ABSTRACT

DISTANCE PRESERVING TRANSFORMATIONS

By

Allen Jay Beadle

Let (M_1,d_1) and (M_2,d_2) be two metric spaces. An injective map $T:M_1 \longrightarrow M_2$ is called distance transforming if for some pair of positive numbers a and b, $d_1(x,y) = a$ implies that $d_2(Tx,Ty) = b$. If $a = b$, we call T distance preserving and say that a is preserved by T.

In 1953, F.S. Beckman and D.A. Quarles proved that if $T:E^n \longrightarrow E^n$, $2 \leq n < \infty$, is distance preserving, then T is an isometry. In 1973, R. Bishop gave a different proof.

In this thesis, we study a number of generalizations of the Beckman-Quarles result. We show that, up to a point, it can be extended to the classical non-Euclidean spaces and to most Minkowski planes. Beckman and Quarles noted that their theorem did not extend to Hilbert space but the example they gave was not continuous. We have been unable to find a continuous distance preserving self map of a Hilbert space which is not an isometry. In the cases of S^n, E^1, and a wide class of Banach spaces (not strictly convex) we have examples of continuous distance preserving self maps which are not isometries.

The thesis is organized into six sections. In Section 1, we develope a few basic lemmas in a general setting.

Using methods similar to those of Bishop, we show in

Section 2 that if M_1 and M_2 are two Minkowski planes and M_2 has no flat spots of length greater than 1 on its unit circle, then the only distance preserving maps $T:M_1 \longrightarrow M_2$ are the isometries. It is shown that in any Banach space with a flat spot of length 2 on its unit sphere, this theorem is not valid. However, the situation in Minkowski spaces of dimension more than 2 is unknown. In dimension 2, the only known execption is the space with max norm.

In Section 3, further exploiting the methods of Bishop, we show that for hyperbolic spaces H^n, $2 \leq n < \infty$, if $T:H^n \longrightarrow H^n$ is distance transforming then T is an isometry. While the methods are similar to those used to prove the corresponding result in E^n, the computational detail is much more involved.

Spherical and elliptic spaces are analyzed in Section 4. An example is given of a map $T:S^n \longrightarrow S^n$ which has two preserved distances, π and $\frac{\pi}{2}$, but is not an isometry. This leads us to impose the condition that the transformed distance a must be "small enough". Specific bounds are found which force the mapping to be an isometry, but they may not be the "best possible".

In Section 5, we attempt to improve on some of the results of D. Greenwell and P. Johnson who considered some directional restrictions on the set of preserved distances. More specifically, let $T:E^2 \longrightarrow E^2$ be a map such that there is a set $\mathcal{D}$ of unit vectors such that $\overrightarrow{xy} \in \mathcal{D}$ implies that $d(Tx,Ty) = 1$. If $\mathcal{D} = \mathcal{S}$, the set of all unit vectors, then T preserves the distance 1 and is thus an isometry. The question considered in this section is how small $\mathcal{D}$ can be and still force T to be an isometry. We show that $\mathcal{D}$ can have arbitrarily small measure and still force T to be an isometry, but this requires that the interior of $\mathcal{D}$, in the relative topology of $\mathcal{S}$, to be non-empty and requires $\mathcal{D}$ to contain certain vectors.

In Section 6, we suggest a number of directions for further research, describe a few partial results and give a few examples of a rather negative character showing bounds on the type of theorems to be expected.

DISTANCE PRESERVING TRANSFORMATIONS

By

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A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1977

to my wife

Anne

ACKNOWLEDGMENTS

I wish to express my gratitude to Professor Leroy M. Kelly, not only for his advice and guidance during the preparation of this theses, but also for the encouragement he has given me and the influence he has been on me during the last dozen years.

I also wish to thank Professor Fritz Herzog for the interest he has shown and the help he has been to me, especially during my undergraduate years.

Finally, I wish to thank Professors H. Davis, T. Yen, J. Zaks, and E. Nordhaus for serving on my guidance committee and for their suggestions concerning this theses.

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LIST OF SYMBOLS

$c_n(c)$: the distance between the centroid of a c-equilateral n-simplex and any of its vertices.

$C_n(c)$: $= \cos(c_n(c))$ in elliptic or spherical spaces,
$= \cosh(c_n(c))$ in hyperbolic spaces.

$d_n(c)$: $= \frac{1}{2}h_n(c) - c_n(c)$.

$D_n(c)$: $= \cos(d_n(c))$ in elliptic or spherical spaces,
$= \cosh(d_n(c))$ in hyperbolic spaces.

E^n: n-dimensional Euclidean space.

$\mathcal{E}^n$: n-dimensional elliptic space.

H^n: n-dimensional hyperbolic space.

$h_n(c)$: the distance between the two remaining points of two c-equilateral n-simplices having n points in common.

$H_n(c)$: $= \cos(h_n(c))$ in elliptic or spherical spaces,
$= \cosh(h_n(c))$ in hyperbolic spaces.

S^n: n-dimensional spherical space.

$\lambda_n(c)$: the angle between two adjacent (n-1)-simplices which are the faces of a c-equilateral n-simplex.

$[xyz]$: y is between x and z.

NOMENCLATURE

between: A point y in a metric space (M,d) is between points x and y if $d(x,y)+d(y,z) = d(x,z)$, and this relation will be denoted by $[xyz]$.

centroid: For $n \geq 1$, the centroid of an n-dimensional equilateral simplex in an elliptic, spherical, hyperbolic, or Euclidean space is the intersection point of the $n+1$ segments which have one end at a vertex of the simplex and the other end at the centroid of the $(n-1)$-dimension simplex which is the face opposite that vertex. For $n = 0$, the simplex is just one point which is its own centroid.

convex metric space: A metric space (M,d) is convex if for every pair of points x,y in M, there is a point z in M such that $[xzy]$.

diameter of M: For a metric space (M,d), we will say that the diameter of M is $\sup_{x,y \in M} \{d(x,y)\}$.

distance preserving: A map $T:M_1 \longrightarrow M_2$ between two metric spaces (M_1,d_1) and (M_2,d_2) is distance preserving if there is some distance a such that x,y in M_1 and $d_1(x,y) = a$ implies $d_2(Tx,Ty) = b$.

distance transforming: A map $T:M_1 \longrightarrow M_2$ between two metric spaces (M_1,d_1) and (M_2,d_2) is distance transforming if there are two distances a and b such that for any x,y in M, $d_1(x,y) = a$ implies $d_2(Tx,Ty) = b$.

r-equilateral n-simplex: A set of n+1 points in a metric space is an r-equilateral n-simplex if the distance between any two of them is equal to r.

equilateral unit lattice: A set of points in E^2 is an equilateral unit lattice if it consists of the points $\{\vec{v}_1+n\vec{v}_2+m\vec{v}_3\}$ where n and m range over all integers, $\vec{v}_1$ is any vector, and $\vec{v}_2$ and $\vec{v}_3$ are two unit vectors at an angle of 60° to each other.

extension property: A metric space (M,d) has the extension property if for each two points x,y of M and each distance r such that $d(x,y) \le r \le$ diameter of M, there exists a point z in M such that [xyz] and $d(x,z) = r$.

externally convex metric space: A metric space (M,d) is externally convex if for any x,y in M, there is some z in M such that $d(x,y)+d(y,z) = d(x,z)$.

flat spot: A Banach space is said to have a flat spot on its unit sphere if there is a segment of length greater than 0 on any unit sphere.

__isometry__: An injective map $T:M_1 \to M_2$ between two metric spaces (M_1,d_1) and (M_2,d_2) is an isometry if for all x,y in M_1, $d_1(x,y) = d_2(Tx,Ty)$.

__length of a segment__: If a segment in a metric space has endpoints x and y, then the length of the segment is $d(x,y)$.

__local isometry__: A map $T:M_1 \to M_2$ between two metric spaces (M_1,d_1) and (M_2,d_2) is a local isometry if for every x in M_1, there is a distance $r(x) > 0$ such that y,z in M_1, $d_1(x,y) < r(x)$ and $d_1(x,z) < r(x)$ implies that $d_2(Ty,Tz) = d_1(y,z)$.

__local isosceles property__: A metric space has the local isosceles property if there is a $\rho(M) > 0$ such that for any positive λ less than ρ and any x,y in M, with $d(x,y) \leq 2\lambda$, there is a point z such that $d(x,z) = d(z,y) = \lambda$.

__Minkowski space__: A Minkowski space is a finite dimensional Banach space.

__T preserves a__: A map $T:M_1 \to M_2$ between two metric spaces preserves a if for all x,y in M, $d_1(x,y) = a$ implies $d_2(Tx,Ty) = a$.

__segment__: A set of points in a metric space (M,d) is a segment if it is the isometric image of a finite closed interval of the real line.

segmentally connected: A metric space is segmentally connected if every two points in it are the endpoints of some segment.

T transforms a into b: A map $T:M_1 \longrightarrow M_2$ between two metric spaces (M_1,d_1) and (M_2,d_2) transforms distance a into distance b if for any x,y in M_1 with $d_1(x,y) = a$, it follows that $d_2(Tx,Ty) = b$.

DISTANCE PRESERVING TRANSFORMATIONS

INTRODUCTION

In 1953, F.S. Beckman and D.A. Quarles [1] proved that if T is a mapping of E^n into E^n for $2 \leq n < \infty$ which preserves one distance, then T is an isometry. In 1973, R. Bishop [3] gave a different proof.

The aim of this thesis is to extend this result to other geometries. It will be shown that this theorem is true in any n-dimensional hyperbolic space for $2 \leq n < \infty$; it is true in any n-dimensional elliptic or spherical space if the preserved distance is small enough; and it is true in any Minkowski plane whose unit circle does not have a flat spot of length greater than 1. In hyperbolic and spherical and elliptic spaces, a further generalization will be shown: if $T:M \rightarrow M$ such that there are two distances a and b such that $x,y \in M$ and $d(x,y) = a$ implies $d(Tx,Ty) = b$, then $a = b$ and T is an isometry (here again in the spherical and elliptic spaces, a and b must be small enough). Note that in Euclidean spaces (and Minkowski spaces) any such map is the product of a similarity and a distance preserving map.

As noted by Beckman and Quarles [1], this theorem is not true for E^1, the Euclidean line, or E^∞, Hilbert space. In E^1 the map T defined by $Tx = x+1$ if x is an integer point and $Tx = x$ otherwise is a counterexample. In E^∞, a counterexample is found as follows:

let $\{y_i\}$ be a countable everywhere dense set of points. Define $R: E^{\infty} \longrightarrow \{y_i\}$ so that $d(x, Rx) < \frac{1}{2}$. Define $S: \{y_i\} \longrightarrow \{a_i\}$ so that $Sy_i = a_i$ where a_i is the point in E^{∞} with coordinates $(a_{i1}, a_{i2}, \ldots)$ such that $a_{ij} = \delta_{ij}/\sqrt{2}$ where δ_{ij} is the Kronecker delta. Then $T = SR$ is a map of E^{∞} into itself which preserves the distance 1. For if $d(x,y) = 1$, then $Rx \neq Ry$ and hence $Tx \neq Ty$, But T is not an isometry.

The theorem is also not true in particular Minkowski spaces, such as any max norm space (see Section 2). It is also untrue in spherical spaces for particular distances. For example, in S^n, if T maps every point onto itself except the north and south poles and maps these two points onto each other, then it is clear that T is not an isometry, but T does preserve the two distances π and $\frac{\pi}{2}$.

§1. PRELIMINARIES

Generally we will be concerned throughout this thesis with an injective mapping, T, from one metric space (M_1,d_1) to another, (M_2,d_2). Such a mapping is said to be distance transforming and to transform distance a into b if whenever $d_1(x,y) = a$, then $d_2(Tx,Ty) = b$. If $a = b$, then T is called distance preserving and we say that T preserves a. If T preserves a_i for each term of the null sequence $\{a_i\}$ then we will frequently say that T preserves arbitrarily small distances. Analogously, if the $\{a_i\}$ approach infinity, T is said to preserve arbitrarily large distances.

Definition. A metric segment is the isometric image of a real segment.

Definition. A metric space is called segmentally connected if every two points of the space are the endpoints of some segment.

Definition. If in a metric space, $d(x,y)+d(y,z) = d(x,z)$, then y is said to be between x and z. This relation is denoted by the symbol $[xyz]$.

Definition. A metric space (M,d) has the extension property if for each two points x,y in M and each real number r such that $d(x,y) < r \leq$ diameter of M, there is a point z in M such that $[xyz]$ and $d(x,z) = r$.

Definition. If in a metric space [xyz], [yzw] and $d(x,z)+d(z,w)$ is less than the diameter of the space, imply [xzw], then the betweenness relation is said to be externaly transitive in this space.

Definition. A metric space (M,d) has the local isosceles property if there is a $\rho(M) > 0$ such that for any positive r less than ρ, and any x,y in M with $d(x,y) \leq 2r$, there is a point z such that $d(x,z) = d(z,y) = r$.

Lemma 1.1. If (M_1,d_1) and (M_2,d_2) are two metric spaces where (M_1,d_1) is segmentaly connected and has the local isosceles property and $T:M_1 \longrightarrow M_2$ transforms a into b, then $d_1(x,y) \leq ka$ implies that $d_2(Tx,Ty) \leq kb$, for k any positive integer greater than 1.

Proof. Let $z_o, z_1, \ldots, z_j$ be points on a segment with endpoints x and y such that $x = z_o$ and $d_1(z_i, z_{i+1}) = a$ for $0 \leq i \leq j-1$, and $a < d_1(z_j, y) \leq 2a$. Since (M_1,d_1) has the local isoscles property, there is a point z_{j+1} such that $d_1(z_j, z_{j+1}) = d_1(z_{j+1}, y) = a$. If we denote y by z_{j+2}, then $d_2(Tx,Ty) = d_2(Tz_o, Tz_{j+2}) \leq \sum_{i=0}^{j+1} d_2(Tz_i, Tz_{i+1}) = (j+2)b$. But $j+1 < k-1$, so $j+2 \leq k$ and $d_2(Tx,Ty) \leq kb$. □

The following is a generalization of a lemma of Bishop [3].

Lemma 1.2. If (M_1,d_1) and (M_2,d_2) are two metric spaces such that (M_1,d_1) is segmentaly connected, has the local isosceles property and the extension property and the diameter of M_1 is not less than that of M_2, and if $T:M_1 \longrightarrow M_2$ preserves a null sequence of distances, then T preserves all distances less than any preserved distance.

Proof. Let a be any preserved distance and let c be less than a. Let x,y be in M_1 with $d_1(x,y) = c$. Let z be in M_1 such that $[xyz]$ and $d_1(x,z) = a$. Such a point z exists by virtue of the fact that (M_1,d_1) has the extension property.

Let $u = d_2(Tx,Ty)$ and $v = d_2(Tx,Tz)$. We wish to show that $c = u$. Let r be a preserved distance under T such that $r < \min(c,a-c)$ and $r < \rho(M_1)$. Then there exist integers k and $m \geq 2$ such that $(m-1)r < c \leq mr$ and $(k-1)r < a-c \leq kr$. By Lemma 1.1, we have $u \leq mr$ and $v \leq kr$. Therefore $u-c < r$ and $v-a+c < r$. Now, by the triangle inequality, we have $a \leq u+v$ and this combined with the previous line gives: $c-u \leq c+v-a < r$. Therefore $|c-u| < r$. Since r can be arbitrarily small it follows that $c = u$.□

Lemma 1.3. If (M_1,d_1), (M_2,d_2) and T are as in Lemma 1.2 and if, in addition, the betweenness relation in (M_2,d_2) is externaly transitive, then T is an isometry.

Proof. Let $c = \sup\{r: r$ is preserved under $T\}$. If $c = \infty$ then for any distance a, there is a preserved distance larger than a, so by Lemma 1.2, a is preserved. Hence T is an isometry.

If (M_1,d_1) is bounded and c is its diameter, then all distances less than c are preserved by Lemma 1.2. If there is a pair of points x and y in M_1 such that $c = d_1(x,y)$, then for any ε such that $c > \varepsilon > 0$, there is a point z such that $d_1(x,z) = \varepsilon$ and $d_1(z,y) = c-\varepsilon$. Both ε and $c-\varepsilon$ are preserved.

