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HOLOMORPHIC SELF-MAPS OF THE UNIT BALL: ITERATION AND COMPOSITION OPERATORS

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ABSTRACT

HOLOMORPHIC SELF-MAPS OF THE UNIT BALL:
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Let f be a holomorphic map of the unit disc $\mathbb{D}$ into itself, which is neither the identity map nor an elliptic automorphism of $\mathbb{D}$. A theorem of A. Denjoy and J. Wolff states that the iterates of f converge, uniformly on compacta, to a point z in $\overline{\mathbb{D}}$. This point z, called the Denjoy-Wolff point for f, will be the fixed point of f if f has an interior fixed point; otherwise z will be a point of $\partial\mathbb{D}$. In this paper we consider analogues of the Denjoy-Wolff theorem for holomorphic self-maps of the unit ball B_N in $\mathbb{C}^N$. For a fixed point free map f we show that the iterates of f converge to a point of the boundary of the ball. Part of our argument will yield a useful description of the automorphisms of B_N which fix no point of B_N. The subsequential limits of iterates of maps with interior fixed points are also described.

Secondly we consider several questions related to composition operators on Hardy spaces of the unit ball. If $\varphi: B_N \to B_N$ is a holomorphic map and f is a holomorphic function on B_N, denote the composition $f \circ \varphi$ by $C_\varphi(f)$. We give examples to show that, in contrast to the situation when $N = 1$, there are holomorphic maps φ

for which C_φ is not a bounded operator on $H^p(B_N)$, where $N > 1$ and $p < \infty$.

Finally, we study C_φ in the case that C_φ is a compact operator on some Hardy space $H^p(B_N)$. In this situation we show that φ fixes a unique point z_0 of B_N and determine the spectrum of C_φ to be all possible products of powers of the eigenvalues of the derivative map $\varphi'(z_0) \cup \{0,1\}$.

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INTRODUCTION

Let B_N denote the open unit ball in $\mathbb{C}^N$, in the Euclidean metric. Thus $B_N = \{(z_1,\ldots,z_N) \in \mathbb{C}^N \text{ with } \sum|z_i|^2 < 1\}$. A holomorphic self-map f of B_N is a map $f\colon B_N \to B_N$ which can be written as $f = (f_1,\ldots,f_N)$ where each f_i is a holomorphic function from B_N into $\mathbb{C}$.

Chapter I deals with the iteration of a holomorphic self-map of B_N. A classical result due to A. Denjoy [6] and J. Wolff [24] states that if f is a holomorphic map of the unit disc $\mathbb{D}$ into itself which is neither the identity nor an elliptic automorphism of $\mathbb{D}$, then the sequence of iterates of f converges, uniformly on compacta, to a point $z \in \overline{\mathbb{D}}$. This point z, called the Denjoy-Wolff point for f, will be the fixed point of f if f has an interior fixed point; otherwise z will be a point of $\partial\mathbb{D}$. In Chapter I we extend the Denjoy-Wolff theorem to the unit ball B_N; in particular we show that the entire sequence of iterates of a fixed point free holomorphic self-map of B_N converges, uniformly on compacta, to a point in ∂B_N. Part of our argument involves a characterization of the automorphisms of B_N which fix no point of B_N. We also describe the subsequential limits of iterates of maps with interior fixed points.

The remaining chapters deal with composition operators on Hardy spaces in the ball. Recall that the Hardy space $H^p(B_N)$, $0 < p < \infty$, is the space of holomorphic functions on B_N satisfying

$$\|f\|_P^P \equiv \sup_{0<r<1} \int_S |f(r\zeta)|^P \, d\sigma(\zeta) < \infty,$$

where $S = \partial B_N$ and σ is the rotation invariant probability measure on S. If φ is a holomorphic map of B_N into B_N and f a holomorphic function in B_N, the composition $C_\varphi(f) = f \circ \varphi$ is again a holomorphic function in B_N. In the case $N = 1$, C_φ is a bounded linear operator, called a composition operator, on $H^P(\mathbb{D})$, for $0 < P \leq \infty$. Composition operators in the disc have been an area of active study for the past fifteen years.

In Chapter II we consider the question of the boundedness of C_φ on $H^P(B_N)$ when $N > 1$. We give, in contrast with the case $N = 1$, some examples of maps φ for which C_φ is not bounded on $H^P(B_N)$, $p < \infty$, and we see that C_φ can even fail to take $H^P(B_N)$ into the Ne anlinna class $N(B_N)$. Some positive results related to the question of the boundedness of C_φ are also discussed.

Chapter III deals with composition operators which are compact on some $H^P(B_N)$. Compact composition operators on the H^P spaces of the unit disc have been extensively studied. For example, in [20], J.H. Shapiro and P.D. Taylor relate the compactness of the operator C_φ to certain geometric properties of the inducing map φ. J. Caughran and H. Schwartz [4] show that if C_φ is compact on $H^2(\mathbb{D})$, then φ has a unique fixed point z_0 in $\mathbb{D}$, and that the spectrum of C_φ is the set $\{\varphi'(z_0)^n\colon\ n = 1,2,\ldots\} \cup \{0,1\}$.

The main object of Chapter III is to determine the spectrum of a compact composition operator on $H^P(B_N)$ for $1 \leq P < \infty$. While the fixed point set of a holomorphic map $\varphi\colon\ B_N \to B_N$ is usually more complicated in the case $N > 1$ than it is when $N = 1$, we nevertheless show that a map which induces a compact composition operator on $H^P(B_N)$ has a unique fixed point in B_N. Suppose that C_φ is compact on $H^P(B_N)$ with z_0 the fixed point of φ. Then we show that the spectrum of C_φ consists of 0 and 1 along with all possible products of powers of the eigenvalues of the derivative map $\varphi'(z_0)$. While the methods of Caughran and Schwartz [4] in the case $N = 1$ and $P = 2$ can be made to work for $P = 2$ in several dimensions, we will take a different approach which is applicable for all P, $1 \leq P < \infty$.

CHAPTER I

In this chapter we consider the iteration of holomorphic self-maps of the unit ball in $\mathbb{C}^N$. The main result of Section 1 is an analogue of the classical Denjoy-Wolff theorem in the unit disc [6, 24], which states that the sequence of iterates of a holomorphic, fixed point free map of $\mathbb{D}$ into $\mathbb{D}$ converges, uniformly on compacta, to a constant of norm 1. In Section 2 we consider the iteration of maps with interior fixed points.

1. <u>Maps with no interior fixed points</u>. From now on we will denote the unit ball in $\mathbb{C}^N$ by B instead of B_N, unless we wish to indicate the dimension explicitly. Let $H(B;B)$ be the family of all holomorphic maps of B into itself. For $f \in H(B;B)$ we denote the iterates of f by f_n:

$$f_1 = f, \qquad f_{n+1} = f \circ f_n \qquad n = 1,2,3,\ldots$$

Since $H(B;B)$ is a normal family, every sequence of iterates of f contains a subsequence which converges, uniformly on compact subsets of B. We will examine the possible subsequential limits of $\{f_n\}$ according to the fixed point character of f. Note that a subsequential limit of iterates of $f \in H(B;B)$ need not belong to $H(B;B)$. However the following lemma shows that this can only happen if the limit is a constant map of norm 1.

Lemma 1.1. Let $F: B \to \overline{B}$ be holomorphic. Then either $F(B) \subseteq B$ or $F(z) \equiv \zeta$ in ∂B, for all z in B.

Proof: Suppose there is a z_0 in B with $F(z_0) = \zeta \in \partial B$. Set $G(z) = (1 + \langle z,\zeta\rangle)/2$, so G belongs to $A(B)$, the algebra of functions holomorphic in B and continuous on $\overline{B}$. Note that $G(\zeta) = 1$ and $|G(z)| < 1$ for all z in $\overline{B}\setminus\{\zeta\}$. Consider the holomorphic function $G \circ F$. Since $G \circ F(z_0) = 1$ and $|G \circ F(z)| \leq 1$ for all z in B, the maximum modulus theorem implies $G \circ F$ is identically 1. So $F(z) \equiv \zeta$, for all z in B, as desired. □

We will find it convenient to use some facts from the theory of topological semigroups. Under the operation of composition and with the topology of uniform convergence on compact subsets of B, $H(B;B)$ becomes a topological semigroup [21]. For $f \in H(B;B)$ denote by $\Gamma(f)$ the closure, in the space of all holomorphic maps from B to $\mathbb{C}^N$ with the topology of uniform convergence on compact subsets of B, of the iterates of f. If $\Gamma(f) \subseteq H(B;B)$, then $\Gamma(f)$ is a compact topological semigroup and as such contains a unique idempotent [23]. Recall g is an idempotent if $g \circ g = g$. An idempotent in $H(B;B)$ is also called a retraction of B.

We next give the statement of a theorem due to W. Rudin, which characterizes the fixed point set of any map in $H(B;B)$ as an affine subset of B. This theorem is the key to the proof of the main theorem of this section.

Theorem 1.2 [17; Sec. 8.2.3, p. 166]. If $F: B \to B$ is holomorphic, then the fixed point set of F is an affine subset of B; that is, the intersection of B with $c + L$, where $c \in \mathbb{C}^N$ and L is a complex linear subspace of $\mathbb{C}^N$.

Denote by Aut B the group of biholomorphic maps (automorphisms) of B onto itself. These maps take affine subsets of B onto affine subsets [17, Sec. 2.4.2, p. 33]. Moreover, since Aut B acts transitively on B [17, Sec. 2.2.3, p. 31], if A is an affine subset of B there is a $\psi \in \text{Aut } B$ so that $\psi(A) = \{(z_1,z_2,\ldots,z_N) \in B$ with $z_i = 0$ for $i = r+1,\ldots,N\}$. To see this, first map some point of A to the origin, so that the image of A is the intersection of B with a complex linear subspace of $\mathbb{C}^N$. Now apply a unitary transformation. Thus $\psi(A) \cong B_r$, the unit ball in $\mathbb{C}^r$. We will say $f \in H(B;B)$ is an automorphism of A if $\psi \circ f \circ \psi^{-1}$ is an automorphism when restricted to $\psi(A)$.

Before stating the main result of this section, we need to develop a several variable analogue of a theorem which in the disc is due to J. Wolff [25]. To facilitate the statement of this theorem we introduce some notation. Let

$$e_1 = (1,0,\ldots,0) = (1,0') \in \partial B.$$

For $\lambda > 0$,

$$E(e_1,\lambda) = \{z = (z_1,z_2,\ldots,z_N) \text{ so that } |1 - z_1|^2 < \lambda(1-|z|^2)\}.$$

Some computation shows that $E(e_1,\lambda)$ is the set of points $(z_1,z_2,\ldots,z_N) = (z_1,z')$ in $\mathbb{C}^N$ satisfying

$$|z_1 - (1 - c)|^2 + c|z'|^2 < c^2$$

where $c = \lambda/(1+\lambda)$. Thus $E(e_1,\lambda)$ is an ellipsoid in B, centered at $e_1/(1+\lambda)$ and containing e_1 in its boundary. For an arbitrary ζ in ∂B, $E(\zeta,\lambda)$ is the analogous ellipsoid in B, centered at $\zeta/(1+\lambda)$ and containing ζ in its boundary.

