PROGRESS ON THE 1/3 – 2/3 CONJECTURE

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ABSTRACT

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Let $(P, \leq)$ be a finite partially ordered set, also called a poset, and let n denote the cardinality of P. Fix a natural labeling on P so that the elements of P correspond to $[n] = \{1, 2, \ldots, n\}$. A linear extension is an order-preserving total order $x_1 \prec x_2 \prec \cdots \prec x_n$ on the elements of P, and more compactly, we can view this as the permutation $x_1 x_2 \cdots x_n$ in one-line notation. For distinct elements $x, y \in P$, we define $\mathbb{P}(x \prec y)$ to be the proportion of linear extensions of P in which x comes before y. For $0 \leq \alpha \leq \frac{1}{2}$, we say (x, y) is an α-balanced pair if $\alpha \leq \mathbb{P}(x \prec y) \leq 1 - \alpha$. The $1/3 - 2/3$ Conjecture states that every finite partially ordered set that is not a chain has a 1/3-balanced pair. This dissertation focuses on showing the conjecture is true for certain types of partially ordered sets. We begin by discussing a special case, namely when a partial order is 1/2-balanced. For example, this happens when the poset has an automorphism with a cycle of length 2. We spend the remainder of the text proving the conjecture is true for some lattices, including Boolean, set partition, and subspace lattices; partial orders that arise from a Young diagram; and some partial orders of dimension 2.

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Chapter 1

Introduction

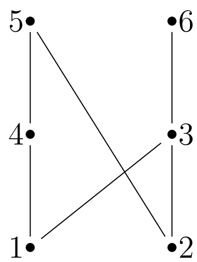
We begin with some background before stating the conjecture. Section 1.1 provides a more complete introduction to partially ordered sets, or posets. Let $(P, \leq)$ be a partially ordered set, and let n be the cardinality of P. Fix a natural labeling on P so that the elements of P correspond to $[n] = \{1, 2, \ldots, n\}$. A *linear extension* is a total order $x_1 \prec x_2 \prec \cdots \prec x_n$ on the elements of P such that $x_i \prec x_j$ if $x_i <_P x_j$; more compactly, we can view this as the permutation $x_1 x_2 \cdots x_n$ in one-line notation. For distinct elements $x, y \in P$, we define $\mathbb{P}(x \prec y)$ to be the proportion of linear extensions of P in which x comes before y. For $0 \leq \alpha \leq \frac{1}{2}$, we say (x, y) is an *α-balanced pair* if

$$\alpha \leq \mathbb{P}(x \prec y) \leq 1 - \alpha,$$

and that P is *α-balanced* if it has some α-balanced pair. Notice that if (x, y) is α-balanced, then (y, x) is α-balanced as well.

Conjecture 1.1 (The $1/3 - 2/3$ Conjecture)**.** *Every finite partially ordered set that is not a chain has a* $1/3$*-balanced pair.*

The conjecture was first proposed by Kislitsyn [Kis68] in 1968, although many resources attribute the conjecture to having been formulated by Fredman [Fre76] in 1976 and later independently by Linial [Lin84] in 1984.



$$\begin{bmatrix} 0 & \mathbf{9} & 15 & 15 & 15 & 15 \\ \mathbf{6} & 0 & 15 & 12 & 15 & 15 \\ 0 & 0 & 0 & \mathbf{6} & 12 & 15 \\ 0 & 3 & \mathbf{9} & 0 & 15 & 13 \\ 0 & 0 & 3 & 0 & 0 & \mathbf{8} \\ 0 & 0 & 0 & 2 & \mathbf{7} & 0 \end{bmatrix}$$

Figure 1.1: A poset P with 6 elements and a matrix counting its linear extensions

We can see, for instance, that the conjecture holds for the poset P depicted in Figure 1.1. This poset has 15 linear extensions, which are

$$\begin{aligned} \{&123456, \quad 123465, \quad 123645, \quad 124356, \quad 124365, \\ &124536, \quad 142356, \quad 142365, \quad 142536, \quad 213456, \\ &213465, \quad 213645, \quad 214356, \quad 214365, \quad 214536\}. \end{aligned}$$

If we want to find the 1/3-balanced pair(s) of P, we must examine the number of linear extensions with $x \prec y$ for each incomparable $x, y \in P$. The matrix in Figure 1.1 has for each entry (i, j) the number of linear extensions with i before j. For instance, P has 9 linear extensions with $1 \prec 2$ and 6 with $2 \prec 1$, and so the entry $(1, 2)$ in the matrix is 9 while the $(2, 1)$ entry is 6. Notice that if $i < j$ in the poset, then the (i, j)th entry in the matrix is the number of linear extensions of P. The bold entries have values that are between 1/3 and 2/3 of the total number of linear extensions, which gives us the 1/3-balanced pairs, namely $(1, 2), (2, 1), (3, 4), (4, 3), (5, 6)$, and $(6, 5)$. We define the *balance constant* of P, denoted $\delta(P)$, to be

$$\delta(P) = \max_{x,y \in P} \min\{\mathbb{P}(x \prec y), \mathbb{P}(y \prec x)\}$$

For any poset P not a chain, it must be that $0 < \delta(P) \leq 1/2$. In the example in Figure 1.1, P has a balance constant of $\frac{7}{15} \approx 0.4667$.

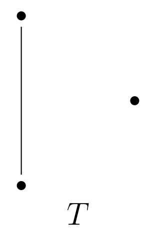


Figure 1.2: The poset T with three elements and one relation.

So far, we have presented two equivalent ideas. That is, it is equivalent to say P has a 1/3-balanced pair and P has a balance constant $\delta(P) \geq 1/3$. We will use these two phrases interchangeably, as does the literature. In [BFT95], Conjecture 1.1 is described as "one of the most intriguing problems in combinatorial theory". If the conjecture is true, the bounds are the best possible, as seen by the poset in Figure 1.2.

1.1 Poset Basics

This section will serve to set up notation and definitions relevant to partially ordered sets later in this text. For a more complete background, see [Sta11].

Some standard notation we use for sets includes:

- $\mathbb{N}$ is the set of nonnegative integers.
- The set $\{1, 2, \ldots, n\}$ will be denoted by $[n]$.
- $\mathfrak{S}_n$ is the set of permutations on n elements.
- The *symmetric difference* of sets S and T is $S\Delta T = (S \cup T) - (S \cap T)$.

Definition 1.2. *A* partially ordered set*, or* poset*, is a set of elements P and binary operation $\leq$ that obey the following properties for all $x, y, z \in P$:*

- *$x \leq x$ (reflexivity),*

- *If $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry),*
- *If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).*

While this definition allows P to be an infinite set, we will assume here that all posets are finite. Throughout the text, we will denote a poset by $(P, \leq)$ or P if the binary relation is clear from context. It may occasionally be useful to distinguish the binary operation between posets, in which case we specify $\leq$ as $\leq_P$. By $x < y$, we mean that $x \leq y$ and $x \neq y$. The *dual* of P, denoted P^*, has the same elements as P and the partial order $x \leq_{P*} y$ if $y \leq_P x$.

For $x, y \in P$, we say x *covers* y, denoted $x \lessdot y$, if $x < y$ and no element $z \in P$ satisfies $x < z < y$. The elements x and y are *comparable* if either $x \leq y$ or $y \leq x$; otherwise, the pair is *incomparable*. An element $x \in P$ is *minimal* if for all y comparable to x, $y \geq x$. Similarly, $z \in P$ is *maximal* if for all y comparable to z, $y \leq z$. If a poset has only one minimal or maximal element, we call it $\hat{0}$ or $\hat{1}$, respectively. For a poset with a $\hat{0}$, the elements that cover $\hat{0}$ are called *atoms*. A poset is *graded* if it has a rank function $r : P \to \mathbb{N}$ such that r respects the partial order and for every $x, y \in P$ with $x \lessdot y$, then $r(y) = r(x) + 1$.

We can create a graphical representation of posets by drawing its cover relations. The *Hasse diagram* of a finite poset P is a graph whose vertices are elements of P and edges are cover relations, such that whenever $x \lessdot y$, x is drawn below y. Figure 2.2 shows two examples of Hasse diagrams of two different posets. A subset of elements $S \subseteq P$ is called a *chain* if for all $u, v \in S$, either $u \leq v$ or $v \leq u$. The size of the largest chain in P is the *height* of P. Similarly, a subset of elements such that any two elements are incomparable in P is called an *antichain* and the size of the largest antichain of P is the *width* of P.

In later sections, we will refer back to the concept of order ideals as it will be useful in a technique to show a poset is 1/3-balanced.

Definition 1.3. *In a poset P, a* lower order ideal *is a subset $I \subset P$ such that if $x \in I$ and $y \leq x$, then $y \in I$. If there is some $x \in P$ which is the unique maximal element of the lower order ideal, we say the ideal is* principal, *and denote it L_x. We similarly define an* upper order ideal, *and denote the* principal upper order ideal generated by x *by U_x.*

Let P and Q be posets. A map $\phi : P \to Q$ is *order preserving* if

$$x \leq_P y \quad \Rightarrow \quad \phi(x) \leq_Q \phi(y).$$

Similarly, a map $\sigma : P \to Q$ is *order reversing* if

$$x \leq_P y \quad \Rightarrow \quad \sigma(y) \leq_Q \sigma(x).$$

An *isomorphism* is an order-preserving bijection whose inverse is also order preserving. An *automorphism* is an isomorphism from P to itself, and an *anti-automorphism* is an order-reversing bijection from P to itself. We say a poset P is a *labeled poset* if there is a bijection $\omega : P \to [n]$. If we want to emphasize the labeling for P, we refer to the poset as (P, ω). A labeling is *natural* if it is order preserving. We can fix a natural labeling on P so that the elements of P correspond to $[n] = \{1, 2, \ldots, n\}$. A *linear extension* is a total order $x_1 \prec x_2 \prec \cdots \prec x_n$ on the elements of P such that $x_i \prec x_j$ if $x_i <_P x_j$. We can view a linear extension as a permutation $x_1 x_2 \cdots x_n$ in one-line notation, which means the permutation maps i to x_i.

