REDUCTION OF THE NUMBER OF INDEPENDENT VARIABLES AND OPTIMIZATION IN SWIRLING FLUID FLOW

Thesis for the Degree of Ph. D.

MICHIGAN STATE UNIVERSITY

MICHAEL JEROME BRINK

1971

THESIS

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Michael Jerome Brink

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mechanical Engineering

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Major professor

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ABSTRACT

REDUCTION OF THE NUMBER OF INDEPENDENT VARIABLES
AND OPTIMIZATION IN SWIRLING FLUID FLOW

By

Michael Jerome Brink

The major objective of this investigation is to determine solutions to the system of fundamental equations governing inviscid, incompressible swirling flow in (R, θ, Z) space by using group theory to reduce the number of independent variables and obtaining the solutions for the resulting system of ordinary differential equations.

Transformations obtained by the application of continuous one-parameter group theory are applied to the fundamental system of non-dimensional equations for conservation of mass, conservation of momentum, and conservation of energy. The number of independent variables is reduced from three to two to one using the absolute invariants determined for each transformation group. The resulting system of non-linear ordinary differential equations is then solved numerically by Hamming's modified predictor-corrector technique. The one-dimensional solutions are transformed back to the three-dimensional space using the transformations obtained from the application of the continuous one-parameter group theory.

Computer curves are presented for both the one-dimensional and the transformed three-dimensional solutions to the conservation of mass and conservation of momentum equations. Two continuous one-parameter transformation groups and their invariants which were discovered and then utilized to accomplish the reduction in the number of

independent variables for cylindrical polar coordinates are also presented.

A Mayer type optimization problem is solved using the one-dimensional representations for conservation of energy and the swirl parameter to optimize the one-dimensional representation for pressure. The one-dimensional optimization results are transformed back to the three-dimensional space using the transformations obtained from the application of the continuous one-parameter group theory.

Computer curves are presented for the optimization results in both the one-dimensional and the three-dimensional space.

REDUCTION OF THE NUMBER OF INDEPENDENT VARIABLES AND OPTIMIZATION IN SWIRLING FLUID FLOW

By

Michael Jerome Brink

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To Marylyn, David, Bridjette, and Cynthia -

for infinite patience and understanding

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Figure Page

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p - Pr

r, θ, z - Ph

u, v, w - Ve

ρ - De

q - Ve

e - In

Q - Th

h - B

λ_j - L

K - A

ϕ_j - F

η_1 - S

η_2 - T

S - S

z_0 -

V_0 -

R =

Z =

U -

V -

W -

P -

NOMENCLATURE

p - Pressure

θ, z - Physical coordinates, independent variables

v, w - Velocity components

ρ - Density

q - Velocity

e - Internal energy and base of the natural logarithm system

Q - Total energy change

h - Bernoulli constant

λ_j - Lagrange multipliers

K - Augmented function

ϕ_j - Functional forms of optimal constraint equations

π_1 - Swirl aspect ratio = r^2/z^2

π_2 - Tangential energy fraction = $v^2/(h/\rho)$

S - Swirl parameter = $\pi_1 \pi_2$

z_o - Axial swirl length

v_o - Tangential velocity at the outer swirl radius

R = Dimensionless independent variable = r/z_o

Z = Dimensionless independent variable = z/z_o

U - Dimensionless radial velocity component = u/v_o

V - Dimensionless tangential velocity component = v/v_o

W - Dimensionless axial velocity component = w/v_o

P - Dimensionless static pressure = $p/(\rho v_o^2)$

H - Dime

t_i - Cons

a - Grou

A_1 - One-

ϕ - Dif

x^i - Arg

η_i - Ind

f_i - Abs

ξ - Gro

α - Gro

F_1 - Two
con

F_2 - Two
co

F_3 - Tw
co

F_4 - Tw

k - Co

Y_i - Co

c - G

B_1 - O

X - I

σ - G

H - Dimensionless Bernoulli constant = $h/(\rho v_o^2)$

$_i$ - Constants

a - Group parameter

A_1 - One-parameter group

Φ - Differential form of k-th order in m independent variables

^{i} - Arguments of the k-th order differential form

η_i - Independent variables, absolute invariants of A_1

F_i - Absolute invariants of A_1

ξ - Group constant = t_2/t_1

α - Group constant = t_4/t_2

F_1 - Two-dimensional representation for the radial velocity component

F_2 - Two-dimensional representation for the tangential velocity component

F_3 - Two-dimensional representation for the axial velocity component

F_4 - Two-dimensional representation for static pressure

k - Constant

$_i$ - Constants

c - Group parameter

$_1$ - One-parameter group

X - Independent variable, absolute invariant of B_1

- Group constant = γ_1/k

g_1 - Abs

G_1 - One
co

G_2 - One
co

G_3 - On
co

G_4 - Or

L^2 - Co

$'$ - De
va

g_i - Absolute invariants of B_1

G_1 - One-dimensional representation for the radial velocity component

G_2 - One-dimensional representation for the tangential velocity component

G_3 - One-dimensional representation for the axial velocity component

G_4 - One-dimensional representation for static pressure

L^2 - Constant $= G_1^2 + G_3^2 = G^2$

$'$ - Denotes differentiation with respect to the independent variable

1.1 Introduc

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I. FUNDAMENTALS

. Introduction

Swirling fluid flows have been of interest to scientists and ;ineers since 1797 when Venturi studied the free vortex. Most re- ntly, it seems that swirling flow may appear in the motion encounter- in Wheeler's theory of gravitational collapse [51] and the so-called lack hole" in space.

Generally, swirling flow may be categorized as any flow in ich the tangential velocity component has a finite magnitude such at the fluid has a macroscopic rotary motion. Typical flow examples cur in Hilsch tubes, vortex amplifiers, tornados, large fuel tanks, d rotating fluid machinery.

The system of fundamental equations describing swirling flow the general case includes three-dimensional, nonlinear partial dif- rential equations which are not amenable to solution by standard ana- ical techniques. Therefore, investigators of swirling type flows e compelled to make various simplifying assumptions. All velocity ponents were assumed to be functions of only one independent varia- [6, 11, 14, 15, 16, 30, 36, 47] or particular velocity components e assumed to have only certain special forms [13, 29, 35].

In this work, the major objective is to determine solutions the fundamental equations governing swirling flow. The solutions obtained by accomplishing a reduction in the number of independent

variables uti

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ables utilizing one-parameter group theory and solving the result-
ordinary differential equations.

The minor objective of this work is to discover two sets of
parameter transformation groups such that the system of partial
erential equations for the model in three independent variables is
sformed into a system of partial differential equations in two in-
ndent variables which is then transformed into a system of ordinary
erential equations.

Scope of this Investigation

The fundamental fluid flow model is posed using the three-
nsional system of equations for conservation of mass, conservation
omentum, and conservation of energy.

One-parameter group theory is applied twice in succession to
ce the number of independent variables from three to one. To ac-
lish this reduction, two sets of one-parameter transformation
ps are discovered since the one-parameter group theory allows a re-
ion by one in the number of independent variables per transforma-
. The resulting system of ordinary differential equations is non-
ar. Hamming's predictor-corrector numerical method is used to de-
ine the solutions to this system of equations.

Calculus of variations techniques are applied to the one-
nsional system of equations to determine an optimum solution con-
ined by a transformed swirl parameter.

The one-dimensional solutions are then transformed from one-
nsional space to two-dimensional space and next to three-
nsional space using the two sets of transformations discovered by
application of the continuous one-parameter group theory.

1.3 Group Th

The

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1. Closure.

2. Associat

3. Left Ide

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Group Theory

The axioms [2, 18] which a set of elements must follow under rticular operation are stated below. A star (*) symbol denotes the icular operation between two elements.

Closure. If a and b are two elements of the set, then $a*b = c$ is also a unique element of the set.

Associativity. When three or more elements are formally manipulated using the defined operation, the order of manipulation makes no difference. Hence, $(a*b)*c = a*(b*c)$.

Left Identity. Among the elements there is an identity element, denoted by I, with the property of leaving the elements unchanged after formal manipulation with the defined operation. Therefore, $I*a = a$.

Left Inverse. Each element, a, in the group has a left inverse a^{-1} such that $a^{-1}*a = I$.

Commutativity. When two or more elements are formally manipulated using the defined operation, the order of manipulation may be reversed. Hence, $a*b = b*a$.

ll groups satisfy the fifth axiom. Those groups that satisfy the axiom are known as Abelian or commutative.

Continuous one-parameter groups are multiplicative groups

The groups of interest in this work are continuous one-parameter formation groups. The defined operation for multiplicative

transformatio

tions. Hence

A set of tra

tion of tran

values a^{-1} a

such that A

and

Having defi

cept is now

1. Closure

2. Associ

3. Left I

sformation groups is the successive application of the transforma-
s. Hence a point x is taken into a point X as follows:

$$X = A_{a^2} X^1 = A_{a^2}(A_{a^1} x) \equiv A_{a^2} A_{a^1} x,$$

where a^2 and a^1 are parameters.

t of transformations is said to be closed under the defined opera-
of transformation multiplication if, given any set of parametric
es a^1 and a^2, a set of parametric values a^3 can always be found
that A_{a^3} is a unique member of the set of given transformations

$$A_{a^1} A_{a^2} = A_{a^3}.$$

ng defined the operation for closure, the transformation group con-
is now formulated in terms of the group axioms.

Closure. $A_{a^1} A_{a^2} = A_{a^3}$ (1.3.1)

Associativity. $A_{a^1}(A_{a^2} A_{a^3}) x = (A_{a^1} A_{a^2}) A_{a^3} x$ (1.3.2)

Left Identity. A transformation I exists such that

$$IA_{a^1} x = A_{a^1} I x = A_{a^1} x \qquad (1.3.3)$$

Any transformation which leaves a point unaltered
is defined as an identity transformation.

4. Left Inve

Def

Q is unaltere

A_r, i.e.

$Q(x^1, \ldots$

then the fun

De

$Q_{m+n}(x, y)$,

following de

$$\begin{vmatrix} \frac{\partial Q_1}{\partial x^1} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial Q_1}{\partial y^1} \end{vmatrix}$$

Following t

now restri

of the form

A_1:

eft Inverse. An inverse transformation $A_{a^1}^{-1}$ belonging to the set exists such that

$$A_{a^1}^{-1} A_{a^1} x = I x = x. \tag{1.3.4}$$

Definition 1.3.1 - If a numerically valued (scalar) function unaltered by an r-parameter continuous group of transformations, .e.

$$(x^1,\ldots, x^m, y^1,\ldots, y^n) = Q(\bar{x}^1,\ldots, \bar{x}^m, \bar{y}^1,\ldots, \bar{y}^n), \tag{1.3.5}$$

the function $Q(x, y)$ is called an absolute invariant of A_r.

Definition 1.3.2 - The m+n functions, $Q_1(x, y),\ldots,$ $x, y)$, are said to be functionally independent if and only if the ving determinant is non-vanishing.

$$\begin{vmatrix} \frac{\partial Q_1}{\partial x^i} & \cdots & \frac{\partial Q_{m+n}}{\partial x^i} \\ \vdots & & \vdots \\ \frac{\partial Q_1}{\partial y^j} & \cdots & \frac{\partial Q_{m+n}}{\partial y^j} \end{vmatrix} \neq 0 (i = 1,\ldots, m;\ j = 1,\ldots, n) \tag{1.3.6}$$

ving the approach developed by A.J.A. Morgan [44], attention is estricted to a one-parameter continuous group of transformations e form:

$$A_1: \quad \begin{aligned} \bar{x}^i &= f^i(x^1,\ldots, x^m; a) \qquad (i = 1,\ldots, m;\ m\geq 2) \\ \bar{y}^j &= f^j(y^1,\ldots, y^n; a) \qquad (j = 1,\ldots, n;\ n\geq 1) \end{aligned} \tag{1.3.7}$$

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re a is a numerical parameter. The transformations

$$\bar{x}^i = f^i(x^1,\ldots, x^m; a) \qquad (1.3.8)$$

a subgroup S_{A_1} of A_1. The x^i and the y^j denote the independent the dependent variables of a system of partial differential equa-s. The y^j are considered to be differentiable functions of the x^i o any required order. If the transformations of the partial deriv-es of the y^j with respect to the x^i are appended to those of .7), the resulting set of transformations also forms a continuous parameter group, A_1^k. New groups constructed in this manner are n as enlargements of the group A_1.

The group A_1 (1.3.7) has m+n-1 functionally independent ab-te invariants

$$\eta_1(x^1,\ldots, x^m),\ldots, \eta_{m-1}(x^1,\ldots, x^m) \qquad (1.3.9)$$

$$g_1(y^1,\ldots, y^n, x^1,\ldots, x^m),\ldots, g_n(y^1,\ldots, y^n, x^1,\ldots, x^m) \qquad (1.3.10)$$

it is possible to choose them such that the Jacobian

$$\frac{\partial(g_1,\ldots, g_n)}{\partial(y^1,\ldots, y^n)} \neq 0 \qquad (1.3.11)$$

$$R\left[\frac{\partial(\eta_1,\ldots, \eta_{m-1})}{\partial(x^1,\ldots, x^m)}\right] = m-1 \qquad (1.3.12)$$

e R denotes the rank of the indicated Jacobian matrix.

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The y^j and $\bar{y}^j$ are considered to be implicitly defined as func-
ons of the x^i and $\bar{x}^i$ by the equations

$$z_j(x^1,\ldots, x^m) = g_j(y^1,\ldots, y^n, x^1,\ldots, x^m) \tag{1.3.13}$$

$$\bar{z}_j(\bar{x}^1,\ldots, \bar{x}^m) = g_j(\bar{y}^1,\ldots, \bar{y}^n, \bar{x}^1,\ldots, \bar{x}^m) \tag{1.3.14}$$

ere the g_j are the absolute invariants of the group (1.3.7) indicated
(1.3.10).

<u>Theorem 1.3.1</u> - A necessary and sufficient condition for the
, implicitly defined as functions of $x^1,\ldots, x^m$ by equation (1.3.13),
be exactly the same functions of $x^1,\ldots, x^m$ as the $\bar{y}^j$, implicitly
fined as functions of $\bar{x}^1,\ldots, \bar{x}^m$ by equation (1.3.14), are of the
$,\ldots, \bar{x}^m$ is that

$$z_j(x^1,\ldots, x^m) = \bar{z}_j(\bar{x}^1,\ldots, \bar{x}^m) = z_j(\bar{x}^1,\ldots, \bar{x}^m). \tag{1.3.15}$$

e condition given by equation (1.3.15) can be replaced by the equiva-
nt condition

$$z_j(x^1,\ldots, x^m) = F_j(\eta_1,\ldots, \eta_{m-1}). \tag{1.3.16}$$

e $\eta_1,\ldots, \eta_{m-1}$ are functionally independent absolute invariants of
e subgroup S_{A_1} of the group (1.3.7). The equivalence of equations
.3.15) and (1.3.16) is seen from the group theory statement that any
solute invariant of a group is expressible in terms of the function-
ly independent absolute invariants.

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Definition 1.3.3 - Solutions of a system of partial differ-
ial equations which have the property that the y^j are exactly the
e functions of the x^i as the $\bar{y}^j$ are of the $\bar{x}^i$ are called invariant
utions of a system of partial differential equations.

When considering invariant solutions of systems of partial
ferential equations, Theorem 1.3.1 demonstrates that it is only nec-
ary to consider equations (1.3.13) or (1.3.14) to be given in the
m,

$$g_j(y^1,\ldots, y^n, x^1,\ldots, x^m) = F_j(\eta_1,\ldots, \eta_{m-1}). \tag{1.3.17}$$

equation (1.3.11) and the assumed continuity and differentiability
perties of the functions F_j and g_j, it follows that the y^j can be
ressed as functions of the F_j and x^i in some neighborhood of the
nt $(x^1,\ldots, x^m)$.

$$y^j = y^j(F_1,\ldots, F_n, x^1,\ldots, x^m) \tag{1.3.18}$$

Definition 1.3.4 - A differential form of the k-th order in
ndependent variables is a function of the form

$$\phi(x^1,\ldots, x^m, y^1,\ldots, y^n,\ldots,\frac{\partial^k y^1}{\partial(x^1)^k},\ldots,\frac{\partial^k y^n}{\partial(x^m)^k}) \tag{1.3.19}$$

se arguments include derivatives of the y^j up to order k. The y^j
considered to be functions of the x^i. For convenience, the p argu-
ts of the differential form will be designated by $z^1,\ldots, z^p$.

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Definition 1.3.5 - A differential form Φ is conformally in-
nt under a one-parameter group A_1, if, under the transformations
e group, it satisfies the relation

$$\Phi(\bar{z}^1,\ldots, \bar{z}^p) = M(z^1,\ldots, z^p; a)\ \Phi(z^1,\ldots, z^p) \tag{1.3.20}$$

e Φ is exactly the same function of the z's as it is of the $\bar{z}$'s and
some function of the z's and the parameter a.

Definition 1.3.6 - A differential form Φ is constant conform-
invariant under a one-parameter group A_1, if, under the transfor-
on of the group, it satisfies the relation

$$\Phi(\bar{z}^1,\ldots, \bar{z}^p) = M(a)\ \Phi(z^1,\ldots, z^p) \tag{1.3.21}$$

e Φ is exactly the same function of the z's as it is of the $\bar{z}$'s
M is some function of the parameter a.

Definition 1.3.7 - A differential form Φ is absolutely in-
nt under a one-parameter group A_1, if, under the transformations
e group, it satisfies the relation

$$\Phi(\bar{z}^1,\ldots, \bar{z}^p) = \Phi(z^1,\ldots, z^p) \tag{1.3.22}$$

e Φ is exactly the same function of the z's as it is of the $\bar{z}$'s.

Definition 1.3.8 - A system of k-th order partial differen-
equations $\Phi_j = 0$ is invariant under a continuous one-parameter
p of transformations A_1 if each of the k-th order differential
s Φ_j is conformally invariant under the transformations of A_1^k. The

group A_1^k is t

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Let

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be reduced

up A_1^k is the k-th enlargement of the group A_1 which has been dis-
sed previously.

Let

$$\Phi_j(x^1,\ldots, x^m, y^1,\ldots, y^n,\ldots,\frac{\partial^k y^1}{\partial(x^1)^k},\ldots,\frac{\partial^k y^n}{\partial(x^m)^k}) = 0 \qquad (1.3.23)$$

a system of partial differential equations of the k-th order in
2) independent variables $x^1,\ldots, x^m$ and $n(\geq 1)$ dependent variables
$\ldots, y^n$ where each of the differential forms is conformally invari-
under the k-th enlargement of the group A_1.

<u>Theorem 1.3.2</u> - If each of the differential forms $\Phi_1,\ldots, \Phi_n$
he system of partial differential equations (1.3.23) is conformally
riant under the k-th enlargement of the group (1.3.7), then the in-
ant solutions of (1.3.23) can be expressed in terms of the solutions
system of the form

$$B_j(\eta_1,\ldots, \eta_{m-1}, F_1,\ldots, F_n,\ldots,\frac{\partial^k F_1}{\partial \eta_1^k},\ldots,\frac{\partial^k F_n}{\partial \eta_{m-1}^k}) = 0 \qquad (1.3.24)$$

h is a system of k-th order partial differential equations in m-1
pendent variables $\eta_1,\ldots, \eta_{m-1}$. In equation (1.3.24), the η's are
m-1 absolute invariants of the group S_{A_1} and the functions F_j are
e of equation (1.3.17).

Theorem 1.3.2 gives the conditions under which the number of
pendent variables in a system of partial differential equations can
educed by one in the process of obtaining invariant solutions.

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Optimization Techniques

The optimization techniques used in this work are the stan- techniques from the calculus of variations as presented by Miele The calculus of variations is concerned with the maxima and a of functional expressions where entire functions must be deter- . Hence, the unknown of the calculus of variations problem is ssembly of an infinite set of points which identify either a curve surface, depending upon the nature of the problem.

A functional form to be optimized may be defined as shown .

$$\Omega \equiv [\tau(x, y_k)]_i^f + \int_{x_i}^{x_f} \sigma(x, y_k, y_k') \, dx \qquad (1.4.1)$$

uation (1.4.1), subscripts i and f denote initial and final points $_k' \equiv \frac{dy_k}{dx}$. The form Ω is optimized by finding that special set of ions

$$y_k(x) \qquad k = 1, \ldots, n \qquad (1.4.2)$$

hich it is either a maximum or a minimum. The class of functions 2) satisfy the constraints

$$\phi_j(x, y_k, y_k') = 0 \qquad j = 1, \ldots, p < n \qquad (1.4.3)$$

involve $\pi = n-p$ degrees of freedom. The number of degrees of om of a differential system is defined to be the difference be- the number of dependent variables and the number of equations. assumed that the functions $y_k(x)$ are consistent with the follow- nd conditions.

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$$\omega_\mu(x_i, y_{ki}) = 0 \qquad \mu = 1,\ldots, q \tag{1.4.4}$$

$$\omega_\mu(x_f, y_{kf}) = 0 \qquad \mu = q + 1,\ldots, s \leq 2n+2 \tag{1.4.5}$$

blem as formulated in the preceding paragraph is known as the roblem.

The Bolza problem can be treated simply and elegantly by in-ng a set of variable Lagrange multipliers

$$\lambda_j(x) \qquad j = 1,\ldots, p \tag{1.4.6}$$

Forming the following expression which is known as the augmented n.

$$K = \sigma + \sum_{j=1}^{p} \lambda_j \phi_j \tag{1.4.7}$$

nown that the optimal arc (the special curve optimizing Ω) must not only equations (1.4.3) but also the following equations e known as the Euler-Lagrange equations.

$$\frac{d}{dx}\left(\frac{\partial K}{\partial y_k'}\right) - \frac{\partial K}{\partial y_k} = 0 \qquad k = 1,\ldots, n \tag{1.4.8}$$

erential system composed of the constraining equations (1.4.3) Euler-Lagrange equations (1.4.8) includes n+p equations and un- Therefore, the solution of this sytem of differential equa- elds the n dependent variables and the p Lagrange multipliers eously.

There are 2n+2 boundary conditions to which equations (1.4.3)

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nd (1.4.8) are subjected. Equations (1.4.4) and (1.4.5) supply s of ne boundary conditions and 2n+2-s of the boundary conditions are sup-lied by the following condition which is known as the transversality ondition.

$$[d\tau + (K - \sum_{k=1}^{n} \frac{\partial K}{\partial y_k'} y_k') \, dx + \sum_{k=1}^{n} \frac{\partial K}{\partial y_k'} dy_k]_i^f = 0 \qquad (1.4.9)$$

e transversality condition is to be satisfied identically for all rstems of infinitesimal displacements consistent with equations (1.4.4) d (1.4.5).

After an optimal arc has been determined, it is necessary to vestigate whether the functional Ω attains a maximum or minimum value. e Legendre-Clebsch necessary condition is used to determine this. e Legendre-Clebsch condition for a relative minimum is

$$\sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 K}{\partial y_k' \, \partial y_j'} \delta y_k' \, \delta y_j' \geq 0. \qquad (1.4.10)$$

r a relative maximum, the inequality (1.4.10) is reversed.

There are two particular cases of the Bolza problem. The rst case is when the integrand of equation (1.4.1) is identically ro; that is, when $\sigma \equiv 0$. This case is known as a Mayer problem. The mber of end conditions s is less than 2n+2 for a Mayer problem. The cond case is when the functional to be optimized appears entirely in egral form; that is, when $\tau \equiv 0$ in equation (1.4.1). This case is own as a Lagrange problem. Equations (1.4.8), (1.4.9), and (1.4.10) e valid for both of these particular cases.

Th

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This section reviews only the optimization relations relevant this work. For a comprehensive treatment on calculus of variations, e reader is referred to the literature on calculus of variations. One cellent reference on calculus of variations is Akhiezer [1].

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II. MATHEMATICAL SWIRLING FLOW MODEL

Type of Fluid Flow and Coordinate System

In swirling flow, the fluid has a macroscopic rotary motion e the velocity has a tangential component of finite magnitude. following assumptions are made for this type of flow:

(1) Viscous effects are negligible.

(2) The dependent variables are not functions of time ($\frac{\partial}{\partial t} = 0$).

(3) The fluid is incompressible (ρ = const.).

(4) Gravitational and electromagnetic body force effects are negligible.

(5) The flow is isentropic.

efore, the class of flows of interest are inviscid, isentropic, in- ressible flows.

Since the fluid has a macroscopic rotary motion, the natural e of coordinates is a cylindrical polar coordinate system. These dinates are related to cartesian coordinates by the familiar rela- shown as equations (2.1.1), (2.1.2), and (2.1.3).

$$r^2 = x^2 + y^2 \tag{2.1.1}$$

$$\tan\theta = y/x \tag{2.1.2}$$

$$z = z \tag{2.1.5}$$

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gure 2.1, the cylindrical polar coordinates are shown referred to ic system of cartesian coordinates. Any point not on the z-axis unique representation (r,θ,z) with the restrictions $r \geq 0$ and π.

The vector form for velocity in cylindrical polar coordinates be written as shown by equation (2.1.4).

$$\vec{q} = u\,\hat{e}_r + v\,\hat{e}_\theta + w\,\hat{e}_z \qquad (2.1.4)$$

elocity components at a point in (r,θ,z) space are shown in e 2.2.

Fundamental Equations

The equations governing the types of swirling flow described ction 2.1 are based on the basic laws of conservation of mass, con- ion of momentum, and conservation of energy.

The conservation of mass is given by the continuity equation id dynamics.

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0 \qquad \text{Conservation of Mass (Continuity Equation)} \qquad (2.2.1)$$

For inviscid, isentropic, incompressible flows, the conserva- f momentum is given by the three Euler equations.

$$u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \text{r-Momentum} \qquad (2.2.2)$$

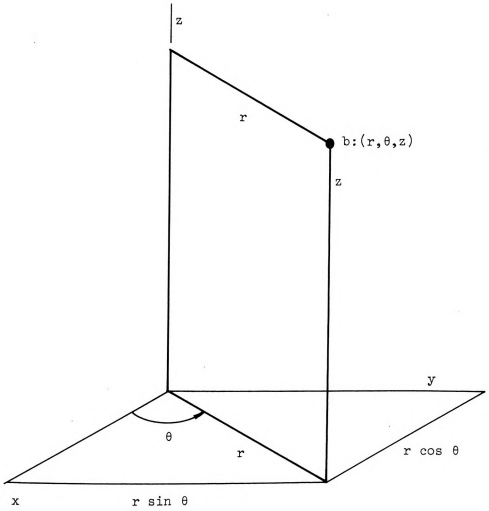

FIGURE 2.1 CYLINDRICAL POLAR COORDINATES

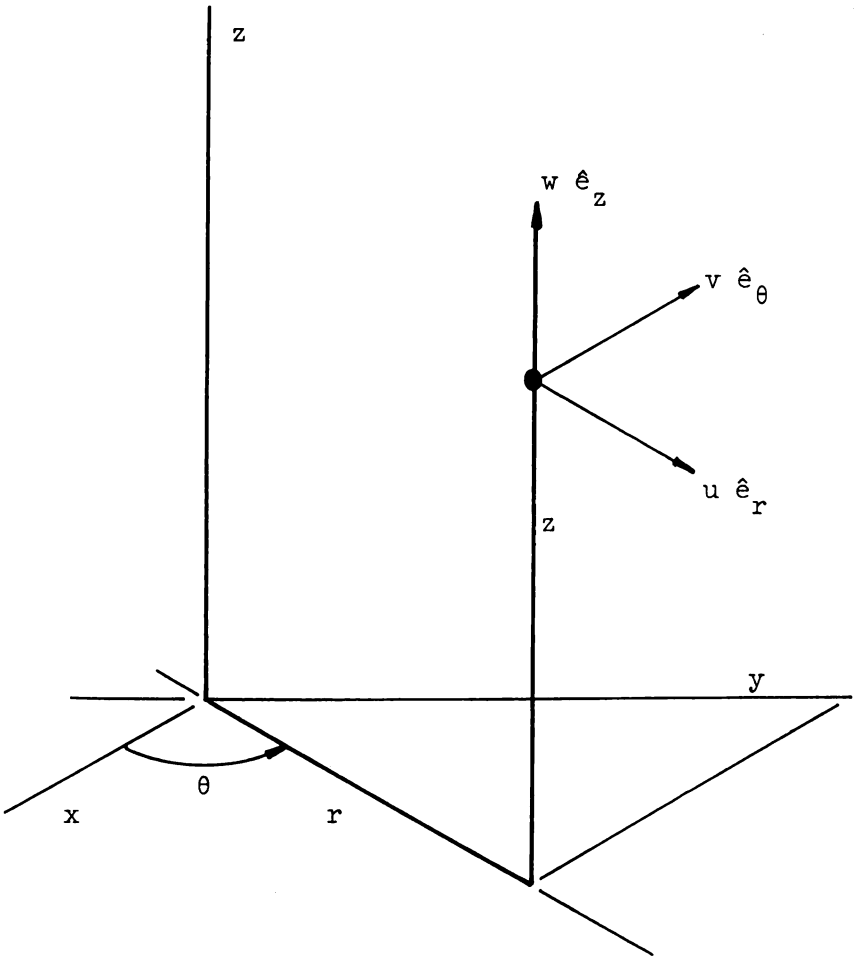

FIGURE 2.2 VELOCITY COMPONENTS

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$$u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \theta} + w \frac{\partial v}{\partial z} + \frac{uv}{r} = - \frac{1}{\rho r} \frac{\partial p}{\partial \theta} \qquad \theta\text{-Momentum} \qquad (2.2.3)$$

$$u \frac{\partial w}{\partial r} + \frac{v}{r} \frac{\partial w}{\partial \theta} + w \frac{\partial w}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial z} \qquad z\text{-Momentum} \qquad (2.2.4)$$

e Euler equations are non-linear, first order, partial differential uations.

The conservation of energy is expressed by the first law of ermodynamics. For the assumed model, the first law of thermodynamics y be written as the familiar steady flow energy equation.

$$Q = \Delta(e + p/\rho + \frac{1}{2}q^2 + gz) \qquad (2.2.5)$$

is the total energy change or difference between the heat energy added the system and the shaft work done by the system. Assuming no shaft rk, isentropic flow, no temperature variations, and negligible gravi- tional body forces, the quantities Q, e, and gz all vanish. For the sumptions mentioned, equation (2.2.5) reduces to Bernoulli's equation:

$$h = p + \frac{1}{2}\rho q^2 = \text{constant}. \qquad (2.2.6)$$

The fundamental governing equations for this investigation e (2.2.1) - conservation of mass; (2.2.2), (2.2.3), and (2.2.4) - nservation of momentum; and (2.2.6) - conservation of energy.

3 Boundary Conditions and Swirl Parameter

The boundary conditions are all specified at the outside swirl dius, r_o, for any θ and any z. The specific values of the dependent riables will be denoted as shown in the following equations. At $= r_o$,

$$u = u_o \tag{2.3.1}$$

$$v = v_o \tag{2.3.2}$$

$$w = w_o \tag{2.3.3}$$

$$p = p_o. \tag{2.3.4}$$

four independent variables have specific values at the outside swirl us and these values are the boundary conditions for this investiga-

.

The swirl parameter is defined to be the product of a geomet- similarity ratio known as the swirl aspect ratio and a kinematic larity ratio known as the tangential energy fraction.

$$S \equiv \Pi_1 \Pi_2 \qquad \text{Swirl Parameter} \tag{2.3.1}$$

$$\Pi_1 \equiv r^2/z^2 \qquad \text{Swirl Aspect Ratio} \tag{2.3.2}$$

$$\Pi_2 \equiv \frac{v^2}{h/\rho} \qquad \text{Tangential Energy Fraction} \tag{2.3.3}$$

ituting equations (2.3.2) and (2.3.3) into equation (2.3.1), the parameter equation becomes

$$S = \frac{r^2}{z^2} \frac{v^2}{h/\rho}. \tag{2.3.4}$$

uantity h in equation (2.3.4) is determined from Bernoulli's equa- (2.2.6).

