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William M. Fitzgerald
Major professor

Dr. William M. Fitzgerald

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ACHIEVEMENT IN SIMILARITY TASKS: EFFECT OF INSTRUCTION, AND RELATIONSHIP WITH ACHIEVEMENT IN SPATIAL VISUALIZATION AT THE MIDDLE GRADES LEVEL

By

Alex Friedlander

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ABSTRACT

ACHIEVEMENT IN SIMILARITY TASKS:
EFFECT OF INSTRUCTION AND RELATIONSHIP WITH
ACHIEVEMENT IN SPATIAL VISUALIZATION AT THE
MIDDLE GRADES LEVEL

BY

ALEX FRIEDLANDER

Purpose

This study had three related purposes: (1) to determine any existing differences in similarity achievement by grade level and by sex, prior to, and after instruction pertaining to similarity, (2) to ascertain any existing differences between a verbal and a visual presentation of similarity tasks, and (3) to study the relationship between performance on similarity tasks and performance on spatial visualization tasks.

The sample (N = 675) was drawn from five schools with a suburban, middle-class, midwestern, predominantly white population and with teachers that volunteered to teach the Similarity Unit--an instructional unit developed by the Middle Grades Mathematics Project for grades six, seven and eight.

Analysis of Variance, and Analysis of Repeated Measures served as the main statistical tools to test the research hypotheses.

Main Findings

Pre, and post instructional performance on four similarity-related topics were analyzed: (1) basic properties of similar shapes, (2) proportional reasoning, (3) area relationships of similar shapes, and (4) applications. At a significance level of .05: no sex differences in pre-instructional achievement or gains were observed; achievement increased as a function of grade level; and seventh graders gained significantly more when compared to the sixth and eighth graders. Pre-instructional performance on area growth tasks was uniformly poor (18-22 percent) and the gains of the sixth graders as a result of instruction were particularly low (5 percent).

Performance on four similarity tasks presented verbally was compared with performance on four equivalent tasks accompanied by figures. Although no overall difference between the two presentation modes could be detected, different presentation modes seemed to be favored for different similarity-related topics.

Student performance on a spatial visualization test before and after instruction on similarity indicated significant gains.

For a restricted sample ($N = 161$) performance in similarity tasks of students that underwent instruction in spatial visualization one year before this study was compared with performance of students that did not undergo this kind of instruction. In this case, instruction in spatial visualization did not have a significant effect on achievement in similarity tasks.

TO:

Sara, Amit, Ronen -- my family

Professors William Fitzgerald (Chair), Richard Houang, Glenda Lappan, Perry Lanier, and Bruce Mitchell -- my doctoral committee.

David Ben-Haim and many other Israeli and American friends.

Sharon Tice -- my typist.

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CHAPTER I

INTRODUCTION

Background

The Concept of Similarity

The acquisition of the similarity concept is important to the development of children's geometrical understanding of their environment, and of proportional reasoning. Phenomena that require familiarity with enlargement, scale factor, projection, area growth, indirect measurement and other similarity-related concepts are frequently encountered by children in their immediate environment and in their studies of natural and social sciences.

Since Felix Klein's Erlanger Programm--a classification of geometrical transformations, similarity has been recognized as an important mathematical concept.

Piaget's (1960) developmental theory of the child's understanding of geometrical concepts added a cognitive aspect to research on the concept of similarity. Piaget viewed the ability to make similarity judgments as an intermediate stage in children's developmental path from a topological to a Euclidean perception of the environment. Many researchers (e.g., Martin, 1976b; Lesh, 1976; Schultz, 1978) raised serious questions about Piaget's hierarchical view of the child's construction of his spatial reality, and about the relevance of his theory to mathematics education. However, researchers seem to agree that around the age of 9, children may make perceptual judgments of similarity (as for example, in comparing two triangles) and that around the age of 11 children may have the mental ability to make a gradually increasing use of proportions (as expressed in judging similarity of rectangles).

A poor performance on similarity tasks (Carpenter et al., 1981) may indicate existing deficiencies in the instruction of geometry in general (Wirszup, 1976), and of the similarity concept in particular. Fuson (1978) points out the need for instruction in similarity:

> Similarity ideas are included in many parts of the school curriculum. Some models for rational number concepts are based on similarity; thus, part of student's difficulty with rationals may stem from problems with similarity ideas. Ratio and proportion are part of the school curriculum from at least the seventh grade on, and they present many difficulties to the student. Standardized tests include many proportion word problems. Verbal analogies (a:b::c:d) form major parts of many intelligence tests. Similar geometric shapes would seem to provide a helpful mental image for other types of proportion analogy situations. Training studies of teaching experiments concerning ways to teach geometric similarities and ways to generalize the solution of geometric proportions to other types of proportion would be valuable. (p. 259)

Proportional Reasoning

The concept of similarity is an instance of proportionality. Proportional reasoning is frequently required in mathematics, natural science and everyday life. Chiapetta and McBride (1978) for example, found among a sample of ninth graders a positive relationship between the ability to reason proportionally and knowledge and understanding of simple machines, concepts on the structure of matter, and applications of equivalent fractions.

The importance of proportional reasoning in a child's intellectual development can hardly be overemphasized. Inhelder and Piaget (1958) consider it one of the six abilities that characterize the formal-operational thinker.

Due to its wide-spread use, the concept of proportionality has been investigated more systematically than similarity. The Karplus studies would be an instance of a thorough analysis of proportional reasoning as a function of age, sex, social status, and nationality. Besides attempts to verify Piaget's cognitive stages in the development of proportionality, more detailed studies suggest that task-related variables are equally important when one investigates performance in proportionality. The variables that have been recommended for consideration are: (1) level of abstractness (Wollman & Karplus, 1974; Portis, 1973), (2) ratio versus fraction (i.e., part-to-part versus part-to-whole) presentation (Wachsmuth, Behr & Post, 1980), (3) level of numerical difficulty (Abramowitz, 1975; Karplus, Pulos & Stage, 1980), (4) task sequencing (Karplus, 1978), and (5) irrelevant information

(Collea, & Numadel 1978). Furthermore, sex differences on performance of proportionality tasks in form of male superiority are also indicated by some studies (Keating & Schaefer, 1975; Stage, Karplus & Pulos, 1980).

Several studies indicate a generally poor performance on proportionality tasks at various ages and stress the need for instructional interventions (e.g., Lovell & Pumfrey, 1966; Renner & Paske, 1977; Pagni, 1983). Consequently, units on proportional reasoning have been designed and evaluated from the middle grades level (Wollman & Lawson, 1978) through high school (Kurz & Karplus, 1977), and even at the college level (Pagni, 1983).

Research indicates that in most cases mastery of abstract proportional reasoning may be expected only above the age of 14-15, if at all (Lovell, 1972). Piagetian research (e.g., Piaget & Inhelder, 1967) shows that similarity tasks may be mastered two or three years earlier than other proportionality schemes. Therefore, the concept of geometrical similarity may be a step towards an understanding of proportionality and research findings in either of these fields should be relevant to the other.

The Middle Grades Mathematics Project

One of the four instructional units designed by the Middle Grades Mathematics Project (MGMP) is on the concept of similarity. The purpose of the MGMP Similarity Unit is to provide a response to the present deficiencies in the instruction of geometry. By offering a rich variety of experiences designed to fit the first two levels in the Van Hiele model of development in geometrical understanding, the unit attempts to build a solid base for further advance in the understanding of geometry in general, and of similarity and proportionality in particular (Lappan, 1983).

At the middle grades level, the importance of teaching informal geometry is not contested. Due to the present deficiencies in the understanding of the most basic geometrical concepts (Carpenter et al., 1981), and to the recommendations mentioned before, a problem-solving oriented, activity-based, informal geometry unit of instruction seems warranted.

The Study

The Purpose

This study is concerned with the performance of middle-grade students in similarity tasks prior to instruction with the MGMP Similarity Unit, and with the impact of instruction of the unit on their performance. The basic goal of the study is to detect the extent to which the MGMP instructional intervention helps to overcome the cognitive difficulties that exist at this critical transitional age.

There are three purposes for this study. The first is to determine any existing differences in similarity achievement in general and in its four identified content components by grade level and by sex, prior to, and after instruction pertaining to similarity. Second, to ascertain any existing grade level and sex differences in performance on similarity tasks presented in a verbal or a visual mode. The third purpose is to study relationships between performance on similarity tasks and performance in spatial visualization. Accordingly, the study will examine performance in similarity tasks from three different aspects:

1. A classification by content will distinguish among tasks requiring: (1) recognition of similar shapes and of their properties, (2) proportional reasoning, (3) use of the area relationship between similar shapes, and (4) applications of the similarity concept.

   Two research questions will be asked on the performance of sixth, seventh, and eighth graders in similarity tasks. The first question is concerned with grade level and sex differences in pre-instructional performance, whereas the second inquires about the effect of instruction (i.e., the existence of gains) on achievement in the four similarity-related topics mentioned above.

2. A classification by presentation mode will distinguish between (1) visual tasks which include drawings of the involved geometric shapes, and (2) verbal tasks which contain only a word description of the geometric situation.

   In this study, two research questions are related to pre-instructional performance and gains on the verbal and the visual subtests. These questions are concerned with grade level and sex differences in achievement on the items presented in the two modes.

3. The relationship with spatial visualization will be examined by analyzing performance on spatial visualization tasks in parallel to performance in similarity tasks.

Two research questions are concerned with the possibility of a relationship between achievement in similarity tasks and spatial visualization. The first question inquires about the effect of instruction in similarity on spatial visualization, whereas the second deals with the effect of instruction in spatial visualization on performance in similarity tasks.

Significance of the Study

One of the goals of this study is to determine patterns in the performance of middle-grade students in geometrical similarity tasks. Investigations on geometrical similarity that were conducted in school settings are rare. Grade level differences (i.e., level of performance increasing with age) would confirm the developmental nature of understanding the concept of similarity. This study may contribute to the present knowledge on the complex issue of sex differences in mathematical performance: male superiority among students of this age has been indicated in spatial visualization (e.g., Ben-Haim, 1983), and in proportional reasoning (e.g., Brendzel, 1978). In regard to geometry, the issue of sex differences is less clear: some studies tend to detect sex differences in relation to informal geometry tasks (Shonberger, 1976; Werdelin, 1961), whereas other studies indicate contradictory results (Olson, 1970; Thomas, 1977; G. D. Peterson, 1973).

Another goal of this study is to determine the effect of instruction with the MGMP Similarity Unit on the performance of middle-grade students in similarity tasks.

Significant gains in performance in similarity and/or spatial visualization would indicate the effectiveness of the instructional intervention and strengthen the claims that deficient geometrical understanding may be improved by using appropriate teaching strategies and materials.

The use of concrete models and of a problem-solving orientation appears to be superior in the instruction of geometry (Bring, 1972; Buchert, 1980; Hempel, 1981). These observations support the educational implications of cognitive research on the acquisition of geometrical concepts. However, cognitive research also sets limitations on the effect of instruction: instructional interventions may accelerate cognitive development, but cannot substitute it (Montangero, 1976). Some studies support this view by indicating a lack of instructional effect as a result of their subjects' low cognitive level (Young, 1975; Wirszup, 1976).

P.M. and Dina Van Hiele identified five cognitive levels in the development of geometrical understanding, and more importantly, showed how to adapt the instruction to the limitations set by these levels. Present deficiencies in the understanding of geometry in the United States are attributed by Wirszup (1976) to instructional, rather than cognitive limitations. He presents as evidence the significant results obtained by a Soviet reform in the geometry curriculum that relied on

the Van Hiele model.

The teaching of geometry in an informal manner is generally accepted at the elementary and middle grades level, and recommended by some mathematics educators even at the high school level (J. C. Peterson, 1973). Transformational geometry was also considered as an alternative for the traditional high school curriculum. However, the results of some related evaluation studies were disappointing (Olson, 1971; Usiskin, 1972; Durapau, 1979)--a fact that can be explained in part by observing that new ways of instruction were used without changing in any significant way the instructional goals or the evaluation tools of the traditional curriculum.

A third goal of this study is to determine the effect of instruction with the MGMP Similarity Unit on the performance of middle-grade students in spatial visualization.

If a relationship between spatial visualization and performance in similarity tasks is indicated by this study, a certain sequence between the MGMP Similarity Unit and Spatial Visualization Unit may be recommended. More general conclusions on the role of spatial visualization in informal geometry tasks may also be drawn.

There are strong cognitive and pedagogical reasons to assume a close relationship between the ability of spatial visualization and the ability to acquire geometrical abilities (Piaget, 1964; Hoffer, 1977). Some studies investigated the relationship between spatial visualization and geometrical abilities in general. In sharp contrast to spatial visualization, geometrical abilities do not have a clear definition,

and have not been widely investigated. Studies that analyzed achievement in plane Euclidean geometry (e.g., Holzinger & Swineford, 1946; Werdelin, 1961; Hanson, 1972) are more likely to find connections with verbal or reasoning abilities rather than with spatial visualization. On the other hand, studies that consider less formal aspects of geometry were able to show a connection between geometry and spatial visualization. This relationship reveals itself, at least from the middle grades level and on, through the improvement in spatial visualization that was shown to occur as a result of instruction in informal geometry (Van Voorhis, 1941; Brinkmann, 1966; Battista, Wheatley & Talsma, 1982).

Population, Sample and Instrumentation

The sample used in this study consisted of sixth, seventh, and eighth graders from five schools in the area of Lansing, Michigan. All five schools have a middle-class, predominatly white student population and may be considered as typical, midwestern suburban schools.

The tests used in this study have been developed as evaluation tools for two MGMP units: The MGMP Similarity Test consists of 25 multiple choice items that assess a variety of similarity-related tasks (for sample items, see Appendix A); the MGMP Spatial Visualization Subtest (Appendix B) consists of 15 multiple choice items chosen from the original test that has been designed to evaluate the MGMP Spatial Visualization Unit.

Both tests were administered to all the sampled students prior to, and after a two to three-week-long instruction with the MGMP Similarity Unit.

Limitations

In order to allow the evaluation of relatively large number of students, this study used paper-and-pencil tests. Although the items are concerned with a variety of concepts related to similarity, the presented situations resemble closely the ones presented in the instructional unit. Consequently, tasks that require proportional reasoning in other than geometrical settings have not been considered.

Research indicates that many task-related variables may influence the performance in proportionality tasks. Not all of them can be controlled in a single study. The level of abstractness varies according to the visual or verbal presentation of items; manipulatory aids were not included in the process of testing. However, all tasks relate to geometrical situations that are considered more concrete than some other proportionality tasks (e.g., the Balance Task, numerical proportions, or some verbal story problems).

The computations involved in the similarity tasks tested are of a numerically moderate level of difficulty. Sequencing of tasks according to level of difficulty or inclusion of irrelevant or redundant information has not been employed in the test items.

Although socio-economical status (SES) was indicated as a variable that influences performance in proportionality tasks (Karplus & Peterson, 1970; Karplus, Karplus & Paulsen, 1977), the sample used by this study is limited mainly to middle-class students from suburban areas.

The conclusions that can be drawn from this study will be therefore limited to the effect of a specific instructional intervention (i.e., the MGMP Similarity Unit), at the middle-grades level, on middle-class students from a typical midwestern suburban area as expressed in two paper-and-pencil achievement tests: (1) the MGMP Similarity Test, and (2) a selection of items from the MGMP Spatial Visualization Test.

CHAPTER II

REVIEW OF LITERATURE

Introduction

In this chapter, literature on different aspects of geometrical similarity will be reviewed. The first three sections examine the development in children's understanding of the similarity concept itself, of proportional reasoning, and of the area relationship of similar shapes. Existing research on the relationship between geometrical ability and spatial visualization will be examined next.

This study examines the influence of grade level and sex on performance in similarity tasks. From the reviewed literature on cognitive development in areas related to similarity, conclusions on grade level (i.e., age) differences may be drawn. Literature on sex differences in performance on similarity-related fields will be reviewed in a separate section.

The rationale behind this study's examination of the effect of instruction on similarity will be presented in the last section of this chapter. This section will review studies that have examined instructional interventions in fields connected to similarity.

