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V. P. Sreedharan

Major professor

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EXISTENCE AND COMPUTATION OF STATIC EQUILIBRIA IN CERTAIN ECONOMIC MODELS, WITH APPLICATION TO THE PIES MODEL OF THE ENERGY SECTOR

By

Paul Arthur Rubin

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ABSTRACT

EXISTENCE AND COMPUTATION OF STATIC EQUILIBRIA IN CERTAIN ECONOMIC MODELS, WITH APPLICATION TO THE PIES MODEL OF THE ENERGY SECTOR

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In this thesis we examine the Project Independence Evaluation System (PIES) integrating model, a static equilibrium model of the energy sector of the national economy. After carefully formulating the equilibrium problem, we establish conditions sufficient to ensure the existence of an equilibrium, as well as conditions sufficient to guarantee uniqueness of that equilibrium. We study the PIES algorithm, prove that in certain cases it must converge to an equilibrium, and exhibit an example on which it fails to converge. The same is done for the PIES-VAR variant algorithm. We pose a minimization problem related to the task of locating equilibria, and propose a subgradient-based algorithm for that problem. Finally, we describe the implementation of the subgradient algorithm and discuss the results of some computational trials.

To all who waited.

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CHAPTER 0

INTRODUCTION

The Project Independence Evaluation System [5] henceforth denoted PIES, is an aggregation of models which describes the energy sector of the national economy, developed by the Federal Energy Administration (now part of the Department of Energy) as a tool for the evaluation of policy decisions. The various components of PIES model the production, refinement, conversion, transportation and consumption of a variety of energy commodities. Of particular importance is the estimation of a static partial equilibrium for the energy sector, a vector of prices at which supply and demand will be in agreement. The component models take as parameters factors such as tax policies and the pricing of crude oil by foreign producers. The static equilibrium predicted by PIES based on specified values for these parameters is taken as an indication of the expected market response to those policies.

Central to the estimation of the static equilibrium is the integrating model [5 ,17], which computes the equilibrium based on supply and demand functions generated by other component models. The integrating model employs an iterative technique which successively refines approximations to the equilibrium demand vector. Although observed to converge rapidly in the examples reported by Hogan and Wagner, the algorithm has evaded to date a complete theoretical analysis regarding convergence, although it has motivated a significant amount of research and several variant algorithms [3 ,6]. In this paper we attempt to shed some light on the existence and determination of equilibria. In particular, we propose a new algorithm for computing the equilibria.

We begin with a mathematical formulation of the integrating model, introducing the necessary concepts from convex analysis. We then prove the existence of an equilibrium under fairly general conditions, conditions consistent with the PIES models. We note that the authors of PIES, in their publications, have assumed rather than demonstrated the existence of equilibria. Uniqueness of the equilibrium has been demonstrated [15] under a number of hypotheses made by the PIES modellers. We show that one of these hypotheses appears to be inconsistent with the form of the demand model, and that in the absence of that hypothesis the equilibrium need not be unique.

We examine the PIES algorithm for computing the equilibrium, exhibiting an example in which the algorithm fails to converge to a solution. This explains the failure of others to produce general convergence results, and suggests the need for either additional hypotheses or a different algorithm. We also examine one variant of the PIES algorithm, the PIES-VAR algorithm [6], and exhibit an example in which it fails to converge.

We suggest an algorithm for solving the equilibrium problem in essentially the same form as that assumed by the PIES algorithm. The algorithm we present is applicable to a general class of problems. The proof of convergence of the algorithm which we present assumes that the demand function has a potential. Implementation of the algorithm does not require the existence of such a potential, which suggests the distinct possibility of proving that the algorithm converges even where the demand function has no potential. We shall return to this problem at a later date.

CHAPTER I

THE INTEGRATING MODEL

We consider the problem of determining a static market equilibrium in a context slightly more general than that of the PIES algorithm [5]. Let us assume that we have a finite collection of goods indexed by the integers $1,\ldots,d$. In the PIES models, goods represent various energy products, differentiated by type of energy, region of production and region of consumption. Wagner [17] suggests that $d = 54$ is typical for the PIES model.

We represent demands, supplies and prices of these goods as vectors of length d, i.e., elements of the d-dimensional euclidean space $\mathbb{R}^d$. Let us pause to establish some notation. We do not, in general, distinguish between row and column vectors: the context determines the shape of the vector. In particular, the usual inner-product of two vectors will be denoted by the juxtaposition of those two vectors; that is, for $x,y \in \mathbb{R}^d$,

$$xy = \sum_{i=1}^{d} x_i y_i .$$

We denote by $\mathbb{R}^d_+$ the nonnegative orthant

$$\{x \in \mathbb{R}^d : x_i \geq 0, \; i = 1,\ldots,d\}.$$

When ordering vectors $x,y \in \mathbb{R}^d$, we adopt the following notation:

$$x \geq y \quad \text{iff} \quad x_i \geq y_i, \quad i = 1,\ldots,d;$$
$$x > y \quad \text{iff} \quad x \geq y \quad \text{and} \quad x \neq y;$$
$$x \gg y \quad \text{iff} \quad x_i > y_i, \; i = 1,\ldots,d.$$

For any subset S of $\mathbb{R}^d$, we denote by $\operatorname{int} S$ and $\operatorname{bd} S$ the interior and boundary respectively of S. The affine subspace of $\mathbb{R}^d$ of least dimension containing S is the <u>affine hull</u> of S, denoted $\operatorname{aff} S$. The interior and boundary of S in the subspace topology of $\operatorname{aff} S$ are the <u>relative interior</u> and <u>relative boundary</u> of S, denoted $\operatorname{rel\,int} S$ and $\operatorname{rel\,bd} S$ respectively. As much of our interest will focus on sequences of vectors rather than components of those vectors, we for convenience index members of a sequence of vectors with subscripts. Thus the symbol x_n may represent the nth component of a vector x or the nth member of a sequence of vectors, depending on context. When such usage is potentially ambiguous, we will exercise greater care.

To continue with our model, we assume that we are given two vector-valued functions of a vector variable,

$$p, p^{-1} : \operatorname{int} \mathbb{R}^d_+ \to \operatorname{int} \mathbb{R}^d_+ ,$$

which are C^1 and are inverses of each other. The functions

p and p^{-1} represent respectively the vector of prices at which a specified vector of quantities is demanded (the <u>indirect demand function</u>) and the vector of quantities demanded at a given vector of price levels (the <u>direct demand function</u>). In the PIES models these functions are predicted by econometric methods and the predictions then approximated: the actual demand model is time-dependent, whereas the input fed to the integrating model represents a cross-section of the demand function with time fixed. The form chosen by the PIES modellers is the log-linear form, expressed by the equations

$$p_i = k_i \prod_{j=1}^{d} q_j^{m_{ij}} , \quad i = 1,\ldots,d \tag{1.0.1}$$

where p_i and q_i are respectively the price of and demand for the ith commodity. Adopting the notation

$$\log x = (\log x_1,\ldots,\log x_d) \quad \text{for} \quad x \in \operatorname{int} \mathbb{R}_+^d ,$$

we can rewrite equations (1.0.1) as

$$\log p = K + M \log q \tag{1.0.2}$$

where $K = (\log k_1,\ldots,\log k_d)$ and M is the $d \times d$ matrix (m_{ij}). The numbers m_{ij} represent the elasticity of the price of good i with respect to demand for good j; the significance of the log-linear form is that all price elasticities are constant.

For the supply side of our model, we employ a linear program. Suppose that there are s activities in which producers may engage. Denote the levels of those activities by a vector $x \in \mathbb{R}^s$. These activities include production, refining, conversion, transportation and storage. We denote by A the $d \times s$ matrix whose ith column represents the output of the d commodities when the ith activity is performed at unit level, so that Ax represents the output from activities x. We assume that the only limitations on the activities are that they be performed at nonnegative levels and that their consumption of r given resources not exceed the availability of these resources. We denote by $b \in \mathbb{R}^r$ the vector of available resources and by B the $r \times s$ matrix whose ith column represents the consumption of resources when the ith activity is performed at unit level. The vector $c \in \mathbb{R}^s$ will represent the costs of performing the various activities at unit levels. We assume that no commodity is both produced and consumed, and that goods produced in excess of demand may be disposed of at no cost. The significance of this last assumption will be discussed later. The supply model is predicated on the assumption that producers will elect to meet a specified demand by adopting a linear program to determine a vector of activities which meets or exceeds that demand at least cost. Given demand q, the linear program is

$$\begin{aligned} Ax &\geq q \\ Bx &\leq b \\ x &\geq 0 \\ cx &\ (\min) \end{aligned} \tag{1.0.3}$$

whose value (least cost) we denote by $v(q)$. Note that q is not required to be nonnegative, although negative demands have no obvious economic interpretation in this model. We extend the definition of v to all of $\mathbb{R}^d$ by setting $v(q) = +\infty$ whenever the demand q is not producible.

The supply model described by Hogan [4 , 5] and Wagner [17] uses equality rather than inequality constraints. In the case of the resource constraints, this is no problem. Constraints on resource consumption are by nature inequalities, but can be made equalities by the addition of slack variables. The so-called "material balance" constraints, such as requirement that material transported to a depot exactly equal material transported from that depot, are by nature equalities but can be written as pairs of inequalities. In the case of output constraints, the distinction is somewhat more critical. In general, the most cost-efficient way to produce at least q (allowing free disposal of excesses) may be strictly more efficient than the least expensive way to produce exactly q. We will return to the matter of free disposal later, at which time we will show that the PIES model tacitly assumes it.

To determine the level of prices at which producers will meet demand q, we adopt marginal pricing, taking the supplier's price for a unit of good i (at the current demand level) to be the marginal cost of producing a unit of good i. Other pricing mechanisms, such as using the average cost of a unit of good i as the supply price, have been mentioned in some of the literature on the PIES model but will not be discussed here. To determine the marginal costs in our formulation of the supply model, we must introduce some concepts from convex analysis. These concepts are treated in detail by Rockafellar [11], whose notation we adopt where practical. We repeat some of the definitions with the aim of establishing relevant notation.

1.1 Definition. A function $f : \mathbb{R}^d \to [-\infty,+\infty]$ is convex iff for any $x,y \in \mathbb{R}^d$ and any $t \in (0,1)$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \qquad (1.1.1)$$

provided the right-hand side makes sense (more precisely, if it does not involve expressions of the form $\infty - \infty$). A convex function is proper iff it never assumes the value $-\infty$ and is not identically $+\infty$.

Throughout this paper we deal only with proper convex functions. It is immediate that the function v defined in (1.0.3), being the value of a linear minimization program, is convex; it will be shown later to be proper under suitable hypotheses.

1.2 Definition. For $f : \mathbb{R}^d \to [-\infty,+\infty]$, convex, the effective domain of f is the set

$$\text{eff dom } f = \{x : f(x) < +\infty\}.$$

1.3 Definition. For f convex on $\mathbb{R}^d$, $u \in \mathbb{R}^d$ is a subgradient of f at a iff $f(a)$ is a finite real number and

$$f(x) \geq f(a) + u(x-a) \quad \text{for all} \quad x \in \mathbb{R}^d . \tag{1.3.1}$$

The collection of all subgradients of f at a is the subdifferential of f at a and is denoted by $\partial f(a)$.

Clearly, $\partial f(a)$ is a closed convex (possibly empty) set. It is known that $\partial f(a)$ is not empty if f is proper and $a \in \text{rel int (eff dom } f)$.

We now return to the matter of marginal cost pricing for our supply model. The function v defined by (1.0.3), while convex and (as will be shown) proper, is in general, not differentiable, the lack of smoothness occurring at demand levels at which an activity begins or ceases to be cost-effective. Our replacement for the marginal cost vector is the subgradient, which provides us, through (1.3.1), with a lower bound for the change in total cost as demand changes. Unfortunately, at demand levels at which v is nondifferentiable the subgradient is not unique, and so in general we have an entire set of possible supply price vectors for a given demand.

A static equilibrium between supply and demand is defined as a vector of goods such that the price at which that vector is demanded is one possible price at which that vector is supplied. In terms of subdifferentials, this becomes the following.

<u>1.4 Definition</u>. $q^* \in \mathbb{R}^d$ is an <u>equilibrium</u> for our models iff

$$p(q^*) \in \partial v(q^*),$$

where p is the indirect demand function and v is defined by (1.0.3).

If our supply model is to be useful from a computational standpoint, we must be able to compute $\partial v(q)$ at least for those q which are producible, i.e., for $q \in$ eff dom v. The following theorems from linear programming provide a useful characterization of $\partial v(q)$.

<u>1.5 Theorem</u>. Let A be an arbitrary $m \times n$ matrix, c an arbitrary n-vector. Denote by $f(z)$ the value of the linear program

$$Ax \geq z, \quad x \geq 0, \quad cx \text{ (min)}, \tag{1.5.1}$$

where the minimum over the empty set is taken to be $+\infty$; then

(i) f is a convex function on $\mathbb{R}^m$,

(ii) f is nondecreasing in the sense that if

$$w \leq z$$

then

$$f(w) \leq f(z),$$

and

(iii) if $f(b)$ is finite then f is subdifferentiable at b.

In the event that $f(b)$ is finite,

(iv) $\bar{y} \in \partial f(b)$ iff $\bar{y}$ is a maximal solution of the dual problem

$$yA \leq c, \quad y \geq 0, \quad yb(\max). \tag{1.5.2}$$

Proof: Conclusions (i) and (ii) are immediate. Though (iii) and (iv) can be derived from the general theory of convex programming, we provide, for the reader's convenience, a self-contained proof using only the duality theorems of linear programming.

Assume $f(b)$ is finite. By the duality theorems, the problem (1.5.2) is feasible, and hence

$$Y = \{y \in \mathbb{R}^m : y \geq 0, \ yA \leq c\}$$

is nonempty. Note that Y is defined independently of z, and so the dual to (1.5.1) is feasible for all z. By the weak duality theorem of linear programming, then,

$$f(z) > -\infty \quad \text{for all} \quad z \in \mathbb{R}^m.$$

Since we have assumed that $f(b)$ is finite, by the strong duality theorem of linear programming the dual problem (1.5.2) has a maximal solution, and any maximal solution $\bar{y}$ of (1.5.2) satisfies

$$f(b) = \bar{y}b. \tag{1.5.3}$$

We now show that every maximal solution $\bar{y}$ of (1.5.2) belongs to $\partial f(b)$. Observe that for any z, $\bar{y}$ is feasible in the problem

$$yA \leq c, \quad y \geq 0, \quad yz(\max) \tag{1.5.4}$$

dual to (1.5.1). By the weak duality theorem, we have that

$$f(z) \geq \bar{y}z \quad \text{for all} \quad z \in \mathbb{R}^m . \tag{1.5.5}$$

In view of (1.5.3) and (1.5.5),

$$f(z) \geq f(b) + \bar{y}(z-b) \quad \text{for all} \quad z \in \mathbb{R}^m , \tag{1.5.6}$$

showing that $\bar{y} \in \partial f(b)$.

We now show that if $\bar{y} \in \partial f(b)$ then $\bar{y}$ is a maximal solution of (1.5.2). We are given that (1.5.6) holds; from (ii), it follows that for $z \leq b$,

$$f(b) \geq f(z) \geq f(b) + \bar{y}(z-b),$$

and so

$$\bar{y}(z-b) \leq 0 \quad \text{for all} \quad z \leq b.$$

This shows that $\bar{y} \geq 0$. Taking $z = Ax$ $(x \geq 0)$ in (1.5.6), we have

$$f(Ax) - f(b) \geq \bar{y}(Ax-b).$$

Also, from (1.5.1)

$$f(Ax) \leq cx,$$

and so

$$cx - f(b) \geq \bar{y}(Ax-b) \quad \text{for all} \quad x \geq 0.$$

Since $f(b)$ is finite, there exists a minimal solution $\bar{x}$ of (1.5.1) when $z = b$, with $f(b) = c\bar{x}$. Thus

$$cx - c\bar{x} \geq \bar{y}(Ax-b) \quad \text{for all} \quad x \geq 0,$$

and so

$$(c - \bar{y}A)x \geq c\bar{x} - \bar{y}b \quad \text{for all} \quad x \geq 0. \tag{1.5.7}$$

If the jth component of $c - \bar{y}A$ were negative, we could violate (1.5.7) by taking x to be a sufficiently large multiple of the jth standard basis vector in $\mathbb{R}^n$. Hence (1.5.7) implies that

$$c - \bar{y}A \geq 0,$$

which, together with the observation that $\bar{y} \geq 0$ made above, shows that $\bar{y}$ is feasible in (1.5.2). Taking $x = 0$ in (1.5.7), we see that

$$c\bar{x} \leq \bar{y}b.$$

The opposite inequality also holds, by the weak duality theorem, and so

$$c\bar{x} = \bar{y}b,$$

indicating that $\bar{y}$ is optimal in (1.5.2).

1.6 Theorem. Let A be an $m \times n$ matrix, B an $r \times n$ matrix, c an n-vector. Denote by $f(z,w)$ the value of the linear program

$$Ax \geq z, \quad Bx \geq w, \quad cx(\min). \tag{1.6.1}$$

Suppose that $f(a,b)$ is a finite real number. For $z \in \mathbb{R}^n$, let

$$g(z) = f(z,b).$$

The function g is subdifferentiable at a, and for any $u \in \mathbb{R}^m$, $u \in \partial g(a)$ iff there exists a $v \in \mathbb{R}^r$ such that $(u,v) \in \partial f(a,b)$.

Proof: Assume that $f(a,b)$ is finite; by Theorem (1.5), $\partial f(a,b)$ is nonempty.

If $(u,v) \in \partial f(a,b)$, then for all $z \in \mathbb{R}^m$

$$\begin{aligned} g(z) = f(z,b) &\geq f(a,b) + u(z-a) + v(b-b) \\ &= g(a) + u(z-a), \end{aligned}$$

and so $u \in \partial g(a)$. In particular, since $\partial f(a,b)$ is nonempty, so is $\partial g(a)$, i.e., g is subdifferentiable at a.

Now let $u \in \partial g(a)$ be given. In view of Theorem 1.5, to show $(u,v) \in \partial f(a,b)$ for some v it suffices to show the existence of a v for which (u,v) is a maximal solution of

$$yA + sB \leq c, \quad y \geq 0, \quad s \geq 0, \quad ya + sb(\max), \tag{1.6.2}$$

the problem dual to (1.6.1) when z and w are a and b respectively. By precisely the same reasoning as in the proof of Theorem 1.5, g is nondecreasing in the sense of part (ii) of that theorem, and so $u \in \partial g(a)$ implies $u \geq 0$. Since, by the weak duality theorem,

$$yA + sB \leq c, \quad y \geq 0, \quad s \geq 0, \quad ya + sb \geq f(a,b)$$

implies (y,s) is optimal in (1.6.2), it suffices to show the existence of a vector v such that

$$uA + vB \leq c, \; v \geq 0, \; ua + vb \geq f(a,b) \tag{1.6.3}$$

Consider the auxiliary linear program

$$Bx \geq b, \quad x \geq 0, \quad cx - uAx(\min) \tag{1.6.4}$$

and its dual

$$sB \leq c - uA, \quad s \geq 0, \quad sb(\max). \tag{1.6.5}$$

Since $f(a,b)$ is finite, (1.6.1) is feasible when $(z,w) = (a,b)$, and so (1.6.4) is feasible. Moreover, for any x feasible in (1.6.4), we have from the definition of g that

$$cx \geq g(Ax).$$

This and the fact that $u \in \partial g(a)$ imply that

$$cx \geq g(Ax) \geq g(a) + u(Ax - a)$$

and so

$$cx - uAx \geq g(a) - ua. \tag{1.6.6}$$

Thus the value of (1.6.4) is bounded below by $g(a) - ua$. It follows that (1.6.4) and (1.6.5) both have optimal solutions, say $\bar{x}$ and v respectively. From the feasibility of v in (1.6.5) we have

$$uA + vB \leq c, \quad v \geq 0,$$

leaving us only the task of showing that

$$ua + vb \geq f(a,b) = g(a). \tag{1.6.7}$$

Since $\bar{x}$ and v are optimal, by the strong duality theorem

$$vb = c\bar{x} - uA\bar{x};$$

applying (1.6.6) with $x = \bar{x}$, we have

$$vb \geq g(a) - ua$$

which is (1.6.7).

