking the adjoint of this equation shows that $T_f T_g = T_{fg}$ whenever f is conjugate analytic. Together these results imply that

$$T_{\bar{u}} T_f T_u = T_f ,$$

for all f in L^∞ and for all inner functions u. In particular, we have

$$T_{\bar{z}} T_f T_z = T_f.$$

This relation can be used to define Toeplitz operators. Given any T in $B(H^2)$, if

$$T_{\bar{z}} T T_z = T,$$

then there exists an f in L^∞ such that $T = T_f$; see [3], p. 95. Using the standard basis for H^2, each T in B(H) has a matrix representation $T \sim (a_{i,j})$, where $a_{i,j} = (Tz^j, z^i)$. For a Toeplitz operator T_f with matrix representation $(a_{i,j})$, we have

$$a_{i,j} = (T_f z^j, z^i) = (T_{\bar{z}} T_f T_z z^j, z^i) = (T_f z^{j+1}, z^{i+1}) = a_{i+1,j+1}.$$

Thus the matrix of a Toeplitz operator is constant on the diagonals.

For g in L^∞, the Hankel operator H_g from H^2 to $H^{2\perp}$ is defined by

$$H_g h = (1-P)(gh),$$

where 1-P is the projection of L^2 onto $H^{2\perp}$. Two general references for Hankel operators are [12] and [15], Chapter Nine. The adjoint of H_g maps $H^{2\perp}$ to H^2 and is given by

$$H_g^* h = P(\bar{g}h),$$

where h is in $H^{2\perp}$. The map $g \to H_g$ is linear, that is

$$H_{ag+bh} = aH_g + bH_h$$

for all g and h in L^∞, and constants a and b. If g is analytic, then $H_g = 0$. Furthermore, Nehari's Theorem [15], p. 100, states that

$$\|H_g\| = \mathrm{dist}(g, H^\infty).$$

The essential norm of H_g (its norm as an element of the Calkin algebra) denoted $\|H_g\|_e$, is

$$\|H_g\|_e = \mathrm{dist}(g, H^\infty + C)$$

[15], p. 101. In particular, if g is in $H^\infty + C$, then H_g is compact.

The key connection between Hankel and Toeplitz operators is given by the equation:

$$T_{gf} - T_g T_f = H_{\bar{g}}^* H_f$$

for all f and g in L^∞.

Since H^∞ is weak-* closed in L^∞, it is not hard to show that for any f in L^∞, there exists a g in H^∞ such that

$$H_f = H_g \text{ and } \|H_f\| = \|H_g\| = \|g\|_\infty = \mathrm{dist}(g, H^\infty)$$

Such a g is a best H^∞ approximation for the coset $f + H^\infty$. If H_g attains its norm on the unit ball of H^2, then the best approximation to H^∞ is unique and unimodular. More precisely, if there exists an h in H^2 such that

$$\|h\|_2 = 1 \text{ and } \|H_g h\|_2 = \|H_g\| = \|g\|_\infty,$$

then g is the unique function of smallest norm in $g + H^{\infty}$. Moreover, gh is in $H^{2\perp}$ and g is unimodular [15], p. 104. This fact will be used in the proof of Theorem 3.

Matrix representations of Hankel operators are similar to the representations of Toeplitz operators; they are constant on the cross diagonals. Let g be in L^{∞} and suppose $H_g \sim (a_{i,j})$, where $a_{i,j} = (H_g z^j, z^{-1-i})$ for $i, j \geq 0$. Then

$$a_{i,j} = \hat{g}(-1-i-j)$$

[15] p. 100.

For H and K Hilbert spaces and an operator T mapping H to K, the Hilbert-Schmidt norm of T, denoted $||T||_2$, is defined to be

$$||T||_2 = [\sum_{n} ||Te_n||^2]^{1/2},$$

where $\{e_n\}$ is any orthonormal basis for H. This definition is independent of the choice of basis; see [4], p. 9. It is easy to show that

$$||T||_2^2 = \sum_{i,j} |a_{i,j}|^2,$$

where $(a_{i,j})$ is any matrix representation of T. The Hilbert-Schmidt norms of Hankel operators with conjugate analytic symbol have an especially nice characterization.

<u>Proposition 2.</u> Let g be in H^{∞}. Then

$$||H_{\bar{g}}||_2 = [(\text{area } g(D))/\pi]^{1/2},$$

where the area of $g(D)$ is counted according to multiplicity. (If g is an N to one mapping, then the area is counted N times.)

Proof: Let $g = \sum_{n=0}^{\infty} c_n z^n$. If $H_g \sim (a_{i,j})$, then

$$a_{i,j} = \bar{c}_{i+j+1},$$

for $i, j \geq 0$. Thus

$$\|H_g\|_2^2 = \sum_{n=1}^{\infty} n |c_n|^2.$$

Now

$$\begin{aligned}
(\text{area } g(D))/\pi &= 1/\pi \int_{g(D)} dA \\
&= 1/\pi \int_D |g'|^2 \, dA \\
&= 2 \int_0^1 (1/2\pi) \int_0^{2\pi} |g'(re^{it})|^2 \, dt \, r dr \\
&= 2 \int_0^1 \sum_{n=1}^{\infty} n^2 |c_n|^2 r^{2n-1} \, dr \\
&= 2 \sum_{n=1}^{\infty} n^2 |c_n|^2 \int_0^1 r^{2n-1} \, dr \\
&= \sum_{n=1}^{\infty} n |c_n|^2.
\end{aligned}$$

This proposition will be used in the proof of Theorem 1.

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$$a_{i,j} = \bar{c}_{i+j+1},$$

for $i,j \geq 0$. Thus

$$\|H_g\|_2^2 = \sum_{n=1}^{\infty} n|c_n|^2.$$

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&= 2 \int_0^1 \sum_{n=1}^{\infty} n^2 |c_n|^2 r^{2n-1} \, dr \\
&= 2 \sum_{n=1}^{\infty} n^2 |c_n|^2 \int_0^1 r^{2n-1} \, dr \\
&= \sum_{n=1}^{\infty} n|c_n|^2.
\end{aligned}$$

This proposition will be used in the proof of Theorem 1.

CHAPTER TWO

The main result of this chapter is Theorem 1.

<u>Theorem 1.</u> For $0 < a < 1$, let v be the Riemann mapping of D onto the interior of the ellipse E with vertices $\pm 1/(1+a)$ and $\pm i/(1-a)$, and let $b = v + a\bar{v}$. Then T_b is a subnormal weighted shift which is neither normal nor analytic.

