EXPLOITING INTERNAL RESONANCE IN MEMS FOR SIGNAL PROCESSING APPLICATIONS

By

Brian Scott Strachan

A DISSERTATION

Submitted to Michigan State University in partial fulfillment of the requirements for the degree of

Mechanical Engineering – Doctor of Philosophy Electrical Engineering – Doctor of Philosophy

2017

ABSTRACT

EXPLOITING INTERNAL RESONANCE IN MEMS FOR SIGNAL PROCESSING APPLICATIONS

By

Brian Scott Strachan

This research focuses on the development and analysis of predictive models for frequency converters and frequency generators that are based on micro-electro-mechanical-system (MEMS) technology. In contrast to applications in which nonlinearity is sought to be avoided, frequency conversion and frequency generation necessarily involve nonlinear processes, and while many existing technologies are available for realizing these operations, MEMS technology offers a potentially advantageous combination of size, power requirements, and noise characteristics.

This dissertation describes a series of investigations related to MEMS frequency conversion and generation, including: (i) an analytical investigation of a class of passive multi-stage frequency dividers, (ii) the design and realization of this behavior in a MEMS device, (iii) the development of a model for nonlinear modal interactions in closed loop MEMS and (iv) the development of a computational method for optimizing their nonlinear resonant response through shape optimization. Items (ii) and (iii) were carried out in close collaboration with experimental groups at the University of California at Santa Barbara and Argonne National Labs, respectively. Item (iv) was carried out in collaboration with the topology optimization group at the Technical University of Denmark.

The subharmonic frequency divider is based on a class of mechanical structures with nonlinearly coupled high *Q* vibration modes with sequential 2:1 internal resonances, for which sequential parametric resonances are used to transfer energy from a high frequency mode down to lower frequency modes. We analyze the normal form for this subharmonic resonance cascade and predict the system response based on system and driving signal parameters. We then show how to design and experimentally implement this subharmonic cascade in MEMS, and we demonstrate frequency division by a factor of eight.

The frequency generator model is based on a closed loop oscillator in which the resonator element has vibration modes with 1:3 frequency ratio and nonlinear intermodal coupling. Experimental observations have shown that the oscillator phase noise performance is significantly improved when operating in a coupled mode regime, in which a flexural mode is nonlinearly coupled to a torsional mode. The device is characterized by comparing its measured open loop response against a model based on 1:3 internal resonance, demonstrating good agreement. The closed loop version of the model is analyzed with a focus on how noise sources are filtered through the system into phase noise. This model predicts the significant drop in phase noise observed when operating with internal resonance. This predictive model provides a basis for future designs that take full advantage of this nonlinear behavior, which has potential for commercialization in the growing area of MEMS oscillators.

Lastly, we describe the development of a computational tool that allows one to tailor the nonlinear resonant response of mechanical structures using a combination of normal forms and structural optimization tools. This approach is used to improve a device's nonlinear modal coupling by nearly an order of magnitude. Such tools will be important for the continuing development of MEMS that utilize nonlinear resonant behavior.

In summary, it is shown that internal resonance, in addition to offering interesting dynamic behavior, can be used to improve the performance of signal processing devices. This work also demonstrates that devices that use internal resonance can be analyzed with generic dynamic models, thereby providing a basis for understanding fundamental device characteristics and future design development.

Copyright by BRIAN SCOTT STRACHAN 2017 For Austin, again.

TABLE OF CONTENTS

5.3	Examp	oles	71
	5.3.1	Single-mode resonator	71
	5.3.2	Coupled-Mode Resonator with Internal Resonance	77
5.4	Outloo	$b\mathbf{k}$	32
СНАРТ	TER 6	CONCLUSION §	84
APPEN	DIX.		87
BIBLIO	GRAPH	IY	95

LIST OF TABLES

- Table 3.1Device parameters: beam lengths, designed and measured resonance
frequencies, and measured quality factors (in vacuum). Direct excita-
tion of each mode is achieved through the actuation beam shown in
Figure 3.3, with signal $V_{dc} = 90$ V and $V_{ac} = 3-10$ V.
 \dots 33
- Table 4.1Model coefficients as measured from the open loop experimental results.45

LIST OF FIGURES

Figure 1.1	Schematic diagram of a frequency generator showing the essential components.	4
Figure 2.1	A mechanical implementation of the subharmonic cascade, consist- ing of rigid bars with elastic hinges at their bases, and coupled by springs with linear stiffnesses. The energy is down-converted from the high frequency beam, parametrically driven by a source at ap- proximately twice its natural frequency, down the chain to a frequency of $\Omega/2^N$ at the terminal beam. For small stiffnesses of the coupling springs the bar displacements u_i are roughly equal to the system modal coordinates q_i , that is, the system modes are localized in the individual beams.	10
Figure 2.2	Analytical predictions for the response regimes in (α, F) space; the destabilization boundaries α_i and F_i for $i = 1, 2$ and ∞ are shown. The fully active regime is shaded grey and the partially activated regime is shaded light grey. The curve labeled r_{eq}^+ corresponds to the stable nontrivial response for the infinite chain. Parameter values for all simulations are as follows, unless specified otherwise: $\zeta = 0.03$, $\gamma = 0.075$, $\delta = 0.064$, and $\beta = 0.008$.	20
Figure 2.3	Simulation of a six resonator chain, showing the sequential activa- tion of the six elements when started with small initial conditions. Resonators 1-6 are shown from top to bottom. The thick lines indi- cate results of the amplitudes obtained by simulating the averaged equations, and the underlying fast oscillations are from simulations of the full equations of motion. The settling time of resonator <i>j</i> is proportional to $Q/2^{j-1}$.	22
Figure 2.4	Steady state amplitudes of a six resonator cascade for $\alpha = 0(> \alpha_{\infty})$ for various forcing amplitudes. The fully activated response is achieved when $F > F_{\infty} = r_{eq}^-$, which is depicted as the grey shaded region. The top two examples (circles and triangles) converge to the infinite lattice amplitude r_{eq}^+ , which is denoted by the row of asterisks, with the amplitude of the final resonator given by a slightly lower value, r_N . The partially activated solutions occur for $F_1 < F < F_{\infty}$ denoted by the light grey region; the diamonds show a sample response. The trivial response occurs for $F < F_1$ and is shown by the unshaded region; the squares show a sample response.	24
		4T

Figure 2.5	Frequency response at $F = 0.6$ for six resonators, computed from the averaged equations. The resonator amplitudes are the thick curves whose darkness indicates place along the chain, that is, the first resonator is light grey and the 6th resonator is black. The vertical lines are the predicted existence region boundaries, and the equal amplitude solution for the infinite chain is shown as a black dashed line. The light dashed line is the activation amplitude for r_j given in terms of r_{j-1} , given in Eqn (2.11); where it crosses the r_{j-1} response line r_j emerges from the trivial response.	25
Figure 2.6	Activation boundaries in (α, F) space computed from the averaged equations. The black lines correspond to the activation boundaries determined from simulations of the original equations of motion for a six resonator cascade. The red dotted lines correspond to the analytical approximations for the boundaries of the first two resonators and the infinite lattice. In general, there is good agreement except near the bottom of the Arnold tongue where modes interact in a complicated manner.	26
Figure 2.7	Activation boundaries with small forcing in the limit of zero damping. While the case with zero backcoupling, $\beta = 0$, has the infinite chain solution as a subset for every other region, the case with backcoupling, $\beta \neq 0$, suggests an aphysical scenario in which the entire chain is activated before second resonator activates. At very low forcing ($F < 0.1$), this incorrectly suggests that the infinite chain will activate before the first resonator activates, indicating a breakdown of the applicability of the infinite chain results.	27
Figure 3.1	Schematic model of a mechanical implementation of a multi-stage frequency divider with four localized modes and coupling spring elements [53]. Driving element x_5 parametrically excites element x_4 , which parametrically excites element x_3 , and so on down to element x_1 , with frequency division by 2 at each stage. Resonant response amplitudes are dictated by system nonlinearities.	30
Figure 3.2	(a) A three stage "candy cane" frequency divider cascade and its lin- ear mode shapes. (b) A four stage "T-bar" frequency divider cascade and its linear mode shapes. For each device, note that the beam ends are fixed and that the modes of interest have resonant frequency ra- tios of 2:1; mode shapes are computed using COMSOL Multiphysics®.	31
Figure 3.3	SEM micrograph of the device, which spans $400 \text{ m} \times 400 \text{ m}$, with uniform depth of 10 m and feature size of 1.85 m .	34

Figure 3.4	The first three vibrational modes from a COMSOL Multiphysics® finite element model. Corresponding modal frequencies are given in Table 3.1.	34
Figure 3.5	Normalized modal parametric instability boundaries for modes 1, 2, and 3 with frequencies normalized by modal number n . Divide-by-eight operation is achieved in the shaded overlap region	36
Figure 3.6	Measured amplitudes extracted from the spectrum at $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{8}$ of the drive frequency at a drive amplitude of $30 V_{ac}$. Displacement values, taken from the envelope of the large amplitude solution, are measured using the LDV, with the laser directed near the point of maximum displacement of each beam. Due to modal coupling, the mode 3 (first to activate) amplitude drops when modes 1 and 2 are active.	37
Figure 3.7	Measured response for each mode (beam) for 824.6 kHz drive signal at $34 V_{ac}$, in the fully cascaded operating mode. Displacement values, taken from the envelope of the large amplitude solution, are measured using the LDV.	39
Figure 3.8	SEM micrographs of two alternative frequency divider designs: the Tbar (left) and ellipse-coupler (right) designs. The orange block represents the drive electrode. The first three modes are concentrated in the red, green, and aqua colored beams, respectively.	39
Figure 4.1	The clamped-clamped flexural-torsional modal coupling device. The fundamental flexural mode has a resonant frequency near 65kHz and the fundamental torsional mode has a resonant frequency near 198kHz. This device is driven and sensed through an interdigitated comb drive mechanism [2].	43
Figure 4.2	Experimental (blue) and model (red) results of the open loop fre- quency sweep response for various drive amplitudes (20mV, 60mV, 200mV, 600mV), noting that the feedthrough capacitance modifies the experimental amplitude results for larger forcing levels. For the model results the dark red curve segments indicate a stable response and the light red segments indicate an unstable response	47
Figure 4.3	Predicted saddle node (green curves, SNB) and Hopf bifurcations (red curves, HB) as a function of drive frequency and drive amplitude, indicating bifurcations from both stable (dark) and unstable (light) response branches. The experimentally measured SNB and HB are denoted by (x) and (o), respectively. The bottom plot is a blowup of the upper plot in the region of interest.	60
Figure 4.4	The feedback loop circuit producing a self-sustained oscillation	61

Figure 4.5	The mean (noise-free) operating frequency as a function of feedback
	amplitude. (Left) The main graph shows predictions from the noise-
	free model. Left inset shows corresponding experimental results where
	red indicates sweeping upward in the amplitude and black indicates
	sweeping downward. The right inset shows a blow up of the fre-
	quency as a function of amplitude in the internally resonance regime
	for different levels of mode two cubic (Duffing) nonlinearity; the green
	point indicates zero slope, that is, a point of zero dispersion. Model
	parameters for this figure and subsequent figures are $\omega_2/\omega_1 \approx 3.067$, $Q_1 =$
	10^4 , $Q_2 = 10^4$, $\gamma_1 = 0.01$, $\beta = 0.00015$

Figure 4.6	This figure shows two measures of the loop phase noise predicted from the model as the forcing amplitude is varied. The three traces from gray to black represent different levels of the mode two cubic nonlinearity, as in Figure 4.5. The upper plot is a measure of the ra- tio of the contribution of the total (modes one and two) amplitude noise to those of the total phase noise, and the lower plot is measure of the ratio of the contributions of mode two to those of mode one. We see that in all three cases there is a significant drop in the contri- butions of the amplitude noise and the overall mode one noise when in the internal resonance operating regime. The inset shows experi- mentally measure standard deviation of the frequency fluctuations as the amplitude is swept up (red) and down (black), showing the drop in frequency fluctuations when operating in the internal resonance regime.	63
Figure 5.1	Initial (uniform width) design and the first two linear vibration modes obtained using COMSOL Multiphysics modal analysis. Left: linear vibration mode 1, right: linear vibration mode 2, with $\omega_2 \approx 2\omega_1$. Color indicates vibration amplitude.	68
Figure 5.2	Optimized design for maximizing the absolute value of the <i>e</i> ssential modal coupling coefficient and the two coupled linear vibration modes obtained using COMSOL modal analysis. Left: linear vibration mode 1, right: linear vibration mode 2 and $\omega_2 \approx 2\omega_1$. The color in online version indicates the vibration amplitude.	68
Figure 5.3	Flowchart of the structural optimization routine for nonlinear dy- namic response.	72
Figure 5.4	Initial (uniform width) design of a clamped-clamped beam and its linear vibration mode. Color indicates vibration amplitude	72

Figure 5.5	Optimized design for maximizing the cubic nonlinearity of a clamped- clamped beam and its linear vibration mode. Color indicates vibra- tion amplitude.	73
Figure 5.6	Optimized design for minimizing the cubic nonlinearity of a clamped- clamped beam and its linear vibration mode. Color indicates vibra- tion amplitude.	73
Figure 5.7	Evolution of the objective function and shapes encountered during the maximization process. The vertical axis is the absolute value of the objective function divided by its initial value and the horizontal axis is the iteration number.	74
Figure 5.8	Evolution of the objective function and shapes encountered during the minimization process. The vertical axis is the absolute value of the objective function divided by its initial value and the horizontal axis is the iteration number.	75
Figure 5.9	Evolution of the objective function and shapes encountered during the optimization process. The vertical axis is the absolute value of the objective function divided by its initial value and the horizontal axis is the iteration number.	80
Figure 5.10	Left: Typical time simulation with added envelope curves showing beats, for the final design with initial first mode amplitude of 7.5×10^{-11} (using eigenvectors normalized by the mass matrix). Right: Data points and fitted exponential curves for the beat period, normalized by the period of the first linear mode, versus the initial amplitude of the first mode, for designs corresponding to iteration numbers 0 (initial design, top curve), 30 (middle curve), and 250 (final design, bottom curve).	81
Figure 5.11	(a) Evolution of the eigenfrequencies of linear vibration modes 1 and 2 encountered during the optimization process. (b) Evolution of the pumping ratio and the mode localization ratio encountered during the optimization process.	83
Figure .1	(a) The mechanical device along with the two mechanical modes and the eigenfrequencies as computed in COMSOL. (b) The layout with the red region being the mechanical device, the dark green indicating the drive and sense electrodes for mode 1, and the light green electrodes allow for modal tuning and coupling bias.	94

CHAPTER 1

INTRODUCTION

In the realm of signal processing, nonlinear behavior is generally viewed as unwanted since it produces spurious harmonics and undesirable system characteristics. As such, much work has been done to operate in the linear regime or in a regime that does not elicit nonlinear phenomena. There are, however, useful devices that operate in a fundamentally nonlinear manner, and the successful design of these devices requires the designer to understand and utilize the nonlinearity in an advantageous way. These include devices such as photonic lasers, superconducting Josephson junctions, electronic diodes, and mechanical nonlinear vibration absorbers [70, 30, 57, 52]. In the field of signal processing, there are two such components that are found in almost every electronic device: frequency dividers and clocks.

The aims of the present work are to develop and examine these type of devices based on internally resonant modal interactions. Below, we provide an overview of frequency dividers, clocks, internally resonant dynamical systems, with MEMS as the technology for the device implementation.

1.1 Frequency Dividers

Frequency converters are essential elements in modern portable electronic devices. Frequency down-conversion and up-conversion, respectively known as frequency division and multiplication, are important to many processes including power conversion, clock distribution, frequency synthesis, radio frequency instrumentation, and optical link frontends. The performance of converters is measured by factors that include size, cost, frequency coverage, frequency resolution, power consumption, spurious harmonic content (including phase and amplitude noise performance), and stability [3]. Frequency conversion can be passive or active, and can be achieved in electrical, optical, or mechanical domains, or combinations thereof, and the strategy chosen for frequency conversion ultimately depends on performance specifications, power requirements, size, and cost.

In the context of portable devices, electronic frequency converters are the most prevalent due to their cost, tunability, integrability, and wide operating range. There are several types of electronic dividers. Digital dividers are useful for low frequencies due to their wide band of operation but are limited at higher frequencies due to energy cost and that they are limited by the analog-to-digital converter clock signal. Resonator based electronic dividers, such as microstrip or cavity based, are narrowband devices useful for higher frequencies (> 10Ghz), but not for lower frequencies due to size constraints. The successful high frequency analog topologies are injection locked, regenerative frequency dividers, and parametric dividers [4, 58, 9] Injection locking, on the one hand, requires two self-sustained oscillation circuits with commensurate resonant frequencies. A weak coupling between the two resonators will, just like Hugyen's clocks, synchronize the clocks if the coupling to noise ratio is large enough [73, 50]. Regenerative dividers, on the other hand, place an over damped or under damped resonator in a feedback loop with an amplifier in the feedback and a mixer that mixes the input and feedback signal. The limitation of these devices is the quality factor and amplifier speed, although these devices have been shown to work in the low Terahertz range [64]. Parametric dividers are generally passive devices that are extremely low power consuming devices and also have the best phase noise characteristics since they do not have amplifiers. The parametric divider topology has also been utilized with amplifiers in order to increase their operating regime [23, 24].

For both second order regenerative dividers and parametric dividers, a nontrivial steady state operation occurs when a resonator with natural frequency ω_0 has a reactive element (stiffness or capacitance) modulated periodically with a frequency $\Omega \approx 2\omega_0$, and the system responds with frequency $\Omega/2 \approx \omega_0$. The archetypal equation govern-

ing this behavior, when the pump is the only input, is the nonlinear Mathieu equation, $\ddot{x} + \omega_0 \dot{x}/Q + \omega_0^2 (1 + \delta \cos(\Omega t))x + \gamma x^3 = 0$, in which *Q* represents dissipation and γx^3 captures nonlinear effects. The threshold for instability, in terms of the modulation amplitude δ , depends on *Q* and the nearness of the excitation frequency to twice the natural frequency, as measured by a detuning parameter $\alpha = (\Omega - 2\omega_0)/(2\omega_0)$ [23, 40]. The effective *Q* factor can also be increased based on the amplifier (if there is one) [51].

Although electronic parametric frequency dividers are useful due to their power savings, their size is a prohibitive dilemma for small devices. MEMS resonators have a much smaller footprint and a much higher Q than electronic resonators at the same frequency, and parametric resonance has already been demonstrated in a variety of MEMS devices [63, 47, 11]. To date, with the exception of a patent [34], the author is unaware of any effort to utilize a MEMS 2:1 parametric resonance device for frequency division, although there has been a general effort to replace many timing application electronic devices with MEMS devices [43]. Frequency dividers are generally used in timing distribution circuits. Since these dividers are slaved to the clock, we turn our attention to the subject of the frequency generator.

1.2 Frequency Generators

Frequency generators are devices that produce a periodic signal that is used for time keeping. A clock is just another name for a frequency generator. The dynamics of these systems can be represented by mathematical models possessing a limit cycle of a given period. The precision of these systems is affected by the inevitable fluctuations of the period of oscillations. Since these devices are not slaved to a reference timing system, they have time translational symmetry, so that any source of noise will cause drift of the phase, resulting in phase noise. The goal of clock designers is to identify and remove any systematic drift which usually entails understanding the source of the noise and eliminating



Figure 1.1 Schematic diagram of a frequency generator showing the essential components.

the clock sensitivity to the noise. Different types of clocks have different phase noise characteristics. For example, an atomic clock, which is often a quartz resonator that is coupled to the resonant frequency of atomic transitions, has a very good long term time stability but the short term time stability is not as good as a purely quartz based clock. A majority of popular single mode clocks can be modeled by the schematic shown in Figure 1.1, where a resonator element is in a closed loop with an amplifier that replaces the energy dissipated by other loop elements, a phase shifter, and an amplitude limiter. The amplifier and phase shifter destabilize the system, resulting in oscillation whose amplitude is fixed by the limiter. The resulting nominal frequency depends on all the loop elements parameters and can be tuned by varying them. A more generic model commonly used lumps all elements in the feedback loop as an impedance $Z = Z_R + jZ_I$. In this case, the amplifier and amplitude limiter affect Z_R and the phase shifter affects Z_I [66]. This linearization impedance approach works well when the resonator element can be regarded as linear. When a mechanical resonator is used, its time-dependent displacement is typically represented by a motional series RLC circuit parallel to a feedthrough capacitor C, that is, the effective capacitance that occurs between voltages across the input and output leads of the transducer. The response is then a second order impedance, where the applied voltage is converted to a current [25]. Usually, the system is linearized about an operating point and analyzed from there. Generally, clocks are made to be as linear as possible and are made to utilize a single second order filter as the frequency selective element. Due to the linearity constraint, they are engineered to be physically large enough in order to have a large enough signal to noise ratio when transducing the mechanical signal to an electronic signal. Furthermore, other "parasitic" modes of the oscillator are engineered to be far away from the primary mode and all of its harmonics, to keep it as isolated as possible. As a result, the resonator models considered in these circuits are primarily single mode and linear.

Integration of a resonator into a circuit topology to create a self-sustaining feedback loop requires specific types of circuits [35]. Due to the fact that increasing the number of elements in the circuit will produce additional noise, it is desirable to minimize the number of elements in the device. Thus, a popular topology used with a mechanical (quartz) resonator is the Pierce oscillator circuit, since it uses only one active element, necessary for counteracting the energy loss in the circuit. There are additional sources of noise in these systems, such as thermo-mechanical noise, internal temperature gradients, and ambient environment fluctuations, and Nyquist-Johnson noise [5, 17]. In this work, we do not consider the details of these noise sources, but rather how to optimize the system based on assumed noise profiles. That is, we take a design engineering point of view.

An important goal in this field is to make extremely small, high frequency resonators with good phase noise characteristics [1]. Since the desired size is small, since frequency is inversely related to the size of the device, the devices may need to be driven into their nonlinear regime in order to overcome measurement noise, and generally to achieve good signal to noise characteristics. Generally, a distinctive feature of operation of an oscillator at an amplitude where the resonator element behaves nonlinearly is that amplitude noise will contribute to the overall phase noise, due to the amplitude-frequency dependence of nonlinear resonators; this does not happen in the linear range of the resonator [13]. As a result, one of the objectives for engineering a generator based on a nonlinear resonator element is to make it as near to a linear system as possible, in terms of phase noise, which is done by eliminating this amplitude to phase (AF) noise coupling. This amplitude noise contribution to the phase noise can be eliminated for certain resonators by operating at a so-called "sweet spot," that is, where $\partial I/\partial \omega \approx 0$ (a point of zero dispersion), thus locally mimicking with a linear oscillator [36, 18]. Another approach is to adjust the phase shifter so that the amplifier noise is orthogonal to the vector that produces phase diffusion [71, 26]. All of these MEMS clock models are based on a single mode response of the resonator element, but recent experimental work has shown that the phase noise performance of a given device can be significantly improved when the resonator responds with two nonlinearly coupled modes. Predictive modeling of these systems requires analysis of internally resonant systems, to which we now turn.