We now introduce a special class of convex functions needed in the sequel.

<u>1.7 Definition</u>. A real-valued function on $\mathbb{R}^d$ is said to be <u>polyhedrally convex</u> iff it is the upper envelope of a finite collection of real affine functions defined on all of $\mathbb{R}^d$. Thus f is polyhedrally convex iff there exist affine functions f_i $(i = 1,\ldots,m)$ defined by

$$f_i(x) = a_i x + b_i \quad \text{for all} \quad x \in \mathbb{R}^d,$$

with $a_i \in \mathbb{R}^d$, $b_i \in \mathbb{R}$, such that

$$f(x) = \max\{f_i(x) : i = 1,\ldots,m\}$$

for all $x \in \mathbb{R}^d$. Note that this is equivalent to saying that f is piecewise-affine.

<u>1.8 Theorem</u>. Let A be an $m \times n$ matrix, B an $r \times n$ matrix, c an n-vector, b an r-vector. Denote by $g(z)$ the value of the linear program

$$Ax \geq z, \quad Bx \geq b, \quad x \geq 0, \quad cx(\min) \tag{1.8.1}$$

If there exists $a \in \mathbb{R}^m$ such that $g(a)$ is finite, then there exists a polyhedrally convex function h on $\mathbb{R}^m$ such that

$$g(z) = h(z) \quad \text{for all} \quad z \in \text{eff dom } g. \tag{1.8.2}$$

<u>Proof</u>: Let a be such that $g(a)$ is finite. Since

$$Ax \geq a, \quad Bx \geq b, \quad x \geq 0, \quad cx(\min)$$

has an optimal solution, so must its dual

$$yA + sB \leq c, \quad y \geq 0, \quad s \geq 0, \quad ya + sb(\max). \tag{1.8.3}$$

Thus the problem

$$yA + sB \leq c, \quad y \geq 0, \ s \geq 0, \quad yz + sb(\max), \qquad (1.8.4)$$

which is dual to (1.8.1), is feasible for all z, the optimal solution to (1.8.3) in particular being feasible in (1.8.4), and so the value of (1.8.1) is bounded below for all z, i.e., $g(z) > -\infty$ for all z. This means g is finite on eff dom g, and so it suffices to find h polyhedrally convex such that

$$g(z) = h(z) \quad \text{whenever} \quad g(z) \quad \text{is finite.}$$

Let F be the set of feasible solutions to (1.8.4), i.e.,

$$F = \{(y,s) : y \geq 0, \ s \geq 0, \ yA + sB \leq c\},$$

which we have shown to be nonempty. Let z be any point where g is finite. Since F is polyhedrally convex and (by virtue of the nonnegativity constraints) line-free, problem (1.8.4) has optimal solutions, at least one of which must be an extreme point of F. F has finitely many extreme points, which we may enumerate as $\{(u_i,v_i) : i = 1,\ldots,T\}$. Let

$$h_i(w) = u_i w + v_i b, \quad i = 1,\ldots,T, \quad w \in \mathbb{R}^m,$$

and let

$$h(w) = \max\{h_i(w) : i = 1,\ldots,T\}, \ w \in \mathbb{R}^m.$$

Since (1.8.4) is optimized at one of the extreme points of F, by the strong duality theorem

$$g(z) = \max\{u_i z + v_i b : i = 1,\ldots,T\} = h(z)$$

whenever $g(z)$ is finite, which is the desired result.

We remark that equality (1.8.2) does not extend beyond eff dom g. Also, in the course of the proof we showed g to be proper if it is ever finite.

<u>1.9 Corollary</u>. The restriction of g to eff dom g is continuous.

<u>Proof</u>: This follows from (1.8.2) and the observation that h is continuous on all of $\mathbb{R}^m$.

<u>1.10 Proposition</u>. Let g be as in Theorem 1.8, with G = eff dom g. The set G is polyhedrally convex and closed.

<u>Proof</u>: Let

$$X = \{x \in \mathbb{R}^n : Bx \geq b,\ x \geq 0\}.$$

X is polyhedrally convex and

$$G = A(X) - \mathbb{R}^m_+ ,$$

so G is polyhedrally convex, and consequently closed.

We have now obtained the desired description of $\partial v(g)$, v the supply cost function defined by (1.0.3). Let $g \in$ eff dom v. Replacing a,B,b and g in Theorem 1.6 with $q,-B,-b$ and v respectively, we find that $\partial v(q)$ is nonempty and $u \in \partial v(q)$ iff there exists $s \in \mathbb{R}^r$ such that

$$u \geq 0,\ s \geq 0,\ \ uA - sB \leq c,\ \ uq - sb = v(q),$$

i.e., $u \in \partial v(q)$ iff (u,s) is an optimal solution to the problem dual to (1.0.3) for some s. Also, by

Theorem 1.8, v is polyhedrally convex on eff dom v, which by Proposition 1.10 is polyhedrally convex and closed. We thus have a procedure for computing the subgradients of v.

The time has come to face the issue of free disposal of excess goods. In view of definition 1.4 and the hypothesis that the indirect demand function p maps int $\mathbb{R}_+^d$ into int $\mathbb{R}_+^d$, we must have some reasonable hope that $\partial v(q)$ contains at least one nonnegative vector for many (preferably all) q in eff dom v. In our formulation, Theorem 1.6 guarantees that

$$\partial v(q) \subset \mathbb{R}_+^d .$$

In the absence of free disposal, however, the output constraints of (1.0.3) become equalities. The corresponding change in Theorem 1.6 would be to write $Ax = z$ in (1.6.1) and drop the restriction $y \geq 0$ in (1.6.2). In this event, we no longer have restrictions on the signs of the marginal prices. Economic arguments can be made for the nonnegativity of the marginal costs, but close scrutiny shows that such arguments require free disposal of excesses if they are to be valid. We thus assume free disposal in order to ensure nonnegative prices.

In the next chapter, we exhibit conditions under which an equilibrium exists.

CHAPTER II

EXISTENCE OF EQUILIBRIUM

We turn to the task of establishing the existence of an equilibrium demand, i.e., a vector q satisfying

$$p(q) \in \partial v(q),$$

under conditions sufficiently broad to include the supply and demand models in PIES. We accomplish this in two stages. Recall that the log-linear indirect demand function p of the PIES model, though continuous on $\operatorname{int} \mathbb{R}^d_+$, is undefined on $\operatorname{bd} \mathbb{R}^d_+$. We begin with the case in which p is continuous on all of $\mathbb{R}^d_+$.

Our original statement of the first theorem used the conjugate function for v [11] to arrive at a formulation under which the Kakutani fixed-point theorem could be invoked. We present instead a version which combines a more general statement with a more direct proof.

<u>2.1 Lemma</u>. Let f be a convex function on $\mathbb{R}^d$; then the set of minimizers of f is a convex (possibly empty) subset of $\mathbb{R}^d$.

<u>Proof</u>: Let

$$m = \inf\{f(x) : x \in \mathbb{R}^d\}.$$

The set of minimizers of f is precisely the set

$$\{x \in \mathbb{R}^d : f(x) \leq m\}.$$

Due to the convexity of f this set is convex.

<u>2.2 Theorem</u>. Let $K \subset \mathbb{R}^d$ be compact, convex and nonempty, $f : K \to \mathbb{R}^d$ continuous and $g : K \to \mathbb{R}$ continuous and convex. Define $g(x) = +\infty$ for $x \notin K$. Then there exists a point $a \in K$ such that

$$f(a) \in \partial g(a).$$

<u>Proof</u>: For each $k \in K$, define $f_k : K \to \mathbb{R}$ by

$$f_k(x) = g(x) - f(k)x \quad \text{for all } x \in K.$$

Each f_k is continuous and convex on K. Since K is compact, the set $F(k)$ of minimizers of f_k over K is nonempty; that is, if

$$m_k = \min_{x \in K} f_k(x)$$

then

$$F(k) = f_k^{-1}(m_k) \neq \emptyset.$$

In view of Lemma 2.1, $F(k)$ is also convex. Since f_k is continuous, $F(k)$ must be closed. Thus for each $k \in K$, $F(k)$ is a nonempty, compact, convex subset of K. We now show that the point-to-set map F satisfies the hypotheses of Kakutani's fixed-point theorem [8]. We need only verify that F is a closed map, i.e., that if $k_i \in K$, $k_i \to k$, $s_i \in F(k_i)$ and $s_i \to s$ then $s \in F(k)$. Since each s_i is a minimizer of f_{k_i}, we have

$$f_{k_i}(x) \geq f_{k_i}(s_i) \quad \text{for all } x \in K,$$

i.e.,

$$g(x) - f(k_i)x \geq g(s_i) - f(k_i)s_i \quad \text{for all } x \in K.$$

As $i \to \infty$, we have by the continuity of f and g that

$$g(x) - f(k)x \geq g(s) - f(k)s \quad \text{for all } x \in K,$$

proving that s is a minimizer of f_k. In other words,

$$s \in F(k).$$

Hence by Kakutani's theorem, F has a fixed-point, so that there exists $a \in K$ such that

$$a \in F(a).$$

This says that

$$g(x) - f(a)x \geq g(a) - f(a)a \quad \text{for all } x \in K,$$

or equivalently

$$g(x) \geq g(a) + f(a)(x - a) \quad \text{for all } x \in K.$$

Since g is infinite outside K, this characterizes $f(a)$ as a subgradient of g at a.

We introduce more assumptions regarding the supply model, so that the hypotheses of the PIES models are met. We then apply Theorem 2.2 to a sequence of approximations to arrive at Theorem 2.6.

<u>2.3 Definition</u>. We define the set Q to be

$$Q = \text{eff dom } v \cap \mathbb{R}^d_+$$
$$= \{q \geq 0 : q \leq Ax \quad \text{for some} \quad x \in X\},$$

where X is given by

$$X = \{x \geq 0 : Bx \leq b\},$$

B and b as in (1.0.3). Q is the set of demand vectors of interest to us from the economic standpoint. We have shown as a consequence of Proposition 1.11 that eff dom v is closed, and so Q is closed. We assume that X is bounded, which is in accord with its economic interpretation. It follows that Q is also bounded, and so is compact. We further assume that Q is nonempty, and more specifically that the linear program (1.0.3) is feasible for some $q \gg 0$.

Under these assumptions, take K to be Q, g to be v, and f to be p in Theorem 2.2. The theorem then tells us that an equilibrium exists if p is continuous on Q. As noted earlier, this condition is not satisfied by the function p in the PIES model. In this case, p is undefined on part of bd Q, namely $Q \cap \text{bd } \mathbb{R}^d_+$.

The following lemmas will be used in the course of the proof of the next theorem.

<u>2.4 Lemma</u>. Let f be a convex function on $\mathbb{R}^d$, x a point at which f is finite; then

$$z \in \partial f(x) \quad \text{iff} \quad f'(x;y) \geq zy \quad \text{for all} \quad y \in \mathbb{R}^d,$$

where

$$f'(x;y) = \lim_{t\downarrow 0} t^{-1}\{f(x+ty) - f(x)\}.$$

Proof: See Rockafellar [11].

2.5 Lemma. Let v be as in (1.0.3) and Q as in Definition 2.3; then as q ranges over Q, $\partial v(q)$ assumes at most a finite number of subsets of $\mathbb{R}^d$.

Proof: As a consequence of Theorems 1.5 and 1.6, $\partial v(q)$ is the set of all $y \in \mathbb{R}^d_+$ for which there exists a vector $z \in \mathbb{R}^r_+$ such that (y,z) is optimal in the dual to (1.0.3); that is, $\partial v(q)$ is the projection into $\mathbb{R}^d$ of the set of optimal solutions to the dual. The set of optimal solutions to the dual is a face of the polyhedrally convex set

$$\{(y,z) : y \geq 0,\ z \geq 0,\ yA - zB \leq c\}.$$

Polyhedrally convex sets have finitely many faces, proving the lemma.

In the following theorem we use the subscripts i,j and k to denote the components of a vector and the subscripts m and n to denote the members of a sequence.

2.6 Theorem. Let v be as in (1.0.3), Q as in Definition 2.3, and p continuous with

$$p : \text{int } \mathbb{R}^d_+ \to \text{int } \mathbb{R}^d_+ .$$

Assume that Q is compact and that p satisfies the following conditions for $i = 1,\ldots,d$ and $q \in Q$:

$$\text{if } p_i(q) \to +\infty \text{ then } q_i \to 0, \tag{2.6.1}$$

where p_i and q_i are the ith components of p and q respectively; and

$$\text{if } q_i \to 0 \text{ then some } p_j(q) \to +\infty. \tag{2.6.2}$$

Under these assumptions there exists an equilibrium demand $q \in Q$, i.e., a vector q such that

$$p(q) \in \partial v(q).$$

Proof: Let $e \in \mathbb{R}^d$ be the vector $(1,1,\ldots,1)$. For $n = 1,2,\ldots$ define $p_n : \mathbb{R}^d_+ \to \operatorname{int} \mathbb{R}^d_+$ by

$$p_n(q) = p(q + n^{-1}e).$$

Since each p_n is continuous on $\mathbb{R}^d_+$ and hence on Q, we have by Theorem 2.2 that for each n there exists a $q_n \in Q$ such that

$$p(q_n + n^{-1}e) = p_n(q_n) \in \partial v(q_n).$$

Since Q is compact, the sequence (q_n) contains a convergent subsequence, which we denote by $(q_n)_{n\in N}$. Let q be the limit of this subsequence. We will show that $q \gg 0$, so that by the continuity of p at q,

$$p(q_n + n^{-1}e) \to p(q) \quad \text{as} \quad n \to \infty,\ n \in N.$$

Once this is shown, since

$$q_n \to q,\ p_n(q_n) \to p(q),\ p_n(q_n) \in \partial v(q_n),$$

and the map $z \mapsto \partial v(z)$ is closed, we see that

$$p(q) \in \partial v(q),$$

which is what we seek.

To show that $q \gg 0$, it suffices to show that the set

$$I = \{i = 1,\ldots,d : q_i = 0\}$$

is empty. Suppose it is not. For $i \in I$,

$$(q_n + n^{-1}e)_i = q_{n,i} + n^{-1} \to 0 \quad \text{as} \quad n \to \infty,\ n \in N.$$

In view of (2.6.2),

$$p_{n,j}(q_n) = p_j(q_n + n^{-1}e) \to +\infty \tag{2.6.3}$$

for some j (not necessarily i).

For each $k \in \{1,\ldots,d\} \setminus I$,

$$q_{n,k} \not\to 0$$

and so by (2.6.1)

$$p_{n,k}(q_n) = p_k(q_n + n^{-1}e) \not\to +\infty.$$

Thus for such k, some subsequence of $(p_k(q_n))$ is bounded. Hence we may extract a subsequence

$$(q_n)_{n\in L}$$

from the subsequence $(q_n)_{n\in N}$ such that for each $k \in \{1,\ldots,d\} \setminus I$ there exists a finite number x_k such that

$$p_{n,k}(q_n) \to x_k \quad \text{as} \quad n \to \infty,\ n \in L. \tag{2.6.4}$$

By Lemma 2.5,

$$\{\partial v(z) : z \in Q\}$$

is a finite collection of sets, and so we may pass to yet another subsequence,

$$(q_n)_{n \in M'}$$

such that

$$\partial v(q_n) = \partial v(q_m) \quad \text{for all} \quad m,n \in M.$$

We have postulated the existence of a vector $y \in Q$ such that $y \gg 0$; since

$$q_{n,i} \to 0 \quad \text{for} \quad i \in I,\ n \to \infty,\ n \in M,$$

surely we can find a number $m \in M$ such that

$$q_{n,i} < y_i \quad \text{for all} \quad i \in I \quad \text{and all} \quad n \in M,\ n \geq m. \tag{2.6.5}$$

Since v is convex,

$$v'(q_m; y - q_m) \leq v(y) - v(q_m) < \infty.$$

Since

$$p_n(q_n) \in \partial v(q_n) = \partial v(q_m) \quad \text{for all} \quad n \in M,\ n > m,$$

by Lemma 2.4

$$p_n(q_n)(y - q_m) \leq v'(q_m; y - q_m) < \infty \quad \text{for all} \quad n \in M,\ n > m.$$

Hence

$$\limsup_{\substack{n \to \infty \\ n \in M}} p_n(q_n)(y - q_m) \leq v'(q_m; y - q_m) < \infty. \tag{2.6.6}$$

Now

$$\begin{aligned} p_n(q_n)(y - q_m) &= \sum_{k \notin I} p_{n,k}(q_n)(y_k - q_{m,k}) \\ &\quad + \sum_{i \in I} p_{n,i}(q_n)(y_i - q_{m,i}), \end{aligned} \tag{2.6.7}$$

noting that a sum taken over an empty index set is zero. For $k \notin I$,

$$p_{n,k}(q_n)(y_k - q_{m,k}) \to x_k(y_k - q_{m,k}) \quad \text{as} \quad n \to \infty,\ n \in M,$$

and so the first sum in (2.6.7) converges to a finite limit as $n \to \infty$. For $i \in I$,

$$y_i - q_{m,i} > 0 \quad \text{by (2.6.5)}$$

and

$$p_{n,i}(q_n) > 0,$$

so the second sum in (2.6.7) consists exclusively of positive terms. By (2.6.3), there exists a j such that

$$p_{n,j}(q_n) \to +\infty \quad \text{as} \quad n \to \infty,\ n \in M;$$

in view of (2.6.1), that j must belong to I, and so

$$p_{n,j}(q_n)(y_j - q_{m,j}) \to +\infty \quad \text{as} \quad n \to \infty.$$

Thus the second sum in (2.6.7) diverges to $+\infty$ as $n \to \infty$, $n \in M$, contradicting (2.6.6). The theorem is proved.

Conditions (2.6.1) and (2.6.2) are not satisfied by every function p of the form (1.0.1), so they represent nontrivial restrictions for the PIES model in particular. Since q is allowed to move only in a bounded set, (2.6.2) will hold if, for instance, the matrix M in (1.0.2) is nonpositive with negative diagonal, a condition satisfied at least by the example published in Nissen and Knapp [9]. Condition (2.6.1) is an economic consequence if p

realistically represents the behavior of actual consumers, since violation of (2.6.1) requires an infinite supply of money in the economy.

The following example shows that (2.6.1), or some similar condition, must be satisfied.

