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By

Micheal Hugh Vernon

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ABSTRACT

SOME ISOPARAMETRIC HYPERSURFACES OF A COMPLEX HYPERBOLIC SPACE AND THEIR COUNTERPARTS IN ANTI-DE SITTER SPACE TIME

By

Micheal Hugh Vernon

In this study, real hypersurfaces of a complex hyperbolic space, i.e. a complex Riemannian manifold of negative constant holomorphic sectional curvature, that satisfy certain tensor equations are classified by utilizing a Lorentzian hyperbolic S^1-fiber bundle over the ambient complex space. All the hypersurfaces classified are isoparametric (have constant principal curvatures), although this hypothesis is used primarily for congruence. As a byproduct of the classification, some information is gained concerning S^1-invariant hypersurfaces of Lorentzian manifolds of negative constant sectional curvature. The major results are as follows:

Theorem 4

A complete, connected contact hypersurface of a complex hyperbolic space of complex dimension n and holomorphic sectional curvature -4 is congruent to one of the following:

i) a tube of radius $r>0$ about a totally geodesic real hyperbolic subspace of dimension n and sectional curvature -1,

ii) a tube of radius $r>0$ about a totally geodesic complex hyperbolic subspace of complex dimension n-1 and holomorphic sectional curvature -4,

iii) a geodesic hypersphere of radius $r>0$, or

iv) a horosphere.

Theorem 5

A complete connected hypersurface of a complex hyperbolic space of complex dimension n and holomorphic sectional curvature -4 whose second fundamental form commutes with the induced almost contact structure is congruent to one of the following:

i) a tube of radius $r>0$ about a totally geodesic complex hyperbolic subspace of complex dimension p, $0 \leq p \leq n-1$, and holomorphic sectional curvature -4, or

ii) a horosphere.

Corollary

A semi-symmetric S^1-invariant hypersurface of anti-De Sitter space time is congruent to an S^1-fiber bundle over one of the hypersurfaces of Theorem 5.

DEDICATION

This dissertation is dedicated to my loving and beloved wife of eleven years. Without her support none of this would have been possible.

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The author is deeply indebted to Professor Ludden for his patience, guidance and many valuable suggestions in the preparation of this study. I cherish the association I have had with Dr. Ludden and hope to maintain it in years to come. I would like to expess my appreciation to Professors Chen and Blair for their suggestions and sharing their vast knowledge in the subject. Thanks go to Dr. Ralph Howard for many worthwhile conversations concerning tubes in Riemannian manifolds.

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INTRODUCTION

The study of hypersurfaces of a given manifold is fundamental to understanding the geometric structure of its submanifolds and ultimately the intrinsic geometry of the ambient space. This has been an especially productive endeavor for hypersurfaces of spaces of constant sectional curvature and more recently for real hypersurfaces of complex manifolds that have constant holomorphic sectional curvature.

In this study, real hypersurfaces of a complex hyperbolic space, i.e. a complex Riemannian manifold of negative constant holomorphic sectional curvature, that satisfy certain tensor equations are classified by utilizing a Lorentzian hyperbolic S^1-fiber bundle over the ambient complex space. All the hypersurfaces classified are isoparametric (have constant principal curvatures), although this hypothesis is used primarily for congruence. As a byproduct of the classification, some information is gained concerning S^1-invariant hypersurfaces of Lorentzian manifolds of negative constant sectional curvature.

The first condition studied is that of a real hypersurface of a complex hyperbolic space being contact with respect to the induced metric. Okumura, [19], studied this condition in 1966 for real hypersurfaces of complex spaces of constant holomorphic sectional

curvature. Kon, [13], found a classification of contact hypersurfaces of complex projective space in terms of Takagi's work on isoparametric hypersurfaces of complex projective space, [27]. However, the classifications of contact real hypersurfaces as tubes occurred in [19] and then in [22], published in 1983. In these papers, a contact hypersurface of complex euclidean space is shown to be either a hypersphere or a certain type of cylinder.

With the publication of [2] in 1982, Kon's classification of contact hypersurfaces of a complex projective space can be made in terms of tubes. However, a complete classification of contact hypersurfaces of complex projective space can also be made by using the techniques of section 3 and the congruences of a sphere. For a contact hypersurface of a complex hyperbolic space the classification is given by the following theorem:

Theorem 4

A complete, connected contact hypersurface of a complex hyperbolic space of complex dimension n and holomorphic sectional curvature -4 is congruent to one of the following:

i) a tube of radius $r>0$ about a totally geodesic real hyperbolic subspace of dimension n and sectional curvature -1,

ii) a tube of radius $r>0$ about a totally geodesic complex hyperbolic subspace of complex dimension n-1 and holomorphic sectional curvature -4,

iii) a geodesic hypersphere of radius $r>0$, or

iv) a horosphere.

A condition related to that of a real hypersurface of a complex Riemannian manifold of constant holomorphic sectional curvature being contact is that of the induced almost contact structure ϕ commuting with the second fundamental form H. Kon obtained a classification of real hypersurfaces satisfying this condition as well in [13], again in terms of [27]. Romero and Montiel, [16], found a complete classification of real hypersurfaces of a complex hyperbolic space satisfying $\phi H=H\phi$ in 1980, in terms of explicitly defined models in the Lorentzian S^1-fiber bundle over the ambient space. In our study, the classification is essentially new and is in terms of hypersurfaces of complex hyperbolic space instead of submersions of S^1-fiber bundles as occurs in [16]. To wit:

<u>Theorem 5</u>

A complete connected hypersurface of a complex hyperbolic space of complex dimension n and holomorphic sectional curvature -4 whose second fundamental form commutes with the induced almost contact structure is congruent to one of the following:

i) a tube of radius $r>0$ about a totally geodesic complex hyperbolic subspace of complex dimension p, $0\le p\le n-1$, and holomorphic sectional curvature -4, or

ii) a horosphere.

Semi-symmetric spaces are those whose curvature tensor annihilates itself when acting as a derivation. Nomizu, [17], in 1967 and Tanno, [15], in 1969 investigated semi-symmetric hypersurfaces

of euclidean space. Tanno and Takahashi, [28], widened the investigation to semi-symmetric hypersurfaces of spheres in 1970. In 1969, [23], and in 1971, [24], Ryan broadened the class of ambient spaces in this type of investigation to those of constant sectional curvature. However, as far as the author knows, no work has been done on semi-symmetric hypersurfaces of an indefinite space which makes the following corollary of more than passing interest.

Corollary

A semi-symmetric S^1-invariant hypersurface of anti-de Sitter space time is congruent to an S^1-fiber bundle over one of the hypersurfaces of Theorem 5.

Other results that are of intrinsic value themselves are generated enroute to the above theorems and corollary. For instance, the construction of the model spaces in section 3 is of interest as the technique is quite general and should yield satisfying characterizations of isoparametric hypersurfaces in other ambient spaces. In section 4, not only is the Lorentzian S^1-fiber bundle over complex hyperbolic space used to obtain information concerning hypersurfaces of the Riemannian complex space, but the Riemannian structure of complex hyperbolic space is used to obtain information concerning hypersurfaces of the Lorentzian S^1-fiber bundle. In section 5, a congruence theorem for a certain type of isoparametric hypersurface in a complex hyperbolic space is proven that enables the classification. Finally, in section 6 a horosphere in complex

[illegible]

Corollary

A semi-symmetric S^1-invariant hypersurface of anti de Sitter space time is congruent to an S^1 fiber bundle over one of the hypersurfaces of Theorem 5.

Other results that are of intrinsic value themselves are generated enroute to the above theorems and corollary. For instance, the [illegible] structure of complex hyperbolic space is used to obtain information [illegible] representations of the S^1-fiber bundles over the hypersurfaces of Theorem 5.

The condition of a certain direction on a real hypersurface of complex hyperbolic space being principal is central to the study. This condition merits an independent study, and hopefully, the following pages will facilitate such a study.

0. Geometric Preliminaries

Throughout this study, all manifolds will be assumed to be smooth (C^∞) and complete. $C^\infty(M)$ will denote the set of smooth real valued functions on a manifold M. The base fields of all manifolds discussed here will be **R** (the field of real numbers) and **C** (the field of complex numbers).

If p is a point of a manifold M, $T_p(M)$ will designate the tangent space to M at p, which is the vector space generated by all tangent vectors to M at p. A vector field on M is a smooth assignment of a tangent vector to each point of M. Hence, on the manifolds under consideration, the set of all vector fields will at each point generate the tangent space. The set of vector fields on a manifold M will be referred to as the tangent bundle, T(M). T(M) forms a fiber bundle over M with $T_p(M)$ as the fiber over a point $p \in M$. A subbundle of T(M) will be called a <u>distribution</u> on M. For each $p \in M$, there is a nieghborhood U of p in M for which we can select n vector fields from T(M) with the property that the corresponding tangent vectors are linearly independent in $T_q(M)$ at each $q \in U$.

As each $T_p(M)$ is an n-dimensional real vector space, the notion of tensor applies. In the same way that a tangent vector at $p \in M$ is extended to a vector field on M, we can extend the notion of a tensor on $T_p(M)$ to that of tensor field on M. For instance, let $\{X_1,...,X_n\}$ be a subset of T(M) such that $\{X_1,...,X_n\}_q$ is linearly independent at each point q in an open nieghborhood U of p in M. Define a positive definite inner product on each $T_q(M)$, $q \in U$, by $\langle X_i, X_j \rangle_q = \delta_{ij}$ (where δ is the Kronecker delta). By extending $\langle\, , \rangle_q$ bilinearly to the remainder of $T_q(M)$, for each $q \in U$, we obtain a tensor field of type (0,2) on U (and consequently a local orthonormal basis of T(M)). In fact tensors of this sort are fundamental in semi-Riemannian geometry and when a certain one is selected to work with on a semi-Riemannian manifold it is often called the first fundamental form of the manifold. In many respects its choice really determines the geometry of a manifold, hence only certain bilinear tensor fields will be admissable as a first fundamental form on a given manifold.

Definition

A metric tensor (or first fundamental form) on a manifold M is a non-degenerate symmetric tensor field of type (0,2) on M, that has the property that the dimension of the negative definite subbundle of T(M) (with respect to the metric tensor) is constant on M.

The dimension of the negative definite subbundle is usually refered to as the index of the manifold. Of course, if the index is constant then the dimensions of the positive definite and neutral subbundles will remain constant as well. If M is a semi-Riemannian manifold, then M_n^m will denote that M is of dimension m and has index n.

The existence of a global metric tensor field on a manifold and its nature will determine the intrinsic geometry of the manifold. This was shown by Gauss for surfaces in $\mathbf{R}^3$ and by Riemann for manifolds that admit a positive definite metric tensor. The full generalization to manifolds with indefinite metrics occured under

the impetus of relativity.

If a manifold admits a metric tensor it is said to be semi-Riemannian. Two important special cases will concern us: A manifold that admits a positive definite metric tensor is called a Riemannian manifold. A semi-Riemannian manifold of index one is called Lorentzian. (Incidentally, Lorentzian geometry is the geometry of special relativity.)

The metric tensor on a manifold is used to define the lengths of vector fields and the angles between them. This allows us to speak of a local orthonormal basis of T(M), which will be called a frame. If a metric is indefinite, then there are nontrivial vectors of both negative and zero length. We shall say that a vector field is spacelike if it has positive length, timelike if it has negative length and lightlike or neutral if it has length zero.

The metric tensor on a manifold also can be used to measure how one vector field may vary with respect to any other as a point is moved about on the manifold. This will ultimately lead to the curvature of a manifold, so we shall define the vector rate of change in any direction on a manifold:

Definition

A connection on a manifold M is a function

$\nabla: T(M) \times T(M) \rightarrow T(M)$

that satisfies the following properties:

(0.1) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$

(0.2) $\nabla_X(aY+bZ) = a\nabla_X Y + b\nabla_X Z$

(0.3) $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$

for all $X,Y,Z \in T(M)$, $a,b \in \mathbf{R}$ and $f,g \in C^\infty(M)$.

$\nabla_X Y$ is called the <u>covariant derivative</u> of Y with respect to X for the connection ∇.

In general there may be many different connections on a semi-Riemannian manifold M. However, we shall be interested in only one, namely the so-called Levi-Civita metric connection:

On a semi-Riemannian manifold M there exists a unique connection ∇ such that

(0.4) $[X,Y] \equiv X \circ Y - Y \circ X = \nabla_X Y - \nabla_Y X$ for all $X, Y \in T(M)$,

i.e. the connection has zero torsion, and

(0.5) $X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ for all $X, Y, Z \in T(M)$,

where $\langle\,,\,\rangle$ is the metric tensor field on M. If (0.1)-(0.5) hold for a connection ∇, then ∇ is called the <u>Levi-Civita</u> metric connection on M.

The notion of covariant derivative of arbitrary tensor fields is crucial in defining aspects of intrinsic geometry of manifolds and the extrinsic geometry of submanifolds, as we will see. It is defined inductively as follows:

Let K be a tensor field of degree (r,s) on a semi-Riemannian manifold M; i.e. K is a multilinear mapping of $\prod_{i=1}^{s} T_x(M)$ into the space of contravariant tensors of degree r at x for each $x \in M$. Define the covariant derivative of K with respect to $X \in T(M)$ by

$$(\nabla_X K)(X_1,\dots,X_s)=\nabla_X(K(X_1,\dots,X_s))-\sum_{i=1}^{s} K(X_1,\dots,X_{i-1},\nabla_X X_i,X_{i+1},\dots,X_s)$$

for any set $\{X_1,\dots,X_s\} \subset T(M)$. By setting

$$(\nabla K)(X_1,\dots,X_s;X)=(\nabla_X K)(X_1,\dots,X_s)$$

we obtain a tensor field ∇K on M of type (r,s+1). In this paper we shall investigate only tensors of type (0,0) (C^∞ functions), (1,0) (vector fields), (1,1) (C^∞ endomorphisms on T(M)), (0,2) (bilinear forms, e.g. metrics), (1,3) (curvature tensors) and their derivatives.

In particular, for $V \in T(M)$ and $f \in C^{\infty}(M)$, $\nabla_V f = Vf$. From this we can obtain a (1,0) tensor field (i.e. a vector field) ∇f from f by setting $\langle \nabla f, X \rangle = Xf$ for all $X \in T(M)$. ∇f is called the gradient of f in M, and is nothing more than the first covariant derivative of f with respect to the metric on M. By taking the second covariant derivative of f we can obtain a (0,2) tensor on M: define $Hess(f;M): T(M) \times T(M) \to \mathbb{R}$ by

$$Hess(f;M)(X,Y) = \langle \nabla_X(\nabla f), Y \rangle = XY(f) - \nabla_X Y(f) = (\nabla^2 f)(X;Y).$$

for all $X,Y \in T(M)$. $Hess(f;M)$ is called the hessian of f on M.

The existence of tensors that are covariant constant will force certain geometric and topological consequences on a manifold, as we will see later in this survey section. An easy example of a tensor that is covariant constant is that of a constant function on a manifold M. Clearly, if f is constant on M then for any vector field V we must have $0 = Vf = \nabla_V f$. (This is easily verified using the partial differential operators obtained fom a local coordinate system as a local orthonormal basis of $T(M)$.) Conversely, if $\nabla_V f=Vf=0$ for all $V\epsilon T(M)$ it is not hard to show that f is constant on M. In general we say that a tensor K on M is parallel if $\nabla K=0$. Notice that (0.5) says that the Levi-Civita metric tensor of a semi-Riemannian manifold is parallel.

The idea of "straight" in a semi-Riemannian manifold is closely linked to the notion of covariant constant. For instance, given any line L in $\mathbf{R}^n$ and $p\epsilon L$, there is a unit tangent vector V in the direction of one of the rays emanating from p along L. Translating the origin of $\mathbf{R}^n$ to p, we can choose a coordinate system of $\mathbf{R}^n$ in such a way that L is a coordinate axis, say span$\{x_1\}$. We may then set $V=\partial/\partial x_1$ so that $\nabla_V V=0$.

The notion of "straight" is also related to the idea of distance in a semi-Riemannian manifold. In an arbitrary manifold, a length minimizing curve is called geodesic. Let $\sigma:[0,r]\to M$ be a curve in a semi-Riemannian manifold M that is parameterized by arclength. Let $V=\sigma'(t)$ be its velocity vector field at $\sigma(t)$ for any $t\epsilon[0,r]$. If σ is geodesic, then its acceleration must vanish, i.e. $\sigma''(t)=0$. Using this

and the trivial observation of the previous paragraph as motivation we shall say that a curve σ is geodesic if $\nabla_V V=0$ along σ where V is the velocity vector field of σ; i.e. a curve σ is geodesic if its velocity vector field is covariant constant with respect to itself.

For each $p \in M^n$, there exist n geodesics through p that are mutually orthogonal at p. Then the velocity vectors of these geodesics at p are mutually orthogonal tangent vectors and hence form a basis for $T_p(M)$. Conversely, given an orthonormal basis of $T_p(M)$, there exist n corresponding geodesics whose velocity vectors at p are the elements of the given basis. As the geodesics are the means of measuring distances in a semi-Riemannian manifold we see that a semi-Riemannian manifold is approximated by its tangent spaces. That is, given any point $p \in M$, there exists a neighborhood of the origin in $T_p(M)$ that is diffeomorphic to a neighborhood of p in M. Not surprisingly, a diffeomorphism can be defined in terms of geodesics.

Definition

Let $p \in M$ and X be a unit vector in $T_p(M)$. Let $\gamma(t)$ be the geodesic emanating from p with velocity vector X at p (i.e. $\gamma'(0)=X$ and $\gamma(0)=p$), with domain (a,b). Set $\exp_p(rX)=\gamma(r)$ for $r \in (a,b)$.

$\exp_p$ carries lines through the origin in $T_p(M)$ to geodesics through $p \in M$. Thus, distances in M near p are approximated by distances in $T_p(M)$. $\exp_p$ is a convenient tool for discussing semi-Riemannian analogues of spheres and tubes (as is done in section 3). Although $T_p(M)$ approximates a neighborhood of $p \in M$, the approximation is in general not very good; that is, the neighborhood may have to be of very small diameter in order to acheive a given degree of accuracy. This is a manifestation of the intrinsic geometry M. In particular, the degree of accuracy will depend upon the curvature of M at p.

The curvature of a curve in $\mathbf{R}^2$ or a surface in $\mathbf{R}^3$ is easily understood intuitively. However, generalizing this concept to

Riemannian and semi-Riemannian manifolds requires the definition of a new tensor field:

Definition

Let M be a semi-Riemannian manifold with Levi-Civita connection ∇. The curvature tensor on M is defined to be a tensor field R of type $(1,3)$ given by:

$$R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z$$

for all $X,Y,Z \in T(M)$.

R seems far removed from the usual idea of Gaussian curvature of curves and surfaces, but is a necessary generalization to discuss the curvature in semi-Riemannian manifolds of arbitrary dimension. However, this generalization gives rise to more than just one notion of curvature. The notion that generalizes Gaussian curvature is that of sectional curvature.

Definition

Let X,Y be linearly independent elements of T(M) where M is semi-Riemannian manifold with metric tensor < , >. The sectional curvature of the plane $\Pi=\mathrm{span}\{X,Y\}$ on M is given by

$$K(\Pi)= \langle R(X,Y)X,Y\rangle/[\langle X,X\rangle\langle Y,Y\rangle-\langle X,Y\rangle^2].$$

In case X and Y are orthonormal, $K(\Pi)=\langle R(X,Y)X,Y\rangle$. Not surprisingly, $K(\Pi)$ is a geometric invariant, i.e. $K(\Pi)$ is independent of the choice of basis of Π. It is not hard to verify that for a surface in $\mathbf{R}^3$, the Gaussian curvature agrees with the sectional curvature.