Non-Dimensional Equations

For greater generality and convenience, all variables are dimensionless. A reference length, z_o, which is the axial length ne swirl, reduces the radial and axial lengths into dimensionless linates. Similarly, the velocity components and static pressure nade dimensionless by comparing them to the tangential velocity, and the tangential dynamic pressure, ρv_o^2. The dimensionless vari- s are defined as follows:

$$R \equiv r/z_o \tag{2.4.1}$$

$$Z \equiv z/z_o \tag{2.4.2}$$

$$U \equiv u/v_o \tag{2.4.3}$$

$$V \equiv v/v_\Theta \tag{2.4.4}$$

$$W \equiv w/v_o \tag{2.4.5}$$

$$P \equiv p/\rho v_o^2. \tag{2.4.6}$$

Application of the dimensionless variables to equation (2.2.2) ls the following intermediate result.

$$Uv_o^2 \frac{\partial U}{z_o \partial R} + \frac{Vv_o^2}{Rz_o}\frac{\partial U}{\partial \theta} + Wv_o^2 \frac{\partial U}{z_o \partial Z} - \frac{v_o^2 V^2}{z_o R} = -\frac{1}{\rho}\rho v_o^2 \frac{\partial P}{z_o \partial R}$$

result reduces to

$$U\frac{\partial U}{\partial R} + \frac{V}{R}\frac{\partial U}{\partial \theta} + W\frac{\partial U}{\partial Z} - \frac{V^2}{R} = -\frac{\partial P}{\partial R}. \qquad \text{R-Momentum} \tag{2.4.7}$$

Similar applications of the dimensionless variables to equations (2.2.3), (2.2.4), (2.2.1), and (2.2.6) yield the following set of non-dimensional equations:

$$U \frac{\partial V}{\partial R} + \frac{V}{R} \frac{\partial V}{\partial \theta} + W \frac{\partial V}{\partial Z} + \frac{UV}{R} = - \frac{1}{R} \frac{\partial P}{\partial \theta} \qquad \theta\text{-Momentum} \qquad (2.4.8)$$

$$U \frac{\partial W}{\partial R} + \frac{V}{R} \frac{\partial W}{\partial \theta} + W \frac{\partial W}{\partial Z} = - \frac{\partial P}{\partial Z} \qquad \text{Z-Momentum} \qquad (2.4.9)$$

$$\frac{\partial U}{\partial R} + \frac{U}{R} + \frac{1}{R} \frac{\partial V}{\partial \theta} + \frac{\partial W}{\partial Z} = 0 \qquad \text{Conservation of Mass} \qquad (2.4.10)$$

$$H = P + 1/2(U^2 + V^2 + W^2) \qquad \text{Bernoulli Equation.} \qquad (2.4.11)$$

The quantity H in equation (2.4.11) is defined to be

$$H \equiv h/\rho v_o^2. \qquad (2.4.12)$$

The dimensionless boundary conditions at $R = R_o$ are:

$$U = U_o \qquad (2.4.13)$$

$$V = V_o = 1 \qquad (2.4.14)$$

$$W = W_o \qquad (2.4.15)$$

$$P = P_o. \qquad (2.4.16)$$

When converted to dimensionless variables, the swirl parameter (2.3.4) becomes

$$S = \frac{R^2}{Z^2} \frac{V^2}{H}. \qquad (2.4.17)$$

Equations (2.4.7), (2.4.8), (2.4.9), (2.4.10), and (2.4.11) are the non-dimensional forms of the fundamental governing equations for this work.

III. REDUCTION OF THE NUMBER OF INDEPENDENT VARIABLES

In this chapter, the ideas outlined in section 1.3 are applied to the non-dimensional model equations given in section 2.4. The number of independent variables is reduced from three to two and then from two to one.

3.1 Three to Two Reduction in the Number of Independent Variables

The three independent variables (R,θ,Z) are reduced to the two independent variables (η_1,η_2). The new independent variables (η_1,η_2) are functionally independent absolute invariants of a subgroup of the one-parameter continuous group of transformations, A_1. The one-parameter continuous group of transformations, A_1, will subsequently be referred to as the group A_1.

The group A_1 is chosen as shown below:

$$A_1: \quad \begin{array}{lll} \bar{R} = e^{at_1} R & \bar{U} = e^{at_4} U & \bar{P} = e^{at_7} P \\ \bar{\theta} = \theta + at_2 & \bar{V} = e^{at_5} V & \\ \bar{Z} = e^{at_3} Z & \bar{W} = e^{at_6} W & \end{array} \qquad (3.1.1)$$

where t_1, t_2, t_3, t_4, t_5, t_6, and t_7 are constants and a is the group parameter. The group labeled as (3.1.1) has the general form specified by equation (1.3.7). The transformed independent variables are functions of the parameter a and the previous independent variables.

ilarly, the transformed dependent variables are functions of the ameter a and the previous dependent variables. There is no unique hod for specifying groups such as (3.1.1) and the form for θ is t proposed by Hall [24, pg. 93]. By specifying θ as shown, the au- r discovered that the group A_1 would satisfy all the required con- ions to affect a reduction by one in the number of independent vari- es for the system of equations given in section 2.4.

The group A_1 (3.1.1) must satisfy the four group axioms given equations (1.3.1), (1.3.2), (1.3.3), and (1.3.4). To demonstrate satisfaction of the group axioms, only the transformations on R and re considered. All the remaining transformations are similar to the nsformation on R and if the R transformation satisfies the group ax- s, so will the remaining transformations.

The closure axiom, equation (1.3.1), is applied using the bol A_{a^i} to denote a particular transformation utilizing the parameter

$$A_{a^1} A_{a^2} R = A_{a^1}(e^{a^2 t_1} R) = A_{a^1} \bar{R} = e^{a^1 t_1} \bar{R} = e^{a^1 t_1} e^{a^2 t_1} R = e^{(a^1 + a^2) t_1} R = e^{a^3 t_1} R \equiv A_{a^3} R \tag{3.1.2}$$

$$A^1_{a^1} A^1_{a^2} \theta = A^1_{a^1}(\theta + a^2 t_2) = A^1_{a^1} \bar{\theta} = \bar{\theta} + a^1 t_2 = \theta + a^2 t_2 + a^1 t_2 = \theta + (a^1 + a^2) t_2 = \theta + a^3 t_2 \equiv A^1_{a^3} \theta \tag{3.1.3}$$

$$\text{Where} \qquad a^1 + a^2 \equiv a^3. \tag{3.1.4}$$

closure axiom is satisfied since A_{a^3} and $A^1_{a^3}$ have been shown to be que members of the set of given transformations.

Equation (1.3.2), the associativity axiom, is applied where denotes a particular transformation utilizing the parameter a^i.

$$a^1(A_{a^2} A_{a^3}) R = A_{a^1} e^{(a^2 + a^3)t_1} R = e^{a^1 t_1} e^{(a^2 + a^3)t_1} R = e^{[a^1 + (a^2 + a^3)]t_1} R \tag{3.1.5}$$

$$(A_{a^1} A_{a^2}) A_{a^3} R = A_{a^1} A_{a^2} e^{a^3 t_1} R = e^{(a^1 + a^2)t_1} e^{a^3 t_1} R = e^{[(a^1 + a^2) + a^3]t_1} R \tag{3.1.6}$$

ce equation (3.1.5) equals (3.1.6), the associativity axiom is satis-d for the type of transformation employed on R.

$$(A^1_{a^2} A^1_{a^3}) \theta = A^1_{a^1}(\theta + (a^2 + a^3)t_2) = \theta + [a^1 + (a^2 + a^3)]t_2 \tag{3.1.7}$$

$$1 A^1_{a^2}) A^1_{a^3} \theta = A^1_{a^1} A^1_{a^2} (\theta + a^3 t_2) = \theta + [(a^1 + a^2) + a^3]t_2 \tag{3.1.8}$$

ation (3.1.7) equals equation (3.1.8) which demonstrates that the ociativity axiom is satisfied also for the type of transformation loyed on θ.

Any transformation which leaves a point unaltered is by defi-ion an identity transformation. For the group A_1, it is seen by in-tion that this definition is satisfied when the parameter a equals). The left identity axiom, equation (1.3.3), is satisfied when

$$a = 0. \tag{3.1.9}$$

= 0,

$$\bar{R} = e^{at_1} R = R \tag{3.1.10}$$

$$\bar{\theta} = \theta + at_2 = \theta. \tag{3.1.11}$$

For the types of transformations employed on R and θ, it is by inspection that the inverse transformation is obtained by allow- the parameter a to be negative. Therefore, the left inverse axiom, tion (1.3.4), is satisfied when

$$a = -a. \tag{3.1.12}$$

example,

$$A_a^{-1} A_a R = e^{-at_1} e^{at_1} R = IR = R \tag{3.1.13}$$

$$A_a \theta = A_a^{-1}(\theta + at_2) = A_a^{-1} \bar{\theta} = \bar{\theta} - at_2 = \theta + at_2 - at_2 = \theta. \tag{3.1.14}$$

group A_1 (3.1.1) satisfies the four group axioms for closure, asso- vity, left identity, and left inverse.

Theorem 1.3.2 gives the conditions under which the number of endent variables in a system of partial differential equations can duced by one in the process of obtaining invariant solutions. The imensional equations given in section 2.4 must be conformally in- nt under the k-th enlargement of the group A_1 (3.1.1). Conformal iance is defined by Definition 1.3.5 and given in equation form by ion (1.3.20). For convenience, equation (1.3.20) for conformal in- nce is displayed in this section.

$$\Phi(\bar{z}^1, \ldots, \bar{z}^p) = M(z^1, \ldots, z^p; a)\, \Phi(z^1, \ldots, z^p) \tag{1.3.20}$$

fy conformal invariance, the group A_1 (3.1.1) is formally sub-
d into an equation and the mathematical manipulations indicated
equation are performed. The resulting form must then be the
equation (1.3.20) for the equation to be conformally invariant.

The group A_1 (3.1.1) is formally substituted into equation
the R-Momentum equation.

$$\bar{U}\frac{\partial \bar{U}}{\partial \bar{R}} + \frac{\bar{V}}{\bar{R}}\frac{\partial \bar{U}}{\partial \bar{\theta}} + \bar{W}\frac{\partial \bar{U}}{\partial \bar{Z}} - \frac{\bar{V}^2}{\bar{R}} + \frac{\partial \bar{P}}{\partial \bar{R}} = e^{a(2t_4-t_1)}\, U\frac{\partial U}{\partial R}$$

$$(t_4+t_5-t_1)\ \frac{V}{R}\frac{\partial U}{\partial \theta} + e^{a(t_6+t_4-t_3)}\, W\frac{\partial U}{\partial Z} - e^{a(2t_5-t_1)}\frac{V^2}{R}$$

$$+ e^{a(t_7-t_1)}\frac{\partial P}{\partial R} \qquad (3.1.15)$$

ential terms in equation (3.1.15) must equal each other to ob-
necessary form for conformal invariance as indicated by equa-
.20). Equating the exponential terms, the following results
ned:

$$_1 = t_4 + t_5 - t_1 = t_6 + t_4 - t_3 = 2t_5 - t_1 = t_7 - t_1, \quad (3.1.16)$$

$$t_4 = t_5,\ t_4 = t_6 + t_1 - t_3,\ t_7 = 2t_4. \qquad (3.1.17)$$

dent from equation (3.1.15) that the function M of equation
is only a function of the parameter a. By Definition 1.3.6,
rential form represented by equation (3.1.15) is constant con-
invariant.

The group A_1 (3.1.1) is formally substituted into equation
he θ-Momentum equation.

$$\bar{U}\frac{\partial\bar{V}}{\partial\bar{R}} + \frac{\bar{V}}{\bar{R}}\frac{\partial\bar{V}}{\partial\bar{\theta}} + \bar{W}\frac{\partial\bar{V}}{\partial\bar{Z}} + \frac{\bar{U}\bar{V}}{\bar{R}} + \frac{1}{\bar{R}}\frac{\partial\bar{P}}{\partial\bar{\theta}} = e^{a(t_4+t_5-t_1)}\, U\frac{\partial V}{\partial R}$$

$$e^{a(2t_5-t_1)}\frac{V}{R}\frac{\partial V}{\partial\theta} + e^{a(t_6+t_5-t_3)}\, W\frac{\partial V}{\partial Z} + e^{a(t_4+t_5-t_1)}\frac{UV}{R}$$

$$+ e^{a(t_7-t_1)}\frac{1}{R}\frac{\partial P}{\partial\theta} \qquad (3.1.18)$$

ponential terms in equation (3.1.18) must equal each other to ob-
he necessary form for conformal invariance as indicated by equa-
.3.20). Equating the exponential terms, the following results
ained:

$$t_4 + t_5 - t_1 = 2t_5 - t_1 = t_6 + t_5 - t_3 = t_7 - t_1, \qquad (3.1.19)$$

$$t_4 = t_5,\ t_5 = t_6 + t_1 - t_3,\ t_7 = 2t_4. \qquad (3.1.20)$$

son of equations (1.3.20) and (3.1.18) indicates that the func-
of equation (1.3.20) is only a function of the parameter a. By
ion 1.3.6, the differential form represented by equation (3.1.18)
tant conformally invariant.

The group A_1 (3.1.1) is formally substituted into equation
, the Z-Momentum equation.

$$\bar{U}\frac{\partial\bar{W}}{\partial\bar{R}} + \frac{\bar{V}}{\bar{R}}\frac{\partial\bar{W}}{\partial\bar{\theta}} + \bar{W}\frac{\partial\bar{W}}{\partial\bar{Z}} + \frac{\partial\bar{P}}{\partial\bar{Z}} = e^{a(t_4+t_6-t_1)}\, U\frac{\partial W}{\partial R}$$

$$+ e^{a(t_5+t_6-t_1)}\frac{V}{R}\frac{\partial W}{\partial\theta} + e^{a(2t_6-t_3)}\, W\frac{\partial W}{\partial Z}$$

$$+ e^{a(t_7-t_3)}\frac{\partial P}{\partial Z} \qquad (3.1.21)$$

obtain the necessary form for conformal invariance as indicated by uation (1.3.20), the exponential terms in equation (3.1.21) must ual each other. The following results are obtained by equating the ponential terms:

$$t_4 + t_6 - t_1 = t_5 + t_6 - t_1 = 2t_6 - t_3 = t_7 - t_3, \tag{3.1.22}$$

$$t_4 = t_5, \; 2t_6 = t_7. \tag{3.1.23}$$

mparison of equations (3.1.17), (3.1.20), and (3.1.23) yields the sult,

$$t_4 = t_5 = t_6. \tag{3.1.24}$$

om equations (3.1.22) and (3.1.24), it follows that

$$t_1 = t_3. \tag{3.1.25}$$

mparison of equations (3.1.20) and (3.1.21) indicates that the func- on M in equation (1.3.20) is only a function of the parameter a. erefore, the differential form represented by equation (3.1.21) is nstant conformally invariant according to Definition 1.3.6.

The group A_1 (3.1.1) is formally substituted into equation .4.10), the Conservation of Mass equation.

$$\frac{\partial \bar{U}}{\partial \bar{R}} + \frac{\bar{U}}{\bar{R}} + \frac{1}{\bar{R}}\frac{\partial \bar{V}}{\partial \bar{\theta}} + \frac{\partial \bar{W}}{\partial \bar{Z}} = e^{a(t_4-t_1)}\frac{\partial U}{\partial R} + e^{a(t_4-t_1)}\frac{U}{R}$$

$$+ e^{a(t_5-t_1)}\frac{1}{R}\frac{\partial V}{\partial \theta} + e^{a(t_6-t_3)}\frac{\partial W}{\partial Z} \tag{3.1.26}$$

e exponential terms in equation (3.1.26) must equal each other to ob-
ain the necessary form for conformal invariance as indicated by equa-
on (1.3.20). Equating the exponential terms, the following results
e obtained:

$$t_4 = t_5,\ t_5 = t_6 + t_1 - t_3,\ t_4 = t_5 = t_6. \tag{3.1.27}$$

uations (3.1.27) agree with the results obtained from the previous
nformal invariance verifications. Again, by comparison of equations
.3.20) and (3.1.26), the function M in equation (1.3.20) is seen to
only a function of the parameter a. Therefore, the differential
rm represented by equation (3.1.26) is constant conformally invariant
cording to Definition 1.3.6.

The group A_1 (3.1.1) is formally substituted into equation
.4.11), the Bernoulli equation:

$$H - \overline{P} - 1/2\ (\overline{U}^2 + \overline{V}^2 + \overline{W}^2) = H - e^{at7}\ P - 1/2\ (e^{a2t_4}\ U^2 + e^{a2t_5}\ V^2 + e^{a2t_6}\ W^2). \tag{3.1.28}$$

r conformal invariance, the exponential terms in equation (3.1.28)
st equal each other as indicated by equation (1.3.20). The results
equating the exponential terms are shown below:

$$t_7 = 2t_4,\ t_4 = t_5 = t_6. \tag{3.1.29}$$

uations (3.1.29) agree with the results obtained previously. Compari-
n of equations (3.1.28) and (1.3.20) indicates that the function M in
uation (1.3.20) is a function only of the parameter a. By Definition
3.6, the differential form represented by equation (3.1.28) is

constant conformally invariant.

By inspection, it is seen that equation (2.4.17) for the Swirl Parameter is constant conformally invariant under the group A_1 (3.1.1).

Since the equations (2.4.7), (2.4.8), (2.4.9), (2.4.10), (2.4.11), and (2.4.17) are all constant conformally invariant under the k-th enlargement of the group A_1 (3.1.1), Theorem 1.3.2 states that the number of independent variables for the system of equations mentioned above can be reduced by one in the process of obtaining invariant solutions.

Using the results (3.1.17), (3.1.24), and (3.1.25) obtained from the conformal invariance verifications, the group A_1 (3.1.1) is modified as shown below:

$$A_1: \quad \begin{array}{lll} \overline{R} = e^{at_1} R & \overline{U} = e^{at_4} U & \overline{P} = e^{2at_4} P \\ \overline{\theta} = \theta + at_2 & \overline{V} = e^{at_4} V & \\ \overline{Z} = e^{at_1} Z & \overline{W} = e^{at_4} W & \end{array} \qquad (3.1.30)$$

where t_1, t_2, and t_4 are constants and a is the group parameter.

The group A_1 (3.1.30) has m+n-1 = 6 functionally independent absolute invariants. Equation (1.3.9) gives the form for m-1 = 2 of the absolute invariants and equation (1.3.10) gives the form for the remaining n = 4 absolute invariants. Definition 1.3.1 and equation (1.3.5) give the definition and general form for an absolutely invariant form. The manner in which absolute invariants are selected is not well defined. The functions η_1 and η_2 are chosen as follows:

$$\eta_1 = e^{-\theta} R^p \tag{3.1.31}$$

$$\eta_2 = e^{-\theta} Z^b. \tag{3.1.32}$$

For the functions in equations (3.1.31) and (3.1.32) to be absolute invariants, the following relations must hold:

$$\eta_1(\overline{\theta},\overline{R}) = \eta_1(\theta,R) \tag{3.1.33}$$

$$\eta_2(\overline{\theta},\overline{Z}) = \eta_2(\theta,Z). \tag{3.1.34}$$

The values of the exponents p and b of equations (3.1.31) and (3.1.32) must be determined such that equations (3.1.33) and (3.1.34) are true. The required tranformations from the group A_1 (3.1.30) are utilized in the following equations to determine p and b:

$$\eta_1 = e^{-\overline{\theta}} \overline{R}^p = e^{-(\theta+at_2)} e^{pat_1} R^p = e^{a(pt_1-t_2)} e^{-\theta} r^p$$

$$p = t_2/t_1 \tag{3.1.35}$$

$$\eta_2 = e^{-\overline{\theta}} \overline{Z}^b = e^{-(\theta+at_2)} e^{bat_1} Z^b = e^{a(bt_1-t_2)} e^{-\theta} Z^b$$

$$b = t_2/t_1 \tag{3.1.36}$$

Equations (3.1.35) and (3.1.36) are substituted into equations (3.1.31) and (3.1.32):

$$\eta_1 = e^{-\theta} R^{t_2/t_1} \tag{3.1.37}$$

$$\eta_2 = e^{-\theta} Z^{t_2/t_1}. \tag{3.1.38}$$

functions f_1, f_2, f_3, and f_4 are chosen to resemble η_1 and η_2:

$$f_1 = Ue^{-\delta_1\theta},\ f_2 = Ve^{-\delta_2\theta},\ f_3 = We^{-\delta_3\theta},\ f_4 = Pe^{-\delta_4\theta}. \tag{3.1.39}$$

the functions in equations (3.1.39) to be absolute invariants, the
lowing relations must hold:

$$f_1(\overline{U}, \overline{\theta}) = f_1(U, \theta),\ f_2(\overline{V}, \overline{\theta}) = f_2(V, \theta), \tag{3.1.40}$$

$$f_3(\overline{W}, \overline{\theta}) = f_3(W, \theta),\ f_4(\overline{P}, \overline{\theta}) = f_4(P, \theta). \tag{3.1.41}$$

values of the exponents δ_1, δ_2, δ_3, and δ_4 of equations (3.1.39)
be determined such that equations (3.1.40) and (3.1.41) are true.
required transformations from the group A_1 (3.1.30) are utilized in
following equations to determine the exponents δ_1, δ_2, δ_3, and δ_4:

$$f_1 = \overline{U}e^{-\delta_1\overline{\theta}} = e^{at_4}\ Ue^{-\delta_1(\theta+at_2)} = e^{a(t_4-\delta_1 t_2)}\ Ue^{-\delta_1\theta}$$

$$\delta_1 = t_4/t_2 \tag{3.1.42}$$

$$f_2 = \overline{V}e^{-\delta_2\overline{\theta}} = e^{at_4}\ Ve^{-\delta_2(\theta+at_2)} = e^{a(t_4-\delta_2 t_2)}\ Ve^{-\delta_2\theta}$$

$$\delta_2 = t_4/t_2 \tag{3.1.43}$$

$$f_3 = \overline{W}e^{-\delta_3\overline{\theta}} = e^{at_4}\ We^{-\delta_3(\theta+at_2)} = e^{a(t_4-\delta_3 t_2)}\ We^{-\delta_3\theta}$$

$$\delta_3 = t_4/t_2 \tag{3.1.44}$$

$$f_4 = \overline{P}e^{-\delta_4\overline{\theta}} = e^{2at_4}\ Pe^{-\delta_4(\theta+at_2)} = e^{a(2t_4-\delta_4 t_2)}\ Pe^{-\delta_4\theta}$$

$$\delta_4 = 2t_4/t_2. \tag{3.1.45}$$

e values of the δ_i's are substituted into equations (3.1.39) to ob-
in the following results:

$$= Ue^{-t_4/t_2\theta},\ f_2 = Ve^{-t_4/t_2\theta},\ f_3 = We^{-t_4/t_2\theta},\ f_4 = Pe^{-2t_4/t_2\theta}. \tag{3.1.46}$$

The condition given by equation (1.3.11) must be satisfied by
e chosen functions f_1, f_2, f_3, and f_4.

$$J = \frac{\partial(f_1,\ f_2,\ f_3,\ f_4)}{\partial(U,\ V,\ W,\ P)} \neq 0 \tag{3.1.47}$$

$$J = \begin{vmatrix} \frac{\partial f_1}{\partial U} & \frac{\partial f_1}{\partial V} & \frac{\partial f_1}{\partial W} & \frac{\partial f_1}{\partial P} \\ \frac{\partial f_2}{\partial U} & \frac{\partial f_2}{\partial V} & \frac{\partial f_2}{\partial W} & \frac{\partial f_2}{\partial P} \\ \frac{\partial f_3}{\partial U} & \frac{\partial f_3}{\partial V} & \frac{\partial f_3}{\partial W} & \frac{\partial f_3}{\partial P} \\ \frac{\partial f_4}{\partial U} & \frac{\partial f_4}{\partial V} & \frac{\partial f_4}{\partial W} & \frac{\partial f_4}{\partial P} \end{vmatrix} \tag{3.1.48}$$

ng equations (3.1.46), equation (3.1.48) becomes:

$$J = \begin{vmatrix} e^{-t_4/t_2\theta} & 0 & 0 & 0 \\ 0 & e^{-t_4/t_2\theta} & 0 & 0 \\ 0 & 0 & e^{-t_4/t_2\theta} & 0 \\ 0 & 0 & 0 & e^{-2t_4/t_2\theta} \end{vmatrix}.$$

The Jacobian determinant J is non-vanishing for the chosen functions f_1, f_2, f_3, and f_4. Therefore, the condition (3.1.47) is satisfied.

The condition given by equation (1.3.12) must be satisfied by the functions η_1 and η_2.

$$R\left[\frac{\partial(\eta_1,\ \eta_2)}{\partial(R,\ \theta,\ Z)}\right] = m-1 = 2 \tag{3.1.49}$$

The rank of the indicated matrix must equal 2. Expanding the matrix given in equation (3.1.49) gives the following result:

$$\begin{bmatrix} \frac{\partial\eta_1}{\partial R} & \frac{\partial\eta_1}{\partial\theta} & \frac{\partial\eta_1}{\partial Z} \\ \\ \frac{\partial\eta_2}{\partial R} & \frac{\partial\eta_2}{\partial\theta} & \frac{\partial\eta_2}{\partial Z} \end{bmatrix} \equiv M. \tag{3.1.50}$$

Using equations (3.1.37) and (3.1.38), equation (3.1.50) becomes:

$$\begin{bmatrix} \frac{\xi}{R}\eta_1 & -\eta_1 & 0 \\ \\ 0 & -\eta_2 & \frac{\xi}{Z}\eta_2 \end{bmatrix} = M, \tag{3.1.51}$$

$$\text{where } \xi \equiv t_2/t_1. \tag{3.1.52}$$

Since M is a 2x3 matrix and the row rank equals the column rank for a matrix, 2 is the maximum rank that M in (3.1.51) can have. Both rows of M in (3.1.51) are linearly independent and hence, M has rank of 2. The condition indicated by equation (3.1.49) is satisfied for the chosen functions η_1 and η_2.

It has been demonstrated that equations (3.1.37), (3.1.38), (3.1.46) represent the functionally independent absolute invariants the group A_1 (3.1.30).

Application of equation (1.3.17) to the invariants of the up A_1 (3.1.30) yields the following result:

$$f_j(U, V, W, P, R, \theta, Z) = F_j(\eta_1, \eta_2) \qquad j = 1, 2, 3, 4. \tag{3.1.53}$$

ations (3.1.46) are rearranged using the symbolic equation (3.1.53):

$$U = e^{\alpha\theta} F_1(\eta_1, \eta_2), \; V = e^{\alpha\theta} F_2(\eta_1, \eta_2), \tag{3.1.54}$$

$$W = e^{\alpha\theta} F_3(\eta_1, \eta_2), \; P = e^{2\alpha\theta} F_4(\eta_1, \eta_2), \tag{3.1.55}$$

$$\text{where } \alpha \equiv t_4/t_2. \tag{3.1.56}$$

For convenience, equations (3.1.44) and (3.1.45) and the par-l derivatives of these equations are listed together.

$$\eta_1 = e^{-\theta} R^{\xi} \tag{3.1.57}$$

$$\eta_2 = e^{-\theta} Z^{\xi} \tag{3.1.58}$$

$$\begin{aligned} \frac{\partial \eta_1}{\partial Z} &= 0 & \frac{\partial \eta_2}{\partial R} &= 0 \\ \frac{\partial \eta_1}{\partial \theta} &= -\eta_1 & \frac{\partial \eta_2}{\partial \theta} &= -\eta_2 \\ \frac{\partial \eta_1}{\partial R} &= \frac{\xi}{R}\eta_1 & \frac{\partial \eta_2}{\partial Z} &= \frac{\xi}{Z}\eta_2 \end{aligned} \tag{3.1.59}$$

The relations (3.1.54) through (3.1.59) are now used to re-
e the number of independent variables in the system of equations
ven in section 2.4. The three independent variables (R, θ, Z) are
luced to the two independent variables (η_1, η_2) in equations (2.4.7),
4.8), (2.4.9), (2.4.10), (2.4.11), and (2.4.17). Figures 3.1
rough 3.6 illustrate the transformation details and give the final
sult for each of these equations. Note that the final result dis-
yed in each figure is obtained by summing all the transformed terms
ven in the figure.

$$\frac{\partial}{\partial R} = e^{2\alpha\theta} F_1 \frac{\partial}{\partial R} = e^{2\alpha\theta} F_1\left(\frac{\partial}{\partial \eta_1}\frac{\partial}{\partial R} + \frac{\partial}{\partial \eta_2}\frac{\partial}{\partial R}\right) = e^{2\alpha\theta} \frac{\xi}{R}\eta_1 F_1 \frac{\partial}{\partial \eta_1} \tag{3.1.60}$$

$$\frac{V}{R}\frac{\partial U}{\partial \theta} = \frac{e^{2\alpha\theta} F_2}{R}\left(\frac{\partial F_1}{\partial \theta} + \alpha F_1\right) = e^{2\alpha\theta}\frac{F_2}{R}\left(\frac{\partial F_1}{\partial \eta_1}\frac{\partial \eta_1}{\partial \theta} + \frac{\partial F_1}{\partial \eta_2}\frac{\partial \eta_2}{\partial \theta} + \alpha F_1\right) = \frac{e^{2\alpha\theta}}{R}\left(F_2\left(-\eta_1\frac{\partial F_1}{\partial \eta_1} - \eta_2\frac{\partial F_1}{\partial \eta_2}\right) + \alpha F_1 F_2\right) \tag{3.1.61}$$

$$W\frac{\partial U}{\partial Z} = e^{2\alpha\theta} F_3 \frac{\partial F_1}{\partial Z} = e^{2\alpha\theta} F_3\left(\frac{\partial F_1}{\partial \eta_1}\frac{\partial \eta_1}{\partial Z} + \frac{\partial F_1}{\partial \eta_2}\frac{\partial \eta_2}{\partial Z}\right) = e^{2\alpha\theta} F_3 \frac{\xi}{Z}\eta_2 \frac{\partial F_1}{\partial \eta_2} \tag{3.1.62}$$

$$-\frac{V^2}{R} = -e^{2\alpha\theta}\frac{F_2^2}{R} \tag{3.1.63}$$

$$-\frac{\partial P}{\partial R} = -e^{2\alpha\theta}\frac{\partial F_4}{\partial R} = -e^{2\alpha\theta}\left(\frac{\partial F_4}{\partial \eta_1}\frac{\partial \eta_1}{\partial R} + \frac{\partial F_4}{\partial \eta_2}\frac{\partial \eta_2}{\partial R}\right) = -e^{2\alpha\theta}\frac{\xi}{R}\eta_1\frac{\partial F_4}{\partial \eta_1} \tag{3.1.64}$$

$$\frac{R}{Z} = \left(\frac{\eta_1}{\eta_2}\right)^{1/\xi} \qquad \text{(From equations (3.1.57) and (3.1.58).)} \tag{3.1.65}$$

$$(\xi F_1 - F_2)\,\eta_1\frac{\partial F_1}{\partial \eta_1} + \left(\xi F_3\left(\frac{\eta_1}{\eta_2}\right)^{1/\xi} - F_2\right)\eta_2\frac{\partial F_1}{\partial \eta_2} - F_2^2 + \alpha F_1 F_2 = \xi\eta_1\frac{\partial F_4}{\partial \eta_1} \tag{3.1.66}$$

FIGURE 3.1 $(R, \theta, Z) \to (\eta_1, \eta_2)$ REDUCTION R-MOMENTUM EQUATION (2.4.7)

$$U\frac{\partial V}{\partial R} = e^{2\alpha\theta} F_1 \frac{\partial F_2}{\partial R} = e^{2\alpha\theta} F_1\left(\frac{\partial F_2}{\partial \eta_1}\frac{1}{R} + \frac{\partial F_2}{\partial \eta_2}\frac{\partial \eta_2}{\partial R}\right) = e^{2\alpha\theta} F_1 \frac{\xi}{R}\eta_1 \frac{\partial F_2}{\partial \eta_1} \tag{3.1.67}$$

$$\frac{V}{R}\frac{\partial V}{\partial \theta} = e^{2\alpha\theta}\frac{F_2}{R}\left(\frac{\partial F_2}{\partial \theta} + \alpha F_2\right) = e^{2\alpha\theta}\frac{F_2}{R}\left(\frac{\partial F_2}{\partial \eta_1}\frac{\partial \eta_1}{\partial \theta} + \frac{\partial F_2}{\partial \eta_2}\frac{\partial \eta_2}{\partial \theta} + \alpha F_2\right) = \frac{e^{2\alpha\theta}}{R}\left(F_2\left(-\eta_1\frac{\partial F_2}{\partial \eta_1} - \eta_2\frac{\partial F_2}{\partial \eta_2}\right) + \alpha F_2^2\right) \tag{3.1.68}$$

$$W\frac{\partial V}{\partial Z} = e^{2\alpha\theta} F_3 \frac{\partial F_2}{\partial Z} = e^{2\alpha\theta} F_3\left(\frac{\partial F_2}{\partial \eta_1}\frac{\partial \eta_1}{\partial Z} + \frac{\partial F_2}{\partial \eta_2}\frac{\partial \eta_2}{\partial Z}\right) = e^{2\alpha\theta} F_3 \frac{\xi}{Z}\eta_2 \frac{\partial F_2}{\partial \eta_2} \tag{3.1.69}$$

$$\frac{UV}{R} = \frac{e^{2\alpha\theta}}{R} F_1 F_2 \tag{3.1.70}$$

$$\frac{-1}{R}\frac{\partial P}{\partial \theta} = \frac{-e^{2\alpha\theta}}{R}\left(\frac{\partial F_4}{\partial \theta} + 2\alpha F_4\right) = \frac{-e^{2\alpha\theta}}{R}\left(\frac{\partial F_4}{\partial \eta_1}\frac{\partial \eta_1}{\partial \theta} + \frac{\partial F_4}{\partial \eta_2}\frac{\partial \eta_2}{\partial \theta} + 2\alpha F_4\right) = \frac{e^{2\alpha\theta}}{R}\left(\eta_1\frac{\partial F_4}{\partial \eta_1} + \eta_2\frac{\partial F_4}{\partial \eta_2} - 2\alpha F_4\right) \tag{3.1.71}$$

$$(\xi F_1 - F_2)\,\eta_1\frac{\partial F_2}{\partial \eta_1} + \left(\xi F_3\left(\frac{\eta_1}{\eta_2}\right)^{1/\xi} - F_2\right)\eta_2\frac{\partial F_2}{\partial \eta_2} + F_1F_2 + \alpha F_2^2 = \eta_1\frac{\partial F_4}{\partial \eta_1} + \eta_2\frac{\partial F_4}{\partial \eta_2} - 2\alpha F_4 \tag{3.1.72}$$