1.3 Internally Resonant Systems

Internal resonance occurs in systems when nonlinear coupling terms become strong given an alignment of the modal resonant frequencies. Internal resonances in mechanical systems have been the subject of many investigations over the past fifty years, and some internally resonant MEMS have been proposed and built in recent years. The nonlinear coupling in these systems is necessarily passive and can thus be derived from a potential. For two modes, the following potential displays a variety of different nonlinear coupling terms

$$\mathcal{V} = \frac{1}{2}\omega_1^2 x_1^2 + \frac{1}{2}\omega_2^2 x_2^2 + \beta_1 x_1 x_2^2 + \beta_2 x_1^3 x_2 + \beta_3 x_1^2 x_2^2 + \dots$$
(1.1)

The β_1 coupling term provides the so-called autoparametric resonance phenomena and is important in the 1:2 case, and the β_2 term provides the essential modal coupling in the 1:3 case. The 1:1 case is considerably more difficult since all of the terms are relevant. The β_3 term is the so-called "diffusive coupling" term, which should be acknowledged as possibly being important in all of the interactions. Fortunately, standard perturbation methods provide a systematic means of determining which coefficients are essential for different resonance conditions [39], and the theory of normal forms gives this a mathematically rigorous foundation [20]. Certain mechanical structures, such as cables and flexural and torsional beams and plates, exhibit internal resonance and have been explored with non-linear coupling between their various modes [27, 32, 48, 38, 41, 7]. Detailed investigations of the response of systems composed of many nearly-identical resonances with parametric excitation, that is, with $1:1:\cdots:1$ internal resonances have been considered with both nearest-neighbor [31] and global [10, 72, 37] coupling. Several micro mechanical devices have been designed for a 2:1 internal resonance [68, 65] and 3:1 internal resonance [2].

1.4 Summary of Topics Considered and Results Obtained

In this work we discuss two main topics. The first topic is an investigation into a novel class of frequency dividers, namely, a cascade of nonlinearly coupled micro-mechanical resonators. This cascade is analyzed in Chapters 2 and 3. Chapter 2 considers an idealized model for the cascade, based on the normal form for sequential 1:2 internally resonant elements. Perturbation methods are used to analyze the system and determine viable operating points in the driving signal parameter space and how these depend on system parameters. In Chapter 3, we describe how one realizes such a cascade in mechanical frames using coupled beam elements, suitable for MEMS implementation. We provide simple designs rules for these systems and show different design configurations. Experimental results, carried out in collaboration with our UCSB partners, and demonstrating a frequency division of eight, are described for one of these designs.

The second topic is an investigation of the class of frequency generators that utilize internally resonant modal interactions to improve phase noise performance. This work is based on close collaboration with an experimental group at Argonne National Labs. In chapter 4, we develop a generic model for this class of problems, including small noise sources, and develop a method for predicting how these noises are filtered through the nonlinear system response. We demonstrate that certain operating points, in particular those where the modal coupling becomes strong, can result in significant phase noise reduction. We develop a detailed model based on an Argonne device with 1:3 internal resonance, and consider its open and closed loop response. The model is shown to capture the experimentally measured responses of the device, and it is used to predict the limits of phase noise reduction using internal resonance.

In Chapter 5 we describe computational tools that allow one to determine the normal form coefficients for a given nonlinear resonance from a nonlinear finite element model. These tools are used with an optimization algorithm to develop structures with tailored nonlinear response, specifically, either maximized or minimized normal form coefficients. This work forms the basis on which future generations of devices will be developed.

The dissertation closes in Chapter 6 with a discussion of the results, their significance, and directions for future work in this area.

CHAPTER 2

SUBHARMONIC CASCADE FREQUENCY DIVIDER

In this chapter, we propose a novel frequency conversion method based on nonlinear dynamics that exploits the robustness of parametric resonance. The primary approach is to use a subharmonic resonance cascade, which involves a chain of internally resonant subsystems of a form that allows energy exchange between the subsystems. Although de-tailed investigations of the response of systems composed of nearly-identical resonances with parametric excitation have been considered with both nearest-neighbor [31] and global [10] coupling, the resonance structure considered here has only been analyzed for a small number of modes (up to three) [42]. As discussed in the introduction, this investigation is motivated by providing an alternative to electronic frequency dividers that are used in many devices [16, 22]. Most of the work presented here is taken from the following published papers [54]. In the following chapter we consider realizations of this cascade in MEMS hardware.

In order to physically intuit the dynamics of interest, consider the system shown in Figure 2.1, consisting of *N* hinged beams (resonators) coupled by springs with linear stiffness. The first beam is parametrically excited and, for a range of driving frequencies and amplitudes, the first resonator will experience parametric resonance; its response will parametrically excite the next resonator through the coupling spring and so on down the cascade. The inter-beam springs provide nonlinear coupling of the desired form via finite deformation kinematics. Note that the coupling is two way, since the *j* + 1 resonator, when it is active, necessarily back-acts on resonator *j*, thereby providing two-way coupling, and this coupling is resonant in both directions.

In this chapter, we first present the generic equations of motion that model such a subharmonic resonant cascade system, from which we derive averaged equations that approximate the slowly varying amplitudes and phases of the system elements. These



Figure 2.1 A mechanical implementation of the subharmonic cascade, consisting of rigid bars with elastic hinges at their bases, and coupled by springs with linear stiffnesses. The energy is down-converted from the high frequency beam, parametrically driven by a source at approximately twice its natural frequency, down the chain to a frequency of $\Omega/2^N$ at the terminal beam. For small stiffnesses of the coupling springs the bar displacements u_i are roughly equal to the system modal coordinates q_i , that is, the system modes are localized in the individual beams.

equations are essentially the normal form for this type of resonance. For the present study we consider special conditions on the parameters that provide a simple form that is amenable for analysis and offers some explicit results. We start the analysis by considering a semi-infinite chain, considering a special solution in which all resonators are active, helping set the stage for further analysis. This approach also offers approximate closed form expressions for the steady state response of most elements of the semi-infinite chain, which are shown to be very useful approximations for many elements in chains of finite length. We then examine the conditions on the forcing amplitude and frequency for nontrivial steady state operation of the first few resonators, specifically computing how elements of the chain become activated as the input parameters are varied. Using the results from these analyses, we delineate the regions in excitation parameter space over which the system is expected to produce a complete subharmonic cascade. This parameter space is then given more resolution using numerical simulations, particularly in the small forcing amplitude case, and other interesting phenomena are discussed. The analytical approximations are confirmed using simulations of a sample system. The chapter closes with some thoughts about applications and extensions of these results.

2.1 The Model and The Averaged Equations

For the generic cascade model we start with a Hamiltonian (per unit mass) for a system of *N* resonators, described in terms of vibrational modal amplitudes q_j and conjugate momenta p_j , each with cubic nonlinear stiffness and quadratic nearest-neighbor coupling, specifically given by,

$$\mathcal{H} = \sum_{j=0}^{N} \left[\frac{1}{2} p_j^2 + \frac{1}{2} \omega_j^2 q_j^2 + \frac{1}{4} \nu_j q_j^4 + \kappa_j q_j q_{j+1}^2 \right],$$
(2.1)

where ω_j and v_j are, respectively, the linear natural frequency and mass-normalized cubic stiffness coefficient for the *j*th resonator, and the κ_j 's are the mass-normalized nonlinear coupling coefficients. The resonator nonlinearity will limit the amplitudes of vibration when undergoing subharmonic resonance, and the coupling nonlinearity is required for energy transfer between resonators. Note that many nonlinear coupling terms may be present, but only one is present in the normal form, that is, only one is needed to capture the energy transfer and the others can be removed by coordinate transformations [20]. In order to account for dissipation, we introduce resonator viscous damping factors, ζ_j directly into the modal differential equations, via generalized forces. We also assume that nonlinear effects and damping are relatively small, consistent with many MEMS resonators [48]. The boundary conditions on the resonator chain consist of: (i) a source with $q_0 = F \sin(\Omega_0 t + \phi_0)$, which parametrically excites the first resonator with frequency $\Omega_0 = 2\omega_1(1 + \alpha)$, where α describes small detuning from resonance; ¹ and (ii) a grounded terminal condition with $q_{N+1} \equiv 0$.

The cascade occurs when each resonator j is parametrically excited by resonator j - 1 via the coupling, which requires that the resonators be tuned such that $\omega_{j-1}/\omega_j \approx 2$. Conditions for complete activation of all N resonators must account for the effects of damping, detuning, and quadratic backaction from resonator j + 1. The mechanical system depicted in Figure 2.1 offers the desired features, and is modeled by the given Hamiltonian.

To simplify the model we rescale time using the natural frequency of the first resonator, $\tau = \omega_1 t$, and the displacement according to $x_j = (\sqrt{v_j}/\omega_j)q_j$, thus normalizing the cubic nonlinearities and the resonators' amplitudes. Linear frequencies of adjacent resonators are set to be close to 2:1 ratios, as required for the cascade to occur. For the present model we make some additional assumptions that simplify the analysis, by making the non-dimensional resonator equations have identical form. It is important to note that these assumptions are taken for convenience in the analysis, but are not required for the desired cascade to occur. These additional assumptions are: cubic nonlinearity and quadratic coupling coefficients for adjacent resonators are taken to be close to 4:1 ratios, that is,

$$\frac{\nu_j}{\nu_{j-1}} = \frac{\kappa_j}{\kappa_{j-1}} = 4$$
(2.2)

and the modal damping values are nearly the same, that is, $\zeta_j \approx \zeta \forall j$.² These assumptions render the nondimensional parameters for each resonator to be conveniently

¹Note that one can use direct excitation of the first resonator, but the present form is taken for convenience.

²Note that the 4:1 ratio for resonator cubic nonlinearity holds for fixed-fixed beams with identical uniform cross-sections where the lengths are varied to account for the 2:1 change of modal frequency.

identical, resulting in equations of motion,

$$\ddot{x}_{j} + 2^{2-j}\zeta\dot{x}_{j} + 2^{2(1-j)}x_{j} = f_{j}$$

$$f_{j} = 2^{2(1-j)} \left[-\sigma_{j}x_{j} - \frac{8}{3}\gamma x_{j}^{3} - 4\delta x_{j-1}x_{j} - 4\beta x_{j+1}^{2} \right]$$
for $j = 0, \cdots, N+1$

$$(2.3)$$

where the non-dimensional parameter definitions are: nonlinearity, $\gamma = 3/8$; forward coupling, $\delta = 2\kappa_j/(\sqrt{\nu_j}\omega_j)$; backward coupling, β , where for passive coupling $\beta = \delta/2$; and detuning $\alpha = (\Omega_j - \omega_j)/\omega_j$, where $\Omega_j = \Omega_0/2^j$. Note that $\sigma_j << 1$ represents deviations from the 2:1 natural frequency ratio conditions. We further simplify the model for our present investigation by taking $\sigma_j = 0 \forall j \neq 1$, that is, we make the resonators perfectly tuned relative to one another. This assumption is also not required for the cascade to occur, but it simplifies the analysis which, for present purposes, focuses on the general features of the cascade. Of course, $\sigma_j \neq 0$ will need to be incorporated in system design studies to account for variabilities arising from fabrication tolerances and other effects. Note that this approach does allow for tuning of the input amplitude *F*, via x_0 , and input frequency Ω_0 , via α .

Due to the nature of the excitation and the internal resonances, $x_i = 0 \forall i$ satisfies the equations of motion, but its stability depends on the input and system parameters. The response of interest will be one that is dynamically stable and has all resonators operating with non-zero amplitudes. To investigate all possible responses and their stability we turn to a perturbation analysis.

To approximate the system response, we use a transformation to rotating coordinates, the van der Pol transformation, by defining amplitude and phase coordinates via $x_j(t) = r_j(t) \cos(\Omega_j t + \phi_j(t))$ and $\dot{x}_j(t) = -r_j(t)\Omega_j \sin(\Omega_j t + \phi_j(t))$. Applying a first order perturbation expansion about the harmonic solution, averaging over the period of the lowest frequency resonator, and ignoring higher order terms, we obtain equations approximating the slowly varying amplitude and phase of each resonator [40]. Under the stated assumptions and transformations, the averaged equations for the slowly varying amplitudes and phases are given by,

$$\dot{r}_{j} = 2^{1-j} \left[-\zeta r_{j} - \delta r_{j-1} r_{j} \sin \Theta_{j} + \beta r_{j+1}^{2} \sin \Theta_{j+1} \right]$$

$$r_{j} \dot{\phi}_{j} = 2^{1-j} \left[-\alpha r_{j} + \delta r_{j-1} r_{j} \cos \Theta_{j} + \beta r_{j+1}^{2} \cos \Theta_{j+1} + \gamma r_{j}^{3} \right]$$

$$(2.4)$$

where $\Theta_j = \phi_{j-1} - 2\phi_j$. The array end conditions are $r_0 = F$, $\phi_0 = 0$, and $r_{N+1} = 0$. Under the given assumptions about the parameters, the averaged equations become identical for each resonator, except for a factor of 2^{1-j} for resonator *j*. This implies that the time scale of resonator *j* is slower than that of j - 1 by a factor of two, resulting in slower transient behavior as one moves down the chain. This is expected, since the resonator Q values are the same and the frequencies of adjacent resonators scale by a factor of 1/2.

To find the steady state solutions, we set $\dot{r}_j = \dot{\phi}_j = 0$, and solve for the resulting amplitudes and phases. The stability of the steady-state responses can then be determined from the averaged equations from the Jacobian in the usual manner. However, these equations do not conveniently uncouple, and no closed form solution is possible, except for some special responses (for example, the trivial response). However, in the case with no back coupling, $\beta = 0$, one can sequentially solve the equations in closed form. While this case is not of primary interest, it is considered in Section 2.4 below, since it allows for a calculation of interest.

We begin by considering a special case that helps set the stage for a more full investigation, namely, a special type of fully activated response for an infinite length chain. This is followed by an analysis that considers sequential activation of the chain as input parameters are varied.

2.2 The Semi-Infinite and Long Chains

Instead of solving the full system of 2*N* equations, which requires numerics, we consider a special response wherein each of the resonators in a semi-infinite chain has the same amplitude. It is shown below that this solution provides a useful approximation for the response of many of the central elements in finite length chains. By setting $r_j = r_{eq}$ for all *j*, and assuming a semi-infinite lattice of resonators, we can solve for the steady state amplitudes from the averaged equations. Regarding the phases, it can be shown that $\sin \Theta_j = \sin \Theta_{j+1}$ and $\cos \Theta_j = \cos \Theta_{j+1}$ for an equal amplitude solution, yielding 3 possible steady-state amplitudes: the trivial solution and

$$r_{eq}^{\pm} = \sqrt{A \pm \sqrt{A^2 - \frac{\alpha^2}{\gamma^2} - \left(\frac{(\beta + \delta)\zeta}{\gamma(\beta - \delta)}\right)^2}},$$
(2.5)

where $A = [(\delta + \beta)^2 + 2\alpha\gamma]/(2\gamma^2)$. It is notable that these solutions are independent of *F* since we assumed $F = r_0 = r_{eq}$. Solutions r_{eq}^{\pm} appear in a saddle-node bifurcation as parameter vary, resulting in a stable response for r_{eq}^+ and an unstable response for r_{eq}^- . The trivial response is always stable. While these equal amplitude responses have explicit solutions, due to the symmetry of the solutions there are 2^N possible phase solutions for a lattice of *N* resonators. The phases for the steady state solutions must satisfy $\Theta_j - \Theta_{j+1} =$ $2\pi n$ for $n \in \mathbb{Z}$, recalling that $\Theta_j = \phi_{j-1} - 2\phi_j$.

It will be seen that for finite length chains, the steady-state amplitude for the bulk of the resonators, not including the first few or the last (j = N) resonator, is approximately equal to either zero or r_{eq}^+ . Whether the bulk of the chain is activated or quiescent depends on the magnitude of *F* relative to r_{eq}^- , as follows: If $F > r_{eq}^-$, $F < r_{eq}^-$, or $F = r_{eq}^-$, then the bulk of the resonators $r_j \approx r_{eq}^+$, $r_j \approx r_{eq}^0$, or $r_j \approx r_{eq}^-$, respectively, where the r_{eq}^- case is unstable.

Returning to the amplitude solutions, we can analytically derive the conditions for which the two nontrivial solutions r_{eq}^{\pm} exist and are real. In particular, we are interested

in the region of (α, F) for which r_{eq}^{\pm} exist. By considering their form it is found that r_{eq}^{\pm} exist when $\alpha \ge \alpha_{\infty}$, where the bifurcation value of α is given by,

$$\alpha_{\infty} = \frac{\gamma \zeta^2}{(\beta - \gamma)^2} - \frac{(\beta + \delta)^2}{4\gamma} \,. \tag{2.6}$$

We can then apply the assertion that when $F \ge r_{eq}^-$, the entire cascade becomes active. In later sections we will utilize the notation that F_{∞} is the critical forcing that makes all the resonators destabilize from the 0 solution, where we have shown that $F_{\infty} = F_{eq}^-$. This provides an approximate activation condition for the excitation amplitude that depends on the other system parameters.

For long chains, the behavior of the terminal resonator N is completely determined by the response of the N - 1 resonator, since it is parametrically driven by the N - 1 resonator from one side and the other side is grounded, $r_{N+1} = 0$. As such, r_N is described by the nonlinear Mathieu equation with excitation provided by r_{N-1} from the infinite chain solutions, which has the trivial solution as well as

$$r_N^{\pm} = \sqrt{\frac{\alpha}{\gamma} \pm \sqrt{\left(\frac{\delta r_{N-1}}{\gamma}\right)^2 - \frac{\zeta^2}{\gamma^2}}}.$$
 (2.7)

These solutions are identical to the solutions of the first resonator when $r_2 = 0$ (given in Equation 2.10 below), except that the effective forcing is r_{N-1} in this case, in place of *F*. In order to approximate r_N , we approximate $r_{N-1} \approx r_{eq}^+$ as in Equation (2.7) and choose the stable solution in the desired parameter space, r_N^+ . The result is found to be quite accurate for chains even with only a few elements, as demonstrated in the examples simulated below.

Since low-power usage is of interest in applications, it is relevant to consider how a semi-infinite chain can respond with equal amplitudes when each resonator is dissipative, here modeled with viscous friction. For an entirely activated semi-infinite chain, we can show that that the total power dissipated is finite, as follows. The calculation of the total

loss through dissipation is given by,

$$P_{diss} = \sum_{j=1}^{N} \frac{1}{T} \int_{0}^{T} (2^{2-j} \zeta \dot{x}_{j}) \dot{x}_{j} dt .$$
(2.8)

Assuming a semi-infinite chain where each element responds with amplitude r_{eq}^+ , the result is found to be,

$$P_{diss} = \frac{2}{7} \zeta \Omega_0^2 \left(r_{eq}^+ \right)^2, \qquad (2.9)$$

since the infinite sum is convergent. We would expect that for $r_{eq}^- < F < r_{eq}^+$, the actual power used would be slightly less than P_{diss} since the first few resonators have amplitudes less than r_{eq}^+ , and yet the chain is fully activated. If one's goal is to limit the power consumption of the device, then one can either decrease the damping or decrease steady state amplitudes, r_{eq}^+ . From Equation 2.5, the latter can be achieved by either decreasing the coupling or by increasing the cubic nonlinearity of the resonators.

2.3 Activating the Cascade

Here we derive conditions related to activation of the cascade as input parameters are varied. We begin by considering the system with parameter conditions such that the fully trivial response is the only stable response, regardless of initial conditions. As the excitation amplitude or frequency is slowly varied, the first step in activation of the cascade occurs when the driven resonator (j = 1) experiences a subharmonic instability, specifically, a period-doubling bifurcation of the system, which corresponds to a pitchfor*k*-bifurcation in the averaged equations. As parameters are varied further, the second resonator activates through another similar bifurcation. This occurs at a point at which the first resonator, which is driving the second, reaches a critical amplitude, and this scenario is then repeated along the entire chain. For these activations, we append the additional subscript to denote the "*k*-bifurcation", that is, when r_k becomes nonzero. As such, $r_{j,k}$ is the amplitude of the *j*th resonator at the *k*-bifurcation and α_k , for example, is the value of the detuning parameter at the *k*-bifurcation. In the desired fully activated response, the zero solution for each resonator is unstable so that all *N* resonators are active. In fact, the only way the entire chain can be activated is by sequential activation of the resonators, since there is no mechanism for stimulating isolated internal elements in the chain. In this section, we obtain expressions for the critical values of the excitation parameters that correspond to this sequence of instabilities, and explicit expressions for the first two instabilities are derived.

We begin by considering a partially activated chain, for which $r_i \neq 0$ for $i \leq j$ and $r_{\ell} = 0$ for $\ell > j$. Since resonator j + 1 is inactive, the response of resonator j is governed by the simple nonlinear Mathieu equation for which the excitation is provided by resonator j - 1. Its nontrivial steady-state response amplitude, where $r_{j+1} = 0$, can be found from the averaged equations and is given by,

$$r_j = \sqrt{\frac{\alpha}{\gamma} \pm \sqrt{\left(\frac{\delta r_{j-1}}{\gamma}\right)^2 - \frac{\zeta^2}{\gamma^2}}}.$$
 (2.10)

From this expression we find a condition for the *j*-bifurcation by based on the amplitude $r_{j-1,j'}$

$$r_{j-1,j} = \sqrt{\left(\frac{\zeta}{\delta}\right)^2 + \left(\frac{\alpha}{\delta}\right)^2}.$$
(2.11)

Note that while useful, this expression does not give an explicit condition for the *j*-bifurcation in terms of the excitation parameters. However, using this result we can obtain such conditions for the first two bifurcations, as follows.