Let $E(P)$ be the set of linear extensions of P and $e(P)$ be the cardinality of $E(P)$. If $(P, \leq)$ is a poset and $x, y \in P$, let $P + xy$ denote the poset $(P, \leq')$, where $\leq'$ is the transitive closure of $\leq$ extended by the relation $x < y$. Therefore, $E(P + xy)$ is the set of linear

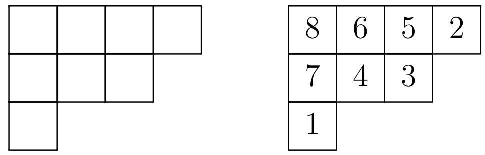


Figure 1.3: A diagram of shape $(4,3,1)$ and a standard Young tableau of the same shape.

extensions of P that have x before y and $e(P+xy)$ is the size of $E(P+xy)$. Notice that $e(P+xy)+e(P+yx)=e(P)$.

We can build new posets by combining two or more posets. One way to do this is through the linear sum. The *linear sum* of posets P and Q, denoted $P+Q$, is obtained by adding the relations $a<b$ for every $a \in P$ and $b \in Q$. Another way to build a new poset is by taking the product. The *product* of P and Q, denoted $P \times Q$, is the set $\{(a,b) : a \in P, b \in Q\}$ with the partial order given by $(a,b) \leq (c,d)$ if $a \leq_P c$ and $b \leq_Q d$.

We end with a discussion about integer partitions and Young diagrams, which we will see again in Sections 2.3 and 2.4. An *integer partition* of n, or *partition*, is a weakly decreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ that sums to n. In this case, we use the notation $\lambda \vdash n$.

Definition 1.4. *Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n. The* Young diagram *corresponding to λ consists of k left-justified rows of cells where the ith row from the top has λ_i cells.*

For an example of a Young diagram, see Figure 1.3. We will always let n be the number of cells in a diagram, and since we often make no distinction between a partition and its Young diagram, the same notation for both suffices. We can also abbreviate by using exponential notation; we write $\lambda = (\lambda_1^{m_1}, \ldots, \lambda_k^{m_k})$ when the first m_1 rows of the diagram have length λ_1, etc. The cells of a Young diagram can be filled with integers from 1 to n using each number exactly once and increasing in the rows and columns to form a *standard Young tableau*. An example of this can be seen in Figure 1.3.

While we have covered some basic background information in this section, we later include definitions relevant to particular sections. In Section 2.2, we see how automorphisms of posets provide insight into Conjecture 1.1. In Section 2.3, we prove the conjecture for many types of posets that are lattices, including rectangular Young diagrams. In Section 2.4, we consider Young diagrams which are not lattices. In Section 2.5, we discuss permutations and how they relate to posets of dimension 2. In Section 2.6, we discuss posets whose balance constants are the smallest known values greater than 1/3.

Chapter 2

The $1/3 - 2/3$ Conjecture

2.1 History of the Conjecture

Given the age of Conjecture 1.1, it should come as no surprise that many partial results have been generated by many mathematicians. One of the first results appeared in 1984 by Linial [Lin84], namely that the $1/3 - 2/3$ Conjecture holds for posets of width 2.

Theorem 2.1 ([Lin84]). *Let $(P, \leq)$ be a poset of width exactly 2. Then, $\delta(P) \geq 1/3$.*

Aigner [Aig85] showed further that posets of width exactly 2 fit into one of two categories: either the poset is a linear sum of copies of the singleton poset and T (the poset from Figure 1.2); or the poset has an α-balanced pair with $1/3 < \alpha < 2/3$. In fact, the only known posets that have a balance constant of $1/3$ are the linear sums of singletons and T. The poset of width 2 in Figure 2.1 has a balance constant of $\frac{16}{45} \approx 0.3556$, and until recently, it was the poset with the smallest balance constant greater than $1/3$ [Bri99]. We found posets of width 2 that have smaller balance constants, and we describe these results in Section 2.6.

The smallest known balance constant for a poset with width strictly greater than 2 is $\frac{14}{39} \approx 0.3590$, as described in [Bri99]. It belongs to the poset with 7 elements in Figure 2.1. We used a variation of the code in the Appendix to search all posets with up to 9 elements and found no posets with balance constant smaller that $\frac{14}{39}$ and width greater than 2.

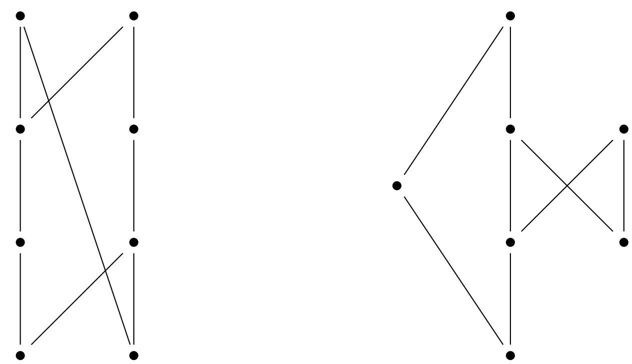


Figure 2.1: Two posets with small balance constants.

There are many types of posets for which the conjecture has already been proven. This includes posets of up to 11 elements [Pec06], posets with height 2 [TGF92], semiorders [Bri89], posets with each element incomparable to at most 6 others [Pec08], N-free posets [Zag12], and posets whose Hasse diagram is a tree [Zag16]. While the proof of the 1/3 bound for a general poset remains elusive, in 1984 Kahn and Saks [KS84] proved that for any poset P, there is some pair $x, y \in P$ such that $\frac{3}{11} < \mathbb{P}(x \prec y) < \frac{8}{11}$. In 1995, Brightwell, Felsner, and Trotter [BFT95] improved the bound to be $\frac{5-\sqrt{5}}{10} \leq \mathbb{P}(x \prec y) \leq \frac{5+\sqrt{5}}{10}$. The interested reader can refer to Brightwell's 1999 survey [Bri99] on the conjecture for more information.

We now wish to recall a result of Zaguia which will be useful to us in the later sections. The following definitions describe concepts introduced in [Zag16], although here we refer to them with different names.

Definition 2.2. *Let P be a poset and x and y be elements of P.*

(a) We call the pair (x, y) twin elements *if $L_x = L_y$ and $U_x = U_y$.*

(b) We call the pair (x, y) almost twin elements *if the following two conditions hold in P or in the dual of P:*

  (i) $L_x = L_y$, and

(ii) $U_x \setminus U_y$ *and* $U_y \setminus U_x$ *are chains (possibly empty).*

We will see that a poset with twin elements is 1/2-balanced in Section 2.2. In [Zag16], Zaguia proves that a poset with an almost twin pair is 1/3-balanced. In fact, he proves something stronger by generalizing the notion of an almost twin pair. He uses the term *good pair* to describe these elements, but we will call them *asymmetric*.

Definition 2.3. *Let* P *be a poset. A pair* (x, y) *of elements of* P *is* asymmetric *if the following two conditions hold in* P *or in the dual of* P*:*

(i) $L_x \subseteq L_y$ *and* $U_y \setminus U_x$ *is a chain (possibly empty), and*

(ii) $\mathbb{P}(x \prec y) \leq \frac{1}{2}$.

Theorem 2.4 ([Zag16])**.** *A finite poset that has an asymmetric pair of elements or an almost twin pair of elements is* 1/3*-balanced.*

In fact, every almost twin pair of elements is also asymmetric, and Zaguia's proof focuses on asymmetric pairs. Since we use almost twin elements later, we include them in Theorem 2.4 for emphasis. It is important to keep in mind that many of the known results for the $1/3-2/3$ Conjecture are existence proofs and do not compute $\mathbb{P}(x \prec y)$ exactly for any pair (x, y) in the given poset. This is particularly true in the case of asymmetric pairs and almost twin pairs of elements. In fact, one can find many examples of posets with asymmetric or almost twin pairs of elements that do not coincide with any 1/3-balanced pairs of the poset.

We can make a few quick observations about the structure of a poset and how it relates to the proportions of its linear extensions. A poset with a $\hat{0}$ has $\mathbb{P}(\hat{0} \prec x) = 1$ for all $x > \hat{0}$, and so the proportion of linear extensions remains the same in $P - \{\hat{0}\}$. For this reason, we usually insist that posets have at least 2 minimal elements and similarly, at least 2 maximal elements. Also, if we consider the linear sum $P + Q$ of posets P and Q, we can easily see that $\delta(P + Q) = \max\{\delta(P), \delta(Q)\}$. For this reason, we need not consider posets that are linear sums.

2.2 Automorphisms of Posets

We first provide a proof of an important observation about the linear extensions of a poset with a nontrivial automorphism.