FIGURE 3.2 $(R, \theta, Z)\rightarrow(\eta_1, \eta_2)$ REDUCTION θ-MOMENTUM EQUATION (2.4.8)

$$\frac{\partial F_1}{\partial R} = e^{\quad} F_1 \frac{\partial F_3}{\partial R} = e^{\quad} F_1\left(\frac{\partial F_3}{\partial \eta_1}\frac{\partial \eta_1}{\partial R} + \frac{\partial F_3}{\partial \eta_2}\frac{\partial \eta_2}{\partial R}\right) = e^{\quad} \frac{\xi}{R} F_1 \eta_1 \frac{\partial F_3}{\partial \eta_1} \tag{3.1.73}$$

$$\frac{V}{R}\frac{\partial W}{\partial \theta} = \frac{e^{2\alpha\theta}}{R}\left(F_2 \frac{\partial F_3}{\partial \theta} + \alpha F_2 F_3\right) = \frac{e^{2\alpha\theta}}{R}\left(F_2\left(\frac{\partial F_3}{\partial \eta_1}\frac{\partial \eta_1}{\partial \theta} + \frac{\partial F_3}{\partial \eta_2}\frac{\partial \eta_2}{\partial \theta}\right) + \alpha F_2 F_3\right) = \frac{e^{2\alpha\theta}}{R}\left(F_2\left(-\eta_1 \frac{\partial F_3}{\partial \eta_1} - \eta_2 \frac{\partial F_3}{\partial \eta_2}\right) + \alpha F_2 F_3\right) \tag{3.1.74}$$

$$W \frac{\partial W}{\partial Z} = e^{2\alpha\theta} F_3 \frac{\partial F_3}{\partial Z} = e^{2\alpha\theta} F_3\left(\frac{\partial F_3}{\partial \eta_1}\frac{\partial \eta_1}{\partial Z} + \frac{\partial F_3}{\partial \eta_2}\frac{\partial \eta_2}{\partial Z}\right) = e^{2\alpha\theta} F_3 \frac{\xi}{Z} \eta_2 \frac{\partial F_3}{\partial \eta_2} \tag{3.1.75}$$

$$-\frac{\partial P}{\partial Z} = -e^{2\alpha\theta} \frac{\partial F_4}{\partial Z} = -e^{2\alpha\theta}\left(\frac{\partial F_4}{\partial \eta_1}\frac{\partial \eta_1}{\partial Z} + \frac{\partial F_4}{\partial \eta_2}\frac{\partial \eta_2}{\partial Z}\right) = -e^{2\alpha\theta} \frac{\xi}{Z} \eta_2 \frac{\partial F_4}{\partial \eta_2} \tag{3.1.76}$$

$$(\xi F_1 - F_2)\, \eta_1 \frac{\partial F_3}{\partial \eta_1} + \left(\xi F_3\left(\frac{\eta_1}{\eta_2}\right)^{1/\xi} - F_2\right) \eta_2 \frac{\partial F_3}{\partial \eta_2} + \alpha F_2 F_3 = \xi \left(\frac{\eta_1}{\eta_2}\right)^{1/\xi} \eta_2 \frac{\partial F_4}{\partial \eta_2} \tag{3.1.77}$$

FIGURE 3.3 $(R, \theta, Z) \rightarrow (\eta_1, \eta_2)$ REDUCTION Z-MOMENTUM EQUATION (2.4.9)

$$\frac{\partial}{\partial R} \qquad \frac{\partial}{\partial R} \qquad \left(\frac{\partial}{\partial \eta_1}\frac{\partial}{\partial R} + \frac{\partial}{\partial \eta_2}\frac{\partial}{\partial R}\right) = \frac{\xi}{R}\,\eta_1\,\frac{\partial}{\partial \eta_1} \tag{3.1.78}$$

$$\frac{U}{R} = \frac{e^{\alpha\theta}}{R}\,F_1 \tag{3.1.79}$$

$$\frac{1}{R}\frac{\partial V}{\partial \theta} = \frac{e^{\alpha\theta}}{R}\left(\frac{\partial F_2}{\partial \theta} + \alpha F_2\right) = \frac{e^{\alpha\theta}}{R}\left(-\eta_1\frac{\partial F_2}{\partial \eta_1} - \eta_2\frac{\partial F_2}{\partial \eta_2} + \alpha F_2\right) \tag{3.1.80}$$

$$\frac{\partial W}{\partial Z} = e^{\alpha\theta}\frac{\partial F_3}{\partial Z} = e^{\alpha\theta}\left(\frac{\partial F_3}{\partial \eta_1}\frac{\partial \eta_1}{\partial Z} + \frac{\partial F_3}{\partial \eta_2}\frac{\partial \eta_2}{\partial Z}\right) = e^{\alpha\theta}\frac{\xi}{Z}\,\eta_2\,\frac{\partial F_3}{\partial \eta_2} \tag{3.1.81}$$

$$\xi\eta_1\frac{\partial F_1}{\partial \eta_1} + F_1 + \alpha F_2 - \eta_1\frac{\partial F_2}{\partial \eta_1} - \eta_2\frac{\partial F_2}{\partial \eta_2} + \xi\left(\frac{\eta_1}{\eta_2}\right)^{1/\xi}\eta_2\frac{\partial F_3}{\partial \eta_2} = 0 \tag{3.1.82}$$

FIGURE 3.4 $(R, \theta, Z)\rightarrow(\eta_1, \eta_2)$ REDUCTION CONSERVATION OF MASS EQUATION (2.4.10)

$$H = e^{2\alpha\theta}(F_4 + 1/2(F_1^2 + F_2^2 + F_3^2)) \tag{3.1.83}$$

FIGURE 3.5 $(R, \theta, Z) \rightarrow (\eta_1, \eta_2)$ REDUCTION BERNOULLI EQUATION (2.4.11)

$$S = \left(\frac{\eta_1}{\eta_2}\right)^{2/\xi} e^{2\alpha\theta} F_2^2/H \tag{3.1.84}$$

FIGURE 3.6 $(R, \theta, Z) \rightarrow (\eta_1, \eta_2)$ REDUCTION SWIRL PARAMETER EQUATION (2.4.17)

A three-to-two reduction in the number of independent varia- has been accomplished for the system of equations given in section The new system of equations for the fluid flow model in (η_1, η_2) consists of equations (3.1.66), (3.1.72), (3.1.77), (3.1.82), 83), and (3.1.84)

Two to One Reduction in the Number of Independent Variables

The two independent variables (η_1, η_2) are reduced to the one endent variable (X). The new independent variable (X) is a func- lly independent absolute invariant of a subgroup of the one- eter continuous group of transformations, B_1. The one-parameter nuous group of transformations, B_1, will subsequently be referred the group B_1.

Taking advantage of the experience gained from choosing the A_1 (3.1.1), the group B_1 is chosen as shown:

$$B_1: \quad \begin{aligned} \bar{\eta}_1 &= c^k \eta_1 & \bar{F}_2 &= c^{\gamma_1} F_2 \\ \bar{\eta}_2 &= c^k \eta_2 & \bar{F}_3 &= c^{\gamma_1} F_3 \\ \bar{F}_1 &= c^{\gamma_1} F_1 & \bar{F}_4 &= c^{2\gamma_1} F_4 \end{aligned} \tag{3.2.1}$$

k and γ_1 are constants and c is the group parameter. The group ed as (3.2.1) has the general form specified by equation (1.3.7). ransformed independent variables are functions of the parameter c ne previous independent variables. Similarly, the transformed de- nt variables are functions of the parameter c and the previous de- nt variables.

The group B_1 (3.2.1) must satisfy the four group axioms given ations (1.3.1), (1.3.2), (1.3.3), and (1.3.4). The form of the ormations in group B_1 (3.2.1) is the same as the form for the R ormation in group A_1 (3.1.1). Since the R transformation in A_1 (3.1.1) satisfies all the group axioms, it follows that the ormations in the group B_1 (3.2.1) satisfy all the group axioms.

The conditions under which the number of independent varia- n a system of partial differential equations can be reduced by the process of obtaining invariant solutions are given by m 1.3.2. The two-dimensional equations (3.1.66), (3.1.72), 7), (3.1.82), (3.1.83), and (3.1.84) must be conformally invari- der the k-th enlargement of the group B_1 (3.2.1). Conformal in- ce is defined by Definition 1.3.5 and given in equation form by on (1.3.20). To verify conformal invariance, the group B_1) is formally substituted into an equation and the mathematical lations indicated by the equation are performed. The resulting ust then be the same as equation (1.3.20) for the equation to be mally invariant.

The group B_1 (3.2.1) is formally substituted into equation 5), the two-dimensional representation of the R-Momentum equation.

$$\text{L} - \bar{F}_2)\,\bar{\eta}_1 \frac{\partial \bar{F}_1}{\partial \bar{\eta}_1} + (\xi \bar{F}_3 (\frac{\bar{\eta}_1}{\bar{\eta}_2})^{1/\xi} - \bar{F}_2)\,\bar{\eta}_2 \frac{\partial \bar{F}_1}{\partial \bar{\eta}_2} - \bar{F}_2^2 + \alpha \bar{F}_1 \bar{F}_2$$

$$\frac{\partial \bar{F}_4}{\partial \bar{\eta}_1} = c^{2\gamma_1}(\xi F_1 - F_2)\,\eta_1 \frac{\partial F_1}{\partial \eta_1} + c^{2\gamma_1}(\xi F_3 (\frac{\eta_1}{\eta_2})^{1/\xi} - F_2)\,\eta_2 \frac{\partial F_1}{\partial \eta_2}$$

$$- c^{2\gamma_1} F_2^2 + c^{2\gamma_1} \alpha F_1 F_2 - c^{2\gamma_1} \xi \eta_1 \frac{\partial F_4}{\partial \eta_1} \qquad (3.2.2)$$

s evident from equation (3.2.2) that the function M of equation .20) is only a function of the parameter c. By Definition 1.3.6, differential form represented by equation (3.2.2) is constant con- ally invariant.

The group B_1 (3.2.1) is formally substituted into equation .72), the two-dimensional representation of the θ-Momentum equation.

$$\xi\bar{F}_1 - \bar{F}_2)\,\bar{\eta}_1 \frac{\partial \bar{F}_2}{\partial \bar{\eta}_1} + (\xi\bar{F}_3(\frac{\bar{\eta}_1}{\bar{\eta}_2})^{1/\xi} - \bar{F}_2)\,\bar{\eta}_2 \frac{\partial \bar{F}_2}{\partial \bar{\eta}_2} + \bar{F}_1\bar{F}_2 + \alpha\bar{F}_2^2$$

$$- \bar{\eta}_1 \frac{\partial \bar{F}_4}{\partial \bar{\eta}_1} - \bar{\eta}_2 \frac{\partial \bar{F}_4}{\partial \bar{\eta}_2} + 2\alpha\bar{F}_4 = c^{2\gamma_1}(\xi F_1 - F_2)\,\eta_1 \frac{\partial F_2}{\partial \eta_1}$$

$$+ c^{2\gamma_1}(\xi F_3(\frac{\eta_1}{\eta_2})^{1/\xi} - F_2)\,\eta_2 \frac{\partial F_2}{\partial \eta_2} + c^{2\gamma_1}F_1F_2 + c^{2\gamma_1}\alpha F_2^2$$

$$- c^{2\gamma_1}\eta_1 \frac{\partial F_4}{\partial \eta_1} - c^{2\gamma_1}\eta_2 \frac{\partial F_4}{\partial \eta_2} + c^{2\gamma_1}2\alpha F_4 \qquad (3.2.3)$$

equation (3.2.3), it is evident that the function M of equation 20) is only a function of the parameter c. By definition 1.3.6, ifferential form represented by equation (3.2.3) is constant con- lly invariant.

The group B_1 (3.2.1) is formally substituted into equation 77), the two-dimensional representation of the Z-Momentum equation.

$$(\xi\bar{F}_1 - \bar{F}_2)\,\bar{\eta}_1 \frac{\partial\bar{F}_3}{\partial\bar{\eta}_1} + (\xi\bar{F}_3(\frac{\bar{\eta}_1}{\bar{\eta}_2})^{1/\xi} - \bar{F}_2)\,\bar{\eta}_2 \frac{\partial\bar{F}_3}{\partial\bar{\eta}_2} + \alpha\bar{F}_2\bar{F}_3$$

$$(\frac{\bar{\eta}_1}{\bar{\eta}_2})^{1/\xi}\,\bar{\eta}_2 \frac{\partial\bar{F}_4}{\partial\bar{\eta}_2} = c^{2\gamma_1}(\xi F_1 - F_2)\,\eta_1 \frac{\partial F_3}{\partial\eta_1} + c^{2\gamma_1}(\xi F_3(\frac{\eta_1}{\eta_2})^{1/\xi} - F_2)\,\eta_2 \frac{\partial F_3}{\partial\eta_2}$$

$$+ c^{2\gamma_1}\alpha F_2 F_3 - c^{2\gamma_1}\xi(\frac{\eta_1}{\eta_2})^{1/\xi}\,\eta_2 \frac{\partial F_4}{\partial\eta_2} \qquad (3.2.4)$$

parison of equations (1.3.20) and (3.2.4) indicates that the func-
n M in equation (1.3.20) is only a function of the parameter c.
refore, the differential form represented by equation (3.2.4) is
stant conformally invariant according to Definition 1.3.6.

The group B_1 (3.2.1) is formally substituted into equation
1.82), the two-dimensional representation of the Conservation of
; equation.

$$\xi\bar{\eta}_1 \frac{\partial\bar{F}_1}{\partial\bar{\eta}_1} + \bar{F}_1 + \alpha\bar{F}_2 - \bar{\eta}_1 \frac{\partial\bar{F}_2}{\partial\bar{\eta}_1} - \bar{\eta}_2 \frac{\partial\bar{F}_2}{\partial\bar{\eta}_2} + \xi(\frac{\bar{\eta}_1}{\bar{\eta}_2})^{1/\xi}\,\bar{\eta}_2 \frac{\partial\bar{F}_3}{\partial\bar{\eta}_2}$$

$$= c^{\gamma_1}\xi\eta_1 \frac{\partial F_1}{\partial\eta_1} + c^{\gamma_1}F_1 + c^{\gamma_1}\alpha F_2 - c^{\gamma_1}\eta_1 \frac{\partial F_2}{\partial\eta_1} - c^{\gamma_1}\eta_2 \frac{\partial F_2}{\partial\eta_2}$$

$$+ c^{\gamma_1}\xi(\frac{\eta_1}{\eta_2})^{1/\xi}\,\eta_2 \frac{\partial F_3}{\partial\eta_2} \qquad (3.2.5)$$

s evident from equation (3.2.5) that the function M of equation
.20) is only a function of the parameter c. By Definition 1.3.6,
differential form represented by equation (3.2.5) is constant con-
ally invariant.

The group B_1 (3.2.1) is formally substituted into equation (3.1.83), the two-dimensional representation of the Bernoulli equation.

$$H - e^{2\alpha\theta}(\bar{F}_4 + 1/2(\bar{F}_1^2 + \bar{F}_2^2 + \bar{F}_3^2)) = H - e^{2\alpha\theta}(c^{2\gamma_1}F_4 + 1/2(c^{2\gamma_1}F_1^2 + c^{2\gamma_1}F_2^2 + c^{2\gamma_1}F_3^2)) \qquad (3.2.6)$$

Comparison of equations (1.3.20) and (3.2.6) indicates that the function M of equation (1.3.20) is only a function of the parameter c. The differential form represented by equation (3.2.6) is constant conformally invariant according to Definition 1.3.6.

The group B_1 (3.2.1) is formally substituted into equation (3.1.84), the two-dimensional representation of the Swirl Parameter equation.

$$S - \left(\frac{\bar{\eta}_1}{\bar{\eta}_2}\right)^{2/\xi} e^{2\alpha\theta}\bar{F}_2^2/H = S - c^{2\gamma_1}\left(\frac{\eta_1}{\eta_2}\right)^{2/\xi} e^{2\alpha\theta}F_2^2/H \qquad (3.2.7)$$

From equation (3.2.7), it is seen that the function M of equation (1.3.20) is a function of only the parameter c. By Definition 1.3.6, equation (3.2.7) is constant conformally invariant.

The group B_1 (3.2.1) has m+n-1 = 5 functionally independent absolute invariants. Equation (1.3.9) gives the form for m-1 = 1 of the absolute invariants and equation (1.3.10) gives the form for the remaining n = 4 absolute invariants. Definition 1.3.1 and equation (1.3.5) give the definition and general form for an absolutely invariant form. As previously mentioned, the manner in which absolute

invariants are selected is not well defined. The function X is chosen as shown below:

$$X = \frac{\eta_1}{\eta_2} \quad . \tag{3.2.8}$$

This choice is suggested by the appearance of this ratio in equations (3.1.66), (3.1.72), (3.1.77), (3.1.82), and (3.1.84). For the function X to be an absolute invariant, the following relation must hold:

$$X(\bar{\eta}_1, \bar{\eta}_2) = X(\eta_1, \eta_2). \tag{3.2.9}$$

The specific condition for the absolute invariance of X is determined by substituting equation (3.2.8) into relation (3.2.9):

$$\frac{\bar{\eta}_1}{\bar{\eta}_2} = \frac{\eta_1}{\eta_2} \quad . \tag{3.2.10}$$

Using relations from group B_1 (3.2.1) for η_1 and η_2, it is evident that equation (3.2.10) is satisfied and that X is an absolute invariant. The functions g_1, g_2, g_3, and g_4 are chosen as shown:

$$g_1 = F_1\eta_1^{-\sigma}, \qquad g_2 = F_2\eta_1^{-\sigma}, \qquad g_3 = F_3\eta_1^{-\sigma}, \qquad g_4 = F_4\eta_1^{-2\sigma}. \tag{3.2.11}$$

For the functions in equations (3.2.11) to be absolute invariants, the following relations must hold:

$$g_1(\bar{F}_1, \bar{\eta}_1) = g_1(F_1, \eta_1), \qquad g_2(\bar{F}_2, \bar{\eta}_1) = g_2(F_2, \eta_1) \tag{3.2.12}$$

$$g_3(\bar{F}_3, \bar{\eta}_1) = g_3(F_3, \eta_1), \qquad g_4(\bar{F}_4, \bar{\eta}_1) = g_4(F_4, \eta_1). \tag{3.2.13}$$

e value of the exponent σ in equations (3.2.11) must be determined ch that symbolic equations (3.2.12) and (3.2.13) are true. The re-ired transformations from the group B_1 (3.2.1) are utilized in the llowing equations to determine σ:

$$
\begin{aligned}
g_1 &= \bar{F}_1\bar{\eta}_1^{-\sigma} = c^{\gamma_1}F_1(c^k\eta_1)^{-\sigma} = c^{\gamma_1-\sigma k}F_1\eta_1^{-\sigma}, \\
g_2 &= \bar{F}_2\bar{\eta}_1^{-\sigma} = c^{\gamma_1}F_2(c^k\eta_1)^{-\sigma} = c^{\gamma_1-\sigma k}F_2\eta_1^{-\sigma}, \\
g_3 &= \bar{F}_3\bar{\eta}_1^{-\sigma} = c^{\gamma_1}F_3(c^k\eta_1)^{-\sigma} = c^{\gamma_1-\sigma k}F_3\eta_1^{-\sigma}, \\
g_4 &= \bar{F}_4\bar{\eta}_1^{-2\sigma} = c^{2\gamma_1}F_4(c^k\eta_1)^{-2\sigma} = c^{2(\gamma_1-\sigma k)}F_4\eta_1^{-2\sigma}.
\end{aligned}
\tag{3.2.14}
$$

om the set of equations (3.2.14), it is evident that the g_i will be solutely invariant if

$$\sigma = \gamma_1/k. \tag{3.2.15}$$

The condition given by equation (1.3.11) must be satisfied by chosen functions g_1, g_2, g_3, and g_4.

$$J = \frac{\partial(g_1, g_2, g_3, g_4)}{\partial(F_1, F_2, F_3, F_4)} \neq 0 \tag{3.2.16}$$

$$J = \begin{vmatrix} \frac{\partial g_1}{\partial F_1} & \frac{\partial g_1}{\partial F_2} & \frac{\partial g_1}{\partial F_3} & \frac{\partial g_1}{\partial F_4} \\ \frac{\partial g_2}{\partial F_1} & \frac{\partial g_2}{\partial F_2} & \frac{\partial g_2}{\partial F_3} & \frac{\partial g_2}{\partial F_4} \\ \frac{\partial g_3}{\partial F_1} & \frac{\partial g_3}{\partial F_2} & \frac{\partial g_3}{\partial F_3} & \frac{\partial g_3}{\partial F_4} \\ \frac{\partial g_4}{\partial F_1} & \frac{\partial g_4}{\partial F_2} & \frac{\partial g_4}{\partial F_3} & \frac{\partial g_4}{\partial F_4} \end{vmatrix} \tag{3.2.17}$$

Using equations (3.2.11), equation (3.2.17) becomes:

$$J = \begin{vmatrix} \eta_1^{-\sigma} & 0 & 0 & 0 \\ 0 & \eta_1^{-\sigma} & 0 & 0 \\ 0 & 0 & \eta_1^{-\sigma} & 0 \\ 0 & 0 & 0 & \eta_1^{-2\sigma} \end{vmatrix} .$$

The Jacobian determinant J is non-vanishing for the chosen functions g_1, g_2, g_3, and g_4. Therefore, the condition (3.2.16) is satisfied.

The function X must satisfy the condition given by equation (1.3.12).

$$R \left[\frac{\partial(X)}{\partial(\eta_1, \eta_2)} \right] = m-1 = 1 \qquad (3.2.18)$$

The rank of the indicated matrix must equal 1. Expanding the matrix given in equation (3.2.18) gives the following result:

$$\left[\frac{\partial X}{\partial \eta_1} \quad \frac{\partial X}{\partial \eta_2} \right] \equiv M. \qquad (3.2.19)$$

sing equation (3.2.8), equation (3.2.19) becomes,

$$\left[\frac{1}{\eta_2} \quad \frac{-\eta_1}{\eta_2^2} \right] = M. \qquad (3.2.20)$$

ince M is a 1x2 matrix and the row rank equals the column rank for a atrix, 1 is the maximum rank that M in (3.2.20) can have. The columns f M in (3.2.20) are not linearly independent (one is a multiple of the ther) and hence, M has rank of 1. The condition indicated by equation 3.2.18) is satisfied for the chosen function X.

It has been demonstrated that equations (3.2.8) and (3.2.11) epresent the functionally independent absolute invariants of the group (3.2.1).

Application of equation (1.3.17) to the invariants of the roup B_1 (3.2.1) yields the following result:

$$g_j(F_1, F_2, F_3, F_4, \eta_1, \eta_2) = G_j(X) \qquad j = 1, 2, 3, 4. \tag{3.2.21}$$

quations (3.2.11) are rearranged using the symbolic equation (3.2.21) o obtain the following general form for the F_i:

$$F_i = \eta_1^{\kappa\sigma} G_i(X) \qquad \begin{array}{l} i = 1, 2, 3, 4 \\ \kappa = \{ \begin{array}{ll} 1 & i \neq 4 \\ 2 & i = 4. \end{array} \end{array} \tag{3.2.22}$$

For convenience, equation (3.2.8) and the partial derivatives the function X are listed together.

$$X = \eta_1/\eta_2 \tag{3.2.23}$$

$$\frac{\partial X}{\partial \eta_1} = \frac{1}{\eta_2} \qquad \frac{\partial X}{\partial \eta_2} = \frac{-\eta_1}{\eta_2^2} \tag{3.2.24}$$

The relations (3.2.22) through (3.2.24) are now used to reduce e number of independent variables in the system of equations given in

tion 3.1. The two independent variables (η_1, η_2) are reduced to the independent variable (X) in equations (3.1.66), (3.1.72), (3.1.77), 1.82), (3.1.83), and (3.1.84). Figures 3.7 through 3.11 illustrate transformation details and give the final result for each of these ations. Note that the final result displayed in each figure is obned by summing all the transformed terms given in the figure.

$$-F_2\eta_1 \frac{\partial F_1}{\partial \eta_1} = -\eta_1^{\sigma} G_2 \eta_1 \left(\eta_1^{\sigma} \frac{\partial G_1}{\partial \eta_1} + G_1 \frac{\partial \eta_1^{\sigma}}{\partial \eta_1}\right) = -\eta_1^{2\sigma}(XG_2G_1' + \sigma G_1G_2) \tag{3.2.26}$$

$$-F_2\eta_2 \frac{\partial F_1}{\partial \eta_2} = -\eta_1^{\sigma} G_2 \eta_2 \eta_1^{\sigma} \frac{\partial G_1}{\partial \eta_2} \qquad \frac{\partial G_1}{\partial \eta_2} = \frac{dG_1}{dX}\frac{\partial X}{\partial \eta_2} = \frac{-X}{\eta_2} G_1' \qquad -F_2\eta_2 \frac{\partial F_1}{\partial \eta_2} = \eta_1^{2\sigma} XG_1'G_2 \tag{3.2.27}$$

$$\xi\left(\frac{\eta_1}{\eta_2}\right)^{1/\xi} \eta_2 F_3 \frac{\partial F_1}{\partial \eta_2} = \xi(X)^{1/\xi}\, \eta_2 \eta_1^{\sigma} G_3 \eta_1^{\sigma} \frac{\partial G_1}{\partial \eta_2} = -\xi X^{1+1/\xi}\, \eta_1^{2\sigma} G_3 G_1' \tag{3.2.28}$$

$$-F_2^2 = -\eta_1^{2\sigma} G_2^2 \tag{3.2.29}$$

$$-\xi\eta_1 \frac{\partial F_4}{\partial \eta_1} = -\xi\eta_1\left(\eta_1^{2\sigma} \frac{\partial G_4}{\partial \eta_1} + G_4 \frac{\partial \eta_1^{2\sigma}}{\partial \eta_1}\right) = -\xi\eta_1^{2\sigma}(XG_4' + 2\sigma G_4) \tag{3.2.30}$$

$$-\alpha F_1 F_2 = -\alpha\eta_1^{2\sigma} G_1 G_2 \tag{3.2.31}$$

$$\xi XG_1G_1' + \xi\sigma G_1^2 - \sigma G_1G_2 - \xi X^{1+1/\xi}\, G_3G_1' - G_2^2 + \alpha G_1G_2 = -\xi XG_4' - 2\xi\sigma G_4 \tag{3.2.32}$$

FIGURE 3.7 $(\eta_1, \eta_2) \to (X)$ REDUCTION EQUATION (3.1.66)

$$F_1\eta_1 \frac{\partial F_2}{\partial \eta_1} = \eta_1^{\sigma} G_2 \eta_1 \left(\eta_1^{\sigma} \frac{\partial G_2}{\partial \eta_1} + G_2 \frac{\partial \eta_1^{\sigma}}{\partial \eta_1}\right) = \eta_1^{2\sigma} G_2 (XG_2' + \sigma G_2) \tag{3.2.33}$$

$$-F_2\eta_1 \frac{\partial F_2}{\partial \eta_1} = -\eta_1^{\sigma} G_2 \eta_1 \left(\eta_1^{\sigma} \frac{\partial G_2}{\partial \eta_1} + G_2 \frac{\partial \eta_1^{\sigma}}{\partial \eta_1}\right) = -\eta_1^{2\sigma} G_2 (XG_2' + \sigma G_2) \tag{3.2.34}$$

$$-F_2\eta_2 \frac{\partial F_2}{\partial \eta_2} = -\eta_1^{2\sigma} G_2 \eta_2 \frac{\partial G_2}{\partial \eta_2} = \eta_1^{2\sigma} XG_2 G_2' \tag{3.2.35}$$

$$\xi\left(\frac{\eta_1}{\eta_2}\right)^{1/\xi} F_3\eta_2 \frac{\partial F_2}{\partial \eta_2} = -\xi (X)^{1/\xi} \eta_1^{2\sigma} XG_3 G_2' \tag{3.2.36}$$

$$F_1 F_2 = \eta_1^{2\sigma} G_1 G_2 \tag{3.2.37}$$

$$\alpha F_2^2 = \alpha \eta_1^{2\sigma} G_2^2 \tag{3.2.38}$$

FIGURE 3.8 $(\eta_1, \eta_2) \rightarrow (X)$ REDUCTION EQUATION (3.1.72)

$$\eta_1 \frac{\partial F_4}{\partial \eta_1} - \eta_1 (\eta_1 \frac{\partial G_4}{\partial \eta_1} + G_4 \frac{\partial \eta_1}{\partial \eta_1}) = \eta_1^{2\sigma} (XG_4' + 2\sigma G_4) \tag{3.2.39}$$

$$\eta_2 \frac{\partial F_4}{\partial \eta_2} = \eta_2 \eta_1^{2\sigma} \frac{\partial G_4}{\partial \eta_2} = -\eta_1^{2\sigma} XG_4' \tag{3.2.40}$$

$$-2\alpha F_4 = -2\alpha \eta_1^{2\sigma} G_4 \tag{3.2.41}$$

$$\xi X G_1 G_2' + (\xi\sigma + 1) G_1 G_2 + (\alpha - \sigma) G_2^2 - \xi X^{1+1/\xi} G_3 G_2' = 2(\sigma - \alpha) G_4 \tag{3.2.42}$$

FIGURE 3.8 (CONT'D.)

$$\xi F_1 \eta_1 \frac{\partial F_3}{\partial \eta_1} = \xi \eta_1^{\sigma} G_1 \eta_1 \frac{\partial}{\partial \eta_1} (\eta_1^{\sigma} G_3) \qquad \frac{\partial G_3}{\partial \eta_1} = \frac{dG_3}{dX} \frac{\partial X}{\partial \eta_1} = \frac{G_3'}{\eta_2} \qquad \xi F_1 \eta_1 \frac{\partial F_3}{\partial \eta_1} = \xi \eta_1^{2\sigma} (X G_1 G_3' + \sigma G_1 G_3) \tag{3.2.43}$$

$$-F_2 \eta_1 \frac{\partial F_3}{\partial \eta_1} = -\eta_1^{\sigma} G_2 \eta_1 \frac{\partial}{\partial \eta_1} (\eta_1^{\sigma} G_3) = -\eta_1^{2\sigma} (X G_2 G_3' + \sigma G_2 G_3) \tag{3.2.44}$$

$$-F_2 \eta_2 \frac{\partial F_3}{\partial \eta_2} = -\eta_1^{2\sigma} G_2 \eta_2 \frac{\partial G_3}{\partial \eta_2} = \eta_1^{2\sigma} X G_2 G_3' \tag{3.2.45}$$

$$F_3 \xi \left(\frac{\eta_1}{\eta_2}\right)^{1/\xi} \eta_2 \frac{\partial F_3}{\partial \eta_2} = \eta_1^{2\sigma} G_3 X^{1/\xi} \eta_2 \frac{\partial G_3}{\partial \eta_2} = -\xi \eta_1^{2\sigma} X^{1+1/\xi} G_3 G_3' \tag{3.2.46}$$

$$\alpha F_2 F_3 = \alpha \eta_1^{2\sigma} G_2 G_3 \tag{3.2.47}$$

$$-\xi \left(\frac{\eta_1}{\eta_2}\right)^{1/\xi} \eta_2 \frac{\partial F_4}{\partial \eta_2} = -\xi X^{1/\xi} \eta_2 \eta_1^{2\sigma} \frac{\partial G_4}{\partial \eta_2} \qquad \frac{\partial G_4}{\partial \eta_2} = \frac{dG_4}{dX} \frac{\partial X}{\partial \eta_2} = -\frac{X}{\eta_2} G_4' \qquad -\xi \left(\frac{\eta_1}{\eta_2}\right)^{1/\xi} \eta_2 \frac{\partial F_4}{\partial \eta_2} = \xi X^{1+1/\xi} \eta_1^{2\sigma} G_4' \tag{3.2.48}$$

$$\xi X G_1 G_3' + \xi \sigma G_1 G_3 + (\alpha - \sigma) G_2 G_3 = \xi X^{1+1/\xi} G_3 G_3' = \xi X^{1+1/\xi} G_4' \tag{3.2.49}$$

FIGURE 3.9 $(\eta_1, \eta_2) \rightarrow (X)$ REDUCTION EQUATION (3.1.77)

$$\eta_1 \frac{\partial F_1}{\partial \eta_1} = \eta_1(\eta_1^{\sigma} \frac{\partial G_1}{\partial \eta_1}) = \xi\eta_1(\sigma G_1 + XG_1') \tag{3.2.50}$$

$$F_1 = \eta_1^{\sigma} G_1 \tag{3.2.51}$$

$$-\eta_1 \frac{\partial F_2}{\partial \eta_1} = -\eta_1(\eta_1^{\sigma} \frac{\partial G_2}{\partial \eta_1} + \sigma G_2 \eta_1^{\sigma-1}) = -\eta_1^{\sigma}(XG_2' + \sigma G_2) \tag{3.2.52}$$

$$-\eta_2 \frac{\partial F_2}{\partial \eta_2} = -\eta_2(\eta_1^{\sigma} \frac{\partial G_2}{\partial \eta_2}) = \eta_1^{\sigma} XG_2' \tag{3.2.53}$$

$$\alpha F_2 = \alpha \eta_1^{\sigma} G_2 \tag{3.2.54}$$

$$\xi(\frac{\eta_1}{\eta_2})^{1/\xi} \eta_2 \frac{\partial F_3}{\partial \eta_2} = \xi X^{1/\xi} \eta_2 \eta_1^{\sigma} \frac{\partial G_3}{\partial \eta_2} = -\xi X^{1+1/\xi} \eta_1^{\sigma} G_3' \tag{3.2.55}$$

$$(1 + \xi\sigma)\, G_1 + (\alpha - \sigma)\, G_2 + \xi XG_1' - \xi X^{1+1/\xi} G_3' = 0 \tag{3.2.56}$$

FIGURE 3.10 $(\eta_1, \eta_2) \to (X)$ REDUCTION EQUATION (3.1.82)

$$G_4 = \frac{H}{e^{2\alpha\theta}\eta_1^{2\sigma}} - 1/2(G_1^2 + G_2^2 + G_3^2) \tag{3.2.57}$$

$$S = X^{2/\xi}\, e^{2\alpha\theta}\, \eta_1^{2\sigma}\, G_2^2/H \tag{3.2.58}$$

FIGURE 3.11 $(\eta_1, \eta_2) \rightarrow (X)$ REDUCTION EQUATION (3.1.83) AND EQUATION (3.1.84)

To obtain a complete reduction to one independent variable in equations (3.2.57) and (3.2.58), the constants α and σ must vanish. Let $\alpha = \sigma = 0$, equations (3.2.32), (3.2.42), (3.2.49), (3.2.56), (3.2.57), and (3.2.58) then become:

$$\xi X G_1 G_1' - \xi X^{1+1/\xi} G_3 G_1' - G_2^2 = -\xi X G_4' \tag{3.2.59}$$

$$\xi X G_1 G_2' + G_1 G_2 - \xi X^{1+1/\xi} G_3 G_2' = 0 \tag{3.2.60}$$

$$\xi X G_1 G_3' - \xi X^{1+1/\xi} G_3 G_3' = \xi X^{1+1/\xi} G_4' \tag{3.2.61}$$

$$G_1 + \xi X G_1' - \xi X^{1+1/\xi} G_3' = 0 \tag{3.2.62}$$

$$G_4 = H - 1/2(G_1^2 + G_2^2 + G_3^2) \tag{3.2.63}$$

$$S = X^{2/\xi} G_2^2 / H. \tag{3.2.64}$$

A two to one reduction in the number of independent variables has been accomplished for the system of equations given in section 3.1. Equations (3.2.59) through (3.2.64) comprise the new system of equations for the fluid flow model in (X) space.