From the triangle inequality we have that

$$d_2(Tx,Ty) \geq d_2(Ty,Tz) - d_2(Tz,Tx) = c-2\varepsilon.$$

Now since ε can be taken arbitrarily small and the diameter of M_2 is no greater than c it follows that $d_2(Tx,Ty) = c$ and T is an isometry.

Now suppose (M_1,d_1) is bounded but c is less than its diameter. Let r be a distance in (M_1,d_1) such that $2r < c$, Note that r and $2r$ are preserved by T because of Lemma 1.2. Consider four points f_1,f_2,f_3,f_4 such that f_2 and f_3 are both between f_1 and f_4 and $c < d_1(f_1,f_4) < c+r$, $d_1(f_3,f_4) = r$ and $d_1(f_2,f_4) = 2r$.

Since all distances less than c are preserved, Tf_2 is between Tf_1 and Tf_3. Similarly Tf_3 is between Tf_2 and Tf_4. Now the diameter of M_2 is not less than that of M_1, so $d_2(Tf_1,Tf_3)+d_2(Tf_3,Tf_4)$ is not greater than the diameter of M_2 since it is equal to $d_1(f_1,f_4) = d_1(f_1,f_3)+d_1(f_3,f_4)$. Since the betweenness relation in M_2 is transitive, we have:

$$d_2(Tf_1,Tf_4) = d_2(Tf_1,Tf_3) + d_2(Tf_3,Tf_4)$$
$$= d_1(f_1,f_3) + d_1(f_3,f_4)$$
$$= d_1(f_1,f_4).$$

Therefore the distance $d_1(f_1,f_4)$, which is greater than c, is preserved under T contradicting the definition of c. Hence c equals the diameter of M_1 and T is an isometry. □

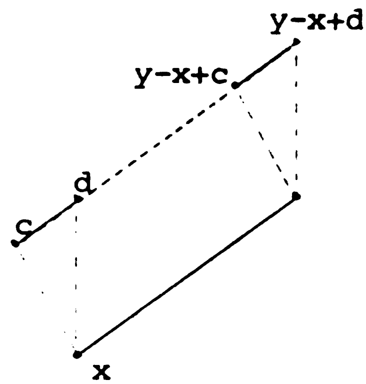
§2. MINKOWSKI PLANES

Throughout this section, M_1 and M_2 will denote two Minkowski planes such that the unit circle in M_2 does not have a flat spot of length > 1. Also, T will denote a mapping of M_1 into M_2 which preserves the distance 1. The objective of this section is to show that T must be an isometry. In this section the distance between points x and y will be denoted by $\|x-y\|$ if they are in M_1 and by $|||x-y|||$ if they are in M_2, and we will use the vector notation in M_1 and M_2. The method used here is similar to that used by Bishop [3].

Lemma 2.1. If $x, y \in M_2$ and $|||x-y||| = 1$ then there exists a unique pair of points a and b such that $|||x-a||| = |||y-a||| = |||y-b||| = |||x-b||| = 1$. For these a and b, $|||a-b||| > 1$ and $x - a = b - y$.

Proof. Let $\mathcal{C}_x$ and $\mathcal{C}_y$ be the unit circles in M_2 with centers x and y. Then $\mathcal{C}_x \cap \mathcal{C}_y \neq \emptyset$ so let $a \in \mathcal{C}_x \cap \mathcal{C}_y$, and set $b = x + y - a$. Then $b \in \mathcal{C}_x \cap \mathcal{C}_y$ and the required norms are = 1.

To show uniqueness, suppose $\mathcal{C}_x \cap \mathcal{C}_y$ contains at least 3 points. Since none of them can be on the line xy, two of them must lie on the same side of line xy; call these two points c and d. Then points $y - x + c$ and $y - x + d$ are on $\mathcal{C}_y$, so the convexity of $\mathcal{C}_y$ requires that c,d, $y - x + c$, and $y - x + d$ be collinear. Then they are on a segment of length > 1. Hence no such c and d exists.



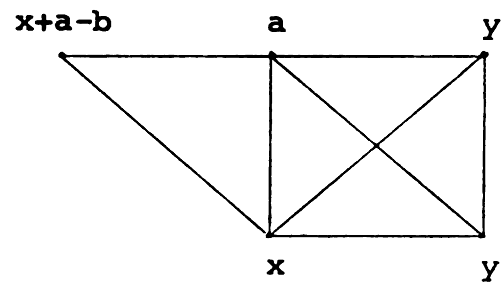


Figure 2-1 Figure 2-2

Finally, $|||a-b||| > 1$ since if $|||a-b||| < 1$, then $x+a-b$ would be inside $\mathcal{C}_x$, which cannot be since $\mathcal{C}_x$ is convex and $a \in \mathcal{C}_x$, $y \in \mathcal{C}_x$; and if $|||a-b||| = 1$, then $x+a-b \in \mathcal{C}_x$ and hence the segment from y to $x+a-b$ is a flat spot of length 2. □

Lemma 2.2. *If* $x,y \in M_1$ *and* $\|x-y\| = 1$ *then there exists* a *and* $b \in M_1$ *such that* $\|x-a\| = \|y-a\| = \|x-b\| = \|b-a\|$ *and* $b-a = x-y$. *Also,* a *and* b *depend continuously on* x *if* y *is held fixed.*

Proof. There are two points a, b on the unit circle with center x such that $b-a = x-y$. Then linearity gives that the needed norms = 1. Note that if a is restricted so that triangle xya has positive orientation, then a and b are unique by the same argument used in Lemma 2.1. This implies that a and b are continuous

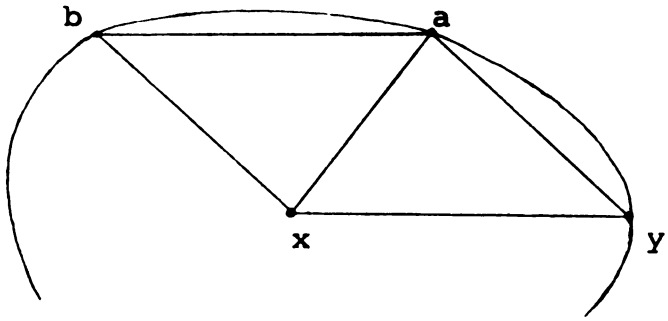


Figure 2-3

functions of x if y is held fixed. □

<u>Lemma 2.3</u>. <u>*If*</u> $a,b,x,y \in M_1$, $b \neq y$, $\|x-y\| = \|y-a\| = \|b-a\| = \|x-b\| = \|x-a\| = 1$, <u>*then there exists*</u> $\tilde a, \tilde b, \tilde x \in M_1$ <u>*such that*</u>

$$\|y-\tilde a\| = \|\tilde a-\tilde b\| = \|\tilde b-\tilde x\| = \|\tilde x-\tilde a\| = \|b-\tilde b\| = \|\tilde x-y\| = 1.$$

<u>Proof</u>. By Lemma 2.2, for any $\tilde x \in M_1$ with $\|\tilde x-y\| = 1$, there is $\tilde a$, $\tilde b$ so that $\|\tilde a-y\| = \|\tilde a-\tilde b\| = \|\tilde a-\tilde x\| = \|\tilde x-\tilde b\| = 1$.

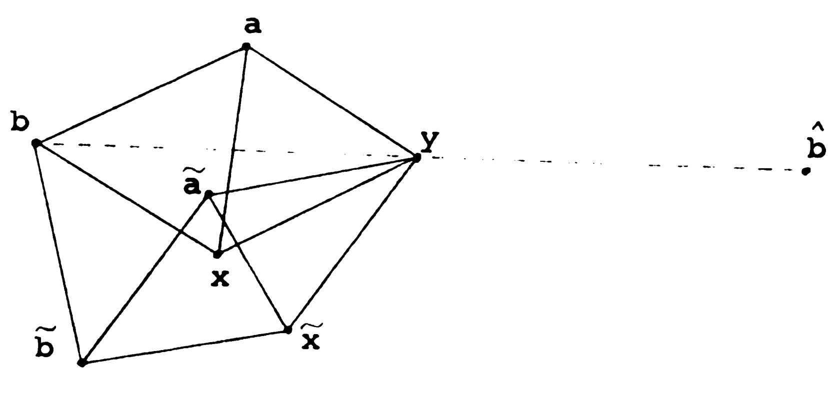


Figure 2-4

But $\tilde b$ is a continuous function of $\tilde x$ and $\tilde b = b$ when $\tilde x = x$ and $\tilde b = \hat b = 2y - b$ when $\tilde x = 2y - x$, then since

$\|b-b\| = 0$ and $\|b-\hat{b}\| = 2\|b-y\| \geq 2$, there is some choice of $\tilde{x}$ so that $\|b-\tilde{b}\| = 1$. □

Lemma 2.4. *If* T *has arbitrarily large and arbitrarily small preserved distances, then* T *is an isometry*.

Proof. This is immediate from Lemma 1.2. □

Lemma 2.5. *If* $a,b,c,d \in M_1$ *such that* $\|a-b\| = \|a-c\| = \|d-b\| = \|b-c\| = \|a-d\| = 1$ *then* $Tc \neq Td$.

Proof. By Lemma 2.3 we have $\tilde{a},\tilde{b},\tilde{d} \in M_1$ so that

$$\|\tilde{b}-c\| = \|\tilde{b}-\tilde{d}\| = \|\tilde{d}-\tilde{a}\| = \|\tilde{a}-c\| = \|\tilde{a}-\tilde{b}\| = \|d-\tilde{d}\| = 1.$$

Now since T preserves the distance 1, we have

$$\begin{aligned} 1 &= |||Ta-Tb||| = |||Ta-Tc||| = |||Tb-Tc||| = |||Ta-Td||| = |||Tb-Td||| \\ &= |||T\tilde{a}-T\tilde{b}||| = |||T\tilde{a}-Tc||| = |||T\tilde{b}-Tc||| = |||T\tilde{a}-T\tilde{d}||| = |||T\tilde{b}-T\tilde{d}||| \\ &= |||Td-T\tilde{d}||| . \end{aligned}$$

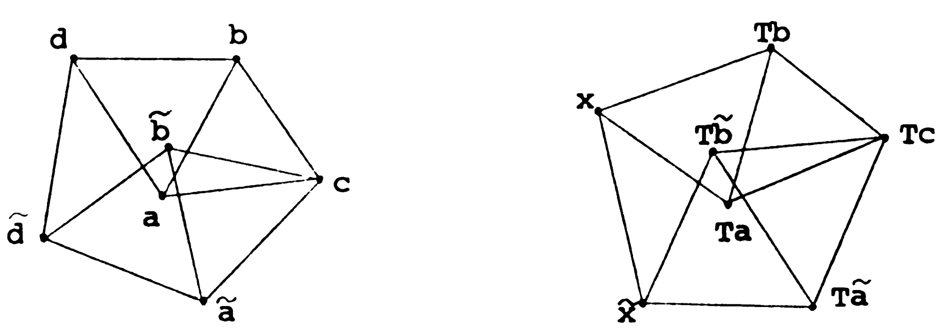


Figure 2-5

Then by Lemma 2.1, either $Td = x$ or $Td = Tc$ (where $x = Ta + Tb - Tc$). Also either $T\tilde{d} = \tilde{x}$ or $T\tilde{d} = Tc$ (where

$\tilde{x} = T\tilde{a} + T\tilde{b} - Tc)$.

But $\|d-\tilde{d}\| = 1 \Longrightarrow |\!|\!|Td-T\tilde{d}|\!|\!| = 1$ while Lemma 2.1 $\Longrightarrow$ $|\!|\!|x-Tc|\!|\!| > 1$ and $|\!|\!|\tilde{x}-Tc|\!|\!| > 1$. Also $|\!|\!|Tc-Tc|\!|\!| = 0$. So the only possibility for Td and $T\tilde{d}$ is $Td = x$ and $T\tilde{d} = \tilde{x}$.

Hence $Tc \neq Td$. □

Lemma 2.6. *If* $x_1, x_2, \ldots, x_n \in M_1$ *such that* $x_1 - x_2 = x_k - x_{k+1}$ *for* $k = 1,2,\ldots,n-1$ *and* $\|x_1-x_2\| = 1$ *then* $|\!|\!|Tx_1-Tx_2|\!|\!| = 1$ *and* $Tx_1 - Tx_2 = Tx_k - Tx_{k+1}$ *for* $k = 1,2,\ldots,n-1$.

Proof. By Lemma 2.2, there is $b_1, b_2 \in M_1$ such that $b_1 - b_2 = x_1 - x_2$ and $\|x_1-b_1\| = \|b_1-b_2\| = \|b_1-x_2\| = \|b_2-x_2\| = 1$. For $k \geq 3$, define $b_k = b_2 + (k-2)(b_2-b_1)$.

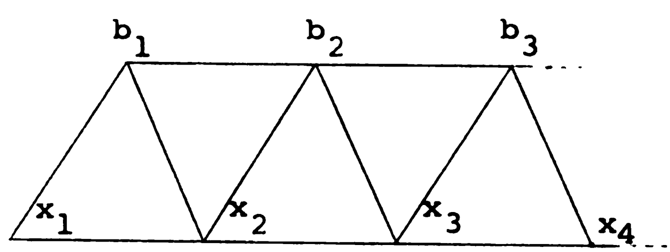


Figure 2-6

Then

$$\|x_k-x_{k+1}\| = \|x_k-b_k\| = \|b_k-x_{k+1}\|$$
$$= \|b_k-b_{k+1}\|$$
$$= \|x_{k+1}-b_{k+1}\| = 1$$

and

$$\|b_k-x_{k+1}\| = \|b_k-b_{k+1}\| = \|x_{k+1}-b_{k+1}\| = \|x_{k+1}-x_{k+2}\|$$
$$= \|b_{k+1}-x_{k+2}\| = 1$$

for $k = 1,2,\cdots,n-2$.

Then by Lemma 2.1, $Tb_2 - Tb_1 = Tx_2 - Tx_1$ and $Tx_3 = Tx_2 = Tb_2 - Tb_1$, therefore $Tx_3 - Tx_2 = Tx_2 - Tx_1$ and by induction on k: $Tx_{k+1} - Tx_k = Tx_2 - Tx_1$ for $k = 1,2,\cdots,n-1$. □

Corollary 2.7. *For every integer* $n \geq 1$, n *is preserved by* T.

Proof. This is immediate from Lemma 2.6 since

$$n = \|x_{n+1}-x_1\| = \|n(x_2-x_1)\| = n\|x_2-x_1\| = n\|Tx_2-Tx_1\|$$
$$= \|n(Tx_2-Tx_1)\| = \|Tx_{n+1}-Tx_1\|. \quad □$$

Lemma 2.8. *For any integer* $n \geq 1$, $\frac{1}{n}$ *is preserved by* T.

Proof. Let c be a point of intersection of the two unit circles with centers x and y. Let $a = x + (n-1)(x-c)$ and $b = y + (n-1)(y-c)$. Then $\|c-y\| = \|c-x\| = 1$, $\|a-b\| = \|nx-ny\| = n\|x-y\| = 1$, $\|a-c\| = \|nx-nc\| = n\|x-c\| = n$, $\|b-c\| = n$. Therefore Lemma 2.6 implies $Tc - Ta = n(Tc-Tx)$ and $Tc - Tb = n(Tc-Ty)$.

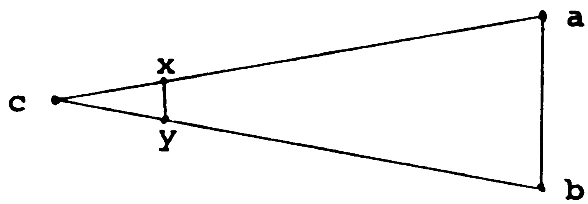


Figure 2-7

Therefore $|||Tx-Ty||| = \frac{1}{n} |||Ta-Tb||| = \frac{1}{n}$ since $\|a-b\| = 1$.