In Lemma 1.2 two proofs of subnormality for the weighted shift in Theorem 1 are given. (Lemma 1.2 actually proves subnormality for a larger class of weighted shifts.) In general the problem of determining whether a hyponormal operator is subnormal is quite difficult. One criterion equivalent to subnormality is due to Bram and Halmos [4], p. 117. An operator S in $B(H)$ is subnormal if and only if for all $x_0, x_1, \ldots, x_n$ in H, we have

$$\sum_{i,j=0}^{n} (S^i x_j, S^j x_i) \geq 0.$$

When S is a weighted shift, the Bram-Halmos condition holds whenever a certain infinite collection of matrices has positive determinant. This is Lemma 1.1 and is the basis for the first proof of Lemma 1.2.

Another way to determine whether a weighted shift is subnormal is due to Berger, Gellar and Wallen; see [4] p. 159 and [8]. A weighted shift W with weights $\{w_n\}$ is subnormal if and only if there exists a probability measure v defined on [0,1], with 1 in the support of v, such that

$$(w_0 w_1 \cdots w_{n-1})^2 = \int_0^1 r^{2n}\, dv(r)$$

for all $n \geq 1$. This is the basis of the second proof of Lemma 1.2.

<u>Lemma 1.1.</u> Let W be a weighted shift on a Hilbert space H with orthonormal basis $\{e_n\}$ and positive weights $\{w_n\}$ for $n \geq 0$. For all i and $k \geq 0$, define $w_k^{[i]}$ by

$$w_k^{[i]} = w_k w_{k+1} \cdots w_{k+i-1}.$$

Let B(n,k) be the (n+1) by (n+1) matrix with (i,j)-th entry $w_{i+k}^{[j]}\, w_{j+k}^{[i]}$, for $0 \leq i,j \leq n$. If the determinant of B(n,k) is positive for all nonnegative n and k, then W is subnormal.

<u>Proof:</u> It suffices to show that the Halmos-Bram condition holds for any $x_0, x_1, \ldots, x_n$ lying in a dense subset of H. Assume then that each x_j is a finite linear combination of basis elements.

Fix $n \geq 0$ and $N \geq 0$. Define e_j and $w_j^{[i]}$ to be zero for $j < 0$, and let $w_k^{[0]}$ equal one. For $0 \leq j \leq n$, let

$$x_j = a_{j,-n} e_{j-n} + a_{j,-n+1} e_{j-n+1} + \cdots + a_{j,N} e_{j+N},$$

where each $a_{j,k}$ is a complex number. Note that $W^i e_k = w_k^{[i]} e_{i+k}$ and $(W^i e_k, W^j e_m) = w_k^{[i]}\, w_m^{[j]}\, (e_{i+k}, e_{j+m})$.

Recall that a matrix $B = (b_{i,j})$ is positive semidefinite if and only if

$$\sum_{i,j=0}^{n} \bar{a}_i b_{i,j} a_j \geq 0$$

for all complex scalars $a_0, a_1, \ldots, a_n$.

With this notation we have

$$\sum_{i,j=0}^{n} (W^i x_j, W^j x_i) =$$

$$= \sum_{i,j=0}^{n} \sum_{k=-n}^{N} \sum_{m=-n}^{N} a_{j,m} \bar{a}_{i,k} (W^i e_{j+m}, W^j e_{i+k})$$

$$= \sum_{i,j=0}^{n} \sum_{k=-n}^{N} \sum_{m=-n}^{N} a_{j,m} \bar{a}_{i,k}\, w_{j+m}^{[i]}\, w_{i+k}^{[j]} (e_{i+j+m}, e_{i+j+k})$$

$$= \sum_{i,j=0}^{n} \sum_{m=-n}^{N} \bar{a}_{i,m} a_{j,m}\, w_{j+m}^{[i]}\, w_{i+m}^{[j]}$$

$$= \sum_{m=0}^{N} \sum_{i,j=0}^{n} \bar{a}_{i,m}\, w_{j+m}^{[i]}\, w_{i+m}^{[j]}\, a_{j,m} + \sum_{m=1}^{n} \sum_{i,j=m}^{n} \bar{a}_{i,-m}\, w_{i-m}^{[j]}\, w_{j-m}^{[i]}\, a_{j,-m}$$

$$= \sum_{m=0}^{N} \sum_{i,j=0}^{n} \bar{a}_{i,m}\, w_{j+m}^{[i]}\, w_{i+m}^{[j]}\, a_{j,m}$$

$$+ \sum_{m=1}^{n} \sum_{i,j=0}^{n-m} \bar{a}_{i+m,-m}\, w_{i}^{[j+m]}\, w_{j}^{[i+m]}\, a_{j+m,-m}.$$

Let B(n,k,m) be the matrix with (i,j)-th entry

$$w_{i+k}^{[j+m]} w_{j+k}^{[i+m]},$$

for $0 \leq i,j \leq n$. The first term in the sum is nonnegative for all $a_{i,j}$ if and only if the matrix B(n,m,0) is positive semidefinite, and the second term is nonnegative if and only if B(n-m,0,m) is positive semidefinite.

In fact, for all n, k and $m \geq 0$, B(n,k,m) is positive definite. Any self-adjoint matrix is positive definite if all its principal (upper left) submatrices have positive determinant. Since the principal submatrices of B(n,k,m) are: B(0,k,m), B(1,k,m), . . . ,B(n-1,k,m), it suffices to prove that each B(n,k,m) has positive determinant. The (i,j)-th entry of B(n,k,m) can be written:

$$w_{i+k}^{[j+m]} \, w_{j+k}^{[i+m]} = w_{i+k}^{[m]} \, w_{i+k+m}^{[j]} \, w_{j+k}^{[m]} \, w_{j+k+m}^{[i]}$$

The first factor is constant in each row and the third factor is constant in each column, and both are positive. Thus B(n,k,m) has positive determinant precisely when B(n,k+m,0) = B(n,k+m) also has positive determinant.

Lemma 1.1 can be used to construct examples of hyponormal operators which are not subnormal. The proof of the lemma shows that W is subnormal if and only if the matrix B(n,k) is positive semidefinite for all n and $k \geq 0$. But if the determinant of B(n,k) is negative for any values of n and k, then B(n,k) is not positive semidefinite. Hyponormality alone implies that B(n,k) has nonnegative determinant for all $k \geq 0$, and $n \leq 2$. For example, any hyponormal shift with initial weights satisfying

$$w_0 = 1;\ w_1 = 2^{1/2};\ w_2 = 2;\ w_3 = (9/2)^{1/2};\ w_4 > (14/3)^{1/2}$$

is not subnormal since $B(3,0)$ has negative determinant.

Gellar and Wallen [8] prove a much stronger version of Lemma 1.1. They show that W is subnormal if and only if the matrices $B(n,0)$ and $B(n,1)$ are positive semidefinite for all $n \geq 0$.

Lemma 1.2. Let W be a weighted shift with weights $\{w_n\}$ satisfying

$$w_n = (1 - a^n c)^{1/2}$$

for $0 < a < 1$ and $0 < c < 1$. Then W is subnormal.