3.3 Summary

The partial differential equations of the fluid flow model in (R, θ, Z) space have been transformed into partial differential equations in (η_1, η_2) space by the application of the continuous one-parameter transformation group A_1 (3.1.30). Subsequently, the partial differential equations of the fluid flow model in (η_1, η_2) space have

been transformed into ordinary differential equations in (X) space by the application of the continuous one-parameter transformation group B_1 (3.2.1).

IV. SOLUTIONS FOR THE CASE $\xi = -1$

4.1 One-Dimensional Solutions of the Ordinary Differential Equations

The group constant ξ which is defined by equation (3.1.52) appears as an exponent in the transformation equations (3.1.57) and (3.1.58). ξ may be chosen to be positive, negative, or zero. The effect of each of these choices is readily determined after substitution of equations (3.1.57) and (3.1.58) into equation (3.2.8). For each choice, the variable X is influenced as shown:

$$\xi>0,\ X = (R/Z)^{\xi}; \qquad \xi<0,\ X = (Z/R)^{|\xi|}; \qquad \xi = 0,\ X = 1.$$

For the choice $\xi = 0$, X becomes a constant and is not a variable and the system of ordinary differential equations (3.2.59) through (3.2.62) becomes meaningless.

For an example in this investigation, ξ is chosen to be negative. By inspection, it is noted that for $\xi = -1$:

1. The system of ordinary differential equations is simplified.
2. The velocity components will be proportional to $1/R$ in (R, θ, Z) space and the pressure will be proportional to $1/R^2$ in (R, θ, Z) space.
3. The system of ordinary differential equations on page 60 is greatly simplified as the exponential term $1 + 1/\xi$ vanishes.

The system of ordinary differential equations consisting of equations (3.2.59), (3.2.60), (3.2.61), and (3.2.62) simplifies to the following system of ordinary differential equations:

$$-XG_1G_1' + G_3G_1' - G_2^2 = XG_4' \tag{4.1.1}$$

$$-XG_1G_2' + G_1G_2 + G_3G_2' = 0 \tag{4.1.2}$$

$$-XG_1G_3' + G_3G_3' = -G_4' \tag{4.1.3}$$

$$-XG_1' + G_1 + G_3' = 0 \tag{4.1.4}$$

where $G_i' \equiv dG_i/dX$. Since this system of ordinary differential equations is non-linear, a numerical analysis approach is required to determine its solutions.

Hamming's modified predictor-corrector numerical method is utilized to obtain solutions for the system consisting of equations (4.1.1), (4.1.2), (4.1.3), and (4.1.4). The method employs the following numerical calculations:

Predictor: $P_{j+1} = G_{j-3} + \frac{4h}{3}(2G_j' - G_{j-1}' + 2G_{j-2}')$

Modifier: $M_{j+1} = P_{j+1} - \frac{112}{121}(P_j - C_j)$

Corrector: $C_{j+1} = \frac{1}{8}(9G_j - G_{j-2} + 3h(M_{j+1}' + 2G_j' - G_{j-1}'))$

Final Value: $G_{j+1} = C_{j+1} + \frac{9}{121}(P_{j+1} - C_{j+1})$

If the results are known at the equidistant points X_{j-3}, X_{j-2}, X_{j-1}, and X_j; the results at point $X_{j+1} = X_j + h$ (where h is the step size) can be computed using the numerical equations given above. Hamming's predictor-corrector method is not self starting; that is, the functional values at a single previous point are not enough to get the functional values ahead. To obtain the starting values, a special Runge-Kutta procedure followed by one iteration step is added to the predictor-corrector method.

It may be demonstrated that the convergence criterion given by Wilf [52, pg. 98] for Hamming's modified predictor-corrector method requires that the step size be chosen such that $h<8/(3|\partial F/\partial M_{j+1}|)$ where $F(X_{j+1}, M_{j+1}) = M'_{j+1}$. For an extensive discussion on Hamming's modified predictor-corrector method, the reader is referred to Carnahan [9, pp. 381-392] or Wilf [52, pp. 96-108].

Appendix A contains the computer printout of the IBM scientific subroutine DHCPG, a double precision arithmetic routine utilizing Hamming's method, which was used in this work to solve the system of non-linear ordinary differential equations. The numerical computations were performed by an IBM 360 computer and the numerical results are displayed in Table A.1.

After rearrangement, equations (4.1.1), (4.1.2), (4.1.3), and (4.1.4) become:

$$G'_4 = C/E \tag{4.1.5}$$

$$G'_3 = G'_4/A \tag{4.1.6}$$

$$G_2' = (G_1 G_2)/A \tag{4.1.7}$$

$$G_1' = (G_1 + G_3')/X \tag{4.1.8}$$

where $A \equiv XG_1 - G_3$, $C \equiv -(AG_1)/X$, and $E \equiv X + 1/X$.

The boundary conditions (2.4.13), (2.4.14), (2.4.15), and (2.4.16) are constants and transform to X space unaltered. The values for the boundary conditions are selected as shown:

$$U_o \to F_{1*} \to G_{1*} = 1.0, \quad V_o \to F_{2*} \to G_{2*} = 1.0,$$
$$W_o \to F_{3*} \to G_{3*} = 1.0, \quad P_o \to F_{4*} \to G_{4*} = 0.5. \tag{4.1.9}$$

In X space, $G_{j*} \equiv G_j(X_*)$ where $j = 1, 2, 3, 4$.

The boundary point X_* is the left limit of the X interval ($X_* \leq X \leq X_2$). Using the result obtained from substituting equations (3.1.57) and (3.1.58) into equation (3.2.8), X_* is defined to transform as indicated by the following equation:

$$X_* \equiv \frac{\eta_{1*}}{\eta_2} = \frac{e^{-\theta} R_o^{-1}}{e^{-\theta} Z^{-1}} = \frac{Z}{R_o}. \tag{4.1.10}$$

For the selected boundary conditions, the numerical solutions to equations (4.1.1), (4.1.2), (4.1.3), and (4.1.4) are plotted for the interval $0.1 \leq X \leq 1.0$ in Figure 4.1. The output from the computer program using the DHCPG subroutine was stored as the input to a subroutine which controls a Calcomp plotter from which the plots were produced. Appendix B contains a printout for the Calcomp plotter routine.

At the point X = 0.46531 in Figure 4.1, the solution func-
:ions G_1, G_2, and G_3 all have a discontinuous derivative. G_4 (the one-
dimensional representation for pressure) reaches its maximum value and
$_2$ (the one-dimensional representation for the tangential velocity com-
onent) reaches its minimum value. Therefore, the point X = 0.46531 is
esignated as the "tangential quasi-stagnation point" for this flow.

The solutions G_1, G_2, G_3, and G_4 were determined by simul-
aneously solving the one-dimensional representations for conservation
f momentum and conservation of mass. To verify these results, the
ne-dimensional representation for conservation of energy, equation
3.2.63), is used to determine the Bernoulli constant H at each point X.

From equation (3.2.63), it is seen that the equation for H is:

$$H = G_4 + 1/2 \sum_{i=1}^{3} G_i^2. \qquad (4.1.11)$$

e value of H is fixed by the selected boundary conditions and for
is case, H = 2. Data values for H were computed at each point X
ing equation (4.1.11) and the solutions G_1, G_2, G_3, and G_4. From the
data values listed in Table A.1, the maximum percent deviation of H
om the value H = 2 is determined to be 0.28%. This maximum percent
viation occurs at only one point. Most of the percent deviations for
are in the range from 0.00% to 0.09% and hence, equation (3.2.63) is
:isfied.

Polynomial expressions for the solutions G_1, G_2, G_3, and G_4
e determined using a Gaussian least squares method known as Polfit.
Polfit computer printout is included in Appendix C. The polynomial
pressions for the interval $0.1 \leq X \leq 0.46531$ are:

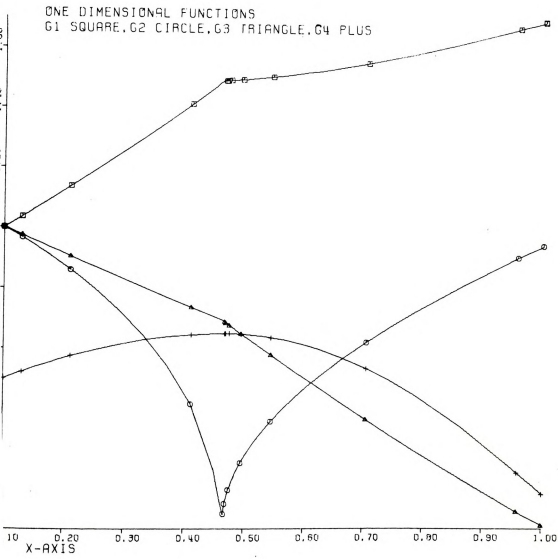

FIGURE 4.1 ONE-DIMENSIONAL FUNCTIONS

$$G_1 = 0.88531 + 1.1040X + 0.44657X^2 \tag{4.1.12}$$

$$G_2 = 1.4366 - 9.5497X + 82.897X^2 - 383.95X^3 + 829.38X^4 - 691.74X^5 \tag{4.1.13}$$

$$G_3 = 1.0909 - 0.91970X + 0.13199X^2 \tag{4.1.14}$$

$$G_4 = 0.40740 + 1.0314X - 1.0941X^2. \tag{4.1.15}$$

The polynomial expressions for the interval $0.46531 \leq X \leq 1.0$ are:

$$G_1 = 1.5338 - 0.29709X + 0.43202X^2 \tag{4.1.16}$$

$$G_2 = -36.184 + 241.04X - 636.39X^2 + 838.23X^3 - 546.94X^4 + 141.25X^5 \tag{4.1.17}$$

$$G_3 = 1.4350 - 1.7466X + 0.30862X^2 \tag{4.1.18}$$

$$G_4 = 0.31091 + 1.5504X - 1.7556X^2 \tag{4.1.19}$$

4.2 Three-Dimensional Pressure and Velocity Curves

Application of equations (3.1.54), (3.1.55), (3.1.57), (3.1.58), (3.2.8), and (3.2.22) to this case results in the following transformation equations:

$$X = \eta_1/\eta_2 = Z/R \tag{4.2.1}$$

$$G_1(X) = F_1(\eta_1, \eta_2) = U(R, \theta, Z),$$
$$G_2(X) = F_3(\eta_1, \eta_2) = V(R, \theta, Z), \tag{4.2.2}$$

$$G_3(X) = F_3(\eta_1, \eta_2) = W(R, \theta, Z),$$
$$G_4(X) = F_4(\eta_1, \eta_2) = P(R, \theta, Z). \tag{4.2.3}$$

The three-dimensional equations for pressure and velocity components are obtained by substituting the transformation relations (4.2.1), (4.2.2), and (4.2.3) into equations (4.1.12) through (4.1.19). For the interval $0.1 \leq Z/R \leq 0.46531$, the polynomial expressions become:

$$U = 0.88531 + 1.1040(Z/R) + 0.44657(Z/R)^2 \tag{4.2.4}$$

$$V = 1.4366 - 9.5497(Z/R) + 82.897(Z/R)^2 - 383.95(Z/R)^3$$
$$+ 829.38(Z/R)^4 - 691.74(Z/R)^5 \tag{4.2.5}$$

$$W = 1.0909 - 0.91970(Z/R) + 0.13199(Z/R)^2 \tag{4.2.6}$$

$$P = 0.40740 + 1.0314(Z/R) - 1.0941(Z/R)^2. \tag{4.2.7}$$

For the interval $0.46531 \leq Z/R \leq 1.0$, the polynomial expressions become:

$$U = 1.5338 - 0.29709(Z/R) + 0.432.02(Z/R)^2 \tag{4.2.8}$$

$$V = -36.184 + 241.04(Z/R) - 636.39(Z/R)^2 + 838.23(Z/R)^3$$
$$- 546.94(Z/R)^4 + 141.25(Z/R)^5 \tag{4.2.9}$$

$$W = 1.4350 - 1.7466(Z/R) + 0.30862(Z/R)^2 \qquad (4.2.10)$$

$$P = 0.31091 + 1.5504(Z/R) - 1.7556(Z/R)^2. \qquad (4.2.11)$$

The absence of the θ coordinate in the transformed equations for U, V, W, and P signifies that for this case an axially symmetric flow is obtained.

The U, V, W, and P curves are plotted as functions of R using as a parameter. For Z as a parameter, the plot intervals become: $.0Z \leq R \leq 2.1491Z$ and $2.1491Z \leq R \leq 10Z$. The Calcomp plotter routine is given nput data from the author's subroutines PC05E1 and PC05E2 which calcu-ate data points from the polynomial expressions for pressure and the elocity components. Appendix D contains the PC05E1 and PC05E2 subrou-ine programs and a sample of the computed data for these curves.

Figures 4.2, 4.3, 4.4, and 4.5 display the curves for pres-ure, radial velocity, tangential velocity, and axial velocity, respec-ively. For the parametric values Z = 0.1, Z = 0.4, and Z = 0.7, the rresponding tangential quasi-stagnation point locations are = 0.21491, R = 0.85964, and R = 1.5044. In each set of curves except e set for axial velocity, the location of the tangential quasi-stagnation int is pronounced by an abrupt slope change for a particular curve. gure 4.6 displays the real space functions U, V, W, and P on a single agram for the parametric value Z = 0.7.

3 Summary and Recommendations for Further Research

Numerical solutions to the non-linear ordinary system of dif-rential equations have been obtained for selected boundary conditions n the group constant $\xi = -1$. These solutions are plotted in Figure

4.1. Polynomial expressions were determined for each of these solutions and the polynomial expressions were transformed from the one-dimensional (X) space back to the three-dimensional (R, θ, Z) space. The three-dimensional polynomial solutions for pressure and the velocity components are plotted in Figures 4.2, 4.3, 4.4 and 4.5.

Solutions to the case $\xi = -1$ for boundary conditions other than those reported on in this chapter were investigated by the author. However, the investigations were not completed as the computer expense was prohibitive. The author's bill for computer time on this investigation is $2,765.47. During this investigation, funds for computer time were generously provided by the Process Engineering Department of General Motors Institute, Flint, Michigan.

Many interesting areas for further research are exposed by this work. The author suggests that profitable results could be obtained from investigation into the case $\xi = -1$ in this work, transformations for a system of equations in spherical coordinates, and transformations for a system of equations including derivative boundary conditions.

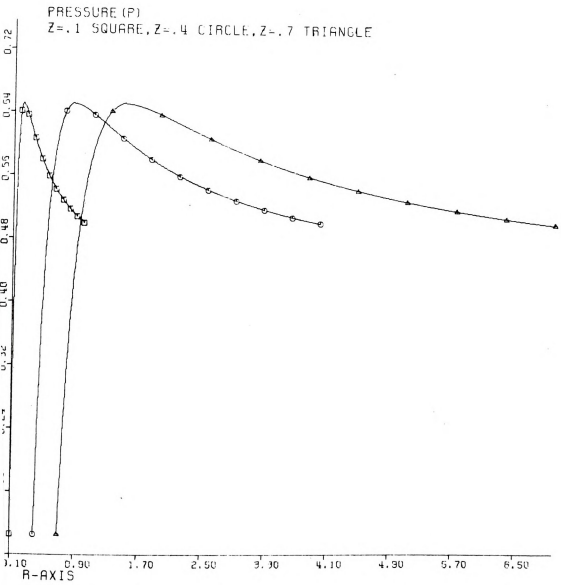

FIGURE 4.2 PRESSURE CURVES

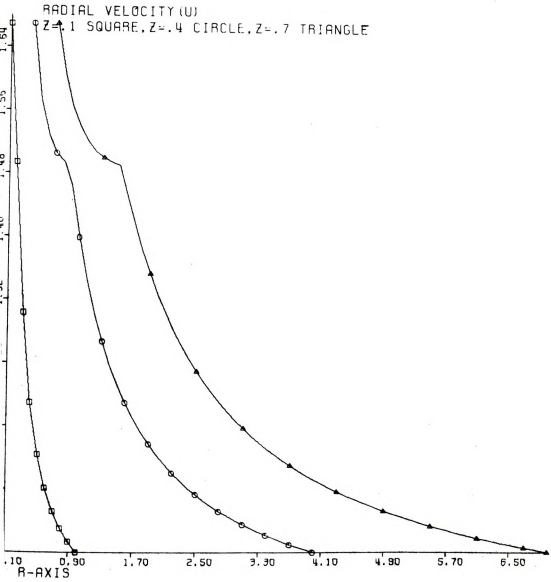

FIGURE 4.3 RADIAL VELOCITY CURVES

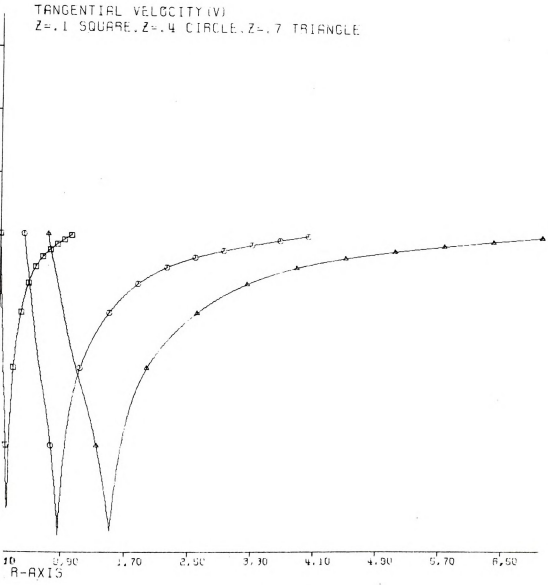

FIGURE 4.4 TANGENTIAL VELOCITY CURVES

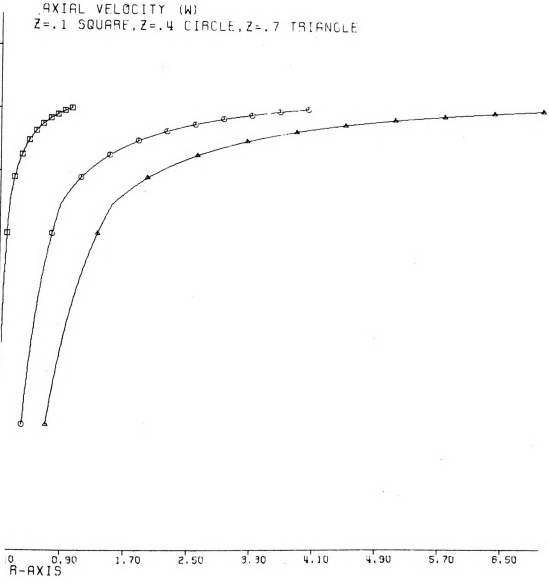

FIGURE 4.5 AXIAL VELOCITY CURVES

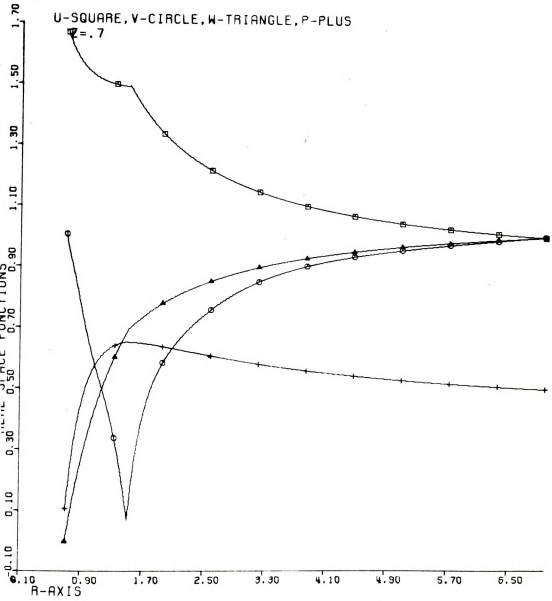

FIGURE 4.6 REAL SPACE FUNCTIONS

V. OPTIMAL SOLUTIONS

1 Application of Optimization Techniques in X Space

The mathematical optimization model employs the one-mensional representations for the conservation of energy (equation .2.63)) and the swirl parameter (equation (3.2.64)). For the case = -1, these equations become:

$$G_4 = H - 1/2(G_2^2 + G^2) \text{ where } G^2 \equiv G_1^2 + G_3^2 \tag{5.1.1}$$

$$S = G_2^2/(HX^2). \tag{5.1.2}$$

e optimization problem is posed to discover those functions G_2 and G at will optimize the function G_4 subject to the restriction imposed the swirl parameter S.

Let

$$\phi_1 = G_4 - H + 1/2(G_2^2 + G^2) = 0 \tag{5.1.3}$$

$$\text{and } \phi_2 = G_2^2 - SHX^2 = 0. \tag{5.1.4}$$

augmented function for this case is written as:

$$= \lambda_1\phi_1 + \lambda_2\phi_2 = \lambda_1 G_4 - \lambda_1 H + \frac{\lambda_1}{2}(G_2^2 + G\ ) + \lambda_2 G_2^2 - \lambda_2 SHX^2. \tag{5.1.5}$$

Comparison of equation (5.1.5) to equation (1.4.7) demonstrates that this problem is formulated as a classical Mayer type problem (refer to section 1.4).

Following the proposition given in reference [41, pg. 33] for solving problems not involving derivatives of the dependent variables, a change of dependent variables is introduced.

$$\text{Let } \alpha' = G_4,\ \beta' = G_2, \text{ and } \gamma' = G. \tag{5.1.6}$$

Substitution of equations (5.1.6) into equations (5.1.3), (5.1.4), and (5.1.5) yields the following expressions for the constraint equations and the augmented function:

$$\phi_1 = \alpha' - H + 1/2(\beta'^2 + \gamma'^2) = 0 \tag{5.1.7}$$

$$\phi_2 = \beta'^2 - SHX^2 = 0 \tag{5.1.8}$$

$$K = \lambda_1\alpha' - \lambda_1 H + \frac{\lambda_1}{2}(\beta'^2 + \gamma'^2) + \lambda_2\beta'^2 - \lambda_2 SHX^2. \tag{5.1.9}$$

A single degree of freedom exists for this problem as there are three dependent variables (α, β,γ) and two constraint equations (5.1.7) and (5.1.8). Hence, one optimum requirement can be imposed on α. The following end conditions are specified for this problem: x_i, x_f, α_f, γ_i, and γ_f. Now the optimum problem is specifically formulated as follows: In the class of functions $\alpha(X)$, $\beta(X)$, and $\gamma(X)$ which are consistent with the constraint equations (5.1.7) and (5.1.8) and the specified end conditions, find that special set which minimizes the difference $\Delta\tau = \tau_f - \tau_i$, where $\tau = \alpha$. Note that for α_f specified, minimizing the difference $\Delta\tau$ corresponds to maximizing α_i.

Applying equation (1.4.8), one Euler-Lagrange equation is obtained for each dependent variable.

$$\frac{d}{dX}\left(\frac{\partial K}{\partial \alpha'}\right) - \frac{\partial K}{\partial \alpha} = 0 \tag{5.1.10}$$

$$\frac{d}{dX}\left(\frac{\partial K}{\partial \beta'}\right) - \frac{\partial K}{\partial \beta} = 0 \tag{5.1.11}$$

$$\frac{d}{dX}\left(\frac{\partial K}{\partial \gamma'}\right) - \frac{\partial K}{\partial \gamma} = 0 \tag{5.1.12}$$

Substitution of equation (5.1.9) into equations (5.1.10), (5.1.11), and and (5.1.12) leads to the following results:

$$\lambda_1' = 0 \text{ or } \lambda_1 = \text{constant} \tag{5.1.13}$$

$$(\lambda_1 + 2\lambda_2)\,\beta'' + (\lambda_1' + 2\lambda_2')\,\beta' = 0 \tag{5.1.14}$$

$$\lambda_1\gamma'' + \lambda_1'\gamma' = 0. \tag{5.1.15}$$

Substitution of equation (5.1.13) into equation (5.1.15) leads to the following result:

$$\gamma' = \frac{L}{\lambda_1} \quad \text{where L is an integration constant.} \tag{5.1.16}$$

Since λ_1 is a constant, an obvious solution to equation (5.1.14) is obtained by choosing λ_2 to be a constant. Let

$$\lambda_2 = \text{constant} = \frac{-\lambda_1}{2}. \tag{5.1.17}$$

The value of λ_1 is determined by applying equation (1.4.9), the transversality condition. The specific transversality condition for this problem becomes:

$$[d\alpha - (\lambda_1(H + \frac{\beta'^2}{2} + \frac{\gamma'^2}{2}) + \lambda_2(\beta'^2 + SHX^2))\, dX + \lambda_1 d\alpha + (\lambda_1 + 2\lambda_2)\, \beta' d\beta$$

$$+ \lambda_1 \gamma' d\gamma]_i^f = 0. \qquad (5.1.18)$$

Equation (5.1.18) is to be satisfied identically and for the specified end conditions this then requires that:

$$\lambda_{1i} = -1. \qquad (5.1.19)$$

From equation (5.1.13),

$$\int_i^f d\lambda_1 = 0.$$

Therefore,

$$\lambda_{1f} = \lambda_{1i} = \lambda_1 \text{ and } \lambda_1 = -1. \qquad (5.1.20)$$

Thus, from the application of the Euler-Lagrange equations and the transversality condition; λ_1, λ_2, and γ' have been determined.

If expression (1.4.10), the Legendre-Clebsch necessary condition, is negative, the optimum obtained is a maximum. For this case, expression (1.4.10) becomes:

$$K_{y'_k y'_j}\, \delta y'_k\, \delta y'_j < 0 \qquad (5.1.21)$$

where repeated indicial subscripts indicate summation.

By inspection of the form of the augmented function, equation (5.1.9), it is seen that

$$K_{y'_k y'_j} = 0 \qquad \text{for } j \neq k. \tag{5.1.22}$$

Utilizing equation (5.1.22), the specific expression for expression (5.1.21) becomes:

$$K_{\alpha'\alpha'}(\delta\alpha')^2 + K_{\beta'\beta'}(\delta\beta')^2 + K_{\gamma'\gamma'}(\delta\gamma')^2 < 0. \tag{5.1.23}$$

From equation (5.1.9), the following relations are determined:

$$K_{\alpha'\alpha'} = 0, \qquad K_{\beta'\beta'} = \lambda_1 + 2\lambda_2, \qquad K_{\gamma'\gamma'} = \lambda_1. \tag{5.1.24}$$

Upon substitution of equations (5.1.24), expression (5.1.23) becomes:

$$(\lambda_1 + 2\lambda_2)\,(\delta\beta')^2 + \lambda_1(\delta\gamma')^2 < 0. \tag{5.1.25}$$

From equations (5.1.17) and (5.1.20) it is indeed evident that expression (5.1.25) holds and hence a maximum is obtained for this case.

The optimum expression for G_4 is obtained by substituting equations (5.1.6), (5.1.8), (5.1.16), and (5.1.20) into equation (5.1.7).

$$G_4 = H - 1/2(SHX^2 + L^2) \tag{5.1.26}$$

Equation (5.1.26) represents the optimum G_4 when the sum $G^2 = G_1^2 + G_3^2 = L^2$ = a constant and G_2 is governed by the swirl parameter S as given by equation (5.1.2). G_4 attains its maximum value at $X = X_i$.

Since G_4 is the one-dimensional representation for pressure, it is restricted to positive values. This restriction on G_4 places the following restriction on the swirl parameter:

$$S<2 - L^2/H. \tag{5.1.27}$$

From Table A.1, the values H = 2.0000 and L^2 = 2.6951 are selected since they occur at the point where the function G_4 is a maximum as determined by the solution of the system of ordinary differential equations in Chapter IV. For the selected values of H and L^2, equation (5.1.27) requires that S<0.6525.

Computing the maximum G_4 at $X = X_1 = 0.1$ using equation (5.1.26) with the above-mentioned values for H and L^2, the computed maximum G_4 for S = 0.25 is found to be within 0.02% of the maximum value given in Table A.1. For S = 0.65, the computed maximum value of G_4 is found to be within 0.63% of the maximum value given in Table A.1. Therefore, the maximum G_4 value determined by equation (5.1.26) compares favorably with the maximum G_4 value determined by the solution of the system of ordinary differential equations in Chapter IV.

Figures 5.1 and 5.2 illustrate the G_2 and G_4 curves generated from equations (5.1.2) and (5.1.26) using S as a parameter. The S values chosen are S = 0.25, S = 0.45, S = 0.55, and S = 0.65. The maximum value of G_4 occurs at the left end of the X interval where G_2 is at a minimum.

The curves were plotted by a Calcomp plotter which is controlled by the output data from an IBM 360 computer. The computer facility used during this investigation is located at General Motors

Institute, Flint, Michigan. Appendix E contains the program for computing the data for these curves and the actual curve data.

5.2 Three-Dimensional Optimal Pressure and Velocity Curves

The three-dimensional equations for optimum pressure and tangential velocity are obtained by substituting the transformation relations (4.2.1), (4.2.2), and (4.2.3) into equations (5.1.26) and (5.1.2):

$$P = H - 1/2(SH(Z/R)^2 + L^2), \tag{5.2.1}$$

$$V = [SH]^{1/2}\ Z/R. \tag{5.2.2}$$

Equation (5.2.1) represents the optimum pressure and equation (5.2.2) represents the tangential velocity for optimum pressure.

The optimum P and V curves are plotted as functions of R using Z as a parameter and S = 0.65. For Z as a parameter, the plot interval becomes: $Z \leq R \leq 10Z$. The maximum value of P occurs at the right end of the R interval where R = 10Z. At R = 10Z, V is a minimum. Therefore, the maximum pressure occurs at the point where the tangential velocity is a minimum. This same result is obtained in Chapter IV for the transformed equations of the solutions to the system of ordinary differential equations.

These curves were also plotted by a Calcomp plotter. The computing program and the actual computed data are contained in Appendix E.

Figures 5.3 and 5.4 display the curves for optimum pressure and tangential velocity, respectively. For the parametric values

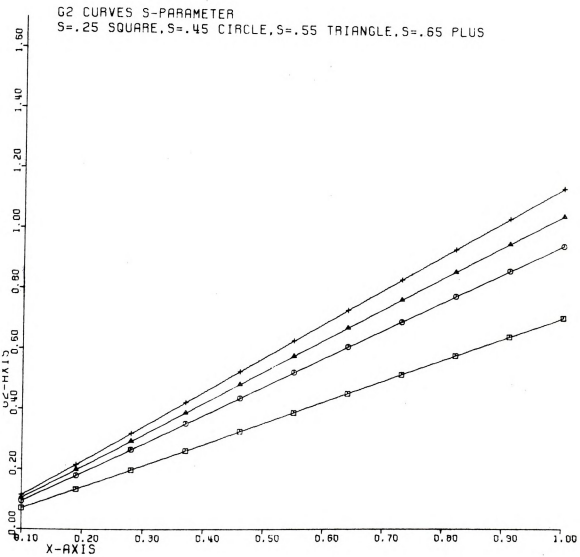

FIGURE 5.1 G_2 CURVES S PARAMETER

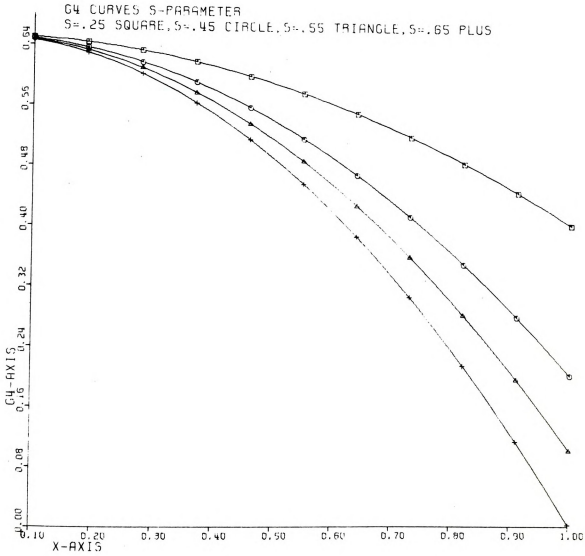

FIGURE 5.2 G_4 CURVES S PARAMETER

$Z = 0.1$, $Z = 0.4$, and $Z = 0.7$, the corresponding maximum pressure locations are $R = 1$, $R = 4$, and $R = 7$. Hence, the maximum pressure value occurs at the right-end point for each pressure curve in Figure 5.3. Similarly, the minimum value of the tangential velocity occurs at the right-end point for each velocity curve in Figure 5.4.