□

This gives us the main result of this section:

<u>Theorem 1</u>. <u>If</u> $T: M_1 \longrightarrow M_2$ <u>is a mapping between two Minkowski planes</u> M_1 <u>and</u> M_2 <u>where the unit circle in</u> M_2 <u>has no flat spot of length</u> > 1 <u>and</u> T <u>preserves the distance</u> 1, <u>then</u> T <u>is an isometry</u>.

<u>Proof</u>. This is immediate from Lemmas 2.4, 2.7, and 2.8. □

This theorem is not true if $M_1 = M_2 = \ell_\infty$ as the following example shows: let $M_1 = M_2 = \{(x,y) \mid x,y \text{ real}\}$ and with max norm, $\|(x_1,y_1) - (x_2,y_2)\| = \max(|x_1-x_2|, |y_1-y_2|)$. Let T be defined by $T(x,y) = ([x]+\{x\}^2, [y]+\{y\}^2)$ where $[z]$ is the largest integer $\leq z$ and $\{z\} = z - [z]$. Then T preserves all integer distances, but is not an isometry.

A similar example can be found in any Minkowski or Banach space which has a flat spot of length 2. This includes all ℓ_1 and ℓ_∞ spaces.

It is not known what the situation is in Minkowski planes with a flat spot of length > 1 and < 2 or in Minkowski spaces of dimension ≥ 3 without flat spots of length 2.

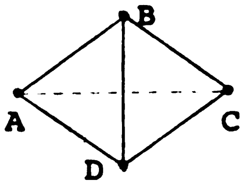
§3. HYPERBOLIC SPACE

Let H^n be n-dimensional hyperbolic space and let $T: H^n \to H^n$ be a mapping so that there are two distances a and b such that $d(x,y) = a \Longrightarrow d(Tx,Ty) = b$. The aim of this section is to show that T must then be an isometry. The method used here is similar to that used by Bishop [3] in E^n for $n \geq 3$, especially the idea behind Lemma 3.6.

In this section, the following notation will be used:

$h_n(c)$ = the distance between the two remaining points of two c-equilateral n-simplices having n points in common;

example: $n = 2$



$AD = AB = BD = BC = CD = c$

$AC = h_2(c)$

Figure 3-1

$H_n(c)$ = $\cosh(h_n(c))$;

$c_n(c)$ = distance between the centroid of a c-equilateral n-simplex and any of the vertices;

$C_n(c)$ = $\cosh(c_n(c))$;

$d_n(c)$ = $\frac{1}{2}h_n(c) - c_n(c)$;

$D_n(c)$ = $\cosh(d_n(c))$;

$\lambda_n(c)$ = the angle between two adjacent (n-1)-simplices which are faces of a c-equilateral n-simplex.

Note: $c_0(c) = 0$; $c_1(c) = \frac{1}{2}c$; so $C_0(c) = 1$, and $C_1(c) = \sqrt{\frac{1 + \cosh c}{2}}$.

When there is no ambiguity in the value of the argument of the functions h_n, H_n, c_n, C_n, d_n, D_n, and λ_n; they will be written without the argument.

Lemma 3.1. $H_n(c) = \frac{2n(\cosh c)^2}{1 + (n-1)\cosh c} - 1$ for $n \geq 1$.

Proof. Let $A = \cosh c$. Let $f_1, e_1, e_2, \ldots, e_n$ and $f_2, e_1, e_2, \ldots, e_n$ be two c-equilateral n-simplices with $f_1 \neq f_2$. Let g_1 be the midpoint of the segment $f_1 f_2$ and let g_2 be the centroid of the simplex $f_1, e_1, e_2, \ldots, e_n$.

example: n = 2

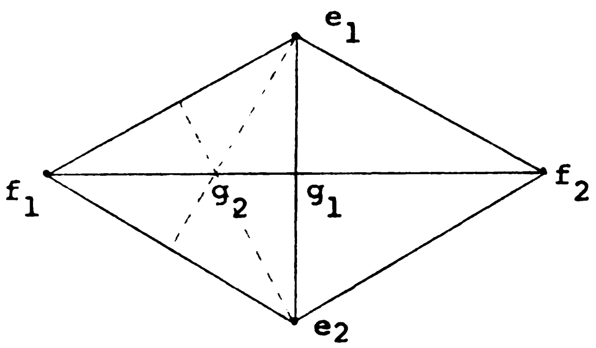


Figure 3-2

Then g_1 is the centroid of the c-equilateral (n-1)-simplex $e_1, e_2, \ldots, e_n$; and $d(e_1, g_1) = c_{n-1}$; $d(e_1, g_2) = d(f_1, g_2) = c_n$; $d(f_1, g_1) = \frac{1}{2}h_n$; $d(g_1, g_2) = d_n$; and angle $f_1 g_1 e_1$ is $\pi/2$.

Hence

$\Delta g_2 g_1 e_1$ is a right triangle $\Longrightarrow$

$\cosh c_n = \cosh c_{n-1} \cosh d_n \Longrightarrow D_n = C_n/C_{n-1}$.

Similarly

$\Delta f_1 g_1 e_1$ is a right triangle $\Longrightarrow$

$\cosh c = \cosh \frac{1}{2} h_n \cosh c_{n-1} \Longrightarrow H_n = \frac{2A^2}{C_{n-1}^2} - 1.$

Therefore

$$d_n = \frac{1}{2} h_n - c_n$$

$$\Longrightarrow \quad \cosh d_n = \cosh \frac{1}{2} h_n \cosh c_n - \sinh \frac{1}{2} h_n \sinh c_n;$$

$$\Longrightarrow \quad C_n/C_{n-1} = D_n = \sqrt{\frac{H_n+1}{2}}\, C_n - \sqrt{\frac{H_n-1}{2}} \sqrt{C_n^2 - 1}\ ;$$

$$\Longrightarrow \quad \frac{H_n-1}{2} (C_n^2 - 1) = \left(\frac{H_n+1}{2} + \frac{1}{C_{n-1}^2} - \frac{\sqrt{2}\, \sqrt{H_n+1}}{C_{n-1}} \right) C_n^2;$$

$$\Longrightarrow \quad C_n^2 = \frac{H_n-1}{2} \Bigg/ \left(-1 - \frac{1}{C_{n-1}^2} + \frac{\sqrt{2}\, \sqrt{H_n+1}}{C_{n-1}} \right)$$

$$= \frac{\frac{A^2}{C_{n-1}^2} - 1}{-1 - \frac{1}{C_{n-1}^2} + 2\frac{A}{C_{n-1}^2}} = \frac{A^2 - C_{n-1}^2}{2A - C_{n-1}^2 - 1}.$$

Since $c_0 = 0$, $C_0 = 1$, by induction on n it follows that:

$$C_n^2 = \frac{nA+1}{n+1} \qquad \text{for } n \geq 0.$$

Then

$$H_n = \frac{2A^2}{C_{n-1}^2} - 1 \implies H_n = \frac{2nA^2}{1+(n-1)A} - 1 \quad \text{for} \quad n \geq 1. \square$$

<u>Lemma 3.2</u>. <u>For all</u> c, $\frac{H_n(c) - 1}{\cosh c - 1} > 2$, <u>in particular</u>, $h_n(c) > c$.

<u>Proof</u>. Let $A = \cosh c$. Then $H_n = \frac{2nA^2}{1+(n-1)A} - 1 \implies$

$$\frac{H_n - 1}{A-1} = 2\,\frac{nA+1}{(n-1)A+1} > 2.$$

So $H_n > 2A - 1 > A$, (since $A > 1$) hence $h_n(c) > c$. $\square$

<u>Lemma 3.3</u>. <u>If</u> $d(x,y) = a \implies d(Tx,Ty) = b$, <u>then</u> $d(x,y) = h_n(a) \implies d(Tx,Ty) = h_n(b)$.

<u>Proof</u>. Let $f_1, e_1, e_2, \ldots, e_n$ and $f_2, e_1, e_2, \ldots, e_n$ be two a-equilateral n-simplices with $f_1 \neq f_2$. Then $h_n(a) = d(f_1, f_2)$.

Now $h_n(a) > a$ implies that the (n-1)-sphere with center f_1 and radius $h_n(a)$ has point $\tilde{f}_2$ so that $d(f_2, \tilde{f}_2) = a$. Then there exists points $\tilde{e}_1, \ldots, \tilde{e}_n$ such that $f_1, \tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_n$ and $\tilde{f}_2, \tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_n$ are a-equilateral n-simplices.

Then Tf_1, Te_1, $Te_2, \ldots, Te_n$ and $Tf_2, Te_1, Te_2, \ldots, Te_n$ are b-equilateral n-simplices, so $d(Tf_1, Tf_2) = 0$ or $h_n(b)$ depending on whether or not $Tf_1 = Tf_2$. Likewise $Tf_1, T\tilde{e}_1, T\tilde{e}_2, \ldots, T\tilde{e}_n$ and $T\tilde{f}_2, T\tilde{e}_1, T\tilde{e}_2, \ldots, T\tilde{e}_n$ are b-equilateral n-simplices and $d(Tf_1, T\tilde{f}_2) = 0$ or $h_n(b)$ depending on whether or not $Tf_1 = T\tilde{f}_2$. But $d(f_2, \tilde{f}_2) = a$

so $d(Tf_2, T\tilde{f}_2) = b$. So it cannot happen that $Tf_1 = Tf_2 = T\tilde{f}_2$ since then $d(Tf_2, T\tilde{f}_2)$ would be 0; also $Tf_1 = Tf_2 \neq T\tilde{f}_2$ cannot be since then $d(Tf_2, T\tilde{f}_2)$ would be $d(Tf_1, T\tilde{f}_2) = h_n(b) > b$; and similarly $Tf_1 = T\tilde{f}_2 \neq Tf_2$ cannot be, so $Tf_1 \neq Tf_2$ and $Tf_1 \neq T\tilde{f}_2$. Hence $d(Tf_1, Tf_2) = h_n(b)$. □

Corollary 3.4. *If* T *has a preserved distance, then* T *has arbitrarily large preserved distances.*

Proof. Let a_0 be a preserved distance of T and define $a_k = h_n(a_{k-1})$ for $k \geq 1$. By Lemma 3.3, a_k is a preserved distance of T for all $k \geq 1$, and by Lemma 3.2; $\cosh a_k - 1 > (\cosh a_0 - 1)2^k$. Hence $\cosh a_k \to \infty$ as $k \to \infty$. Hence $a_k \to \infty$ as $k \to \infty$. □

Lemma 3.5. $\cos \lambda_n(c) = \dfrac{\cosh c}{(n-1)\cosh c + 1}$.

Proof. $\lambda_n(c)$ is the angle opposite the base of a triangle whose sides are $\frac{1}{2}h_{n-1}(c)$, $\frac{1}{2}h_{n-1}(c)$, and c.

The law of cosines yields:

$$\cosh c = \cosh^2 \tfrac{1}{2} h_{n-1}(c) - \sinh^2 \tfrac{1}{2} h_{n-1}(c) \cos \lambda_n(c).$$

Let $A = \cosh c$, $H_{n-1} = \cosh h_{n-1}(c)$; then

$$A = \frac{H_{n-1}+1}{2} + \frac{1-H_{n-1}}{2} \cos \lambda_n.$$

Therefore

$$\cos \lambda_n = \frac{H_{n-1}+1-2A}{H_{n-1}-1} = \frac{A}{(n-1)A+1}. \quad \square$$

example: $n = 3$

$\frac{1}{2}h_2(c) = d(e_1,e_3)$

$= d(e_2,e_3)$

$c = d(e_1,e_2)$

$\lambda_3 = \angle e_1 e_3 e_2.$

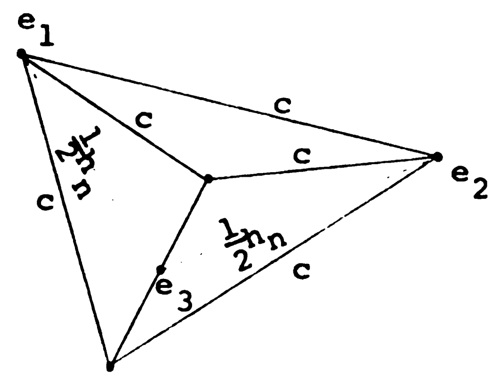


Figure 3-3

Lemma 3.6. If $d(x,y) = a \Longrightarrow d(Tx,Ty) = b$, and $\frac{2\pi}{\lambda_n(b)}$ is not an integer, then there are $\tilde{a}$ and $\tilde{b}$ such that $d(x,y) = \tilde{a} \Longrightarrow d(Tx,Ty) = \tilde{b}$ and $\cosh \tilde{b} - 1 <$ $0.6(\cosh b - 1)$.

Proof. Let $e_1,\ldots,e_{n-1},f_0,f_1$ and $e_1,\ldots,e_{n-1},f_1,f_2$ be a-equilateral n-simplices so that $f_0 \neq f_2$; and by

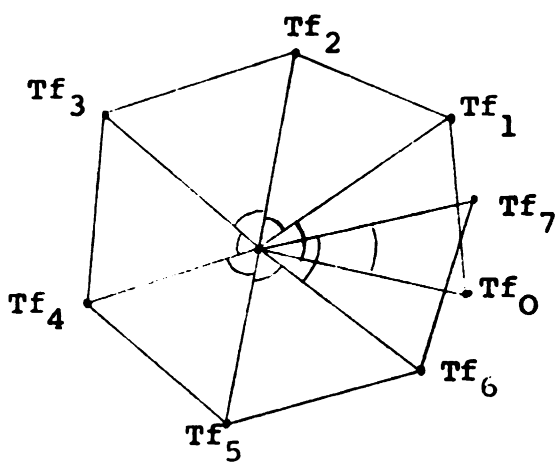


Figure 3-4

induction let $e_1,\ldots,e_{n-1},f_{k-1},f_k$ be another such simplex so that $f_{k-2} \neq f_k$ for $k \geq 2$. Since $d(x,y) = a \Rightarrow d(Tx,Ty) = b$; $Te_1,\ldots,Te_{n-1},Tf_{k-1},Tf_k$ is a b-equilateral n-simplex for all $k \geq 1$. Then the angle between the (n-1)-simplices $Te_1,\ldots,Te_{n-1},Tf_0$ and $Te_1,\ldots,Te_{n-1},Tf_k$ is $k\lambda_n(b)$. Now, if $\frac{2\pi}{\lambda_n(b)}$ is not an integer, there exists k so that $0 < |2\pi - k\lambda_n(b)| \leq \frac{1}{2}\lambda_n(b)$.

Now, $d(f_0,f_2) = h_n(a)$, therefore $d(Tf_0,Tf_2) = h_n(b)$ by Lemma 3.3. Likewise, for all i, $0 \leq i \leq k-2$ we have $d(Tf_i,Tf_{i+2}) = h_n(b)$.

Hence $d(x,y) = d(f_0,f_k) \Longrightarrow d(Tx,Ty) = d(Tf_0,Tf_k)$ for all x, y in H^n since any such x and y can be made the f_0 and f_k points of some such $e_1,\ldots,e_{n-1},f_0,\ldots,f_k$. Let $\tilde{a} = d(f_0,f_k)$ and $\tilde{b} = d(Tf_0,Tf_k)$. Let β be the (smallest) angle between the (n-1)-simplices $Te_1,\ldots,Te_{n-1},Tf_0$ and $Te_1,\ldots,Te_{n-1},Tf_k$. Then $0 < \beta = |2\pi - k\lambda_n(b)| \leq \frac{1}{2}\lambda_n(b)$ and β is the angle opposite the base of a triangle whose sides are $\frac{1}{2}h_{n-1}(b)$,

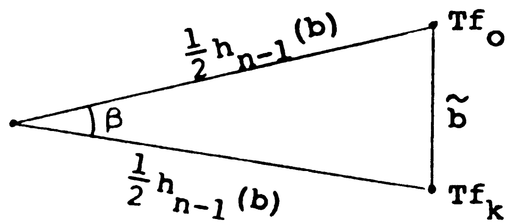


Figure 3-5

$\frac{1}{2} h_{n-1}(b)$, and $\tilde{b}$.