The Development of the Similarity Concept

The Mathematical Aspect

In 1872, Klein established a new structure for the analysis of geometrical concepts, which is known also as the Erlanger Programm.

The Programm includes a classification of geometrical transformations according to the transformation groups involved and the invariant properties under that group (Fig. 2.1). The similarity transformation determines one of the "middle geometries", becoming thus an important stage in Klein's hierarchy of geometrical concepts.

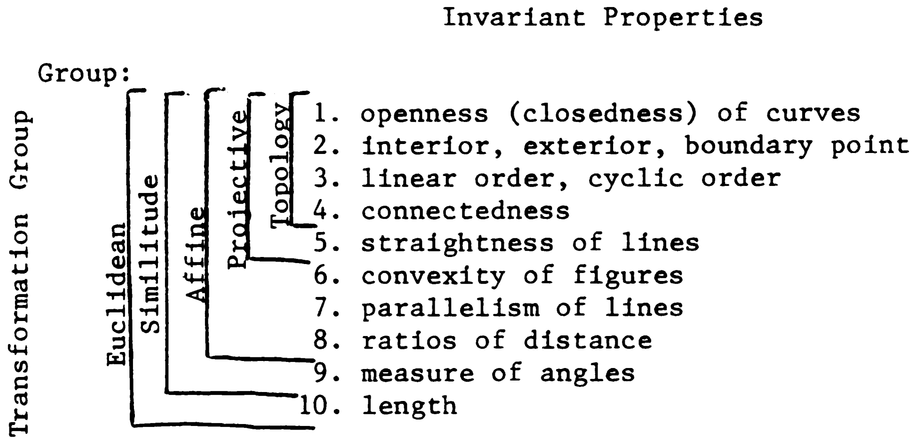


Figure 2.1 Properties invariant under transformation group. (adapted from Martin, 1976b, p.95)

Piagetian Research

Piaget's research on the child's perception of space fits Klein's model of classification quite smoothly. Both, Klein and Piaget emphasized invariability under transformation. Piaget (1970) opposed the view that knowledge is a passive copy of reality. According to him, to know reality, one must assimilate reality into a system of transformations which attempts to model isomorphically the transformations of reality.

In his space books, Piaget (1960, 1967) asserts that projective and Euclidean concepts develop concurrently. However, the design of

the chapters, the research reported, and other comments suggest that after the development of certain basic topological notions, spatial concepts tend to develop from projective, to affine, to similarity, to Euclidean. Fuson (1978) observes that most of Piaget's analyses focus on topological concepts, or on Euclidean concepts, leaving thus the "middle geometries" relatively neglected. Nevertheless, Piaget (1967) describes five experiments related to the concept of similarity: (1) drawing similar triangles, (2) sorting similar cardboard triangles, (3) choosing and drawing similar rectangles, (4) drawing a similar configuration of line segments, and (5) copying supplementary angles. Piaget's categorization of results indicates that at the age of 9-11, his subjects were able to perform perceptual comparisons (considering slopes, parallelism, or angles), began to use simple proportions (1:2), and made measurements, whereas they could use proportions at the age of 11 or above. The failure to perform well in these tasks at earlier ages is attributed to the child's perceptually centered nature of preoperational thought, and later to the inability to use mental constructs (i.e., ratios and proportions) which have no direct concrete-empirical representation.

Mathematics educators generally recognize both the advantages and the limitations of Piaget's research in the field of mathematics education. In a mathematical analysis of some of Piaget's topological tasks, Martin (1976a) concludes:

> Piaget's objective is not to study the development of mathematical concepts of the child but to study the

> development of the child's concept of space. The models provided by mathematics are merely the means to an end. As a developmental psychologist, Piaget does not always use mathematical language as precisely as the mathematician might desire. (p.24)

The issue of topological-to-Euclidean spatial development was also questioned. Martin (1976b) considers the evidence presented by Piaget's research and states that "it is premature to claim that any particular hierarchy models the sequence or the structure of the child's construction of his spatial reality" (p.112). Martin further suggests that "on the basis of evidence now available" it seems that topological, projective, and Euclidean concepts develop in parallel rather than sequentially. Relating to the concept of similarity, Martin asks whether it is psychologically sound to expect a child to develop the concept of a variable constant of proportion (i.e., similarity) before the concept of a fixed constant of proportion of one (i.e., congruence).

Variables Related to the Acquisition of Similarity Concepts

Schultz (1978) showed in her experiments that operational structure is not the only determining factor in tasks that involved geometrical transformations. Attributes of fixed states (e.g., familiarity and size of the involved figures) and the features of the operation itself tend to have a significant influence. Martin (1976a) suggests that the child's ability to order, organize and coordinate his actions might offer a better framework for his spatial development.

The model built by Pascual-Leone (1970) considers the informational content and the processing load required by a task as determinant variables in the logical thinking of a child. He defines the number of schemes, rules, or ideas that a child can handle simultaneously as M-capacity. According to him, this capacity increases regularly in an all-or-none manner from age 3 through 16 at a linear rate of one chunk every two years. Thus, the derivation of the proportionality rule that requires simultaneous manipulation of four variables, is not possible before the age of 9.

Leon (1982) applied the information processing theory to the child's development of some geometrical concepts such as similarity, and area. He reexamined Piaget's experiment on similarity judgments of rectangles and suggests that the emphasis should be not on the cognitive structures necessary to use proportions, but rather on the prerequisite quantitative concepts, i.e., the logically implicit response rule requiring proportions.

Another limitation on Piaget's theory lies in observed differences in ages at which children come to master operationally equivalent tasks. Piaget calls this phenomenon decalage. According to many researchers (e.g, Laurendeau and Pinard, 1970; Fuson, 1978), Piaget makes a too frequent use of the decalage as an explanation of arising difficulties. Referring to the "topological to projective to Euclidean" developmental path, Lesh (1976) states:

> The most important challenge to Piagetian theory is not the possibility of a different hierarchy-but that often

> operationally isomorphic tasks vary so much in difficulty that it may be meaningless to classify concepts on the basis of operational structure. (p.235)

To summarize, not much research has been done on the growth of the similarity concept. There is disagreement on the exact nature or the sequential order of the cognitive development of geometrical concepts in general, and of similarity concepts in particular. However, researchers agree that at the age of 11, a more general and conceptual use of proportions (as expressed in judgments on similar rectangles) occurs. Perceptual judgments of similarity (e.g., judgments on similar triangles) is possible about two years earlier.

Performance on Similarity Tasks

Most of the above mentioned studies were based on intensive interviews conducted with a small number of children. Statistical support for many of these claims is even scarcer. Young (1975) examined the performance of 791 children from grades K through 3 on tasks related to ten geometrical concepts--one of them being similarity. In view of the poor results on this specific task, Young recommends introducing the concept of similarity "above the third grade".

Carpenter et al. (1981) analyze the results of the 1977-78 second mathematics assessment of the National Assessment of Educational Progress (NAEP). The answers of 70,000 students, ages 9,13, and 17 on the similarity items follow the general pattern of performance in geometry: Students have some knowledge of certain basic concepts, but have little knowledge of the properties associated with these concepts, and little

abililty to apply these properties. In the case of similarity, 94 percent of the 13-year-olds recognized two similar triangles. However, only about 33 percent of the 13 year-olds, and 44 percent of the 17-year-olds knew that similarity does not require congruent sides, and less than 33 percent of the subjects knew that the angles of similar shapes must be congruent; a third of the 13-year-olds and half of the 17-year-olds could use indirect (shadow) measurement to determine the height of a tree.

This gap between the potential achievements shown by cognitive research and the observed deficiencies in the acquisiton of geometrical concepts implies that instruction may play a crucial role, particularly if it is conducted at the transitional stages. Research on cognition shows that children at the middle grades level are at such a transitional stage in the growth of concepts related to similarity. This suggests that more detailed research, particularly at this age, is needed.

The Development of Proportional Reasoning

Similarity and Proportional Reasoning

The ability to handle metric proportions and the concept of geometric similarity are related to each other: As mentioned in the previous section, at the mastery level of the similarity concept, proportions are recognized as the inherent rule that governs the situations in which similarity is encountered. Piaget, Inhelder and Szeminska (1960) argue that it is easier to study the growth of the concept of proportion

in geometric, rather than in non-geometric form, since before a child can think about proportions, he can perceive whether two different shapes are similar or not. On the other hand, Lunzer and Pumfrey (1966) showed in their study, that success in geometrical situations that require proportional reasoning (i.e., building a "wall" of a given length with Cuisinaire rods, or using a pantograph) should not be interpreted as mastery of the underlying proportional rule. Fifty percent of the nine-year-olds could handle the Pantograph Task with simple proportions, whereas less than 50 percent of the 15-year-old subjects were successful in the Balance Task that required the recognition and the application of proportionality in an abstract way.

The section on the development of proportional reasoning will follow the same path as the review of the similarity studies. Piagetian research that analyzed mental operational structures needed in proportional reasoning will be followed by studies that consider other variables that may influence performance on proportionality tasks. Studies on children's level of performance in proportional reasoning will be reviewed next, whereas attempts to improve proportional reasoning through instruction will be presented in a different section.

Piagetian Research

Inhelder and Piaget (1958) identify the understanding of proportionality with the stage of formal operational reasoning, which according to them emerges at 12 to 13 years of age. They investigated the child's acquisition of proportionality by examining children's reactions to situations such as equilibrium on a balance and prediction of shadow size. On the basis of this inquiry the claim was made that proportional reasoning appears at the age of 11-12 years, but initially only in a qualitative form, and later (ages 13-17) it evolves into abstract reasoning involving the formulation of the law of proportions and the ability to operate on this law in a quantitative form. They identified proportional reasoning as one of the eight schemes that develop at the stage of formal thought. Martorano (1974) tested six out of these eight schemes and found that the proportionality scheme was more difficult than all others, except mechanical equilibrium. Linn and Swiney (1978) found a strong relationship between proportional reasoning and general ability. Lunzer (1968) argued that the equality of two ratios always constitutes a second order relation (i.e., a relation between relations), and even a system of such relations (i.e., $a/b = c/d$ implies $ad = bc$ and $a/c = b/d$).

Studies with British children (Lovell, 1961; Lunzer, 1965; Lovell & Butterworth, 1966) and with American children (Steffe & Parr, 1968; Gray, 1972) confirmed basically the developmental categories identified by Piaget, but noticed a difference in age distribution. Lovell (1972) sums up these findings as following:

> Apart from very able twelve-years-olds, it is from 12 years of age onwards, the actual age depending on the ability of the pupil, that facility is acquired in handling metric proportion. Many pupils may not be able to do this until 14 or 15 years of age, and some never. (p.8)

In order to check the linkage between proportional reasoning and the formal operational stage, Chapman (1975) compared the performance of first, third, and fifth graders on conservation of ratio tasks (presented in a probabilistic manner) with that of college students. He concluded that even 10, or 11-year-old children do not discriminate proportions, whereas most of the items were anwered correctly by college students.

Proportional reasoning has been widely investigated by Robert Karplus, who devised a test to determine the level of reasoning children use (Karplus & Peterson, 1970). In his Mr. Tall/Mr. Short task, the subject is presented two drawings: Mr. Short, whose height is measured in large and in small paper clips, and Mr. Tall, measured in small paper clips only. The subject is asked to predict Mr. Tall's height as expressed in large paper clips. Later, Karplus modified the task, by eliminating the drawing of Mr. Tall in order to prevent children from relying on perceptual cues.

Karplus and Peterson (1970) categorized students' explanations (strategies) given in solving the Mr. Tall/Mr. Short proportional reasoning problem in the following way:

N. no explanation or statement "I can't explain" given.

I. <u>intuition</u>. Referring to estimates, guesses, appearances or extraneous factors, without data.

IC. <u>Intuitive computation</u>. Use of data haphazardly and in an illogical way.

A. <u>addition</u>. Applying the difference rather than the ratio.

S. <u>scaling</u>. Not relating to the scale inherent in the data, thereby failing to see the whole problem-displays a tentative attitude toward his/her estimate.

AS. <u>addition and scaling</u>. Focuses on the difference rather than ratio, but scales it up.

P. <u>proportional reasoning</u>. Uses proportionality. May or may not use word ratio.

In a later follow-up study, Karplus and Karplus (1972) found it convenient to collapse these seven categories into three categories representing conceptual levels:

Intuitive	Level I	= I + IC
Concrete-operational	Level II	= A + S
Formal-Operational	Level III	= AS + P

In this latter study, the performance of 153 students was compared to their performance on the same task, two years earlier. It was found that during this period, 65 percent of those who were at level I moved up, whereas more than one third of the subjects showed no change in category. In the investigators' view the results prove the developmental nature of the acquisition of the concept of proportionality. (No clear order could be established between the substages of level II.)

Further Karplus studies reveal that the additive strategy (defined above as A) is used systematically by many seventh and eighth graders (Wollman & Karplus, 1974), and even by 20 percents of the college students that participated in another experiment (Karplus, Adi & Lawson, 1980). The use of this strategy is attributed to a concrete-operational way of thinking combined with inadequate instruction.

Variables Related to Proportional Reasoning

Many alternative factors in the development of proportional reasoning have been suggested in addition to or instead of Piaget's scheme of cognitive development.

A comprehensive study in this direction has been conducted by Pulos, Stage and Karplus (1980). Eighty-seven sixth and eighth graders were tested for: (1) proportional reasoning (measured by their Lemonade Task that involved ratios of sugar to concentrated lemon juice), (2) M-capacity (the number of schemes that a child is able to handle simultaneously), (3) fluid intelligence (measured by a wide range series completion test), (4) crystallized intelligence (measured by a wide range vocabulary test), (5) cognitive restructuring (measured by the FASP Embedded Figures Test) (6) field dependency (measured by Pascual-Leone's Water Level Task), (7) formal reasoning (measured by Piaget's Conservation of Volume Task), and (8) divergent thinking (measured by an alternative uses test). Stepwise multiple regression analyses were employed to measure the relationship between proportional reasoning and the other seven cognitive variables. The results of the study suggest that proportional reasoning is significantly related to only two of the variables: (1) information processing (M) capacity (as suggested also by de Ribaupierre and Pascual-Leone, 1979; Case, 1979; and Furman, 1980), and (2) encoding, or cognitive restructuring (as suggested also in Siegler, 1978). In view of the lack of relationship between proportional reasoning and conservation of volume, Pulos, Stage and Karplus conclude that "the results do not

seem to support the hypothesis that a formal reasoning structure, as defined by Inhelder and Piaget, is a necessary prerequisite for proportional reasoning" (p.148).

Other variables that seem to influence performance in proportional reasoning tasks are:

1. Level of abstractness. In a comparison of six proportionality tasks, Wollman and Karplus (1974) report that seventh and eighth graders (N = 450) were more successful in concrete tasks (Mr. Tall/Mr. Short, Ruler, Shading Fractions) as opposed to more abstract ones (Candy, Numerical). Similarly, Portis (1973) indicates a significantly better performance of fourth, fifth, and sixth graders (N = 138) in a proportionality test using physical and pictorial aids, as compared to an equivalent test that used only symbols. Lunzer and Pumfrey (1966) report a much better performance on the concrete Cuisenaire Rods, and Pantograph Tasks than in the Balance Task. They also remark that in the first two tasks, their subjects rarely applied specifically the proportionality rule. Moreover, Wollman and Karplus (1974) found that in abstract tasks most subjects who reason proportionally in an incomplete manner (i.e., incorrectly or concretely) "regressed to additive reasoning". On the other hand, increasing the level of abstraction may have also a positive effect: when Karplus's subjects were denied the opportunity to see Mr. Tall in the Mr. Tall/Mr. Short Task, the number of intuitive-perceptual guesses dropped significantly.

2. Ratio versus fraction. Wachsmuth, Behr and Post (1980) contrasted performance of fifth grade students (N = 15) on Ink-Mixture

Tasks presented in a ratio (i.e., part-to-part) format with the performance in the same tasks presented in a fraction (i.e., part-whole) format. They indicated a greater success (72 percent) for the ratio format than for fractions (58 percent). Noelting and Gagne (1980) found low correlations between the Orange Juice Experiment bearing on ratios and the Sharing Cookies Experiment bearing on fractions.