Theorem 1.3. If f is in $H(B;B)$ and fixed point free, then there is a unique point $\zeta \in \partial B$ such that each ellipsoid $E(\zeta,\lambda)$ is mapped into itself by f and every iterate of f.

Proof: Choose $r_n \uparrow 1$. Let $a_n \in B$ be a fixed point of the map $r_n f\colon r_n\overline{B} \to r_n\overline{B}$. Passing to a subsequence if necessary, assume $a_n \to \zeta \in \overline{B}$. Since f has no fixed points in B, $\zeta \in \partial B$. Without loss of generality assume $\zeta = e_1$. Then $a_n \to e_1$, $f(a_n) = a_n/r_n \to e_1$, and

$$\frac{1-|f(a_n)|}{1 - |a_n|} = \frac{1 - (|a_n|/r_n)}{1 - |a_n|} < 1 .$$

Again passing to a subsequence if necessary we have

$$\lim_{n\to\infty} \frac{1 - |f(a_n)|}{1 - |a_n|} = \alpha \leq 1.$$

By Julia's lemma [17; Sec. 8.5.3, p. 175]

$$\frac{|1-f_1(z)|^2}{1-|f(z)|^2} \leq \alpha \frac{|1-z_1|^2}{1-|z|^2}$$

(here $f = (f_1, f_2, \ldots, f_N)$). Geometrically this means $f(E(e_1,\lambda)) \subseteq E(e_1,\alpha\lambda) \subseteq E(e_1,\lambda)$ since $\alpha \leq 1$, as desired.

To see that ζ is unique suppose we have another point ζ' in ∂B with the property that each ellipsoid $E(\zeta',\lambda)$ is mapped into itself by f. Choose λ_1 and λ_2 so that $E(\zeta',\lambda_1)$ and $E(\zeta,\lambda_2)$ are tangent to each other at the point z in B. Then $f(z)$ is in $\overline{E}(\zeta',\lambda_1) \cap \overline{E}(\zeta,\lambda_2) = \{z\}$, contradicting the hypothesis that f is fixed point free. □

<u>Notation</u>: We call the point ζ of Theorem 1.3 the Denjoy-Wolff point of f. The constant map $g(z) \equiv \zeta$ for z in B will be denoted $\zeta(f)$.

A consequence of Theorem 1.3 is the following:

<u>Corollary 1.4</u>. Let $f \in H(B;B)$ be fixed point free. Then $\Gamma(f)$ contains at most one constant map, which can only be $\zeta(f)$.

<u>Proof</u>: Let ζ be the Denjoy-Wolff point of f. Suppose there is a sequence $\{n_i\}$ so that $f_{n_i} \to w \in \overline{B}$. If $w \neq \zeta$ we can find a small neighborhood V of w in B disjoint from some ellipsoid $E(\zeta,\lambda)$. By Theorem 1.3, if z is any point in $E(\zeta,\lambda)$, then the image point $f_n(z)$ is in $E(\zeta,\lambda)$ for all $n \geq 1$. Thus $f_{n_i}(z) \notin V$ for any i,

so $f_{n_i}(z) \neq w$. Therefore the only constant function which may appear in $\Gamma(f)$ is $\zeta(f)$. $\square$

We can now state the main theorem of this section.

Theorem 1.5. Let f be in $H(B;B)$ and suppose f has no fixed points in B. Then $f_n \to \zeta(f)$.

We give the proof of Theorem 1.5 in several steps, beginning with the following proposition.

Proposition 1.6. Let f be an arbitrary map in $H(B;B)$. If there is a nonconstant map among the subsequential limits of $\{f_n\}$, then $\Gamma(f)$ contains a nonconstant idempotent.

Proof: We suppose there is a nonconstant map g and a sequence $\{n_i\}$ so that $f_{n_i} \to g$. Note that $g(B) \subseteq B$. Set $m_i = n_{i+1} - n_i$. Choose a convergent subsequence of $\{f_{m_i}\}$, say $f_{m_{i_k}} \to h$. On the one hand $f_{m_{i_k}} \circ f_{n_{i_k}} \to h \circ g$. But also $f_{m_i} \circ f_{n_i} = f_{n_{i+1}} \to g$. So $h \circ g = g$ which implies that h is the identity map on the range of g, which consists of more than one point. By Theorem 1.2 the fixed point set of h is an affine subset A of B. The dimension of A is ≥ 1 and the range of h contains A.

Now suppose that the range of h properly contains A. Then the above argument, applied to h instead of g, produces another subsequential limit of $\{f_n\}$ which is the identity on an affine subset

A' of B containing the range of h. Moreover the dimension of A' is strictly greater than the dimension of A. Choose from among the subsequential limits of $\{f_n\}$ a map H with fixed point set of maximal dimension. For this map H we must have range H = fixed point set of H, since otherwise there would be another subsequential limit with a fixed point set of larger dimension. Thus H is an idempotent, and since the dimension of the fixed point set of H is ≥ 1, H is nonconstant. $\square$

Our next goal is to establish Theorem 1.5 for automorphisms of B with no fixed points in B. If f is in Aut B then f is continuous from $\overline{B}$ to $\overline{B}$ and thus has a fixed point in $\overline{B}$. The automorphisms of B with no fixed points in B fix either exactly one or exactly two points of ∂B [17, Sec. 2.4.6, p. 33]. The case of two fixed points in ∂B is easy to handle:

Proposition 1.7. Let $f \in$ Aut B fix precisely two points of ∂B. Then f_n converges to one of these fixed points.

Proof: Suppose that f fixes ζ_1 and ζ_2 in ∂B. Consider the complex line L through ζ_1 and ζ_2. Since an automorphism takes complex lines to complex lines, f maps $L \cap B$ onto $L \cap B$. Now the Denjoy-Wolff theorem in one variable implies that the iterates of f restricted to $L \cap B$ converge to one of the fixed points, say ζ_1. By Lemma 1.1, every convergent subsequence of $\{f_n\}$ must converge to ζ_1. This implies that $f_n \to \zeta_1$, since $H(B;B)$ is a normal family. Clearly ζ_1 must be the Denjoy-Wolff point of f. $\square$

The case of one fixed point in ∂B requires more work. We will assume, without loss of generality, that the fixed point is $e_1 = (1,0')$. To study automorphisms of the disc it is convenient to transfer to the upper half plane via the biholomorphic map $z \to i(1 + z)/(1 - z)$. A similar device is available in several variables. Let $\Omega \subset \mathbb{C}^N$ be the region (<u>The Siegel upper half-space</u>) consisting of those points (w_1,w') with $\operatorname{Im} w_1 > |w'|^2$, where $w' = (w_2,\ldots,w_N)$, $|w'|^2 = |w_2|^2 + \ldots + |w_N|^2$. Define Φ, the Cayley transform, on $\mathbb{C}^N\setminus\{z_1 = 1\}$ by $\Phi(z) = i(e_1 + z)/(1 - z_1)$. Then Φ is a biholomorphic map of B onto Ω [17; Sec. 2.3.1, p. 31]. Moreover if $\overline{\Omega} = \Omega \cup \partial\Omega$, where $\partial\Omega = \{(w_1,w')$ such that $\operatorname{Im} w_1 = |w'|^2\}$, and $\overline{\Omega} \cup \{\infty\}$ is the one-point compactification of $\overline{\Omega}$, then defining $\Phi(e_1) = \infty$ extends Φ to a homeomorphism of $\overline{B}$ onto $\overline{\Omega} \cup \{\infty\}$. The automorphisms of B with fixed point set $\{e_1\}$ correspond to the automorphisms of Ω with fixed point set $\{\infty\}$.

An example of a class of such automorphisms are the Heisenberg translations, defined as follows. For each $b = (b_1,b')$ in $\partial\Omega$ set $h_b(w_1,w') = (w_1 + b_1 + 2i\langle w',b'\rangle, w' + b')$. The Heisenberg translations form a subgroup of $\operatorname{Aut}\Omega$, and for $b \neq 0$ each h_b fixes ∞ only [17; Sec. 2.3.3, p. 32]. By a Heisenberg translation of B we shall mean an automorphism of B of the form $\Phi^{-1} \circ h_b \circ \Phi$, where Φ is the Cayley transform and h_b is as above.

It is easy to see that the iterates of a Heisenberg translation converge to e_1, since for any $0 \neq b \in \partial\Omega$, $(h_b)_n \to \infty$. However, in contrast to the situation in one variable, not every automorphism of Ω with fixed point set precisely $\{\infty\}$ is a Heisenberg translation.

For example, if $\lambda = (\lambda_2,\ldots,\lambda_N)$ where $|\lambda_i| = 1$ and if $b \neq 0$ is real then $g_{b,\lambda}(w_1,w') \equiv (w_1 + b,\lambda_2 w_2,\ldots,\lambda_N w_N)$ is an automorphism of Ω fixing $\{\infty\}$ only. Note that $g_{b,\lambda}$ fixes setwise the image under Φ of the complex line through 0 and e_1, namely $\{(w_1,w') \in \Omega$ with $w' = 0\}$. We will see that any automorphism of B with fixed point set $\{e_1\}$ which is not a Heisenberg translation of B must fix setwise some nonempty, proper affine subset of B. A map f is said to fix a set S <u>setwise</u> if $f(S) \subseteq S$. In this situation we will also say f fixes S <u>as a set</u>.

<u>Theorem 1.8</u>. Let $G \in \mathrm{Aut}\, B$ fix e_1 only. Write $G = (G_1,G_2,\ldots,G_N)$. If

$$(*) \qquad |1 - G_1(z)|^2/(1 - |G(z)|^2) = |1 - z_1|^2/(1 - |z|^2)$$

holds for every z in B, then either G is a Heisenberg translation of B or G fixes as a set a proper, nonempty, affine subset of B.

<u>Remark</u>. Professor David Ullrich has pointed out to me that condition (*) of Theorem 1.8 <u>must</u> hold for an automorphism of B with fixed point set precisely $\{e_1\}$. We give a proof of this fact at the end of this section. Note that (*) has a simple geometric meaning: the boundary of each ellipsoid $E(e_1,\lambda)$ is mapped into itself by G.

Before giving a proof of Theorem 1.8 we will establish the following corollary.

<u>Corollary 1.9</u>. If $G \in \mathrm{Aut}\, B_N$ fixes e_1 only then $G_n \to e_1$.

Proof. If condition (*) of Theorem 1.8 fails to hold for some point w in B then by Theorem 1.3 we must have

$$|1 - G_1(w)|^2/(1 - |G(w)|^2) = \beta|1 - w_1|^2/(1 - |w|^2)$$

for some $\beta < 1$. Suppose further that G_n does not converge to $e_1 = \zeta(G)$. By Corollary 1.4 and Proposition 1.6 $\Gamma(G)$ contains a nonconstant idempotent. Moreover, by a theorem of H. Cartan [15; p. 78] the nonconstant subsequential limits of the iterates of an automorphism must again be automorphisms. Since the only idempotent which is an automorphism is the identity map I on B, $\Gamma(G)$ contains I. Thus there is a sequence $\{n_i\}$ so that $G_{n_i} \to I$. In particular $G_{n_i}(w) \to w$. But this cannot be, for w lies in the boundary of $E(e_1,\lambda)$ where $\lambda = |1 - w_1|^2/(1 - |w|^2)$ and $G_n(w)$ is in $\overline{E}(e_1,\beta\lambda) \subseteq \operatorname{int} E(e_1,\lambda)$ for every $n \geq 1$. This contradicts our assumption that G_n does not converge to e_1.