First Proof: Consider the matrices $B(n,k)$ as defined in Lemma 1.1. We will show by induction on n that each matrix has positive determinant.

For $n = 0$, we have $B(0,k) = 1$.

For fixed $n > 0$, let $R = (r_{i,j})$ be the $(n+1)$ by $(n+1)$ matrix with (i,j)-th entry $r_{i,j}$ defined by:

$$r_{j,j} = 1 \text{ for } 0 \leq j \leq n, \text{ and}$$

$$r_{j+1,j} = -w_{k+j} \text{ for } 0 \leq j \leq n-1.$$

Set $r_{i,j} = 0$ elsewhere. Then $RB(n,k)$ has first column $[1,0,0\ .\ .\ .0]^t$, and for i and $j > 0$ the (i,j)-th entry is

$$w_{k+i}^{[j-1]}\, w_{k+j}^{[i-1]}\, (w_{k+i+j-1}^2 - w_{k+i-1}^2).$$

The formula for the w_n implies that

$$(w_{k+i+j-1}^2 - w_{k+i-1}^2) = (1-a^j)a^{k+i-1}c.$$

Thus, for i and j > 0, the (i,j)-th entry of RB(n,k) is:

$$ca^{k+i-1}(1-a^j)w_{k+i}^{[j-1]}\ w_{k+j}^{[i-1]}.$$

Now ca^{k+i-1} is constant for each row and $(1-a^j)$ is constant for each column. Thus the determinant of B(n,k) is positive if the matrix with (i,j)-th entry $w_{k+i}^{[j-1]}\ w_{k+j}^{[i-1]}$ for $1 \le i,j \le n$ has positive determinant. But this is the matrix B(n-1,k+1), and the proof follows by induction.

<u>Second Proof:</u> This proof is due to Steven Power. The use of the q-binomial theorem was suggested by Richard Askey.

The q-binomial theorem [2], p. 350, formula (3.3) is

$$1 + \sum_{k=1}^{\infty} (r;q)_k x^k/(q;q)_k = (rx;q)_\infty/(x;q)_\infty,$$

where $(r;q)_k = (1-r)(1-rq)\ .\ .\ .(1-rq^{k-1})$ and $(r;q)_\infty$ is the infinite product. Let r = 0, $x = a^n c$ and q = a. Let $p_0 = 1$, $p_j = (c;a)_j$ for j > 0 and let p_∞ be the corresponding infinite product. This gives

$$1 + \sum_{k=1}^{\infty} c^k a^{nk}/p_k = [(1-a^n c)(1-a^{n+1}c)\ .\ .\ .\ ]^{-1} = p_n/p_\infty.$$

Then

$$(w_0 w_1\ .\ .\ .\ w_{n-1})^2 = p_n$$

$$= p_\infty \sum_{k=0}^{\infty} c^k a^{kn}/p_k$$

$$= \int_0^1 t^{2n}\, dv(t),$$

where v is the discrete measure on [0,1] with mass p_∞ at 1 and mass $p_\infty c^k/p_k$ at $a^{k/2}$ for $k \geq 1$. The conclusion now follows from the Berger-Gellar-Wallen condition [4], p. 159.

<u>Theorem 1.</u> For $0 < a < 1$, let v be the Riemann mapping of D onto the interior of the ellipse E with vertices $\pm 1/(1+a)$ and $\pm i/(1-a)$, and let $b = v + a\bar{v}$. Then T_b is a subnormal weighted shift which is neither normal nor analytic.

<u>Proof:</u> We will show that T_b is a weighted shift with respect to some orthonormal basis $\{e_n : n \geq 0\}$ with weights $\{w_n\}$ which satisfy the hypothesis of Lemma 1.2. If T_b is to be a weighted shift, then we must have

$$(1 - T_{\bar{b}}T_b)e_n = (1 - w_n^2)e_n.$$

Thus the basis which T_b shifts can be characterized as a basis of eigenvectors for the compact operator $(1 - T_{\bar{b}}T_b)$.

Since v is a Riemann mapping, v is continuous on ∂D and maps ∂D onto ∂E. The map $z \longrightarrow z + a\bar{z}$ sends the boundary of E back onto the boundary of D and preserves the winding number of curves in E about zero. Thus b is continuous and unimodular. Furthermore, ker $T_{\bar{b}}$ has dimension one.

Let $K = 1 - T_{\bar{b}}T_b$. Then $K = H_b^*H_b$, which is compact. Note that $b - a\bar{b} = (1-a^2)v$, which is an analytic function. This gives

$$K = H_b^* H_b = a^2 H_{\bar{b}}^* H_{\bar{b}} = a^2(1 - T_b T_{\bar{b}}).$$

Then

$$\begin{aligned} KT_b &= a^2(1 - T_b T_{\bar{b}})T_b \\ &= a^2 T_b(1 - T_{\bar{b}} T_b) \\ &= a^2 T_b K. \end{aligned}$$

Let e_0 be in ker $T_{\bar{b}}$ with $\|e_0\| = 1$. For $n \geq 0$, define

$$e_{n+1} = T_b e_n / \|T_b e_n\|,$$

and let $w_n = \|T_b e_n\|$. Then $T_b e_n = w_n e_{n+1}$ for all $n \geq 0$ and each e_n is of norm one.

The next step is to show by induction on n, that each e_n is an eigenvector for K with corresponding eigenvalue a^{2n+2}. For n = 0, we have

$$Ke_0 = a^2(1 - T_b T_{\bar{b}})e_0 = a^2 e_0.$$

Assume that $Ke_k = a^{2k+2} e_k$. This gives

$$\begin{aligned} Ke_{k+1} &= KT_b e_k / w_k \\ &= a^2 T_b K e_k / w_k \\ &= a^{2k+4} T_b e_k / w_k \\ &= a^{2k+4} e_{k+1}, \end{aligned}$$

which is the claim for n = k+1.

Thus $\{a^{2n+2} : n \geq 0\}$ are eigenvalues of K, and since K is self-adjoint, $\{e_n : n \geq 0\}$ is an orthonormal set. This set will be a basis for H^2 if each a^{2n+2} has multiplicity one and there are no other eigenvalues.

The trace of $H_b^* H_b$ is the sum of its eigenvalues [4], p. 16. Thus

$$\sum_{n=0}^{\infty} a^{2n+2} \leq \operatorname{tr}(H_b^* H_b)$$
$$= \|H_b\|_2^2,$$

where $\| \ \|_2$ is the Hilbert-Schmidt norm. Now

$$\|H_b\|_2^2 = \|H_v + aH_{\bar{v}}\|_2^2$$
$$= a^2 \|H_{\bar{v}}\|_2^2$$
$$= a^2 [\text{area } v(D)] / \pi$$
$$= a^2/(1 - a^2).$$

But $\sum_{n=0}^{\infty} a^{2n+2} = a^2/(1 - a^2)$, so $\{a^{2n+2} : n \geq 0\}$ is a complete set of non-zero eigenvalues and each has multiplicity one.