3. Numerical context. Abramowitz (1975) raised the possibility of a relationship between the numbers employed in a proportion and the rate of success, since children may "have an intuitive understanding of proportionality without concurrently having the mathematical facility to solve proportion problems "(p.25). She checked three numerical characteristics: (1) equal or unequal differences (i.e., presence or absence of a repeated difference among the three given numbers), (2) size of the unknown number (i.e., whether the unknown number is larger or smaller than the three given numbers), and (3) types of ratio (i.e., whole numbers or fractions). The results obtained from 32 seventh graders indicate that type of ratio and size of the unknown number have a significant effect, but the difference between numbers does not. Abramowitz (1975) also observed that

> subjects, especially those transitional between concrete and formal operational thinking...may be quite capable of reasoning through proportions of moderate difficulty. However, when faced with a more demanding task, these same subjects may revert to the use of patterned concrete strategies. (p.26)

Karplus, Karplus, and Wollman (1974) also report that the value of the ratio influences performance: a ratio between 1 and 2 attracts more errors (specifically, more additive responses) as compared to ratios smaller than 1 or greater than 2. Quintero (1983) tested the performance of 36 fifth, sixth, and seventh grade Puerto-Rican children on verbal problems involving the basic proportion $a/b = c/x$. She found that one third of her subjects could solve problems in which both a/b and a/c are integers, whereas 20 percent could handle problems in which only one of these ratios is integer. The same phenomenon was reported by Karplus, Pulos and Stage (1980) with 120 sixth and eighth graders: a success rate of about 60 percent was observed in the Lemonade Task when both a/b and a/c or only a/b were integers, but the rate of success was much lower when only a/c or no ratio were integers (38 and 18 percents respectively).

The British project "Concepts in Secondary Mathematics and Science" (CSMS) administered a Ratio and Proportion Test to over 2200 eleven to sixteen-year-old students. In analyzing the results, Hart (1982) identified five levels of performance: At level zero, the student is unable to make a coherent attempt at any of the questions. At the next four levels, the student is able to solve items of increasing difficulty. Her categorization of the items was:

I. No rate is needed or given; the answer may be obtained through multiplication by 2, 3 or taking half.

II. Rate is easy to find or answer can be obtained by taking an amount and then add half again as much.

III. Rate must be found and is harder to find than above; fraction operation may also be needed.

IV. Ratio is needed; the questions are complex in either numbers needed or setting.

4. Sequencing. Karplus (1978) reported a 51 percent rate of success when a proportional reasoning task was preceeded by a simple ratio task, while only 12 percent answered correctly when the first task was more difficult.

5. Irrelevant and redundant information. Collea and Numadel (1978) found that when a problem contained both irrelevant and redundant information, a significantly poorer performance resulted.

All these findings suggest that "consistent use of proportional reasoning is not a developmental outcome, but depends instead on overcoming task related obstacles" (Karplus, Pulos & Stage, 1980, p.141). Reaching the same conclusions, Abramowitz (1975) contemplates that it may be that "the use of proportionality occurs in a developmental sequence across a certain set of tasks." This argument "has implications for when it would be best to teach various concepts requiring an understanding of proportionality" (p.27). These instructional implications will be discussed in a separate section.

Socio-economic status (SES) and sex are other variables that have to be considered in performance on proportionality tasks. Particularly poor proportional reasoning by low SES children was shown in two Karplus studies: Karplus and Peterson (1970) indicate that although urban and suburban children perform equally low on the Mr. Tall/Mr. Short Task

at the sixth grade level, by the end of high school most of the suburban subjects (80%) had mastered the task, whereas only 9 percent of their urban counterparts did so. Thirty-six hundred seventh and eighth graders from seven countries were tested on the same task (Karplus, Karplus & Paulsen, 1977), and only small differences in overall achievement among countries were detected. However, the results show that groups of low SES American students performed at the lowest level (i.e., primarily intuitive).

Difficulties in Proportional Reasoning

In the section on the concept of similarity, it was shown that generally, at the age of 11 or 12 years, children are able to handle proportion tasks if they are accompanied by physical action. Studies that used paper-and-pencil tests or tasks that required the use of the proportional rule in a generalized form, report that the ages at which a significant portion of the population could master the tasks is higher--about 15 years.

Lunzer and Pumfrey (1966) report a success rate of less than 50 percent for his 15-year-old subjects in the Balance Task (see also Lovell & Butterworth, 1966). Similar trends are reported by the Karplus studies: Karplus and Peterson (1970) found that 32 percent of the suburban 8-10th graders and 80 percent of the suburban 11-12th graders could reason proportionally on the Mr. Tall/Mr. Short Task. The corresponding numbers for their urban peers were respectively five and nine percent. Wollman and Karplus (1974) found that only 20 percent of the 450 seventh and eighth graders tested on five proportionality tasks applied proportional

reasoning in a consistent way. In their survey of seven countries, Karplus, Karplus and Paulsen (1977) discovered that 25 percent of their subjects (N = 3600, grades 7-8) used proportional reasoning. The same rate of success among American seventh graders was reported by Abramowitz (1975). Moreover, the rate of development seems to be very slow, at least before the age of 15 years (Karplus & Karplus, 1972). Sudden "jumps" in the level of performance were noticed later in high school (Karplus, Adi & Lawson, 1980; Karplus & Peterson, 1970).

Studies report a poor level of proportional reasoning even at the college level. According to Lawson and Wollman (1976), Fuller and Thornton (reported in Pagni, 1983), and Renner and Paske (1977), only about 50 percent of their samples of college students were able to reason proportionally in a formal way. The figures reported by Karplus, Adi and Lawson (1980) for college freshmen, and by Karplus and Peterson (1970) for suburban 12th graders are higher: 74 and 80 percent respectively.

Conclusion

There are various development models for the acquistion of the concepts of geometrical similarity and of proportional reasoning; many of them are complementary rather than contradictory. By attempting to superimpose the literature on similarity and on proportionality, it may be concluded that, the developmental stages in the acquisition of the similarity concept and of proportional reasoning are quite similar, and any explanation that seems valid in one field can be applied in the other. Chronologically, however, the concept of similarity (and possibly some other concrete applications of proportionality) seem to precede a more general and abstract use of proportionality.

The Concepts of Area and Area Growth

Definition of the Problem

The area growth of similar shapes, requires the recognition of the fact that the enlargement of a figure by a scale factor of n will increase its area by n^2. The concepts of similarity and of area are, therefore, prerequisites to an understanding of area growth. In this section, the literature on the concept of area will precede the description of a few studies that can be related directly to the topic of area growth.

Concept of Area

Conservation of area. According to Piaget, Inhelder, and Szeminska (1960), area conservation (i.e., considering area as a stable attribute independently of the shape of a figure) is attained at the early concrete-operational stage--around the age of 7 1/2 years. Beilin (1964), however, objected to the fact that in Piaget's experiments on area conservation, the transformations of the figures were performed in front of the subjects. He defines "quasiconservation" as the ability to determine that two noncongruent regions have equal areas when the child is presented with the end product of a rearrangement. Beilin tested 316 children in grades K through 4 and found a correct response level of less than 50 percent of his fourth-grade subjects. Two other larger-scale studies

(Renner, 1971; Hademenos, 1974) indicate that area conservation as defined by Piaget is generally achieved at a later age of about ten years. Similarly, in studying the conception of area measure with 75 eight, ten, and eleven-year-old children, Wagman (1975) concludes that "about a third of the ten, and 11-year-olds in the sample either failed to apply at least one of the neglected axioms (area, congruence, and additivity) even in perceptually easy cases, or did not conserve area" (p.109).

Calculation of area. The quantification skills necessary to derive the area (e.g., height x width for rectangles) seem to evolve also at a later age than Piaget's theory of cognitive development indicates. Anderson and Cuneo (1978) showed that the adding rule (i.e., height + width) describes five, and six-year-olds' judgment of rectangular area. Leon (1982) shows that seven-year-old children use the "linear extent" rule to judge area--which in the case of rectangles would be the length of the diagonal, and that only by the ages 8-9 is the linear extent rule replaced by the multiplicative rule.

The difficulties in the calculation of area seem to persist even at a later age. Carpenter et al. (1981) analyzed the results on the area items of the second mathematics assessment of the NAEP and observed that (1) few nine-year-olds have any knowledge of even basic area concepts (28 percent could find the area of a rectangle divided in square units); (2) among the 13-year-olds, 51 percent could find the area of a rectangle by the dimension of its sides, 12 percent could find the area of a square with a given side, and even fewer could find the area of a right triangle; (3) 74 percent of the 17-year-olds in the sample calculated correctly the area of a rectangle, 42 percent--the area of a square, and 20 percent--the area of a parallelogram or of a right triangle; (4) the success rate in application exercises was even lower--16 percent of the 17-year-olds could find the area of a region made up of two rectangles. Carpenter et al. (1981) conclude:

> Performance of perimeter, area, and volume exercises was among the poorest of any content area on the assessment. Not only was performance extremely low on exercises at the application level, but many students at all ages appeared to have no understanding of the most basic concepts of perimeter, area, and volume. (p.98)

In a survey of geometric concepts possessed by 198 sixth graders on leaving elementary school, Schnur and Callahan (1973) report similar results: finding areas of rectangles was "marginally easy" (i.e., a difficulty index between .50 and .69), areas of squares and parallelograms were "marginally difficult" (i.e., a difficulty index between .30 and .49), and calculating the area of a triangle was "very difficult"(i.e., a difficulty level of less than .10).

Perimeter and area. The confusion of area and perimeter has been observed in all the studies that related to the acqusition of these concepts. Piaget, Inhelder, and Szeminska (1960) observe that in their area tasks, some subjects related to the perimeters of the two regions, and consider the reliance on linear terms characteristic of the concrete level (III A). Wagman (1975) reports that approximately one third of

their subjects (ages 8, 10, and 11 years) confused area and perimeter at some time during their interview. In their analysis of the second NAEP results, Carpenter et al. (1981) indicate that 23 percent of the 13-year-olds and 12 percent of the 17-year-olds calculated the perimeter of a rectangle instead of its area, whereas half of the 13-year-olds, and a third of the 17-year-olds did the same in the case of a square and a right triangle.

Area Growth of Similar Shapes

After three weeks of instruction that focused on growth and shrinking of figures and objects, Fitzgerald and Shroyer (1979) report a low level of performance among their sample of 350 sixth graders: 35 percent could master a task on the growth of a square presented in a concrete mode, but only half of these answered correctly when the task was presented in an abstract mode; the rate of success for tasks that required the application of the area growth principle varied between 3 and 13 percent.

McGillicudy-DeLisi (1977) related the performance of six through 13-year-old children on tasks that involved enlarging rubberband figures on a pegboard with the cognitive level of her subjects ($N = 75$). By a qualitative analysis of her subjects' strategies and types of movement, she found that the rate of success increased with operative level.

An explanation for the difficulties encountered in area growth tasks can be deduced from the research conducted by Bang in France (reported in English by Montangero, 1976) and by Lunzer (1968, 1973) in England. The tasks involved inquiries about area and perimeter under two kinds of transformations: (1) a series of rectangles with a fixed perimeter

(but a decreasing area) created from an initial square, and (2) a series of figures with a fixed area (but an increasing perimeter) created by cutting a triangular section from a lower corner of a square and transferring it vertically until it is attached to the upper corner of the original figure.

Lunzer (1968) argues that at the concrete-operational level of reasoning, children employ "false conservation": realizing that something was preserved, they resist the evidence of perception, and regard the conservation of both area and perimeter as logically necessary in both cases (1) and (2). They reason about the figures as objects and about area and perimeter as essential characteristics of the object that must "go together". According to Lunzer, only at the formal level, area and perimeter are clearly disassociated, and both are assumed to vary according to some (but not necessarily the same) law. He argues that since area snd perimeter are well-defined relations existing in figures, the discrimination between them requires the ability to use second-order relations--an ability that can be found only at the formal level of reasoning.

This argument may provide a valid explanation to the reported difficulties in the acquisiton of the concepts of perimeter, area, and area growth. The understanding of the area growth requires the recognition of the fact that for similar shapes linear dimensions and area vary according to different rules.

Geometry and Spatial Visualization

Spatial ability

Although descriptions and measurements of aspects of intelligence started earlier, only in the 1940's did spatial ability start to gain more attention. Wolfle (1940) reviewed the factorial studies up to 1940, and stated that verbal ability and space ability were the two most frequently identified factors. Factor analyses conducted by a group of American psychologists in the Aviation Psychology Program (Fruchter, 1954; Zimmerman, 1954; Michael, 1954) identified in spatial ability at least two factors: (1) Spatial Relations and Orientation, and (2) Spatial Visualization. Ekstrom et al. (1976) observe that there has been some difficulty in explaining the difference between these two factors. They suggest that the figure is perceived as a whole in spatial orientation, but must be mentally restructured into components for manipulation in spatial visualization. More specifically, they define spatial visualization as "the ability to manipulate or transform the image of spatial patterns into other arrangements" (p.173). In his survey of studies on spatial visualization, Ben-Haim (1983) argues that the nature of this ability is still a controversial issue--especially for children younger than 11 years. He points out that "the picture in the eighties is still unclear," and quotes from Harris (1981):

> Our attempts to identify the critical components of various "spatial" tests are still part guess work, particularly where we lack factor analyses involving both standard and

> nonstandard tasks...Furthermore, consensus is still lacking on the meaning of the two factors--orientation and visualization...factor analysts continue to have difficulty in differentiating and interpreting these factors. (p.23)

There is an abundance of spatial visualization tests (see for example Smith, 1964; Ben-Haim, 1983), but only two will be mentioned here: (1) the Spatial Relations of the Differential Aptitude Test (DAT) which requires the subject to identify the figure of a solid that can be obtained when a given pattern is folded (Bennett et al., 1966), and (2) the Middle Grades Mathematics Projects (MGMP) Spatial Visualization Test which requires the subject to rotate cube constructions mentally and to identify them from a different perspective (Lappan, 1983). The DAT test was frequently used in studies that compare spatial visualization with geometric abilities, whereas data from the MGMP test will be used in this study to assess the relationship between spatial visualiztion and achievement in similarity concepts.

Geometry and spatial visualization

One of the goals of this study is to examine the relationship between the acquisition of similarity concepts and spatial visualization. Since studies on this specific topic have not been reported, the issue will be extended to include the relationship between spatial visualization and geometrical abilities in general. Research in this area, however, has limitations of its own. In his study on the relationship between geometrical, and spatial ability, Werdelin (1961) observes:

> Very few studies have dealt with the structure of geometrical ability, especially in comparison with the number of investigations of, for example, the verbal and the visual-perceptual field. With a few exceptions, the investigators of the field seem to have considered it to be an unitary part of mathematics, or devised only one test to cover it. Quite often, geometrical ability and spatial ability have been confused with each other, which of course is easily done, but which does not facilitate the solution of the problem. (p.33)

The source of this confusion is the fact that there is no clear definition of geometrical ability. Most of the relevant factor analyses and regression studies relate to achievement in plane Euclidean geometry at the high school level. Euclidean geometry is a clearly defined field, but on the other hand, it also requires a great deal of formal knowledge and of logical and verbal ability. Whether conclusions can be drawn from these studies to the relationship between spatial visualization and a more informal geometrical ability, is a debatable issue.

There are strong cognitive and pedagogical reasons to assume a close relationship between informal geometry and spatial ability. Some of the visual aspects of similarity and proportional reasoning tasks have been already discussed. From a pedagogical point of view, Hoffer (1977) lists seven aspects of the visual perception ability that are essential in the geometrical development of a child at the elementary and middle grades level: (1) visual-motor coordination (i.e., coordinating vision with movement of the body), (2) figure-ground perception (i.e., distinguishing foreground from background), (3) perceptual constancy (i.e., recognizing an object out of its original context or from a different viewpoint), (4) position in space (i.e., determining the relationship of an object as compared to the observer), (5) spatial relationship (i.e., perceiving the position of two or more objects both in relation to the observer and in relation to each other), (6) visual discrimination (i.e., distinguishing similarities and differences between objects), and (7) visual memory (i.e., recalling accurately an object no longer in view and relating its similarities and differences to other items in view or not in view).