We suppose now that G satisfies (*) at every point of B and apply Theorem 1.8. If G is a Heisenberg translation $\Phi^{-1} \circ h_b \circ \Phi$, then $G_n \to e_1$ since $(h_b)_n \to \infty$. We finish the remaining case by induction. Note that the corollary is true for $N = 1$ and assume it holds for $k < N$. We are left to consider the possibility that G fixes setwise a nonempty, proper, affine subset A of B of dimension $k < N$. Now $\tilde{G} = G|_A$ is an automorphism of $A \cong B_k$ fixing e_1 only. By induction $\tilde{G}_n \to e_1$ and by Lemma 1.1 $G_n \to e_1$. □

In the proof of Theorem 1.8 we will transfer back and forth between the ball B and the Siegel upper half-space Ω via the Cayley transform Φ. For the proof of Theorem 1.8 we will use lower case letters to denote automorphisms of Ω and the corresponding capital letters for the associated automorphism of B obtained by composition on the right and left by Φ and Φ^{-1} respectively.

Proof of Theorem 1.8. Let $G \in \text{Aut } B_N$ fix e_1 only and satisfy (*). If $\Phi(z) = w$ then $\text{Im } w_1 - |w'|^2 = (1 - |z|^2)/|1 - z_1|^2$. Thus the boundary of the ellipsoid $E(e_1,\lambda)$ is mapped by Φ to $\{(w_1,w') \in \Omega$ such that $\text{Im } w_1 - |w'|^2 = 1/\lambda\}$. Condition (*) for $G \in \text{Aut } B_N$ becomes, for the function $g = \Phi \circ G \circ \Phi^{-1}$,

$$(**) \qquad \text{Im } g_1(w) - |g'(w)|^2 = \text{Im } w_1 - |w'|^2$$

where $g = (g_1,g_2,\ldots,g_N) = (g_1,g')$.

Set $G(0) = \alpha$ so $g(i,0') = \Phi(\alpha) = (a_1,a')$. Now $\text{Im } a_1 - |a'|^2 = 1$ since g satisfies (**). Write $a_1 = c + i(1 + |a'|^2)$ where c is real. We claim that there is a Heisenberg translation of Ω taking (a_1,a') to $(i,0')$. To see this consider the point $(c + i|a'|^2,a')$ in $\partial\Omega$. The Heisenberg translation associated to this point takes $(i,0')$ to (a_1,a'). Its inverse is a Heisenberg translation having the desired property; we denote it simply by h_b. (A computation shows that $b = (-c + i|a'|^2, -a')$).

Now $h_b \circ g$ is an automorphism of Ω fixing ∞ and $(i,0')$. The corresponding automorphism F of B fixes 0 and e_1, and is just $H_b \circ G$. Note that F is unitary. Moreover, since F fixes e_1, F fixes as a set the orthogonal complement of the complex line through e_1, namely the set $\{z_1 = 0\}$. Thus $F(z_1,z_2,\ldots,z_N) = F(z_1,z')= (z_1,Uz')$ where U is a unitary operator on $\mathbb{C}^{N-1}$. An easy computation shows that

$$F \circ \Phi^{-1}(w_1,w') = \left(\frac{w_1-i}{w_1+i}\,,\ \frac{2}{w_1+i}\,Uw'\right) = \Phi^{-1} \circ F(w_1,w')$$

on $\mathbb{C}^N \backslash \{w_1 = -i\}$. Therefore the automorphism f of Ω defined by $f = \Phi \circ F \circ \Phi^{-1}$ coincides with the original unitary map F on Ω;

$$f(w) = (w_1,Uw') \qquad (w = (w_1,w') \in \Omega)\ .$$

At this point we consider two cases. If every eigenvalue of U is 1 then U, and hence F, is the identity. Thus $G = H_b^{-1}$ is a Heisenberg translation of B and we are done. So we suppose that U has an eigenvalue $e^{i\theta} \neq 1$. We will show that this implies that G fixes setwise a proper affine subset of B. It is sufficient to show that $g = \Phi \circ G \circ \Phi^{-1}$ fixes setwise a proper affine subset of Ω, since Φ preserves affine sets.

Choose $0 \neq \Lambda = (\lambda_2,\lambda_3,\ldots,\lambda_N)$ so that $\Lambda(U) = e^{i\theta}\Lambda$ where (U) denotes the matrix of the operator U relative to the standard basis on $\mathbb{C}^{N-1}$. Recall that $g = \Phi \circ G \circ \Phi^{-1} = \Phi \circ H_b^{-1} \circ F \circ \Phi^{-1} = h_b^{-1} \circ f$, where h_b^{-1} is the Heisenberg translation associated to the point $(c + i|a'|^2, a_2,\ldots,a_N)$ in $\partial\Omega$. Let A be the column vector $(a_2,\ldots,a_N)^t$ so that $\Lambda A = \sum_{i=2}^{N} \lambda_i a_i$. Now consider the set

$$\mathfrak{F} = \{(w_1,w_2,\ldots,w_N) \in \Omega \ \text{ with } \ \sum_{i=2}^{N} \lambda_i w_i = \Lambda A/(1-e^{i\theta})\}\ .$$

$\mathfrak{F}$ is a nonempty, proper affine subset of Ω. We claim that g fixes $\mathfrak{F}$ as a set. To see this choose $(w_1,w_2,\ldots,w_N)$ in $\mathfrak{F}$. Now

$g(w_1,w_2,\ldots,w_N) = h_b^{-1} \circ f(w_1,w') = h_b^{-1}(w_1,Uw')$. Writing $W' = (w_2,w_3,\ldots,w_N)^t$ we see that the last $N-1$ coordinates of $h_b^{-1}(w_1,Uw')$ are $((U)W' + A)^t$. To check that $g(w_1,w_2,\ldots,w_N)$ is in $\mathfrak{F}$ we compute

$$\begin{aligned}\Lambda((U)W' + A) &= e^{i\theta}\Lambda W' + \Lambda A \\ &= e^{i\theta}\Lambda A/(1 - e^{i\theta}) + \Lambda A \\ &= \Lambda A/(1 - e^{i\theta}) .\end{aligned}$$

Therefore $g(w_1,w')$ is in $\mathfrak{F}$, as desired. □

A final observation before the proof of Theorem 1.5 is the following.

Lemma 1.10. If $f \in H(B;B)$ is such that $f_{n_i} \to I$, the identity map on B, for some sequence $\{n_i\}$, then $f \in \text{Aut } B$.

Proof: We may assume $f_{n_i-1} \to g$. Then $f_{n_i-1} \circ f \to g \circ f$. Since $f_{n_i} \to I$ we have $g \circ f = I$. In particular g is in $H(B;B)$ and therefore we also have $f_{n_i} = f \circ f_{n_i-1} \to f \circ g$. So $f \circ g = g \circ f = I$ as desired. □

Proof of Theorem 1.5. Proposition 1.7 and Corollary 1.9 together establish Theorem 1.5 for automorphisms of B with no fixed points in B. Now suppose f is an arbitrary fixed point free map in $H(B;B)$. If every subsequential limit of $\{f_n\}$ is constant then by Corollary 1.4 $f_n \to \zeta(f)$, uniformly on compact subsets of B, and we are done.

Hence we suppose there is a nonconstant map among the subsequential limits of $\{f_n\}$. By Proposition 1.6 there is a sequence $\{n_i\}$ and a nonconstant idempotent h so that $f_{n_i} \to h$. Let A be the fixed point set of h, an affine set of dimension ≥ 1.

We claim that f maps A into A. To see this choose z_0 in A. Now $f_{n_i}(z_0) \to h(z_0) = z_0$ and thus $f(f_{n_i}(z_0)) \to f(z_0)$. But $f(f_{n_i}(z_0)) = f_{n_i}(f(z_0)) \to h(f(z_0))$. So $f(z_0) = h(f(z_0))$; that is, $f(z_0)$ is in the fixed point set A of h, as desired.

Moreover, f_{n_i} restricted to A converges to the identity on A. Lemma 1.10, with A replacing B, implies that $\tilde{f} \equiv f|_A$ is an automorphism of A, which clearly has no interior fixed points. By Corollary 1.9, $\tilde{f}_n$ converges to a constant in ∂B. But this contradicts the fact that $\tilde{f}_{n_i}$ converges to the identity map on A. Thus the subsequential limits of $\{f_n\}$ must all be constant and we are done by Corollary 1.4. □

We finish this section with a proof of the fact that condition (*) of Theorem 1.8 must hold for any $G \in \text{Aut } B$ with fixed point set $\{e_1\}$. As previously remarked this is equivalent to the following:

<u>Theorem 1.11</u>. Let $g \in \text{Aut } \Omega$ fix ∞ only. Then for every $w = (w_1, w')$ in Ω

$$(**) \qquad \text{Im } g_1(w) - |g'(w)|^2 = \text{Im } w_1 - |w'|^2 .$$

<u>Proof</u>. Suppose $g(i,0') = (a_1,a')$. Set $t = \text{Im } a_1 - |a'|^2$. Since (a_1,a') is in Ω, t is positive. For $s > 0$ define $\delta_s \in \text{Aut } \Omega$ by

$\delta_s(w_1,w') = (s^2w_1,sw')$. If $s \neq 1$ the fixed point set of δ_s is $\{0,\infty\}$. Consider the automorphism $\delta_s \circ g$ where $s = 1/\sqrt{t}$. The image of $(i,0')$ under this map is $(t^{-1}a_1,t^{-1/2}a')$ and $\mathrm{Im}(t^{-1}a_1) - t^{-1}|a'|^2 = 1$. Thus, as in the proof of Theorem 1.8, there is a Heisenberg translation h_c^{-1} so that $h_c^{-1} \circ \delta_s \circ g$ fixes $(i,0')$ and ∞. Moreover we must have, for some unitary operator U,

$$h_c^{-1} \circ \delta_s \circ g(w_1,w') = (w_1,Uw')$$

so that

$$\begin{aligned} g(w_1,w') &= \delta_{\sqrt{t}} \circ h_c(w_1,Uw') \\ &= (t(w_1 + c_1 + 2i\langle Uw',c'\rangle),\sqrt{t}(Uw' + c')). \end{aligned}$$

If $t = 1$ we have $g(w_1,w') = h_c(w_1,Uw')$ and an easy computation shows that g satisfies (**). Suppose that $t \neq 1$. We will show that this contradicts the hypothesis that g fixes ∞ only by producing a point in $\partial\Omega$ fixed by g.