For the activation condition of the first resonator, we consider Equation (2.10) with j = 1, recognizing that $r_0 = F$ is the excitation amplitude. One can solve r_1 in terms of F, which is the usual solution of the Mathieu equation. The point where r_1 becomes nonzero yields a condition that can be solved for a relationship between the critical values of the excitation parameters (amplitude and frequency) and the system parameters. This initial activation is the 1-bifurcation, and is simply that for a simple Mathieu equation. For the

next activation condition, the 2-bifurcation, we use the fact that r_1 is known from Equation (2.10) in closed form in terms of the input and system parameters, so long as $r_2 = 0$. We again use Equation (2.10), this time with j = 2, $r_2 = 0$, and the known expression for r_1 to determine the 2-bifurcation condition in closed form. These calculations result in the following expressions for the parameter conditions for the j-bifurcations for j = 1, 2, which are expressed in terms of both the force amplitude and frequency,

$$\alpha_1 = \sqrt{(\delta F)^2 - \zeta^2} \tag{2.12}$$

$$F_1 = \sqrt{\left(\frac{\zeta}{\delta}\right)^2 + \left(\frac{\alpha}{\delta}\right)^2}$$
(2.13)

$$\alpha_2 = \frac{\delta^2}{2\gamma} - \frac{1}{2\gamma} \sqrt{\delta^4 - 4\gamma^2 \zeta^2 + 4\gamma^2 \zeta^2} \sqrt{\frac{(\alpha_1)^2}{\gamma^2}}$$
(2.14)

$$F_2 = \sqrt{\left(\frac{\zeta}{\delta}\right)^2 + \left(\frac{\alpha}{\delta} - \frac{\gamma}{\delta} \left(F_1\right)^2\right)^2}.$$
(2.15)

These conditions describe boundaries in the (α , F) input parameter space, shown as the indicated solid lines in Figure 2.2, which delineate where each of the first two resonators become active. Note that bifurcation conditions for j > 2 cannot be obtained explicitly in this manner since no closed form expression is available for the r_j 's when two or more resonators are active.

2.4 Period Doubling Cascade Accumulation

In Section 2.3 we found conditions for the critical input values for destabilizing the *j*th resonator, which corresponds to a period-doubling bifurcation. Explicit expressions were obtained for α_j and F_j for j = 1, 2. Due to the fact that the infinite cascade is fully activated at finite values of the input parameters, there must be an accumulation of period-doubling bifurcations. Period-doubling cascades and the attendant accumulation ratios for the amplitudes and parameter values at bifurcation points are well understood for uni-modal maps [6]. The universal nature of these sequences was described by Feigenbaum, who



Figure 2.2 Analytical predictions for the response regimes in (α , F) space; the destabilization boundaries α_i and F_i for i = 1, 2 and ∞ are shown. The fully active regime is shaded grey and the partially activated regime is shaded light grey. The curve labeled r_{eq}^+ corresponds to the stable nontrivial response for the infinite chain. Parameter values for all simulations are as follows, unless specified otherwise: $\zeta = 0.03$, $\gamma = 0.075$, $\delta = 0.064$, and $\beta = 0.008$.

computed values for the accumulation ratios [19]. It is important to note that the present system differs from those satisfying Feigenbaum's "universal" conditions, since here degrees of freedom are added at each instability, and the resulting response is not chaotic. In all these cases, however, accumulation ratios of the following form can be determined,

$$a_p = \lim_{j \to \infty} \frac{p_j - p_{j-1}}{p_{j+1} - p_j}$$
(2.16)

where *p* represents some parameter value or response amplitude of the system.

For the present system we take p_j as the square of the forcing parameter, F^2 , and formulate the accumulation ratio problem for the general cascade. By taking zero backcoupling, $\beta = 0$, we can obtain an explicit solution for the accumulation constant for F^2 in terms of the system parameters. While not completely general, this special solution demonstrates that the accumulation occurs and that the accumulation ratio is not universal, but rather depends on the system parameters. For $\beta = 0$ amplitude r_{j-1} depends on r_{j} , but not on r_{j+1} , and the steady state averaged equations give

$$r_{j-1}^2 = \left(\frac{\zeta}{\delta}\right)^2 + \left(\frac{\alpha}{\delta} - \frac{\gamma}{\delta}r_j^2\right)^2.$$
(2.17)

The expressions for F_1 and F_2 are, even in this special case, given in Equations (2.13) and Equation (2.15), respectively. The following pattern emerges emerges for all F_j ,

$$(F_j)^2 = \left(\frac{\zeta}{\delta}\right)^2 + \left(\frac{\alpha}{\delta} - \frac{\gamma}{\delta}\left(F_{j-1}\right)^2\right)^2.$$
(2.18)

We can then find a_{F^2} by writing F_j^2 and F_{j+1}^2 in terms of F_{j-1}^2 . Since we are interested in the accumulation ratio as $j \to \infty$, we set $F_{j-1} = r_{eq}^-$, which is the known critical forcing amplitude for destabilizing the entire cascade, thus yielding the accumulation ratio for F^2 in the infinite limit,

$$a_{F^2} = \frac{1}{1 - \sqrt{1 + 4\left(\frac{\alpha\gamma}{\delta^2} - \frac{\gamma^2\zeta^2}{\delta^4}\right)}}.$$
(2.19)

This validity of this result is confirmed by simulations of the system with $\beta = 0$.

2.5 Numerical Simulations

In this section, we perform a set of numerical simulations in order to validate and extend the analysis. A set of N = 6 resonators is considered, which is sufficiently large for the infinite lattice approximation to be useful. We take the resonator nonlinearity to be hardening, that is, $\gamma > 0$. For a softening nonlinearity, the results are simply flipped in terms of the sign of α .

First, the averaged equations and the original equations of motion are numerically integrated and the results compared. As seen in Figure 2.3, the amplitudes from the averaged equations, $r_i(t)$, match the transient and steady state response envelopes of the original system of equations quite well in terms of decay times, transient beat frequencies, and amplitudes, albeit with time shifts for the transient beating response of the resonator



Figure 2.3 Simulation of a six resonator chain, showing the sequential activation of the six elements when started with small initial conditions. Resonators 1-6 are shown from top to bottom. The thick lines indicate results of the amplitudes obtained by simulating the averaged equations, and the underlying fast oscillations are from simulations of the full equations of motion. The settling time of resonator *j* is proportional to $Q/2^{j-1}$.

furthest down the chain. These time shifts arise due to variations in the startup transients, which are very sensitive to the initial conditions selected. The steady-state conditions are very accurately captured.

This numerical simulation, as well as others carried out but not shown here, confirm the use of the averaged equations as a basis for analytical predictions of the system response, and confirms our intuitive predictions about the system dynamics. It is important to note that the periodicity of each resonator is on the slowest timescale of the system, namely, on the period of the last resonator in the chain. This is not visible at the resolution shown, since the low frequency amplitude modulations, which arise from back coupling, are quite subtle, and the steady-state response each resonator is dominated by Ω_{j} .

Figure 2.4 shows the steady state amplitudes of all resonators, r_i for i = 1 - 6, for 4 different values of the forcing strength *F*, for the case of zero detuning, $\alpha = 0$. Re-

call that when $F = r_{eq}^+$, the finite lattice is well approximated by the infinite lattice; this corresponds to the solution depicted by the row of asterisks in the figure. The critical amplitude $F_{\infty}^- = r_{eq}^-$ is depicted by the row of asterisks that differentiates the fully activated (shaded) and partially activated (unshaded) regions. The resonant solutions described by the circle and triangle symbols correspond to two cases with $F \ge F_{\infty}$, for which the entire chain is activated; we refer to this as a fully activated response. The two solutions represented by diamond and square symbols correspond to cases with $F < F_{\infty}$. The solution with diamonds shows a partially activated chain, which occurs when the forcing is strong enough to activate only the first two resonators, since $F_2 < F < F_3$. For F slightly lower, shown as squares in the Figure, not even the first resonator is activated, resulting in a fully trivial response. The simulation results shown also confirm the assertion that for $F \ge F_{\infty}$, the steady state amplitude profile is well approximated by $r_i = r_{eq}^+$ for all but the first few and the final resonators. As discussed above, the amplitude of the final resonator, i = N, indeed shows a slight decrease in amplitude from the infinite chain amplitude, and matches its predicted value, given in Equation (2.7), very well.

The system frequency response, determined by sweeping the detuning parameter α for fixed levels of forcing, is also of interest. As seen in Figure 2.5, as the frequency is increased, the resonators sequentially activate, and the amplitude profile of many elements of the lattice, specifically, all except for the first few, are well approximated by r_{eq}^+ when the chain is fully activated. Also, these responses are quite insensitive to the forcing strength, so long as $F \ge r_{eq}^-$; recall that this follows since r_{eq}^+ is independent of F. Note that r_{eq}^+ has the form $r_{eq}^+ \approx \sqrt{\alpha - a} + b$ where a and b are constants, which dictates how the response amplitude depends on α , an observation confirmed by Figure 2.5. Also, it is interesting to note that the critical values α_j at which individual resonators activate accumulate onto α_{∞} as j becomes large. In fact, this accumulation must be geometric and, although in Section 2.4 we only discussed the accumulation ratio for F^2 for the $\beta = 0$ case, due to its closed form availability, an accumulation ratio must exist for α as well, and for


Figure 2.4 Steady state amplitudes of a six resonator cascade for $\alpha = 0(> \alpha_{\infty})$ for various forcing amplitudes. The fully activated response is achieved when $F > F_{\infty} = r_{eq}^{-}$, which is depicted as the grey shaded region. The top two examples (circles and triangles) converge to the infinite lattice amplitude r_{eq}^{+} , which is denoted by the row of asterisks, with the amplitude of the final resonator given by a slightly lower value, r_N . The partially activated solutions occur for $F_1 < F < F_{\infty}$ denoted by the light grey region; the diamonds show a sample response. The trivial response occurs for $F < F_1$ and is shown by the unshaded region; the squares show a sample response.

the more interesting case with backcoupling.

Lastly, Figure 2.6 shows numerically calculated (in solid black) activation boundaries for each of the six resonators, along with the analytical predictions (in dashed red) for the first two resonators and for the infinite cascade . Note that this figure focuses on the lower central portion of the region shown in Figure 2.2, since this is where the most interesting features occur. As expected, the analytical solutions correctly predict the first and second resonator activation boundaries. The figure also shows that for all but a small range of forcing amplitudes the infinite lattice activation boundary is a good predictor for the finite lattice. However, due to secondary bifurcations, discussed below, the infinite lattice approximation becomes invalid near the bottom of the shaded region. Other features of this Figure as also described below.



Figure 2.5 Frequency response at F = 0.6 for six resonators, computed from the averaged equations. The resonator amplitudes are the thick curves whose darkness indicates place along the chain, that is, the first resonator is light grey and the 6th resonator is black. The vertical lines are the predicted existence region boundaries, and the equal amplitude solution for the infinite chain is shown as a black dashed line. The light dashed line is the activation amplitude for r_j given in terms of r_{j-1} , given in Eqn (2.11); where it crosses the r_{i-1} response line r_i emerges from the trivial response.

2.6 Other Phenomena

In this section, we consider two other types of phenomena that occur in this system: multistability, and the behaviors that arise near the bottom of the shaded region in Figure 2.6. We do not pursue an investigation of these issues in this thesis, since they are not relevant to the applications we have in mind, although they are no doubt interesting.

2.6.1 Multistability

So far, we have focused on the activation solution from the near zero initial conditions and have defined this region to be localized in the (α , F) parameter space as a generalized type of Arnold tongue. However, as one would expect from the single resonator case, there does exist a response regime for $\alpha > -\alpha_1$ (for the hardening case), where multiple stable solutions occur, namely, the zero solution, the r_{eq}^+ solution (which approximates the fully activated chain), and partially activated solutions. Simulated frequency sweeps



Figure 2.6 Activation boundaries in (α, F) space computed from the averaged equations. The black lines correspond to the activation boundaries determined from simulations of the original equations of motion for a six resonator cascade. The red dotted lines correspond to the analytical approximations for the boundaries of the first two resonators and the infinite lattice. In general, there is good agreement except near the bottom of the Arnold tongue where modes interact in a complicated manner.

have confirmed the existence of the fully active solution and the zero solution in the multistable regime.

2.6.2 Small Forcing and Small Detuning

As indicated in Figure 2.6, there is a region where the infinite lattice solution breaks down as a predictor for the activation of the entire chain. The main dynamics observed in this region involve quasi-periodic energy transfer between the first two resonators, indicating the presence of a secondary bifurcation not captured by the present analysis. In these responses, simulations show that the first resonator approaches an amplitude large enough to excite the second resonator, but when it transfers its energy to the second resonator, the first resonator amplitude decays, and does so to an amplitude where it no longer excites the second resonator, at which point it begins to gain amplitude, starting the cycle over again. This energy exchange phenomena does not occur for zero back-action ($\beta = 0$).

In addition to the energy exchange phenomena, Figure 2.7 shows that in the limit of



Figure 2.7 Activation boundaries with small forcing in the limit of zero damping. While the case with zero backcoupling, $\beta = 0$, has the infinite chain solution as a subset for every other region, the case with backcoupling, $\beta \neq 0$, suggests an aphysical scenario in which the entire chain is activated before second resonator activates. At very low forcing (F < 0.1), this incorrectly suggests that the infinite chain will activate before the first resonator activates, indicating a breakdown of the applicability of the infinite chain results.

small damping, $\zeta = 0$, the (α_{∞} , F_{∞}) curve, corresponding to full activation, intersects with both the (α_1 , F_1) and (α_2 , F_2) curves near the bottom of the Arnold tongue. This presents an impossible situation, since the first and second resonators must destabilize before the rest of the cascade activates. This observation indicates a breakdown of the applicability of the infinite chain results. This may be an important region in the parameter space to consider for a given physical implementation of a cascade, since the effective forcing strength depends on many design variables and is often relatively small.

2.7 Outlook

The results of this investigation demonstrate that cascades of subharmonic resonances will occur in systems designed with the appropriate forms of internal tuning and coupling. The infinite chain approximation was shown to be very useful for predicting the fully activated response for chains of finite lengths, even as small as the six element example considered. It was also found that the analytical perturbation results are very effective in predicting the transient and steady-state response of these systems, thereby providing a useful tool for investigating system parameter dependence, which will be a significant aid for device design. Also, while it was found that the response of this system can be quite complicated, especially near the bottom of the Arnold tongue, there exist large open sets of system and input parameter values over which the system behaves quite simply, specifically, in the desired mode of a robust frequency dividing cascade.

Ongoing work on this class of systems includes the following topics: robustness to parameter variations such as the effects of inter-resonator mistuning; design realization of MEMS devices with the desired properties; the possibility of using this arrangement for frequency up-conversion; and issues related to practical performance measures of subharmonic cascades, including phase noise characteristics and power requirements.

CHAPTER 3

A MEMS FREQUENCY DIVIDER: DESIGN AND EXPERIMENTAL RESULTS

As discussed in the Chapter 2, the frequency divider cascade combines the benefits of cascading, internal resonance, and mechanical coupling in a single micro-device. The operation is based on nonlinear modal coupling and exploits the robustness of parametric resonance. This narrow-band approach uses a subharmonic resonance cascade in a chain of internally resonant subsystems with specific coupling that allows energy exchange between successive divide-by-two stages. In this chapter we develop realistic devices based on the generic model of a mechanical implementation of such a system consisting of four stages, with the frequencies of consecutive elements (vibration modes) having 2:1 ratios, as shown in Figure 3.1.

This work was done in close collaboration with Kamala Qalandar and Kim Turner of the University of California at Santa Barbara. The author was primarily responsible for the MEMS preliminary design and simulation, while the UCSB group fabricated and set up the testing of the device. The experimental testing of the device was done jointly. Much of the second half of this chapter is verbatim from the publication [46].

3.1 Designs

The design objectives for the subharmonic cascade are: a single mechanical structure, localized modes each with a modal nonlinearity and high Q, the desired modal coupling, simple modal frequency tuning, and capacitive based parametric driving of the highest mode in the cascade. The single mechanical structure is for simplicity and to guarantee the passive coupling between the modes. Due to the large number of modes in the subharmonic cascade, in order for a conceptual design (as opposed to, for example, an optimization based design), we need localized linear modes. Thin flexural-based beams



Figure 3.1 Schematic model of a mechanical implementation of a multi-stage frequency divider with four localized modes and coupling spring elements [53]. Driving element x_5 parametrically excites element x_4 , which parametrically excites element x_3 , and so on down to element x_1 , with frequency division by 2 at each stage. Resonant response amplitudes are dictated by system nonlinearities.

that each can represent a mode will provide the mode localization. The modal nonlinearity is necessary for amplitude saturation. The high Q is necessary for several reasons: a large response is usually necessary for a MEMS based readout in order to overcome the measurement noise floor and the effect of noise is weaker in a large amplitude solution, and, since we do not want to drive the system hard, we need to be able to destabilize the trivial solution for small forcing levels. Modal coupling nonlinearities come about for many reasons, and they often will have a different form depending on the geometry of



Figure 3.2 (a) A three stage "candy cane" frequency divider cascade and its linear mode shapes. (b) A four stage "T-bar" frequency divider cascade and its linear mode shapes. For each device, note that the beam ends are fixed and that the modes of interest have resonant frequency ratios of 2:1; mode shapes are computed using COMSOL Multiphysics®.

the device; in this case, due to the desired 2:1 parametric coupling, we need to ensure that the dominant modal coupling nonlinearity is of the quadratic form. In addition, due to fabrication tolerances, the proper alignment of the resonant frequencies will likely not occur in the nominal equilibrium state of the device so the resonators need to be tunable. The tuning must be small so that the integrity of the desired geometry which produces the desired nonlinearity is not affected. Finally, the mode localization allows for driving the first mode and being nearly tangent to all other modes. In this section we review two designs and discuss, qualitatively, their operation. We then look at the eigensystem, namely the linear modes, as calculated in COMSOL Multiphysics®software. We then simulate the excitation force in COMSOL Multiphysics®and discuss the results.

In practice, the schematic in Figure 3.1 is not feasible in a MEMS device for high frequency operation. Instead, more realistic implementations of the device are shown in Figure 3.2. Since the modes are localized on a particular beam, we will discuss the dynamics in terms of beam modes.

As seen in Figure 3.2 (a), the "candy cane" cascade is composed of the several beams

whose lengths increase by factors near $\sqrt{2}$ since the resonant frequencies need to be in a 2:1 ratio. It is clear that the eigenmodes are relatively localized, although there are small deflections in the connecting beams, which are semi-circular springs. The coupling springs are relatively weak so that the modes are spatially localized in the beams, but the coupling is also sufficiently strong to provide parametric pumping from one stage to the next. The eigenmodes have a mode shape similar to a beam with one fixed end and one end with a spring. The j^{th} mode is excited by the j-1 beam through a modulation of the j^{th} beam's length. This modulation of a length is the source of the parametric excitation. Since the coupling is passive, there is also back coupling present, which implies that a lengthening of the j^{th} happens twice per j mode oscillation and each lengthening has the same effect on the j - 1 beam. As shown in 3.2 (b), the "Tbar" cascade is also composed of several beams whose length increase by factors near $\sqrt{2}$. In contrast to the candy cane model, the Tbar divider is composed of modes that are nearly symmetric. Except for the first beam, the end that is connected to the previous mode' main beam is free to rotate, with a grounded spring attached. Due to the ability to rotate the j^{th} mode contains some small anti-symmetric deflections in the j - 1 beam. Even so, the individual modes can be approximated as fixed-pinned beams, since the j^{th} mode is still concentrated on the j^{th} beam. For each of these designs, each main modal beam will undergo midline stretching for large deflections, which results in a hardening cubic nonlinearity for each mode.

At this point, the devices satisfy the three of the desired criteria: a single mechanical structure, localized modes with a modal nonlinearity, and the desired modal coupling. The other criteria are satisfied via the surrounding electronic components, fabrication, and packaging, which are standard features of MEMS devices. In this design, we use electrostatic fields in both devices for driving, tuning, and readout. The devices are driven by exciting the first mode of the structure. In the candy cane cascade, the first beam can be driven directly at the frequency near ω_1 , with a fluctuating potential across a small gap along the length of the beam. The first beam can also be driven parametrically, by quasi-

statically exciting the 0^{th} beam near the frequency $2\omega_1$. The Tbar divider can only be excited directly, in this case, although the design can be slightly altered to also be driven parametrically.

3.2 Experimental Results

We realize the "candy cane" design, as shown in the SEM micrograph in Figure 3.3, using a single-crystal silicon structure fabricated with deep reactive ion etching (DRIE) and released with HF vapor. The microbeam elements are uniformly 10^{-m} in depth, 1.85^{-m} wide, and the beam lengths L_n are determined to satisfy the required 2:1 frequency ratios. Table 3.1 shows measured device parameters and characteristics. Note that only modes 1-3, which are localized in the response of beams 1-3, of the device shown in Figure 3.3 are considered since, in this particular device they exhibit the necessary modal frequency ratio alignments. As such, beam 5 is directly actuated and it directly drives beam 4 which acts as the parametric driver to beam 3, the highest frequency mode in this cascade. Finite element modeling, shown in Figure 3.4, is used to refine the basic device design and provide the desired modal frequencies and ratios.

	Beam	Design	Measured	Quality
	Length	Frequency	Frequency	Factor
	[µm]	[kHz]	[kHz]	Q
Mode 1	376	105	103	858
Mode 2	263	210	206	1035
Mode 3	183	420	413	1315

Table 3.1 Device parameters: beam lengths, designed and measured resonance frequencies, and measured quality factors (in vacuum). Direct excitation of each mode is achieved through the actuation beam shown in Figure 3.3, with signal $V_{dc} = 90$ V and $V_{ac} = 3-10$ V.

Actuation for both characterization and operation is provided through the beam shown at the top left in Figure 3.3, which is driven using capacitive plate actuation with an applied voltage $V_{in} = V_{dc} + V_{ac}\sin(\omega_d t)$. The electrostatic force f_e is approximated by an



Figure 3.3 SEM micrograph of the device, which spans $400 \text{ m} \times 400 \text{ m}$, with uniform depth of 10 m and feature size of 1.85 m.



Figure 3.4 The first three vibrational modes from a COMSOL Multiphysics® finite element model. Corresponding modal frequencies are given in Table 3.1.

effective parallel plate capacitance and assuming that $V_{ac} \ll V_{dc}$, the force is approximately linear in V_{ac} . The velocity and phase of the in-plane motion of a point on each beam are detected using a Polytec MSA-400 Laser Doppler Vibrometer (LDV). To direct the laser beam perpendicular to the in-plane motion, 45° angled mirrors are etched into anchored regions adjacent to the beams using FIB milling[62]. Modal parameters are characterized by sweeping ω_d for different levels of V_{ac} and measuring the response of individual beams, which requires a separate run for each mode due to the use of a single point laser. The semi-circular coupling beams allow direct excitation of each mode individually through the actuation beam. The modal frequencies and quality factors listed in Table 3.1 are measured at a pressure of 350 mTorr with drive signal of $V_{dc} = 90$ V and $V_{ac} = 3-10$ V.