Now $\ker K = \ker H_b^* H_b = \ker H_b = uH^2$, where u is an inner function or $u = 0$. Since $KT_b = a^2 T_b K$, if f is in $\ker K$, then $T_b f$ is also in $\ker K$. Apply these observations to u: since u is in $\ker K$, $T_b u = bu - H_b u = bu$ is in $\ker K$, and thus $bu = ug$ for some g in H^2. But b is not in H^2, so u must be zero. Thus $\ker K = \{0\}$ and 0 is not an eigenvalue of K. Therefore $\{e_n : n \geq 0\}$ is a basis for H^2 and T_b is a weighted shift.

For each $n \geq 0$,

$$w_n^2 = \|T_b e_n\|^2$$
$$= (T_{\bar{b}} T_b e_n, e_n)$$
$$= (e_n - Ke_n, e_n)$$

$= 1 - (Ke_n, e_n)$

$= (1 - a^{2n+2}).$

Thus $w_n = (1 - a^{2n+2})^{1/2}$ and this is the formula for the weights in Lemma 1.2 (with a and c replaced by a^2). Then T_b is subnormal, b is clearly not analytic, and the range of b is not contained in a line segment so T_b is not normal. This proves the theorem.

Unfortunately, this result does not give a general description of subnormal Toeplitz operators. If T_b is a subnormal Toeplitz operator, what can be said about the symbol b? In the above example, b is continuous, but this is not a necessary condition. This follows from Carl Cowen's result [5];

<u>Theorem</u> Suppose f is in L^∞ and u is an inner function of order n (where n is a positive integer or ∞). Then $T_{f \circ u}$ is a direct sum of copies of T_f. The number of copies is the order of u.

Let u be an inner function of infinite order and let b be as in Theorem 1. Then $T_{b \circ u}$ is a direct sum of subnormal operators, hence $T_{b \circ u}$ is subnormal, but $b \circ u$ is not continuous, or even in $H^\infty + C$. (An easy way to see this is to note that

$$K = 1 - T_{b \circ u} T_{\overline{b \circ u}} = H^*_{b \circ u} H_{b \circ u}$$

is not compact if the order of u is infinite.)

In view of this result, perhaps the right question to consider is, "What are the irreducible subnormal Toeplitz operators?" Do the symbols of such operators lie in $H^\infty + C$? Are the symbols either in H^∞ or in C?

Theorem 2 is a direct proof of the above theorem for the weighted shift in Theorem 1. The proof follows the outline of Theorem 1.

<u>Theorem 2.</u> Let $g = v + a\bar{v}$, where v is as in Theorem 1, and let u be an inner function. Then $T_{g\circ u}$ is unitarily equivalent to the direct sum of copies of T_g. The number of direct summands is the order of u.

<u>Proof:</u> Let $b = g\circ u$, and let $N = \text{ord}(u)$. We will show that H^2 has a basis $\{e_{n,j} : n \geq 0, 1 \leq j \leq N\}$ such that for each fixed j, we have $T_b e_{n,j} = w_n e_{n+1,j}$, where $w_n = (1 - a^{2n+2})^{1/2}$. Then T_b is a direct sum of weighted shifts of the type in Theorem 1.

As in Theorem 1, b is unimodular and

$$b - a\bar{b} = (1 - a^2)v\circ u,$$

which is analytic. Let $K = 1 - T_{\bar{b}}T_b$. Then

$$K = H_b^* H_b = a^2 H_{\bar{b}}^* H_{\bar{b}} \text{ and } KT_b = a^2 T_b K$$

as before.

The first step is to show that the dimension of ker $T_{\bar{b}}$ equals N. If N is finite, then u is continuous which implies that b is continuous. Then dim ker $T_{\bar{b}}$ = winding number of b about zero, which equals the winding number $v \circ u$ about zero, and this is N since v is a Riemann mapping. Conversely, suppose dim ker $T_{\bar{b}}$ is finite. Now $\|K\| = a < 1$, so $1 - K = T_{\bar{b}}T_b$ is invertible. Thus $T_{\bar{b}}$ is right invertible and must be Fredholm. Let $i(T_{\bar{b}}) = n$ and let $f = v \circ u$. We have

$$\|1 - (1-a^2)\bar{b}f\|_\infty = \|b - (1-a^2)f\|_\infty = a\|\bar{b}\|_\infty = a < 1.$$

Thus $T_{\bar{b}}T_f$ is invertible and T_f is Fredholm. The index of T_f is $-n$. Now f is in H^∞, so the winding number of f about zero is n. Since $f = v \circ u$ the winding number of u about zero is also n. Therefore dim ker $T_{\bar{b}}$ is finite if and only if N is finite and in this case, they are equal.

Let $\{e_{0,j} : 1 \le j \le N\}$ be an orthonormal basis for ker $T_{\bar{b}}$. For $n \geq 0$, define $e_{n+1,j} = T_b e_{n,j}/w_{n,j}$, where $w_{n,j} = \|T_b e_{n,j}\|$. Then $T_b e_{n,j} = w_{n,j}e_{n+1,j}$, and each $e_{n,j}$ has norm one.

<u>Claim 1:</u> Each $e_{n,j}$ is an eigenvector for K with corresponding eigenvalue a^{2n+2}.

The proof is by induction on n. For $n = 0$,

$$Ke_{0,j} = a^2(1-T_bT_{\bar{b}})e_{0,j} = a^2e_{0,j}.$$

Assume that $Ke_{k,j} = a^{2k+2}e_{k,j}$. This gives

$$\begin{aligned} Ke_{k+1,j} &= KT_b e_{k,j}/w_{k,j} \\ &= a^2T_bKe_{k,j}/w_{k,j} \\ &= a^{2k+4}T_b e_{k,j}/w_{k,j} \\ &= a^{2k+4}e_{k+1,j}, \end{aligned}$$

which is the claim for $n = k+1$.

Claim 2: For all $n \geq 0$ and $j \geq 1$, we have $w_{n,j} = (1-a^{2n+2})^{1/2}$. Let $w_n = w_{n,j}$ for all $n \geq 0$. Then $T_{\bar{b}} e_{n+1,j} = w_n e_{n,j}$.