If $t \neq 1$ we may solve $\sqrt{t}(Uw' + c') = w'$, since $U - t^{-1/2}I$ is nonsingular. Let v' denote the solution. If $v_1 = \alpha + i|v'|^2$ where α is real, then (v_1,v') will be in $\partial\Omega$. We claim we may choose α so that $g(v_1,v') = (v_1,v')$. By our choice of v' we have

$$g(v_1,v') = (t(\alpha + i|v'|^2 + c_1 + 2i\langle Uv',c'\rangle),v') .$$

We wish to have $t(\alpha + i|v'|^2 + c_1 + 2i\langle Uv',c'\rangle) = \alpha + i|v'|^2$. Since (v_1,v') is in $\partial\Omega$ and g is an automorphism, $g(v_1,v')$ lies in $\partial\Omega$.

Thus for any real α,

$$\operatorname{Im} t(\alpha + i|v'|^2 + c_1 + 2i\langle Uv',c'\rangle) = |v'|^2 = \operatorname{Im}(\alpha + i|v'|^2).$$

Thus (v_1,v') will be a fixed point of g if α is chosen in $\mathbb{R}$ to satisfy

$$\operatorname{Re} t(\alpha + i|v'|^2 + c_1 + 2i\langle Uv',c'\rangle) = \alpha = \operatorname{Re}(\alpha + i|v'|^2)$$

or

$$t\alpha + \operatorname{Re} t(c_1 + 2i\langle Uv',c'\rangle) = \alpha .$$

Since $t \neq 1$ we may solve this equation for real α. Thus the assumption that $t \neq 1$ implies that the fixed point set of g contains more than one point, contradicting the hypothesis. □

2. <u>Maps which fix an interior point</u>. We consider now the case of $f \in H(B;B)$ fixing at least one point of B. Several remarks can be made about the sequence of iterates of f; we collect these comments together in:

<u>Theorem 1.12</u>. Let $f \in H(B;B)$ have a fixed point in B. Then <u>either</u>

(1) There is a constant function $g(z) \equiv z_0 \in B$ in $\Gamma(f)$. In this case $f_n \to g$, and the fixed point set of f is of course precisely $\{z_0\}$.

<u>or</u>

(2) There is a sequence $\{m_i\}$ such that f_{m_i} converges to

a nonconstant idempotent h. The fixed point set of h is an affine subset of B which may be strictly larger than the fixed point set of f, even if f is not in Aut B. Moreover, if f is not an automorphism of B then every subsequential limit of $\{f_n\}$ is degenerate in the sense that its range is contained in an affine subset of B of lower dimension than B.

<u>Proof</u>: Suppose there is a sequence $\{n_i\}$ such that f_{n_i} converges to a constant function g. Then clearly the fixed point set of f is precisely the range of g. We claim $f_n \to g$, for otherwise there is a sequence $\{m_i\}$ such that $f_{m_i} \to h$, where h is not a constant map. Without loss of generality $f_{m_i-n_i} \to k \in H(B;B)$. Then $f_{m_i-n_i} \circ f_{n_i} \to k \circ g$ and also $f_{m_i-n_i} \circ f_{n_i} = f_{m_i} \to h$. But $k \circ g$ is constant and h is not, which is a contradiction. This proves (1).

If there is no constant map in $\Gamma(f)$, then Proposition 1.6 shows that there is a nonconstant idempotent among the subsequential limits of $\{f_n\}$. Moreover, the proof of Proposition 1.6 shows that given any nonconstant subsequential limit G there is a subsequential limit H which is the identity map on the range of G. Thus if the affine subset of B of smallest dimension containing the range of G is all of B, then the identity map on B is a subsequential limit of $\{f_n\}$. This implies that f is an automorphism of B, by Lemma 1.10. $\square$

For an example where the fixed point set of the limit function is strictly larger than the fixed point set of f, let g be a holomorphic function on the unit disc, with $|g| < 1$. Define f on

B_2 by $f(z_1,z_2) = (-z_1, g(z_1)z_2)$. Thus $f \in H(B_2;B_2)$ and the fixed point set of f is $\{(0,0)\}$. Now $f_{2k}(z_1,z_2) = (z_1, g^k(z_1)g^k(-z_1)z_2)$ and $f_{2k} \to h$, where $h(z_1,z_2) = (z_1,0)$.

We remark that case (2) of Theorem 1.12 can only occur if f acts as an automorphism on some affine set in B of dimension ≥ 1.

Remarks on Theorems 1.5 and 1.12. Some similar results have been obtained by Yoshisha Kubota [13], using different methods. He does not consider the fixed point free maps as a separate case, and his result does not show that in this situation the entire sequence of iterates converges to a point in ∂B.

Corollary 1.9 has also been independently obtained by David Ullrich. His argument, while similar in spirit to ours, uses the Iwasawa decomposition for $g \in \operatorname{Aut}\Omega$ as $g = \psi \circ \delta_\lambda \circ h_b$ where h_b is a Heisenberg translation, $\delta_\lambda(w_1,w') = (\lambda^2 w_1, \lambda w')$ and ψ is an automorphism of Ω fixing $(i,0')$ in Ω. He shows that $\lambda = 1$ if g fixes ∞ only and that $\psi(w_1,w') = (w_1, Uw')$ for some unitary operator U on $\mathbb{C}^{N-1}$. The remainder of the argument proceeds as before.

CHAPTER II

In this chapter we introduce composition operators on the Hardy space $H^p(B_N)$. For $N > 1$, we give some examples of maps which fail to induce bounded composition operators on any $H^p(B_N)$ for $p < \infty$, and discuss some other results related to the question of boundedness.

1. Notation and Preliminaries.

For φ a holomorphic map of B_N into B_N and f a holomorphic function on B_N, the composition $f \circ \varphi$ is denoted by $C_\varphi(f)$. In the case $N = 1$, Littlewood's subordination principle [8] shows that C_φ is a bounded linear operator, called a <u>composition operator</u>, on the Hardy space $H^p(\mathbb{D})$, for each $p > 0$. However for $N > 1$ this need no longer be the case; in fact we will show that there are maps $\varphi\colon B_N \to B_N$ so that for each $p < \infty$ there exist functions $f \in H^p(B_N)$ for which $f \circ \varphi$ is not even in the Nevanlinna class $N(B_N)$.

2. Examples of composition operators which are not bounded on $H^p(B_N)$.

It is convenient at this point to introduce certain spaces of holomorphic functions in the unit disc $\mathbb{D}$, and examine their connection with the spaces $H^p(B_N)$.

<u>Definition 2.1</u>. For $\alpha > -1$, the weighted Bergman space $A^{p,\alpha}(\mathbb{D})$ consists of all holomorphic functions f on $\mathbb{D}$ for which

$$\|f\|_{p,\alpha}^{p} \equiv \int_0^{2\pi} \int_0^1 |f(re^{i\theta})|^p (1 - r^2)^\alpha \, r dr d\theta < \infty .$$

Functions in certain weighted Bergman spaces arise naturally in the study of H^p functions in the ball in several variables. In particular, we have the following result:

Lemma 2.2. [17; Sec. 1.4.4, p. 14]. If f is a holomorphic function in B_N which depends only on the variable z_1, then, if f_{e_1} denotes the slice function on $\mathbb{D}$ defined by $f_{e_1}(\lambda) = f(\lambda e_1)$,

$$\|f\|_{H^p} = c\|f_{e_1}\|_{A^{p,N-2}}$$

where c is a constant depending only on N and p.

We wish to obtain an extension of Lemma 2.2 to arbitrary functions in $H^p(B_N)$. To do this we need the following preliminary result:

Lemma 2.3. Let F be in $H^p(B_N)$ and define f on B_N by $f(z_1,z') = F(z_1,0)$. Then f is in $H^p(B_N)$ with $\|f\|_p \leq \|F\|_p$.

Proof. The argument we give is basically that given in [19; p. 247]. For ζ in ∂B_N and z in $\bar{\Delta}^N$, the closed polydisc in $\mathbb{C}^N$, define

$$w(z,\zeta) = |F(z_1\zeta_1,\ldots,z_N\zeta_N)|^p .$$

For each ζ in ∂B_N, $w_\zeta(z) \equiv w(z,\zeta)$ is an N-subharmonic function in Δ^N, that is, w_ζ is subharmonic in each variable separately.

Define $W(z) = \int_{\partial B_N} w(z,\zeta)d\sigma(\zeta)$. Since σ is unitarily invariant, $W(z_1,\ldots,z_N) = W(|z_1|,\ldots,|z_N|)$. This, combined with the N-subharmonicity of W, shows that for all $r < 1$

$$W(r,0,\ldots,0) \leq W(r,r,0,\ldots,0) \leq \ldots \leq W(r,r,\ldots,r) .$$

But $W(r,0,\ldots,0) = \int_{\partial B_N} |F(r\zeta_1,0,\ldots,)|^p d\sigma(\zeta) = \int_{\partial B_N} |f(r\zeta)|^p d\sigma(\zeta)$ and $W(r,r,\ldots,r) = \int_{\partial B_N} |F(r\zeta)|^p d\sigma(\zeta)$, so that $\|f\|_{H^p} \leq \|F\|_{H^p}$ as desired. □

Lemmas 2.2 and 2.3 together yield:

<u>Corollary 2.4</u>. If F is in $H^p(B_N)$, $N \geq 2$, and F_{e_1} is the slice function defined on $\mathbb{D}$ by $F_{e_1}(\lambda) = F(\lambda e_1)$, then F_{e_1} is in $A^{p,N-2}(\mathbb{D})$ and $\|F\|_{H^p} \geq c\|F_{e_1}\|_{A^{p,N-2}}$.

We can now give some examples of holomorphic maps Φ: $B_N \to B_N$, where $N > 1$, which do not induce bounded composition operators on any $H^p(B_N)$, for $1 \leq p < \infty$. Our first result concerns maps Φ defined as follows. Let α be any multi-index $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_N)$, where α_i is a non-negative integer and at least two of the α_i's are nonzero. Define $\Phi(z) = (c(\alpha)z^\alpha,0,\ldots,0)$ where

$$(*) \qquad c(\alpha) = |\alpha|^{|\alpha|/2} \Big/ \prod_{\alpha_i \neq 0} \alpha_i^{\alpha_i/2} \qquad (\textstyle\sum \alpha_i = |\alpha|) .$$

Note that $\Phi(B) \subseteq B$.

Theorem 2.5. For Φ as defined above, C_Φ is not bounded on $H^p(B_N)$, $1 \leq p < \infty$.

Proof. We will exhibit functions g_n in the unit ball of $H^p(B_N)$ for which $\|g_n \circ \Phi\|_p \to \infty$ as $n \to \infty$.

A computation based on [17; Sec. 1.4.4, p. 14] shows that if $f(z) = z_1^n$, then

$$\|f\|_{H^p}^p \leq C(N)\Gamma(np + 2)/\Gamma(np + N + 1)$$

where $C(N)$ is a constant depending only on the dimension N. Thus if

$$g_n(z) = [C(N)^{-1}\Gamma(np + N + 1)/\Gamma(np + 2)]^{1/p}\, z_1^n$$

then g_n is in the unit ball of $H^p(B_N)$.