2.7 Example. Let $d = 2$, $r = 1$, $s = 2$,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\quad B = (1\ 1),\ b = (1),\ c = (1,1) \text{ and}$$

$$p(q) = (q_1^{-1}q_2^{-1}, q_2^{-1}) \quad \text{for} \quad q \gg 0. \tag{2.7.1}$$

Here

$$Q = \{q \geq 0 : q_1 + q_2 \leq 1\}.$$

Observe that the inverse of $p(\cdot)$ is

$$q(p) = (p_1^{-1}p_2, p_2^{-1});$$

if we let $p_1 \to +\infty$ and maintain $p_2 = p_1^{-1}$, we see that $q_1 \to 1$, violating (2.6.1). In searching for an equilibrium, we need only consider $q \gg 0$, since p is undefined for $q_1 = 0$ or $q_2 = 0$. For $q \gg 0$, we obtain by direct calculation:

$$\partial v(q) = \begin{cases} \{(1,1)\} & \text{if } q_1 + q_2 < 1,\ q_1 > 0, \\ & \qquad\qquad\qquad\ \ q_2 > 0; \\ \{(t,t) : t \geq 1\} & \text{if } q_1 + q_2 = 1,\ q_1 > 0, \\ & \qquad\qquad\qquad\ \ q_2 > 0. \end{cases} \tag{2.7.2}$$

Now from $q_1 > 0$ and (2.7.1), we infer that

$$p_1(q) \neq p_2(q) \quad \text{for} \quad q \in Q,\ q \gg 0,$$

and so from (2.7.2) we see that

$$p(q) \notin \partial v(q) \quad \text{for all} \quad q \in Q,\ q \gg 0,$$

i.e., no equilibrium exists.

We close the chapter by noting that the use of equality demand constraints in the PIES version of (1.0.3), while presenting difficulties mentioned previously, foreshadows the following observation: at an equilibrium, the most economical production program must meet all demands exactly. We prove this below.

<u>2.8 Proposition</u>. Let v be given by (1.0.3), let

$$p : \text{int } \mathbb{R}^d_+ \to \text{int } \mathbb{R}^d_+$$

be continuous, and suppose that there exists a vector q such that

$$p(q) \in \partial v(q).$$

Let $\bar{x}$ be an optimal solution of (1.0.3) for that q; then

$$A\bar{x} = q.$$

<u>Proof</u>: Note that for $p(q)$ to be defined, we must have $q \gg 0$. Also, the optimal solution $\bar{x}$ must exist, since $\partial v(q) \neq \emptyset$ implies $v(q)$ is finite which in turn implies that the linear program is feasible and bounded. If the ith demand is satisfied with slack, i.e., if $(A\bar{x})_i > q_i$, by the complementary slackness principle of linear programming the corresponding component of any optimal solution to the dual of (1.0.3) must be zero. As was shown

in Chapter I, there exists a vector z such that $(p(q),z)$ is optimal in the dual to (1.0.3), and so

$$p_i(q) = 0 \quad \text{if} \quad (A\bar{x})_i > q_i.$$

Since the range of p lies by assumption in the interior of the positive orthant, we must have

$$A\bar{x} = q.$$

Note that this result is not surprising, as we have legislated it by requiring that both the argument and the value of $p(\cdot)$ be strictly positive.

CHAPTER III

UNIQUENESS OF EQUILIBRIA

In this chapter, we address the question of whether an equilibrium for our models, if one exists, must be unique. We require a monotonicity condition on p, a higher-dimensional generalization of the idea that, for one commodity, demand is a decreasing function of price. We make the following definition.

<u>3.1 Definition</u>. Let $X \subset \mathbb{R}^n$, $f : X \to \mathbb{R}^n$. We say f is <u>strictly monotonically decreasing</u> iff

$$(f(x) - f(y))(x - y) < 0 \quad \text{for all} \quad x,y \in X,\ x \neq y.$$

We now prove uniqueness of the equilibrium when p is strictly decreasing.

<u>3.2 Theorem</u>. If $x,y \in Q$, $p(x) \in \partial v(x)$, $p(y) \in \partial v(y)$ and p is strictly monotonically decreasing on Q, then $x = y$.

<u>Proof</u>: Since $p(x) \in \partial v(x)$,

$$v(y) \geq v(x) + p(x)(y - x). \tag{3.2.1}$$

Similarly, since $p(y) \in \partial v(y)$,

$$v(x) \geq v(y) + p(y)(x-y). \tag{3.2.2}$$

Adding (3.2.1) and (3.2.2), we have

$$0 \geq (p(x) - p(y))(y-x).$$

Since p is strictly monotonically decreasing,

$$0 < (p(x) - p(y))(y-x) \quad \text{if} \quad x \neq y,$$

and so x must equal y.

Sweeney [15], among others, has observed that under the assumption that p' is globally negative definite, the equilibrium, if one exists, is unique. Note that here and in the sequel we apply the terms negative definite and negative semidefinite to asymmetric as well as symmetric matrices. An arbitrary $n \times n$ matrix A is negative definite iff for all $y \in \mathbb{R}^n$,

$$y \neq 0 \quad \text{implies} \quad yAy < 0.$$

This is equivalent to defining A to be negative definite iff $A + A^t$ is negative definite, where A^t is the transpose of A. The following proposition shows that p' negative definite implies p is strictly decreasing, so that Theorem 3.2 applies. The proposition also includes a partial converse.

<u>3.3 Proposition</u>. Let X be a convex subset of $\mathbb{R}^n$ and let $f : X \to \mathbb{R}^n$ be of class C^1 on X. If $f'(x)$ is negative definite for all $x \in X$, then f is strictly

monotonically decreasing on X. Conversely, if f is monotonically decreasing and C^1 on an open superset G of X, then $f'(y)$ is negative semidefinite for all $y \in G$.

Proof: Suppose first that f' is negative definite on X and $x,y \in X$. Set $h = y - x$. We have

$$f(y) - f(x) = \int_0^1 f'(x+th)h \, dt$$

and so

$$h(f(y) - f(x)) = \int_0^1 hf'(x+th)h \, dt.$$

The real-valued function

$$g(t) = hf'(x+th)h$$

is continuous and negative on $[0,1]$, and so

$$(y-x)(f(y) - f(x)) = \int_0^1 g(t)dt < 0,$$

proving strict monotonicity of f.

Now suppose that f is strictly decreasing and C^1 on an open superset G of X, and suppose $y \in G$. We wish to show that for all $h \in \mathbb{R}^n$,

$$hf'(y)h \leq 0.$$

Suppose not, i.e., that there exists $h \in \mathbb{R}^n$ for which

$$hf'(y)h > 0.$$

Since G is open and f' is continuous at y, there exists $\epsilon > 0$ such that

$$hf'(y+th)h > 0 \quad \text{for all} \quad t \in [-\varepsilon,\varepsilon].$$

Again using continuity of f', we have

$$\int_0^{\epsilon} hf'(y+th)h\,dt > 0;$$

but

$$\begin{aligned}\int_0^{\epsilon} hf'(y+th)h\,dt &= h(f(y+\epsilon h) - f(y)) \\ &= \frac{1}{\epsilon}((y+\epsilon h) - y)(f(y+\epsilon h) - f(y)) \\ &< 0,\end{aligned}$$

a contradiction. This completes the proof.

Monotonicity of p is central to the proof of uniqueness of the equilibrium. In the PIES model, monotonicity is a consequence of the negative definiteness of p', as shown above. The authors of the PIES model made negative definiteness of p' a standing assumption [5]. We shall produce arguments to the effect that, for the log-linear form of p, this assumption is exceedingly restrictive <u>when enforced globally</u>, and will exhibit an example with multiple equilibria in which p' is not globally negative definite.

We begin with the case $d = 2$.

<u>3.4 Lemma</u>. For p given by (1.0.1) with $d = 2$, if $p'(q)$ is negative definite for all $q \gg 0$, either

$$m_{12} = m_{21} = 0 \qquad (3.4.1)$$

or

$$m_{12} = m_{22} + 1,\ m_{21} = m_{11} + 1 \quad \text{and}$$
$$m_{11} + m_{22} + 1 < 0. \tag{3.4.2}$$

<u>Proof</u>: We remark first that a necessary condition for a 2×2 matrix H to be negative definite is that

$$\frac{1}{4}\det(H + H^t) = \det\left(\frac{H + H^t}{2}\right) > 0. \tag{3.4.3}$$

We compute p':

$$p'(q) = \begin{pmatrix} k_1 m_{11} q_1^{m_{11}-1} q_2^{m_{12}} & k_1 m_{12} q_1^{m_{11}} q_2^{m_{12}-1} \\ k_2 m_{21} q_1^{m_{21}-1} q_2^{m_{22}} & k_2 m_{22} q_1^{m_{21}} q_2^{m_{22}-1} \end{pmatrix}.$$

Let $q = (s,1)$, $s > 0$; then (3.4.3), with $H = p'$, becomes

$$k_1k_2(m_{11}m_{22} - \tfrac{1}{2} m_{12}m_{21}) - \tfrac{1}{4} k_1^2 m_{12}^2 s^{m_{11}-m_{21}+1}$$
$$- \tfrac{1}{4} k_2^2 m_{21}^2 s^{m_{21}-m_{11}-1} > 0 \tag{3.4.4}$$

after division by $s^{m_{11}+m_{21}-1}$. For this inequality to hold both as $s \to +\infty$ and as $s \to 0$, we require that either (3.4.1) or

$$m_{11} - m_{21} + 1 = 0 \tag{3.4.5}$$

hold. A similar analysis with $q = (1,s)$ shows that either (3.4.1) or

$$m_{22} - m_{12} + 1 = 0 \tag{3.4.6}$$

must hold. Assuming that (3.4.1) fails to hold, we can reduce (3.4.4) to

$$0 < k_1 k_2 m_{11} m_{22} - \frac{1}{4}\,[k_1(m_{22}+1) + k_2(m_{11}+1)]^2$$
$$= -k_1 k_2 (m_{11} + m_{22} + 1) - \frac{1}{4}\,[k_1(m_{22}+1) - k_2(m_{11}+1)]^2,$$

and so $p'(q)$ is negative definite only if

$$m_{11} + m_{22} + 1 < 0,$$

completing the proof.

For higher dimensions, global negative definiteness is even more restrictive.

<u>3.5 Proposition</u>. For p as in (1.0.1), $d > 2$, if $p'(q)$ is negative definite for all $q \gg 0$ then for all $i \neq j$, either

$$m_{ij} = m_{ji} = 0 \tag{3.5.1}$$

or

$$\begin{aligned} &m_{ij} = m_{jj} + 1, \quad m_{ji} = m_{ii} + 1, \quad \text{and} \\ &m_{ii} + m_{jj} + 1 < 0. \end{aligned} \tag{3.5.2}$$

In addition, if (3.5.1) fails, then for all k such that $i \neq k \neq j$,

$$m_{ik} = m_{jk}. \tag{3.5.3}$$

<u>Proof</u>: Assume that $p'(q)$ is negative definite for all $q \gg 0$. This in particular requires that the 2×2 submatrix

$$\begin{pmatrix} d_{ii} & d_{ij} \\ d_{ji} & d_{jj} \end{pmatrix} \tag{3.5.4}$$

of

$$p' = (d_{k\ell}),$$

formed by the intersection of rows i and j with columns i and j in p' $(i \neq j)$, must be negative definite, and so (3.5.1) and (3.5.2) follow from Lemma 3.4, taking $q_k = 1$ for $i \neq k \neq j$. Now suppose (3.5.1) fails to hold for some $i \neq j$, and select h such that $i \neq h \neq j$. Let $q_k = 1$ for all $k \neq h$, $q_h = s > 0$, and examine the submatrix (3.5.4), which reduces to

$$\begin{pmatrix} k_i m_{ii} s^{m_{ih}} & k_i m_{ij} s^{m_{ih}} \\ k_j m_{ji} s^{m_{jh}} & k_j m_{jj} s^{m_{jh}} \end{pmatrix} .$$

Condition (3.4.3) for this matrix becomes

$$k_i k_j m_{ii} m_{jj} s^{m_{ih}+m_{jh}} - \frac{1}{4} [k_i m_{ij} s^{m_{ih}} + k_j m_{ji} s^{m_{jh}}]^2 > 0,$$

or equivalently

$$4k_i k_j m_{ii} m_{jj} > k_i^2 m_{ij}^2 s^{m_{ih}-m_{jh}} + 2k_i k_j m_{ij} m_{ji} + k_j^2 m_{ji}^2 s^{m_{jh}-m_{ih}}.$$

Since one of m_{ij}, m_{ji} is nonzero, this cannot hold for all $s \in (0,+\infty)$ unless (3.5.3) holds.

In economic terms, m_{ij} is the elasticity of price i with respect to demand j, a measure of consumer reaction to prices. Proposition 3.5 places exceptionally tight restrictions on what these elasticities could be. In practice, the demand model is constructed to fit actual consumer behavior, and it is quite unlikely that the conclusions of Proposition 3.5 would be satisfied, assuming the specified commodities bear some relation to each other. We are thus led to abandon the assumption that $p'(q)$ is negative definite for all $q \gg 0$. We might ask that $p'(q)$ be negative definite at least for $q \in Q$, but if the proofs of Lemma 3.4 and Proposition 3.5 are revised to allow s to go to 0 (as it certainly may in practice as q moves in Q) but not to go to $+\infty$, we arrive at more complicated but still highly restrictive necessary conditions. Hogan [5] describes negative definiteness of p' as "a weak economic assumption and an observed property of the PIES demand functions over the relevant regions," but does not explicitly describe the "relevant regions."

Proposition 3.3 indicates that we will be hard pressed to relax the assumption of negative definiteness significantly while retaining monotonicity. Our next example shows that without monotonicity, we may well have multiple equilibria.

<u>3.6 Example</u>. Let $d = 2$, $r = 1$, $s = 2$,

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad B = (1\ 1), \quad b = (2), \quad c = (1,1) \text{ and}$$

$$p(q) = (32q_1^{-1}q_2^{-2}, 32q_1^{-2}q_2^{-1}) \quad \text{for} \quad q \gg 0.$$

We assert that $(4,2)$, $(2,4)$ and $(3,3)$ are equilibrium values for q. Observe that

$$p(4,2) = (2,1),$$

$$p(2,4) = (1,2),$$

and

$$p(3,3) = (32/27, 32/27).$$

The dual to problem (1.0.3) is

$$\begin{aligned} 2z_1 + z_2 - w &\leq 1 \\ z_1 + 2z_2 - w &\leq 1 \\ z_1, z_2, w &\geq 0 \\ q_1 z_1 + q_2 z_2 - 2w & \ (\max) \end{aligned} \qquad (3.6.1)$$

The triples (2,1,4), (1,2,4) and (32/27,32/27,23/9) are optimal in (3.6.1) when q is (4,2), (2,4) and (3,3) respectively, which, in light of Theorems 1.5 and 1.6, proves the assertion.

CHAPTER IV

THE PIES ALGORITHM

In the remainder of this paper we will assume the existence of at least one equilibrium for our models, and pursue methods for locating an equilibrium. The previous chapter indicated that multiple equilibria are a possibility when p is not strictly monotonic. From the standpoint of utilization of such models as tools in policy making, we should be able to find all equilibria, but as yet we cannot. Hogan and others sought equilibria under the assumption that $p'(q)$ is negative definite for each q. We make the comparable assumption that p is strictly decreasing, ensuring that the model has a unique equilibrium.

With an eye toward simplifying future calculations, and with Proposition 2.8 in hand, we revise (1.0.3) using equality demand constraints, obtaining

$$\begin{aligned} Ax &= q \\ Bx &\leq b \\ x &\geq 0 \\ cx &\ (\min). \end{aligned} \tag{4.0.1}$$

We define the set Q^*, corresponding to Q in Definition 2.3, to be

$$Q^* = A(X) \cap \mathbb{R}^d_+ , \qquad (4.0.2)$$

where X is as in Definition 2.3. Moreover, we will assume, unless otherwise stated, that p is strictly monotonically decreasing, so that the equilibrium is unique.

Lastly, we must face the fact that $p(\cdot)$ is not defined on all of Q^*. We have two options. We could work on a compact subset of Q^* bounded away from the coordinate hyperplanes, or we could assume $p(\cdot)$ is defined on all of Q^*. The former alternative is plausible, since equilibria are strictly positive vectors, but we have no a priori knowledge of where to truncate Q^*, and in addition, we would be adding constraints to (4.0.1). We therefore choose the latter alternative.

The PIES algorithm is motivated by the observation that if p has a potential f, i.e., if there exists a function f on Q such that $p = \nabla f$, and if all the other assumptions above are met, then q is an equilibrium demand if and only if q minimizes $v - f$.

<u>4.1 Lemma</u>. If f and g are proper convex functions and there exists $a \in \text{rel int (eff dom } f) \cap \text{rel int (eff dom } g)$, then $f + g$ is convex and

$$\partial(f+g)(x) = \partial f(x) + \partial g(x)$$

for all $x \in$ eff dom $f \cap$ eff dom g.

Proof: See Rockafellar [11].

We remark that for $S, T \subset \mathbb{R}^d$, the expression $S + T$ denotes the vector sum

$$\{s+t : s \in S,\ t \in T\}.$$

4.2 Proposition. If p is strictly monotonically decreasing on Q^* and $f : \mathbb{R}^d_+ \to \mathbb{R}$ is a potential for p, i.e., $\nabla f(x) = p(x)$ for all $x \in \text{int } \mathbb{R}^d_+$, then q is an equilibrium iff q minimizes $v - f$.

Proof: Assume that the hypotheses hold, and let $F = v - f$. Since p is strictly monotonically decreasing, $-f$ is convex, and so F is convex on Q^*. Since Q^* is a convex set and $F = +\infty$ on $\mathbb{R}^d_+ \setminus Q^*$, F is convex on $\mathbb{R}^d_+$. From Definition 1.4, q is an equilibrium iff

$$p(q) \in \partial v(q). \tag{4.2.1}$$

Since f is finite on $\mathbb{R}^d_+$, F is proper, and so q minimizes F iff

$$0 \in \partial F(q). \tag{4.2.2}$$

We need only show the equivalence of (4.2.1) and (4.2.2). We have assumed that there exists a vector $y \in Q$ such that $y \gg 0$; without loss of generality $y \in Q^*$ and so rel int (eff dom v) $\cap$ rel int (eff dom $-f$) $\neq \emptyset$. Using Lemma 4.1,

$$\partial F(q) = \partial v(q) + \partial(-f)(q) = \partial v(q) - p(q)$$

for all $q \in Q^*$. Thus $0 \in \partial F(q)$ iff $p(q) \in \partial v(q)$, proving the proposition.

Necessary and sufficient conditions for the potential f of a C^1 function p on $\mathbb{R}^d_+$ to exist are that

$$\partial p_i/\partial q_j = \partial p_j/\partial q_i \quad \text{for all } i,j = 1,\ldots,d.$$

Hogan [5] notes that these conditions are not met by the PIES indirect demand function p, and so p is not integrable independently of path, which implies that p has no potential. We will nonetheless be guided at times by Proposition 4.2.

The computationally easiest case to handle is when each component function p_i is a function only of the corresponding variable q_i, i.e.,

$$p(q) = (g_1(q_1),\ldots,g_d(q_d)).$$

This occurs precisely when $p'(q)$ is a diagonal matrix for all q (with negative diagonal entries, since p is strictly decreasing). Proposition 4.2 applies here, and in fact the potential f is given by

$$f(q) = k + \sum_{i=1}^{d} \int_{a_i}^{q_i} g_i(t)dt$$

for an arbitrary constant k and arbitrary $a \in \operatorname{int} \mathbb{R}^d_+$. Wagner [17] describes a method in which the problem of minimizing F, as in Proposition 4.2, can be approximately solved by solving a linear program which is an expansion of (4.0.1). We will elucidate this now.

Let $I_1, \ldots, I_d$ be closed intervals in $[0,+\infty)$ such that the rectangle $I = I_1 \times \ldots \times I_d$ contains both a and the unique equilibrium point q. Partition each interval I_j into a finite number of subintervals with partition points

$$w_{j,-n} < \ldots < w_{j,0} < \ldots < w_{j,n}$$

such that $w_{j,0} = a_j$. There is no need to use the same number of points in each interval or even to use equal number of points on either side of $w_{j,0}$, but it simplifies the notation to do so. Let

$$g_{i,j} = \begin{Bmatrix} g_i(w_{i,j}) & j = 0,\ldots,n \\ \\ -g_i(w_{i,j}) & j = -n,\ldots,-1 \end{Bmatrix} \quad , \quad i = 1,\ldots,d$$

and

$$\Delta_{i,j} = w_{i,j+1} - w_{i,j}, \; j = -n,\ldots,n-1, \; i = 1,\ldots,d.$$

The following result appears in Wagner's paper.