5.3 Summary

A one-dimensional optimal expression for G_4 has been determined for the case where the G_2 expression is governed by the swirl parameter and the expression G^2 is required to be a constant. Optimal curves for G_2 and G_4 are plotted in Figures 5.1 and 5.2. The expressions for G_2 and G_4 were transformed from the one-dimensional (X) space back to the three-dimensional (R, θ, Z) space omitting the passage to the two-dimensional (η_1, η_2) space. The three-dimensional expressions for optimal pressure and tangential velocity are plotted in Figures 5.3 and 5.4.

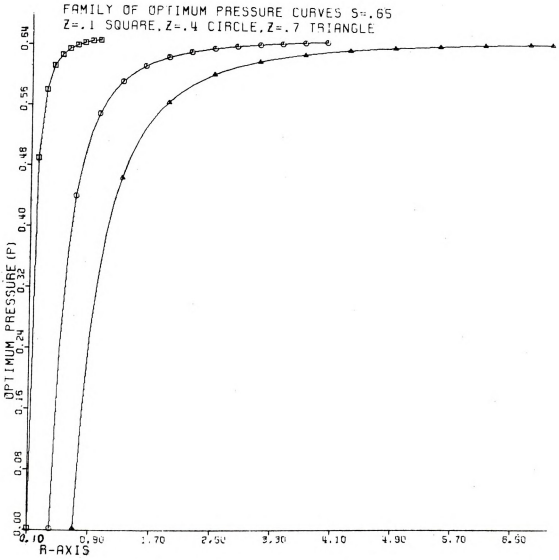

FIGURE 5.3 OPTIMUM PRESSURE CURVES

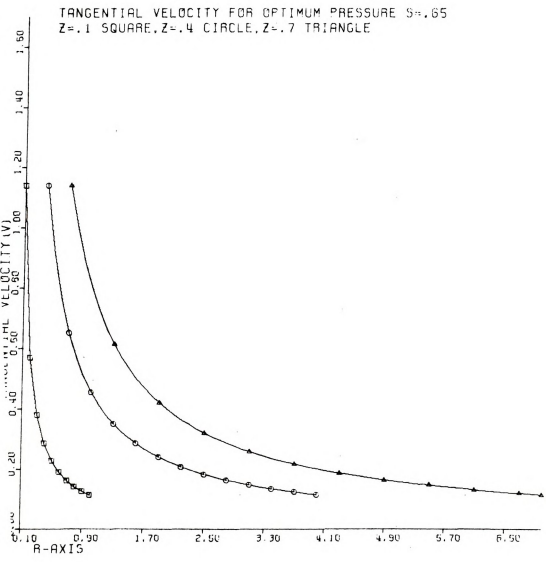

FIGURE 5.4 VELOCITY CURVES FOR OPTIMUM PRESSURE

BIBLIOGRAPHY

BIBLIOGRAPHY

1. Akhiezer, Naum I., The Calculus of Variations, Blaisdell Publishing Co., New York, 1962.

2. Alexandroff, P. S., Einführung in Die Gruppentheorie, Deutscher Verlag der Wissenschaften, Berlin, 1954.

3. Batchelor, G. K., "On Steady Laminar Flow with Closed Streamlines at Large Reynolds Number", Journal of Fluid Mechanics, Vol. 12, Part 2, pp. 177-190, 1956.

4. Beckett, Royce and Hurt, James, Numerical Calculations and Algorithms, McGraw-Hill Book Co., New York, 1967.

5. Benjamin, T. B., "Significance of the Vortex Breakdown Phenomenon", Transactions of the ASME, pp. 518-524, June 1965.

6. Bichara, R. T., and Orner, P. A., "Analysis and Modeling of the Vortex Amplifier", ASME Publication, 69-FLCS-20, 1969.

7. Birkhoff, Garrett, Hydrodynamics, Second Edition, Princeton University Press, Princeton, 1960.

8. Bookston, J. M. and McKeachie, D. D., "Notes on the Numerical Solution of Differential Equations", General Motors Corporation Research Laboratories Report GMR 161, July 1958.

9. Carnahan, Brice, Luther, H. A., and Wilkes, James O., Applied Numerical Methods, John Wiley & Sons, Inc., New York, 1969.

10. Chow, Chuen-Yen, "Swirling Flow in Tubes of Nonuniform Cross-Sections", Journal of Fluid Mechanics, Vol. 38, Part 4, pp. 843-854, October 1969.

11. Deissler, Robert G. and Perlmutter, Morris, "An Analysis of the Energy Separation in Laminar and Turbulent Compressible Vortex Flows", Proceedings Heat Transfer and Fluid Mechanics Institute, 1958.

12. Dickson, L. E., "Differential Equations from the Group Standpoint", Annals of Mathematics, Second Series, Vol. 25, No. 4, pp. 287-378, June 1924.

13. Donaldson, Coleman duP. and Sullivan, Roger D., "Behavior of Solutions of the Navier-Stokes Equations for a Complete Class of Three Dimensional Viscous Vortices", 1960 Proceedings of the Heat Transfer and Fluid Mechanics Institute, Stanford University Press, pp. 16-30.

14. Dussourd, J. L., and Schivley, G. P., "An Analytical and Experimental Study of a Vortex Pump", ASME Publication, 70-FE-32, 1970.

15. Eckert, E. R. G., and Hartnett, J. P., "Experimental Study of the Velocity and Temperature Distribution in a High Velocity Vortex Type Flow", University of Minnesota Heat Transfer Laboratory Technical Report No. 6, September 1955.

16. Einstein, Hans Albert and Li, Huon, "Steady Vortex Flow in a Real Fluid", Heat Transfer and Fluid Mechanics Proceedings, Stanford University Press, 1951.

17. Ewing, George M., Calculus of Variations with Applications, W. W. Norton & Co., New York, 1969.

18. Frame, J. S., "Matrix Theory and Linear Algebra with Applications", Lecture notes - Michigan State University, East Lansing, Michigan, 1968.

19. Gartshore, I. S., "Recent Work in Swirling Incompressible Flow", National Research Council of Canada Aeronautical Report LR-343, June, 1962.

20. Greenspan, Donald, "Numerical Studies of Prototype Cavity Flow Problems", The Computer Journal, Vol. 12, No. 1, pp. 89-94, February 1969.

21. Greenspan, Donald, "Numerical Studies of Viscous, Incompressible Flow for Arbitrary Reynolds Number", University of Wisconsin Computer Sciences Technical Report No. 11, January 1968.

22. Greenspan, Donald, "Numerical Studies of Two Dimensional, Steady State Navier-Stokes Equations for Arbitrary Reynold's Number", University of Wisconsin Computer Sciences Technical Report No. 9, October 1967.

23. Greenspan, H. P., The Theory of Rotating Fluids, Cambridge University Press, London, 1968

24. Hall, G. G., Applied Group Theory, American Elsevier Publishing Co., New York, 1967.

25. Hansen, Arthur G., Similarity Analysis of Boundary Value Problems in Engineering, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1964.

26. Hildebrand, Francis B., Finite Difference Equations and Simulations, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1968.

27. Hilsch, R., "The Use of the Expansion of Gases in a Centrifugal Field as Cooling Process", The Review of Scientific Instruments, Vol. 18, No. 2, pp. 108-113, February 1947.

28. Kawaguti, M., "Numerical Solutions of the Navier-Stokes Equations for the Flow in a Two Dimensional Cavity", Journal of the Physical Society of Japan, Vol. 16, pp. 2307-2318, April 1961.

29. Kolf, R. C., "Vortex Flow from Horizontal Thin-Plate Orifices", Ph.D. Thesis, University of Wisconsin, Madison, Wisconsin, 1956.

30. Kwok, C. C. K., "A Theoretical Study of Vortex Flow in a Thin Cylindrical Chamber", Bendix Technical Journal, Vol. 1, No. 4, 1969.

31. Lamb, Sir Horace, Hydrodynamics, Sixth Edition, Dover Publications, New York, 1932.

32. Ledermann, Walter, Introduction to the Theory of Finite Groups, Interscience Publishers, Inc., New York, 1957.

33. Leitmann, George, Optimization Techniques, Academic Press, New York, 1962.

34. Lomont, J. S., Applications of Finite Groups, Academic Press, New York, 1961.

35. Lugt, Hans J. and Schwiderski, Ernst W., "Birth and Decay of Vortices", Physics of Fluids, Vol. 9, No. 5, pp. 851-859, May 1966.

36. Mack, Leslie M., "The Compressible, Viscous, Heat-Conducting Vortex", Cal. Tech. Jet Propulsion Laboratory Progress Report No. 20-382, 1959.

37. Maker, Paul, and Mayer, Endre A., "Control Characteristics of Vortex Valves", Proceedings of the Fluid Amplification Symposium, Vol. II, U. S. Army Material Command, pp. 61-83, May 1964.

38. Mayer, Endre A., "Large-Signal Vortex Valve Analysis", Advances in Fluidics, Proceedings of the 1967 ASME Fluidics Symposium, ASME Publication, pp. 233-249, May 1967.

39. Mellor, G. L., Chapple, P. J., and Stokes, V. K., "On the Flow Between a Rotating and a Stationary Disk", Journal of Fluid Mechanics, Vol. 31, Part 1, pp. 95-112, January 1968.

40. Michal, A. D., "Differential Invariants and Invariant Partial Differential Equations in Normed Linear Spaces", Proceedings of the National Academy of Sciences, Vol. 37, No. 9, pp. 623-627, September 1952.

41. Miele, Angelo, Theory of Optimum Aerodynamic Shapes, Academic Press, New York, 1965.

42. Milne, William Edmund, Numerical Solution of Differential Equations", John Wiley & Sons, Inc., New York, 1953.

43. Milne-Thomson, L. M., Theoretical Hydrodynamics, Fourth Edition, The Macmillan Co., New York, 1966.

44. Morgan, A. J. A., "The Reduction by One of the Number of Independent Variables in Some Systems of Partial Differential Equations", Quarterly Journal of Mathematics, Oxford Second Series, Vol. 3, No. 12, pp. 250-259, December 1952.

45. Olmsted, John M. H., Advanced Calculus, Appleton-Century-Crofts, Inc., New York, 1961.

46. Pearson, Carl E., "A Computational Method for Viscous Flow Problems", Journal of Fluid Mechanics, Vol. 21, Part 4, pp. 611-622, 1965.

47. Pengelley, C. Desmond, "Flow in a Viscous Vortex", Journal of Applied Physics, Vol. 28, No. 1, pp. 86-92, January 1957.

48. Protter, Murray H. and Morrey, Charles B., Modern Mathematical Analysis, Addison-Wesley Publishing Co., Reading, Mass., 1964.

49. Springer, E. Kent, and Patterson, F. M., "Experimental Investigation of Critical Submergence for Vortexing in a Vertical Cylindrical Tank", ASME Publication, 69-FE-49, 1969.

50. Torda, T. P., and Agarwal, Urmila, "Numerical Investigation of Unsteady Laminar Incompressible Boundary Layer Flows with Cylindrical Symmetry", ASME Publication, 69-FE-47, 1969.

51. Wheeler, John A., and Ruffini, Remo, "Introducing the Black Hole", Physics Today, Vol. 24, No. 1, January 1971.

52. Wilf, Herbert S. and Ralston, Anthony, Mathematical Methods for Digital Computers, John Wiley & Sons, Inc., New York, 1960.

APPENDICES

APPENDIX A

APPENDIX A

NUMERICAL DATA FOR THE SOLUTIONS TO THE SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

A computer program utilizing Hamming's modified predictor-corrector method for numerically solving a general system of ordinary differential equations is presented in this appendix. This program is written in Fortran IV language and includes a file save such that the computed output data can be stored and used as input data to the subroutine which controls a Calcomp plotter. The IBM scientific subroutine DHCPG mentioned in Chapter IV is included in this particular program.

Table A.1 presents the numerical data for the solutions to the system of non-linear ordinary differential equations given in Chapter IV. In addition to values for G_1, G_2, G_3, and G_4, values for H, S, L^2, and π_2 are listed. The values of these functions at each value of X are displayed in the order shown in the following 2X5 matrix:

$$\text{X value} \begin{bmatrix} G_1 & G_2 & G_3 & G_4 & H \\ S & L^2 & \pi_2 & - & - \end{bmatrix}.$$

```
/1D PC05,BRINK,PC05,DIFFERENTIAL EQUATIONS
/JOB GO
/FILE DISK=(3,NREC=960,PTPTPT),VOL=RAXFL1,DISP=(OLD,KEEP)
/JOB GO,TIME=10
/FTC LIST
      COMMON IYY,ITERBT,IPLOT,IONCE,IJTIME,ITIME
C     THIS PROGRAM REPLACES THE BASIC DIFFEQ
      DOUBLE PRECISION OPT
      DOUBLE PRECISION X
      DIMENSION PRMT(5),Y(23),DERY(20),AUX(16,20)
      DIMENSION YTEMP(20)
      IONCE=0
      IJTIME=240
      J1=0
      PRMT(4)=.02
      READ(5)ITERBT,IPLOT,IYY
      READ(5)NDIM,NN,PRMT(3),PRMT(1),PRMT(2),OPT
      WRITE(6,160)
  160 FORMAT('1')
      WRITE(6,800)NDIM
  800 FORMAT(' NO OF DIFFERENTIAL EQUATIONS', 7X,I3)
      WRITE(6,810)NN
  810 FORMAT(' VARIABLE(=0) OR FIXED(=1) STEP SIZE',I3)
      WRITE(6,820)PRMT(3)
  820 FORMAT(' INITIAL STEP SIZE',6X,F15.5)
      WRITE(6,830)PRMT(1),PRMT(2)
  830 FORMAT(' INITIAL AND FINAL VALUES OF X',2F15.5)
      WRITE(6,840)OPT
  840 FORMAT(' PRINT OUT INTERVAL',5X,F15.5///)
      Y(1)=1.
      Y(2)=1.
      Y(3)=1.
      Y(4)=.5
      DO 8310 I=1,NDIM
 8310 YTEMP(I)=Y(I)
      IF(NN)90,80,90
   90 DO 100 I=1,NDIM
```

```
  100 DERY(I)=0.
      GO TO 120
   80 DO 110 I=1,NDIM
  110 DERY(I)=1./NDIM
  120 CALL HPCG(PRMT,Y,DERY,NDIM,IHLF,AUX,OPT,J1,X)
      IF(IONCE-1)8300,8090,8300
 8090 IONCE=2
      PRMT(2)=X
      IJTIME=ITIME/300+240
      IF(NN)190,180,190
  190 DO 1000 I=1,NDIM
 1000 DERY(I)=0.
      GO TO 1200
  180 DO 1100 I=1,NDIM
 1100 DERY(I)=1./NDIM
 1200 CONTINUE
      DO 8120 I=1,NDIM
 8120 Y(I)=YTEMP(I)
      CALL HPCG(PRMT,Y,DERY,NDIM,IHLF,AUX,OPT,J1,X)
 8300 X=99999.
      CALL OUTP(X,Y,DERY,IHLF,NDIM,PRMT,OPT,J1)
      IF(IHLF-31)130,200,210
  200 WRITE(6,220)
  220 FORMAT(' MORE THAN 30 BISECTIONS OF THE INITIAL INCREMENT ARE'/
     1'NECESSARY TO GET SATISFACTORY ACCURACY')
      GO TO 130
  210 WRITE(6,230)
  230 FORMAT('INITIAL INCREMENT IS EQUAL TO 0 OR HAS WRONG SIGN')
  130 STOP
      END
/BAP
ELAPSE START 0
       USING *,15
       STM   14,12,12(13)
       L     2,0(1)
       L     3,X'268'
       L     4,40(3)
```

```
      ST     4,0(2)
      LM     14,12,12(13)
      BCR    15,14
      END
/FTC LIST
      SUBROUTINE FCT(X,Y,D)
      COMMON IYY,ITERBT, IPLOT,IONCE,IJTIME,ITIME
      DOUBLE PRECISION X
      DIMENSION Y(1),D(1)
      DIMENSION ZZ(4)
      A=X*Y(1)-Y(3)
      C=-(A*Y(1))/X-Y(2)**2
      E=X+1./X
      D(4)=C/E
      D(3)=D(4)/A
      D(2)=(Y(1)*Y(2))/A
      D(1)=(Y(1)+D(3))/X
      Y(5)=Y(4)+(Y(1)**2+Y(2)**2+Y(3)**2)/2.
      Y(6)=(Y(2)**2)/(Y(5)*X**2)
      Y(7)=Y(1)**2+Y(3)**2
      Y(8)=(Y(2)**2)/Y(5)
      IF(Y(21)-1)9111,9112,9111
 9112 IJ=0
      KKK=KKK+1
      NOZ=Y(22)
      IN=Y(23)
      DO 9113 IJJI=IN,IYY,NOZ
      INOZ=IJJI+NOZ-1
      IJ=IJ+1
      Z=ZZ(IJ)
 9113 WRITE(7,9114)(X,Y(I),Z,KKK,I,I=IJJI,INOZ)
 9114 FORMAT(6X,'GO TO/',E13.6,',',E13.6,',',E13.6,22X,I3,I2)
      RETURN
 9111 CONTINUE
      RETURN
      END
      SUBROUTINE OUTP(X,Y,DERY,IHLF,NDIM,PRMT,OPT,J1)
```

```
      COMMON IYY,ITERBT,IPLOT,IONCE,IJTIME,ITIME
      DIMENSION Y(1),DERY(1),PRMT(1)
      DIMENSION ICT(20),JCT(20),YYMAX(20),YYMIN(20)
      DIMENSION NY(20),YSAVE(100,20),YSCLE(20)
      DIMENSION DSAVE(20),YYMX(20),YYMN(20)
      DIMENSION YSTOP(20)
      DOUBLE PRECISION OPT
      IF(X-PRMT(1))5,84,5
   84 PP=PRMT(1)
      IJJJ=0
      KKKC=0
      IF(NDIM-IYY)93,93,94
   93 IYMAX=IYY
      GO TO 96
   94 IYMAX=NDIM
   96 XX=X
      DO 850 I=1,NDIM
      ICT(I)=1
      JCT(I)=1
  850 DSAVE(I)=DERY(I)
      DO 730 I=1,IYMAX
      YYMX(I)=-1.E75
      YYMN(I)=1.E75
      YYMAX(I)=-1.E75
  730 YYMIN(I)=1.E75
      IF(IPLOT)333,333,71
   71 IF(IONCE-2)8100,726,8100
 8100 INPTS=40
      IF(ITERBT)726,726,8101
 8101 WRITE(6,725)
  725 FORMAT(' HOW MANY PTS DO YOU WISH TO PLOT?')
      READ(9)INPTS
      IF(INPTS-20)851,726,726
  851 INPTS=20
  726 XSTEP=(PRMT(2)-PRMT(1))/INPTS
      XPLOT=PRMT(1)
      ICOTT=0.
```

```
 333 IF(IONCE-2)8110,5,8110
8110 WRITE(6,140)
 140 FORMAT(//3H NO,14X,1HX,9X,4HY(1),9X,4HY(2),9X,4HY(3),9X,4HY(4))
     READ(5)IPUNCH,NOZ,IN
     Y(22)=NOZ
     Y(23)=IN
     READ(5)IONEY,XSTART
   5 CONTINUE
     IF(X-99999.)3099,3091,3099
3099 IF(X-XSTART)3091,3091,3092
3092 IF(DERY(IONEY))3091,3093,3093
3093 IONONE=IONONE+1
     IF(IONONE-2)3096,3091,3091
3096 DO 3095 II=1,IYMAX
3095 Y(II)=YSTOP(II)
     X=XXXYYY
     PRMT(5)=1.
     RETURN
3091 DO 3094 II=1,IYMAX
3094 YSTOP(II)=Y(II)
     IF(IJJJ-5)9222,223,9224
9222 IJJJ=IJJJ+1
     GO TO 222
 223 SIIJ=(PRMT(2)-PRMT(1))/PRMT(3)
     IIJ=SIIJ/1000.+1.
     GO TO 9223
9224 KKKC=KKKC+1
     IF(KKKC-IIJ)224,9223,224
9223 IJJJ=IJJJ+1
     KKKC=0
 222 CONTINUE
     WRITE(3)X,(Y(I),I=1,IYMAX)
 224 CONTINUE
     IF(IPLOT)540,540,740
 740 CONTINUE
     IF(X-99999.)540,550,540
 550 WRITE(6,551)
```

```
 551 FORMAT(////)
     IYMMX=IYMAX+1
     WRITE(3)(Y(I),I=1,IYMMX)
     CALL PTPLOT(YSAVE,ICOTT,YYMAX,YYMIN,PRMT(1),XSTEP)
     RETURN
 540 IF(X-99999.)541,35,541
  35 X=XXXYYY
     GO TO 38
 541 DO 900 I=1,IYMAX
     IF(Y(I)-YYMAX(I))920,920,910
 910 YYMAX(I)=Y(I)
 920 IF(Y(I)-YYMIN(I))930,900,900
 930 YYMIN(I)=Y(I)
 900 CONTINUE
     DO 600 I=1,NDIM
     IF(DSAVE(I))602,332,601
 332 IF(X-PRMT(1))335,600,335
 335 IF(DERY(I))600,334,600
 334 JCT(I)=JCT(I)+1
     ICT(I)=ICT(I)+1
     GO TO 600
 601 IF(DERY(I))603,603,600
 603 XY=ABS(YYMX(I)/100.)
     IF(XY)6000,6010,6010
6000 XY=.01
6010 IF(ABS(Y(I)-YYMX(I))-XY)604,604,605
 604 ICT(I)=ICT(I)+1
     IF(Y(I)-YYMX(I))600,600,608
 605 ICT(I)=1
 608 YYMX(I)=Y(I)
     GO TO 600
 602 IF(DERY(I))600,610,610
 610 XY=ABS(YYMN(I)/100.)
     IF(XY)6020,6030,6030
6020 XY=.01
6030 IF(ABS(Y(I)-YYMN(I))-XY)611,611,612
 611 JCT(I)=JCT(I)+1
```

```
      IF(Y(I)-YYMN(I))613,600,600
  612 JCT(I)=1
  613 YYMN(I)=Y(I)
  600 CONTINUE
      DO 800 I=1,NDIM
  800 DSAVE(I)=DERY(I)
  720 CONTINUE
      IF(IPLOT)6,6,8
    8 IF( ABS(XPLOT-X)-(X-XX)/2.+.0001)500,510,510
  510 IF(X-XPLOT)6,500,500
  500 XPLOT=XPLOT+XSTEP
   51 ICOTT=ICOTT+1
      YSAVE(ICOTT,1)=X
      DO 53 I=1,IYMAX
   53 YSAVE(ICOTT,I+1)=Y(I)
    6 IF(X-PRMT(2))8041,85,8041
 8041 CONTINUE
      IF( ABS(PP-X)-(X-XX)/2.+.0001)85,87,87
   87 IF(X-PP)20,85,85
   85 PP=PP+DPT
   10 J1=J1+1
      IF(IONCE)20,62,20
   62 WRITE(6,50)J1,X,(Y(I),I=1,IYY)
   50 FORMAT(I4,E14.5,4E13.5/(18X,4E13.5))
      IF(IPUNCH-1)20,8223,20
 8223 Y(21)=1.
      CALL FCT(X,Y,D)
      Y(21)=0.
   20 XX=X
      IF(IPLOT)8000,8000,34
   34 CALL ELAPSE(ITIME)
      IF(IONCE-2)2081,2080,2080
 2081 IF(JTIME-1)2090,2091,2080
 2090 IF(ITIME-1200)2080,3010,3010
 2091 IF(ITIME-2400)2080,3010,3010
 3010 JTIME=JTIME+1
      IF(JTIME-1)3011,3011,3012
```

```
 3011 XTIME=(X-PRMT(1))*60.+PRMT(1)
      GO TO 3013
 3012 XTIME=(X-PRMT(1))*30.+PRMT(1)
 3013 IF(XTIME-PRMT(2))3020,2080,2080
 3020 XSTEP=(XTIME-PRMT(1))/INPTS
 2080 CONTINUE
      IF(ITIME/300-IJTIME)8000,8000,36
   36 WRITE(6,37)
   37 FORMAT(//' 4 MINUTES EXECUTION TIME HAS BEEN USED'/
     1' BRANCH TO PLOT ROUTINE BEING MADE')
   38 WRITE(6,746)X
  746 FORMAT(/' FINAL X =',E15.7,//)
      IYMMX=IYMAX+1
      WRITE(3)(Y(I),I=1,IYMMX)
      WRITE(6,747)
  747 FORMAT('     Y  CURRENT VALUE                MIN                MAX')
      DO 748 I=1,IYMAX
  748 WRITE(6,749)I,Y(I),YYMIN(I),YYMAX(I)
  749 FORMAT(I5,3E15.7)
      PRMT(5)=1.
 8000 XXXYYY=X
      RETURN
      END
      SUBROUTINE HPCG(PRMT,Y,DERY,NDIM,IHLF,AUX,OPT,J1,X)
      COMMON IYY,ITERBT,IPLOT,IONCE,IJTIME,ITIME
C
C
      DIMENSION PRMT(1),Y(1),DERY(1),AUX(16,1)
      DOUBLE PRECISION OPT
      DOUBLE PRECISION X
      N=1
      IHLF=0
      X=PRMT(1)
      H=PRMT(3)
      PRMT(5)=0.
      DO 1 I=1,NDIM
      AUX(16,I)=0.
```

```
      AUX(15,I)=DERY(I)
    1 AUX(1,I)=Y(I)
      IF(H*(PRMT(2)-X))3,2,4
C
C     ERROR RETURNS
    2 IHLF=32
      GO TO 4
    3 IHLF=33
C
C     COMPUTATION OF DERY FOR STARTING VALUES
    4 CALL FCT(X,Y,DERY)
C
C     RECORDING OF STARTING VALUES
      CALL OUTP(X,Y,DERY,IHLF,NDIM,PRMT,OPT,J1)
      IF(PRMT(5))6,5,6
    5 IF(IHLF)7,7,6
    6 RETURN
    7 DO 8 I=1,NDIM
    8 AUX(8,I)=DERY(I)
C
C     COMPUTATION OF AUX(2,I)
      ISW=1
      GO TO 100
C
    9 X=X+H
      DO 10 I=1,NDIM
   10 AUX(2,I)=Y(I)
C
C     INCREMENT H IS TESTED BY MEANS OF BISECTION
   11 IHLF=IHLF+1
      X=X-H
      DO 12 I=1,NDIM
   12 AUX(4,I)=AUX(2,I)
      H=.5*H
      N=1
      ISW=2
      GO TO 100
```

```
C
   13 X=X+H
      CALL FCT(X,Y,DERY)
      N=2
      DO 14 I=1,NDIM
      AUX(2,I)=Y(I)
   14 AUX(9,I)=DERY(I)
      ISW=3
      GO TO 100
C
C     COMPUTATION OF TEST VALUE DELT
   15 DELT=0.
      DO 16 I=1,NDIM
   16 DELT=DELT+AUX(15,I)*ABS(Y(I)-AUX(4,I))
      DELT=.06666667*DELT
      IF(DELT-PRMT(4))19,19,17
   17 IF(IHLF-30)11,18,18
C
C     NO SATISFACTORY ACCURACY AFTER 10 BISECTIONS. ERROR MESSAGE.
   18 IHLF=31
      X=X+H
      GO TO 4
C
C     THERE IS SATISFACTORY ACCURACY AFTER LESS THAN 11 BISECTIONS.
   19 X=X+H
      CALL FCT(X,Y,DERY)
      DO 20 I=1,NDIM
      AUX(3,I)=Y(I)
   20 AUX(10,I)=DERY(I)
      N=3
      ISW=4
      GO TO 100
C
   21 N=1
      X=X+H
      CALL FCT(X,Y,DERY)
      X=PRMT(1)
```

```
      DO 22 I=1,NDIM
      AUX(11,I)=DERY(I)
   22 Y(I)=AUX(1,I)+H*(.375*AUX(8,I)+.7916667*AUX(9,I)
     1-.2083333*AUX(10,I)+.04166667*DERY(I))
   23 X=X+H
      N=N+1
      CALL FCT(X,Y,DERY)
      CALL OUTP(X,Y,DERY,IHLF,NDIM,PRMT,OPT,J1)
      IF(PRMT(5))6,24,6
   24 IF(N-4)25,200,200
   25 DO 26 I=1,NDIM
      AUX(N,I)=Y(I)
   26 AUX(N+7,I)=DERY(I)
      IF(N-3)27,29,200
C
   27 DO 28 I=1,NDIM
      DELT=AUX(9,I)+AUX(9,I)
      DELT=DELT+DELT
   28 Y(I)=AUX(1,I)+.3333333*H*(AUX(8,I)+DELT+AUX(10,I))
      GO TO 23
C
   29 DO 30 I=1,NDIM
      DELT=AUX(9,I)+AUX(10,I)
      DELT=DELT+DELT+DELT
   30 Y(I)=AUX(1,I)+.375*H*(AUX(8,I)+DELT+AUX(11,I))
      GO TO 23
C
C     THE FOLLOWING PART OF SUBROUTINE HPCG COMPUTES BY MEANS OF
C     RUNGE-KUTTA METHOD STARTING VALUES FOR THE NOT SELF-STARTING
C     PREDICTOR-CORRECTOR METHOD.
  100 DO 101 I=1,NDIM
      Z=H*AUX(N+7,I)
      AUX(5,I)=Z
  101 Y(I)=AUX(N,I)+.4*Z
C     Z IS AN AUXILIARY STORAGE LOCATION
C
      Z=X+.4*H
```

```
      CALL FCT(Z,Y,DERY)
      DO 102 I=1,NDIM
      Z=H*DERY(I)
      AUX(6,I)=Z
  102 Y(I)=AUX(N,I)+.2969776*AUX(5,I)+.1587596*Z
C
      Z=X+.4557372*H
      CALL FCT(Z,Y,DERY)
      DO 103 I=1,NDIM
      Z=H*DERY(I)
      AUX(7,I)=Z
  103 Y(I)=AUX(N,I)+.2181004*AUX(5,I)-3.050965*AUX(6,I)+3.832865*Z
C
      Z=X+H
      CALL FCT(Z,Y,DERY)
      DO 104 I=1,NDIM
 1040Y(I)=AUX(N,I)+.1747603*AUX(5,I)-.5514807*AUX(6,I)
     1+1.205536*AUX(7,I)+.1711848*H*DERY(I)
      GO TO (9,13,15,21),ISW
C
C     POSSIBLE BREAK-POINT FOR LINKAGE
C
C     STARTING VALUES ARE COMPUTED.
C     NOW START HAMMINGS MODIFIED PREDICTOR-CORRECTOR METHOD.
  200 ISTEP=3
  201 IF(N-8)204,202,204
C
C     N=8 CAUSES THE ROWS OF AUX TO CHANGE THEIR STORAGE LOCATIONS
  202 DO 203 N=2,7
      DO 203 I=1,NDIM
      AUX(N-1,I)=AUX(N,I)
  203 AUX(N+6,I)=AUX(N+7,I)
      N=7
C
C     N LESS THAN 8 CAUSES N+1 TO GET N
  204 N=N+1
C
```

```
C     COMPUTATION OF NEXT VECTOR Y
      DO 205 I=1,NDIM
      AUX(N-1,I)=Y(I)
  205 AUX(N+6,I)=DERY(I)
      X=X+H
  206 ISTEP=ISTEP+1
      DO 207 I=1,NDIM
     ODELT=AUX(N-4,I)+1.333333*H*(AUX(N+6,I)+AUX(N+6,I)-AUX(N+5,I)+
     1AUX(N+4,I)+AUX(N+4,I))
      Y(I)=DELT-.9256198*AUX(16,I)
  207 AUX(16,I)=DELT
C     PREDICTOR IS NOW GENERATED IN ROW 16 OF AUX, MODIFIED PREDICTOR
C     IS GENERATED IN Y. DELT MEANS AN AUXILIARY STORAGE.
C
      CALL FCT(X,Y,DERY)
C     DERIVATIVE OF MODIFIED PREDICTOR IS GENERATED IN DERY
C
      DO 208 I=1,NDIM
     ODELT=.125*(9.*AUX(N-1,I)-AUX(N-3,I)+3.*H*(DERY(I)+AUX(N+6,I)+
     1AUX(N+6,I)-AUX(N+5,I)))
      AUX(16,I)=AUX(16,I)-DELT
  208 Y(I)=DELT+.07438017*AUX(16,I)
C
C     TEST WHETHER H MUST BE HALVED OR DOUBLED
      DELT=0.
      DO 209 I=1,NDIM
  209 DELT=DELT+AUX(15,I)*ABS(AUX(16,I))
      IF(DELT-PRMT(4))210,222,222
C
C     H MUST NOT BE HALVED. THAT MEANS Y(I) ARE GOOD.
  210 CALL FCT(X,Y,DERY)
      CALL OUTP(X,Y,DERY,IHLF,NDIM,PRMT,OPT,J1)
      IF(PRMT(5))212,211,212
  211 IF(IHLF-31)213,212,212
  212 RETURN
  213 IF(H*(X-PRMT(2)))214,1212,1212
  214 IF(DABS(X-PRMT(2))-1.*ABS(H))1212,215,215
```

```
 1212 H=PRMT(2)-X
      IF(H)201,212,201
  215 IF(DELT-.10*PRMT(4))216,216,201
C
C
C     H COULD BE DOUBLED IF ALL NECESSARY PRECEEDING VALUES ARE
C     AVAILABLE
  216 IF(H-.1)217,217,201
  217 IF(N-7)201,218,218
  218 IF(ISTEP-4)201,219,219
  219 IMOD=ISTEP/2
      IF(ISTEP-IMOD-IMOD)201,220,201
  220 H=H+H
      IHLF=IHLF-1
      ISTEP=0
      DO 221 I=1,NDIM
      AUX(N-1,I)=AUX(N-2,I)
      AUX(N-2,I)=AUX(N-4,I)
      AUX(N-3,I)=AUX(N-6,I)
      AUX(N+6,I)=AUX(N+5,I)
      AUX(N+5,I)=AUX(N+3,I)
      AUX(N+4,I)=AUX(N+1,I)
      DELT=AUX(N+6,I)+AUX(N+5,I)
      DELT=DELT+DELT+DELT
 2210AUX(16,I)=8.962963*(Y(I)-AUX(N-3,I))-3.361111*H*(DERY(I)+DELT
     1+AUX(N+4,I))
      GO TO 201
C
C
C     H MUST BE HALVED
  222 IHLF=IHLF+1
      IF(IHLF-30)223,223,210
  223 H=.5*H
      ISTEP=0
      DO 224 I=1,NDIM
     0Y(I)=.00390625*(80.*AUX(N-1,I)+135.*AUX(N-2,I)+40.*AUX(N-3,I)+
     1AUX(N-4,I))-.1171875*(AUX(N+6,I)-6.*AUX(N+5,I)-AUX(N+4,I))*H
```

```
     OAUX(N-4,I)=.00390625*(12.*AUX(N-1,I)+135.*AUX(N-2,I)+
     1108.*AUX(N-3,I)+AUX(N-4,I))-.0234375*(AUX(N+6,I)+18.*AUX(N+5,I)-
     29.*AUX(N+4,I))*H
      AUX(N-3,I)=AUX(N-2,I)
224   AUX(N+4,I)=AUX(N+5,I)
      X=X-H
      DELT=X-(H+H)
      CALL FCT(DELT,Y,DERY)
      DO 225 I=1,NDIM
      AUX(N-2,I)=Y(I)
      AUX(N+5,I)=DERY(I)
225   Y(I)=AUX(N-4,I)
      DELT=DELT-(H+H)
      CALL FCT(DELT,Y,DERY)
      DO 226 I=1,NDIM
      DELT=AUX(N+5,I)+AUX(N+4,I)
      DELT=DELT+DELT+DELT
     OAUX(16,I)=8.962963*(AUX(N-1,I)-Y(I))-3.361111*H*(AUX(N+6,I)+DELT
     1+DERY(I))
226   AUX(N+3,I)=DERY(I)
      GO TO 206
      END
```