Proposition 2.5. *A poset P with a nontrivial automorphism α has a nontrivial bijection on its linear extensions. Further, $\mathbb{P}(x \prec y) = \mathbb{P}(\alpha(x) \prec \alpha(y))$ for all $x, y \in P$.*

Proof. Let $\alpha : P \to P$ be a nontrivial automorphism. This means that for $x, y \in P$, $x \leq_P y$ if and only if $\alpha(x) \leq_P \alpha(y)$. Now, let $\pi = a_1 a_2 \cdots a_n$ be a linear extension of P, and by the definition of linear extensions, we know that if $a_i \leq_P a_j$, then $i \leq j$. As α is an automorphism, then we also have that if $\alpha(a_i) \leq_P \alpha(a_j)$, then $i \leq j$. This gives us, by definition, that $\alpha(\pi) = \alpha(a_1)\alpha(a_2) \cdots \alpha(a_n)$ is a linear extension of P. Therefore, α induces a bijection on the linear extensions of P. Further, this bijection is nontrivial as α is nontrivial.

We can also observe that the linear extensions with x before y map bijectively via α to the linear extensions with $\alpha(x)$ before $\alpha(y)$. Hence, $\mathbb{P}(x \prec y) = \mathbb{P}(\alpha(x) \prec \alpha(y))$, as desired. □

In [GHP87], Ganter, Hafner, and Poguntke describe a proof of the fact that a poset with a non-trivial automorphism will satisfy the $1/3 - 2/3$ Conjecture. We present their short argument here to illustrate a popular yet insightful approach to the conjecture using proof by contradiction.

Theorem 2.6 ([GHP87]). *If a poset P has a non-trivial automorphism, then P is $1/3$-balanced.*

Proof. Let $\alpha : P \to P$ be a nontrivial automorphism of a poset P. Assume there are no $x, y \in P$ with $1/3 \leq \mathbb{P}(x \prec y) \leq 2/3$. We can then create a new relation $\ll$ on P by defining $u \ll v$ if $\mathbb{P}(u \prec v) > 2/3$.

Observe that $\ll$ is transitive. Indeed, for $u, v, w \in P$, assume $u \ll v$ and $v \ll w$. This means $\mathbb{P}(u \prec v) > 2/3$ and $\mathbb{P}(v \prec w) > 2/3$. We can see that since

$$\mathbb{P}(u \prec v \prec w) + \mathbb{P}(v \prec u \prec w) + \mathbb{P}(v \prec w \prec u) = \mathbb{P}(v \prec w) > 2/3$$

and

$$\mathbb{P}(v \prec u \prec w) + \mathbb{P}(v \prec w \prec u) \leq \mathbb{P}(v \prec u) < 1/3,$$

it must be that $\mathbb{P}(u \prec v \prec w) > 1/3$. Therefore, $\mathbb{P}(u \prec w) > 1/3$, and since we assumed there are no $1/3$-balanced pairs in P, then $\mathbb{P}(u \prec w) > 2/3$. So, $u \ll w$, and hence $\ll$ is a transitive binary relation.

As $\ll$ is transitive, it gives a linear order on P. Now, if $u \ll v$, then $P(u \prec v) > 2/3$, and so by Proposition 2.5, $P(\alpha(u) \prec \alpha(v)) > 2/3$. This means that $\alpha(u) \ll \alpha(v)$, and hence α respects $\ll$. But as $\ll$ is a linear order on P and α respects the linear order, it must be that α is the identity. This contradicts our assumption. □

Next, we present a result about when a poset has a balance constant of $1/2$.

Proposition 2.7. *If a poset P has an automorphism with a cycle of length 2, then P is $1/2$-balanced. Further, if x and y are the elements in the cycle of length 2, then (x, y) is a $1/2$-balanced pair.*

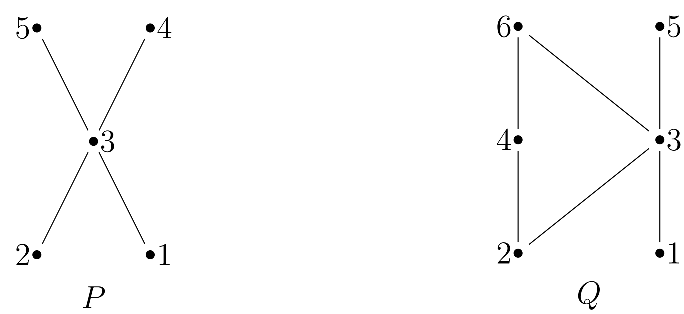


Figure 2.2: The poset P has an automorphism with cycle length 2 and balance constant 1/2, while Q has no nontrivial automorphisms and balance constant 1/2.

Proof. Let $\alpha : P \to P$ be an automorphism and $x, y \in P$ be such that $\alpha(x) = y$ and $\alpha(y) = x$. Then we can see that

$$e(P + xy) = e(P + \alpha(x)\alpha(y)) = e(P + yx),$$

where the first equality comes from Proposition 2.5. So we have

$$\begin{aligned} e(P) &= e(P + xy) + e(P + yx) \\ &= 2e(P + xy), \end{aligned}$$

and so $e(P + xy) = e(P)/2$. Hence, (x, y) is a 1/2-balanced pair, as desired. □

An example of a poset with an automorphism of cycle length 2 is given in Figure 2.2. Poset P has a balance constant of 1/2. A counterexample to the converse of Proposition 2.7 is also provided in Figure 2.2. Poset Q has a balance constant of 1/2, as it has 12 linear extensions and $e(P + 34) = 6$. However, we can see by inspection it has no nontrivial automorphisms.

The following is a corollary to Proposition 2.7.

Corollary 2.8. *A poset P with a twin pair of elements is* $1/2$*-balanced.*

Proof. Let P be a poset with x and y a twin pair of elements. We can see that P has a non-trivial automorphism which fixes all elements except for x and y and maps x to y and y to x. So, this poset has an automorphism is a cycle of length 2. By Proposition 2.7, then P is 1/2-balanced and (x, y) is a 1/2-balanced pair. □

While the above results depend on an automorphism of a poset, it is natural to ask if we can obtain results from other types of maps. Next, we consider an anti-automorphism σ on a poset. Observe that the linear extensions with x before y in P map bijectively via σ to the linear extensions with $\sigma(y)$ before $\sigma(x)$. Hence, $\mathbb{P}(x \prec y) = \mathbb{P}(\sigma(y) \prec \sigma(x))$, as desired. Also observe that any even number of iterations of σ give an automorphism of P. Hence, if σ^2 is not the identity, then we know that P has a non-trivial automorphism, and by Theorem 2.6, P is 1/3-balanced. We state this corollary to Theorem 2.6 here for completeness.

Corollary 2.9. *If σ is an anti-automorphism on P and σ^2 is a non-trivial automorphism, then P is* $1/3$*-balanced.*

We can also ask when an anti-automorphism guarantees a poset to be 1/2-balanced. Once such case is described in Proposition 2.10.

Proposition 2.10. *Let $\sigma : P \to P$ be an anti-automorphism. If σ has* 2 *fixed points, then P is* $1/2$*-balanced.*

Proof. Let P be a poset and $\sigma : P \to P$ be an anti-automorphism with fixed points x and y. Then, $\sigma(x) = x$ and $\sigma(y) = y$. Since

$$\mathbb{P}(x \prec y) = \mathbb{P}(\sigma(y) \prec \sigma(x)) = \mathbb{P}(y \prec x),$$

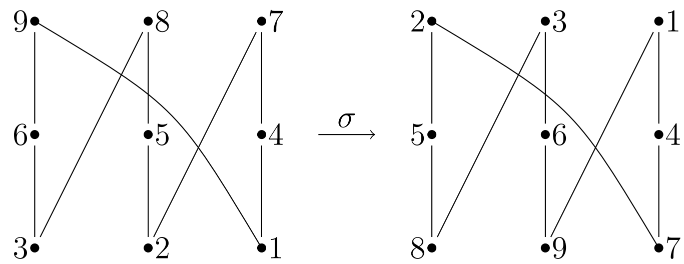


Figure 2.3: P and anti-automorphism σ with 1 fixed point.

and $\mathbb{P}(x \prec y) + \mathbb{P}(y \prec x) = 1$, we can see that $\mathbb{P}(x \prec y) = 1/2$. Hence, (x, y) is a 1/2-balanced pair in P, and we are done. $\square$

We cannot weaken the assumption in Proposition 2.10, since a unique fixed point in an anti-automorphism is not enough to guarantee that the poset is 1/2-balanced. For an example, consider the poset P and anti-automorphism σ in Figure 2.3 where for $x \in P$ we place $\sigma(x)$ on the right in the same position as x on the left. Any nontrivial anti-automorphism of P, including σ shown here, will have exactly 1 fixed point, and computer calculations give us that $\delta(P) = \frac{711}{1431} \neq \frac{1}{2}$.

We can also see that the converse of Proposition 2.10 is not true, as evidenced by the counterexample in Figure 2.2. Any anti-automorphism of the poset P will have exactly 1 fixed point, and yet it is 1/2-balanced.

2.3 Lattices

One commonly studied type of poset is a lattice. Here, we discuss the general definition of a lattice before considering the conjecture for specific lattices in this section.