<u>4.3 Lemma</u>. The optimal solution $(\bar{x},\bar{y})$ to the linear program

$$\begin{aligned} & Bx \leq b \\ & (Ax)_i = a_i + \sum_{j=0}^{n-1} y_{i,j} - \sum_{j=-n}^{-1} y_{i,j}, \quad i = 1,\ldots,d \\ & x \geq 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.3.1) \\ & 0 \leq y_{i,j} \leq \Delta_{i,j}, \; i = 1,\ldots,d, \; j = -n,\ldots,n-1 \\ & cx - \sum_{i=1}^{d} \sum_{j=-n}^{n-1} g_{i,j} y_{i,j} \; (\min) \end{aligned}$$

exists and satisfies the following properties for $i = 1,\ldots,d$:

if $y_{i,m} > 0$ for some $m > 0$, then $y_{i,j} = 0$ for all $j < 0$ and $y_{i,j} = \Delta_{i,j}$ for $j = 0,\ldots,m-1$; (4.3.2)

if $y_{i,m} > 0$ for some $m < 0$, then $y_{i,j} = 0$ for all $j \geq 0$ and $y_{i,j} = \Delta_{i,j}$ for $j = m+1,\ldots,-1$. (4.3.3)

In addition, the component π_i corresponding to demand constraint i in any optimal solution to the dual program satisfies one of the following:

$$0 < y_{i,j} < \Delta_{i,j},\ j \geq 0,\ \pi_i = g_{i,j}; \tag{4.3.4}$$

$$0 < y_{i,j} < \Delta_{i,j},\ j < 0,\ \pi_i = -g_{i,j}; \tag{4.3.5}$$

$$y_{i,j} = \Delta_{i,j},\ y_{i,j+1} = 0,\ j \geq 0,\quad g_{i,j} \geq \pi_i \geq g_{i,j+1}; \tag{4.3.6}$$

$$y_{i,j} = \Delta_{i,j},\ y_{i,j-1} = 0,\ y < 0,\quad -g_{i,j-1} \geq \pi_i \geq -g_{i,j}. \tag{4.3.7}$$

<u>Remarks</u>. The key to the proof is that each component function g_i is monotonically decreasing and positive. The partition essentially estimates the integral of each g_i (i.e., the area under the graph of g_i) by a sum of rectangles, taking into account the sign reversal when integrating from right to left. Letting

$$z_i = a_i + \sum_{j=0}^{n-1} \bar{y}_{i,j} - \sum_{j=-n}^{-1} \bar{y}_{i,j},$$

the sum

$$\sum_{j=-n}^{n-1} g_{i,j}\bar{y}_{i,j}$$

is just a Riemann-type sum for

$$\int_{a_i}^{z_i} g_i(t)dt.$$

(4.3.2) and (4.3.3) state that the approximation by rectangles is consistent with the integral, i.e., that rectangles do not occur on both sides of a_i and that no gaps appear between rectangles. The proof consists of showing that when this is not the case, a redistribution of weight among the variables $y_{i,j}$ can be made to reduce the value of the objective function while preserving the value of $\sum_{j=-n}^{n-1} y_{i,j}$. (4.3.4)-(4.3.7) are proved by characterizing an optimal dual solution as a subgradient of the value of (4.3.1) and then observing the effect on the objective function of a perturbation in the most extreme nonzero $y_{i,j}$ for each i.

<u>4.4 Notation</u>. To connect problem (4.3.1) to the problem

$$\begin{aligned} Ax &= z \\ Bx &\leq b \\ x &\geq 0 \\ v(z) - f(z) = cx - f(a) &- \sum_{i=1}^{d} \int_{a_i}^{z_i} g_i(t)dt \text{ (min)}, \end{aligned} \tag{4.4.1}$$

we establish a correspondence between vectors (x,y) feasible in (4.3.1) and vectors (x,z) feasible in (4.4.1) with $z \in I$, via the following:

$$z_i = a_i + \sum_{j=0}^{n-1} y_{i,j} - \sum_{j=-n}^{-1} y_{i,j}. \tag{4.4.2}$$

Each y clearly produces a z feasible for (4.4.1); each z produces a y feasible for (4.3.1) if we also require that y satisfy (4.3.2) and (4.3.3) as well as the constraints $0 \leq y_{i,j} \leq \Delta_{i,j}$ for all i,j.

<u>4.5 Lemma</u>. Let (x,y) be feasible in (4.3.1) and let z be given by (4.4.2); then (x,z) is feasible in (4.4.1) and the values of the objective functions of (4.3.1) and (4.4.1), evaluated at those x,y and z, differ by no more than

$$\sum_{i=1}^{d} [\max_j \Delta_{i,j}][g_i(w_{i,-n}) - g_i(w_{i,n})]. \tag{4.5.1}$$

<u>Proof</u>: Feasibility of (x,z) in (4.4.1) is clear. Since each g_i is strictly decreasing,

$$g_i(w_{i,j})y_{i,j} \geq \int_{w_{i,j}}^{w_{i,j}+y_{i,j}} g_i(t)dt \geq g_i(w_{i,j+1})y_{i,j}$$

for $i = 1,\ldots,d$ and $j \geq 0$, and so

$$\begin{aligned} 0 &\leq \int_{w_{i,j}}^{w_{i,j}+y_{i,j}} g_i(t)dt - g_i(w_{i,j+1})y_{i,j} \\ &\leq [g_i(w_{i,j}) - g_i(w_{i,j+1})]y_{i,j}. \end{aligned}$$

A similar estimate holds for $j < 0$. If we add these inequalities as j runs from $-n$ to $n-1$ and over-estimate the right side by replacing $y_{i,j}$ with $\max_k \Delta_{i,k}$, the right side telescopes, and after summing over i we arrive at (4.5.1).

We remark in (4.5.1) that the value of g_j at the endpoints of I_j are independent of the mesh size, and so once the rectangle I is established, the accuracy of the value of (4.3.1) in estimating the value of (4.4.1) depends on the mesh sizes $\max_j \Delta_{i,j}$ $(i = 1,\ldots,d)$ with (4.5.1) providing an explicit estimate.

A more critical issue is the accuracy of the optimal $\bar{y}$ of (4.3.1) in estimating the optimal z of (4.4.1). The following proposition shows that arbitrary accuracy between $\bar{y}$ and z can be obtained by taking a sufficiently fine partition, but the crucial estimate relies on a number which we cannot calculate in practice.

<u>4.6 Proposition</u>. Let $(\bar{x},\bar{y})$ be optimal in (4.3.1), let $\bar{z}$ correspond to $\bar{y}$ as in (4.4.2), and let $(\tilde{x},\tilde{z})$ be optimal in (4.4.1). Given $\epsilon > 0$, there exists a $\delta > 0$ such that $|\tilde{z} - \bar{z}| < \epsilon$ if $\max_{i,j} \Delta_{i,j} < \delta$.

<u>Proof</u>: We denote by (M) the problem obtained from (4.4.1) by adding the constraint

$$|z - \tilde{z}| \geq \epsilon,$$

where ϵ is assumed to be small enough such that (M) has a feasible solution. Let the minimal values of (4.4.1) and (M) be C and D respectively. Let $\sigma = D - C$; $\sigma > 0$ since all vectors feasible in (M) are feasible in (4.4.1) and any optimal solution (x,z) of (4.4.1) must have $z = \tilde{z}$ and hence cannot be feasible in (M).

According to Lemma 4.5, there is a $\delta > 0$ such that if $\max_{i,j} \Delta_{i,j} < \delta$, then for every (x,y) feasible in (4.3.1), the value produced by (x,y) in the objective function of (4.3.1) and the value produced by the corresponding (x,z) in the objective function of (4.4.1) differ by at most $\sigma/3$. It follows that the optimal value E of

(4.3.1) differs from the optimal value C of (4.4.1) by at most $\sigma/3$, and in particular

$$C - E \geq -\sigma/3. \tag{4.6.1}$$

Now suppose we choose the mesh size to be less than δ and still find that $|\bar{z} - \tilde{z}| \geq \epsilon$; then $(\bar{x},\bar{z})$ is feasible in (M). Problems (M) and (4.4.1) have the same objective function. Let F be the value of that function at $(\bar{x},\bar{z})$. Since D is the optimal value of (M),

$$F \geq D = C + \sigma. \tag{4.6.2}$$

From our choice of δ and the observation that $(\bar{x},\bar{y})$ produces the value E in (4.3.1),

$$E - F \geq -\sigma/3. \tag{4.6.3}$$

Adding (4.6.1)-(4.6.3),

$$C \geq C + \sigma/3,$$

which contradicts $\sigma > 0$.

Note that the choice of δ depends, through σ, on D, which is unknown in practice. Proposition 4.6 guarantees however, that if we were to repeatedly solve (4.3.1) with mesh sizes tending to 0, $\bar{z}$ would converge to $\tilde{z}$.

Before moving to the more general case, we note that if the partitioned rectangle I fails to contain the optimal solution of (4.4.1), the vector $\bar{z}$ corresponding to the $\bar{y}$ optimal in (4.3.1) will lie on bd I. In this eventuality, we may partition a new rectangle about $\bar{z}$ and repeat the process.

For the more general case, in which p' is not even symmetric, let alone diagonal, the PIES algorithm attempts to exploit problem (4.3.1) by approximating $p(\cdot)$ by a function with diagonal derivative matrix. We now present the algorithm.

4.7 PIES Algorithm. Start with a vector $p_0 \in \mathbb{R}^d_+$, $k = 0$, $\theta \in [0,1]$.

Step 1: Calculate $q = p^{-1}(p_k)$.

Step 2: Define an approximation $g(\cdot;q)$ to $p(\cdot)$ by the following rule:

$$g_i(w;q) = p_i(q_1,\ldots,q_{i-1},w_i,q_{i+1},\ldots,q_d) \quad \text{for } i = 1,\ldots,d.$$

Step 3: Solve approximately problem (4.4.1), taking $g_i(\cdot)$ to be $g_i(\cdot;q)$, by solving (4.3.1), obtaining a solution (x,z), and set $\pi_k = g(z;q)$.

Step 4: If $\pi_k = p_k$ (to within predetermined tolerances), stop with (approximate) equilibrium demand q. Otherwise, set

$$p_{k+1} = \theta p_k + (1-\theta)\pi_k,$$

increment k by 1, and repeat from Step 1.

Both Hogan [5] and Wagner [17] report that the algorithm typically converges within ten, and often within six, iterations. Wagner states that termination occurs when the maximum deviation (presumably in p) is within 1%.

He further states, and Hogan concurs, that taking $\theta = 1/2$ rather than 0 accelerates convergence. Despite these findings, the algorithm has to date evaded complete analysis, due in part to the fact that the function defined in Step 2 does not approximate $p(\cdot)$ in a fashion that we find tractable. The results we have on this algorithm are fragmentary. We first introduce another concept from convex analysis.

<u>4.8 Definition</u>. The conjugate v^* of a convex function $v : \mathbb{R}^d \to [-\infty, +\infty]$ is given by

$$v^*(x) = \sup\{xy - v(y) : y \in \mathbb{R}^d\} \quad \text{for all} \quad x \in \mathbb{R}^d.$$

We note that since $v(y) = +\infty$ for $y \notin \text{eff dom } v$,

$$v^*(x) = \sup\{xy - v(y) : y \in \text{eff dom } v\}.$$

When v is the value of (1.0.3) or (4.0.1), we have the following result.

<u>4.9 Lemma</u>. v^* is a proper convex function and $v = (v^*)^*$.

<u>Proof</u>: Since v is both lower semicontinuous and proper, the lemma follows from a result in Rockafellar [11].

<u>4.10 Lemma</u>. $y \in \partial v(x)$ iff $x \in \partial v^*(y)$.

<u>Proof</u>: See Rockafellar [11].

We note that $\partial v^*(p)$ is precisely the set of demands q for which p is a possible supply price.

In all subsequent analysis we will assume that in Step 3 of the PIES algorithm, problem (4.4.1) is solved exactly. We denote its optimal solution z by $z = Z(q)$ to show the dependence on q.

<u>4.11 Lemma</u>. $Z : Q^* \to Q^*$ is continuous.

<u>Proof</u>: We note first that $g_i(\cdot;\cdot)$ is jointly continuous for each i. Let $g = (g_1,\dots,g_d)$. Since $g'(\cdot;q)$ is negative definite, there is a unique z optimal in (4.4.1), and so

$$z = Z(q) \quad \text{iff} \quad g(z;q) \in \partial v(z).$$

Let $q \in Q^*$ and suppose $(q_n) \subset Q^*$ such that $q_n \to q$. Let $y_n = Z(q_n) \in Q^*$. Since Q^* is compact, (y_n) clusters at some $y \in Q^*$. Passing to a subsequence, we may assume $y_n \to y$. By the closedness of the point-to-set map $\partial v(\cdot)$, we have $g(y;q) \in \partial v(y)$ and so $y = Z(q)$. This holds for any cluster point of the original sequence (y_n), so by compactness of Q^*, $Z(q_n) \to Z(q)$, i.e., Z is continuous.

<u>4.12 Proposition</u>. Let $\bar{q}$ be an equilibrium demand, $\bar{p} = p(\bar{q})$. If v^* is differentiable at $\bar{p}$, then the PIES algorithm converges to equilibrium in one step when started in a suitable neighborhood of $\bar{p}$.

<u>Proof</u>: We begin by noting that since $\partial v(q)$ is closed for all q and only finitely many sets $\partial v(q)$ exist [Lemma 2.5], there exists a neighborhood M of

$\bar{p}$ such that for all $q \in Q^*$,

$$\bar{p} \notin \partial v(q) \text{ implies } \partial v(q) \cap M = \emptyset. \qquad (4.12.1)$$

Since $g(\cdot;\cdot)$ is jointly continuous and $g(\bar{q};\bar{q}) = p(\bar{q}) = \bar{p} \in M$, there exists a neighborhood N of $\bar{q}$ such that $y,z \in N$ implies $g(y;z) \in M$. Since Z is continuous and $Z(\bar{q}) = \bar{q}$ [because $g(\bar{q};\bar{q}) = \bar{p} \in \partial v(\bar{q})$], we can find a neighborhood V of $\bar{q}$ such that $V \subset N$ and $Z(V \cap Q^*) \subset N$. Suppose that we start the algorithm with $p_0 \in p(V)$, noting that $p(V)$ is a neighborhood of $\bar{p}$. Let $q_0 = p^{-1}(p_0)$, $y = Z(q_0)$. $q_0 \in V$, so q_0 and y belong to N and thus $g(y;q_0) \in M$. Since $g(y;q_0) \in \partial v(y)$, $\partial v(y) \cap M \neq \emptyset$, and so by (4.12.1) $\bar{p} \in \partial v(y)$, which implies by Lemma 4.10 that $y \in \partial v^*(\bar{p})$. Since v^* is differentiable at $\bar{p}$ and $\bar{q} \in \partial v^*(\bar{p})$, $\partial v^*(\bar{p}) = \{\bar{q}\}$. Thus $y = \bar{q}$.

We remark without proof that $\partial v^*(\bar{p}) = \{\bar{q}\}$ when $\bar{q}$ is an extreme point of Q and $\bar{p} \in \text{int}\, \partial v(\bar{q})$.

<u>4.13 Lemma</u>. For $x,y \in \partial v^*(z)$,

$$v(x) - v(y) = z(x-y).$$

<u>Proof</u>: Using Lemma 4.10, x and y in $\partial v^*(z)$ implies that $z \in \partial v(x)$ and $z \in \partial v(y)$, and so

$$v(y) \geq v(x) + z(y-x)$$

and

$$v(x) \geq v(y) + z(x-y).$$

Combining these inequalities proves the lemma.

<u>4.14 Lemma</u>. $\partial v(x) = \{y\}$ for all $x \in \operatorname{int} \partial v^*(y)$.

<u>Proof</u>: If $\operatorname{int} \partial v^*(y) = \emptyset$, there is nothing to prove. Suppose $x \in \operatorname{int} \partial v^*(y)$. Let $u \in \mathbb{R}^d$ be arbitrary. There exists $t > 0$ such that $x + tu \in \partial v^*(y)$, and so by Lemma 4.13

$$v(x+tu) - v(x) = tyu. \qquad (4.14.1)$$

On the other hand, for any $w \in \partial v(x)$

$$v(x+tu) - v(x) \geq twu. \qquad (4.14.2)$$

Combining (4.14.1) and (4.14.2),

$$(y-w)u \geq 0 \quad \text{for all} \quad u \in \mathbb{R}^d,$$

and so $y = w$. Since $w \in \partial v(x)$ was arbitrary, $\partial v(x) = \{y\}$.

<u>4.15 Proposition</u>. If $\bar{q}$ is an equilibrium, $\bar{p} = p(\bar{q})$ and $\bar{q} \in \operatorname{int} \partial v^*(\bar{p})$, then the PIES algorithm with $\theta < 1$ converges to $\bar{q}$ when started in a suitable neighborhood of $\bar{p}$.

<u>Proof</u>: Let $N = Z^{-1}(\operatorname{int} \partial v^*(\bar{p}))$, a neighborhood of $\bar{q}$ since $\bar{q} \in \operatorname{int} \partial v^*(\bar{p})$, $Z(\bar{q}) = \bar{q}$ and Z is continuous by Lemma 4.11. Since p and p^{-1} are both continuous on $\operatorname{int} \mathbb{R}^d_+$, they are homeomorphisms there, and so we can find a ball M about $\bar{p}$ such that $p^{-1}(M) \subset N$. Suppose that at some iteration k of the PIES algorithm, $p_k \in M$; then $q = p^{-1}(p_k) \in N$ and so $y = Z(q) \in \operatorname{int} \partial v^*(\bar{p})$. By Lemma 4.14, $\partial v(y) = \{\bar{p}\}$.

Since

$$\pi_k = g(y;q) \in \partial v(y) = \{\bar{p}\},$$

we must have $\pi_k = \bar{p}$. Thus

$$|p_{k+1} - \bar{p}| = |\theta p_k + (1-\theta)\pi_k - \bar{p}|$$
$$= \theta |p_k - \bar{p}|.$$

Either $p_k = \bar{p}$ (in which case the algorithm terminates) or $|p_{k+1} - \bar{p}| < |p_k - \bar{p}|$. In the latter case, $p_{k+1} \in M$ and the argument can be repeated, so that by induction

$$|p_{k+m} - \bar{p}| = \theta^m |p_k - \bar{p}| \to 0$$

as $m \to \infty$.

We now present an example in which the indirect demand function, while always possessing a negative definite derivative matrix, contains a parameter we are free to set. For large values of the parameter, the PIES algorithm has failed to converge in over 100 iterations, and is believed by us not to converge at all. We shall give a heuristic argument for this and describe the observed behavior.

<u>4.16 Example</u>. Let $d = 2$, $r = 1$, $s = 2$, $T > 1$,

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad B = (1\ 1), \quad b = (2), \quad c = (1,1), \quad \text{with}$$

p defined by (1.0.2) with

$$M = \begin{pmatrix} -T-1 & -T \\ -T & -T-1 \end{pmatrix}, \qquad K = \begin{pmatrix} 3(T+1)\log 2 \\ (3T+1)\log 2 \end{pmatrix}.$$

We note that the example is constructed to have the unique equilibrium

$$\bar{q} = (4,2), \quad \bar{p} = (2,1),$$

and so that the hypothesis $\partial v^*(\bar{p}) = \{\bar{q}\}$ of Proposition 4.12 holds.