TABLE A.1 NUMERICAL DATA FOR SOLUTIONS TO THE SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

NO	X	Y(1)	Y(2)	Y(3)	Y(4)	Y(5)
1	0.10000E 00	0.10000E 01	0.10000E 01	0.10000E 01	0.50000E 00	0.20000E 01
		0.50000E 02	0.20000E 01	0.50000E 00		
2	0.10500E 00	0.10060E 01	0.99441E 00	0.99560E 00	0.50394E 00	0.20000E 01
		0.44846E 02	0.20033E 01	0.49443E 00		
3	0.11000E 00	0.10120E 01	0.98876E 00	0.99120E 00	0.50783E 00	0.20000E 01
		0.40399E 02	0.20067E 01	0.48882E 00		
4	0.11500E 00	0.10181E 01	0.98304E 00	0.98681E 00	0.51167E 00	0.20000E 01
		0.36536E 02	0.20103E 01	0.48318E 00		
5	0.12000E 00	0.10242E 01	0.97725E 00	0.98242E 00	0.51547E 00	0.20000E 01
		0.33160E 02	0.20140E 01	0.47751E 00		
6	0.12500E 00	0.10302E 01	0.97139E 00	0.97803E 00	0.51922E 00	0.20000E 01
		0.30195E 02	0.20179E 01	0.47180E 00		
7	0.13000E 00	0.10364E 01	0.96547E 00	0.97364E 00	0.52292E 00	0.20000E 01
		0.27578E 02	0.20220E 01	0.46606E 00		
8	0.14000E 00	0.10487E 01	0.95339E 00	0.96487E 00	0.53019E 00	0.20000E 01
		0.23188E 02	0.20307E 01	0.45448E 00		
9	0.15000E 00	0.10610E 01	0.94102E 00	0.95612E 00	0.53725E 00	0.20000E 01
		0.19678E 02	0.20400E 01	0.44276E 00		
10	0.16000E 00	0.10735E 01	0.92833E 00	0.94738E 00	0.54413E 00	0.20000E 01
		0.16832E 02	0.20499E 01	0.43090E 00		
11	0.17000E 00	0.10861E 01	0.91532E 00	0.93865E 00	0.55080E 00	0.20000E 01
		0.14495E 02	0.20606E 01	0.41890E 00		
12	0.19000E 00	0.11114E 01	0.88824E 00	0.92124E 00	0.56355E 00	0.20000E 01
		0.10928E 02	0.20839E 01	0.39449E 00		
13	0.21000E 00	0.11371E 01	0.85965E 00	0.90390E 00	0.57548E 00	0.20000E 01
		0.83788E 01	0.21100E 01	0.36951E 00		
14	0.23000E 00	0.11631E 01	0.82939E 00	0.88662E 00	0.58657E 00	0.20000E 01
		0.65019E 01	0.21389E 01	0.34395E 00		
15	0.25000E 00	0.11895E 01	0.79725E 00	0.86942E 00	0.59682E 00	0.20000E 01
		0.50849E 01	0.21707E 01	0.31781E 00		
16	0.29000E 00	0.12431E 01	0.72629E 00	0.83527E 00	0.61471E 00	0.20000E 01
		0.31362E 01	0.22431E 01	0.26375E 00		

TABLE A.1 (CONT'D.)

17	0.33000E 00	0.12981E 01	0.64385E 00	0.80148E 00	0.62905E 00	0.20000E 01
		0.19033E 01	0.23273E 01	0.20727E 00		
18	0.37000E 00	0.13542E 01	0.54460E 00	0.76809E 00	0.63975E 00	0.20000E 01
		0.10832E 01	0.24238E 01	0.14830E 00		
19	0.41000E 00	0.14116E 01	0.41622E 00	0.73517E 00	0.64674E 00	0.19999E 01
		0.51533E 00	0.25331E 01	0.86626E-01		
20	0.45000E 00	0.14725E 01	0.19415E 00	0.70384E 00	0.64992E 00	0.20007E 01
		0.93040E-01	0.26638E 01	0.18841E-01		
21	0.45500E 00	0.14824E 01	0.14255E 00	0.70096E 00	0.65003E 00	0.20046E 01
		0.48961E-01	0.26888E 01	0.10136E-01		
22	0.46000E 00	0.14885E 01	0.93135E-01	0.69633E 00	0.65012E 00	0.20047E 01
		0.20448E-01	0.27006E 01	0.43268E-02		
23	0.46500E 00	0.14934E 01	0.53592E-01	0.69115E 00	0.65014E 00	0.20056E 01
		0.66230E-02	0.27080E 01	0.14321E-02		
24	0.46531E 00	0.14900E 01	0.47948E-01	0.68910E 00	0.65013E 00	0.19988E 01
		0.53122E-02	0.26951E 01	0.11502E-02		
25	0.46562E 00	0.14900E 01	0.50620E-01	0.68864E 00	0.65013E 00	0.19986E 01
		0.59134E-02	0.26944E 01	0.12821E-02		
26	0.46594E 00	0.14899E 01	0.52570E-01	0.68811E 00	0.65012E 00	0.19982E 01
		0.63709E-02	0.26933E 01	0.13831E-02		
27	0.46625E 00	0.14901E 01	0.57925E-01	0.68776E 00	0.65012E 00	0.19986E 01
		0.77227E-02	0.26935E 01	0.16788E-02		
28	0.46656E 00	0.14902E 01	0.61595E-01	0.68732E 00	0.65012E 00	0.19986E 01
		0.87206E-02	0.26931E 01	0.18983E-02		
29	0.46687E 00	0.14903E 01	0.65260E-01	0.68690E 00	0.65011E 00	0.19986E 01
		0.97758E-02	0.26928E 01	0.21308E-02		
30	0.46750E 00	0.14900E 01	0.77124E-01	0.68582E 00	0.65010E 00	0.19983E 01
		0.13619E-01	0.26904E 01	0.29766E-02		
31	0.46812E 00	0.14901E 01	0.83619E-01	0.68492E 00	0.65009E 00	0.19983E 01
		0.15967E-01	0.26894E 01	0.34991E-02		
32	0.46875E 00	0.14901E 01	0.89673E-01	0.68402E 00	0.65008E 00	0.19983E 01
		0.18314E-01	0.26884E 01	0.40240E-02		
33	0.46937E 00	0.14902E 01	0.94929E-01	0.68314E 00	0.65006E 00	0.19983E 01
		0.20469E-01	0.26875E 01	0.45096E-02		

TABLE A.1 (CONT'D.)

34	0.47062E 00	0.14905E 01	0.10404E 00	0.68140E 00	0.65003E 00	0.19984E 01	
		0.24455E-01	0.26859E 01	0.54165E-02			
35	0.47187E 00	0.14907E 01	0.11286E 00	0.67963E 00	0.64999E 00	0.19984E 01	
		0.28624E-01	0.26841E 01	0.63736E-02			
36	0.47312E 00	0.14909E 01	0.12145E 00	0.67784E 00	0.64995E 00	0.19984E 01	
		0.32976E-01	0.26821E 01	0.73816E-02			
37	0.47437E 00	0.14910E 01	0.12948E 00	0.67605E 00	0.64990E 00	0.19984E 01	
		0.37282E-01	0.26802E 01	0.83895E-02			
38	0.47687E 00	0.14916E 01	0.14251E 00	0.67258E 00	0.64977E 00	0.19985E 01	
		0.44686E-01	0.26771E 01	0.10162E-01			
39	0.47937E 00	0.14919E 01	0.15575E 00	0.66903E 00	0.64962E 00	0.19985E 01	
		0.52823E-01	0.26735E 01	0.12139E-01			
40	0.48187E 00	0.14923E 01	0.16780E 00	0.66549E 00	0.64945E 00	0.19985E 01	
		0.60678E-01	0.26699E 01	0.14090E-01			
41	0.48437E 00	0.14927E 01	0.17920E 00	0.66195E 00	0.64924E 00	0.19985E 01	
		0.68489E-01	0.26664E 01	0.16069E-01			
42	0.48937E 00	0.14933E 01	0.20092E 00	0.65476E 00	0.64875E 00	0.19982E 01	
		0.84357E-01	0.26586E 01	0.20202E-01			
43	0.49437E 00	0.14941E 01	0.21991E 00	0.64769E 00	0.64816E 00	0.19982E 01	
		0.99020E-01	0.26518E 01	0.24201E-01			
44	0.49937E 00	0.14949E 01	0.23728E 00	0.64064E 00	0.64747E 00	0.19982E 01	
		0.11299E 00	0.26452E 01	0.28176E-01			
45	0.50437E 00	0.14958E 01	0.25342E 00	0.63361E 00	0.64667E 00	0.19982E 01	
		0.12634E 00	0.26389E 01	0.32140E-01			
46	0.51437E 00	0.14976E 01	0.28332E 00	0.61953E 00	0.64476E 00	0.19982E 01	
		0.15183E 00	0.26266E 01	0.40173E-01			
47	0.52437E 00	0.14995E 01	0.31013E 00	0.60556E 00	0.64244E 00	0.19982E 01	
		0.17506E 00	0.26153E 01	0.48136E-01			
48	0.53437E 00	0.15016E 01	0.33492E 00	0.59163E 00	0.63971E 00	0.19982E 01	
		0.19659E 00	0.26047E 01	0.56138E-01			
49	0.54437E 00	0.15037E 01	0.35799E 00	0.57777E 00	0.63658E 00	0.19982E 01	
		0.21643E 00	0.25950E 01	0.64136E-01			
50	0.56437E 00	0.15085E 01	0.39979E 00	0.55028E 00	0.62913E 00	0.19982E 01	
		0.25112E 00	0.25784E 01	0.79987E-01			

TABLE A.1 (CONT'D.)

51	0.58437E 00	0.15136E 01	0.43809E 00	0.52296E 00	0.62008E 00	0.19982E 01	
		0.28126E 00	0.25643E 01	0.96049E-01			
52	0.60437E 00	0.15190E 01	0.47341E 00	0.49589E 00	0.60948E 00	0.19982E 01	
		0.30706E 00	0.25533E 01	0.11216E 00			
53	0.62437E 00	0.15249E 01	0.50643E 00	0.46905E 00	0.59733E 00	0.19982E 01	
		0.32923E 00	0.25452E 01	0.12835E 00			
54	0.66437E 00	0.15376E 01	0.56710E 00	0.41602E 00	0.56848E 00	0.19980E 01	
		0.36467E 00	0.25373E 01	0.16097E 00			
55	0.70437E 00	0.15519E 01	0.62235E 00	0.36401E 00	0.53379E 00	0.19979E 01	
		0.39075E 00	0.25409E 01	0.19387E 00			
56	0.74437E 00	0.15676E 01	0.67350E 00	0.31300E 00	0.49343E 00	0.19979E 01	
		0.40975E 00	0.25553E 01	0.22704E 00			
57	0.78437E 00	0.15846E 01	0.72151E 00	0.26296E 00	0.44756E 00	0.19979E 01	
		0.42351E 00	0.25801E 01	0.26056E 00			
58	0.86437E 00	0.16220E 01	0.81063E 00	0.16556E 00	0.33974E 00	0.19974E 01	
		0.44032E 00	0.26582E 01	0.32899E 00			
59	0.94437E 00	0.16639E 01	0.89268E 00	0.72077E-01	0.21199E 00	0.19973E 01	
		0.44737E 00	0.27737E 01	0.39898E 00			
60	0.95133E 00	0.16680E 01	0.89992E 00	0.63766E-01	0.20016E 00	0.19982E 01	
		0.44782E 00	0.27863E 01	0.40529E 00			
61	0.95828E 00	0.16720E 01	0.90697E 00	0.55612E-01	0.18774E 00	0.19983E 01	
		0.44826E 00	0.27986E 01	0.41164E 00			
62	0.96523E 00	0.16755E 01	0.91292E 00	0.48681E-01	0.17672E 00	0.19982E 01	
		0.44767E 00	0.28095E 01	0.41708E 00			
63	0.97219E 00	0.16793E 01	0.91964E 00	0.40860E-01	0.16425E 00	0.19981E 01	
		0.44784E 00	0.28219E 01	0.42328E 00			
64	0.98609E 00	0.16868E 01	0.93162E 00	0.26939E-01	0.14256E 00	0.19995E 01	
		0.44640E 00	0.28459E 01	0.43407E 00			
65	0.10000E 01	0.16947E 01	0.94480E 00	0.11529E-01	0.11680E 00	0.19992E 01	
		0.44649E 00	0.28723E 01	0.44649E 00			
66	0.10000E 01	0.16950E 01	0.94522E 00	0.11034E-01	0.11599E 00	0.19993E 01	
		0.44689E 00	0.28731E 01	0.44689E 00			
67	0.10000E 01	0.16951E 01	0.94534E 00	0.10897E-01	0.11576E 00	0.19993E 01	
		0.44700E 00	0.28734E 01	0.44700E 00			

TABLE A.1 (CONT'D.)

68	0.10000E 01	0.16947E 01	0.94477E 00	0.11562E-01	0.11686E 00	0.19992E 01
		0.44646E 00	0.28722E 01	0.44646E 00		
69	0.10000E 01	0.16947E 01	0.94472E 00	0.11620E-01	0.11695E 00	0.19992E 01
		0.44642E 00	0.28721E 01	0.44642E 00		

APPENDIX B

APPENDIX B

CALCOMP PLOTTER ROUTINE

A special library subroutine named L9MIKE was used to control the Calcomp plotter which plotted all the curves displayed in this thesis. This subroutine allows the user to title curves, title axes, select scale lengths, plot parametric curves where the plot interval is a function on the parameter, specify the order and number (up to 20) of variables to be plotted per diagram, and identify each curve by at least 10 identification marks (squares, circles, triangles, etc.). The input data for the L9MIKE subroutine is taken from a stored file and all data points in the file are plotted, but only 10 or 11 identification points are marked on each curve. The L9MIKE subroutine is written in Fortran IV language.

This appendix presents the computer printout listing of the L9MIKE subroutine used for controlling the Calcomp plotter.

```
/JOB GO,TIME=10      L9MIKE CALCOMP PLOTTER ROUTINE
/FILE DISK=(3,PTPTPT),VOL=RAXFL1,DISP=(OLD,KEEP)
/FILE DISK=(2,NREC=960,OFFWAL),VOL=RAXFL1,DISP=(NEW,DELETE)
/FILE DISK=(1,NREC=960,TEMPRY),VOL=RAXFL1,DISP=(NEW,DELETE)
      READ(5)NVAR
      NVAR=NVAR+1
      CALL PLOTTT(NVAR)
      STOP
      END
      SUBROUTINE PLOTTT(NVAR)
      DIMENSION IBUF(1000),NYY(21),YY(150),Y(2002),X(2002),NAME(15),
     1X1(10),Y1(10),NN(4),Z(5),XXMI(21),XXMA(21)
      DIMENSION NNAME(15)
      DIMENSION NNME(8)
      ICOMP='P400'
      CALL STARTR(ICOMP,99,1,2)
      CALL PLOTS(IBUF,1000)
      NX=1
      READ(5)IGOZZ,KKKK
      IF(IGOZZ-5)1207,1205,1207
 1205 READ(5)NZ
      DO 1650 I=1,NZ
 1650 READ(5)Z(I)
      READ(5)IFST,IINC,A1,B1
      IJK=0
      NVRR=NVAR-1
      DO 1210 IIJ=IFST,NVRR,IINC
      IJK=IJK+1
      IKJ=IIJ+IINC-1
      DO 1210 IIJSY=IIJ,IKJ
      XXMI(IIJSY)=Z(IJK)*A1
      XXMA(IIJSY)=Z(IJK)*B1
 1210 CONTINUE
 1207 CONTINUE
      READ(5)ISCLE
      IF(ISCLE-1)9031,9032,9031
 9032 READ(5)XLN,YLN
```

```
9031 CONTINUE
   2 IF(ISCLE-1)914,914,913
 913 READ(5)NXX
 914 READ(5)NY
     ITWO=1
     DO 10 I=1,NY
  10 READ(5)NYY(I)
     READ(5,5)(NNAME(I),I=1,15)
     READ(5,5)(NAME(I),I=1,15)
     READ(5,5)(NNME(I),I=1,8)
   5 FORMAT(15A4)
     AMAX=-1.E68
     AMIN=1.E68
     AAMAX=-1.E68
     AAMIN=1.E68
     INTEQ=0
     KONE=KONE+1
     CALL PLOT(3.,-15.,-3)
     CALL PLOT(0.,.5,-3)
     JJJ=NYY(1)
     REWIND 3
     I=0
 333 I=I+1
     READ(3)(YY(J),J=1,NVAR)
     IF(YY(1)-99999.)334,335,334
 334 IF(IGOZZ-5)1230,1610,1230
1610 IF(ISCLE-1)1820,1820,1810
1810 X10=YY(NXX+1)
     GO TO 1830
1820 X10=YY(NX)
1830 IF(X10-AAMAX)1850,1850,1840
1840 AAMAX=X10
1850 IF(X10-AAMIN)1860,1870,1870
1860 AAMIN=X10
1870 CONTINUE
     IF(YY(1)-XXMI(JJJ)+.0001)1215,1220,1220
1215 I=I-1
```

```
      GO TO 819
 1220 IF(YY(I)-XXMA(JJJ))1230,1230,1225
 1225 I=I-1
      GO TO 819
 1230 Y(I)=YY(JJJ+1)
      IF(ISCLE-1)20,20,912
  912 X(I)=YY(NXX+1)
      GO TO 799
   20 X(I)=YY(NX)
  799 IF(ISCLE-1)333,819,333
  819 DO 821 II=2,NVAR
      DO 851 IJ=1,NY
      NNNY=NYY(IJ)
      IF(IGOZZ-5)1936,9221,1936
 9221 CONTINUE
      IF(YY(I)-XXMA(NNNY))1934,1934,821
 1934 IF(YY(I)-XXMI(NNNY)+.0001)821,1936,1936
 1936 CONTINUE
      IF(II-1-NYY(IJ))851,852,851
  852 IF(AMAX-YY(II))822,823,823
  822 AMAX=YY(II)
  823 IF(AMIN-YY(II))851,851,824
  824 AMIN=YY(II)
      GO TO 821
  851 CONTINUE
  821 CONTINUE
 1935 IF(ITWO-2)333,1265,333
  335 NOB=I-1
      IINN=NOB/10
      IF(IINN)8112,8112,8113
 8112 IINN=1
 8113 CONTINUE
      IF(ISCLE-1)826,798,826
  798 YY(1)=AMIN
      YY(2)=AMAX
      CALL SCALE(YY,YLN,2,1)
      Y(NOB+1)=YY(3)
```

```
      Y(NOB+2)=YY(4)
      GO TO 827
  826 CALL SCALE(Y,YLN,NOB,1)
  827 IF(IGOZZ-5)1890,1880,1890
 1880 YY(1)=AAMIN
      YY(2)=AAMAX
      CALL SCALE(YY,XLN,2,1)
      X(NOB+1)=YY(3)
      X(NOB+2)=YY(4)
      GO TO 1900
 1890 CALL SCALE(X,XLN,NOB,1)
 1900 CONTINUE
      XXXX1=X(NOB+1)
      XXXX2=X(NOB+2)
      YYYY1=Y(NOB+1)
      YYY2=Y(NOB+2)
      CALL AXIS(0.,0.,NNAME,-60,XLN,0.,X(NOB+1),X(NOB+2))
      CALL AXIS(0.,0.,NNME(1),32,YLN,90.,Y(NOB+1),Y(NOB+2))
      CALL LINE(X,Y,NOB,1,IINN,INTEQ)
      CALL LINE(X(NOB),Y(NOB),1,1,1,INTEQ)
      IDCB=' =  '
      XX=-.5
      YW=4.
      YYY=0.
      XXX=0.
      IBCD='VAR '
      AA=NYY(1)
      CALL SYMBOL(.5,YLN,.14,NAME,0.,60)
      IF(KKKK-1)2010,2020,2010
 2020 READ(5,5)(NAME(I),I=1,15)
      CALL SYMBOL(.5,YLN-.25,.14,NAME,0.,60)
 2010 CONTINUE
      IF(ISCLE-2)1014,1013,1014
 1013 AB=NXX
      CALL SYMBOL(4.,.5,.14,IBCD,0.,4)
      CALL NUMBER(999.,999.,.14,AB,0.,-1)
 1014 CONTINUE
```

```
      CALL SYMBOL(XX,YW,.14,IBCD,90.,4)
      CALL NUMBER(999.,999.,.14,AA,90.,-1)
      CALL SYMBOL(999.,999.,.14,IDCB,90.,3)
      CALL SYMBOL(999.,999.,.14,INTEQ,90.,-1)
      IF(NY-1)9382,9882,9382
 9382 CONTINUE
      AMAX=-1.E68
      AMIN=1.E68
      DO 50 JJ=2,NY
      XX=XX-1.
      JJJ=NYY(JJ)+1
      REWIND 3
      INTEQ=INTEQ+1
      AA=JJJ-1
      CALL SYMBOL(XX,YW,.14,IBCD,90.,4)
      CALL NUMBER(999.,999.,.14,AA,90.,-1)
      CALL SYMBOL(999.,999.,.14,IDCB,90.,3)
      CALL SYMBOL(999.,999.,.14,INTEQ,90.,-1)
      ITWO=2
      NOB=0
 1265 NOB=NOB+1
      READ(3)(YY(J),J=1,NVAR)
      IF(YY(1)-99999.)1300,1320,1300
 1300 IF(IGOZZ-5)1275,1630,1275
 1630 IF(YY(1)-XXMI(JJJ-1)+.0001)1255,1260,1260
 1255 NOB=NOB-1
      GO TO 819
 1260 IF(YY(1)-XXMA(JJJ-1))1275,1275,1270
 1270 NOB=NOB-1
      GO TO 819
 1275 Y(NOB)=YY(JJJ)
      IF(ISCLE-1)1280,1280,1285
 1285 X(NOB)=YY(NXX+1)
      IF(ISCLE-1)1265,1310,1265
 1280 X(NOB)=YY(NX)
      IF(ISCLE-1)1265,819,1265
 1310 GO TO 819
```

```
1320 NOB=NOB-1
     X(NOB+1)=XXXX1
     X(NOB+2)=XXXX2
     Y(NOB+1)=YYYY1
     Y(NOB+2)=YYY2
     IINN=NOB/10
     IF(IINN)1240,1240,1250
1240 IINN=1
1250 CONTINUE
     YYY=YYY-1.
     XXX=XXX+1.
     IF(ISCLE-1)829,50,829
 829 CALL PLOT( YYY,0.,-3)
     CALL SCALE(Y,YLN,NOB,1)
     CALL AXIS(0.,0.,NN(1),6,XLN,90.,Y(NOB+1),Y(NOB+2))
     CALL PLOT(XXX,0.,-3)
  50 CALL LINE(X,Y,NOB,1,IINN,INTEQ)
     CALL LINE(X(NOB),Y(NOB),1,1,1,INTEQ)
9882 CONTINUE
     CALL PLOT(16.,0.,-3)
     READ(5)ISCLE
     REWIND 3
     IF(ISCLE-4)2,70,70
  70 CALL ENDIT
     CALL PLOTTR(2)
     RETURN
     END
     SUBROUTINE PLOTTR(IDS)
     DIMENSION ISPOOL(31), IC(10), IOUT(118), IH(6) ,IGET(8)
     REWIND IDS
     IC(1)=20987968
     IC(2)=37765184
     IC(3)=71319616
     IC(4)=138428480
     IC(5)=272646208
     IC(6)=339755072
     IC(7)=406863936
```

```
      IC(8)=541081664
      IC(9)=608190528
      IC(10)=675299392
      IDOL1=1530937408
      IH(1)=-489171242
      IH(2)=-750729019
      IH(3)=-1044068903
      IH(4)=1088013763
      IH(5)=-690371520
      IH(6)=-909847069
      LIM=8
      IBLK=31
      READ (IDS) ISPOOL
      IHOLD=ISPOOL(7)
      DO 1 I=1,6
      IF (ISPOOL(I)-IH(I)) 14,1,14
    1 CONTINUE
      NREC=ISPOOL(8)/(256*256)
      IF (NREC) 14,14,2
    2 ISPOOL(8)=ISPOOL(8)-256*256*NREC
      INAME=ISPOOL(8)/256
       ISPOOL(8)=ISPOOL(8)-256*INAME
      M=1
      N=1
      I=9
      GO TO 4
    3 I=1
      N=2
      DO 11 J=2,NREC
      READ (IDS) ISPOOL
    4 DO 10 K=I,IBLK
      CALL TESTED (ISPOOL(K), IGET(1))
   17 DO 9 L=1,LIM
      I1=IGET(L)
   19 IF (I1) 14,14,5
    5 IF (I1-10) 6,6,14
    6 IOUT(M)=IC(I1)
```

```
   15 M=M+1
      IF (M-118) 9,9,7
    7 WRITE (6,100) IDOL1,IDOL1,IOUT
      M=1
    9 CONTINUE
   10 CONTINUE
      GO TO (3,11),N
   11 CONTINUE
      DO 12 I=M,118
   12 IOUT(I)=IC(2)
      WRITE (6,100) IDOL1,IDOL1,IOUT
   13 STOP
   14 WRITE (6,200)
      STOP
  100 FORMAT (120A1)
  200 FORMAT (' DATA SET HAS AN INVALID HEADER OR ID')
      END
/BAP
TESTED START 0
       USING *,15
       STM   14,12,12(13)
       L     10,0(1)
       L     7,0(10)
       L     9,4(1)
       LA    11,8
A      SR    6,6
       SLDL  6,4
       ST    6,0(9)
       LA    9,4(9)
       BCT   11,A
       LM    14,12,12(13)
       BCR   15,14
       END
/DATA
/END
```

APPENDIX C

APPENDIX C

LEAST SQUARES POLYNOMIAL CURVE FITTING PROGRAM

A computer program utilizing the method of least squares for determining a polynomial approximation of tabular data is presented in this appendix. The program is written in Fortran IV language and the least squares portion is a part of a library of standard scientific subroutines. The program is named PC05MB and it will fit 2 through 100 points with a polynomial of degree 1 through 11. For this program, the standard error of estimate is defined as:

$$\text{Standard error of estimate} \equiv \left[\frac{\sum_{i=1}^{m} (G^i - G^i_{est.})^2}{m - (n+1)} \right]^{1/2}$$

In the equation for the standard error of estimate, m represents the total number of observations and n represents the degree of the approximating polynomial.

Table C.1 displays the polynomial coefficients and the curve fitting data for the functions G_1, G_2, G_3, and G_4 in the interval $0.10000 \leq X \leq 0.46531$.