The possibility of a connection between the acquisition of similarity concepts and spatial visualization is strengthened by results from a study that revealed a relationship between proportional reasoning and spatial ability: Brendzel (1981) used a sample of 400 ninth and eleventh graders, and by conducting an analysis of covariance with the I.Q. as a covariant, she found that ability in spatial visualization accounted for 62 percent of the variation in achievement in proportional reasoning (as measured by the Karplus, Rund, and Piaget's tasks).

The findings of studies on the relationship between high school geometry achievement and spatial visualization are inconclusive. Holzinger and Swineford (1946) report a high multiple correlation (.71) for predicting plane geometry achievement by the best three predictors out of an initial battery of nine tests: one of them was connected to the general factor (as measured by a series completion test) and

two others to spatial ability (as measured by the Visual Imagery, and Punched Holes-Verbal tests). Werdelin (1961) conducted a comprehensive factorial analysis of mathematical abilities in three highly selective Swedish schools, and concludes that "the space factor or factors are of essential importance to the different aspects of geometry; primarily to geometrical construction and abstraction and also to problem solving" (p.122). In an earlier study, Werdelin (1958) reanalyzed the data published by Rogers (1918), and showed that contrary to the initial findings of this study, geometrical ability depends on spatial ability and also, but to a smaller degree on verbal ability. Werdelin (1958) notes that the geometrical tests in Rogers' study were "quite similar" to space tests. This similarity may be considered a potential source of confusion between geometrical and spatial abilities, but on the other hand, it may be considered a valid evaluation mean of a less formal geometrical ability.

In an investigation of the informal aspects of geometrical abilities at the college level, Blade and Watson (1955) indicate that achievement in spatial visualization among engineering freshmen correlates positively with grades in descriptive geometry ($r=.54$) and in engineering drawing ($r=.34$).

On the other hand, there are many studies (Murray, 1949; Weiss, 1955; Werdelin, 1958; French, 1964; Bennet et al., 1966) that seem to indicate that geometrical ability is independent of the space factors.

The ambivalence of these findings is reflected also in a study by Hanson (1972): he found some connection between visual perception ability and geometry for his ninth-grade subjects (11 percent of the variance in geometry achievement could be accounted for), but did not find a significant correlation for the tenth graders. Werdelin (1961) observes that "the question whether mathematical (geometrical) ability is dependent on the visual factor(s) has not been definitely answered" (p.38). This observation seems to be valid in view of the contradictory findings in the case of Euclidean geometry, and of lack of evidence in the case of informal geometry.

Sex Differences

Sex Differences in Geometric ability

Sex differences in mathematical abilities is a widely debated issue. For many years it was accepted that sex differences in mathematical abilities, favoring males emerge at adolescence (Maccoby and Jacklin, 1974). More recently, counter-evidence has been provided by Fennema and Sherman (1978) who investigated the achievement of high school students in light of their mathematical backgrounds and found that males and females with similar course enrollment had similar achievements.

As compared with studies on sex differences in spatial visualization, the research related to geometric abilities is significantly scarcer. Moreover, the two fields are sometimes hardly distinguishable. For example, a study on the effect of instruction in Tangram puzzles (Smith & Shroeder, 1979; Smith & Litman, 1979) is discussed as a study in spatial visualization, but could be related to informal geometry as

well.

The issue of sex differences related to the concept of similarity has not been considered as a separate topic. An indirect attempt to address it was made in 1925 by Cameron. By using rather rudimentary statistical methods and a battery of nine tests, she analyzed sex differences in mathematical aptitudes in a sample of 13 through 17-year-olds. One of her tasks required the completion of a construction of a rectangle similar to a given one, when one of its sides is also a given. Cameron (1925) concludes that the performance of boys and girls was slightly in the boys' favor , but "fairly equal". Due to the lack of systematic research on the concept of similarity, studies on sex differences in geometric abilities in general and in proportional reasoning may be considered relevant.

Three studies that relate to informal, or to transformational geometry have detected no sex differences in performance: Olson (1970) at high school level, and Thomas (1977) at the grade levels 1, 3, 6, 9, and 11, reached this conclusion in studies on transformation geometry; G.D. Peterson (1973) made this observation after administrating a 50-item geometry achievement test to 725 fourth, fifth, and sixth graders.

When the connection with spatial ability is made, the results become less clear. Shonberger (1976) indicates that practical and geometrical problems involving spatial components are the source for sex differences in mathematical abilities. In his study on geometric and spatial abilities of Swedish boys and girls at the high school level, Werdelin

(1961) found a male superiority in less formal aspects of geometry such as geometrical construction and abstraction, and geometrical problem solving, whereas girls had a better ability to prove theorems. These findings remained basically unchanged, when he compared 143 boys with the same number of girls matched by general reasoning, numerical ability, and age. Werdelin also refers to another Swedish study conducted in 1944 by Siegwald. This study indicated male superiority in dynamic visualization (i.e., mental imagery manipulations), space relations, geometrical imagination, construction of difficult, unfamiliar geometrical figures, and in geometrical problems, but not in perception of form, construction of simple, familiar shapes, and in static visualization.

With regard to the effect of geometry instruction on spatial abilities as a function of sex, Brinkmann (1966) found that eighth-grade boys and girls performed and gained similarly on a spatial visualization test after an intervention of ten programmed units in informal geometry.

A four-hour-long instruction based on Tangram puzzles caused similarly good results for both sexes at the fourth grade level (Smith & Schroeder, 1979). However, the same intervention erased the pre-interventional female superiority at the sixth and seventh grade level implying that at the stage of early adolescence, girls profit less from this kind of instruction (Smith & Litman, 1979).

Most of the studies on sex differences in high school Euclidean geometry imply that as compared to boys, girls have similar or higher abilities in tasks that require proving theorems (Touton, 1924; Perry, 1929; Werdelin, 1961; Senk & Usiskin, 1982).

To conclude, the findings in geometry seem to confirm a trend that has been observed in other mathematical fields: when experience is controlled (as for example, in Euclidean geometry or in a tranformational but formal approach to geometry) girls and boys perform equally well. On the other side, in mathematical tasks that also rely on out-of-school experience (e.g., some spatial tasks) sex differences tend to show up.

Sex Differences in Proportional Reasoning

Studies on sex differences in proportionality tasks suggest that boys tend to outperform girls at any age.

Chapman (1975) showed that even at the third and fifth grade levels, boys scored significantly ($p<.01$) higher than girls in ratio comparison tasks and exhibited greater frequency of verbal explanations referring to proportions.

In his proportional reasoning study of seven countries, Karplus et al. (1977) observe that eighth and ninth grade female subjects tended to use additive responses (which are of a lower order) more often than males.

Brendzel (1981) refers to a study conducted in 1977 by Piburn at Rutgers University. Piburn detected a male superiority in proportional reasoning tasks among his subjects ranging in age from 13 to 23.

Keating and Schaefer (1975) found that fifth and seventh-grade psychometrically bright boys outperformed sixth and eighth-grade bright girls on three Piagetian tasks--one of them being the Balance Task.

Some studies also indicated sex differences as a function of task-related variables. Besides a general male superiority, Stage, Karplus, and Pulos (1980) found, that among their sixth, and eight-grade subjects, younger and female students have additional difficulties with structural changes that make tasks more difficult for all students (such as a more difficult numerical context). At the ninth grade level, Brendzel (1978) found evidence of sex differences favoring males in proportional reasoning tasks, particularly when unfamiliar measuring units were employed. Garrard (1982) concludes that at the eighth grade level, visual adjuncts were more effective ($p<.10$) for girls in the solution of twenty non-routine proportionality tasks.

The male superiority in proportional reasoning, and the inconclusive findings on sex differences in geometrical tasks, do not allow for predictions on the existence of, or lack of sex differences in geometrical similarity tasks that combine these two aptitudes.

The Effect of Instruction

Influencing Cognitive Development

Relatively few attempts have been made to study the influence of instruction on the acquisition of geometrical concepts, and among these studies few could show positive results. Since no studies have

concentrated specifically on instruction related to the concept of similarity, three related topics will be discussed: (1) the teaching of geometrical concepts in general, (2) the teaching of proportional reasoning, and (3) the effect of instruction in geometry on spatial ability.

The Piagetian research on these topics is mainly descriptive, and its instructional implications are usually stated as speculations in separate studies. More recently, Piagetian researchers recognize the potential of instruction as an accelerator of cognitive processes. Montangero (1976) states:

> Learning must be subordinated to the laws of development since operational structures do not derive from structures that might exist outside the child, but stem from the coordination of internalized actions...In brief, training, according to the Genevan conception, consists in trying to accelerate cognitive development. (p. 126)

Some empirical studies showed that instruction on a certain concept at a certain age failed to bring the desired results, and the failure was related to the subjects' functioning at a lower mental-operational stage than the level requested by the presented task. Young (1975) for example, examined the relationship between performance on tasks related to ten geometrical concepts and instruction. By use of Chi-Square scores in a sample of 791 kindergarten through third graders, Young concluded that performance was independent from received instruction. Boulanger (1974) attempted to teach proportionality to 51 third graders that were at the concrete-operational level, and concluded that the training was neither retained over time, nor did it transfer to other simple proportion tasks. Lawson and Wollman (1980) also experimented with teaching proportional reasoning in a class of seventh graders, and concluded that the extent of success depended on each subject's operational level, and that formal operational level is a prerequisite for teaching proportional reasoning.

An important model that takes into account both the cognitive and the instructional aspects in children's understanding of geometry, was designed by P.M. and Dina van Hiele (for a good description in English, see Wirszup, 1976). The Van Hieles identified five stages of geometrical development (Table 2.1) and propose that the levels cannot be skipped.

Unlike Piaget, they argue that the levels develop primarily under the influence of school instruction. Therefore, instruction should be

Table 2.1 THE VAN HIELE LEVELS OF DEVELOPMENT IN GEOMETRY (adapted from Burger et al. 1981)

LEVEL	CONTEXT	FORM OF REASONING
0	Basic shapes and figures in some geometrical space	Visual identification and comparison of shapes
I	Properties of shapes in the space	Informal analysis of shapes in terms of their properties
II	Relationships among properties of shapes	Logical partial ordering of properties
III	An abstract geometrical system	Formal deduction (proof) of theorems within the system
IV	Various geometries	Rigorous mathematical study of the geometries

geared to lead students deliberately from one level to the next. Wirszup (1976) reports on two Soviet researchers who adapted a program of geometry instruction to the Van Hiele model with striking success.

Wirszup (1976) also uses the Van Hiele model to explain the present deficiencies in the teaching of geometry in the United States:

> Only a very small number of the elementary schools offer any organized studies in visual geometry, and where they are done, they begin with measurements and other concepts which correspond to Levels II and III of thought development in geometry. Since Level I is passed over, the material that is taught even in these schools does not promote any deeper understanding and is soon completely forgotten. Then, in the 10th grade, 15 and 16 year old youngsters are confronted with geometry for almost the first time in their lives. The whole unknown and complex world of plane and space is given to them in a passive axiomatic or pseudo-axiomatic treatment. The majority of our high school students are at the <u>first</u> level of development in geometry, while the course they take demands the <u>fourth</u> level of thought. It is no wonder that high school graduates have hardly any knowledge of geometry, and that this irreparable deficiency haunts them continually later on. (p.96)

Thus, according to this view, instructional, rather than cognitive deficiencies are the main reason for a poor performance in geometrical

tasks.

Instructional Interventions

Geometry. Most of the studies about the effect of instruction on achievement in geometry compare the effect of different modes of instruction on a wide variety of concepts, rather than examine the effect of instruction on the acquisition of a specific concept. The topics of reported comparisons are: (1) concrete-activity orientation versus paper-and-pencil approach, (2) problem-solving orientation versus presentation of the end-product, and (3) formal (i.e., Euclidean) versus informal geometry.

1. Concreteness. In a survey of studies that compared the use of concrete materials with reliance on symbolic representations in the teaching of mathematics, Fennema (1972) concludes:

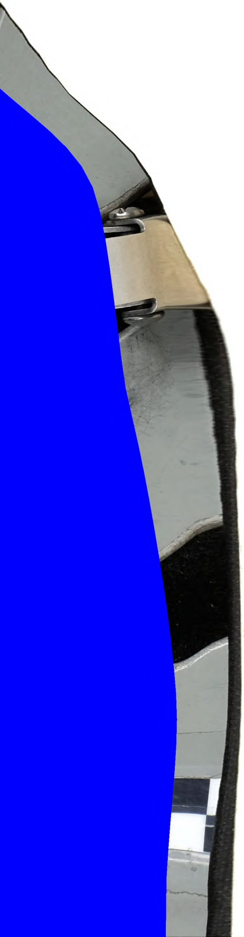
> Learning of mathematical ideas is likely to be facilitated by a predominance of concrete models in the early grades and a gradually increasing proportion of symbolic models as children move through the elementary school...Piaget placed children up to twelve years of age in the concrete-operational stage of cognitive development. Children in this stage are capable of learning with symbols, but only if these symbols represent actions the learners have done previously. (p. 637)

Relating specifically to geometry, Bring (1972) showed that a unit for fifth and sixth graders on volume, congruence, symmetry and isometrics gave significantly better results when presented in a concrete-activity mode as compared to a control group that used only paper and pencil.

2. Problem solving. Mathematics educators agree on the advantages of emphasizing the evolution of mathematical models from real-life situations, as opposed to an expository teaching that presents a ready-made product. Two studies that addressed this issue in relation to the teaching of informal geometry were conducted by Scott, Fryer, and Klausmeyer (1971), and by Hempel (1981). Both indicate that expository teaching gives better short-term results, but the discovery approach proved superior in retention and transfer. Buchert (1980) also compared the two modes of instruction of informal geometry with 108 seventh graders, and found that the "mathematized" (i.e., process-oriented) instruction was clearly superior ($p<.0001$) for students of all arithmetical abilities.

3. Informality. In the last eighty years, many committees recommended that geometry be taught informally at the K-8 grade levels. J. C. Peterson (1973) suggests to extend this recommendation to the high school level as well. Several studies conducted in high schools compared the teaching of formal Euclidean geometry with a less formal approach, but their findings are not clear.

From a pedagogical point of view, Peterson presents the advantages of teaching informal geometry at any grade level. According to him, "informal geometry is more than a list of topics--it is a method of teaching geometry. Informal geometry at its best makes use of discovery methods of teaching, inductive reasoning and the student's



inquisitiveness" (p.60).

However, difficulties arose when more rigorous studies compared achievement of students who learned an experimental informal geometry course for the 10th, and 11th grade level with a control group that studied the traditional high school geometry course. Cox (1980) concludes that his results are "not clear": the control group did better on the ETS Cooperative Test in Geometry-Part 1 and also on problems in solving multiple concepts and properties, and more complex constructions. On the other side, the experimental group showed better attitudes towards geometry.

Three studies experimented with interventions based on transformation geometry at the high school level: Olson (1971) reports disappointing results--especially on a four-problem proof test; Usiskin (1972) found that his control group outperformed ($p<.01$) the experimental group in standard geometry content; Durapau (1979) indicates similar achievements for his control and experimental groups, but a better performance of the latter ($p<.01$) on transfer questions.

A possible explanation of these results is the fact that in order to allow for comparisons, new experimental approaches have been evaluated with tools that measured skills characteristic to the content and the goals of a traditional geometry course.