Definition 2.11. *Let $S \subseteq P$ be a nonempty set of elements. A* lower bound *of S is an element z such that $z \leq u$ for all $u \in S$. The* greatest lower bound*, or* meet*, of S is a lower* *bound w of S such that if z is another lower bound of S, then $z \leq w$. If $S = \{x, y\}$, then the meet, if it exists, is denoted by $x \wedge y$.*

Similarly, an upper bound *of S is an element z such that $z \geq u$ for all $u \in S$. The* least upper bound*, or* join*, of S is an upper bound w of S such that if z is another upper bound of S, then $z \geq w$. If $S = \{x, y\}$, then the join, if it exists, is denoted by $x \vee y$.*

Note that in general, a set $S \subseteq P$ could have zero, one, or many lower bounds or upper bounds. However, if a meet or join exists, it must be unique. We call a poset a *lattice* if every nonempty subset $S \subseteq P$ has a meet and a join. One type of lattice we will consider later is a distributive lattice.

Definition 2.12. *A* distributive lattice *P is a lattice that obeys the following two equivalent properties for all $a, b, c \in P$:*

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \text{ and } a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

2.3.1 Boolean Lattices

Consider the set of all subsets of $[n]$, and define an order by $S \leq T$ when $S \subseteq T$ for $S, T \subseteq [n]$. This is a lattice as it is closed under joins and meets, specifically, $S \vee T = S \cup T$ and $S \wedge T = S \cap T$. We call this poset the *Boolean lattice* of size n, denoted B_n. This poset has the empty set as its $\hat{0}$ and $[n]$ is its $\hat{1}$. An example when $n = 3$ is given in Figure 2.4.

In the case that $n = 1$, the poset B_1 is a chain, and so the $1/3 - 2/3$ Conjecture only applies to B_n when $n \geq 2$. We present the following as a corollary to Proposition 2.7.

Corollary 2.13. *For all $n \geq 2$, the Boolean lattice B_n has an automorphism with a cycle of length 2, and so the Boolean lattice is 1/2-balanced.*

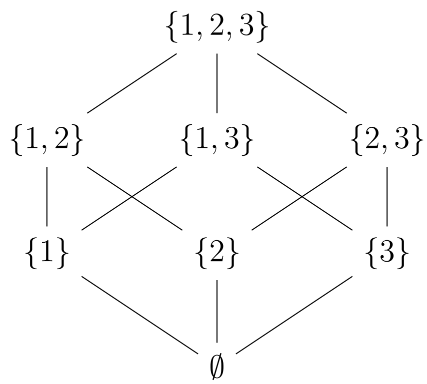


Figure 2.4: The Boolean lattice B_3

Proof. Let $n \geq 2$. We will first describe a map from B_n to itself that is a poset isomorphism. For $S \subseteq [n]$, consider $\phi : B_n \to B_n$ defined by

$$\phi(S) = \begin{cases} S\Delta\{1,2\}, & \text{if } S \cap \{1,2\} = \{1\} \text{ or } S \cap \{1,2\} = \{2\} \\ S, & \text{otherwise.} \end{cases}$$

We can easily see that ϕ is a poset automorphism. Let $A = \{1\}$ and $B = \{2\}$ in B_n. Then, $\phi(A) = B$ and $\phi(B) = A$. Hence, by Proposition 2.7, B_n has a 1/2-balanced pair. □

2.3.2 Set Partition Lattices

Consider the set Π_n of all partitions of $[n]$. A *partition* is composed of pairwise disjoint, non-empty subsets $B_1, \ldots, B_k$ whose union is $[n]$, and we write $B_1/\cdots/B_k \vdash [n]$. Each B_i is called a *block* of the partition. For $\pi, \tau \in \Pi_n$, we say π is a *refinement* of τ if every block of π is contained in a block of τ. This produces a partial order on Π_n given by $\pi \leq \tau$ when π is a refinement of τ. This is a lattice as it is closed under joins and meets, specifically $\pi \vee \tau$ is the minimal partition that refines both π and τ and $\pi \wedge \tau$ is the maximal partition that is a refinement of both π and τ. Hence, Π_n is called the *set partition lattice*. For brevity, we

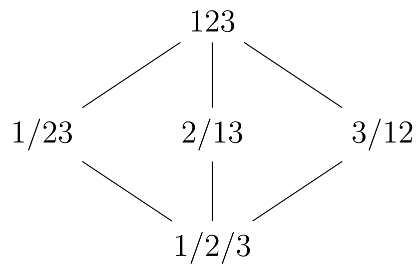


Figure 2.5: The set partition lattice Π_3

will write 12/3/4 for the set partition $\{\{1, 2\}, \{3\}, \{4\}\}$ and similarly for other partitions. An example of Π_3 can be seen in Figure 2.5.

For $n = 1, 2$, Π_n is a chain, and so the $1/3 - 2/3$ Conjecture only applies to Π_n when $n \geq 3$. We present the following as a corollary to Proposition 2.7.

Corollary 2.14. *For $n \geq 3$, the set partition lattice Π_n has an automorphism with a cycle of length 2, and so the set partition lattice is 1/2-balanced.*

Proof. Let $n > 2$. We consider the map that sends a partition π to the partition π', where π' has the same blocks as π with the elements 1 and 2 interchanged. This is an automorphism of the lattice. Indeed, it is a bijection because it is an involution and swapping 1 and 2 preserves ordering by refinement. To see that this automorphism has a 2-cycle, notice that the lattice contains partitions $\pi_1 = 13/2/4/\cdots/n$ and $\pi_2 = 1/23/4/\cdots/n$ as $n \geq 3$. Under the automorphism described above, π_1 and π_2 form a 2-cycle. Hence, by Proposition 2.7, the set partition lattice on n elements is 1/2-balanced when $n \geq 3$. □

2.3.3 Subspace Lattices

Let $q = p^m$ be some power of a prime p, and consider the finite vector space $V = \mathbb{F}_q^n$, which is the set of n-tuples of elements from the finite field $\mathbb{F}_q$. The *subspace lattice* $L_n(q)$ consists of the set of all subspaces of $\mathbb{F}_q^n$ ordered by inclusion. We can see that for two subspaces

$W, W' \subseteq \mathbb{F}_q^n$, their meet $W \wedge W'$ is $W \cap W'$ and their join $W \vee W'$ is $W + W'$. As $L_n(q)$ is closed under meets and joins, $L_n(q)$ is a lattice. Further, we can note that $L_n(q)$ is ranked by the dimension of the subspace. If $U \subseteq \mathbb{F}_q^n$ is a subspace spanned by $\{v_1, \ldots, v_k\}$, then we write $U = \langle v_1, \ldots, v_k \rangle$. Let e_i be the standard basis vector of $\mathbb{F}_q^n$ that has zeros everywhere except for a 1 in the ith position. We present the following as a corollary to Proposition 2.7.

Corollary 2.15. *For $n \geq 2$, the subspace lattice $L_n(q)$ has an automorphism with a cycle of length 2, and so the subspace lattice is 1/2-balanced.*

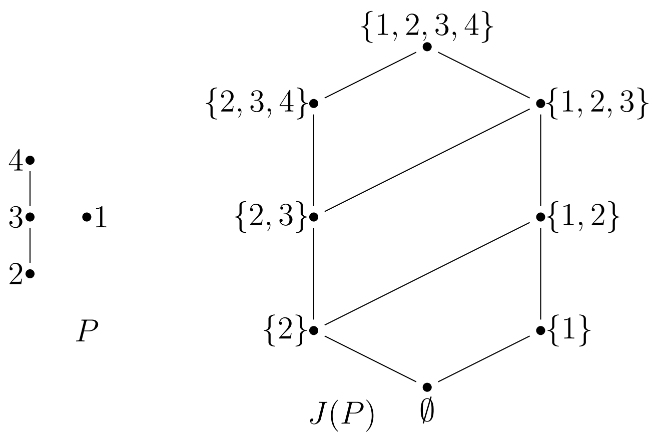
Proof. Let $B = \{e_1, \ldots, e_n\}$ be the standard basis of $L_n(q)$. Since $n \geq 2$, then there are at least 2 distinct elements in B, and $\langle e_1 \rangle$ and $\langle e_2 \rangle$ are two 1-dimensional subspaces of $\mathbb{F}_q^n$. Consider the linear transformation on $\mathbb{F}_q^n$ defined by the $n \times n$ matrix M that has 1s in the $(1, 2)$, $(2, 1)$, and $(i, i), 3 \leq i \leq n$, positions and zeros elsewhere. We can see that M sends e_1 to e_2, e_2 to e_1, and fixes all other basis elements of $\mathbb{F}_q^n$.

We can create an automorphism of $L_n(q)$ as follows: for $U \subseteq \mathbb{F}_q^n$, $\phi(U) = M \cdot U$, where $M \cdot U$ multiplies every element $u \in U$ by M. Indeed, this is an automorphism of the lattice, since if $U \leq U'$ in $L_n(q)$, then $M \cdot U \leq M \cdot U'$ as well.

Since $\phi(\langle e_1 \rangle) = \langle e_2 \rangle$ and $\phi(\langle e_2 \rangle) = \langle e_2 \rangle$, we have that $\langle e_1 \rangle$ and $\langle e_2 \rangle$ form a 2-cycle. By Proposition 2.7, $L_n(q)$ is 1/2-balanced. □

2.3.4 Distributive Lattices

For a given poset P, consider the lower order ideals of P. Recall the definition of order ideals is given in Definition 1.3. The lower order ideals are partially ordered by $I \leq I'$ if $I \subseteq I'$. This gives us the *distributive lattice* $J(P)$ created from P. It is a lattice as meets and joins are given by intersections and unions, respectively. An example of P and $J(P)$ is



$x \diagdown y$	$\{2\}$	$\{2,3\}$	$\{2,3,4\}$
$\{1\}$	5	10	13
$\{1,2\}$		4	10
$\{1,2,3\}$			5

Figure 2.6: A 4 element poset P, its corresponding $J(P)$, and a chart with values $e(P+xy)$.

given in Figure 2.6. In fact, by the Fundamental Theorem on Distributive Lattices, every distributive lattice is isomorphic to the lattice of lower order ideals of some poset P. As a result, we were motivated to find results about $J(P)$ that depend on properties of P.