By Claim 1,

$$(1-T_{\bar{b}}T_b)e_{n,j} = a^{2n+2}e_{n,j}.$$

Thus

$$\begin{aligned}(1-a^{2n+2})e_{n,j} &= T_{\bar{b}}T_b e_{n,j} \\ &= w_{n,j}T_{\bar{b}}e_{n+1,j}.\end{aligned}$$

Therefore,

$$[(1-a^{2n+2})/w_{n,j}]e_{n,j} = T_{\bar{b}}e_{n+1,j}.$$

Then

$$\begin{aligned}[(1-a^{2n+2})/w_{n,j}] &= (T_{\bar{b}}e_{n+1,j} , e_{n,j}) \\ &= (e_{n+1,j} , T_b e_{n,j}) \\ &= w_{n,j},\end{aligned}$$

and the claim follows. Define w_{-1} to be zero, so that

$$T_{\bar{b}}e_{n,j} = w_{n-1}e_{n-1,j}$$

for all $n \geq 0$.

Claim 3: The set $\{e_{n,j} : n \geq 0, 1 \leq j \leq N\}$ is orthonormal.

In view of Claim 1, and since K is self-adjoint, we have $e_{n,j} \perp e_{m,k}$ whenever $n \neq m$. It must be shown that $e_{n,j} \perp e_{n,k}$ whenever $j \neq k$. This is true by definition when $n = 0$. For any nonnegative n,

$$w_n(e_{n+1,k}\ ,\ e_{n+1,j}) = (T_b e_{n,k}\ ,\ e_{n+1,j})$$
$$= (e_{n,k}\ ,\ T_{\bar{b}} e_{n+1,j})$$
$$= w_n(e_{n,k}\ ,\ e_{n,j})$$

and the claim follows by induction on n.

The last step is to show that $\{e_{n,j} : n \geq 0,\ 1 \leq j \leq N\}$ is a basis for H^2. If $N < \infty$, then the method used in the proof of Theorem 1 works here. The operator K is compact and

$$N \sum_{n=0}^{\infty} a^{2n+2} \leq a^2 \|H_{\bar{v}\circ u}\|_2^2$$
$$= a^2(\text{area } v\circ u(D))/\pi$$
$$= Na^2/(1 - a^2).$$

(Recall that the area of $v\circ u(D)$ is counted according to multiplicity.) Thus $\{a^{2n+2} : n \geq 0\}$ is a complete set of eigenvalues and each has multiplicity N.

Note that zero is not an eigenvalue for K. The proof is the same as in Theorem 1.

<u>Claim 4:</u> Let M be the orthogonal complement of the set $\{e_{n,j} : n \geq 0,\ 1 \leq j \leq N\}$. Then $T_b(M) = M$.

Let h be an element of M. Then for all $n \geq 0$,

$$(T_b h\ ,\ e_{n,j}) = (h\ ,\ T_{\bar{b}} e_{n,j})$$
$$= w_{n-1}(h\ ,\ e_{n-1,j})$$
$$= 0.$$

Thus $T_b h$ is also in M. If h is in M, then $h \perp \ker T_{\bar{b}}$, which implies that h is in the range of T_b. (The range is closed, since T_b is left invertible.) Let $h = T_b f$. Then for all $n \geq 0$,

$$\begin{aligned}(f, e_{n,j}) &= (f, T_{\bar{b}} e_{n+1,j})/w_n \\ &= (T_b f, e_{n+1,j})/w_n \\ &= (h, e_{n+1,j})/w_n \\ &= 0.\end{aligned}$$

Thus f is also in M and therefore $T_b(M) = M$.

<u>Claim 5:</u> $M = \{0\}$.

Given an h in M, there exists a sequence $\{f_n\}$ in H^2 such that $h = T_b^n f_n$. Then for all positive n, it follows from $KT_b = a^2 T_b K$ that

$$\begin{aligned}Kh &= KT_b^n f_n \\ &= a^{2n} T_b^n K f_n.\end{aligned}$$

Taking the norm of this equation yields $\|Kh\| \leq a^{2n+2} \|f_n\|$. If there exists a constant C, such that $\|f_n\| \leq C$, for all $n \geq 1$, then $Kh = 0$ which implies $h = 0$. Since T_b is left invertible, it suffices to show that $\|S^n\| \leq C$, where S is a left inverse for T_b.

Let $S = (T_{\bar{b}} T_b)^{-1} T_{\bar{b}}$. Now

$$(T_{\bar{b}} T_b)^{-1} = (1-K)^{-1} = \sum_{j=0}^{\infty} K^j.$$

Let $L_n = \sum_{j=0}^{\infty} a^{2nj}K^j$, for $n \geq 0$. Note that

$$\|L_n\| \leq \sum_{j=0}^{\infty} a^{2nj}\|K\|^j \leq \sum_{j=0}^{\infty} a^{(2n+2)j} = 1/(1-a^{2n+2}).$$

Since $KT_b = a^2 T_b K$, we have $T_{\bar{b}}K = a^2 K T_{\bar{b}}$ and hence

$$\begin{aligned} T_{\bar{b}}L_n &= \sum_{j=0}^{\infty} a^{2nj}T_{\bar{b}}K^j \\ &= \sum_{j=0}^{\infty} a^{(2n+2)j}K^jT_{\bar{b}} \\ &= L_{n+1}T_{\bar{b}}. \end{aligned}$$

Thus $S = L_0T_{\bar{b}}$ and $S^2 = L_0(T_{\bar{b}}L_0)T_{\bar{b}} = L_0L_1T_{\bar{b}}^2$.
For any n, this implies that

$$S^n = L_0 \cdot\cdot\cdot L_{n-1}T_{\bar{b}}^n.$$

Then

$$\begin{aligned} \|S^n\| &= \|L_0 \cdot\cdot\cdot L_{n-1}T_{\bar{b}}^n\| \\ &\leq \|L_0\| \cdot\cdot\cdot \|L_{n-1}\| \\ &\leq 1/[(1-a^2) \cdot\cdot\cdot (1-a^{2n})] \\ &\leq 1/[(1-a^2)(1-a^4)(1-a^6) \cdot\cdot\cdot] \end{aligned}$$

which is a convergent infinite product since $a < 1$.

Therefore, $M = \{0\}$ and $\{e_{n,j} : n \geq 0,\ 1 \leq j \leq N\}$ is a basis for H^2. Thus T_b is a direct sum of weighted shifts.

CHAPTER THREE

Theorem 2 gives examples of direct sums of subnormal weighted shifts which are unitarily equivalent to Toeplitz operators. Are there any hyponormal weighted shifts or direct sums of such which are unitarily equivalent to Toeplitz operators? In this chapter we prove that any Toeplitz operator unitarily equivalent to a finite direct sum of hyponormal weighted shifts is either an analytic Toeplitz operator or one of the form in Theorem 2. We also obtain some similar conclusions for infinite direct sums. These results are based on Theorem 3.

Theorem 3. Let $\{W_j\}$ be a finite or countably infinite collection of hyponormal weighted shifts; each W_j has weights $\{w_{n,j}\}$ and orthonormal basis $\{e_{n,j}\}$ for $n \geq 0$ and $1 \leq j \leq N$. Let $\sup\{\|W_j\| : 1 \leq j \leq N\} = 1$, and assume that not all the weights are one. Suppose there exists b in L^∞ such that T_b is unitarily equivalent to the direct sum of $\{W_j : 1 \leq j \leq N\}$. Assume that:

i) b is unimodular, and

ii) $\inf\{w_{n,j} : 1 \leq j \leq N\} = w_{0,1}$.