In operation, a measurable response of mode *n* is observed when the amplitude of the drive signal is sufficiently large and the drive frequency is sufficiently close to $2\omega_n$. This induces a parametric resonant response in the corresponding beam. The regions in which these responses occur are mapped in the (V_{ac} , ω_d) parameter space by sweeping these input parameters for the three modes of interest (Figure 3.5). The instability regions observed in Figure 3.5 are not the usual single mode parametric resonance regions due to the fact that the modal responses include the dynamics of other modes. Specifically, mode *n* is parametrically driven by mode n+1, which, due to the complicated geometry, is in turn driven by both parametric and direct excitation from the drive signal. The various mode *n* instability wedges are dictated by these subtle effects. For example, modes 3 and 1 are parametrically driven by modes 4 and 2, respectively. Since both beams 4 and 2 are perpendicular to the drive beam, the drive signal will have a dominant parametric force and a much smaller direct forcing term on beams 4 and 2. In contrast, mode 2 is parametrically driven by mode 3, which has a parallel orientation to the drive beam and, as a result, the drive signal has a much larger direct force on beam 3 than either beams 4 or 2. As such, the effective parametric forcing on modes 3 and 1 is considerably smaller

than that on mode 2 which is why the second instability region is much larger than the other two instability regions.



Figure 3.5 Normalized modal parametric instability boundaries for modes 1, 2, and 3 with frequencies normalized by modal number *n*. Divide-by-eight operation is achieved in the shaded overlap region.

To determine the input conditions for which the entire cascade is activated, the required drive frequencies of the individual modal activation regions are normalized by the modal frequency and modal number n. The entire cascade activates when the drive amplitude V_{ac} and frequency are in the region delineated by the intersection of the three individual modal activation regions, shown shaded in Figure 3.5, resulting in frequency division by 2, 4 and 8 in sequential beam elements.



Figure 3.6 Measured amplitudes extracted from the spectrum at $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{8}$ of the drive frequency at a drive amplitude of 30 V_{ac}. Displacement values, taken from the envelope of the large amplitude solution, are measured using the LDV, with the laser directed near the point of maximum displacement of each beam. Due to modal coupling, the mode 3 (first to activate) amplitude drops when modes 1 and 2 are active.

Figure 3.6 shows the measured vibration amplitudes from beams 1-3 as the drive frequency is swept up across the range of interest. As the drive frequency crosses the left boundary of the instability region for mode *n*, a jump to a finite amplitude response of that mode (beam) is observed; conversely, the amplitude drops to the stable zero solution as the drive frequency crosses the right boundary. There is no observed bi-stability in this sweep. The response amplitudes are determined by modal frequency detuning mechanisms, such as mechanical and electrostatic nonlinearities captured by parameters γ_n in the model, as well as by the modal interactions described by the κ_n terms. Note that the observed drop in the mode 3 amplitude between 825 kHz and 826 kHz results from the activation of mode 1 in that range. Figure 3.7 shows the scaled FFT of the responses of the three beams in full cascade operation when the system is driven through the actuation beam with a signal of 34 V_{ac} at 824.6 kHz, which is well inside the full cascade activation region. In this response, vibration amplitudes reach up to 70% of the gap size.

3.3 Outlook

The results show demonstration of a multi-stage micro-mechanical frequency divider which takes an input signal and provides division by 2, 4 and 8 in different output elements. Ongoing work includes testing of packaged devices, experimental characterization of system nonlinearities, development of devices optimized for nonlinear response, and consideration of phase noise and power usage in comparison with more traditional devices. The phase noise of all passive, open loop, divider topologies will track the input frequency at very low frequencies, with a drop of 6dB per divide-by-two stage. A variety of designs that exhibit this type of frequency division in the UHF range are also in development, which are expected to have reduced power requirements when compared with digital-based dividers operating in that range. A SEM micrograph of a couple of these devices are shown in Figure 3.8.



Figure 3.7 Measured response for each mode (beam) for 824.6 kHz drive signal at $34 V_{ac}$, in the fully cascaded operating mode. Displacement values, taken from the envelope of the large amplitude solution, are measured using the LDV.



Figure 3.8 SEM micrographs of two alternative frequency divider designs: the Tbar (left) and ellipse-coupler (right) designs. The orange block represents the drive electrode. The first three modes are concentrated in the red, green, and aqua colored beams, respectively.

CHAPTER 4

COUPLED MODE FREQUENCY GENERATOR

In this chapter, our ultimate goal is to understand and describe how various noise sources contribute to phase noise in closed loop oscillators when the electromechanical frequency selective element operates with internal resonance, specifically, with two nonlinearly coupled vibration modes. The closed loop produces a self-sustained oscillation, which can be used as a frequency generator, that is, a clock. The performance of a clock is related to measure of the frequency fluctuations during operation. Note that this is also related to the phase fluctuations or the phase noise, but we will use the frequency fluctuations as the main matric for clock performance in this chapter. The preferred operation of such oscillators has the resonator element maintaining vibration in its linear regime, since nonlinear operation converts amplitude noise into frequency noise [18]. However, aggressive size and power requirements require smaller resonator elements, which necessitates consideration of nonlinear resonator effects. There have been two recent experimental demonstrations of reducing frequency fluctuations in oscillators based on nonlinear resonator behavior, one using a so-called parametric feedback oscillator in which the output has its frequency doubled and pumped back into the loop [67], and a device in which nonlinear activation of a secondary mode through a 1:3 internal resonance results in significant drop in frequency fluctuations [2]. In both cases, the main effect appears to be that the oscillator improves, in terms of frequency fluctuations, when the resonator element is coupled to a signal that is off from the main oscillator frequency. Although much recent work has been directed towards understanding how single mode nonlinear resonator frequency generators work [36, 28], coupled mode oscillators and the frequency fluctuation reduction observed in them have not been adequately theoretically explained and understood.

The contribution of the present work focuses on modeling and predictive analysis, done in close collaboration with David Czaplewski and Daniel Lopez from the Nanofabrication and Devices group at Argonne National Labs (ANL) who first observed internal resonance in a closed loop system [2], using the frequency selective resonator device shown in Figure 4.1. Relevant experimental results from ANL are interspersed throughout the chapter to demonstrate the validity and utility of our modeling and analytical efforts.

The analytical work is based on a normal form two-mode model that captures the experimental results observed in [2]. We begin with an analysis of the open loop response of a system with a 1:3 frequency ratio for the two modes, corresponding to the fundamental flexural and torsional modes in the ANL device. Open loop frequency sweep techniques and bifurcation diagrams are used to develop a quantitative model with parameter values that provide a good match with the experimental open response. Since the ANL device follows closely the dynamics of the 1:3 internal resonance model, we have confidence in the subsequent investigation of the closed loop system.

The closed loop model with the coupled mode resonator element provides a means of predicting the spectrum and the statistics of frequency fluctuations. The analysis provides a good qualitative match with experimental results, but a quantitative match is not possible without better models of the noise sources in the experimental device. The model also takes into account the important effect of feedthrough capacitance. A key to understanding the system response is a nonlinear version of anti-crossing, where the crossing occurs at the point of perfect tuning for the internal resonance. We conclude the chapter with a discussion of how one might improve the frequency fluctuation performance in this class of devices.

4.1 The Open Loop Analysis

4.1.1 The Open Loop Model

In this section, we provide a derivation and analysis of a two-mode model for the open loop response of the MEMS device under consideration when the primary mode is subjected to an external harmonic drive. The mechanical device, shown in Figure 4.1, is a clamped-clamped beam structure composed of three thin beams, and comb-finger paddles on each side that are used for transduction, namely, one for actuation and one for sensing. The parallel beams are designed to reduce the internal strain which is associated with thermo-elastic damping [33], which can cause additional noise terms. This device acts as a relatively simple clamped-clamped beam with a central lumped mass (the comb structures). This structure has many modes, but we are interested in the fundamental flexural mode at 65kHz, the primary mode of interest, and the fundamental torsional mode at slightly more than three times that frequency, at 198kHz. The stiffnesses of these modes are dominated by an elastic potential, with a small contribution from the electrostatic fields from the transduction mechanisms. Thus, we model the two vibration modes and their interaction using a potential, which will generally contains all possible cubic and quartic modal coupling terms. However, only some of these terms will be resonant, depending on the frequency ratio of the modal coupling, are they are determined by the normal form [20]. Here, we observe a 1:3 frequency ratio, which suggests the following for the interaction potential

$$V_{int} = \alpha x_1^2 x_2^2 + \beta x_1^3 x_2 \tag{4.1}$$

If one assumes symmetry in the electrostatic fields and the mechanical structure, the modes are pure flexure and torsion, and the only mechanism for a 1 : 3 modal interaction is through the $\alpha x_1^2 x_2^2$ term, which provides only a very narrow 2:3 parametric resonance



Figure 4.1 The clamped-clamped flexural-torsional modal coupling device. The fundamental flexural mode has a resonant frequency near 65kHz and the fundamental torsional mode has a resonant frequency near 198kHz. This device is driven and sensed through an interdigitated comb drive mechanism [2].

tongue [8]. A more viable model that matches the experiments includes a bias that creates asymmetry in the equations of motion, which can be represented by a simple linear coupling of flexure and torsion. This force can originate physically from many sources including the vertical asymmetry in the electrostatic transduction forces, from pretension in the flexural beam producing a vertical force on the structure, from vertical asymmetries due to the crystal orientation, or from other physical asymmetries. The resulting potential, with terms for the isolated modes and their coupling, has the form

$$\mathcal{V} = \left[\sum_{i=1}^{2} \frac{1}{2}\omega_{i}^{2}q_{i}^{2} + \frac{1}{4}\gamma_{i}q_{i}^{4}\right] + \beta_{q}q_{1}^{3}q_{2}$$
(4.2)

where the q_1 are modal coordinates. These coordinates are dominated by flexural and torsional motions of the device, but are combinations of the two, due to the asymmetry.

The open loop internally resonant response is governed by dynamic variables S_i , i = 1, 2, which represent modal amplitudes scaled by a some complex transduction gain, that is, $S_i = g_i q_i$, where S_i is the output signal and has units volts (*V*). From the Hamiltonian whose potential is described by Equation (4.2), the model for the 1:3 internal resonance is

then given by

$$\ddot{S}_{1} + 2\Gamma_{1}'\dot{S}_{1} + \omega_{1}^{2}S_{1} + \alpha_{1}'S_{2}^{2}S_{1} + \gamma_{1}'S_{1}^{3} + 3\beta_{1}'S_{1}^{2}S_{2} = F'\cos(\Omega t)$$
(4.3)

$$\ddot{S}_{2} + 2\Gamma'_{2}\dot{S}_{2} + \omega_{2}^{2}S_{2} + \alpha'_{2}S_{1}^{2}S_{2} + \gamma'_{2}S_{2}^{3} + \beta'_{2}S_{1}^{3} = 0$$
(4.4)

which includes coefficients for the individual modal damping, Γ_i , frequencies ω_i , and Duffing nonlinearities γ'_i , along with coefficients for nonlinear dispersive coupling α'_1 and α'_2 and modal coupling β'_i , and external drive of amplitude F' and frequency Ω . We refer to the forward and back coupling coefficients as β'_2 and β'_1 , respectively. While important in some applications [44], dispersive coupling is not required in the present analysis, so we take $\alpha'_1 = \alpha'_2 = 0$ in the subsequent development. The complex amplitudes associated with the S_i are more relevant to experimentally measured quantities, reflecting the amplitude and phase, or the quadratures, of the system response. As such, we utilize averaging to obtain a more useful dynamic system.

The complex amplitude equations are found by expressing $S_1 = s_1 \exp(i\Omega t) + c.c.$, $S_2 = s_2 \exp(i3\Omega t) + c.c.$, $\dot{S}_1 = i\Omega s_1 \exp(i\Omega t) + c.c.$, and $\dot{S}_2 = i3\Omega s_2 \exp(i3\Omega t) + c.c.$. Differentiating these and utilizing the constraints on (S_1, \dot{S}_1) and (S_2, \dot{S}_2) , we obtain conditions $\ddot{S}_1 = -\Omega^2 S_1 + 2i\Omega \dot{s}_1 \exp(i\Omega t)$ and $\ddot{S}_2 = -9\Omega^2 S_2 + i6\Omega \dot{s}_2 \exp(i3\Omega t)$. Substitution of these into Equations (4.3) and (4.4) and integrating over the timescale of the slowest period of oscillation, $T = 2\pi/\Omega$, we obtain the complex amplitude equations of interest:

$$\dot{s}_{1} = \left[-\Gamma_{1} - i \left(\delta \omega_{1} - \frac{\alpha_{1}'}{\Omega} |s_{2}|^{2} \right) \right] s_{1} + \frac{i}{2\Omega} 3\beta_{1}' s_{1}^{*(2)} s_{2} + i \frac{3}{2\Omega} \gamma_{1}' |s_{1}|^{2} s_{1} - \frac{iF'}{4\Omega}$$
(4.5)

$$\dot{s}_{2} = \left[-\Gamma_{2} - i \left(\delta \omega_{2} - \frac{\alpha_{2}'}{6\Omega} |s_{1}|^{2} \right) \right] s_{2} + \frac{\beta_{2}' i}{6\Omega} s_{1}^{3} + i \frac{3}{6\Omega} \gamma_{2}' |s_{2}|^{2} s_{2}$$
(4.6)

where

$$\delta\omega_1 = \frac{(\Omega^2 - \omega_1^2)}{2\Omega} \tag{4.7}$$

$$\delta\omega_2 = \frac{(9\Omega^2 - \omega_2^2)}{6\Omega} \tag{4.8}$$

represent the frequency detuning of the drive relative to the fundamental mode and also from the internal resonance condition. These model equations provide the basis for the analysis of the open loop system. Periodic steady states in which the primary mode responds at frequency Ω and the secondary mode responds are 2Ω are given by the constant solutions of Equations (4.5) and (4.6), denoted as (\bar{s}_1, \bar{s}_2) . The steady state amplitudes and phases are determined in the usual manner from the complex amplitudes. Furthermore, the local stability of a steady state (\bar{s}_1, \bar{s}_2) is determined in the usual manner using linearization of Equations (4.5) and (4.6) about (\bar{s}_1, \bar{s}_2) . As parameters are varied, bifurcations of (\bar{s}_1, \bar{s}_2) occur, and these correspond to qualitative changes in the steady-state response. Details about the method of averaging for harmonically driven systems can be found in many standard text on nonlinear vibrations, for example [20].

From the experimentally measured open loop response of the system, we utilize frequency sweep data and associated bifurcation diagrams in order to derive coefficients in equations (4.5) and (4.6) that provide a match between the model and the measured response. The results are found in Table 4.1. Here we review the model predictions and how they compare with the experimental results.

coeff	values	units
ω_1	$64580 \cdot 2\pi$	s^{-1}
ω_2	$3 \cdot 66533 \cdot 2\pi$	s^{-1}
Q_1	$7.2 \cdot 10^5$	-
Q ₂	$4 \cdot 10^{5}$	-
γ'_1	$2.45\cdot 10^{15}$	$V^{-2}s^{-2}$
γ_2	0	$V^{-2}s^{-2}$
β'_1	$2\cdot 10^{16}$	$V^{-2}s^{-2}$
β'_2	$3\cdot 10^{10}$	$V^{-2}s^{-2}$
$F_{1}'(20mV)$	$3.198 \cdot 10^3$	Vs^{-2}

Table 4.1 Model coefficients as measured from the open loop experimental results.

4.1.2 The Open Loop Frequency Sweeps

In Figure 4.2, four upward frequency sweeps are shown, with different drive amplitudes, with experimental data in blue and the model predictions in red. The experimental data was taken by fixing the drive amplitude and sweeping the drive frequency, and recording the mode 1 complex amplitude, s_1 , as a function of these drive parameters. Mode 1 is deeply into the non-linear regime before any modal interaction occurs. As a result, the lowest drive amplitude ($V_{in} = 20mV$), provides a sweep where mode 1 is governed by a single mode Duffing model. For modest drive amplitudes, $V_{in} = 60mV$ and $V_{in} =$ 200*mV*, mode 1 is affected by mode 2, saturating the upper frequency limit of the Duffing (upper) branch response. For these two drive levels, the model predicts that the upper branch solution continues after a small gap near the interaction frequency, although such a gap cannot be found by one-way frequency sweeps. For the frequency range plotted, the model predicts that the continued branch is unstable and stable for the drive amplitudes $V_{in} = 60mV$ and $V_{in} = 200mV$, respectively. Note that experimental observations can be made only for stable responses, and while unstable responses exist and important to the overall frequency response, they cannot be directly observed. Interestingly, for $V_{in} =$ 600mV, the solution is able to jump the frequency gap and continue on the stable upper branch solution. The reason for this is related to the structure of the solution stability, and is better understood by looking at bifurcation diagram, shown subsequently.

4.1.3 The Open Loop Bifurcation Diagram

In addition to the frequency response approach, we also consider the structure of the bifurcations as both the drive amplitude and frequency are varied. The resulting bifurcation diagrams clearly show the effects of modal coupling and form an important part of the present analysis. In the single mode case, the system experiences only stable saddle node bifurcations (SNB), that is SNB in which a stable and an unstable response merge and



Figure 4.2 Experimental (blue) and model (red) results of the open loop frequency sweep response for various drive amplitudes (20mV, 60mV, 200mV, 600mV), noting that the feedthrough capacitance modifies the experimental amplitude results for larger forcing levels. For the model results the dark red curve segments indicate a stable response and the light red segments indicate an unstable response.

annihilate one another. However, modal interactions caused by internal resonance gives rise to both stable and unstable SNBs and Hopf bifurcations (HB), where unstable bifurcations are associated with changes that involve only unstable response branches. We first review these bifurcations and then examine the bifurcation diagram for the present model.

We refer to the frequency sweeps, in Figure 4.2, to help explain the bifurcations. The formal criteria for identifying these bifurcations are reviewed in the Appendix, which are related to changes in the signs of the real parts of the eigenvalues of the model linearized about a steady state operating point. The appearance and disappearance of solutions is the hallmark of SNBs. As a result, when traversing the upper response branch (for any of

the frequency sweeps where $V_{in} < 600mV$), when the response jumps off the branch and decays to a lower branch response, this is due to a SNB that annihilates the upper branch. The SNB can also create solutions, such as the solutions after the "gap", as seen when $V_{in} > 20mV$. The HBs, on the other hand, appear in this system when solutions continue to exist but switch stability as a complex conjugate pair of eigenvalues cross the imaginary axis. There is a HB on the upper branch after the SNB bifurcation since that branch turns from unstable ($V_{in} = 60mV$) to stable ($V_{in} = 200mV$). The hallmark of a (supercritical) HB is that when the solution branch loses stability, a stable oscillatory response is created. This is a preliminary explanation for how the solution jumps the "gap" for $V_{in} = 600mV$, since after the initial SNB, as the frequency is increased, the response is first attracted to the oscillatory response, then returns to the upper branch solution after the HB restabilizes the upper branch. (Note that here an "oscillatory" response corresponds to a beating type response in the time domain, since it is the slowly varying amplitude and phase that oscillate.)

In addition to considering the four frequency sweep responses, we can combine the bifurcation data from many sweeps at different drive amplitudes into one plot, resulting in the bifurcation diagram of interest. Utilizing the parameters in Table 4.1, we see that the model predicts a bifurcation structure, as shown in Figure 4.3, consisting of stable and unstable SNB (solid and translucent green curves), and stable and unstable HB (solid and translucent red curves). Bifurcations from a stable response branch are shown as solid curves while those from unstable response branches, which are not experimentally observable, are shown as translucent curves. For these bifurcation diagrams, the horizontal axis is the frequency of operation and the vertical axis is the forcing voltage, so that, for a given open loop frequency sweep, a horizontal line is traced out. The lower green branch is the single mode standard Duffing SNB where the stable upper branch solution disappears by merging with an unstable response branch. The upper green branch is also a standard Duffing SNB, where the lower stable branch disappears by merging with the un-

stable branch, as one sweeps in the other direction. The *x* and *o* points indicated in Figure 4.3 are the measured points where the experimental device undergoes a SNB sweeping up or a HB sweeping up (jumping the gap) or sweeping down. Of course, only those bifurcations involving a stable response branch are observed. From this diagram is it clear that the modal interactions begin to occur above the AC drive voltage $V_{in} = 35mV$, and the SNB occurs near ω_2 , rather than at the valued predicted by the single mode Duffing model. This saturation in the SNB frequency is caused by the internal resonance and is captured by the nearly vertical segment of the solid green SNB line, which provides a close match with the experimental results. The SNB trend near ω_2 is sensitive to the non-linearity of the mode 2. The model here has a linear mode 2, for ease of calculation, but the data shows a slight softening trend.

As one approaches $V_{in} = 500mV$, as seen in the frequency sweep case for $V_{in} = 600mV$, the resonator is able to jump the gap and maintain on the upper amplitude response branch. This is because of the HB structure that appears at these higher voltages, as predicted.

With this successful model for the open loop system in hand, and confidence in the type of modal coupling it experiences, we turn our attention to a closed loop oscillator with this device as the frequency selective element. The goal is to understand how internally resonant coupled mode operation improves frequency fluctuation performance in these systems, and estimate the limits one can achieve using this mode of operation.

4.2 The Closed Loop Analysis

In this section, we investigate the dynamics of a closed loop oscillator that employs an internally resonant coupled mode system for its frequency selective element, with a focus on the system frequency stability characteristics. We develop a model and analytical results that explain the significant frequency fluctuation improvements reported in [2] and

[67] and allow one to predict how this effect depends on system parameters and noise characteristics. The device described in [2] is used to benchmark the analytical treatment, and the results are used to explore the limits of this approach for frequency stabilization. The experimental results are incorporated into the discussion to provide qualitative verification of the analytical predictions. Quantitative frequency fluctuation predictions are not available, due to the lack of quantitive measures of the various noise sources in the Argonne device, and the simplicity of the closed loop model, which ignores feedthrough capacitance and some nuances of the phase shifter.

The oscillator model is based on a simple oscillator loop consisting of the standard components, namely, an amplifier, and phase shifter, an amplitude limiter, and a resonator element, with the novel feature being that the resonator operates with two modes of vibration that are internally resonant and nonlinearly coupled, as discussed above. In the closed loop system the amplitude is fixed by the limiter, and the frequency of operation is set by this amplitude, the phase shift, and the resonator parameters. The dynamics of the deterministic system are used to compute the operating conditions, and the effects of small sources of noise in the resonator modes and the oscillator loop are determined by linearizing about these operating points. The predictions show that indeed one can tune the response to achieve dramatic reduction in the frequency fluctuations of the oscillator, which is ultimately limited only by the intrinsic noise of mode 2.