Unfortunately, it is not true that if P is 1/2-balanced, then $J(P)$ is 1/2-balanced as well. An example can be seen in Figure 2.6. While P is 1/2-balanced by the pair $(1, 3)$, $J(P)$ is not 1/2-balanced, as evidenced in the chart in Figure 2.6 whose entries are $e(P + xy)$ for every x and y not comparable in $J(P)$. Since $J(P)$ has 14 linear extensions, we can see no pair is $1/2-$balanced.

Adding an extra condition, namely that P has an automorphism with a 2-cycle, allows us to prove that $J(P)$ is 1/2-balanced.

Proposition 2.16. *If P has an automorphism with cycle of length 2, then $J(P)$ is 1/2-balanced.*

Proof. Let $\alpha : P \to P$ be an automorphism with $\alpha(x) = y$ and $\alpha(y) = x$ for some $x, y \in P$. This induces an automorphism $\bar{\alpha}$ of $J(P)$, given by $\bar{\alpha}(I) = \{\alpha(w) : w \in I\}$ for $I \in J(P)$. We claim that $\bar{\alpha}$ has a cycle of length 2, namely that $\bar{\alpha}(L_x) = L_y$ and $\bar{\alpha}(L_y) = L_x$.

We will show $\bar{\alpha}(L_x) = L_y$, as the proof of the other equality is similar. Let $z \in \bar{\alpha}(L_x)$, so $z = \alpha(w)$ for some $w \in L_x$. This means that $w \leq x$, and so $\alpha(w) \leq \alpha(x) = y$. Therefore, $z \leq y$ and we have $z \in L_y$. Hence, $\bar{\alpha}(L_x) \subseteq L_y$. The proof of the other set containment is similar. Thus, we have proven the claim. Since $\bar{\alpha}$ has a cycle of length 2, by Proposition 2.7, $J(P)$ is 1/2-balanced. □

This leads to another proof that Boolean lattices are 1/2-balanced.

Corollary 2.17. *For $n \geq 2$, the Boolean lattice B_n is 1/2-balanced.*

Proof. Let $n \geq 2$. The Boolean lattice B_n is the distributive lattice corresponding to the poset P with n elements and no relations. There is an automorphism on P that swaps elements 1 and 2 and is the identity on the remaining elements. Since P has an automorphism with a cycle of length 2, then by Proposition 2.16, B_n is 1/2-balanced. □

2.3.5 Products of Two Chains

Let C_n be the chain with n elements. This section will be concerned with the product of two chains C_m and C_n, with $m, n \geq 2$. The poset $C_m \times C_n$ is a lattice with grid structure, as in Figure 2.7. Recall Young diagrams from Definition 1.4 and notation from Section 1.1. Figure 2.7 depicts the rectangular Young diagram of shape $\lambda = (4^3)$ corresponding to $C_3 \times C_4$. We can see the correspondence as m gives us the number of rows of the Young diagram, while n gives us the length of each row. The number of linear extensions of $C_m \times C_n$ is the same as the number of standard Young tableaux (SYT) of the Young diagram of shape (n^m).

Unlike many other demonstrations that a poset is 1/3-balanced, our proof for $C_m \times C_n$ finds the exact value of $\mathbb{P}(a \prec b)$ for a pair of elements (a, b). Let a be the atom from the chain $C_m \times \hat{0}$ and b the atom from the chain $\hat{0} \times C_n$, as labeled in Figure 2.7. In order to

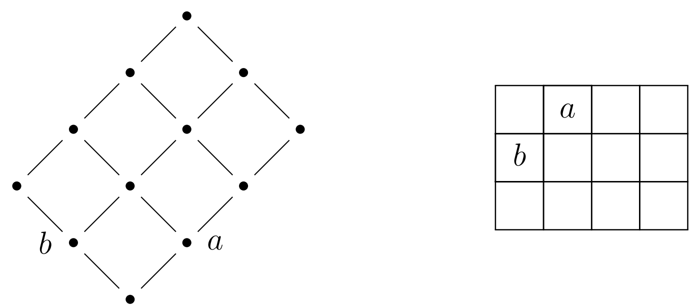


Figure 2.7: The poset $C_3 \times C_4$ and its corresponding diagram.

compute how many linear extensions of $C_m \times C_n$ have $a \prec b$, we will compute how many SYT have $(1, 2)$ filled with a smaller number than $(2, 1)$. Since the entry 2 must go in one of these two cells, this assumption forces the SYT to have the $(1, 1)$ cell filled with a 1 and the $(1, 2)$ cell filled with a 2. Flipping and rotating by 180 degrees, one sees that this is equivalent to counting the SYT of shape $(n^{m-1}, n-2)$.

To prove Lemma 2.18, we will need the hooklength formula for f^λ, the number of SYT of a diagram of shape λ. For a given cell (i, j) in a diagram of shape λ, its hook is the set of all the cells weakly to its right together with all cells weakly below it, and its hooklength $h_\lambda(i, j)$ is the number of cells in its hook. The hooklength formula for a diagram with n cells is

$$f^\lambda = \frac{n!}{\prod h_\lambda(i, j)},$$

where the product is over all cells (i, j) in λ. A diagram of shape $(4, 4, 2)$ is given in Figure 2.8, and each cell is labeled by its hooklength. Using the formula, we can see that

$$f^{(4,4,2)} = \frac{10!}{6 \cdot 5^2 \cdot 4 \cdot 3 \cdot 2^3 \cdot 1^2} = 252,$$

and so the diagram has 252 SYT.

Lemma 2.18. *Let $m \geq 1$ and $n \geq 3$. We can relate the number of standard Young tableaux*

6	5	3	2
5	4	2	1
2	1		

Figure 2.8: The diagram of shape $(4,4,2)$ where each cell is labeled by its hooklength.

of shape (n^m) *and of shape* $(n^{m-1}, n-2)$ *by the following equality:*

$$f^{(n^{m-1},n-2)} = \frac{(n-1)(m+1)}{2(mn-1)} f^{(n^m)}.$$

Proof. Let $\lambda = (n^m)$ and $\mu = (n^{m-1}, n-2)$ be diagrams and f^λ be the number of SYT of shape λ. We will proceed by first describing which factors differ between f^λ and f^μ. We can observe that the hooklengths only disagree between λ and μ in those cells in the last two columns and those in the last row. The last two columns of λ have hooklengths of $m+1, m, \ldots, 2$ and $m, m-1, \ldots, 1$, while in μ the last two columns have hooklengths $m, m-1, \ldots, 2$ and $m-1, m-2, \ldots, 1$. Overall, f^μ is missing a factor of $(m+1)m$ which appears in the denominator of f^λ. Similarly, the hooklength values of the last row of λ, excluding the ones in the last two columns which have already been accounted for, are $n, n-1, \ldots, 3$, while those in μ are $n-2, n-3, \ldots, 1$. So our formula for f^μ is missing a factor of $n(n-1)$ from the denominator and a factor of 2 from the numerator. Finally f^λ has a numerator of $(mn)!$ while μ has a numerator of $(mn-2)!$, so there is a factor of $(mn)(mn-1)$ we need to remove from the numerator of f^λ. Overall, our hooklength formula for μ derived from f^λ is

$$f^\mu = \frac{n(n-1)(m+1)m}{2(mn)(mn-1)} f^\lambda = \frac{(n-1)(m+1)}{2(mn-1)} f^\lambda,$$

as desired. □

Theorem 2.19. *Let C_m and C_n be chains of lengths $m \geq 2$ and $n \geq 2$, respectively. Then their product $C_m \times C_n$ has a 1/3-balanced pair.*

Proof. Without loss of generality, we can let $n \geq m$. Let $P = C_m \times C_n$. If $m = 2$, then P has width 2, and so by Theorem 2.1, we know that $\delta(P) \geq 1/3$. If $m = n = 3$, then P has a non-trivial automorphism, and so by Theorem 2.6, P has a 1/3-balanced pair.

Next, let $m \geq 3$ and $n \geq 4$. We can see that P has exactly two atoms. Let the atoms be labeled a and b, as in Figure 2.7. We claim that a, b are a 1/3-balanced pair. Notice that the linear extensions of P begin with either $\hat{0}a\ldots$ or $\hat{0}b\ldots$, and the number that begin with $\hat{0}a\ldots$ is the same as $e(P + ab)$. We can also see that $e(P + ab)$ is the same as the number of standard Young tableaux of shape $(n^{m-1}, n-2)$. Hence, by Lemma 2.18, we know that

$$e(P + ab) = \frac{(n-1)(m+1)}{2(mn-1)} e(P).$$

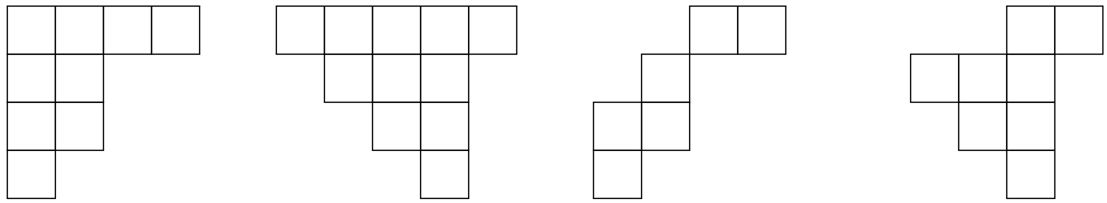
It remains to be shown that

$$\frac{1}{3} \leq \frac{(n-1)(m+1)}{2(mn-1)} \leq \frac{2}{3} \tag{2.1}$$

for all $m \geq 3$, $n \geq 4$. For the first inequality, cross multiply and bring everything to one side to get the equivalent inequality $(mn-1) + 3(n-m) \geq 0$. This inequality is true since $n \geq m$ and $mn \geq 1$.