A second computation, entirely similar to that in [17; Sec. 1.4.9, p. 16] shows that

$$\int_S |\zeta^\alpha|^p d\sigma(\zeta) = \frac{N! \prod_{j=1}^{N} \Gamma(p\alpha_j/2 + 1)}{N\, \Gamma(N + |\alpha|p/2)}$$

where σ is the rotation invariant probability measure on $S = \partial B_N$.

For g_n as defined above, we have $\|g_n \circ \Phi\|_p^p =$

$$C(N)^{-1} \frac{\Gamma(np + N + 1)}{\Gamma(np + 2)}\; c(\alpha)^{np} \int_S |\zeta^\alpha|^{np}\, d\sigma(\zeta) =$$

$$(**) \qquad C(N)^{-1} \frac{\Gamma(np + N + 1)}{\Gamma(np + 2)}\; c(\alpha)^{np} \frac{\prod_{j=1}^{N} \Gamma(p\alpha_j/2 + 1)}{\Gamma(N + |\alpha|p/2)}\, .$$

Note that $\Gamma(np + N + 1)/\Gamma(np + 2) = (np + N)(np + N - 1) \ldots (np + 2)$ and for n large this is approximately $(np)^{N-1}$. Using this, the definition of $c(\alpha)$, and Stirling's formula in (**) we see that $\|g_n \circ \Phi\|_p^p \approx n^{(m-1)/2}$ where m is the number of indices α_j which are nonzero. The symbol $\approx$ means that the two terms being compared have positive finite limit as $n \to \infty$. Thus if $m > 1$ (at least two of the indices α_j are nonzero) then $\|g_n \circ \Phi\|_p^p \to \infty$ as $n \to \infty$. Thus C_Φ is not bounded on $H^p(B_N)$, since g_n is in the unit ball of $H^p(B_N)$. □

Remarks on the proof of Theorem 2.5. In the case $p = 2$ we may explicitly exhibit functions F in $H^2(B_N)$ for which $F \circ \Phi$ is not in $H^2(B_N)$. Choose constants C_k satisfying

$$(1) \qquad \sum_{k=0}^{\infty} |C_k|^2 (k + 1)^{-N + 1} < \infty$$

and

$$(2) \qquad \sum_{k=0}^{\infty} |C_k|^2 k^{(m+1-2N)/2} = \infty$$

where, as above, m is the number of nonzero α_j's.

Condition (1) guarantees that the function $f(z) = \sum_{k=0}^{\infty} C_k z^k$ is in the weighted Bergman space $A^{2,N-2}(\mathbb{D})$. Thus, by Lemma 2.2 f extends to a function F in $H^2(B_N)$ defined by $F(z_1, z') = f(z_1)$. The same estimates as above show that

$$\int_S |(c(\alpha)\zeta^\alpha)^k|^2 \, d\sigma(\zeta) \approx k^{(m+1-2N)/2}.$$

Thus condition (2) guarantees that $F \circ \Phi(z) = \sum_{k=0}^{\infty} C_k(c(\alpha)z^{\alpha})^k$ is not in $H^2(B_N)$.

We remark that the map $\psi\colon B_N \to \mathbb{D}$ given by $\psi(z_1,\dots,z_N) = N^{N/2} \prod_{j=1}^{N} z_j$ appears in the work of P. Ahern [1], where it is shown that if h is in the weighted Bergman space $A^{p,(N-3)/2}(\mathbb{D}) \equiv A^{p,(N-3)/2}$, then $h \circ \psi$ is in $H^p(B_N)$. From this fact it follows that the map $\Phi\colon B_N \to B_N$, defined by $\Phi = (\psi,0,\dots,0)$, (this is the case $\alpha = (1,1,\dots,1)$ in (*)), does take $H^p(B_N)$ boundedly into $H^{p/2-\varepsilon}(B_N)$, for every $\varepsilon > 0$. To see this note first that $A^{p,N-2} \subseteq A^{p/2-\varepsilon,\ (N-3)/2}$. (See [10] for containment relations between weighted Bergman spaces). If f is in $H^p(B_N)$, we have $f \circ \Phi = \tilde{f} \circ \psi$, where $\tilde{f}$ is the restriction of f to the complex line $[e_1]$ through 0 and $e_1 = (1,0')$. Since f is in $H^p(B_N)$, $\tilde{f}$ is in $A^{p,N-2}$, and thus also in $A^{p/2-\varepsilon,(N-3)/2}$. Ahern's result now shows that $f \circ \Phi = \tilde{f} \circ \psi$ is in $H^{p/2-\varepsilon}(B_N)$, as desired.

Recently A.B. Aleksandrov [2] and E. Low [12] have independently shown the existence of non-constant inner functions on B_N, for $N > 1$. An inner function in B is a function $f \in H^{\infty}(B)$ whose radial limits f^* satisfy $|f^*(\zeta)| = 1$ for almost every $\zeta \in S$. The existence of such functions gives a way of constructing maps $\Phi\colon B_N \to B_N$ $(N > 1)$ for which C_Φ is not bounded on $H^p(B_N)$, and moreover, $C_\Phi(H^p(B_N)) \not\subseteq N(B_N)$, where $N(B_N)$ is the Nevanlinna class consisting of all holomorphic functions in B_N satisfying

$$\sup_{0<r<1} \int_S \log^+ |f(r\zeta)|\,d\sigma(\zeta) < \infty .$$

Specifically, we have the following result.

Theorem 2.6. Let $u(z)$ be a nonconstant inner function on B_N, $N > 1$, and define Φ to be an inner map of B_N into B_N by $\Phi(z) = (u(z), 0,\dots,0)$. Then for any $p < \infty$, $C_\Phi(H^p(B_N)) \not\subset N(B_N)$.

Proof. By a theorem of Bagemihl, Erdos and Seidel [14, Theorem 4], for any $p < \infty$, there is a function $g \in A^p(\mathbb{D})$ such that $|g|$ assumes arbitrarily large values along any curve in $\mathbb{D}$ which tends to $\partial\mathbb{D}$. Extend g to a function defined on B_N by setting $G(z_1,z') = g(z_1)$. Note that G is in $H^p(B_N)$. For almost every $\zeta \in S$, $\Phi(r\zeta)$ is a curve in $[e_1] \cap B_N \approx \mathbb{D}$ tending to a point of $\partial\mathbb{D}$ as $r \uparrow 1$. For each such ζ, $\sup\limits_{0<r<1} |G \circ \Phi(r\zeta)| = \infty$. Since a function in the Nevanlinna class has finite radial limits at almost every point of S, $G \circ \Phi \notin N(B_N)$. □

Remarks on Theorem 2.6. a) If f is any nonconstant function in $H^\infty(B_N)$ with $\|f\|_\infty = 1$ which has $|f^*(\zeta)| = 1$ on a subset of S of positive measure, then the above argument shows that for the map $\Phi = (f,0,\dots,0)$ we have $C_\Phi(H^p(B_N)) \not\subset N(B_N)$. It is an open question [17; Sec. 11.4.1, p. 247] whether or not there exists such a function f in $A(B_N) = H(B_N) \cap C(\overline{B}_N)$.

b) Examples similar to those of Theorem 2.6 can be given with the polydisc $\Delta^N = \{(z_1,\dots,z_N) \in \mathbb{C}^N\colon \ |z_i| < 1\}$ replacing the unit ball. To see this, let $u(z)$ be any non-constant inner function on Δ^N. Construct $\varphi\colon \Delta^N \to \Delta^N$ by $\varphi(z) = (u(z),\dots,u(z))$, so that for almost every $\zeta \in T^N = \{(z_1,\dots,z_N)\colon \ |z_i| = 1\}$, $\varphi(r\zeta)$ is a curve in $\operatorname{diag}\Delta^N = \{(z_1,\dots,z_N) \in \Delta^N\colon \ z_1 = \dots = z_N\}$ tending to a point of the boundary of diag Δ^N. Let f be any function in $A^{p,N-2}(\mathbb{D})$ which assumes arbitrarily large values along every curve in $\partial\mathbb{D}$ tending to $\mathbb{D}$, as in the proof of Theorem 2.6. C. Horowitz and D. Oberlin [11] have shown that restriction to the diagonal takes $H^p(\Delta^N)$ onto $A^{p,N-2}(\mathbb{D})$. Thus there is a function F in $H^p(\Delta^N)$ with $F = f$ on diag Δ^N. The composition $F \circ \varphi$ fails to have finite radial limits at almost every point of T^N, hence it is not in the Nevanlinna class $N(\Delta^N)$, the collection of all functions g holomorphic in Δ^N satisfying

$$\sup_{0<r<1} \int_{T^N} \log^+|f(rw)|\ dm_N(w) < \infty.$$

In contrast with the situation in B_N, inner functions on Δ^N can be nice on $\partial\Delta^N$. In particular, there are non-constant inner functions in $A(\Delta^N)$, the class of holomorphic functions on Δ^N which are continuous on $\overline{\Delta^N}$ [see 16].

The sort of maps which appeared in Theorems 2.5 and 2.6, that is, maps of B_N into B_N whose range is contained in a one-dimensional affine subset of B_N, do take H^p spaces boundedly into certain Bergman spaces of functions in B_N. Weighted Bergman spaces in the disc have already been defined (see Definition 2.1); more generally we have

Definition 2.7. For $0 < p < \infty$ and $\alpha > -1$ let $A^{p,\alpha}(B_N)$ be the space of all functions f holomorphic in B_N satisfying

$$\|f\|_{p,\alpha}^p \equiv c_N \int_{B_N} |f(z)|^p\ (1-|z|^2)^\alpha\ dv(z) < \infty,$$

where v is Lebesgue measure on $\mathbb{C}^N = \mathbb{R}^{2N}$, normalized so that

$v(B_N) = 1$, and c_N is a constant depending on N, whose value will not interest us. When $\alpha = 0$ we write $A^p(B)$ instead of $A^{p,\alpha}(B)$.

We have the following lemma.

Lemma 2.8. Let g be a holomorphic function in B_N. If the slice functions g_ζ, defined by $g_\zeta(\lambda) = g(\lambda\zeta)$ $(\lambda \in \mathbb{D}, \zeta \in S)$ form a bounded family in $A^{p,\alpha}(\mathbb{D})$ as ζ runs through S, then g is in $A^{p,\alpha}(B_N)$.

Proof. Changing to polar coordinates we have

$$\int_{B_n} |g(z)|^p (1-|z|^2)^\alpha \, dv(z) = 2N \int_0^1 r^{2N-1}(1-r^2)^\alpha dr \int_S |g(r\zeta)|^p d\sigma(\zeta)$$

$$\leq 2N \int_0^1 r(1-r^2)^\alpha dr \int_S |g(r\zeta)|^p d\sigma(\zeta).$$

Using slice integration [17; Sec. 1.4.7, p. 15] we have

$$\int_S |g(r\zeta)|^p \, d\sigma(\zeta) = \int_S d\sigma(\zeta) \int_{-\pi}^{\pi} |g(re^{i\theta}\zeta)|^p d\theta/2\pi.$$

Thus

$$\int_{B_n} |g(z)|^p(1-|z|^2)^\alpha \, dv(z) \leq c(N) \int_S d\sigma(\zeta) \int_{-\pi}^{\pi}\int_0^1 |g_\zeta(re^{i\theta})|^p r(1-r^2)^\alpha dr d\theta$$

$$= c(N) \int_S d\sigma(\zeta) \int_{\mathbb{D}} |g_\zeta(z)|^p(1-|z|^2)^\alpha dv(z).$$

By hypothesis, $\|g_\zeta\|_{p,\alpha} \leq K$, for all $\zeta \in S$, so from the last line we see that g is in $A^{p,\alpha}(B_N)$. □

We use this lemma in the proof of the following result.

