SET FUNCTIONS AND LOCAL CONNECTIVITY

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ABSTRACT

SET FUNCTIONS AND LOCAL CONNECTIVITY

By

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This is a study of the closure, Y, with respect to continua with connected interior. Chapter one develops the elementary properties of Y; Chapter two develops the basic relationships between Y and T, where T denotes the closure with respect to continua; Chapter three develops relationships between Y and monotone maps.

In Chapter one the usual hypothesis is that S is a compact Hausdorff space. The main theorems are:

S _is locally connected if and only if for_ $A \subset S$ $Y(A) = Cl(A)$.

S _is locally connected if and only if_

1. $Y(p) = \{p\}$ _for all_ $p \in S$ _and_
2. S _is Y-additive_.

_If_ S _is Y-symmetric, then_ S _is locally connected at_ p, _if and only if_ $Y(p) = \{p\}$.

_If_ C _is a subcontinuum of the continuum_ S, _then_ $Y(C)$ _is a continuum_.

In Chapter two S denotes a Hausdorff continuum. The main theorems are:

_If_ S _is weakly irreducible, then_ S _is locally connected at_ p, _if and only if_ S _is connected Im Kleinen at_ p;

moreover, if S *is also aposyndetic, then* S *is locally connected.*

S *is weakly irreducible if and only if for any subcontinuum,* W, *of* S, S - W *has a finite number of components.*

S *is locally connected if and only if* $S \times S$ *is* Y-*additive.*

If $A \subset \mathrm{Int}(B) \subset B \subset S$ *and* $T(B) = B$, *then* $Y(A) \subset B$. (Here S need not be connected.)

This last result generalizes the theorem which states: S is locally connected if and only if S is connected Im Kleinen.

In Chapter three S denotes a compact Hausdorff space, which need not be connected. The main theorems are:

Let f *be a monotone map of* S *onto* Z, *then* $Y(f^{-1}(A)) \subset f^{-1}(Y(A))$ *for all* $A \subset Z$.

Let f *be an open monotone map of* S *onto* Z, *then* $Y(A) = f(Y(f^{-1}(A))$ *for all* $A \subset Z$.

Let f *be an open monotone map of* S *onto* Z. *Then*

1. *If* S *is* Y-*additive, then* Z *is* Y-*additive*
2. *If* S *is* Y-*symmetric, then* Z *is* Y-*symmetric.*

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By

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CHAPTER I

ELEMENTARY PROPERTIES OF Y

<u>Definition</u>. Let S be a set and let $P(S)$ denote the collection of all subsets of S. Let S be a topological space and let $\eta \subset P(S)$. θ <u>is the closure with respect to</u> η if and only if $\theta: P(S) \to P(S)$ by the following rule: x is not an element of $\theta(A)$ if and only if there exist N and element of η such that x is an element of the interior of N and N and A are disjoint.

The following are directly verifiable.

<u>Formulas</u>. <u>Let</u> S <u>be a topological space</u>, $\eta \subset P(S)$ <u>and let</u> θ <u>be the closure with respect to</u> η, <u>then</u>

i. $A \subset \theta(A)$

ii. $\theta(A \cap B) \subset \theta(A) \cap \theta(B)$

iii. $\theta(A) \cup \theta(B) \subset \theta(A \cup B)$

iv. $\theta(S) = S$

v. $\theta(A)$ is closed

vi. If $A \subset B$, then $\theta(A) \subset \theta(B)$.

In [1] the set function T was defined and its basic properties were discussed. T is the closure with respect to the collection of continua.

This paper is a study of the closure with respect to continua with connected interiors.

Definition. Let S be a topological space and let W be a subset of S, then W is called a strong continuum if and and only if

1. W is a closed compact connected subset of S
2. The interior of W is connected.

The set-function under consideration is defined as follows:

Definition. Let S be a topological space, then Y is the closure with respect to the collection of strong continua.

Definition. Let S be a topological space and let p be an element of S, the $\mathcal{M}(p)$ is the set of all strong continua W in S such that p is an element of the interior of W.

Following are some immediate results for all topological spaces S.

Theorem 1. *Let $W \subset S$ be a strong continuum then the closure of the interior of W is a strong continuum of S.*

Theorem 2. *Let A be a subset of S, then*

$$Y(A) = \{x \mid W \in \mathcal{M}(x) \Rightarrow W \cap A \neq \emptyset\} \cup \{x \mid W(x) = \emptyset\} = \{x \mid W \in \mathcal{M}(x) \Rightarrow W \cap A \neq \emptyset\}.$$

A sequence of related set-functions can be defined as follows.

Definition. Let n be a positive integer and let A be a subset of a topological space S, then $Y^1(A) = Y(A)$ and $Y^{n+1}(A) = Y(Y^n(A))$.

Formulas for Y.

1. $A \subset Y(A)$
2. $Y(A \cap B) \subset Y(A) \cap Y(B)$
3. $Y(A) \cup Y(B) \subset Y(A \cup B)$
4. $Y(S) = S$

5. $Y(A)$ is closed

6. If $A \subset B$ then $Y(A) \subset Y(B)$

7. Let $m \le n$ then $Y^m(A) \subset Y^n(A)$.

The following example shows that $\mathcal{M}(p)$ can be empty and that the inequality in formula 1 may be proper for closed sets.

<u>Example</u> 1. Let $S = \{(\frac{1}{n},0) | n$ is a positive integer$\} \cup \{(0,0)\}$ with the relative plane topology, then $\mathcal{M}((0,0)) = \phi$ and $Y((\frac{1}{n},0)) = \{(\frac{1}{n},0), (0,0)\}$.

The following example shows that the inequality in formula 1 may be proper for a closed set when $\mathcal{M}(p) \neq \phi$ for all $p \in S$.

<u>Example</u> 2. Let S be the subcontinuum of the plane defined by the union of the closed line segments between $(0,1)$ and $(\frac{1}{n},0)$ for $n \ge 1$ and the closed line segment between $(0,0)$ and $(0,1)$; topologically this is the cone over example 1. In S, $Y((0,1))$ is the closed line segment between $(0,0)$ and $(0,1)$.

The following example shows that formula 3 need not be an equality.