If $K(\Pi_p)=K_p$ is constant for any choice of non-degenerate plane section Π_p at p, then M is said to have constant sectional curvature at p. F. Schur showed that if M is a connected semi-Riemannian manifold that has constant sectional curvature at each point p of M, then K_p is a constant over M; i.e. $K_p=K$ for all $p\in M$. In this case, M is said to be of constant sectional curvature K.

For example, a euclidean space with a semi-Riemannian metric tensor and Levi-Civita metric connection has $K(\Pi)=0$ for every plane distribution Π. As such, $\mathbf{R}_n^m$ is said to be flat, that is, a space of constant sectional curvature 0. If we consider the sphere

$$S^n(r^{-2})=\{(x_0,\ldots,x_n)\,|\,x_0^2+x_1^2+\ldots+x_n^2=r^2\},$$

with the metric induced by the ambient euclidean space, it is not hard to show that $K(\Pi)=r^{-2}$ for every plane distribution Π on $S^n(r^{-2})$. This is an example of a space of positive constant sectional curvature r^{-2}. The hyperboloid

$$H^n(-r^{-2})=\{(x_0,x_1,\ldots,x_n)\,|\,-x_0^2+\sum_{i=1}^{n} x_i^2=-r^2\}$$

endowed with the metric

$$ds^2=[\sum_{i=0}^{n} dx_i\otimes dx_i]/\{1-(1/4r^2)\sum_{i=0}^{n}(x_i)^2\}$$

is an example of a space of constant negative sectional curvature $-r^{-2}$.

Complete, simply connected spaces of constant sectional curvature are called <u>real space forms</u>. In [32], any Riemannian space form is shown to be isometric to one of the examples in the previous paragraph as part of a classification of semi-Riemannian space forms. We shall have the opportunity to work with the semi-Riemannian euclidean space $\mathbf{R}_2^{2(n+1)}$ equipped with the metric

$$\langle x,y\rangle=-x_0y_0-x_1y_1+\sum_{i=2}^{2n+1}x_iy_i$$

and an imbedded Lorentzian hypersurface of $\mathbf{R}_2^{2(n+1)}$

$$H_1^{2n+1}(-1)=\{(x_0,x_1,\ldots,x_{2n+2})\mid -x_0^2-x_1^2+x_2^2+\ldots+x_{2n+2}^2=-1\},$$

(with the appropriate metric, $H_1^{2n+1}(-1)$ is a Lorentzian space form of sectional curvature -1).

A real space form will have a particularly simple form for its curvature tensor. If $M(c)$ has constant sectional curvature c, then

$$R(X,Y)Z=c(\langle Z,Y\rangle X-\langle Z,X\rangle Y).$$

(As we already knew, the curvature tensor of a euclidean space vanishes.) On $S^n(1)$,

$$R(X,Y)Z=\langle Z,Y\rangle X-\langle Z,X\rangle Y$$

and on $H^n(-1)$

$$R(X,Y)Z=\langle Z,X\rangle Y-\langle Z,Y\rangle X.$$

If we compute the covariant derivative of R on a space form we will find that $\nabla R=0$. So constant sectional curvature is linked to the idea of covariant constancy.

In general we will say that a semi-Riemannian space form is <u>locally symmetric</u> if its curvature tensor is parallel. Thus, local symmetry is a generalization of constant sectional curvature. Local symmetry has a generalization, as well. Define a new tensor field $R\cdot R$ by letting R act on itself as a derivation: let $X,Y\epsilon T(M)$. Define a (1,3) tensor field $R(X,Y)\cdot R$ by setting

$$(R(X,Y)\cdot R)(V,W)Z=[R(X,Y),R(V,W)]Z-R(R(X,Y)V,W)Z-R(V,R(X,Y)W)Z$$

for all $V,W,Z\in T(M)$. Applying (0.4) and the definition of R, we find that $\nabla R=0$ implies that $R(X,Y)\cdot R=0$ for all $X,Y\in T(M)$, or simply $R\cdot R=0$. A manifold whose curvature tensor satisfies $R\cdot R=0$ is said to be semi-symmetric ([25]). Hence, semi-symmetry is a generalization of local symmetry. A great deal of research has been performed on semi-symmetric Riemannian manifolds, (see [15], [17], [23]-[26] and [28]-[30]), but little if any on semi-symmetric Lorentzian spaces as is done in section 4.

All of the preceeding geometric concepts are aspects of what is called the intrinsic geometry of a semi-Riemannian manifold, as they arise from the structures intrinsic to the manifold and not from any external considerations. However, by immersing one semi-Riemannian manifold into another, we can study the geometry of the image of the immersion as viewed from the ambient manifold. This geometry is called the extrinsic geometry of the immersion.

An immersed manifold in a semi-Riemannian manifold is called a submanifold. (We shall blur the distinction between an immersion and its image.) It must form a semi-Riemannian manifold with structures compatible with those on the ambient manifold; e.g. the metric induced by the ambient manifold forms a semi-Riemannian metric on the submanifold with index at most that of the ambient manifold.

As the extrinsic geometry of a submanifold is intricately linked to the intrinsic geometries of the ambient space and the submanifold, it can be determined by "comparing" the two, i.e. by finding mathematical relations between similar aspects of the two geometries. Hence, given information about either the intrinsic or extrinsic geometry of a submanifold or the intrinsic geometry of the ambient manifold, one can usually determine some information concerning an unknown geometry.

Let M be an immersed semi-Riemannian submanifold of a semi-Riemannian manifold $\check{M}$. The Levi-Civita connection $\check{\nabla}$ on $\check{M}$ will induce a connection ∇ on M that is Levi-Civita with respect to the metric induced on M by $\check{M}$. For $X, Y \epsilon T(M)$ we can write

$$(0.6)\quad \check{\nabla}_X Y = \nabla_X Y + B(X,Y)$$

where the first term is the component tangent to M and the second term is normal. B forms a symmetric, normal-valued, bilinear form on T(M)xT(M) and is called the <u>second fundamental form</u> of the submanifold M. (0.6) is referred to as the <u>Gauss formula</u> for M in $\check{M}$.

If ξ is a normal field to M in T($\check{M}$), then we can write the <u>Weingarten formula</u> for M in $\check{M}$:

$$(0.7)\quad \check{\nabla}_X \xi = -A_\xi X + \nabla^\perp{}_X \xi$$

where as before the first term is tangential and the second normal to M. A_ξ is called the <u>Weingarten map</u> associated to ξ and forms a self-adjoint tangent bundle endomorphism on T(M). $\nabla^\perp$ is called the <u>normal connection</u> on M and actually satisfies all the axioms of a connection if tangent vectors and normal vectors are used appropriately ([3]). Due to the unique nature of B, its covariant derivative, $\mathring{\nabla}B$, has a separate definition and is defined (as in [3]) by

$$(\mathring{\nabla}_X B)(Y,Z) = \nabla^{\perp}_X(B(Y,Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$$

for all $X,Y,Z \in T(M)$. $\mathring{\nabla}$ is called the connection of Van der Waerden-Bortolotti.

The tensors $B: T(M) \times T(M) \to T(M)^{\perp}$ and $A: T(M) \times T(M)^{\perp} \to T(M)$ obtained by (0.6) and (0.7) contain all the information necessary to determine the extrinsic geometry of M in $\check{M}$, and are intimately related via

$$(0.8) \quad \langle \xi, B(X,Y) \rangle = \langle A_{\xi} X, Y \rangle$$

for all $X,Y \in T(M)$ and $\xi \in T(M)^{\perp}$. For instance, the curvature tensor $\check{R}$ of $\check{M}$ is related by B to the curvature tensor R of M by the <u>Gauss equation</u>:

$$(0.9) \quad \langle R(X,Y)Z,W \rangle = \langle \check{R}(X,Y)Z,W \rangle + \langle B(X,W),B(Y,Z) \rangle - \langle B(X,Z),B(Y,W) \rangle$$

and the Codazzi equation:

(0.10) $(\check{R}(X,Y)Z)^{\perp}=(\dot{\nabla}_X B)(Y,Z)-(\dot{\nabla}_Y B)(X,Z)$

for all $X,Y,Z,W \in T(M)$, where $^{\perp}$ denotes the normal component relative to M. In these equations we see that the second fundamental form provides a measure of the difference between the curvature of the submanifold and that of the ambient space. In case the ambient manifold is a space form $\check{M}(k)$, the Gauss and Codazzi equations are simpler and more explicit:

$$\langle R(X,Y)Z,W\rangle = k(\langle Z,Y\rangle\langle X,W\rangle-\langle Z,X\rangle\langle Y,W\rangle) + \langle B(Y,Z),B(X,W)\rangle-\langle B(X,Z),B(Y,W)\rangle$$

and

$$(\dot{\nabla}_Y B)(X,Z)=(\dot{\nabla}_X B)(Y,Z)$$

for all $X,Y,Z,W \in T(M)$.

The second fundamental form can be used to define a normal field that is, in a sense, a measure of how M curves relative to $\check{M}$. Let $X_1,\ldots,X_n$ be a local orthonormal basis of $T(M)$ consisting of non-neutral vector fields and set $\epsilon_i=1$ if X_i is spacelike and $\epsilon_i=-1$ if X_i is timelike. The normal field

$$\xi_M=(1/n)\sum \epsilon_i B(X_i,X_i)$$

is called the <u>mean curvature vector</u> of M in $\check{M}$. It will lengthen and twist in $T(\check{M})$ according to the relative curvature of M. M is said to be <u>minimal</u> in $\check{M}$ if ξ_M vanishes on M.

Straightness and distance are concepts that serve as useful tools for comparing the intrinsic and extrinsic geometry of a submanifold. For example, geodesics in $S^2(r)$, $r>0$, are great circles isometric to $S^1(r)$ which are certainly not geodesic in the ambient space $\mathbf{R}^3$. Hence, $S^2(r)$ is not flat in $\mathbf{R}^3$ although locally it may

appear to be so. In contrast, an extrinsically flat surface in $\mathbf{R}^3$ is forced to be a plane as all its geodesics would have to be extrinsically straight, that is, lines. The notion of extrinsic flatness is formulated as follows:

Definition

If all the geodesics of a submanifold of a semi-Riemannian manifold are also geodesic in the ambient manifold, then the submanifold is said to be totally geodesic.

It is easy to see geometrically that the totally geodesic submanifolds of a euclidean space are euclidean spaces of lower dimensions, and that the totally geodesic submanifolds of spheres are merely great spheres of lower dimensions. However, for spaces in which geometric intuition fails us, it would be nice to have an analytic criterion of this condition. It happens that there is a nice characterization of this in terms of the second fundamental form. In a semi-Riemannian manifold, it can be shown that a submanifold is totally geodesic if and only if its second fundamental form vanishes.

From these examples we catch a glimpse of why the intrinsic geometry of a manifold is determined by its totally geodesic submanifolds. On the other side of the coin, we can see that the length of the second fundamental form serves as a measure of how far a submanifold deviates from being extrinsically flat and the length of the mean curvature vector serves as a measure of how far the submanifold deviates from being minimal.

From (0.8) we see that for a vector field ξ normal to a totally geodesic submanifold M, the associated Weingarten map vanishes, i.e. $A_\xi = 0 \cdot I_{T(M)}$. This is an example of a more general condition that can be imposed on a submanifold, that of requiring the existence of a normal field to have an associated Weingarten map that is proportional to the identity map on T(M). In general, a normal field is said to be <u>umbillic</u> on M if A_ξ is proportional to the identity map on T(M). If $B(X,Y) = \langle X,Y \rangle \xi_M$ for all $X, Y \in T(M)$, then M is said to be <u>totally umbillic</u> in the ambient semi-Riemannian space.

For example, a sphere $S^2(r)$ of radius $1/\sqrt{r}$ in $\mathbf{R}^3$ is totally umbillic with a single global Weingarten map $A_\xi = rI_{T(S^2(r))}$. (Notice that there is only one choice for ξ on $S^2(r)$ and that $\xi = \xi_{S^2(r)}$.) In general, $S^n(r)$ is totally umbillic in $\mathbf{R}^m$, $m>n$. Small spheres $S^m(r')$ are also totally umbillic in $S^n(r)$, for $m<n$ and $r<r'$.

The submanifolds that will be of primary concern in the following sections will be those of codimension one in the ambient space, i.e. hypersurfaces. As there is only one uniquely (up to choice of orientation) determined normal direction on a hypersurface, many of the preceeding formulae that express the extrinsic geometry will simplify considerably.

Let M^{n-1} be a hypersurface of the semi-Riemannian manifold $\check{M}$. Let ξ be a global unit normal field on M in $\check{M}$. Then the Gauss and Weingarten fomulae can be written as

(0.11) $\quad \check{\nabla}_X Y = \nabla_X Y + b(X,Y)\xi \qquad$ and $\qquad \check{\nabla}_X \xi = -HX$

for all $X,Y\in T(M)$. Notice that $H=A_\xi$ and that the second fundamental form can be expressed as a (0,2) tensor. However, even b can be discarded when (0.8) is applied: $b(X,Y)=\langle HX,Y\rangle$. Hence, almost all the information concerning the extrinsic geometry of the hypersurface is embodied in the single (1,1) tensor H. Henceforth, we shall refer to H as the second fundamental form of the hypersurface.

The Gauss and Codazzi equations for a hypersurface M of a real space form $\check{M}(k)$ are:

$$R(X,Y)Z=k(\langle Z,Y\rangle X-\langle Z,X\rangle Y)+\langle HY,Z\rangle HX-\langle HX,Z\rangle HY$$

and

$$(\nabla_X H)Y=(\nabla_Y H)X$$

for all $X,Y,Z\in T(M)$, which means that the curvature of a hypersurface is (relatively) easy to analyse. We immediately see that the task of determining the extrinsic geometry of a hypersurface reduces to analysing the behavior of the $T(M)$ endomorphism H.

Of course, many different types of hypersurfaces of Riemannian space forms have already been classified. However, this is not at all the case where the ambient space form is semi-Riemannian or in particular if the ambient space form is merely Lorentzian. The primary reason for this is that H, although self-adjoint, may not be diagonalizable (i.e. have real eigenvalues) on $T(M)$ if M has non-zero index; a stark contrast to the Riemannian case. In sections 4 and 5, an initial attack on this problem is made when semi-symmetric Lorentzian hypersurfaces of a certain ambient Lorentzian space form are classified.

However, the main purpose of the subsequent sections is to classify some hypersurfaces in a certain Riemannian manifold that does not have constant sectional curvature. Yet, this ambient space will have a specific notion of curvature being constant if we view it as a manifold with base field **C** instead of **R**. So a short discussion of complex manifolds should ensue.

Let M be a 2n-dimensional Riemannian manifold with $J \in End(T(M))$

that has the property that $J^2=-1_{T(M)}$. Then at each point $p\epsilon M$, $T_p(M)$ forms a complex vector space that is isomorphic to $\mathbf{C}^n$ and J can be associated with the endomorphism obtained on $\mathbf{C}^n \cong \mathbf{R}^{2n}$, viewed as a real vector space, from multiplication by $i=\sqrt{-1}$. J is called an almost complex structure on M. Clearly, if such a structure exists on a Riemannian manifold, the manifold is necessarily even-dimensional.

Of course, $\mathbf{C}^{n+1}$ itself forms such a manifold. In fact, $\mathbf{C}^{n+1}$ can lead us to the correct generalization of semi-Riemannian manifolds over $\mathbf{R}$ to those with base field $\mathbf{C}$: by defining a symmetric bilinear form

$$F_q(z,w)=-\sum_{j=0}^{q} z_j\bar{w}_j + \sum_{k=q+1}^{n} z_k\bar{w}_k$$

for $z=(z_0,z_1,...,z_n)$ and $w=(w_0,w_1,...,w_n)$, we obtain a non-degenerate Hermitian metric on $\mathbf{C}^{n+1}$ that turns $\mathbf{C}^{n+1}$ into a semi-Riemannian complex manifold of index q. $\mathbf{C}^{n+1}$ with the metric F_q will usually be written $\mathbf{C}_q^{n+1}$. $\mathbf{C}_q^{n+1}$ can be made into a real even-dimensional

semi-Riemannian euclidean space $\mathbf{R}_q^{2n+2}$ by using as a metric the

(0,2) tensor $\langle\, ,\, \rangle = \mathrm{Re}(F_q)$. Now the almost complex structure J is

Hermitian with respect to $\langle\, ,\, \rangle$, that is

$$\langle JX, JY \rangle = \langle X, Y \rangle$$

for all real vector fields X and Y on $\mathbf{R}_q^{2n+2}$, so J is both orthogonal and skew-adjoint. Furthermore, J will be parallel with respect to the Levi-Civita connection induced by this metric. $\mathbf{R}_q^{2n+2}$ is an example of what is called a Kaehler semi-Riemannian manifold.

Definition

A semi-Riemannian manifold M with an almost complex structure J and metric $\langle\, ,\, \rangle$ is Kaehler provided that

i) $\langle JX, JY \rangle = \langle X, Y \rangle$ for all $X, Y \in M$, and

ii) $\nabla J = 0$.

On a Kaehler manifold, J is often referred to simply as a complex structure.

Other examples can be derived from $\mathbf{C}_q^{n+1}$. For instance, consider the sphere

$$S^{2n+1}(1)=\{z\in \mathbf{C}^{n+1} \mid |z_0|^2+|z_1|^2+\ldots+|z_n|^2=1\}.$$

We can form a Riemannian submersion of $S^{2n+1}(1)$ onto an complex n-dimensional Riemannian manifold $\mathbf{C}P^n$ by identifying all points on $S^{2n+1}(1)$ that lie on a complex line through the origin of $\mathbf{C}^{n+1}$. The metric and almost complex structure of $\mathbf{C}P^n$ can be induced from the natural complex structure of $\mathbf{C}^{n+1}$. Notice that $S^{2n+1}(1)$ forms a principal fiber bundle over $\mathbf{C}P^n$ with fiber $S^1(1)$. $\mathbf{C}P^n$ is called complex projective space.

If M is a Kaehler manifold, then for any $p\in M$ and $X\in T(M)$, the plane $\Pi_p=\text{span}\{X_p, JX_p\}$ is invariant under J and is said to be a holomorphic section at p. The sectional curvature $K(\Pi_p)$ is called the holomorphic sectional curvature of M by Π at p. If the sectional curvature is a constant for all J-invariant planes Π_p at $p \in M$, i.e. $K(\Pi_p)=K_p$ for all holomorphic sections Π_p, then M is said to be of constant holomorphic sectional curvature at p. As in the real case, it is well known that if M is of constant holomorphic sectional curvature K_p at each $p \in M$, then K_p is a constant over M. In this case M is said to be of constant holomorphic sectional curvature. If, in addition, M is simply connected, M is called a complex space form.

C_q^{n+1} is a complex space form of zero holomorphic sectional curvature whereas on CP^n a metric (namely the Fubini-Study metric tensor) can be constructed that turns CP^n into a complex space form of holomorphic sectional curvature 4. CP^n is compact and has diameter $\pi/2$ under this metric. The primary ambient space in the following sections is a complex hyperbolic space, CH^n, that has constant holomorphic sectional curvature -4. Sections 1, 2 and 3 only use the abstract properties derived from the constant holomorphic sectional curvature of CH^n. However, CH^n can also be

constructed in a fashion similar to that of CP^n, only in this case the submersion is from a Lorentzian hyperbolic space form. This construction is crucial to obtaining the geometric results of the later sections.