Proportional reasoning. Lunzer (1973) tries to strike a balance between the limitations set by the cognitive-operational ability of a child in terms of proportionality and the need for instruction. He states:

> It has not been shown that the realization of proportionality comes about spontaneously as a result of maturing logic...It is more probable that these relations need to be taught. But the evidence suggests that even given good teaching, they will not be applied spontaneously to new problems until the student's powers of reasoning have reached an advanced level of development. (p.13)

After reporting SES- rather than nation-related differences of performance in proportional reasoning, Karplus et al. (1977) conclude their international survey by arguing that "teaching makes a difference at this age" (i.e., 13-15 years), and recommend an instruction that provides concrete experiences and emphasizes active participation in learning. Similar conclusions are drawn by Wollman and Lawson (1978) in a comparison of a verbal approach with a physical-action orientation in teaching seventh graders to reason proportionally. By using a three-week-long instruction unit, Kurz and Karplus (1977) showed that proportional reasoning can be taught effectively at the high school prealgebra level: after instruction, more than 50 percent of the experimental students, but less than ten percent of the control group scored highly on proportional reasoning tasks. However, in a comparison of two instructional modes for the same unit, they could not detect a significant difference in achievement: the rate of success in the "manipulative" group went up as a result of instruction from 15 percent to 70 percent, whereas the "paper-and-pencil" group advanced from 17 to 62 percent.

Spatial ability. Presently there is no evidence that instruction in geometrical similarity can cause improvement in spatial visualization, the issue can be addressed indirectly by analyzing the studies that attempted to relate instruction in geometry in general to spatial visualization. The studies that deal with transformational geometry are particularly relevant, since the concept of similarity fits naturally into a transformational approach to geometry.

Theoretically, the connection between transformational geometry and spatial ability is quite straightforward: the study of slides, flips, and turns on objects or drawings should increase the ability to perform mental manipulations of figures. Piaget (1964) argues that imagery manipulations can be described in terms of geometrical transformation. However, findings of some empirical studies, do not support this argumentation.

Perham (1977) reports that after an 11-session-long instruction in transformation geometry, his experimental group of first graders gained significantly more than the control group (N = 72) in transformation geometry but not in spatial visualization (as measured by a test on horizontial-vertical and left-right orientation, and on figure folding). A similar failure to achieve transfer from transformation geometry to mental imagery manipulation after a 12-session-long instruction was reported by Williford (1972) at the second and the third grade level. Contrary results were reported by Ekman (1968), who detected significant improvement in spatial ability during the summer, as result of instruction in geometrical measurement and calculations conducted towards the end of the fourth grade. This study, however, has no control group that would assure that the improvement is not a result of maturation.

Brinkmann (1966) experimented at the eighth grade level with a 10-session-long programmed course in geometry that emphasized figure differentiation or discrimination, pattern folding, and object manipulation in a problem-solving approach. Wolfe (1970) basically replicated Brinkmann's study at the grade levels seven, eight, and nine. Both studies show significant gains in spatial visualization (as measured by the DAT Space Relations Test). Wolfe, however notes that little transfer was observed in tasks that were different from the ones encountered during instruction.

At high school level, Ranucci (1952) did not detect any significant differences in space perception abilities between a group that studied a course in solid geometry and a control group that did not take the solid geometry course. Similarly, Brown (1955) compared performance in spatial visualization (as measured by the DAT Space Relations Test) of a control group that studied one year of plane geometry, with a group that studied a combined course of plane and solid geometry. He concluded that the addition of solid geometry did not prove itself as a facilitator for spatial visualization. Studies by Myers (1958) and Sedgwick (1961) indicate respectively that neither a high school course in mechanical drawing nor a college course in descriptive geometry could cause a significantly different performance in spatial visualization tasks as compared to control groups that did not study these courses.

However, instruction that uses a less formal approach to geometry has a stronger influence on spatial visualization. Van Voorhis (1941) conducted a first-year college course that included estimating linear extent, angles and areas, three-dimensional tick-tack-toe, and various other visualizing experiences. A study by Battista, Wheatley, and Talsma (1982) involved a geometry course for prospective elementary teachers that included manipulating concrete models, paper folding, tracing, using Miras, constructing polyhedra, and transformational geometry. Both studies report a significantly better performance in spatial visualization: the first as compared to a control group (on the "Cards" and "Figures" sections of the Thurstone Test for Primary Mental Abilities), and the second as compared to pre-interventional performance (on the Purdue Spatial Visualization Test: Rotations).

To summarize the presented evidence, at the elementary level, instruction in geometry did not have a significant effect on spatial visualization. Starting from the middle grades level, this effect manifests itself in interventions that emphasize the visual aspect of geometry. Studies on traditional courses in geometry could not show similar effects. Brinkmann (1966) offers an explanation for this phemomenon:

> When one realizes that the emphasis in the teaching of geometry is usually on development of formal proofs based on a certain type of "givens", the failure to add to the performance on spatial visualization should not be surprising. The behaviors demanded are simply different. (p. 180)

CHAPTER III

DESIGN OF THE STUDY

Purposes of the Study

This study was designed to investigate three areas that are related to the concept of similarity: (1) performance in four similarity-related areas, (2) performance on tasks presented in a visual or a verbal mode, and (3) relationship between performance on similarity and spatial visualization.

1. Performance in four similarity-related areas.

1.1 To investigate prior to instruction in similarity the possibility of grade level, or sex differences in performance of sixth, seventh, and eighth graders in tasks that require the use of (1) basic properties of similar shapes, (2) proportional reasoning related to similarity, (3) the principle of area growth, and (4) applications of similarity in scaled drawings and in indirect measurement.

2. Performance on tasks presented in a visual or a verbal mode.

1.2 To investigate the effect of instruction of the MGMP _Similarity Unit_ on the performance in the four topics mentioned above.

2.1 To investigate grade level and sex differences in pre-instructional performance on similarity tasks presented in a verbal mode, and on equivalent tasks accompanied by drawings.

2.2 To investigate the effect of instruction pertaining to similarity on similarity tasks presented in a verbal mode, and on their visual counterparts.

3. Relationship between performance on similarity tasks and spatial visualization.

   3.1 To explore any possible effect of instruction in similarity on performance in spatial visualization tasks.

   3.2 To explore any possible effect of instruction of the MGMP Spatial Visualization (one year before this study) on performance in similarity tasks.

Population and Sample

The participant subjects in this study were students from five suburban middle schools in the Lansing, Michigan area. All five schools have a middle-class, predominantly white student population. The selection of the schools was made according to the teachers' willingness to participate in the experimental instruction of the MGMP Similarity Unit: schools that had more than one volunteering teacher have been selected to participate in the study. Table 3.1 presents the distribution of classes and teachers by school and by grade level.

Since School 5 participated a year before this study (Winter, 1982-83) in the instruction of the MGMP Spatial Visualization Unit, the sample was divided into two parts:

Sample A consisted of sixth, seventh and eighth-grade students from Schools 1, 2, 3, and 4. This sample was used to explore issues related to students who did not undergo an instruction in spatial visualization. Although most of the questions were concerned with performance in similarity, School 5, has been omitted from this sample to prevent a contamination of the results due to a potential influence of instruction in spatial visualization on performance in similarity tasks. Table 3.2 presents the distribution of the subjects in Sample A by grade and by sex.

Sample B consisted of seventh and eigtht-grade students from School 5. This sample was used to explore issues related to the comparison of students who learned the Spatial Visualization Unit and those who did not. Students of both kinds could be found in this sample. The distribution of the subjects from this sample by grade level and by experience in instruction in spatial visualization is presented in Table 3.3.

Table 3.1 DISTRIBUTION OF PARTICIPATING CLASSES AND TEACHERS BY GRADE LEVEL AND BY SCHOOL

	N Classes			N Teachers		
	Gr. 6	Gr. 7	Gr. 8	Gr. 6	Gr. 7	Gr. 8
School 1	6	2	-	2	2	-
School 2	3	3	-	3	3	-
School 3	-	-	3	-	-	2
School 4	2	2	1	1	1	1
School 5	-	3	5	-	2	2
Total	11	10	9	6	8	5

Table 3.2 SAMPLE A--DISTRIBUTION OF STUDENTS BY GRADE LEVEL AND BY SEX

	N Male	N Female	N Total
Grade 6	137	111	248
Grade 7	87	83	170
Grade 8	54	42	96
Total	278	236	514

Table 3.3 SAMPLE B--DISTRIBUTION OF STUDENTS BY GRADE LEVEL AND BY EXPERIENCE IN SPATIAL VISUALIZATION (SV) INSTRUCTION

	N Students with SV instruction	N Students without SV instruction	N Total
Grade 7	29	29	58
Grade 8	69	34	103
Total	98	63	161

Instrumentation

The instruments used in this study were a similarity performance test and a spatial visualization performance test.

Similarity

The similarity performance was measured by the Middle Grades Mathematics Project Similarity Test (MGMP SIMT). This test was developed by the Middle Grades Mathematics Project staff, including this investigator, in the Mathematics Department of Michigan State University. During Fall 1982, a pilot test was administered to approximately one hundred sixth and seventh grade students in two schools, prior to, and after instruction with the first draft of the MGMP Similarity Unit.

Test validity and reliability were considered: the test was revised in view of student results and observations made by the project's staff and the participating teachers.

The MGMP Similarity Test consists of 25 multiple choice items with five options for each item. The presented options have been selected according to the most popular misconceptions indicated by cognitive research on similarity concepts and on proportional reasoning. The items were scored by assigning a 1 for a correct response and a 0 otherwise; no correction was made for guessing. The test was not timed, but usually did not exceed 25-30 minutes. The total score on the test was considered as a general indicator of the level of performance in similarity tasks.

The Cronbach reliability coefficients calculated for the test ranged from .53 to .82. Table C.1 in Appendix C includes the reliability coefficients for the MGMP SIMT by grade and by sex.

Another indicator of the quality of the instrument was the significant high correlations ($P < .001$) between the pretest and the postest scores. The corresponding Pearson correlation coefficients (Table C.2 in Appendix C) ranged from .46 to .79.

Twenty items out of the 25 were clustered into four subtests (Appendix A), in order to analyze student performance on four topics: (1) basic properties of similar shapes (i.e., recognition of similar shapes and their properties), (2) proportional reasoning (i.e., finding the fourth number in a proportion that represents lengths of sides of similar shapes), (3) the principle of area growth in similar shapes, and (4) applications of similarity in scaled drawings and in indirect measurement. Subtests (1) and (4) include six items each, and subtests (2) and (3) include four items each. In the original test, the items were not presented in a clustered form; the item numbers in Appendix A represent their original sequence. The scores on these subtests were used as indicators of student pre, and post-instructional performance on the considered similarity-related topics.

In order to examine student performance in similarity tasks presented in a verbal or in a visual mode, eight items were clustered into a visual and a verbal subtest of four items each (Appendix A). Although twenty out of the 25 items were accompanied by drawings, only four of these were selected so that the selected visual items had strictly equivalent counterparts stated in a purely verbal form. The four pairs of equivalent items included one pair on the concept of similarity, one pair on indirect (shadow) measurement, and two pairs on area growth.

Spatial Visualization

The performance in spatial visualization was measured by a 15-item subtest of the MGMP Spatial Visualization Test. The original 32-item test was developed by the MGMP staff during summer and fall 1981 and was administered to about 1500 students in two evaluation studies of the MGMP Spatial Visualization Unit (Ben-Haim, 1983 ; Lappan, 1983).

Fifteen representative items have been selected to form the MGMP Spatial Visualization Subtest (MGMP SVST) used in this study. The SVST consists of items requiring three-dimensional visualization and is unrelated to topics or notations developed in the Similarity Unit. Appendix B presents a representative sample of six items from the SVST. The items of this test are also multiple choice, with five options for each item, and with a 0/1 scoring. The total score on the test was considered as an indicator of the level of performance in spatial visualization.

The Cronbach α reliability coefficients of the SVST ranged from .59 to .80, whereas the pre-post Pearson correlation coefficients ranged from .66 to .80 (Tables C.1 and C.2 in Appendix C).

The SIMT and the SVST were both administered during the same class period, and the total administration time did not exceed 45-50 minutes. In order to neutralize a possible effect due to a certain sequencing of the two tests, about half of the classes received the SIMT first, and the SVST second, while the other half received the two tests in reverse order. The order of testing was randomly attributed to whole classes.

Instructional Material

The MGMP Instructional Model

The Middle Grades Mathematics Project developed four instructional units. Each of these units requires two to three weeks of instructional time, and concentrates on a specific concept that is considered to be important in the mathematical development of the middle grades level student. Two MGMP units will be briefly described here: (1) the Similarity Unit that formed the basis for the instructional intervention in this study, and (2) the Spatial Visualization Unit that was taught in several experimental classes in School 5 one year before this study.

Each of the MGMP instructional units provides a carefully sequenced set of challenging and exploratory activities designed to lead the student to the understanding of a mathematical concept, and a carefully developed and very detailed instructional guide for the teacher. The units utilize an instructional model developed by Fitzgerald and Shroyer (1979). The model consists of three phases: launch (introducing new concepts, clarifying definitions, reviewing old concpets, and issuing a challenge), exploration (individual work in gathering data, sharing ideas looking for patterns, making conjectures, or developing other types of problem-solving strategies), and summary (demonstrating ways to organize data discussing the used strategies, and refining these strategies into efficient problem-solving techniques). The launch and the summary are conducted in a whole-class mode, while at the exploration stage, the teacher becomes a fellow investigator who helps individual students to follow their own exploratory path of thinking.

The MGMP Similarity Unit

The concrete-activity orientation of the MGMP Similarity Unit follows recommendations of the Piagetian research on geometrical development, and the instructional implications of the Van Hiele studies.

The unit presents the concept of similarity at the Van Hiele levels I (properties of shapes) and II (relationships among properties of shapes) of geometrical development. The present neglect in the consideration of the Van Hiele levels in the teaching of geometry in the United States is believed to be a reason for present deficiencies in many children's understanding of geometrical concepts (Wirszup, 1976; Carpenter et al., 1981).

The MGMP Similarity Unit is a response to deficiencies in the teaching of geometry. By offering a rich variety of experiences designed to fit the Van Hiele levels I and II, the unit contributes to the building of a solid base for further advance in the understanding of geometrical concepts. Moreover, the launching-exploration-summary model corresponds to Van Hiele's three steps required to help the child move to a higher developmental level: information, directed orientation, explanation.

The unit contains nine activities that require 1-1 1/2 class periods each:

-figure enlargement by rubber bands

-figure enlargement by point coordinates

-comparing rectangles

-families of similar rectangles

-comparing triangles

-creating similar figures with repeating tiles

-figure enlargement by point projections

-enlarging pictures

-applications of similar triangles.

A more detailed overview of the unit is presented in Appendix D. The nine activities use concrete manipulatives to help provide the transition from the student's concrete to abstract thinking. The four similarity-related topics of this study (basic properties, proportional reasoning, area growth, and applications) are encountered throughout the unit in a spiral way.

<u>The MGMP Spatial Visualization Unit</u>

The Spatial Visualization Unit includes ten carefully sequenced activities on the representation of three-dimensional objects in two dimensional drawings, and vice versa, on the construction of three-dimensional objects with blocks from their two dimensional representations. The activities deal with the flat views of buildings as well as with the isometric drawings on dot paper (paper with dots arranged on diagonals rather than rows). In most of the activities the students are asked to perform some fairly demanding orientation and visualization tasks. They are asked to mentally rotate a building and draw either flat views of the other sides or isometric drawings from other corners. Cubes are always available to help a student who needs to see the concrete object to be successful.

Procedure and Data Collection

The experimental instruction of the Similarity Unit and the data collection were conducted from January to May 1982. The whole sample of 695 students was tested on the Similarity Test and on the Spatial Visualization Subtest both before, and after instruction.

The classroom teachers were provided with all the instructional materials, the testing material and written instructions concerning the administration of the tests. The tests were administered during the regular school day by the classroom math teachers. For all the teachers, this was the first time that they taught the Similarity Unit, although many of them have already taught another MGMP instructional unit. Prior to instruction, a two-hour workshop for the teachers was conducted in each of the five participating school, but no direct intervention by the staff of the project has been provided during instruction. The workshop acquainted the teachers with the nine activities in the unit, recommended instructional strategies, and presented a more in-depth view of the geometrical concepts involved.

The Statistical Design of the Study

To analyze the data collected during the study, analyses of means, standard deviations, correlations, Multivariate and Univariate Analyses of Variance and ANOVA with Repeated Measures were conducted. A detailed description of the hypotheses tested and the statistical procedures used for each purpose of the study will be given in the next chapter, where they will be followed by a description of the findings.