For the second inequality, proceed in the same manner to get $mn + 3(m-n) - 1 \geq 0$. By the lower bounds for m, n we have $(m-3)(n-4) \geq 0$. So it suffices to prove $mn + 3m - 3n - 1 \geq (m-3)(n-4)$. Moving everything to one side yet again gives the equivalent inequality $7m - 13 \geq 0$ which is true since $m \geq 3$.

Therefore, we have shown that (2.1) holds, and so (a, b) is a 1/3-balanced pair in P. $\square$



(a) (b) (c) (d)

Figure 2.9: (a) A Young diagram of shape $(4, 2^2, 1)$, (b) a shifted diagram of shape $(5, 3, 2, 1)$, (c) a skew left-justified diagram of shape $(4, 2^2, 1)$ / $(2, 1)$, and (d) a shifted skew diagram of shape $(5, 3, 2, 1)$ / (3).

2.4 Other Diagrams

In Section 2.3.5, we considered the product of two chains as a rectangular Young diagram, and the linear extensions of the poset corresponded to the standard Young tableaux of that Young diagram. Given Theorem 2.19, it is natural to consider other posets that come from other diagrams. Recall the definition of Young diagram given in Definition 1.4, and let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a weakly decreasing partition of n. An example of the Young diagram of shape $(4, 2^2, 1)$ is given in Figure 2.9(a). To generalize the notion of Young diagram, next let $\lambda_1 > \lambda_2 > \cdots > \lambda_k$, in which case λ is called a *strict partition*. The *shifted diagram* corresponding to a strict partition λ indents row i so that it begins on the diagonal cell (i, i). An example is given in Figure 2.9(b). A third type of diagram is a *skew diagram*, λ/μ, which is the set-theoretic difference between diagrams $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_l)$ such that $\mu \subseteq \lambda$, that is, $l \leq k$ and $\mu_i \leq \lambda_i$ for each $1 \leq i \leq l$. A skew diagram can be either *left-justified*, as seen in Figure 2.9(c), or *shifted*, as seen in Figure 2.9(d).

We would like to point out that if we allow μ to be an empty partition, then λ/μ could refer to a Young, shifted, or skew diagram. To avoid ambiguity, we will always specify the type of diagram we intend, and to be clear, when referring to Young diagrams, we exclusively mean left-justified and non-skew diagrams.

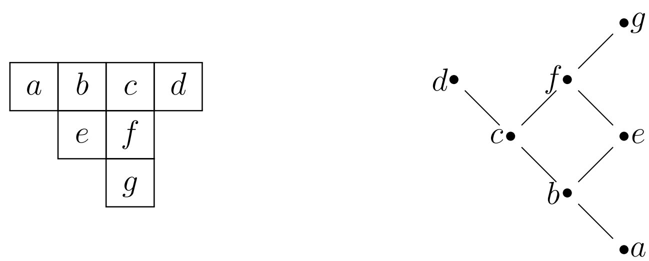


Figure 2.10: The shifted diagram of shape $(4, 2, 1)$ and its corresponding poset.

For each diagram, there is a corresponding poset formed by letting each cell be an element of P. For $x, y \in P$, $x \leq y$ exactly when the cell for y is weakly to the right and/or weakly below the cell for x. See Figure 2.10 for an example of the poset corresponding to the shifted diagram $(4, 2, 1)$. We will use the notation λ/μ to refer to a diagram and $P_{\lambda/\mu}$ for the corresponding poset.

As we have already seen, the cells of a Young diagram can be filled with integers from 1 to n using each number exactly once and increasing in the rows and columns to form a standard Young tableau. In a similar manner, we define *shifted standard Young tableaux* and *skew standard Young tableaux.* The standard (shifted) (skew) Young tableaux for a given diagram are in bijective correspondence with linear extensions of the poset. Therefore, when we want to discuss linear extensions, we can discuss standard (shifted) (skew) Young tableaux instead.

We next present a generalized version of Theorem 2.19. For any type of diagram, we use the notation (i, j) to refer to the cell in the ith row and jth column. For instance, the $(2, 2)$ cell in the shifted diagram in Figure 2.10 is labeled with an e.

Theorem 2.20. *Let $P_{\lambda/\mu}$ be the poset corresponding to the diagram λ/μ, where μ could be an empty partition, and assume $P_{\lambda/\mu}$ is not a linear order. If λ/μ is a Young diagram, shifted diagram, or skew diagram, $P_{\lambda/\mu}$ is a 1/3-balanced poset.*

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_l)$ be partitions such that $\mu \subseteq \lambda$. Assume the diagram λ/μ does not correspond to a linear order. Assume first that μ is empty; we will show that when P_λ corresponds to a Young diagram or shifted non-skew diagram, it has an almost twin pair of elements. Hence, by Theorem 2.4, P_λ is 1/3-balanced.

When λ is a Young diagram, let x correspond to the $(1,2)$ cell and y correspond to the $(2,1)$ cell of λ. So, (x,y) is an almost twin pair of elements in P_λ.

When λ is a shifted diagram which is not skew, then $\lambda_1 \geq 3$, as P_λ is not a linear order. Let x correspond to the $(1,3)$ cell and y correspond to the $(2,2)$ cell. So, (x,y) is an almost twin pair of elements in P_λ.

Next, we consider skew diagrams. If λ/μ is a disconnected diagram, observe that an almost twin pair in a connected component of $P_{\lambda/\mu}$ remains an almost twin pair in the entire poset. Therefore, we can assume λ/μ is a connected skew diagram that does not correspond to a poset that is a chain. First consider left-justified skew diagrams. By removing any empty columns on the left of the diagram, we can assume without loss of generality that $k \geq l+1$. For ease in discussing the first and last rows of μ, define $\mu_0 = \lambda_1$ and $\mu_{l+1} = 0$. We have the following cases:

(i) If there exists $i \in [l]$ such that

$$\mu_{i-1} - 1 \geq \mu_i = \mu_{i+1} + 1$$

then $(i, \mu_i + 1)$ and $(i+1, \mu_{i+1}+1)$ is an almost twin pair. For an example of this case, see the pair (a,b) in Figure 2.11.

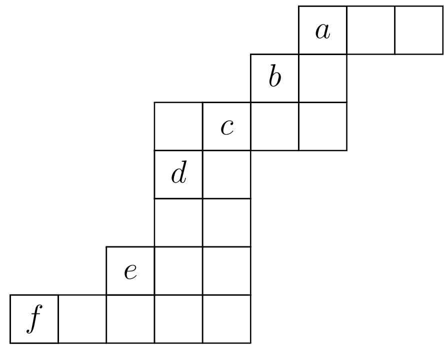


Figure 2.11: The skew diagram $(9, 7^2, 5^4)$ / $(6, 5, 3^2, 2)$

(ii) If there exists $i \in [l-1]$ such that

$$\mu_{i-1} - 2 \geq \mu_i = \mu_{i+1}$$

then $(i, \mu_i + 2)$ and $(i+1, \mu_{i+1} + 1)$ is an almost twin pair. Note that $(i, \mu_i + 2)$ exists in the diagram since λ/μ is connected. For an example of this, see the pair (c, d) in Figure 2.11.

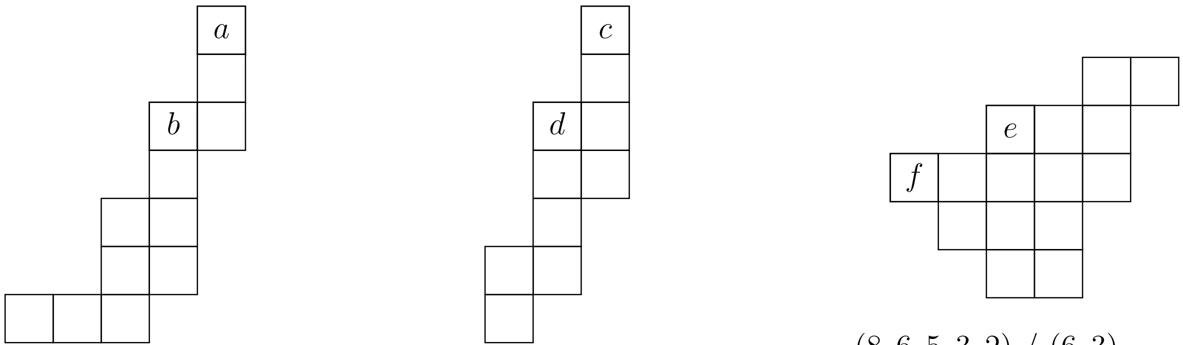
(iii) If $k = l + 1$ and $\mu_{l-1} - 1 \geq \mu_l$, then $(l, \mu_l + 1)$ and $(l+1, 1)$ are an almost twin pair. For an example of this, see the pair (e, f) in Figure 2.11.

