VOLUMES, DETERMINANTS, AND MERIDIAN LENGTHS OF HYPERBOLIC LINKS

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A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

Mathematics - Doctor of Philosophy

2017

ABSTRACT

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We study relationships between link diagrams and link invariants arising from hyperbolic geometry. The *volume density* of a hyperbolic link K is defined to be the ratio of the hyperbolic volume of K to the crossing number of K. We show that there are sequences of non-alternating links with volume density approaching v_8, where v_8 is the volume of the regular ideal hyperbolic octahedron. We show that the set of volume densities is dense in $[0, v_8]$. The *determinant density* of a link K is $2\pi \log \det(K)/c(K)$. We prove that the closure of the set of determinant densities contains the set $[0, v_8]$. We examine the conjecture, due to Champanerkar, Kofman, and Purcell [20] that $\mathrm{vol}(K) < 2\pi \log \det(K)$ for alternating hyperbolic links, where $\mathrm{vol}(K) = \mathrm{vol}(S^3 \backslash K)$ is the hyperbolic volume and $\det(K)$ is the determinant of K. We prove that the conjecture holds for 2-bridge links, alternating 3-braids, and various other infinite families. We show the conjecture holds for highly twisted links and quantify this by showing the conjecture holds when the crossing number of K exceeds some function of the twist number of K. We derive bounds on the length of the meridian and the cusp volume of hyperbolic knots in terms of the topology of essential surfaces spanned by the knot. We provide an algorithmically checkable criterion that guarantees that the meridian length of a hyperbolic knot is below a given bound. As applications we find knot diagrammatic upper bounds on the meridian length and the cusp volume of hyperbolic adequate knots and we obtain new large families of knots with meridian lengths bounded above by four. We also discuss applications of our results to Dehn surgery.

Dedicated to my wonderful wife, Tina.

ACKNOWLEDGMENTS

I would like to thank my wife, Tina, for all the support she has given me. She kept me motivated, and listened to me no matter what I was talking about.

I would like to thank my adviser, Effie Kalfagianni, for her work and help with preparing my research. She helped me find tractable problems and I learned a lot by studying with her.

I would also like to thank my Masters degree adviser, Jessica Purcell, for introducing me to hyperbolic 3-manifolds and planting the seed of interest that lead me to continue researching and learning about the field. I appreciate my work with Darrin Doud and Jeff Humpherys as an undergraduate, who sparked my interest in advanced mathematics.

Conversations with Colin Adams, Abhijit Champanerkar, Dave Futer, Effie Kalfagianni, Ilya Kofman, and Jessica Purcell were helpful in producing the research contained in this work.

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Chapter 1

Introduction

Knot theory is the study of piecewise linear embeddings $f : S^1 \hookrightarrow S^3$ of the circle into the 3-sphere. A *knot* $K \subset S^3$ is the image of such an embedding. A *link* $L \subset S^3$ is a union of disjoint knots. We say that two links L_1 and L_2 are equivalent if there exists an orientation preserving piecewise linear homeomorphism $h : S^3 \to S^3$ such that $h(L_1) = L_2$. Throughout this work, we will consider equivalent links to be the same link. Knot theory is a classical subject of topology and there is a large library of literature devoted to its study. See [47] and [62] for an introduction to the subject.

A natural way of studying knots and links is via diagrams. A *diagram* of a link L is a projection of L onto S^2 together with "over and under" crossing information. For example, a diagram of the knot known as the 7_7 knot is given in Figure 1.1. There are many different diagrams of the same knot. It is a difficult problem in general to determine whether or not two diagrams represent the same link. For example, the diagrams in Figure 1.2 were long thought to represent different knots until Perko [59] proved otherwise.

Much of knot theory is centered on the study of link invariants. A link invariant is a rule f that assigns to each link L some mathematical object $f(L)$ such that if L_1 and L_2 are equivalent links then $f(L_1) = f(L_2)$. A simple example of a link invariant is the *crossing number*, denoted $c(L)$, which is the minimum number of crossings in a diagram of L. Another example of an invariant is whether a link has an alternating diagram – a diagram where the

Figure 1.1: A diagram of the 7_7 knot.

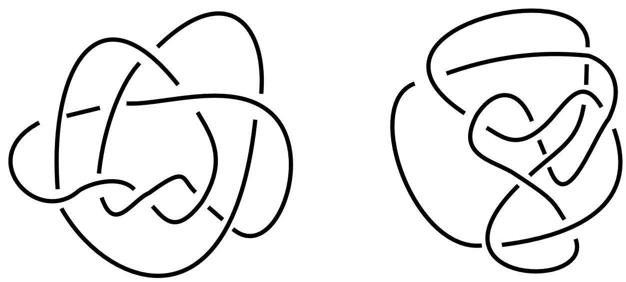

Figure 1.2: Two diagrams of the same knot.

over and under nature of the crossings alternate as one travels along the link. A link that has an alternating diagram is called *alternating*.

Some link invariants, such as the crossing number, are defined in terms of diagrams of the link. Another interesting class of link invariants arises from the study of hyperbolic geometry. Many knot complements $S^3 \backslash K$ admit a finite-volume hyperbolic structure, meaning that $S^3 \backslash K$ admits a Riemannian metric of constant curvature -1. Such knots are called *hyperbolic*. See Section 2.1.1 for further details. In fact, Thurston [66] showed that any knot which is not a satellite knot or torus knot is hyperbolic. Further, a celebrated theorem of Mostow [57] and Prasad [60] states that any two finite volume hyperbolic structures are unique up to isometry. Therefore properties of the hyperbolic structure of $S^3 \backslash K$, such as the volume or lengths of certain geodesics, are invariants of the knot K. The hyperbolic volume of $S^3 \backslash K$ is of special interest in this work, and we will denote it by $\mathrm{vol}(K)$.

The following compelling question has drawn significant attention. "How can one relate the hyperbolic invariants of a knot to their diagrams?" Lackenby [44] showed that one may provide bounds on the volume of a link in terms of a diagrammatic property known as the twist number (see Theorem 2.3.2 for more details). Futer, Kalfagianni, and Purcell [30] found bounds on the volume of a hyperbolic link that depend on coefficients of the Jones polynomial, a polynomial that may be combinatorially defined.

In this work, we will further study the relationship between hyperbolic invariants of knots and their diagrams. In Chapter 2, we will present necessary background material. In Chapter 3 we will explore relationships between volume, crossing number, and determinant through invariants called volume and determinant densities. In Chapter 4 we will further study the relationship between determinant and volume showing that the determinant may be used in some cases to find an upper bound on the volume of a link. In Chapter 5 we will explore how topologically and diagrammatically defined essential spanning surfaces relate with the hyperbolic invariants of meridian length and cusp volume. Sections 1.1, 1.2, and 1.3 outline the major results of Chapters 3, 4, and 5 respectively.

1.1 Volume and Determinant Densities

We will begin our study of the relationship between diagrams and the hyperbolic geometry of links by examining the relationship between the hyperbolic volume of a link and its crossing number via a quantity known as *volume density* $d_{\mathrm{vol}}(K)$ which is defined by

$$d_{\mathrm{vol}}(K) = \frac{\mathrm{vol}(K)}{c(K)} \tag{1.1}$$

where $\mathrm{vol}(K)$ is the hyperbolic volume of $S^3 \backslash K$ and $c(K)$ is the crossing number.

D. Thurston [64] showed that $d_{\text{vol}}(K) \leq v_8$ for all knots K, where $v_8 \approx 3.66286$ is the volume of the regular ideal hyperbolic octahedron. Champanerkar, Kofman, and Purcell [20] showed that the upper bound of v_8 on volume density is asymptotically sharp. More precisely, there exist sequences of links $\{K_n\}_{n=0}^{\infty}$ such that

$$\lim_{n\to\infty} d_{\text{vol}}(K_n) = v_8$$

(see Theorem 3.2.3). Such sequences of links are called *geometrically maximal*. All the geometrically maximal sequences of links constructed in [20] were alternating, and did not contain a cycle of tangles (see Definition 3.2.2). Theorem 3.3.1 will show that there exist geometrically maximal sequences of links having a cycle of tangles. We then use this to prove Theorem 3.3.3 which states that there exist geometrically maximal sequences of non-alternating links.

Let $\mathcal{C}_{\text{vol}} = \{\text{vol}(K)/c(K) : K \text{ is a hyperbolic link}\} \subseteq \mathbb{R}$ and let Spec_{vol} be the set of limit points of $\mathcal{C}_{\text{vol}}$. We call Spec_{vol} the *spectrum of volume densities*. A question that arose in [20] was how one may describe the sets $\mathcal{C}_{\text{vol}}$ and Spec_{vol}. (See also [19] for discussion and related questions.) In particular, what numbers occur as volume densities? We answer this question with the following theorem.

Theorem 3.4.3. *The set $\mathcal{C}_{\text{vol}}$ of volume densities of hyperbolic links is a dense subset of $[0, v_8]$, and $\text{Spec}_{\text{vol}} = [0, v_8]$. In other words, given $x \in [0, v_8]$ there exists a sequence $\{K_n\}_{n=1}^{\infty}$ of hyperbolic links such that the volume densities satisfy $\lim_{n\to\infty} d_{\text{vol}}(K_n) = x$.*

Another invariant that was studied in [19] and [20] was the *determinant density*, $d_{\text{det}}(K)$

of a link K, which is defined by the equation

$$d_{\text{det}}(K) := \frac{2\,\pi\;\log(\det(K))}{c(K)}$$

The authors of [20] conjectured that

$$\frac{2\pi \log(\det(K))}{c(K)} < v_8 \tag{1.2}$$

for any knot or link K. It is shown in [20] that there are sequences $\{K_n\}_{n=1}^{\infty}$ of links such that $\lim_{n\to\infty} d_{\text{det}}(K_n) = v_8$ (see Theorem 3.2.4 of this paper). Such sequences are called *diagrammatically maximal.* We define $\mathcal{C}_{\text{det}} = \{d_{\text{det}(K)} : K$ is a (not necessarily hyperbolic) link$\} \subseteq \mathbb{R}$ and define Spec_{vol} to be the set of limit points of $\mathcal{C}_{\text{det}}$. We call Spec_{det} the *spectrum of determinant densities.*

We explore the relationship between Spec_{vol} and Spec_{det}, thus connecting a hyperbolic invariant to a diagrammatic invariant. While we know that v_8 is an upper bound on volume density, it is unknown whether v_8 forms an upper bound for Spec_{det}. However, it is a conjecture [20, Conjecture 1.1] that v_8 is an upper bound for Spec_{det}. We will prove the following theorem in Chapter 3 Section 3.5.

Theorem 3.5.3. *The spectrum of determinant densities* Spec_{det} *contains* $[0, v_8]$. *In other words, given any* $x \in [0, v_8]$ *there exists a sequence* $\{K_n\}_{n=1}^{\infty}$ *of links satisfying*

$$\lim_{n\to\infty} d_{\text{det}}(K_n) = x \tag{3.26}$$

In [19] and [20] it is shown that $0, v_8 \in \text{Spec}_{\text{vol}} \cap \text{Spec}_{\text{det}}$, and the authors ask what this intersection is. As a corollary of Theorems 3.4.3 and 3.5.3 we obtain:

Corollary 3.1.2. *The intersection* $\mathrm{Spec}_{\mathrm{vol}} \cap \mathrm{Spec}_{\mathrm{det}}$ *is equal to* $[0, v_8]$.

Since $\mathrm{Spec}_{\mathrm{vol}}$ considers only the densities of hyperbolic links, it is useful to know what determinant densities are realized by *hyperbolic* links. We show in Theorem 3.5.4 that the set of determinant densities of hyperbolic links is also dense in $[0, v_8]$.

1.2 Relating Determinant and Volume

The Alexander polynomial $\Delta_L(t)$ of a link L is a classical link invariant. It can be defined by considering the infinite cyclic covering space of $S^3 \backslash K$, or combinatorially via Skein relations. One invariant stemming from the Alexander polynomial is the *determinant*, which is defined to be $|\Delta_L(-1)|$. For further information about the Alexander polynomial and determinant, see [47].

Dunfield noted a relationship between the volume and determinant of a knot in an online post [26]. He observed that there is a nearly linear relationship between the hyperbolic volume of an alternating knot and $\log(J(-1))$ where J denotes the Jones polynomial. After further study of this relationship and some experimentation, Champanerkar, Kofman and Purcell [20] made the following conjecture.

Conjecture 4.1.1. Let K be a hyperbolic alternating knot. Then $\mathrm{vol}(K) < 2\pi \log \det(K)$.

Conjecture 4.1.1 is false if we do not require K to be alternating. Kalfagianni [41] showed that there are knots with trivial Alexander polynomial (hence having determinant 1) which have arbitrarily large volume. One can use the data from Knotscape [36] and SnapPy [22] to verify this conjecture for all alternating knots with up to 16 crossings. Champanerkar, Kofman and Purcell in [20] computationally verified Conjecture 4.1.1 for many examples of an infinite family of links known as weaving knots. Using these weaving knots, they showed that the constant 2π is sharp in the sense that, given $\alpha < 2\pi$, there exists an alternating link K with $\alpha \log(\det(K)) < \mathrm{vol}(K)$.

In Chapter 4, we will verify that Conjecture 4.1.1 holds for various infinite families of links. In particular, we have the following theorem.

Theorem 4.1.2. *If K is a 2-bridge link or an alternating 3-braid, then*

$$\mathrm{vol}(K) \leq 2\pi \log(\det(K)) \tag{4.3}$$

The proof of Theorem 4.1.2 is a combination of Theorems 4.3.5 and 4.4.1. We also show that the conjecture holds for "highly twisted" links, *i.e.* those having many crossings and few twist regions (see Section 2.3 for a definition of twist region). We further quantify this by the following theorem.

Theorem 4.5.1. *Let K be an alternating hyperbolic link with t twist regions and c crossings. If*

$$c > t + \xi^{t-1} - 2\gamma^{t-1} \tag{4.42}$$

where $\gamma \approx 1.4253$ is the unique positive real number satisfying $\gamma^{-5} + 2\gamma^{-4} + \gamma^{-3} - 1 = 0$ and $\xi = e^{5v_4/\pi} \approx 5.0296$, then $\mathrm{vol}(K) < 2\pi \log(\det(K))$.

It follows that if some or all of the twist regions of a link have many crossings, the link will satisfy Conjecture 4.1.1. An immediate corollary of this is:

Corollary 4.1.3. *For any given integer $t_0 > 0$ there are only finitely many links with t_0 twist regions which fail to satisfy Conjecture 4.1.1.*

This can be used, for example (as demonstrated in Section 4.5), to show that all alternating pretzel links with a fixed number t_0 of twist regions satisfy Conjecture 4.1.1. The idea is that if an alternating pretzel link has, for example 3 twist regions, then by Theorem 4.5.1, if it also has more than 25 total crossings, it must satisfy Conjecture 4.1.1. One can then enumerate all alternating pretzel knots with no more than 25 crossings in 3 twist regions, compute the determinants, and estimate the volumes via other methods to show that the remaining links also satisfy Conjecture 4.1.1. Using a computer to aid in calculation, we have shown that all alternating pretzel links with no more than 13 twist regions satisfy Conjecture 4.1.1. See Section 4.5.1 for details.

1.3 Meridians, Cusp Area, and Essential Spanning Surfaces

Let K be a hyperbolic knot in S^3. Then there is a well-defined notion of a maximal cusp neighborhood C of K in S^3 homeomorphic to a solid torus (see Section 2.1.2 for a formal definition). The area of ∂C is called the *cusp area* and is an invariant of K. A *meridian* of K is a non-separating simple closed curve μ in ∂C such that μ bounds a disk in C. A meridian of K is isotopic to a unique geodesic, so we will refer to μ as *the* meridian. The length of the meridian $\ell(\mu)$ is therefore a well-defined quantity and an invariant of a hyperbolic knot. Related to the meridian is the notion of a λ-curve, which is a non-separating simple closed curve in ∂C which intersects the meridian exactly once.

We relate the geometric invariants of meridian length and cusp area to topologically defined essential spanning surfaces. A *spanning surface* S of a knot K is one such that its boundary ∂S is equal to K. A surface S in a 3-manifold M is *essential* if it is incompressible and ∂-incompressible (see Section 2.2). Some essential spanning surfaces are defined combinatorially via diagrams (for example the checkerboard and state surfaces of Section 2.2), so the relationships between meridian length, cusp area, and essential spanning surfaces uncover another connection between diagrams of knots and hyperbolic invariants. The following theorem demonstrates a relationship between these invariants.

Theorem 5.1.1. *Let K be a hyperbolic knot with meridian length $\ell(\mu)$. Suppose that K admits essential spanning surfaces S_1 and S_2 such that*

$$|\chi(S_1)| + |\chi(S_2)| \leq \frac{b}{6}\, i(\partial S_1, \partial S_2) \tag{5.1}$$

where b is a positive real number and $i(\partial S_1, \partial S_2)$ is the minimal intersection number of ∂S_1 and ∂S_2 on ∂M. Then the meridian length satisfies $\ell(\mu) \leq b$. Moreover, given a hyperbolic knot K and $b > 0$, there is an algorithm to determine if there are essential surfaces S_1 and S_2 satisfying (5.1)

An open question is whether meridian lengths are bounded above by 4. Agol [11] and Purcell [61] have produced examples of knots whose meridian lengths approach 4 from below. The best known general bound can be found as follows. A slope σ on ∂M is called *exceptional* if the 3-manifold $M(\sigma)$ is not hyperbolic. The Gromov-Thurston "2π-theorem" [12] asserts that if $\ell(\sigma) > 2\pi$ then $M(\sigma)$ admits a Riemannian metric of negative curvature. This combined with the proof of Thurston's geometrization conjecture [56] implies that actually $M(\sigma)$ is hyperbolic. The work of Agol [10] and Lackenby [43], that has improved 2π to 6, asserts that exceptional slopes must have length less or equal to six. Examples of exceptional slopes with length six are given in [10] and in [4]. Since the meridian curve of every hyperbolic knot in S^3 is an exceptional slope, we have $\ell(\mu) \leq 6$. The work of Adams, Colestock, Fowler, Gillam, and Katerman [6] shows that that $\ell(\mu) < 6$. Examples of knots whose meridian length approaches four from below are given in [10] and by Purcell in [61].

We will show that meridian lengths are bounded above by 4 for broad families of knots. One broad family can be defined through an invariant called Turaev genus (see Section 5.2.2).

Theorem 5.3.3. *Let K be an adequate hyperbolic knot in S^3 with crossing number $c = c(K)$ and Turaev genus g_T. Let C denote the maximal cusp of $S^3 \setminus K$ and let* $\mathrm{Area}(\partial C)$ *denote the cusp area. Finally let $\ell(\mu)$ and $\ell(\lambda)$ denote the length of the meridian and the shortest λ-curve of K. Then we have*

1. $\ell(\mu) \leq 3 + \dfrac{6g_T - 6}{c}$
2. $\ell(\lambda) \leq 3c + 6g_T - 6$
3. $\mathrm{Area}(\partial C) \leq 9c\left(1 + \dfrac{2g_T - 2}{c}\right)^2$

A knot is alternating if and only if $g_T = 0$. In this case, the bounds of Theorem 5.3.3 agree with the bounds of [6]. The technique of the proof of Theorems 5.1.1 and 5.3.3, as well as the proof of results in [6], is reminiscent of arguments with pleated surfaces that led to the proof of the "6-Theorem" [10, 43]. The algorithm for checking the criterion of Theorem 2.1.2 involves normal surface theory and in particular the work of Jaco and Sedgwick [39].

Note that Theorem 5.3.3 implies that whenever the Turaev genus $g_T \leq 3$ then $\ell(\mu) \leq 4$. Another broad family for which $\ell(\mu) \leq 4$ is "highly twisted" knots, *i.e.* those with many crossing per twist region (see Section 2.3 for a definition of twist region), as shown in the following theorem.

Theorem 5.3.6. *Let K be a hyperbolic knot with an adequate diagram with c crossings and* *t twist regions. Then we have*

$$\ell(\mu) \leq 3 + \frac{3t}{c} - \frac{6}{c}.$$

In particular if $c \geq 3t$ then we have $\ell(\mu) < 4$.

Let K be a hyperbolic knot with maximal cusp C and slopes σ, σ' on ∂C (for definitions of maximal cusp and slopes see Section 2.1.2). Calculating area in Euclidean geometry on ∂C (see for example the proof of [10, Theorem 8.1]), we have

$$\ell(\sigma)\ell(\sigma') \geq \mathrm{Area}(\partial C)\Delta(\sigma, \sigma') \tag{1.3}$$

where $\Delta(\sigma, \sigma')$ denotes the absolute value of the intersection number of σ and σ'. Work of Cao and Meyerhoff [16, Proposition 5.8] shows that $\mathrm{Area}(\partial C) \geq 3.35$. Given an adequate hyperbolic knot K, we will apply (1.3) for $\sigma' = \mu$. Using the upper bound for $\ell(\mu)$ from Theorem 5.3.3, we have

$$\ell(\sigma) > \frac{3.35\Delta(\mu, \sigma)c}{3c + 6g_T - 6} = \frac{3.35}{3} \cdot \frac{\Delta(\mu, \sigma)}{1 + \delta} \tag{1.4}$$

where $\delta = \dfrac{2g_T - 2}{c}$. Note that δ is an invariant of K that can be calculated from any adequate diagram (see Theorem 5.2.4). Now (1.4) implies that if

$$\Delta(\mu, \sigma) > \frac{18}{3.35}\,(1 + \delta) > 5.37\,(1 + \delta)$$

then $\ell(\sigma) > 6$ and thus σ cannot be an exceptional slope.

Note that if σ is a slope represented by $p/q \in \mathbb{Q}$ in $H_1(\partial C)$ then $\Delta(\mu, \sigma) = |q|$. Hence if $|q| > 6(1 + \delta)$, inequality (5.3) implies that $\ell(\sigma) > \dfrac{3.35}{3} \cdot 6 > 2\pi$. In this case, we may apply a result of Futer, Kalfagianni and Purcell [31, Theorem 1.1] to estimate the change of volume under Dehn filling of adequate knots. This produces the following theorem.

Theorem 5.1.3. *Let K be a hyperbolic adequate knot and let δ be as above. If $|q| \geq 6(1+\delta)$, then the 3-manifold N obtained by p/q surgery along K is hyperbolic and the volume satisfies the following*

$$\mathrm{vol}(S^3 \setminus K) \; > \; \mathrm{vol}(N) \; \geq \; \left(1 - \frac{36(1+\delta)^2}{q^2}\right)^{3/2} \mathrm{vol}(S^3 \setminus K).$$

The assertion that N is hyperbolic follows immediately from the discussion above. The left hand side inequality is due to the result of Thurston that the hyperbolic volume drops under Dehn filling (Theorem 2.1.2). The right hand side follows by [31, Theorem 1.1].

Theorem 5.14 of [30], and its corollaries, give diagrammatic bounds for $\mathrm{vol}(S^3 \setminus K)$ in terms any adequate diagram of K. This combined with Theorem 5.1.3 implies that the volume of N can be estimated from any adequate diagram of K. For example, Montesinos knots with a reduced diagrams that contains at least two positive tangles and at least two negative tangles are adequate and have $\delta \leq 0$. Combining Theorem 5.1.3 with [30, Theorem 9.12] and [29, Theorem 1.2] we have the following.

Corollary 5.1.4. *Let $K \subset S^3$ be a Montesinos link with a reduced diagram $D(K)$ that contains at least two positive tangles and at least two negative tangles. If $|q| \geq 6$, then the 3-manifold N obtained by p/q surgery along K is hyperbolic and we have*

$$2v_8\, t \; > \; \mathrm{vol}(N) \; \geq \; \left(1 - \frac{36}{q^2}\right)^{3/2} \frac{v_8}{4}\,(t-9)\,,$$

where $t = t(D)$ is the twist number of $D(K)$, and $v_8 = 3.6638...$ is the volume of a regular ideal octahedron.

Chapter 2

Background

In this chapter we detail relevant background information that will be used throughout this work. In Section 2.1 we recall definitions and results that are pertinent to the study of hyperbolic invariants of 3-manifolds. In Section 2.2 we review some properties of knot diagrams and some notions from the classical study of 3-manifolds. In Section 2.3 we discuss work that has been done relating hyperbolic volume to diagrammatic properties of links.

2.1 Hyperbolic Geometry

In this section we will define what it means for a 3-manifold to be hyperbolic. We will see that the choice of hyperbolic metric on a 3-manifold is unique provided the manifold has finite volume. We will also recall results demonstrating how the topological operation of Dehn filling affects the geometric properties of a 3-manifold. For an introduction and survey of work done in the study of hyperbolic 3-manifolds, see [49].

2.1.1 Hyperbolic Structures on Manifolds

Throughout this work, we will consider the upper half-space model of hyperbolic geometry. We let $\mathbb{H}^3 = \{(z,t) \in \mathbb{C} \times \mathbb{R} : t > 0\}$ and endow $\mathbb{H}^3$ with the metric $ds^2 = \dfrac{|dz|^2 + dt^2}{t^2}$. It is well-known that the isometries of $\mathbb{H}^3$ correspond to $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\pm I$ where

$\mathrm{SL}(2,\mathbb{C})$ is the group of 2×2 matrices of determinant one and I is the identity matrix. The group $\mathrm{PSL}(2,\mathbb{C})$ corresponds to Möbius transformations via

$$A(z) = \frac{az+b}{cz+d} \longleftrightarrow \pm \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2.1}$$

These Möbius transformations extend to maps of $\mathbb{H}^3$ via

$$(z,t) \mapsto \begin{cases} \left(-\dfrac{\overline{z+d/c}}{c^2(|z+d/c|^2+t^2)} + \dfrac{a}{c}, \dfrac{t}{|c|^2(|z+d/c|^2+t^2} \right) & \text{when } c \neq 0 \\ \left(\dfrac{a}{d}\left(z+\dfrac{b}{a}\right), \left|\dfrac{a}{d}\right| t \right) & \text{when } c = 0 \end{cases} \tag{2.2}$$

A group Γ of Möbius transformations is *discrete* if one of the following equivalent conditions holds:

1. No infinite sequence of distinct elements of Γ converges to a Möbius transformation.

2. Γ acts properly discontinuously on $\mathbb{H}^3$, *i.e.* given any closed ball $B \subseteq \mathbb{H}^3$, the set $\{g \in \Gamma : g(B) \cap B \neq \emptyset\}$ is finite.

3. Γ has no limit points in $\mathbb{H}^3$, *i.e.* given $x \in \mathbb{H}^3$ there is no point $y \in \mathbb{H}^3$ with an infinite sequence of distinct elements $\{g_n\}$ in Γ such that $\lim_{n\to\infty} g_n(y) = x$.

A discrete group of Möbius transformations is called a *Kleinian group.* Note that while a Kleinian has no limit points in $\mathbb{H}^3$, it may have limit points on $\partial\mathbb{H}^3 = S^2$. The set $\Omega(\Gamma) = \{x \in S^2 : x \text{ is not a limit point of } \Gamma\}$ is known as the *ordinary set.*

Given torsion free Kleinian group Γ, one may form a quotient manifold $\mathcal{M}(\Gamma) = (\mathbb{H}^3 \cup$

$\Omega(\Gamma))/\Gamma$, namely the set of equivalence classes

$$\mathcal{M}(\Gamma) = \{[x] : x \in \mathbb{H}^3 \cup \Omega(\Gamma) \text{ where } x \equiv y \text{ if and only if } x = g(y) \text{ for some } g \in \Gamma\} \quad (2.3)$$

A manifold M such that $M = \mathcal{M}(\Gamma)$ for some torsion-free Kleinian group Γ is called a *hyperbolic manifold*. Note that the hyperbolic metric of $\mathbb{H}^3$ descends to a hyperbolic metric on the quotient manifold $\mathcal{M}(\Gamma)$.

For a given manifold M, there may be many Kleinian groups Γ such that $M = \mathcal{M}(\Gamma)$. Indeed, if g is a Möbius transformation then $M = \mathcal{M}(g^{-1}\Gamma g)$. Nor is it necessarily the case that if $\mathcal{M}(\Gamma_1)$ is homeomorphic to $\mathcal{M}(\Gamma_2)$ for torsion-free Kleinian groups Γ_1 and Γ_2 then $\mathcal{M}(\Gamma_1)$ is isometric to $\mathcal{M}(\Gamma_2)$. However, when $M = \mathcal{M}(\Gamma)$ has finite volume, the following theorem of Mostow and Prasad shows that the hyperbolic metric on M is independent of the choice of Γ.