<u>Example</u> 3. Let S be the subcontinuum of the plane defined by the union of closed line segments between $(0,1)$ and $(\frac{1}{n},0)$ for $n \ge 1$, the closed line segments between $(0,-1)$ and $(\frac{1}{n},0)$ for $n \ge 1$ and the closed line segment between $(0,1)$ and $(0,-1)$; this is topologically the suspension over example 1 with vertices $(0,1)$ and $(0,-1)$. In S, $Y((0,1)) = (0,1)$ and $Y((0,-1)) = (0,-1)$ but $Y(\{(0,1), (0,-1)\})$ is the

closed line segment between $(0,1)$ and $(0,-1)$.

The following example shows that $Y(A)$ need not be $Y^2(A)$ and similar examples can be found which have the property that for m and n distinct positive integers $Y^m(A)$ need not be $Y^n(A)$.

<u>Example</u> 4. Let $S = \{(x,\frac{1}{4} + \frac{1}{4}\sin(\frac{1}{x})) \mid 0 < x \leq 1\} \cup \{(x,-\frac{1}{4} + \frac{1}{4}\sin(\frac{1}{x})) \mid 0 < x \leq 1\} \cup \{(0,y) \mid -\frac{1}{2} \leq y \leq \frac{1}{2}\}$ with the topology induced by the plane, then $Y((0,\frac{1}{2})) = \{(0,y) \mid 0 \leq y \leq \frac{1}{2}\}$ and $Y^2((0,\frac{1}{2})) = \{(0,y) \mid -\frac{1}{2} \leq y \leq \frac{1}{2}\}$.

<u>Definition</u>. A space S is called Y-<u>additive</u> if and only if for any collection $\{A_\alpha\}$ of closed subsets whose union is closed $Y(\cup\{A_\alpha\}) = \cup\{Y(A_\alpha)\}$.

The space in example 3 is not Y-additive since $Y(\{(0,1)\}) \cup Y(\{(0,-1)\}) \neq Y(\{(0,1), (0,-1)\})$.

For the remainder of this chapter S will denote a compact Hausdorff space.

<u>Theorem</u> 3. $Y(\phi) = \phi$ <u>if and only if</u> S <u>has a finite number of components</u>.

Proof. Let S have a finite number of components. Then each component is both open and closed, hence $\mathcal{M}(p) \neq \phi$ for all $p \in S$ and thus if $p \in S$ then $p \notin Y(\phi)$.

Let $Y(\phi) = \phi$. Then $\mathcal{M}(p) \neq \phi$ for all $p \in S$, hence each component of S is open. Since S is compact and each component of S is open, S has only a finite number of components.

The following theorems show the relationships between the concept of locally connected and the set function Y.

<u>Theorem</u> 4. S <u>is locally connected at a point</u> p <u>if and only if for all subsets</u> A <u>of</u> S, <u>if</u> p <u>is an element of</u> Y(A), <u>then</u> p <u>is an element of the closure of</u> A.

Proof. Let S be locally connected at p and suppose p is not an element of the closure of A. There exists an open set U such that $p \in U$ and $\mathrm{Cl}(U) \cap A = \emptyset$. Since S is locally connected at p, there exist an open connected set V such that $p \in V \subset U$. $\mathrm{Cl}(V) \cap A = \emptyset$ and $\mathrm{Cl}(V) \in \mathcal{M}(p)$. Therefore p is not an element of Y(A) and it follows that if p is an element of Y(A), then p is an element of the closure of A.

Let p be an element of S such that for all $A \subset S$, if p is an element of Y(A), then p is an element of the closure of A. Let U be an open set containing p, then S - U is a closed set and p is not an element of S - U. There exist $W \in \mathcal{M}(p)$ such that $W \cap (S - U) = \emptyset$, hence $p \in \mathrm{Int}(W) \subset W \subset U$ and thus, S is locally connected at p. The theorem is proven.

<u>Corollary</u> 5. S <u>is locally connected if and only if for</u> $A \subset S$, $Y(A) = \mathrm{Cl}(A)$.

The next theorem shows the relation between Y-additivity and the locally connected spaces.

<u>Theorem</u> 6. S <u>is locally connected if and only if</u>

1. $Y(p) = \{p\}$ <u>for all</u> $p \in S$ <u>and</u>
2. S <u>is</u> Y-<u>additive</u>.

Proof. Let $Y(p) = \{p\}$ for all $p \in S$ and let S be Y-additive. Let $A \subset S$, then $\mathrm{Cl}(A) \subset Y(A) \subset Y(\mathrm{Cl}(A)) =$

$Y(\cup\{\{p\}|p \in Cl(A)\}) = \cup\{Y(p)|p \in Cl(A)\} = \cup\{\{p\}|p \in Cl(A)\} = Cl(A)$. Hence $Y(A) = Cl(A)$ and thus by corollary 5 S is locally connected.

Let S be locally connected and let $\{A\alpha\}$ be a set of closed sets such that $\cup\{A_\alpha\}$ is closed. By corollary 5 $Y(A_\alpha) = A_\alpha$ and $Y(\cup\{A_\alpha\}) = \cup\{A_\alpha\}$. Hence $\cup\{Y(A_\alpha)\} = \cup\{A_\alpha\} = Y(\cup\{A_\alpha\})$ and S is Y-additive. Since S is Hausdorff, for $p \in S$ $\{p\}$ is closed and by corollary 5 $Y(p) = \{p\}$. The theorem is proven.

The following two examples show that neither Y-additivity or $Y(p) = \{p\}$ for all $p \in S$ implies the other.

In example 3 S was not Y-additive, but $Y(p) = \{p\}$ for all p element of S (hence S is not locally connected).

In example 2 S was Y-additive, but $Y((0,1)) = \{(0,y)|0 \leq y \leq 1\}$ (hence S is not locally connected).

Definition. S is called Y-symmetric if and only if for any two closed subsets A and B of S, if $Y(A)$ is disjoint from B, then $Y(B)$ is disjoint from A.

Theorem 7. Let S be Y-symmetric, then S is Y-additive.

Proof. Let $\{A_\alpha\}$ be a set of closed sets such that $\cup\{A_\alpha\}$ is closed, then $\cup\{Y(A_\alpha)\} \subset Y(\cup\{A_\alpha\})$. Hence all that needs to be shown is that $Y(\cup\{A_\alpha\}) \subset \cup\{Y(A_\alpha)\}$. Let S be Y-symmetric. Let $p \in Y(\cup\{A_\alpha\})$, then $Y(\cup\{A_\alpha\}) \cap \{p\} \neq \phi$, hence $Y(p) \cap (\cup\{A_\alpha\}) \neq \phi$. Hence there exists β such that $Y(p) \cap A_\beta \neq \phi$. Therefore $\{p\} \cap Y(A_\beta) \neq \phi$, hence $p \in Y(A_\beta)$ and therefore $p \in \cup\{Y(A_\alpha)\}$. Therefore $Y(\cup\{A_\alpha\}) \subset \cup\{Y(A_\alpha)\}$ and the theorem is proven.