Our heuristic argument against convergence of the PIES algorithm on this example when T is large proceeds as follows. Suppose that the sequence (p_k) generated by the algorithm converges to $\bar{p}$, and so $q_k = p^{-1}(p_k) \to \bar{q}$. Recall from the proof of Proposition 4.12 that when $\partial v^*(\bar{p}) = \{\bar{q}\}$ and q is sufficiently near $\bar{q}$, $Z(q) = \bar{q}$. Thus for k large, $Z(q_k) = \bar{q}$ and so

$$\pi_k = g(Z(q_k);q_k) = g(\bar{q};q_k).$$

Let

$$M_d = \begin{pmatrix} -T-1 & 0 \\ 0 & -T-1 \end{pmatrix}, \quad M_o = \begin{pmatrix} 0 & -T \\ -T & 0 \end{pmatrix};$$

then

$$\log \pi_k = K + M_d \log \bar{q} + M_o \log q_k$$

and

$$\log \bar{p} = K + M_d \log \bar{q} + M_o \log \bar{q},$$

and so

$$\log \pi_k - \log \bar{p} = M_o(\log q_k - \log \bar{q}).$$

Now M is invertible, so from (1.0.2)

$$\log q = M^{-1} \log p - M^{-1}K$$

and hence

$$\log q_k - \log \bar{q} = M^{-1}(\log p_k - \log \bar{p}).$$

Therefore

$$\log \pi_k - \log \bar{p} = M_O M^{-1}(\log p_k - \log \bar{p}).$$

The eigenvalues of $M_O M^{-1}$ are $-T$ and $\frac{T}{2T+1}$. If we write $\log p_k - \log \bar{p}$ as a linear combination of the eigenvectors of $M_O M^{-1}$, we arrive at $\log \pi_k - \log \bar{p}$ by stretching one component by a factor of $-T$ and shrinking the other by a factor of $\frac{T}{2T+1}$, which approaches 1/2 as T grows large. If θ were 0 (so that $p_{k+1} = \pi_k$), this would prevent $\log p_k - \log \bar{p}$ from converging to 0 unless by some chance $\log p_k - \log \bar{p}$ were an eigenvector of $M_O M^{-1}$ with eigenvalue $\frac{T}{2T+1}$. The situation when $\theta > 0$ is less clear; we present this only to motivate the choice of the particular example.

A modification of the PIES algorithm was tried on the example, using a program written and executed on a Tektronix 4051 microcomputer, which holds numbers to fourteen decimal digit accuracy. The modification occurred in Step 3. Rather than using the linear program (4.3.1), we solved (4.4.1) directly, using the method of steepest descents. It was expected that, if anything, this would be more accurate than the use of the linear program approximation.

When run with $\theta = 0.5$ and T either 1 or 2, the sequence appeared to be converging to $\bar{q}$, even from fairly poor starts. When run with $\theta = 0.5$ and $T = 3$, however, convergence did not occur. Table 1 shows the distance (in the euclidean norm) between p_k and $\bar{p}$ at various points during a number of runs. In the first run, the program was started close to $\bar{p}$. After a few iterations, it began to exhibit oscillatory behavior, appearing to be approaching two distinct limit points. To test this hypothesis, the program was restarted at what appeared to be one of the two limit points. As expected, the algorithm oscillated between

$$p = (1.9999995,\ 1.00000025)$$

and

$$p = (2.0000005,\ 0.99999975)$$

with the norm of the error constant to within machine tolerances. To test the effect of roundoff errors, the program was next started exactly at the solution. Ideally, the sequence generated should be identically $\bar{p}$, but in practice we would anticipate errors due to the computation of logarithms and exponentials and the finite accuracy of the machine. After ten iterations, the error was less than 3×10^{-13}, which lends credence to the claim that the oscillatory behavior observed earlier is due to the algorithm and not to machine errors. Yet another attempt was made with $T = 3$, this time with a

TABLE 1

T	Iteration	$\lvert p-\bar{p}\rvert$
3	0	1.4142×10^{-6}
	10	5.2628×10^{-7}
	20	5.5781×10^{-7}
	30	5.5907×10^{-7}
3	0	5.5902×10^{-7}
	5	5.5902×10^{-7}
3	0	0
	10	2.6671×10^{-13}
3	0	2.0616
	2	6.8113×10^{10}
	10	2.6790×10^{8}
	20	2.6524×10^{5}
	30	3.6686×10^{2}
	50	5.9964×10^{-2}
	100	1.0114×10^{-7}
	120	1.0288×10^{-7}
	130	1.0288×10^{-7}
4	0	1.4142×10^{-6}
	10	3.2196×10^{-5}
	20	1.2403×10^{-3}
7	0	1.4142×10^{-6}
	10	3.2593×10^{-2}
	20	9.8588×10^{-1}

T	Iteration	$\lvert p - \bar{p} \rvert$
7	40	6.7905×10^{10}
	50	6.6315×10^{7}
	60	1.2958×10^{5}
10	0	0
	7	7.2116×10^{-9}

mediocre start. The results were somewhat startling. In two iterations the error went up ten orders of magnitude, suggesting divergence. It then fell off for the next one hundred iterations, suggesting convergence. Even while apparently converging, the sequence (p_k) appeared to oscillate, but after one hundred twenty iterations it appeared to be clustering at both

$$(1.99999990798,\ 1.00000004601)$$

and

$$(2.00000009202,\ 0.99999995399).$$

Attempts with $3 < T < 10$ indicated neither convergence nor a limit oscillation as in the case $T = 3$. Errors did rise and fall, which is to be expected, since the sequence $(p^{-1}(p_k))$, contained in the compact set Q^*, must cluster, and so (p_k) must also cluster.

We did observe that when T was large, the algorithm tended to work better with θ nearer to 1. In fact, an attempt with $T = 10$, $\theta = 0.999$ and initial error 1.4142×10^{-6} was still within 1.4254×10^{-6} of the correct solution after ten iterations. This would tend to confirm our heuristic reasoning, i.e., that π_k is further from $\bar{p}$ than p_k is.

The results of our tests indicate that even for fairly small values of T, the PIES algorithm fails to converge in a reasonable number of iterations (certainly not within six to ten iterations, the published figures). Moreover, when T is moderately large, the algorithm is numerically unstable. The last entry in Table 1 shows that in a benchmark run ($\theta = 0.5$, $T = 10$, initial error 0), round-off errors of 7×10^{-9} accumulated in just seven iterations. Values of T above 10 are in no way unnatural; the data published by Nissen and Knapp [9] include elasticities as large as 25 in absolute value.

While we cannot prove analytically that the PIES algorithm fails on Example 4.16, all computational evidence indicates that it does.

CHAPTER V

THE PIES-VAR ALGORITHM

Irwin [6] has proposed a variant of the PIES algorithm, which he has named PIES-VAR, that can be proved to converge to the unique equilibrium when several hypotheses are met. Among these hypotheses is the existence of a "selection" of subgradients, a function $f : Q^* \to \mathbb{R}^d$ such that $f(q) \in \partial v(q)$ for all $q \in Q^*$, with differentiable component functions. The following result shows that for v defined by (1.0.3), this hypothesis is generally not met.

5.1 Proposition. Let $K \subset \mathbb{R}^d$ be a convex set, $F : K \to \mathbb{R}$ a convex function, $x \in \text{int}\, K$ and $f : K \to \mathbb{R}^d$ continuous at x. If

$$f(y) \in \partial F(y) \quad \text{for all} \quad y \in K , \qquad (5.1.1)$$

then F is differentiable at x and $\nabla F(x) = f(x)$.

Proof: It suffices to show that

$$\lim_{y \to x} |y - x|^{-1} \, |F(y) - F(x) - f(x)(y - x)| = 0 . \qquad (5.1.2)$$

Now (5.1.1) implies that for $y \in K$,

$$F(y) - F(x) \geq f(x)(y - x)$$

and

$$F(x) - F(y) \geq f(y)(x - y)$$

and so

$$0 \leq F(y) - F(x) - f(x)(y - x) \leq [f(y) - f(x)](y - x) \;.$$

Therefore

$$0 \leq |F(y) - F(x) - f(x)(y - x)| \leq |f(y) - f(x)| \; |y - x| \qquad (5.1.3)$$

for all $y \in K$. Since $x \in \text{int } K$, (5.1.2) need only be verified for $y \in K$, in which case we have, due to (5.1.3) and the continuity of f, that

$$\lim_{y \to x} |y - x|^{-1} |F(y) - F(x) - f(x)(y - x)| = \lim_{y \to x} |f(y) - f(x)| = 0 \;.$$

Taking $K = Q^*$ and $F = v$ in Proposition 5.1, we see that Irwin's hypothesis that the component functions of f be differentiable would require that v be differentiable on $\text{int } Q^*$, which in general is not the case.

Despite the failure of our supply model to fit the hypotheses of Irwin's convergence result, the PIES-VAR algorithm will still converge when some stringent conditions are met. This algorithm is similar to, though much more straight forward than, the PIES algorithm, and so analysis of PIES-VAR might suggest directions of study for the PIES algorithm. With this in mind, we present the PIES-VAR algorithm.

5.2 PIES-VAR Algorithm. Start with $q_0 \in Q^*$, $k = 0$.

Step 1: Construct $g(\cdot;q_k)$ as in Step 2 of algorithm 4.7.

Step 2: Solve problem (4.4.1) or its approximation (4.3.1), obtaining the solution (x,q_{k+1}).

Step 3: If $q_{k+1} = q_k$ (to within predetermined tolerances), stop with (approximate) equilibrium q_k. Otherwise, increase k by one and repeat from Step 1.

From the standpoint of analysis, the major advantages of PIES-VAR are that there is no shifting back and forth between p and q, and no averaging. Even with these advantages, we are able to prove convergence only in special cases.

5.3 Proposition. If $\bar{q}$ is an equilibrium, $\bar{p} = p(\bar{q})$, and the conjugate v^* of v is differentiable at $\bar{p}$, then the PIES-VAR algorithm converges to $\bar{q}$ in one step when started sufficiently near $\bar{q}$.

Proof: This is an immediate consequence of the proof of Proposition 4.12: for q_0 in the neighborhood N of $\bar{q}$ defined in the proof of 4.12, $q_1 = Z(q_0) = \bar{q}$ and the algorithm terminates.

The next result suggests that the structure of the elasticity matrix M in (1.0.2) is relevant to convergence. We decompose M as $M = M_d + M_0$, where M_d is a diagonal matrix and M_O has zeros along its diagonal. We do not require that p' be negative definite, but instead require that M_d have strictly negative diagonal entries, which guarantees both that M_d is invertible and that $g(\cdot;\cdot)$ is monotone in its first argument. This requirement corresponds to the economic condition of negative own-elasticities, which is a standing hypothesis of the PIES demand model.

<u>5.4 Proposition</u>. Let p be as in (1.0.2), $M = M_d + M_O$ as above, $\bar{q}$ an equilibrium, and $\bar{p} = p(\bar{q})$. If $\bar{q} \in \operatorname{int} \partial v^*(\bar{p})$ and $M_d^{-1}M_O$ has spectral radius strictly less than unity, then the PIES-VAR sequence (q_k) converges to $\bar{q}$ when started sufficiently near $\bar{q}$.

<u>Proof</u>: The set $\operatorname{int} \partial v^*(\bar{p})$ is by assumption a neighborhood of $\bar{q}$, and $\partial v(z) = \{\bar{p}\}$ for all $z \in \operatorname{int} \partial v^*(\bar{p})$ by Lemma 4.14. Define $E : \operatorname{int} \mathbb{R}_+^d \to \mathbb{R}^d$ by $E(x) = \log x - \log \bar{q}$, where the notation $\log x$ is as in (1.0.2). Let $N = \operatorname{int} \partial v^*(\bar{p})$. We assert that for any member $y = q_k$ of a PIES-VAR sequence and any $x \in N$,

$$M_d E(x) + M_O E(y) = 0 \text{ implies } x = q_{k+1} \, . \tag{5.4.1}$$

Since $M_d \log \bar{q} + M_O \log \bar{q} + K = \log \bar{p}$ and $M_d \log x + M_O \log y + K = \log g(x;y)$, $M_d E(x) + M_O E(y) = 0$ implies that $g(x;y) = \bar{p}$. Moreover, $x \in N$ implies that

$\partial v(x) = \bar{p}$, and hence $g(x;y) \in \partial v(x)$. This last relation implies, by the definition of Z (preceding Lemma 4.11), that $x = Z(y) = Z(q_k) = q_{k+1}$. Thus (5.4.1) is established.

Let $H = -M_d^{-1}M_O$. It is well known (cf. Varga [16]) that $H^n \to 0$ as $n \to \infty$ iff the spectral radius $\rho(H)$ of H is less than 1. Since we have assumed $\rho(H) < 1$, $H^n \to 0$ and hence there exists m such that

$$\|H^n\| < 1 \quad \text{for all} \quad n \geq m , \tag{5.4.2}$$

where $\|\cdot\|$ denotes the spectral norm (cf. Varga [16] for the definition of the spectral norm). Since E is a homeomorphism, the set $E(N)$ is an open neighborhood of 0 in $\mathbb{R}^d$. Let S_0 be the largest open ball with center 0 contained in $E(N)$ and let $S_i = H^{-1}(S_{i-1}) = \{s : Hs \in S_{i-1}\}$, $i = 1,\ldots,m$. Each S_i is an open neighborhood of 0, and hence so is $S = \bigcap_{i=0}^{m} S_i$. Finally, set $U = E^{-1}(S)$, a neighborhood of $\bar{q}$ contained in N. Suppose that the sequence (q_k) is generated by PIES-VAR with $q_0 \in U$. Let $r_0 = q_0$ and

$$r_{k+1} = E^{-1}(H^{k+1}E(r_0)), \quad k = 0,1,\ldots \quad . \tag{5.4.3}$$

We will show that (r_k) and (q_k) are in actuality the same sequence. We will prove this by mathematical induction. To begin, note that in view of (5.4.3) we have

$$E(r_{k+1}) = H^{k+1}\, E(r_0) = H^{k+1}\, E(q_0) \ . \tag{5.4.4}$$

Observe that

$$M_dE(r_{k+1}) + M_0E(r_k) = M_dH\ E(r_k) + M_0E(r_k)$$
$$= -M_0E(r_k) + M_0E(r_k) = 0 \tag{5.4.5}$$

for $k \geq 0$. We will now show that $r_k = q_k$ for $k \geq 0$. By definition, $r_0 = q_0$. We show that if $r_k = q_k$ then $r_{k+1} = q_{k+1}$. Taking $r_k = q_k$ in (5.4.5) we have

$$M_dE(r_{k+1}) + M_0E(q_k) = 0\ ; \tag{5.4.6}$$

in view of (5.4.1) and (5.4.6) we can conclude that $r_{k+1} = q_{k+1}$ if we show that $r_{k+1} \in N$. For $k < m$, $E(r_0) = E(q_0) \in S \subset S_{k+1}$, since $q_0 \in U$, and so $E(r_{k+1}) = H^{k+1}\ E(r_0) \in H^{k+1}\ S_{k+1} \subset S_0 \subset E(N)$. Therefore $r^{k+1} \in N$. For $k \geq m$, $|E(r_{k+1})| = |H^{k+1}\ E(r_0)| \leq |E(r_0)|$, since $\|H^{k+1}\| < 1$. Since $E(r_0) \in S_0$, a ball centered at 0, $E(r_{k+1})$ also belongs to S_0, and again we find that $r_{k+1} \in N$.

We can now show that $q_k \to \bar{q}$ when $q_0 \in U$. Since $E(r_k) = H^k\ E(r_0)$ and $H^k \to 0$ as $k \to \infty$, $E(r_k) \to 0$, and so $q_k = r_k \to \bar{q}$.

The next proposition guarantees convergence of PIES-VAR from any start in Q^*, when the indirect demand function satisfies a very strong condition. We will use the symbol $g'(\cdot;\cdot)$ to denote the derivative of g with respect to its first argument. We again require monotonicity only of g, not of p. For p as in (1.0.2), this reduces to M_d being negative definite.

<u>5.5 Lemma</u>. Let g be as in Step 2 of Algorithm 4.7, and let $g'(x;y)$ be negative definite for all $x,y \in Q^*$. Then for all $x \in Q^*$ there is a number $m(x) > 0$ such that

$$g(a;x)(b-a) - \int_a^b g(s;x) \cdot ds \geq m(x)\ |b-a|^2 \qquad (5.5.1)$$

for all $a,b \in Q^*$.

<u>Proof</u>: We note first that the integral on the left hand side of (5.5.1) is well-defined, since $g'(\circ;x)$ diagonal implies that its integral is independent of the path. Fix $a,x \in Q^*$. Define $G : [0,1] \to \mathbb{R}$ by $G(t) = tg(a;x)(b-a) - \int_0^t g(\tau b + (1-\tau)a;x)(b-a)d\tau$. Clearly $G(0) = 0$ and $G'(0) = g(a;x)(b-a) - g(a;x)(b-a) = 0$. Moreover, $G''(t) = -(b-a)g'(tb+(1-t)a;x)(b-a)$ for $0 < t < 1$. Using the second Mean Value Theorem,

$$G(1) = -(b-a)g'(\xi b + (1-\xi)a;x)(b-a)$$

for some $\xi \in (0,1)$. Since $g'(\cdot;x)$ is negative definite and continuous and Q^* is compact, $-g'(\cdot;x)$ is uniformly positive definite on Q^*, i.e. there exists $m(x) > 0$ such that $-ug'(w;x)u \geq m(x)|u|^2$ for all $w \in Q^*$ and all $u \in \mathbb{R}^d$. Thus

$$\begin{aligned} g(a;x)(b-a) - \int_a^b g(s;x) \cdot ds &= G(1) \\ &= -(b-a)g'(\xi b + (1-\xi)a;x)(b-a) \\ &\geq m(x)|b-a|^2 \quad . \end{aligned}$$

<u>5.6 Proposition</u>. Suppose that $p \in C^1$ such that $g(\cdot;\cdot)$ is monotone in the first argument and satisfies the following Lipschitz condition:

$$|g(x;y) - g(x;z)| \leq \lambda \; |y - z| \quad \text{for all} \quad x,y,z \in Q^*.$$

Let $\bar{q}$ be an equilibrium. If $\lambda < m(\bar{q})$, then the PIES-VAR sequence converges to $\bar{q}$ from any start in q^*.