4.2.1 Overview of Oscillators with Nonlinear Resonator Elements

The open loop analysis given above provides important insight into the behavior of the closed loop system. The main difference between open loop and closed loop operation is the locking of the phase operating point; in open loop this phase takes on a value determined by the system dynamics, while in closed loop it is manipulated by the phase shifter. With the single mode linear open loop frequency response in mind, if the feedback phase is $\pi/2$, then the oscillator will operate at the peak of the response; this is also

true for the single mode nonlinear (Duffing) response. For a linear resonator, fluctuations in the drive amplitude will cause fluctuations in the amplitude of the response, but since the peak occurs at the same frequency for all amplitudes, these fluctuations will not affect the frequency. However, for a nonlinear resonator, both the response amplitude and frequency depend on the drive amplitude, so that amplitude fluctuations are converted into frequency fluctuations. This is the intuition behind the so-called amplitude to frequency (AF) fluctuations which often dominate the performance of oscillators that use a nonlinear resonator element, and why linear resonator elements are preferred for oscillator loops. Note that this AF conversion occurs for any feedback phase, not just one that sets the operating point at the peak.

We continue to use the open loop bifurcation diagram, Figure 4.3, to phenomenologically explain what happens in the closed loop. We are interested in what happens to the peak operating frequency, Ω_{peak} as we vary *F*, as would be the case for a feedback phase of $\pi/2$. While obtain the peak is easy in the linear regim, experimentally obtaining the peak in the nonlinear regime is difficult; it is far easier to obtain the bifurcations. Fortunately, the peak operating point for a nonlinear resonator is very close to the SNB that occurs on the upper branch of the Duffing response curve. We will subsequently use Ω_{SN} , denoting the driving frequency where the SNB occurs, in place of Ω_{peak} when in the nonlinear regime. When the system is operated in the single mode nonlinear regime, such as with V_{in} < 35mV in Figure 4.3, the AF conversion will be nearly identical to that of the single mode resonator. The AF conversion is measured local to the operating point by $d\Omega_{SN}/dF$. As such, considering the Duffing SNB branch in Figure 4.3, flatter response curves correspond to poor clock performance, while steeper response curves relate to better performance, due to their AF characteristics. For example, the AF conversion degrades as one operates further into the single mode non-linear regime. However, as V_{in} continues to increase, the resonator begins to operate in the coupled mode regime, and the bifurcations begin to have a very steep slope, suggesting that $d\Omega_{SN}/dF \approx 0$ for $V_{in} = 600mV$. This is analogous to the linear case when $d\Omega_{peak}/dF = 0$. This is known as a "sweet spot" operating condition, where AF conversion is significantly reduced [36, 18]. In addition, there is a considerable range of operating points, from 50mV to 1V, where $d\Omega_{SN}/dF$ is very small. Simulations of the ensuing analysis show this to be the case for the present system. The unique feature of the present model is that the sweet spot is generated by modal coupling, in contrast to the single mode approaches previously considered [36, 18].

Using sweet spot operation for cancellation of the AF conversion is only one effect of the nonlinear modal interaction. Another consequence of the interaction is that the frequency stability of the (undriven) mode 2 dominates the clock performance when operating in this regime. This is due to the mode 2 frequency ultimately controlling the frequency at which the modal interaction takes place; the interaction frequency is not determined at all by the fluctuations of the mode 1 frequency. The consequences of this are considerable: one can evade electronic and feedthrough capacitance noise completely, something that cannot be done in a single mode device.

In the next section, we provide an analysis of this closed loop system, showing how the coupled mode response improves frequency stability. This model and predictive analysis provide an essential step towards designing oscillators that take advantage of nonlinear modal coupling from internal resonance.

4.2.2 The Closed Loop Oscillator Model

By assuming that there is no feedthrough capacitance, and that the amplifier, phase shifter, and amplitude limiter are ideal, that is, their characteristics are static and independent of frequency and amplitude (an idealization made for simplicity in the analysis), small adjustments to the open loop model equations can represent the closed loop model. From Equations (.1) and (.2), we first set $\Omega \rightarrow \omega_1$ and $\phi_1 \rightarrow \phi_f$, where ϕ_f is the feedback phase, dictated by the phase shifter. Here, the equations assume that $|\Omega - \omega_1|/\omega_1$ is small. In contrast to the open loop equations, this closed loop model will cause a small deviations from the actual solution since ω_1 is not the actual operating frequency, whereas Ω is the operating frequency for the open loop. As long as Ω_{CL} , or the closed loop operating frequency, is near ω_1 , this error will be small. Also, the forcing amplitude *F* is now the set by the amplitude limiter, rather than by the amplitude of an external AC drive. In both cases, *F* is the amplitude of the drive signal as seen from the perspective of the resonator. Now, it is important to note that we are ignoring two aspects of the actual system in our model: phase shifter dependence on operating frequency and feedthrough capacitance. As such, we only expect qualitative agreement with experimental results.

Due to the qualitative nature of this model, we revert to looking at the modal equations with the unscaled potential for simplicity. This is in contrast to the open loop characterization, which accounts for transduction gains for the modal amplitudes. Under these assumptions, the model equations of motion for closed loop operation have the form

$$\ddot{q}_1 + 2\Gamma_1 \dot{q}_1 + \omega_1^2 q_1 + \gamma_1 q_1^3 + 3\beta q_1^2 q_2 = \mathcal{F}(q_1) + \xi_{a1}(t) + \xi_{m1}(t) q_1$$
(4.9)

$$\ddot{q}_2 + 2\Gamma_2 \dot{q}_2 + \omega_2^2 q_2 + \gamma_2 q_2^3 + \beta q_1^3 = \xi_{a2}(t) + \xi_{m2}(t)q_2$$
(4.10)

with q_i as the modal coordinates and noise sources, ξ_{ai} and ξ_{mi} , representing the additive and mulitplicative noises for the i^{th} mode, respectively. As in the open loop model, we assume that the feedback drive, $\mathcal{F}(q_1)$, acts only on mode 1. While the drive clearly acts on mode 2 at some level, it can be ignored since it is non-resonant, specifically at 1/3 of the resonant frequency of that mode. The drive is also dependent on the phase of q_1 through the phase shifter, and the limiter sets the feedback amplitude at *F*. The intrinsic and feedback noises are represented, to first order, as the additive and multiplicative noises, and stem from a variety of sources, including amplifier noise, feedthrough capacitance noise, intrinsic thermo-mechanical fluctuations, extrinsic thermal drift, and measurement noise. It is assumed that the overall noise additive and multiplicative noises are small relative to the feedback force. These noise sources ultimately act to produce oscillator frequency fluctuations, the focus of the present investigation. There are additional assumptions that are usually made on the noise sources for the sake of applying analytical tools, such assuming white noise with a flat power spectrum. While this is necessary for traditional methods of averaging [55, 49], recent work has suggested that the results are more broadly applicable to noises with a generic spectrum [12, 36].

Since we are interested in the frequency fluctuations over a long time period, it is convenient to convert these equations to complex amplitude equations using the rotating wave approximation, $q_1 = a_1 \exp(i\omega_1 t) + c.c.$ and $\dot{q}_1 = \omega_1 (a_1 \exp(i\omega_1 t) - c.c)$, and ignoring fast fluctuating terms (equivalent to the averaging procedure carried out for the open loop system):

$$\dot{a}_1 = -\Gamma_1 a_1 + \frac{i}{2\omega_1} 3\beta(a_1^*)^2 a_2 + i\frac{3}{2\omega_1}\gamma_1 |a_1|^2 a_1 - \frac{iFa_1}{4\omega_1 |a_1|} e^{i\phi_S} + \Xi_1(t, a_1)$$
(4.11)

$$\dot{a}_2 = -\Gamma_2 a_2 - i \frac{(9\omega_1^2 - \omega_2^2)}{6\omega_1} a_2 + \frac{\beta i}{6\omega_1} a_1^3 + i \frac{1}{2\omega_1} \gamma_2 |a_2|^2 a_2 + \Xi_2(t, a_2)$$
(4.12)

which, as mentioned above, are similar to the open loop Equations (4.5) and (4.6), with the already noted modifications. Note that the noise terms are converted in the averaging process, resulting in the terms Ξ_1 and Ξ_2 , which are considered in more detail below. Also, the averaging here is only a first order averaging, ignoring the stochastic drift terms. Since the noises are small, its effects are small, and including them in the analysis detracts from the simplicity in describing how the internal resonance affects the noise characteristics of the system.

The complex modal amplitudes are represented as $a_i = r_i \exp(i\phi_i)/2$. To solve for the steady state operating point, we transform the system to a four dimensional system with r_1 , r_2 , ϕ_1 , and ϕ_2 as the state variables. As it typical for internal resonances, the dynamic system actually depends on the modal amplitudes, r_i , and only the phase difference, $\phi_2 - 3\phi_2$, described here in the three-dimensional state vector $R = (r_1, r_2, \phi_2 - 3\phi_2)$. With

this notation the model equations can be expressed as

$$\dot{R} = f(R) + g(R)\Xi, \tag{4.13}$$

where the noises acting on the modal amplitudes and phases are represented as $\Xi = [\Xi_{r_1} \Xi_{r_2} \Xi_{\phi_1} \Xi_{\phi_2}]^T$, which consist of components near the resonant frequencies and DC parts of their spectra, respectively. Note here that f(R) is a three-to-three mapping and g(R) is a 4 × 3 matrix.

The deterministic steady state operating point about which the noise acts is found by setting the noise terms to zero and solving $f(\bar{R}) = 0$ for \bar{R} . (Fluctuations in the the steady state operating point caused by noise contribute only at higher orders due to the small noise assumption, and are therefore neglected.) The local stability of these steady states is determined by considering the local coordinate $\delta R = R - \bar{R}$ and linearization. The linearized model that includes the behavior of noise near \bar{R} is given by:

$$\delta \dot{R} = A(\bar{R})\delta R + B(\bar{R})\Xi(t) \tag{4.14}$$

Here, the matrices *A* and *B* are the Jacobians for the deterministic and stochastic forces, respectively, that is, they are the linearizations of f(R) and g(R) about \overline{R} . The deterministic local stability of \overline{R} is dictated in the usual manner by the real parts of the eigenvalues of the Jacobian matrix *A*. We next consider the deterministic operation of the oscillator, which sets the stage for the analysis for the noise response, which is considered at the end of the chapter.

4.2.3 The Deterministic Operating Frequency

The deterministic closed loop operating frequency, Ω_{CL} , which is the output of interest, is obtained by substitution of \bar{R} into the $\dot{\phi}_1$ equation and evaluating the expression $\Omega_{CL} = \omega_1 + \dot{\phi}_1$. Plots of the noise-free steady state operating frequency as a function of feedback amplitude *F*, from the model and for the experimental device, are shown in

Figure (4.5), showing the qualitative success of the model. Figure (4.5) indicates that in single mode operation the frequency changes as a function of amplitude, as determined by the cubic nonlinearity of mode 1. When $\Omega_{CL} \approx \omega_2/3$, however, the internal resonance stabilizes the oscillator frequency as a function of feedback amplitude, resulting in an additional response branch that coexists with the single mode response over a range of amplitudes beyond the $\Omega_{CL} \approx \omega_2/3$ point. On this branch the resonator frequency is stabilized to near 1/3 that of the mode 2 frequency and therefore depends on the mode 2 cubic nonlinearity. This is indicated by the three different curves in the inset, which represent different levels of modal softening in mode 2. This type of mode 2 dependency, as mentioned in the previous section, arises from an anti-crossing behavior, known to occur in systems with linear coupling. Here, that notion is generalized to nonlinear modal coupling. As shown in the figure, when the mode 2 cubic nonlinearity has an opposite sign to that of mode 1, and with sufficient strength, the "sweet spot" appears, where $d\Omega_{CL}/dF = 0$, which is analogous to the $d\Omega_{SN}/dF = 0$ condition in the bifurcation diagram of the open loop system. Note that there is a range of amplitudes/frequencies with two stable operating points, representing single mode and coupled mode responses, and the behavior will exhibit hysteresis as the amplitude *F* is varied up and down. The coupled mode regime is only reached by increasing F from a small value, while sweeping down from a large value will experience only the single mode branch, except for a small region near the anti-crossing point. We now turn our attention to the noise response.

4.2.4 The Oscillator Frequency Fluctuations

The eigenvalues of *A* not only determine the local stability of the operating point, they also dictate how the system is perturbed by small levels of noise. The real parts of the eigenvalues provide a relaxation time and the imaginary parts of the eigenvalues set the frequency at which the slow complex amplitude will spiral around the steady state operating point. It is not surprising, then, that the noise analysis relies heavily on *A*. Specifically, the behavior of closed loop frequency fluctuations due to the noises are found by looking at the spectrum of the model linearized for small noise near the deterministic operating point. We define the output for the linear system by $\dot{\delta}\phi$, which is the variation of the oscillator frequency from its deterministic value, Ω_{CL} . Using Equation (4.14) for the underlying dynamics, this output is governed by $\dot{\delta}\phi = C\delta R + D\Xi$ where *C* and *D* map the local deterministic (*R*) and noise (Ξ) dynamics onto the local frequency behavior. This is solved in the frequency domain in conjunction with Equation (4.14), yielding the transfer function between the noise spectra and the spectrum of the frequency. The resulting spectrum is given by

$$S_{y}(\omega) = M^{*T}(\omega)S_{\Xi}(\omega)M(\omega)$$
(4.15)

where $M = C(i\omega - A)^{-1}B + D$. The coefficients of $M = [m_{r_1} m_{r_2} m_{\phi_1} m_{\phi_2}]$ represent the weighting of how each of the noise sources, given by the elements of Ξ , map onto the output frequency.

To illustrate the predictive capabilities of this model, we focus on the how the *M* coefficients behave as a function of the feedback forcing amplitude, *F*, since the actual strengths of the spectrum S_{Ξ} , are highly system dependent, and not known for the Argonne device. While the model is capable of predicting the spectrum, to show the frequency stabilization due to internal resonance we consider only the long term effects of frequency stability by evaluating *M* at $\omega = 0$. In Figure 4.6, we consider two revealing quantities that demonstrate what happens when the system enters into the internal resonance regime: the contribution ratio of amplitude noise to frequency fluctuations, and the contribution ratio of mode 2 noise to mode 1 noise.

The amplitude noise to phase noise ratio is obtained by taking the collective noise weights from the modal amplitudes, $|m_{r_1}|$ and $|m_{r_2}|$, which is the AF contribution, and comparing them to the collective noise weights from the modal phases, $|m_{\phi_1}|$ and $|m_{\phi_2}|$. In the single mode regime, as expected, mode 1 is the dominant mode and the contribu-

tion of the mode 1 AF noise dominates. In fact, the AF contribution is several orders of magnitude larger than the phase noise contribution, and increases as a function of amplitude due to the increasing effect of the nonlinearity. This trend is expected based on our observations from the open loop, as discussed in the previous section. In the coupled mode regime, however, the AF noise contribution drops by several orders of magnitude and becomes comparable to the contribution from the phases. Experimentally, the improvement in frequency fluctuations qualitatively follows this trend as shown in the inset in Figure 4.6. The gray curves represent the various amounts of the mode 2 nonlinearity, as in Figure 4.5, which has little effect on the elimination of the amplitude to frequency conversion. Although one might expect the AF contribution to become zero at the "sweet spot", as indicated by the green point, there is a small AF contribution from mode 2.

The contribution of the two modal noises to the overall frequency stability of the device is measured by taking the collective noise weights of mode 2, $|m_{r_2}|$ and $|m_{\phi_2}|$, and comparing it to the collective noise weights of mode 1, $|m_{r_1}|$ and $|m_{\phi_1}|$. Not surprising, outside of the internal resonance regime, the mode 1 noise dominates the contribution to the frequency stability due to the large mode 1 AF contribution. However, as the system enters into internal resonance operation, we see that the contributions from the two modes become comparable. The most interesting case is that when one approaches the "sweet spot", the stability is dominated by mode 2. This does not occur unless there is a "sweet spot".

The consequences of this shift in the noise filtering of the system are only made clear when you consider the sources of the noise. Since mode 1 is the driven mode, the measurement, feedthrough, and feedback noises are all lumped into Ξ_{r_1} and Ξ_{ϕ_1} . If mode 2 is perfectly insulated from the electronics, then none of these noises will contribute to the mode 2 noises. As such, perhaps the most important contribution provided by this class of oscillator is that it has is the ability to evade all of the noises associated with the feedback loop. Consequently, it is possible to obtain an oscillator whose stability is governed by thermal (both internal and external) noises acting on mode 2. If the device was ovenized or temperature compensated, then one can operate at the fundamental intrinsic thermal limit, or thermo-mechanical noise limit, of the oscillator [5]. This assumes, how-ever, that the mode 2 noises from other sources are smaller than the thermo-mechanical noise, which would require careful engineering. In any case, this approach offers a new avenue for oscillator development.

4.3 Outlook

This chapter presented modeling and analysis of a frequency generator whose internal resonant characteristics stabilize its frequency fluctuations by two to three orders of magnitude. At the outset of this chapter, we characterized an open loop model that produces excellent quantitative agreement with the open loop experimental results obtained at ANL. We then expanded the resonator model to embody the dynamics of a simplified closed loop circuit. This closed loop circuit model shows excellent qualitative agreement with the closed loop experimental results in terms of both deterministic response and noise characteristics. Further work in this area includes alternative designs, such as that described in the Appendix, that allow for more thorough characterization of both modes, and more detailed examinations of the noise sources and feedback impedance in order to provide a better quantitative match. These will be crucial for development of devices optimized to take advantage of couple mode response.

In terms of the more general impact on oscillator development, this work demonstrates that the decrease in frequency fluctuations due to internal resonance comes from two sources: (i) a decrease in the AF noise conversion and (ii) an evasion of the feedback noise. It may be possible to design devices utilizing internal resonance that outperform traditional oscillators when the noise is dominated by either of these two sources.


Figure 4.3 Predicted saddle node (green curves, SNB) and Hopf bifurcations (red curves, HB) as a function of drive frequency and drive amplitude, indicating bifurcations from both stable (dark) and unstable (light) response branches. The experimentally measured SNB and HB are denoted by (x) and (o), respectively. The bottom plot is a blowup of the upper plot in the region of interest.



Figure 4.4 The feedback loop circuit producing a self-sustained oscillation.



Figure 4.5 The mean (noise-free) operating frequency as a function of feedback amplitude. (Left) The main graph shows predictions from the noise-free model. Left inset shows corresponding experimental results where red indicates sweeping upward in the amplitude and black indicates sweeping downward. The right inset shows a blow up of the frequency as a function of amplitude in the internally resonance regime for different levels of mode two cubic (Duffing) nonlinearity; the green point indicates zero slope, that is, a point of zero dispersion. Model parameters for this figure and subsequent figures are $\omega_2/\omega_1 \approx 3.067$, $Q_1 = 10^4$, $Q_2 = 10^4$, $\gamma_1 = 0.01$, $\beta = 0.00015$.



Figure 4.6 This figure shows two measures of the loop phase noise predicted from the model as the forcing amplitude is varied. The three traces from gray to black represent different levels of the mode two cubic nonlinearity, as in Figure 4.5. The upper plot is a measure of the ratio of the contribution of the total (modes one and two) amplitude noise to those of the total phase noise, and the lower plot is measure of the ratio of the contributions of mode one. We see that in all three cases there is a significant drop in the contributions of the amplitude noise and the overall mode one noise when in the internal resonance operating regime. The inset shows experimentally measure standard deviation of the frequency fluctuations as the amplitude is swept up (red) and down (black), showing the drop in frequency fluctuations when operating in the internal resonance regime.

CHAPTER 5

DESIGN OPTIMIZATION FOR NONLINEARITY

Results from the previous chapters describe applications where an internal resonance is utilized in a MEMS application. All the devices were designed in an ad hoc manner, with the required frequency ratios and modal coupling, but without any means of quantifying system parameters at the design stage. The purpose of this chapter is to describe a method for using finite element models to determine the linear and nonlinear modal parameters that arise from the kinetic energy and the elastic potential, and to use this characterization to optimize features of the nonlinear system response. This work will be important for next generation devices, and has applicability to devices that operate using one and two mode nonlinear response. This work was done in close collaboration with Suguang Dou and Jakob Jensen of the Technical University of Denmark.

The optimization routine and single mode example were done primarily by our DTU collaborators, and are included here for completeness. The author was primarily responsible for the general formulation of the normal forms from the nonlinear finite element models and the coupled mode example. Much of this chapter is verbatim from the forth-coming publication [14].

5.1 Characterization

In order to demonstrate the methodology we consider systems for which the dominant nonlinear effects arise from non-inertial conservative effects and we neglect the nonlinear inertial terms that might be important in, for example, cantilevered structures. This includes planar frame structures for which mid-plane stretching is the primary nonlinearity. For convenience in implementation, the derivation is formulated in matrix-vector form instead of tensor form. The full set of coefficients for individual mode nonlinear stiffness and for nonlinear modal coupling are first derived for the Hamiltonian of the system, and then the attendant essential coefficients in the reduced-order model are obtained in a straightforward manner.