As with real space forms, complex space forms and their submanifolds admit relatively simple expressions for their curvature tensors. If $M(c)$ is a complex space form of holomorphic sectional curvature c, then the curvature tensor R of M is given by

$$(0.12)\quad R(X,Y)Z=(c/4)[\langle Y,Z\rangle X-\langle X,Z\rangle Y \\ +\langle JY,Z\rangle JX-\langle JX,Z\rangle JY+2\langle X,JY\rangle JZ]$$

for all $X,Y,Z\epsilon T(M)$.

If M is a real semi-Riemannian manifold immersed in a complex space form $\check{M}(c)$ of complex dimension n, then the Gauss and Weingarten formulae hold for M as a submanifold of the real 2n-dimensional semi-Riemannian manifold $\check{M}$. Now (0.9) and (0.10) can be combined with (0.12) to obtain the Gauss and Codazzi

equations of M in $\check{M}$.

In the sequel we shall be interested in the particular case where M is a real hypersurface of the Riemannian complex hyperbolic space form $\mathbf{C}H^n(-4)$.

1. Real Hypersurfaces of $\mathbf{C}H^n(-4)$

Let $\mathbf{C}H^n(-4)$, $n\geq 2$, denote a complex hyperbolic space with the Bergman metric tensor, i.e. a complex space form of constant holomorphic sectional curvature -4. Let M^{2n-1} be a real hypersurface of $\mathbf{C}H^n$, ∇ and $\boldsymbol{\nabla}$ be the metric connections on M and $\mathbf{C}H^n$, respectively, so that the Gauss and Weingarten formulae can be written as:

$$(1.1)\qquad \boldsymbol{\nabla}_X Y = \nabla_X Y + \langle HX, Y\rangle \xi \ , \quad \boldsymbol{\nabla}_X \xi = -HX \quad \text{for all } X, Y \in T(M),$$

where ξ is a unit normal field on M in $\mathbf{C}H^n$ and H denotes the second fundamental form (in this case the Weingarten map of ξ in End[T(M)]). We shall refer to the eigenvalues and eigenvectors of H in $\mathbf{R}$ and T(M), respectively, as principal curvatures and principal directions.

If J is the complex structure of the ambient complex space form, it induces an endomorphism ϕ of rank 2n-2 and a linear functional f on T(M) given by setting at each point p of M

(1.2) $JX = \phi X + f(X)\xi$

for all X in $T_p(M)$. Set $U = -J\xi$. As M is of codimension one we have $U \epsilon T(M)$. The following equations now hold for all X,Y in $T(M)$:

(1.3) $f(X) = \langle X,U \rangle$

(1.4) $f(\phi X) = 0$

(1.5) $\phi U = 0$

(1.6) $\phi^2 X = -X + f(X)U$

(1.7) $\langle X,\phi Y \rangle = -\langle \phi X,Y \rangle$

(1.8) $\langle \phi X,\phi Y \rangle = \langle X,Y \rangle - f(X)f(Y)$.

(ϕ,f,U) is an example of what is called an almost contact structure on M. The tensor fields ϕ and U have the following derivatives:

(1.9) $\nabla_X U = \phi HX$

(1.10) $(\nabla_X \phi)Y = f(Y)HX - \langle HX,Y\rangle U.$

We also have the usual Gauss and Codazzi equations for a real hypersurface of a complex space form (of holomorphic sectional curvature -4) in terms of ϕ and H:

(1.11) $$R(X,Y)Z = \langle X,Z\rangle Y - \langle Y,Z\rangle X + \langle \phi X,Z\rangle \phi Y - \langle \phi Y,Z\rangle \phi X - 2\langle X,\phi Y\rangle \phi Z + \langle HY,Z\rangle HX - \langle HX,Z\rangle HY$$

(1.12) $$(\nabla_X H)Y - (\nabla_Y H)X = -f(X)\phi Y + f(Y)\phi X - 2\langle X,\phi Y\rangle U$$

for all $X,Y,Z \epsilon T(M)$, where R is the curvature tensor on M.

An important special case for us will be when U is a principal direction on M. Under this assumption, more information can be gained concerning the structure of M. For example, by applying (1.9) and assuming that $HU=\alpha U$ for some $\alpha \epsilon C^{\infty}(M)$,

$$\nabla_U U = \phi HU = \phi(\alpha U) = \alpha \cdot \phi U = 0$$

by (1.5). Thus, if U is principal the trajectories of U are geodesics in M. Conversely, if the trajectories of U are geodesics, then

$$0 = \nabla_U U = \phi HU$$

which shows that $HU \epsilon \ker(\phi) = \mathrm{span}\{U\}$ as ϕ is of rank $2n-2$ on $T(M)$. This shows that for a real hypersurface M of $CH^n(-4)$, the direction U is principal if and only if the trajectories of U on M are geodesic, [14].

The assumption that U is a principal direction will also force a strong relationship to hold between H and ϕ:

<u>Lemma 1</u>-[14]

Suppose that U is a principal direction on M with principal curvature α. Then

$$(1.13) \qquad 2(H\phi H+\phi)=\alpha(\phi H+H\phi) \qquad \text{on } M.$$

Proof:

Assume that $HU=\alpha U$ for $\alpha\epsilon C^{\infty}(M)$ on M. Applying (1.9), for all $X\epsilon T(M)$,

$$(\nabla_X H)U = \nabla_X HU - H\nabla_X U = \nabla_X(\alpha U) - H\nabla_X U$$

$$= (X\alpha)U + \alpha\nabla_X U - H\phi HX = (X\alpha)U + \alpha\phi HX - H\phi HX.$$

As $\nabla_X H$ is self-adjoint, for all $Y\epsilon T(M)$,

$$<(\nabla_X H)Y,U> = <Y,(\nabla_X H)U>$$

$$= (X\alpha)f(Y) + \alpha<\phi HX,Y> - <H\phi HX,Y>.$$

Now using (1.4) and (1.12),

$$\langle(\nabla_X H)Y-(\nabla_Y H)X,U\rangle = -2\langle X,\phi Y\rangle.$$

Combining these equations yields

$$-2\langle X,\phi Y\rangle=(X\alpha)f(Y)+\alpha\langle Y,\phi HX\rangle-\langle Y,H\phi HX\rangle$$
$$-(Y\alpha)f(X)-\alpha\langle X,\phi HY\rangle+\langle X,H\phi HY\rangle$$
$$=(X\alpha)f(Y)-(Y\alpha)f(X)+\alpha\langle(\phi H+H\phi)X,Y\rangle-2\langle H\phi HX,Y\rangle$$

where (1.7) is used to combine terms. Rewrite the above equation as

$$(1.14)\quad 2\langle(H\phi H+\phi)X,Y\rangle=(X\alpha)f(Y)-(Y\alpha)f(X)+\alpha\langle(\phi H+H\phi)X,Y\rangle.$$

Replacing X by U and then Y by U in (1.14) yields $Y\alpha=(U\alpha)f(Y)$ and $X\alpha=(U\alpha)f(X)$. Substituting these values back into (1.14) gives

$$2\langle(H\phi H+\phi)X,Y\rangle=(U\alpha)f(X)f(Y)-(U\alpha)f(Y)f(X)+\alpha\langle(\phi H+H\phi)X,Y\rangle$$
$$=\alpha\langle(\phi H+H\phi)X,Y\rangle.$$

As X and Y are arbitrary we have the assertion. //

Once U is principal it is clear that we can extend to a local orthonormal basis of principal directions $\{X_1,\ldots,X_{2n-2},U\}$ of $T(M)$ with $\{X_1,\ldots,X_{2n-2}\}$ forming a local orthonormal basis of ker(f). Assume that each X_i has as principal curvature λ_i, $i=1,\ldots,2n-2$. The next question to ask, since ker(f) is ϕ-invariant, are there nontrivial ϕ-invariant subspaces of ker(f) that are also H-invariant? Not surprisingly, the answer is yes.

Lemma 2-[14]

If λ is a principal curvature on M, let D_λ denote the distribution of principal directions on M with principal curvature λ. If $X\in D_\lambda\cap\ker(f)$ and $\lambda^2-1\neq0$, then ϕX is also principal.

Proof:

$$2(H\phi H+\phi)X=2\lambda H\phi X+2\phi X \quad \text{and} \quad \alpha(\phi H+H\phi)X=\alpha\lambda\phi X+\alpha H\phi X.$$

So by lemma 1,

$H\phi X=(\alpha\lambda-2)/(2\lambda-\alpha)\cdot\phi X$

if $\alpha\neq 2\lambda$. If $\alpha=2\lambda$ then $2\lambda^2-2=0$ as X and hence ϕX can be chosen to be nontrivial in ker(f). This is equivalent to $\lambda^2=1$ and $\alpha^2=4$. //

For the remaining case assume first that $\alpha=2$ and $\lambda=-1$. Then, for $X\in D_{-1}$, (1.13) yields ϕX principal with principal curvature 1. However, if $\alpha=2$ and $\lambda=1$, then for $X\in D_1$, (1.13) is an identity on the distribution span$\{X,\phi X,U\}$.

A major consequence of Lemma 2 is that whenever U is principal on M, we can choose a frame $\{X_1,\dots,X_{n-1},\phi X_1,\dots,\phi X_{n-1},U\}$ on M that consists entirely of principal directions with the property that ϕ interchanges the distributions $\mathrm{span}\{X_1,\ldots,X_{n-1}\}$ and $\mathrm{span}\{\phi X_1,\ldots,\phi X_{n-1}\}$. The set $\{X_1,\ldots,X_{n-1},\phi X_1,\ldots,\phi X_{n-1}\}$ forms a local basis of the distribution $\ker(f)$ on which ϕ acts as a complex structure. The principal curvatures of a ϕ-invariant plane $\mathrm{span}\{X_i,\phi X_i\}$ will be related by

$$(1.15)\qquad \gamma_i=(\alpha\lambda_i-2)/(2\lambda_i-\alpha)$$

where $HX_i=\lambda_i X_i$ and $H\phi X_i=\gamma_i\phi X_i$, whenever $\alpha\neq 2\lambda_i$.

The principal curvatures need not be constant even if U is principal as we will see in example 4 of section 3. This is contrary to the situation of the ambient space being CP^n (see [14]). However, there are two classes of real hypersurfaces in CH^n that do have U principal with all the principal curvatures constant and these are the subjects of section 2.

2. Contact Hypersurfaces: Algebraic Consequences of the Contact Condition

Let M be a 2n-1 dimensional Riemannian manifold that admits a triple of tensor fields (ϕ, f, U), (where $\phi \in End[T(M)]$, f is a linear functional on T(M) and $U \in T(M)$), satisfying (1.3) and (1.6). As remarked in the preceeding section, such a triple (ϕ, f, U) forms an <u>almost contact structure</u> on M. In general, a Riemannian manifold that admits an almost contact structure also admits a metric satisfying (1.8). From these formulae (1.4), (1.5) and (1.7) can be obtained. This is a generalization of another intrinsic condition that can be imposed on a Riemannian manifold: M is said to be a <u>contact manifold</u> if it admits a linear functional f that satisfies $f \wedge (df)^{n-1} \neq 0$. (Such a manifold also admits an almost contact structure (ϕ, f, U), [1].)

In section 1, we saw that a real hypersurface M of CH^n (in fact of any complex space form) automatically admits an almost contact structure that is already compatible with the metric induced from

the ambient space. In [19], Okumura showed that if M^{2n-1} is a contact real hypersurface of a complex space form of complex dimension n, then

$$(2.1)\quad \phi H+H\phi=2\rho\phi$$

on M, where ρ can be shown to be a constant. By selecting an appropriate orientation of M, ρ may be assumed to be positive. In particular, (2.1) is equivalent to $\rho^{-n}f\wedge(df)^{n-1}\neq 0$. In the following discussion we shall assume that M is a complete, connected, contact hypersurface of $CH^n(-4)$, with $n\geq 3$, and obtain algebraic consequences of (2.1). Specifically, the principal curvatures and directions will be determined using (2.1).

Combining (1.5) and (2.1) we have $\phi HU=0$, which shows that $HU\in\mathrm{span}\{U\}$, i.e. U is a principal direction. Set $HU=\alpha U$.

Lemma 3

On a contact hypersurface of $\mathbf{C}H^n$, $H|_{ker(f)}$ satisfies the polynomial

$$(2.2)\quad \lambda^2-2\rho\lambda+\alpha\rho-1=0.$$

Proof:

Combining Lemma 1 and formula (2.1), we have $H\phi H+\phi=\alpha\rho\phi$. Applying (2.1) again,

$$\alpha\rho\phi=\phi+H(2\rho\phi-H\phi)=\phi+2\rho H\phi-H^2\phi.$$

So,

$$0=H^2\phi-2\rho H\phi+(\rho\alpha-1)\phi$$

on M. Choosing $X\epsilon ker(f)$ we can write $X=\phi Y$ for some Y (namely $Y=-\phi X$) to see that

$0=H^2X-2\rho HX+(\alpha\rho-1)X$ for all $X\epsilon\ker(f)$. //

From this we have

<u>Lemma 4</u>-[13]

α is constant on M, if M is contact.

Proof:

From the proof of Lemma 1, we have $X\alpha=(U\alpha)f(X)$ for all $X\epsilon T(M)$. Thus, for $X\epsilon\ker(f)=\{U\}^{\perp}$ we have $X\alpha=0$. So it suffices to show that $U\alpha=0$.

Since $0=X\alpha=<\nabla\alpha,X>$, where $\nabla\alpha$ denotes the gradient field of the function α on M, we have $\nabla\alpha\epsilon\mathrm{span}\{U\}$. Thus,

$$\nabla\alpha=<\nabla\alpha,U>U=(U\alpha)U.$$

Hence,

$$\nabla_X(\nabla\alpha)=X(U\alpha)U+(U\alpha)\nabla_X U$$
$$=X(U\alpha)U+(U\alpha)\phi HX.$$

This means that

$$\langle\nabla_X(\nabla\alpha),Y\rangle=X(U\alpha)f(Y)+(U\alpha)\langle\phi HX,Y\rangle$$

and similarly

$$\langle\nabla_Y(\nabla\alpha),X\rangle=Y(U\alpha)f(X)+(U\alpha)\langle\phi HY,X\rangle.$$

But, we also know that

$$\langle\nabla_X(\nabla\alpha),Y\rangle=X\langle\nabla\alpha,Y\rangle-\langle\nabla\alpha,\nabla_X Y\rangle$$
$$=X(Y\alpha)-\nabla_X Y(\alpha)$$

and

$\langle \nabla_Y(\nabla\alpha),X\rangle = Y(X\alpha)-\nabla_Y X(\alpha).$

Then,

$$\langle \nabla_X(\nabla\alpha),Y\rangle - \langle \nabla_Y(\nabla\alpha),X\rangle = X(Y\alpha)-Y(X\alpha)+\nabla_Y X(\alpha)-\nabla_X Y(\alpha)$$

$$=[X,Y](\alpha)+[Y,X](\alpha)$$

$$=0.$$

Now

$$0=\langle \nabla_X(\nabla\alpha),Y\rangle - \langle \nabla_Y(\nabla\alpha),X\rangle$$

$$=X(U\alpha)f(Y)+(U\alpha)\langle \phi HX,Y\rangle - Y(U\alpha)f(X)-(U\alpha)\langle \phi HY,X\rangle$$

which yields

$$(2.3)\quad X(U\alpha)f(Y)-Y(U\alpha)f(X)=(U\alpha)\langle \phi HY,X\rangle-(U\alpha)\langle \phi HX,Y\rangle$$

$$=(U\alpha)\langle (\phi H+H\phi)Y,X\rangle$$

$$=2\rho(U\alpha)\langle\phi Y,X\rangle.$$

Substituting U for Y and then for X in (2.3) gives

$$X(U\alpha)=U(U\alpha)f(X)$$

and

$$Y(U\alpha)=U(U\alpha)f(Y).$$

Substituting these values into the left hand side of (2.3) yields

$$0=2\rho(U\alpha)\langle\phi Y,X\rangle.$$

Choosing $X=\phi Y\neq 0$ shows that $U\alpha=0$. //

Thus, M has at most three distinct principal curvatures, all of which must be constant by (2.2) and Lemma 4. Since $CH^n(-4)$ has no

complete totally umbillic hypersurfaces (see [5]), we are left with only two cases to consider:

A) (2.2) has two distinct solutions $\lambda_1 \neq \lambda_2$, or

B) (2.2) has only one solution $\lambda \neq \alpha$.

Case B) is the easiest to analyse. Let D_λ and D_α denote the eigendistributions of λ and α on M. Of course, D_λ is of dimension $2n-2$ and D_α has dimension 1. It is immediate that ϕ acts as a complex structure on D_λ. Requiring (2.2) to have only one solution forces $\lambda=\rho$ and $\rho^2-\alpha\rho+1=0$. The latter equation has real solutions only when $\alpha^2-4\geq 0$. In case $\alpha^2-4>0$, we may regard α as a parameter. By selecting the orientation of M appropriately, we may assume that $\alpha=2\coth(2r)$ and $\lambda=\rho=\tanh(r)$ or $\coth(r)$, for some $r>0$. Otherwise, set $\alpha=2$ and $\lambda=\rho=1$. So with respect to the frame consisting of principal directions $\{X_1,\ldots,X_{n-1},\phi X_1,\ldots,\phi X_{n-1},U\}$ of $T(M)$, (where for each $i=1,\dots,n-1$, $HX_i=\lambda X_i$ and $H\phi X_i=\lambda\phi X_i$), H has only three possible matrix representations:

i) $\mathrm{diag}(\tanh(r)I_{2n-2}, 2\coth(2r))$,

ii) $\mathrm{diag}(\coth(r)I_{2n-2}, 2\coth(2r))$, or

iii) $\mathrm{diag}(I_{2n-2}, 2)$.

Notice that in each of these cases, $HX=\lambda X+\alpha f(X)U$, i.e. M is <u>totally U-umbillic</u>. These hypersurfaces also satisfy the condition $\phi H=H\phi$. In fact it is not hard to show that a contact hypersurface is totally U-umbillic if and only if $\phi H=H\phi$. Real hypersurfaces satisfying this condition in CH^n have been classified in [16]. However, the classification for hypersurfaces satisfying B) in this paper will involve a new geometric characterization.

The analysis of case A) is considerably more laborious and basically new. We first notice that if $\alpha^2-4<0$, then M must satisfy A). In the following we will show that if M satisfies A) and $n\geq 3$, then $\alpha^2-4<0$.

Let D_1 and D_2 be the eigendistributions of λ_1 and λ_2, respectively. Since (2.2) has two distinct solutions $\lambda_1 \neq \lambda_2$, $\lambda_1+\lambda_2=2\rho$. Now if $X\epsilon D_1$, (2.1) shows that $\phi X\epsilon D_2$. Therefore, ϕ interchanges the distributions D_1 and D_2 from which it follows that each distribution is of dimension $n-1$.

<u>Lemma 5</u>

If M satisfies A) and $n\geq 3$, then $\lambda_1\lambda_2=1$.

Proof:

This is established in a series of steps. First assume that $\alpha\neq\lambda_i$ for $i=1,2$.

Step 1-

If X and Y are in D_i then so is $\nabla_X Y$.