<u>Proof</u>: Since q_{k+1} solves (4.4.1) at iteration k,

$$v(q_{k+1}) - \int_{\bar{q}}^{q_{k+1}} g(s;q_k) \cdot ds \leq v(\bar{q})$$

noting that the solution to (4.4.1) does not depend on what we take to be the lower limit of integration. Consequently

$$\begin{aligned} v(q_{k+1}) - v(\bar{q}) - \int_{\bar{q}}^{q_{k+1}} g(s;\bar{q}) \cdot ds \\ \leq \int_{\bar{q}}^{q_{k+1}} [g(s;q_k) - g(s;\bar{q})] \cdot ds. \end{aligned} \tag{5.6.1}$$

Since $\bar{p} = p(\bar{q}) \in \partial v(\bar{q})$ and $g(\bar{q};\bar{q}) = p(\bar{q})$,

$$v(q_{k+1}) - v(\bar{q}) \geq g(\bar{q};\bar{q})(q_{k+1} - \bar{q}) \; . \tag{5.6.2}$$

In view of (5.6.2) and Lemma 5.5, we see that

$$\begin{aligned} & v(q_{k+1}) - v(\bar{q}) - \int_{\bar{q}}^{q_{k+1}} g(s;\bar{q}) \cdot ds \\ & \geq g(\bar{q};\bar{q})(q_{k+1} - \bar{q}) - \int_{\bar{q}}^{q_{k+1}} g(s;\bar{q}) \cdot ds \\ & \geq m(\bar{q})|q_{k+1} - \bar{q}|^2 \; . \end{aligned} \tag{5.6.3}$$

Now

$$\int_{\bar{q}}^{q_{k+1}} [g(s;q_k) - g(s;\bar{q})] \cdot ds$$

$$\leq \sup_{z \in [\bar{q}, q_{k+1}]} |g(z;q_k) - g(z;\bar{q})| \; |q_{k+1} - \bar{q}| \; . \qquad (5.6.4)$$

Since $|g(z;q_k) - g(z;\bar{q})| \leq \lambda \; |q_k - \bar{q}|$, we have by combining (5.6.1), (5.6.3) and (5.6.4) that

$$m(\bar{q}) \; |q_{k+1} - \bar{q}|^2 \leq \lambda \; |q_k - \bar{q}| \; |q_{k+1} - \bar{q}|$$

and so

$$|q_{k+1} - \bar{q}| \leq m(\bar{q})^{-1} \lambda \; |q_k - \bar{q}| \; .$$

If $\lambda < m(\bar{q})$, we have $q_k \to \bar{q}$ at least as fast as a geometric sequence.

Turning back now to Proposition 5.4, we might ask what happens if all other hypotheses are satisfied but $M_d^{-1} M_O$ has spectral radius greater than unity. This question, which leads us to the following example, also prompted example 4.16 and the corresponding remarks there.

<u>5.7 Example</u>. Let $d = 2$, $r = 1$, $s = 2$,
$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $B = (1 \;\; 1)$, $b = (2)$, $c = (1, 1)$, with
p as in (1.0.2) with

$$K = \begin{pmatrix} 1.1 & \log 2 - \log 3 \\ 1.1 & \log 2 - \log 3 \end{pmatrix}, \quad M = \begin{pmatrix} -2 & 0.9 \\ -1 & -0.1 \end{pmatrix},$$

$\bar{q} = (2 \;, \; 2)$, $\bar{p} = (1/3 \;, \; 1/3)$.

This example is constructed with p' negative definite throughout Q^* (guaranteeing a unique equilibrium, namely $\bar{q}$) and with $\bar{q} \in \text{int}\, \partial v^*(\bar{p})$ and $\partial v(\bar{q}) = \{\bar{p}\}$. To verify this last claim, we note that for all $q \in Q^*$, the unique solution to (1.0.3) is $x(q) = ((2q_1 - q_2)/3,$ $(-q_1 + 2q_2)/3)$ with $v(q) = cx(q) = (q_1 + q_2)/3$. Thus $\partial v(\bar{q}) = \{(1/3, 1/3)\} = \{\bar{p}\}$. Moreover, for all $p \in \mathbb{R}^2$ we have

$$v^*(p) = \sup\, \{pq - v(q) : q \in \mathbb{R}^2\} = \sup\, \{pq - v(q) : q \in Q^*\}$$
$$= \sup\, \{pq - (q_1 + q_2)/3 : q \in Q^*\} = \sup\, \{(p - \bar{p})q : q \in Q^*\},$$

and in particular $v^*(\bar{p}) = 0$. Since $\bar{q} \in \text{int}\, Q^*$,

$$v^*(p) - v^*(\bar{p}) = v^*(p) \geq (p - \bar{p})q$$

for all $p \in \mathbb{R}^2$ and any q in a neighborhood of $\bar{q}$ in Q^*, from which it follows that $q \in \partial v^*(\bar{p})$ for all q near $\bar{q}$ and hence $\bar{q} \in \text{int}\, \partial v^*(\bar{p})$.

<u>5.8 Lemma</u>. In example 5.7, $g(\bar{q};q) \in \partial v(\bar{q})$ implies that $q = \bar{q}$.

<u>Proof</u>: Since $\partial v(\bar{q}) = \{\bar{p}\}$, $g(\bar{q};q) \in \partial v(\bar{q})$ implies that $g(\bar{q};q) = \bar{p}$. Now $p(x) = (kx_1^{-2}x_2^{0.9},$ $kx_1^{-1}x_2^{-0.1})$ for all $x \gg 0$, where $k = 3^{-1}2^{1.1}$. Hence $g(\bar{q};q) = (k2^{-2}q_2^{0.9}, k2^{-0.1}q_1^{-1})$, since $\bar{q} = (2,2)$. Therefore $g(\bar{q};q) = \bar{p}$ iff $q_2^{0.9} = 2^{0.9}$ and $q_1^{-1} = 2^{-1}$, i.e. iff $q = \bar{q}$.

<u>5.9 Proposition</u>. If $q_0 \neq \bar{q}$, the PIES-VAR sequence (q_k) cannot converge to $\bar{q}$.

<u>Proof</u>: By Lemma 5.8, if $q_{k+1} = \bar{q}$ then $g(\bar{q};q_k) \in \partial v(\bar{q})$ and so $q_k = \bar{q}$. Since $q_0 \neq \bar{q}$, it follows that $q_k \neq \bar{q}$ for all k. Now suppose $q_k \to \bar{q} \in \text{int}\, \partial v^*(\bar{p})$. By Lemma 4.14, $\partial v(q) = \{\bar{p}\}$ for all $q \in \text{int}\, \partial v^*(\bar{p})$. Since $q_k \to \bar{q}$, there exists $m > 0$ such that $q_k \in \text{int}\, \partial v^*(\bar{p})$ for all $k \geq m$. Using the decomposition $M = M_d + M_0$, for $k \geq m$ we have $g(q_{k+1};q_k) = \bar{p}$ and so $K + M_d \log q_{k+1} + M_0 \log q_k = \log \bar{p}$. Also $K + M_d \log \bar{q} + M_0 \log \bar{q} = \log \bar{p}$, and so for $k \geq m$

$$\log q_{k+1} - \log \bar{q} = -M_d^{-1} M_0[\log q_k - \log \bar{q}] .$$

It follows that for $k \geq m$

$$\log q_{k+2} - \log \bar{q} = (M_d^{-1} M_0)^2[\log q_k - \log \bar{q}] .$$

Now

$$(M_d^{-1} M_0)^2 = \begin{pmatrix} -4.5 & 0 \\ 0 & -4.5 \end{pmatrix}$$

and so for $k \geq m$ $| \log q_{k+2} - \log \bar{q} | = 4.5 \, | \log q_k - \log \bar{q}|$. From this we deduce that for $k = 1,2,\ldots$

$$| \log q_{m+2k} - \log \bar{q} | = 4.5^k \, | \log q_m - \log \bar{q} | .$$

Since $q_k \to \bar{q}$ by assumption, $| \log q_{m+2k} - \log \bar{q} | \to 0$ as $k \to \infty$, and so we must have $q_m = \bar{q}$, a contradiction.

CHAPTER VI

A SUBGRADIENT PROJECTION ALGORITHM

The lack of theoretical justification for convergence of the PIES algorithm, and the inherent difficulties in analyzing it, have led a number of researchers, including Eaves [3] and Irwin [6], to propose other algorithms for locating equilibria in models of the PIES form. The failure of the PIES algorithm on example 4.16 underscores the need for a more generally convergent method. In this chapter we state an algorithm for minimizing a convex function of a particular type over a polytope. The algorithm and a proof of convergence were proposed by Sreedharan. When the indirect demand function p has a concave potential f, the function $v - f$ is of the specified type, and we will prove that our algorithm produces a sequence converging to the minimizer of $v - f$ over Q^*. The extremal point located is, by virtue of proposition 4.2, an equilibrium. In addition, the function f is used as a tool to prove convergence but is never specifically evaluated by the algorithm, leaving open the possibility that the algorithm will prove efficacious on some problems in which p does not have a potential f.

The problem of minimizing a convex function over a linearly constrained set has been the subject of much study. We single out the approach taken by Rosen [12] where the objective function is smooth. Rosen attempts to exploit the known convergence of the method of steepest descent in the unconstrained case. As is typical of so-called "feasible direction" methods, Rosen computes at each iteration a direction of descent which points into the constrained region from the current point. He searches in that direction until he reaches either a relative minimum along the ray of search or the boundary of the constrained region. The process then repeats. Rosen's contribution is the choice of direction. When possible, he uses the negative gradient as the direction; when this direction points out of the set, he projects it onto a face of the set. Rosen's method is susceptible to a phenomenon known variously as "jamming" or "zigzagging", in which the sequence generated clusters at, or even converges to, nonoptimal points. The trouble lies in the possibility that the sequence is alternating among two or more faces of the constrained region in such a way that the distance along the direction of search from the current point to the boundary is going to zero. Various modifications have been proposed to avoid this. In particular, Polak [10] has adopted a technique which prohibits the sequence generated from approaching arbitrarily close to a face when conditions for a constrained minimum are not being met at the limit.

Rosen's method, and much of the other work in the area, requires that the objective function be differentiable. Even when p has a potential f, our objective function $v - f$ is not differentiable everywhere because v is not. Attention has recently been focused on algorithms for optimizing nondifferentiable convex functions. The algorithms of Wolfe [19] and Lemarechal [7], which generalize classical methods for unconstrained optimization by replacing the gradient with a carefully chosen subgradient, do not treat the constrained case. The algorithm of Bertsekas and Mitter [1], which does handle constrained problems, requires computation of the "ϵ-subdifferential" of the objective function, which is prohibitive in the problem we consider here. The algorithm proposed here is prompted by those of Sreedharan [13,14], Rosen [12] and Polak [10]. It resembles the Bertsekas - Mitter algorithm but requires the computation of only a manageable portion of the ϵ-subdifferential. In this chapter we pose the algorithm and prove its convergence; in the next chapter, we discuss the actual implementation and report on some trial applications.

<u>6.1 Problem</u>. Let $X \subset \mathbb{R}^d$ be a nonempty convex polytope given by $X = \{x \in \mathbb{R}^d \mid a_i x \leq b_i,\ i = 1,\ldots,m\}$. Note that polytopes are, by definition, bounded. Let $v_j : \mathbb{R}^d \to \mathbb{R}$ $(j = 1,\ldots,r)$ be given by $v_j(x) = g_j x + c_j$, $g_j \in \mathbb{R}^d$, $c_j \in \mathbb{R}$, and let the

polyhedrally convex function $v : \mathbb{R}^d \to \mathbb{R}$ be given by

$$v(x) = \max\{v_j(x) \mid j = 1,\dots,r\}, \quad x \in \mathbb{R}^d. \tag{6.1.1}$$

Let $f : \mathbb{R}^d \to \mathbb{R}$ be strictly convex and of class C^1 on a neighborhood of X. Finally, denote by δ the <u>indicator function</u> (cf. Rockafellar [11]) of X, i.e. $\delta(x) = 0$ if $x \in X$ and $\delta(x) = +\infty$ if $x \notin X$. We consider the problem

$$\left\{\begin{array}{ll} a_i x \le b_i, & i = 1,\dots,m \\ & \\ f(x) + v(x) & (\min) \end{array}\right\} , \tag{6.1.2}$$

which is equivalent to the problem of locating an unconstrained minimizer of $F = f + v + \delta$ over $\mathbb{R}^d$.

We note before proceeding that the problem of locating the equilibrium of the PIES model fits the form of (6.1.2) when the indirect demand function p has a strictly concave potential, for if we take $X = Q^*$ and take f to be the negative of the potential of p, then by theorem 1.8 the value v of the linear program (4.0.1) is polyhedrally convex on Q^*, and by proposition 4.2 q is an equilibrium iff q solves (6.1.2).

We introduce some needed notation.

<u>6.2 Notation</u>. For $x \in X$ and $\varepsilon \ge 0$, we define the following sets of indices:

$$I_\varepsilon(x) = \{1 \le i \le m \mid a_i x \ge b_i - \varepsilon\}; \tag{6.2.1}$$

$$J_\varepsilon(x) = \{1 \le j \le r \mid v_j(x) \ge v(x) - \varepsilon\}. \tag{6.2.2}$$

Note that

$$I_0(x) = \{1 \le i \le m \mid a_i x = b_i\} \tag{6.2.3}$$

and

$$J_0(x) = \{1 \le j \le r \mid v_j(x) = v(x)\}. \tag{6.2.4}$$

We also define two convex subsets of $\mathbb{R}^d$, namely

$$C_\varepsilon(x) = \text{cone}\{a_i \mid i \in I_\varepsilon(x)\} \tag{6.2.5}$$

and

$$K_\varepsilon(x) = \text{conv}\{g_j \mid j \in J_\varepsilon(x)\}, \tag{6.2.6}$$

where for any set S we use cone S and conv S to denote respectively the convex cone, with apex at the origin, generated by S and the convex hull of S.

For any nonempty closed convex set $S \subset \mathbb{R}^d$ there is a unique point $x \in S$ nearest to the origin, which we denote by $N[S]$. The point $a = N[S]$ is characterized by the following:

$$a = N[S] \quad \text{iff} \quad a(x-a) \ge 0 \quad \text{for all} \quad x \in S. \tag{6.2.7}$$

We now present a subgradient projection algorithm for problem (6.1.2).

<u>6.3 Algorithm</u>. Begin with arbitrary $x_0 \in X$, $\varepsilon_0 > 0$ and with $k = 0$.

<u>Step 1</u>: Compute $y_0 = N[\nabla f(x_k) + K_0(x_k) + C_0(x_k)]$. If $y_0 = 0$, stop: x_k solves (6.1.2). If $y_0 \neq 0$, set $\varepsilon = \varepsilon_0$.

<u>Step 2</u>: Compute $y_\varepsilon = N[\nabla f(x_k) + K_\varepsilon(x_k) + C_\varepsilon(x_k)]$.

<u>Step 3</u>: If $|y_\varepsilon|^2 > \varepsilon$, set $\varepsilon_k = \varepsilon$, $s_k = y_\varepsilon$ and go to step 5.

<u>Step 4</u>: Replace ε with $\varepsilon/2$ and go to step 2.

<u>Step 5</u>: Compute $\bar{\alpha}_k = \max\{\alpha \geq 0 \mid x_k - \alpha s_k \in X\}$. (It will be shown that $\bar{\alpha}_k > 0$). Find $\alpha_k \in [0,\bar{\alpha}_k]$ such that there exists $z_k \in \nabla f(x_k - \alpha_k s_k) + K_0(x_k - \alpha_k s_k)$ satisfying $z_k s_k = 0$; if no such α_k exists, set $\alpha_k = \bar{\alpha}_k$.

<u>Step 6</u>: Set $x_{k+1} = x_k - \alpha_k s_k$, increase k by 1, and go to step 1.

The implementation of steps 1 and 2, which can be treated as quadratic programs, and of step 5, which requires a special line search procedure, are discussed in the next chapter.

We next state a sequence of lemmas leading to a proof that algorithm 6.3 solves problem (6.1.2), or equivalently locates an unconstrained minimizer of $F = f + v + \delta$. We must first define the "ε-subdifferential."

<u>6.4 Definition</u>. Let $G : \mathbb{R}^n \to [-\infty, \infty]$ be a convex function. The <u>ε-subdifferential</u> of G at the point x, denoted $\partial_\varepsilon G(x)$, is defined by

$$\partial_\varepsilon G(x) = \{u \in \mathbb{R}^n \mid G(y) \geq G(x) + u(y - x) - \varepsilon \text{ for all } y \in \mathbb{R}^n\}.$$

The usual subdifferential $\partial G(x)$ of G at x is just $\partial_0 G(x)$.

We now state a sequence of lemmas, using the earlier notation.

<u>6.5 Lemma</u>. For all $\varepsilon \geq 0$ and all $x \in \mathbb{R}^d$, $K_\varepsilon(x) \subset \partial_\varepsilon v(x)$.

<u>Proof</u>: If $u \in K_\varepsilon(x)$, then by (6.2.6) there exist $\lambda_j \geq 0$, $j \in J_\varepsilon(x)$ such that $\sum \lambda_j = 1$ and

$$u = \sum_{j \in J_\varepsilon(x)} \lambda_j g_j.$$

For $j \in J_\varepsilon(x)$, we have

$$v_j(y) = v_j(x) + g_j(y - x) \geq v(x) - \varepsilon + g_j(y - x). \qquad (6.5.1)$$

Therefore for every $j \in J_\varepsilon(x)$,

$$v(y) = \max_{i=1,\ldots,r} v_i(y) \geq v(x) - \varepsilon + g_j(y - x),$$

and so

$$v(y) = \sum_{j \in J_\varepsilon(x)} \lambda_j v(y) \geq v(x) - \varepsilon + \sum_{j \in J_\varepsilon(x)} \lambda_j g_j(y-x)$$
$$= v(x) - \varepsilon + u(y-x). \tag{6.5.2}$$

Since (6.5.2) holds for all $y \in \mathbb{R}^d$, the lemma is proved.

<u>6.6 Lemma</u>. Given $x \in \mathbb{R}^d$, there exists a neighborhood V of x such that $J_0(y) \subset J_0(x)$ for all $y \in V$.

<u>Proof</u>: The functions $w_j = v - v_j$, $j = 1,\ldots,r$ are continuous with $w_j(x) > 0$ iff $j \notin J_0(x)$. Thus there exists a neighborhood V of x such that w_j is positive throughout V for each $j \notin J_0(x)$. If $j \notin J_0(x)$ and $y \in V$, $w_j(y) > 0$, and so $j \notin J_0(y)$, proving the lemma.

<u>6.7 Definition</u>. Let $S \subset \mathbb{R}^n$ be a nonempty set. The <u>support</u> <u>function</u> φ of S is defined by

$$\varphi(x) = \sup\{xy \mid y \in S\}, \quad x \in \mathbb{R}^n.$$

<u>6.8 Lemma</u>. Two closed convex subsets of $\mathbb{R}^n$ are identical iff their support functions are identical.

<u>Proof</u>: See Rockafellar [11].

<u>6.9 Lemma</u>. $\partial v(x) = K_0(x)$ for every $x \in X$.

<u>Proof</u>: Let $x \in X$. Since both $\partial v(x)$ and $K_0(x)$ are closed and convex, it suffices, in view of lemma 6.8, to show that they have the same support function, namely

$v'(x;\cdot)$ [v' defined as in lemma 2.4]. It is well-known that since v is everywhere finite-valued,

$$v'(x;y) = \sup \{yu \mid u \in \partial v(x)\} \text{ for all } y \in \mathbb{R}^d, \qquad (6.9.1)$$

i.e. $v'(x;\cdot)$ is the support function of $\partial v(x)$. Given $y \in \mathbb{R}^d$, by lemma 6.6 there exists an $\varepsilon > 0$ such that $J_0(x+\alpha y) \subset J_0(x)$ for all $\alpha \in [0,\varepsilon]$. Since for $0 \le \alpha \le \varepsilon$

$$v(x+\alpha y) = \max_{j \in J_0(x+\alpha y)} v_j(x+\alpha y)$$

and

$$v(x) = \max_{j \in J_0(x)} v_j(x),$$

we see that

$$v(x+\alpha y) - v(x) = \max_{j \in J_0(x)} \{v_j(x+\alpha y) - v_j(x)\}$$

$$= \max_{j \in J_0(x)} \alpha g_j y.$$

This shows that

$$v'(x;y) = \max_{j \in J_0(x)} g_j y = \max \{uy \mid u \in K_0(x)\}, \qquad (6.9.2)$$

and so $v'(x;\cdot)$ is also the support function of $K_0(x)$.

We note that (6.9.2) proves the following statement:

<u>6.10 Corollary</u>. $v'(x;s) = \max \{su \mid u \in K_0(x)\}$.