Theorem 2.1.1 ([57] and [60])**.** *Suppose for some Kleinian group Γ_1 that $\mathcal{M}(\Gamma_1)$ is has finite volume. Then any isomorphism $\varphi : \Gamma_1 \to \Gamma_2$ onto another Kleinian group Γ_2 is realized by an isometry $\mathcal{M}(\Gamma_1) \to \mathcal{M}(\Gamma_2)$.*

In particular, if $M = \mathcal{M}(\Gamma_1)$ is homeomorphic to $\mathcal{M}(\Gamma_2)$, then $\pi_1(M) \cong \Gamma_1$ and $\pi_1(M) \cong \Gamma_2$. A homeomorphism $h : \mathcal{M}(\Gamma_1) \to \mathcal{M}(\Gamma_2)$ induces an isomorphism $\varphi : \Gamma_1 \to \Gamma_2$, and by Theorem 2.1.1, $\mathcal{M}(\Gamma_1)$ is isometric to $\mathcal{M}(\Gamma_2)$. It follows that the geometric properties of a hyperbolic 3-manifold M do not depend on the choice of Kleinian group used to form the quotient manifold. Therefore geometric properties of a finite volume hyperbolic 3-manifold (such as a link complement) are invariants of the manifold. In particular, the volume of a finite volume hyperbolic 3-manifold is well-defined. A knot K such that $S^3 \backslash K$ is a hyperbolic manifold with finite volume is said to be *hyperbolic*.

2.1.2 Maximal Cusps and Dehn Filling

Suppose that M is a 3-manifold with torus boundary. One can form a closed 3-manifold from M by gluing in a solid torus. Formally, let T be a solid torus and $h : \partial T \to \partial M$ a homeomorphism. Then one can form the manifold $N = M \cup_h T$ by identifying $x \sim h(x)$. We say that N is obtained from M by *Dehn filling*. It is well-known (see for example [47]) that if h and g are isotopic homeomorphisms then $M \cup_h T$ is homeomorphic to $M \cup_g T$.

Suppose that $M = S^3 \backslash N(K)$ where $N(K)$ is a regular open neighborhood of a knot K. Recall that a meridian μ of a solid torus T is a non-separating curve in ∂T that bounds a disk in T. Any two meridians of T are equivalent by ambient isotopy of T (see [62]) so we can refer to *the* meridian of T. Let T be the closure of $N(K)$ and let μ be the meridian of T. There is a curve λ in ∂T, unique up to ambient isotopy of T, that is homologically trivial in M. We can choose to orient μ and λ so that they algebraic intersection number 1. Then any closed curve c in ∂T is isotopic in ∂T to a curve with homology class $a[\mu] + b[\lambda]$ for some $a, b \in \mathbb{Z}$. The ratio a/b is the *slope* of c. One may show that the homeomorphism type of $M \cup_h T$ depends only on the slope a/b of the curve $h(\mu)$ (see [62]). We define $M(a/b)$ to be the manifold obtained by Dehn filling sending a meridian to a curve of slope a/b.

We would like to study the effect that Dehn filling has on the geometry of a hyperbolic 3-manifold. In order to do this, we need to define some terminology. An element

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2, \mathbb{C}) \tag{2.4}$$

is *parabolic* if one of the following equivalent properties hold:

1. A is conjugate to $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

2. A has exactly one fixed point in $S^2 = \partial\mathbb{H}^3$

3. $\mathrm{tr}(A) = \pm 2$ and $A \neq I$ where I is the identity matrix.

A *horosphere* is a set $X \subseteq \mathbb{H}^3$ such that there exists a parabolic element g of $\mathrm{PSL}(2, \mathbb{C})$ for which $g(X) = X$. For example, the plane $\mathbb{C} \times \{1\} \subseteq \mathbb{H}^3$ is a horosphere since it is preserved by the Möbius transformation $g(z) = z + 1$. In general, horospheres in $\mathbb{H}^3$ are Euclidean spheres tangent to a point in $\mathbb{C}$ and horizontal planes. The point of tangency of the former type of horosphere is called the center of the horosphere. The center of a horosphere that is a horizontal plane is said to be ∞.

Suppose that K is a knot and $S^3 \backslash K$ is a hyperbolic manifold. The boundary of a regular neighborhood C of K is homeomorphic to $T^2 \times [1, \infty)$ where T^2 is a torus. Let $\rho : \mathbb{H}^3 \to S^3 \backslash K$ be a covering map. One may choose $C = T^2 \times [1, \infty)$ carefully so that $\rho^{-1}(T^2 \times \{1\})$ is a collection $\mathcal{H}$ of disjoint horospheres in $\mathbb{H}^3$. One may expand or contract C until the horospheres of $\rho^{-1}(C)$ intersect tangentially. This choice of C is called a *maximal cusp neighborhood* of K.

Each horosphere H in $\rho^{-1}(C)$ inherits a Euclidean structure by restricting the metric of $\mathbb{H}^3$ to H. The *cusp area* of K, denoted by $\mathrm{Area}(\partial C)$ is the Euclidean area of ∂C. The *cusp volume*, denoted by $\mathrm{Vol}(C)$ is the volume of C. It is known that $\mathrm{Area}(\partial C) = 2\mathrm{Vol}(C)$. Given a slope s in ∂C, the a curve with slope s is isotopic to a unique geodesic in ∂C. We let $\ell(s)$ denote the length of this geodesic. We define $\ell(\mu)$ to be the length of the meridian of C, and $\ell(\lambda)$ to be the length of the shortest curve with geometric intersection number 1 with the meridian. It is known that cusp volume is bounded above by $\ell(\mu)\ell(\lambda)$.

Dehn filling of $S^3 \backslash K$ is performed by removing the interior of a maximal cusp neighborhood C of K and then attaching a solid torus to the boundary of $S^3 \backslash C$. Depending on the choice of slope, s, for the Dehn filling, $M(s)$ may or may not be hyperbolic. For example, $M(1/0) = S^3$ which is not hyperbolic. A slope s for which $M(s)$ is not hyperbolic is said to be *exceptional*. The Gromov-Thurston 2π theorem [12] showed that if a curve of slope s is isotopic to a geodesic of length greater than 2π, then $M(s)$ will be hyperbolic. This bound was further improved by Agol [10], who showed that if the slope length exceeds 6 then $M(s)$ is hyperbolic. In particular, a meridian must always have length less than 6.

The following theorem, due to Thurston, details the effect that Dehn filling has on the volume of a hyperbolic 3-manifold.

Theorem 2.1.2 ([65]). *Let M be a hyperbolic manifold with torus boundary components and N the result of Dehn filling along one of the boundary components of M. Then vol$(M) >$ vol(N).*

By Theorem 2.1.2 we know that if the result of Dehn filling is a hyperbolic manifold, then the volume must drop. A question that one may ask is how much the volume drops after Dehn filling. Futer, Kalfagianni, and Purcell proved the following theorem in answer to this question.

Theorem 2.1.3 ([31]). *Let M be a complete, finite volume hyperbolic manifold with torus boundary components $C_1, \ldots, C_k$. are disjoint horoball neighborhoods of some subset of the cusps. Let $s_1, \ldots, s_k$ be slopes on $\partial C_1, \ldots, \partial C_k$, each with length greater than 2π. Denote the minimal slope length by ℓ_{min}. Let $M(s_1, \ldots, s_k)$ be the result of Dehn filling with slopes*

$s_1, \ldots, s_k$ *on* $\partial C_1, \ldots, \partial C_k$. *Then* $M(s_1, \ldots, s_k)$ *is a hyperbolic manifold, and*

$$\mathrm{vol}(M(s_1, \ldots, s_k)) \geq \left(1 - \left(\frac{2\pi}{\ell_{\min}}\right)^2\right)^{3/2} \mathrm{vol}(M) \tag{2.5}$$

2.2 Essential Surfaces

A surface S is said to be properly embedded in a 3-manifold M if ∂S is embedded in ∂M and the interior of S is embedded in the interior of M. A *compressing disk* D for S in M is a disk embedded in M such that $\partial S \cap D = \partial D$. A properly embedded surface S in M is *incompressible* if, given a compressing disk D for S in M, the curve ∂D bounds a disk in S. A ∂*-compressing disk* D for S in M is one such that

1. $D \cap S$ is an arc α in ∂D
2. $D \cap M$ is an arc β in ∂M
3. $\partial D = \alpha \cup \beta$
4. $\alpha \cap \beta = \partial \alpha = \partial \beta$
5. α does not co-bound a disk with an arc in ∂S

A surface is ∂*-incompressible* if there are no ∂-compressing disks for S in M. A surface that is both incompressible and ∂-incompressible is called *essential*.

We now provide some examples of essential surfaces in knot complements that will be used in this work. One example is the *checkerboard surface* which is described as follows. Given a diagram D for a knot K, one may color the regions of D black and white so that each black (and respectively white) region meets a black (respectively white) region at the

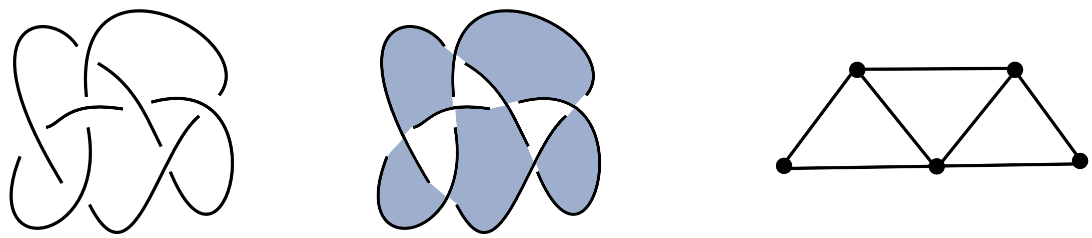

Figure 2.1: Left: The 7_7 knot. Center: Checkerboard coloring of the 7_7 knot diagram. Right: The checkerboard graph associated to the black coloring.

crossings. See for example Figure 2.1. One can form a checkerboard surface by placing a disk in each black region of the coloring and then replacing a neighborhood of each crossing with a twisted band. Note that the boundary of the checkerboard surface is equal to the knot, and that the surface is not necessarily orientable. One can form a checkerboard surface in a similar manner by considering the white regions instead of the black regions.

Related to checkerboard surfaces for a knot are checkerboard graphs. A checkerboard graph is formed by checkerboard coloring the knot diagram, placing each vertex in each black region (alternatively in each white region), and connecting vertices with one edge per crossing of D connecting the corresponding shaded (respectively white) regions. See the right of Figure 2.1 for an example.

An alternating diagram of a link is reduced if it does not contain a nugatory crossing, that is a crossing of the form shown in Figure 2.2. A diagram D of a knot is *prime* if a circle S^1 in the diagram meeting the knot transversely in two points bounds an unknotted arc on one side. It was shown by Menasco and Thistlethwaite [51] that when D is a reduced alternating prime diagram for a knot K, then the checkerboard surfaces are essential.

Another type of essential surface that arises is related to the notion of adequacy for links. Let D be a diagram for a knot K. At each crossing of the diagram D one may resolve the

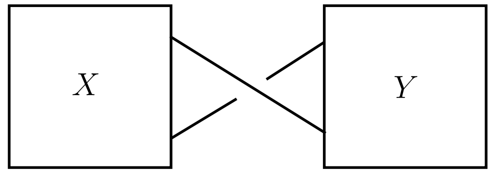

Figure 2.2: A nugatory crossing.

crossing in one of two ways: the A-resolution and the B-resolution as depicted in Figure 2.3. One often records the resolution by replacing the crossing with an arc, for example the red arc in Figure 2.3. A choice of resolution at every crossing of D is called a *state* σ. The result of applying the state σ to D, denoted $s_\sigma(D)$, is a collection of disjoint circles called state circles. See for example Figure 2.4, where the thick lines of the center figure are the state circles arising from the all-A resolution of the link on the left. One may then form the state graph G_σ where vertices correspond to state circles of $s_\sigma(D)$ and and edges correspond to former crossings in D.

Definition 2.2.1. A diagram D is called A-*adequate* (respectively B-*adequate*) if the state graph of the all-A resolution (respectively all-B resolution) has no 1-edge loops. A diagram that is both A and B adequate is called *adequate*. A knot is called *adequate* if it has an adequate diagram.

Given a diagram D of a knot K, one may form a surface S_A as follows. The state circles of the all-A resolution of D bound disks on the projection plane. Isotope these disks slightly off the projection plane so they become disjoint. For each crossing of D, attach a half-twisted band so that the resulting surface S_A has boundary $\partial S_A = K$. One may form the surface S_B similarly. See the right of Figure 2.4, for an example of a state surface.

The following theorem is due to Ozawa [58]. A different proof is given by Futer, Kalfagianni, and Purcell [30, Theorem 3.19].

Figure 2.3: The two resolutions of a crossing, the arcs recording them, and their contribution to state surfaces. The left frame depicts the A-resolution, and the right depicts the B-resolution.

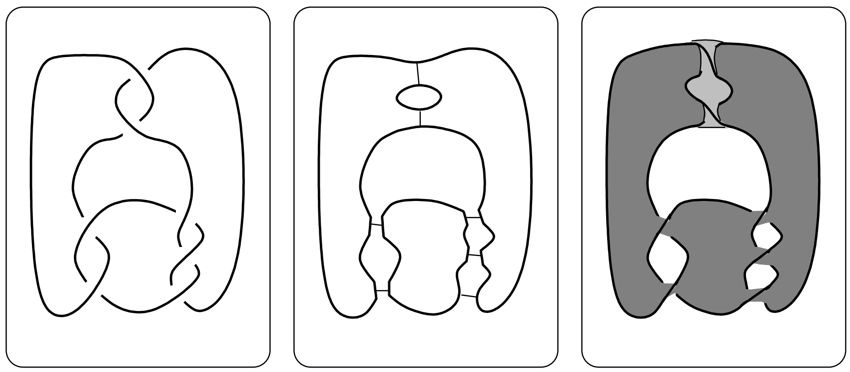

Figure 2.4: Left: A link L. Center: The all-A resolution. The thick lines are the state circles, and the thin lines are arcs recording the resolutions. Right: The all-A state surface of L. Figure taken from [30].

Theorem 2.2.2. *Let $D(K)$ be an adequate link diagram of a knot K. Then the all-A state and the all-B state surfaces corresponding to $D(K)$ are essential in $S^3 \backslash K$.*

2.3 Twist Number and Volume

A topic explored by researchers is finding bounds on the hyperbolic volume of a knot in terms of information easily read from a diagram. This section recalls some results in this direction, in particular volume bounds in terms of the number of twist regions of a diagram, which we now define.

Definition 2.3.1. A *twist region* of a diagram D is either a connected collection of bigon regions of D arranged in a row, which is maximal in the sense that it is not part of a longer row of bigons, or a single crossing adjacent to no bigon regions. The *twist number* of a diagram D is the number of twist regions in the diagram. A diagram is *twist reduced* whenever a simple closed curve in the diagram intersects the link projection transversely in four points disjoint from the crossings, and two of these points are adjacent to some crossing, and the remaining two points are adjacent to some other crossing, then this curve bounds a subdiagram that consists of a (possibly empty) collection of bigons arranged in a row between these two crossings.

Theorem 2.3.2 ([44])**.** *Let K be an alternating hyperbolic link with t twist regions in a prime alternating diagram. Then*

$$\frac{v_4}{2}(t-2) < \mathrm{vol}(K) < 10v_4(t-1) \tag{2.6}$$

where $v_4 \approx 1.01494$ is the volume of a regular ideal tetrahedron.

The upper bound of Theorem 2.3.2 can be improved in the case of Montesinos links using work of Futer, Kalfagianni, and Purcell.

Theorem 2.3.3 ([30, Theorem 9.12])**.** *Let K be a hyperbolic Montesinos link. Then*

$$\mathrm{vol}(K) < 2v_8 t \tag{2.7}$$

where t is the number of twists in some diagram of K and $v_8 \approx 3.66386237$ is the volume of a regular ideal hyperbolic octahedron.

Note that while the statement of Theorem 2.3.3 in [30] requires the link to have at least three positive tangles and at least three negative tangles, this condition was only necessary to prove the lower bound stated in that theorem.

Chapter 3

Volume and Determinant Densities

3.1 Introduction

Recent work of Champanerkar, Kofman, and Purcell ([19], [20], and [21]) investigated a relationship between the volume of a knot, its determinant, and its crossing number. In this chapter we continue to explore this relationship. We recall the definition of the following invariant relating volume and crossing number.

Definition 3.1.1. Given a hyperbolic link K, let $\mathrm{vol}(K)$ be its hyperbolic volume, and let $c(K)$ be the crossing number of K. The *volume density* of K is defined to be

$$d_{\mathrm{vol}}(K) := \frac{\mathrm{vol}(K)}{c(K)}$$

D. Thurston [64] showed that $d_{\mathrm{vol}}(K) \leq v_8$ for all knots K, where $v_8 \approx 3.66286$ is the volume of the regular ideal hyperbolic octahedron. This was done by decomposing $S^3 \backslash K$ into octahedra, placing one octahedron at each crossing, and pulling the remaining vertices to $\pm\infty$. Adams [2] showed that $\mathrm{vol}(K) \leq (c(K) - 5)v_8 + 4v_4$ for any link having $c(K) \geq 5$, where $v_4 \approx 1.01494$ is the volume of the ideal hyperbolic tetrahedron. Therefore v_8 is a strict upper bound for the volume density of any finite link.

Champanerkar, Kofman, and Purcell [20] showed that the upper bound of v_8 on volume

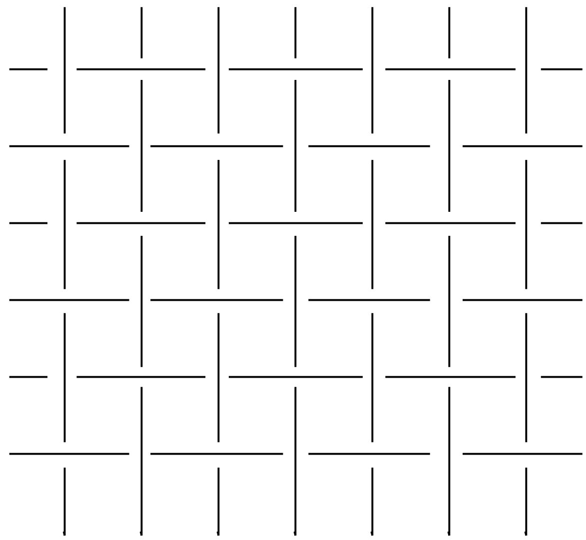

Figure 3.1: The infinite weave $\mathcal{W}$.

density is asymptotically sharp. More precisely, there exist sequences of links $\{K_n\}_{n=0}^{\infty}$ such that

$$\lim_{n\to\infty} d_{\mathrm{vol}}(K_n) = v_8$$

(see Theorem 3.2.3). Such sequences of links are called *geometrically maximal*. The authors of [20] produce examples of geometrically maximal sequences of links by constructing links approaching the *infinite weave* $\mathcal{W}$, the infinite alternating link with the square lattice projection depicted in Figure 3.1. All the links in the examples constructed were alternating, and do not contain a cycle of tangles (see Definition 3.2.2). In Section 3.3 we will show that there exist geometrically maximal sequences of links having a cycle of tangles (see Theorem 3.3.1). We then use this to prove Theorem 3.3.3 which states that there exist geometrically maximal sequences of non-alternating links.

Let $\mathcal{C}_{\mathrm{vol}} = \{\mathrm{vol}(K)/c(K) : K \text{ is a hyperbolic link}\} \subseteq \mathbb{R}$ and let $\mathrm{Spec}_{\mathrm{vol}}$ be the set of limit points of $\mathcal{C}_{\mathrm{vol}}$. We call $\mathrm{Spec}_{\mathrm{vol}}$ the *spectrum of volume densities*. A question that arose in [20] was how one may describe the sets $\mathcal{C}_{\text{vol}}$ and Spec_{vol}. (See also [19] for discussion and related questions.) In particular, what numbers occur as volume densities? Since v_8 forms an upper bound on volume density, it is clear that $\mathcal{C}_{\text{vol}}$ and Spec_{vol} are subsets of $[0, v_8]$. The existence of geometrically maximal knots implies that $v_8 \in \text{Spec}_{\text{vol}}$. In Lemma 3.4.1, we describe *geometrically minimal* sequences of links, i.e. those with volume density approaching 0. Hence $0 \in \text{Spec}_{\text{vol}}$. It was shown by Champanerkar, Kofman, and Purcell in [21] that $2v_4 \in \text{Spec}_{\text{vol}}$. We are able to prove the following theorem.

Theorem 3.4.3. *The set $\mathcal{C}_{\text{vol}}$ of volume densities of hyperbolic links is a dense subset of $[0, v_8]$, and $\text{Spec}_{\text{vol}} = [0, v_8]$. In other words, given $x \in [0, v_8]$ there exists a sequence $\{K_n\}_{n=1}^{\infty}$ of hyperbolic links such that the volume densities satisfy $\lim_{n\to\infty} d_{\text{vol}}(K_n) = x$.*

Another invariant that was studied in [19] and [20] was the *determinant density*, $d_{\text{det}}(K)$ of a link K, which is defined by the equation

$$d_{\text{det}}(K) := \frac{2\,\pi\, \log(\det(K))}{c(K)}$$

It is shown in [20] that there are sequences $\{K_n\}_{n=1}^{\infty}$ of links such that $\lim_{n\to\infty} d_{\text{det}}(K_n) = v_8$ (see Theorem 3.2.4 of this paper). Such sequences are called *diagrammatically maximal*. We define $\mathcal{C}_{\text{det}} = \{d_{\text{det}(K)} : K \text{ is a (not necessarily hyperbolic) link}\} \subseteq \mathbb{R}$ and define Spec_{vol} to be the set of limit points of $\mathcal{C}_{\text{det}}$. We call Spec_{det} the *spectrum of determinant densities*.

It is interesting to study the relationships between Spec_{vol} and Spec_{det}. While we know that $\text{Spec}_{\text{vol}} \subseteq [0, v_8]$, it is unknown whether v_8 forms an upper bound for Spec_{det}. However, it is a conjecture [20, Conjecture 1.1] that v_8 is an upper bound for Spec_{det}. We are able to prove the following theorem.

Theorem 3.5.3. *The spectrum of determinant densities* $\mathrm{Spec}_{\mathrm{det}}$ *contains* $[0, v_8]$*. In other words, given any* $x \in [0, v_8]$ *there exists a sequence* $\{K_n\}_{n=1}^{\infty}$ *of links satisfying*

$$\lim_{n\to\infty} d_{\mathrm{det}}(K_n) = x \tag{3.26}$$

In [19] and [20] it is shown that $0, v_8 \in \mathrm{Spec}_{\mathrm{vol}} \cap \mathrm{Spec}_{\mathrm{det}}$, and the authors ask what this intersection is. As a corollary of Theorems 3.4.3 and 3.5.3 we obtain:

Corollary 3.1.2. *The intersection* $\mathrm{Spec}_{\mathrm{vol}} \cap \mathrm{Spec}_{\mathrm{det}}$ *is equal to* $[0, v_8]$.

Corollary 3.1.3. *The intersection* $\mathrm{Spec}_{\mathrm{vol}} \cap \mathrm{Spec}_{\mathrm{det}}$ *is equal to* $[0, v_8]$.

Since $\mathrm{Spec}_{\mathrm{vol}}$ considers only the densities of hyperbolic links, it is useful to know what determinant densities are realized by *hyperbolic* links. We show in Theorem 3.5.4 that the set of determinant densities of hyperbolic links is also dense in $[0, v_8]$.

The results of this chapter appeared in [14]. Shortly after the author posted the results of this paper, Colin Adams, Aaron Calderon, Xinyi Jiang, Alexander Kastner, Gregory Kehne, Nathaniel Mayer, and Mia Smith [5] independently proved results similar to those in this paper. In particular, they proved that the set of volume densities is dense in $[0, v_8]$ and that the set of determinant densities is also dense in $[0, v_8]$. Their proof is similar to the one found in this paper. They consider knots similar to the knots $N(W_{n,n})$ of Definition 3.3.2 of this paper, and they cite work of Futer, Kalfagianni, and Purcell [31] and Thurston [65] to obtain volume bounds. We likewise cite [65], and we cite a result of Futer, Kalfagianni, and Purcell [32], which was derived from [31], to obtain our volume bounds.

3.2 Background

3.2.1 Geometrically and Diagrammatically Maximal Links

In this subsection we recall definitions and results from [20]. Given a link $K \subset \mathbb{R}^3$ we define the projection graph $G(K)$ to be the 4-valent graph obtained by projection of K onto $\mathbb{R}^2 \times \{0\}$.

Definition 3.2.1. Let G be a possibly infinite graph. For any finite subgraph H, let ∂H be the set of vertices of H that share an edge with a vertex not in H. Let $|\cdot|$ denote the number of vertices in a finite graph. An exhaustive nested sequence of finite, connected subgraphs $\{H_n \subseteq G : H_n \subseteq H_{n+1}, \bigcup_{n=0}^{\infty} H_n = G\}$ is a *Følner sequence* for G if

$$\lim_{n\to\infty} \frac{|\partial H_n|}{|H_n|} = 0.$$

The graph G is *amenable* if a Følner sequence for G exists. In particular, the infinite square lattice (i.e. the projection graph of $\mathcal{W}$) is amenable.

Given a link K in S^3, a *Conway sphere* is a 2-sphere in S^3 intersecting K transversely in four points. Using Conway spheres, we will define a cycle of tangles, which is an important definition for the statement of Theorem 3.2.3.

Definition 3.2.2. A Conway sphere is called *visible* if it is parallel to one dividing the diagram into two tangles, as in Figure 3.2. A tangle is called *knotty* if it is nontrivial, and not a (portion of a) single twist region; i.e. not a rational tangle of type n or $1/n$ for $n \in \mathbb{Z}$. We will say that K_n contains a *cycle of tangles* if K_n contains a visible Conway sphere with a knotty tangle on each side.

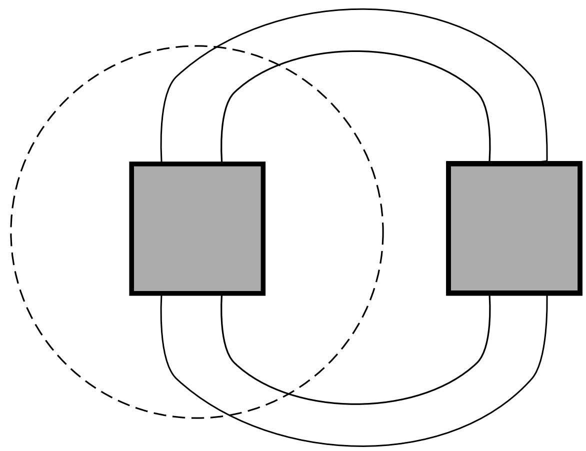

Figure 3.2: The dashed line represents a visible Conway sphere.

The following theorem was proved by Champanerkar, Kofman, and Purcell.

Theorem 3.2.3 ([20, Theorem 1.4])**.** *Let K_n be any sequence of hyperbolic alternating link diagrams that contain no cycle of tangles, such that*

1. *there are subgraphs $G_n \subseteq G(K_n)$ that form a Følner sequence for $G(\mathcal{W})$, and*
2. $\displaystyle\lim_{n\to\infty} \frac{|G_n|}{c(K_n)} = 1.$

Then $\{K_n\}_{n=0}^{\infty}$ is geometrically maximal: $\displaystyle\lim_{n\to\infty} \frac{vol(K_n)}{c(K_n)} = v_8.$

The following theorem, proved in [20, Theorem 1.5], gives a similar statement for diagrammatically maximal links.