For the significance tests, the assumptions were that the subjects were responding independently of one another and that the error vectors had a multivariate normal distribution with mean zero and a common variance covariance matrix. The fact that this study was conducted in a regular school setting made the assumption of independence and normality feasible.

The effects of instructional intervention were examined by using design 6 from Campbell and Stanley (1963): 0_1 X 0_2, where 0_1 and 0_2 are the scores collected before and after instruction, and is the treatment (instructional intervention). A control (no-intervention) group was not employed in this study, due to (1) physical difficulties of recruiting and testing a group of equivalent size and structure, and (2) the relatively small possibility of confounding effects as a result of the short duration of instruction and the wide variety of sampled subjects.

The collapsing of students from four different school buildings into one sample would probably increase to a certain degree the heterogeneity of the population. However, the purpose of the study was to examine the effect of instruction pertaining to similarity in heterogeneous, middle-class suburban school conditions, rather than in an artifically homogeneous environment.

There are two alternative units of analysis for this study: teacher versus student. Hopkins (1982) compares these two possibilities and concludes that taking the teacher as a unit ignores the fact that the observational unit is the student and actually only exchanges the assumption of independence among students with another not necessarily more true assumption of independence among teachers. Moreover, post-hoc examinations of the results did not reveal any outstanding discrepancies in the results by school building or by teacher. Thus, this study will consider the student as unit of analysis.

All the analyses were carried out on the CYBER 170, Model 750 computer at the Michigan State University Computer Center, using SPSS (Statistical Package for the Social Sciences) programs.

CHAPTER IV

ANALYSIS OF DATA AND RESULTS

Introduction

This chapter presents the findings that correspond to the six research questions of this study. Each of the six questions that were posed in the introductory chapter will be restated here as a hypothesis. Each hypothesis will be followed by a description of: (1) the statistical design and procedures used to examine the data, (2) a description of the obtained results, and (3) a short summary of results related to the initial question.

Basically, the six hypotheses stated in this chapter have been tested by employing Analysis of Variance. The means, standard deviations and correlation coefficients computed allowed for further interpretations of the results.

The Analyses of Variance employed in this chapter have two limitations:

1. Since for Sample A the grade effect has three levels (i.e., two degrees of freedom), only two orthogonal contrasts were formed: grade six versus grade seven, and grade six and seven combined versus grade eight.

2. Since the design is unbalanced (i.e., the cells do not contain equal numbers of cases), the testing of hypotheses within a certain ANOVA is ordered. The order of testing is presented in the ANOVA tables

from the bottom to the top: each effect controls for all the effects that are presented in the summary table above it, but is confounded with the effects that appear below it. Failure to reject a null hypothesis implies that the parameter corresponding to that effect is assumed to be zero, allowing thus to test the next hypothesis (presented in the table above it). The rejection of a certain hypothesis, means that the corresponding parameter may not be assumed to be zero. In this case, the testing of the next hypothesis is not possible because of the confounding effect of the non-zero parameter tested before.

These difficulties may be sometimes overcome by changing the order in which the effects are introduced, or by conducting a secondary analysis with a different partition of the effects. However, by conducting a multiple number of tests the probability of a Type I error would be increased accordingly.

Further inferences can be drawn by an examination of group means. Cohen (1969) suggests using the estimated standardized difference of population means[1] as a primary index for the effect size. He suggests that in behavioral sciences, standardized differences of .2, .5, and .8 standard deviation units may serve as operational definitions for respectively "small", "medium", and "large" effects. In order to allow for judgments of effect sizes, this study reports occasionally estimated standardized differences of means.

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[1]The estimated standardized difference of population means is $(m_1 - m_2)/MS_e^{1/2}$ where m_1, m_2 are the group means, and MS_e is the mean square error.

Pre-instructional Peformance in Four Similarity-Related Areas.

Question 1.1: Prior to instruction in similarity, what patterns can be observed in the performance of sixth, seventh, and eighth graders in tasks that require the use of: (1) basic properties of similar shapes, (2) proportional reasoning related to similarity, (3) the principle of area growth, and (4) applications of similarity in scaled drawings and in indirect measurements?

Hypothesis 1.1: In each of the four similarity-related areas, there will be no sex or grade level differences in the SIMT performance, and there will be no grade by sex interaction.

The hypothesis has been tested by a Two-Way Multivariate Analysis of Variance. Table 4.1 presents the design for the Analysis of Variance, and Table 4.2 is a summary of results. Mean scores of student performance and correlation coefficients among the four topics (Tables E.1 and E.2, Appendix E) allow for further interpretation of the data.

The following inferences may be drawn from the Multivariate and Univariate Analysis of Variance for Hypothesis 1.1 at a significance level of .05:

1. There are no significant grade by sex interactions for the pre-instructional performance on the four similarity-related topics

either when tested together in a multivariate form, or when considered separately in a univariate form.

2. No significant sex differences in performance on the four topics were detected in the multivariate, or in the univariate form. Analysis of the means in Table E.1 reveals, however, that in a majority of cases (eight out of twelve) the performance of the boys slightly exceeds that of the girls.

3. There are significant grade differences in the level of performance on the four topics (both in the multivariate and in the univariate

Table 4.1 DESIGN OF THE 3 x 2 MULTIVARIATE ANALYSIS OF VARIANCE FOR HYPOTHESIS 1.1

		Basic Properties	Proportional Reasoning	Area Growth	Applications
Grade 6	Males	x_1	y_1	z_1	u_1
	Females	x_1	y_1	z_1	u_1
Grade 7	Males	x_1	y_1	z_1	u_1
	Females	x_1	y_1	z_1	u_1
Grade 8	Males	x_1	y_1	z_1	u_1
	Females	x_1	y_1	z_1	u_1

Table 4.2 SUMMARY OF THE 3 x 2 MULTIVARIATE AND UNIVARIATE ANALYSIS OF VARIANCE FOR HYPOTHESIS 1.1

Source of Variation	D.F.	Multivariate[a] F	Multivariate[a] P<		Univariate F	Univariate P<
Constant	1	602.12	.0001	BCPT[b]	1530.59	.0001
				PROP[c]	622.13	.0001
				AREA[d]	455.40	.0001
				APPL[e]	1444.51	.0001
Grade	2					
Gr2 (6, 7 vs. 8)	(1)	58.82	.0001	BCPT	174.13	.0001
				PROP	128.89	.0001
				AREA	.29	.60
				APPL	92.61	.0001
Gr1 (6 vs. 7)	(1)	13.87	.0001	BCPT	29.02	.0001
				PROP	42.43	.0001
				AREA	4.05	.05
				APPL	10.86	.001
Sex	1	.98	.42	BCPT	3.54	.06
				PROP	.31	.58
				AREA	.62	.44
				APPL	.88	.35
Grade by Sex	2	1.53	.15	BCPT	1.88	.16
				PROP	.20	.82
				AREA	2.63	.08
				APPL	1.20	.31
Between Groups	6					
Within Groups	508	Univariate MS error:		BCPT	464.52	
				PROP	935.94	
				AREA	442.30	
				APPL	435.57	

[a] Multivariate D.F. = 4 and 505 for main effects
8 and 1012 for interaction

[b] Basic Concepts

[c] Proportional Reasoning

[d] Area Growth

[e] Applications

form) between sixth, and seventh graders favoring the latter. Mean score differences of .53, .65, .20 and .33 in standard deviation units were observed in the topics of basic properties, proportional reasoning, area growth, and applications respectively.

4. Grade level differences between seventh, and eighth graders (in favor of the latter in three out of four cases) may be inferred from Table E.1 (Appendix E) by arguing that the seventh-eighth grade performance gap is even larger than the performance gap between sixth and seventh graders. In the same order as in 3. the standardized mean score differences in this case were 1.17, .90, .06, and .89 standard deviation units. An additional Multivariate Analysis of Variance that partitioned the grade effect into the "7 vs. 8", and the "6 vs. 7, 8" contrasts also revealed significant differences between seventh and eighth graders (Multivariate $F = 27.67$, $p < .0001$). Both tests combined together are within the overall level of significance.

5. The univariate analysis of the performance on the topic of area growth reveals nonsignificant differences between seventh and eighth graders. An analysis of the mean scores (Table E.1, Appendix E) reveals that in this topic eighth graders performed at a slightly lower level (.06 standard deviations) than seventh graders, and actually all three grades performed in an almost uniformly poor manner (an average of 18-22 percent). The outstanding status of the area growth topic is also revealed by analyzing the correlation

coefficients among topics (Table E.2, Appendix E): all the coefficients except those related to area growth, are significant ($p < .01$) and range from .255 to .529. On the other hand, the correlation coefficients that involve the topic of area growth are either nonsignificant or low (less than .25).

To summarize the findings based on the pre-instructional performance in four similarity-related areas, at a significance level of .05, there are no sex by grade interactions and no significant sex differences; in most cases, level of performance increased significantly with level of grade; performance on tasks related to area relationships of similar shapes was lower than in other similarity tasks, and did not show the same increasing pattern as a function of grade level.

Effect of Instruction in Four Similarity-Related Areas

Question 1.2: What is the effect of instruction pertaining to similarity on the performance of sixth, seventh, and eighth graders in tasks that require the use of: (1) basic properties of similar shapes, (2) proportional reasoning related to similarity, (3) the principle of area growth, and (4) applications of similarity in scaled drawings and in indirect measurements?

Hypothesis 1.2: There will be no differences among gain scores as a function of grade level or sex, and there will be no grade by sex interactions.

The hypothesis has been tested by a Two-Way Analysis of Repeated Measures. Table 4.3 presents the design that was used for this analysis, and Table 4.4 is a summary of results. The pre, and post-instructional mean scores presented in Table E.1 (Appendix E) and in Figure 4.1 allowed for further interpretations of the data.

The following trends could be observed at a significance level of .05:

1. There are no significant grade by sex interactions or sex differences in gain scores in any of the four similarity-related topics.

Table 4.3 3 x 2 MULTIVARIATE REPEATED MEASURES DESIGN FOR HYPOTHESIS 1.2

		Basic Properties		Proportional Reasoning		Area Growth		Applications	
		Pre	Post	Pre	Post	Pre	Post	Pre	Post
Grade 6	Males	x_1	x_2	y_1	y_2	z_1	z_2	u_1	u_2
	Females	x_1	x_2	y_1	y_2	z_1	z_2	u_1	u_2
Grade 7	Males	x_1	x_2	y_1	y_2	z_1	z_2	u_1	u_2
	Females	x_1	x_2	y_1	y_2	z_1	z_2	u_1	u_2
Grade 8	Males	x_1	x_2	y_1	y_2	z_1	z_2	u_1	u_2
	Females	x_1	x_2	y_1	y_2	z_1	z_2	u_1	u_2

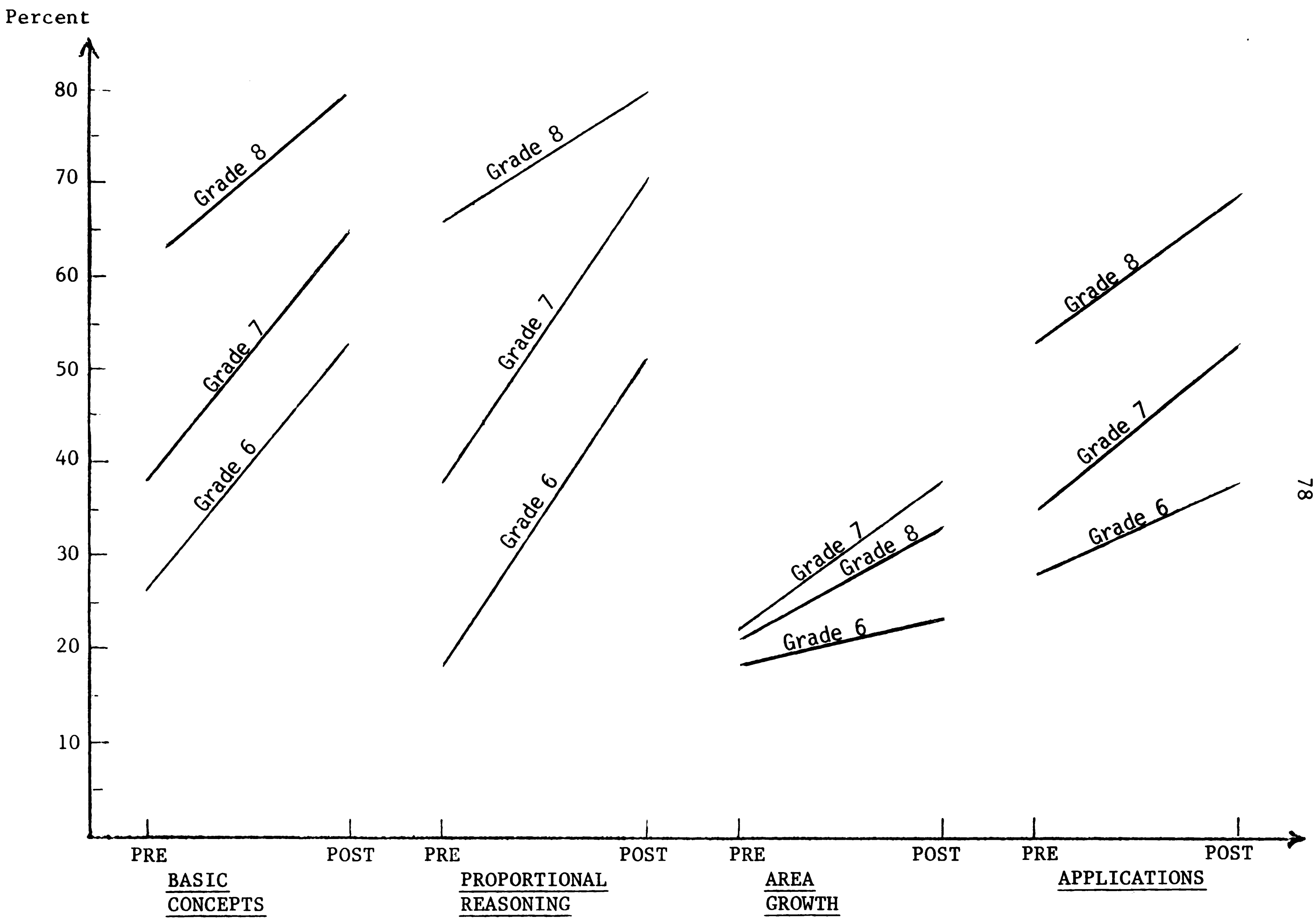


Figure 4.1 Performance by grade level in four similarity-related areas, prior to, and after instruction.