The characterization theory is derived for general finite element models with quadratic and cubic nonlinearities arising from nonlinear strain-displacement relations. With the finite element discretization of the continuous structure, the Hamiltonian for the system, i.e. the sum of kinetic energy T and potential energy U, can be expressed as

$$\mathcal{H} = T + U = \frac{1}{2} \dot{\mathbf{u}}^T \mathbf{M} \dot{\mathbf{u}} + \sum_{e=1}^{N_e} \frac{1}{2} \int_{V_e} \boldsymbol{\epsilon}^T \boldsymbol{\sigma} \, dV$$
(5.1)

where **M** is the mass matrix of the finite element model, **u** is the global vector of nodal displacements, ϵ and σ are element-wise vectors of strain and stress components, respectively, and V_e indicates that the volume integration is performed within the element e. Assuming a linear strain-stress relation, the potential energy U can be written as

$$U = \sum_{e=1}^{N_e} \frac{1}{2} \int_{V_e} \epsilon^T \sigma \ dV = \sum_{e=1}^{N_e} \frac{1}{2} \int_{V_e} \epsilon^T \mathbb{C} \epsilon \ dV$$
(5.2)

where \mathbb{C} is a symmetric constitutive matrix. We may now divide the strain into a linear and nonlinear part as

$$\boldsymbol{\epsilon} = \mathbf{B}_0 \, \mathbf{u}^e + \frac{1}{2} \, \mathbf{B}_1(\mathbf{u}^e) \, \mathbf{u}^e \tag{5.3}$$

where \mathbf{B}_0 is the linear strain-displacement matrix and $\mathbf{B}_1(\mathbf{u}^e)$ is a function of the elementwise vector of nodal displacements \mathbf{u}^e . Substituting Eq. (5.3) into Eq. (5.2) and using the symmetry of \mathbb{C} , the potential energy U can be divided into three parts, representing its expansion at leading orders, as

$$U^{(2)} = \sum_{e=1}^{N_e} \frac{1}{2} \int_{V_e} (\mathbf{u}^e)^T \mathbf{B}_0^T \mathbb{C} \mathbf{B}_0 \mathbf{u}^e \, dV,$$

$$U^{(3)} = \sum_{e=1}^{N_e} \frac{1}{2} \int_{V_e} (\mathbf{u}^e)^T \mathbf{B}_0^T \mathbb{C} (\mathbf{B}_1(\mathbf{u}^e)) \mathbf{u}^e \, dV$$
(5.4)

$$U^{(4)} = \sum_{e=1}^{N_e} \frac{1}{8} \int_{V_e} (\mathbf{u}^e)^T (\mathbf{B}_1(\mathbf{u}^e))^T \mathbb{C} (\mathbf{B}_1(\mathbf{u}^e)) \mathbf{u}^e \, dV$$

The first step in setting up the modal equations is to solve the corresponding linear eigenproblem

$$(\omega^2 \mathbf{M} - \mathbf{K})\mathbf{\Phi} = \mathbf{0} \tag{5.5}$$

where **K** is the linear stiffness matrix of the finite element model, to yield a set of modal frequencies and mode shapes (ω_p , Φ_p); these are normalized w.r.t. **M** such that $\Phi_p^T \mathbf{M} \Phi_p = 1$ and $\Phi_p^T \mathbf{K} \Phi_p = \omega_p^2$. We then express the displacements as a superposition of N_m linear modes as

$$\mathbf{u}^e = \sum_{p=1}^{N_m} q_p \, \boldsymbol{\Phi}_p^e \tag{5.6}$$

where q_p are the modal coordinates and Φ_p^e are the element-wise mode shape vectors extracted from the global vector. Substituting Eq. (5.6) into Eq. (5.4), we obtain the strain energy expressed in terms of mode shapes and modal coordinates as

$$U^{(2)} = \sum_{i=1}^{N_m} \sum_{j=1}^{N_m} \alpha_{ij}^{(2)} q_i q_j,$$

$$U^{(3)} = \sum_{i=1}^{N_m} \sum_{j=1}^{N_m} \sum_{k=1}^{N_m} \alpha_{ijk}^{(3)} q_i q_j q_k$$

$$U^{(4)} = \sum_{i=1}^{N_m} \sum_{j=1}^{N_m} \sum_{k=1}^{N_m} \sum_{l=1}^{N_m} \alpha_{ijkl}^{(4)} q_i q_j q_k q_l$$
(5.7)

where the linear modal coupling coefficients $\alpha_{ij}^{(2)}$ and the non-linear modal coupling coefficients, $\alpha_{ijk}^{(3)}$ and $\alpha_{ijkl}^{(4)}$, are explicitly expressed as

$$\begin{aligned} \alpha_{ij}^{(2)} &= \sum_{e=1}^{N_e} \frac{1}{2} \int_{V_e} (\boldsymbol{\Phi}_i^e)^T \, \mathbf{B}_0^T \, \mathbb{C} \, \mathbf{B}_0 \, \boldsymbol{\Phi}_j^e \, dV \\ \alpha_{ijk}^{(3)} &= \sum_{e=1}^{N_e} \frac{1}{2} \, \int_{V_e} (\boldsymbol{\Phi}_i^e)^T \, \mathbf{B}_0^T \, \mathbb{C} \left(\mathbf{B}_1(\boldsymbol{\Phi}_j^e) \right) \, \boldsymbol{\Phi}_k^e \, dV \\ \alpha_{ijkl}^{(4)} &= \sum_{e=1}^{N_e} \frac{1}{8} \, \int_{V_e} (\boldsymbol{\Phi}_i^e)^T \, \left(\mathbf{B}_1(\boldsymbol{\Phi}_j^e) \right)^T \, \mathbb{C} \, \left(\mathbf{B}_1(\boldsymbol{\Phi}_k^e) \right) \, \boldsymbol{\Phi}_l^e \, dV \end{aligned}$$
(5.8)

Since we use the linear eigenmodes normalized with respect to the mass matrix, we have $\alpha_{ij}^{(2)} = 0$ for $i \neq j$ and $\alpha_{ii}^{(2)} = \omega_i^2/2$. With these modal coupling coefficients, we can now

write the Hamiltonian for the system in Eq. (5.1) in modal coordinates (q_i, p_i) (where $p_i = \dot{q}_i$) as

$$\mathcal{H} = \sum_{i=1}^{N_m} \left(\frac{1}{2} p_i^2 + \frac{1}{2} \omega_i^2 q_i^2 \right) + \sum_{i=1}^{N_m} \sum_{j=1}^{N_m} \sum_{k=1}^{N_m} \alpha_{ijk}^{(3)} q_i q_j q_k + \sum_{i=1}^{N_m} \sum_{j=1}^{N_m} \sum_{k=1}^{N_m} \sum_{l=1}^{N_m} \alpha_{ijkl}^{(4)} q_i q_j q_k q_l$$
(5.9)

where the relation $\frac{1}{2}p_i^2 = \frac{1}{2}(p_i\Phi_i)^T \mathbf{M}(p_i\Phi_i)$ has been used. With the Hamiltonian of the system, it is straightforward to derive a set of ordinary differential equations in modal coordinates as

$$\ddot{q}_p + 2\xi_p \omega_p \dot{q}_p + \omega_p^2 q_p + \sum_{i=1}^{N_m} \sum_{j=1}^{N_m} g_{ij}^p q_i q_j + \sum_{i=1}^{N_m} \sum_{j=1}^{N_m} \sum_{k=1}^{N_m} h_{ijk}^p q_i q_j q_k = f_p(t)$$
(5.10)

where $p = 1, ..., N_m$, q_p is the modal coordinate corresponding to Φ_p , and we have introduced the modal force f_p obtained from a projection of the load vector **f** in the full finite element model onto the p^{th} mode, i.e., $f_p = \Phi_p^T \mathbf{f}$, as well as modal damping ratios (damping as a ratio of critical damping) expressed as ξ_p . The modal coupling coefficients g_{ij}^p and h_{ijk}^p are

$$g_{ij}^{p} = \alpha_{pij}^{(3)} + \alpha_{ipj}^{(3)} + \alpha_{ijp}^{(3)}$$

$$h_{ijk}^{p} = \alpha_{pijk}^{(4)} + \alpha_{ipjk}^{(4)} + \alpha_{ijpk}^{(4)} + \alpha_{ijkp}^{(4)}$$
(5.11)

It is noted that the differential equation representation of Eq. (5.10) with the modal coupling coefficients in Eq. (5.11) is equivalent to the formulation obtained using the principle of virtual work in previous studies [61].

In this work we apply the framework to structures modeled by beam elements [15]. However, it should be noted that the characterization and optimization procedure in this paper works independent of the choice of elements.

5.2 Optimization and sensitivity analysis

In this section we present the general formulation for optimizing an objective function that depends on non-linear coefficients of interest for a given model. For example, we can



Figure 5.1 Initial (uniform width) design and the first two linear vibration modes obtained using COMSOL Multiphysics modal analysis. Left: linear vibration mode 1, right: linear vibration mode 2, with $\omega_2 \approx 2\omega_1$. Color indicates vibration amplitude.



Figure 5.2 Optimized design for maximizing the absolute value of the *e*ssential modal coupling coefficient and the two coupled linear vibration modes obtained using COMSOL modal analysis. Left: linear vibration mode 1, right: linear vibration mode 2 and $\omega_2 \approx 2\omega_1$. The color in online version indicates the vibration amplitude.

maximize/minimize the hardening behaviour in a clamped-clamped beam by maximizing/minimizing the coefficient of the cubic non-linearity.

For generality, we consider an objective function *c* that may be an explicit function of the nonlinear coefficients, as well as the eigenvectors and associated eigenvalues, and

formulate our optimization problem as the following minimization problem:

where the subscripts *i*, *j*, *k*, *l*, $p = 1, ..., N_m$. The optimization problem is subjected to a set of constraints associated with the beam shape parametrization: h_e is the thickness of a beam element which is bounded by $[h_{\min}, h_{\max}]$ in the optimization and ρ_e is the normalized design variable bounded between 0 and 1. In practice, the lower bound h_{\min} is dictated by fabrication tolerances, and the upper bound h_{\max} is used to keep the beam relatively slender. For coupled -mode resonators, such as the one treated in the second example of this paper, we impose an additional constraint such that the ratio of two associated eigenvalues stays in a sufficiently small neighbourhood of $n_1\omega_{p_1} = n_2\omega_{p_2}$, where, for internal resonance conditions, n_1 and n_2 are selected integers and p_1 and p_2 are the orders of the two modes of interest.

For efficient structural optimization we will use a gradient-based approach. The sensitivity of the objective function can be calculated by using direct differentiation [45, 29], but a more efficient approach for many design variables is the adjoint method [59] where we merely need to solve N_m groups of adjoint equations. To derive the adjoint equation, the objective function is first rewritten with adjoint variables λ_p and η_p as

$$c = c(\omega_p, \boldsymbol{\Phi}_p, \boldsymbol{\alpha}_{ijk}^{(3)}(\boldsymbol{\Phi}_p), \boldsymbol{\alpha}_{ijkl}^{(4)}(\boldsymbol{\Phi}_p)) + \sum_{p=1}^{N_m} \left[\lambda_p^T(\omega_p^2 \mathbf{M} \boldsymbol{\Phi}_p - \mathbf{K} \boldsymbol{\Phi}_p) + \eta_p(\boldsymbol{\Phi}_p^T \mathbf{M} \boldsymbol{\Phi}_p - 1) \right]$$
(5.13)

where it is noted that the terms in the two sets of parentheses that appear in the appended term both vanish identically. Differentiation of the objective function with respect to design variable ρ_e is then expressed as

$$\frac{\mathrm{d}c}{\mathrm{d}\rho_{e}} = \frac{\partial c}{\partial\rho_{e}} + \sum_{p=1}^{N_{m}} \left[\frac{\partial c}{\partial \Phi_{p}^{T}} \frac{\mathrm{d}\Phi_{p}}{\mathrm{d}\rho_{e}} + \frac{\partial c}{\partial\omega_{p}} \frac{\mathrm{d}\omega_{p}}{\mathrm{d}\rho_{e}} \right] + \sum_{i=1}^{N_{m}} \sum_{j=1}^{N_{m}} \sum_{k=1}^{N_{m}} \frac{\partial c}{\partial\alpha_{ijk}^{(3)}} \left(\frac{\partial \alpha_{ijk}^{(3)}}{\partial\rho_{e}} + \sum_{p=1}^{N_{m}} \frac{\partial \alpha_{ijk}^{(3)}}{\partial\Phi_{p}^{T}} \frac{\mathrm{d}\Phi_{p}}{\mathrm{d}\rho_{e}} \right) \\
+ \sum_{i=1}^{N_{m}} \sum_{j=1}^{N_{m}} \sum_{k=1}^{N_{m}} \sum_{l=1}^{N_{m}} \frac{\partial c}{\partial\alpha_{ijkl}^{(4)}} \left(\frac{\partial \alpha_{ijkl}^{(4)}}{\partial\rho_{e}} + \sum_{p=1}^{N_{m}} \frac{\partial \alpha_{ijkl}^{(4)}}{\partial\Phi_{p}^{T}} \frac{\mathrm{d}\Phi_{p}}{\mathrm{d}\rho_{e}} \right) + \sum_{p=1}^{N_{m}} \eta_{p} \left(2\Phi_{p}^{T}\mathbf{M}\frac{\mathrm{d}\Phi_{p}}{\mathrm{d}\rho_{e}} + \Phi_{p}^{T}\frac{\partial \mathbf{M}}{\partial\rho_{e}}\Phi_{p} \right) \\
+ \sum_{p=1}^{N_{m}} \lambda_{p}^{T} \left[\left(\omega_{p}^{2}\mathbf{M} - \mathbf{K} \right) \frac{\mathrm{d}\Phi_{p}}{\mathrm{d}\rho_{e}} + 2\omega_{p}\mathbf{M}\Phi_{p}\frac{\mathrm{d}\omega_{p}}{\mathrm{d}\rho_{e}} + \left(\omega_{p}^{2}\frac{\partial \mathbf{M}}{\partial\rho_{e}} - \frac{\partial \mathbf{K}}{\partial\rho_{e}} \right) \Phi_{p} \right]$$
(5.14)

Since the direct computation of $\frac{d\Phi_p}{d\rho_e}$ and $\frac{d\omega_p}{d\rho_e}$ is computationally expensive, the adjoint variables are selected such that the coefficients of $\frac{d\Phi_p}{d\rho_e}$ and $\frac{d\omega_p}{d\rho_e}$ vanish. This leads to the adjoint equations in terms of λ_p and η_p as

$$\frac{\mathrm{d}c}{\mathrm{d}\boldsymbol{\Phi}_p} + \left(\omega_p^2 \mathbf{M} - \mathbf{K}\right) \boldsymbol{\lambda}_p + \left(2\mathbf{M}\boldsymbol{\Phi}_p\right) \boldsymbol{\eta}_p = \mathbf{0}$$
(5.15)

$$\frac{\partial c}{\partial \omega_p} + \left(2\omega_p \mathbf{\Phi}_p^T \mathbf{M}\right) \boldsymbol{\lambda}_p \qquad = 0 \qquad (5.16)$$

where we use the symmetry of M and K, and

$$\frac{\mathrm{d}c}{\mathrm{d}\boldsymbol{\Phi}_p} = \frac{\partial c}{\partial\boldsymbol{\Phi}_p} + \sum_{i=1}^{N_m} \sum_{j=1}^{N_m} \sum_{k=1}^{N_m} \frac{\partial c}{\partial \alpha_{ijk}^{(3)}} \frac{\partial \alpha_{ijk}^{(3)}}{\partial \boldsymbol{\Phi}_p} + \sum_{i=1}^{N_m} \sum_{j=1}^{N_m} \sum_{k=1}^{N_m} \sum_{l=1}^{N_m} \frac{\partial c}{\partial \alpha_{ijkl}^{(4)}} \frac{\partial \alpha_{ijkl}^{(4)}}{\partial \boldsymbol{\Phi}_p} \tag{5.17}$$

In this case, the sensitivity of the objective function writes

$$\frac{\mathrm{d}c}{\mathrm{d}\rho_{e}} = \frac{\partial c}{\partial\rho_{e}} + \sum_{i=1}^{N_{m}} \sum_{j=1}^{N_{m}} \sum_{k=1}^{N_{m}} \frac{\partial c}{\partial\alpha_{ijk}^{(3)}} \frac{\partial\alpha_{ijk}^{(3)}}{\partial\rho_{e}} + \sum_{i=1}^{N_{m}} \sum_{j=1}^{N_{m}} \sum_{k=1}^{N_{m}} \sum_{l=1}^{N_{m}} \frac{\partial c}{\partial\alpha_{ijkl}^{(4)}} \frac{\partial\alpha_{ijkl}^{(4)}}{\partial\rho_{e}} + \sum_{p=1}^{N_{m}} \left[\lambda_{p}^{T} \left(\omega_{p}^{2} \frac{\partial \mathbf{M}}{\partial\rho_{e}} - \frac{\partial \mathbf{K}}{\partial\rho_{e}} \right) \mathbf{\Phi}_{p} + \eta_{p} \left(\mathbf{\Phi}_{p}^{T} \frac{\partial \mathbf{M}}{\partial\rho_{e}} \mathbf{\Phi}_{p} \right) \right]$$
(5.18)

It is noted that the two quantities $\frac{\partial \alpha_{ijk}^{(3)}}{\partial \Phi_p}$ and $\frac{\partial \alpha_{ijkl}^{(4)}}{\partial \Phi_p}$ are assembled from the corresponding element-wise quantities $\frac{\partial \alpha_{ijk}}{\partial \Phi_p^{(e)}}$ and $\frac{\partial \beta_{ijkl}}{\partial \Phi_p^{(e)}}$. For convenience of computational implementation, the adjoint equations are expressed in matrix form as

$$\begin{bmatrix} \left(\omega_p^2 \mathbf{M} - \mathbf{K}\right) & \left(2\mathbf{M}\boldsymbol{\Phi}_p\right) \\ \left(2\omega_p \boldsymbol{\Phi}_p^T \mathbf{M}\right) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_p \\ \boldsymbol{\eta}_p \end{bmatrix} = -\begin{bmatrix} \frac{\mathrm{d}c}{\mathrm{d}\boldsymbol{\Phi}_p} \\ \frac{\partial c}{\partial \omega_p} \end{bmatrix}$$
(5.19)

Based on the objective function and calculated sensitivities, the update of design variables is performed using the mathematical optimization tool called the method of moving asymptotes (MMA) [56], which solves a series of convex approximating subproblems. The algorithm has proven efficient for large-scale structural optimization. A new system analysis is then performed with the updated design variables. These steps are repeated until the design variables no longer change within some prescribed small tolerance. To summarize the procedure, a flow chart of the proposed optimization is displayed in Fig. 5.3. We now demonstrate the approach with examples.

5.3 Examples

Two examples are presented. The first is to maximize/minimize the cubic nonlinearity of the fundamental mode in a nonlinear resonator consisting of a clamped-clamped beam, a common element in MEMS applications. The second example shows how one can maximize the essential modal coupling nonlinearity in a T-bar frame with a 2:1 internal resonance, a structure that has been proposed as a MEMS frequency divider.

5.3.1 Single-mode resonator

A crucial feature of a nonlinear resonator is the hardening and softening behaviour associated with a given vibration mode. For a lightly damped single-mode resonator, we will focus on the hardening and softening behaviour of its free responses, i.e., without damping and external loads. This problem has been investigated from a structural dynamics/finite element perspective; see, for example, [60]. For its applicability in general structures, including symmetric structures like clamped-clamped beams and asymmetric



Figure 5.3 Flowchart of the structural optimization routine for nonlinear dynamic response.



Figure 5.4 Initial (uniform width) design of a clamped-clamped beam and its linear vibration mode. Color indicates vibration amplitude.

structures like arches or shells, the model development includes both quadratic and cubic terms. For single-mode resonator, the reduced order model based on a single linear mode is written as

$$\ddot{q}_p + \omega_p^2 q_p + g_{pp}^p q_p^2 + h_{ppp}^p q_p^3 = 0$$
(5.20)



Figure 5.5 Optimized design for maximizing the cubic nonlinearity of a clamped-clamped beam and its linear vibration mode. Color indicates vibration amplitude.

Figure 5.6 Optimized design for minimizing the cubic nonlinearity of a clamped-clamped beam and its linear vibration mode. Color indicates vibration amplitude.

with
$$g_{pp}^p = 3\alpha_{ppp}^{(3)}$$
 and $h_{ppp}^p = 4\alpha_{pppp}^{(4)}$. The frequency-amplitude relation is derived as

$$\omega_{\rm NL} = \omega_p (1 + \Gamma a^2) \tag{5.21}$$

where *a* is the amplitude of the corresponding linear mode, and the effective coefficient Γ is

$$\Gamma = \frac{3}{8} \frac{h_{ppp}^p}{\omega_p^2} - \frac{5}{12} \frac{(g_{pp}^p)^2}{\omega_p^4}$$
(5.22)

A more accurate model for the hardening/softening behaviour of these structures can be created with the dynamics projected onto a single nonlinear normal mode and the corresponding equation is then written as

$$\ddot{R}_{p} + \omega_{p}^{2}R_{p} + (A_{ppp}^{p} + h_{ppp}^{p})R_{p}^{3} + B_{ppp}^{p}R_{p}\dot{R}_{p}^{2} = 0$$
(5.23)

with the new coefficients A_{ppp}^{p} and B_{ppp}^{p} given as explicit functions of nonlinear modal coupling coefficients g_{ij}^{l} and h_{ijk}^{l} , [60]. Based on Eq. (5.23), the frequency-amplitude relation is derived as

$$\omega_{\rm NL} = \omega_p (1 + \Gamma^* A^2) \tag{5.24}$$



Figure 5.7 Evolution of the objective function and shapes encountered during the maximization process. The vertical axis is the absolute value of the objective function divided by its initial value and the horizontal axis is the iteration number.

where *A* is the amplitude of the corresponding nonlinear normal mode in curved, normal coordinates, and the effective coefficient Γ^* is

$$\Gamma^* = \frac{3 \left(A_{ppp}^p + h_{ppp}^p \right) + \omega_p^2 B_{ppp}^p}{8 \, \omega_p^2} \tag{5.25}$$

as approximated by averaging methods. Note that in this case Γ^* is not only linked to the p^{th} linear mode explicitly through ω_p and h^p_{ppp} and implicitly through g^p_{pp} which is in A^p_{ppp} and B^p_{ppp} , but also linked to other linear modes, whose contributions are taken into account through A^p_{ppp} and B^p_{ppp} .



Figure 5.8 Evolution of the objective function and shapes encountered during the minimization process. The vertical axis is the absolute value of the objective function divided by its initial value and the horizontal axis is the iteration number.