Proof of step 1:

We shall prove this in the case $i=1$. Let $X,Y\epsilon D_1$. Then,

$$\langle\nabla_X Y,U\rangle=X\langle Y,U\rangle-\langle Y,\nabla_X U\rangle$$

$$=-\langle Y,\phi HX\rangle$$

$$=-\lambda_1\langle Y,\phi X\rangle$$

$$=0.$$

This shows that $\nabla_X Y\epsilon \ker(f)$ so that $H\nabla_X Y\epsilon \ker(f)$. Let $Z\epsilon \mathrm{Ker}(f)$. Then

$$\langle H\nabla_X Y,Z\rangle=\langle\nabla_X HY,Z\rangle-\langle(\nabla_X H)Y,Z\rangle$$

$$=\lambda_1\langle\nabla_X Y,Z\rangle-\langle Y,(\nabla_X H)Z\rangle$$

$$=\lambda_1\langle\nabla_X Y,Z\rangle-\langle Y,(\nabla_Z H)X-f(X)\phi Z+f(Z)\phi X-2\langle X,\phi Z\rangle U\rangle$$

$$=\lambda_1\langle\nabla_X Y,Z\rangle-\langle Y,(\nabla_Z H)X\rangle$$

$$=\lambda_1 <\nabla_X Y,Z>-<Y,\nabla_Z HX>+<Y,H\nabla_Z X>$$

$$=\lambda_1 <\nabla_X Y,Z>-\lambda_1 <Y,\nabla_Z X>+\lambda_1 <Y,\nabla_Z X>$$

$$=\lambda_1 <\nabla_X Y,Z>.$$

Thus, $H\nabla_X Y=\lambda_1 \nabla_X Y$ which completes the proof of step 1.

Step 2-

If $X\epsilon D_1$ and $Y\epsilon D_2$, then $\nabla_X Y\epsilon D_2 \oplus \mathrm{span}\{U\}$ and $\nabla_Y X\epsilon D_1 \oplus \mathrm{span}\{U\}$.

Proof of step 2:

Let $Z\epsilon D_1$. Then,

$<Y,Z>=0 \Rightarrow 0=<\nabla_X Y,Z>+<Y,\nabla_X Z> \Rightarrow <\nabla_X Y,Z>=0$ by step 1.

This yields the first inclusion. The second follows in exactly the same way by choosing $Z\epsilon D_2$.

Step 3-

If $X\epsilon D_1$ and $Y\epsilon D_2\cap\{\phi X\}^\perp$, then $\nabla_X Y\epsilon D_2$ and $\nabla_Y X\epsilon D_1$. (Here we see the reason for the stipulation $n\geq 3$.)

Proof of step 3:

From the hypotheses,

$$\begin{aligned}\langle\nabla_X Y,U\rangle &= X\langle Y,U\rangle-\langle Y,\nabla_X U\rangle \\ &= -\langle Y,\phi HX\rangle \\ &= -\lambda_1\langle Y,\phi X\rangle \\ &= 0.\end{aligned}$$

Combining this with step 2 we have step 3.

Let $X\epsilon D_1$ and $Y\epsilon D_2\cap\{\phi X\}^\perp$ with $\|X\|=\|Y\|=1$. Applying steps 1,2 and 3 we find that

$$R(X,Y)Y=\nabla_X(\nabla_Y Y)-\nabla_Y(\nabla_X Y)-\nabla_{[X,Y]}Y\epsilon D_2\oplus\mathrm{span}\{U\}$$

by writing $[X,Y]=\nabla_X Y-\nabla_Y X$. However, a direct computation using the Gauss equation reveals that $R(X,Y)Y=(\lambda_1\lambda_2-1)X\epsilon D_1$. Therefore, $R(X,Y)Y=0$ and since X is nontrivial we must have $\lambda_1\lambda_2-1=0$.

Now if $\alpha=\lambda_i$ for i=1 or 2, the same statements hold if D_1 and D_2 are replaced by $D_1\cap\ker(f)$ and $D_2\cap\ker(f)$. //

Because of (2.2) we must have $\lambda_1\lambda_2=\alpha\rho-1$. So by lemma 5, $\alpha\rho=2$ when $n\geq 3$. (2.2) can now be written as $\lambda^2-2\rho\lambda+1=0$. This has two distinct solutions only when $4\rho^2-4>0$, i.e. when $\alpha^2-4<0$. Notice that $\alpha=0$ is ruled out by $\alpha\rho=2$.

Hence, if M is a contact hypersurface satisfying A), α can be viewed as a parameter. So set $\alpha=2\tanh(2r)$, $r>0$. Then the solutions of (2.2) are $\lambda_1=\tanh(r)$ and $\lambda_2=\coth(r)$. (Note that $\rho=(\tanh(r)+\coth(r))/2$ in this case.) So with respect to a suitably chosen basis of $T(M)=D_1\oplus D_2\oplus\mathrm{span}\{U\}$, H has the matrix representation:

$\mathrm{diag}(\tanh(r)I_{n-1}, \coth(r)I_{n-1}, 2\tanh(2r))$.

Remark: M. Okumura in [19] treats the case $\alpha=\lambda_i$ for i=1 or 2 as a separate case. But from our work so far we see that this occurs in case A) for a specific r, namely $r=\ln(2+\sqrt{3})$ so that $\alpha=\lambda_2=\sqrt{3}$ and $\lambda_1=1/\sqrt{3}$.

A quick glance at the classification results in [16] will convince the reader that not all hypersurfaces satisfying

(2.4) $\phi H=H\phi$

are contact. Yet, in section 5 we will still be able to obtain the same sort of characterization for these hypersurfaces as in the contact case. Also in section 4 we shall analyse hypersurfaces of a Lorentzian space that submerse into hypersurfaces of CH^n satisfying (2.4). So we should spend a little time on algebraic consequences of

(2.4). (For more detail see [16].)

Combining (2.4) with (1.4) yields U principal; say with principal curvature α. An argument similar to that of Lemma 4 shows that α is constant. By combining Lemma 1 with (2.4) we have

$$(H|_{\ker(f)})^2-\alpha H|_{\ker(f)}+I_{\ker(f)}=0.$$

That is, H satisfies the polynomial

$$(2.5)\quad \lambda^2-\alpha\lambda+1=0$$

on ker(f).

This equation has real solutions only if $\alpha^2-4\geq 0$. In case $\alpha^2-4>0$ we can again regard α as a parameter and set $\alpha=2\coth(2r), r>0$. The solutions of (2.5) are now $\lambda=\tanh(r)$ or $\coth(r)$. In case $\alpha=\pm 2$ set $\lambda=\pm 1$.

If D_λ is a proper subspace of ker(f), then (2.4) ensures that D_λ is

ϕ-invariant. Since $\ker(f)=D_\lambda \oplus D_{1/\lambda}$, $D_{1/\lambda}$ is also ϕ-invariant so that ϕ acts as a complex structure on each of the even dimensional distributions D_λ and $D_{1/\lambda}$. From our analysis of case A) for contact hypersurfaces we see that a hypersurface satisfying (2.4) is not in general contact.

The possible matrix representations for the second fundamental form on real hypersurfaces satisfying (2.4) with respect to a suitably chosen basis of $T(M)=D_\lambda \oplus D_{1/\lambda} \oplus \mathrm{span}\{U\}$ are now

i) $\mathrm{diag}(\tanh(r)I_{2p}, \coth(r)I_{2n-2-2p}, 2\coth(2r))$, $p=0,\ldots,n-1$, or

ii) $\mathrm{diag}(I_{2n-2}, 2)$.

Of course, if $p=0$ or $p=n-1$ in i) then M is contact. ii) is obviously contact.

3. Tubes in Complex Hyperbolic Space: the Model Hypersurfaces

In this section, hypersurfaces of $CH^n(-4)$ are constructed that have second fundamental forms with the algebraic properties set forth in section 2. Hence, we are provided with an ample supply of contact hypersurfaces as well as those satisfying (2.4). Our initial discussion will be of a more general nature: tubes in Riemannian manifolds. (For more detail, see [6], [9], [11] and [31].)

Recall first the notions of cut point and cut locus. (A detailed and analytic discussion of cut loci can be found in Vol II of [12].) A <u>cut point</u> of a point p in a Riemannian manifold M is a point $c=\gamma(t)$, where γ is a geodesic emanating from $p=\gamma(0)$, with the property that for $s>t$, the length of the curve $\gamma(J)$, $J=[0,s]$, is greater than the distance $d_M(p,\gamma(s))$. For instance, if $p\in S^2(r)$, its only cut point is its antipodal point.

The <u>cut locus</u> of a point $p\in M$, written Cut(p), is the set of all cut points of p. The cut locus of a point on a sphere is a singleton, whereas for a point p on a cylinder, Cut(p) is the axial line opposite

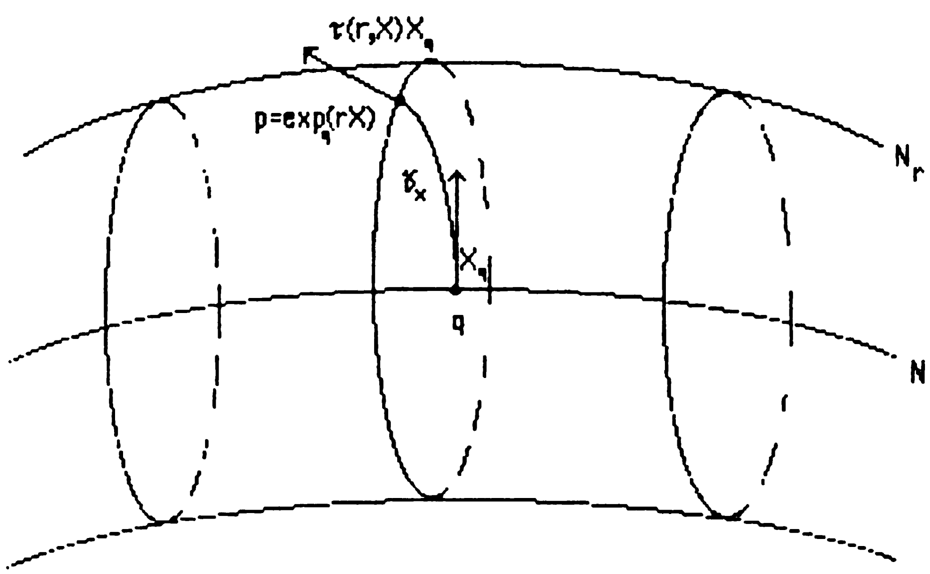
p. Define $c(p)=\min\{d(p,q) \mid q\in Cut(p)\}$.

Let N^m be an immersed submanifold of a Riemannian manifold M^n. Define the unit normal sphere bundle of N by:

$$S^{\perp}(N)=\{X\in T(N)^{\perp}: \|X\|=1\}.$$

Set $c(N)=\inf\{c(p) \mid p\in N\}$. Now for each $r\in(0,c(N))$, define the <u>tube of radius r about N in M</u> to be the hypersurface given by

$$N_r=\{\exp_q(rX): q\in N, X\in S^{\perp}(N)\}.$$



Let $\tau(t,X_q)$ be parallel translation of vector fields along the geodesic

$(\gamma_X)_q: t \to \exp_q(tX)$. For $p=\exp_q(rX)\in N_r$, $\tau(t,X_q): T_q(M) \to T_p(M)$ is a linear isometry. Because parallel translation preserves the fibers of vector bundles in a Riemannian manifold, ([20], p.66), we have

$$(3.1) \quad T_p(N_r)=\tau(r,X_q)(\{X_q\}^\perp)=\{\tau(r,X_q)X_q\}^\perp$$

and

$$(3.2) \quad T_p(N_r)\cong T_q(N) \bullet [\{X_q\}^\perp \cap T_q(N)^\perp],$$

where $\cong$ denotes the isomorphism of parallel translation.

For $X\in S^\perp(N)$ and $q\in N$, define $\tilde{R}_X(t)\in \mathrm{End}[T_q(N)]$, for each $t>0$, by

$$\tilde{R}_X(t)Y_q=\tau(t,X_q)^{-1}\{R(\tau(t,X_q)Y_q,\tau(t,X_q)X_q)\tau(t,X_q)X_q\}$$

where R is the curvature tensor of M. As we are primarily interested

in the tangent space of the tube N_r, set

$$R_X(t)=\tilde{R}_X(t)\,|\,_{\{X\}^\perp}.$$

Finally, define $F(t,X)\epsilon End(\{X\}^\perp)$, for each $X\epsilon S^\perp(N)$ to be the solution of the initial value problem:

$$(3.3)\quad (d^2/dt^2)[F(t,X_q)]+R_X(t)\circ F(t,X_q)=0$$

$$F(0,X_q)=P$$

$$(d/dt)[F(t,X_q)]\,|_{t=0}=-A_X\circ P+P^\perp$$

for each $q\epsilon N$, where $P:\{X\}^\perp\rightarrow T(N)$ and $P^\perp:\{X\}^\perp\rightarrow T(N)^\perp\cap\{X\}^\perp$ are orthogonal projections of the vector bundle $\{X\}^\perp=T(N)\oplus[T(N)^\perp\cap\{X\}^\perp]$ onto the indicated component distributions, and A_X is the Weingarten map of X on N in M. To simplify notation we shall write $F'(s,X_q)$ for $(d/dt)[F(t,X_q)]\,|_{t=s}$ and $F''(s,X_q)$ for $(d^2/dt^2)[F(t,X_q)]\,|_{t=s}$.

Theorem 1 -[11]

The second fundamental form of N_r at $p=\exp_q(rX)$ is given by

(3.4) $H_r=\tau(r,X_q)\circ F'(r,X_q)\circ F(r,X_q)^{-1}\circ\tau(r,X_q)^{-1}$. //

Hence, in order to find an explicit representation of the second fundamental form of a tube, we need merely select a suitable basis of $T(N_r)$ using (3.1) and (3.2), solve (3.3) and then compute (3.4). Of course (3.4) says that $H_r\epsilon\mathrm{End}[T(N_r)]$ at $p=\exp_q(rX)$ is nothing more than parallel displacement of the endomorphism

$F'(r,X_q)\circ F(r,X_q)^{-1}\,\epsilon\mathrm{End}[\{X_q\}^\perp]$

along the geodesic γ_X emanating from q and passing through p.

Now we will assume that M is not an arbitrary Riemannian manifold, but the ambient space discussed in sections 1 and 2, namely a complex hyperbolic space. Let N be an immersed submanifold of $CH^n(-4)$. As CH^n is a symmetric space, once a suitable basis of $\{X_q\}^\perp$ is selected (where $q \in N$ and $X \in S^\perp(N)$), parallel displacement along the geodesic γ_X will preserve the basis and the respective orthogonality relations between its elements. Furthermore, H_r will have the same matrix representation with respect to the displaced basis as $F'(r,X_q) \circ F(r,X_q)^{-1}$ has with respect to the chosen basis of $\{X_q\}^\perp$. This simplifies the calculation of (3.4) considerably.

An additional feature of CH^n is that $R_X(r)$ is of a particularly simple form. Let $X \in S^\perp(N)$ and $Y \in \{X, JX\}^\perp$. Direct computations using the Gauss equation show that:

$$(R_X(t)Y)_q=\tau(t,X_q)^{-1}[R(\tau(t,X_q)Y_q,\tau(t,X_q)X_q)\tau(t,X_q)X_q]$$

$$=\tau(t,X_q)^{-1}[-\tau(t,X_q)Y_q]$$

$$=-Y_q$$

and

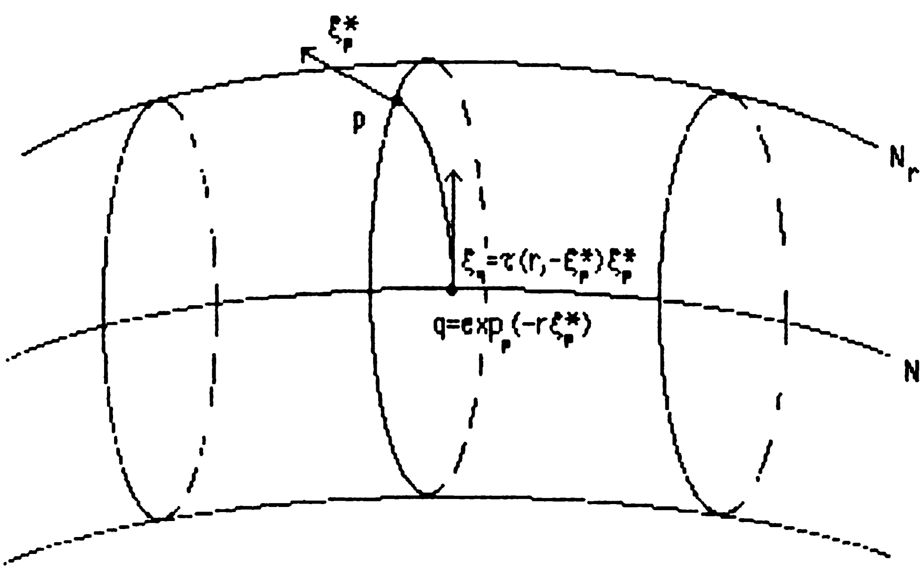
$$(R_X(t)JX)_q=\tau(t,X_q)^{-1}[R(\tau(t,X_q)JX_q,\tau(t,X_q)X_q)\tau(t,X_q)X_q]$$

$$=\tau(t,X_q)^{-1}[-4\tau(t,X_q)JX_q]$$

$$=-4JX_q$$

for all $t>0$. In conjunction with the following observations the task of computing a representation of H_r will be greatly simplified.

As a hypersurface, N_r has a single well defined global unit normal ξ. From the earlier discussion on tubes, at any point $p=\exp_q(rX)\epsilon N_r$, we can write $\xi_p=\tau(r,X_q)X_q$. In this way a unique point q and a unique direction in $S^{\perp}(N_q)$ can be associated to each point $p \in N_r$. In order to simplify notation, set $Y^*=\tau(r,X_q)Y \in T_p(N_r)$ for any $Y \in \{X_q\}^{\perp}$. In particular, we shall write ξ^* for the global normal on N_r, and ξ will refer to the associated direction in $S^{\perp}(N)$; i.e. $\xi^*_p=\exp_q(r\xi_q)$.



In terms of section 1,

$$U^*_p=-J\xi^*_p=-\tau(r,X_q)J\xi_q$$

and

$$\{U^*_p\}^\perp \cap T_p(N_r)=\tau(r,\xi_q)[(T_q(N)\cap(J\xi_q)^\perp)\oplus(\{J\xi_q,\xi_q\}^\perp\cap T_q(N)^\perp)]$$

is the ϕ-invariant subspace, ker(f), of $T_p(N_r)$.

Also, $CH^n(-4)$ as a space of constant negative holomorphic sectional curvature, is a space of negative sectional curvature. Since $CH^n(-4)$ is simply connected, by Theorem 8.1, Chapter VIII, Vol. 2 of [12], all cut loci will be empty. This means that tubes of any radii may be constructed about smooth submanifolds. In particular, a certain class of submanifolds will give us tubes that are hypersurfaces of the type discussed in section 1.

Proposition 1-

A tube around a proper totally geodesic submanifold of CH^n is a hypersurface that has U as a principal direction.

Proof:

Let N^m be a totally geodesic submanifold of CH^n of dimension $m<2n$. Let N_r be the tube of radius r about N and $p=\exp_q(r\xi_q)\in N_r$.

From the previous discussion we can select a local orthonormal basis B_q of $\{\xi_q\}^\perp$ that has $U_q=-J\xi_q$ as an element and contains a basis of $T_q(N)$.

By Theorem I of [5], N is either a totally real or a complex submanifold of CH^n. Hence, U_q is either tangential or normal to N. Thus, $R_\xi(t)$, P and $P^\perp$ each have diagonal matrix representations with respect to B_q. Because $A_\xi=0$, (3.3) is now

(3.5) i) $F''(t,\xi_q)+R_\xi(t)\circ F(t,\xi_q)=0,\ t\in(0,r]$

ii) $F(0,\xi_q)=P$ and iii) $F'(0,\xi_q)=P^\perp$.