In example 2 S is Y-additive but S is not Y-symmetric since $Y((0,1)) = \{(0,y) | 0 \leq y \leq 1\} \supset \{(0,0)\}$ and $Y((0,0)) = \{(0,0)\}$.

<u>Theorem</u> 8. <u>Let</u> S <u>be</u> Y-<u>symmetric</u>, <u>then</u> S <u>is locally connected at</u> p, <u>if and only if</u> $Y(p) = \{p\}$.

Proof. Let S be Y-symmetric and locally connected at p. Let $q \in Y(p)$, then $Y(p) \cap \{q\} \neq \phi$, hence $Y(q) \cap \{p\} \neq \phi$, hence $p \in Y(q)$; since S is locally connected at p, $p \in Cl(\{q\}) = \{q\}$ and hence $p = q$. Therefore $\{p\} = Y(p)$.

Let S be Y-symmetric and $\{p\} = Y(p)$. Let U be an open set containing p. If $p \in Y(S - U)$, then $(S - U) \cap Y(p) \neq \phi$, but $Y(p) = \{p\}$ and hence $(S - U) \cap \{p\} \neq \phi$, a contradiction. Therefore $p \notin Y(S - U)$ and hence there exists $W \in \mathfrak{M}(p)$ such that $W \cap (S - U) = \phi$. Thus $p \in Int(W) \subset W \subset U$ and the theorem is proven.

Following the convention in [4] page 6, $\mathfrak{F}$ is called a filter-base in a topological space S if and only if

1. $\mathfrak{F} \subset P(S)$
2. $\mathfrak{F} \neq \phi$
3. $A, B \in \mathfrak{F}$ implies that there exist $C \in \mathfrak{F}$ such that C is a subset of A intersect B.

$\mathfrak{F}$ is said to be <u>proper</u> if and only if $\phi \notin \mathfrak{F}$.

$\mathfrak{F}$ is said to be <u>closed</u> if and only if $A \in \mathfrak{F}$ implies A is closed.

<u>Theorem</u> 9. <u>If</u> $\mathfrak{F}$ <u>is a proper closed filter-base in</u> S, <u>then</u> $Y(\bigcap\{A | A \in \mathfrak{F}\}) = \bigcap\{Y(A) | A \in \mathfrak{F}\}$.

Proof. Let $x \in Y(\cap\{A \mid A \in \mathfrak{F}\})$, then for all $W \in \mathcal{M}(x)$ $W \cap (\cap\{A \mid A \in \mathfrak{F}\}) \neq \phi$. Therefore for all $A \in \mathfrak{F}$ and all $W \in \mathcal{M}(x)$, $W \cap A \neq \phi$, hence $x \in Y(A)$ for all $A \in \mathfrak{F}$, hence $x \in \cap\{Y(A) \mid A \in \mathfrak{F}\}$ and $Y(\cap\{A \mid A \in \mathfrak{F}\}) \subset \cap\{Y(A) \mid A \in \mathfrak{F}\}$.

Let $x \notin Y(\cap\{A \mid A \in \mathfrak{F}\})$, then there exists $W \in \mathcal{M}(x)$ such that $W \cap (\cap\{A \mid A \in \mathfrak{F}\}) = \phi$. Hence $W \subset S - \cap\{A \mid A \in \mathfrak{F}\}$ and $\{S - A \mid A \in \mathfrak{F}\}$ is an open covering for W. Since W is compact there exist $A_1, \ldots, A_n$ such that $W \subset \cup\{S - A_i \mid 1 \leq i \leq n\} = S - \cap\{A_i \mid 1 \leq i \leq n\}$. Since $\mathfrak{F}$ is a proper closed filter base, there exists an element, A, of $\mathfrak{F}$ such that $A \subset \cap\{A_i \mid 1 \leq i \leq n\}$, hence $W \cap A = \phi$ and $x \notin Y(A)$. Therefore $x \notin \cap\{Y(A) \mid A \in \mathfrak{F}\}$ and $\cap\{Y(A) \mid A \in \mathfrak{F}\} \subset Y(\cap\{A \mid A \in \mathfrak{F}\})$. The theorem is proven.

<u>Theorem</u> 10. S <u>is</u> Y-<u>additive if and only if for each pair</u> A <u>and</u> B <u>of closed subsets of</u> S $Y(A \cup B) = Y(A) \cup Y(B)$.

Proof. Let S be Y-additive and let A and B be closed subsets, then $Y(A) \cup Y(B) = Y(A \cup B)$.

Let $Y(A) \cup Y(B) = Y(A \cup B)$ for any two closed subsets of S. Let $\{A_\alpha \mid \alpha \in \mathcal{Q}\}$ be a set of closed sets such that $\cup\{A_\alpha \mid \alpha \in \mathcal{Q}\}$ is closed.

Since $\cup\{Y(A_\alpha) \mid \alpha \in \mathcal{Q}\} \subset Y(\cup\{A_\alpha \mid \alpha \in \mathcal{Q}\})$ all that needs to be shown is that $Y(\cup\{A_\alpha \mid \alpha \in \mathcal{Q}\}) \subset \cup\{Y(A_\alpha) \mid \alpha \in \mathcal{Q}\}$.

For each $\alpha \in \mathcal{Q}$ let $\mathfrak{F}(A_\alpha)$ be the collection of closed subsets B of S such that $A_\alpha \subset \mathrm{Int}(B)$. If $A_\alpha = \phi$ then $Y(A_\alpha) = \cap\{Y(B) \mid B \in \mathfrak{F}(A_\alpha)\}$. If $A_\alpha \neq \phi$, then $\mathfrak{F}(A_\alpha)$ is a closed proper filter base of S and since $\cap\{B \mid B \in \mathfrak{F}(A_\alpha)\} = A_\alpha$, $Y(A_\alpha) = \cap\{Y(B) \mid B \in \mathfrak{F}(A_\alpha)\}$.