Then $b = c(v \circ u + a\bar{v} \circ u)$, where c is a constant of modulus one, u is an inner function, and v is the Riemann mapping of Theorem 1. Thus T_b is of the form in Theorem 2.

<u>Corollary 3.1.</u> Let T_b be unitarily equivalent to a finite direct sum of the shifts $\{W_j\}$. Then the conclusion of Theorem 3 holds.

<u>Corollary 3.2.</u> Let T_b be unitarily equivalent to an infinite direct sum of the shifts in Theorem 3. For $n \geq 0$, define

$$c_n = \inf\{w_{n,j} : j = 1,2,3\ldots\}.$$

Then the conclusion of Theorem 3 holds whenever $w_{0,1} = c_0$ and $\lim c_n = 1$.

The hypothesis in Theorem 3 that not all the weights are one eliminates the analytic case. Direct sums of weighted shifts are analytic Toeplitz operators if and only if all the weights are equal. This is Theorem 4. The other assumptions i) and ii) actually concern the infinite case, since these conditions are always satisfied by finite direct sums. Corollary 3.2 shows that the extra assumptions needed in the infinite case can be restated in terms of the of the given weight sequences. The constant c in the conclusion of Theorem 3 corresponds to a rotation of the ellipse E given in Theorem 1.

<u>Proof of Corollary 3.1</u> Since T_b is a finite direct sum of the shifts $\{W_j\}$, the essential spectrum of T_b is a union of the circles $\{z : |z| = \|W_j\|\}$. But the essential spectrum of a Toeplitz operator is connected, thus each of the shifts W_j has norm one and $sp_e(T_b) = \partial D$. The essential range of b lies in $sp_e(T_b)$, so b is unimodular.

Relabel the shifts (if necessary) so that $w_{0,1} \leq w_{0,j}$ for all j, then ii) holds.

If T_b is an infinite direct sum of hyponormal weighted shifts, then the technique used in Corollary 3.2 fails to show that b is unimodular. The connectedness of the $sp_e(T_b)$ only implies that the lim sup$\|W_j\|$ is one and thus the essential spectrum of T_b is the entire unit disk. But the essential range of b is also contained in the left essential spectrum of T_b. Since T_b is one to one, its left essential spectrum is equal to its left spectrum. The conditions on the weight sequences in Corollary 3.2 imply that the left spectrum of T_b is contained in the unit circle.

<u>Proof of Corollary 3.2.</u> It must be shown that b is a unimodular function.

Since T_b is a direct sum of weighted shifts, for any vector f,

$$\|T_b^k f\| \geq \inf\{w_{0,j} w_{1,j} \cdots w_{k-1,j} : j \geq 1\} \|f\|$$
$$\geq (c_0 c_1 \cdots c_{k-1}) \|f\|.$$

Now $0 < w_{0,1} = c_0 \leq c_1 \cdots \leq c_n$ and $\lim c_n = 1$, thus

$$(c_0 c_1 \cdots c_{k-1})^{1/k}$$

also converges to 1. Given w in D, choose k so large that

$$(c_0 \cdots c_{k-1}) - |w|^k > 0.$$

Then

$$\|(T_b^k - w^k)f\| \geq \|T_b^k f\| - |w|^k \|f\|$$
$$\geq [(c_0 \ldots c_{k-1}) - |w|^k] \|f\|$$

Thus $T_b^k - w^k$ is bounded away from zero, and this together with the equation

$$T_b^k - w^k = (T_b^{k-1} + wT_b^{k-2} \ldots + w^k)(T_b - w)$$

implies that $(T_b - w)$ is bounded away from zero. Thus w is not contained in $sp_l(T_b)$. Since w was an arbitrary point in D, the left spectrum of T_b lies in ∂D. Therefore b is unimodular and the hypotheses of Theorem 3 are satisfied.

Before proving Theorem 3, we will eliminate the case where b is analytic. If T_b is analytic and a direct sum of weighted shifts, then these shifts are easily characterized.

<u>Theorem 4.</u> Suppose T_b is unitarily equivalent to a direct sum of hyponormal weighted shifts, W_j, with basis $\{e_{n,j}\}$ and weights $\{w_{n,j}\}$. Let $\|T_b\| = 1$. Then the following are equivalent:

1) b is an inner function.

2) T_b is analytic.

3) For all n and j, $w_{n,j} = 1$.

4) T_b is hyponormal and there is a weight $w_{n,j}$ equal to one.

1) implies 2): Obvious.

2) implies 3): Let $f = |b|^2$. Since b is analytic, we have $T_{\bar{b}}T_b = T_f$ and $T_f e_{n,j} = w_{n,j}^2 e_{n,j}$ for all n and j. Thus the Toeplitz operator with symbol $f - w_{n,j}^2$ has nontrivial kernel. But this operator is self-adjoint; by Coburn's proposition $f - w_{n,j}^2 = 0$, for all n and j. Since $\|b\|_\infty = 1$, all the weights must be equal to one.

3) implies 4): Clear.

4) implies 1): Suppose $w_{n,j} = 1$. Then

$$1 = \|T_b e_{n,j}\| = \|P(be_{n,j})\| \leq \|be_{n,j}\| \leq \|e_{n,j}\| = 1.$$

Since the last inequality is actually an equality, b is unimodular.

Now

$$\begin{aligned} \ker H_b &= \{h : bh \text{ is in } H^2\} \\ &= \{h : T_b h = bh\} \\ &= \{h : \|T_b h\| = \|h\|\} \quad \text{(since b is unimodular)} \\ &= \text{Cl span}\{e_{n,j} : w_{n,j} = 1\}. \end{aligned}$$

This last set is invariant for T_b since $w_{n,j} \leq w_{n+1,j}$ for all $n \geq 0$ and $j \geq 1$. Hypothesis 4) implies that $\ker H_b$ is non-trivial, hence $\ker H_b = uH^2$, where u is an inner function. Now u is in $\ker H_b$, so

$$T_b u = bu - H_b u = bu$$

is also in ker H_b. Thus bu = uh for some h in H^2 and b is analytic as well as unimodular. Therefore 1) holds.

To prove Theorem 3, two lemmas are needed. These lemmas make use of the H^2 inner - outer factorization. A function g in H^2 is called an outer function if the set gH^∞ is dense in H^2. An outer function g is determined (up to a constant) by its modulus. Given two outer functions f and g such that $|f| = |g|$, we have f = cg for some constant c of modulus one. If h is in H^2, then h can be written as h = ug, where u is an inner function and g is an outer function [7], pp. 158-159. Another fact about H^2 used in Lemma 3.1 is that no nonzero H^2 function can be zero on a set of positive measure [7], p. 154. In the proofs that follow, let H^2_0 denote the space zH^2.