Table 4.4 SUMMARY OF MULTIVARIATE AND UNIVARIATE ANALYSIS OF REPEATED MEASURES FOR HYPOTHESIS 1.2

Source of Variation	Multivariate[a] D.F.	F	P<	Univariate[a]	F	P<
Constant	1	1128.08	.0001	Con. Avg.[b]	3753.32	.0001
				Prop. Avg.[c]	1786.92	.0001
				Area Avg.[d]	891.65	.0001
				Appl. Avg.[e]	1895.50	.0001
Grade	2					
Gr2 (6, 7 vs. 8)	(1)	55.91	.0001	Con. Avg.	167.42	.0001
				Prop. Avg.	109.70	.0001
				Area Avg.	1.69	.20
				Appl. Avg.	141.89	.0001
Gr1 (6 vs. 7)	(1)	19.16	.0001	Con. Avg.	40.22	.0001
				Prop. Avg.	58.65	.0001
				Area Avg.	24.75	.0001
				Appl. Avg.	38.46	.0001
Sex	1	1.18	.32	Con. Avg.	1.73	.19
				Prop. Avg.	1.60	.21
				Area Avg.	1.04	.31
				Appl. Avg.	4.52	.04
Grade by Sex	2	.93	.49	Con. Avg.	.81	.44
				Prop. Avg.	.58	.55
				Area Avg.	.20	.82
				Appl. Avg.	1.09	.34
Between Groups	6					
Time	1	220.53	.0001	Con. Diff	577.79	.0001
				Prop. Diff	369.53	.0001
				Area Diff	53.14	.0001
				Appl. Diff	162.21	.0001

Table continued

Table 4.4 (Cont'd)

Source of Variation	D.F.	Multivariate[a] F	P<		Univariate F	P<
Grade by Time						
Gr2 (6, 7 vs. 8) by Time	2	8.68	.0001	Con. Diff	15.44	.001
				Prop. Diff	20.21	.0001
				Area Diff.	.71	.40
				Appl. Diff	.65	.42
Gr1 (6 vs. 7) by Time	(1)	5.13	.0005	Con. Diff.	.01	.96
				Prop. Diff.	.01	.93
				Area Diff.	10.80	.001
				Appl. Diff	10.70	.001
Sex by Time	1	1.42	.23	Con. Diff.	1.93	.17
				Prop. Diff.	.83	.36
				Area Diff.	.03	.86
				Appl. Diff.	2.07	.15
Grade by Sex by Time	2	1.27	.26	Con. Diff.	1.54	.22
				Prop. Diff.	.41	.67
				Area Diff.	2.72	.07
				Appl. Diff.	.32	.73

Within Subjects	508			
Univariate MS Error:	Con Avg.	676.76	Con. Diff.	278.72
	Prop Avg.	1333.86	Prop. Diff.	584.42
	Area Avg.	710.80	Area Diff.	490.28
	Appl. Avg.	627.46	Appl. Diff.	315.22

[a] Multivariate D.F. = 4 and 505 for main effects
8 and 1012 for interactions

[b] Basic Concepts averaging over time (Pre + Post)

[c] Proportional Reasoning averaging over time (Pre + Post)

[d] Area Growth averaging over time (Pre + Post)

[e] Applications averaging over time (Pre + Post)

[f] Basic Concepts time effect over subjects (Pre - Post)

[g] Proportional Reasoning time effect over subjects (Pre - Post)

[h] Area Growth time effect over subjects (Pre - Post)

[i] Applications time effect over subjects (Pre - Post)

2. Seventh graders gained significantly more than the sixth graders. These differences are due to the topics of area growth and applications(standardized gain differences of .46 in both topics); in the other two topics no significant gain differences could be detected.

3. A comparison of gains between seventh and eighth graders can be made by observing the graphs in Figure 4.1, and by conducting an additional Analysis of Repeated Measures with "7 vs. 8" as one of the grade level contrasts. Both strategies reveal an overall significant (Multivariate $F = 6.18$, $P < .0008$) difference in gains favoring the seventh graders. The univariate analysis shows that this difference is due to gain differences in the topics of basic similarity properties and proportional reasoning (gain differences of .63 and .71 standard deviation units respectively). As in the case of Hypothesis 1.1, the combined level of significance of the analyses conducted in 2. and 3. is still very low.

To summarize the findings based on performance in four similarity-related areas prior to and after instruction, at a significance level of .05, no sex differences in gains could be detected, and seventh graders seem to have profited the most from the instruction: their gains were similar to the sixth graders' and higher than the gains of the eighth graders on area growth and applications; on the other side, they gained as much as the eighth graders and more than the sixth graders on the basic properties of similar shapes and on proportional reasoning related to similarity tasks.

Pre-instructional Performance on Tasks Presented in a Verbal and a Visual Mode

Question 2.1: Prior to instruction on similarity, what patterns can be observed in the performance of sixth, seventh, and eighth graders in tasks that are presented in a verbal mode or in a visual mode?

Hypothesis 2.1: By analyzing the level of performance on the two modes combined together, and the difference in performance between the two modes: there will be no grade by sex interaction, and there will by no sex or grade differences.

In order to test this hypothesis, a Bivariate Analysis of Variance has been used, by using the pretest scores on the items presented in a verbal mode and a visual mode of each student as variables (Table 4.5). The sum of the two scores has been used as an indicator of the overall level of performance in the two modes, and the difference of the scores has been used as an indicator of differences in performance between the two modes. Table 4.6 presents a summary of the Analysis of Variance results. Mean scores (Table E.3, Appendix E) have been examined to detect further trends in the obtained data.

Table 4.5 DESIGN OF THE 3 x 2 BIVARIATE ANALYSIS OF VARIANCE FOR HYPOTHESIS 2.1

		Verbal Mode (Pretest)	Visual Mode (Pretest)
Grade 6	Males	x_1	y_1
	Females	x_1	y_1
Grade 7	Males	x_1	y_1
	Females	x_1	y_1
Grade 8	Males	x_1	y_1
	Females	x_1	y_1

The following inferences may be drawn from the Analysis of Variance for Hypothesis 2.1 at a significance level of .05:

1. There are no significant sex by grade interactions in the analysis of the combined performance in the visual and the verbal modes, or of the score difference between modes of presentation.

2. There are no significant sex differences in the analysis of the combined performance in the visual and the verbal modes or the score difference between the two modes of presentation.

Table 4.6 SUMMARY OF THE 3 x 2 BIVARIATE ANALYSIS OF VARIANCE FOR HYPOTHESIS 2.1

SOURCE OF VARIATION	D.F.	F	P<	
Constant	1	1103.48	.0001	
Grade	2			
Gr2 (6, 7 vs. 8)	(1)	77.81	.0001	
Gr1 (6 vs. 7)	(1)	15.63	.0009	
Sex	1	1.57	.21	
Grade by Sex	2	.41	.67	
Between Groups	6		MS error[a]:	592.49
Mode	1	2.33	.13	
Grade by Mode	2			
Gr2 (6, 7 vs. 8) by Mode	(1)	.05	.83	
Gr1 (6 vs. 7) by Mode	(1)	.38	.54	
Sex by Mode	1	.01	.95	
Grade by Sex by Mode	2	.65	.53	
Within Subjects	508		MS error[b]:	357.01

[a] for Verbal (pre) + Visual (pre) score

[b] for Verbal (pre) - Visual (pre) score

3. There is a significant difference between the performance of sixth, and seventh graders (in favor of grade seven) in their performance on the tasks included in one of the two modes, but there is no significant grade effect in the analysis of the differences between the two modes. The means in Table E.3 (Appendix E) reveal that for all grade levels, the mean differences between pre-instructional performance in the verbal and visual modes are uniformly low (less than 4 percent) with a slightly higher level of performance in the verbal mode.

4. A second Analysis of Variance with the grade effect partitioned in the "7 vs. 8" and "6 vs. 7, 8" contrasts shows a significant ($F = 35.89$, $p < .0001$) difference favoring the seventh graders in the verbal and visual mode combined but, as shown also in Table E.3 of means, not in the score difference between the verbal and the visual mode ($F = .01$, $p < .93$)

To summarize, at a significance level of .05, no sex by grade interaction, sex or grade level effects could be detected in the analysis of the difference between performances on tasks presented in a visual, and a verbal mode. If the pre-instructional performances on the two modes are combined, the same pattern of grade differences that was detected in the test of Hypothesis 1.1 reappear: the level of performance increases significantly with grade level.

Effect of Instruction on Performance in Verbal and in Visual Tasks

Question 2.2: What is the effect of instruction pertaining to similarity on the performance of sixth, seventh, and eighth graders in similarity tasks that are presented (1) in a verbal mode, and (2) in a visual mode?

Hypothesis 2.2: a) In each of the two modes, there will be no differences among gain scores as a function of grade level or sex, and there will be no sex by grade interaction.

b) Comparing the gain scores in the two modes one with the other, there will be no significant differences by grade level or by sex, and there will be no grade by sex interaction.

Table 4.7 presents the Repeated Measure Design that has been used to test Hypothesis 2.2, and Table 4.8 is a summary of results of the Analysis of Repeated Measures that has been conducted to test this hypothesis. The gain scores are defined as the pretest-posttest difference of scores for each presentation mode. For testing Hypothesis 2.2a, the gain scores are combined (i.e., Verbal gain + Visual gain) and the analyzed effects are Grade by Time, Sex by Time, and Grade by Sex by Time. For testing Hypothesis 2.2b, the gain scores are compared (i.e., Verbal gain-Visual gain) under the Mode by Time interaction:

Grade by Mode by Time, Sex by Mode by Time, and Grade by Sex by Mode by Time. Table E.3 (Appendix E) of means and standard deviations, and Figure 4.2 of graphs that illustrate pre, and post-instructional performance in the verbal and the visual modes allowed for further interpretations of the data.

The analyses rendered the following results at a significance level of .05:

1. There are no significant grade by sex interactions in gain scores of the verbal, and visual scores either when combined together or when compared one versus the other.

Table 4.7 3 x 2 MULTIVARIATE REPEATED MEASURE DESIGN FOR HYPOTHESIS 2.2

		Verbal Mode		Visual Mode	
		Pre	Post	Pre	Post
Grade 6	Males	x_1	x_2	y_1	y_2
	Females	x_1	x_2	y_1	y_2
Grade 7	Males	x_1	x_2	y_1	y_2
	Females	x_1	x_2	y_1	y_2
Grade 8	Males	x_1	x_2	y_1	y_2
	Females	x_1	x_2	y_1	y_2

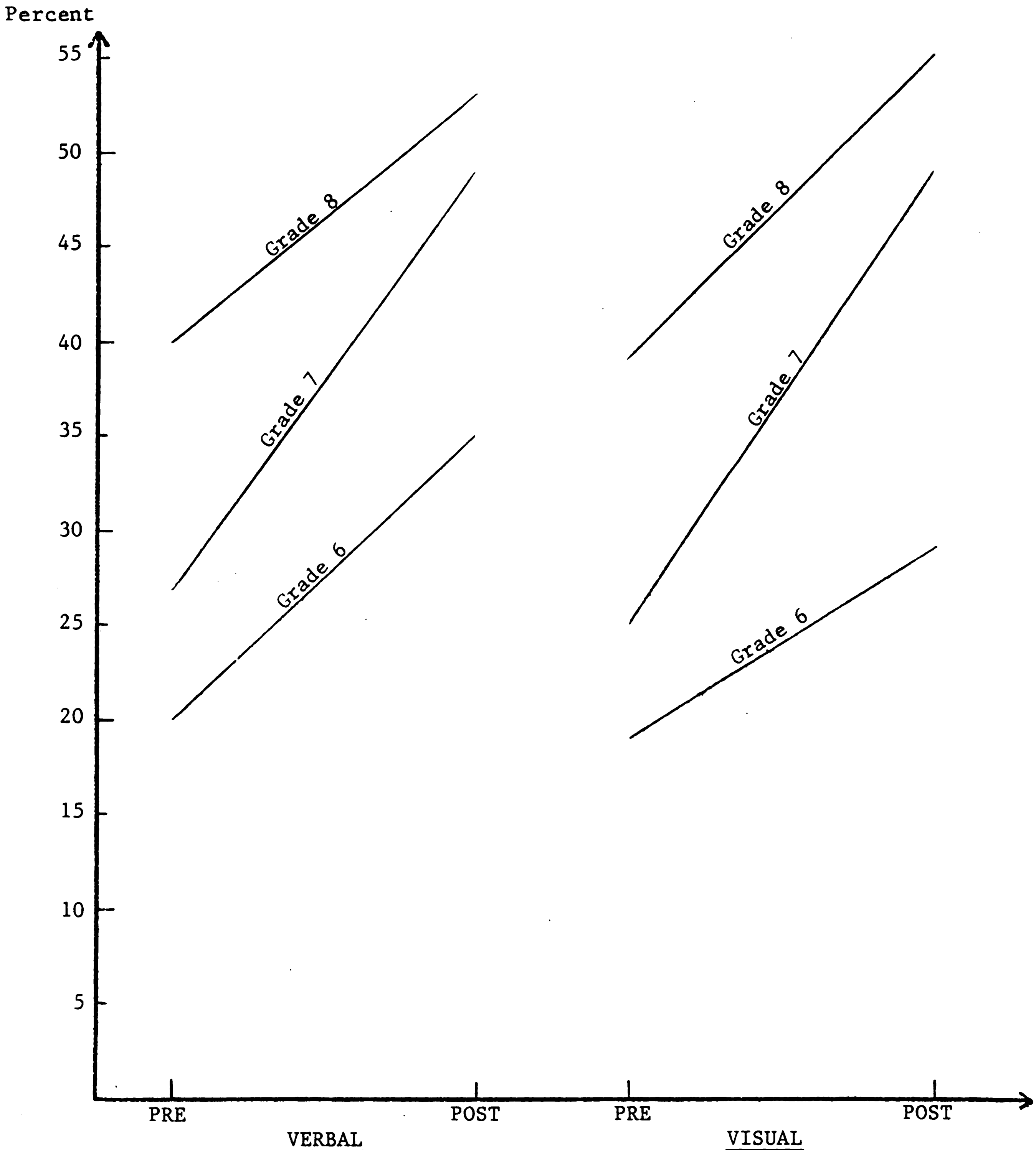


Figure 4.2 Performance by grade level in tasks presented in a visual and and a verbal mode, prior to, and after instruction.

Table 4.8 SUMMARY OF ANALYSIS OF REPEATED MEASURES FOR HYPOTHESIS 2.2

Source of Variation	D.F.	F	P<
Constant	1	2128.53	.0001
Grade	2		
Gr2 (6, 7 vs. 8)	(1)	73.45	.0001
Grl (6 vs. 7)	(1)	49.94	.0001
Sex	1	2.85	.10
Grade by Sex	2	.03	.98
Between Groups	6		(MS error [a]: 1083.98)
Time	1	292.35	.0001
Grade by Time	2		
Gr2 (6, 7 vs. 8) by Time	(1)	.96	.33
Grl (6 vs. 7) by Time	(1)	19.34	.0001
Sex by Time	2	.32	.57
Grade by Sex by Time	2	.86	.43
			(MS error [b] : 444.13)
Mode	1	5.18	.03
Grade by Mode	2		
Gr2 (6, 7 vs. 8) by Mode	(1)	1.50	.22
Grl (6 vs. 7) by Mode	(1)	.81	.37
Sex by Mode	1	1.02	.32
Grade by Sex by Mode	2	.51	.61
			(MS error [c]: 292.35)

Table continued

Table 4.8 (Cont'd)

Source of Variation	D.F.	F	P<
Mode by Time	1	.16	.69
Grade by Mode by Time	2		
Gr2 (6,7vs.8) by Mode by Time	(1)	3.15	.08
Gr1 (6vs.7) by Mode by Time	(1)	3.91	.05
Sex by Mode by Time	1	1.19	.28
Grade by Sex by Mode by Time	2	.42	.66
			(MS error[d]: 319.91)
Within Subjects	508		

[a] for overall means (Verbal Pre + Verbal Post + Visual Pre + Visual Post)

[b] for combined gains (Verbal Pre - Verbal Post + Visual Pre - Visual Post)

[c] for comparison of modes (Verbal Pre + Verbal Post - Visual Pre - Visual Post)

[d] for comparison of gains (Verbal Pre - Verbal Post - Visual Pre + Visual Post)

2. There are no significant sex differences in gain scores of the verbal and visual presentation modes either when combined together or when compared one versus the other.

3. In both modes, seventh graders gained significantly more than the sixth graders.

4. In the examination of gain differences between the two modes, there are significant differences between sixth and seventh grade level. An analysis of the means Table E.3 (Appendix E) and the graphs in Figure 4.2 reveals that seventh graders gained slightly more (by about 2 percent) in the visual mode as compared to their gain in the verbal mode, whereas the sixth graders gained 5 percent (or .27 standard deviations) more in the verbal mode as compared to their gain in the visual mode.

5. By conducting a second test that compares seventh, with eighth graders, significant differences ($F = 8.42$, $p < .004$) were detected: seventh grades gained more than the eight graders in both presentation modes combined together (see also slope of the corresponding lines in Figure 4.2).

6. The second test did not detect significant gain differences between modes ($F = .43$, $p < .52$) for the seventh graders compared with the eighth graders: at both grade levels the gains in the visual mode were higher than in the verbal mode (see Table E.3). The combined level of significance of the two tests is still very low.

To summarize, the analysis of the gain scores (i.e, the effect of instruction) in the two presentation modes at a significance level of .05, detected no sex by grade interaction and no sex differences; seventh graders gained significantly more than sixth or eighth graders; sixth graders gained more in their verbal scores than in the visual scores, whereas seventh and eighth graders revealed the opposite pattern of gains (i.e. higher gains in visual scores than in the verbal ones).