Suppose $x \notin \cup\{Y(A_\alpha) | \alpha \in \mathcal{A}\}$. Then for each $\alpha \in \mathcal{A}$ there exists $B_\alpha \in \mathfrak{J}(A_\alpha)$ such that $x \notin Y(B_\alpha)$. $\{\mathrm{Int}(B_\alpha) | \alpha \in \mathcal{A}\}$ is an open covering of the compact set $\cup\{A_\alpha | \alpha \in \mathcal{A}\}$ and hence there exists $B_1, \ldots, B_n$ such that $\cup\{A_\alpha | \alpha \in \mathcal{A}\} \subset \cup\{\mathrm{Int}(B_i) | 1 \leq i \leq n\}$. By hypothesis $Y(\cup\{B_i | 1 \leq i \leq n\}) = \cup\{Y(B_i) | 1 \leq i \leq n\}$, hence $x \notin Y(\cup\{B_i | 1 \leq i \leq n\}) \supset Y(\cup\{A_\alpha | \alpha \in \mathcal{A}\})$. Therefore $Y(\cup\{A_\alpha | \alpha \in \mathcal{A}\}) \subset \cup\{Y(A_\alpha) | \alpha \in \mathcal{A}\}$ and the theorem is proven.

The following example shows that $Y(\{p,q\}) = Y(p) \cup Y(q)$ for any two elements of Z and that Z is not Y-additive.

<u>Example</u> 5. Let S be the space in example 3 and let $Z = S \times I$. Then $Y(p) \cup Y(q) = Y(\{p,q\})$ for all $p,q \in Z$, but $Y((0,1) \times I) = (0,1) \times I$ and $Y((0,-1) \times I) = (0,-1) \times I$ but $Y(((0,1) \times I) \cup ((0,-1) \times I)) = \{(0,y) | -1 \leq y \leq 1\} \times I$.

<u>Theorem</u> 11. <u>If for all</u> $p \in S$ <u>and any finite collection</u>, $\{W_i\}$, <u>of elements of</u> $\mathcal{M}(p)$, <u>there exists</u> $W \in \mathcal{M}(p)$ <u>such that</u> $W \subset \cap\{W_i\}$, <u>then</u> S <u>is</u> Y-<u>additive</u>.

Proof. By Theorem 10 all that is needed to be shown is that $Y(A) \cup Y(B) = Y(A \cup B)$ for any two closed subsets of S. By formula 3 all that needs to be shown is that $Y(A \cup B) \subset Y(A) \cup Y(B)$.

Let A and B be any two closed subsets of S and let $p \notin Y(A) \cup Y(B)$. There exists $W_1, W_2 \in \mathcal{M}(p)$ such that $W_1 \cap A = \phi = W_2 \cap B$. By hypothesis there exists $W \in \mathcal{M}(p)$ such that $W \subset W_1 \cap W_2$. Hence $W \cap (A \cup B) = \phi$ and $p \notin Y(A \cup B)$, thus $Y(A \cup B) \subset Y(A) \cup Y(B)$. Compare with Chapter 1, paragraph 2 of [5].

<u>Theorem</u> 12. <u>Let</u> C <u>be a subcontinuum of the continuum</u> S, <u>then</u> $Y(C)$ <u>is a continuum</u>.

Proof. Since $Y(C)$ is closed and hence compact, all that needs to be proven is that $Y(C)$ is connected.

Suppose $Y(C) = A \underset{sep}{\cup} B$ and $C \subset A$. Since S is normal, there exists U and V open set with disjoint closures such that $A \subset U$ and $B \subset V$. Since $C \subset A \subset U$ and $B \subset V$, $Y(C)$ is disjoint from $Fr(V)$, where $Fr(V)$ denotes the boundary of V. Therefore for all $y \in Fr(V)$, there exists $Wy \in \mathcal{M}(y)$ such that $Wy \cap C = \emptyset$. Since $\{Int(Wy)\}$ forms an open covering of $Fr(V)$, there exists strong continua $W_1, \ldots, W_n$ such that $W_i \cap C = \emptyset$ and $\cup\{Int(W_i) | 1 \leq i \leq n\} \supset Fr(V)$. If K is a component of V then $Cl(K) \cap Fr(V) \neq \emptyset$, hence $V \cup (\cup\{Int(W_i) | 1 \leq i \leq n\})$ has only a finite number of components. Therefore each component of $V \cup (\cup\{Int(W_i) | 1 \leq i \leq n\})$ is open. Let $b \in B$ and K_b be the component of $V \cup (\cup\{Int(W_i) | 1 \leq i \leq n\})$ containing b, then $Cl(K_b) \in \mathcal{M}(b)$ and $Cl(K_b) \cap C = \emptyset$. Therefore $b \notin Y(C)$, but this contradicts the fact that $B \subset Y(C)$. Therefore $Y(C)$ is a continuum.

CHAPTER 2

RELATIONSHIPS BETWEEN Y AND T

This chapter develops some relationships between T and Y.

Definition. Let S be a topological space, then T is the closure with respect to continua.

Theorem 13. *Let A be a subset of the topological space S, then T(A) is a subset of Y(A).*

Proof. Suppose $x \notin Y(A)$. Then there exists $W \in \mathcal{M}(p)$ such that $W \cap A = \emptyset$. Since $W \in \mathcal{M}(p)$, $x \in \mathrm{Int}(W)$ and therefore $x \notin T(A)$. Hence $T(A) \subset Y(A)$.

The following example shows that T(A) need not be Y(A).

Example 6. Let S be the subcontinuum of the plane defined as follows. S is the closed line segments between (1,0) and $(0,\frac{1}{n})$ for $n \geq 1$ union with the closed line segments between $(1,\frac{1}{n})$ and (2,0) for $n \geq 1$ union with the closed line segment between (0,0) and (2,0). $T((2,0)) = \{(x,0) \mid 1 \leq x \leq 2\}$ and $Y((2,0)) = \{(x,0) \mid 0 \leq x \leq 2\}$.

The following is an example of a space that is Y-additive, but not T-additive.

Example 7. Let S_0 be the closed line segments between $(0,\frac{1}{n},0)$ and (1,0,0), between $(1,\frac{1}{n},0)$ and (2,0,0), between

$(2,\frac{1}{n},0)$ and $(3,0,0)$, between $(1,-\frac{1}{n},0)$ and $(0,0,0)$, between $(2,-\frac{1}{n},0)$ and $(1,0,0)$, between $(3,-\frac{1}{n},0)$ and $(2,0,0)$, between $(1\frac{1}{2},0,\frac{1}{n})$ and $(0,0,0)$, between $(3,0,0)$ and $(1\frac{1}{2},0,\frac{1}{n})$ for $n \geq 1$ unioned with the closed line segment $(0,0,0)$ and $(3,0,0)$. $T((0,0,0)) = \{(x,0,0) | 0 \leq x \leq 1\}$ and $T((3,0,0)) = \{(x,0,0) | 2 \leq x \leq 3\}$, but $T(\{(0,0,0), (3,0,0)\}) = \{(x,0,0) | 0 \leq x \leq 3\}$.