(iv) If $k \geq l + 2$ and $\mu_l \geq 2$, then $(l+1, 2)$ and $(l+2, 1)$ are an almost twin pair. Notice this is similar to case (ii), only it occurs at the bottom of the skew diagram.

We can now decide what types of diagrams do not fall into cases (i)-(iv) above. We claim that any remaining diagram has μ of the form

$$(s^{m_1}, (s-1)^{m_2}, \ldots, (s-p+1)^{m_p})$$

where $m_i \geq 2$ for all $i \in [p]$ and $s \in \mathbb{N} \backslash \{0\}$. We call this case (v).



$(5^3, 4^3, 3)\ /\ (4^2, 3^2, 2^2)$ $(3^4, 2^2, 1)\ /\ (2^2, 1^3)$ $(8, 6, 5, 3, 2)\ /\ (6, 3)$

Figure 2.12: Two skew left-justified diagrams and one skew shifted diagram.

Indeed, all consecutive μ_i values differ by 1 or 0 since if there is some r with $\mu_{r-1} - 2 \geq \mu_r$, then to avoid cases (i) and (ii) above, it must be that $\mu_{s-1} - 2 \geq \mu_s$ for all $s \in [r, l+1]$. In particular, this means $\mu_l \geq 2$, and this diagram will fall into case (iii) or (iv). So, any consecutive μ_i values differ by 1 or 0. Further, if $m_i = 1$ for any $i \in [p]$, the diagram would fall into case (i). Hence, μ must have the form above.

It also must be true that λ/μ has $\lambda_1 = \mu_1 + 1$, in order to avoid case (ii) above. Two examples of diagrams λ/μ that do not fall into cases (i)-(iv) are given in Figure 2.12. In these remaining diagrams, $(1, \mu_1 + 1)$ and $(m_1 + 1, \mu_{(m_1+1)} + 1)$ is an almost twin pair. Examples of this pair are (a, b) and (c, d) in Figure 2.12. Hence every skew left-justified diagram λ/μ satisfies one of these five cases, and so $P_{\lambda/\mu}$ not a chain has an almost twin pair.

Finally, we consider the skew shifted diagrams. Notice that that the first l rows of the diagram can be viewed as a skew left-justified diagram. Therefore, if any of the first $l-1$ rows are of the forms found in cases (i) or (ii), or if the first rows correspond to case (v), then the almost twin pairs in those cases remain almost twin in this poset, and we are done.

If none of cases (i), (ii), or (v) apply, then consider μ_l. In particular, it must be the $\mu_l > 1$, else case (ii) or (v) applies. If $\mu_l > 3$, then the last $k - l$ rows of the diagram are a shifted diagram, and so we have the same almost twin pair as in the shifted case. If $\mu_l \in \{2, 3\}$, then $(l, \mu_l + l)$ and $(l+1, l+1)$ are an almost twin pair, as seen by the pair (e, f) in Figure 2.12. Hence, for skew diagrams λ/μ, if $P_{\lambda/\mu}$ is not a chain, it has an almost twin pair of elements, as claimed. □

2.5 Posets of Dimension 2

The set of linear extensions $E(P)$ of a labeled poset P with n elements can be considered a subset of $\mathfrak{S}_n$, where permutations are written in one-line notation. The full set $\mathfrak{S}_n$ is a poset under the weak Bruhat ordering. For background on the weak Bruhat order, see [BB05, Chapter 3]. Since we use natural numbers to discuss the elements of a labeled poset P, we will use $<_{\mathbb{N}}$ and $<_P$ to distinguish between the linear order on the natural numbers and the partial order on P. The following definitions for permutations are relevant to this section and are thus provided here:

Definition 2.21. *Let $\pi = \pi_1\pi_2 \ldots \pi_n$ be a permutation in one-line notation. An* inversion *in π is a pair (i, j) such that $i < j$ and $\pi_i > \pi_j$.*

Definition 2.22. *For π and σ permutations, π is said to* contain σ as a pattern *if some subsequence of π has the same relative order as σ. Otherwise we say that π* avoids σ*. We can also consider a subsequence $S = \pi_{i_1}\pi_{i_2} \cdots \pi_{i_m}$ of elements from $\pi = \pi_1 \cdots \pi_n$. We say that S* is contained in the pattern σ *if some instance of σ in π contains S. Otherwise, we say that S* avoids σ*.*

An example of Definition 2.22 can be seen with $\pi = 23154$. We can see that π contains the pattern 123 in the subsequence 235, while it avoids 321. If $S = 14$, then S is contained in the pattern 132 in the instance 154 and avoids 123.

For the remainder of this section, we will consider $E(P)$ for any poset P to be a subposet of $\mathfrak{S}_n$. We might ask what can be said about $E(P)$ as a poset, and Björner and Wachs [BW91] showed that $E(P)$ has especially nice structure when P has dimension 2.

The *dimension* of a poset is the least k such that there is some $U \subseteq E(P)$ of size k such that $\cap U = (P, \leq)$. An equivalent definition is that the dimension of P is the least k such that P can be embedded as a subset into the product $\mathbb{N}^k$. For our purposes, we discuss posets with dimension 2, which Björner and Wachs [BW91] characterize in the following proposition.

Proposition 2.23 ([BW91])**.** *A poset P has dimension 2 if and only if it has a labeling ω such that $E(P, \omega)$ forms an interval in the weak Bruhat ordering of $\mathfrak{S}_n$.*

If the labeling ω for a poset P of dimension 2 is natural, then $E(P, \omega)$ forms a lower order interval. Since the characterization by Björner and Wachs is necessary and sufficient, we know there is a correspondence between naturally labeled posets and lower-order intervals in $\mathfrak{S}_n$. The interval contains permutations between the identity permutation e and some maximal permutation π. When $E(P, \omega)$ forms the interval $[e, \pi]$, the two linear extensions required to describe P as a poset of dimension 2 are exactly e and π. For the reverse correspondence, we can start with a permutation π and create the naturally labeled poset (P_π, ω) with the property that $E(P_\pi, \omega)$ is the set of permutations in the interval $[e, \pi]$. Here, the subscript emphasizes that P_π was created from π. To create P_π, add labeled elements by reading π left to right. A relation is added between two elements a and b exactly when a is to the left of b in π and $a <_{\mathbb{N}} b$.

As a brief example, consider the poset P and labeling ω in Figure 2.13. This poset has 8 linear extensions, and using the weak Bruhat order, we can create the poset $E(P, \omega)$, which

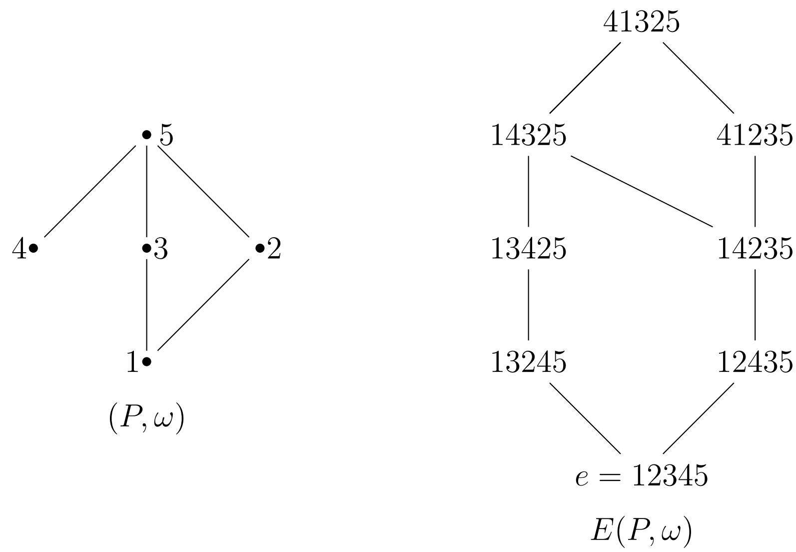


Figure 2.13: The set of linear extensions $E(P,\omega)$ of the poset (P,ω) forms the interval $[e, 41325]$ in the weak Bruhat order.

is a subposet of $\mathfrak{S}_5$. For the reverse correspondence, we construct (P_π, ω) from the maximal element of the interval $[e, 41325]$ as follows: Add labeled elements by reading π left to right, so first add 4, then 1, etc. Add the relations $1 <_P 3$, $1 <_P 2$, and finally add 5 as a maximal element. Notice that 1 and 4 are minimal elements as no values to the left of 1 or 4 in 41325 are smaller than them. By this process, we can see that for every inversion (i, j) of π gives us a pair of elements in P_π that are not comparable.

Our goal is to use properties of the maximal permutation in $E(P,\omega)$, where P has dimension 2, to determine if the poset has a 1/3-balanced pair. Our result, in fact, deals with a case when the poset has a 1/2-balanced pair.

Proposition 2.24. *Let $\pi = \pi_1\pi_2\ldots\pi_n$ be an element of $\mathfrak{S}_n$, and assume that π has an inversion (i, j) such that $\pi_i\pi_j$ avoid the patterns* 312 *and* 231 *in π. Then (π_i, π_j) is a 1/2-balanced pair of P_π.*

Before we proceed to the proof, we can observe an example of this in Figure 2.13. We can see $(3, 4)$ is inversion of $\pi = 41325$, and $\pi_3\pi_4 = 32$ avoids 312 and 231 in π. So,

$(\pi_3, \pi_4) = (3, 2)$ is a 1/2-balanced pair in P.