Table C.2 displays the polynomial coefficients and the curve fitting data for the functions G_1, G_2, G_3, and G_4 in the interval $0.46531 \leq X \leq 1.0000$.

```
/JOB GO     PC05MB LEAST SQUARES POLYNOMIAL FIT
/FILE DISK=(3,KEEPIT),VOL=RAXFL1,DISP=(OLD,KEEP)
      DIMENSION YYY(20)
      DIMENSION A(15),B(15),S(15),G(15),U(15)
      DIMENSION Q(1600),P(1600),X(1600),Y(1600),C(1600)
      WRITE(6,1)
    1 FORMAT(' WHAT ARE THE N AND NDIM OF THE DATA')
      READ(9)N,NDIM
      IXXX=N
  790 WRITE(6,2)
    2 FORMAT(' WHICH G VARIABLE DO YOU WISH TO USE?')
      READ(9)KK
      Z=0
      O=1
      K=12
      N=N+1
      IF(N-11)55,55,576
   55 CONTINUE
   70 T7=Z
      T8=Z
      W7=Z
      I=1
   54 CONTINUE
      READ(3)X(I),(YYY(J),J=1,NDIM)
      IF(X(I)-99999)60,311,60
   60 CONTINUE
      Y(I)=YYY(KK)
      W7=W7+X(I)
      T7=T7+Y(I)
      T8=T8+Y(I)**2
      I=I+1
      GO TO 54
  311 M=I-1
      T9=(M*T8-T7**2)/(M**2-M)
      IF(NNN-1)740,730,740
  740 CONTINUE
      WRITE(6,316)
```

```
316 FORMAT(/' L E A S T - S Q U A R E S  P O L Y N O M I A L S',/)
    WRITE(6,320)M
320 FORMAT('            NUMBER OF POINTS =',I12)
    AMEAN=W7/M
    WRITE(6,322)AMEAN
322 FORMAT(/'             MEAN VALUE OF X =',E12.5)
    YMEAN=T7/M
    WRITE(6,324)YMEAN
324 FORMAT(/'             MEAN VALUE OF G =',E12.5)
    STDE=SQRT(T9)
    WRITE(6,326)STDE
326 FORMAT(/'              STD ERROR OF G =',E12.5)
    WRITE(6,330)
330 FORMAT(/'   NOTE. . .CODE FOR ''WHAT NEXT?'' IS',/,9X,
   1'0 = STOP PROGRAM',/,9X,'1 = COEFFICIENTS ONLY',/,9X,
   2'2 = ENTIRE SUMMARY',/,9X,'3 = FIT NEXT HIGHER DEGREE',//)
730 CONTINUE
    DO 352 I=1,M
    P(I)=Z
    Q(I)=0
352 CONTINUE
    DO 362 I=1,11
    A(I)=Z
    B(I)=Z
    S(I)=Z
362 CONTINUE
    E1=Z
    F1=Z
    W1=M
    N4=K
    I=1
    K1=2
    IF(N)378,380,378
378 K1=N4
380 W=Z
    DO 386 L=1,M
    W=W+Y(L)*Q(L)
```

```
386 CONTINUE
    S(I)=W/W1
    IF(I-N4)392,428,428
392 E1=Z
    DO 398 L=1,M
    E1=E1+X(L)*Q(L)*Q(L)
398 CONTINUE
    E1=E1/W1
    A(I+1)=E1
    W=Z
    DO 416 L=1,M
    V=(X(L)-E1)*Q(L)-F1*P(L)
    P(L)=Q(L)
    Q(L)=V
    W=W+V*V
416 CONTINUE
    F1=W/W1
    B(I+2)=F1
    W1=W
    I=I+1
    GO TO 380
428 DO 432 L=2,11
    G(L)=Z
432 CONTINUE
    G(1)=0
    DO 464 J=1,N
    S1=Z
    DO 448 L=1,N
    IF(L-1)444,446,444
444 G(L)=G(L)-A(L)*G(L-1)-B(L)*G(L-2)
446 S1=S1+S(L)*G(L)
448 CONTINUE
    U(J)=S1
    L=N
    DO 460 I2=2,N
    G(L)=G(L-1)
    L=L-1
```

```
460 CONTINUE
462 G(1)=Z
464 CONTINUE
    T=Z
    DO 488 L=1,M
    C(L)=Z
    J=N
    DO 482 I2=1,N
    C(L)=C(L)*X(L)+U(J)
    J=J-1
482 CONTINUE
    T3=Y(L)-C(L)
    T=T+T3**2
488 CONTINUE
    IF(M-N)496,492,496
492 T5=0
    GO TO 498
496 T5=T/(M-N)
498 Q7=1-(T5/T9)
    NN=N-1
    WRITE(6,502)NN,Q7
502 FORMAT(//' POLYFIT OF DEGREE',I3,'  INDEX OF DETERM =',F10.4,
   1//,'WHAT NEXT ')
    READ(9)IR
512 IF(IR)514,628,514
514 IF(IR-3)516,564,516
516 WRITE(6,517)
517 FORMAT(//,11X,'TERM',4X,'COEFFICIENT',/)
520 DO 526 J=1,N
    I2=J-1
    WRITE(6,524)I2,U(J)
524 FORMAT(I15,E15.5)
526 CONTINUE
    READ(5,829)
829 FORMAT('                                                    ')
    WRITE(6,829)
    IF(IR-1)531,558,531
```

```
531 WRITE(6,530)
530 FORMAT(//6X,'X-ACTUAL',6X,'G-ACTUAL',8X,'G-CALC',7X,'DIFF',8X,
   1'PCT-DIFF',/)
    DO 550 L=1,M
    Q8=Y(L)-C(L)
    IF(C(L))544,548,544
548 WRITE(6,540)X(L),Y(L),C(L),Q8
540 FORMAT(4E15.6,4X,'INFINITE')
    GO TO 550
544 CL=100.*Q8/C(L)
    WRITE(6,547)X(L),Y(L),C(L),Q8,CL
547 FORMAT(5E14.6)
550 CONTINUE
    ST5=SQRT(T5)
    WRITE(6,554)ST5
554 FORMAT(//'          STD ERROR OF ESTIMATE FOR G =',F10.4/)
    IF(K-N)558,628,558
558 WRITE(6,560)
560 FORMAT(' WHAT NEXT?')
    READ(9)IR
    GO TO 512
564 N=N+1
    IF(M-N)616,568,568
568 GO TO 428
570 WRITE(6,572)
572 FORMAT('PROGRAM SIZE LIMIT IS 100 DATA POINTS.')
    GO TO 628
576 WRITE(6,578)
578 FORMAT('ELEVENTH DEGREE IS THE LIMIT')
    GO TO 628
616 NN=N-1
    WRITE(6,618)NN
618 FORMAT(' TOO FEW POINTS FOR FITTING DEGREE',I3)
    GO TO 628
628 WRITE(6,710)
710 FORMAT(' TYPE 1 IF ANOTHER RUN IS DESIRED'/)
    READ(9)IONES
```

```
      N=IXXX
      REWIND 3
      NNN=1
      IF(IONES-1)720,790,720
  720 STOP
      END
/DATA
/END
```

TABLE C.1 CURVE FITTING DATA FOR THE INTERVAL 0.10000≤X≤0.46531

G1 ONE-DIMENSIONAL REPRESENTATION FOR RADIAL VELOCITY

TERM	COEFFICIENT
0	0.88531E 00
1	0.11040E 01
2	0.44657E 00

X-ACTUAL	G-ACTUAL	G-CALC	DIFF	PCT-DIFF
0.100000E 00	0.100000E 01	0.100017E 01	-0.170708E-03	-0.170679E-01
0.105000E 00	0.100601E 01	0.100615E 01	-0.142097E-03	-0.141229E-01
0.110000E 00	0.101203E 01	0.101215E 01	-0.114441E-03	-0.113067E-01
0.115000E 00	0.101808E 01	0.101817E 01	-0.858307E-04	-0.842989E-02
0.120000E 00	0.102416E 01	0.102422E 01	-0.600815E-04	-0.586610E-02
0.125000E 00	0.103025E 01	0.103028E 01	-0.333786E-04	-0.323975E-02
0.130000E 00	0.103636E 01	0.103637E 01	-0.858307E-05	-0.828185E-03
0.140000E 00	0.104866E 01	0.104862E 01	0.400543E-04	0.381973E-02
0.150000E 00	0.106104E 01	0.106095E 01	0.848770E-04	0.800008E-02
0.160000E 00	0.107350E 01	0.107337E 01	0.126839E-03	0.118168E-01
0.170000E 00	0.108605E 01	0.108589E 01	0.163078E-03	0.150180E-01
0.190000E 00	0.111141E 01	0.111118E 01	0.224113E-03	0.201689E-01
0.210000E 00	0.113710E 01	0.113684E 01	0.262260E-03	0.230693E-01
0.230000E 00	0.116312E 01	0.116284E 01	0.275612E-03	0.237015E-01
0.250000E 00	0.118947E 01	0.118921E 01	0.258446E-03	0.217326E-01
0.290000E 00	0.124313E 01	0.124302E 01	0.119209E-03	0.959033E-02
0.330000E 00	0.129806E 01	0.129825E 01	-0.192642E-03	-0.148386E-01
0.370000E 00	0.135420E 01	0.135491E 01	-0.710487E-03	-0.524379E-01
0.410000E 00	0.141159E 01	0.141300E 01	-0.141335E-02	-0.100024E 00
0.450000E 00	0.147254E 01	0.147252E 01	0.209808E-04	0.142482E-02
0.455000E 00	0.148240E 01	0.148006E 01	0.233364E-02	0.157672E 00

TABLE C.1 (CONT'D.)

0.460000E 00	0.148851E 01	0.148763E 01	0.887871E-03	0.596837E-01
0.465000E 00	0.149344E 01	0.149521E 01	-0.177574E-02	-0.118762E 00

STD ERROR OF ESTIMATE FOR G = 0.0008

G2 ONE-DIMENSIONAL REPRESENTATION FOR TANGENTIAL VELOCITY

TERM	COEFFICIENT
0	0.14366E 01
1	-0.95497E 01
2	0.82897E 02
3	-0.38395E 03
4	0.82938E 03
5	-0.69174E 03

X-ACTUAL	G-ACTUAL	G-CALC	DIFF	PCT-DIFF
0.100000E 00	0.100000E 01	0.100270E 01	-0.270081E-02	-0.269353E 00
0.105000E 00	0.994412E 00	0.995365E 00	-0.952661E-03	-0.957097E-01
0.110000E 00	0.988759E 00	0.988469E 00	0.289798E-03	0.293178E-01
0.115000E 00	0.983039E 00	0.981934E 00	0.110513E-02	0.112546E 00
0.120000E 00	0.977251E 00	0.975686E 00	0.156546E-02	0.160447E 00
0.125000E 00	0.971394E 00	0.969657E 00	0.173712E-02	0.179148E 00
0.130000E 00	0.965466E 00	0.963787E 00	0.167972E-02	0.174283E 00
0.140000E 00	0.953393E 00	0.952306E 00	0.108695E-02	0.114139E 00
0.150000E 00	0.941019E 00	0.940871E 00	0.147939E-03	0.157236E-01

TABLE C.1 (CONT'D.)

0.160000E 00	0.928332E 00	0.929190E 00	-0.858247E-03	-0.923651E-01
0.170000E 00	0.915317E 00	0.917046E 00	-0.172901E-02	-0.188541E 00
0.190000E 00	0.888241E 00	0.890827E 00	-0.258547E-02	-0.290233E 00
0.210000E 00	0.859655E 00	0.861656E 00	-0.200075E-02	-0.232198E 00
0.230000E 00	0.829392E 00	0.829637E 00	-0.244737E-03	-0.294992E-01
0.250000E 00	0.797253E 00	0.795271E 00	0.198203E-02	0.249227E 00
0.290000E 00	0.726288E 00	0.721877E 00	0.441188E-02	0.611168E 00
0.330000E 00	0.643846E 00	0.643291E 00	0.554621E-03	0.862162E-01
0.370000E 00	0.544596E 00	0.550659E 00	-0.606215E-02	-0.110089E 01
0.410000E 00	0.416225E 00	0.415973E 00	0.252247E-03	0.606402E-01
0.450000E 00	0.194148E 00	0.183534E 00	0.106146E-01	0.578348E 01
0.455000E 00	0.142546E 00	0.143481E 00	-0.935674E-03	-0.652123E 00
0.460000E 00	0.931352E-01	0.100297E 00	-0.716174E-02	-0.714053E 01
0.465000E 00	0.535923E-01	0.537481E-01	-0.155836E-03	-0.289938E 00

STD ERROR OF ESTIMATE FOR G = 0.0039

TABLE C.1 (CONT'D.)

G3 ONE-DIMENSIONAL REPRESENTATION FOR AXIAL VELOCITY

TERM	COEFFICIENT
0	0.10909E 01
1	-0.91970E 00
2	0.13199E 00

X-ACTUAL	G-ACTUAL	G-CALC	DIFF	PCT-DIFF
0.100000E 00	0.100000E 01	0.100029E 01	-0.287056E-03	-0.286974E-01
0.105000E 00	0.995601E 00	0.995824E 00	-0.223279E-03	-0.224215E-01
0.110000E 00	0.991204E 00	0.991367E 00	-0.163674E-03	-0.165100E-01
0.115000E 00	0.986809E 00	0.986917E 00	-0.108302E-03	-0.109737E-01
0.120000E 00	0.982417E 00	0.982474E 00	-0.571609E-04	-0.581805E-02
0.125000E 00	0.978027E 00	0.978037E 00	-0.100732E-04	-0.102994E-02
0.130000E 00	0.973640E 00	0.973607E 00	0.329614E-04	0.338549E-02
0.140000E 00	0.964874E 00	0.964766E 00	0.108242E-03	0.112195E-01
0.150000E 00	0.956121E 00	0.955952E 00	0.169098E-03	0.176890E-01
0.160000E 00	0.947381E 00	0.947164E 00	0.216544E-03	0.228623E-01
0.170000E 00	0.938654E 00	0.938403E 00	0.251114E-03	0.267598E-01
0.190000E 00	0.921244E 00	0.920959E 00	0.285089E-03	0.309557E-01
0.210000E 00	0.903898E 00	0.903621E 00	0.277400E-03	0.306987E-01
0.230000E 00	0.886622E 00	0.886388E 00	0.233769E-03	0.263733E-01
0.250000E 00	0.869421E 00	0.869261E 00	0.159800E-03	0.183834E-01
0.290000E 00	0.835267E 00	0.835324E 00	-0.572205E-04	-0.685009E-02
0.330000E 00	0.801477E 00	0.801809E 00	-0.331938E-03	-0.413987E-01
0.370000E 00	0.768092E 00	0.768717E 00	-0.625014E-03	-0.813062E-01
0.410000E 00	0.735170E 00	0.736047E 00	-0.876963E-03	-0.119145E 00
0.450000E 00	0.703835E 00	0.703799E 00	0.363588E-04	0.516608E-02
0.455000E 00	0.700963E 00	0.699798E 00	0.116533E-02	0.166524E 00

TABLE C.1 (CONT'D.)

0.460000E 00	0.696333E 00	0.695803E 00	0.530660E-03	0.762659E-01
0.465000E 00	0.691151E 00	0.691815E 00	-0.663817E-03	-0.959529E-01

STD ERROR OF ESTIMATE FOR G = 0.0004

G4 ONE-DIMENSIONAL REPRESENTATION FOR PRESSURE

TERM	COEFFICIENT
0	0.40740E 00
1	0.10314E 01
2	-0.10941E 01

X-ACTUAL	G-ACTUAL	G-CALC	DIFF	PCT-DIFF
0.100000E 00	0.500000E 00	0.499600E 00	0.400126E-03	0.800893E-01
0.105000E 00	0.503937E 00	0.503635E 00	0.301898E-03	0.599437E-01
0.110000E 00	0.507828E 00	0.507616E 00	0.212133E-03	0.417900E-01
0.115000E 00	0.511673E 00	0.511542E 00	0.130534E-03	0.255178E-01
0.120000E 00	0.515470E 00	0.515414E 00	0.565648E-04	0.109746E-01
0.125000E 00	0.519221E 00	0.519231E 00	-0.995398E-05	-0.191706E-02
0.130000E 00	0.522923E 00	0.522993E 00	-0.692010E-04	-0.132317E-01
0.140000E 00	0.530185E 00	0.530353E 00	-0.167310E-03	-0.315470E-01
0.150000E 00	0.537253E 00	0.537494E 00	-0.240147E-03	-0.446791E-01
0.160000E 00	0.544126E 00	0.544416E 00	-0.289857E-03	-0.532419E-01
0.170000E 00	0.550800E 00	0.551119E 00	-0.318944E-03	-0.578721E-01
0.190000E 00	0.563547E 00	0.563870E 00	-0.323057E-03	-0.572929E-01
0.210000E 00	0.575475E 00	0.575745E 00	-0.270069E-03	-0.469077E-01

TABLE C.1 (CONT'D.)

0.230000E 00	0.586570E 00	0.586745E 00	-0.175416E-03	-0.298965E-01
0.250000E 00	0.596816E 00	0.596870E 00	-0.541806E-04	-0.907746E-02
0.290000E 00	0.614707E 00	0.614493E 00	0.213504E-03	0.347447E-01
0.330000E 00	0.629047E 00	0.628616E 00	0.431248E-03	0.686014E-01
0.370000E 00	0.639749E 00	0.639237E 00	0.512242E-03	0.801334E-01
0.410000E 00	0.646739E 00	0.646357E 00	0.381649E-03	0.590461E-01
0.450000E 00	0.649924E 00	0.649976E 00	-0.526309E-04	-0.809735E-02
0.455000E 00	0.650032E 00	0.650182E 00	-0.150859E-03	-0.232026E-01
0.460000E 00	0.650120E 00	0.650334E 00	-0.214458E-03	-0.329765E-01
0.465000E 00	0.650137E 00	0.650431E 00	-0.293434E-03	-0.451137E-01

STD ERROR OF ESTIMATE FOR G = 0.0003

TABLE C.2 CURVE FITTING DATA FOR THE INTERVAL $0.46531 \leq X \leq 1.0000$

G1 ONE-DIMENSIONAL REPRESENTATION FOR RADIAL VELOCITY

TERM	COEFFICIENT
0	0.15338E 01
1	-0.29709E 00
2	0.43202E 00

X-ACTUAL	G-ACTUAL	G-CALC	DIFF	PCT-DIFF
0.465312E 00	0.149004E 01	0.148913E 01	0.913620E-03	0.613526E-01
0.470312E 00	0.149063E 01	0.148967E 01	0.965118E-03	0.647876E-01
0.475312E 00	0.149105E 01	0.149022E 01	0.831604E-03	0.558040E-01
0.480312E 00	0.149158E 01	0.149080E 01	0.778198E-03	0.522000E-01
0.485312E 00	0.149214E 01	0.149140E 01	0.739098E-03	0.495572E-01
0.490312E 00	0.149267E 01	0.149202E 01	0.641823E-03	0.430169E-01
0.495312E 00	0.149324E 01	0.149267E 01	0.573158E-03	0.383983E-01
0.500312E 00	0.149350E 01	0.149402E 01	-0.522614E-03	-0.349804E-01
0.515312E 00	0.149456E 01	0.149546E 01	-0.901222E-03	-0.602640E-01
0.525312E 00	0.149591E 01	0.149698E 01	-0.107670E-02	-0.719246E-01
0.535312E 00	0.149747E 01	0.149859E 01	-0.112724E-02	-0.752200E-01
0.555312E 00	0.150102E 01	0.150208E 01	-0.105572E-02	-0.702839E-01
0.575312E 00	0.150485E 01	0.150590E 01	-0.104904E-02	-0.696620E-01
0.595312E 00	0.150905E 01	0.151008E 01	-0.102806E-02	-0.680801E-01
0.615312E 00	0.151367E 01	0.151459E 01	-0.923157E-03	-0.609508E-01
0.655312E 00	0.152442E 01	0.152467E 01	-0.246048E-03	-0.161378E-01
0.695312E 00	0.153632E 01	0.153612E 01	0.196457E-03	0.127891E-01
0.735312E 00	0.154973E 01	0.154896E 01	0.763893E-03	0.493164E-01
0.775312E 00	0.156445E 01	0.156318E 01	0.126648E-02	0.810192E-01
0.855312E 00	0.159709E 01	0.159577E 01	0.131130E-02	0.821733E-01
0.935312E 00	0.163463E 01	0.163389E 01	0.732422E-03	0.448267E-01

TABLE C.2 (CONT'D.)

0.951484E 00	0.164308E 01	0.164227E 01	0.810623E-03	0.493598E-01
0.967656E 00	0.165142E 01	0.165088E 01	0.546455E-03	0.331009E-01
0.983828E 00	0.165956E 01	0.165971E 01	-0.149727E-03	-0.902128E-02
0.100000E 01	0.166809E 01	0.166876E 01	-0.667572E-03	-0.400040E-01
0.100000E 01	0.166837E 01	0.166876E 01	-0.389099E-03	-0.233166E-01
0.100000E 01	0.166845E 01	0.166876E 01	-0.314713E-03	-0.188590E-01
0.100000E 01	0.166808E 01	0.166876E 01	-0.685692E-03	-0.410898E-01
0.100000E 01	0.166804E 01	0.166876E 01	-0.719070E-03	-0.430900E-01

STD ERROR OF ESTIMATE FOR G = 0.0009

G2 ONE-DIMENSIONAL REPRESENTATION FOR TANGENTIAL VELOCITY

TERM	COEFFICIENT
0	-0.36184E 02
1	0.24104E 03
2	-0.63639E 03
3	0.83823E 03
4	-0.54694E 03
5	0.14125E 03

X-ACTUAL	G-ACTUAL	G-CALC	DIFF	PCT-DIFF
0.465312E 00	0.479479E-01	0.772400E-01	-0.292921E-01	-0.379234E 02
0.470312E 00	0.106286E 00	0.106171E 00	0.114977E-03	0.108295E 00
0.475312E 00	0.141378E 00	0.133530E 00	0.784856E-02	0.587777E 01
0.480312E 00	0.170483E 00	0.159332E 00	0.111511E-01	0.699866E 01
0.485312E 00	0.193925E 00	0.183701E 00	0.102249E-01	0.556609E 01
0.490312E 00	0.215562E 00	0.206635E 00	0.892752E-02	0.432044E 01

TABLE C.2 (CONT'D.)

0.495312E 00	0.235046E 00	0.228363E 00	0.668252E-02	0.292627E 01
0.505312E 00	0.271852E 00	0.268158E 00	0.369370E-02	0.137743E 01
0.515312E 00	0.303545E 00	0.303665E 00	-0.119984E-03	-0.395120E-01
0.525312E 00	0.331882E 00	0.335373E 00	-0.349081E-02	-0.104087E 01
0.535312E 00	0.357847E 00	0.363937E 00	-0.609070E-02	-0.167356E 01
0.555312E 00	0.404325E 00	0.413040E 00	-0.871551E-02	-0.211009E 01
0.575312E 00	0.446494E 00	0.454086E 00	-0.759202E-02	-0.167193E 01
0.595312E 00	0.485214E 00	0.489456E 00	-0.424224E-02	-0.866725E 00
0.615312E 00	0.521174E 00	0.521332E 00	-0.157356E-03	-0.301835E-01
0.655312E 00	0.586157E 00	0.580185E 00	0.597239E-02	0.102939E 01
0.695312E 00	0.645593E 00	0.637848E 00	0.774491E-02	0.121422E 01
0.735312E 00	0.700313E 00	0.696426E 00	0.388664E-02	0.558083E 00
0.775312E 00	0.751559E 00	0.754303E 00	-0.274372E-02	-0.363743E 00
0.855312E 00	0.846603E 00	0.857346E 00	-0.107429E-01	-0.125304E 01
0.935312E 00	0.933529E 00	0.933289E 00	0.240147E-03	0.257313E-01
0.951484E 00	0.950668E 00	0.947662E 00	0.300568E-02	0.317168E 00
0.967656E 00	0.967415E 00	0.963150E 00	0.426483E-02	0.442800E 00
0.983828E 00	0.983073E 00	0.980560E 00	0.251234E-02	0.256214E 00
0.100000E 01	0.999268E 00	0.100145E 01	-0.218183E-02	-0.217867E 00
0.100000E 01	0.999801E 00	0.100145E 01	-0.164884E-02	-0.164646E 00
0.100000E 01	0.999942E 00	0.100145E 01	-0.150752E-02	-0.150534E 00
0.100000E 01	0.999237E 00	0.100145E 01	-0.221252E-02	-0.220932E 00
0.100000E 01	0.999174E 00	0.100145E 01	-0.227559E-02	-0.227229E 00

STD ERROR OF ESTIMATE FOR G = 0.0088

TABLE C.2 (CONT'D.)

G3 ONE-DIMENSIONAL REPRESENTATION FOR AXIAL VELOCITY

TERM	COEFFICIENT
0	0.14350E 01
1	-0.17466E 01
2	0.30862E 00

X-ACTUAL	G-ACTUAL	G-CALC	DIFF	PCT-DIFF
0.465312E 00	0.689105E 00	0.689145E 00	-0.407696E-04	-0.591596E-02
0.470312E 00	0.681928E 00	0.681856E 00	0.717640E-04	0.105248E-01
0.475312E 00	0.674674E 00	0.674583E 00	0.908375E-04	0.134657E-01
0.480312E 00	0.667468E 00	0.667325E 00	0.143349E-03	0.214812E-01
0.485312E 00	0.660281E 00	0.660082E 00	0.199080E-03	0.301598E-01
0.490312E 00	0.653075E 00	0.652855E 00	0.219882E-03	0.336800E-01
0.495312E 00	0.645894E 00	0.645643E 00	0.250876E-03	0.388568E-01
0.500312E 00	0.631077E 00	0.631265E 00	-0.187933E-03	-0.297709E-01
0.515312E 00	0.616679E 00	0.616950E 00	-0.270128E-03	-0.437845E-01
0.525312E 00	0.602428E 00	0.602696E 00	-0.267088E-03	-0.443156E-01
0.535312E 00	0.588290E 00	0.588503E 00	-0.213563E-03	-0.362893E-01
0.555312E 00	0.560165E 00	0.560304E 00	-0.138700E-03	-0.247544E-01
0.575312E 00	0.532268E 00	0.532351E 00	-0.833273E-04	-0.156527E-01
0.595312E 00	0.504590E 00	0.504645E 00	-0.554323E-04	-0.109844E-01
0.615312E 00	0.477169E 00	0.477186E 00	-0.174642E-04	-0.365982E-02
0.655312E 00	0.423108E 00	0.423010E 00	0.976920E-04	0.230945E-01
0.695312E 00	0.369946E 00	0.369821E 00	0.125408E-03	0.339105E-01
0.735312E 00	0.317819E 00	0.317618E 00	0.200808E-03	0.632230E-01
0.775312E 00	0.266677E 00	0.266404E 00	0.272870E-03	0.102427E 00
0.855312E 00	0.166948E 00	0.166939E 00	0.971556E-05	0.581983E-02
0.935312E 00	0.713677E-01	0.714226E-01	-0.548959E-04	-0.768607E-01

TABLE C.2 (CONT'D.)

0.951484E 00	0.523065E-01	0.525951E-01	-0.288595E-03	-0.548709E 00
0.967656E 00	0.334678E-01	0.339279E-01	-0.460166E-03	-0.135631E 01
0.983828E 00	0.157426E-01	0.154219E-01	0.320721E-03	0.207965E 01
0.100000E 01	-0.263858E-02	-0.292110E-02	0.282528E-03	-0.967194E 01
0.100000E 01	-0.324369E-02	-0.292110E-02	-0.322583E-03	0.110432E 02
0.100000E 01	-0.340454E-02	-0.292110E-02	-0.483434E-03	0.165497E 02
0.100000E 01	-0.260535E-02	-0.292110E-02	0.315750E-03	-0.108093E 02
0.100000E 01	-0.253397E-02	-0.292206E-02	0.388088E-03	-0.132813E 02

STD ERROR OF ESTIMATE FOR G = 0.0003

G4 ONE-DIMENSIONAL REPRESENTATION FOR PRESSURE

TERM	COEFFICIENT
0	0.31091E 00
1	0.15504E 01
2	-0.17556E 01

X-ACTUAL	G-ACTUAL	G-CALC	DIFF	PCT-DIFF
0.465312E 00	0.650129E 00	0.652227E 00	-0.209773E-02	-0.321625E 00
0.470312E 00	0.650045E 00	0.651766E 00	-0.172091E-02	-0.264037E 00
0.475312E 00	0.649853E 00	0.651217E 00	-0.136465E-02	-0.209553E 00
0.480312E 00	0.649554E 00	0.650581E 00	-0.102729E-02	-0.157903E 00
0.485312E 00	0.649149E 00	0.649857E 00	-0.708163E-03	-0.108972E 00
0.490312E 00	0.648634E 00	0.649045E 00	-0.411034E-03	-0.633289E-01
0.495312E 00	0.648015E 00	0.648146E 00	-0.130951E-03	-0.202040E-01
0.500312E 00	0.646397E 00	0.646083E 00	0.314116E-03	0.486186E-01

TABLE C.2 (CONT'D.)

0.515312E 00	0.644402E 00	0.643669E 00	0.732422E-03	0.113788E 00
0.525312E 00	0.642002E 00	0.640905E 00	0.109738E-02	0.171224E 00
0.535312E 00	0.639195E 00	0.637789E 00	0.140649E-02	0.220526E 00
0.555312E 00	0.632351E 00	0.630504E 00	0.184745E-02	0.293011E 00
0.575312E 00	0.623878E 00	0.621814E 00	0.206381E-02	0.331901E 00
0.595312E 00	0.613816E 00	0.611720E 00	0.209552E-02	0.342562E 00
0.615312E 00	0.602191E 00	0.600222E 00	0.196874E-02	0.328002E 00
0.655312E 00	0.574334E 00	0.573012E 00	0.132209E-02	0.230727E 00
0.695312E 00	0.540477E 00	0.540184E 00	0.292540E-03	0.541555E-01
0.735312E 00	0.500874E 00	0.501738E 00	-0.863850E-03	-0.172172E 00
0.775312E 00	0.455682E 00	0.457674E 00	-0.199234E-02	-0.435319E 00
0.855312E 00	0.348888E 00	0.352694E 00	-0.380647E-02	-0.107926E 01
0.935312E 00	0.222055E 00	0.225243E 00	-0.318843E-02	-0.141555E 01
0.951484E 00	0.195123E 00	0.196749E 00	-0.162536E-02	-0.826110E 00
0.967656E 00	0.166266E 00	0.167336E 00	-0.106996E-02	-0.639412E 00
0.983828E 00	0.137811E 00	0.137004E 00	0.806868E-03	0.588936E 00
0.100000E 01	0.107390E 00	0.105756E 00	0.163400E-02	0.154507E 01
0.100000E 01	0.106399E 00	0.105756E 00	0.643849E-03	0.608809E 00
0.100000E 01	0.106132E 00	0.105756E 00	0.376225E-03	0.355749E 00
0.100000E 01	0.107449E 00	0.105756E 00	0.169325E-02	0.160110E 01
0.100000E 01	0.107566E 00	0.105755E 00	0.181121E-02	0.171265E 01

STD ERROR OF ESTIMATE FOR G = 0.0017

APPENDIX D

APPENDIX D

PRESSURE AND VELOCITY PROGRAMS AND NUMERICAL DATA

This appendix presents the computer program utilizing the subroutines PC05E1 and PC05E2 to calculate the pressure and velocity component data for the polynomial equations given in Chapter IV.

Table D.1 displays some sample data for these functions at each R value in the order shown in the following 3X5 matrix:

$$\text{R value} \begin{bmatrix} U_{.1} & V_{.1} & W_{.1} & P_{.1} & H_{.1} \\ U_{.4} & V_{.4} & W_{.4} & P_{.4} & H_{.4} \\ U_{.7} & V_{.7} & W_{.7} & P_{.7} & H_{.7} \end{bmatrix} .$$

The subscripts indicate the value of the Z parameter used to calculate each of the functions U, V, W, P, and H. Only a single page sample of the data is included in this appendix as it would take 76 pages to display it all.

```
/1D PC05,BRINK,PC05,VELOCITY AND PRESSURE Z-PARAMETER
/JOB GO
/FILE DISK=(3,PTPTPT),VOL=RAXFL1,DISP=(OLD,KEEP)
      DIMENSION Y(21),ZZ(10)
      J1=0
      WRITE(6,9015)
 9015 FORMAT(//3H NO,14X,1HX,9X,4HY(1),9X,4HY(2),9X,4HY(3),9X,
     14HY(4))
      READ(5)XMIN,XMAX,XSTEP,NYYYY
      X=XMIN
    1 CONTINUE
      ZZ(1)=.1
      ZZ(2)=.4
      ZZ(3)=.7
      IJK=0
      DO 10 I=1,11,5
      IJK=IJK+1
      Z=ZZ(IJK)
      IF(Z-.1) 11,4,11
   11 IF(Z-.4) 2,5,2
    2 IF(Z-.7) 3,6,3
    3 CONTINUE
    4 IF(X-.21491) 7,7,8
    5 IF(X-.85964) 7,7,8
    6 IF(X-1.5044) 7,7,8
    7 Y(I)=1.5338-.29709*(Z/X)+.43202*(Z/X)**2
      Y(I+1)=-36.184+241.04*(Z/X)-636.39*(Z/X)**2+838.23*(Z/X)**3-546.94
     1*(Z/X)**4+141.25*(Z/X)**5
      Y(I+2)=1.435-1.7466*(Z/X)+.30862*(Z/X)**2
      Y(I+3)=.31091+1.5504*(Z/X)-1.7556*(Z/X)**2
      GO TO 10
    8 Y(I)=.88531+1.104*(Z/X)+.44657*(Z/X)**2
      Y(I+1)=1.4366-9.5497*(Z/X)+82.897*(Z/X)**2-383.95*(Z/X)**3+829.38*
     1(Z/X)**4-691.74*(Z/X)**5
      Y(I+2)=1.0909-.9197*(Z/X)+.13199*(Z/X)**2
      Y(I+3)=.4074+1.0314*(Z/X)-1.0941*(Z/X)**2
   10 Y(I+4)=Y(I+3)+(Y(I)**2+Y(I+1)**2+Y(I+2)**2)/2.
```

```
      J1=J1+1
      WRITE(6,9013)J1,X,(Y(III),III=1,NYYYY)
 9013 FORMAT(I4,E14.5,4E13.5/(18X,4E13.5))
      WRITE(3)X,(Y(IIII),IIII=1,NYYYY)
      X=X+XSTEP
      IF(X-XMAX)1,1,9014
 9014 X=99999.
      WRITE(3)X,(Y(IIII),IIII=1,NYYYY)
      STOP
      END
/DATA
 .1 7. .01 15
/END
```

TABLE D.1 SAMPLE DATA PRESSURE AND VELOCITY FUNCTIONS

NO	X	Y(1)	Y(2)	Y(3)	Y(4)	Y(5)
1	0.10000E 00	0.16687E 01	0.10063E 01	-0.29798E-02	0.10571E 00	0.20044E 01
		0.72578E 01	0.49016E 05	-0.61348E 00	-0.21577E 02	0.12013E 10
		0.20623E 02	0.13188E 07	0.43312E 01	-0.74861E 02	0.86957E 12
2	0.11000E 00	0.16208E 01	0.91386E 00	0.10224E 00	0.26945E 00	0.20057E 01
		0.61661E 01	0.26908E 05	-0.83534E 00	-0.17266E 02	0.36201E 09
		0.17138E 02	0.76886E 06	0.28181E 01	-0.60917E 02	0.29558E 12
3	0.12000E 00	0.15862E 01	0.83405E 00	0.19382E 00	0.38374E 00	0.20084E 01
		0.53437E 01	0.15346E 05	-0.95789E 00	-0.14028E 02	0.11775E 09
		0.14501E 02	0.46686E 06	0.17482E 01	-0.50384E 02	0.10898E 12
4	0.13000E 00	0.15609E 01	0.74713E 00	0.27408E 00	0.46471E 00	0.19996E 01
		0.47098E 01	0.90307E 04	-0.10173E 01	-0.11540E 02	0.40777E 08
		0.12460E 02	0.29327E 06	0.97839E 00	-0.42243E 02	0.43004E 11
5	0.14000E 00	0.15420E 01	0.66631E 00	0.34489E 00	0.52262E 00	0.19930E 01
		0.42117E 01	0.54541E 04	-0.10359E 01	-0.95908E 01	0.14874E 08
		0.10849E 02	0.18961E 06	0.41750E 00	-0.35827E 02	0.17975E 11
6	0.15000E 00	0.15277E 01	0.59726E 00	0.40776E 00	0.56424E 00	0.19927E 01
		0.38137E 01	0.33661E 04	-0.10280E 01	-0.80390E 01	0.56654E 07
		0.95558E 01	0.12564E 06	0.52595E-02	-0.30687E 02	0.78933E 10
7	0.16000E 00	0.15169E 01	0.53646E 00	0.46393E 00	0.59413E 00	0.19961E 01
		0.34912E 01	0.21154E 04	-0.10026E 01	-0.67856E 01	0.22375E 07
		0.85032E 01	0.85054E 05	-0.29919E 00	-0.26509E 02	0.36171E 10
8	0.17000E 00	0.15085E 01	0.47768E 00	0.51438E 00	0.61544E 00	0.19996E 01
		0.32265E 01	0.13497E 04	-0.96602E 00	-0.57607E 01	0.91090E 06
		0.76354E 01	0.58656E 05	-0.52422E 00	-0.23071E 02	0.17203E 10
9	0.18000E 00	0.15021E 01	0.41373E 00	0.55992E 00	0.63039E 00	0.20009E 01
		0.30070E 01	0.87217E 03	-0.92228E 00	-0.49134E 01	0.38034E 06
		0.69121E 01	0.41117E 05	-0.68993E 00	-0.20211E 02	0.84531E 09
10	0.19000E 00	0.14971E 01	0.33850E 00	0.60123E 00	0.64059E 00	0.19993E 01
		0.28231E 01	0.56953E 03	-0.87421E 00	-0.42061E 01	0.16218E 06
		0.63032E 01	0.29241E 05	-0.81081E 00	-0.17807E 02	0.42752E 09
11	0.20000E 00	0.14933E 01	0.24758E 00	0.63885E 00	0.64721E 00	0.19968E 01
		0.26677E 01	0.37513E 03	-0.82372E 00	-0.36107E 01	0.70363E 05
		0.57862E 01	0.21063E 05	-0.89750E 00	-0.15769E 02	0.22182E 09

APPENDIX E

APPENDIX E

PROGRAMS AND NUMERICAL DATA FOR OPTIMUM FUNCTIONS

This appendix presents the computer programs utilized to generate the data for the functions G_2, G_4, V, and P as given in Chapter V.