Let $L = \{(x,0,0) | 0 \leq x \leq 3\}$. If $A \subset S$ is closed then $Y(A) = A$ if $A \cap L = \emptyset$ and $Y(A) = A \cup L$ if $A \cap L \neq \emptyset$. Thus S is Y-additive.

For the remainder of this chapter S will be a compact Hausdorff continuum.

Definition. S is called weakly irreducible if and only if given $C_1, C_2, \ldots, C_n$ subcontinua of S, $S - \cup\{C_i | 1 \leq i \leq n\}$ has a finite number of components.

Lemma 14. *Let S be weakly irreducible and let C be a subcontinuum of S, then $\mathrm{Int}(C)$ has only a finite number of components.*

Proof. Let $S - C$ have components $K_1, \ldots, K_n$, then $\mathrm{Int}(C) = S - \cup\{\mathrm{Cl}(K_i) | 1 \leq i \leq n\}$ which has only a finite number of components.

Theorem 15. *Let S be weakly irreducible and let A be a subset of S, then $T(A) = Y(A)$.*

Proof. Since $T(A) \subset Y(A)$, all that needs to be shown is that $Y(A) \subset T(A)$. Let $x \notin T(A)$, then there exists a continuum W such that $x \in \mathrm{Int}(W)$ and $W \cap A = \emptyset$. Since $\mathrm{Int}(W)$ has only a finite number of components each component is open;

let K be the component containing x, then $Cl(K) \in \mathcal{M}(p)$ and $Cl(K) \cap A = \phi$, therefore $x \notin Y(A)$; hence $Y(A) \subset T(A)$ and the theorem is proven.

Corollary 16. *If S is weakly irreducible, then*

1. *S is Y-symmetric*
2. *S is locally connected at p, if and only if S is connected Im Kleinen at p.*
3. *If S is also aposynedetic, then S is locally connected.*

Proof. This follows from the previous theorem and from Theorem 6 of [2].

The following develops a weaker statement which is equivalent to weakly irreducible.

Definition. Let S be a continuum and let A be a subset of S. S is called *irreducible about* A if and only if for C a subcontinuum of S such that $A \subset C$, then $C = S$. Notation $S = [A]$.

Definition. Let S be a topological space, let A and B be two disjoint closed subsets of S and let M be a subcontinuum of S. M *is called irreducible from* A *to* B if and only if M intersects both A and B non-voidly and no proper subcontinuum of M intersects both A and B.

The following two theorems are from [6].

Theorem A (Theorem 43). *Let A and B be two disjoint closed subsets of S, then S contains a continuum irreducible from A to B.*

<u>Theorem B</u> (Theorem 47). <u>Let</u> A <u>and</u> B <u>be two disjoint closed subsets of</u> S <u>and let</u> M <u>be an irreducible continuum from</u> A <u>to</u> B, <u>then</u> $M - (A \cup B)$ <u>and</u> $M - A$ <u>are connected</u>.

<u>Theorem 17</u>. <u>Let</u> $C_1, C_2, \ldots, C_n$ <u>be disjoint subcontinua of</u> S, <u>then there exists a component</u> K <u>of</u> $S - \cup\{C_i | 1 \leq i \leq n\}$ <u>such that</u> $Cl(K) \cap C_1 \neq \emptyset$ <u>and</u> $Cl(K) \cap (\cup\{C_i | 2 \leq i \leq n\}) \neq \emptyset$.

Proof. Since C_1 and $\cup\{C_i | 2 \leq i \leq n\}$ are closed disjoint subsets of S, S contains a continuum, M, irreducible from C_1 to $\cup\{C_i | 2 \leq i \leq n\}$.

Let $L = M - \cup\{C_i | 1 \leq i \leq n\}$, then $Cl(L) \cap C_1 \neq \emptyset$, $Cl(L) \cap (\cup\{C_i | 2 \leq i \leq n\}) \neq \emptyset$ and L is connected. Let K be the component of $S - \cup\{C_i | 1 \leq i \leq n\}$ containing L, then $Cl(K) \cap C_1 \neq \emptyset$ and $Cl(K) \cap (\cup\{C_i | 2 \leq i \leq n\}) \neq \emptyset$.

<u>Corollary 18</u>. <u>Let</u> $C_1, C_2, \ldots, C_n$ <u>be disjoint subcontinua of</u> S, <u>then there exists</u> $K_1, \ldots, K_m$, <u>components of</u> $S - \cup\{C_i | 1 \leq i \leq n\}$, $m \leq n$, <u>such that</u> $(\cup\{K_i | 1 \leq i \leq m\}) \cup \{\cup\{C_i | 1 \leq i \leq n\}\}$ <u>is a subcontinuum of</u> S.

<u>Theorem 19</u>. S <u>is weakly irreducible if and only if given any</u> W <u>a subcontinuum of</u> S, $S - W$ <u>has a finite number of components</u>.

Proof. Let S be weakly irreducible and let C be a subcontinuum of S, then $S - C$ has a finite number of components by definition.

Let S be such that for any subcontinuum W of S, $S - W$ has a finite number of components. Let $C_1, C_2, \ldots, C_n$ be subcontinua of S. Then $\cup\{C_i | 1 \leq i \leq n\} = \cup\{M_i | 1 \leq i \leq m\}$ where the M_i are disjoint components of $\cup\{C_i | 1 \leq i \leq n\}$.

The M_i are disjoint subcontinua of S. Therefore there exist $k_1,\ldots,k_\ell$ components of $S - \cup\{M_i | 1 \le i \le m\}$ such that $C = (\cup\{k_i | 1 \le i \le \ell\}) \cup (\cup M_i | 1 \le i \le m)$ is a continuum. Therefore $S - C$ has a finite number of components $t_1,\ldots,t_0$. Therefore $S - \cup\{C_i | 1 \le i \le n\}$ has less than or equal to $\ell + 0$ components. The theorem is proven.

<u>Theorem 20</u>. <u>Let</u> $S = [\{x_1,x_2,\ldots,x_n\}]$, then S is weakly <u>irreducible</u>.