Proof. Let $\pi \in \mathfrak{S}_n$, and assume π has an inversion (i, j). Let $\pi_i = y$ and $\pi_j = x$, and further assume yx avoids the patterns 312 and 231. Therefore, π has the form

$$\pi = \pi_1 \cdots \; y \; \cdots \; x \; \cdots \pi_n$$

where $y >_{\mathbb{N}} x$. Now, since yx avoids 312 and 231, there are no elements between x and y in π that are larger than y or smaller than x. Also, no elements to the right of x or left of y have values between x and y. To put this description another way, if yx avoids 312 and 231 in π, the elements between y and x in π are exactly those in the set $\{a \mid x <_{\mathbb{N}} a <_{\mathbb{N}} y\}$.

We claim that $U_x = U_y$ and $L_x = L_y$ in P_π. We will show that $U_x = U_y$ as the proof of $L_x = L_y$ is nearly identical. If $z \in U_x$, then z is to the right of x in π and thus also to the right of y in π. Since yx avoids 312 in π and $x <_{\mathbb{N}} z$, it must be that $y <_{\mathbb{N}} z$. Hence, $y <_P z$ and so $z \in U_y$.

If $z \in U_y$, then z is to the right of y in π and $y <_{\mathbb{N}} z$. Since yx avoids 231 in π, then z must also be to the right of x in π. Also, $x <_{\mathbb{N}} y <_{\mathbb{N}} z$. Thus $x <_P z$, which means $z \in U_x$. Hence, we have that $U_x = U_y$.

Now, because $U_x = U_y$ and $L_x = L_y$, (x, y) is a twin pair of elements. By Corollary 2.8, as P_π has a twin pair, then P_π is 1/2-balanced, as desired. □

It is important to note that not every permutation has an inversion that satisfies the conditions of Proposition 2.24. This means we have not shown the $1/3 - 2/3$ Conjecture for every poset of dimension 2; however, we hope that other helpful properties of permutations will emerge to show every dimension 2 poset satisfies the conjecture.

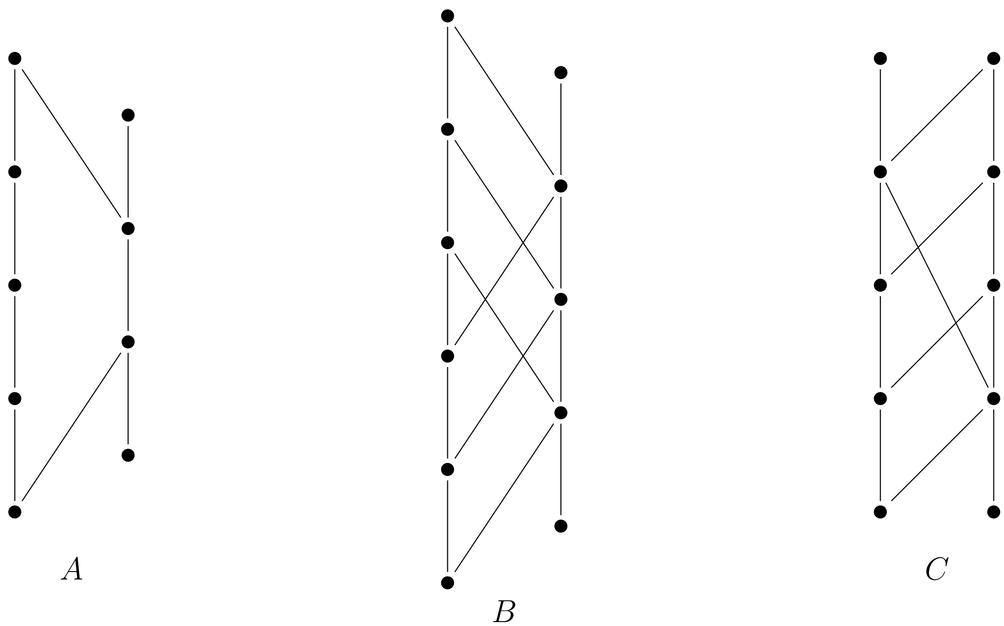


Figure 2.14: Posets with the smallest balance constants greater than 1/3.

2.6 Posets with Small Balance Constants

As previously mentioned, Aigner [Aig85] proved that a poset of width 2 has a balance constant of 1/3 if and only if it is the linear sum of the singleton poset and T from Figure 1.2. One can then ask: how close to 1/3 can a balance constant be for a poset not of this form? The posets with the previous closest values can be found in Figure 2.1.

We have found posets of width 2 that have balance constants closest known values to 1/3. In Figure 2.14, poset A has $\delta(A) = \frac{6}{17} \approx 0.35294$. Poset B has $\delta(B) = \frac{60}{171} \approx 0.350877$. Poset C has $\delta(C) = \frac{37}{106} \approx 0.349057$.

This analysis was done through the code found in the Appendix , which can be adapted to find the balance constants of many posets at a time.

2.7 Future Work

In Section 2.3.4, we discussed one special case when a distributive lattice is 1/2-balanced. Future work on the conjecture could include showing that all distributive lattices are 1/3-balanced.

In Section 2.3.5, we discussed the product of two chains. We were motivated to study products of chains as they are isomorphic to divisor lattices. A *divisor lattice* is the set of divisors of n partially ordered by $x \leq y$ if x divides y. It is a lattice as the meet of two elements is their greatest common divisor and the join of two elements is their least common multiple. One future direction this work could take is to prove that a product of k chains is 1/3-balanced, for $k \geq 3$.

In Section 2.6, we discussed finding posets of width 2 with small balance constants. Future work in this direction could continue to search for posets with balance constants approaching 1/3, and it would be ideal to find a pattern among these posets.

APPENDIX

Appendix

Code to Count $\mathbb{P}(x \prec y)$

In this section, we include a simplified version of the code we developed in the `Python` programming language [Ros95] to calculate the proportions of linear extensions for a given poset. The input is a representation of the poset as an upper-triangular binary matrix M, and the output is the total number of linear extensions and a matrix L whose (i, j)th entry is the number of linear extensions in which $i \prec j$.

We originally implemented the algorithm in `Sage`, but when we attempted to compute the linear extensions for all posets of size 7, it became necessary to incorporate modules found in `Python`, such as `numpy`, abbreviated as `np` in the code, for faster matrix computations.

Matrix Representations

In order to obtain an upper-triangular binary matrix for a poset P, first give the poset a natural labeling, which ensures the desired matrix M is upper-triangular. The entries of M are $M_{ij} = 1$ if and only if $i <_P j$, and zeros elsewhere. As as example, consider the poset and natural labeling in Figure 1.1. The following is a matrix representation of this poset.

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.1}$$

Finding the Linear Extensions

Given a matrix M, we recursively construct the set of permutations that are linear extensions of the given poset in the function `linear_extensions(M)`. The function operates by starting with the identity permutation and considers if the largest value k can be moved one position to the left. If it can, it calculates all the permutations recursively with k in that position before considering if k can again be moved to the left. We can see that $M_{kk} = 0$ for all $k \in [n]$, and so in the kth iteration of the recursion, we can always append k to the end of the permutation in consideration.

```
def linear_extensions(M, k=None):
    if k is None:
        k = len(M) - 1

    if k < 0:
        yield []
    else:
        # Recursively generate permutations with fewer elements
        # and insert k at all allowed positions in each
        for p in linear_extensions(M, k - 1):
            yield p + [k]

            # Attempt to insert k as we move left through p
            for i in reversed(xrange(k)):
                if not M[p[i]][k]:
                    # If allowed, insert and continue
                    yield p[:i] + [k] + p[i:]
                else:
                    # All smaller i insertions forbidden
                    break
```

Counting the Proportions

Now that we have all the linear extensions of P, we need a way to keep track of how many times $i \prec j$ for all $i, j \in P$. We do this through a matrix L whose entries are the number of linear extensions of P with $i \prec j$.

The function `count_extensions(M)` loops over the linear extensions found by `linear_extensions(M)` and computes the matrix L. It returns not only the matrix, but also the maximum value of L, which is the total number of linear extensions of P. To compute L, we use the inverse h of each linear extension p, since we want to count when the values i and j satisfy $i \prec j$, not when the positions i and j satisfy $i \prec j$.

To do the count, the function creates a matrix `h_as_matrix`, whose ith row is n copies of the ith element of h. We compare this to the transpose of `h_as_matrix`, whose jth column is n copies of the jth element of h. When we compare the (i, j)th elements of these matrices, we obtain a 1 if $h_i < h_j$ and 0 otherwise. The matrix L increments any position which obtains a 1 in the comparison.

```
def count_extensions(M):
    n = len(M)
    rng = np.arange(n, dtype=np.int_)
    h = np.zeros(n, dtype=np.int_)

    # Preallocate indexing matrix
    Lind = np.array([range(n)]*n).T

    # Use 64bit integers for counting to avoid overflow
    # (as we learned the hard way)
    L = np.zeros((n, n), dtype=np.int64)

    for p in linear_extensions(M):
        # Compute h, the inverse of p
        h[p] = rng

        # Copy h into the columns of a matrix
        h_as_matrix = h[Lind]

        # Increment L by doing a comparison for each entry
        L += (h_as_matrix < h_as_matrix.T)

    return L.max(), L
```

As an example, by inputting the matrix in (A.1) for the poset found in Figure 1.1, we obtain the matrix from Figure 1.1. The bold values in the matrix were added for emphasis.

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