Table E.1 displays the data for the functions G_2 and G_4 at each X value for different values of the swirl parameter S in the order shown in the following 2X5 matrix:

$$\text{X value} \begin{bmatrix} G_2^{.25} & G_4^{.25} & G_2^{.45} & G_4^{.45} & G_2^{.55} \\ G_4^{.55} & G_2^{.65} & G_4^{.65} & - & - \end{bmatrix}.$$

The superscripts indicate the value of the S parameter used to calculate each of the functions G_2 and G_4.

Table E.2 displays the data for the functions V and P at each R value for different values of the Z parameter at S = 0.65. The data for these functions is displayed in the order shown in the following 2X5 matrix:

$$\text{R value} \begin{bmatrix} V_{.1} & P_{.1} & V_{.4} & P_{.4} & V_{.7} \\ P_{.7} & - & - & - & - \end{bmatrix}.$$

The subscripts indicate the value of the Z parameter used to calculate each of the functions V and P.

```
/1D PC05,BRINK,PC05,ONE DIMENSIONAL OPTIMIZATION G2 AND G4  WITH S PAR.
/JOB GO
/FILE DISK=(3,PTPTPT),VOL=RAXFL1,DISP=(OLD,KEEP)
      DIMENSION Y(21),ZZ(10)
      J1=0
      WRITE(6,9015)
 9015 FORMAT(//3H NO,14X,1HX,9X,4HY(1),9X,4HY(2),9X,4HY(3),9X,
     14HY(4))
      READ(5)XMIN,XMAX,XSTEP,NYYYY
      X=XMIN
    1 CONTINUE
      H=2.
      G=2.6951
      ZZ(1)=.25
      ZZ(2)=.45
      ZZ(3)=.55
      ZZ(4)=.65
      IJK=0
      DO 10 I=1,7,2
      IJK=IJK+1
      S=ZZ(IJK)
      Y(I)=X*SQRT(S*H)
   10 Y(I+1)=H*(1.-(S/2.)*X**2)-G/2.
      J1=J1+1
      WRITE(6,9013)J1,X,(Y(III),III=1,NYYYY)
 9013 FORMAT(I4,E14.5,4E13.5/(18X,4E13.5))
      WRITE(3)X,(Y(IIII),IIII=1,NYYYY)
      X=X+XSTEP
      IF(X-XMAX)1,1,9014
 9014 X=99999.
      WRITE(3)X,(Y(IIII),IIII=1,NYYYY)
      STOP
      END
/DATA
 .1 1. .01 8
/END
```

```
/1D PC05,BRINK,PC05,OPTIMUM PRESSURE AND VELOCITY Z PARAMETER
/JOB GO
/FILE DISK=(3,PTPTPT),VOL=RAXFL1,DISP=(OLD,KEEP)
      DIMENSION Y(21),ZZ(10)
      J1=0
      WRITE(6,9015)
 9015 FORMAT(//3H NO,14X,1HX,9X,4HY(1),9X,4HY(2),9X,4HY(3),9X,
     14HY(4))
      READ(5)XMIN,XMAX,XSTEP,NYYYY
      X=XMIN
    1 CONTINUE
      ZZ(1)=.1
      ZZ(2)=.4
      ZZ(3)=.7
      IJK=0
      DO 10 I=1,5,2
      IJK=IJK+1
      Z=ZZ(IJK)
      H=2.
      S=.65
      G=2.6951
      Y(I)=(Z/X)*SQRT(S*H)
   10 Y(I+1)=H*(1.-(S/2.)*(Z/X)**2)-G/2.
      J1=J1+1
      WRITE(6,9013)J1,X,(Y(III),III=1,NYYYY)
 9013 FORMAT(I4,E14.5,4E13.5/(18X,4E13.5))
      WRITE(3)X,(Y(IIII),IIII=1,NYYYY)
      X=X+XSTEP
      IF(X-XMAX)1,1,9014
 9014 X=99999.
      WRITE(3)X,(Y(IIII),IIII=1,NYYYY)
      STOP
      END
/DATA
 .1 7. .1 6
/END
```

TABLE E.1 NUMERICAL DATA FOR OPTIMUM ONE-DIMENSIONAL FUNCTIONS

NO	X	Y(1)	Y(2)	Y(3)	Y(4)	Y(5)
1	0.10000E 00	0.70711E-01	0.64995E 00	0.94868E-01	0.64795E 00	0.10488E 00
		0.64695E 00	0.11402E 00	0.64595E 00		
2	0.11000E 00	0.77782E-01	0.64943E 00	0.10436E 00	0.64701E 00	0.11537E 00
		0.64579E 00	0.12542E 00	0.64459E 00		
3	0.12000E 00	0.84853E-01	0.64885E 00	0.11384E 00	0.64597E 00	0.12586E 00
		0.64453E 00	0.13682E 00	0.64309E 00		
4	0.13000E 00	0.91924E-01	0.64822E 00	0.12333E 00	0.64485E 00	0.13634E 00
		0.64316E 00	0.14822E 00	0.64147E 00		
5	0.14000E 00	0.98995E-01	0.64755E 00	0.13282E 00	0.64363E 00	0.14683E 00
		0.64167E 00	0.15962E 00	0.63971E 00		
6	0.15000E 00	0.10607E 00	0.64682E 00	0.14230E 00	0.64233E 00	0.15732E 00
		0.64008E 00	0.17103E 00	0.63783E 00		
7	0.16000E 00	0.11314E 00	0.64605E 00	0.15179E 00	0.64093E 00	0.16781E 00
		0.63837E 00	0.18243E 00	0.63581E 00		
8	0.17000E 00	0.12021E 00	0.64523E 00	0.16128E 00	0.63945E 00	0.17830E 00
		0.63656E 00	0.19383E 00	0.63367E 00		
9	0.18000E 00	0.12728E 00	0.64435E 00	0.17076E 00	0.63787E 00	0.18879E 00
		0.63463E 00	0.20523E 00	0.63139E 00		
10	0.19000E 00	0.13435E 00	0.64342E 00	0.18025E 00	0.63621E 00	0.19927E 00
		0.63260E 00	0.21663E 00	0.62899E 00		
11	0.20000E 00	0.14142E 00	0.64245E 00	0.18974E 00	0.63445E 00	0.20976E 00
		0.63045E 00	0.22803E 00	0.62645E 00		
12	0.21000E 00	0.14849E 00	0.64143E 00	0.19922E 00	0.63261E 00	0.22025E 00
		0.62819E 00	0.23944E 00	0.62379E 00		
13	0.22000E 00	0.15556E 00	0.64035E 00	0.20871E 00	0.63067E 00	0.23074E 00
		0.62583E 00	0.25084E 00	0.62099E 00		
14	0.23000E 00	0.16263E 00	0.63923E 00	0.21820E 00	0.62864E 00	0.24123E 00
		0.62335E 00	0.26224E 00	0.61806E 00		
15	0.24000E 00	0.16971E 00	0.63805E 00	0.22768E 00	0.62653E 00	0.25171E 00
		0.62077E 00	0.27364E 00	0.61501E 00		
16	0.25000E 00	0.17678E 00	0.63683E 00	0.23717E 00	0.62432E 00	0.26220E 00
		0.61808E 00	0.28504E 00	0.61182E 00		

TABLE E.1 (CONT'D.)

17	0.26000E 00	0.18385E 00	0.63555E 00	0.24666E 00	0.62203E 00	0.27269E 00
		0.61527E 00	0.29645E 00	0.60851E 00		
18	0.27000E 00	0.19092E 00	0.63422E 00	0.25614E 00	0.61965E 00	0.28318E 00
		0.61236E 00	0.30785E 00	0.60507E 00		
19	0.28000E 00	0.19799E 00	0.63285E 00	0.26563E 00	0.61717E 00	0.29367E 00
		0.60933E 00	0.31925E 00	0.60149E 00		
20	0.29000E 00	0.20506E 00	0.63142E 00	0.27512E 00	0.61460E 00	0.30415E 00
		0.60620E 00	0.33065E 00	0.59778E 00		
21	0.30000E 00	0.21213E 00	0.62995E 00	0.28460E 00	0.61195E 00	0.31464E 00
		0.60295E 00	0.34205E 00	0.59395E 00		
22	0.31000E 00	0.21920E 00	0.62843E 00	0.29409E 00	0.60921E 00	0.32513E 00
		0.59960E 00	0.35345E 00	0.58998E 00		
23	0.32000E 00	0.22627E 00	0.62685E 00	0.30358E 00	0.60637E 00	0.33562E 00
		0.59613E 00	0.36486E 00	0.58589E 00		
24	0.33000E 00	0.23335E 00	0.62523E 00	0.31307E 00	0.60345E 00	0.34611E 00
		0.59256E 00	0.37626E 00	0.58167E 00		
25	0.34000E 00	0.24042E 00	0.62355E 00	0.32255E 00	0.60043E 00	0.35659E 00
		0.58887E 00	0.38766E 00	0.57731E 00		
26	0.35000E 00	0.24749E 00	0.62183E 00	0.33204E 00	0.59733E 00	0.36708E 00
		0.58508E 00	0.39906E 00	0.57283E 00		
27	0.36000E 00	0.25456E 00	0.62005E 00	0.34153E 00	0.59413E 00	0.37757E 00
		0.58117E 00	0.41046E 00	0.56821E 00		
28	0.37000E 00	0.26163E 00	0.61823E 00	0.35101E 00	0.59085E 00	0.38806E 00
		0.57716E 00	0.42186E 00	0.56347E 00		
29	0.38000E 00	0.26870E 00	0.61635E 00	0.36050E 00	0.58747E 00	0.39855E 00
		0.57303E 00	0.43327E 00	0.55859E 00		
30	0.39000E 00	0.27577E 00	0.61443E 00	0.36999E 00	0.58401E 00	0.40903E 00
		0.56880E 00	0.44467E 00	0.55359E 00		
31	0.40000E 00	0.28284E 00	0.61245E 00	0.37947E 00	0.58045E 00	0.41952E 00
		0.56445E 00	0.45607E 00	0.54845E 00		
32	0.41000E 00	0.28991E 00	0.61042E 00	0.38896E 00	0.57681E 00	0.43001E 00
		0.56000E 00	0.46747E 00	0.54319E 00		
33	0.42000E 00	0.29698E 00	0.60835E 00	0.39845E 00	0.57307E 00	0.44050E 00
		0.55543E 00	0.47887E 00	0.53779E 00		

TABLE E.1 (CONT'D.)

34	0.43000E 00	0.30406E 00	0.60623E 00	0.40793E 00	0.56925E 00	0.45099E 00	
		0.55076E 00	0.49027E 00	0.53227E 00			
35	0.44000E 00	0.31113E 00	0.60405E 00	0.41742E 00	0.56533E 00	0.46148E 00	
		0.54597E 00	0.50168E 00	0.52661E 00			
36	0.45000E 00	0.31820E 00	0.60183E 00	0.42691E 00	0.56133E 00	0.47196E 00	
		0.54108E 00	0.51308E 00	0.52083E 00			
37	0.46000E 00	0.32527E 00	0.59955E 00	0.43639E 00	0.55723E 00	0.48245E 00	
		0.53607E 00	0.52448E 00	0.51491E 00			
38	0.47000E 00	0.33234E 00	0.59723E 00	0.44588E 00	0.55305E 00	0.49294E 00	
		0.53096E 00	0.53588E 00	0.50887E 00			
39	0.48000E 00	0.33941E 00	0.59485E 00	0.45537E 00	0.54877E 00	0.50343E 00	
		0.52573E 00	0.54728E 00	0.50269E 00			
40	0.49000E 00	0.34648E 00	0.59243E 00	0.46485E 00	0.54440E 00	0.51392E 00	
		0.52040E 00	0.55869E 00	0.49639E 00			
41	0.50000E 00	0.35355E 00	0.58995E 00	0.47434E 00	0.53995E 00	0.52440E 00	
		0.51495E 00	0.57009E 00	0.48995E 00			
42	0.51000E 00	0.36062E 00	0.58743E 00	0.48383E 00	0.53541E 00	0.53489E 00	
		0.50940E 00	0.58149E 00	0.48339E 00			
43	0.52000E 00	0.36770E 00	0.58485E 00	0.49331E 00	0.53077E 00	0.54538E 00	
		0.50373E 00	0.59289E 00	0.47669E 00			
44	0.53000E 00	0.37477E 00	0.58222E 00	0.50280E 00	0.52605E 00	0.55587E 00	
		0.49796E 00	0.60429E 00	0.46987E 00			
45	0.54000E 00	0.38184E 00	0.57955E 00	0.51229E 00	0.52123E 00	0.56636E 00	
		0.49207E 00	0.61569E 00	0.46291E 00			
46	0.55000E 00	0.38891E 00	0.57683E 00	0.52178E 00	0.51632E 00	0.57684E 00	
		0.48608E 00	0.62710E 00	0.45583E 00			
47	0.56000E 00	0.39598E 00	0.57405E 00	0.53126E 00	0.51133E 00	0.58733E 00	
		0.47997E 00	0.63850E 00	0.44861E 00			
48	0.57000E 00	0.40305E 00	0.57123E 00	0.54075E 00	0.50625E 00	0.59782E 00	
		0.47376E 00	0.64990E 00	0.44127E 00			
49	0.58000E 00	0.41012E 00	0.56835E 00	0.55024E 00	0.50107E 00	0.60831E 00	
		0.46743E 00	0.66130E 00	0.43379E 00			
50	0.59000E 00	0.41719E 00	0.56542E 00	0.55972E 00	0.49581E 00	0.61880E 00	
		0.46100E 00	0.67270E 00	0.42619E 00			

TABLE E.1 (CONT'D.)

51	0.60000E 00	0.42426E 00	0.56245E 00	0.56921E 00	0.49045E 00	0.62928E 00
		0.45445E 00	0.68410E 00	0.41845E 00		
52	0.61000E 00	0.43133E 00	0.55943E 00	0.57870E 00	0.48501E 00	0.63977E 00
		0.44780E 00	0.69551E 00	0.41059E 00		
53	0.62000E 00	0.43841E 00	0.55635E 00	0.58818E 00	0.47947E 00	0.65026E 00
		0.44103E 00	0.70691E 00	0.40259E 00		
54	0.63000E 00	0.44548E 00	0.55323E 00	0.59767E 00	0.47385E 00	0.66075E 00
		0.43416E 00	0.71831E 00	0.39447E 00		
55	0.64000E 00	0.45255E 00	0.55005E 00	0.60716E 00	0.46813E 00	0.67124E 00
		0.42717E 00	0.72971E 00	0.38621E 00		
56	0.65000E 00	0.45962E 00	0.54683E 00	0.61664E 00	0.46233E 00	0.68172E 00
		0.42008E 00	0.74111E 00	0.37783E 00		
57	0.66000E 00	0.46669E 00	0.54355E 00	0.62613E 00	0.45643E 00	0.69221E 00
		0.41287E 00	0.75251E 00	0.36931E 00		
58	0.67000E 00	0.47376E 00	0.54023E 00	0.63562E 00	0.45045E 00	0.70270E 00
		0.40556E 00	0.76392E 00	0.36067E 00		
59	0.68000E 00	0.48083E 00	0.53685E 00	0.64510E 00	0.44437E 00	0.71319E 00
		0.39813E 00	0.77532E 00	0.35189E 00		
60	0.69000E 00	0.48790E 00	0.53343E 00	0.65459E 00	0.43821E 00	0.72368E 00
		0.39060E 00	0.78672E 00	0.34299E 00		
61	0.70000E 00	0.49497E 00	0.52995E 00	0.66408E 00	0.43195E 00	0.73417E 00
		0.38295E 00	0.79812E 00	0.33395E 00		
62	0.71000E 00	0.50205E 00	0.52643E 00	0.67356E 00	0.42561E 00	0.74465E 00
		0.37520E 00	0.80952E 00	0.32479E 00		
63	0.72000E 00	0.50912E 00	0.52285E 00	0.68305E 00	0.41917E 00	0.75514E 00
		0.36733E 00	0.82093E 00	0.31549E 00		
64	0.73000E 00	0.51619E 00	0.51923E 00	0.69254E 00	0.41265E 00	0.76563E 00
		0.35936E 00	0.83233E 00	0.30607E 00		
65	0.74000E 00	0.52326E 00	0.51555E 00	0.70202E 00	0.40603E 00	0.77612E 00
		0.35127E 00	0.84373E 00	0.29651E 00		
66	0.75000E 00	0.53033E 00	0.51183E 00	0.71151E 00	0.39933E 00	0.78661E 00
		0.34308E 00	0.85513E 00	0.28683E 00		
67	0.76000E 00	0.53740E 00	0.50805E 00	0.72100E 00	0.39253E 00	0.79709E 00
		0.33477E 00	0.86653E 00	0.27701E 00		

TABLE E.1 (CONT'D.)

68	0.77000E 00	0.54447E 00	0.50423E 00	0.73049E 00	0.38565E 00	0.80758E 00
		0.32636E 00	0.87793E 00	0.26707E 00		
69	0.78000E 00	0.55154E 00	0.50035E 00	0.73997E 00	0.37867E 00	0.81807E 00
		0.31783E 00	0.88934E 00	0.25699E 00		
70	0.79000E 00	0.55861E 00	0.49643E 00	0.74946E 00	0.37161E 00	0.82856E 00
		0.30920E 00	0.90074E 00	0.24679E 00		
71	0.80000E 00	0.56568E 00	0.49245E 00	0.75895E 00	0.36445E 00	0.83905E 00
		0.30045E 00	0.91214E 00	0.23645E 00		
72	0.81000E 00	0.57276E 00	0.48843E 00	0.76843E 00	0.35721E 00	0.84953E 00
		0.29160E 00	0.92354E 00	0.22599E 00		
73	0.82000E 00	0.57983E 00	0.48435E 00	0.77792E 00	0.34987E 00	0.86002E 00
		0.28263E 00	0.93494E 00	0.21539E 00		
74	0.83000E 00	0.58690E 00	0.48023E 00	0.78741E 00	0.34245E 00	0.87051E 00
		0.27356E 00	0.94634E 00	0.20467E 00		
75	0.84000E 00	0.59397E 00	0.47605E 00	0.79689E 00	0.33493E 00	0.88100E 00
		0.26437E 00	0.95775E 00	0.19381E 00		
76	0.85000E 00	0.60104E 00	0.47183E 00	0.80638E 00	0.32733E 00	0.89149E 00
		0.25508E 00	0.96915E 00	0.18283E 00		
77	0.86000E 00	0.60811E 00	0.46755E 00	0.81587E 00	0.31963E 00	0.90197E 00
		0.24567E 00	0.98055E 00	0.17171E 00		
78	0.87000E 00	0.61518E 00	0.46323E 00	0.82535E 00	0.31185E 00	0.91246E 00
		0.23616E 00	0.99195E 00	0.16047E 00		
79	0.88000E 00	0.62225E 00	0.45885E 00	0.83484E 00	0.30397E 00	0.92295E 00
		0.22653E 00	0.10034E 01	0.14909E 00		
80	0.89000E 00	0.62932E 00	0.45443E 00	0.84433E 00	0.29601E 00	0.93344E 00
		0.21680E 00	0.10148E 01	0.13759E 00		
81	0.90000E 00	0.63640E 00	0.44995E 00	0.85381E 00	0.28795E 00	0.94393E 00
		0.20695E 00	0.10262E 01	0.12595E 00		
82	0.91000E 00	0.64347E 00	0.44543E 00	0.86330E 00	0.27981E 00	0.95441E 00
		0.19700E 00	0.10376E 01	0.11419E 00		
83	0.92000E 00	0.65054E 00	0.44085E 00	0.87279E 00	0.27157E 00	0.96490E 00
		0.18693E 00	0.10490E 01	0.10229E 00		
84	0.93000E 00	0.65761E 00	0.43623E 00	0.88227E 00	0.26325E 00	0.97539E 00
		0.17676E 00	0.10604E 01	0.90266E-01		

TABLE E.1 (CONT'D.)

85	0.94000E 00	0.66468E 00	0.43155E 00	0.89176E 00	0.25483E 00	0.98588E 00	
		0.16647E 00	0.10718E 01	0.78112E-01			
86	0.95000E 00	0.67175E 00	0.42683E 00	0.90125E 00	0.24633E 00	0.99637E 00	
		0.15608E 00	0.10832E 01	0.65826E-01			
87	0.96000E 00	0.67882E 00	0.42205E 00	0.91074E 00	0.23773E 00	0.10069E 01	
		0.14557E 00	0.10946E 01	0.53411E-01			
88	0.97000E 00	0.68589E 00	0.41723E 00	0.92022E 00	0.22905E 00	0.10173E 01	
		0.13496E 00	0.11060E 01	0.40866E-01			
89	0.98000E 00	0.69296E 00	0.41235E 00	0.92971E 00	0.22027E 00	0.10278E 01	
		0.12423E 00	0.11174E 01	0.28192E-01			
90	0.99000E 00	0.70004E 00	0.40743E 00	0.93920E 00	0.21141E 00	0.10383E 01	
		0.11340E 00	0.11288E 01	0.15387E-01			
91	0.10000E 01	0.70711E 00	0.40245E 00	0.94868E 00	0.20245E 00	0.10488E 01	
		0.10245E 00	0.11402E 01	0.24509E-02			

TABLE E.2 NUMERICAL DATA FOR OPTIMUM PRESSURE AND TANGENTIAL VELOCITY

NO	X	Y(1)	Y(2)	Y(3)	Y(4)	Y(5)
1	0.10000E 00	0.11402E 01	0.24500E-02	0.45607E 01	-0.97475E 01	0.79812E 01
		-0.31198E 02				
2	0.20000E 00	0.57009E 00	0.48995E 00	0.22803E 01	-0.19475E 01	0.39906E 01
		-0.73101E 01				
3	0.30000E 00	0.38006E 00	0.58023E 00	0.15202E 01	-0.50310E 00	0.26604E 01
		-0.28864E 01				
4	0.40000E 00	0.28504E 00	0.61182E 00	0.11402E 01	0.24500E-02	0.19953E 01
		-0.13382E 01				
5	0.50000E 00	0.22803E 00	0.62645E 00	0.91214E 00	0.23645E 00	0.15962E 01
		-0.62155E 00				
6	0.60000E 00	0.19003E 00	0.63439E 00	0.76012E 00	0.36356E 00	0.13302E 01
		-0.23227E 00				
7	0.70000E 00	0.16288E 00	0.63918E 00	0.65153E 00	0.44021E 00	0.11402E 01
		0.24500E-02				
8	0.80000E 00	0.14252E 00	0.64229E 00	0.57009E 00	0.48995E 00	0.99765E 00
		0.15479E 00				
9	0.90000E 00	0.12669E 00	0.64443E 00	0.50674E 00	0.52406E 00	0.88680E 00
		0.25924E 00				
10	0.10000E 01	0.11402E 00	0.64595E 00	0.45607E 00	0.54845E 00	0.79812E 00
		0.33395E 00				
11	0.11000E 01	0.10365E 00	0.64708E 00	0.41461E 00	0.56650E 00	0.72557E 00
		0.38923E 00				
12	0.12000E 01	0.95015E-01	0.64794E 00	0.38006E 00	0.58023E 00	0.66510E 00
		0.43127E 00				
13	0.13000E 01	0.87706E-01	0.64860E 00	0.35082E 00	0.59091E 00	0.61394E 00
		0.46399E 00				
14	0.14000E 01	0.81441E-01	0.64913E 00	0.32576E 00	0.59939E 00	0.57009E 00
		0.48995E 00				
15	0.15000E 01	0.76012E-01	0.64956E 00	0.30405E 00	0.60623E 00	0.53208E 00
		0.51089E 00				
16	0.16000E 01	0.71261E-01	0.64991E 00	0.28504E 00	0.61182E 00	0.49883E 00
		0.52804E 00				

TABLE E.2 (CONT'D.)

17	0.17000E 01	0.67069E-01	0.65020E 00	0.26828E 00	0.61646E 00	0.46948E 00	
		0.54224E 00					
18	0.18000E 01	0.63343E-01	0.65044E 00	0.25337E 00	0.62035E 00	0.44340E 00	
		0.55415E 00					
19	0.19000E 01	0.60009E-01	0.65065E 00	0.24004E 00	0.62364E 00	0.42007E 00	
		0.56422E 00					
20	0.20000E 01	0.57009E-01	0.65083E 00	0.22804E 00	0.62645E 00	0.39906E 00	
		0.57282E 00					
21	0.21000E 01	0.54294E-01	0.65098E 00	0.21718E 00	0.62887E 00	0.38006E 00	
		0.58023E 00					
22	0.22000E 01	0.51826E-01	0.65111E 00	0.20731E 00	0.63096E 00	0.36278E 00	
		0.58664E 00					
23	0.23000E 01	0.49573E-01	0.65122E 00	0.19829E 00	0.63279E 00	0.34701E 00	
		0.59224E 00					
24	0.24000E 01	0.47507E-01	0.65132E 00	0.19003E 00	0.63439E 00	0.33255E 00	
		0.59715E 00					
25	0.25000E 01	0.45607E-01	0.65141E 00	0.18243E 00	0.63581E 00	0.31925E 00	
		0.60149E 00					
26	0.26000E 01	0.43853E-01	0.65149E 00	0.17541E 00	0.63707E 00	0.30697E 00	
		0.60533E 00					
27	0.27000E 01	0.42229E-01	0.65156E 00	0.16892E 00	0.63818E 00	0.29560E 00	
		0.60876E 00					
28	0.28000E 01	0.40721E-01	0.65162E 00	0.16288E 00	0.63918E 00	0.28504E 00	
		0.61182E 00					
29	0.29000E 01	0.39317E-01	0.65168E 00	0.15727E 00	0.64008E 00	0.27522E 00	
		0.61458E 00					
30	0.30000E 01	0.38006E-01	0.65173E 00	0.15202E 00	0.64089E 00	0.26604E 00	
		0.61706E 00					
31	0.31000E 01	0.36780E-01	0.65177E 00	0.14712E 00	0.64163E 00	0.25746E 00	
		0.61931E 00					
32	0.32000E 01	0.35631E-01	0.65182E 00	0.14252E 00	0.64229E 00	0.24941E 00	
		0.62135E 00					
33	0.33000E 01	0.34551E-01	0.65185E 00	0.13820E 00	0.64290E 00	0.24186E 00	
		0.62320E 00					

TABLE E.2 (CONT'D.)

34	0.34000E 01	0.33535E-01	0.65189E 00	0.13414E 00	0.64345E 00	0.23474E 00
		0.62490E 00				
35	0.35000E 01	0.32577E-01	0.65192E 00	0.13031E 00	0.64396E 00	0.22804E 00
		0.62645E 00				
36	0.36000E 01	0.31672E-01	0.65195E 00	0.12669E 00	0.64443E 00	0.22170E 00
		0.62787E 00				
37	0.37000E 01	0.30816E-01	0.65198E 00	0.12326E 00	0.64485E 00	0.21571E 00
		0.62918E 00				
38	0.38000E 01	0.30005E-01	0.65200E 00	0.12002E 00	0.64525E 00	0.21003E 00
		0.63039E 00				
39	0.39000E 01	0.29235E-01	0.65202E 00	0.11694E 00	0.64561E 00	0.20465E 00
		0.63151E 00				
40	0.40000E 01	0.28504E-01	0.65204E 00	0.11402E 00	0.64595E 00	0.19953E 00
		0.63254E 00				
41	0.41000E 01	0.27809E-01	0.65206E 00	0.11124E 00	0.64626E 00	0.19466E 00
		0.63350E 00				
42	0.42000E 01	0.27147E-01	0.65208E 00	0.10859E 00	0.64655E 00	0.19003E 00
		0.63439E 00				
43	0.43000E 01	0.26516E-01	0.65210E 00	0.10606E 00	0.64683E 00	0.18561E 00
		0.63522E 00				
44	0.44000E 01	0.25913E-01	0.65211E 00	0.10365E 00	0.64708E 00	0.18139E 00
		0.63600E 00				
45	0.45000E 01	0.25337E-01	0.65213E 00	0.10135E 00	0.64731E 00	0.17736E 00
		0.63672E 00				
46	0.46000E 01	0.24787E-01	0.65214E 00	0.99146E-01	0.64754E 00	0.17351E 00
		0.63740E 00				
47	0.47000E 01	0.24259E-01	0.65216E 00	0.97037E-01	0.64774E 00	0.16981E 00
		0.63803E 00				
48	0.48000E 01	0.23754E-01	0.65217E 00	0.95015E-01	0.64794E 00	0.16628E 00
		0.63863E 00				
49	0.49000E 01	0.23269E-01	0.65218E 00	0.93076E-01	0.64812E 00	0.16288E 00
		0.63918E 00				
50	0.50000E 01	0.22804E-01	0.65219E 00	0.91214E-01	0.64829E 00	0.15963E 00
		0.63971E 00				

TABLE E.2 (CONT'D.)

51	0.51000E 01	0.22356E-01	0.65220E 00	0.89426E-01	0.64845E 00	0.15650E 00
		0.64020E 00				
52	0.52000E 01	0.21927E-01	0.65221E 00	0.87706E-01	0.64860E 00	0.15349E 00
		0.64067E 00				
53	0.53000E 01	0.21513E-01	0.65222E 00	0.86051E-01	0.64875E 00	0.15059E 00
		0.64111E 00				
54	0.54000E 01	0.21114E-01	0.65223E 00	0.84458E-01	0.64888E 00	0.14780E 00
		0.64153E 00				
55	0.55000E 01	0.20731E-01	0.65224E 00	0.82922E-01	0.64901E 00	0.14511E 00
		0.64192E 00				
56	0.56000E 01	0.20360E-01	0.65224E 00	0.81441E-01	0.64913E 00	0.14252E 00
		0.64229E 00				
57	0.57000E 01	0.20003E-01	0.65225E 00	0.80013E-01	0.64925E 00	0.14002E 00
		0.64265E 00				
58	0.58000E 01	0.19658E-01	0.65226E 00	0.78633E-01	0.64936E 00	0.13761E 00
		0.64298E 00				
59	0.59000E 01	0.19325E-01	0.65226E 00	0.77300E-01	0.64946E 00	0.13528E 00
		0.64330E 00				
60	0.60000E 01	0.19003E-01	0.65227E 00	0.76012E-01	0.64956E 00	0.13302E 00
		0.64360E 00				
61	0.61000E 01	0.18691E-01	0.65228E 00	0.74766E-01	0.64966E 00	0.13084E 00
		0.64389E 00				
62	0.62000E 01	0.18390E-01	0.65228E 00	0.73560E-01	0.64974E 00	0.12873E 00
		0.64416E 00				
63	0.63000E 01	0.18098E-01	0.65229E 00	0.72392E-01	0.64983E 00	0.12669E 00
		0.64443E 00				
64	0.64000E 01	0.17815E-01	0.65229E 00	0.71261E-01	0.64991E 00	0.12471E 00
		0.64467E 00				
65	0.65000E 01	0.17541E-01	0.65230E 00	0.70165E-01	0.64999E 00	0.12279E 00
		0.64491E 00				
66	0.66000E 01	0.17275E-01	0.65230E 00	0.69102E-01	0.65006E 00	0.12093E 00
		0.64514E 00				
67	0.67000E 01	0.17018E-01	0.65231E 00	0.68070E-01	0.65013E 00	0.11912E 00
		0.64536E 00				

TABLE E.2 (CONT'D.)

68	0.68000E 01	0.16767E-01	0.65231E 00	0.67069E-01	0.65020E 00	0.11737E 00
		0.64556E 00				
69	0.69000E 01	0.16524E-01	0.65231E 00	0.66097E-01	0.65027E 00	0.11567E 00
		0.64576E 00				
70	0.70000E 01	0.16288E-01	0.65232E 00	0.65153E-01	0.65033E 00	0.11402E 00
		0.64595E 00				

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