Proof. Let C be a subcontinuum of S and let k_i be the component of $S - C$ containing x_i if $x_i \notin C$. $(\cup\{k_i | 1 \le i \le n$ and $x_i \notin C\}) \cup C$ is a subcontinuum and $\{x_1,x_2,\ldots,x_n\} \subset (\cup\{k_i | 1 \le i \le n$ and $x_i \notin C\}) \cup C$; therefore $S = (\cup\{k_i | 1 \le i \le n$ and $x_i \notin C\}) \cup C$. Hence $S - C$ has less than or equal to n components. Therefore S is weakly irreducible.

<u>Corollary 21</u>. <u>If</u> $S = [\{x_1,x_2,\ldots,x_n\}]$, <u>then</u>

1. S <u>is connected Im Kleinen at</u> p <u>if and only if</u> S <u>is locally connected at</u> p
2. S <u>is Y-symmetric</u>.

Proof. This follows from Corollary 16 and Theorem 20.

In Example 2 $S = [\{(\frac{1}{n},0) | n > 0\} \cup \{(0,0), (0,1)\}]$, but S is not weakly irreducible since $S - \{(0,0)\}$ has an infinite number of components.

<u>Definition</u>. B is called a <u>compact separator</u> of the topological space S if and only if

1. B is compact
2. $S - B = H \cup K$ sep.

Theorem 22. _If_ A _is a subset of_ S, _then_ $Y(A)$ _intersects any compact separator of_ S _that separates_ A _from any point of_ $Y(A)$.

Proof. Suppose the theorem is false and there exists a compact set B such that $S - B = H \cup K$ sep, $A \subset H$, $p \in Y(A) \cap K$ and $S - B \supset Y(A)$. Since B is closed, both H and K are open and since $Cl(K) \cap H = \emptyset$, if k is a component of K then $Cl(k) \cap B \neq \emptyset$. Since $B \subset S - Y(A)$, there exists a finite number of strong continua, W_i, such that $W_i \cap A = \emptyset$ and $B \subset \cup\{Int(W_i)\}$. Therefore $K \cup (\cup\{Int(W_i)\})$ has a finite number of components and each component is open. Let k be the component of $K \cup (\cup\{Int(W_i)\})$ containing p, then $Cl(k) \in \mathcal{M}(p)$ and $Cl(k) \cap A = \emptyset$. Therefore $p \notin Y(A)$, contradiction. The theorem is proven.

Corollary 23. _If_ $A \subset S$, _then every component of_ $Y(A)$ _intersects_ $\bar{A}$.

Proof. See Corollary 1.1 of [7].

Corollary 24. _If_ A _is a closed subset of_ S _and_ $Y(A)$ _is totally disconnected, then_ $Y(A) = A$.

Proof. See Corollary 1.2 of [7].

Corollary 25. _Let_ A _and_ B _be closed, totally disconnected subsets, where_ $A \subset S_1$ _and_ $B \subset S_2$, _then for any closed subset_ $K \subset A \times B \subset S_1 \times S_2$, $Y(K) = K = T(K)$.

Proof. See Theorem 2 of [7].

Theorem 26. _Let_ $p \in S \times S$, _then_ $Y(p) = \{p\}$.

Proof. Let $q = (a,b)$ and $p = (c,d)$ be two distinct points of $S \times S$, then $a \neq c$ or $b \neq d$. It may be assumed that $a \neq c$. There exists U and V open set such that $a \in U$, $c \notin \overline{U}$ and $d \notin \overline{V}$. Then (a,b) is an element of the open set $(U \times S) \cup (S \times V)$ and (c,d) is not an element of the closed set $(Cl(U) \times S) \cup (S \times Cl(V))$.

To show that $(U \times S) \cup (S \times V)$ is connected, it need only be shown that for (e,f) and (h,g) elements of $(U \times S) \cup (S \times V)$ that there are elements of a connected subset of $(U \times S) \cup (S \times V)$.

Case 1. (e,f) and (h,g) are elements of $U \times S$ or $S \times V$. It may be assumed that there are elements of $U \times S$. Let $t \in V$ then $(\{e\} \times S) \cup (\{h\} \times S) \cup (S \times \{t\})$ is a connected set since $(\{e\} \times S) \cap (S \times \{t\}) = (e,t)$, $(\{h\} \times S) \cap (S \times \{t\}) = (n,t)$ and $(\{e\} \times S) \cup (\{h\} \times S) \cup (S \times \{t\}) \subset U \times S \cup S \times V$.

Case 2. $(e,f) \in U \times S$ and $(h,g) \in S \times V$ then $(\{e\} \times S) \cup (S \times \{g\})$ is a connected subset of $(U \times S) \cup (S \times V)$. The theorem is proven.

Corollary 27. *The following are equivalent*

1. S *is locally connected*
2. $S \times S$ *is Y-additive*
3. $S \times S$ *is T-additive*.

In the following theorem the hypothesis that S is connected is not necessary.

Theorem 28. *If* $A \subset Int(B) \subset B \subset S$ *and* $T(B) = B$, *then* $Y(A) \subset B$.

Proof. First we show that each component of $S - B$ is open. Let K be a component of $S - B$ and let $x \in K$.

Since $x \notin T(B)$, there exists a continuum W such that $x \in \text{Int}(W)$ and $W \cap B = \emptyset$. Hence $W \subset K$ and $x \in \text{Int}(W)$, therefore $x \in \text{Int}(K)$. Since x could be any element of K, K is open.

Now we show that $Y(A) \subset B$. Let $p \in S - B$ and let K be the component of $S - B$ containing p. Then $\text{Cl}(K) \in \mathcal{M}(p)$ and $\text{Cl}(K) \cap A = \emptyset$, therefore $p \notin Y(A)$ and thus $Y(A) \subset B$.

Corollary 29. S is locally connected if and only if S is connected Im Kleinen.

Proof. If S is locally connected then S is connected Im Kleinen.

Let S be connected Im Kleinen, then $T(A) = \text{Cl}(A)$ and S is T-additive [2]. Let U be an open set containing A, then $T(\text{Cl}(U)) = \text{Cl}(U)$, hence $Y(A) \subset \text{Cl}(U)$ by Theorem 28. $\text{Cl}(A) \subset Y(A) \subset \bigcap\{\text{Cl}(U) \mid A \subset U \text{ and } U \text{ is open}\} = \text{Cl}(A)$, hence $T = Y$. Therefore S is Y-additive and $Y(p) = \{p\}$ for all $p \in S$, thus S is locally connected.