Order B_q in such a way that $R_\xi(t)$ is represented by the matrix $\mathrm{diag}(-I_{2n-2},-4)$. We shall regard (3.5) as a matrix valued differential equation and will write its matrix solution as $F(t,\xi_q)=[f_{ij}(t)]$ where

$i,j=1,\ldots,2n-1$. So (3.5) i) yields

$$f''_{ij}(t)-f_{ij}(t)=0 \text{ for } i\neq 2n-1, \text{ and}$$

$$f''_{2n-1,j}(t)-4f_{2n-1,j}(t)=0 \text{ otherwise.}$$

These ordinary differential equations have solutions of the form:

$$f_{ij}(t)=a_{ij}e^{-t}+b_{ij}e^{t} \text{ for } i\neq 2n-1, \text{ and}$$

$$f_{2n-1,j}=a_{2n-1,j}\,e^{-2t}+b_{2n-1,j}\,e^{2t} \text{ otherwise,}$$

where the a_{ij}'s and the b_{ij}'s are constants with respect to t.

Both P and $P^{\perp}$ are diagonal with 0's and 1's on the diagonal (of course: $P+P^{\perp}=I_{2n-1}$). Thus, from (3.5) ii) and iii), $f_{ij}(t)=0$ for $i\neq j$ for any $t>0$, which shows that $F(t,\xi_q)$ is diagonal with respect to B_q for all $t>0$. Hence, $F'(t,\xi_q)$ is also diagonal with respect to B_q. Therefore, H_r is diagonal with respect to the basis of $T_p(N_r)$ obtained by parallel translating B_q along the geodesic γ_ξ from q to p. Since $(U^*)_p$ is an element of this basis, U^* is principal at p. As p is arbitrary in N_r, we are done. //

Actually, Proposition 1 is true for any complex space form. The nature of the solution to (3.5) will depend upon the holomorphic sectional curvature of the ambient space and the dimension of the core of the tube, as we will see in the following examples. In the case at hand, i.e. the ambient space being CH^n, if N is a totally geodesic submanifold, $f_{ii}(t)=\sinh(t)$ or $\cosh(t)$ depending upon whether the ii^{th} entry of P is 0 or 1, for $i=1,\ldots,2n-2$, and $f_{2n-1,2n-1}(t)=\sinh(2t)$ or $\cosh(2t)$ depending upon whether the coresponding entry of P is 0 or 1.

Example 1:

Let $N=H^n(-1)$ be a real space form of constant sectional curvature -1 immersed in CH^n as a totally geodesic and totally real submanifold. (See the proof of Theorem 1 in [5].) Let N_r be the tube of radius r about N in CH^n. If ξ^* is a unit normal to N_r, at each point $p=\exp_q(r\xi)\epsilon N_r$ we can write

$$T_p(N_r) = \{\xi^*{}_p\}^\perp \cong T_q(N) \bullet [\{\xi_q\}^\perp \cap T_q(N)^\perp]$$

where the isomorphism is parallel translation. As N is totally real of dimension n, $U_q=-J\xi_q \epsilon T_q(N)$, so $U^*{}_p \epsilon T_p(N_r)$. So let $\{X_1, X_2, \ldots, X_{n-1}, U_q\}$ be an orthonormal basis of $T_q(N)$. If (ϕ, f) is the almost contact structure induced by J on N_r, then

$$B_p=\{X_1{}^*, \ldots, X_{n-1}{}^*, \phi(X_1{}^*), \ldots, \phi(X_{n-1}{}^*), U^*{}_p\}$$

forms an orthonormal basis of $T_p(N_r)$. Setting $\phi X_i=\tau(r,\xi_q)^{-1}[\phi(X_i^*)]$ for each $i=1,\ldots,n-1$ allows us to write

$$B_q=\{X_1,\ldots,X_{n-1},\phi X_1,\ldots,\phi X_{n-1},U_q\}$$

for an orthonormal basis of $\{\xi_q\}^\perp$. With respect to this basis P, $P^\perp$ and $R_\xi(t)$ have matrix representations:

$$P=\mathrm{diag}(I_{n-1},0_{n-1},1),$$

$$P^\perp=\mathrm{diag}(0_{n-1},I_{n-1},0) \text{ and}$$

$$R_\xi(t)=\mathrm{diag}(-I_{2n-2},-4),$$

for all $t\epsilon(0,r]$, as endomorphisms on $\{\xi_q\}^\perp$. Straightforward calculations yield that (3.5) has the matrix solution:

$F(t,\xi_q)=\mathrm{diag}(\cosh(t)I_{n-1}, \sinh(t)I_{n-1}, \cosh(2t))$.

Now, from (3.4), the second fundamental form, H_r, of the tube will have the following matrix representation with respect to the basis B_p:

(3.6) $H_r=\mathrm{diag}(\tanh(r)I_{n-1}, \coth(r)I_{n-1}, 2\tanh(2r))$.

By selecting a suitable frame on N, we see that the representation of H_r depends only upon r and is hence constant on N_r. N_r is obviously contact and in fact satisfies A): $\alpha^2-4<0$.

Example 2:

Let $N=CH^k$, $k=0,1,\ldots,n-1$, be a complex space form immersed in $CH^n(-4)$ as a totally geodesic submanifold, (see [5]). In case k=0, we are regarding a point to be a trivial complex space form. Otherwise, from [5], the CH^k will have constant holomorphic sectional curvature

-4. Let N_r be the tube of radius r about N in $CH^n(-4)$. Let ξ^* be a global unit normal to N_r so that at each $p=\exp_q(r\xi_q)$ we can write

$$T_p(N_r)=\{(\xi^*)_p\}^\perp \cong T_q(N) \oplus [T_q(N)^\perp \cap \{\xi_q\}^\perp].$$

As N is a complex submanifold of CH^n, its tangent space is invariant under J; so $\tau(r,\xi_q)[T_q(N)]$ is a ϕ-invariant subspace of $T_p(N_r)$. Let $\{X_1,\ldots,X_k,JX_1,\ldots,JX_k\}$ be an orthonormal basis of $T_q(N)$ and extend to an orthonormal basis

$$B_q=\{X_1,\ldots,X_k,JX_1,\ldots,JX_k,X_{k+1},\ldots,X_{n-1},JX_{k+1},\ldots,JX_{n-1},U_q\}$$

of $\{\xi_q\}^\perp$. The last $2n-1-2k$ elements of B_q form a basis of $T_q(N)^\perp \cap \{\xi_q\}^\perp$. Due to the invariance under J of $T_q(N)$, we have $\phi(X_i^*)=\tau(r,\xi_q)[JX_i]$ for all $i=1,\ldots,n-1$. So let

$$B_p=\{X_1{}^*,\dots,X_k{}^*,\phi X_1{}^*,\dots,\phi X_k{}^*,$$

$$X_{k+1}{}^*,\dots,X_{n-1}{}^*,\phi X_{k+1}{}^*,\dots,\phi X_{n-1}{}^*,(U^*)_p\}$$

be the orthonormal basis of $T_p(N_r)$ obtained by parallel translation of B_q. With respect to B_q, P, $P^\perp$ and $R_\xi(t)$ have the following matrix representations:

$$P=\mathrm{diag}(I_{2k},0_{2n-1-2k}),$$

$$P^\perp=\mathrm{diag}(0_{2k},I_{2n-1-2k})\text{ and}$$

$$R_\xi(t)=\mathrm{diag}(-I_{2n-2},-4)\text{ for all }t>0.$$

The radially symmetric matrix solution to (3.5), with respect to B_q, has matrix representation:

$F(t,\xi_q)=\mathrm{diag}(\cosh(t)I_{2k},\sinh(r)I_{2n-2k-2},\sinh(2r))$.

Computing (3.4) for this case yields the following matrix representation for the second fundamental form on N_r with respect to B_p:

(3.7) $H_r=\mathrm{diag}(\tanh(r)I_{2k},\coth(r)I_{2n-2k-2},2\coth(2r))$.

In the context of section 2, the geodesic hypersphere (k=0) and the tube around a maximal complex space form (k=n-1) are contact hypersurfaces and satisfy (2.4). These are our models for totally U-umbillic hypersurfaces of $\mathbf{CH}^n$. The remaining cases k=1,...,n-2 satisfy (2.4), but are not contact.

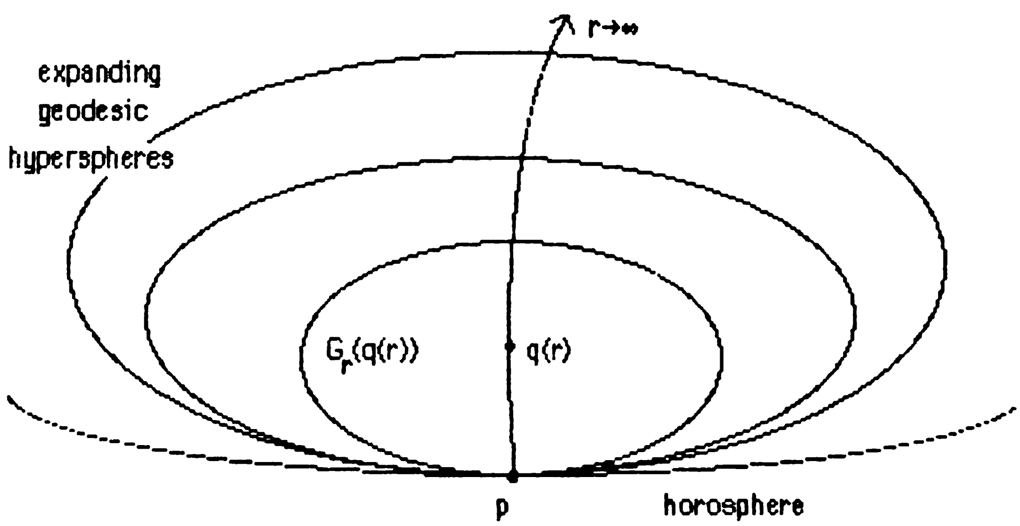
Example 3:

Each of the previous examples, despite their obvious differences as tubes with different cores, do have one thing in common. If the bases B_p, $p=\exp_q(r\xi_q)$, are chosen for each $r>0$ to be compatible (via τ) with B_q, then we can discuss the limit of the tensor H_r, viewed as acting on span(B_q), as $r\to\infty$. Clearly, this matrix limit is given by

$$(3.8)\qquad \lim_{r\to\infty} H_r = \mathrm{diag}(I_{2n-2}, 2).$$

The geometric significance of this is obscure from our view of these tensors as acting on the tangent space of the core of the tube. However, in section 6 (originally in [16]), a hypersurface is constructed in a very abstract way that has a second fundamental form of the form (3.8). So we know that such a hypersurface exists, the question is whether we can obtain a more geometric characterization. In this example, we shall construct a model that can be thought of as a limit of expanding geodesic hyperspheres, called the horosphere, that also has second fundamental form with representation (3.8).

Choose a point $p \epsilon CH^n$ and any direction $\xi \epsilon T(CH^n)$. For each $r>0$, let $q(r)=\exp_p(r\xi_p)$ and γ_ξ be geodesic with initial direction ξ_p that joins p to q(r). Then p is on each geodesic hypersphere, $G_r(q(r))$, centered at q(r) with radius r. It is known that as q(r) recedes from p ($r\to\infty$) the $G_r(q(r))$ approach a limiting hypersurface, M^∞, called the horosphere (see [7] and [10]).



Furthermore, the horosphere will have an extrinsic geometry that is obtained as a limiting hypersurface of these expanding geodesic hyperspheres. That is, M^∞ will have a second fundamental form with a representation (3.8) with respect to a suitable basis of

$T_p(M^\infty)$. This last point will be proven analytically in section 6, when, subsequent to considerably more theory, we will be able to show the convergence of the geodesic hyperspheres to a hypersurface with a second fundamental form of the type (3.8).

Example 4:

In this example a relatively simple hypersurface is constructed that has U principal, but no principal curvature is constant.

Consider a two dimensional real hyperbolic space form $N=H^2(-1)$ immersed in CH^3 as a totally real, totally geodesic submanifold. Let N_r be the tube of radius r about N. Then, by Proposition 1, N_r is a hypersurface that has the direction U as a principal direction. For each $q \in N$, let $S_q(r)=\{x \in N_r \mid d(x,q)=r\}$ denote a cross section of N_r over q.

Let $\{X_1, X_2\}_q$ be an orthonormal basis of the tangent plane $T_q(N)$ that extends to an orthonormal basis $\{X_1, X_2, JX_1, JX_2, Y, JY\}$ of $T(CH^3)|_N$. If ξ^* is a unit normal field to N_r, then for any $p \in S_q(r)$ we can write

$$(\xi^*)_p=a_1(p)(JX_1)^*+a_2(p)(JX_2)^*+b(p)Y^*+c(p)(JY)^*$$

where the values a_1, a_2, b and c are C^∞ functions on $S_q(r)$ that satisfy

$$1=a_1^2+a_2^2+b^2+c^2$$

at any point $p\epsilon S_q(r)$. Two quick computations will convince the reader that the principal curvatures are not constant on $S_q(r)$.

Select p such that $a_1=1$ and $a_2=b=c=0$. Then U_p has principal curvature 2tanh(2r) and the other principal curvatures at p are tanh(r) of multiplicity one and coth(r) of multiplicity three. However, if p is chosen so that $a_1=a_2=c=0$ and $b=1$, then U_p has principal curvature 2coth(2r) and the other other principal curvatures at p are tanh(r) and coth(r), each of multiplicity two.

In fact it can be shown on $S_q(r)$ that H_r is a linear combination (with coefficients the functions a_1, a_2, b and c) of four linearly independent diagonal matrices that have their non-zero entries drawn from the set {tanh(r),coth(r),2tanh(2r),2coth(2r)}.

The conjecture is now that the hypersurfaces of examples 1, 2 and 3 are the only hypersurfaces that are contact or satisfy (2.4). It turns out that this is indeed the case, but this cannot be shown by viewing CH^n only as a complex space form and applying the techniques of sections 1 and 2. To make further progress CH^n must be thought of as the base manifold of a certain Lorentzian S^1-fiber bundle.

4. The Lorentzian Circle Bundle over $CH^n(-4)$ and its S^1-invariant Hypersurfaces

The best understood of all non-Euclidean complex space forms, complex projective space, (usually written CP^n), is constructed using a natural equivalence on an odd dimensional sphere, S^{2n+1}, itself immersed in $\mathbf{C}^{n+1}$. Complex hyperbolic space can be constructed in a similar way (see [4], and [12], vol II). In this case CH^n is formed by taking the equivalence on a Lorentzian hyperbolic space form in $\mathbf{C}^{n+1}$, instead of on a real Riemannian space form.

Define a hermitian bilinear form F on $\mathbf{C}^{n+1}$ by

$$F(z,w)=-z_0\bar{w}_0+\sum_{j=1}^{n} z_j\bar{w}_j$$

for all $z=(z_0,z_1,\dots,z_n)$ and $w=(w_0,w_1,\dots,w_n)$ in $\mathbf{C}^{n+1}$. F forms a complex Lorentzian metric so that $\mathbf{C}^{n+1}$ with the metric F forms a complex Lorentzian space $\mathbf{C}_1^{n+1}$. $\mathbf{C}^{n+1}$ can also be regarded as a real semi-Riemannian euclidean space, $\mathbf{R}_2^{2n+2}$, if it is equipped with the metric Re(F). Anti-De Sitter spacetime is the hyperquadric defined

by

$$H_1^{2n+1}=\{z\epsilon\mathbf{C}^{n+1}\,|\,F(z,z)=-1\}.$$

The tangent space of anti-De Sitter space is determined by its immersion into the ambient complex euclidean space; explicitly:

$$T_z(H_1^{2n+1})=\{w\epsilon\mathbf{C}^{n+1}\,|\,\mathrm{Re}[F(z,w)]=0\}$$

for any $z\epsilon H_1^{2n+1}$. As a real hypersurface of $\mathbf{R}_2^{2n+2}$, H_1^{2n+1} has $\mathrm{Re}(F)|_{H_1^{2n+1}}$ as a natural Lorentzian metric that is of constant sectional curvature -1.

An S^1-action can be defined on H_1^{2n+1} (in fact on $\mathbf{C}_1^{n+1}$) by $z\to\lambda z$ for any $z\epsilon H_1^{2n+1}$ and $\lambda\epsilon\mathbf{C}$ with $|\lambda|=1$. At each point $z\epsilon H_1^{2n+1}$, the vector $V=iz$ is tangent to the flow and has length -1. Given a point $z\epsilon H_1^{2n+1}$, the flow of V through z in H_1^{2n+1} will be given by the orbit

$O_z=\{x_t=e^{it}z \mid t\in R\}$

that satisfies the differential equation $dx_t/dt=ix_t$, which in turn shows that O_z lies in the intersection of the negative definite plane span$\{z,V\}$ with H_1^{2n+1}. Let $\sim$ be the equivalence given by the orbits of the action, i.e. $w\sim z$ if $w\in O_z$. Then the natural projection

$$\pi: H_1^{2n+1} \to H_1^{2n+1}/\sim = \mathbf{C}H^n$$

is a Riemannian submersion with fundamental tensor the natural complex structure J on $\mathbf{C}_1^{n+1}$, (see [21],), and with time-like totally geodesic fibers, each of which is a trajectory of the vertical vector $V=iz$ at any point $z\in H_1^{2n+1}$. The complex Riemannian space, $\mathbf{C}H^n$, obtained in this way has its complex structure induced from that on $\mathbf{C}_1^{n+1}$, and has constant holomorphic sectional curvature -4, with the metric induced from Re(F). Then the differential of the submersion, π_*, is a linear isometry ([21]); i.e. it preserves the metric tensor on the horizontal distribution as it projects onto $T(\mathbf{C}H^n)$. So we shall make no distinction between the metric on H_1^{2n+1} and that on $\mathbf{C}H^n$. We can now write

$$T_Z(H_1^{2n+1}) \cong T_{\pi(Z)}(\mathbf{C}H^n) \oplus \mathrm{span}\{V\}$$

where the isomorphism is given by $\pi_* \oplus 1|_{\mathrm{span}\{V\}}$.

π_*, as a linear isometry of the distribution H of horizontal vector fields on H_1^{2n+1} onto $T(\mathbf{C}H^n)$, induces relations between the connections $\dot\nabla$ and $\bar\nabla$ of H_1^{2n+1} and $\mathbf{C}H^n$, respectively. (See [8] and [21].) These are:

(4.1) $\dot\nabla_{(X^\sim)} Y^\sim = (\bar\nabla_X Y)^\sim + \langle J(X^\sim), Y^\sim\rangle V$

(4.2) $\dot\nabla_V(X^\sim) = \dot\nabla_{(X^\sim)} V = (JX)^\sim = J(X^\sim)$

for all $X, Y \in T(CH^n) \equiv \mathcal{H}$, and where $X^\sim$ or $(X)^\sim$ denotes the unique horizontal lift of a vector field $X \in T(CH^n)$.

If M^{2n-1} is a real hypersurface of $CH^n(-4)$ then the hypersurface $\tilde{M}_1^{2n} = \pi^{-1}(M)$ of H_1^{2n+1}, is invariant under the S^1-action, and $\pi|_{\tilde{M}}: \tilde{M} \to M$ is a Riemannian submersion with timelike totally geodesic fibers. Conversely, if $\tilde{M}_1^{2n}$ is an S^1-invariant hypersurface of H_1^{2n+1}, then $\pi|_{\tilde{M}}$ is a Riemannian submersion of $\tilde{M}$ onto $M^{2n-1} = \pi(\tilde{M})$ with timelike totally geodesic fibers. Hence, we have the following commutative diagram of immersions and submersions:

$$\begin{tikzcd} \tilde{M}_1^{2n} \arrow[r, dashed, "\tilde{j}"] \arrow[d, "\pi|_{\tilde{M}}"'] & H_1^{2n+1}(-1) \arrow[d, "\pi"] \\ M^{2n-1} \arrow[r, dashed, "j"] & CH^n(-4) \end{tikzcd}$$

where $j: M^{2n-1} \to CH^n(-4)$ and $\tilde{j}: \tilde{M}_1^{2n} \to H_1^{2n+1}(-1)$ are immersions compatible with the fibration. If a global normal ξ to M is selected, its unique lift, $\xi^\sim$, is horizontal and forms a global normal to $\tilde{M}$.