Theorem 3.2.4. *Let $\{K_n\}_{n=0}^{\infty}$ be a sequence of alternating link diagrams such that*

1. *there are subgraphs $G_n \subseteq G(K_n)$ that form a Følner sequence for $G(\mathcal{W})$, and*
2. $\displaystyle\lim_{n\to\infty} \frac{|G_n|}{c(K_n)} = 1.$

Then $\{K_n\}_{n=0}^{\infty}$ is diagrammatically maximal: $\displaystyle\lim_{n\to\infty} \frac{\det(K_n)}{c(K_n)} = v_8.$

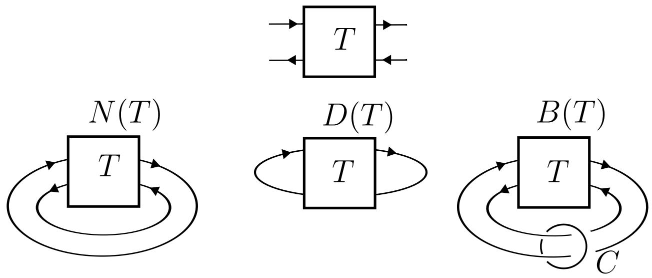

Figure 3.3: A tangle T, its two closures $N(T)$ and $D(T)$, and its corresponding belted tangle $B(T)$.

Note that the cycle of tangles condition is not necessary in the construction of *diagrammatically* maximal links. This is one reason motivating Theorem 3.3.1, which states that there are *geometrically* maximal sequences of links containing a cycle of tangles.

3.2.2 Conway Sums and Belted Sums of Links

Let T be a 2-tangle. There are two ways to close the tangle T to form a link. Denote these closures $N(T)$ and $D(T)$ as depicted in Figure 3.3. The closure $N(T)$ is called the *numerator closure* and $D(T)$ is called the *denominator closure*. Next, one may add an extra component, C, called the *belt component*, to the link $N(T)$ in the following manner. Let C be an unknotted circle that lies in a plane orthogonal to the projection plane and encircles the arcs added to T to form $N(T)$, as in Figure 3.3. The resulting link, $B(T)$, is the *belted link corresponding to T* or simply the *belted tangle*.

Given two tangles T_1 and T_2 such that $B(T_1)$ and $B(T_2)$ are hyperbolic, one may form the *belted sum* of $B(T_1)$ and $B(T_2)$ as follows. Let M_i be the link complement $S^3 \backslash B(T_i)$, for $i = 1, 2$. Work of Adams [8] shows that the belt component C_i of T_i bounds a totally geodesic, twice-punctured disk in M_i. Cut each M_i along this twice-punctured disk to form a manifold M_i'. By Adams [8], there is a unique hyperbolic structure on a twice-punctured

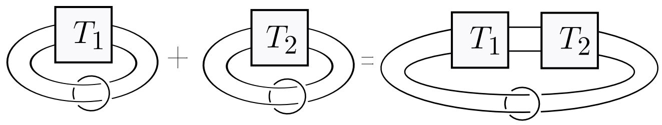

Figure 3.4: The belted sum of two belted tangles.

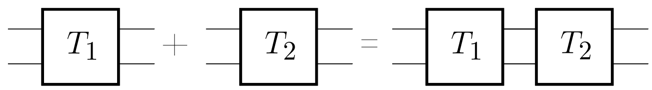

Figure 3.5: The Conway sum of two tangles. The tangle $T_1 + T_2$ is pictured on the right.

disk. Therefore we may glue M_1' to M_2' via an isometry of twice-punctured disks that maps C_1 to C_2. The result is a link, denoted $B(T_1) + B(T_2)$, which we call the belted sum of $B(T_1)$ and $B(T_2)$. See Figure 3.4.

The following theorem about belted sums of links will be important in the proofs of Theorems 3.4.3, 3.3.1, 3.3.3.

Lemma 3.2.5 ([8, Corollary 5.2])**.** *Suppose that n belted tangles $B(T_1), \ldots B(T_n)$ are hyperbolic with finite volume. Then the belted sum $B(T_1) + \ldots + B(T_n)$ is also a hyperbolic link and has volume* $\mathrm{vol}(B(T_1)) + \ldots + \mathrm{vol}(B(T_n))$.

In addition to the belted sum of belted tangles, we can sum tangles in the manner described below.

Definition 3.2.6. Let T_1 and T_2 be 2-tangle diagrams. One may connect these 2-tangle diagrams as indicated in Figure 3.5 to form the 2-tangle $T_1 + T_2$. We call $T_1 + T_2$ the *tangle sum*, or *Conway sum* of T_1 and T_2.

Lemma 3.2.5 showed that volume is additive under belted sums of tangles. Similarly, we have the following lemma describing the determinant of a Conway sum of tangles.

Lemma 3.2.7. *Let $T_1, \ldots, T_n$ be tangles and let $T = T_1 + \ldots + T_n$ be the tangle sum. Then*

$$\det(D(T)) = \prod_{i=1}^{n} \det(D(T_i)).$$

Proof. When $n = 2$ we have the following well-known fact [47, Proposition 6.12]:

$$\det(D(T_1 + T_2)) = \det(D(T_1)\#D(T_2)) = \det(D(T_1))\det(D(T_2))$$

The result now follows by induction.

□

We now recall a result of Futer, Kalfagianni, and Purcell [32] which will play an important role in the proofs of Theorems 3.3.1, 3.3.3, and 3.4.3. We begin with the following definition. A tangle T is an *east-west* twist if $N(T)$ is the standard diagram of a $(2, q)$-torus link. To simplify notation throughout, we define ξ_n as follows:

$$\xi_n := \left(1 - \left(\frac{8\pi}{11.524 + n\sqrt[4]{2}}\right)^2\right)^{3/2} \tag{3.1}$$

Note that $\lim\limits_{n\to\infty} \xi_n = 1$.

Theorem 3.2.8. *Let $T_1, \ldots, T_n$, $n \geq 12$, be tangles admitting prime, alternating diagrams, none of which is an east-west twist. Let K be the numerator closure of the Conway sum $N(T_1 + \ldots + T_n)$. Let L be the belted sum $B(T_1) + \ldots + B(T_n)$. Then K is hyperbolic and*

$$\mathrm{vol}(K) \geq \xi_n \mathrm{vol}(L) \tag{3.2}$$

Proof. This follows directly from the proof of [32, Theorem 1.5] as follows. In the second to last line of that proof, the authors showed that

$$\mathrm{vol}(K) \geq \left(1 - \left(\frac{2\pi}{\ell_{\min}}\right)^2\right)^{3/2} \mathrm{vol}(L) \tag{3.3}$$

where $\ell_{\min}$ is the minimum possible length of the meridian of the belt. They also showed in the same proof that $\ell_{\min} \geq \dfrac{11.524 + n\sqrt[4]{2}}{4}$, hence $\mathrm{vol}(K) \geq \xi_n \mathrm{vol}(L)$. □

3.2.3 Adequate Tangles

We state some results about adequate link diagrams that will play an important role in the proofs of Theorems 3.3.1, 3.3.3, 3.4.3, and 3.5.3.

Definition 3.2.9. A link diagram $D \subset S^2$ is *prime* if any simple closed curve in S^2 that meets D transversely at two points bounds, on one side of it, a disk that intersects D in a diagram U of the unknotted ball-arc pair. The diagram D is *strongly prime* if, in addition, U is the zero-crossing diagram.

We will use the following theorem, which follows from [30, Corollary 3.21] to show that a link diagram is prime.

Theorem 3.2.10. *Suppose that K is a non-split, prime link. Then every adequate diagram of K without nugatory crossings is prime.*

The following proposition gives a way to show that a link is non-alternating. We will use this in the proof of Theorem 3.3.3 where we show that there exist geometrically maximal sequences of non-alternating links.

Proposition 3.2.11. *If L has a prime, non-alternating, adequate diagram, then L is non-alternating.*

Proof. Let L be a link and D a diagram for L having n crossings. Let breadth(L) be the breadth of the Jones polynomial for L, in other words, the difference between the maximal and minimal degrees of the Jones polynomial for L. We recall the following facts which may be found in [47, Proposition 5.3, Theorem 5.9, and Corollary 5.14].

1. If D is reduced and alternating, then D is adequate.

2. If D is an adequate diagram, then $c(L) = n$.

3. If D is reduced and alternating, then breadth$(L) = n$.

4. If D is non-alternating and prime, then breadth$(L) < n$.

Suppose D_1 is a prime, non-alternating, adequate diagram for L, and that D_2 is an alternating diagram for L. By removing nugatory crossings, we may assume that D_2 is a reduced, alternating diagram. Therefore both D_1 and D_2 are adequate diagrams and have the same number of crossings by facts (1) and (2) above. Let n be the number of crossings in D_1 (or D_2). Since D_1 is non-alternating, fact (4) above implies breadth$(L) < n$. On the other hand, since D_2 is reduced and alternating, fact (3) implies that breadth$(L) = n$, a contradiction. □

Definition 3.2.12. A tangle diagram T is *adequate* if the diagrams of both $N(T)$ and $D(T)$ are adequate. In the event that both closures produce reduced, alternating diagrams, the tangle T is said to be *strongly alternating*. A strongly alternating tangle is adequate.

Proposition 3.2.13. *The tangle sum of n adequate tangles is adequate for $n \geq 2$.*

Proof. It follows from work of Lickorish and Thistlethwaite [48, Proposition 4] that the tangle sum of two adequate tangles is adequate. The proposition readily follows by induction on n. □

We close this section with the following corollary showing that crossing number of adequate tangles is additive. This plays a key role in the proof of Theorems 3.4.3 and 3.5.3.

Corollary 3.2.14. *Let $T_1, \ldots, T_n$ be adequate tangle diagrams and let $c(N(T_i))$ be the crossing number of the closure for $i = 1, \ldots n$. Let $T_1 + \ldots + T_n$ be the tangle sum. Then the crossing numbers satisfy*

$$c(N(T_1 + \ldots + T_n)) = c(N(T_1)) + \ldots + c(N(T_n))$$

$$c(D(T_1 + \ldots + T_n)) = c(D(T_1)) + \ldots + c(D(T_n))$$

Proof. The natural diagram representing the link $N(T_1 + \ldots + T_n)$ as a Conway sum has $c(N(T_1)) + \ldots + c(N(T_n))$ crossings. Since $T_1, \ldots, T_n$ are adequate, we know that $N(T_1 + \ldots + T_n)$ is adequate by Proposition 3.2.13. Note that an adequate diagram of a link has the minimal number of crossings by Proposition 3.2.11. Therefore $c(N(T_1 + \ldots + T_n)) = c(N(T_1)) + \ldots + c(N(T_n))$. The proof that $c(D(T_1 + \ldots + T_n)) = c(D(T_1)) + \ldots + c(D(T_n))$ is similar. □

3.3 Non-alternating Geometrically Maximal Knots

We now address the question of whether all the conditions in Theorem 3.2.3 are necessary. We begin with a construction of a geometrically maximal sequence of links for which each link contains a cycle of tangles.

Theorem 3.3.1. *Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of tangles admitting prime, alternating diagrams, none of which is an east-west twist. Let $K_i = N(T_i)$ for all i. Suppose that each K_i is hyperbolic and that the sequence $\{K_i\}_{i=1}^{\infty}$ is geometrically maximal. For each $n \in \mathbb{N}$, let*

$T(n)$ be the Conway sum of n copies of T_n, and define $K(n) = N(T(n))$. Then $K(n)$ is hyperbolic whenever $n \geq 12$, and the sequence $\{K(n)\}_{n=1}^{\infty}$ is a geometrically maximal sequence of links, all but one of which contains a cycle of tangles.

Proof. By construction, whenever $n \geq 2$, we have that $K(n)$ contains a cycle of tangles. Moreover, whenever $n \geq 12$ we have from Theorem 3.2.8 that $K(n)$ is hyperbolic. We now show that the sequence $\{K(n)\}_{n=1}^{\infty}$ is geometrically maximal. Let $L_i = B(T_i)$, the belted link corresponding to L_i, and for each $n \in \mathbb{N}$ let $L(n)$ be the belted sum of n copies of L_n. By Theorem 3.2.8 we have that $\mathrm{vol}(K(n)) \geq \xi_n \mathrm{vol}(L(n))$ where ξ_n was defined in (3.1). We obtain

$$
\begin{aligned}
\mathrm{vol}(L(n)) &= \sum_{i=1}^{n} \mathrm{vol}(L_n) && \text{by Lemma 3.2.5} \\
&= n\,\mathrm{vol}(L_n) \\
&> n\,\mathrm{vol}(K_n) && \text{by Theorem 2.1.2} \qquad (3.4)
\end{aligned}
$$

Observe that $c(K(n)) \leq n\, c(K_n)$ since the natural diagram demonstrating $K(n)$ as a Conway sum has $n\, c(K_n)$ crossings. Therefore

$$
\begin{aligned}
\frac{\mathrm{vol}(K(n))}{c(K(n))} &\geq \frac{\mathrm{vol}(K(n))}{n\, c(K_n)} \\
&\geq \xi_n \frac{\mathrm{vol}(L(n))}{n\, c(K_n)} && \text{by Theorem 3.2.8} \\
&> \xi_n \frac{n\,\mathrm{vol}(K_n)}{n\, c(K_n)} && \text{by (3.4)} \qquad (3.5)
\end{aligned}
$$

Note that $\lim_{n\to\infty} \xi_n = 1$. Since $\{K_n\}_{n=1}^{\infty}$ is geometrically maximal, we know that

$$\lim_{n\to\infty} \frac{\mathrm{vol}(K_n)}{c(K_n)} = v_8$$

hence by (3.5) we obtain

$$\lim_{n\to\infty} d_{\mathrm{vol}}(K(n)) = \lim_{n\to\infty} \frac{\mathrm{vol}(K(n))}{c(K(n))} \geq \lim_{n\to\infty} \xi_n \frac{\mathrm{vol}(K_n)}{c(K_n)} = v_8$$

Moreover, $d_{\mathrm{vol}}(K(n)) \leq v_8$ for all n, as this is true for any link. Therefore $\{K(n)\}_{n=1}^{\infty}$ is geometrically maximal. $\square$

We can use a method similar to the proof of Theorem 3.3.1 to prove that there exist geometrically maximal sequences of non-alternating links. We will construct a specific example of such a sequence. We begin by considering a family of weaving tangles in Definition 3.3.2. By taking a Conway sum of these weaving tangles and their reflections (i.e. the result of changing all the over-crossings to under-crossings and vice versa), we will obtain a geometrically maximal sequence of non-alternating links.

Definition 3.3.2. Let B_m be the braid group on m strings. For $i = 1, \ldots m-1$, let $\sigma_i \in B_m$ be the the ith Artin generator, *i.e.* the group element corresponding to twisting the ith strand under the $(i+1)$st strand. Let $L_{m,n}$ be the element $(\sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\ldots\sigma_{m-1}^{(-1)^{m-1}})^n \in B_m$ depicted on the left hand side of Figure 3.6. Let $W_{m,n}$ be the result of closing all but the second and third strands of the braid as in the right hand side of Figure 3.6. We will call $W_{m,n}$ the *weaving tangle* of order n on m strands. The knots $N(W_{m,n})$ are the weaving knots of [21].

Theorem 3.3.3. *There exist geometrically maximal sequences of non-alternating links.*

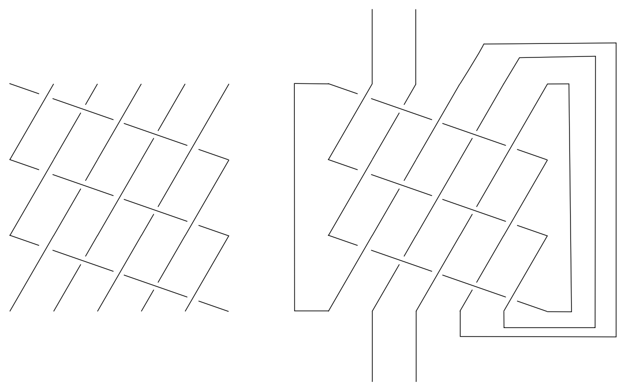

Figure 3.6: LEFT: The braid element $L_{6,3}$. RIGHT: The weaving tangle $W_{6,3}$.

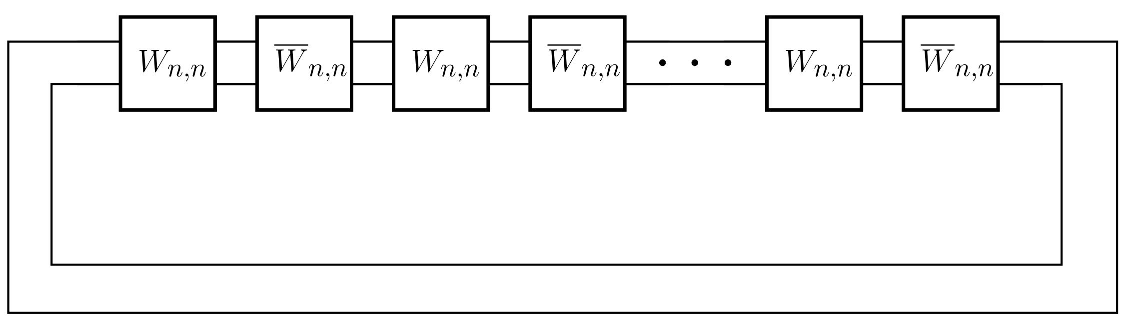

Figure 3.7: The link $K(n)$ for the proof of Theorem 3.3.3. It contains n copies of $W_{n,n}$ and n copies of $\overline{W}_{n,n}$.

Proof. Let $W_{k,k}$ be the weaving tangle and let $\overline{W}_{k,k}$ be its reflection, i.e. the result of changing each over-crossing to an under-crossing and vice versa. For $n \geq 1$ define

$$K(n) = N(W_{n,n} + \overline{W}_{n,n} + \ldots + W_{n,n} + \overline{W}_{n,n})$$

to be the closure of the Conway sum of n copies of $W_{n,n}$ with n copies of $\overline{W}_{n,n}$ as indicated in Figure 3.7. Let $K_n = N(W_{n,n})$ and $\overline{K}_n = N(\overline{W}_{n,n})$, and $L_n = B(W_{n,n})$ and $\overline{L}_n = B(\overline{W}_{n,n})$. Note that K_n and $\overline{K}_n$ are Dehn fillings of L_n and $\overline{L}_n$, respectively.

We prove that the sequence $\{K(n)\}_{n=1}^{\infty}$ is geometrically maximal. First, observe that when $n \geq 6$, Theorem 3.2.8 implies that $K(n)$ is hyperbolic. Let

$$L(n) = B(W_{n,n}) + B(\overline{W}_{n,n}) + \ldots + B(W_{n,n}) + B(\overline{W}_{n,n})$$

be the belted sum of n copies of $B(W_{n,n})$ with n copies of $B(\overline{W}_{n,n})$. Similar to the proof of Theorem 3.3.1 we have

$$\begin{aligned}
\mathrm{vol}(L(n)) &= \sum_{i=1}^{n}(\mathrm{vol}(L_n) + \mathrm{vol}(\overline{L}_n)) && \text{by Lemma 3.2.5} \\
&= n[\mathrm{vol}(L_n) + \mathrm{vol}(\overline{L}_n)] \\
&> n[\mathrm{vol}(K_n) + \mathrm{vol}(\overline{K}_n)] && \text{by Theorem 2.1.2} \qquad (3.6)
\end{aligned}$$

Since K_n and $\overline{K_n}$ are reduced and alternating, by counting the number of crossings in Figure 3.6 and using Proposition 3.2.11, we find that $c(K_n) = c(\overline{K}_n) = n(n-1)$. Now the diagram of $K(n)$ depicted in Figure 3.7 has $2n^2(n-1)$ crossings, implying $c(K(n)) \leq 2n^2(n-1)$.

Therefore

$$
\begin{aligned}
\frac{\operatorname{vol}(K(n))}{c(K(n))} &\geq \frac{\operatorname{vol}(K(n))}{2n^2(n-1)} \\
&\geq \xi_{2n}\frac{\operatorname{vol}(L(n))}{2n^2(n-1)} && \text{by Theorem 3.2.8} \\
&> \xi_{2n}\frac{n[\operatorname{vol}(K_n)+\operatorname{vol}(\overline{K}_n)]}{2n^2(n-1)} && \text{by (3.6)} \\
&= \frac{\xi_{2n}}{2}\left(\frac{\operatorname{vol}(K_n)}{c(K_n)}+\frac{\operatorname{vol}(\overline{K}_n)}{c(\overline{K}_n)}\right) && (3.7) \\
&= \frac{\xi_{2n}}{2}[d(K_n)+d(\overline{K}_n)] && (3.8)
\end{aligned}
$$

Both K_n and $\overline{K}_n$ are geometrically maximal by Theorem 3.2.3, so

$$\lim_{n\to\infty} d(K_n) = \lim_{n\to\infty} d(\overline{K}_n) = v_8 \qquad (3.9)$$

Moreover, $\lim_{n\to\infty}\xi_{2n} = 1$, so (3.8) implies that

$$\lim_{n\to\infty} d(K(n)) = v_8.$$

We show that $K(n)$ is non-alternating. Let D be the diagram shown in Figure 3.7. Observe that D is non-alternating. Since $W_{k,k}$ and $\overline{W}_{k,k}$ are strongly alternating (hence adequate) tangle diagrams, Proposition 3.2.13 implies that D is adequate. Next, we establish the conditions needed to apply Theorem 3.2.10. Since an adequate diagram has the minimal number of crossings, there are no nugatory crossings in D. An adequate (hence minimal crossing) diagram of a split link must also be split. Since D is non-split, it follows that $K(n)$ is non-split. Now $K(n)$ is hyperbolic when $n \geq 6$, as noted above. Therefore $K(n)$ is a prime link. It now follows from Theorem 3.2.10 that the diagram D is prime. Proposition 3.2.11

now implies that $K(n)$ is non-alternating. $\square$

Observe that the sequence of links $\{K(n)\}_{n=1}^{\infty}$ in the proof of Theorem 3.3.3 is geometrically maximal and each link of the sequence contains a cycle of tangles.

3.4 The Spectrum of Volume Densities

We now turn to the proof that the set of volume densities is dense in $[0, v_8]$. In Lemma 3.4.1, we prove the existence of tangles whose closures form a *geometrically minimal* sequence of links, *i.e.* a sequence links with volume density approaching 0. On the other hand, Lemma 3.4.2 produces a sequence of tangles whose closures form a geometrically maximal sequence of links.

The idea of the proof of Theorem 3.4.3 is to combine geometrically maximal tangles with geometrically minimal tangles via Conway sum. Key to the proof is the fact that hyperbolic volume is additive under belted sum (see Lemma 3.2.5), and that the crossing number of adequate tangles is additive under Conway sum (see Corollary 3.2.14). The ratio of geometrically maximal to geometrically minimal tangles may then be controlled so that one may find a sequence of links with volume density approaching any number in the interval $[0, v_8]$.

Lemma 3.4.1. *There exists a sequence of strongly alternating tangles $\{T_m\}_{m=1}^{\infty}$ such that*

$$\lim_{m\to\infty} d_{\text{vol}}(N(T_m)) = 0$$

and $N(T_m)$ is hyperbolic for all $m \geq 7$.

Proof. Let $P_{\ell,m,n}$ be the pretzel tangle shown in Figure 3.8. It is well-known that many

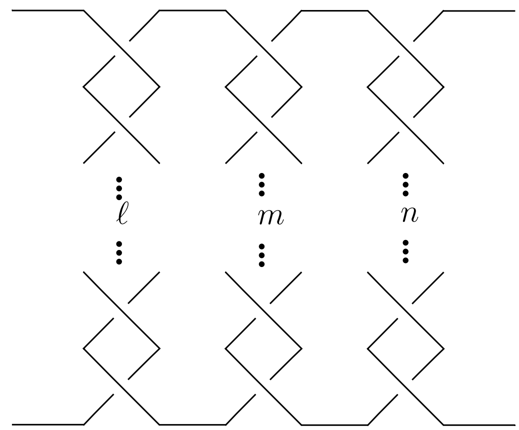

Figure 3.8: The pretzel tangle $P_{\ell,m,n}$ has ℓ, m, and n crossings in the respective twist regions.

pretzel links are hyperbolic. For example, whenever $\ell, m, n \geq 7$, work of Futer, Kalfagianni, and Purcell [31, Theorem 1.2] implies that $N(P_{\ell,m,n})$ is hyperbolic. Moreover, $P_{\ell,m,n}$ is strongly alternating whenever ℓ, m, and $n \geq 2$. Fix $\ell \geq 7$ and $n \geq 7$ and let $K_m = N(P_{\ell,m,n})$. By work of Lackenby, Agol, and D. Thurston [44], we know that $\mathrm{vol}(K_m) \leq 10v_4(\mathrm{tw}(K_m) - 1)$, where $\mathrm{tw}(K_m)$ is the number of twist regions in a diagram of K_m. In particular, $\mathrm{tw}(K_m) = 3$ for all values of m so $\mathrm{vol}(K_m) \leq 20v_4$. It follows that

$$\lim_{m\to\infty} d_{\mathrm{vol}}(K_m) = \lim_{m\to\infty} \frac{\mathrm{vol}(K_m)}{c(K_m)} \leq \lim_{m\to\infty} \frac{20v_4}{\ell + m + n} = 0$$

$\square$

Lemma 3.4.2. *There exists a sequence of strongly alternating (hence adequate) tangles $\{T_n\}_{n=1}^{\infty}$ such that $\{N(T_n)\}_{n=1}^{\infty}$ is geometrically maximal.*

Proof. Consider the weaving tangles $W_{m,n}$ from Definition 3.3.2 and depicted in Figure 3.6. Let $T_n = W_{n,n}$. It was shown in [21] that $\{N(T_n)\}_{n=1}^{\infty}$ is a geometrically maximal sequence of links. Note that $N(T_n)$ and $D(T_n)$ have reduced, alternating diagrams, whenever $n \geq 4$. $\square$

We have established examples of geometrically minimal and geometrically maximal sequences of adequate links. We restate and prove Theorem 3.4.3.