Proposition 2.9. Suppose Φ is a holomorphic mapping of B_n into B_N of the form $\Phi = (\varphi, 0, \ldots, 0)$. Then C_Φ takes $H^p(B_N)$ into $A^{p,N-2}(B_N)$, with

$$\|C_\Phi(f)\|_{p,N-2} \leq c(N,p) \left(\frac{1 + \Phi|(0)|}{1-|\Phi(0)|}\right)^{N/p} \|f\|_p .$$

Proof. Let f be in $H^p(B_N)$, and let F denote the restriction of f to $[e_1]$, the complex line through 0 and e_1. Then F is in $A^{p,N-2}(\mathbb{D})$, with $\|F\|_{p,N-2} \leq c(N,p)\,\|f\|_p$, where $c(N,p)$ is a constant depending only on N and p.

Note that $(f \circ \Phi)_\zeta = F \circ \varphi_\zeta$. Moreover, since φ_ζ is a holomorphic map of $\mathbb{D}$ into $\mathbb{D}$, and F is in $A^{p,N-2}(\mathbb{D})$, we see that $F \circ \varphi_\zeta$ is in $A^{p,N-2}(\mathbb{D})$, with

$$\|F \circ \varphi_\zeta\|_{p,N-2} \leq \left(\frac{1 + |\varphi_\zeta(0)|}{1-|\varphi_\zeta(0)|}\right)^{N/p} \|F\|_{p,N-2}$$

$$= \left(\frac{1 + |\varphi(0)|}{1-|\varphi(0)|}\right)^{N/p} \|F\|_{p,N-2} .$$

(This estimate, in the case $p = 2$ and $N-2 = 0$ appears in [3]. A similar argument yields the result in the more general form we need.) Thus we have shown that $\{(f \circ \Phi)_\zeta : \zeta \in S\}$ is a bounded family in $A^{p,N-2}(\mathbb{D})$. Lemma 2.8 now shows that $f \circ \Phi$ is in $A^{p,N-2}(B_N)$, with

$$\|f \circ \Phi\|_{p,N-2} \leq c(N,p) \left(\frac{1 + |\Phi(0)|}{1-|\Phi(0)|}\right)^{N/p} \|f\|_p . \qquad \square$$

3. Sufficient conditions for boundedness of C_φ

In spite of the examples of the last section, there are still many interesting examples of bounded composition operators on $H^p(B_N)$. In particular, if φ is an automorphism of B_N, then C_φ is a bounded operator on every $H^p(B_N)$ [17; Sec. 5.6, p. 85]. The next proposition gives a necessary and sufficient condition for C_φ to be a Hilbert-Schmidt operator on $H^2(B_N)$. In particular C_φ will be bounded, and in fact compact. The case $N = 1$ of this result is in [20, Theorem 3.1].

Proposition 2.10. C_φ is a Hilbert-Schmidt operator on $H^2(B_N)$ if and only if φ satisfies

$$\int_S [1-|\varphi(\zeta)|]^{-N}\, d\sigma(\zeta) < \infty .$$

Proof. The functions $e_\alpha = c(\alpha)z^\alpha$ form an orthonormal basis for $H^2(B_N)$, where α is a multi-index $\alpha = (\alpha_1,\dots,\alpha_N)$ of non-negative integers, z^α denotes $z_1^{\alpha_1}\dots z_N^{\alpha_N}$ and

$$c(\alpha)^2 = \frac{(N-1+|\alpha|)!}{(N-1)!\,\alpha!} \qquad (|\alpha| = \textstyle\sum \alpha_j,\ \alpha! = \alpha_1!\dots\alpha_N!) .$$

Thus C_φ is Hilbert-Schmidt on $H^2(B_N)$ if and only if

$$\infty > \sum_\alpha \|e_\alpha \circ \varphi\|_2^2 = \sum_\alpha \int_S |C(\alpha)\varphi^\alpha|^2\, d\sigma$$

$$= \sum_{n=0}^{\infty} \int_S \sum_{|\alpha|=n} |c(\alpha)\varphi^\alpha|^2\, d\sigma.$$

If $\varphi = (\varphi_1, \ldots, \varphi_N)$, by φ^α we mean $\varphi_1^{\alpha_1} \ldots \varphi_N^{\alpha_N}$. Using the definition of $c(\alpha)$ and the multi-nomial theorem we see that

$$\sum_{|\alpha| = n} |c(\alpha)\varphi^\alpha|^2 = \frac{(N-1+n)!}{(N-1)!\, n!} (|\varphi|^2)^n .$$

Thus we have

$$\infty > \int_S \sum_{n=0}^{\infty} \frac{(N-1+n)!}{(N-1)!\, n!} (|\varphi|^2)^n \, d\sigma$$

$$= \int_S (1-|\varphi|^2)^{-N} \, d\sigma .$$

Therefore C_φ is a Hilbert-Schmidt operator on $H^2(B_N)$ if and only if $\int_S (1-|\varphi|)^{-N} \, d\sigma < \infty$. □

The remaining results of this section deal with situations in which the boundedness of C_φ on $H^p(B_N)$ for one value of p allows one to conclude that C_φ is bounded on $H^p(B_N)$ for some other values of p.

Proposition 2.11. If C_φ is bounded on $H^p(B_N)$, then C_φ is bounded on $H^{np}(B_N)$ for $n = 1,2,\ldots$.

Proof. If f is in $H^{np}(B_N)$, then f^n is in $H^p(B_N)$. Moreover $\|f \circ \varphi\|_{np}^{np} = \|f^n \circ \varphi\|_p^p \le \|C_\varphi\|^p \, \|f^n\|_p^p$, where $\|C_\varphi\|$ denotes the norm of C_φ as a bounded linear operator on $H^p(B_N)$. Since $\|f^n\|_p^p = \|f\|_{np}^{np}$, we have $\|f \circ \varphi\|_{np} \le \|C_\varphi\|^{1/n} \, \|f\|_{np}$, giving the desired result. □

Proposition 2.12. Suppose C_φ is bounded on $H^{p_1}(B)$ and $H^{p_2}(B)$, where $1 < p_1 < p_2 < \infty$. Then C_φ is bounded on $H^q(B)$, for all $p_1 < q < p_2$.

Proof. Denote the Cauchy transform of a function $f \in L^1(\sigma)$ by $C[f]$. For $1 < p < \infty$, the map T: $f \to C[f]^*$ is a bounded linear projection of $L^p(\sigma)$ onto $H^p(S) \cong H^p(B)$ [17; Sec. 6.3.1, p. 99]. The hypothesis on C_φ implies that $A \circ C_\varphi \circ A^{-1} \circ T$ is a bounded linear map of $L^{p_i}(\sigma)$ into $H^{p_i}(S) \subseteq L^{p_i}(S)$ $(i = 1,2)$, where A denotes the linear isometry of $H^p(B)$ and $H^p(S)$ given by $A(f) = f^*$, the radial limit function. By the M. Riesz convexity theorem [22, p. 179], $A \circ C_\varphi \circ A^{-1} \circ T$ is a bounded linear operator on $L^q(\sigma)$, $p_1 < q < p_2$.

Suppose f is in $H^q(B)$. Then $f^* \in H^q(S)$, and since T is onto, $f^* = T(g)$ for some $g \in L^q(\sigma)$. Thus $f \circ \varphi = C_\varphi \circ A^{-1} \circ T(g)$. Since $[C_\varphi AT(g)]^*$ is in $L^q(\sigma)$, and $C_\varphi AT(g)$ is holomorphic, $f \circ \varphi$ is in $H^q(B)$. The closed graph theorem now shows that C_φ is bounded on $H^q(B)$, as desired.

Propositions 2.11 and 2.12 together yield:

Corollary 2.13. If C_φ is bounded on $H^p(B)$ for some $p > 1$, then C_φ is bounded on $H^q(B)$ for all $q \geq p$.

In one variable a standard technique for extending results from one Hardy space to another is to use a factorization theorem. While this technique is generally not available in several variables (for example, the set of functions in $H^1(B_N)$ which can be factored as a product of two functions in $H^2(B_N)$ is a set of first category in $H^1(B_N)$ [9]), the following substitute for factorization, due to Coifman, Rochberg and Weiss, is sometimes useful.

Theorem 2.14. [5, Sec. 3] If f is in $H^1(B_N)$, then there exist functions g_i and h_i in $H^2(B_N)$ such that

$$f = \sum_{i=1}^{\infty} g_i h_i$$

and

$$\sum_{i=1}^{\infty} \|g_i\|_2 \, \|h_i\|_2 \leq c \, \|f\|_1$$

for some constant c depending only on the dimension N.

With this result we can prove the following proposition.

Proposition 2.15. Suppose C_φ is bounded linear operator on $H^2(B_N)$. Then C_φ is a bounded operator on $H^1(B_N)$.

Proof. Let f be in $H^1(B_N)$. Write $f = \sum_{i=1}^{\infty} g_i h_i$ as in Theorem 2.14. Then $f \circ \varphi = \sum_{i=1}^{\infty} (g_i \circ \varphi)(h_i \circ \varphi)$ and we have $\|f \circ \varphi\|_1 \leq$ $\sum_{i=1}^{\infty} \|g_i \circ \varphi\|_2 \, \|h_i \circ \varphi\|_2 \leq c\|C_\varphi\|^2 \, \|f\|_1$. □

We remark that Propositions 2.11, 2.12 and 2.15 do not completely answer the question of whether one can conclude that C_φ is bounded for all $p < \infty$ whenever C_φ is bounded for one value of $p < \infty$. In particular, we do not know in general whether H^p-boundedness implies $H^{p/2}$-boundedness, except in the case $p = 2$ (Proposition 2.15).

CHAPTER III

In this chapter we consider composition operators which are compact on some $H^p(B_N)$. We show that if φ induces a compact operator C_φ, then φ has a unique fixed point z_0 in B_N. Moreover we show that the spectrum of the compact operator C_φ can then be described as the set consisting of all products of powers of the eigenvalues of the derivative map $\varphi'(z_0) \cup \{0,1\}$.

1. Existence of a fixed point for the inducing map. Our goal in this section is to show that a map φ inducing a compact operator C_φ on some space $H^p(B_N)$ has a unique fixed point in B_N. The motivation for the argument we give is a result due to J. Shapiro and P. Taylor [20] which shows that a holomorphic map of the disc $\mathbb{D}$ into itself with an angular derivative at some point of $\partial\mathbb{D}$ does not induce a compact composition operator. We begin with the following lemma.