<u>6.11 Lemma</u>. For each $x \in X$ and $\varepsilon > 0$ there exists a $\gamma > 0$ such that

$$J_0(x) \subset J_\varepsilon(y) \quad \text{whenever} \quad |x-y| < \gamma.$$

<u>Proof</u>. Choose $\gamma > 0$ such that $|g_j|\gamma < \frac{\epsilon}{2}$ for $j = 1,\ldots,r$ and $|v(x) - v(y)| < \frac{\epsilon}{2}$ if $|x - y| < \gamma$. Now if $j \in J_0(x)$ and $|x - y| < \gamma$, then

$$\begin{aligned} v(y) - v_j(y) &= v(y) - v(x) + v_j(x) - v_j(y) \\ &< \frac{\epsilon}{2} + |g_j(x - y)| < \epsilon, \end{aligned}$$

and so $j \in J_\epsilon(y)$.

<u>6.12 Lemma</u>. $\partial F(x) = \nabla f(x) + K_0(x) + C_0(x)$ for all $x \in X$.

<u>Proof</u>: The indicator function δ is clearly proper and convex, while f and v are everywhere finite valued. It is well-known that for $x \in X$, $\partial\delta(x) = C_0(x)$. Moreover, any $a \in$ rel int X belongs to rel int (eff dom f) $\cap$ rel int (eff dom v) $\cap$ rel int (eff dom δ). The result now follows from lemma 4.1.

The next lemma shows that the stopping criterion in step 1 of the algorithm is well chosen.

<u>6.13 Lemma</u>. If $y_0 = 0$ in step 1 of algorithm 6.3, then x_k is the minimizer of F.

<u>Proof</u>: $y_0 = 0$ implies that $0 \in \partial F(x_k)$, a necessary and sufficient condition for x_k to minimize F. The strict convexity of f ensures that the minimizer of F is unique.

<u>6.14 Lemma</u>. Step 4 of algorithm 6.3 is not executed infinitely often in any one iteration.

Proof: If step 4 is executed infinitely often, then $\varepsilon \to 0$ and $y_\varepsilon \to 0$. Now

$\varepsilon_1 > \varepsilon_2 \geq 0$ implies that $K_{\varepsilon_2}(x_k) \subset K_{\varepsilon_1}(x_k)$ and

$$C_{\varepsilon_2}(x_k) \subset C_{\varepsilon_1}(x_k)$$

and hence that

$$|y_{\varepsilon_1}| \leq |y_{\varepsilon_2}|,$$

so that $y_\varepsilon \to 0$ as $\varepsilon \to 0$ implies that $y_\varepsilon = 0$ for every $\varepsilon \geq 0$; but then $y_0 = 0$, and we cannot have reached step 4, a contradiction.

We now show the practicability of step 5 of the algorithm.

6.15 Lemma. If $s_k \neq 0$, then $-s_k$ is a feasible direction of strict descent at the point x_k.

Proof: From the definition of s_k in step 3 of the algorithm,

$$s_k = N[\nabla f(x_k) + K_{\varepsilon_k}(x_k) + C_{\varepsilon_k}(x_k)].$$

Let $i \in I_0(x_k) \subset I_{\varepsilon_k}(x_k)$; then $a_i \in C_{\varepsilon_k}(x_k)$ and so

$$s_k + a_i \in \nabla f(x_k) + K_{\varepsilon_k}(x_k) + C_{\varepsilon_k}(x_k),$$

using the fact that $C_{\varepsilon_k}(x_k)$ is a convex cone. By (6.2.7) we have $s_k(s_k + a_i - s_k) \geq 0$. Thus $a_i s_k \geq 0$ for every $i \in I_0(x_k)$. Since $a_i x_k < b_i$ for $i \notin I_0(x_k)$, there exists $\alpha > 0$ such that $a_i(x_k - \alpha s_k) \leq b_i$ for all $i = 1,\ldots,r$. Hence $-s_k$ is a feasible direction at x_k.

To show that $-s_k$ is a direction of strict descent, we show that

$$F'(x_k;-s_k) = \lim_{\alpha\downarrow 0} [F(x_k - \alpha s_k) - F(x_k)]/\alpha < 0. \tag{6.15.1}$$

From the first part of the proof, there exists $\bar{\alpha} > 0$ such that $x_k - \alpha s_k \in X$ for $0 \leq \alpha \leq \bar{\alpha}$. For α in this range, $F(x_k - \alpha s_k) = f(x_k - \alpha s_k) + v(x_k - \alpha s_k)$ and so by corollary 6.10

$$\begin{aligned} F'(x_k;-s_k) &= f'(x_k;-s_k) + v'(x_k;-s_k) \\ &= -\nabla f(x_k)s_k + \max\{-s_k y \mid y \in K_0(x_k)\} \\ &= -\min\{(\nabla f(x_k) + y)s_k \mid y \in K_0(x_k)\}. \end{aligned} \tag{6.15.2}$$

When $y \in K_0(x_k) \subset K_{\epsilon_k}(x_k)$ we have

$$\nabla f(x_k) + y \in \nabla f(x_k) + K_{\epsilon_k}(x_k) + C_{\epsilon_k}(x_k)$$

and so by (6.2.7) $s_k(\nabla f(x_k) + y - s_k) \geq 0$ and consequently $(\nabla f(x_k) + y)s_k \geq |s_k|^2 > 0$. Combining this with (6.15.2), we have

$$F'(x_k;-s_k) \leq -|s_k|^2 < 0, \tag{6.15.3}$$

completing the proof.

From the first half of lemma 6.15 we have the following corollary.

6.16 Corollary. The number $\bar{\alpha}_k$ defined in step 5 of algorithm 6.3 is positive.

The next lemma shows that in the relevant case the vector z_k in step 5 of the algorithm exists.

6.17 Lemma. Let $s_k \neq 0$ and define φ on $[0,\bar{\alpha}_k]$ by $\varphi(\alpha) = F(x_k - \alpha s_k)$. If $\bar{\alpha}_k$ is not a minimizer of φ on $[0,\bar{\alpha}_k]$, then z_k satisfying step 5 of algorithm 6.3 exists.

Proof: By lemma 6.15, $\varphi'(0) = F'(x_k;-s_k) < 0$, so that there is some $\alpha \in (0,\bar{\alpha}_k]$ such that $\varphi(\alpha) < \varphi(0)$. Since we have hypothesized that $\bar{\alpha}_k$ does not minimize φ, there exists $\alpha_k \in (0,\bar{\alpha}_k)$ minimizing φ over $[0,\bar{\alpha}_k]$. Set $y = x_k - \alpha_k s_k$. There exists $\epsilon > 0$ such that $F(y) \leq F(y + \lambda s_k)$ for $|\lambda| \leq \epsilon$. It follows that

$$[F(y + \lambda s_k) - F(y)]/\lambda \geq 0 \tag{6.17.1}$$

and

$$[F(y - \lambda s_k) - F(y)]/\lambda \geq 0, \tag{6.17.2}$$

$0 < \lambda \leq \epsilon$. Since F is convex, the directional derivatives $F'(y;s_k)$ and $F'(y;-s_k)$ both exist, and from (6.17.1) and (6.17.2) we conclude that $F'(y;s_k) \geq 0$ and $F'(y;-s_k) \geq 0$. Using corollary 6.10,

$$\begin{aligned} F'(y; \pm s_k) &= f'(y; \pm s_k) + v'(y; \pm s_k) \\ &= \pm \nabla f(y) s_k + \max\{ \pm u s_k \mid u \in K_0(y) \}. \end{aligned}$$

Since $K_O(y)$ is compact, there exist $u,w \in K_O(y)$ such that

$$\nabla f(y)s_k + us_k = F'(y;s_k) \geq 0$$

and

$$\nabla f(y)s_k + ws_k = -F'(y;-s_k) \leq 0,$$

and so for an appropriately chosen convex combination h of u and w we have $h \in K_O(y)$ and

$$\nabla f(y)s_k + hs_k = 0.$$

Taking $z_k = \nabla f(y) + h \in \nabla f(y) + K_O(y)$ satisfies the requirement in step 5 of the algorithm.

The number α_k determined in step 5 of algorithm 6.3 has the following property.

6.18 Lemma. Let $s_k \neq 0$ and φ be as in the previous lemma. Then α_k is the unique minimizer of φ on $[0,\bar{\alpha}_k]$. Moreover, α_k is positive.

Proof: Since $F'(x_k;-s_k) < 0$ by lemma 6.15, the conclusion that $\alpha_k > 0$ follows immediately once we show that α_k minimizes φ over $[0,\bar{\alpha}_k]$. Uniqueness of this minimizer follows from the strict convexity of F.

If z_k satisfying step 5 of the algoirthm cannot be found, then by lemma 6.17 $\bar{\alpha}_k$ minimizes φ over

$[0,\bar{\alpha}_k]$, and in step 5 we set $\alpha_k = \bar{\alpha}_k$. Suppose then that $\alpha_k \in (0,\bar{\alpha}_k]$ is located such that an appropriate vector z_k exists. Set $y = x_k - \alpha_k s_k$. Since $z_k \in \nabla f(y) + K_0(y) \subset \partial F(y)$, for any $\alpha \in [0,\bar{\alpha}_k]$ we have by the subgradient inequality that

$$\varphi(\alpha) = F(x_k - \alpha s_k) \geq F(y) + (\alpha_k - \alpha) z_k s_k = F(y) = \varphi(\alpha_k),$$

so that α_k minimizes φ over $[0,\bar{\alpha}_k]$.

<u>6.19 Corollary</u>. Let $s_k \neq 0$ and $x_{k+1} = x_k - \alpha_k s_k$ as in step 6 of algorithm 6.3; then $F(x_{k+1}) < F(x_k)$.

<u>Proof</u>: This follows from lemma 6.18 and the observation that $F'(x_k;-s_k) < 0$.

The lemmas stated up to this point prove that the algorithm is feasible and that F decreases at each iteration. We now turn to lemmas leading to a convergence proof.

<u>6.20 Lemma</u>. Let $\bar{x} \in X$ be the minimizer of F and $\bar{x}$ be a cluster point of (x_k). Then x_k converges to $\bar{x}$.

<u>Proof</u>: Let $\hat{x}$ be any cluster point of (x_k). We have $F(\hat{x}) = F(\bar{x})$. Since F is strictly convex, $\bar{x}$ is the unique minimizer, so that $\hat{x} = \bar{x}$. Thus (x_k) is a sequence, from a compact set X, having only one cluster point $\bar{x}$, and so $x_k \to \bar{x}$.

6.21 Lemma. Let 0 be a cluster point of (s_k). Then the sequence (x_k) converges to $\bar{x}$, the minimizer of F.

Proof: We pass to corresponding subsequences $(s_{k'})$ and $(x_{k'})$ such that $s_{k'} \to 0$ and $x_{k'} \to \hat{x} \in X$. We shall show that $\hat{x}$ minimizes F, so that by the previous lemma $x_k \to \bar{x}$. Since the restriction of F to X is continuous from within X, to prove that $\hat{x}$ is a minimizer of F, it suffices to show that $F(y) \geq F(\hat{x})$ for all $y \in \text{rel int } X$. Let $y \in \text{rel int } X$. For all $i \in I_0(\hat{x})$, $a_i y < b_i = a_i \hat{x}$, and so for k' sufficiently large

$$a_i(y - x_{k'}) < 0 \quad \text{for all} \quad i \in I_0(\hat{x}). \tag{6.21.1}$$

Since $s_{k'} \to 0$ and $\varepsilon_{k'} \leq |s_{k'}|^2$, $\varepsilon_{k'} \to 0$, so that for k' sufficiently large

$$I_{\varepsilon_{k'}}(x_{k'}) \subset I_0(\hat{x}). \tag{6.21.2}$$

Now there exist $u_{k'} \in K_{\varepsilon_{k'}}(x_{k'})$ and $w_{k'} \in C_{\varepsilon_{k'}}(x_{k'})$ such that $s_{k'} = \nabla f(x_{k'}) + u_{k'} + w_{k'}$. By lemma 6.5, $K_{\varepsilon_k}(x_k) \subset \partial_{\varepsilon_k} v(x_k)$, so that

$$v(y) - v(x_{k'}) \geq u_{k'}(y - x_{k'}) - \varepsilon_{k'}. \tag{6.21.3}$$

Since f is convex, it follows that

$$\begin{aligned} F(y) - F(x_{k'}) &\geq \nabla f(x_{k'})(y - x_{k'}) + u_{k'}(y - x_{k'}) - \varepsilon_{k'} \\ &= s_{k'}(y - x_{k'}) - w_{k'}(y - x_{k'}) - \varepsilon_{k'}. \end{aligned} \tag{6.21.4}$$

Assume that k' is large enough that (6.21.1) and (6.21.2) hold. Since $w_{k'}$ belongs to the convex cone generated by $\{a_i \mid i \in I_{\epsilon_{k'}}(x_{k'})\}$, in view of (6.21.1) and (6.21.2) we have $w_{k'}(y - x_{k'}) \leq 0$, and so from (6.21.4)

$$F(y) - F(x_{k'}) \geq s_{k'}(y - x_{k'}) - \epsilon_{k'} \tag{6.21.5}$$

when k' is sufficiently large. In the limit (6.21.5) gives

$$F(y) - F(\hat{x}) \geq 0,$$

proving the lemma.

<u>6.22 Lemma</u>. If 0 is a cluster point of the sequence (ϵ_k) defined in algorithm 6.3, then (x_k) converges to $\bar{x}$, the minimizer of F.

<u>Proof</u>: Passing to corresponding subsequences $(\epsilon_{k'})$ and $(x_{k'})$, we may assume that $\epsilon_{k'} \to 0$ and $x_{k'} \to \hat{x} \in X$. By lemma 6.14, step 4 of the algorithm is executed finitely often per iteration, and hence the subsequence $(\epsilon_{k'})$ can be chosen such that

$$|y_\epsilon|^2 \leq \epsilon, \quad y_\epsilon = N[\nabla f(x_{k'}) + K_\epsilon(x_{k'}) + C_\epsilon(x_{k'})]$$

and

$$|y_{\epsilon/2}|^2 > \frac{\epsilon}{2} = \epsilon_{k'}, \quad y_{\epsilon/2} = N[\nabla f(x_{k'}) + K_{\frac{\epsilon}{2}}(x_{k'}) + C_{\frac{\epsilon}{2}}(x_{k'})].$$

From these we see that $|y_{2\epsilon_{k'}}|^2 \leq 2\epsilon_{k'}$, showing that $y_{2\epsilon_{k'}} \to 0$. We now repeat the proof of lemma 6.21, replacing $s_{k'}$ with $y_{2\epsilon_{k'}}$, concluding that $x_k \to \bar{x}$.

6.23 Lemma. The sequence (s_k) is bounded.

Proof: Note that

$$\nabla f(x_k) + K_0(x_k) \subset \nabla f(x_k) + K_{\epsilon_k}(x_k) + C_{\epsilon_k}(x_k),$$

so that

$$|s_k| \leq |\nabla f(x_k)| + |N[K_0(x_k)]|. \tag{6.23.1}$$

$K_0(x_k)$ is one of a finite number of possible polytopes, so that there is an upper bound on $|N[K_0(x_k)]|$ independent of k. As f is of class C^1 on the compact set X, the right hand side of (6.23.1) is bounded, proving the lemma.

6.24 Lemma. If the sequence (s_k) is bounded away from 0, then (α_k) converges to 0.

Proof: Suppose that (s_k) is bounded away from 0 and that $\alpha_k \not\to 0$. Since $\alpha_k|s_k|$ is bounded above by the diameter of X and (s_k) is bounded away from 0, (α_k) is bounded. Given this and the compactness of X, we can pass to corresponding subsequences $(s_{k'})$, $(\alpha_{k'})$ and $(x_{k'})$ such that $s_{k'} \to s \neq 0$, $\alpha_{k'} \to \alpha > 0$ and $x_{k'} \to x \in X$. By lemma 6.19, the sequence $(F(x_{k'}))$ is monotone decreasing, so that all of its subsequences have the same limit, namely $F(x)$. In particular, $F(x_{k'+1}) \to F(x)$;

but $x_{k'+1} = x_{k'} - \alpha_{k'} s_{k'} \to x - \alpha s$, so that

$$F(x - \alpha s) = F(x). \tag{6.24.1}$$

Since F is convex and $F(x_{k'} - \alpha_{k'} s_{k'}) \leq F(x_{k'} - \lambda s_{k'})$ for all $\lambda \in [0, \bar{\alpha}_{k'}]$, we have

$$F(x_{k'} - \alpha_{k'} s_{k'}) \leq F(x_{k'} - \alpha_{k'} s_{k'}/2) \leq F(x_{k'})$$

and so in the limit

$$F(x - \alpha s) \leq F(x - \alpha s/2) \leq F(x). \tag{6.24.2}$$

Since $\alpha > 0$ and $s \neq 0$, (6.24.1) and (6.24.2) taken together contradict the strict convexity of F.

6.25 Lemma. Let the sequence (ϵ_k) defined in algorithm 6.3 be such that there exists $\epsilon \geq 0$ satisfying $\epsilon_k \geq \epsilon$ for every k. For any index i, the inequality

$$b_i - a_i x_k \leq \epsilon \tag{6.25.1}$$

implies the inequality

$$b_i - a_i x_k \leq b_i - a_i x_{k+1} . \tag{6.25.2}$$

Proof: If (6.25.1) holds, then $i \in I_{\epsilon_k}(x_k)$, and so $a_i \in C_{\epsilon_k}(x_k)$. As was noted in the proof of lemma 6.15, it follows that $a_i s_k \geq 0$. Since $x_{k+1} = x_k - \alpha_k s_k$, (6.25.2) must hold.

6.26 Lemma. Assume that the following hold.

(i) The sequence (ϵ_k) in algorithm 6.3 is such that there exists $\epsilon > 0$ with $\epsilon_k \geq \epsilon$ for all k.

(ii) The sequence (α_k) converges to 0.

(iii) Some subsequence $(x_{k'})$ of (x_k) converges to the point x.

Then there exists a subsequence of $(x_{k'})$, again denoted $(x_{k'})$, such that $I_0(x_{k'}) = I_0(x)$ for every index k'.

<u>Proof</u>: Assume that (i), (ii) and (iii) hold. Since the index sets $I_0(x_{k'})$ are subsets of the finite set $\{1,\dots,m\}$, we can pass to a subsequence of $(x_{k'})$, again denoted $(x_{k'})$, such that for some subset I of $\{1,\dots,m\}$ we have $I_0(x_{k'}) = I$ for all k'. We must show that $I_0(x) = I$. If $i \in I$, then $a_i x_{k'} = b_i$ for all k', so that in the limit $a_i x = b_i$. Therefore $I \subset I_0(x)$. Now suppose that $i \in I_0(x) \setminus I$. We derive a contradiction. Since $x_{k+1} = x_k - \alpha_k s_k$, with (s_k) shown bounded in lemma 6.23 and $\alpha_k \to 0$, we see that $|x_{k+1} - x_k| \to 0$ as $k \to \infty$. Hence there exists k_0 such that

$$a_i(x_{k+1} - x_k) < \frac{\varepsilon}{2} \quad \text{for all} \quad k \geq k_0. \tag{6.26.1}$$

Choose $p \geq k_0$ such that $I_0(x_p) = I$ and

$$\varepsilon_* = b_i - a_i x_p < \varepsilon. \tag{6.26.2}$$

Such an index p exists because $i \in I_0(x)$ implies that $b_i - a_i x_{k'} \to b_i - a_i x = 0$. Also $\varepsilon_* > 0$, since $i \notin I$. Let q be the first index such that $q > p$ and

$$b_i - a_i x_q \leq \varepsilon_*/2 . \tag{6.26.3}$$

Now by (6.26.1), (6.26.2) and (6.26.3)

$$b_i - a_i x_{q-1} = b_i - a_i x_q + a_i(x_q - x_{q-1})$$

$$< \epsilon_*/2 + \epsilon/2 < \epsilon ,$$

and so by lemma 6.25

$$b_i - a_i x_{q-1} \leq b_i - a_i x_q \leq \epsilon_*/2 . \qquad (6.26.4)$$

Note that $q-1 \geq p$. If $q-1 = p$, then (6.26.4) contradicts (6.26.2). If $q-1 > p$, then (6.26.4) contradicts the choice of q as the smallest index greater than p such that (6.26.3) holds.