Theorem 3.4.3. *The set $\mathcal{C}_{\mathrm{vol}}$ of volume densities of hyperbolic links is a dense subset of $[0, v_8]$, and $\mathrm{Spec}_{\mathrm{vol}} = [0, v_8]$. In other words, given $x \in [0, v_8]$ there exists a sequence $\{K_n\}_{n=1}^{\infty}$ of hyperbolic links such that the volume densities satisfy $\lim_{n\to\infty} d_{\mathrm{vol}}(K_n) = x$.*

Proof. It suffices to show that for any non-negative integers a and b with a and b not both zero, and any $\epsilon > 0$, there exists a link K such that

$$\frac{b}{a+b}v_8 - \epsilon < d_{\mathrm{vol}}(K) < \frac{b}{a+b}v_8 + \epsilon.$$

Let $\epsilon > 0$ be arbitrary. Let $P_{\ell,m,n}$ be the pretzel tangles from Lemma 3.4.1. Let $\widetilde{T}_1 = P_{7,\widetilde{m},7}$ and $\widetilde{K}_1 = N(\widetilde{T}_1)$, where $\widetilde{m}$ is chosen to be large enough that

$$\frac{40\, a\, v_4}{(a+b)(\widetilde{m}+14)} < \frac{\epsilon}{2} \tag{3.10}$$

In Lemma 3.4.2, it was pointed out that the sequence of links $\{N(W_{k,k})\}_{k=1}^{\infty}$, where $W_{k,k}$ is the weaving tangle, is a geometrically maximal sequence of links. Therefore we may choose $T_2 = W_{k,k}$ and $K_2 = N(T_2)$ with k sufficiently large that

$$\frac{\mathrm{vol}(K_2)}{c(K_2)} > v_8 - \left(\frac{a+b}{b}\right)\frac{\epsilon}{2} \quad \text{and} \quad c(K_2) \geq c(\widetilde{K}_1) \tag{3.11}$$

Since $\widetilde{K}_1$ and K_2 have reduced alternating diagrams, by counting the number of crossings in these diagrams, we see that

$$k(k-1) = c(K_2) \geq c(\widetilde{K}_1) = \widetilde{m} + 14. \tag{3.12}$$

Let $m = k(k-1) - 14$. Define $T_1 = P_{7,m,7}$ and $K_1 = N(T_1)$. This choice of m implies that

$$c(K_1) = c(K_2) \tag{3.13}$$

Observe that $m \geq \widetilde{m}$, $\mathrm{tw}(K_1) = 3$ and $c(K_1) = m + 14$, so (3.10) guarantees that

$$\frac{10\, a\, v_4 (\mathrm{tw}(K_1) + 1)}{(a+b)\, c(K_1)} = \frac{40\, a\, v_4}{(a+b)(m+14)} \leq \frac{40\, a\, v_4}{(a+b)(\widetilde{m}+14)} < \frac{\epsilon}{2} \tag{3.14}$$

Choose $n \geq 12$ so that ξ_n as defined in (3.1) satisfies

$$\xi_n > \max\left\{1 - \frac{(a+b)\epsilon}{2\, b\, v_8 - \epsilon(a+b)}, \frac{2\, b\, v_8}{2\, v_8 + \epsilon(a+b)}\right\} \tag{3.15}$$

Since $0 < \xi_n < 1$, we need to check that such a choice of n is possible. Since a, b, v_8, and ϵ are positive we know that

$$0 < \frac{2bv_8}{2bv_8 + \epsilon(a+b)} < 1$$

Moreover, as long as $\epsilon < 2\ v_8/(a+b)$ we have $2\, b\, v_8 > \epsilon(a+b)$, hence

$$1 - \frac{(a+b)\,\epsilon}{2\, b\, v_8 - \epsilon(a+b)} < 1$$

Let $L_1 = B(T_1)$ and $L_2 = B(T_2)$ be the belted links corresponding to T_1 and T_2 respectively. Form the link L by taking the belted sum of $a \cdot n$ copies of L_1 with $b \cdot n$ copies of L_2. Let K be the result of filling the belt of L via the meridional Dehn filling. Note that K is simply the result of taking the closure of the tangle sum of $a \cdot n$ copies of T_1 with $b \cdot n$ copies of T_2 as shown in Figure 3.9. We obtain

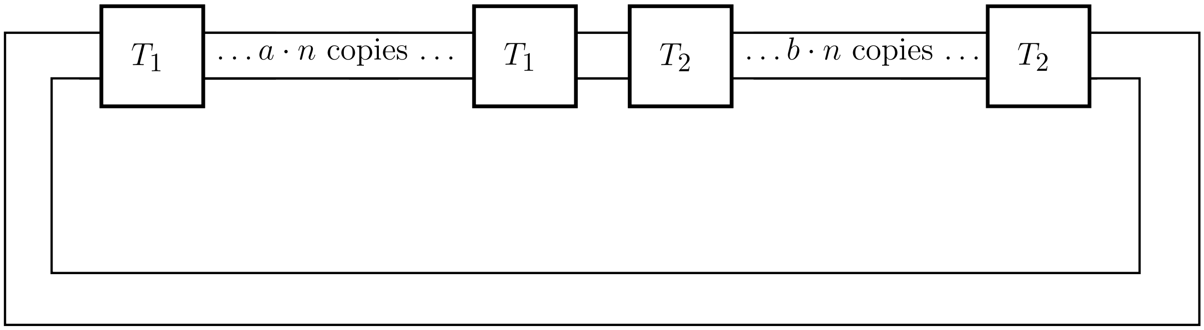

Figure 3.9: The link K is the closure of the tangle sum of $a \cdot n$ copies of T_1 with $b \cdot n$ copies of T_2 shown above.

$$\mathrm{vol}(K) \geq \xi_n \mathrm{vol}(L) \qquad \text{by Theorem 3.2.8} \tag{3.16}$$

$$= \xi_n(a \cdot n \cdot \mathrm{vol}(L_1) + b \cdot n \cdot \mathrm{vol}(L_2)) \qquad \text{by Lemma 3.2.5} \tag{3.17}$$

$$> \xi_n(a \cdot n \cdot \mathrm{vol}(K_1) + b \cdot n \cdot \mathrm{vol}(K_2)) \qquad \text{by Theorem 2.1.2} \tag{3.18}$$

Since T_1 and T_2 are strongly alternating tangles, it follows from Corollary 3.2.14 that

$$c(K) = n \cdot a \cdot c(K_1) + n \cdot b \cdot c(K_2) \tag{3.19}$$

This produces the following inequalities to form a lower bound for the density $d_{\text{vol}}(K)$.

$$
\begin{aligned}
d(K)_{\text{vol}} &= \frac{\text{vol}(K)}{c(K)} \\
&> \frac{\xi_n(a \cdot n \cdot \text{vol}(K_1) + b \cdot n \cdot \text{vol}(K_2))}{c(K)} && \text{by (3.18)} \\
&= \frac{\xi_n(a \cdot n \cdot \text{vol}(K_1) + b \cdot n \cdot \text{vol}(K_2))}{a \cdot n \cdot c(K_1) + b \cdot n \cdot c(K_2)} && \text{by (3.19)} \\
&= \xi_n \frac{\text{vol}(K_1)}{c(K_1)} \frac{a}{a+b} + \xi_n \frac{\text{vol}(K_2)}{c(K_2)} \frac{b}{a+b} && \text{since } c(K_1) = c(K_2) \\
&\geq \xi_n \frac{\text{vol}(K_2)}{c(K_2)} \frac{b}{a+b} \\
&> \left(1 - \frac{\epsilon(a+b)}{2\,b\,v_8 - \epsilon(a+b)}\right)\left(v_8 - \frac{a+b}{b}\frac{\epsilon}{2}\right)\frac{b}{a+b} && \text{by (3.11) and (3.15)} \\
&= \left(1 - \frac{\epsilon(a+b)}{2\,b\,v_8 - \epsilon(a+b)}\right)\left(v_8 \frac{b}{a+b} - \frac{\epsilon}{2}\right) \\
&= v_8 \frac{b}{a+b} - \frac{\epsilon\, v_8\, b}{2\,b\,v_8 - \epsilon(a+b)} + \frac{\epsilon^2(a+b)}{2(2\,b\,v_8 - \epsilon(a+b))} - \frac{\epsilon}{2} \\
&= v_8 \frac{b}{a+b} + \frac{\epsilon}{2}\left(\frac{-2\,v_8\,b}{2\,b\,v_8 - \epsilon(a+b)} + \frac{\epsilon(a+b)}{2\,b\,v_8 - \epsilon(a+b)}\right) - \frac{\epsilon}{2} \\
&= v_8 \frac{b}{a+b} - \epsilon
\end{aligned}
$$

We now find an upper bound for $d_{\text{vol}}(K)$. Since K is obtained from L by Dehn filling, Theorem 2.1.2 implies that $\text{vol}(K) < \text{vol}(L)$. Moreover, Lemma 3.2.5 implies that $\text{vol}(L) = a \cdot n \cdot \text{vol}(L_1) + b \cdot n \cdot \text{vol}(L_2)$. Therefore

$$
\text{vol}(K) < a \cdot n \cdot \text{vol}(L_1) + b \cdot n \cdot \text{vol}(L_2) \tag{3.20}
$$

Theorem 2.3.2 showed that $a \cdot n \cdot \text{vol}(L_1) < 10\,a \cdot n \cdot v_4(\text{tw}(L_1) - 1)$. Consider the diagrams of L_1 and K_1. Adding the belt circle to K_1 to form L_1 adds no more than two twist regions, as indicated in Figure 3.10. Therefore $\text{tw}(L_1) \leq \text{tw}(K_1) + 2$, hence

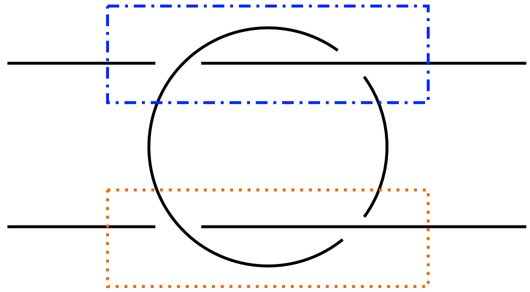

Figure 3.10: The belt circle adds no more than the two twist regions indicated in the figure.

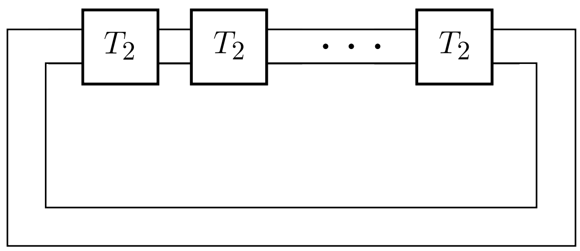

Figure 3.11: The link K_2^n is the closure of the sum of n copies of T_2 as indicated above.

$$a \cdot n \cdot \mathrm{vol}(L_1) < 10\, a\, n\, v_4(\mathrm{tw}(K_1) + 1) \tag{3.21}$$

Define L_2^n to be the belted sum of n copies of L_2 and let K_2^n be the result of filling the belt of L_2^n via the meridional filling. Then $K_2^n = N(T_2 + \ldots + T_2)$ where $+$ denotes Conway sum and the sum consists of n copies of T_2 (see Figure 3.11). Then Lemma 3.2.5 implies $\mathrm{vol}(L_2^n) = n\, \mathrm{vol}(L_2)$. Theorem 3.2.8 implies $\mathrm{vol}(L_2^n) \leq \mathrm{vol}(K_2^n)/\xi_n$. Therefore

$$b \cdot n \cdot \mathrm{vol}(L_2) = b \cdot \mathrm{vol}(L_2^n) \leq \frac{b}{\xi_n} \mathrm{vol}(K_2^n) \tag{3.22}$$

Since v_8 forms an upper bound on volume density, we know that $\mathrm{vol}(K_2^n) \leq v_8\, c(K_2^n)$. It follows that

$$\frac{b}{\xi_n}\,\mathrm{vol}(K_2^n) \leq \frac{b}{\xi_n}\,v_8\,c(K_2^n) \tag{3.23}$$

The diagram for K_2^n shown in Figure 3.11 has $n{\cdot}c(K_2)$ crossings. Therefore $c(K_2^n) \leq n{\cdot}c(K_2)$. This yields

$$\frac{b}{\xi_n}\,v_8\,c(K_2^n) \leq \frac{b\,n\,v_8}{\xi_n}\,c(K_2) \tag{3.24}$$

Summarizing (3.22), (3.23), and (3.24) we see that

$$b \cdot n \cdot \mathrm{vol}(L_2) \leq \frac{b\,n\,v_8}{\xi_n}c(K_2). \tag{3.25}$$

This allows us to produce an upper bound on the volume density $d(K)$ as follows.

$$\begin{aligned}
d_{\mathrm{vol}}(K) = \frac{\mathrm{vol}(K)}{c(K)} &< \frac{a \cdot n \cdot \mathrm{vol}(L_1) + b \cdot n \cdot \mathrm{vol}(L_2)}{c(K)} && \text{by (3.20)}\\
&= \frac{a \cdot n \cdot \mathrm{vol}(L_1) + b \cdot n \cdot \mathrm{vol}(L_2)}{a \cdot n \cdot c(K_1) + b \cdot n \cdot c(K_2)} && \text{by (3.19)}\\
&< \frac{10\,a\,n\,v_4(\mathrm{tw}(K_1)+1) + \frac{b}{\xi_n}(n\,v_8\,c(K_2))}{a \cdot n \cdot c(K_1) + b \cdot n \cdot c(K_2)} && \text{by (3.21) and (3.25)}\\
&= \frac{10\,a\,v_4(\mathrm{tw}(K_1)+1)}{(a+b)c(K_1)} + \frac{b\,v_8\,c(K_2)}{\xi_n(a+b)\,c(K_2)} && \text{since } c(K_1) = c(K_2)\\
&< \frac{\epsilon}{2} + \frac{1}{\xi_n}\frac{bv_8}{a+b} && \text{by (3.14)}\\
&< \frac{\epsilon}{2} + \frac{2bv_8 + \epsilon(a+b)}{2bv_8}\frac{bv_8}{a+b} && \text{by (3.15)}\\
&= \frac{\epsilon}{2} + \frac{2bv_8 + \epsilon(a+b)}{2(a+b)}\\
&= \frac{b}{a+b}v_8 + \epsilon
\end{aligned}$$

Therefore

$$\frac{b}{a+b}v_8 - \epsilon < d_{\mathrm{vol}}(K) < \frac{b}{a+b}v_8 + \epsilon$$

as desired.

$\square$

3.5 The Spectrum of Determinant Densities

We now turn to studying the spectrum of determinant densities. The proof that $\mathrm{Spec}_{\det}$ contains $[0, v_8]$ will follow a similar method as the proof of Theorem 3.4.3. Namely, we note that there exist sequences of diagrammatically maximal links with determinant density approaching v_8, and there exist sequences of *diagrammatically minimal* links, i.e. those with determinant density near 0. We combine these diagrams via Conway sum, but instead of taking the numerator closure as in the proof of Theorem 3.4.3, we consider the denominator closure. We will then use the facts that $\log \det$ is additive under the denominator closure of Conway sums (see Lemma 3.2.7), and that the crossing number is additive under Conway sums of adequate diagrams (see Corollary 3.2.14). Then by controlling the ratio of diagrammatically minimal links to diagrammatically maximal links, we may obtain a link that has determinant density near any number $x \in [0, v_8]$.

Lemma 3.5.1. *There exists a sequence of adequate tangles $\{T_m\}_{m=1}^{\infty}$ such that the sequence of closures $\{D(T_m)\}_{m=1}^{\infty}$ satisfies $\lim_{n\to\infty} d_{\det}(D(T_n)) = 0$.*

Proof. Let $P_{\ell,m,n}$ be the pretzel tangles shown in Figure 3.8. Let $T_m = P_{3,m,3}$ and $K_m = D(T_m)$. Then the tangles are strongly alternating, hence adequate. It follows from [47, Page 100] (see also work of Dasbach, Kalfagianni, Futer, Lin, and Stoltzfus [25, Example 4.3][1])

[1] The author thanks Jessica Purcell for pointing out this reference.

that $\det(K_m) = 6m + 9$. Now

$$d_{\det}(K_m) = \lim_{m\to\infty} \frac{2\pi \log \det(K_m)}{c(K_m)} = \lim_{m\to\infty} \frac{2\pi \log(6m+9)}{m+6} = 0$$

$\square$

Lemma 3.5.2. *There exists a sequence of adequate tangles* $\{T_m\}_{m=1}^{\infty}$ *such that the sequence of closures* $\{D(T_m)\}_{m=1}^{\infty}$ *satisfies* $\lim_{n\to\infty} d_{\det}(D(T_n)) = v_8$.

Proof. Let $W_{m,m}$ be the weaving tangles from Definition 3.3.2 and depicted in Figure 3.6. Then the sequence $\{D(W_{m,m})\}_{m=1}^{\infty}$ satisfies the conditions of Theorem 3.2.4, hence is diagrammatically maximal. $\square$

Having now established examples of diagrammatically minimal and diagrammatically maximal sequences of links, we can now restate and prove Theorem 3.5.3.

Theorem 3.5.3. *The spectrum of determinant densities* $\mathrm{Spec}_{\det}$ *contains* $[0, v_8]$. *In other words, given any* $x \in [0, v_8]$ *there exists a sequence* $\{K_n\}_{n=1}^{\infty}$ *of links satisfying*

$$\lim_{n\to\infty} d_{\det}(K_n) = x \tag{3.26}$$

Proof. It suffices to show that for any non-negative integers a and b with a and b not both zero, and any $\epsilon > 0$, there exists a link K such that

$$\frac{b}{a+b} v_8 - \epsilon < d_{\det}(K) < \frac{b}{a+b} v_8 + \epsilon.$$

Let $\epsilon > 0$ be arbitrary. Let $P_{\ell,m,n}$ be the pretzel tangles. Let $\widetilde{T}_1 = P_{3,\widetilde{m},3}$ and $\widetilde{K}_1 = D(\widetilde{T}_1)$, where $\widetilde{m}$ is chosen large enough that

$$\frac{2\pi\log(6m+9)}{m+6} < \left(\frac{a+b}{a}\right)\frac{\epsilon}{2} \tag{3.27}$$

Since the sequence $\{D(W_{k,k})\}_{k=1}^{\infty}$ is diagrammatically maximal, we may choose $T_2 = W_{k,k}$ and $K_2 = D(T_2)$ with k sufficiently large that

$$\left|\frac{2\pi\log\det(K_2)}{c(K_2)} - v_8\right| < \left(\frac{a+b}{b}\right)\frac{\epsilon}{2} \quad \text{and} \quad c(K_2) \geq c(\widetilde{K}_1) \tag{3.28}$$

The choice of K_2 ensures that the crossing numbers $c(\widetilde{K}_1)$ and $c(K_2)$ satisfy the inequality

$$k(k-1) = c(K_2) \geq c(\widetilde{K}_1) = \widetilde{m} + 6 \tag{3.29}$$

Let $m = k(k-1) - 6$. Define $T_1 = P_{3,m,3}$ and $K_1 = D(T_1)$. This choice of m implies that

$$c(K_1) = c(K_2)$$

Using the fact that $m \geq \widetilde{m}$ we see that

$$d_{\det}(K_1) = \frac{2\pi\log(6m+9)}{m+6} \leq \frac{2\pi\log(6\widetilde{m}+9)}{\widetilde{m}+6} < \left(\frac{a+b}{a}\right)\frac{\epsilon}{2} \tag{3.30}$$

Let T be the tangle sum of a copies of T_1 with b copies of T_2, and let $K = D(T)$ (see Figure 3.12). Then by Lemma 3.2.7 we have that $\det(K) = (\det K_1)^a(\det K_2)^b$. Since T_1 and T_2 are strongly alternating tangles, we know from Corollary 3.2.14 that $c(K) = a\cdot c(K_1) + b\cdot c(K_2)$.

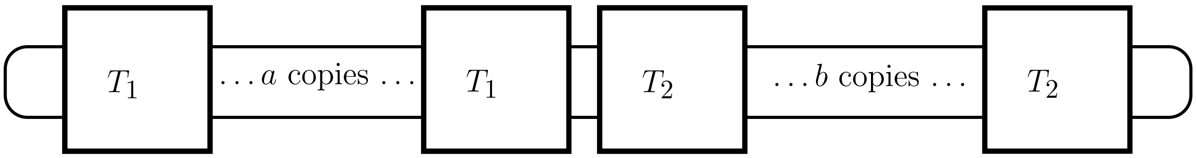

Figure 3.12: The link K is the closure of the tangle sum of a copies of T_1 with b copies of T_2 shown above.

Therefore, we obtain the upper bound on $d_{\det}(K)$:

$$\begin{aligned}
d_{\det}(K) &= \frac{2\pi \log \det(K)}{c(K)} \\
&= \frac{2\pi \log[(\det K_1)^a (\det K_2)^b]}{a \cdot c(K_1) + b \cdot c(K_2)} \\
&= \frac{a}{a+b}\frac{2\pi \log \det(K_1)}{c(K_1)} + \frac{b}{a+b}\frac{2\pi \log \det(K_2)}{c(K_2)} \qquad \text{since } c(K_1) = c(K_2) \qquad (3.31) \\
&< \frac{a}{a+b}\left(\frac{a+b}{a}\right)\frac{\epsilon}{2} + \frac{b}{a+b}\left[v_8 + \left(\frac{a+b}{b}\right)\frac{\epsilon}{2}\right] \qquad \text{by (3.28) and (3.30)} \\
&= \frac{b}{a+b}v_8 + \epsilon
\end{aligned}$$

Now we obtain a lower bound on $d_{\det(K)}$:

$$\begin{aligned}
d_{\det}(K) &= \frac{a}{a+b}\frac{2\pi \log \det(K_1)}{c(K_1)} + \frac{b}{a+b}\frac{2\pi \log \det(K_2)}{c(K_2)} \qquad \text{by (3.31)} \\
&\geq \frac{b}{a+b}\frac{2\pi \log \det(K_2)}{c(K_2)} \\
&> \frac{b}{a+b}\left(v_8 - \left(\frac{a+b}{b}\right)\frac{\epsilon}{2}\right) \qquad \text{by (3.28)} \\
&> \frac{b}{a+b}v_8 - \epsilon
\end{aligned}$$

□

Note that the links constructed in the proof of Theorem 3.5.3 are not hyperbolic. It is interesting to study the relationship between $\mathrm{Spec}_{\det}$ and $\mathrm{Spec}_{\mathrm{vol}}$. Since $\mathrm{Spec}_{\mathrm{vol}}$ considers only the densities of hyperbolic links, it is useful to know what determinant densities are realized by *hyperbolic* links. To this end, we have the following result.

Theorem 3.5.4. *The set of determinant densities of hyperbolic links is dense in* $[0, v_8]$. *In other words, given* $x \in [0, v_8]$, *and* $\epsilon > 0$, *there exists a hyperbolic link* K *such that* $|d_{\det}(K) - x| < \epsilon$.

Proof. Let $\epsilon > 0$ be arbitrary, and let a and b be non-negative integers with a and b not both zero. Let $\overline{W}_{m,n}$ denote the reflection of the weaving tangle of Definition 3.3.2. Note that the sequence $\{\overline{W}_{\ell,\ell}\}_{\ell=1}^{\infty}$ is diagrammatically maximal by Theorem 3.2.4. Then by following an argument similar to the proof of Theorem 3.5.3, there exist integers k and ℓ such that the link $L = D(P_{3,k,3} + \ldots + P_{3,k,3} + \overline{W}_{\ell,\ell} + \ldots + \overline{W}_{\ell,\ell})$, i.e. the denominator closure of the tangle sum of a copies of $P_{3,k,3}$ with b copies of $\overline{W}_{\ell,\ell}$, satisfies

$$\left| d_{\det}(L) - \frac{b}{a+b} v_8 \right| < \frac{\epsilon}{2} \tag{3.32}$$

Let T be the tangle obtained by the tangle sum of a copies of $P_{3,k,3}$ with b copies of $\overline{W}_{\ell,\ell}$. For each positive integer m, let $K_m = N(T + \ldots + T)$ be the numerator closure of the tangle sum of m copies of T, and let $L_m = D(T + \ldots + T)$ be the denominator closure of this tangle sum. Note that T is prime, alternating, and not an east-west twist. Therefore Theorem 3.2.8 implies K_m is hyperbolic for $m \geq 12$. Moreover, the links K_m and L_m are reduced and alternating. Now the proof of [19, Theorem 1.7] implies that

$$\lim_{m\to\infty} \frac{2\pi \log \det(K_m)}{c(K_m)} = \lim_{m\to\infty} \frac{2\pi \log \det(L_m)}{c(L_m)} = \lim_{m\to\infty} \frac{2\pi\, m \log \det(L_1)}{m \cdot c(L_1)} = d_{\det}(L_1) \tag{3.33}$$

Since $L_1 = L$, it follows that we may choose $m \geq 12$ sufficiently large that $|d_{\text{det}}(K_m) - d_{\text{det}}(L)| < \epsilon/2$. Since $\left| d_{\text{det}}(L) - \frac{b}{a+b} v_8 \right| < \frac{\epsilon}{2}$ by (3.32), it follows that

$$\left| d_{\text{det}}(K_m) - \frac{b}{a+b} v_8 \right| < \epsilon \tag{3.34}$$

□

Chapter 4

Relating Determinant and Volume

4.1 Introduction

In this chapter, we explore the relationship between the hyperbolic volume $\mathrm{vol}(K) = \mathrm{vol}(S^3 \backslash K)$ of an alternating hyperbolic knot and its determinant $\det(K)$. Dunfield noted a relationship between the volume and determinant of a knot in an online post [26]. He observed that there is a nearly linear relationship between the hyperbolic volume of an alternating knot and $\log(J(-1))$ where J denotes the Jones polynomial. After further study of this relationship and some experimentation, Champanerkar, Kofman and Purcell [20] made the following conjecture.

Conjecture 4.1.1. Let K be a hyperbolic alternating knot. Then $\mathrm{vol}(K) < 2\pi \log \det(K)$.

One can use the data from Knotscape [36] and SnapPy [22] to verify this conjecture for all alternating knots with up to 16 crossings. Champanerkar, Kofman and Purcell in [20] computationally verified Conjecture 4.1.1 for many examples of an infinite family of links known as weaving knots. Using these weaving knots, they showed that the constant 2π is sharp, in the sense that given $\alpha < 2\pi$ there exists an alternating link K with $2\pi \log(\det(K)) < \mathrm{vol}(K)$.

Stoimenow [63] also explored the relationship between volume and determinant, and showed that if K is a non-trivial, non-split, alternating hyperbolic link then

$$\det(K) \geq 2(1.0355)^{\mathrm{vol}(K)} \tag{4.1}$$

He further demonstrated that there exist constants $C_1, C_2 > 0$ such that for any hyperbolic link K

$$\det(K) \leq \left(\frac{C_1 c(K)}{\mathrm{vol}(K)}\right)^{C_2 \mathrm{vol}(K)} \tag{4.2}$$

where $c(K)$ denotes the crossing number of K.

In this chapter, we will verify that Conjecture 4.1.1 holds for various infinite families of knots including 2-bridge knots and 3-braids. In particular, we have the following theorem.

Theorem 4.1.2. *If K is a 2-bridge link or an alternating 3-braid, then*

$$\mathrm{vol}(K) \leq 2\pi \log(\det(K)) \tag{4.3}$$

The proof of Theorem 4.1.2 is a combination of Theorems 4.3.5 and 4.4.1. To obtain upper bounds on the volumes of knots we largely rely on work of Adams [3] who gave an upper bound in terms of volumes of bipyramids. Adams *et al.* [7] used this upper bound to study the volume densities of 2-bridge knots. Other useful upper bounds in the case of highly twisted knots are due to Lackenby, Agol and Thurston [44] and Futer, Kalfagianni and Purcell [30].

To study the determinant of knots, we will count the number of spanning trees of a graph associated to the checkerboard coloring of a diagram of the knot. We rely on a recurrence equation due to Kauffman [42] as well as two well-known combinatorial theorems for counting spanning trees. In the case of highly twisted knots, we utilize work of Stoimenow [63] who provided a lower bound on the number of spanning trees in certain graphs.

We also show that the conjecture holds for "highly twisted" links, *i.e.* those having many crossings and few twist regions (see Section 2.3 for a definition of twist region). We further quantify this by the following theorem.

Theorem 4.5.1. *Let K be an alternating hyperbolic link with t twist regions and c crossings. If*

$$c > t + \xi^{t-1} - 2\gamma^{t-1} \tag{4.42}$$

where $\gamma \approx 1.4253$ is the unique positive real number satisfying $\gamma^{-5} + 2\gamma^{-4} + \gamma^{-3} - 1 = 0$ and $\xi = e^{5v_4/\pi} \approx 5.0296$, then $\mathrm{vol}(K) < 2\pi \log(\det(K))$.

It follows that if some or all of the twist regions of a link have many crossings, the link will satisfy Conjecture 4.1.1. An immediate corollary of this is:

Corollary 4.1.3. *For any given integer $t_0 > 0$ there are only finitely many links with t_0 twist regions which fail to satisfy Conjecture 4.1.1.*

An outline of the chapter is as follows. In Section 4.2, we outline the technology used in this paper to estimate volumes and determinants of knots. In Section 4.3, we will prove Conjecture 4.1.1 for 2-bridge links. In Section 4.4, we will prove the conjecture for alternating 3-braids and an infinite family of 4-braids. We discuss a general result about highly twisted links in Section 4.5 and include an application to alternating pretzel links.

4.2 Background

In this section, we discuss the relevant theorems that will be used throughout this chapter. We begin with a discussion of how one may find upper bounds on volumes of alternating

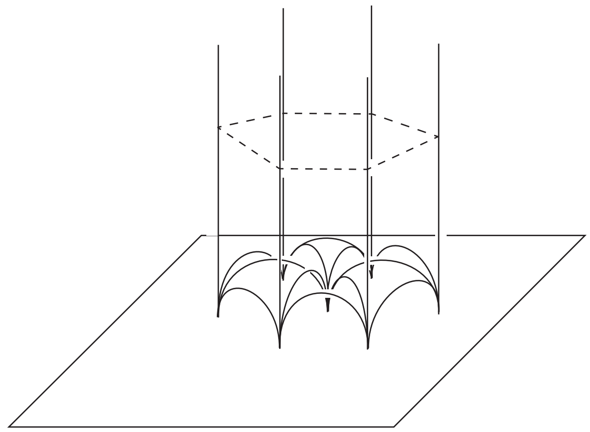

Figure 4.1: A regular ideal 6-bipyramid. Figure taken from [3].

hyperbolic links, and conclude with results on how one may calculate the determinant.

4.2.1 Hyperbolic Volumes

Adams in [3] developed a method for finding an upper bound of the volume of an alternating hyperbolic link given an alternating diagram of the link. We will recall his notation and results.

Definition 4.2.1. The regular ideal n-bipyramid can be formed as follows. Begin with n ideal tetrahedra each having dihedral angles $\frac{2\pi}{n}, \frac{(n-2)\pi}{2n}, \frac{(n-2)\pi}{2n}$. Let e be an edge running from a point in $\partial\mathbb{H}^3$ to ∞. Glue an edge of each ideal tetrahedron with dihedral angle $2\pi/n$ to the edge e. The resulting polyhedron is called a *regular ideal n-bipyramid* and will be denoted by B_n.