Lemma 3.1. Let N be an integer ≥ 2. Then for f in $A^{p,N-2}$ we have

$$\int_0^1 \int_{-r}^{r} |f(y)|^p (1-r)^{N-2} dy dr \leq \pi \|f\|_{p,N-2}^p .$$

Proof. Recall the Fejer-Riesz inequality for a function g in $H^p(\mathbb{D})$ [8; p. 46]:

$$\int_{-1}^{1} |g(x)|^p dx \leq \pi \, \|g\|_{H^p}^p$$

Now suppose that f is in $A^{p,N-2}(\mathbb{D})$. For $r < 1$ the function $f_r(z) \equiv f(rz)$ is in $H^p(\mathbb{D})$. Thus the Fejer-Riesz inequality gives

$$\int_{-1}^{1} |f_r(x)|^p dx \leq \pi \, \|f_r\|_{H^p}^p .$$

Multiplying this inequality by $r(1 - r)^{N-2}$ and integrating with respect to r yields the desired result:

$$\int_0^1 \int_{-r}^{r} |f(y)|^p (1 - r)^{N-2} dy dr \leq \pi \, \|f\|_{A^{p,N-2}}^p . \qquad \square$$

Now suppose that φ is a holomorphic map of B into B with no fixed points in B. Then φ has a (unique) Denjoy-Wolff point ζ in ∂B. Recall that this is the point to which the iterates of φ converge [see Chapter I, Section 1]. Without loss of generality assume that the Denjoy-Wolff point is the point $e_1 = (1,0')$. In this case we have (by Theorem 1.3):

$$\liminf_{z \to e_1} \; (1 - |\varphi(z)|^2)/(1 - |z|^2) = \alpha \leq 1$$

and

$$\frac{|1 - \varphi_1(z)|^2}{1 - |\varphi(z)|^2} \leq \alpha \; \frac{|1 - z_1|^2}{1 - |z|^2}$$

where $\varphi = (\varphi_1, \varphi_2, \ldots \varphi_N)$.

The next lemma shows that $(1 - \varphi_1(re_1))/(1 - r)$ is bounded for $-1 < r < 1$. A similar result appears in [17; Sec. 8.5.6, p. 177].

<u>Lemma 3.2</u>. Let φ: $B \to B$ be a holomorphic, fixed point free map with Denjoy-Wolff point e_1. Then there is an $M < \infty$ so that

$$\left|\frac{1 - \varphi_1(re_1)}{1 - r}\right| \leq M \quad \text{for} \quad -1 < r < 1.$$

<u>Proof</u>. Let $\sup\limits_{z \in B} |1 - \varphi_1(z)|^2(1 - |z|^2)/|1 - z_1|^2(1 - |\varphi(z)|^2) = A$

By the preceeding paragraph we have $A \leq \alpha \leq 1$. Note that

$$(1) \qquad |1 - \varphi_1(re_1)|^2 \leq A(1 - |\varphi(re_1)|^2)(1 - r)^2/(1 - r^2) .$$

Thus

$$\frac{1 - |\varphi_1(re_1)|}{1 - r} \cdot \frac{1 + r}{1 + |\varphi_1(re_1)|} \leq \frac{|1 - \varphi_1(re_1)|^2}{1 - |\varphi_1(re_1)|^2} \cdot \frac{1 - r^2}{(1 - r)^2}$$

$$\leq A \, \frac{1 - |\varphi(re_1)|^2}{1 - |\varphi_1(re_1)|^2}$$

$$\leq A .$$

Thus we have

$$\limsup_{r \to 1} \frac{1 - |\varphi_1(re_1)|}{1 - r} \leq A$$

$$(2) \qquad \limsup_{r \to 1} \frac{1 - |\varphi(re_1)|}{1 - r} \leq A$$

and

$$\liminf_{r \to 1} \frac{1 - |\varphi_1(re_1)|}{1 - r} \leq A$$

Now since $\liminf_{z\to e_1} \frac{1-|\varphi(z)|^2}{1-|z|^2} = \alpha \geq A$ we have

$$\liminf_{r\to 1} \frac{1-|\varphi_1(re_1)|}{1-r} \geq \liminf_{z\to e_1} \frac{1-|\varphi_1(z)|}{1-|z|} \geq A$$

So we must have

$$A = \lim_{r\to 1} \frac{1-|\varphi_1(re_1)|}{1-r} \leq \liminf_{r\to 1} \frac{|1-\varphi_1(re_1)|}{1-r} \leq \limsup_{r\to 1} \frac{|1-\varphi_1(re_1)|}{1-r} \leq A.$$

where the last inequality follows from (1) and (2). Therefore we have equality throughout the last line of inequalities. In particular

$$\lim_{r\to 1} \frac{|1-\varphi_1(re_1)|}{1-r} = A$$

and thus $(1-\varphi_1(re_1))/(1-r)$ is bounded on $(-1,1)$. □

We can now show that a map which induces a compact composition operator has a fixed point. This result is known for $N = 1$ [4], so we assume that $N \geq 2$ in Theorem 3.3. The argument given here is very similar to that in [20, Theorem 2.1, p. 478].

Theorem 3.3. Suppose $\varphi\colon B_N \to B_N$ is holomorphic and assume that C_φ is a compact operator on some $H^p(B_N)$, $1 \leq p < \infty$. Then φ has a fixed point in B_N.

Proof. Suppose φ has no fixed point in B_N. Then, without loss of generality, φ has Denjoy-Wolff point e_1 in ∂B_N. For $\frac{1}{2} < \alpha < 1$ define

$$f_\alpha(z_1,z') = [(1-\alpha)/(1-z_1)^{\alpha+N-1}]^{\frac{1}{p}}.$$

Let ρf_α be the restriction of f_α to the complex line through e_1: $\rho f_\alpha(\lambda) = f_\alpha(\lambda e_1)$. A computation based on [8, p. 65] shows that $\{\rho f_\alpha\}$ forms a bounded family in $A^{p,N-2}(\mathbb{D})$. Thus, by Lemma 2.2, $\{f_\alpha\}$ forms a bounded family in $H^p(B_N)$. Moreover, $f_\alpha \to 0$, uniformly on compact subsets of B_N, as $\alpha \uparrow 1$. Since C_φ is compact, we conclude that $f_\alpha \circ \varphi$ tends to 0 in $H^p(B_N)$ (see Proposition 3.6).

Now

$$f_\alpha \circ \varphi(z) = [(1-\alpha)/(1-\varphi_1(z))^{\alpha+N-1}]^{\frac{1}{p}},$$

where $\varphi = (\varphi_1,\ldots,\varphi_N)$. Setting g_α to be the restriction of $f_\alpha \circ \varphi$ to the complex line through e_1 we have, by Corollary 2.4 and Lemma 3.1,

$$\|f_\alpha \circ \varphi\|_{H^p}^p \geq C\|g_\alpha\|_{A^{p,N-2}}^p \geq C\int_0^1 \int_{-r}^{r} |g_\alpha(y)|^p (1-r)^{N-2}\,dy\,dr.$$

Since $|1-\varphi_1(re_1)|/(1-r) \leq M < \infty$ by Lemma 3.2, a computation shows that the right-hand integral above is $\geq C\, M^{1-\alpha-N}/(N-1)$. Thus $\|f_\alpha \circ \varphi\|_{H^p}$ is bounded away from 0 as $\alpha \to 1$, contradicting the compactness of C_φ. Therefore φ must have a fixed point in B_N, as desired. □

<u>Remark</u>. The fixed point of φ is necessarily unique. This follows from the fact that a holomorphic self-map of the unit ball B which fixes more than one point of the ball must fix an entire affine subset of B [17; Sec. 8.2.3, p. 166]. Thus φ would be the identity on at least a complex line in B. Since the identity map on $\mathbb{D}$ does

not induce a compact operator on any Bergman space $A^{p,N-2}(\mathbb{D})$ (see [3] for the case $p = 2$, $N = 2$), the operator C_φ, where φ is the identity on (at least) a complex line in B, cannot be compact on $H^p(B)$.

2. <u>The spectrum of C_φ</u>. We will use Theorem 3.3 to identify the spectrum of a compact composition operator C_φ on $H^p(B)$. Our main theorem is the following:

<u>Theorem 3.4</u>. Let $\varphi\colon B \to B$ be a holomorphic map such that C_φ is a compact operator on $H^p(B)$ for some p, $1 \le p < \infty$. Let z_0 be the fixed point of φ in B. Then $\sigma(C_\varphi)$ consists of all possible products of the eigenvalues of $\varphi'(z_0)$, together with 0 and 1.

We prove two lemmas before giving the proof of Theorem 3.4. Recall that for $\varphi\colon B_N \to B_N$ a holomorphic map and z_0 in B_N, the derivative $\varphi'(z_0)$ is a linear operator represented by a matrix (a_{ij}) where

$$a_{ij} = D_j\varphi_i(z_0) \quad 1 \le i,j \le N, \quad \varphi = (\varphi_1,\ldots,\varphi_N)$$

and
$$D_j = \frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\,\frac{\partial}{\partial y_j}\right) .$$

The next lemma shows that for the purpose of proving Theorem 3.4 there is no loss of generality in assuming the fixed point z_0 to be 0 and $\varphi'(0)$ to be upper triangular.

<u>Lemma 3.5</u>. Suppose $\varphi\colon B \to B$ is holomorphic with fixed point z_0 in B. Then there is a map $\psi\colon B \to B$ with $\psi(0) = 0$ and $\psi'(0)$

upper triangular such that C_φ is similar to C_ψ.

Proof. Let $\tau \in \mathrm{Aut}\, B$ have $\tau(z_0) = 0$. Then $\rho \equiv \tau \circ \varphi \circ \tau^{-1}$ fixes 0. There is a unitary matrix U so that $U\rho'(0)U^{-1}$ is an upper triangular matrix T. Set $\psi = U \circ \rho \circ U^{-1}$ so that $\psi: B \to B$ is holomorphic and fixes 0. Then $\psi'(0) = T$ and C_ψ is similar to C_φ as desired. □

In the case $N = 1$, if the composition operator C_φ is compact on $H^p(\mathbb{D})$ for some $p < \infty$, then C_φ is compact on $H^p(\mathbb{D})$ for all $p < \infty$ [20]. We prove next a weaker result along these lines for $N > 1$. The proof uses a several variable analogue of a criterion due to H. Schwartz for a composition operator to be compact.

Proposition 3.6. [18; Theorem 2.5]. C_φ is compact on $H^p(B)$ $(1 \leq p < \infty)$ if and only if for every sequence $\{f_n\}$ bounded in $H^p(B)$ with $f_n \to f$ uniformly on compact subsets of B, then $f_n \circ \varphi \to f \circ \varphi$ in $H^p(B)$.

Proof. The result follows, exactly as in the one variable case, from the fact that $\{f_n \circ \varphi\}$ is a normal family if $\{f_n\}$ is a bounded sequence in $H^p(B)$. To see this, use the estimate [17; Sec. 7.2.5, p. 128]

$$|f_n \circ \varphi(z)| \leq 2^{N/p} \|C_\varphi\| \, \|f_n\|_p \, (1 - |z|^{-N/p}) \qquad \square$$

where N is the dimension of B.

Lemma 3.7. If C_φ is compact on $H^p(B_N)$, then C_φ is compact on $H^{np}(B_N)$ for all $n \geq 1$.