The law of cosines yields:

$$\cosh \tilde{b} = \cosh^2 \tfrac{1}{2} h_{n-1}(b) - \sinh^2 \tfrac{1}{2} h_{n-1}(b) \cos \beta .$$

Let B denote $\cosh b$. Then:

$$\cosh \tilde{b} - 1 = \left(\sinh^2 \tfrac{1}{2} h_{n-1}(b)\right)(1 - \cos \beta)$$

$$\cosh \tilde{b} - 1 \le \left(\sinh^2 \tfrac{1}{2} h_{n-1}(b)\right)\left(1 - \cos \tfrac{1}{2} \lambda_n(b)\right)$$

$$= \frac{H_{n-1}(b) - 1}{2}\left(1 - \sqrt{\frac{1 + \cos\lambda_n(b)}{2}}\right)$$

$$= \frac{\frac{2(n-1)\cosh^2 b}{1 + (n-2)\cosh b} - 2}{2}\left(1 - \sqrt{\frac{1 + \cos\ _n(b)}{2}}\right)$$

$$= \frac{\frac{2(n-1)B^2}{1 + (n-2)B} - 2}{2}\left(1 - \sqrt{\frac{1 + \frac{B}{1 + (n-1)B}}{2}}\right)$$

$$= \frac{(B-1)((n-1)B + 1)}{(n-2)B+1}\left(1 - \frac{1}{\sqrt{2}}\sqrt{\frac{nB + 1}{(n-1)B + 1}}\right).$$

Now if $n > 2$:

$$\cosh \tilde{b} - 1 \le (B-1)\frac{n-1}{n-2}\left(1 - \sqrt{\tfrac{1}{2}}\right) < (B-1)\,2\left(1 - \sqrt{\tfrac{1}{2}}\right)$$

$$< 0.6(B-1).$$

If $n = 2$, then:

$$\cosh \tilde{b} - 1 \leq (B-1)(B+1)\left(1 - \sqrt{\frac{1}{2}}\sqrt{\frac{2B+1}{B+1}}\right)$$

$$= (B-1)\left(B + 1 - \sqrt{\frac{1}{2}}\ \sqrt{2B^2 + 3B + 1}\right).$$

Define $\rho(B) = B + 1 - \sqrt{\frac{1}{2}}\ \sqrt{2B^2 + 3B + 1}$. Then

$$\frac{d\rho}{dB} = 1 - \frac{1}{2\sqrt{2}}\ \frac{4B+3}{\sqrt{2B^2+3b+1}}.$$

Now: $16B^2 + 24B + 8 < 16B^2 + 24B + 9$; so

$$8(2B^2+3B+1) = (4B+3)^2;$$

from which it follows that

$$1 < \frac{1}{2\sqrt{2}}\ \frac{4B+1}{\sqrt{2B^2+3B+1}};$$

hence $\frac{d\rho}{dB} < 0$ for all $B > 0$;

$$B > 1 \Longrightarrow \rho(B) < \rho(1) = 2 - \sqrt{\frac{1}{2}}\ \sqrt{6} = 2 - \sqrt{3} < 0.6;$$

therefore $\cosh \tilde{b} - 1 < 0.6(B-1)$. □

<u>Corollary 3.7</u>. <u>If</u> $d(x,y) = a \Longrightarrow d(Tx,Ty) = b$, <u>then either there is some</u> $\tilde{a}$ <u>and</u> $\tilde{b}$ <u>such that</u> $d(x,y) = \tilde{a} \Longrightarrow d(Tx,Ty) = \tilde{b}$, $\tilde{b} \leq b$ <u>and</u> $\frac{2\pi}{\lambda_n(\tilde{b})}$ <u>is an integer, or there are arbitrarily small</u> $\tilde{a}$ <u>and</u> $\tilde{b}$ <u>such that</u> $d(x,y) = \tilde{a} \Longrightarrow d(Tx,Ty) = \tilde{b}$.

<u>Proof</u>. If we assume that there is no $\tilde{a}$ and $\tilde{b}$ such

that $d(x,y) = \tilde{a} \Longrightarrow d(Tx,Ty) = \tilde{b}$, $\tilde{b} \le b$ and $\frac{2\pi}{\lambda_n(\tilde{b})}$ is an integer, then we can define a sequence of a_k's and b_k's by letting $a_0 = a$ and $b_0 = b$ and then define a_k and b_k inductively for $k \ge 1$ by setting $a_k = \tilde{a}_{k-1}$ and $b_k = \tilde{b}_{k-1}$ where $\tilde{a}_{k-1}$ and $\tilde{b}_{k-1}$ are the results obtained from Lemma 3.6 using a_{k-1} and b_{k-1} in place of a and b.

Then $\cosh b_k - 1 < 0.6(\cosh b_{k-1} - 1)$ for $k \ge 1$, so

$$\cosh b_k - 1 < (0.6)^k(\cosh b_0 - 1),$$

hence

$\cosh b_k - 1 \to 0$ as $k \to \infty$; therefore $b_k \to 0$ as $k \to \infty$.

For any $k > 0$, $d(x,y) = a_k \Longrightarrow d(Tx,Ty) = b_k$. Let x and y be two points in H^n such that $d(x,y) = a$. Let i_k be the smallest integer greater than $\frac{a}{a_k}$; then by Lemma 1.1, $b = d(Tx,Ty) \le i_k b_k$, and so

$i_k \ge \frac{b}{b_k}$, which $\to \infty$ as $k \to \infty$ since $b_k \to 0$ as $k \to \infty$.

But then

$$\frac{a}{a_k} \to \infty \quad \text{as} \quad k \to \infty;$$

therefore

$$a_k \text{ also} \to 0 \text{ as } k \to \infty. \quad \square$$

Lemma 3.8. If $d(x,y) = a \Longrightarrow d(Tx,Ty) = b > 0$ and either $\frac{2\pi}{\lambda_n(a)}$ or $\frac{2\pi}{\lambda_n(b)}$ are integers, then $a = b$.

Proof. Let $k = \frac{2\pi}{\lambda_n(a)}$, $i = \frac{2\pi}{\lambda_n(b)}$. Then either k or i (or both) is an integer. Let $x, y_0, y_1, \ldots \in H^n$ such that $d(x,y_j) = a$ for all j and $d(y_j, y_{j+1}) = a$ and $y_j \neq y_{j+2}$ for all j.

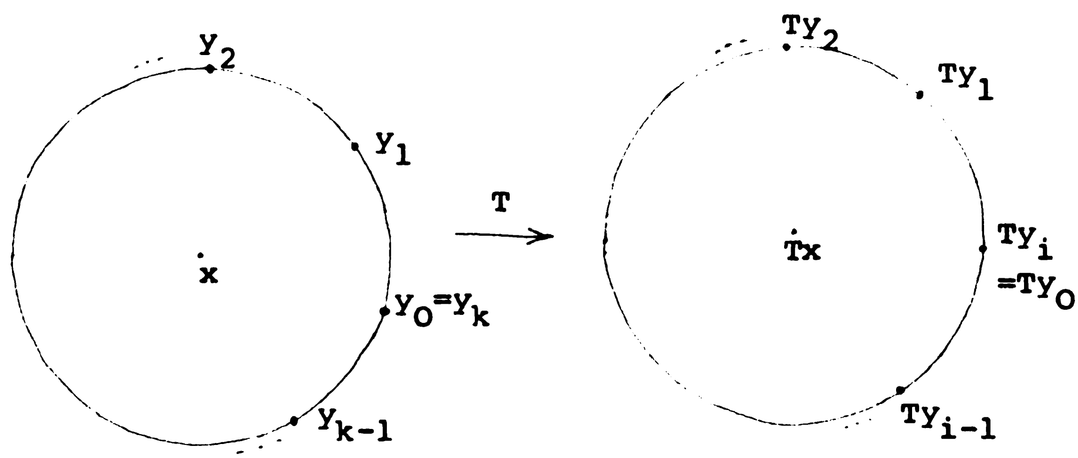


Figure 3-6

We have two cases:

Case 1: i is an integer, k is not. Then $Ty_0 = Ty_i$ and $y_0 \neq y_i$.

Case 2: k is an integer. Then $y_0 = y_k$, and hence $Ty_0 = Ty_k$. Thus, i is an integer and $i \leq k$ since i is the minimum integer such that $Ty_0 = Ty_i$. If $i = k$, then $a = b$, since $\lambda_n(c)$ is a monotonic function of c. If $i < k$, then $y_0 \neq y_i$, but $Ty_0 = Ty_i$.

So in either case we have either $a = b$ or for any two points $\tilde{y}_0$ and $\tilde{y}_i$ such that $d(\tilde{y}_0,\tilde{y}_i) = d(y_0,y_i)$; $T\tilde{y}_0 = T\tilde{y}_i$, hence $d(T\tilde{y}_0,T\tilde{y}_i) = 0$. But then by Lemma 1.1; for any $w,z \in H^n$, $d(Tw,Tz) = 0$, hence the image of T is one point and hence $b = 0$. Therefore $b > 0 \Longrightarrow a = b$. □

Lemma 3.9. *If* $d(x,y) = a \Longrightarrow d(Tx,Ty) = b$, *then the image under* T *of a circle of radius* $\frac{1}{2}h_{n-1}(a)$ *is contained in some circle of radius* $\frac{1}{2}h_{n-1}(b)$.

Proof. Let $H^2 \subset H^n$ be some plane, and let $\mathcal{C} \subset H^2$ be a circle in that plane with center x and radius $\frac{1}{2}h_{n-1}(a)$. There are points $e_1,e_2,\ldots,e_{n-1}$ such that $d(e_i,e_j) = a$ for $i \neq j$, and point x is the centroid of the $(n-2)$-simplex $e_1,e_2,\ldots,e_{n-1}$, and plane H^2 is normal to the $(n-2)$-plane formed by $e_1,\ldots,e_{n-1}$. Then $\mathcal{C}$ is the locus of points which can be appended to $e_1,\ldots,e_{n-1}$ to form an

example: $n = 3$

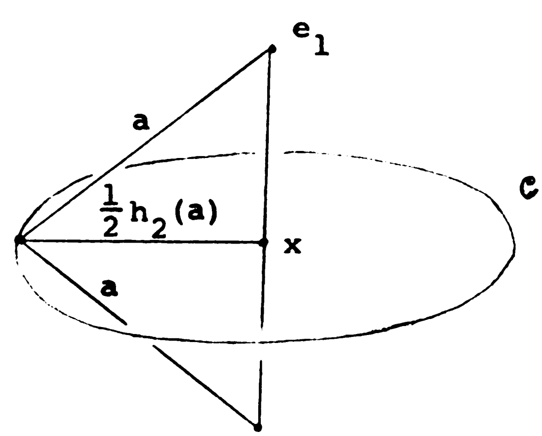


Figure 3-7

a-equilateral $(n-1)$-simplex. But the image of an a-equilateral $(n-1)$-simplex is b-equilateral $(n-1)$-simplex, so the image of $\mathcal{C}$ under T is contained in the circle whose center is the centroid of the $(n-2)$-simplex $Te_1, Te_2, \ldots, Te_{n-1}$, whose radius is b, and whose plane is normal to the $(n-2)$-plane formed by $Te_1, Te_2, \ldots, Te_{n-1}$. □

Lemma 3.10. If there are two sequences of distances $\{a_i\}_{i=0}^{\infty}$ and $\{b_i\}_{i=0}^{\infty}$ such that for all i, $d(x,y) = a_i \Longrightarrow d(Tx,Ty) = b_i$ and $a_i \to 0$ and $b_i \to 0$ as $i \to \infty$, then $\dfrac{\sinh \frac{1}{2} h_{n-1}(a_i)}{\sinh \frac{1}{2} h_{n-1}(b_i)}$ is constant over i.

Proof. T is continuous since $a_i \to 0$ and $b_i \to 0$ as $i \to \infty$. Let $x \in H^n$ and let $\mathcal{C}$ be some circle with center x and radius $\frac{1}{2} h_{n-1}(a_i)$. Fix $y \in \mathcal{C}$. Consider the function given by $f(w) = \angle Twx'Ty$ for $w \in \mathcal{C}$, where x' is the center of circle $T\mathcal{C}$. Then T continuous $\Longrightarrow f$ is continuous. But for any $y_1, y_2 \in \mathcal{C}$ such that $d(y_1,y_2) = a_i$, we have $d(Ty_1,Ty_2) = b_i$, so that $f(\tilde{y}_1) - f(\tilde{y}_2)$ is always $\pm \angle Ty_1x'Ty_2 = \pm(f(y_1) - f(y_2))$ as $\tilde{y}_1$ and $\tilde{y}_2$ vary over $\mathcal{C}$ with $\angle \tilde{y}_1x\tilde{y}_2 = \angle y_1xy_2$. But continuity of $f \Longrightarrow f(\tilde{y}_1) - f(\tilde{y}_2) = f(y_1) - f(y_2)$ for all such $\tilde{y}_1$, $\tilde{y}_2$. Hence if $y_1, y_2, y_3, \ldots$ is a sequence of points on $\mathcal{C}$ such that $d(y_k,y_{k+1}) = a_i$ and $\angle y_kxy_{k+1} = \angle y_1xy_2$, then $Ty_1, Ty_2, \ldots$ are on $T\mathcal{C}$ such that $d(Ty_k,Ty_{k+1}) = b_i$ and $\angle Ty_kx'Ty_{k+1} = \angle Ty_1x'Ty_2$.

For any $i \geq 0$, let $y_1, y_2, \ldots, y_{m_i}$ be points on $\mathcal{C}$ such that for all $k < m_i$, $d(y_k,y_{k+1}) = a_i$, $\angle y_kxy_{k+1} = \angle y_1xy_2$ and $d(y_1,y_k) > a_i$, and $d(y_1,y_{m_i}) \leq a_i$; then $d(Ty_1,Ty_{m_i}) \leq 2b_1$ by Lemma 1.1.

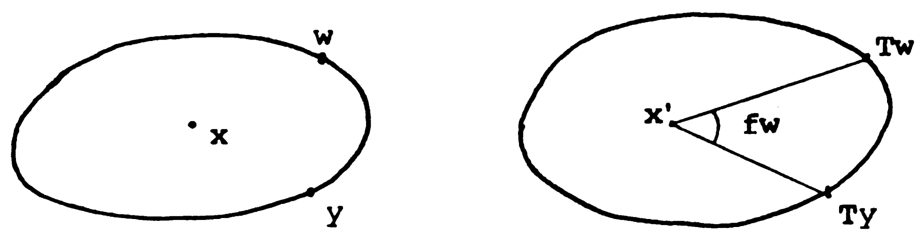


Figure 3-8

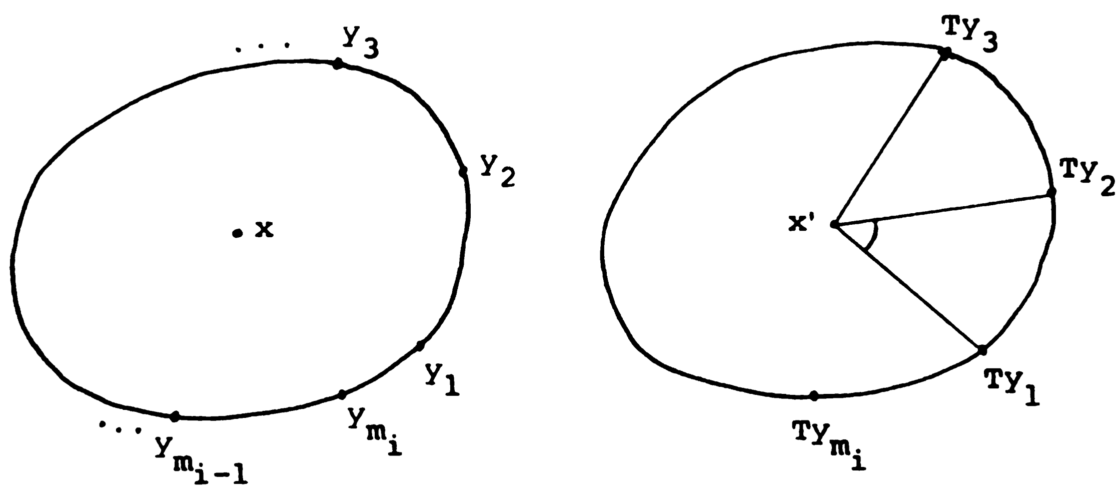


Figure 3-9

Let

$$c \;=\; \text{circumference of } \mathcal{C} \;=\; \lim_{i\to\infty} m_i a_i$$

$$= \; 2\pi \sinh \tfrac{1}{2} h_{n-1}(a_0),$$

and let

$$\tilde{c} \;=\; \text{circumference of } T\mathcal{C} \;=\; \lim_{i\to\infty} m_i b_i$$

$$= \; 2\pi \sinh \tfrac{1}{2} h_{n-1}(b_0).$$

So that $\dfrac{\sinh \frac{1}{2} h_{n-1}(a_0)}{\sinh \frac{1}{2} h_{n-1}(b_0)} = \lim\limits_{i\to\infty} \dfrac{a_i}{b_i}$; likewise,

$\dfrac{\sinh \frac{1}{2} h_{n-1}(a_i)}{\sinh \frac{1}{2} h_{n-1}(b_i)} = \lim\limits_{i\to\infty} \dfrac{a_i}{b_i}$ by using a_i and b_i instead of a_0 and b_0 above. Therefore $\dfrac{\sinh \frac{1}{2} h_{n-1}(a_i)}{\sinh \frac{1}{2} h_{n-1}(b_i)}$ is constant over i. □

<u>Lemma 3.11</u>. <u>If there are two sequences of distances</u> $\{a_i\}_{i=1}^{\infty}$, <u>and</u> $\{b_i\}_{i=1}^{\infty}$ <u>such that for all</u> i, $d(x,y) = a_i \Longrightarrow d(Tx,Ty) = b_i$ <u>and</u> $a_i \to 0$ <u>and</u> $b_i \to 0$ <u>as</u> $i \to \infty$, <u>then there is an integer</u> N <u>such that for</u> $i > N$, $a_i = b_i$.