Henceforth, we will drop this distinction between ξ and $\xi^{\sim}$.

Let $\tilde{\nabla}$ be the metric connection of $\tilde{M}$ as a hypersurface in H_1^{2n+1} and $\tilde{H}$ be its second fundamental form. The Gauss and Wiengarten formulae in this case are given by

(4.3) $\dot{\nabla}_X Y=\tilde{\nabla}_X Y+\langle\tilde{H}X,Y\rangle\xi$ and $\dot{\nabla}_X\xi=-\tilde{H}X$

for all $X,Y\epsilon T(\tilde{M})$. Now combining (1.1), (4.1), (4.2) and (4.3) we have the following relations between $\tilde{H}$ and H:

(4.4) $\tilde{H}(X^{\sim})=(HX)^{\sim}-f(X)V$

for all $X\epsilon T(M)\cong\{V\}^{\perp}\cap T(\tilde{M})$, and

(4.5) $\tilde{H}V=U^{\sim}$.

Let $\tilde{R}$ be the curvature tensor of $\tilde{M}$. We shall have the opportunity to use the Gauss and Codazzi equations for $\tilde{M}$ in H_1^{2n+1}:

(4.6) $\tilde{R}(X,Y)Z = \langle X,Z\rangle Y - \langle Y,Z\rangle X + \langle \tilde{H}Y,Z\rangle \tilde{H}X - \langle \tilde{H}X,Z\rangle \tilde{H}Y$

(4.7) $(\tilde{\nabla}_X \tilde{H})Y = (\tilde{\nabla}_Y \tilde{H})X$

for all $X,Y \epsilon T(\tilde{M})$. These and the preceeding formulae can be used to prove the following useful identities:

(4.8) $(\tilde{\nabla}_W \tilde{R})(X,Y)Z = \langle(\tilde{\nabla}_W \tilde{H})Y,Z\rangle \tilde{H}X - \langle(\tilde{\nabla}_W \tilde{H})X,Z\rangle \tilde{H}Y$

$$+\langle \tilde{H}Y,Z\rangle(\tilde{\nabla}_W \tilde{H})X - \langle \tilde{H}X,Z\rangle(\tilde{\nabla}_W \tilde{H})Y$$

for all $X,Y,Z,W \epsilon T(\tilde{M})$, and

(4.9) $(\tilde{\nabla}_{(X^\sim)} \tilde{H})V = [(\phi H - H\phi)X]^\sim$

for any $X \epsilon T(M)$.

Let $\{X_1,\ldots,X_{2n-1}\}$ be a frame on M consisting of principal directions in $T(M)$ with corresponding principal curvatures $\{\lambda_1,\ldots,\lambda_{2n-1}\}$. Then $\{X_1^{\sim},\ldots,X_{2n-1}^{\sim},V\}$ forms a frame on $\tilde{M}$ with respect to which $\tilde{H}$ is represented by the matrix:

$$\begin{bmatrix} \lambda_1 & & O & f(X_1) \\ & \ddots & & \vdots \\ O & & \lambda_{2n-1} & f(X_{2n-1}) \\ -f(X_1) & \ldots & -f(X_{2n-1}) & 0 \end{bmatrix},$$

all of whose entries are functions on $\tilde{M}$. Hence, even the stipulation that $\tilde{M}$ be S^1-invariant does not guarantee an easy analysis of the structure of $\tilde{M}$ via an investigation of those subbundles of $T(\tilde{M})$ held invariant by $\tilde{H}$, for we are not even assured the existence of principal directions that have real principal curvatures. However, in case U is a principal direction on M, we can choose $U=X_{2n-1}$ so that $\tilde{H}$ is represented by the matrix:

$\mathrm{diag}(\lambda_1,\ldots,\lambda_{2n-2}, \begin{bmatrix} \alpha & 1 \\ -1 & 0 \end{bmatrix})$.

As we have seen, there are a number of conditions on M that force U to be a principal direction on M. In section 2, we saw this for the conditions

(2.1) $\phi H+H\phi=2\rho\phi$ (M is contact), and

(2.4) $\phi H=H\phi$.

In [16], (2.4) is shown to be equivalent to $\tilde{H}$ being parallel on $\tilde{M}$. Thus, (2.4) satisfied on M shows that $\tilde{M}$ is parallel, which in turn yields $\tilde{M}$ locally symmetric, which finally implies that $\tilde{M}$ is semi-symmetric. Symbollically:

$(2.4)\Leftrightarrow \tilde{\nabla}\tilde{H}=0\Rightarrow\tilde{\nabla}\tilde{R}=0\Rightarrow\tilde{R}\cdot\tilde{R}=0.$

Surprisingly, <u>all</u> the implications are equivalences for

S^1-invariant hypersurfaces of H_1^{2n+1}, not just the first, which will enable us to contribute a little toward the work began in [17], [23]-[26] and [28]-[30]. In order to begin the proof of this fact, we must see that when an S^1-invariant hypersurface $\tilde{M}$ is semi-symmetric its submersion $M=\pi(\tilde{M})$ is a hypersurface that has U as a principal direction.

<u>Lemma 6</u>

Let $\tilde{M}$ be an S^1-invariant hypersurface of H_1^{2n+1} and $M=\pi(\tilde{M})$. Then $\tilde{R}\cdot\tilde{R}=0$ on $\tilde{M} \Rightarrow$ U is principal on M.

Proof:

Let $\{X_1,\dots,X_{2n-1}\}$ be a basis of principal directions on M for T(M) with corresponding principal curvatures $\{\lambda_1,\dots,\lambda_{2n-1}\}$. A direct computation shows that at any point $p\in\tilde{M}$

$$(4.10)\quad (\tilde{R}(X_i^{\sim},X_j^{\sim})\cdot\tilde{R})(X_k^{\sim},V)V = f(X_j)f(X_k)(\lambda_i^2-\lambda_i\lambda_j+1)X_i^{\sim}$$

$$-f(X_i)f(X_k)(\lambda_j^2-\lambda_i\lambda_j+1)X_j^{\sim}$$

$$+2\lambda_k(\lambda_j-\lambda_i)f(X_j)f(X_i)X_k^{\sim}$$

at p if i,j and k are distinct indices.

Assume that U is not principal on M. Then there exist at least two principal directions not in $\ker(f)$, say X_1 and X_2. For $k>2$, we have $\lambda_k^2-\lambda_k\lambda_1+1=0$, by setting $i=1$, $j=2$ then switching j with k in (4.10) and reading off the second term in the right hand side. This shows that $\lambda_k\neq 0$ when $k>2$. So, after switching the indices back, from the third term in the right hand side of (4.10), we have $\lambda_k(\lambda_1-\lambda_2)=0$, i.e. $\lambda_1=\lambda_2=\lambda$.

If there is a third direction, say X_3, not in $\ker(f)$, then again from the right hand side of (4.10), we must have $\lambda_1^2-\lambda_1\lambda_2+1=0$, which is impossible in light of $\lambda_1=\lambda_2$. Hence, $U\in\mathrm{span}\{X_1,X_2\}\subset D_\lambda$, an absurdity.

Therefore, U must be principal on M. //

For the remainder of this section we will assume that $\tilde{M}$ is an S^1-invariant hypersurface of H_1^{2n+1} that satisfies the condition $\tilde{R}\cdot\tilde{R}=0$. Let $M=\pi(\tilde{M})$ and $\{X_1,\dots,X_{2n-2},U\}$ be a local orthonormal basis of $T(M)$ consisting entirely of principal directions that have corresponding principal curvatures $\{\lambda_1,\dots,\lambda_{2n-2},\alpha\}$. In order to achieve a classification of the S^1-invariant hypersurfaces that satisfy this condition, we will need to obtain algebraic consequences for the curvatures associated to the spacelike distribution on $\tilde{M}$ that is invariant under $\tilde{A}$, namely:

$$\mathrm{span}\{X_1{}^\sim,\dots,X_{2n-2}{}^\sim\}=\mathcal{H}\cap(\ker(f))^\sim.$$

<u>Lemma 7</u>

If $\tilde{R}\cdot\tilde{R}=0$ on an S^1-invariant hypersurface $\tilde{M}$ of H_1^{2n+1}, let λ_i,λ_j and λ_k be principal curvatures on $\tilde{M}$ with distinct spacelike principal directions $X_i{}^\sim,X_j{}^\sim$ and $X_k{}^\sim\in T(\tilde{M})$. Then $(\lambda_i\lambda_j-1)(\lambda_i-\lambda_j)\lambda_k=0$.

Proof:

By direct computation using (4.6)

$$(\tilde{R}(X_i^{\sim},X_j^{\sim})\cdot\tilde{R})(X_i^{\sim},X_k^{\sim})X_j^{\sim}$$

$$=[(1-\lambda_i\lambda_j)(1-\lambda_i\lambda_k)+(\lambda_i\lambda_j-1)(1-\lambda_j\lambda_k)]X_k^{\sim}$$

so that $(1-\lambda_i\lambda_j)(1-\lambda_i\lambda_k)=(1-\lambda_i\lambda_j)(1-\lambda_j\lambda_k)$. If $1-\lambda_i\lambda_j\neq 0$ then we must have $1-\lambda_i\lambda_k=1-\lambda_j\lambda_k$; that is $\lambda_k(\lambda_i-\lambda_j)=0$. //

From Lemma 7, if there is a nonzero principal curvature λ_k with direction $X_k \in \mathcal{H}$, then for curvatures λ_i and λ_j with distinct spacelike directions X_i and X_j different from X_k either

a) $\lambda_i=\lambda_j$ or b) $\lambda_i=1/\lambda_j$.

If case a) holds, then every basis direction different from X_k has the same principal curvature and a simple argument similar to the one employed in the proof of Lemma 6 yields all the nonzero principal curvatures the same value. If case b) is true, all the principal curvatures that have spacelike principal directions must be nonzero and once a nonzero curvature λ has been given, only the values λ and $1/\lambda$ are allowable as principal curvatures. Hence, we have only four possible matrix representations with respect to the orthonormal basis $\{X_1, \ldots, X_{2n-2}, U\}$ for the second fundamental form of a real hypersurface, $M=\pi(\tilde{M})$, of $\mathbf{C}H^n$, where $\tilde{R}\cdot\tilde{R}=0$ on $\tilde{M}$:

i) $H=\mathrm{diag}(0_{2n-2}, \alpha)$,

ii) $H=\mathrm{diag}(\lambda I_{2n-2}, \alpha)$,

iii) $H=\mathrm{diag}(\lambda, 0_{2n-3}, \alpha)$, or

iv) $H=\mathrm{diag}(\lambda I_p, (1/\lambda) I_{2n-2-p}, \alpha)$

where $\lambda \neq 0$ and the basis may need to be reordered for case iii).

(Notice that in each case, $U^{\sim}$ is <u>not</u> principal on $\tilde{M}$.)

Cases i) and iii) are ruled out by the following lemma:

<u>Lemma 8</u>

If λ is a principal curvature on a semi-symmetric S^1-invariant hypersurface $\tilde{M}$ of $H_1{}^{2n+1}$ that has a spacelike principal direction, then $\lambda^2-\alpha\lambda+1=0$.

Proof:

Let X be a spacelike principal direction of λ. Then

$$(\tilde{R}(X,U^{\sim})\cdot\tilde{R})(X,V)U^{\sim}=(\lambda^2-\alpha\lambda+1)V. \quad //$$

In particular, we see that no principal curvature can be zero if it has a spacelike principal direction. Furthermore, the principal curvature α must satisfy $\alpha^2-4\geq 0$ on M. However, an analysis of the values λ can attain cannot yet proceed as in section 2 as we do not know if α and λ are constant. As far as we are concerned the principal curvatures α and λ are merely functions on M that satisfy

(2.5) $\lambda^2-\alpha\lambda+1=0.$

Lemma 9

If $\tilde{R}\cdot\tilde{R}=0$ on $\tilde{M}$ then the principal curvature α with direction U is constant on M.

Proof:

The proof is similar to that of Lemma 4.

From the proof of Lemma 1, we have $X\alpha=0$ for all $X\epsilon\ker(f)$. Hence, it is sufficient to show that $U\alpha=0$ on M.

Following the same steps as in the proof of Lemma 4, we recall the equation:

(2.3) $X(U\alpha)f(Y)-Y(U\alpha)f(X)=(U\alpha)<(\phi H+H\phi)Y,X>.$

Again replacing X by U and then Y by U and then substituting the results back into (2.3) yields

$0=(U\alpha)\langle(\phi H+H\phi)Y,X\rangle$

for all X and Y in $T(M)$.

Suppose that $\alpha^2-4>0$ on an open set $\mathcal{U}$ in M. (Otherwise, α is a constant ± 2 on M.) Then by (2.5) we have the curvature λ satisfying $\lambda^2\neq 1$ on $\mathcal{U}$. Choose $Y\epsilon D_\lambda \cap \ker(f)$. From the proof of lemma 2,

$$0=(U\alpha)\langle(\lambda+\frac{\alpha\lambda-2}{2\lambda-\alpha})\phi Y,X\rangle.$$

By choosing $X=\phi Y$ we have $U\alpha=0$ or

$$0=\lambda+\frac{\alpha\lambda-2}{2\lambda-\alpha}=(2\lambda^2-\alpha\lambda+\alpha\lambda-2)/(2\lambda-\alpha)=2(\lambda^2-1)/(2\lambda-\alpha).$$

The latter equation is ruled out by hypothesis on $\mathcal{U}$, hence $U\alpha=0$ on any open subset of M with $\alpha^2-4>0$.

Therefore, α is constant on M. //

We now know that an S^1-invariant hypersurface $\tilde{M}$ of $H_1{}^{2n+1}$ on which $\tilde{R}\cdot\tilde{R}=0$, admits a frame with respect to which its second fundamental form has the following possible matrix representations:

i) $\tilde{H}=\mathrm{diag}(\lambda I_{2n-2}, \begin{bmatrix}\alpha & 1\\ -1 & 0\end{bmatrix})$, or

ii) $\tilde{H}=\mathrm{diag}(\lambda I_p, (1/\lambda) I_{2n-2-p}, \begin{bmatrix}\alpha & 1\\ -1 & 0\end{bmatrix})$

and the second fundamental form of its submersion will have the corresponding representations with respect to a suitable basis:

i) $H=\mathrm{diag}(\lambda I_{2n-2}, \alpha)$, or

ii) $H=\mathrm{diag}(\lambda I_p, (1/\lambda) I_{2n-2-p}, \alpha)$

where α and λ satisfy (2.5) and $\alpha^2-4\geq 0$. Case i) is obviously a subcase of ii) with $p=0$ or $2n-2$. As in section 2, if $\alpha^2-4>0$, we can set $\alpha=2\coth(2r)$ and $\lambda=\tanh(r)$ or $\coth(r)$. Of course if $\alpha=\pm 2$ we may set $\lambda=\pm 1=1/\lambda$.

Assume that $H=\mathrm{diag}(\lambda I_p, (1/\lambda) I_{2n-2-p}, \alpha)$ with respect to a suitably selected orthonormal basis of principal directions. There are two cases to consider:

I. Suppose that p is odd.

Then there exists $X \epsilon D_\lambda$ with $\phi X \epsilon D_{1/\lambda}$. From the proof of Lemma 1

$$\frac{1}{\lambda} = \frac{\alpha\lambda-2}{2\lambda-\alpha}$$

which implies that

$$0=\alpha\lambda^2-4\lambda+\alpha.$$

Using (2.5)

$$0=\alpha(\alpha\lambda-1)-4\lambda+\alpha=\lambda(\alpha^2-4)$$

which shows that either $\lambda=0$ or $\alpha^2-4=0$, neither of which hold by hypothesis.

This only leaves the single case:

II. p is even.

From the above argument we must have D_λ, and hence $D_{1/\lambda}$, invariant under ϕ. Thus, $\phi H = H\phi$ on M whenever its lift is semi-symmetric in H_1^{2n+1}.

Remark:

As the only models we have for S^1-invariant hypersurfaces satisfying $\tilde{R}\cdot\tilde{R}=0$ are lifts of certain tubes, one might conjecture that the lift of any tube would satisfy this condition. However, a tube of radius $r>0$ about a totally geodesic real hyperbolic space form (example 1) has $\alpha=2\tanh(2r)$ and $\lambda=\tanh(r)$ or $\coth(r)$ which do not satisfy (2.5). Thus, $\tilde{R}\cdot\tilde{R}\neq 0$ for the lift of this tube.

Incorporating the preceeding discussion with known results in this area, including those in [4] and [16], yields the following :

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Theorem 2

Let $\tilde{M}_1{}^{2n}$ be an S^1-invariant hypersurface of $H_1{}^{2n+1}(-1)$ and $M^{2n-1}=\pi(\tilde{M})$ be its submersion in $CH^n(-4)$. Then the following statements are equivalent:

1) $\tilde{\nabla}\tilde{H}=0$ on $\tilde{M}$.

2) $\tilde{R}\cdot\tilde{R}=0$ on $\tilde{M}$.

3) $\phi H=H\phi$ on M.

4) H is cyclic parallel on M.

Proof:

1) $\Rightarrow$ 2) is obvious.

2) $\Rightarrow$ 3) follows from the preceeding discussion.

3) $\Leftrightarrow$ 1) is in [16].

4) $\Leftrightarrow$ 1) is in [4]. //

A classification of semi-symmetric S^1-invariant hypersurfaces is now possible and will be accomplished in the next section.

5. Congruence and Classifications

Submanifold theory, as with other branches of mathematics, has its own notion of equivalence. For instance, two hyperspheres of the same radius with different centers in a euclidean space are different merely by location, yet obviously have the same extrinsic geometry and as such are extrinsically equivalent submanifolds of the ambient euclidean space. We would say that these spheres are <u>congruent</u> and would see the extrinsic equivalence by observing that one sphere can be mapped isometrically onto the other by making a rigid motion of the ambient space. In this case, a rigid motion of a euclidean space is a translation or a rotation or any combination thereof. (We shall exclude reflections from rigid motions in this study as these reverse orientation.)

We generalize to semi-Riemannian manifolds:

<u>Definition</u>:

Let $\hat{M}$ be a semi-Riemannian manifold with semi-Riemannian submanifolds M and N. M and N are congruent if there exists an isometry Ω of $\hat{M}$ such that $\Omega|_M$ is an isometry of M onto N.

We have seen that for submanifolds of euclidean space, congruences are given by translations and rotation. However, in the ambient space we have been studying, $CH^n(-4)$, the characterization of congruence is not so simple. The principal S^1-fiber bundle over CH^n will be taken into account as well as the fiber bundle's own imbedding in C_1^{n+1}, i.e. from section 4:

$$\begin{array}{ccc} H_1^{2n+1} & \to & C_1^{n+1} \\ \downarrow\pi & & \\ CH^n & & \end{array} \quad .$$

We see that rigid motions of C_1^{n+1} will induce rigid motions of CH^n.