We find that for the flexural mode of a clamped-clamped beam with geometric nonlinearity from mid-plane stretching, the dominating coefficient is h_{ppp}^{p} and we can therefore simplify our optimization problem considerably by approximating Γ^* as:

$$\Gamma^* \approx \Gamma = \frac{3}{8} \frac{h_{ppp}^p}{\omega_p^2} = \frac{3}{2} \frac{\alpha_{pppp}^{(4)}}{\omega_p^2}$$
(5.26)

since $g_{pp}^p \equiv 0$, due to the symmetry. Based on the coefficient Γ^* in Eq. (5.26) and omitting

the constant factor, the objective function is now selected as

$$\min_{\rho_e} \quad c = \pm \frac{\alpha_{pppp}^{(4)}}{\omega_p^2} \tag{5.27}$$

where the plus/minus sign corresponds to minimizing/maximizing the hardening behaviour, respectively. This simplification is possible since we consider the fundamental mode of a clamped-clamped beam, which has a strictly hardening nonlinearity, that is $\alpha_{pppp}^{(4)} > 0$. Substituting the objective function *c* in Eq. (5.27) into Eq. (5.18), its sensitivity with respect to design variables ρ_e is obtained as

$$\frac{\mathrm{d}c}{\mathrm{d}\rho_e} = \pm \frac{1}{\omega_p^2} \frac{\partial \alpha_{pppp}^{(4)}}{\partial \rho_e} + \lambda_p^T \left(\omega_p^2 \frac{\partial \mathbf{M}}{\partial \rho_e} - \frac{\partial \mathbf{K}}{\partial \rho_e} \right) \mathbf{\Phi}_p + \eta_p \mathbf{\Phi}_p^T \frac{\partial \mathbf{M}}{\partial \rho_e} \mathbf{\Phi}_p$$
(5.28)

with adjoint variables λ_p and η_p solved from adjoint equation in Eq. (5.19) with

$$\frac{\mathrm{d}c}{\mathrm{d}\boldsymbol{\Phi}_p} = \pm \frac{1}{\omega_p^2} \frac{\partial \alpha_{pppp}^{(4)}}{\partial \boldsymbol{\Phi}_p}, \quad \frac{\partial c}{\partial \omega_p} = \pm 2 \frac{\alpha_{pppp}^{(4)}}{\omega_p^3} \tag{5.29}$$

where

$$\frac{\partial \alpha_{pppp}^{(4)}}{\partial \boldsymbol{\Phi}_{p}^{e}} = \frac{1}{2} \int_{V_{e}} (\boldsymbol{\Phi}_{p}^{e})^{T} \left(\mathbf{B}_{1}(\boldsymbol{\Phi}_{p}^{e}) \right)^{T} \mathbb{C} \left(\mathbf{B}_{1}(\boldsymbol{\Phi}_{p}^{e}) \right) dV$$
(5.30)

The beam has a fixed length *L* of 300 µm and a fixed out-plane width of 20 µm. The initial design has a uniform in-plane thickness of 4 µm, and is discretized with 400 beam elements as described in [15]. During shape optimization the in-plane thickness *h* is varied to tailor the cubic nonlinearity in the reduced order model. We set $h_{min} = 2 \mu m$, and $h_{max} = 6 \mu m$. The material properties are assumed for Si, that is, mass density $\rho = 2329 \text{ kg/m}^3$, and Young's modulus E = 170 GPa. The vibration modes of the initial design and two optimized designs are shown in Fig. 5.4, Fig. 5.5 and Fig. 5.6, respectively. Evolution of the objective function and shapes obtained during the evolution are shown in Fig. 5.7 and Fig. 5.8. In these optimizations, the objective function is increased by a factor of 13 and reduced by a factor of 4, respectively. The optimized designs are in accordance with the results in [15], obtained using the incremental harmonic balance

method, where we found the nonlinear strain energy due to mid-plane stretching reaches its local maximum around $x = \frac{1}{4}L$ and $x = \frac{3}{4}L$, which is precisely where the optimized structures are altered most significantly relative to their general thickness. Furthermore, the eigenfrequency of the first flexural mode decreases during optimization of maximizing the cubic nonlinearity, and increases during optimization of minimizing the cubic nonlinearity. This follows from the fact that the structure is made generally thinner when maximizing the nonlinearity, so that the critical sections can be made relatively thick, and the opposite trend occurs when minimizing the nonlinearity. It should be emphasized that, no guarantee can be made that the obtained designs are global optima and the found solutions will in general depend on the chosen initial design. However, repeated optimization runs reveal that the formulation is quite robust and we believe only a marginal improvement in performance could be possibly found with another design.

5.3.2 Coupled-Mode Resonator with Internal Resonance

We present an example of two-to-one internal resonance for which the normal form has a single important inter-modal coupling term. This example is motivated by a MEMS frequency divider that makes use of internal resonance, for which a general theory is presented in [53]. The structure considered is also similar to other proposed MEMS devices developed for filtering [68, 69]. In these structures two vibrational modes can exchange energy during free vibration. For convenience, we will refer to modes p_1 and p_2 as modes 1 and 2, respectively, in our discussion. The divider device consists of two localized modes with $\omega_{p_2} \approx 2\omega_{p_1}$, which provides a division by two in frequency, as measured in mode 1, when mode 2 is driven near its resonance. The term "localized mode" indicates that the dominant vibration associated with the mode occurs in a localized part of the structure, even though the entire structure is generally involved in the modal vibration. For instance, for the T-bar structure shown in Fig. 5.1, the vibration of mode 1 is localized in the vertical beam and the vibration of mode 2 is localized in the horizontal beam. In operation, a harmonic load with frequency close to ω_2 is applied to drive mode 2 into resonance. When the amplitude of mode 2 is sufficiently large, it will induce the vibration of mode 1 due to the parametric pumping, wherein the transverse vibration of the horizontal beam provides an axial force in the vertical beam, which in turn induces transverse motion of the vertical beam when the frequency of the horizontal beam is approximately twice that of the vertical beam. The response of the structure is governed by a model in which these two modes are coupled through resonant terms. In application, *n* such elements are linked to achieve division by 2^n ; division by eight has been experimentally achieved [46]. The reduced order dynamic model with its conservative dynamics projected onto two nonlinear normal modes in curved, normal coordinates is written as

$$\ddot{R}_{1} + \omega_{1}^{2}R_{1} + (g_{12}^{1} + g_{21}^{1})R_{2}R_{1} + (A_{111}^{1} + h_{111}^{1})R_{1}^{3} + B_{111}^{1}R_{1}\dot{R}_{1}^{2} + \left[(A_{212}^{1} + A_{122}^{1} + h_{122}^{1} + h_{212}^{1} + h_{221}^{1})R_{2}^{2} + B_{122}^{1}\dot{R}_{2}^{2} \right]R_{1} + \left(B_{212}^{1}R_{2}\dot{R}_{2}\right)\dot{R}_{1} = 0$$

$$\ddot{R}_{2} + \omega_{2}^{2}R_{2} + g_{11}^{2}R_{1}^{2} + (A_{222}^{2} + h_{222}^{2})R_{2}^{3} + B_{222}^{2}R_{2}\dot{R}_{2}^{2} + \left[(A_{112}^{2} + A_{211}^{2} + h_{112}^{2} + h_{211}^{2} + h_{211}^{2})R_{1}^{2} + B_{211}^{2}\dot{R}_{1}^{2} \right]R_{2} + \left(B_{112}^{2}R_{1}\dot{R}_{1} \right)\dot{R}_{2} = 0$$
(5.31)

where the explicit expressions of A_{ijk}^p and B_{ijk}^p are given in [60]. The key term of interest is that associated with the *e*ssential modal coupling coefficients g_{12}^1 , g_{21}^1 and g_{11}^2 , since they are the terms in the normal form that describes the resonant nonlinear coupling terms that promote energy exchange between the modes. These modal coupling coefficients can also be observed from the governing equation with its dynamics projected onto two linear modes, which is written as

$$\ddot{q}_{1} + \omega_{1}^{2}q_{1} + (g_{12}^{1} + g_{21}^{1})q_{1}q_{2} + g_{11}^{1}q_{1}^{2} + h_{111}^{1}q_{1}^{3} + \text{others} = f_{1}(t)$$

$$\ddot{q}_{2} + \omega_{2}^{2}q_{2} + g_{11}^{2}q_{1}^{2} + g_{22}^{2}q_{2}^{2} + h_{222}^{2}q_{2}^{3} + \text{others} = f_{2}(t)$$
(5.33)

where the most important coefficients are g_{12}^1 , g_{21}^1 and g_{11}^2 , between which there is an exact relation $g_{12}^1 + g_{21}^1 = 2g_{11}^2$. The reason for this exact relation is that the coefficients of q_1q_2 and q_1^2 in Eq. (5.33) arise from differentiation of the term $\beta q_1^2 q_2$ in the Hamiltonian \mathcal{H} with respect to q_1 and q_2 , where $\beta = \alpha_{112}^{(3)} + \alpha_{121}^{(3)} + \alpha_{211}^{(3)}$. In fact, this example is

particularly attractive for demonstrating the present approach since energy exchange between the modes can be described (to leading order) by this single nonlinear term. Also, as observed in the equation for q_1 in the reduced order model, β is essentially the amplitude of the parametric excitation provided to mode 1 from mode 2, given by $2\beta_1q_2q_1$, and is therefore related to the stability region of the linear parametric resonance of q_1 . The q_1^2 term in the equation for q_2 captures the back-coupling of the driven mode onto the driving mode, as required for passive coupling, which is beneficial in frequency dividers [46]. The combined effect of this coupling is the possibility of energy transfer between the modes when there is a two-to-one internal resonance.

The optimization problem for this resonance is therefore formulated as

$$\min_{\rho_e} \quad c = -\left| \alpha_{112}^{(3)} + \alpha_{121}^{(3)} + \alpha_{211}^{(3)} \right|$$
s.t. $|\omega_1/\omega_2 - 1/2| \le \epsilon$

$$(5.34)$$

where $\epsilon = 0.001$ and additional constraints are imposed as in Eq. (5.12). The sensitivity is computed with Eq. (5.18) written as

$$\frac{\mathrm{d}c}{\mathrm{d}\rho_{e}} = S \cdot \left(\frac{\partial \alpha_{112}^{(3)}}{\partial \rho_{e}} + \frac{\partial \alpha_{121}^{(3)}}{\partial \rho_{e}} + \frac{\partial \alpha_{211}^{(3)}}{\partial \rho_{e}} \right) + \sum_{p=1}^{2} \left[\lambda_{p}^{T} \left(\omega_{p}^{2} \frac{\partial \mathbf{M}}{\partial \rho_{e}} - \frac{\partial \mathbf{K}}{\partial \rho_{e}} \right) \mathbf{\Phi}_{p} + \eta_{p} \mathbf{\Phi}_{p}^{T} \frac{\partial \mathbf{M}}{\partial \rho_{e}} \mathbf{\Phi}_{p} \right]$$
(5.35)

where $S = -\text{sign} \left(\alpha_{112}^{(3)} + \alpha_{121}^{(3)} + \alpha_{211}^{(3)} \right)$. It is noted that there are two groups of adjoint variables, λ_1 and η_1 , λ_2 and η_2 , corresponding to the two vibration modes. These two groups of adjoint variables are solved using adjoint equation in Eq. (5.19) with p = 1, 2.

For a specific example the length of the horizontal beam is taken to be 300 µm and the length of the vertical beam is taken to be 195.5 µm, so that $\omega_2 \approx 2\omega_1$. The lengths of the two beams are fixed during the optimization. The initial in-plane thickness is uniformly 4 µm along both beams, and the in-plane thickness is bounded between 2 µm and 6 µm during the optimization. The material properties are the same as in example 1. An opti-

mized design and its two important vibration modes are displayed in Fig. 5.2. Evolution of the objective function and optimized designs over iterations are displayed in Fig. 5.9.



Figure 5.9 Evolution of the objective function and shapes encountered during the optimization process. The vertical axis is the absolute value of the objective function divided by its initial value and the horizontal axis is the iteration number.

In order to demonstrate the effects of tuning and optimizing the coupling nonlinearity in this example, we carry out simulations of the structure for different shapes encountered during the optimization iteration process. With this internal resonance and zero damping, free vibrations of the system will exhibit energy exchange between the modes in a beating type response, whose features depend crucially on the magnitude of the coupling coefficient and the initial conditions, as well as the other system parameters. For each structural shape and initial amplitude of the starting mode, the exchange of energy has a particular beat period; this period will be shorter for higher starting amplitudes and for larger values of the coupling coefficient, since both these effects enhance the nonlinear modal coupling. The left panel of Figure 5.10 shows a typical response obtained from the finite element model of the final (optimized) shape, obtained by initiating the response using the second linear mode at a moderate amplitude. The right panel of Figure 5.10 shows the normalized beat period for several values of the initial energy for three different designs, that is, for three different values of the coupling coefficient. The predicted trends are evident as both the coupling coefficient and initial amplitude are varied. Note that it would be also worthwhile to compare results from the finite element model with simulations and analysis of the reduced two mode model, but this requires a more detailed study of the effects of all nonlinear coefficients that vary during the optimization process.



Figure 5.10 Left: Typical time simulation with added envelope curves showing beats, for the final design with initial first mode amplitude of 7.5×10^{-11} (using eigenvectors normalized by the mass matrix). Right: Data points and fitted exponential curves for the beat period, normalized by the period of the first linear mode, versus the initial amplitude of the first mode, for designs corresponding to iteration numbers 0 (initial design, top curve), 30 (middle curve), and 250 (final design, bottom curve).

Evolution of the eigenfrequencies of linear vibration modes 1 and 2 encountered over iterations of the optimization process is displayed in Fig. 5.11(a). Other measures of interest for this system are (i) the degree of spatial energy localization in the vertical beam of the first vibration mode (note that from symmetry the second vibration mode has perfect localization) and (ii) the effectiveness of the horizontal beam in parametrically pumping the vertical beam in the second mode. The localization of the first mode is measured by the maximum transverse amplitude of the horizontal beam divided by the maximum transverse amplitude of the vertical beam. Likewise, the pumping effectiveness of the horizontal beam in the second mode is measured by the ratio of the transverse vibration at the midspan of the horizontal beam to the maximum transverse vibration of the same beam, which occurs near the quarter spans. The results in Fig. 5.11(b) show that the localization ratio decreases during optimization, which indicates improved localization, and the pumping ratio increases, which indicates enhanced coupling of the two modes. The intuition of the optimized design is that the axial deformation along the vertical beam increases with a larger end mass and a thinner cross section, both of them occur in the optimized design.

5.4 Outlook

This work provides a framework for shape optimization of a mechanical structure based on nonlinear coefficients of a reduced order model based on normal forms [20]. The first step is to extract the nonlinear coefficients of a reduced order model from a finite-element model of the mechanical structure. These nonlinear parameters are then available to use as objective functions for optimization routines. In the examples, we considered the shape optimization for minimizing and maximizing the Duffing nonlinearity associated with the fundamental mode of a clamped-clamped beam, and the in-plane flexural vibration of a frame with 1:2 internal resonance, in which the quadratic modal coupling coefficient



Figure 5.11 (a) Evolution of the eigenfrequencies of linear vibration modes 1 and 2 encountered during the optimization process. (b) Evolution of the pumping ratio and the mode localization ratio encountered during the optimization process.

is maximized. This approach creates designs that alter the relevant nonlinear coefficients by about an order of magnitude.

While this work focuses on mechanical nonlinearities that arise from a potential, future work is geared to adding nonlinearities due to inertial effects and electrostatic nonlinearities, as well as other relevant physics. In addition, although this work focused on maximizing the nonlinearity, future work will focus more on minimization of nonlinearity, thereby increasing the linear range of devices. This approach will also be useful when designing dynamic MEMS devices such as frequency dividers, oscillators, and disk resonator gyroscopes.

CHAPTER 6

CONCLUSION

As electronic and MEMS devices are pushed towards miniaturization, they are often driven in the nonlinear o'perating regime in order to obtain a good signal to noise ratio. Most previous studies of nonlinear behavior in MEMS resonators have focused on single mode operation. In this work, we demonstrate the use of nonlinear modal coupling in MEMS frequency generators and converters.

In Chapter 2, we examined the extent to which a conceptual model of a passive *N*-stage frequency divider could lead to a stable solution. This conceptual model consists of a chain of parametrically coupled resonator modes whose consecutive modes are arranged in a 2:1 frequency ratio. Since the energy is allowed to flow forwards and backwards in the cascade, one must consider the fully coupled response of the entire cascade, as opposed to merely considering sequentially driven individual stages. We found that the system could be tuned so that the cascade would settle to a steady state operation wherein all stages are active.

In Chapter 3, we developed MEMS designs that exhibit the frequency division characteristics that were described by the models in Chapter 2. The designs consist of perpendicular beams connected by springs, ultimately providing a network of localized modes with quadratic nonlinear modal interactions. The designs were fabricated and tested by our collaborators at University of California at Santa Barbara. The main results were a device that achieved successful division by 8. The devices did not exactly match the theory in Chapter 1 due to the parameter ratios not being as specified for the equal amplitude solution, and the modal coupling coefficients could not be quantified for the device. There is further work to be done in the design and characterization of these devices. Also, additional design topologies were fabricated, but have yet to be tested.

In Chapter 4, we investigated the role of internal resonance in stabilizing the frequency

fluctuations of a frequency generator. This work was motivated by the experimental observations from Argonne National Labs, as described in [2]. The internal resonance is associated with a modal frequency ratio of 1 : 3, where the fundamental mode is the driven mode. The resonator was characterized through an open loop investigation, and we found excellent agreement between the model and the experiments, in terms of coupled mode frequency response bifurcation diagrams. The closed loop model, which is more complex than the open loop, provides good a qualitative match with the experiments, and clearly indicates the model is sufficient to capture the phenomena for the frequency stabilization. To achieve quantitative agreement, one must have specific knowledge about the noise sources, which is lacking for the experimental device. The main result of this work is that the model shows that a frequency generator can move beyond the limits imposed by the measurement and feedback noise in the main loop, due to the nonlinear coupling to the second mode, which is nearly insulated from the actuation and detection and thus is not subjected to the noise felt by the driven mode. If the device is operated where there is a strong modal coupling, the precision of the clock is dependent on the noise associated with the secondary mode, which is limited by its own thermomechanical noise. Future work is necessary to reach this metric, which may be feasible, particularly by investigating systems with a 1:2 modal frequency ratio.

In Chapter 5, we developed and applied a tool that allows for shape optimization for nonlinear response, where the cost function is associated with nonlinear phenomena, specifically coefficients in a normal form for the resonance of interest. The first step was to extract the required normal form coefficient (or combination thereof) from a nonlinear computational model. This coefficient is maximized or minimized using a gradient-based optimization algorithm, by changing the geometry of the beam elements, specifically, the local beam thickness. Two examples wer considered: the Duffing nonlinearity in a clamped-clamped beam and the modal coupling coefficient for a 1:2 internal resonance of a T-frame structure, of the type developed as a frequency converter. The results show that one can increase/decrease the coefficient of interest by about an order of magnitude. The results from the modal coupling example were verified by examining modal energy exchange during transient motion using time-simulations of finite element models. These tools open up the possibility for shape optimization based on dynamic nonlinearities. Future work in this area includes extending the method to physics beyond mechanics, such as electrostatics and thermal quantities, and implementation in a variety of MEMS resonator applications.

In summary, the body of work described in this dissertation indicates that, in addition to providing rich dynamic behavior that is of academic interest, nonlinear modal coupling in MEMS holds promise for improving the performance of a variety of devices relevant to signal processing technologies.

APPENDIX

APPENDIX

In this Appendix, we provide supplementary material to support the results presented in Chapter 4. We begin with some solutions for a simple two resonator interaction with nonlinear modal coupling and review how to identify various bifurcations that appear in the open loop system as the drive strength and frequency are varied. Finally, we provide an alternative resonator design for future studies of internal resonances, one that has certain advantages over the Argonne device.

A.1 The 1 : 3 Modal Amplitude Equations

In this section, we derive the averaged equations, look at a specific case where the second mode is linear and has faster dynamics than the first mode, and obtain the closed form solution for the single mode Duffing resonator.

A.1.1 The Averaged Equations

The most basic equations of motion that capture the phenomena of interest, expressed in terms of the flexural and torsional degrees of freedom coordinates, S_i , are:

$$\ddot{S}_1 + 2\Gamma_1 \dot{S}_1 + \omega_1^2 S_1 + \alpha_1 S_2^2 S_1 + \gamma_1' S_1^3 + 3\beta_1' S_1^2 S_2 = F' \cos(\Omega t)$$
(.1)

$$\ddot{S}_2 + 2\Gamma_2 \dot{S}_2 + \omega_2^2 S_2 + \alpha_2 S_1^2 S_2 + \gamma_2' S_2^3 + \beta_2' S_1^3 = 0$$
(.2)

noting that in addition to the Duffing and 1 : 3 modal interaction nonlinearity, we also include the dispersive coupling nonlinearity with coefficients α_1 and α_2 ./ We ignore the effects of the frequency shift and noise contributions of the dispersive coupling terms, since they are not needed to capture the behavior of current interest.

The external drive clearly acts on all modes, but is assumed here to act only on the primary mode. This is justified when the drive acts dominantly on the primary mode and/or it is nonresonant with the secondary mode. The effects of the drive and the form of nonlinear modal coupling depend on the system details, including any symmetries, and must be considered on a case-by-case basis.

Approximate techniques, both analytical and/or numerical, are required to investigate the above set of coupled differential equations. It is convenient to consider the dynamics of the slowly varying complex amplitudes, which describe the modal amplitude and phase dynamics. To this end we express $S_1 = s_1 \exp(i\Omega t) + c.c.$, $S_2 = s_2 \exp(in\Omega t) + c.c.$, $\dot{S}_1 = i\Omega s_1 \exp(i\Omega t) + c.c.$, and $\dot{S}_2 = i\Omega s_2 \exp(i\Omega t) + c.c.$. Differentiating the above again and utilizing the constraints from \dot{S}_1 and \dot{S}_2 , we obtain $\ddot{S}_1 = -\Omega^2 S_1 + 2i\Omega \dot{s}_1 \exp(i\Omega t)$ and $\ddot{S}_2 = -9\Omega^2 S_2 + 6i\Omega \dot{s}_2 \exp(i3\Omega t)$. Substitution of these into Equations (.1) and (.2) and integrating over a timescale of the slowest period of oscillation, $T = 2\pi/\Omega$, we obtain complex amplitude equations:

$$\dot{s}_{1} = \left[-\Gamma_{1} - i \left(\delta \omega_{1} - \frac{\alpha_{1}'}{\Omega} |s_{2}|^{2} \right) \right] s_{1} + \frac{i}{2\Omega} 3\beta_{1}' s_{1}^{*2} s_{2} + i \frac{3}{2\Omega} \gamma_{1}' |s_{1}|^{2} s_{1} - \frac{iF'}{4\Omega} \quad (.3)$$

$$\dot{s}_{2} = \left[-\Gamma_{2} - i \left(\delta \omega_{2} - \frac{\alpha_{2}'}{6\Omega} |s_{1}|^{2} \right) \right] s_{2} + \frac{\beta_{2}' i}{6\Omega} s_{1}^{3} + i \frac{3}{6\Omega} \gamma_{2}' |s_{2}|^{2} s_{2}$$
(.4)

with detuning parameters defined as

$$\delta\omega_1 = \frac{(\Omega^2 - \omega_1^2)}{2\Omega} \tag{.5}$$

$$\delta\omega_2 = \frac{(3\Omega^2 - \omega_2^2)}{2\Omega} \tag{.6}$$

These equations are the normal form for this resonance and describe its dynamics, which are known to be very complicated, as demonstrated by the Argonne device.