<u>Lemma 3.1.</u> Let f be in L^∞, with $f \neq 0$, and let h be in H^2. Suppose ker $T_f \neq \{0\}$ and M is a nontrivial subspace of ker T_f. Then:

1) For all inner functions u, if uh is in ker T_f, then h is in ker T_f. In particular, ker T_f contains an outer function.

2) If M is invariant for T_z, then f is conjugate analytic.

3) If u is an inner function and M is invariant under multiplication by u, then u is constant.

Proof of 1. Let $g = uh$. Then

$$T_f h = P(f\bar{u}g) = T_{f\bar{u}}g = T_{\bar{u}}T_f g = 0.$$

Proof of 2. If h is in $\ker T_f$, then $P(fh) = 0$ and fh is in $H^{2\perp}$. Thus $\overline{fh}$ is in H_o^2, which means that fh and zfh are conjugate analytic.

Choose h in M of the form

$$h = z^k + c_{k+1}z^{k+1} + \ldots$$

for some $k \geq 0$. Apply $T_{\bar{z}}$ k times to h to obtain a function in M of the form

$$1 + c_{k+1}z + c_{k+2}z^2 + \ldots$$

Thus we can assume that M contains an h such that

$$h = 1 + zg,$$

where g is in H^2. Then $T_{\bar{z}}h = g$ is also in M. Since M is contained in $\ker T_f$, both fh and zfg are conjugate analytic. Therefore, $f = fh - zfg$ is conjugate analytic.

Proof of 3. Let h be in M with $h \neq 0$. Then $\overline{fh}$ is in H_o^2 and for all $n \geq 0$, $\overline{fhu^n}$ is in H_o^2. Let $\overline{fh} = u^n g_n$, where g_n is in H^2. Divide out the common outer factor of $\overline{fh}$ and g_n to obtain $v_0 = u^n v_n$, where v_n is the inner factor of g_n for all $n \geq 0$. Then for any w such that $|w| < 1$, we have

$$|v_0(w)| \leq |u(w)|^n.$$

If u is nonconstant, then by the maximum modulus principle, $|u(w)| < 1$, for all w in D. This implies that $v_0 = 0$. Then $fh = 0$, but h cannot be zero on a set of positive measure, so f must be zero almost everywhere - contradiction. Therefore u is a constant.

<u>Lemma 3.2.</u> Let f and g be unimodular. If $\{0\} \neq \ker T_g$ and $\ker T_g$ is contained in $\ker T_f$, then there exists an inner function u such that $g = uf$. Moreover multiplication by u maps $\ker T_g$ into $\ker T_f$.

<u>Proof:</u> Define $A_f h = \overline{fzh}$ for all h in L^2. If h is in $\ker T_f$, then $\overline{fh}$ is in H_o^2 and so $A_f h$ is in H^2. Hence

$$T_f A_f h = P(f\overline{fzh}) = P(\overline{zh}) = 0.$$

Therefore, A_f maps $\ker T_f$ into itself. Since f is unimodular, A_f^2 is the identity map on $\ker T_f$ and

$$|A_f h| = |\overline{fzh}| = |h|.$$

This last equation implies that the outer factor of $A_f h$ is the outer factor of h. The same properties hold for the analogously defined A_g.

By Lemma 3.1.1 there is an outer function h in $\ker T_g$. Then $A_g h = vh$ for some inner function v. Thus $A_g vh = h$. Since h and vh are also in $\ker T_f$, we have $A_f vh = uh$ for some inner function u. Then

$$uA_g vh = uh = A_f vh$$

or

$$u\overline{gzvh} = \overline{fzvh}.$$

Therefore $g = uf$. The map $A_f A_g$ sends $\ker T_g$ into $\ker T_f$, and for all h in $\ker T_g$,

$$A_f A_g h = A_f(\overline{gzh}) = \overline{f}gh = uh.$$

Thus multiplication by u maps $\ker T_g$ into $\ker T_f$.

Theorem 3. Let $\{W_j\}$ be a finite or countably infinite collection of hyponormal weighted shifts; each W_j has weights $\{w_{n,j}\}$ and orthonormal basis $\{e_{n,j}\}$ for $n \geq 0$ and $1 \leq j \leq N$. Let $\sup\{\|W_j\| : 1 \leq j \leq N\} = 1$, and assume that not all the weights are one. Suppose there exists b in L^∞ such that T_b is unitarily equivalent to the direct sum of $\{W_j : 1 \leq j \leq N\}$. Assume that:

i) b is unimodular, and

ii) $\inf\{w_{n,j} : 1 \leq j \leq N\} = w_{0,1}$.

Then $b = c(v \circ u + a\overline{v} \circ u)$, where c is a constant of modulus one, u is an inner function, and v is the Riemann mapping of Theorem 1. Thus T_b is of the form in Theorem 2.

Proof: The key idea in the proof is as follows. Recall from Theorem 2 that $H_b^* H_b = a^2 H_{\overline{b}}^* H_{\overline{b}}$ and $\|H_{\overline{b}}\| = 1$. This means that $a\overline{b}$ is the unique function of smallest norm in the coset of $b + H^\infty$. The technique here is to start with the function of smallest norm in $b + H^{\infty}$ and argue that this is indeed $a\bar{b}$. Assumption ii) guarantees that this function is unimodular.

Since T_b is hyponormal, the equivalence of conditions 3) and 4) in Theorem 4 shows that no weight is equal to one. From the proof of 4) implies 1) in Theorem 4, we can conclude that $\ker H_b = \{0\}$.

By Coburn's proposition and the fact that $\ker T_{\bar{b}}$ is nontrivial, $\ker T_b = \{0\}$ and thus $w_{0,1} > 0$. Let

$$a_{n,j} = (1-w_{n,j}^2)^{1/2}$$

and let $a = a_{0,1}$. Since b is unimodular,

$$1-T_{\bar{b}}T_b = T_{b\bar{b}} - T_{\bar{b}}T_b = H_b^*H_b.$$

Then $H_b^*H_b e_{n,j} = a_{n,j}^2 e_{n,j}$. The norm of H_b is equal to a and is attained by $e_{0,1}$. Now, there exists a g in L^{∞} such that $b-a\bar{g}$ is analytic and $\|g\|_{\infty} = 1$. Thus $H_b = aH_{\bar{g}}$ and g must be unimodular since $\|H_{\bar{g}}e_{0,1}\|_2 = 1 = \|g\|_{\infty}$. We will show that $b = c^2g$, where c is a constant of modulus one, from which the conclusion will follow. Now

$$\begin{aligned}\ker T_{\bar{g}} &= \ker T_g T_{\bar{g}} \\ &= \ker (1 - H_{\bar{g}}^* H_{\bar{g}}) \\ &= \ker (a^2 - H_b^* H_b) \\ &= \text{Cl span}\{e_{n,j} : w_{n,j} = w_{0,1}\}.\end{aligned}$$

This last set contains $e_{0,1}$ and is invariant for $T_{\bar{b}}$ since $w_{n,j} \leq w_{n+1,j}$.