Recall that the bilinear form F defined in section 4 forms a semi-Riemannian metric on $\mathbf{C}^{n+1}$ that turns the complex euclidean space into a complex Lorentzian space, $\mathbf{C}_1^{n+1}$, where at the origin $(z_0,0,\ldots,0)$ is a timelike vector if $z_0 \neq 0$. The bilinear form Re(F) on $\mathbf{C}^{n+1}$, forms a semi-Riemannian metric that turns the complex euclidean space into the real semi-Riemannian euclidean space $\mathbf{R}_2^{2n+2}$, where at the origin $(1,0,\ldots,0)$ and $(i,0,\ldots,0)$ form a basis of the negative definite subspace of $T_0(\mathbf{R}_2^{2n+2})$. The isometries of $\mathbf{C}_1^{n+1}$ are precisely the group

$$U(1,n)=\{A\in GL(n+1;\ \mathbf{C}): F(Az,Aw)=F(z,w)\ \forall\ z,w\in\mathbf{C}_1^{n+1}\}.$$

Furthermore, $U(1,n)$ holds H_1^{2n+1} invariant and acts transitively on H_1^{2n+1}. Hence, the elements of $U(1,n)$ will induce isometries of $\mathbf{C}H^n$ via π. Using these ideas, we shall see that isoparametric hypersurfaces of $\mathbf{C}H^n$ that have U principal and the "same" second fundamental form are congruent. Consequently, we will obtain nice geometric characterizations of contact hypersurfaces and hypersurfaces satisfying (2.4) in $\mathbf{C}H^n$ as well as a characterization of semi-symmetric S^1-invariant hypersurfaces of H_1^{2n+1}.

Let M and N be isoparametric hypersurfaces of $\mathbf{C}H^n$ that each has the distinguished direction of the induced almost contact structure as a principal direction. (Henceforth, we shall write U for this direction indiscriminately on M and N. Hopefully, the domain of U will remain clear in context.) Suppose that M and N have second fundamental forms with the same matrix representation with respect to suitably chosen local orthonormal bases of principal directions of T(M) and T(N). Let $\hat{M},\rho$ and $\hat{N},\sigma$ be the simply connected covering spaces of the lifts $\tilde{M}=\pi^{-1}(M)$ and $\tilde{N}=\pi^{-1}(N)$, respectively. If $i:M\to \mathbf{C}H^n$ and $j:N\to \mathbf{C}H^n$ are the isometric immersions in complex hyperbolic space and $\tilde{i}:\tilde{M}\to H_1^{2n+1}$ and $\tilde{j}:\tilde{N}\to H_1^{2n+1}$ are the induced immersions of the respective lifts, we have the following commutative diagram of immersions, submersions and covering maps:

$$\begin{array}{ccccc}
\hat{M} & & & & \hat{N} \\
\downarrow \rho & & & & \downarrow \sigma \\
\tilde{M} & \xrightarrow{\tilde{i}} & H_1^{2n+1} & \xleftarrow{\tilde{j}} & \tilde{N} \\
\downarrow \pi|_{\tilde{M}} & & \downarrow \pi & & \downarrow \pi|_{\tilde{N}} \\
M & \xrightarrow{i} & CH^n & \xleftarrow{j} & N
\end{array}$$

The maps $\tilde{i}\circ\rho:\hat{M}\to H_1^{2n+1}$ and $\tilde{j}\circ\sigma:\hat{N}\to H_1^{2n+1}$ are now isometric immersions of simply connected Lorentzian spaces of codimension 1 into the Lorentzian symmetric space H_1^{2n+1}, that have the same constant matrix representation for their second fundamental forms with respect to cannonically chosen orthonormal bases of $T(\hat{M})$ and $T(\hat{N})$.

<u>Lemma 10</u>

Let N and M be real hypersurfaces of $CH^n(-4)$ that have:

i) U as a principal direction, and

ii) the same constant matrix representation for the second fundamental form with respect to suitably chosen bases of principal directions.

If $\hat{N}$ and $\hat{M}$ are the simply connected covering spaces of $\tilde{N}=\pi^{-1}(N)$ and $\tilde{M}=\pi^{-1}(M)$, then $\hat{N}$ and $\hat{M}$ are isometric.

Proof:

We shall use the notation of the preceeding paragraph and diagram.

Let $x\epsilon\hat{N}$ and $y\epsilon\hat{M}$. By hypothesis and using the ideas of sections 1 and 4 we can choose local orthonormal bases

$\{X_1,\ldots,X_{n-1},\phi X_1,\ldots,\phi X_{n-1},U\}$ of $T_{\pi(\sigma(x))}(N)$ and

$\{Y_1,\ldots,Y_{n-1},\phi Y_1,\ldots,\phi Y_{n-1},U\}$ of $T_{\pi(\rho(y))}(M)$

that consist entirely of principal directions, with respect to which the second fundamental forms have the same representations. (We are allowing ϕ to denote the almost contact structure of both N and M. The domain of ϕ will, hopefully, always be clear in context.) To simplify notation set $X\hat{}=[(\sigma_*)^{-1}(X\tilde{})]_x$ for any $X\epsilon T(N)$ and similarly set $Y\hat{}=[(\rho_*)^{-1}(Y\tilde{})]_y$ for any $Y\epsilon T(M)$. Let V denote a unit timelike vector on H_1^{2n+1} and $V\hat{}$ denote either $[(\sigma_*)^{-1}(V|_N)]_x$ or $[(\rho_*)^{-1}(V|_M)]_y$, depending upon the context. Similarly we shall write $U\hat{}$ for either $[(\sigma_*)^{-1}(U\tilde{})]_x$ or $[(\rho_*)^{-1}(U\tilde{})]_y$ depending upon the context.

We now have local orthonormal bases

$$B_x(\hat{N})=\{(X_1)\hat{},\ldots,(X_{n-1})\hat{},(\phi X_1)\hat{},\ldots,(\phi X_{n-1})\hat{},U\hat{},V\hat{}\} \text{ of } T_x(\hat{N}), \text{ and}$$

$$B_y(\hat{M})=\{(Y_1)\hat{},\ldots,(Y_{n-1})\hat{},(\phi Y_1)\hat{},\ldots,(\phi Y_{n-1})\hat{},U\hat{},V\hat{}\} \text{ of } T_y(\hat{M}).$$

Define a linear isometry $\psi: T_x(\hat{N}) \to T_y(\hat{M})$ by

$$\psi((X_i)^\wedge)=(Y_i)^\wedge \text{ and } \psi((\phi X_i)^\wedge)=(\phi Y_i)^\wedge \text{ for } i=1,\dots,n-1,$$

$$\psi((U^\wedge)_x)=(U^\wedge)_y \text{ and } \psi((V^\wedge)_x)=(V^\wedge)_y,$$

and extend linearly. Let $\hat{H}$ and $\hat{H}'$ denote the second fundamental forms of the isometric immersions $\tilde{j}\circ\sigma$ and $\tilde{i}\circ\rho$, and similarly let $\tilde{H}$ and $\tilde{H}'$ denote the second fundamental forms of the immersions $\tilde{j}$ and $\tilde{i}$. As σ and ρ are local isometries, $\hat{H}_x$ and $\hat{H}'_y$ will agree with $\tilde{H}_{\sigma(x)}$ and $\tilde{H}'_{\rho(y)}$. Applying (4.4) and (4.5) we have that $\psi(\hat{H}_x(Z))=\hat{H}'_y(\psi(Z))$ for any $Z\in T_x(\hat{N})$. Let $\hat{R}$ and $\hat{R}'$ denote the curvature tensors of $\hat{N}$ and $\hat{M}$. By (4.6) $\psi(\hat{R}_x(X,Y)Z)=\hat{R}'_y(\psi X,\psi Y)\psi Z$ for all $X,Y,Z\in T_x(\hat{N})$.

Let $\hat{\nabla}$ and $\hat{\nabla}'$ denote the connections on $\hat{N}$ and $\hat{M}$. Using (4.7) and (4.9), it follows that ψ maps the tensor $(\hat{\nabla}\hat{H})_x$ to the tensor $(\hat{\nabla}'\hat{H}')_y$.

Hence by (4.8), ψ maps $(\hat{\nabla}\hat{R})_x$ to $(\hat{\nabla}'\hat{R}')_y$. Define the m^{th} covariant differential $\hat{\nabla}^m\hat{R}$ at x, (as on p.125, vol. I of [12]) by:

$$(\hat{\nabla}\hat{R})_x(X,Y;W)Z = \hat{\nabla}_W(\hat{R}(X,Y)Z) - \hat{R}(\hat{\nabla}_W X,Y)Z - \hat{R}(X,\hat{\nabla}_W Y)Z$$

for all $X,Y,W,Z \in T_x(\hat{N})$, and then for $m=1,2,\ldots$ and any set of vectors $\{W_1,\ldots,W_{m-1}\} \subset T_x(\hat{N})$, set

$$\begin{aligned}
(\hat{\nabla}^m\hat{R})_x&(X,Y;W_1;\ldots;W_{m-1};W)Z \\
&= \hat{\nabla}_W((\hat{\nabla}^{m-1}\hat{R})_x(X,Y;W_1;\ldots;W_{m-1})Z) \\
&\quad -(\hat{\nabla}^{m-1}\hat{R})_x(\hat{\nabla}_W X,Y;W_1;\ldots;W_{m-1})Z \\
&\quad -(\hat{\nabla}^{m-1}\hat{R})_x(X,\hat{\nabla}_W Y;W_1;\ldots;W_{m-1})Z \\
&\quad -\sum_{i=1}^{m-1}(\hat{\nabla}^{m-1}\hat{R})_x(X,Y;W_1;\ldots;W_{i-1};\hat{\nabla}_W W_i;W_{i+1};\ldots;W_{m-1})Z.
\end{aligned}$$

Assume that ψ maps the tensor $(\hat{\nabla}^{m-1}\hat{R})_x$ to the tensor $(\hat{\nabla}'^{m-1}\hat{R}')_y$.

Then in order to have an inductive proof that ψ will map $(\hat{\nabla}^m \hat{A})_x$ to $(\hat{\nabla}'^m \hat{A}')_y$ for any m, it is sufficient to show that $\psi(\hat{\nabla}_X Y) = \hat{\nabla}'_{(\psi X)}(\psi Y)$ for all $X, Y \epsilon T_x(\hat{N})$. But this is indeed the case as $\hat{N}$ and $\hat{M}$ are isoparametric and ψ preserves the metric.

Now by Corollary 7.3, Chap. VI , Vol. I of [12], there is a unique isometry $\Omega: \hat{N} \to \hat{M}$ such that $(d\Omega)_x = \psi$. //

The point of Lemma 10 is to get into position to establish a general congruence theorem for real hypersurfaces of complex space forms. It is well known that isometric submanifolds of a real space form that have the same second fundamental form are congruent. So congruence results for submanifolds of complex space forms can be established by using known congruence results for the real space forms that are principal circle bundles over complex space forms.

<u>Theorem 3</u>

Let M and N be real hypersurfaces of $CH^n(-4)$ that each have U as a principal direction and all the principal curvatures are constant. If M and N have the same second fundamental form, i.e. the corresponding principal curvatures are the same, then M and N are congruent.

Proof:

Let ψ and Ω be as in the proof of lemma 10. Our commutative diagram of the immersions, submersions and covering maps, with the addition of the isometry Ω, is now:

$$\begin{tikzcd}
\hat{M} \arrow[d, "\rho"] & & \hat{N} \arrow[ll, "\Omega"'] \arrow[d, "\sigma"] \\
\tilde{M} \arrow[r, "\tilde{i}"] \arrow[d, "\pi|_{\tilde{M}}"] & H_1^{2n+1} \arrow[d, "\pi"] & \tilde{N} \arrow[l, "\tilde{j}"'] \arrow[d, "\pi|_{\tilde{N}}"] \\
M \arrow[r, "i"] & CH^n & N \arrow[l, "j"']
\end{tikzcd}$$

As H_1^{2n+1} is a totally umbillic hyperquadric of the semi-Riemannian manifold $\mathbf{R}_2^{2n+2} \approx \mathbf{C}_1^{n+1}$ with second fundamental form $-I$ on $T(H_1^{2n+1})$, we can choose local orthonormal bases of $T(\hat{N})$ and $T(\hat{M})$ as in lemma 10 with respect to which the two second fundamental forms have the same constant matrix representations.

Let $x \in \hat{N}$ and $y=\Omega(x) \in \hat{M}$. We can identify

$$C_1^{n+1} \cong T_{\tilde{j}(\sigma(x))}(H_1^{2n+1}) \oplus \mathrm{span}\{\tilde{j}(\sigma(x))\}$$

$$\cong T_{\tilde{i}(\rho(y))}(H_1^{2n+1}) \oplus \mathrm{span}\{\tilde{i}(\rho(y))\}$$

as $\tilde{j}(\sigma(x))$ and $\tilde{i}(\rho(y))$ are unit normals to H_1^{2n+1} at $\tilde{j}(\sigma(x))$ and $\tilde{i}(\rho(y))$, respectively. Choose ξ and ξ' to be unit normals to $\hat{N}$ and $\hat{M}$ in H_1^{2n+1} and write

$$T_{\tilde{j}(\sigma(x))}(H_1^{2n+1}) \cong T_x(\hat{N}) \oplus \mathrm{span}\{\xi_x\}$$

$$\cong ((\ker(f))_{\pi(\tilde{j}(\sigma(x)))})^{\sim} \oplus \mathrm{span}\{(U^{\sim})_{\sigma(x)}, V_{\sigma(x)}, \xi_{\sigma(x)}\}$$

and

$$T_{\tilde{i}(\rho(y))}(H_1^{2n+1}) \cong T_y(\hat{M}) \oplus \mathrm{span}\{\xi'_y\}$$

$$\cong ((\ker(f))_{\pi(\tilde{i}(\rho(y)))})^{\sim} \oplus \mathrm{span}\{(U^{\sim})_{\rho(y)}, V_{\rho(y)}, \xi'_{\rho(y)}\}.$$

Hence, we can regard $\hat{M}$ and $\hat{N}$ as Lorentzian submanifolds of the semi-Riemannian manifold $\mathbf{R}_2^{2n+2}$. Let α be a curve in $\hat{N}$ beginning at x and denote normal parallel translation of vector fields along the curves α and $\Omega(\alpha)$ by P_α and $P_{\Omega(\alpha)}$, respectively. As H_1^{2n+1} is a totally umbillic hyperquadric of $\mathbf{R}_2^{2n+2}$, parallel translation in H_1^{2n+1} preserves the second fundamental form of H_1^{2n+1}.

Define $\psi^\perp: T_x(\hat{N})^\perp \to T_y(\hat{M})^\perp$ by $\psi^\perp(x)=y$ and $\psi^\perp(\xi_x)=\xi'_y$ and then extend linearly to obtain a linear isometry of the two dimensional normal spaces. A linear isometry $\psi^\perp_{\alpha(s)}: T_{\alpha(s)}(\hat{N})^\perp \to T_{\Omega(\alpha(s))}(\hat{M})^\perp$ can be defined by setting $\psi^\perp_{\alpha(s)} = P_{\Omega(\alpha(s))} \circ \psi^\perp \circ P_{\alpha(s)}^{-1}$. At each s in the domain of α, $(d\Omega)_{\alpha(s)} = \psi_{\alpha(s)}$ is an isometry that maps the second fundamental form of $\hat{N}$ at $\alpha(s)$ onto that of $\hat{M}$ at $\Omega(\alpha(s))$. Therefore, the linear isometry $\psi^\perp_{\alpha(s)}$ will map the second fundamental form of

$\hat{N}$ in C_1^{n+1} at $\alpha(s)$ to that of $\hat{M}$ in C_1^{n+1} at $\Omega(\alpha(s))$, i.e.

$$\psi^{\perp}{}_{\alpha(s)}B(X,Y)=B'((d\Omega)_{\alpha(s)}X,(d\Omega)_{\alpha(s)}Y)$$

where

$$\dot{\nabla}_X Y=\hat{\nabla}_X Y+B(X,Y) \quad \text{for all } X,Y\in T_{\alpha(s)}(\hat{N}),$$

and

$$\dot{\nabla}_X Y=\hat{\nabla}'_X Y+B'(X,Y) \quad \text{for all } X,Y\in T_{\Omega(\alpha(s))}(\hat{M})$$

are the Gauss formulae of $\hat{N}$ and $\hat{M}$ in C_1^{n+1}.

(In fact, $B_x(X,Y)=<\hat{A}X,Y>\xi_x-<X,Y>x$ for all $X,Y\in T_x(\hat{N})$ with the analogous statement for $T_y(\hat{M})$.)

Now by Theorem 41, chap. 4 of [20], there exists an isometry $\bar{\Psi}$

of C_1^{n+1} such that $\bar{\Psi}|_{\hat{N}}=\Omega$. $\bar{\Psi}$ induces a rigid motion of CH^n that maps N isometrically to M.//

With Theorem 3 we can classify the hypersurfaces of section 2 in terms of the examples of section 3.

Theorem 4

Let M be a complete connected contact hypersurface of $CH^n(-4)$, $n\geq 3$. Then M is congruent to one of the following:

i) A tube of radius $r>0$ around a totally geodesic, totally real hyperbolic space form $H^n(-1)$,

ii) A tube of radius $r>0$ around a totally geodesic complex hyperbolic space form $CH^{n-1}(-4)$,

iii) A geodesic hypersphere of radius $r>0$, or

iv) A horosphere.

Proof:

The proof of cases i)-iii) is obvious. An analytic proof of the existence of a horosphere in $CH^n(-4)$ is postponed until the next section. //

Theorem 5

Let M be a complete connected real hypersurface of $CH^n(-4)$ that satisfies

(2.4) $\phi H=H\phi$.

Then M is congruent to one of the following:

i) A tube of radius $r>0$ around a totally geodesic $CH^p(-4)$, $0\leq p\leq n-1$, or

ii) A horosphere.

Proof:

Again i) and ii) are obvious if we grant the existence of a horosphere in $CH^n(-4)$. //

In section 4, semi-symmetric hypersurfaces that are also S^1-invariant are characterized as lifts of real hypersurfaces in CH^n

that satisfy (2.4). Hence:

Corollary 1

Let M_1^{2n} be a semi-symmetric hypersurface of H_1^{2n+1} that is S^1-invariant. Then, M_1^{2n} is congruent to an S^1-fiber bundle over either a tube about a complex hyperbolic space $\mathbf{C}H^p(-4)$, $p=0,1,\ldots,n-1$, imbedded as a totally geodesic complex submanifold of $\mathbf{C}H^n(-4)$, or a horosphere. //

6. An Analytic Construction of a Horosphere

Using the congruence results of section 5, we can now place the model spaces used in [16] into the context of the preceeding geometric classification. First, recall an elegant and well-known , (e.g. [16] and [18]), analytic method of determining the extrinsic geometry of a level hypersurface of a C^∞ function on a space form imbedded as a hypersurface in a euclidean space, modified to fit the particular needs of this section.

Let $f: \mathbf{R}_2^{n+1} \to \mathbf{R}$ be a C^∞ function and $M_1^n(c)$ be an imbedded Lorentzian space form in $\mathbf{R}_2^{n+1}$ of sectional curvature c. Let ∇f denote the gradient of f in T(M) as a function on M, and $\tilde{\nabla} f$ denote the gradient of f as a function on $\mathbf{R}_2^{n+1}$. Let S be the set of all $s \epsilon \mathbf{R}$ such that $\tilde{M}_S = f^{-1}(s)$ is a hypersurface of $\mathbf{R}_2^{n+1}$. Then for any $s \epsilon S$, $\tilde{M}_S$ has $\tilde{\nabla} f / \| \tilde{\nabla} f \|$ as a unit normal field in $T(\mathbf{R}_2^{n+1})$. Similarly, let T be the set of all $s \epsilon S$ such that $M_S = \tilde{M}_S \cap M_1^n(c)$ is a hypersurface of $M_1^n(c)$. Then M_S will have $\nabla f / \| \nabla f \|$ as a unit normal in $T(M_1^n(c))$, for each

$s \in T$.