<u>Proof</u>. For any $i \geq 1$, we have by Lemma 3.3 that $d(x,y) = h_n(a_i) \Longrightarrow d(Tx,Ty) = h_n(b_i)$, so if we let $a_0 = h_n(a_i)$, $b_0 = h_n(b_i)$, then $\{a_i\}_{i=0}^{\infty}$, $\{b_i\}_{i=0}^{\infty}$ satisfies the hypothesis of Lemma 3.10 so we have

$$\frac{\sinh \frac{1}{2} h_{n-1}(a_i)}{\sinh \frac{1}{2} h_{n-1}(b_i)} \;=\; \frac{\sinh \frac{1}{2} h_{n-1}(h_n(a_i))}{\sinh \frac{1}{2} h_{n-1}(h_n(b_i))}.$$

Therefore

$$\frac{\sinh\frac{1}{2}\, h_{n-1}(a_i)}{\sinh\frac{1}{2}\, h_{n-1}(h_n(a_i))} = \frac{\sinh\frac{1}{2} h_{n-1}(b_i)}{\sinh\frac{1}{2} h_{n-1}(h_n(b_i))} \quad \text{for all } i.$$

Let $A = \cosh a$, and define ρ by

$$\rho(A) = \left(\frac{\sinh\frac{1}{2} h_{n-1}(a)}{\sinh\frac{1}{2} h_{n-1}(h_n(a))}\right)^2 = \frac{\cosh h_{n-1}(a) - 1}{\cosh h_{n-1}(h_n(a)) - 1}.$$

By Lemma 3.1, $\cosh h_{n-1}(a) = \frac{2(n-1)A^2}{1+(n-2)A} - 1,$

$$\cosh h_{n-1}(h_n(a)) = \frac{2(n-1)(H_n(a))^2}{1+(n-2)H_n(a)} - 1, \text{ and}$$

$H_n(a) = \frac{2nA^2}{1+(n-1)A} - 1,$ so it is easily seen that $\rho(A)$ is a rational fraction in A. There are only a finite number of roots of $\frac{d\rho}{dA}$ which are > 1, so there is some $\widetilde{A}$ such that $\rho(A)$ is monotone in the interval $(1,\widetilde{A})$. For i sufficiently large, $\cosh a_i$ and $\cosh b_i$ are in $(1,\widetilde{A})$, hence there is an N such that for $i > N$, $\rho(\cosh a_i) = \rho(\cosh b_i)$ only if $\cosh a_i = \cosh b_i$. Therefore $a_i = b_i$ for all $i > N$. □

<u>Corollary 3.12</u>. <u>If</u> $d(x,y) = a \Longrightarrow d(Tx,Ty) = b$, <u>then there is a distance</u> $\tilde{a} \leq a$ <u>such that</u> $d(x,y) = \tilde{a} \Longrightarrow d(Tx,Ty) = \tilde{a}$.

<u>Proof</u>. This is immediate from Corollary 3.7 and Lemmas 3.8 and 3.11. □

For the remainder of this section we assume that a is a preserved distance, that is, that

$d(x,y) = a \Longrightarrow d(Tx,Ty) = a.$

Corollary 3.13. *If* T *preserves* a *and* $n \geq 5$ *or* $n = 3$, *then* T *is an isometry*.

Proof. If $n \geq 5$, by Lemma 3.5, $\cos \lambda_n(a) = \frac{\cosh a}{1 + (n-1)\cosh a}$. Now $\cosh a > 1$, so $0 < \frac{1}{n} < \cos \lambda_n(a) < \frac{1}{n-1} \leq \frac{1}{4}$. Then

$$\frac{2\pi}{4} > \lambda_n(a) > \cos^{-1}\frac{1}{4} > \frac{2\pi}{5}.$$

Hence

$$4 < \frac{2\pi}{\lambda_n(a)} < 5 \quad \text{for all } a, \text{ so } \frac{2\pi}{\lambda_n(a)} \text{ is non-integer.}$$

Therefore by Corollary 3.7 and Lemma 3.8, T has arbitrarily small preserved distances. Then by Corollary 3.4 and Lemma 1.2, T is an isoemtry.

If $n = 3$, then $\cos \lambda_3(a) = \frac{\cosh a}{1 + 2\cosh a}$. So

$$\frac{1}{3} < \cos \lambda_3(a) < \frac{1}{2}.$$

So that

$$\frac{2\pi}{5} > \cos^{-1}\frac{1}{3} > \lambda_3(a) > \cos^{-1}\frac{1}{2} = \frac{2\pi}{6}.$$

Therefore $5 < \frac{2\pi}{\lambda_3(a)} < 6$, and $\frac{2\pi}{\lambda_3(a)}$ is non-integer.

Again T has arbitrarily small preserved distances and is an isometry. □

Lemma 3.14. *If* $n = 4$ *and* T *preserves* a, *then* T *is an isometry*.

Proof. For any $\hat{a}$: $\cos \lambda_4(\hat{a}) = \dfrac{\cosh \hat{a}}{1 + 3\cosh \hat{a}}$. Then

$$\frac{1}{4} < \cos \lambda_4(\hat{a}) < \frac{1}{3}.$$

Hence

$$\cos^{-1}\frac{1}{4} > \lambda_4(\hat{a}) > \cos^{-1}\frac{1}{3}.$$

So that $\dfrac{2\pi}{\lambda_4(\hat{a})}$ is an integer, $\lambda_4(\hat{a}) = \dfrac{2\pi}{5}$, and

$$\cos \lambda_4(\hat{a}) \approx 0.30902,$$

$$\cosh \hat{a} \approx 4.2361$$

$$\hat{a} \approx 2.1226.$$

Now if $a < \hat{a}$, then by Corollary 3.7 and Lemma 3.11, T has arbitrarily small preserved distances. Also if $a \geq \hat{a}$, then either T has arbitrarily small preserved distances or T preserves $\hat{a}$.

But if $\hat{a}$ is preserved by T, then by Lemma 3.3, $h_4(\hat{a})$ is preserved by T, and by using the construction of Lemma 3.6 with $h_4(\hat{a})$ replacing a and b in the lemma, we get a preserved distance $\tilde{a}$ which by calculation is about 0.241226. So $\tilde{a}$ is a preserved distance smaller than $\hat{a}$, so by Corollary 3.7 and Lemma 3.11, T has arbitrarily small preserved distances.

Therefore T is an isometry by Lemma 1.3. □

Lemma 3.15. *If $n = 2$ and a is preserved by T, and $\dfrac{2\pi}{\lambda_n(a)}$ is an integer, then T is an isometry.*

<u>Proof</u>. Let $k = \frac{2\pi}{\lambda_n(a)}$.

<u>Case 1</u>: k is even. Let $\tilde{k} = \frac{1}{2}k$. Let e_i, f_{ij} be points for $i \geq 1$, $1 \leq j \leq \tilde{k}-1$ so that for all i, and for $2 \leq j \leq \tilde{k}-1$, $e_i e_{i+1} f_{i1}$, $e_{i+1} e_{i+2} f_{i\tilde{k}-1}$, and $e_{i+1} f_{ij-1} f_{ij}$ are equilateral triangles.

example:

$k = 8$

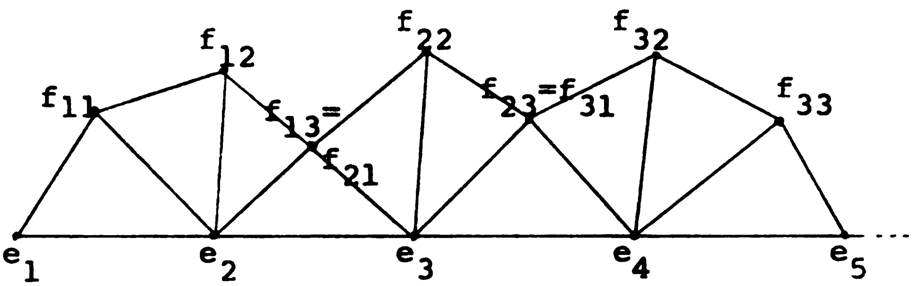


Figure 3-10

Then $h_2(a)$ is preserved by $T \Rightarrow T$ is a motion on this set of points. Also $e_1, e_2, \ldots$ are collinear since k is even. Hence there exists a distance $\tilde{a}$ (= a) such that for any collinear set of points $\{e_i\}$ such that $d(e_i, e_{i+1}) = \tilde{a}$, then T is a motion on the set $\{e_i\}$; so that $m\tilde{a}$ is a preserved distance for all positive integers m.

<u>Case 2</u>: k is odd. Let $\tilde{k} = \frac{1}{2}(k+1)$. For $i \geq 1$, $i \leq j \leq k+1$, let e_i, f_{ij} be points so that $e_i f_{i1} f_{i2}$ and $f_{i1} f_{ij} f_{ij+1}$ for $2 \leq j \leq \tilde{k}$, and $e_{i+1} f_{ij} f_{ij+1}$ for $\tilde{k} \leq j \leq k$ are a-equilateral triangles and $e_{i+1} \neq f_{i1}$.

Then $h_2(a)$ is preserved by $T \Rightarrow T$ is a motion on this set of points. Also k is odd $\Rightarrow e_1, e_2, \ldots$ are

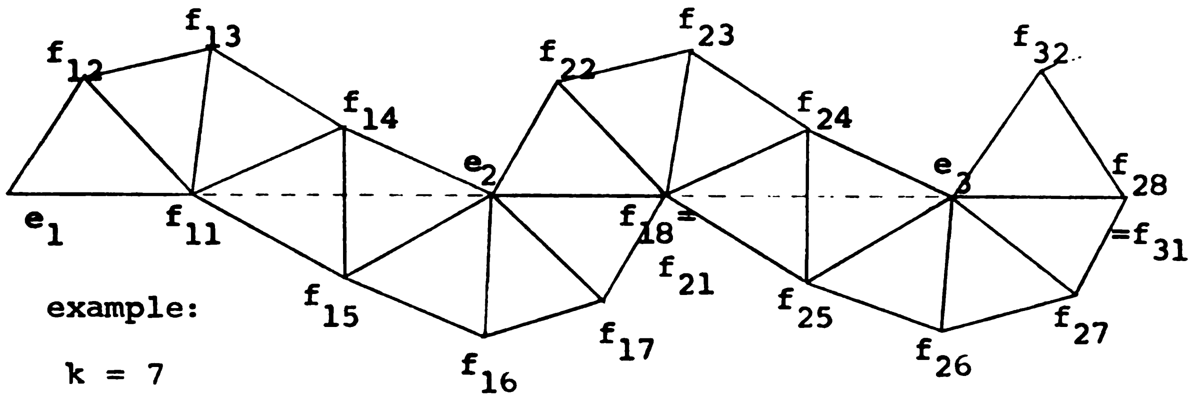


Figure 3-11

collinear. Hence there exists a distance a $(= a + h_2(a))$ such that for any collinear set of points $\{e_i\}$ with $d(e_i, e_{i+1}) = \tilde{a}$, then T is a motion of the set $\{e_i\}$.

So whether k is even or odd, there is a preserved distance $\tilde{a}$ such that $m\tilde{a}$ is preserved by T for all positive integers m.

Consider five points e, f_1, f_2, g_1, g_2 such that $d(e,f_1) = d(e,f_2) = d(g_1,g_2) = \tilde{a}$ and $d(e,g_1) = d(e,g_2) = m\tilde{a}$ and $d(f_1,g_1) = d(f_2,g_2) = (m-1)\tilde{a}$ for an integer $m \geq 2$.

Then T is a motion on this point set, hence $\hat{a} = d(f_1,f_2)$ is a preserved distance. Let β be the angle $\angle g_1 e g_2$. The law of cosines for triangle ef_1f_2 yields:

$$\cosh \hat{a} = \cosh^2 \tilde{a} - \sinh^2 \tilde{a} \cos\beta,$$

and applied to the triangle eg_1g_2 it yields:

$$\cosh \tilde{a} = \cosh^2 m\tilde{a} - \sinh^2 m\tilde{a} \cos \beta.$$

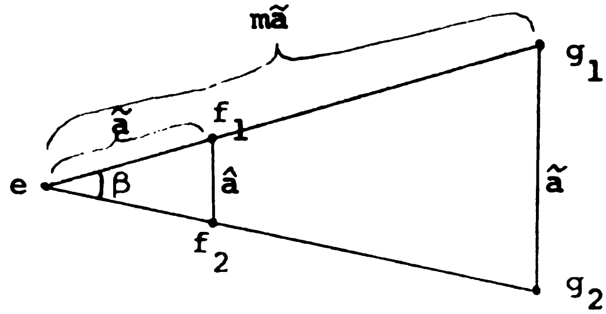


Figure 3-12

So

$$\cosh \tilde{a} - 1 = \sinh^2 m\tilde{a}\,(1 - \cos \beta)$$

and

$$\cosh \hat{a} - 1 = \sinh^2 \tilde{a}\,(1 - \cos \beta).$$

Therefore

$$\cosh \hat{a} = 1 + \frac{\sinh^2 \tilde{a}\,(\cosh \tilde{a} - 1)}{\sinh^2 m\tilde{a}}.$$

Now as $m \to \infty$,

$$1 + \frac{\sinh^2 \tilde{a}\,(\cosh \tilde{a} - 1)}{\sinh^2 m\tilde{a}} \to 1,$$

hence $\hat{a} \to 0$ as $m \to \infty$, and T is an isometry by Lemma 1.3. □

Corollary 3.16. *If* $n = 2$ *and* a *is a preserved distance of* T, *then* T *is an isometry.*

Proof. By Corollaries 3.7 and 3.12, either T has arbitrarily small preserved distances (making T an isometry by Lemma 1.3) or there is a preserved distance $\tilde{a}$ such that $\frac{2\pi}{\lambda_2(\tilde{a})}$ is an integer, again making T an isometry by Lemma 3.15. □

Therefore:

Theorem 2. *If* $n \geq 2$, H^n *is n-dimensional hyperbolic space,* $T: H^n \to H^n$, *and* $d(x,y) = a \Longrightarrow d(Tx,Ty) = b \neq 0$, *then* $a = b$ *and* T *must be an isometry.*

§4. SPHERICAL AND ELLIPTIC SPACES

Let $T: M \to M$ where M is either S^n, the n-dimensional sphere in E^{n+1} (with the shortest arc metric), or $\mathcal{E}^n$, the n-dimensional elliptic space. Let T have the property that there are two distances a and b such that $d(x,y) = a \Longrightarrow d(Tx,Ty) = b$. Our aim in this section is to show that T must be an isometry if a is small enough. Let a be restricted so that $a \leq \frac{\pi}{2}$ if M is a spherical space and $a < \frac{\pi}{3}$ if M is an elliptic space, then $\cos a \geq 0$ and a-equilateral n-simplices have unique centroids.