6.27 Corollary. Suppose that the following hold.

(i) There exists $\epsilon > 0$ such that $\epsilon_k \geq \epsilon$ for all k.

(ii) There exists $\eta > 0$ such that $|s_k| \geq \eta$ for all k.

(iii) Some subsequence $(x_{k'})$ of (x_k) converges to x.

Then there is a subsequence of $(x_{k'})$, again denoted $(x_{k'})$, such that $I_0(x_{k'}) = I_0(x)$ for all k'.

Proof: Hypothesis (ii) of this corollary implies hypothesis (ii) of lemma 6.26 by lemma 6.24.

We are at last prepared to prove the convergence of our algorithm.

6.28 Theorem. Algorithm 6.3 generates either a terminating sequence whose last term solves problem (6.1.2) or an infinite sequence converging to the solution of (6.1.2).

Proof: In view of lemma 6.13, we need only consider the case in which algorithm 6.3 generates an infinite sequence (x_k). In this case, $s_k \neq 0$ for every k. We assume that (x_k) fails to converge to the solution of (6.1.2), and derive a contradiction.

By lemma 6.21 we may suppose that there exists $\eta > 0$ such that $|s_k| \geq \eta$ for all k. Similarly, by lemma 6.22 we may assume that there exists $\epsilon > 0$ such that $|\epsilon_k| \geq \epsilon$ for all k. Since X is compact and, by lemma 6.23, (s_k) is bounded, we may pass to a subsequence (k') of positive integers such that

$$x_{k'} \to x \in X \quad \text{and} \quad s_{k'} \to s \neq 0. \tag{6.28.1}$$

From step 6 of the algorithm, $x_{k'+1} = x_{k'} - \alpha_{k'} s_{k'}$. Since $|s_k| \geq \eta$ for all k, lemma 6.24 ensures that $\alpha_k \to 0$, and so $x_{k'+1} \to x$. Passing to a subsequence of (k'), again denoted (k'), we may suppose that there exist sets I, J and J' of indices such that

$$I_{\epsilon_{k'}}(x_{k'}) = I, \quad J_{\epsilon_{k'}}(x_{k'}) = J, \quad J_0(x_{k'+1}) = J' \tag{6.28.2}$$

for all k'. We assert that $J' \subset J$. Since $x_{k'+1} \to x$,

$J_0(x_{k'+1}) \subset J_0(x)$ for k' sufficiently large. Moreover, since $x_{k'} \to x$ and $\varepsilon > 0$, by lemma 6.11 $J_0(x) \subset J_\varepsilon(x_{k'})$ for k' large enough. As $\varepsilon_{k'} \geq \varepsilon$, we must have $J_\varepsilon(x_{k'}) \subset J_{\varepsilon_{k'}}(x_{k'})$. Thus for k' sufficiently large

$$J' = J_0(x_{k'+1}) \subset J_0(x) \subset J_\varepsilon(x_{k'}) \subset J_{\varepsilon_{k'}}(x_{k'}) = J,$$

and so $J' \subset J$.

Using corollary 6.27, since $x_{k'+1} \to x$ we can pass to yet another subsequence, again denoted (k') such that,

$$I_0(x_{k'}) = I_0(x) = I_0(x_{k'+1}) \tag{6.28.3}$$

for all k'. Now set

$$K = \operatorname{conv}\{g_j \mid j \in J\} \text{ and } C = \operatorname{cone}\{a_i \mid i \in I\}. \tag{6.28.4}$$

Due to (6.28.2) and (6.28.4), we see that for all k'

$$K_{\varepsilon_{k'}}(x_{k'}) = K \text{ and } C_{\varepsilon_{k'}}(x_{k'}) = C. \tag{6.28.5}$$

From (6.28.3) we deduce that $\alpha_{k'} < \bar{\alpha}_{k'}$ for each k'; for if $\alpha_{k'} = \bar{\alpha}_{k'}$, some constraint inactive at $x_{k'}$ becomes active at $x_{k'+1}$, and so $I_0(x_{k'}) \neq I_0(x_{k'+1})$. Thus for each k', the vector $z_{k'}$ specified in step 5 of algorithm 6.3 must exist, i.e.

$$z_{k'} \in \nabla f(x_{k'+1}) + K_0(x_{k'+1})$$

and

$$z_{k'} s_{k'} = 0. \tag{6.28.6}$$

Taking into account (6.28.2) and (6.28.4), we have

$$K_0(x_{k'+1}) = \mathrm{conv}\{g_j \mid j \in J'\} \subset \mathrm{conv}\{g_j \mid j \in J\}$$

and so

$$z_{k'} \in \nabla f(x_{k'+1}) + K. \tag{6.28.7}$$

Since $\nabla f(x_{k'+1}) \to \nabla f(x)$ and K is compact, by passing to still another subsequence (k') and applying (6.28.7) we may assume that there exists $z \in \nabla f(x) + K$ such that $z_{k'} \to z$.

From steps 2 and 3 of the algorithm, we have that

$$s_{k'} = N[\nabla f(x_{k'}) + K_{\epsilon_{k'}}(x_{k'}) + C_{\epsilon_{k'}}(x_{k'})]$$

and so, in view of (6.28.2) and (6.28.4),

$$s_{k'} = N[\nabla f(x_{k'}) + K + C].$$

Since $x_{k'} \to x$ and $s_{k'} \to s$, it follows easily that

$$s = N[\nabla f(x) + K + C]. \tag{6.28.8}$$

Now $z \in \nabla f(x) + K \subset \nabla f(x) + K + C$, and so by (6.2.7) we have $s(z - s) \geq 0$, i.e.

$$zs \geq |s|^2. \tag{6.28.9}$$

As $|s_{k'}| \geq \eta$ for all k', clearly (6.28.9) implies that

$$zs \geq \eta^2 > 0. \tag{6.28.10}$$

On the other hand, letting $k' \to \infty$ in (6.28.6) yields

$$zs = 0, \tag{6.28.11}$$

contradicting (6.28.10). Thus our assumption that (x_k) fails to converge to the solution of (6.1.2) cannot be valid. The proof that the algorithm generates a sequence converging to the optimal solution is now complete.

CHAPTER VII

IMPLEMENTATION OF THE SUBGRADIENT PROJECTION ALGORITHM

In this chapter we propose methods for implementing algorithm 6.4 and describe computational experiences. The algorithm was coded in FORTRAN on a CDC 6500 computer and tested on several problems of the form described in section 6.1. We present here the results of those tests.

The program used a value of 10^{-5} for the parameter ϵ_0 in algorithm 6.4. For computational efficiency, we divided ϵ by 10 rather than by 2 in step 4 of the algorithm. Step 4 can be rewritten so that ϵ is replaced by ϵ/a for any fixed $a > 1$. The algorithm terminated when the euclidean length of the projection y_0 in step 1 was less than a specified figure, usually 10^{-10}.

The two major problems in implementation were the nearest-point projection subroutine, required in steps 1 and 2 of the algorithm, and the line search procedure used in step 5. We first describe the line search.

In step 5, it is required to determine $\alpha \in [0,\bar{\alpha}]$ such that $zs = 0$ for some $z \in \nabla f(x - \alpha s) + K_0(x - \alpha s)$. Using the notation established in section 6.3, the orthogonality condition becomes

$$\sum_{i \in J_0(x-\alpha s)} \beta_i g_i s + \nabla f(x - \alpha s)s = 0 \ , \qquad (7.0.1)$$

$$\beta_i \geq 0 \quad \text{for all} \quad i, \quad \sum_{i \in J_0(x-\alpha s)} \beta_i = 1 \ .$$

Rewriting (7.0.1) as

$$\sum_{i\in J_0(x-\alpha s)} \beta_i [g_i s + \nabla f(x-\alpha s)s] = 0 , \qquad (7.0.2)$$

we note that a solution $(\beta_i)_{i\in J_0(x-\alpha s)}$ exists iff for some $i,j \in J_0(x-\alpha s)$ either $g_i s + \nabla f(x-\alpha s)s = 0$ or $g_i s + \nabla f(x-\alpha s)s$ and $g_j s + \nabla f(x-\alpha s)s$ have opposite signs. We can summarize the characterization of α as follows.

<u>7.1 Lemma</u>. $\alpha \in [0,\bar{\alpha}]$ is the desired solution in step 5 of algorithm 6.4 if and only if either

(a) there exist $i,j \in J_0(x-\alpha s)$ such that

$$[g_i s + \nabla f(x-\alpha s)s][g_j s + \nabla f(x-\alpha s)s] \leq 0 \qquad (7.1.1)$$

or

(b) $\alpha = \bar{\alpha}$ and no triple (α,i,j) with $0 \leq \alpha \leq \bar{\alpha}$ and $i,j \in J_0(x-\alpha s)$ satisfies (7.1.1).

Our line search proceeds as follows. We begin with $\tilde{\alpha} = 0$ and locate the first value $\hat{\alpha} > \tilde{\alpha}$ at which $J_0(x-\hat{\alpha}s)\setminus J_0(x-\tilde{\alpha}s) \neq \emptyset$. We check whether there exist $i,j \in J_0(x-\hat{\alpha}s)$ satisfying (7.1.1). If not, we check whether there exists $i \in J_0(x-\hat{\alpha}s) \cap J_0(x-\tilde{\alpha}s)$ such that $g_i s + \nabla f(x-\tilde{\alpha}s)s$ and $g_i s + \nabla f(x-\hat{\alpha}s)s$ have opposite signs. If so, we locate the value of α between $\tilde{\alpha}$ and $\hat{\alpha}$ for which $g_i s + \nabla f(x-\alpha s) = 0$ and terminate the search. This value of α can be found by Newton's method when f is sufficiently smooth. If no such i exists and $\hat{\alpha} < \bar{\alpha}$, we replace $\tilde{\alpha}$ with $\hat{\alpha}$ and repeat the process.

<u>7.2 Line search procedure</u>. Begin with $x, s, \bar{\alpha}$ given.

<u>Step 0</u>: Set $\tilde{\alpha} = 0$ and consider all indices in $J_0(x)$ as untested.

<u>Step 1</u>: Choose an untested index $i \in J_0(x - \tilde{\alpha}s)$.

<u>Step 2</u>: Compute $\hat{\alpha} = \max\{\alpha \in [\tilde{\alpha}, \bar{\alpha}] : v_i(x - \alpha s) = v(x - \alpha s)\}$. If $\hat{\alpha} = \tilde{\alpha}$, go to step 1.

<u>Step 3</u>: For each $j \in J_0(x - \hat{\alpha}s)$ do the following:

<u>3a</u>: If

$$[g_i s + \nabla f(x - \hat{\alpha}s)s][g_j s + \nabla f(x - \hat{\alpha}s)s] \le 0 \ ,$$

terminate the search with $\alpha = \hat{\alpha}$;

<u>3b</u>: If $j \in J_0(x - \tilde{\alpha}s)$ and

$$[g_j s + \nabla f(x - \tilde{\alpha}s)s][g_j s + \nabla f(x - \hat{\alpha}s)s] < 0 \ ,$$

go to step 5.

<u>Step 4</u>: If $\hat{\alpha} < \bar{\alpha}$, replace $\tilde{\alpha}$ with $\hat{\alpha}$ and go to step 1, treating all indices as untested. Otherwise, terminate the search with $\alpha = \bar{\alpha}$.

<u>Step 5</u>: Locate the value $\alpha \in [\tilde{\alpha}, \hat{\alpha}]$ for which $g_j s + \nabla f(x - \alpha s)s = 0$ and terminate the search.

7.3 Projection procedure. The other major problem in implementing the algorithm is the calculation of the nearest point to the origin in a convex set. The convex set in question is $\nabla f(x) + K_\varepsilon(x) + C_\varepsilon(x)$ with $\varepsilon \geq 0$. For convenience, let $u_j = g_j + \nabla f(x)$, $j = 1,\ldots,r$. Since $\nabla f(x) + K_\varepsilon(x) = \text{conv}\,\{u_j : j \in J_\varepsilon(x)\}$, the projection problem can be posed as:

$$\left\{\begin{array}{l} \alpha \in \mathbb{R}_+^h, \quad \beta \in \mathbb{R}_+^k \\ \\ \displaystyle\sum_{\nu=1}^{k} \beta_\nu = 1 \\ \\ \displaystyle\frac{1}{2}\left|\sum_{\mu=1}^{h} \alpha_\mu a_{i_\mu} + \sum_{\nu=1}^{k} \beta_\nu u_{j_\nu}\right|^2 \quad (\min) \end{array}\right\} \tag{7.3.1}$$

where $I_\varepsilon(x) = \{i_1,\ldots,i_h\}$ and $J_\varepsilon(x) = \{j_1,\ldots,j_k\}$. The objective function is a quadratic form in α and β, and so the projection problem is simply a quadratic programming problem. Letting $\xi = (\alpha,\beta)$ and $H = (a_{i_1}\ldots a_{i_h}\; u_{j_1}\ldots u_{j_k})$, we can write the objective function as $1/2\;\xi\, H^tH\,\xi$, where (using superscript t to denote transposes)

$$H^tH = \begin{pmatrix} G^tG & G^tU \\ U^tG & U^tU \end{pmatrix},$$

$$G = (a_{i_1}\ldots a_{i_h}) \quad \text{and} \quad U = (u_{i_1}\ldots u_{i_k}).$$

H^tH is positive semidefinite but not positive definite, and problem (7.3.1) can in general have multiple optimal solutions. Despite this, there exists a unique nearest point $N[\nabla f(x) + K_\epsilon(x) + C_\epsilon(x)]$, since $\nabla f(x) + K_\epsilon(x) + C_\epsilon(x)$ is a convex set and the euclidean norm is a strictly convex norm.

Our approach is to locate a vector $\xi = (\alpha,\beta)$ satisfying the Kuhn-Tucker necessary and sufficient conditions for optimality in the quadratic programming problem (7.3.1). These conditions are:

$$\left\{\begin{array}{l} \xi,\ y \in \mathbb{R}_+^{h+k},\ w \in \mathbb{R} \\ e\xi = 1 \\ H^tH\xi - y + we = 0 \\ \quad \xi y = 0 \end{array}\right\} \tag{7.3.2}$$

where $e = (0,\ldots,0) \times (1,\ldots,1) \in \mathbb{R}^h \times \mathbb{R}^k$ and w is a sign-unrestricted scalar. We initially employed the first phase of the Dantzig Two-Phase linear programming procedure [2], with a restricted basis-entry rule to maintain the complementarity condition $\xi y = 0$. This method can encounter nonoptimal tableaus in which further pivoting is blocked by the complementarity restriction, even when H^tH is positive definite. A simple modification, however, solves this. Since all equality constraints, with one exception, are homogeneous, we can construct an initial Phase I - feasible solution by setting any one of the β_ν equal to unity and completing the basis with artificial variables (which may require multiplication of some of the homogeneous equations by -1). From this point onward, the modified Phase I procedure reduces to an algorithm of Wolfe [18]. Our elimination of all linear terms in the objective function, by means of the substitution $u_j = \nabla f(x) + g_j$, causes our problem to satisfy conditions under which Wolfe's proof of convergence of the algorithm applies with only minor alterations, even through H^tH is only semidefinite. Wolfe assumes that the system of equality constraints is nondegenerate, i.e. that the constraint equations are linearly independent and that no basic feasible solutions with a basic variable equal to zero occur. The problem before us clearly satisfies the assumption of independent constraints, while Wolfe's proof can be modified to obviate the assumption that no basic variable vanishes.

7.4 Computational results: the PIES counterexample.

Algorithm 6.4 was first tested on example 4.16, although convergence in this case has not been proved due to the lack of a potential for $-p$. When tested with the parameter $T = 3$ and a convergence criterion of less than 10^{-10} error in the euclidean norm of q, the algorithm converged in one to three iterations from a variety of starts. The general pattern was one step from the starting point to an edge of Q^* containing the equilibrium vector (4,2), and then another step to the solution. Where a third step was required, it appeared to be a very short step, perhaps correcting rounding errors. The only difficulties occurred when the initial point was too close to the origin, where we believe the size of the components of $-p(q)$ is so much greater than that of the generators a_i and g_j as to cause severe roundoff errors. The algorithm was also tested with parameter values $T = 7$ and $T = 10$, with similar convergence results.

7.5 Computational results: the PIES-VAR counterexample.

We next tested algorithm 6.4 on example 5.7. As with the previous example, the "gradient" $-p$ does not actually have a potential, and so convergence of the algorithm is not guaranteed. Example 5.7 was attempted with three different starts. When started at $q = (1,1)$, the algorithm reached the equilibrium (2,2), to within 10^{-8} euclidean norm, in one iteration. From a start at (4,2), convergence was oscillatory: the error was approximately 2×10^{-7} after 40 iterations and 2.2×10^{-10} after 60 iterations. Started from (1,2), however, the algorithm appeared to fall into a four-iteration cycle. Unlike the previous attempt, in which the norm of the error decreased monotonically, the norm of the error in the four step cycle fluctuated between .511 and .961.

7.6 Computational results: Wolfe's example.

Wolfe [19] considers an example in two variables with f identically zero:

$$v(x) = \max \{v_1, v_2, v_3\} \quad (\min)$$

where $v_1(x) = -x_1$, $v_2(x) = x_1 + x_2$ and $v_3(x) = x_1 - 2x_2$. The level sets of v are triangles nested about the origin. The example was solved under the added constraints $-10 \leq x_1, x_2 \leq 10$. The global minimizer $(0,0)$, interior to the constraint region, was reached in at most two iterations from a variety of starts. Similar results were obtained using constraints which placed $(0,0)$ on the boundary of the feasible region. The method also reached the constrained optimum in at most two iterations using constraints which put $(0,0)$ exterior to the feasible region.

We note that Wolfe's example (and Powell's example, which follows) can be posed as linear programs, but with no gain in speed of convergence.

7.7 Computational results: Powell's example.

Wolfe [19] reports the following example due to Powell, on which the conjugate gradient method converges only linearly:

$$v(x) = \max \{v_j(x) : j = 0,\dots,4\} \quad (\min)$$

where $v_j(x) = x_1 \cos(2\pi j/5) + x_2 \sin(2\pi j/5)$. Linear convergence of the conjugate gradient method is observed when started at any point of the form $(\rho \cos(\pi j/5), \rho \sin(\pi j/5))$. The contours of v are regular pentagons centered at the minimizer $(0,0)$. Algorithm 6.4 converged to the solution in two iterations from any feasible starting point using constraints which placed the origin in the interior of the constraint region. Similar results occurred when constraints were used which placed the origin on the boundary of the feasible region. Using the constraints $x_1 \leq 0$, $x_2 \leq 0$, $x_1 + x_2 \leq -1$, for which the origin is infeasible, the algorithm typically took approximately five to ten iterations to reach the constrained minimum.

7.8 Computational results: a larger example.

The algorithm was tested on an example having nine variables. The feasible region was a hypercube with the optimal solution at one vertex. The smooth part f of the objective was a strictly convex quadratic, and there were three affine functions v_i. The algorithm converged to within 10^{-13} of the solution (which was a unit vector), in at most ten iterations, from a variety of starts.

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