An example of a regular ideal n-bipyramid is shown in Figure 4.1. The volume of B_n is given by

$$\mathrm{vol}(B_n) = n\left(\int_0^{\frac{2\pi}{n}} -\ln|2\sin(\theta)|\, d\theta + 2\int_0^{\frac{\pi(n-2)}{2n}} -\ln|2\sin(\theta)|\, d\theta\right). \tag{4.4}$$

Adams [3] proved the following theorem about the volumes of regular ideal n-bipyramids.

Theorem 4.2.2 ([3]). *The volume of a regular ideal n-bipyramid satisfies the inequality*

$$\mathrm{vol}(B_n) < 2\pi \log\left(\frac{n}{2}\right).$$

Moreover, this inequality is asymptotically sharp.

Adams used regular ideal $n-$bipryamids to give an upper bound on the volume of hyperbolic, alternating links. The following theorem directly follows from [3, Theorem 4.1].

Theorem 4.2.3. *Let K be a hyperbolic link with a reduced alternating projection D. Let b_n be the number of faces of D having n edges. Suppose that there are two distinct faces of D having respectively r and s edges. Then*

$$\mathrm{vol}(K) \leq -\mathrm{vol}(B_r) - \mathrm{vol}(B_s) + \sum b_n \mathrm{vol}(B_n) \tag{4.5}$$

Combining Theorems 4.2.2 and 4.2.3 we get the following corollary.

Corollary 4.2.4. *Let K be a hyperbolic knot having an alternating projection D. Let b_n be the number of faces of D having n edges. Suppose that there are two distinct faces of D having respectively r and s edges. Then*

$$\mathrm{vol}(K) < 2\pi \log\left(\frac{\prod n^{b_n}}{2^m}\frac{4}{rs}\right) \tag{4.6}$$

Proof. This is a straightforward calculation obtained by inserting the inequality of Theorem

4.2.2 into Theorem 4.2.3.

$$\begin{aligned}
\mathrm{vol}(K) &\leq -\mathrm{vol}(B_r) - \mathrm{vol}(B_s) + \sum b_n \mathrm{vol}(B_n) \\
&= (b_r - 1)\mathrm{vol}(B_r) + (b_s - 1)\mathrm{vol}(B_s) + \sum_{n \neq r,s} b_n \mathrm{vol}(B_n) \\
&< 2\pi \left[(b_r - 1) \log\left(\frac{r}{2}\right) + (b_s - 1) \log\left(\frac{s}{2}\right) + \sum_{n \neq r,s} b_n \log\left(\frac{n}{2}\right) \right] \\
&= 2\pi \log\left(\frac{r^{b_r - 1}}{2^{b_r - 1}} \frac{s^{b_s - 1}}{2^{b_s - 1}} \prod_{n \neq r,s} \frac{n^{b_n}}{2^{b_n}} \right) \\
&= 2\pi \log\left(\frac{\prod n^{b_n}}{2^m} \frac{4}{rs} \right)
\end{aligned}$$

□

The volume bound of Corollary 4.2.4 is insufficient in certain cases involving links with a large number of crossings in a twist region. To handle this case, we appeal to Theorems 2.3.2 and 2.3.3.

4.2.2 Determinants of Links

The determinant of a link K is defined by $\det(K) = |\Delta_K(-1)|$ where $\Delta_K(t)$ is the Alexander polynomial. It is well-known when K is alternating, the determinant is equal to the number of spanning trees of any of the checkerboard graphs for K. See, for example, [63, Lemma 3.14] for a proof of this fact, and see Section 2.2 for a definition of the checkerboard graph.

We recall two methods that one may use to compute the number of spanning trees of a graph. First we present the Matrix Tree Theorem proved by Kirchoff in 1847. One can find a modern proof in [18].

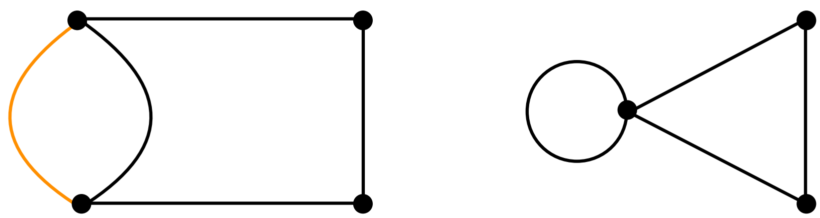

Figure 4.2: LEFT: Initial graph. RIGHT: Result of collapsing the orange edge.

Theorem 4.2.5 (Matrix Tree Theorem)**.** *Let G be a graph and let $v_1, \ldots, v_n$ be the vertices of G. Let $\alpha(i,j)$ be the number of edges with endpoints on both of the vertices v_i and v_j. Define L to be the matrix (known as the Laplacian) where the (i,j) entry ℓ_{ij} of L is given by*

$$\ell_{ij} = \begin{cases} \deg(v_i) - 2\alpha(i,i) & \text{if } j = i \\ -\alpha(i,j) & \text{if } j \neq i \end{cases} \tag{4.7}$$

Then $\tau(G)$ is given by the determinant of any of the $(n-1) \times (n-1)$ minors of L.

For the next lemma, we introduce some notation. Let G be a graph and e an edge of the graph. We define $G - e$ to be the graph obtained by removing the edge e from G. We define G/e to be the graph obtained by contracting the edge e and identifying the endpoints of e to a single vertex as shown in Figure 4.2. With this notation, we recall the following well-known result.

Lemma 4.2.6. *Let G be a graph. Then $\tau(G) = \tau(G - e) + \tau(G/e)$.*

Using spanning trees, Stoimenow [63] was able to give a lower bound on the determinant of an alternating knot.

Theorem 4.2.7 ([63, Theorem 4.3])**.** *Let t be the number of twist regions in a twist-reduced*

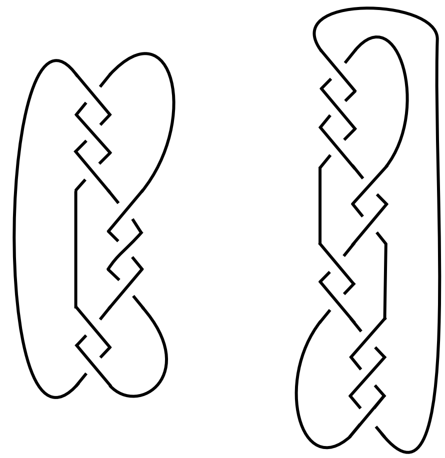

Figure 4.3: LEFT: The knot $R(3,3,2)$. RIGHT: The link $R(3,2,2,3)$.

alternating diagram D of a link K. Then

$$\det(K) \geq 2 \cdot \gamma^{t-1}$$

where $\gamma \approx 1.4253$ is the unique positive real number satisfying $\gamma^{-5} + 2\gamma^{-4} + \gamma^{-3} - 1 = 0$.

4.3 Two Bridge Links

It is known that any 2-bridge link has an alternating projection of one of the forms shown in Figure 4.3. We will denote 2-bridge links by $R(a_1, a_2, \ldots, a_n)$ where the sequence $a_1, a_2, \ldots, a_n$ denotes the number of half-twists in each crossing region. Examples of $R(3,3,2)$ and $R(3,2,2,3)$ are provided in Figure 4.3. We begin the proof that Conjecture 4.1.1 holds for 2-bridge links by studying the determinant of a 2-bridge link. Kauffman and Lopes [42] gave the following recursive method of calculating the determinant of rational links.

Theorem 4.3.1 ([42]). *Let $K = R(a_1, a_2, \ldots, a_n)$ be a two-bridge link. Then $det(K) = T(n)$*

where $T(n)$ is defined by the recursion

$$\begin{cases} T(0) & = & 1 \\ T(1) & = & a_1 \\ T(k+1) & = & a_{k+1}T(k) + T(k-1) \end{cases} \tag{4.8}$$

It is interesting to note that when $a_k = 1$ for all k, then the recursion yields the Fibonacci sequence. We now introduce some notation that will aid the exposition. Define

$$V(a_1, \ldots, a_n) = \prod_{i=1}^{n} \frac{(a_i + 2)}{2} \tag{4.9}$$

Let $K = R(a_1, \ldots, a_n)$. Note that by Corollary 4.2.4 we have

$$\mathrm{vol}(K) < 2\pi \log\left(\frac{(a_1+1)(a_n+1)}{4} \prod_{i=2}^{n-1} \frac{(a_i+2)}{2}\right) < 2\pi \log(V(a_1, \ldots, a_n)) \tag{4.10}$$

We will obtain a lower bound on $T(n)$ from the the recurrence of Theorem 4.3.1 and then show that it exceeds $V(a_1, \ldots, a_n)$. The most problematic cases for obtaining a lower bound on $T(n)$ are when $a_i = 1$ for many values of i. The following lemma allows us to reduce to the case where $a_1, \ldots, a_n$ contains no long sequences of consecutive ones.

Lemma 4.3.2. *Let $K = R(a_1, a_2, \ldots, a_n)$ and let $T(i)$ be the recurrence (4.8). Fix $k \geq 2$. Let*

$$K' = R(a_1, \ldots, a_{k-1}, a_{k+m}, \ldots, a_n)$$

and define another recurrence $\widehat{T}(i)$ by

$$\widehat{T}(i) = \begin{cases} T(i) & \text{if} \quad i < k \\ a_{i+1}\widehat{T}(i-1) + \widehat{T}(i-2) & \text{if} \quad i \geq k \end{cases} \tag{4.11}$$

Then the following are true:

(a) *If $T(k) > \frac{3}{2}T(k-1)$ and $T(k-1) > \frac{3}{2}T(k-2)$, then $T(n) > \frac{3}{2}\widehat{T}(n-1)$ and*

$$\det(R(a_1, \ldots, a_n)) > \frac{3}{2}\det(R(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n))$$

(b) *Suppose $k \geq 4$ and $a_{k-2} = a_{k-1} = a_k = \ldots = a_{k+m-1} = 1$ for some $m \geq 1$. Then*

$$\det(K) > \left(\frac{3}{2}\right)^m \det(K')$$

(c) *Suppose $k \geq 4$ and $a_{k-2} = a_{k-1} = a_k = \ldots = a_{k+m-1} = 1$ for some $m \geq 1$. If*

$$2\pi\log(V(a_1, \ldots, a_{k-1}, a_{k+m}, \ldots, a_n)) \leq 2\pi\log(\det(K'))$$

then $\mathrm{vol}(K) < 2\pi\log(\det(K))$.

Proof. To begin the proof of part (a) by proving the following claim:

$$T(k+i+1) > (3/2)\widehat{T}(k+i) \text{ for all } i \geq -1 \tag{4.12}$$

The case where $i = -1$ holds since

$$T(k) > \frac{3}{2}T(k-1) = \frac{3}{2}\widehat{T}(k-1)$$

To prove the case where $i = 0$, note that

$$\begin{aligned} T(k+1) &= a_{k+1}T(k) + T(k-1) \\ &> \frac{3}{2}[a_{k+1}T(k-1) + T(k-2)] \\ &= \frac{3}{2}\widehat{T}(k) \end{aligned}$$

We now proceed by induction. Assume that $T(k+i+1) > \frac{3}{2}\widehat{T}(k+i)$ and $T(k+i) > \frac{3}{2}\widehat{T}(k+i-1)$. Then

$$\begin{aligned} T(k+i+2) &= a_{k+i+2}T(k+i+1) + T(k+i) \\ &> \frac{3}{2}[a_{k+i+2}\widehat{T}(k+i) + \widehat{T}(k+i-1)] \\ &= \frac{3}{2}\widehat{T}(k+i+1) \end{aligned}$$

This proves (4.12). Observe that

$$\begin{aligned} T(n) &= \det(R(a_1, \ldots, a_n)) \\ \widehat{T}(n-1) &= \det(R(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n)) \end{aligned}$$

thus completing the proof of part (a).

We prove part (b) by induction on m. Note that

$$\begin{aligned}
T(k-1) &= T(k-2) + T(k-3) \\
&= 2T(k-3) + T(k-4) && \text{since } T(k-2) = T(k-3) + T(k-4) \\
&\geq \frac{3}{2}T(k-3) + \frac{3}{2}T(k-4) && \text{since } T(k-3) \geq T(k-4) \\
&= \frac{3}{2}T(k-2)
\end{aligned}$$

Similarly, $T(k) \geq \frac{3}{2}T(k-1)$. Then by part (a)

$$\det(R(a_1, \ldots, a_n)) = T(n) > \frac{3}{2}\widehat{T}(n-1) = \frac{3}{2}\det(R(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n)) \tag{4.13}$$

This proves the case where $m = 1$. Assume that

$$\det(K) > \left(\frac{3}{2}\right)^{m-1} \det(R(a_1, \ldots, a_{k-1}, a_{k+m-1}, \ldots, a_n) \tag{4.14}$$

Since $a_{k-2} = a_{k-1} = a_{k+m-1} = 1$ we may use part (a) to obtain

$$\begin{aligned}
\det(K) &> \left(\frac{3}{2}\right)^{m-1} \det(R(a_1, \ldots, a_{k-1}, a_{k+m-1}, a_{k+m}, \ldots, a_n)) && \text{by (4.14)} \\
&> \left(\frac{3}{2}\right)^{m} \det(R(a_1, \ldots, a_{k-1}, a_{k+m}, \ldots, a_n)) && \text{by part (a)}
\end{aligned}$$

which finishes the proof of part (b). To prove part (c), observe that if

$$2\pi \log(V(a_1, \ldots, a_{k-1}, a_{k+m}, \ldots a_n)) < \det(R(a_1, \ldots, a_{k-1}, a_{k+m}, \ldots, a_n)) \tag{4.15}$$

then

$$
\begin{aligned}
\mathrm{vol}(K) &< 2\pi \log(V(a_1, \ldots, a_n)) \\
&= 2\pi \log\left(\left(\frac{3}{2}\right)^m V(a_1, \ldots, a_{k-1}, a_{k+m}, \ldots, a_n)\right) \\
&< 2\pi \log\left(\left(\frac{3}{2}\right)^m \det(R(a_1, \ldots, a_{k-1}, a_{k+m}, \ldots, a_n))\right) && \text{by (4.15)} \\
&< 2\pi \log(\det(K)) && \text{by part (b)}
\end{aligned}
$$

□

Using Lemma 4.3.2 we can reduce to the case where we do not have $a_{k-2} = a_{k-1} = a_k = 1$ for any $k \geq 4$, *i.e.* there is no subsequence of three or more consecutive ones. Next we will prove Lemma 4.3.3 which empowers us to bound $\det(R(a_1, \ldots, a_n))$ by breaking up the sequence $a_1, \ldots, a_n$ into shorter subsequences.

Lemma 4.3.3. *Let* $1 \leq k \leq n-1$. *Then*

$$\det R(a_1, \ldots, a_n) > \det R(a_1, \ldots, a_k) \det(R(a_{k+1}, \ldots, a_n))$$

Proof. Let $T(i)$ be the recursion defined in Theorem 4.3.1. Define the following recursive sequences $T'(i)$ and $T''(i)$:

$$\begin{cases} T'(0) &= 1 \\ T'(1) &= a_{k+1} \\ T'(i) &= a_{k+i}T'(i-1) + T'(i-2) \end{cases} \tag{4.16}$$

$$\begin{cases} T''(0) &= 1 \\ T''(1) &= a_{k+2} \\ T''(i) &= a_{k+i+1}T''(i-1) + T''(i-2) \end{cases} \tag{4.17}$$

Note that

$$T'(n-k) = \det(R(a_{k+1}, a_{k+2}, \ldots, a_n)) \tag{4.18}$$

We will show that

$$T(k+m) = T'(m)T(k) + T''(m-1)T(k-1) \tag{4.19}$$

for $m \geq 1$. We proceed by induction on m. When $m = 1$ we have

$$\begin{aligned} T(k+1) &= a_{k+1}T(k) + T(k-1) && \text{by (4.8)} && (4.20) \\ &= T'(1)T(k) + T''(0)T(k-1) && \text{since } T'(1) = a_{k+1} \text{ and } T''(0) = 1 \text{ by definition} && (4.21) \end{aligned}$$

When $m = 2$ we have

$$\begin{aligned}
T(k+2) &= a_{k+2}T(k+1) + T(k) && \text{by (4.8)} \\
&= a_{k+2}T'(1)T(k) + a_{k+2}T''(0)T(k-1) + T'(0)T(k) && \text{by (4.21) and } T'(0) = 1 \\
&= [a_{k+2}T'(1) + T'(0)]T(k) + a_{k+2}T(k-1) && \text{since } T''(0) = 1 \\
&= T'(2)T(k) + T''(1)T(k-1) && \text{by (4.16) and } T''(1) = a_{k+2}
\end{aligned}$$

Now assume that

$$T(k+m) = T'(m)T(k) + T''(m-1)T(k-1) \tag{4.22}$$

for $m \geq 2$. Then

$$T(k+m+1) = a_{k+m+1}T(k+m) + T(k+m-1)$$

Which by applying (4.22) to $T(k+m)$ and $T(k+m-1)$ becomes

$$\begin{aligned}
T(k+m+1) = a_{k+m+1}T'(m)T(k) + a_{k+m+1}T''(m-1)T(k-1) \\
+ T'(m-1)T(k) + T''(m-2)T(k-1)
\end{aligned} \tag{4.23}$$

By collecting like terms (4.23) simplifies to

$$\begin{aligned}
T(k+m+1) = [a_{k+m+1}T'(m) + T'(m-1)]T(k) \\
+ [a_{k+m+1}T''(m-1) + T''(m-2)]T(k-1)
\end{aligned} \tag{4.24}$$

By (4.16) we have that

$$a_{k+m+1}T'(m) + T'(m-1) = T'(m+1) \tag{4.25}$$

and by (4.17) we also know that

$$a_{k+m+1}T''(m-1) + T''(m-2) = T''(m) \tag{4.26}$$

Combining (4.24), (4.25), and (4.26) we see that

$$T(k+m+1) = T'(m+1)T(k) + T''(m)T(k-1)$$

Finally, we observe that

$$\begin{aligned}
\det(R(a_1,\ldots,a_n)) &= T(n) \\
&= T(k+n-k) \\
&= T'(n-k)T(k) + T''(n-k-1)T(k-1) && \text{by (4.19)} \\
&> T'(n-k)T(k) \\
&= \det(R(a_{k+1},\ldots,a_n))\det(R(a_1,\ldots,a_k)) && \text{by (4.18)}
\end{aligned}$$

□

Using Lemmata 4.3.2 and 4.3.3 we will break up the sequence $a_1, \ldots, a_n$ into smaller subsequences which will have one of the eleven special types listed in the following lemma.

Lemma 4.3.4. *Let $a_1, \ldots, a_n$ be a sequence of one of the following eleven types:*

1. *a_1 where $a_1 \geq 2$*

2. *$1, a_2$ where $a_2 \geq 2$*

3. *$a_1, 1$ where $a_1 \geq 2$*

4. *$1, 1, a_3$ where $a_3 \geq 2$*

5. *$a_1, 1, 1$ where $a_1 \geq 2$*

6. *$1, 1, 1, a_4$ where $a_4 \geq 2$*

7. *$1, a_2, 1, 1$ where $a_2 \geq 2$*

8. *$1, 1, a_3, 1$ where $a_3 \geq 2$*

9. *$1, 1, a_3, 1, 1$ where $a_3 \geq 2$*

10. *$1, a_2, 1, a_4, 1$ where $a_2 \geq 2$ and $a_4 \geq 2$*

11. *$1, 1, a_3, 1, a_5, 1$ where $a_3 \geq 2$ and $a_5 \geq 2$*

Let $T(i)$ be the recurrence of Theorem 4.3.1. Then $V(a_1, \ldots, a_n) \leq T(n)$.

Proof. For type (1), one readily obtains that $a_1 + 2 \leq 2a_1$ implying

$$V(a_1) = \frac{a_1 + 2}{2} \leq \frac{2a_1}{2} = T(1)$$

For type (2), we see that $T(2) = a_2 + 1$ and

$$V(1, a_2) = \frac{3(a_2 + 2)}{4}$$

When $a_2 \geq 2$, one readily obtains $V(1, a_2) < T(2)$.

For type (3), we have $T(2) = a_1 + 1$ and

$$V(a_1, 1) = \frac{3(a_1 + 2)}{4}$$

and the proof proceeds in a similar manner to type (2).

For type (4), $T(3) = 2a_3 + 1$ and

$$V(1, 1, a_3) = \frac{9(a_3 + 2)}{8}$$

When $a_3 \geq 2$ one readily obtains

$$\frac{9(a_3 + 2)}{8} \leq 2a_3 + 1$$

For type (5), we have $T(3) = 2a_1 + 1$ and

$$V(a_1, 1, 1) = \frac{9(a_1 + 2)}{8}$$

and the proof proceeds similarly to type (4).

For type (6), we have $T(4) = 3a_4 + 2$ and

$$V(1,1,1,a_4) = \frac{27(a_4+2)}{16}$$

When $a_4 \geq 2$, one may show that

$$\frac{27(a_4+2)}{16} \leq 3a_4 + 2$$

For type (7), we have $T(4) = 2a_2 + 3$ while

$$V(1,a_2,1,1) = \frac{27}{16}(a_2+2)$$

Then

$$T(4) - V(1,a_2,1,1) = \frac{5}{16}a_2 - \frac{3}{8} \geq 0$$

since $a_2 \geq 2$.

For type (8), we have $T(4) = 2a_3 + 3$ and

$$V(1,1,a_3,1) = \frac{27}{16}(a_3+2)$$

The proof is now similar to type (7).

For type (9), we have $T(5) = 4a_3 + 4$ and

$$V(1,1,a_3,1,1) = \frac{81}{32}(a_3+2)$$

Then

$$T(5) - V(1,1,a_3,1,1) = \frac{47}{32}a_3 - \frac{17}{16} \geq 0$$

since $a_3 \geq 2$.

For type (10), we have $T(5) = a_2a_4 + 2a_2 + 2a_4 + 3$ and

$$V(1,a_2,1,a_4,1) = \frac{27}{32}(a_2a_4 + 2a_2 + 2a_4 + 4)$$

Then

$$T(5) - V(1,a_2,1,a_4,1) = \frac{5}{32}(a_2a_4 + 2a_2 + 2a_4) - \frac{3}{8} \geq 0$$

since $a_2 \geq 2$ and $a_4 \geq 2$.

For type (11), we have $T(6) = 2a_3a_5 + 4a_3 + 3a_5 + 4$ an

$$V(1,1,a_3,1,a_5,1) = \frac{81}{64}(a_3a_5 + 2a_3 + 2a_5 + 4)$$

Then

$$T(6) - V(1,1,a_3,1,a_5,1) = \frac{47}{64}a_3a_5 + \frac{47}{32}a_3 + \frac{15}{32}a_5 - \frac{17}{16} \geq 0$$

since $a_3 \geq 2$ and $a_5 \geq 2$. $\square$

We are now prepared to present the proof of Theorem 4.3.5.

Theorem 4.3.5. *Let K be the 2-bridge link $R(a_1, a_2, \ldots, a_n)$. Then* $\mathrm{vol}(K) < 2\pi \log \det(K)$.

Proof. We consider three cases:

1. $n = 3, a_1 = 1, a_2 \geq 2, a_3 = 1$

2. $a_1 = a_2 = \ldots = a_n = 1$

3. $a_i \neq 1$ for some i and not case 1

Case 1:

We can calculate that $\det(K) = a_2 + 2$. Using Corollary 4.2.4 we see that

$$\mathrm{vol}(K) < 2\pi \log \left(\frac{a_2 + 2}{2} \right) < 2\pi \log(a_2 + 2) = 2\pi \log(\det(K)) \tag{4.27}$$

For the remaining cases, it suffices to show that $V(a_1, \ldots, a_n) \leq \det(K)$.

Case 2:

If $n = 1$, 2, or 3 then K is $R(1)$, $R(1,1)$, or $R(1,1,1)$ respectively, none of which is hyperbolic. If $n = 4$ then $\det(K) = T(4) = 5$. Corollary 4.2.4 implies that

$$\mathrm{vol}(K) < 2\pi \log \left(\frac{2 \cdot 3 \cdot 3 \cdot 2}{2^4} \right) = 2\pi \log \left(\frac{9}{4} \right) < 2\pi \log(5) = 2\pi \log(\det(K))$$

If $n = 5$ then $\det(K) = T(5) = 8$ and while $V(1,1,1,1,1) = 243/32 < 8$. If $n \geq 6$ then we use Lemma 4.3.2 part (c). We can let $k = 6$ and then the link K' defined in Lemma 4.3.2 part (c) will be $R(1,1,1,1,1)$. The result now follows from the case $n = 5$ above.

Case 3:

We may use Lemma 4.3.2 part (c) to assume that we do not have $a_{k-2} = a_{k-1} = a_k = 1$ for any $k \geq 4$, *i.e.* $a_1, \ldots, a_n$ has no subsequences of three or more consecutive ones except possibly $a_1 = a_2 = a_3 = 1$. Let m be the cardinality $|\{a_k \in \{a_i\}_{i=1}^n : a_k \geq 2\}|$. We will partition the sequence $\{a_1, \ldots, a_n\} = \{b_1^{(1)}, \ldots, b_{n_1}^{(1)}, b_1^{(2)}, \ldots b_{n_2}^{(2)}, \ldots, b_1^{(m)}, \ldots b_{nm}^{(m)}\}$ into subsequences of the types found in Lemma 4.3.4 according to one of the cases below.

Case 3a: $a_1 \geq 2$

Let $b_1^{(k)}$ be the kth element of the sequence $a_1, \ldots, a_n$ that is greater than or equal to 2. Then each subsequence $b_1^{(k)}, \ldots, b_{n_k}^{(k)}$ is either type 1, 3, or 5 from Lemma 4.3.4.

Case 3b: $a_1 = 1$ and $a_n \geq 2$

Let $b_{n_k}^{(k)}$ be the kth element of $a_1, \ldots, a_n$ that is greater than or equal to 2. Then each $b_1^{(k)}, \ldots, b_{n_k}^{(k)}$ is either of type 1, 2, 4, or 6 in 4.3.4.

Case 3c: $a_1 = a_{n-1} = a_n = 1$

For $1 \leq k \leq m-1$ let $b_{n_k}^{(k)}$ be the kth element of $a_1, \ldots, a_n$ that is greater than or equal to 2. Then each $b_1^{(k)}, \ldots, b_{n_k}^{(k)}$ is either of type 1, 2, 4, or 6 in Lemma 4.3.4. Let $b_1^{(m)}, \ldots, b_{nm}^{(m)}$ be the remaining elements of the sequence $a_1, \ldots, a_n$. Then $b_1^{(m)}, \ldots, b_{nm}^{(m)}$ is either of type 7 or 9 from Lemma 4.3.4.

Case 3d: $a_1 = a_n = 1$ and $a_{n-1} \neq 1$

For $1 \leq k \leq m-2$ let $b_{n_k}^{(k)}$ be the kth element of $a_1, \ldots, a_n$ that is greater than or equal to 2. Then each $b_1^{(k)}, \ldots, b_{n_k}^{(k)}$ is either of type 1, 2, 4, or 6 in Lemma 4.3.4. Let $b_1^{(m)}, \ldots, b_{nm}^{(m)}$

be the remaining elements of the sequence $a_1, \ldots, a_n$. Since case 3 excludes cases 1 and 2, $\{a_1, a_2, a_3\} \neq \{1, a_2, 1\}$. Therefore $b_1^{(m)}, \ldots, b_{n_m}^{(m-1)}$ is either of type 8, 10, or 11 from Lemma 4.3.4. For notational purposes, take $b_1^{(m-1)}, \ldots, b_{n_{m-1}}^{(m-1)}$ to be the empty sequence.