In this section, the following notation will be used:

$h_n(c)$ = the distance between the two remaining points of two c-equilateral n-simplices having n points in common;

$H_n(c)$ = $\cos(h_n(c))$;

$c_n(c)$ = distance between the centroid of a c-equilateral n-simplex and any of the vertices;

$C_n(c)$ = $\cos(c_n(c))$;

$d_n(c)$ = $\frac{1}{2}h_n(c) - c_n(c)$;

$D_n(c)$ = $\cos(d_n(c))$;

$\lambda_n(c)$ = angle between two adjacent $(n-1)$-simplices which are faces of a c-equilateral n-simplex.

Note: $c_0(c) = 0$; $c_1(c) = \frac{1}{2}c$; so $C_0(c) = 1$, and $C_1(c) = \sqrt{\frac{1 + \cos c}{2}}$.

When there is no ambiguity in the value of the argument of the functions h_n, H_n, c_n, C_n, d_n, D_n, and λ_n, they will be written without argument.

Lemma 4.1. $H_n(c) = \dfrac{2n(\cos c)^2}{1 + (n-1)\cos c} - 1$ for $n \geq 1$.

Proof. Let $A = \cos c$ and let $f_1, f_2, g_1, g_2, e_1, \ldots, e_n$ be as in Lemma 3.1.

Then $C_n/C_{n-1} = D_n$ and $H_n = 2A^2/C_{n-1}^2 - 1$, the same as in Lemma 3.1.

Also

$$d_n = \frac{1}{2}h_n - c_n$$

$$\Longrightarrow \quad \cos d_n = \cos\frac{1}{2}h_n \cos c_n + \sin\frac{1}{2}h_n \sin c_n;$$

$$\Longrightarrow \quad \frac{C_n}{C_{n-1}} = D_n = \sqrt{\frac{H_n+1}{2}}\; C_n + \sqrt{\frac{1-H_n}{2}}\sqrt{1-C_n^2}$$

$$\Longrightarrow \quad \frac{H_n-1}{2}(C_n^2-1) = \left(\frac{H_n+1}{2} + \frac{1}{C_{n-1}^2} - \frac{\sqrt{2}\sqrt{H_n+1}}{C_{n-1}}\right)C_n^2 .$$

From this point on, the proof is the same as in Lemma 3.1. □

Lemma 4.2. If $8n^2A^3 + (-n^2+10n-1)A^2 + (-2n+2)A - 1 \geq 0$ then a circle of radius h_n has two points at distance c from each other.

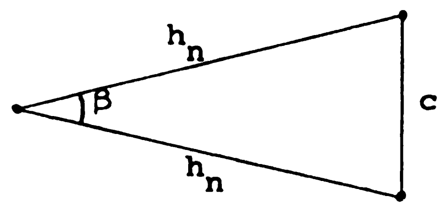


Figure 4-1

<u>Proof</u>. What we need is a sufficient condition that there exists an angle β such that

$$\cos a = \cos^2 h_n + \sin^2 h_n \cos \beta .$$

That condition is that $-1 \le \cos \beta \le 1$ where $\cos \beta = \dfrac{\cos a - \cos^2 h_n}{\sin^2 h_n}$.

Then $\cos \beta = \dfrac{A-H_n^2}{1-H_n^2}$ so that $H_n < 1$ and $A < 1 \Longrightarrow \cos \beta < 1$. Now

$$\cos \beta = \frac{A-H_n^2}{1-H_n^2}$$

$$= \frac{{A - \frac{2nA^2}{1+(n-1)A} - 1}^{2}}{{1 - \frac{2nA^2}{1+(n-1)A} - 1}^{2}}$$

$$= \frac{-4n^2A^4+(5n^2-6n+1)A^3+(-n^2+8n-3)A^2+(-2n+3)A-1}{-4n^2A^4+(4n^2-4n)A^3+4nA^2}$$

$$= \frac{(1-A)(4n^2A^3+(-n^2+6n-1)A^2+(-2n+2)A-1)}{(1-A)(4n^2A^3+4nA^2)} .$$

All we need now is $-1 \le \cos\beta$.

But

$$0 \le 8n^2A^3 + (-n^2+10n-1)A^2 + (-2n+2)A - 1$$

$$\Longrightarrow \quad -4n^2A^3 - 4nA^2 \le 4n^2A^3 + (-n^2+6n-1)A^2 + (-2n+2)A-1$$

$$\Longrightarrow \quad -1 \le \cos\beta . \quad \square$$

<u>Remark</u>: For any positive integer $n \ge 1$, the cubic (in A): $p(A) = 8n^2A^3+(-n^2+10n-1)A^2+(-2n+2)A-1$ has exactly one positive root r_n. For all $n \ge 2$, $r_n > \frac{1}{8}$ and $r_n < \frac{3}{10}$. In fact: $r_2 \approx .2553$, r_n is monotone decreasing as n increases, and $\lim_{n\to\infty} r_n = \frac{1}{8}$.

Thus, $p(A) \ge 0 \iff A \ge r_n$. Therefore if $n \ge 2$, a sufficient condition that $p(A) \ge 0$ is $A \ge r_2 \approx .2553$, that is, $c \le \arccos r_2 \approx 1.313$.

For the rest of this section we assume that $a \le \arccos r_n$ $(\le \arccos \frac{1}{8} \approx 1.445)$.

<u>Lemma 4.3</u>. <u>If</u> $d(x,y) = a \Longrightarrow d(Tx,Ty) = b$ <u>and</u> $a < \arccos r_n$, <u>then</u> $d(x,y) = h_n(a) \Longrightarrow d(Tx,Ty) = h_n(b)$.

<u>Proof</u>. The proof is the same as that of Lemma 3.3, using Lemma 4.2 in place of Lemma 3.2. $\square$

<u>Lemma 4.4</u>. $\cos\lambda_n(c) = \frac{\cos c}{1+(n-1)\cos c}$.

Proof. $\lambda_n(c)$ is the angle opposite the base of a triangle whose sides are $\frac{1}{2}h_{n-1}$, $\frac{1}{2}h_{n-1}$, and c (see Figure 3.3 at Lemma 3.5).

The law of cosines yields:

$$\cos c = \cos^2 \tfrac{1}{2}h_{n-1} + \sin^2 \tfrac{1}{2}h_n \cos \lambda_n$$

$$\Longrightarrow \cos c = \frac{H_{n-1}+1}{2} - \frac{H_{n-1}-1}{2} \cos \lambda_n$$

$$\Longrightarrow \cos \lambda_n = \frac{H_{n-1} + 1 - 2\cos c}{H_{n-1}-1} = \frac{\cos c}{1 + (n-1)\cos c} . \quad \square$$

Lemma 4.5. *If* $d(x,y) = a \Longrightarrow d(Tx,Ty) = b$ *and* $a < \arccos r_n$ *and* $\frac{2\pi}{\lambda_n(b)}$ *is not an integer, then there are* $\tilde{a}$ *and* $\tilde{b}$ *such that* $d(x,y) = \tilde{a} \Longrightarrow d(Tx,Ty) = \tilde{b}$ *and* $(1 - \cos \tilde{b}) < 0.6(1 - \cos b)$.

Proof. Constructing the points $e_1,\ldots,e_{n-1},f_0,\ldots,f_k$ as in the proof of Lemma 3.6, we get $\tilde{a} = d(f_0,f_k)$, $\tilde{b} = d(Tf_0,Tf_k)$ and the law of cosines yields:

$$\cos \tilde{b} = \cos^2 \tfrac{1}{2}h_{n-1}(b) + \sin^2 \tfrac{1}{2}h_{n-1}(b) \cos \beta$$

$$\Longrightarrow 1 - \cos \tilde{b} = \sin^2 \tfrac{1}{2}h_{n-1}(b)(1 - \cos \beta)$$

$$\leq \frac{1 - H_{n-1}(b)}{2} (1 - \cos \tfrac{1}{2}\lambda_n(b))$$

$$\Longrightarrow 1 - \cos \tilde{b}$$

$$\leq \frac{(1 - \cos b)(1 + (n-1)\cos b)}{1 + (n-2)\cos b} \left(1 - \frac{1}{\sqrt{2}} \sqrt{\frac{1 + n\cos b}{1 + (n+1)\cos b}}\right).$$

Then just as in Lemma 3.6, we get:

if $n > 2$: $(1 - \cos \tilde{b}) < (1 - \cos b) 2 \left(1 - \sqrt{\frac{1}{2}}\right) < 0.6(1 - \cos b)$;

if $n = 2$: $\cos b > 0 \Longrightarrow \rho(\cos b) < \rho(0) = 1 - \sqrt{\frac{1}{2}} < 0.6$;

therefore $1 - \cos \tilde{b} < 0.6(1 - \cos b)$. □

Corollary 4.6. If $d(x,y) = a \Longrightarrow d(Tx,Ty) = b$ and $a < \arccos r_n$, then either there is some $\tilde{a}$ and $\tilde{b}$ such that $d(x,y) = \tilde{a} \Longrightarrow d(Tx,Ty) = \tilde{b}$ and $\frac{2\pi}{\lambda_n(\tilde{b})}$ is an integer or there are arbitrarily small $\tilde{a}$ and $\tilde{b}$ such that $d(x,y) = \tilde{a} \Longrightarrow d(Tx,Ty) = \tilde{b}$.

Proof. The proof is the same as that of Corollary 3.7 with the 'cos' function replacing the 'cosh' function. □

Lemma 4.7. If $d(x,y) = a \Longrightarrow d(Tx,Ty) = b$ and either $\frac{2\pi}{\lambda_n(a)}$ or $\frac{2\pi}{\lambda_n(b)}$ are integers, then $a = b$.

Proof. The proof is the same as that of Lemma 3.8. □

Lemma 4.8. If $d(x,y) = a \Longrightarrow d(Tx,Ty) = b$, then the image under T of a circle of radius $\frac{1}{2}h_{n-1}(a)$ is contained in some circle of radius $\frac{1}{2}h_{n-1}(b)$.

Proof. The proof is the same as that of Lemma 3.9. □

Lemma 4.9. If there are two sequences of distances $\{a_i\}_{i=0}^{\infty}$, and $\{b_i\}_{i=0}^{\infty}$ such that for all i, $d(x,y) = a_i \Longrightarrow d(Tx,Ty) = b_i$ and $a_i \to 0$ and $b_i \to 0$ as $i \to \infty$, then $\dfrac{\sin \frac{1}{2} h_{n-1}(a_i)}{\sin \frac{1}{2} h_{n-1}(b_i)}$ _is constant over_ i.

Proof. The proof is the same as that of Lemma 3.10 with the function 'sin' replacing the function 'sinh.' □

Lemma 4.10. If there are two sequences of distances $\{a_i\}_{i=1}^{\infty}$, _and_ $\{b_i\}_{i=1}^{\infty}$ _such that for all_ i, $d(x,y) = a_i \Longrightarrow d(Tx,Ty) = b_i$ _and_ $a_i \to 0$ _and_ $b_i \to 0$ _as_ $i \to \infty$, _then there is an integer_ N _such that for_ $i > N$, $a_i = b_i$.

Proof. The proof is the same as that of Lemma 3.11, with 'sin' and 'cos' replacing 'sinh' and 'cosh' and noting that $\frac{d\rho}{dA}$ has only finitely many roots < 1. □

Corollary 4.11. If $d(x,y) = a \Longrightarrow d(Tx,Ty) = b$, _and_ $a < \arccos r_n$, _then there is a distance_ $\tilde{a} \le a$ _such that_ $d(x,y) = \tilde{a} \Longrightarrow d(Tx,Ty) = \tilde{a}$.

Proof. This is immediate from Lemmas 4.6, 4.7 and 4.10. □

For the remainder of this section we assume that a is a preserved distance, that is, that $d(x,y) = a \Longrightarrow d(Tx,Ty) = a$.

Corollary 4.12. If $n \ge 4$, a _is a preserved distance under_ T, _and_ $a < \arccos r_n$, _then_ T _is an isometry._

Proof. $\cos a < 1$ and $\cos \lambda_n = \dfrac{\cos a}{1 + (n-1)\cos a}$;

$$\Longrightarrow \quad 0 < \cos \lambda_n < \frac{1}{n} \leq \frac{1}{4}$$

$$\Longrightarrow \quad \frac{2\pi}{4} > \lambda_n > \frac{2\pi}{5}$$

$$\Longrightarrow \quad 4 < \frac{2\pi}{\lambda_n} < 5.$$

Hence $\frac{2\pi}{\lambda_n}$ is not an integer.

So by Corollary 4.6 and Lemmas 4.7 and 4.10, T has arbitrarily small preserved distances; hence T is an isometry by Lemma 1.3. □

Corollary 4.13. *If $n = 3$, a is a preserved distance under T, and $a < \frac{\pi}{5}$, then T is an isometry.*

Proof. If $a < \frac{\pi}{5}$, then $\cos a > \cos \frac{\pi}{5} = \dfrac{\cos \frac{2\pi}{5}}{1 - 2\cos \frac{2\pi}{5}} = \frac{1}{4} - \frac{\sqrt{5}}{4}$. Then $\frac{1}{2} > \frac{1}{3} > \cos \lambda_3 = \dfrac{\cos a}{1 + 2\cos a} > \cos \frac{2\pi}{5} = \dfrac{\cos \frac{\pi}{5}}{1 + 2\cos \frac{\pi}{5}}$. Therefore

$$\frac{2\pi}{6} < \lambda_3 < \frac{2\pi}{5}.$$

Also $\arccos r_3 \approx 1.3396 > \frac{\pi}{5}$ so $a < \arccos r_3$.

By Corollary 4.6 and Lemmas 4.7 and 4.10, T has arbitrarily small preserved distances; hence T is an isometry by Lemma 1.3. □

Corollary 4.14. *If $n = 2$, a is a preserved distance under T, and $a < \arccos \dfrac{\cos \frac{2\pi}{5}}{1 - \cos \frac{2\pi}{5}} \approx 1.108$, then T is an isometry.*

Proof. If $a < \arccos \dfrac{\cos \frac{2\pi}{5}}{1 - \cos \frac{2\pi}{5}}$, then $\frac{2\pi}{4} > \lambda_2 > \frac{2\pi}{5}$ and $a < \arccos r_2 \approx 1.313$; so T is again an isometry by Corollary 4.6 and Lemmas 4.7, 4.10, and 1.3. □

Then we get:

Theorem 3. *If $T: S^n \to S^n$, and there are two distances a and b such that $d(x,y) = a \Longrightarrow d(Tx,Ty) = b$ and*

either $n \geq 4$ *and* $a < \arccos r_n$,

or $n = 3$ *and* $a < \frac{\pi}{5}$,

or $n = 2$ *and* $a < \arccos \dfrac{\cos \frac{2\pi}{5}}{1 - \cos \frac{2\pi}{5}}$,

then $a = b$ and T is an isometry.

Since $r_n > \frac{\pi}{3}$, for all $n \geq 2$, the following theorem holds for elliptic spaces:

Theorem 4. *If $\mathcal{E}^n$ is n-dimensional elliptic space, $n \geq 2$, $T: \mathcal{E}^n \to \mathcal{E}^n$, and a and b are two distances such that $d(x,y) = a \longrightarrow d(Tx,Ty) = b$, and*

either $n = 3$ *and* $a < \frac{\pi}{5}$,

or $n \neq 3$ *and* $a < \frac{\pi}{3}$,

then T is an isometry.

The situation for large values of a is not so clear.

An example was given in the introduction to show that the theorem is not true for $a = \frac{\pi}{2}$ or π in the case of spherical spaces. However, these are the only distances for which I have counterexamples in S^n. However, this counterexample does not carry over to $\mathcal{E}^n$. In elliptic spaces, the situation for large distances is unknown.