Proof. Fix $n \geq 1$ and choose $\{f_m\}$ a bounded sequence in $H^{np}(B_N)$. We need to show that $\{f_m \circ \varphi\}$ has a subsequence which converges in $H^{np}(B_N)$. Since $\{f_m\}$ is bounded in $H^{np}(B_N)$, $\{f_m\}$ has a subsequence which converges uniformly on compact subsets of B_N. Without loss of generality assume $f_m \to f$, almost uniformly. Since $\{f_m\}$ is bounded in $H^{np}(B_N)$, $\{f_m^n\}$ is bounded in $H^p(B_N)$. The Schwartz criterion for compact composition operators shows that $f_m^n \circ \varphi \to f^n \circ \varphi$ in $H^p(B_N)$. In particular $\|f_m^n \circ \varphi\|_p \to \|f^n \circ \varphi\|_p$, which implies

$$\|f_m \circ \varphi\|_{np} \to \|f \circ \varphi\|_{np} . \tag{1}$$

Moreover there is a subsequence $f_{m_k}^n \circ \varphi$ which converges almost everywhere on ∂B to $f^n \circ \varphi$. Hence

$$f_{m_k} \circ \varphi \to f \circ \varphi \quad \text{a.e.} \tag{2}$$

Thus (1) and (2) together show that $f_{m_k} \circ \varphi \to f \circ \varphi$ in $H^{np}(B_N)$, as desired. $\square$

Proof of Theorem 3.4. We can now give the proof of the main theorem of this section. Suppose that C_φ is compact on $H^p(B_N)$. By Lemma 3.5 there is no loss of generality in assuming that $\varphi(0) = 0$ and that $\varphi'(0)$ is given by an upper triangular matrix:

$$A \equiv \varphi'(0) = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & * \\ & & \ddots & \\ & 0 & & \\ & & & a_{NN} \end{bmatrix}$$

Thus if $\varphi = (\varphi_1,\ldots,\varphi_N)$ we have $D_i\varphi_j(0) = 0$ if $j > i$, and $D_i\varphi_i(0) = a_{ii}$.

Step 1. Recall that the nonzero points in the spectrum of a compact operator are always eigenvalues. We will first show that if

$$(*) \qquad f \circ \varphi = \lambda f$$

for some holomorphic function f, when $\lambda \neq 0,1$, is not a product of powers of the eigenvalues a_{ii} of $\varphi'(0)$, then $f \equiv 0$. Suppose f satisfies (*), and write f in its homogeneous expansion:

$$f(z) = \sum_{s=0}^{\infty} F_s(z); \; F_s(z) = \sum_{|\alpha|=s} C_\alpha z^\alpha$$

where if is the multi-index $(j_1,\ldots,j_N)$, then $z^\alpha = z_1^{j_1} \ldots z_N^{j_N}$ and $|\alpha| = \sum_{i=1}^{N} j_i$. We will show by induction that $F_s \equiv 0$ for every $s = 0,1,\ldots$. Note that evaluation of both sides of (*) at 0 gives

$$f(0) = \lambda f(0)$$

and, since $\lambda \neq 1$ by hypothesis, we have $f(0) = F_0 = 0$.

Suppose now that $F_s \equiv 0$ for $s < n$. Thus

$$f(z) = \sum_{|\alpha|=n} C_\alpha z^\alpha + \sum_{s=n+1}^{\infty} F_s(z)$$

Since $\varphi(z) = \varphi'(0)z + O(|z|^2)$ in a neighborhood of 0, equation (*) yields

$$(**) \qquad \sum_{|\alpha|=n} C_\alpha (Az)^\alpha = \lambda \sum_{|\alpha|=n} C_\alpha z^\alpha$$

where A is the matrix of $\varphi'(0)$ as above.

Using the hypotheses on λ we will show inductively that $C_\alpha = 0$ for every multi-index α with $|\alpha| = n$. To do this we begin by describing an ordering on the multi-indices with total order n. Suppose

$$\alpha = (j_1, j_2, \ldots, j_N), \; \beta = (k_1, k_2, \ldots, k_N)$$

where j_i, k_i are nonnegative integers and $\sum j_i = \sum k_i = n$. We say $\alpha < \beta$ if there is a positive integer i_0 such that $j_i = k_i$ for $i < i_0$ and $j_{i_0} > k_{i_0}$. In particular the first multi-index of total order n in this ordering is $(n,0,\ldots,0)$. Comparison of the coefficients of z^α, where $\alpha = (n,0,\ldots,0)$, on both sides of (**) yields

$$C_{(n,0,\ldots,0)} a_{11}^n = \lambda C_{(n,0,\ldots,0)}$$

By hypotheses $\lambda \neq a_{11}^n$, so that $C_{(n,0,\ldots,0)} = 0$.

Now suppose $C_\alpha = 0$ for all $\alpha < \beta$, where $|\alpha| = |\beta| = n$. We will show that $C_\beta = 0$ by comparing the coefficients of z^β on both sides of (**). Let $\beta = (j_1, j_2, \ldots, j_N)$. Since C_α is assumed to be 0 for $\alpha < \beta$ the left hand side of (**) is just

$$C_\beta (Az)^\beta + \sum C_\eta (Az)^\eta \qquad (\eta > \beta, \ |\eta| = n)$$

For no η with $\eta > \beta$ and $|\eta| = n$ does $(Az)^\eta$ contain a term in z^β. This follows from the definition of "<" on multi-indices and the fact that A is upper triangular. Thus comparing the coefficients of z^β on both sides of (**) we obtain

$$C_\beta a_{11}^{j_i} a_{22}^{j_2} \ldots a_{NN}^{j_N} = \lambda C_\beta .$$

This implies $C_\beta = 0$, since $\lambda \neq a_{11}^{j_i} a_{22}^{j_2} \ldots a_{NN}^{j_N}$. Thus we see that $C_\alpha = 0$ for all α with $|\alpha| = n$, and hence $F_n(z) \equiv 0$. The induction on s now shows that $F_s(z) \equiv 0$ for all s and therefore $f \equiv 0$, as desired.

<u>Step 2</u>. We complete the proof of Theorem 3.4 by showing that 0,1 and all products of powers of the eigenvalues of $\varphi'(0)$ are in the spectrum of C_φ. Since C_φ is compact, $0 \in \sigma(C_\varphi)$ and since $f \equiv 1$ is in $H^p(B)$, 1 is in $\sigma(C_\varphi)$.

Next we show that $a_{ii} = D_i \varphi_i(0)$ is in $\sigma(C_\varphi)$, for $i = 1, \ldots, N$. The argument is by induction on i. To see that a_{11} is in $\sigma(C_\varphi)$, we may assume that $a_{11} \neq 0$. We claim that the function $g(z) = z_1$ is not in the range of $(C_\varphi - a_{11})$. For suppose that

$$f \circ \varphi - a_{11} f = z_1$$

has a solution $f \in H^p(B)$. Differentiation gives

$$D_1 f(0) D_1 \varphi_1(0) + D_2 f(0) D_1 \varphi_2(0) + \ldots + D_N f(0) D_1 \varphi_N(0) - a_{11} D_1 f(0) = 1$$

Since $D_1 \varphi_\ell(0) = 0$ for $\ell > 1$ this becomes

$$D_1 f(0) D_1 \varphi_1(0) - a_{11} D_1 f(0) = 1$$

which is impossible since $D_1 \varphi_1(0) = a_{11}$. Thus $a_{11} \in \sigma(C_\varphi)$.

Suppose that $a_{jj} \in \sigma(C_\varphi)$ for $j = 1,\ldots,i-1$. We will show that a_{ii} is in (C_φ) by showing that $g(z) = z_i$ is not in the range of $(C_\varphi - a_{ii})$. Without loss of generality we may assume $a_{ii} \neq a_{jj}$ for any j, $1 \leq j < i$. Apply the differential monomials $D_1, D_2, \ldots, D_i$ to the equation $f \circ \varphi - a_{ii} f = z_i$ and evaluate both sides of the resulting equation at 0. This yields the following:

(1) $\quad D_1 f(0)[D_1 \varphi_1(0) - a_{ii}] = 0$

(2) $\quad D_1 f(0) D_2 \varphi_1(0) + D_2 f(0)[D_2 \varphi_2(0) - a_{ii}] = 0$

$\vdots \qquad \vdots$

(i) $\quad D_1 f(0) D_i \varphi_1(0) + \ldots + D_{i-1} f(0) D_i \varphi_{i-1}(0) + D_i f(0)[D_i \varphi_i(0) - a_{ii}] = 1$

By the assumption that $a_{ii} \neq a_{jj}$ for any j, $1 \leq j < i$ equation (1) implies $D_1 f(0) = 0$. Substituting this in (2) shows $D_2 f(0) = 0$. Continuing in this manner equation (i) becomes $D_i f(0)[D_i \varphi_i(0) - a_{ii}] = 1$

which is a contradiction. Thus every a_{ii}, $1 \leq i \leq N$, is in the spectrum of C_φ. Moreover, since C_φ is compact, if $a_{ii} \neq 0$, then a_{ii} is an eigenvalue of C_φ

To finish we show that all possible products of the a_{ii}'s are in $\sigma(C_\varphi)$. Suppose that $\lambda_1,\ldots,\lambda_m$ are a collection of a_{ii}'s, with repeats allowed, and assume that no $\lambda_i = 0$. We wish to show that $\Pi\lambda_i$ is in $\sigma(C_\varphi)$. By Lemma 3.7, C_φ is compact on $H^{mp}(B)$ and the above argument shows that $\lambda_i \in \sigma(C_\varphi)$, relative to $H^{mp}(B)$. So there is an $0 \not\equiv f_i \in H^{mp}(B)$ satisfying

$$f_i \circ \varphi = \lambda_i f_i \qquad 1 \leq i \leq m$$

Since $f_i \in H^{mp}(B)$, $f \equiv \prod_{i=1}^{m} f_i \in H^p(B)$ and

$$f \circ \varphi = \Pi(f_i \circ \varphi) = \Pi\lambda_i f_i = (\Pi\lambda_i)f \quad .$$

Thus $\Pi\lambda_i \in \sigma(C_\varphi)$ as desired. This completes the proof of Theorem 3.4. □

Remark. Suppose C_φ is a power compact operator, i.e., C_φ^M is compact on some $H^p(B)$, for some positive integer M. Since $C_\varphi^M = C_{\varphi_M}$, where φ_M denotes the M^{th} iterate of φ, φ_M must fix exactly one point of B. Suppose the fixed point of φ_M is z_0. We claim that φ fixes z_0. If not, then φ fixes no point of B, since z_0 is the only fixed point of φ_M. But then the entire sequence $\{\varphi_n\}$ of iterates of φ converges to a point ζ in ∂B (Theorem 1.5). This contradicts the fact that $\varphi_{Mk}(z_0) = z_0$ for all positive integers k. Thus a power compact composition operator is induced by a map with exactly one fixed point in B. As with compact operators, the non-zero spectrum of a power compact operator consists of eigenvalues [7]. So Theorem 3.4 holds, with exactly the same proof, for power compact composition operators.

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