Davis has the following theorem in [9].

Let S be a compact Hausdorff space, then the following are equivalent

1. $T(A) \cap B = \emptyset$ where A and B are closed
2. There exist closed subsets M and N such that $A \subset \text{Int}(M)$, $B \subset \text{Int}(N)$ and $T(M) \cap N = \emptyset$.

Corollary 30. If $T^2(A) = T(A)$ for all $A \subset S$, then $T(A) = Y(A)$ for all $A \subset S$.

Proof. Let $x \notin T(A)$, then there exists an open set U such that $A \subset U$ and $x \notin T(\text{Cl}(U))$, [9]. Since $T(T(\text{Cl}(U))) =$

$T(Cl(U))$ and since $A \subset U \subset Int(T(Cl(U)))$, $Y(A) \subset T(Cl(U))$ by Theorem 28, therefore $x \notin Y(A)$. Thus $T(A) = Y(A)$.

CHAPTER 3

Y AND MONOTONE MAPS

This chapter is a study of the relations between Y and monotone functions.

For the following theorems all spaces are Hausdorff.

Definition. A function f from S onto Z is called monotone if and only if f is continuous and $f^{-1}(z)$ is connected for all $z \in Z$.

Theorem. *Let f be an open monotone map of S onto Z and let A be a connected subset of Z, then $f^{-1}(A)$ is connected.*

Proof. See Chapter VI, section 3, problem 1 of [8].

Theorem 31. *Let f be a closed monotone map of S onto Z and let A be a connected subset of Z, then $f^{-1}(A)$ is connected.*

Proof. Suppose the theorem is false and let C be a connected subset of Z such that $f^{-1}(C) = M \cup N$ sep. $f(M) \cap f(N) = \phi$, for if $p \in f(M) \cap f(N)$, then $f^{-1}(p) \cap N \neq \phi$ and $f^{-1}(p) \cap M \neq \phi$, but $f^{-1}(p)$ is connected, hence $f^{-1}(C) \neq M \cap N$ sep. Therefore $f(N) \cap f(M) = \phi$.

$C = f(f^{-1}(C)) = f(M \cup N) = f(M) \cup f(N)$ and C is connected. Therefore $Cl(f(N)) \cap f(M) \neq \phi$ or $f(N) \cap Cl(f(M)) \neq \phi$. We may assume that $Cl(f(N)) \cap M \neq \phi$. Let $p \in Cl(f(N)) \cap f(M)$, then $f^{-1}(p) \cap M \neq \emptyset$. $f^{-1}(p)$ is connected, hence $f^{-1}(p) \subset M$. $\{p\} \subset Cl(f(N)) = f(Cl(N))$ since f is a closed map. Therefore $f^{-1}(p) \cap Cl(N) \neq \emptyset$, hence $Cl(N) \cap M \neq \emptyset$. This is a contradiction of separability. Thus $f^{-1}(C)$ is connected.

For the following theorems all spaces are compact Hausdorff spaces.

Theorem 32. Let f be a monotone map of S onto Z and let W be a strong subcontinuum of Z, then $Cl(f^{-1}(Int(W)))$ is a strong continuum.

Proof. Since f is a continuous map of a compact Hausdorff space onto a compact Hausdorff space f is a closed map. Therefore by Theorem 29 $f^{-1}(Int(W))$ is connected. Since f is continuous, $f^{-1}(Int(W))$ is open. Thus, $Cl(f^{-1}(Int(W)))$ is a strong continuum.

The following example shows that even if f is a closed monotone map, f^{-1} of a strong continuum need not be a strong continuum.

Example 8. Let $S = I^2$, $Z = \{(x,y) | \frac{1}{2} \leq x \leq 1$ and $0 \leq y \leq 1\}$ and let $A = \{(x,y) | 0 \leq x \leq \frac{1}{2}$ and $0 \leq y \leq 1\}$. define $f: S \to Z$ by $f(x,y) = (x,y)$ if $\frac{1}{2} \leq x \leq 1$ and $f(x,y) = (\frac{1}{2},y)$ if $0 \leq x < \frac{1}{2}$. Let $W = \{(x,y) | (x - 3/4)^2 + (y - 1/4)^2 \leq 1/4\} \cup \{(\frac{1}{2},y) | 0 \leq y \leq 1\}$. $f^{-1}(W) = A \cup W$, $Int(A) \cap W = \emptyset$ and $A \cap Int(W) = \emptyset$.

Theorem 33. Let f be a monotone map of S onto Z, then $Y(f^{-1}(A)) \subset f^{-1}(Y(A))$.

Proof. Let A be a subset of Z and suppose $p \notin f^{-1}(Y(A))$, then $f(p) \notin Y(A)$. Hence there exist $W \in \mathcal{M}(f(p))$ such that

$W \cap A = \phi$. Therefore $f^{-1}(W) \cap f^{-1}(A) = \phi$. By Theorem 30 $Cl(f^{-1}(Int(W))) \in \mathcal{M}(p)$, hence $p \notin Y(A)$ and thus $Y(f^{-1}(A)) \subset f^{-1}(Y(A))$.

Theorem 34. *Let* f *be a monotone map of* S *onto* Z, *then* $f(Y(A)) \subset Y(f(A))$.

Theorem 35. *Let* f *be an open monotone map of* S *onto* Z, *then* $Y(f^{-1}(A)) = f^{-1}(Y(A))$.

Proof. By Theorem 33 $Y(f^{-1}(A)) \subset f^{-1}(Y(A))$, hence all that needs to be shown is that $f^{-1}(Y(A)) \subset Y(f^{-1}(A))$. Suppose $p \notin Y(f^{-1}(A))$, then there exist $W \in \mathcal{M}(p)$ such that $W \cap f^{-1}(A) = \phi$, hence $f(W) \cap A = \phi$. Since f is open $f(W) \in \mathcal{M}(f(p))$ and $f(p) \notin Y(A)$. Thus $p \notin f^{-1}(Y(A))$ and $f^{-1}(Y(A)) \subset Y(f^{-1}(A))$.

Theorem 36. *Let* f *be an open monotone map of* S *onto* Z, *then* $Y(A) = f(Y(f^{-1}(A)))$.

Corollary 37. *Let* f *be an open monotone map of* S *onto* Z, *then*

1. *If* S *is Y-additive then* Z *is Y-additive*
2. *If* S *is Y-symmetric, then* Z *is Y-symmetric*.