At present, the only result I have concerning large distances in S^n is the following:

Theorem 5. *If* $T: S^n \rightarrow S^n, n \geq 2$, *and* $a = \pi(\frac{2k}{2k+1})$ *is a preserved distance under* T, *then* T *is an isometry for positive integer* $k \geq 5$.

Proof. For any x in S^n, let $\tilde{x}$ denote the point antipodal to x. Let $x_0 \in S^n$, and let $\mathcal{C}$ be a great circle passing through x_0. Let $x_0, \tilde{x}_1, x_2, \tilde{x}_3, \ldots, x_{2k}, \tilde{x}_{2k+1}, \tilde{x}_0, x_1, \tilde{x}_2, x_3, \ldots, \tilde{x}_{2k}, x_{2k+1}$ be points around $\mathcal{C}$ such that $x_0 = x_{2k+1}$ and the distance between adjacent points in the order listed above is $\frac{\pi}{2k+1}$. Then for each i, x_i and $\tilde{x}_i$ are antipodal and $d(x_i, x_{i+1}) = a$ for $0 \leq i \leq 2k$. Then: $d(Tx_i, Tx_{i+1}) = a$ for $0 \leq i \leq 2k$. Also $0 \leq d(Tx_i, Tx_{i+1}) < 2\frac{\pi}{2k+1}$ for $0 \leq i \leq 2k-1$ since

$$\begin{aligned} d(Tx_i, Tx_{i+1}) &\leq d(Tx_i, \widetilde{Tx}_{i+1}) + d(\widetilde{Tx}_{i+1}, Tx_{i+2}) \\ &\leq d(\widetilde{Tx}_i, Tx_{i+1}) + d(\widetilde{Tx}_{i+1}, Tx_{i+2}) \\ &\leq \frac{\pi}{2k+1} + \frac{\pi}{2k+1} . \end{aligned}$$

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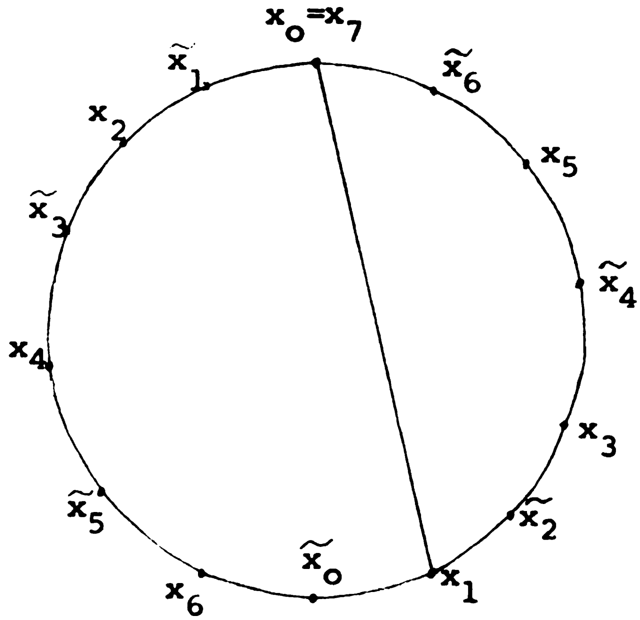

Figure 4-2

Now $d(x_{2k}, x_0) = d(x_{2k}, x_{2k+1}) = a$, so $d(x_{2k}, \tilde{x}_0) = \frac{\pi}{2k+1}$, hence $d(x_0, x_{2k}) = \pi - \frac{\pi}{2k+1} = \frac{2k\pi}{2k+1}$. But if for any even i, $0 \le i < 2k-1$, $d(Tx_i, Tx_{i+2}) < 2\frac{\pi}{2k+1}$, then

$$d(Tx_0, Tx_{2k}) \le \sum_{\substack{i \text{ even} \\ 0 \le i \le 2k-2}} d(Tx_i, Tx_{i+2}) < \frac{2k\pi}{2k+1},$$

which is a contradiction. Hence $d(Tx_i, Tx_{i+2}) = \frac{2\pi}{2k+1} = d(x_i, x_{i+2})$ for all even i, $0 \le i \le 2k-2$.

Hence $\frac{2\pi}{2k+1}$ is also a preserved distance under T. So by the previous theorems, if $k \ge 5$, then $a = \frac{2\pi}{2k+1}$ is small enough so T must be an isometry. □

§5. DIRECTIONAL RESTRICTIONS

Greenwell and Johnson [7] considered the question of restricting the hypothesis so that if $d(x,y) = 1$ and vector xy is in one of certain directions, then $d(Tx,Ty) = 1$. Let $\mathcal{D}$ be any subset of $\mathcal{S}$, where $\mathcal{S}$ is the set of all unit vectors in E^n. Greenwell and Johnson [7] showed the following two results:

Theorem. If the cardinality of $\mathcal{D}$ is less than that of $\mathcal{S}$, then there exists $T: E^n \to E^n$ such that T is not an isometry, but for any $x,y \in E^n$ such that $xy \in \mathcal{D}$, $\|Tx - Ty\| = 1$.

Theorem. If the cardinality of $\mathcal{S} - \mathcal{D}$ is less than that of $\mathcal{S}$ then if $T: E^n \to E^n$ such that $xy \in \mathcal{D} \Longrightarrow \|Tx - Ty\| = 1$, then T is an isometry.

This section gives a little more information on how small $\mathcal{D}$ can be and still force T to be an isometry. For the remainder of this section we consider E^2 and assume that $\mathcal{D}$ is a subset of $\mathcal{S}$ so that the interior of $\mathcal{D}$ is non-void (in the topology of $\mathcal{S}$ relative to E^2). Let L be an equilateral unit lattice, for example the points $n\vec{x} + m\vec{y}$ where n and m are any integers and $\|\vec{x}\| = \|\vec{y}\| = 1$ and $\vec{x}$ and $\vec{y}$ are at an angle of $\pi/3$.

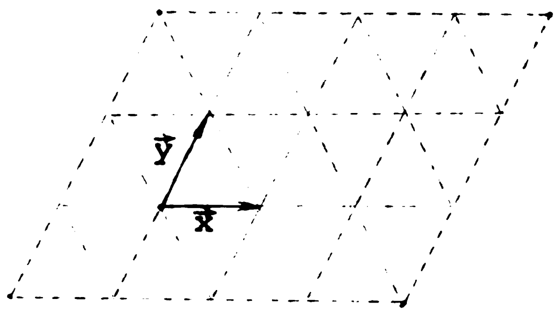

Figure 5-1

Let $\widetilde{\mathcal{D}}$ denote the interior of $\mathcal{D}$.

<u>Lemma 5.1</u>. <u>Let a lattice point</u> $\in L$ <u>be chosen as origin, then the set of unit vectors</u>

$$V = \{\frac{\vec{B}}{\|B\|} \mid B \in L \text{ and } \|\vec{B}\| \text{ is irrational}\}$$

<u>is dense in</u> S.

<u>Proof</u>. Consider $\vec{x}$ and $\vec{y}$ so that

$$L = \{n\vec{x} + m\vec{y} \mid n, m \text{ are any integers}\}.$$

Then $\|n\vec{x} + m\vec{y}\| = \sqrt{n^2 + nm + m^2}$.

Then if $n \equiv m \equiv 1 \pmod 4$, we get that $n^2 + nm + m^2 \equiv 3 \pmod 4$ hence $\sqrt{n^2 + nm + m^2}$ is irrational. It is clear that any point in the plane is within distance 4 of a point $n\vec{x} + m\vec{y}$ with $n \equiv m \equiv 1 \pmod 4$.

Let any unit vector $\vec{v}$ and any $\varepsilon > 0$ be given. Choose a point $n\vec{x} + m\vec{y}$ such that $n \equiv m \equiv 1 \pmod 4$ and $\|n\vec{x} + m\vec{y} - (4 + \frac{8}{\varepsilon})\,\vec{v}\| < 4$. Then $\left\| \frac{n\vec{x} + m\vec{y}}{\sqrt{n^2 + nm + m^2}} - \vec{v} \right\| < \varepsilon$, since:

$$\left| \sqrt{n^2 + nm + m^2} - (4 + \frac{8}{\varepsilon}) \right| < 4,$$

so

$$0 < \sqrt{n^2 + nm + m^2} - \frac{8}{\varepsilon} < 8,$$

and

$$\frac{8}{\varepsilon} < \sqrt{n^2 + nm + m^2} .$$

Therefore

$$\left\| \frac{n\vec{x} + m\vec{y}}{\sqrt{n^2 + nm + m^2}} - \vec{v} \right\|$$

$$= \frac{1}{\sqrt{n^2 + nm + m^2}} \| n\vec{x} + m\vec{y} - \sqrt{n^2 + nm + m^2}\, \vec{v} \|$$

$$\le \frac{\varepsilon}{8} \| n\vec{x} + m\vec{y} - \sqrt{n^2 + nm + m^2}\, \vec{v} \|$$

$$\le \frac{\varepsilon}{8} \| n\vec{x} + m\vec{y} - (4 + \frac{8}{\varepsilon})\, \vec{v} \| + \frac{\varepsilon}{8} \| (\sqrt{n^2 + nm + m^2} - 4 - \frac{8}{\varepsilon})\, \vec{v} \|$$

$$\le \frac{\varepsilon}{8}\, 4 + \frac{\varepsilon}{8} (\sqrt{n^2 + nm + m^2} - 4 - \frac{8}{\varepsilon})$$

$$\le \frac{\varepsilon}{8} (\sqrt{n^2 + nm + m^2} - \frac{8}{\varepsilon})$$

$$\le \frac{\varepsilon}{8}\, 8. \quad \square$$

Lemma 5.2. If T restricted to L is the identity, and $a \in \mathcal{D}$ then $x \in L$ and $x - y = a$ implies $Tx - Ty = a$.

Proof. Let the point $x \in L$ be used as the origin, then we must show that $Ta = a$. Assume the contrary, that $Ta \neq a$.

Now $a \in \tilde{\mathcal{D}}$ implies that $\|Ta\| = 1$. Let z be the foot of the perpendicular to line xa passing through point Ta.

Let $b \in L$ be chosen so that β = angle axb is small enough so that any vector within an angle of 2β of a will be in $\mathcal{D}$ and so that Ta is on the opposite side of xa as b, and $\|\vec{b}\|$ is irrational and $\|\vec{b}\| > 2$. This can be done by Lemma 5.1 since $\mathcal{D}$ is open.

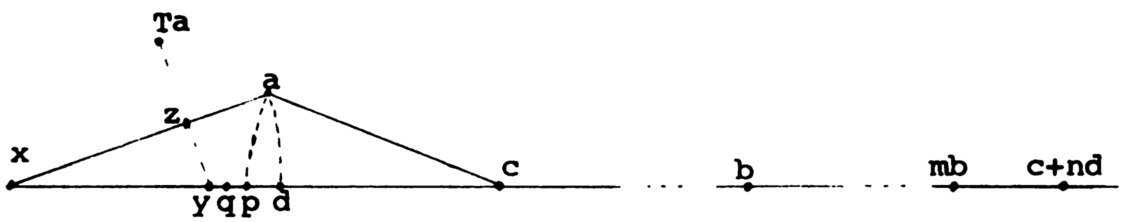

Figure 5-2

Let c be the point on line xb, $\neq$ x so that $\|\vec{c}-\vec{a}\| = 1$. Let $\vec{d} = \vec{b}/\|\vec{b}\|$. Then $\vec{c}-\vec{a}$ and $\vec{d}$ are in $\tilde{\mathcal{D}}$ by our choice of $\vec{b}$. Let $\vec{p} = \vec{c}-\vec{d}$, y = the intersection of line xc and line zTa (if Ta = z, this is just the perpendicular to xa at z).

Then $\|\vec{p}\| = (\sqrt{2 - 2\cos 2\beta} - 1)$ and $\|\vec{y}\| = \|z\| \sec \beta$. In particular, Ta $\neq$ a $\Longrightarrow$ z $\neq$ a $\Longrightarrow$ y is on the same side of p as x is for β small enough (since $\lim_{\beta\to 0} \|\vec{y}\| = \|\vec{z}\|$ and $\lim_{\beta\to 0} \|\vec{p}\| = \|\vec{a}\| = 1$ and $\|\vec{z}\| < 1$ unless $\vec{z} = -a$).

Then p is between y and mb for any $m \geq 1$. Let n and m be positive integers such that $\frac{1}{m} < \|\vec{y}-\vec{p}\|$ and $\|\vec{c}\| + n > m\|\vec{b}\| > \|\vec{c}\| + n - \frac{1}{m}$. Then $\|\vec{c}+n\vec{d}\| > m\|\vec{b}\| > \|\vec{c}+n\vec{d}\| - \frac{1}{m}$.

Let q be the point on segment xc such that $\|m\vec{b}-\vec{q}\| = n + 1$. Then q is between y and p since:

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$$\|\vec{y}\| + n + 1 \;<\; \|\vec{p}\| - \frac{1}{m} + n + 1 \;=\; \|\vec{c}\| + n - \frac{1}{m}$$

$$< \; m\|\vec{b}\| \;<\; \|\vec{c}\| + n \;=\; \|\vec{p}\| + n + 1,$$

so

$$\|\vec{y}\| \;=\; \|\vec{y}\| + n + 1 - n - 1 \;<\; m\|\vec{b}\| - n - 1 \;=\; \|\vec{q}\|$$

and

$$\|\vec{q}\| \;=\; m\|\vec{b}\| - n - 1 \;<\; \|\vec{p}\| + n + 1 - n - 1 \;=\; \|\vec{p}\|.$$

Now $\|m\vec{b}-\vec{a}\| < \|\vec{c}+n\vec{d}-\vec{a}\| < n+1$ and $m\vec{b} - \vec{a}$ is a sum of elements of $\tilde{\mathcal{D}}$, so $\|m\vec{b}-T\vec{a}\| \le n+1$ hence Ta must lie on the same side of the perpendicular to xc at q as does mb. But the choice of b, n and m require that Ta be on the other side of this line. Hence $Ta \neq a$ is impossible. □

<u>Lemma 5.3</u>. <u>If</u> T <u>restricted to</u> L <u>is the identity</u>, <u>then</u> T <u>is the identity on all of</u> E^2.

<u>Proof</u>. By Lemma 5.2, T is the identity on all points of the form $\vec{x} + \vec{a}$ where $\vec{x} \in L$ and $\vec{a} \in \mathcal{D}$. Then replacing the set L with the set $a + L = \{\vec{x}+\vec{a} \mid \vec{x} \in L\}$, Lemma 5.2 gives us that T is the identity on all points $\vec{x} + \vec{a} + \vec{a}'$ where $\vec{x} \in L$ and $\vec{a},\vec{a}' \in \mathcal{D}$. By induction we get that T is the identity on all points of the form $\vec{x} + \sum_{i=1}^{n} \vec{a}_i$ where $\vec{x} \in L$ and $\vec{a}_i \in \tilde{\mathcal{D}}$.

It is easily seen that for any point b in the plane, the set $\{\vec{b}+\vec{a}_1+\vec{a}_2 \mid \vec{a}_1,\vec{a}_2 \in \tilde{\mathcal{D}}\}$ contains a disc. If the radius of this disc is δ and k is any integer larger than $1/\delta$, then the set $\{\vec{b} + \Sigma_{i=1}^{k} \vec{a}_i \mid \vec{a}_i \in \tilde{\mathcal{D}}\}$ will contain a point $y \in L$. Then there are $\vec{a}_i$ in $\tilde{\mathcal{D}}$ so that $\vec{b} = \vec{y} + \Sigma_{i=1}^{k} (-\vec{a}_i)$. But from the definition of $\tilde{\mathcal{D}}$ it is clear that $\vec{a} \in \tilde{\mathcal{D}}$ implies $-\vec{a} \in \tilde{\mathcal{D}}$, hence T is the identity on b. Therefore T is the identity on all of E^2. □

Corollary 5.4. *If T restricted to any lattice is a motion then T is a motion on all of E^2.*

Proof. If T restricted to a lattice is the motion M, then $M^{-1}T$ is the identity on that lattice and so by Lemma 5.3, $M^{-1}T$ is the identity on all of E^2. Hence $T = M$. □

Theorem 6. *If $\mathcal{D}$ contains an open set and there are seven points a,b,c,d,e,f,g such that $a - b \in \mathcal{D}$, $a - g \in \mathcal{D}$, $b - g \in \mathcal{D}$, $a - f \in \mathcal{D}$, $a - e \in \mathcal{D}$, $e - f \in \mathcal{D}$, $c - d \in \mathcal{D}$, then T is an isometry.*

Proof. The same argument as in Lemma 2.5 shows that $Ta \neq Tc$, hence if T restricted to a,b,g is the motion M, then $\|\vec{b}-\vec{c}\| = \|\vec{g}-\vec{c}\| = 1$ requires that $Tc = Mc$. Repeating this argument on the equilateral rhombi contained in the lattice which contains points a,b,c,g gives that T restricted to this lattice is a motion, and hence by Corollary 5.4, T is a motion and hence an isometry. □

It seems likely that a similar theorem can be found in E^n for $n \geq 3$, but the proof is likely to be more difficult since for $n \geq 3$, equilateral n-simplices do not tesselate as simply as triangles in E^2.