Now that we have partitioned the sequence

$$\{a_1, \ldots, a_n\} = \{b_1^{(1)}, \ldots, b_{n_1}^{(1)}, b_1^{(2)}, \ldots b_{n_2}^{(2)}, \ldots, b_1^{(m)}, \ldots b_{n_m}^{(m)}\}$$

into subsequences of the types found in Lemma 4.3.4, we observe that

$$\begin{aligned}
\det(K) &> \det(R(b_1^{(1)}, \ldots, b_{n_1}^{(1)})) \det(R(b_1^{(2)}, \ldots, b_{n_2}^{(2)})) \ldots \det(R(b_1^{(m)}, \ldots, b_{n_m}^{(m)})) \\
&\geq V(b_1^{(1)}, \ldots, b_{n_1}^{(1)}) V(b_1^{(2)}, \ldots, b_{n_2}^{(2)}) \ldots V(b_1^{(m)}, \ldots, b_{n_m}^{(m)}) \\
&= V(a_1, \ldots, a_n)
\end{aligned}$$

where the first line follows by Lemma 4.3.3 and the second follows by Lemma 4.3.4. □

The proof of Theorem 4.3.5 also verified the following fact which will be used in Section 4.4.

Corollary 4.3.6. *Let $K = R(a_1, \ldots, a_n)$ be a 2-bridge link. If $K \neq R(1, 1, 1, 1)$, $R(1, a_2, 1)$, $R(1, 1)$ or $R(1)$ for any $a_2 \geq 1$ then*

$$\det(R(a_1, \ldots, a_n)) \geq V(a_1, \ldots, a_n)$$

4.4 Alternating Braids

We will show in this section that Conjecture 4.1.1 holds for alternating 3-braids and for an infinite family of 4-braids. The former fact will rely on the result of Theorem 4.3.5, while the latter fact will be proved by bounding the hyperbolic volume and explicitly computing the determinant.

4.4.1 3-Braids

Let $B(a_1, b_1, \ldots, a_n, b_n)$ denote the alternating 3-braid with a_1 positive crossings in the first twist region, b_1 negative crossings in the second twist region, and so on. See for example Figure 4.4. Note that up to reflection, this considers all alternating 3-braids, and that up to isomorphism all alternating 3-braids have an even number of twist regions. We state the main theorem for this section.

Theorem 4.4.1. *If K is an alternating 3-braid then* $\mathrm{vol}(K) < 2\pi \log \det(K)$.

The proof of Theorem 4.4.1 will follow immediately from from Lemmata 4.4.2 and 4.4.3. Note that by using Corollary 4.2.4 it is sufficient to show that $V(a_1, b_1, \ldots, a_n, b_n) \leq \det(B(a_1, b_1, \ldots, a_n, b_n))$.

Lemma 4.4.2. *Let $K = B(a_1, b_1, \ldots, a_n, b_n)$. If $a_i \neq 1$ or $b_i \neq 1$ for some i then* $\mathrm{vol}(K) < 2\pi \log \det(K)$.

Proof. Suppose $b_i \neq 1$. Let σ_1 and σ_2 be generators of the 3-braid, where σ_i denotes a positive half-twist of the ith and $(i+1)$st strands. Then K is the closure of $\sigma_1^{a_1}\sigma_2^{-b_1} \ldots \sigma_1^{a_n}\sigma_2^{-b_n}$. Let $\alpha = (\sigma_2^{b_n}\sigma_1^{-a_n}\sigma_2^{b_{n-1}}\sigma_1^{a_{n-1}} \ldots \sigma_2^{b_{i+1}}\sigma_1^{-a_{i+1}})$. Then

$$\alpha^{-1}(\sigma_1^{a_1}\sigma_2^{-b_1} \ldots \sigma_1^{a_n}\sigma_2^{-b_n})\alpha = \sigma_1^{a_{i-1}}\sigma_2^{b_{i-1}} \ldots \sigma_1^{a_n}\sigma_2^{-b_n}\sigma_1^{a_1}\sigma_2^{-b_1} \ldots \sigma_1^{a_i}\sigma_1^{-b_i} \qquad (4.28)$$

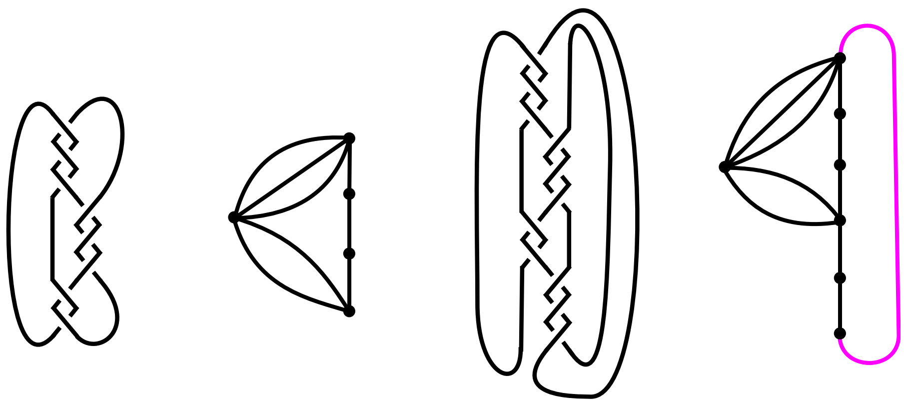

Figure 4.4: LEFT: The two-bridge knot $R(3,3,2)$. CENTER LEFT: The checkerboard graph for $R(3,3,2)$. CENTER RIGHT: The closed alternating 3-braid $B(3,3,2,3)$. RIGHT: The checkerboard graph for $B(3,3,2,3)$. Note that the graph that results from deleting the highlighted edge has the same number of spanning trees as the checkerboard graph for $R(3,3,2)$.

The braid closure of the right hand side of (4.28) corresponds to the three-braid

$$K' = B(a_{i-1}, b_{i-1}, \ldots, a_n, b_n, a_1, b_1, \ldots, a_i, b_i)$$

so K is equivalent to K'. Therefore we may assume in this case that $b_n \neq 1$. We will now reduce the problem to the case of 2-bridge links, and the result will follow from Theorem 4.3.5. Observe that one of the checkerboard graphs for $B(a_1, b_1, \ldots, a_n, b_n)$ and $R(a_1, b_1, \ldots, b_{n-1}, a_n)$ will be of the form shown in Figure 4.4. Apply Lemma 4.2.6 by contracting and deleting the highlighted right edge in the far right of Figure 4.4. The result of deleting the edge yields a graph with the same number of spanning trees as one of the checkerboard graphs of $R(a_1, b_1, \ldots, b_{n-1}, a_n)$. If $b_n \geq 2$ then contraction of the highlighted edge is the checkerboard graph of $B(a_1, b_1, \ldots, a_n, b_n - 1)$. Further, contracting the highlighted edge of a checkerboard graph of $B(a_1, b_1, \ldots, a_n, 1)$ yields a graph isomorphic to a checkerboard graph of $B(a_1 + a_n, b_1, a_2, b_2, \ldots, a_{n-1}, b_{n-1})$. Therefore we see inductively that

$$\begin{aligned}
\det(B(a_1, b_1, \ldots, a_n, b_n)) &= \det(R(a_1, b_1, \ldots, b_{n-1}, a_n)) + \det(B(a_1, b_1, \ldots, a_n, b_n - 1)) \\
&= 2\det(R(a_1, b_1, \ldots, b_{n-1}, a_n)) + \det(B(a_1, b_1, \ldots, a_n, b_n - 2)) \\
&\vdots \\
&= (b_n - 1)\det(R(a_1, b_1, \ldots, b_{n-1}, a_n)) + \det(B(a_1, b_1, \ldots, a_n, 1)) \\
&= b_n \det(R(a_1, b_1, \ldots, b_{n-1}, a_n)) \\
&\qquad + \det(B(a_1 + a_n, b_1, \ldots, a_{n-1}, b_{n-1})) \qquad (4.29) \\
&\geq b_n \det(R(a_1, b_1, \ldots, b_{n-1}, a_n)) \qquad (4.30)
\end{aligned}$$

Suppose that $\{a_1, b_1, \ldots, a_n, b_n\} \neq \{1, b_1, 1, b_2\}$ or $\{1, b_1\}$. Then since $b_n \geq 2$, we have $b_n \geq (b_n + 2)/2$ and we can apply Corollary 4.3.6:

$$\begin{aligned}
\det(B(a_1, b_1, \ldots, a_n, b_n)) &\geq b_n \det(R(a_1, b_1, \ldots, b_{n-1}, a_n)) && \text{by 4.30} \\
&\geq b_n V(a_1, b_1, \ldots, b_{n-1}, a_n) && \text{by Corollary 4.3.6} \qquad (4.31) \\
&= b_n \frac{\prod_{i=1}^{n}(a_i + 2)\prod_{i=1}^{n-1}(b_i + 2)}{2^{2n-1}} \\
&\geq \frac{\prod_{i=1}^{n}(a_i + 2)\prod_{i=1}^{n}(b_i + 2)}{2^{2n}} \\
&= V(a_1, b_1, \ldots, a_n, b_n)
\end{aligned}$$

If $\{a_1, b_1, \ldots, a_n, b_n\} = \{1, b_1, 1, b_2\}$ or $\{1, b_1\}$ then we cannot apply Corollary 4.3.6 to obtain (4.31). If $\{a_1, b_1, \ldots, a_n, b_n\} = \{1, b_1\}$ then $B(1, b_1)$ is seen to be the $(2, b_1)$-torus link which is not hyperbolic. If $\{a_1, b_1, \ldots, a_n, b_n\} = \{1, b_1, 1, b_2\}$ then using (4.29) we see that

$$\det(B(1, b_1, 1, b_2)) = b_2 \det(R(1, b_1, 1)) + \det(B(2, b_1)) \tag{4.32}$$

$$= b_2(b_1 + 2) + 2b_1 \tag{4.33}$$

On the other hand

$$V(1, b_1, 1, b_2) = \frac{9}{16}(b_1 b_2 + 2b_1 + 2b_2 + 4) \tag{4.34}$$

Since $b_2 \geq 2$ we see that

$$\begin{aligned} \det(B(1, b_1, 1, b_2)) - V(1, b_1, 1, b_2) &= \frac{7}{16}(b_1 b_2 + 2b_1 + 2b_2) - \frac{36}{16} \\ &\geq \frac{56}{16} - \frac{36}{16} \\ &> 0 \end{aligned}$$

If $b_i = 1$ for all i, then $a_i \neq 1$ for some i. Then K is equivalent to the link

$$B(a_i, b_i, \ldots, a_n, b_n, a_1, b_1, \ldots, a_{i-1}, b_{i-1})$$

and therefore we assume that $a_1 \neq 1$. Then we can repeat the argument above by considering the other checkerboard surface (*i.e.* considering the checkerboard graph obtained from the white regions instead of the shaded regions). □

Lemma 4.4.3. *Let* $K = B(1, 1, \ldots, 1, 1)$ *where there are* $2n$ *copies of* 1. *Then* $\mathrm{vol}(K) < 2\pi \log \det(K)$.

Proof. We will use the Matrix Tree Theorem, Theorem 4.2.5, to compute the number of spanning trees of the checkerboard graph, and hence the determinant of K. If $n \geq 3$, the associated checkerboard graph has Laplacian

$$L = \begin{bmatrix} n & -1 & -1 & -1 & -1 & -1 & \dots & -1 & -1 & -1 \\ -1 & 3 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & -1 \\ -1 & -1 & 3 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 0 & -1 & 3 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 3 & -1 & & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \\ -1 & 0 & 0 & 0 & 0 & 0 & \dots & -1 & 3 & -1 \\ -1 & -1 & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 3 \end{bmatrix} \tag{4.35}$$

where L is an $(n+1) \times (n+1)$ matrix. Let L' be the minor of L obtained by eliminating the first row and first column. Then it is known (see for example [55]) that

$$\det(L') = -2 + \operatorname{tr}\left(\prod_{i=1}^{n} \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \right) \tag{4.36}$$

By diagonalizing, we can compute an explicit formula:

$$\begin{aligned} \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}^n &= \begin{bmatrix} \frac{3+\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{3+\sqrt{5}}{2} & 0 \\ 0 & \frac{3-\sqrt{5}}{2} \end{bmatrix}^n \begin{bmatrix} \frac{3+\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \frac{3+\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{3+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{3-\sqrt{5}}{2}\right)^n \end{bmatrix} \begin{bmatrix} \frac{3+\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}^{-1} \end{aligned}$$

Therefore

$$\det(L') = -2 + \operatorname{tr}\left(\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}^n\right)$$
$$= -2 + \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n$$

The volume of K is bounded above by

$$V(1,1,\ldots,1,1) = \left(\frac{3}{2}\right)^{2n} = \left(\frac{9}{4}\right)^n \tag{4.37}$$

It is straightforward to check that

$$-2 + \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n > \left(\frac{9}{4}\right)^n \qquad \text{if } n \geq 3 \tag{4.38}$$

If $n = 1$, then K is not hyperbolic. If $n = 2$ then K is the figure-eight knot, which is a 2-bridge knot and therefore satisfies Conjecture 4.1.1 by Theorem 4.3.5. □

4.4.2 A Family of 4-braids

Let $\sigma_1, \sigma_2, \sigma_3$ be the generators of the 4-braid, where σ_i denotes a positive half-twist of the ith and $(i+1)$st strands. Let W_n be the closure of $(\sigma_1\sigma_3\sigma_2^{-1})^n$. We note that these links correspond to the weaving links $W(4,n)$ of [20] and [21]. A checkerboard graph associated with W_n is the maximal planar lantern graph $\mathcal{E}_{n+2}$ on $(n+2)$ vertices as shown in Figure 4.5. Work of Modabish, Lotfi, and El Marraki [54] shows that

$$\tau(\mathcal{E}_{n+2}) = \frac{n}{2+2\sqrt{3}}\left[(2+\sqrt{3})^n - (2-\sqrt{3})^n\right] \leq \frac{n+2}{2\sqrt{3}}(2+\sqrt{3})^n \tag{4.39}$$

On the other hand, Corollary 4.2.4 shows that

$$\operatorname{vol}(W_n) \le 2\pi \log\left(\frac{3^{2n}4^n}{2^{3n}}\right) = 2\pi \log\left[\left(\frac{9}{2}\right)^n\right] \tag{4.40}$$

Since $2+\sqrt{3} < 9/2$ it follows from equations (4.39) and (4.40) that the volume bound of Corollary 4.2.4 is insufficient to prove Conjecture 4.1.1 for these links. However, one may instead use Theorem 4.2.3 to find that

$$\begin{aligned}\exp\left(\frac{\operatorname{vol}(W_n)}{2\pi}\right) &\le \exp\left[2n\operatorname{vol}(B_3) + n\operatorname{vol}(B_4)\right]\\ &\le (3.418677233748620053022)^n\end{aligned}$$

This bound may also be obtained from [21, Theorem 1.1]. On the other hand, equation (4.39) and the fact that $(2-\sqrt{3})^n < \frac{1}{2}(2+\sqrt{3})^n$ for $n \ge 1$ together imply that

$$\tau(\mathcal{E}_{n+2}) = \frac{n+2}{2\sqrt{3}}\left[(2+\sqrt{3})^n - (2-\sqrt{3})^n\right] \ge \frac{n+2}{4\sqrt{3}}(2+\sqrt{3})^n \tag{4.41}$$

It is straightforward to show that

$$(3.418677)^n < \frac{n+2}{4\sqrt{3}}(2+\sqrt{3})^n \text{ for } n \ge 4$$

so Conjecture 4.1.1 holds for all W_n with $n \ge 4$. Note that the case $n \le 3$ has been verified in [20].

One can use this method to find many more infinite families of links for which Conjecture 4.1.1 holds. Given a planar graph G, one may create an alternating link K for which G is the checkerboard graph of K. This is done by replacing each edge with a crossing and connecting

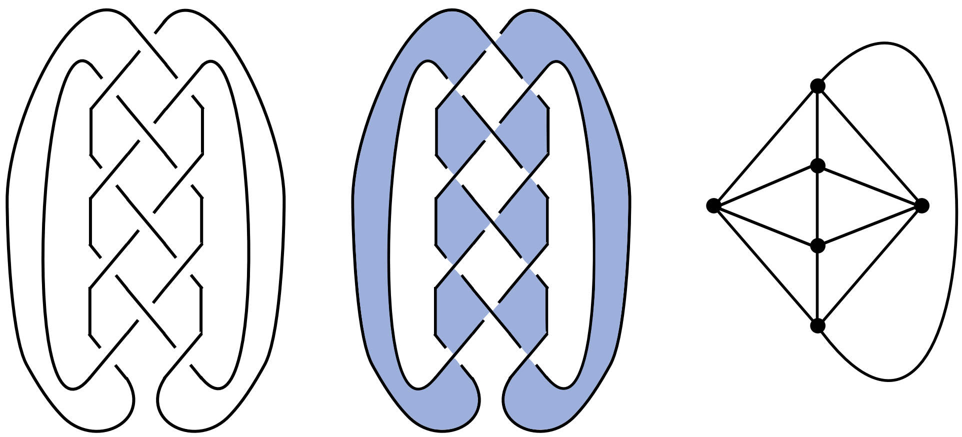

Figure 4.5: LEFT: The link W_n. CENTER: The checkerboard coloring of W_4. RIGHT: The graph $\mathcal{E}_6$ corresponding to the white checkerboard surface.

ends of crossings so that each vertex is on the shaded part of the checkerboard surface. One can then calculate the volume estimates and then if the number of spanning trees of the graph is known test whether the conjecture holds. This method works for the wheel, fan, crystal, star-flower graphs of [53] and [54] as well as the grid graphs and triangulated grid-graphs of [52].

4.5 Highly Twisted Knots

We consider the situation where a link has a twist region with many crossings. It is known by [44] that the volume of an alternating link is bounded by the number of twist regions in the diagram. Therefore, increasing the number of crossings in a twist region of an alternating hyperbolic link has a bounded effect on the hyperbolic volume. On the other hand, the number of spanning trees in the checkerboard graph will increase by adding crossings to a twist region. It follows that highly twisted links must satisfy Conjecture 4.1.1. We quantify this in the following theorem.

Theorem 4.5.1. *Let K be an alternating hyperbolic link with t twist regions and c crossings. If*

$$c > t + \xi^{t-1} - 2\gamma^{t-1} \tag{4.42}$$

where $\gamma \approx 1.4253$ is the unique positive real number satisfying $\gamma^{-5} + 2\gamma^{-4} + \gamma^{-3} - 1 = 0$ and $\xi = e^{5v_4/\pi} \approx 5.0296$, then $\mathrm{vol}(K) < 2\pi \log(\det(K))$.

Proof. Let $a_1, \ldots, a_t$ be the crossing numbers of the t twist regions of K. Let $G(x_1, \ldots, x_t)$ be the checkerboard graph of the link obtained by placing x_i crossings in the ith twist region of K. Since a checkerboard graph has the same number of spanning trees as its dual, we may assume that the first twist region of K corresponds to a path on a_1 vertices in $G(a_1, \ldots, a_t)$. By Lemma 4.2.6 we then obtain

$$\begin{aligned}
\tau(G(a_1, a_2, \ldots, a_t)) &= \tau(G(a_1 - 1, a_2 \ldots, a_t)) + \tau(G(0, a_2, \ldots, a_t)) \\
&= \tau(G(1, a_2, \ldots, a_t)) + (a_1 - 1)\tau(G(0, \ldots, a_t)) \\
&\geq \tau(G(1, a_2, \ldots, a_t)) + (a_1 - 1) \\
&\geq \tau(G(1, 1, \ldots, 1)) + \sum_{i=1}^{t}(a_i - 1) \\
&= \tau(G(1, 1, \ldots, 1)) + c - t
\end{aligned}$$

Theorem 4.2.7 then implies that

$$\det(K) = \tau(G(a_1, \ldots, a_t)) \geq 2\gamma^{t-1} + c - t \tag{4.43}$$

By Theorem 2.3.2 we know that $\mathrm{vol}(K) \leq 10v_4(t - 1)$. It is then straightforward to check that if (4.42) holds then

$$\mathrm{vol}(K) < 10v_4(t-1) \le 2\pi \log(2\gamma^{t-1} + c - t) \le 2\pi \log(\det(K)) \tag{4.44}$$

□

Corollary 4.5.2. *Let K be an alternating hyperbolic Montesinos link with t twist regions and c crossings. If*

$$c > t + \zeta^t - 2\gamma^{t-1} \tag{4.45}$$

where $\zeta = e^{v_8/\pi} \approx 3.2099$ then $\mathrm{vol}(K) > 2\pi \log(\det(K))$.

Proof. The proof is the same as for Theorem 4.5.1, except we replace the upper bound on volume with the bound $2v_8t$ of Theorem 2.3.3. □

4.5.1 Application to Pretzel Knots

We give an application of Theorem 4.5.1 and Corollary 4.5.2 to alternating pretzel links. We begin by calculating the determinant of an alternating pretzel knot.

Proposition 4.5.3. *Let $P(a_1, a_2, \ldots, a_n)$ be the alternating pretzel knot having $a_1, a_2, \ldots, a_n$ crossings in the first, second, and so on to the nth twist region. Then*

$$\det(P(a_1, a_2, \ldots, a_n)) = \sum_{i=1}^{n} \prod_{j \neq i} a_j \tag{4.46}$$

Proof. We will use Lemma 4.2.6. An example checkerboard graph for $P(a_1, a_2, \ldots, a_n)$ is given on the left of Figure 4.6. Deleting an edge in the nth twist region produces a graph with the same number of spanning trees as the checkerboard graph of $P(a_1, \ldots, a_{n-1})$. For

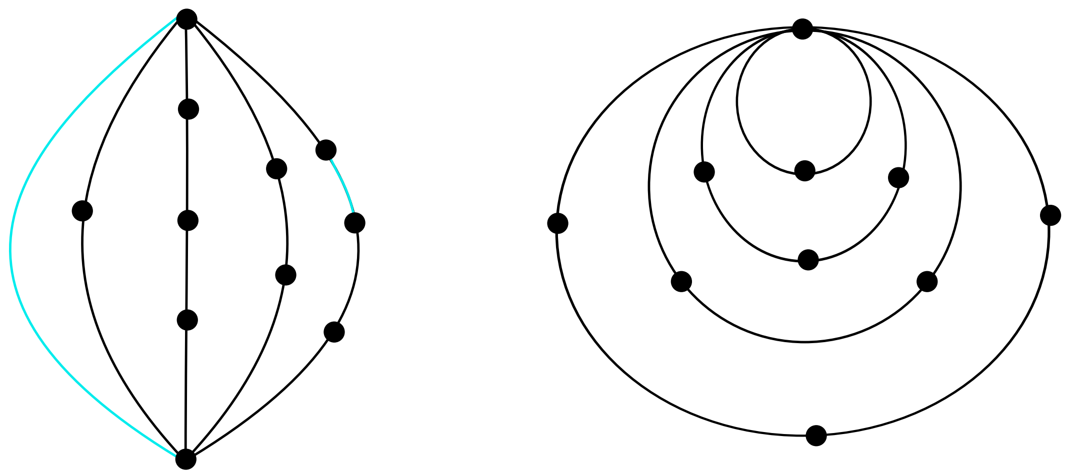

Figure 4.6: LEFT: The checkerboard graph of the Pretzel knot $P(1,2,4,3,4)$. Deletion of the red edge produces a graph with the same number of spanning trees as the checkerboard graph of $P(1,2,4,3)$. Contraction of the cyan edge yields the graph on the right. RIGHT: Result of contracting the cyan edge of the graph on the left.

example one may delete the red edge in Figure 4.6. On the other hand, if $a_n \geq 2$, then contracting that edge results in the checkerboard graph of $P(a_1, a_2 \ldots, a_n - 1)$. If $a_n = 1$ then the resulting graph is the join of $(n-1)$ cycles. For example, contracting the blue edge in Figure 4.6 produces the graph on the right of Figure 4.6) which has $a_1 a_2 \ldots a_{n-1}$ spanning trees. Therefore by Lemma 4.2.6 we see that

$$\det(P(a_1, \ldots, a_n)) = a_1 \ldots a_{n-1} + a_n \det(P(a_1, a_2, \ldots, a_{n-1})) \tag{4.47}$$

Applying the above method to $P(a_1, \ldots, a_{n-1})$ *et cetera* we obtain the desired result. $\square$

Given a fixed number t of twist regions and a pretzel link K with t twist regions, Corollary 4.5.2 can be used to say that if K has more than t $+\zeta^t - 2\gamma^{t-1}$ crossings in any twist region, then it satisfies Conjecture 4.1.1. Therefore for a given t, there are only finitely many links which may fail to satisfy Conjecture 4.1.1. We may enumerate these links, use Theorem 4.2.3

to compute an upper bound on volume, and check that this upper bound is less than the determinant. Using this method, we have shown with computer assistance that Conjecture 4.1.1 holds for all alternating pretzel links with no more than 13 twist regions.

The number of links for which we need to evaluate the determinant and use the upper bound of Theorem 4.2.3 grows exponentially in terms of the number of twist regions. It is therefore prudent to use further techniques to make the computation feasible. We now explain the techniques used to verify the conjecture for alternating pretzel links with up to 13 twist regions. Let $P(a_1, \dots, a_t)$ denote the alternating pretzel link with a_i crossings in the ith twist region. A volume bound that proves useful in this context is due to Dasbach and Tsvietkova [23].

Theorem 4.5.4 ([23]). *Given a diagram D of a hyperbolic alternating link K, with t_i twist regions of precisely i half-twists, and g_i twist regions of at least i half-twists, then*

$$\mathrm{Vol}(S^3 \backslash K) \leq (10g_4 + 8t_3 + 6t_2 + 4t_1 - a)v_4 \tag{4.48}$$

where $a = 10$ if g_4 is non-zero, $a = 7$ if t_3 is non-zero, and $a = 6$ otherwise.

Let $\mathrm{DT}(a_1, \dots, a_t)$ be the right hand side of (4.48). This bound proves particularly useful, since if $a_i \geq 4$ then

$$\mathrm{DT}(a_1, \dots, a_t) = \mathrm{DT}(a_1, \dots, a_i + k, \dots, a_t) \tag{4.49}$$

for any integer k. Moreover $\mathrm{DT}(a_1, \dots, a_t)$ remains unchanged if the a_i's are permuted.

One can use the arithmetic-geometric mean inequality in conjunction with Theorem 4.2.3 to derive another bound on the volume of $P(a_1, \dots, a_t)$ that is unchanged under permutation of the a_i's.

Lemma 4.5.5. *The following inequality is true:*

$$\operatorname{vol}(P(a_1,\ldots,a_t)) \leq \left(\frac{\sum_{i=1}^{t} a_i}{t}\right)^t \tag{4.50}$$

Proof. This follows from Theorem 4.2.3 and the geometric-arithmetic mean inequality:

$$\begin{aligned}
\operatorname{vol}(P(a_1,\ldots,a_t)) &\leq \frac{a_1+a_t}{2}\prod_{i=1}^{t-1}\frac{a_i+a_{i+1}}{2} \\
&\leq \left(\frac{\frac{a_1+a_t}{2}+\sum_{i=1}^{t-1}\frac{a_i+a_{i+1}}{2}}{t}\right)^t \\
&= \left(\frac{\sum_{i=1}^{t} a_i}{t}\right)^t
\end{aligned}$$

□

We consequently define

$$\operatorname{AG}(a_1,\ldots,a_t) = \left(\frac{\sum_{i=1}^{t} a_i}{t}\right)^t \tag{4.51}$$

We can use Corollary 4.5.2 to find the maximum number of crossings in any twist region. Suppose $a_i = 1$ for $i \geq 2$. Then if c is the number of crossings in $P(a_1,\ldots,a_t)$ then $c = a_1 + t - 1$, hence by Corollary 4.5.2 $a_1 > 1 + \zeta^{t-1} - 2\gamma^{t-1}$. Let

$$M = 1 + \zeta^{t-1} - 2\gamma^{t-1} \tag{4.52}$$

By Corollary 4.5.2, we will only need to verify that $P(a_1,\ldots,a_t)$ satisfies Conjecture 4.1.1 if $a_i \leq M$ for all i.

Combining all the above, we produce Algorithm 1 to check Conjecture 4.1.1 for alternating pretzel links with 3 twist regions. If the algorithm does not return any links, then it has verified the conjecture. Otherwise it returns a problem case. The algorithm works by using the AG and DT bounds to verify $P(a_1, \ldots, a_t)$ satisfies Conjecture 4.1.1 for all permutations of the a_i's. It utilizes the fact that DT remains unchanged when a twist region goes from k to $k+1$ crossings for $k \geq 4$ to eliminate many cases. When the AG and DT bounds are insufficient, it computes the volume bound of Corollary 4.2.4. We do this only as a last resort since it requires $t!$ computations because we must consider all permutations of the a_i's. The algorithm utilizes the fact that $\det(P(a_1, \ldots, a_n)) < \det(P(a_1, \ldots, a_n + k))$ for any integer $k \geq 1$. The algorithm can be generalized to check for t twist regions by increasing the number of **for** loops, and generalizing line 17 to compute the volume bound of Corollary 4.2.4.