For a given f, $\tilde{\nabla} f$ is usually easy to calculate. Once this is done,

(6.1) $\nabla f = \tilde{\nabla} f + \langle \tilde{\nabla} f, \zeta \rangle \zeta$

where $\langle\ ,\ \rangle$ is the standard metric of $\mathbf{R}_2^{n+1}$ and ζ is a unit timelike normal field to $M_1^n(c)$ in $\mathbf{R}_2^{n+1}$. (Notice that the choice of positive coefficient of ζ is necessitated by the causal nature of ζ). Let Hess($f; \mathbf{R}_2^{n+1}$) denote the hessian of f as an operator on $\mathbf{R}_2^{n+1}$. For each $s \in S$, the second fundamental form $\tilde{H}$ of $\tilde{M}_s$ in $\mathbf{R}_2^{n+1}$ is given by

(6.2) $\langle \tilde{H}X, Y \rangle = \mathrm{Hess}(f; \mathbf{R}_2^{n+1})(X,Y)/\| \tilde{\nabla} f \|$

for all $X, Y \in T(\tilde{M}_s)$ which for a given f is usually easy to compute. For each $s \in T$, the second fundamental form H of M_s in $M_1^n(c)$ can be obtained similarly:

(6.3) $\langle HX,Y\rangle = \mathrm{Hess}(f;M_1^n(c))(X,Y)/\|\nabla f\|$

for all $X,Y \in T(M_s)$ and where $\mathrm{Hess}(f;M_1^n(c))$ denotes the hessian of f as an operator on $T(M_1^n(c))$. Once (6.2) has been computed, (6.1) can be used to obtain a representation of (6.3), thereby yielding an explicit calculation of the second fundamental form of M_s in $M_1^n(c)$.

Example 5

Consider the function $G_p:\mathbf{C}_1^{n+1} \to \mathbf{R}$, for each $p=0,1,\ldots,n$, defined by

$$G_p(z)=-|z_0|^2+\sum_{i=1}^{p} |z_i|^2$$

where $z=(z_0,z_1,\ldots,z_n)$. For $r>0$, define a level hypersurface of $\mathbf{C}_1^{n+1}$ by

$$\tilde{M}_p(r)=\{z\in \mathbf{C}_1^{n+1} \mid G_p(z)=-\cosh^2(r)\}.$$

The gradient of G_p in $\mathbf{C}_1^{n+1}$ is computed to be

$$\tilde{\nabla}G_p(z)=2(z_0,z_1,\dots,z_p,0,\dots,0)$$

for all $z\in \mathbf{C}_1^{n+1}$, so that

$$\tilde{\nabla}G_p(z)/2\cosh(r)=(\mathrm{sech}(r)z_0,\mathrm{sech}(r)z_1,\dots,\mathrm{sech}(r)z_p,0,\dots,0)$$

is a unit (time-like) normal to $\tilde{M}_p(r)$ in $\mathbf{C}_1^{n+1}$, for all $z\in\tilde{M}_p(r)$.

The level hypersurface

$$M_p(r)=\tilde{M}_p(r)\cap H_1^{2n+1}$$

is nothing more than the model $M_{2p+1,2q+1}$ $(\tanh^2(r))$ of example 4.1 in [16]. Notice that $M_p(r)$ is isometric to the product $H_1^{2p+1}(-\cosh^2(r))\times S^{2(n-p)-1}(\sinh^2(r))$. The gradient of G_p on $M_p(r)$ in $T(H_1^{2n+1})$ is given by (6.1):

$$\nabla G_p(z)=-2\cosh^2(r)(\tanh^2(r)z_0,\dots,\tanh^2(r)z_p,z_{p+1},\dots,z_n)$$

for all $z\in M_p(r)$. Thus,

$$\xi(z)=-(\tanh(r)z_0,\dots,\tanh(r)z_p,\coth(r)z_{p+1},\dots,\coth(r)z_n)$$

is a unit normal to $M_p(r)$ in H_1^{2n+1}, for all $z\in M_p(r)$.

At this point we can see that the second fundamental form of $M_p(r)$ in H_1^{2n+1} is diagonalizable with respect to a real basis of $T_z(M_p(r))=\{z,\xi(z)\}^\perp$ and has constant principal curvatures $\tanh(r)$ and $\coth(r)$ of real multiplicities $2p+1$ and $2n-2p-1$, respectively.

Let $N_p(r)=\pi(M_p(r))$ and U denote the distinguished vector on $N_p(r)$ viewed as a real hypersurface of $\mathbf{C}H^n$. Let H' and H denote the second fundamental forms of $M_p(r)$ and $N_p(r)$, respectively. We can write

$$U_{\pi(z)}=-J(\pi_*(\xi(z)))=\pi_*(-i\xi(z)),\quad \text{i.e. } (U^\sim)_z=-i\xi(z).$$

An explicit calculation of $H'(U^{\sim})_z$ using (6.3) followed by an application of (4.4) shows that $U_{\pi(z)}$ is principal in $T_{\pi(z)}(N_p(r))$ with curvature 2coth(2r). Subsequent calculations yield the other principal curvatures tanh(r) and coth(r) of multiplicities 2p and 2n−2p−2, respectively, each of which having a ϕ-invariant eigendistribution. From the work of sections 3 and 5 we see that $N_p(r)$ is congruent to a tube of radius r about a totally geodesic complex space form in $\mathbf{C}H^n$ isometric to a $\mathbf{C}H^p$. In particular, we shall need that $N_0(r)$ is congruent to a geodesic hypersphere of radius r.

<u>Example 6</u>

Consider the function $G:\mathbf{C}_1^{n+1} \to \mathbf{R}$ given by $G(z)=|z_0-z_1|^2$, where $z=(z_0,\ldots,z_n)$, and the level hypersurface of $\mathbf{C}_1^{n+1}$, $\tilde{M}=G^{-1}(1)$. The gradient of G in $\mathbf{C}_1^{n+1}$ can be written as

$$\tilde{\nabla}G(z)=2(z_1-z_0,z_1-z_0,0,\ldots,0)$$

for all $z\epsilon\tilde{M}$.

The level hypersurface $N=\tilde{M}\cap H_1^{2n+1}$ is the model hypersurface N of example 4.2 of [16]. The gradient of G in H_1^{2n+1} is given by

$$\nabla G(z)=2(z_1,2z_1-z_0,z_2,\ldots,z_n)$$

for all $z\epsilon N$. Hence,

$$\xi(z)=(z_1,2z_1-z_0,z_2,\ldots,z_n)$$

is a unit normal to N in H_1^{2n+1}.

Set $M_n^*=\pi(N)$, as in example 4.2 of [16]. Explicit calculations using (6.3) and (4.4) show that U is principal on M_n^* with curvature 2, and that 1 is a principal curvature of multiplicity $2n-2$, i.e. the second fundamental form of M_n^* acts as the identity transformation on $\ker(f)$. Thus, M_n^* is our candidate for a horosphere. In order to see this and thereby complete the classification analytically we will show that M_n^* is a limiting hypersurface of a specific family of geodesic hyperspheres.

Other than the fact that M_n^* and a horosphere have the same second fundamental form, it is not clear that M_n^* can be realized as the limiting hypersurface of a certain family of expanding geodesic hyperspheres. In the following discussion we shall use the hypersurfaces of H_1^{2n+1} constructed in example 5 to show that this is indeed the case.

Let $P=(1,0,\ldots,0)\epsilon H_1^{2n+1}$ and consider the geodesic emanating from $\pi(P)$ in $\mathbf{C}H^n$ given by $\gamma(r)=\pi(\cosh(r),\sinh(r),0,\ldots,0)$. (See p. 285 of [12], Vol II.) As in example 3, each geodesic hypersphere of radius r centered at $\gamma(r)$ contains the point $\pi(P)$. We will see that these hyperspheres converge to a limiting hypersurface, namely M_n^*.

Earlier in this section, we discovered that the hypersurface of H_1^{2n+1} defined by

$$M_0(r)=\{z\epsilon H_1^{2n+1} \mid \tanh^2(r)|z_0|^2=\sum_{i=1}^{n} |z_i|^2\}$$

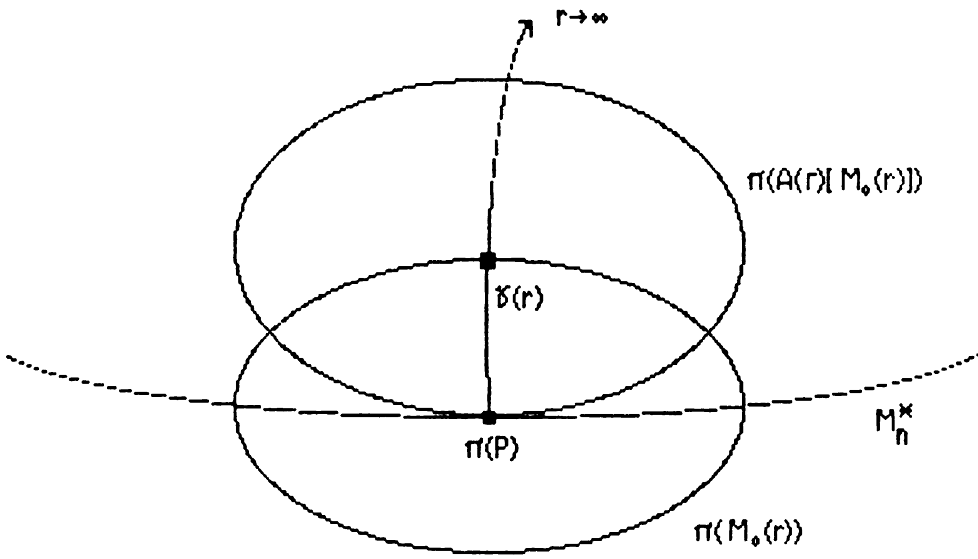
is actually the lift (up to a congruence, of course) of a geodesic hypersphere of radius r. Notice that $\gamma(r)\epsilon M_0(r)$ and that $\pi(P)$ is equidistant from every point on $M_0(r)$. So, $\pi(P)$ plays the role of center of $\pi(M_0(r))$ in $\mathbf{C}H^n$.

In particular, we see that the family of hypersurfaces $\{\pi(M_0(r)) \mid r>0\}$ is not our candidate for the convergent family. However, all is not lost, for we should be able to find a rigid movement of $\mathbf{C}H^n$, induced by an $A(r)\epsilon U(1,n)$, that for each $r>0$ will translate $\pi(M_0(r))$ to a geodesic hypersphere of radius r and center $\gamma(r)$ in such a way that the family $\{\pi(A(r)[(M_0(r))]) \mid r>0\}$ will converge to a limiting hypersurface. This limiting hypersurface will be $M_n^*=\pi^{-1}(N)$, which must then be a horosphere through $\pi(P)$.

For each $r>0$, let $A(r)\epsilon U(1,n)$ be defined by

$$A(r)=\mathrm{diag}\left\{\begin{bmatrix}\cosh(r) & -\sinh(r)\\ \sinh(r) & -\cosh(r)\end{bmatrix}, I_{n-1}\right\}.$$

$A(r)$ is a rigid motion that maps $M_0(r)$ onto the lift of the geodesic hypersphere of radius r centered at $\gamma(r)$, and therefore induces a rigid motion of CH^n that moves the geodesic hypersphere $\pi(M_0(r))$ that has radius r and center $\pi(P)$ onto the geodesic hypersphere that has radius r and center $\gamma(r)$ and contains $\pi(P)$.



Proposition 2

N is the limiting hypersurface of the family $\{A(r)[M_0(r)] \mid r>0\}$ of hypersurfaces in H_1^{2n+1}, i.e.

$$\lim_{r\to\infty} \{A(r)[M_0(r)]\} = N$$

and is therefore an S^1-fiber bundle over a horosphere.

Proof:

For any $z=(z_0,z_1,\ldots,z_n)\epsilon M_0(r)$ we have

$$|z_0|^2=\cosh^2(r)$$

and

$$\sum_{j=1}^{n} |z_j|^2 = \sinh^2(r).$$

Let $w=(w_0,w_1,\ldots,w_n)\epsilon A(r)[M_0(r)]$. Then

$w=(\cosh(r)z_0-\sinh(r)z_1, \sinh(r)z_0-\cosh(r)z_1, z_2,\dots,z_n)$

for some $z\epsilon M_0(r)$. In particular, we have

$|w_0-w_1|=(\cosh(r)-\sinh(r))|z_0+z_1|=e^{-r}|z_0+z_1|$.

Thus,

$|w_0-w_1|\le e^{-r}(|z_0|+|z_1|)\le e^{-r}(\cosh(r)+\sinh(r))=1$

which shows that the limiting hypersurface $\lim_{r\to\infty}\{A(r)[M_0(r)]\}$ must satisfy

$|z_0-z_1|\le 1$ for any $z=(z_0,\dots,z_n)\epsilon \lim_{r\to\infty}\{A(r)[M_0(r)]\}$.

To see the reverse inequality, let $R>0$ be given. For each $r>R$, consider the disc

$S(r,R)=\{w\epsilon A(r)[M_0(r)]\,|\,d(P,w)<R\}$

on the translated lift of the geodesic hypersphere $\pi(M_0(r))$. Let $w \epsilon S(r,R)$. Since $A(r)$ is an isometry, there is a $z \epsilon M_0(r)$ such that $w=A(r)z$ and

$$d((\cosh(r),\sinh(r),0,\ldots,0),z)<R.$$

Hence, $|z_0-\cosh(r)|<R$, $|z_1-\sinh(r)|<R$, and $|z_k|<R$ for $k\geq 2$. Thus, for r sufficiently large,

$$|w_0-w_1| = e^{-r}|z_0+z_1| \geq e^{-r}(\cosh(r)+\sinh(r)-R)=1-e^{-r}R.$$

Now we see that for a point z in the lift of the horosphere within R units of P, we must have $|z_0-z_1|\geq 1$. But R was an arbitrary choice so that

$$N=\bigcup_{R>0}\{\lim_{r\to\infty} S(r,R)\} = \lim_{r\to\infty}\{A(r)[M_0(r)]\}. \quad //$$

This establishes the existence of a horosphere analytically in

$\mathbf{CH}^n$ and thereby completes the preceeding classification.

Notice that the representation of a horosphere as a submersed level hypersurface depends both on the choice of $P \epsilon H_1^{2n+1}$ and on the geodesic emanating from P; equivilantly: upon the choice of normal to the lift of a horosphere at P. It is interesting to note that we obtain different bounds for $|z_0 - z_1|^2$ for a limiting hypersurface of a convergent family of S^1-fiber bundles over geodesic hyperspheres if a different geodesic emanating from P is selected.

In [2], the converse to Proposition 1 is proved for the ambient space $\mathbf{CP}^n$, which allows a classification of its real hypersurfaces that have the direction U principal. If we enlarge the class of tubes to include horospheres (as hyperspheres of infinite radius and centered at points at infinity), I believe the converse to Proposition 1 is also true if the ambient space is $\mathbf{CH}^n$. However, in order to achieve a clasification of hypersurfaces of $\mathbf{CH}^n$ that have U as a principal direction, more general congruence results than those of section 5 must be found, in light of example 4.

LIST OF REFERENCES

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[1] Blair, Contact Manifolds in Riemannian Geometry, Lecture Notes in Mathematics, 509, Springer-Verlag

[2] Cecil and Ryan, Focal Sets and Real Hypersurfaces in Complex Projective Space, Trans. Amer. Math. Soc., Vol 269, #2, Feb. 1982

[3] Chen, Geometry of Submanifolds, Marcel Dekker, Inc. 1973

[4] Chen, Ludden and Montiel, Real Submanifolds of a Kaehler Manifold, Algebras, Groups and Geometries I (1984), 176-212

[5] Chen and Ogiue, Two Theorems on Kaehler Manifolds, Mich. Math. J., (21)1974, 225-229

[6] Chen and Vanhecke, Differential Geometry of Geodesic Spheres, J. Reine und Angewandte Math., Band 325, 1981, 28-67

[7] Eschenburg, Horospheres and the Stable Part of the Geodesic Flow, Math. Z., 153, 237-251(1977)

[8] Escobales, Riemannian Submersions with Totally Geodesic Fibers, J. Diff. Geom. 10(1975), 253-276

[9] Gray and Vanhecke, The Volumes of Tubes About Curves in a Riemannian Manifold, Proc. London Math. Soc., (3), 44(1982),215-243

[10] Heintze and Im Hof, Geometry of Horospheres, J. Diff. Geom., 12(1977), 481-491

[11] Ralph Howard, The Weingarten Map of a Tube, Personal Communication

[12] Kobayashi and Nomizu, Foundations of Differential Geometry, Vol I & II, John Wiley and Sons, 1969

[13] Kon, Pseudo-Einstein Hypersurfaces in Complex Space Forms, J. Diff. Geom., 14(1979), 339-354

[14] Maeda, On Some Real Hypersurfaces of a Complex Projective Space, J. Math. Soc. Japan, Vol 28, #3, 1976

[15] Matsuyama, Complete Hypersurfaces with R·S=0 in E^{n+1}, Proc. Amer. Math. Soc., vol 88, no. 1, May 1983

[16] Montiel and Romero, On Some Real Hypersurfaces of Complex Hyperbolic Space, Preprint

[17] Nomizu, On Hypersurfaces Satisfying a Certain Condition on the Curvature Tensor, Tohoku Math. J., 20(1968), 46-59

[18] Nomizu, Elie Cartan's Work on Isoparametric Families of Hypersurfaces, Proc. Symp. in Pure Math.,Vol 27, 1975

[19] Okumura, Contact Hypersurfaces in Certain Kaehlerian Manifolds, Tohoku Math. J., Vol 18, No. 1, 1966, 74-102

[20] O'Neill, Semi-Riemannian Geometry, Academic Press, 1983

[21] O'Neill, The Fundamental Equations of a Submersion, Michigan Math. J., 13(1966)459-469

[22] Olszak, Contact Metric Hypersurfaces in Complex Spaces, Demonstratio Mathematica, Vol XVI, No. 1, 1983

[23] Ryan, Homogeneity and Some Curvature Conditions, Tohoku Math. Jour., 21(1969), 363-388

[24] Ryan, Hypersurfaces with Parallel Ricci Tensor, Osaka J. Math., 8(1971), 251-259

[25] Szabo, Structure Theorems on Riemannian Spaces Satisfying $R(X,Y)\cdot R=0$, J. Diff. Geom., 17(1982)531-582

[26] Takagi, An Example of a Riemannian Manifold Satisfying $R(X,Y)\cdot R=0$ but not $\nabla R=0$, Tohoku Math. J., 24(1972), 105-108

[27] Takagi, Real Hypersurfaces in a Complex Projective Space with Constant Principal Curvatures, J. Math. Soc. Japan, Vol 27, No. 1, 1975

[28] Takahashi and Tanno, Some Hypersurfaces of a Sphere, Tohoku Math. J., 22(1970), 212-219

[29] Tanno, Hypersurfaces Satisfying a Certain Condition on the Ricci Tensor, Tohoku Math. J., 21(1969),297-219

[30] Tanno, A Class of Riemannian Manifolds Satisfying $R(X,Y)\cdot R=0$, Nagoya Math. J., vol. 42(1971),67-77

[31] Vanhecke and Willmore, Jacobi Fields and Geodesic Spheres, Proc. Roy. Soc. Edin., 82A, 233-240, 1979

[32] Wolf, Spaces of Constant Curvature, McGraw Hill, 1967