<u>Claim:</u> The kernel of T_g is contained in ker $T_{\bar{b}}$.

If T_b is a finite direct sum of subnormal shifts, then this claim is easily proven. For a subnormal weighted shift, if the first two weights are equal, then all the weights are equal [18], Theorem 6. This cannot occur here since no weight equals one. Thus $w_{0,j} < w_{1,j}$ for all j, and the claim holds.

Let $M = T_{\bar{b}}(\ker T_g)$. By the preceeding argument, M is contained in ker T_g. We will show that $T_z(M)$ is contained in M, and then by Lemma 3.1.2, M = {0} or g is analytic. Let h be in M. Then there exists an f_1 in ker T_g, such that $T_{\bar{b}}f_1 = h$. Let e be the outer factor of $e_{0,1}$. By Lemma 3.1.1, e is contained in ker $T_{\bar{b}}$ and in ker T_g. Let $zf = f_1 - de$, where d is a constant chosen so that $f_1 - de$ is in H_o^2. Since e is in ker $T_{\bar{b}}$ we have

$$T_{\bar{b}}zf = T_{\bar{b}}f_1 = h.$$

Now zf is in ker T_g which implies that f is in ker T_g. Thus

$$T_{\bar{z}}h = T_{\bar{z}}T_{\bar{b}}T_z f = T_{\bar{b}}f,$$

which is in M. Therefore $T_{\bar{z}}(M)$ is contained in M.

If g is analytic, then $g(b-a\bar{g}) = bg-a$ is also analytic, which implies that g is in ker H_b. But ker H_b = {0}, therefore M = {0} and the claim is proven.

Now apply Lemma 3.2 to $\bar{g}$ and $\bar{b}$ obtaining an inner function u such that $\bar{g} = \bar{b}u$ and multiplication by u sends ker T_g into ker $T_{\bar{b}}$. Suppose u is not constant. Then by Lemma 3.1.3, there exists an e in ker T_g such that ue is contained in ker $T_{\bar{b}}$ but not in ker $T_{\bar{g}}$. Let $f = ue$ and $h = T_{\bar{g}}f \neq 0$. Then

$$T_{\bar{u}}h = T_{\bar{u}}T_{\bar{g}}f = T_{\bar{u}}T_{\bar{g}}ue = T_{\bar{g}}e = 0.$$

Thus $\bar{u}h \perp H^2$. Now

$$T_g h = T_g T_{\bar{g}} f = f - H_{\bar{g}}^* H_{\bar{g}} f = f - H_b^* H_b f/a^2,$$

which is contained in ker $T_{\bar{b}}$. Let

$$b\bar{u}h = gh = T_g h + \overline{zk},$$

where k is in H^2. Multiplying by $\bar{b}$ gives

$$\bar{u}h - \bar{b}(T_g h) = \overline{bzk}.$$

Since $T_g h$ is in ker $T_{\bar{b}}$, we have $\bar{b}(T_g h) \perp H^2$. Then $\overline{bzk}$ is orthogonal to H^2, and hence bzk lies in H_0^2. Now bk is in H^2 which implies that k is in ker H_b. Thus k = 0. But then $b\bar{u}h = T_g h$ which is in H^2, implying that bh is in H^2 and h is in ker H_b. Thus h = 0 - contradiction. Hence $u = b\bar{g}$ is constant.

Let $b\bar{g} = c^2$, where $|c| = 1$. Then

$$b - a\bar{g} = b - ac^2\bar{b} = c(\bar{c}b - ac\bar{b})$$

is analytic. Let $(1-a^2)f = \bar{c}b - ac\bar{b}$, so f is analytic. Solving for $\bar{c}b$ yields $(f + a\bar{f}) = \bar{c}b$. Since b is unimodular, $f + a\bar{f}$ is also unimodular. Writing $\bar{c}b$ as the composition of f and the map $z \rightarrow z + a\bar{z}$ shows that the essential range of f (as a function on the unit circle) lies in the boundary of the ellipse E with vertices $\pm 1/(1+a)$ and $\pm i/(1-a)$. As a function on D, $f + a\overline{f}$ is the Poisson extension of $\overline{c}b$, hence $|f(w) + a\overline{f(w)}| < 1$ for all w in D. Thus f(D) is contained in E. Let v be the Riemann mapping of D onto E. Now v^{-1} is a conformal mapping of E onto D which extends to a homeomorphism of $\overline{E}$ onto $\overline{D}$. Let $u = v^{-1} \circ f$. The function u is analytic on D and since v^{-1} is continuous on ∂E, u is unimodular. Thus

$$b = c(v \circ u + a\overline{v} \circ u)$$

as required.

The following result is a weakening of the hypotheses of Corollary 3.2.

Corollary 3.3. Let T_b be unitarily equivalent to an infinite direct sum of the shifts in Theorem 3. Let c_0 be defined as in Corollary 3.2. For $n \geq 0$, define

$$d_n = \liminf_{j \to \infty} w_{n,j}.$$

Then the conclusion of Theorem 3 holds whenever $w_{0,1} = c_0$, $\lim d_n = 1$ and $\|W_j\| = 1$, for all $j \geq 1$.

Proof: As in Corollary 3.2, we will show that the left spectrum of T_b is the unit circle. Choose a sequence of positive numbers $\{s_n\}$ such that $\lim s_n = 0$ and $s_n < d_n$ for all $n \geq 0$. Let $r_n = d_n - s_n$. Then for each fixed n, there are only finitely many j's such that $w_{n,j} < r_n$. Now $r_0 > 0$ and $\lim r_n = 1$, so

$$(r_0 \cdot \cdot \cdot r_{k-1})^{1/k}$$

also converges to one. Fix w in D. Choose k such that

$$(r_0 \cdot \cdot \cdot r_{k-1}) - |w|^k > 0.$$

Write $T_b = V_1 \oplus V_2$, where

$$V_1 = \oplus\{W_j : w_{n,j} \geq r_n \text{ whenever } 0 \leq n \leq k-1\}$$

and $V_2 = T_b \ominus V_1$. Then the proof of Corollary 3.2 applies to V_1 and thus $w-V_1$ is bounded away from zero. The second summand is a finite direct sum of hyponormal weighted shifts, each of which has norm one, hence $w-V_2$ is also bounded away from zero. Therefore, $w-T_b$ is left invertible, and $sp_l(T_b)$ is contained in the unit circle.

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