§6. COUNTEREXAMPLES AND OPEN QUESTIONS

In the introduction, counterexamples were given to show that in the spaces E^1 and E^∞, a transformation with a preserved distance need not be an isometry. But what happens if the mapping is required to be continuous?

In E^1, the transformation $x \to [x] + \{x\}^2$ (where $[x]$ is the integer part of x and $\{x\} = x - [x]$) is continuous and preserves the distance 1 but is not an isometry. However, in E^∞, it is not so clear what happens; it seems that the additional restriction of continuity should force the map to be an isometry but no proof is known.

Also, in the introduction an example was given in S^n that preserved the distances $\frac{\pi}{2}$ and π but was not an isometry. However, this transformation was not continuous. It is easy to construct a continuous map of S^n into itself which preserves just the distance π, since preserving the distance π just means preserving the relation of two points being antipodal and this can be done with a continuous map. However, it is still not known whether or not the distance $\frac{\pi}{2}$ can be preserved by a continuous map which is not an isometry. Also in spherical and elliptic spaces, it is not known whether or not the additional requirement of continuity placed on a map which preserves large distances forces that map to be an isometry.

The combination of continuity and distance preserving seems to be powerful enough to make the following conjecture seem likely:

CONJECTURE. If $T: M \to M$ where M is a convex, finitely compact metric space with unique segments and any segment in M has a prolongation, and M has topological dimension > 1 and T is continuous and preserves some distance, then T is an isometry.

The following shows an example of a map $T: M \to M$ where M is a convex, finitely compact metric space of topological dimension 2 and T is continuous and preserves all distances less than or equal to 2. M also has the property that segments have prolongations, but they need not have unique prolongations, nor are segments unique. Let $M \subset E^3$ be defined by:

$$M = \{(x,y,0) \mid x,y \text{ real}\} \cup \{(x,y,1) \mid x,y, \text{ real}\}$$
$$\cup \bigcup_{k=1}^{\infty} \{(k,y,z) \mid y \text{ real}, \ 0 < z < 1\}$$

with the metric defined by the length of the shortest path between two points which is contained in M. Let T be the map $T(x,y,z) = (x+1,y,z)$. Then in Figure 6-1, $d(\tilde{a},\tilde{b}) \neq d(T\tilde{a}, T\tilde{b})$ but it is clear that T is continuous any distance ≤ 2 is preserved by T but T is not an isometry.

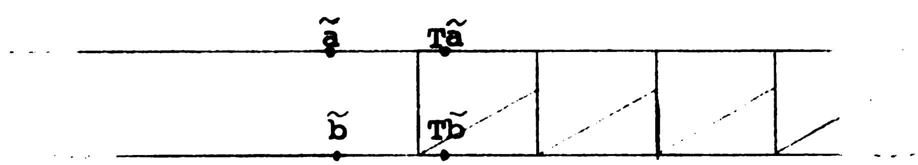

Figure 6-1

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It does seem that the above conjecture would be very difficult to show when M is not one of the special types of spaces considered so far.

In spherical and elliptic spaces, it seems as if continuity combined with distance preserving yields an isometry with the exception of the case when the distance π is preserved in a spherical space, hence the following conjecture seems reasonable:

<u>CONJECTURE</u>. If $T: M \to M$ where M is either a spherical or elliptic space and T is a continuous map which preserves some distance not $= \pi$, then T is an isometry.

Once again, no proof is known.

The following conjecture would be an interesting generalization of the results of Sections 3 and 4:

<u>CONJECTURE</u>. If M is a locally Euclidean manifold of finite dimension ≥ 2, then there is a distance a such that for any $b < a$, and any map $T: M \to M$, T preserves b implies that T is an isometry.

A few examples of the form $T: M_1 \to M_2$ where $M_1 \neq M_2$ are of interest.

If M_1 is the Euclidean line and M_2 is the unit circle with arc metric and T is the "wrapping" function $Tx = e^{ix}$, then T preserves any distance $\leq \pi$, but is not an isometry.

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If $T: E^n \to E^m$ preserves some distance, then clearly $n \le m$ (since E^m has equilateral n-simplices if and only if $n \le m$). Beckman and Quarles [1] showed that T is an isometry if $1 < n = m < \infty$. But if $n < m$, T might not be an isometry if m is too large.

Theorem 7. For any integer $n \ge 1$, there exists an integer $M(n)$ such that $m \ge M(n)$ implies that there exists a map $T: E^n \to E^m$ which preserves the distance 1 but is not an isometry.

Proof. Let E^n be partitioned into a set of regions $\{R_i\}_{i=1}^{\infty}$ such that each region R_i has diameter ≤ 1 (with the distance 1 not being assumed) and so that any closed n-sphere of radius 1 intersects $\le k$ of these regions. Then an integer $M(n)$ can be found so that the regions R_i can be partitioned into $M(n)+1$ sets $\{S_{\tilde{k}}\}_{\tilde{k}=1}^{M(n)+1}$ such that if $x \in R_i$, $y \in R_j$ and R_i and R_j are in the same $S_{\tilde{k}}$, then $d(x,y) \ne 1$. Then define $T: E^n \to E^m$ for $m \ge M(n)$ by mapping each set $\bigcup_{R_i \in S_{\tilde{k}}} R_i$ into a different vertex of a unit equilateral $M(n)$-simplex in E^m. Then $d(x,y) = 1 \Longrightarrow x$ and y are not in the same set $\bigcup_{R_i \in S_{\tilde{k}}} R_i$ hence $d(Tx,Ty) = 1$. Then T is a map from E^n to E^m which preserves the distance 1 but is not an isometry. □

Hadwige[8] gave the following partitioning of the plane into hexagon shaped regions where each region has diameter 1 and contains only its lower boundary (see Figure 6-1A).

Then let these regions be partitioned into 7 sets according to the pattern shown in Figure 6-1B. It is then easily seen that if $d(x,y) = 1$, then x and y are not assigned to the same one of the seven sets. Hence, if $\{a_i\}_{i=1}^{7}$ is an equilateral unit simplex in E^6, then T defined by

A: B:

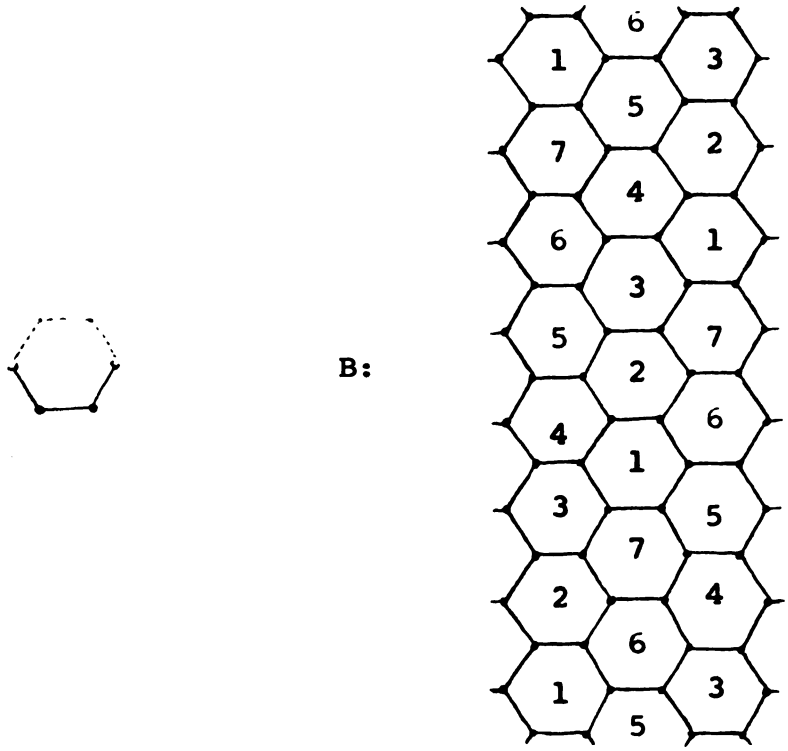

Figure 6-2

$Tx = a_i$ (where x lies in set number i) will preserve the distance 1. However, T is not an isometry.

It is not known if $M(2) = 6$ is the smallest dimension for which such a T exists. In particular, it is not even known whether or not there is a distance preserving map $T: E^2 \to E^3$ which is not an isometry.

Also, it is still an open question whether or not there is a <u>continuous</u> map $T: E^n \to E^m$ for $m > n$ which is distance preserving but not an isometry.

Another possible way to place an added restriction on the mapping is to require it to preserve two distances. The example of a mapping of S^2 onto S^2 given in the introduction shows that in at least one case, preserving two distances does not imply an isometry. However, the following theorem does hold:

<u>Theorem 8</u>. <u>If</u> $T: M_1 \to M_2$ <u>where</u> M_1 <u>and</u> M_2 <u>are Banach spaces such that</u> M_2 <u>is strictly convex, and</u> T <u>preserves the two distances</u> a <u>and</u> ka <u>for some integer</u> $k \geq 2$, <u>then</u> T <u>is an isometry</u>.

Note that this theorem includes the case $M_1 = M_2 = E^\infty$, which is not true if T preserves only one distance.

<u>Proof</u>. Let $x,y,z \in M_1$ such that $\|x-y\| = \|x-z\| = ka$, and $\|y-z\| = a$. Define $w = \frac{1}{k}x + \frac{k-1}{k}y$ and $v = \frac{1}{k}x + \frac{k-1}{k}z$. Then $\|x-w\| = \|x-v\| = a$ and $\|w-v\| = \frac{1}{k}\|y-z\| = \frac{1}{k}a$. Then T preserves a and $ka \Longrightarrow \|Tx-Tw\| = \|Tx-Tv\| = \|Ty-Tz\| = ka$ and $\|Tx-Tw\| = \|Tx-Tv\| = \|Ty-Tz\| = a$. Since $\|y-w\| = (k-1)a$ and $\|z-v\| = (k-1)a$, then $\|Ty-Tw\| \leq (k-1)a$ and $\|Tz-Tv\| \leq (k-1)a$ by Lemma 1.1. But since M_2 is strictly convex, $\|Tx-Ty\| = ka$, $\|Tx-Tw\| = a$ and $\|Ty-Tw\| \leq (k-1)a \Longrightarrow Tw = \frac{1}{k}Tx + \frac{k-1}{k}Ty$. Likewise, $Tv = \frac{1}{k}Tx + \frac{k-1}{k}Tz$. Hence $\|Tw-Tv\| = \frac{1}{k}\|Ty-Tz\| = \frac{1}{k}a$. Hence T preserves $\frac{1}{k}a$ since for any w, v with $\|w-v\| = \frac{1}{k}a$,

there are x, y, z as defined above. Then by induction, T preserves $(\frac{1}{k})^i a$ for all integers $i \geq 0$, and hence T is an isometry by Lemma 1.3. □

The assumption that the two preserved distances have integer ratio was important in the proof of this theorem. However, when T preserves two distances with a non-integer ratio and M_1 and M_2 are as above, it is not known whether or not T must be an isometry, except for those cases covered by the theorems of Section 2 or the work of Beckman and Quarles [3], or the case when $M_1 = M_2$ has a flat spot on the unit sphere of length 2 (when an example similar to that given for ℓ_∞ at the end of Section 2 gives a counterexample).

CONJECTURE. If M_2 is any Banach space without a flat spot of length 2 on the unit sphere, or any elliptic or hyperbolic space (including spaces of dimension ∞), and $T: M_1 \to M_2$ has two preserved distances, then T is an isometry.

CONJECTURE. If $T: S^n \to S^n$ (where n may be ∞), and T preserves two distances not $\frac{\pi}{2}$ and π, then T is an isometry.

A similar idea is to consider maps of the type $T: M_1 \to M_2$ where $d(x,y) < 1$, $x,y \in M_1 \Longleftrightarrow d(Tx,Ty) < 1$. This type of map may come up in some types of psychological or biological measurement where the quantity under

consideration can only be measured indirectly, that is, only after some transformation has been applied to it; and even then the only measurement available is to determine whether or not some threshold is equaled or surpassed.

For example, in an experiment where an individual is asked to judge whether or not two given color samples are the same color, the two colors must differ by some threshold amount or they will be seen as the same color. Then the responses of the subject are the result of transforming the physical colors of the samples according to the characteristics of the subject's eye. Now if there is a map $T: M_1 \to M_2$ where M_1 is a metric space representing the actual physical colors and M_2 is a metric space representing the responses of the subject's eye to colors, such that $x,y \in M_1$, $d(x,y) < 1 \iff d(Tx,Ty) < 1$, but T is <u>not</u> an isometry, then the problem of deducing the nature of M_1 from the responses of the subject can become more complicated.

Similar restriction on T are:

(1) $d(x,y) > 1 \iff d(Tx,Ty) > 1,$

(2) $d(x,y) > 1 \implies d(Tx,Ty) > 1,$

and $d(x,y) < 1 \implies d(Tx,Ty) < 1,$

or even

(3) $\alpha < d(x,y) < \beta \implies \gamma < d(Tx,Ty) < \delta.$

The only progress on the question of preserving inequalities is the following two lemmas:

Lemma 6.3. *If* $T: M_1 \to M_2$ *where* M_1 *and* M_2 *satisfy the hypothesis of Lemma* 1.1 *and* $d(x,y) < 1$, $x,y \in M_1 \Longrightarrow d(Tx,Ty) < 1$ *and* $d(x,y) > 1 \Longrightarrow d(Tx,Ty) > 1$ *and* T *is onto, then* $k < d(x,y) < k+1$, $x,y \in M_1 \Longrightarrow k \le d(Tx,Ty) < k+1$ *for integer* $k \ge 0$.

Proof. Clearly $d(x,y) < k+1 \Longrightarrow d(Tx,Ty) < k+1$ by an argument similar to Lemma 1.1. If $d(Tx,Ty) < k$, then there exist $z_0, z_1, \ldots, z_k \in M_1$ such that $z_0 = x$, $z_k = y$ and $d(Tz_i, Tz_{i+1}) < 1$ for all i, $0 \le i \le k-1$. Then $d(x,y) \le \sum_{i=0}^{k-1} d(z_i, z_{i+1}) \le k$ contrary to the hypothesis. Therefore $k \le d(Tx,Ty) < k+1$. □

Lemma 6.4. *If* $T: E^n \to E^n$ *for* $2 \le n < \infty$ *and* $d(x,y) = 1 \Longrightarrow d(Tx,Ty) < 1$ *and* $d(x,y) > 1 \Longrightarrow d(Tx,Ty) > 1$ *and* T *is onto, then* T *is an isometry*.

Proof. If $u,v \in E^n$ and $d(u,v) = 1$ and T is onto then there exists $x,y \in E^n$ such that $u = Tx$ and $v = Ty$. Then $d(x,y)$ must be 1 by the hypothesis. Hence T^{-1} is a (possibly multivalued) map from E^n to E^n which preserves the distance 1 and is then an isometry (Beckman and Quarles [1]). Therefore T is an isometry. □

However, nothing seems to be known about the particular inequality restriction that seems most relavent to the physics of perception theory: that is, $d(x,y) < 1 \Leftrightarrow d(Tx,Ty) < 1$.

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