Algorithm 1 Verify Conjecture 4.1.1 for Pretzel Links

1: Let $t \geq 3$ be an integer representing the number of twist regions in the pretzel link.
2: Set $M = 1 + \zeta^t - 2\gamma^{t-1}$.
3: **for** $x \in \{1, 2, \ldots, M\}$ **do**
4:     **if** $\mathrm{DT}(x, 4, 4) < \det(P(x, x, x))$ **then**
5:         **break**
6:     **for** $y \in \{x, x+1, \ldots, M\}$ **do**
7:         **if** $\mathrm{DT}(x, y, 4) < \det(P(x, y, y))$ **then**
8:             **break**
9:         **for** $z \in \{y, y+1, \ldots, M\}$ **do**
10:             **if** $\mathrm{DT}(x, y, z) < \det(P(x, y, z))$ **then**
11:                 $P(x, y, z)$ satisfies Conjecture 4.1.1
12:             **else if** $\mathrm{AG}(x, y, z) < \det(P(x, y, z))$ **then**
13:                 $P(x, y, z)$ satisfies Conjecture 4.1.1
14:             **else**
15:                 Let S be the set of all permutations of (x, y, z)
16:                 **for** $(s_1, s_2, s_3) \in S$ **do**
17:                     **if** $\det(P(s_1, s_2, s_3)) \geq (s_1 + s_2)(s_2 + s_3)(s_3 + s_1)/8$ **then**
18:                         Algorithm fails
19:                         **return** $P(s_1, s_2, s_3)$

Chapter 5

Meridians, Cusp Areas, and Essential Spanning Surfaces

5.1 Introduction

In this chapter we derive bounds of slope lengths on the maximal cusp and of the cusp area of hyperbolic knots in terms of the topology of essential surfaces spanned by the knots. Our results are partly motivated by the open question of whether there exist hyperbolic knots in S^3 whose meridian length exceeds four. We show that there is an algorithmically checkable criterion to decide whether a hyperbolic knot has meridian length less than a given bound, and we use it to we obtain large families of knots with meridian lengths bounded above by four. Our results are particularly interesting in the case of knots that project on closed embedded surfaces in an alternating fashion and admit essential checkerboard surfaces. In this case, our bounds are purely combinatorial and can be read directly from a knot diagram. We also discuss applications of our results to Dehn surgery.

Given a hyperbolic knot K in S^3, there is a well-defined notion of a maximal cusp C of the complement $M = S^3 \setminus K$ (see Section 2.1.2). The interior of C is neighborhood of the missing K and the boundary ∂C is a torus that inherits a Euclidean structure from the hyperbolic metric. Each slope σ on ∂C has a unique geodesic representative. The length of

σ, denoted by $\ell(\sigma)$, is the length of its geodesic representative. By Mostow-Prasad Rigidity (Theorem 2.1.1), these lengths are topological invariants of K.

By abusing notation and terminology, we will also refer ∂C as the boundary of M, and we will sometimes use the alternative notation ∂M. For a slope σ on ∂M let $M(\sigma)$ denote the 3-manifold obtained by Dehn filling M along σ. By the knot complement theorem of Gordon and Luecke [35], there is a unique slope μ, called the meridian of K, such that $M(\mu)$ is S^3. A λ-curve of K is a slope on ∂M that intersects μ exactly once and a spanning surface of K is a properly embedded surface in M whose boundary is a λ-curve.

Theorem 5.1.1. *Let K be a hyperbolic knot with meridian length $\ell(\mu)$. Suppose that K admits essential spanning surfaces S_1 and S_2 such that*

$$|\chi(S_1)| + |\chi(S_2)| \leq \frac{b}{6}\, i(\partial S_1, \partial S_2) \tag{5.1}$$

where b is a positive real number and $i(\partial S_1, \partial S_2)$ is the minimal intersection number of ∂S_1 and ∂S_2 on ∂M. Then the meridian length satisfies $\ell(\mu) \leq b$. Moreover, given a hyperbolic knot K and $b > 0$, there is an algorithm to determine if there are essential surfaces S_1 and S_2 satisfying (5.1)

A slope σ on ∂M is called *exceptional* if the 3-manifold $M(\sigma)$ is not hyperbolic. The Gromov-Thurston "2π-theorem" [12] asserts that if $\ell(\sigma) > 2\pi$ then $M(\sigma)$ admits a Riemannian metric of negative curvature. This combined with the proof of Thurston's geometrization conjecture [56] implies that actually $M(\sigma)$ is hyperbolic. The work of Agol [10] and Lackenby [43], that has improved 2π to 6, asserts that exceptional slopes must have length less or equal to six. Examples of exceptional slopes with length six are given in [10] and in [4]. Since the meridian curve of every hyperbolic knot in S^3 is an exceptional slope we have

$\ell(\mu) \leq 6$. The work of Adams, Colestock, Fowler, Gillam, and Katerman [6] shows that that $\ell(\mu) < 6$. Examples of knots whose meridian length approaches four from below are given in [10] and by Purcell in [61]. An open conjecture in the area is that $\ell(\mu) \leq 4$ for all hyperbolic knots in S^3.

Theorem 5.1.1 provides a criterion for checking algorithmically whether a given knot satisfies this conjecture. Indeed, given a hyperbolic knot K there is an algorithm using normal surface theory to decide whether K admits essential spanning surfaces S_1, S_2 such that

$$|\chi(S_1)| + |\chi(S_2)| \leq \frac{4}{6} \cdot i(\partial S_1, \partial S_2),$$

and thus whether $\ell(\mu) \leq 4$.

We will then discuss applications of Theorem 5.1.1. As an example, we first mention the hyperbolic 3-pretzel knots $P(a, -b, -c)$ with $a, b, c > 1$ and all odd. For these knots, Theorem 5.1.1 applies to give $\ell(\mu) \leq 3$. See Example 5.3.2 for details and for generalizations.

5.1.1 Knots with essential checkerboard surfaces

Theorem 5.1.1 can be applied to knots that admit alternating projections on closed surfaces so that they define essential checkerboard surfaces. A large such class of knots is the class of *adequate knots*, that admit alternating projections with essential checkerboard surfaces on certain *Turaev surfaces*. In this case we have the following theorem, where the terms involved are defined in detail in Chapter 2, Section 5.2.1, and Section 5.2.2.

Theorem 5.3.3. *Let K be an adequate hyperbolic knot in S^3 with crossing number $c = c(K)$ and Turaev genus g_T. Let C denote the maximal cusp of $S^3 \setminus K$ and let* $\mathrm{Area}(\partial C)$ *denote the cusp area. Finally let $\ell(\mu)$ and $\ell(\lambda)$ denote the length of the meridian and the shortest*

λ-curve of K. Then we have

1. $\ell(\mu) \leq 3 + \dfrac{6g_T - 6}{c}$

2. $\ell(\lambda) \leq 3c + 6g_T - 6$

3. $\operatorname{Area}(\partial C) \leq 9c\left(1 + \dfrac{2g_T - 2}{c}\right)^2$

A knot is alternating if and only if $g_T = 0$. In this case, the bounds of Theorem 5.3.3 agree with the bounds of [6]. The technique of the proof of Theorems 5.1.1 and 5.3.3, as well as the proof of results in [6], is reminiscent of arguments with pleated surfaces that led to the proof of the "6-Theorem" [10, 43]. The algorithm for checking the criterion of Theorem 2.1.2 involves normal surface theory and in particular the work of Jaco and Sedgwick [39].

5.1.2 Knots with meridian length bounded by four

As mentioned earlier, it has been conjectured that the meridian length of every hyperbolic knot in S^3 is at most four. The conjecture is known for several classes of knots. Adams [9] showed that the meridian of a 2-bridge hyperbolic knot has length less than 2. By [6], when K is an alternating hyperbolic knot then $\ell(\mu) < 3$. Agol [10] found families of knots whose meridian lengths approach four from below and Purcell [61] generalized his construction to construct families of knots whose meridian length approach four from below. She also showed that "highly twisted" knots meridian lengths are less than four. The results of this chapter allow us to verify the meridian length conjecture for additional broad classes of hyperbolic knots. We provide two results for adequate knots. Note that, by Theorem 5.3.3, if $c \geq 6g_T - 6$ then $\ell(\mu) \leq 4$. Thus, for every Turaev genus there can be at most finitely many adequate knots with $\ell(\mu) > 4$. In particular if $g_T \leq 3$, then $\ell(\mu) \leq 4$ unless $c \leq 12$.

Since the knots with up to 12 crossings are known to have meridian lengths less than two (see [17]), in fact, we have the following result.

Corollary 5.1.2. *Given $g_T > 0$, there can be at most finitely many hyperbolic adequate knots of Turaev genus g_T and with $\ell(\mu) > 4$. In particular, if K is a hyperbolic adequate knot with $g_T \leq 3$, then we have $\ell(\mu) < 4$.*

Note that for $g_T = 1$, we actually get $\ell(\mu) \leq 3$. This case includes Conway sums of strongly alternating tangles (see Section 3.2.2). Therefore, if a knot K is a Conway sum of strongly alternating links, then the length of the meridian of K is less or equal to three.

Another instance where our length bounds work well is to show that knots admitting diagrams with large ratio of crossings to twist regions have small meridian length. We have the following result which in particular applies to closed positive braids. See Corollary 5.3.7.

Theorem 5.3.6. *Let K be a hyperbolic knot with an adequate diagram with c crossings and t twist regions. Then we have*

$$\ell(\mu) \leq 3 + \frac{3t}{c} - \frac{6}{c}.$$

In particular if $c \geq 3t$ then we have $\ell(\mu) < 4$.

5.1.3 Slope length bounds, Dehn filling and volume

Let K be a hyperbolic knot with maximal cusp C and slopes σ, σ' on ∂C. Calculating area in Euclidean geometry on ∂C (see for example the proof of [10, Theorem 8.1]), we have

$$\ell(\sigma)\ell(\sigma') \geq \mathrm{Area}(\partial C)\Delta(\sigma, \sigma') \tag{5.2}$$

where $\Delta(\sigma, \sigma')$ denotes the absolute value of the intersection number of σ and σ'. Work of

Cao and Meyerhoff [16, Proposition 5.8] shows that Area(∂C) ≥ 3.35. Given an adequate hyperbolic knot K, we will apply (5.2) for $\sigma' = \mu$. Using the upper bound for $\ell(\mu)$ from Theorem 5.3.3, we have

$$\ell(\sigma) > \frac{3.35\Delta(\mu,\sigma)c}{3c + 6g_T - 6} = \frac{3.35}{3} \cdot \frac{\Delta(\mu,\sigma)}{1+\delta} \tag{5.3}$$

where $\delta = \dfrac{2g_T - 2}{c}$. Note that δ is an invariant of K that can be calculated from any adequate diagram (see Theorem 5.2.4). Now (5.3) implies that if

$$\Delta(\mu,\sigma) > \frac{18}{3.35}\,(1+\delta) > 5.37\,(1+\delta)$$

then $\ell(\sigma) > 6$ and thus σ cannot be an exceptional slope.

Note that if σ is a slope represented by $p/q \in \mathbb{Q}$ in $H_1(\partial C)$ then $\Delta(\mu,\sigma) = |q|$. Hence if $|q| > 6(1+\delta)$, inequality (5.3) implies that $\ell(\sigma) > \dfrac{3.35}{3} \cdot 6 > 2\pi$. In this case, we may apply a result of Futer, Kalfagianni and Purcell [31, Theorem 1.1] to estimate the change of volume under Dehn filling of adequate knots. This produces the following theorem.

Theorem 5.1.3. *Let K be a hyperbolic adequate knot and let δ be as above. If $|q| \geq 6(1+\delta)$, then the 3-manifold N obtained by p/q surgery along K is hyperbolic and the volume satisfies the following*

$$\mathrm{vol}(S^3 \setminus K) \;>\; \mathrm{vol}(N) \;\geq\; \left(1 - \frac{36(1+\delta)^2}{q^2}\right)^{3/2} \mathrm{vol}(S^3 \setminus K).$$

The assertion that N is hyperbolic follows immediately from the discussion above. The left hand side inequality is due to the result of Thurston that the hyperbolic volume drops under Dehn filling (Theorem 2.1.2). The right hand side follows by [31, Theorem 1.1].

Theorem 5.14 of [30], and its corollaries, give diagrammatic bounds for $\mathrm{vol}(S^3 \setminus K)$ in terms any adequate diagram of K. This combined with Theorem 5.1.3 implies that the volume of N can be estimated from any adequate diagram of K. For example, Montesinos knots with a reduced diagrams that contains at least two positive tangles and at least two negative tangles are adequate and have $\delta \leq 0$. Combining Theorem 5.1.3 with [30, Theorem 9.12] and [29, Theorem 1.2] we have the following.

Corollary 5.1.4. *Let $K \subset S^3$ be a Montesinos link with a reduced diagram $D(K)$ that contains at least two positive tangles and at least two negative tangles. If $|q| \geq 6$, then the 3-manifold N obtained by p/q surgery along K is hyperbolic and we have*

$$2v_8\, t \;>\; \mathrm{vol}(N) \;\geq\; \left(1 - \frac{36}{q^2}\right)^{3/2} \frac{v_8}{4}\,(t-9)\,,$$

where $t = t(D)$ is the twist number of $D(K)$, and $v_8 = 3.6638...$ is the volume of a regular ideal octahedron.

5.1.4 Organization

In Section 5.2.1 we recall the hyperbolic geometry terminology we need for this chapter, and the results and facts about pleated surfaces we will use. In Section 5.2.2 we recall results and terminology about Turaev surfaces we need in subsequent sections. In Section 5.3 we derive the bound of the meridian length in Theorem 5.1.1 and corresponding bounds for the length of the shortest λ-curve and cusp volume. Then we prove Theorem 5.3.3 and its corollaries. In Section 5.4 we show that given K and $b > 0$ there is an algorithm which determines if there are essential spanning surfaces S_1 and S_2 satisfying inequality of Theorem 2.1.2 . This completes the proof of Theorem 5.1.1.

The material of this chapter was developed jointly with the author and Efstratia Kalfagianni.

5.2 Background

In this section we will review results that will be used throughout the chapter. See Section 2.1.2 for a review of the notions of maximal cusp and slope length.

5.2.1 Pleated Surfaces

Consider a (possibly non-connected) surface S (possibly with boundary) and a singular continuous map $f : S \longrightarrow M$ that embeds each component of ∂S in ∂C. We will say that f is *homotopically-essential*

1. The image of no essential simple closed loop on S is homotopically trivial in M.

2. The image of no essential embedded arc on S can be homotoped (relative to its endpoints) on ∂C.

If $S \subset M$ is an essential embedded surface, the inclusion map is homotopically-essential. We now recall Thurston's notion of *pleated surface*. See Thurston's notes [65] or the exposition by Canary, Epstein and Green [15] for more details. A singular continuous map $f : (S, \partial S) \longrightarrow (M, \partial C)$ is called *pleated* if the following are true:

1. The components of ∂S map to geodesics on ∂C.

2. The interior of S, denoted by $\mathrm{int}(S)$, is triangulated so that each triangle maps under f to a subset of M that lifts to an ideal hyperbolic geodesic triangle in $\mathbb{H}^3$.

3. The 1-skeleton of the triangulation forms a lamination on S.

Given a pleated map f we may pull-back the path metric from M by f to obtain a hyperbolic metric on $\mathrm{int}(S)$, where the 1-skeleton lamination is geodesic.

The following lemma will be used to make the essential spanning surfaces considered in this chapter pleated. For a proof the reader is referred to [15, 65] or to [10, Lemma 4.1].

Lemma 5.2.1. *Let $M = S^3 \setminus K$ be a hyperbolic knot complement and let S be a surface with boundary and $\chi(S) < 0$. Let $f : (S,\ \partial S) \longrightarrow (M, \partial C)$ be a homotopically essential map. Suppose that each component of ∂S is mapped to a geodesic in ∂C. Then there is a pleated map $g : (S,\ \partial S) \longrightarrow (M, \partial C)$, such that $g|\mathrm{int}(S)$ is homotopic to $f|\mathrm{int}(S)$ and there exists a hyperbolic metric on S so that $g|\partial S$ is an isometry.*

Let $M = S^3 \setminus K$ be a hyperbolic knot complement with maximal cusp C and let $f : (S,\ \partial S) \longrightarrow (M, \partial C)$ be a homotopically essential map that is pleated. In this paper we are interested in the case that S is the disjoint union of spanning surfaces of K. Suppose that ∂S has s components. The geometry of $f(S) \cap C$ can be understood using arguments of [10, Theorem 5.1] and [43, Lemma 3.3]. By the argument in the proof of [10, Theorem 5.1], we can find disjoint horocusp neighborhoods $H = \bigcup_{i=1}^{s} H_i$ of S, such that $f(H_i) \subset C$, $\ell(\partial H_i) = \mathrm{Area}(H_i)$ and such that $\ell(\partial H_i)$ is at least as big as the length of $f(\partial H_i)$ measured on C. Thus we have

$$\ell_C(S) \leq \sum_{i=1}^{s} \ell(\partial H_i) = \mathrm{Area}(H),$$

where $\ell_C(S)$ denotes the total length of the intersection curves in $f(S) \cap \partial C$. Since for all $i \neq j$, we have $H_i \cap H_j \neq \emptyset$, a result of Böröczky [13] on horocycle packings in the hyperbolic plane applies. Using this result one obtains

$$\sum_{i=1}^{s} \mathrm{Area}(H_i) \leq \frac{6}{2\pi}\mathrm{Area}(S) = \frac{6}{2\pi}(2\pi|\chi(S)|),$$

where the last equation follows by the Gauss-Bonnet theorem. The above inequality is also proved in [43, Lemma 3.3]. Combining all these leads to the following theorem which is a special case of [10, Theorem 5.1] and [43, Lemma 3.3].

Theorem 5.2.2. *Let $M = S^3 \setminus K$ be a hyperbolic knot complement with maximal cusp C. Suppose that $f : (S,\ \partial S) \longrightarrow (M, \partial C)$ is a homotopically essential map that is pleated and let $\ell_C(S)$ denote the total length of the intersection curves in $f(S) \cap \partial C$. Then we have*

$$\ell_C(S) \ \leq \ 6|\chi(S)|.$$

5.2.2 Turaev Surfaces

The *Turaev genus* of a knot diagram $D = D(K)$ with c crossings is defined by $g_T(D) = (2 - v_A - v_B + c)/2$, where v_A, v_B denotes the number of the state circles in the all-A and all-B resolutions of D respectively. The Turaev genus of a knot K is defined by

$$g_T(K) = \min\left\{g_T(D) \mid D = D(K)\right\}.$$

The genus $g_T(D)$ is the genus of the Turaev surface $F(D)$ corresponding to D. This surface is constructed as follows. Let $\Gamma \subset S^2$ be the planar, 4–valent graph defined by D. Thicken the (compactified) projection plane to $S^2 \times [-1, 1]$, so that Γ lies in $S^2 \times \{0\}$. Outside a neighborhood of the vertices (crossings), $\Gamma \times [-1, 1]$ will be part of $F(D)$.

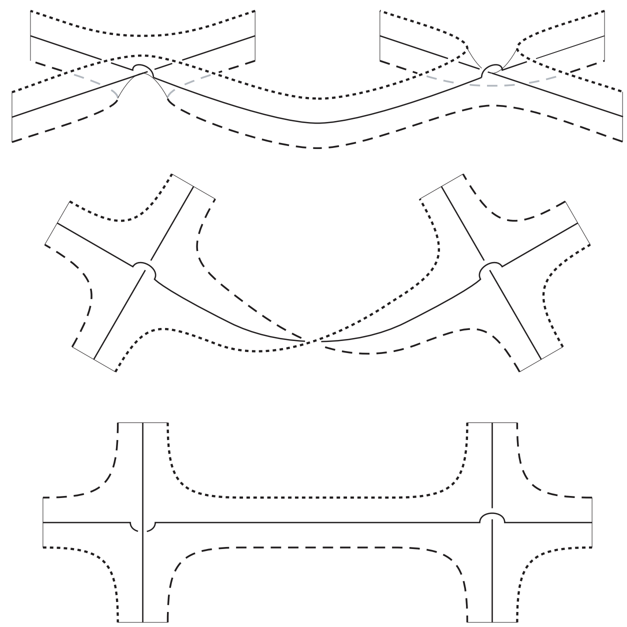

Figure 5.1: Saddles of $F(D)$ corresponding to two successive over-crossings of D. The third picture illustrates how D is is alternating on $F(D)$. The figure is taken from [24].

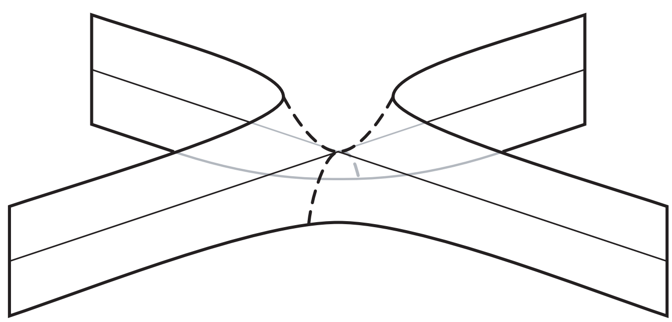

Figure 5.2: The saddle of the Turaev surface at a crossing of a link. Taken from [33].

In the neighborhood of each vertex, we insert a saddle, positioned so that the boundary circles on $S^2 \times \{1\}$ are the components of the A–resolution and the boundary circles on $S^2 \times \{-1\}$ are the components of the B–resolution. See Figure 5.2.

The following is proved in [24].

Lemma 5.2.3. *The Turaev surface $F(D)$ has the following properties:*

(i) It is a Heegaard surface of S^3.

(ii) D is alternating on $F(D)$; in particular D is an alternating diagram if and only if

$g_T(F(D)) = 0$. See Figure 5.1.

(iii) The 4-valent graph underlying D defines a cellulation of $F(D)$ for which the 2-cells can be colored in a checkerboard fashion.

(iv) The checkerboard surfaces defined by D on $F(D)$ are the state surfaces S_A and S_B.

We note that an adequate diagram realizes the crossing number of the knot, hence it is a knot invariant. The following result of Abe [1, Theorem 3.2] shows that the same is true for the Turaev genus.

Theorem 5.2.4. *Suppose that D is an adequate diagram of a knot K. Then,*

$$2g_T(K) = 2g_T(D) = 2 - v_A(D) - v_B(D) + c(D).$$

□

5.3 Lengths of Curves on the Maximal Cusp Boundary

In this section, we prove the main results of this paper. We begin by giving a general bound for lengths of curves in the boundary of a maximal cusp neighborhood of a hyperbolic knot. We then apply this bound to the special cases of adequate knots and three-string pretzel knots.

Theorem 5.3.1. *Let K be a hyperbolic knot with maximal cusp C. Suppose that S_1 and S_2 are essential spanning surfaces in $M = S^3 \setminus K$ and let $i(\partial S_1, \partial S_2) \neq 0$ denote the minimal intersection number of $\partial S_1, \partial S_2$ in ∂C. Let $\ell(\mu)$ and $\ell(\lambda)$ denote the length of the meridian and the shortest λ-curve of K, respectively. Then we have:*

1. $\ell(\mu) \leq \dfrac{6(|\chi(S_1)| + |\chi(S_2)|)}{i(\partial S_1, \partial S_2)}$

2. $\ell(\lambda) \leq 3(|\chi(S_1)| + |\chi(S_2)|)$

3. $\text{Area}(\partial C) \leq 18\dfrac{(|\chi(S_1)| + |\chi(S_2)|)^2}{i(\partial S_1, \partial S_2)}$

Proof. Consider S to be the disjoint union of S_1, S_2, and let $f : S \longrightarrow M$, where $f(S)$ is the union of S_1, S_2 in the complement of K. Since $f|S_i$ is an embedding for $i = 1, 2$, and each S_i is essential, f is a homotopically essential map. Hence, by Lemma 5.2.1, we may pleat f and then apply Theorem 5.2.2. With the notation as in that theorem we have

$$\ell_C(S) \ \leq \ 6|\chi(S)|,$$

where $\ell_C(S)$ is the total length of the curves $f(S) \cap \partial C$.

To find bounds of this total length, we orient $\partial S_1, \partial S_2$ and μ so that $\partial S_1, \partial S_2$ have opposite algebraic intersection numbers with μ. Let $[\partial S_1], [\partial S_2]$, and $[\mu]$ denote their classes in $\pi_1(\partial C) = H_1(\partial C)$. Since S_1 is a spanning surface, we know that $[\partial S_1]$ and $[\mu]$ generate $\pi_1(\partial C)$.

Recall the covering $\pi := \rho|R_H : R_H \longrightarrow \partial C$, where R_H is the boundary of a horoball at infinity, say $H \subset \cup\rho^{-1}(C)$. To fix ideas, assume that ∂S_1 lifts to the horizontal lines $\pi^{-1}(\partial S_1) = \{(x, n) : x \in \mathbb{R}\}$ for each $n \in \mathbb{Z}$ and where μ lifts to the vertical lines $\pi^{-1}(\mu) = \{(n, y) : y \in \mathbb{R}\}$ for each $n \in \mathbb{Z}$. We may apply a homotopy to μ so that $\partial S_1 \cap \partial S_2 \cap \mu = \{x_0\}$, where $\pi^{-1}(x_0) = \mathbb{Z}^2$.

Since $[\partial S_1]$ and $[\mu]$ generate $\pi_1(\partial C)$, we can write $[\partial S_2] = \alpha[\mu] + \beta[\partial S_1]$ for some $\alpha, \beta \in \mathbb{Z}$. The fact that S_2 is a spanning surface implies $|\beta| = 1$ and $|\alpha| = i(\partial S_1, \partial S_2)$. Therefore $[\partial S_2]$ can be represented as a curve which lifts to the segment $\{(x, \alpha x) : x \in [0, 1]\} \subset \mathbb{R}^2 = R_H$.

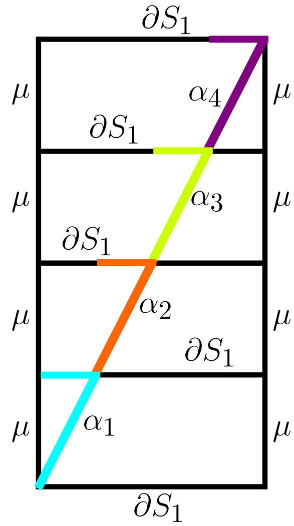

Figure 5.3: The arcs α_k are each homotopic to the meridian, and their union projects to $\partial S_1 \cup \partial S_2$.

The collection of arcs

$$\alpha_k = \{(x, \alpha x) : x \in [k/\alpha, (k+1)/\alpha]\} \cup \{(x, k+1) : x \in [k/\alpha, (k+1)/\alpha]\}$$

for $k = 0, 1, \ldots, \alpha - 1$ is mapped to $\partial S_1 \cup \partial S_2$ by π. Moreover, each $\pi(\alpha_k)$ is a loop in ∂C homotopic to a meridian. See Figure 5.3, where each α_k is indicated in a different color. Therefore $\partial S_1 \cup \partial S_2$ can be decomposed into a collection of simple closed curves that contain $|\alpha|$ meridians. Hence we obtain

$$i(\partial S_1, \partial S_2)\ell(\mu) \leq \ell_C(S) \leq 6|\chi(S_1)| + 6|\chi(S_2)|.$$

The decomposition of $\partial S_1 \cup \partial S_2$ described above can be also seen by resolving all the intersections of $\partial S_1, \partial S_2$ in a way consistent with the orientations chosen above.

To prove part (2), once consider ∂S_1 and ∂S_2 oriented as above in ∂C. By resolving the crossings of ∂S_1 with ∂S_2 in a manner not consistent consistent with the orientations of

∂S_1 and ∂S_2, one obtains two ℓ-curves in ∂C. Thus $2\ell(\lambda) \leq \ell_C(S)$ and Theorem 5.2.2 now implies that

$$2\ell(\lambda) < 6|\chi(S_1)| + 6|\chi(S_2)|.$$

To prove part (3), observe that $\text{Area}(\partial C) \leq \ell(\mu)\ell(\lambda)$. □

As an example, we apply Theorem 5.3.1 to 3-string pretzel knots. Note that non-alternating 3-string pretzel knots are not adequate as it follows from the work of Lee and van der Veen [46].