A.1.2 A Special Case

In order to streamline the analysis, we assume that the second mode is linear, $\gamma'_2 = 0$, and ignore diffusive coupling, $\alpha'_1 = \alpha'_2 = 0$, so that the second mode complex amplitude is governed by

$$\dot{s}_2 = \left[-\Gamma_2 - i\delta\omega_2\right]s_2 + \frac{\beta_2'i}{6\Omega}s_1^3.$$
(.7)

The behavior of interest is captured by this simplified model with the benefit of a less complicated analysis, but at the expense of some accuracy when comparing the model to experiments. As shown in the main text, for example, the experimental results suggest that the second mode Duffing nonlinearity does come into play. Fortunately, the effect on the response is a predictable deviation from the system with a linear second mode. Another convenient simplification is to assume that $\omega_1\Gamma_1 << \omega_2\Gamma_2$, which implies that the second mode dynamics decays much faster than the first mode. In this case the second mode adiabatically tracks the first mode. This assumption provides a single degree of freedom model for the slow mode that complicated by coupling to the fast mode. This model provides the same steady-state amplitude structure as the model given in Equations (.3) and (.4), but it fails to capture Hopf bifurcations in the system. With these assumptions, we can solve for the steady state response of the second mode response in terms of the first mode, by assuming $\dot{s}_2 = 0$, yielding

$$s_2 = \frac{\beta_2^{\prime\prime}}{(i\Gamma_2 + \delta\omega_2)} s_1^3 \tag{.8}$$

where we define parameter β_2'' as

$$\beta_2'' = \frac{\beta_2'}{6\Omega(\Gamma_2^2 + \delta\omega_2^2)}.$$
(.9)

A model for the slow mode dynamics in this case can be obtained by substitution of Equation (.8) into Equation (.3), resulting in a dynamic model for s_1 . The steady state response of the first mode can then obtained from $\dot{s}_1 = 0$, and an equation for steady

state involving only s_1 is obtained by employing Equation (.8). This results in a fifth order equation for s_1 ,

$$\left[-\Gamma_{1} - i\delta\omega_{1}\right]s_{1} + \beta\left(-\Gamma_{2} + i\delta\omega_{2}\right)|s_{1}|^{4}s_{1} + i\frac{3}{2\Omega}\gamma_{1}'|s_{1}|^{2}s_{1} - \frac{iF'}{4\Omega} = 0$$
(.10)

where

$$\beta = \frac{n\beta_1'\beta_2''}{2\Omega} \tag{.11}$$

Re-organizing the terms, we obtain

$$\left[\left(-\Gamma_1 - \Gamma_2 \beta |s_1|^4 \right) - i\delta\omega_1 \right] s_1 + i \left[\frac{3}{2\Omega} \gamma_1' + \beta \delta\omega_2 |s_1|^2 \right] |s_1|^2 s_1 - \frac{iF'}{4\Omega} = 0.$$
 (.12)

This equation shows that the coupling term has the effect of renormalizing the effective damping and cubic nonlinearity of the first mode, resulting in higher order nonlinearities in stiffness and damping.

An expression for the mode 1 steady state real amplitude is found by letting $a_1^2 = |s_1|^2$, and solving for a_1 ,

$$a_{1}^{2}\left[\left(-\Gamma_{1}-\Gamma_{2}\beta a_{1}^{4}\right)^{2}+\left(-\delta\omega_{1}+\frac{3}{2\Omega}\gamma_{1}^{\prime}a_{1}^{2}+\beta\delta\omega_{2}a_{1}^{4}\right)^{2}\right]-\frac{F^{\prime2}}{16\Omega^{2}}=0$$
(.13)

which is a fifth order polynomial in a_1^2 .

A.1.3 The Single Mode Duffing Case

We can also reduce the above to the single mode duffing case by letting $\beta = 0$. Letting $z_1 = a_1^2$, the equation for the amplitude becomes

$$z_1 \left[(-\Gamma_1)^2 + \left(-\delta\omega_1 + \frac{3}{2\Omega}\gamma_1' z_1 \right)^2 \right] - \frac{F'^2}{16\Omega^2} = 0$$
 (.14)

which can be explicitly solved for $\Omega(z_1)$, providing two branches of solutions:

$$\Omega^{2} = \frac{2z_{1}(-2\Gamma_{1}^{2} + 3\gamma_{1}'z_{1} + \omega_{1}^{2}) \pm \sqrt{(F'^{2} - 16\Gamma_{1}^{2}z_{1}(-\Gamma_{1}^{2} + 3\gamma_{1}'z_{1} + \omega_{1}^{2}))z_{1}}}{2z_{1}}$$
(.15)

which map out the Duffing frequency response [39]. The phase is obtained from

$$\phi_1 = \tan^{-1} \left(\frac{-8\Gamma_1 \Omega}{-8\Omega \delta \omega_1 + 3\gamma_1' z_1} \right) \tag{.16}$$

This single mode response is useful for characterization since one can obtain the mode 1 parameters by operating the first mode away from the internal resonance condition.

A.2 Determining Stability and Bifurcation Criteria

In this section, we describe the stability and types of bifurcations of the steady state responses in the four-dimensional dynamical system representing the slowly varying amplitudes and phases of the modes in the open loop response, Equations (.3) and (.4). This analysis can be found in standard texts such as in [21], and was used to generate the bifurcation curves shown in Figure 4.3.

The stability of the solutions is determined by the eigenvalues of the Jacobian, J, computed for the four dimensional system with states u, $u^T = [r_1, r_2, \phi_1, \phi_2]$, where $a_i = r_i/2$, the real amplitudes. The Jacobian describes the linear dynamics for small perturbations about a given steady state operating point \bar{u} , specifically, $\delta u = J\delta u$, where $\delta u = u - \bar{u}$. The characteristic equation of the Jacobian is $Det(\lambda I - J) = 0$, which is a fourth order polynomial and has the following form

$$p(\lambda) = \lambda^4 + p_3 \lambda^3 + p_2 \lambda^2 + p_1 \lambda + p_0 \tag{.17}$$

The characteristic polynomial can be written in terms of the trace, tr(J), and determinant, det(J), as:

$$p_0 = \det(J) \tag{.18}$$

$$p_1 = -\frac{1}{3} \left(tr(J^3) - \frac{3}{2} tr(J^2) tr(J) + tr(J)^3 \right)$$
(.19)

$$p_2 = \frac{1}{2} \left(tr(J)^2 - tr(J^2) \right)$$
(.20)

$$p_3 = -tr(J) \tag{.21}$$

When $Re[\lambda_j] < 0 \ \forall j, \bar{u}$ is locally stable. When there is one or more $Re[\lambda_j] > 0, \bar{u}$ is unstable. The transition between stable and unstable conditions correspond to bifurcations of \bar{u} .

The modal coupling gives rise to additional bifurcations, beyond the simple saddle node bifurcations (SNB) that occur in the usual Duffing model. As one sweeps through a SNB, the response jumps from the stable branch and will be attracted to another response, generally far away in phase space. SNBs correspond to a zero eigenvalue and are thus determined from the characteristic polynomial in Equation .17 using the condition:

$$p_0 = \det(J) = 0 \tag{.22}$$

Since this is SNB in a four dimensional phase space, we need to consider the stability of the bifurcating response, since only SNBs from stable response branches will be observable in the experiments, although both types contribute to the overall bifurcation diagram. For a stable branch to exist the polynomial will need to satisfy the Routh-Hurwitz criterion, which states that all $p_i \ge 0$, in addition to

$$\Delta_3 = p_1(p_2p_3 - p_1) - p_3^2 p_0 \ge 0 \tag{.23}$$

We are also interested in Hopf bifurcations (HB) since these occur in the open loop system as it sweeps through the internal resonance region. For a super(sub)-critical HB the steady state response branch will (de)stabilize and create a(n) (un)stable limit cycle. The conditions for a HB can be determined by expanding the assumed form of the characteristic equation, $(\lambda + \Omega_H^2)(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$, where Ω_H is the Hopf frequency, and where $\lambda_H = 0 + i\Omega_H$, yielding conditions $\Omega_H = p_1/p_3$ and $\Delta_3 = 0$. As for the SN bifurcations, only HBs from stable responses will be visible in the experiment.



Figure .1 (a) The mechanical device along with the two mechanical modes and the eigenfrequencies as computed in COMSOL. (b) The layout with the red region being the mechanical device, the dark green indicating the drive and sense electrodes for mode 1, and the light green electrodes allow for modal tuning and coupling bias.

A.3 An Alternative 1 : 3 Internally Resonant Device Design

Figure .1 shows the layout of a flexural-flexural in-plane device with dynamic features similar to those of the ANL device, namely, a 1:3 internal resonance and nonlinear modal coupling. The linear mode shapes shown are for the mechanical structure without electrostatic transduction or tuning effects. These modes have distinct symmetries and are, of course, uncoupled. The light green electrodes can apply different types of asymmetric lateral bias forces, which result in in changes in the linear modes and thus in the nonlinear coupling, as described above. In contrast to the bias and coupling in the flexural-torsional device, in the proposed design thes effects are tunable. This is just one of several possible designs that allow for controllable parameters in internal resonance. COMSOL simulations verify the existence of the tunable nonlinear coupling.

BIBLIOGRAPHY
BIBLIOGRAPHY

- [1] Dynamics-enabled frequency sources. Technical report, DARPA, August 2009.
- [2] Dario Antonio, Damián H Zanette, and Daniel López. Frequency stabilization in nonlinear micromechanical oscillators. *Nature Communications*, 3:806, 2012.
- [3] A. Chenakin. Frequency synthesis: Current solutions and new trends. *Microwave Journal*, 50(5):256, 2007.
- [4] Y.H. Chuang, S.H. Lee, R.H. Yen, S.L. Jang, J.F. Lee, and M.H. Juang. A wide locking range and low voltage CMOS direct injection-locked frequency divider. *Microwave* and Wireless Components Letters, IEEE, 16(5):299–301, 2006.
- [5] AN Cleland and ML Roukes. Noise processes in nanomechanical resonators. *Journal of Applied Physics*, 92(5):2758–2769, 2002.
- [6] P. Collet and J.P. Eckmann. *Iterated maps on the interval as dynamical systems*. Birkhäuser Boston, 1980.
- [7] MRM Crespo da Silva and CC Glynn. Nonlinear flexural-flexural-torsional dynamics of inextensional beams. i. equations of motion. *Journal of Structural Mechanics*, 6(4):437–448, 1978.
- [8] MRM Crespo Da Silva and CL Zaretzky. Nonlinear flexural-flexural-torsional interactions in beams including the effect of torsional dynamics. i: Primary resonance. *Nonlinear Dynamics*, 5(1):3–23, 1994.
- [9] S. Daneshgar, O. De Feo, and M.P. Kennedy. Observations concerning the locking range in a complementary differential lc injection-locked frequency divider: part i: qualitative analysis. *IEEE Transactions on Circuits and Systems Part I: Regular Papers*, 57(1):179–188, 2010.
- [10] P. Danzl and J. Moehlis. Weakly coupled parametrically forced oscillator networks: existence, stability, and symmetry of solutions. *Nonlinear Dynamics*, 59(4):661–680, 2010.
- [11] B.E. DeMartini, J.F. Rhoads, K.L. Turner, S.W. Shaw, and J. Moehlis. Linear and nonlinear tuning of parametrically excited mems oscillators. *Journal of Microelectromechanical Systems*, 16(2):310–318, 2007.
- [12] Alper Demir. Phase noise and timing jitter in oscillators with colored-noise sources. *Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on*, 49(12):1782–1791, 2002.
- [13] Alper Demir, Amit Mehrotra, and Jaijeet Roychowdhury. Phase noise in oscillators: a unifying theory and numerical methods for characterization. *Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on*, 47(5):655–674, 2000.

- [14] S. Dou, B.S. Strachan, S.W. Shaw, and J.S. Jensen. Structural optimization for nonlinear dynamic response. *Philosophical Transactions of the Royal Society of London. Series A*, 2015.
- [15] Suguang Dou and Jakob Søndergaard Jensen. Optimization of nonlinear structural resonance using the incremental harmonic balance method. *Journal of Sound and Vibration*, 334:239 254, 2015.
- [16] Michael M. Driscoll. Phase noise performance of analog frequency dividers. *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control*, 37:295–301, 1990.
- [17] MI Dykman and MA Krivoglaz. Classical theory of nonlinear oscillators interacting with a medium. *physica status solidi* (*b*), 48(2):497–512, 1971.
- [18] MI Dykman, R Mannella, Peter VE McClintock, SM Soskin, and NG Stocks. Noiseinduced narrowing of peaks in the power spectra of underdamped nonlinear oscillators. *Physical Review A*, 42(12):7041, 1990.
- [19] M.J. Feigenbaum. Universal behavior in nonlinear systems. *Physica D: Nonlinear Phenomena*, 7(1-3):16–39, 1983.
- [20] J. Guckenheimer and P. Holmes. *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields,* volume 42. Springer-Verlag New York, 1983.
- [21] John Guckenheimer and Philip Holmes. *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*. New York Springer Verlag, 1983.
- [22] A. Hajimiri. Noise in phase-locked loops [invited]. *Proc. SSMSD*, 1:1–6, 2001.
- [23] Robert G Harrison. A broad-band frequency divider using microwave varactors. *Microwave Theory and Techniques, IEEE Transactions on,* 25(12):1055–1059, 1977.
- [24] Zainabolhoda Heshmati, Ian C Hunter, and Roger D Pollard. Microwave parametric frequency dividers with conversion gain. *Microwave Theory and Techniques, IEEE Transactions on*, 55(10):2059–2064, 2007.
- [25] Ville Kaajakari et al. Practical mems: Design of microsystems, accelerometers, gyroscopes, rf mems, optical mems, and microfluidic systems. *Las Vegas, NV: Small Gear Publishing*, 2009.
- [26] Eyal Kenig, MC Cross, LG Villanueva, RB Karabalin, MH Matheny, Ron Lifshitz, and ML Roukes. Optimal operating points of oscillators using nonlinear resonators. *Physical Review E*, 86(5):056207, 2012.
- [27] Christopher L Lee and Noel C Perkins. Nonlinear oscillations of suspended cables containing a two-to-one internal resonance. *Nonlinear Dynamics*, 3(6):465–490, 1992.
- [28] Thomas H Lee and Ali Hajimiri. Oscillator phase noise: a tutorial. *Solid-State Circuits*, *IEEE Journal of*, 35(3):326–336, 2000.

- [29] Li Li, Yujin Hu, and Xuelin Wang. Design sensitivity and hessian matrix of generalized eigenproblems. *Mechanical Systems and Signal Processing*, 43(1-2):272–294, 2014.
- [30] Konstantin Konstantinovic Licharev. *Dynamics of Josephson junctions and circuits*. CRC Press, 1986.
- [31] R. Lifshitz and M. C. Cross. Response of parametrically driven nonlinear coupled oscillators with application to micromechanical and nanomechanical resonator arrays. *Physical Review B*, 67, 2003.
- [32] R. Lifshitz and M.C. Cross. Nonlinear dynamics of nanomechanical and micromechanical resonators. *Review of Nonlinear Dynamics and Complexity*, 1:1–52, 2008.
- [33] Ron Lifshitz and Michael L Roukes. Thermoelastic damping in micro-and nanomechanical systems. *Physical review B*, 61(8):5600, 2000.
- [34] Robert Lutwak, William J Riley Jr, and Kenneth D Lyon. Mems analog frequency divider, August 12 2003. US Patent 6,605,849.
- [35] Robert J Matthys. Crystal oscillator circuits. *New York, Wiley-Interscience, 1983, 244 p., 1, 1983.*
- [36] Nick Miller. Noise in nonlinear micro-resonators, 2012.
- [37] N.J. Miller and S.W. Shaw. Frequency sweeping with concurrent parametric amplification. ASME, 2008.
- [38] DT Mook, RH Plaut, and N HaQuang. The influence of an internal resonance on non-linear structural vibrations under subharmonic resonance conditions. *Journal of Sound and Vibration*, 102(4):473–492, 1985.
- [39] A. H. Nayfeh. *Nonlinear Interactions: Analytical, Computational, and Experimental Methods*. Wiley Series in Nonlinear Science. Wiley-Interscience, New York, 2000.
- [40] A. H. Nayfeh and D. T. Mook. *Nonlinear Oscillations*. Wiley-Interscience, 1979.
- [41] Ali H Nayfeh and Dean T Mook. Nonlinear oscillations. Wiley. com, 2008.
- [42] TA Nayfeh, W. Asrar, and AH Nayfeh. Three-mode interactions in harmonically excited systems with quadratic nonlinearities. *Nonlinear Dynamics*, 3(5):385–410, 1992.
- [43] CT-C Nguyen. Mems technology for timing and frequency control. Ultrasonics, Ferroelectrics and Frequency Control, IEEE Transactions on, 54(2):251–270, 2007.
- [44] Sarah H Nitzan, Valentina Zega, Mo Li, Chae H Ahn, Alberto Corigliano, Thomas W Kenny, and David A Horsley. Self-induced parametric amplification arising from nonlinear elastic coupling in a micromechanical resonating disk gyroscope. *Scientific reports*, 5, 2015.

- [45] Niels L. Pedersen. Design of cantilever probes for atomic force microscopy (afm). *Engineering Optimization*, 32(3):373–392, 2000.
- [46] KR Qalandar, BS Strachan, B Gibson, M Sharma, A Ma, SW Shaw, and KL Turner. Frequency division using a micromechanical resonance cascade. *Applied Physics Letters*, 105(24):244103, 2014.
- [47] J. F. Rhoads, S. W. Shaw, K. L. Turner, J. Moehlis, B. E. DeMartini, and W. Zhang. Generalized parametric resonance in electrostatically actuated microelectromechanical oscillators. *Journal of Sound and Vibration*, 296(4-5):797–829, 2006.
- [48] Jeffrey F. Rhoads, Steven W. Shaw, and Kimberly L. Turner. Nonlinear dynamics and its applications in micro- and nanoresonators. *Journal of Dynamic Systems, Measurement, and Control*, 132(3):034001, 2010.
- [49] Hannes Risken. Fokker-Planck Equation. Springer, 1984.
- [50] Michael Rosenblum and Arkady Pikovsky. Synchronization: from pendulum clocks to chaotic lasers and chemical oscillators. *Contemporary Physics*, 44(5):401–416, 2003.
- [51] Kaushik Sengupta, TK Bhattacharyya, and Hossein Hashemi. A nonlinear transient analysis of regenerative frequency dividers. *Circuits and Systems I: Regular Papers*, *IEEE Transactions on*, 54(12):2646–2660, 2007.
- [52] Jinsiang Shaw, Steven W Shaw, and Alan G Haddow. On the response of the nonlinear vibration absorber. *International Journal of Non-Linear Mechanics*, 24(4):281–293, 1989.
- [53] B. Scott Strachan, Steven W. Shaw, and Oleg Kogan. Subharmonic resonance cascades in a class of coupled resonators. *Journal of Computational and Nonlinear Dynamics*, 8(4):041015, 2013.
- [54] B Scott Strachan, Steven W Shaw, and Oleg Kogan. Subharmonic resonance cascades in a class of coupled resonators. *Journal of Computational and Nonlinear Dynamics*, 8(4):041015, 2013.
- [55] Rouslan L Stratonovich. *Topics in the theory of random noise,* volume 2. CRC Press, 1967.
- [56] Krister Svanberg. The method of moving asymptotes—a new method for structural optimization. *International Journal for Numerical Methods in Engineering*, 24(2):359–373, 1987.
- [57] Simon M Sze and Kwok K Ng. *Physics of semiconductor devices*. Wiley. com, 2006.
- [58] M. Tiebout. A CMOS direct injection-locked oscillator topology as high-frequency low-power frequency divider. *Solid-State Circuits, IEEE Journal of*, 39(7):1170–1174, 2004.

- [59] Daniel A Tortorelli and Panagiotis Michaleris. Design sensitivity analysis: overview and review. *Inverse problems in Engineering*, 1(1):71–105, 1994.
- [60] C Touzé, O Thomas, and Antoine Chaigne. Hardening/softening behaviour in nonlinear oscillations of structural systems using non-linear normal modes. *Journal of Sound and Vibration*, 273(1):77–101, 2004.
- [61] C. Touzé, M. Vidrascu, and D. Chapelle. Direct finite element computation of nonlinear modal coupling coefficients for reduced-order shell models. *Computational Mechanics*, pages 1–14, 2014.
- [62] K.L. Turner, P.G. Hartwell, and N.C. MacDonald. Multi-dimensional mems motion characterization using laser vibrometry. In *IEEE Transducers Digest of Technical Papers*, pages 1144–1147, 1999.
- [63] K.L. Turner, S.A. Miller, P.G. Hartwell, N.C. MacDonald, S.H. Strogatz, and S.G. Adams. Five parametric resonances in a microelectromechanical system. *Nature*, 396(6707):149–152, 1998.
- [64] M Urteaga, M Seo, J Hacker, Z Griffith, A Young, R Pierson, P Rowell, A Skalare, and MJW Rodwell. Inp hbt integrated circuit technology for terahertz frequencies. In *Compound Semiconductor Integrated Circuit Symposium (CSICS)*, 2010 IEEE, pages 1–4. IEEE, 2010.
- [65] C Van der Avoort, R Van der Hout, JJM Bontemps, PG Steeneken, K Le Phan, RHB Fey, J Hulshof, and JTM van Beek. Amplitude saturation of mems resonators explained by autoparametric resonance. *Journal of Micromechanics and Microengineering*, 20(10):105012, 2010.
- [66] John R Vig. Quartz crystal resonators and oscillators for frequency control and timing applications. a tutorial. *NASA STI/Recon Technical Report N*, 95:19519, 1994.
- [67] L Guillermo Villanueva, Rassul B Karabalin, Matthew H Matheny, Eyal Kenig, Michael C Cross, and Michael L Roukes. A nanoscale parametric feedback oscillator. *Nano letters*, 11(11):5054–5059, 2011.
- [68] Ashwin Vyas, Dimitrios Peroulis, and Anil K Bajaj. Dynamics of a nonlinear microresonator based on resonantly interacting flexural-torsional modes. *Nonlinear Dynamics*, 54(1-2):31–52, 2008.
- [69] Ashwin Vyas, Dimitrios Peroulis, and Anil K Bajaj. A microresonator design based on nonlinear 1: 2 internal resonance in flexural structural modes. *Microelectromechanical Systems, Journal of*, 18(3):744–762, 2009.
- [70] Carl O Weiss and Ramon Vilaseca. Dynamics of lasers. *NASA STI/Recon Technical Report A*, 92:39875, 1991.

- [71] B Yurke, DS Greywall, AN Pargellis, and PA Busch. Theory of amplifier-noise evasion in an oscillator employing a nonlinear resonator. *Physical Review A*, 51(5):4211, 1995.
- [72] N.J. Miller Z. Yie, K.L. Turner and S.W. Shaw. Sensitivity enhancement using parametric amplification in a resonant sensing array. *Hilton Head*, 2010: A Solid State Sensor, Actuator, and Microsystems Workshop, 2010.
- [73] Xiangdong Zhang, Xuesong Zhou, B Aliener, and AS Daryoush. A study of subharmonic injection locking for local oscillators. *Microwave and Guided Wave Letters, IEEE*, 2(3):97–99, 1992.