Proof. Let S be Y-symmetric and let A and B be closed subsets of Z such that $Y(A) \cap B = \phi$, then $f^{-1}(Y(A)) \cap f^{-1}(B) = \phi$. Since $f^{-1}(Y(A)) = Y(f^{-1}(A))$, $Y(f^{-1}(A)) \cap f^{-1}(B) = \phi$. Since S is Y-symmetric $Y(f^{-1}(B)) \cap f^{-1}(A) = \phi$, thus $Y(B) \cap A = \phi$ and Z is Y-symmetric.

Let S be Y-additive and let A and B be closed subsets of Z, then $f^{-1}(Y(A) \cup Y(B)) = f^{-1}(Y(A)) \cup f^{-1}(Y(B)) = Y(f^{-1}(A)) \cup Y(f^{-1}(B)) = Y(f^{-1}(A) \cup f^{-1}(B)) = Y(f^{-1}(A \cup B)) =$

$f^{-1}(Y(A \cup B))$, thus $Y(A) \cup Y(B) = Y(A \cup B)$ and Z is Y-additive.

Corollary 38. Let S_1 and S_2 be continua, then $S_1 \times S_2$ is locally connected if and only if S_1 and S_2 are locally connected.

Proof. Let P_1 and P_2 be the two projection maps, then P_1 and P_2 are open monotone maps.

Let $S_1 \times S_2$ be locally connected, then S_1 and S_2 are Y-additive and hence it suffices to show that $Y(a) = \{a\}$ for all $a \in S_1$. Let $a \in S_1$, then $Y(a) = P_1(Y(P_1^{-1}(a)) = P_1(Y(\{a\} \times S_2)) = P_1(\{a\} \times S_2) = a$. Thus S_1 and S_2 are locally connected.

Let S_1 and S_2 be locally connected. Let $p \in S_1 \times S_2$ and let A be any subset of $S_1 \times S_2$ such that $p \notin Cl(A)$, then there exist $O_1 \subset S_1$ and $O_2 \subset S_2$, both open, such that $p \in O_1 \times O_2$ and $(O_1 \times O_2) \cap A = \emptyset$. Since S_1 and S_2 are locally connected, there exist $W_1 \in \mathcal{M}(P_1(p))$ and $W_2 \in \mathcal{M}(P_2(p))$ such that $W_1 \subset O_1$ and $W_2 \subset O_2$, hence $p \in W_1 \times W_2$ and $(W_1 \times W_2) \cap A = \emptyset$. Therefore $p \notin Y(A)$. Thus by Theorem 4 $S_1 \times S_2$ is locally connected.

Theorem 39. Let H and K be closed subsets of S, then the following are equivalent.

a. $H \cap Y(K) = \emptyset$

b. There exists a finite collection C of strong continua such that H is contained in the union of the interiors of the elements of C and the intersection of each element of C with K is empty.

c. <u>There exist two closed subsets</u> M <u>and</u> N <u>such that</u> H <u>is a subset of the interior of</u> M, K <u>is a subset of the interior of</u> N <u>and</u> M <u>intersect</u> Y(N) <u>is empty</u>.

Proof. a implies b.

Let L be the set of strong continua of S such that $\mathrm{Int}(W) \cap H \neq \emptyset$ and $W \cap K = \emptyset$. Since $H \cap Y(K) = \emptyset$, $H \subset (\cup\{\mathrm{Int}(W) | W \in L\})$. Since H is compact, there exist $W_1, \ldots, W_n$ such that $W_i \in L$ and $H \subset (\cup\{\mathrm{Int}(W_i) | 1 \leq i \leq n\})$. Let $C = \{W_i | 1 \leq i \leq n\}$.

b implies c.

Since S is normal there exist V such that V is open $H \subset V \subset \mathrm{Cl}(V) \subset \cup\{\mathrm{Int}(W) | W \in C\}$ and there exist U such that U is open $K \subset U$ and $\mathrm{Cl}(U) \cap (\cup\{W | W \in C\}) = \emptyset$. Therefore $\mathrm{Cl}(V) \cap Y(\mathrm{Cl}(U)) = \emptyset$. Let $\mathrm{Cl}(V) = M$ and $\mathrm{Cl}(U) = N$.

c implies a.

$K \subset N$, therefore $Y(K) \subset Y(N)$, $H \subset M$ and $M \cap Y(N) = \emptyset$, thus $H \cap Y(K) = \emptyset$.

The closed monotone image of a T-symmetric compact Hausdorff space is a T-symmetric compact Hausdorff space. This final example shows that the closed monotone image of a Y-symmetric space need not be Y-symmetric and that $f(Y(A))$ need not be $Y(f(A))$.

<u>Example 9</u>. Let S be the closed line segments between $(-\frac{1}{n},0)$ and $(0,1)$ and the closed line segments between $(-\frac{1}{n},2)$ and $(0,3)$ unioned with $\{(x,\frac{1}{2} + \sin(\frac{1}{x}) | 0 \leq x \leq 1\} \cup \{(x,2\frac{1}{2} + \sin(\frac{1}{x}) | 0 < x \leq 1\}$ unioned with the closed line segment

between $(0,0)$ and $(0,3)$, then S is Y-symmetric and $Y((0,3)) = \{(0,y) | 2 \le y \le 3\}$.

Let Z be the closed line segments between $(-\frac{1}{n},0)$ and $(0,1)$ unioned with those between $(-\frac{1}{n},1)$ and $(0,2)$ for $n \ge 1$ unioned with $\{(x,\frac{1}{2} + \sin(\frac{1}{x})) | 1 < x \le 1\} \cup \{(x,1\frac{1}{2} + \sin(\frac{1}{x})) | 0 < x \le 1\}$ unioned with the closed line segment between $(0,0)$ and $(0,2)$.

Let $f((x,y)) = (x,y)$ if $y \le 1$, $f((x,y)) = (0,1)$ if $1 \le y \le 2$ and $f((x,y)) = (x,y-1)$ if $2 \le y \le 3$.

Z is not Y-symmetric since $Y((0,0)) = \{(0,y) | 0 \le y \le 1\}$ and $Y((0,2)) = \{(0,y) | 0 \le y \le 2\}$. $f(Y(0,3)) \ne Y(f(0,3))$ since $f(Y((0,3))) = f(\{(0,y) | 2 \le y \le 3\}) = \{(0,y) | 1 \le y \le 2\}$ and $Y(f((0,3))) = Y((0,2)) = \{(0,y) | 0 \le y \le 2\}$.

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