Example 5.3.2. Let K be the pretzel knot $P(a, -b, -c)$ with a, b, c all positive and odd. The standard 3-pretzel diagram of K is A-adequate. Hence the corresponding all-A state surface S_A is essential in the complement of K. Moreover, the 3-pretzel surface S_P is a minimum genus Seifert surface for K and thus also essential. The boundary slope of the spanning surface S_A of K is given by $s(S) = 2c_+^B - 2c_-^A$ where c_+^B is the number of positive crossings resolved in the B direction. We then see that $s(S_A) = -2b - 2c$. On the other hand, $s(S_P) = 0$. The difference in slopes of two surfaces is equal to the geometric intersection number, so we obtain that $i(\partial S_A, \partial S_P) = 2b + 2c$. An easy calculation shows that $\chi(S_A) = 1 - b - c$ and $\chi(S_P) = -1$. Using Theorem 5.3.1 we have $\ell(\mu) \leq 3$.

The same process will apply to any knot that admits an essential state surface that has non-zero slope. Large familes of such knots are the semi-adequate knots or more generally the σ-adequate and σ-homogeneous knots [30, Definition 2.22].

We now consider an application of Theorem 5.3.1 to the case of adequate knots, and we derive Theorem 5.3.3 stated in the introduction. For the convenience of the reader, we restate the theorem.

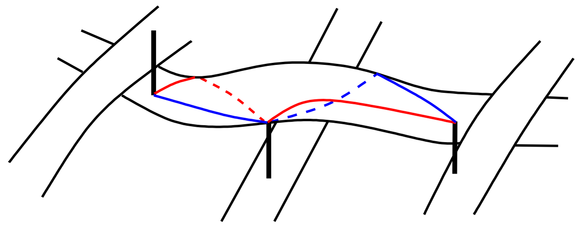

Figure 5.4: The intersection of the surfaces S_A (red) and S_B (blue) with ∂C. Taken from [45].

Theorem 5.3.3. *Let K be an adequate hyperbolic knot in S^3 with crossing number $c = c(K)$ and Turaev genus g_T. Let C denote the maximal cusp of $S^3 \setminus K$ and let* $\mathrm{Area}(\partial C)$ *denote the cusp area. Finally let $\ell(\mu)$ and $\ell(\lambda)$ denote the length of the meridian and the shortest λ-curve of K. Then we have*

1. $\ell(\mu) \leq 3 + \dfrac{6g_T - 6}{c}$
2. $\ell(\lambda) \leq 3c + 6g_T - 6$
3. $\mathrm{Area}(\partial C) \leq 9c\left(1 + \dfrac{2g_T - 2}{c}\right)^2$

Proof. Let D be an adequate diagram for K and let S_A and S_B be the corresponding all-A and all-B state surfaces respectively. By Theorem 2.2.2, S_A, S_B are essential in $M = S^3 \setminus K$. Now ∂S_A and ∂S_B intersect transversely exactly twice per crossing in D. We show that this number of intersections is in fact minimal. To do so, we use the well-known "bigon criterion" (see for example [28, Proposition 1.7]) which states that two transverse simple closed curves in a surface are in minimal position if and only if they do not form a bigon.

Consider the curves ∂S_A and ∂S_B near two consecutive crossings of D. If one crossing is an over-crossing and the other crossing is an under-crossing in the diagram D, then the intersection curves will be as in Figure 5.4. Note that this forms a diamond pattern on ∂C near alternating crossings, hence there are no bigons near alternating crossings.

Consider the Turaev surface $F(D)$ corresponding to D. Recall that D is alternating on $F(D)$ and that S_A, S_B are the checkerboard surfaces of this projection (Lemma 5.2.3).

We turn to the case where two consecutive crossings in D are over-crossings. The Turaev surface T of K in a neighborhood of these two crossings may be visualized as in Figure 5.1. The neighborhood may be straightened as shown in Figure 5.1, and we then see that the intersection of ∂C with $S_A \cup S_B$ in a neighborhood of these two crossings is as in Figure 5.4. Therefore we get an intersection pattern similar to that of 5.4 near pairs of consecutive over-crossings, and it follows that there are no bigons near pairs of over-crossings. Similarly there are no bigons near pairs of under-crossings. Thus we have $i(\partial S_1, \partial S_2) = 2c$.

On the other hand, by construction of the state surface and using the notation of §3.2, we have $\chi(S_A) = v_A - c$ and $\chi(S_B) = v_B - c$. Note that if $\chi(S_A) = 0$ or $\chi(S_B) = 0$ then S_A or S_B is a Möbius band. But then D is a diagram of the $(2, p)$ torus knot contradicting the assumption that K is hyperbolic. Thus $\chi(S_A), \chi(S_B) < 0$. Now by the definition of $g_D(T)$ and Theorem 5.2.4 we have

$$|\chi(S_A)| + |\chi(S_B)| = 2c - v_A - v_B = c + 2g_T - 2.$$

Using these observations, claims (1)-(3) of the statement follow immediately from Theorem 5.3.1. □

An immediate consequence of Theorem 5.3.3 is that the meridian length of a knot with Turaev genus 1 never exceeds 3. Also as noted in Corollary 5.1.2 for every Turaev genus there can be at most finitely many adequate knots where $\ell(\mu) \geq 4$.

Theorem 5.3.3 gives a bound on meridian lengths in terms of the Turaev genus and crossing number. We wish to understand the relationship between these invariants, so we

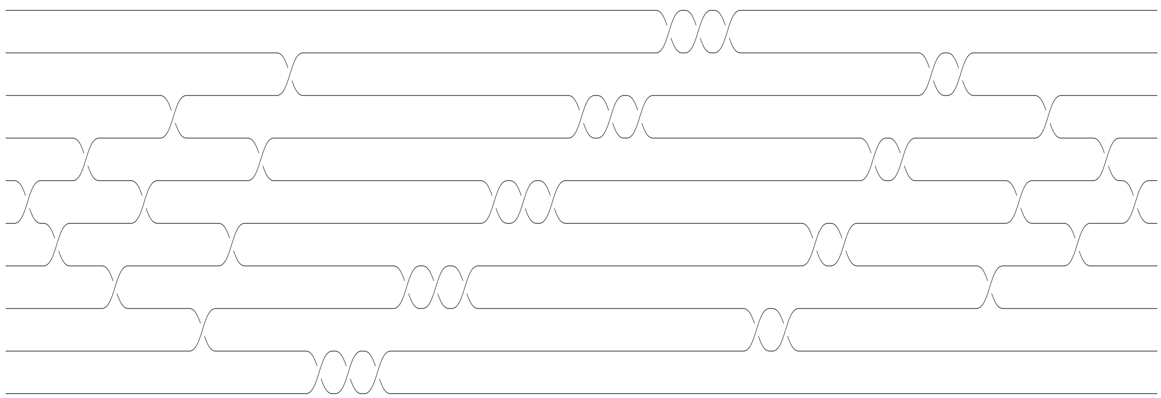

Figure 5.5: The braid of example 5.3.5 in the case $n = 5$.

make the following definition.

Definition 5.3.4. The *Turaev genus density* of a link K is defined to be $\dfrac{2g_T - 2}{c}$ where g_T is the Turaev genus of K and c is the crossing number of K.

We know that there are many families of knots with meridian lengths approaching 4 from below. However the bound of Theorem 5.3.3 does not require that the meridian length of all knots be less than 4, as shown in the following example.

Example 5.3.5. Let $n \geq 2$ be an integer and consider the following braid on $2n$ strings:

$$\begin{aligned} b_n = & \sigma_n \sigma_{n-1} \sigma_{n+1} \sigma_{n-2} \sigma_n \sigma_{n+2} \dots \sigma_2 \sigma_4 \dots \sigma_{2n-4} \sigma_{2n-2} \sigma_1^3 \sigma_3^3 \dots \sigma_{2n-3}^3 \sigma_{2n-1}^3 \sigma_2^2 \sigma_4^2 \\ & \dots \sigma_{2n-4}^2 \sigma_{2n-2}^2 \sigma_3 \sigma_5 \dots \sigma_{2n-5} \sigma_{2n-3} \dots \sigma_{n-1} \sigma_{n+1} \sigma_n \end{aligned}$$

See Figure 5.5.

One can readily verify that the closure K_n of b_n is an adequate knot. The A-resolution has $2n$ state circles. The B-resolution has $(3n - 1)$ state circles coming from bigon regions, and n circles coming from non-bigon regions. Thus $v_A(K_n) + v_B(K_n) = 6n - 1$. The number of crossings in the braid diagram for K_n is equal to $n^2 + 3n - 1$, and since this diagram is adequate, we know that the crossing number $c(K_n)$ is equal to $n^2 + 3n - 1$. Therefore

$$\lim_{n\to\infty} \frac{v_A(K_n) + v_B(K_n)}{c(K_n)} = \lim_{n\to\infty} \frac{6n-1}{n^2+3n-1} = 0 \tag{5.4}$$

Let $g(K_n)$ be the Turaev genus of K_n. Since the braid diagram is adequate, we know that the Turaev genus of the braid diagram is equal to the Turaev genus of K_n by [1]. Therefore $2g(K_n) = c(K_n) + 2 - v_A(K_n) - v_B(K_n)$ implying

$$\frac{2g(K_n) - 2}{c(K_n)} = 1 - \frac{v_A(K_n) + v_B(K_n)}{c(K_n)} \tag{5.5}$$

Therefore the meridian bound of Theorem 5.3.3 is

$$3 + 3\left(\frac{2g(K_n) - 2}{c(K_n)}\right) = 3 + 3\left(1 - \frac{v_A(K_n) + v_B(K_n)}{c(K_n)}\right) \tag{5.6}$$

which by (5.4) approaches 6 as n approaches ∞.

On the other hand, the next result, stated in the introduction, shows that in a certain sense "most" adequate hyperbolic knots have meridian length less than 4.

Theorem 5.3.6. *Let K be a hyperbolic knot with an adequate diagram with c crossings and t twist regions. Then we have*

$$\ell(\mu) \leq 3 + \frac{3t}{c} - \frac{6}{c}.$$

In particular if $c \geq 3t$ then we have $\ell(\mu) < 4$.

Proof. Let g_T be the Turaev genus of K and let v_A and v_B be the number of A and B state circles arising from D. Recall that $2g_T - 2 = c - v_A - v_B$. Now $v_A + v_B = v_{bi} + v_{nb}$ where

v_{bi} is the number of bigon regions in D and v_{nb} is the number of non-bigon regions. Then

$$c - v_{bi} = t \tag{5.7}$$

Since D is adequate and hyperbolic, both the A and B resolutions must have a state circle corresponding to a non-bigon region. For if all the regions in one of the resolutions are bigons then D represents a $(2, p)$ torus knots, which is not hyperbolic. Therefore $v_{nb} \geq 2$ and it follows that

$$2g_T - 2 = c - v_{bi} - v_{nb} = t - v_{nb} \leq t - 2$$

Now by Theorem 5.3.3 we see that

$$\ell(\mu) < 3 + 3\left(\frac{2g_T - 2}{c}\right) \leq 3 + 3\left(\frac{t-2}{c}\right) \leq 3 + \frac{3t}{c} - \frac{6}{c}.$$

Now if $c \geq 3t$, say for example if D has at least three crossings per twist region, then $3t/c \leq 1$, so we see that

$$\ell(\mu) < 3 + 1 - \frac{6}{c} < 4.$$

□

Theorem 5.3.6 applies to positive/negative closed braids. Let B_n be the braid group on n strands, with $n \geq 3$, and let $\sigma_1, \ldots, \sigma_{n-1}$ be the elementary braid generators. Let $b = \sigma_{i_1}^{r_1}\sigma_{i_2}^{r_2}\cdots\sigma_{i_k}^{r_k}$ be a braid in B_n. It is straightforward to check that if either $r_j \geq 2$ for all j, or else $r_j \leq -2$ for all j, then the braid closure D_b of b is an adequate diagram. In particular we have the following.

Corollary 5.3.7. *Suppose that a knot K is represented by a braid closure D_b such that either $r_j \geq 3$ for all j, or else $r_j \leq -3$ for all j. Additionally, suppose D_b is a prime diagram. Then K is hyperbolic and the meridian length satisfies $\ell(\mu) < 4$.*

Proof. The fact that K is hyperbolic follows by [34, Corollary 1.2] and the claim about the meridian follows from Theorem 5.3.6.

□

The twist number of any diagram of a hyperbolic knot K bounds Area(∂C) from above. More precisely, if a hyperbolic knot with maximal cusp C admits a diagram with t twist regions then Area$(\partial C) \leq 10\sqrt{3} \cdot (t-1) \approx 17.32 \cdot (t-1)$. The derivation of this bound is explained for example in [6]. Note that if $c >> t$, this general bound does better than the one of Theorem 5.3.3. On the other hand if $c = t$ and g_T is small the upper bound of Theorem 5.3.3 is sharper than the general bound. For instance if $g_T \leq 1$ and $c = t$, then Theorem 5.3.3 gives Area$(\partial C) \leq 9t$ which for $t \geq 3$ is sharper than the general bound.

Theorem 5.3.1 more generally applies to knots that admit alternating projections on surfaces so that they define essential checkerboard surfaces. Specifically, let F be closed surface that is embedded in S^3 in a standard or non-standard way. Let K be a knot and suppose that there is a projection $p : S^3 \longrightarrow F$ such that: (i) $p(K)$ is alternating and it separates F; (ii) the components of $F \setminus p(K)$ are disks that can be colored in two different colors so that the colors at each crossing of $p(K)$ meet in a checkerboard fashion; and (iii) the surface $F \setminus p(K)$ is essential in $S^3 \setminus K$. For instance results similar to Theorem 5.3.3 and Corollary 5.1.2 should also hold for *weakly alternating knots* considered by Ozawa [58] and further discussed in [38]. In this case one should replace g_T with the genus of the surface F and the crossing number of the knot with the number of crossings of the alternating projection on F.

5.4 Algorithm

In this section we will finish the proof of Theorem 5.1.1. The proof of the first part of the Theorem follows from part (a) of Theorem 5.3.1. That is, if a hyperbolic knot K in S^3 admits essential spanning surfaces S_1, S_2 such that

$$|\chi(S_1)| + |\chi(S_2)| < \frac{b \cdot i(\partial S_1, \partial S_2)}{6}, \tag{5.8}$$

for some real number $b > 0$, then

$$\ell(\mu) < \frac{6(|\chi(S_1)| + |\chi(S_2)|)}{i(\partial S_1, \partial S_2)} < b.$$

The proof of Theorem 5.1.1 will be complete once we show the following.

Theorem 5.4.1. *Given any hyperbolic knot K and positive real number b, there is an algorithm which determines if there are spanning surfaces S_1 and S_2 satisfying inequality (5.8).*

Proof. We now show that the condition of equation (5.8) is algorithmically checkable. Start with a triangulation of the complement $M = S^3 \setminus K$. There is an algorithm [40] to turn the triangulation to one that has a single vertex that lies on the boundary of M. Moreover, by Jaco and Sedgwick [39] there is an algorithm that "layers" this triangulation so that a meridian of K is a single edge on ∂M that is connected to the vertex of the triangulation. Call the latter triangulation $\mathcal{T}$. For normal surface background and terminology the reader is referred to Matveev [50] or the introduction of [39].

Lemma 5.4.2. *Suppose that there are essential spanning surfaces S_1, S_2 that satisfy (5.8). Then we can find essential spanning surfaces that satisfy condition (5.8) and, in addition, are normal fundamental surfaces with respect to $\mathcal{T}$.*

Proof. Suppose that one of S_1, S_2, say S_1 is not connected. Then since S_1 is a spanning surface, and hence has a single boundary component, one of the connected components must be a closed surface F. Since K is hyperbolic and F is essential $\chi(F) \leq 0$, so taking $S = S_1 \backslash F$ we see that $|\chi(S)| \leq |\chi(S_1)|$, and $i(\partial S, \partial S_2) = i(\partial S_1, \partial S_2)$. Replacing S_1 with S, we may assume S_1 (and likewise S_2) is connected.

Any essential surface in $S^3 \backslash K$ may be isotoped to a normal surface with respect to $\mathcal{T}$. Moreover, this normal surface may be be taken to be minimal in the sense of [50, Definition 4.1.6]. This means that the number of intersections of the surface with the edges of $\mathcal{T}$ is minimal in the (normal) isotopy class of the surface. We will show that S_1 and S_2 may be taken to be *fundamental* normal surfaces.

Suppose that S_1 is not fundamental. Then S_1 can be represented as a *Haken sum* $S_1 = \Sigma_1 \oplus \ldots \oplus \Sigma_n \oplus F_1 \oplus \ldots \oplus F_k$ where each Σ_i is a fundamental normal surface with boundary, and each F_i is a closed fundamental normal surface. A theorem of Jaco and Sedgwick [39] states that each Σ_i has the same slope. Since S_1 is a spanning surface, and hence it has a single boundary component, this implies that $n = 1$. Since K is hyperbolic, we know that either $\chi(F_i) < 0$ or F_i is a boundary parallel torus for all i. In the latter case, it is known, as noted in [37] that $\Sigma_1 \oplus F_i$ is isotopic in $S^3 \backslash N(K)$ to Σ_1. In the event that $\chi(F_i) < 0$, we note that $|\chi(\Sigma_1)| < |\chi(S_1)|$ and equation (5.8) will hold with S_1 replaced by Σ_1. Moreover Matveev [50, Corollary 4.1.37] shows that Σ_1 must be incompressible. Therefore we can ignore the other terms of the Haken sum and assume that S_1 is fundamental.

Similarly, we can assume that S_2 is fundamental. $\square$

By Lemma 5.4.2, in order to decide whether there are spanning surfaces that satisfy (5.8), it is enough to decide whether there are fundamental normal spanning surfaces with the same property. Given K, there are only finitely many fundamental surfaces in M, and there is an algorithm, due to Haken, to find them. Let $\mathcal{F}$ denote the list of all fundamental surfaces. Since one of the boundary edges of the triangulation is a meridian, we may create a subset $\mathcal{F}_{\text{Span}} \subset \mathcal{F}$ of fundamental normal spanning surfaces which are spanning by finding the surfaces that intersect the meridian exactly once. There is an algorithm to compute $\chi(F)$ for all surfaces $F \in \mathcal{F}$, and to compute the minimal intersection number of two fundamental normal surfaces [40]. The algorithm now works by computing $|\chi(S_1)| + |\chi(S_2)|$ and $i(\partial S_1, \partial S_2)$ for all pairs of surfaces $S_1, S_2 \in \mathcal{F}_{\text{Span}}$ and checking whether inequality (5.8) holds. If the condition holds, then use the algorithm of Haken to check that S_1 and S_2 are incompressible. If the condition fails for all pairs $S_1, S_2 \in \mathcal{F}_{\text{Span}}$, then inequality (5.8) does not hold for any pair of essential spanning surfaces of K.

Knots with pairs of essential spanning surfaces S_1, S_2 with $i(\partial S_1, \partial S_2) \neq 0$ are abundant. Note however that not all knots have distinct essential spanning surfaces S_1, S_2 for which $i(\partial S_1, \partial S_2) \neq 0$. An example of such a knot is given by Dunfield in [27]. In this case, the algorithm outlined above will return that inequality (5.8) cannot be satisfied. This may be seen as follows. In this case, either

1. the set $\mathcal{F}_{\text{Span}}$ contains only one member, in which case there are no pairs to test, or
2. the intersection number $i(\partial S_1, \partial S_2) = 0$ for all pairs $S_1, S_2 \in \mathcal{F}_{\text{Span}}$, and inequality (5.8) will always fail since K is hyperbolic implies $|\chi(S_1)| > 0$.

$\square$

APPENDIX

Appendix

Python Code for Algorithm 1 of Section 4.5

This is the Python code used to verify Conjecture 4.1.1 for alternating pretzel links with 4 twist regions.

```
# Number of twist regions
regions = 4
import numpy as np
import itertools


# Computes the determinant
# Input: List
# Output: Integer
def det(x):
  partial_sum = 0
  for a in x:
    partial_sum = partial_sum + np.prod(x)/a
  return partial_sum
```

```
# Computes a lower bound on determinant
# Input: List
# Output: integer
def partial_det(x):
  if len(x) == 1:
    return x[0]*(regions-1) + 1
  if len(x) == 2:
    return x[0]*x[1]*(regions - 2) + x[0] + x[1]
  else:
    return np.prod(x)*(regions - len(x)) + det(x)


# Computes an upper bound of the e^(volume of an x-bipyramid/(2pi))
def bipyramid_vol(x):
  if x == 2:
    return 1.0
  elif x == 3:
    return 1.382
  elif x == 4:
    return 1.792
  elif x == 5:
    return 2.212
  elif x == 6:
    return 2.636
  elif x == 7:
```

```
    return 3.063
  elif x == 8:
    return 3.491
  elif x == 9:
    return 3.921
  elif x == 10:
    return 4.351
  elif x == 11:
    return 4.781
  elif x == 12:
    return 5.211
  elif x == 13:
    return 5.643
  elif x == 14:
    return 6.074
  elif x < 20 and x > 14:
    return 8.664/20*x
  elif x == 20:
    return 8.664
  elif x > 20 and x < 100:
    return .43265*x
  elif x == 100:
    return 43.265
  elif x > 100 and x <= 1000000:
```

```
    return .44*x
  else:
    return x/2.0


#Some code I took from ShreevatsaR on Stack Overflow to permute more
#intelligently. It sorts a sorted list l in place and then returns
# false only when sorted in reverse
def next_permutation(l):
    #Changes a list to its next permutation, in place.
    #Returns true unless wrapped around so result is lexicographically
    #smaller.
    n = len(l)
    #Step 1: Find tail
    last = n-1 #tail is from ‘last‘ to end
    while last>0:
        if l[last-1] < l[last]: break
        last -= 1
    #Step 2: Increase the number just before tail
    if last>0:
        small = l[last-1]
        big = n-1
        while l[big] <= small: big -= 1
        l[last-1], l[big] = l[big], small
    #Step 3: Reverse tail
```

```
    i = last
    j = n-1
    while i < j:
        l[i], l[j] = l[j], l[i]
        i += 1
        j -= 1
    return last>0


# Computes the Adams volume estimate for all configurations of strings
# with lengths specified in a list x
# Input: x = list of string lengths
#  regions = number of strings (probably not really needed, oh well)
# Output: float volume = Adams volume upper bound for configuration
# with highest volume
def vol(x, regions):
  # We loop over all the permutations of x to find highest volume
  #configuration
  volume = 0
  #configurations = list(itertools.permutations(x))
  perm = x
  #for perm in configurations:
  stay_in_loop = True
  while stay_in_loop:
    v = []
```

```
    # First we add in all the polygons with twists on the sides
    v.append(perm[0]+perm[len(perm)-1])
    for n in range(0, len(perm)-1):
      v.append(perm[n]+perm[n + 1])
    # Next we add the top and bottom polygons
    v.append(regions)
    v.append(regions)
    # Take the product of all but the two largest polygons
    v = np.sort(v)
    w = []
    for poly in v:
      w.append(bipyramid_vol(poly))
    temp_volume = np.prod(w[0:len(w)-2])
    if temp_volume > volume:
      volume = temp_volume
    stay_in_loop = next_permutation(perm)
  return volume


def dt_vol(twists):
  x = np.array(twists)
  twist_counts = []
  for n in range(1, 5):
    twist_counts.append((x == n).sum())
  twist_counts.append((x>4).sum())
```

```
  if twist_counts[4] > 0:
    a = 12.111
  elif twist_counts[3] > 0:
    a = 10.2873
  elif twist_counts[2] > 0:
    a = 10.088
  elif twist_counts[1] > 0:
    a = 11*1.01494
  else:
    a = 15.4972
  volume = twist_counts[0]*3.6639+\
twist_counts[1]*6*1.01495+\
twist_counts[2]*7.8549+\
twist_counts[3]*9.2375+\
twist_counts[4]*10*1.01495-a
  volume = np.exp(volume/(2*np.pi))
  return volume


def agm_vol(twists):
  return np.mean(twists)**len(twists)


# Lackenby Agol-Thurston-Storm  volume bound
twist_bound = 5.04**(regions - 1)
```

```
# Maximum number of twists in a region
max_twist = int(regions + 5.0297**(regions - 1) - \
2*(1.4254)**(regions-1))+1

# Number of problem cases
problem = 0
problem_list = []

# Total Adams volume checks:
total_checks = 0

# Maximum volume found and the case where it arises
max_volume = 0
max_volume_twists = []

for n in range(0, regions):
  max_volume_twists.append(n)

twists = []
for x in range(1,max_twist+1):
  twists.append(x)
  determinant = partial_det(twists)
  if determinant > twist_bound:
    twists.pop()
```

```
  break
# If the current spot is a 5 or higher then the dt_vol of all the
#successive links
# can be computed and will be constant, while determinant will
#continue to increase
last_tw = twists[len(twists)-1]
rem_tw = regions - len(twists)
if last_tw > 4:
  partial_volume = dt_vol(twists+rem_tw*[5])
  partial_determinant = partial_det(twists + rem_tw*[last_tw])
  if partial_volume < partial_determinant:
    twists.pop()
    break
for x in range(twists[len(twists)-1],max_twist+1):
  twists.append(x)
  determinant = partial_det(twists)
  if determinant > twist_bound:
    twists.pop()
    break
  # If the current spot is a 5 or higher then the dt_vol of all the
  #successive links
  # can be computed and will be constant, while determinant will
  #continue to increase
  last_tw = twists[len(twists)-1]
```

```
rem_tw = regions - len(twists)
if last_tw > 4:
  partial_volume = dt_vol(twists+rem_tw*[5])
  partial_determinant = partial_det(twists + rem_tw*[last_tw])
  if partial_volume < partial_determinant:
    twists.pop()
    break
for x in range(twists[len(twists)-1],max_twist+1):
  twists.append(x)
  determinant = partial_det(twists)
  if determinant > twist_bound:
    twists.pop()
    break
  # If the current spot is a 5 or higher then the dt_vol of all the
  #successive links
  # can be computed and will be constant, while determinant will
  #continue to increase
  last_tw = twists[len(twists)-1]
  rem_tw = regions - len(twists)
  if last_tw > 4:
    partial_volume = dt_vol(twists+rem_tw*[5])
    partial_determinant = partial_det(twists + rem_tw*[last_tw])
    if partial_volume < partial_determinant:
      twists.pop()
```

```
      break
  prev = 0
  for x in range(twists[len(twists)-1],max_twist+1):
    twists.append(x)
    determinant = partial_det(twists)
    if determinant > twist_bound:
      twists.pop()
      break
    # If the current spot is a 5 or higher then the dt_vol of all
    #the successive links
    # can be computed and will be constant, while determinant will
    #continue to increase
    last_tw = twists[len(twists)-1]
    rem_tw = regions - len(twists)
    if last_tw > 4:
      partial_volume = dt_vol(twists+rem_tw*[5])
      partial_determinant = partial_det(twists + rem_tw*[last_tw])
      if partial_volume < partial_determinant:
        twists.pop()
        break
    else:
      #In this case det(twists) < twist_bound
      total_checks = total_checks+1
      agm_volume = agm_vol(twists)
```

```
          if agm_volume < determinant:
            twists.pop()
            break
          volume = vol(twists,regions)
          if volume > max_volume:
            max_volume = volume
            for n in range(0, regions):
              max_volume_twists[n] = twists[n]
          if volume > determinant:
            # In this case the algorithm has failed
            print "Algorithm failed."
            problem = problem + 1
            problem_twist = []
            for n in range(0, len(twists)):
              problem_twist.append(twists[n])
            problem_list.append([problem_twist, det(twists), volume])
        twists.pop()
      twists.pop()
    twists.pop()
  twists.pop()
print "Problem cases: ", problem
print "Cases checked: ", total_checks
print "Maximum volume:", max_volume
print "Max twist volume:", max_volume_twists
```

```
np.save("./problem_list"+str(regions)+".npy", problem_list)
for stuff in problem_list:
  print "Problem twist: ", stuff[0]
  print "det: ", stuff[1], "vol: ", stuff[2]
```

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BIBLIOGRAPHY

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