Effect of Instruction in Similarity on Spatial Visualization

Question 3.1: What effect does instruction in similarity have on performance in spatial visualization tasks?

Hypothesis 3.1: There will be no gains in performance on the Spatial Visualization Subtest as a result of instruction in similarity --i.e., there will be no significant differences in SVST scores prior to, and after instruction, or if there are such gains, they will be unrelated to performance on the Similarity Test.

In order to test this hypothesis, pre, and post-instructional performance on the SIMT and on the SVST has been analyzed for Sample A (Table 4.9). As mentioned before, this sample consists of students that underwent instruction with the MGMP Similarity Unit, but not with the MGMP Spatial Visualization Unit. A Multivariate Analysis of Repeated Measures has been employed to investigate the gains in similarity and in spatial visualization in one test. The correlation coefficients among similarity and spatial visualization scores (Table E.4) range from .270 to .583 ($p < .001$), indicating a considerable relationship between performances in the two fields. Therefore, a Univariate Stepdown Analysis of Variance was also conducted to account for this relationship. In other words, gains in spatial visualization have been examined after gains in similarity (attributed directly to instruction) have been taken into account.

Table 4.9 3 x 2 MULTIVARIATE REPEATED MEASURE DESIGN FOR HYPOTHESIS 3.1

		Similarity Test		Spatial Visualization Subtest	
		Pre	Post	Pre	Post
Grade 6	Males	x_1	x_2	y_1	y_2
	Females	x_1	x_2	y_1	y_2
Grade 7	Males	x_1	x_2	y_1	y_2
	Females	x_1	x_2	y_1	y_2
Grade 8	Males	x_1	x_2	y_1	y_2
	Females	x_1	x_2	y_1	y_2

Table 4.10 presents a summary of results of the Multivariate, Univariate, and Univariate Stepdown Analyses of Variance that were conducted to test Hypothesis 3.1. Mean scores and standard deviations (Table E.5) and their graphic representation (Figure 4.3) will facilitate interpretations of the data.

The results of the analyses indicate the following trends, at a significance level of .05:

1. There is no grade by sex interaction and there are no sex differences in gain scores neither in similarity nor in spatial visualization.

2. Generally, seventh graders gained significantly more than sixth graders. From the univariate ANOVA and Figure 4.3, it is clear that the difference in gains is due to similarity scores (a fact already revealed in the testing of the four similarity-related topics), but not to performance in spatial visualization.

3. A second test employing the "7 vs. 8" contrast in grade level revealed a similar pattern: the test detected an overall significant difference (Multivariate $F = 5.76$, $p < .003$) between seventh, and eighth graders, with the seventh graders gaining significantly more in similarity (Univariate $F = 11.15$, $p < .001$), but not in spatial visualization (Univariate $F = 1.22$, $p < .28$).

4. The Stepdown Analysis of Variance for gains in spatial visualization indicates that after accounting for gains in similarity, there are

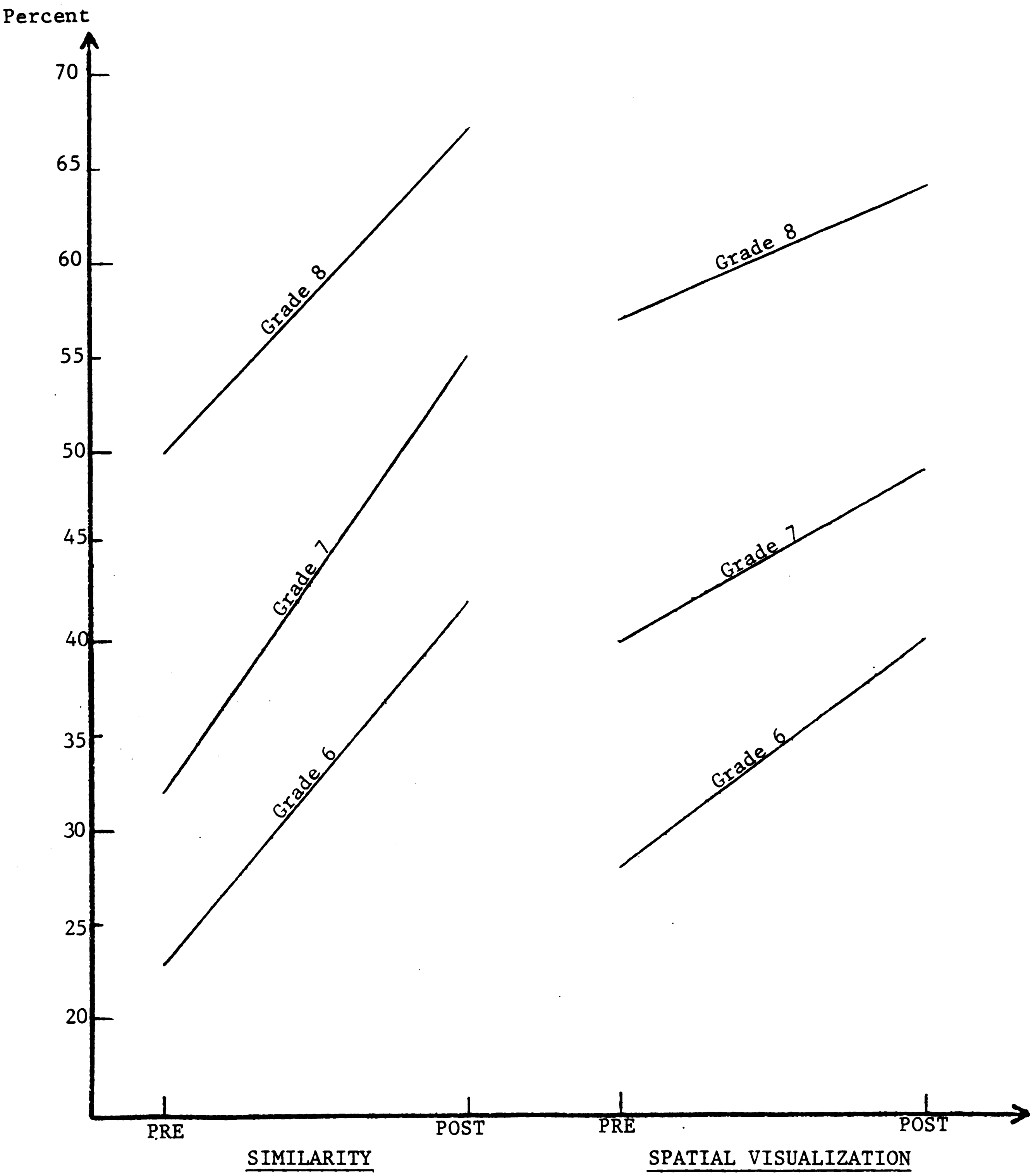


Figure 4.3 Performance by grade level on the Similarity Test and on the Spatial Visualization Subtest, prior to, and after instruction in similarity.

Table 4.10 SUMMARY OF ANALYSIS OF REPEATED MEASURES FOR HYPOTHESIS 3.1

Source Variation	D.F.	Multivariate[a] F	Multivariate[a] P<		Univariate F	Univariate P<	Stepdown F	Stepdown P<
Constant	1	2500.58	.0001	Sim. Avg.[b]	4694.83	.0001	4694.83	.0001
				SV Avg.[e]	2580.70	.0001	30.81	.0001
Grade	2							
Gr2 (6, 7 vs. 8)	(1)	106.05	.0001	Sim Avg.	201.13	.0001	201.13	.0001
				SV Avg.	105.13	.0001	8.14	.005
Gr1 (6 vs 7)	(1)	34.01	.0001	Sim Avg.	66.63	.0001	66.63	.0001
				SV Avg.	28.04	.0001	1.34	.25
Sex	1	2.72	.07	Sim Avg.	3.00	.09	3.00	.09
				SV Avg.	5.02	.03	2.44	.12
Grade by Sex	2	.50	.70	Sim Avg.	.45	.64	.45	.64
				SV Avg.	.99	.38	.65	.52
Between Groups	6							
Time	1	514.76	.0001	Sim Diff[d]	945.00	.0001	945.00	.0001
				SV Diff[e]	187.48	.0001	30.20	.0001
Grade by Time	2							
Gr2 (6, 7 vs. 8) by Time	(1)	3.99	.02	Sim Diff	5.18	.03	5.18	.03
				SV Diff	3.97	.05	2.79	.10
Gr1 (6 vs. 7) by Time	(1)	5.81	.003	Sim Diff.	8.15	.004	8.15	.004
				SV Diff.	2.05	.16	3.43	.07
Sex by Time	1	1.48	.23	Sim Diff.	1.29	.26	1.29	.26
				SV Diff.	2.10	.15	1.68	.20
Grade by Sex by Time	2	.33	.86	Sim Diff.	.37	.70	.37	.70
				SV Diff.	.22	.80	.29	.75

Table continued

Table 4.10 (Cont'd)

Within Subjects	508			
	Univariate MS Error:	Sim Avg.	365.22	
		SV Avg.	714.13	
		Sim Diff.	107.27	
		SV Diff.	133.54	

[a] Multivariate D.F. = 2 and 507 for main effects
4 and 1014 for interactions

[b] Similarity averaging over time (Pre + Post)

[c] Spatial Visualization averaging over time (Pre + Post)

[d] Similarity time effect over subjects (Pre - Post)

[e] Spatial Visualization time effect over subjects (Pre - Post)

no significant grade or sex differences in spatial visualization gains. The significant time effect shows the existence of gains in spatial visualization as expressed by the difference of scores prior and after instruction in similarity. However, not all of these gains could be explained by the gains in similarity.

To summarize, there were significant gains in SVST scores measured before, and after instruction in similarity (a standardized increase of .85 standard deviation units). After accounting for gains in similarity, no grade or sex differences in spatial visualization gains could be detected. Since the overall gains in spatial visualization stayed significant even after controlling for gains in similarity, it is possible that additional factors caused the growth in spatial visualization.

Effect of Instruction in Spatial Visualization on Performance in Similarity Tasks

Question 3.2: What effect does instruction in spatial visualization have on performance in similarity tasks?

Hypothesis 3.2: There will be no significant effect of instruction in spatial visualization on performance in SIMT; i.e., for grade levels seven and eight, there are no differences in mean scores and in gain scores on the SIMT between students who studied the MGMP Spatial Visualization Unit and those who did not.

The investigation of this question was limited by a restricted sample: Sample B consisted of a relatively small number of students (N = 161) at grade levels seven and eight in one school (see Table 3.3). Moreover, the instruction in spatial visualization occured one year before the instruction in similarity. Comparisons of results from the analysis of this question with previously obtained results should be, therefore, conducted in view of these restrictions.

Hypothesis 3.2 was tested by analyzing pre, and post-instructional performance on SIMT of seventh and eighth graders who did undergo instruction in spatial visualization and of students who did not. Performance in similarity was analyzed in two ways: (1) by employing a Univariate Analysis of Repeated Measures, and (2) by using the same analysis as in (1) with the performance in spatial visualization as a covariate. The second test was designed to detect the extent to which patterns in similarity scores can be attributed to spatial visualization.

Table 4.11 presents the design of the data that has been used to test Hypothesis 3.2, and Table 4.12 is a summary of results of the Analyses of Repeated Measures. Further details on the data are provided in Table E.6 (Appendix E) of means and standard deviations on Sample B's performance in similarity and in spatial visualization.

The following conclusions can be drawn from the results of the analyses, at a significance level of .05:

1. If the level of performance in similarity (as expressed by the average

Table 4.11 2 x 2 x 2 UNIVARIATE REPEATED MEASURE DESIGN FOR HYPOTHESIS 3.2

		Similarity Test		Spatial Visualization Test (covariate)	
		Pre	Post	Pre	Post
SV Unit[a]					
Grade 7	Males	x_1	x_2	y_1	y_2
	Females	x_1	x_2	y_1	y_2
Grade 8	Males	x_1	x_2	y_1	y_2
	Females	x_1	x_2	y_1	y_2
No SV Unit[b]					
Grade 7	Males	x_1	x_2	y_1	y_2
	Females	x_1	x_2	y_1	y_2
Grade 8	Males	x_1	x_2	y_1	y_2
	Females	x_1	x_2	y_1	y_2

[a] Studied the MGMP Spatial Visualization Unit.

[b] Did not study the MGMP Spatial Visualization Unit.

Table 4.12 SUMMARY OF ANALYSES OF REPEATED MEASURES FOR HYPOTHESIS 3.2

Source of Variation	D.F.	MANOVA[a] F	MANOVA[a] P<	MANCOVA[b] F	MANCOVA[b] P<
Constant	1	1407.88	.0001	42.24	.0001
SV Unit[c]	1	.75	.39	1.52	.28
Grade	1	.06	.82	.01	.96
Sex	1	4.63	.04	.01	.94
Grade by Sex	1	.69	.41	.27	.61
Grade by SV Unit	1	.02	.90	2.25	.14
Sex by SV Unit	1	.11	.75	.15	.71
Grade by Sex by SV Unit	1	1.53	.22	.47	.50
Between Groups	8	(MS error[d]:	362.54		268.41)
Time	1	150.31	.0001	119.11	.0001
SV Unit by Time	1	.06	.81	.04	.85
Grade by Time	1	3.18	.08	4.13	.04
Sex by Time	1	.06	.81	.01	.95
Grade by Sex by Time	1	.16	.70	.03	.88
Grade by SV Unit by Time	1	.03	.87	.02	.89
Sex by SV Unit by Time	1	2.48	.12	2.79	.10
Grade by Sex by SV Unit by Time	1	.57	.46	.37	.55
Within Subjects	153	(MS error[e]:	98.19		86.02)

[a]Analysis of Repeated Measures for SIMT scores with D.F.=1 and 153.

[b]Analysis of Repeated Measure for SIMT scores with SVST scores as covariate; D.F.=1 and 152.

[c]Effect of instruction with MGMP Spatial Visualization Unit.

[d]Mean Square error for Similarity averaging over time (Pre + Post).

[e]Mean Square error for Similarity time effect over subjects (Pre - Post).

of pre, and post-instructional scores) is examined, there is no significant interaction, influence of grade level or effect of instruction pertaining to spatial visualization. The only marginally significant effect ($p < .04$) is the effect of sex. However, when sex differences in spatial visualization are taken into account, sex differences in level of performance in similarity become nonsignificant.

2. If gain scores in similarity are examined, there is no significant interaction, and no significant influence of instruction pertaining to spatial visualization, of grade level, or of sex. However, when differences in gains in spatial visualization are taken into account, the gains of the seventh graders in similarity appear significantly higher than those of the eighth graders--a fact that was detected also in Sample A.

To summarize, the most relevant finding in this section is that instruction in spatial visualization, one year before instruction in similarity did not have any significant effect on the level of performance or gains in similarity. The following results seem to contradict previous findings for sample A: (1) nonsignificant grade differences in level of performance in similarity, (2) nonsignificant grade differences in gain scores in similarity, and (3) significant sex differences (i.e., male superiority) in level of performance in similarity. The last two of these contradictions can be settled by removing the variation that can be explained by spatial visualization: in that case, the results show that, as for sample A, the seventh graders gained significantly more than the eighth graders, and the sex differences in similarity become nonsignificant. The fact that eighth graders and seventh graders achieved about the same on the SIMT may be attributed to the use of a sample restricted to relatively few students from one school, arranged in a post-hoc blocking according to their experience with instruction in spatial visualization.

CHAPTER V

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

Significance of the Study

The concept of similarity is essential in the development of children's geometrical understanding of their environment. Moreover, studying similarity provides children with concrete images of proportional reasoning, and has, therefore, important pedagogical implications.

One of the four instructional units designed by the Middle Grades Mathematics Project (MGMP) deals with the concept of similarity. The purpose of the MGMP Similarity Unit is to provide a response to the present deficiencies in the instruction of geometry. By offering a rich variety of experiences designed to fit the first two levels in the Van Hiele model for development of geometrical understanding, the unit attempts to build a solid base for further advance in the understanding of geometry in general, and of similarity and proportionality in particular (Lappan, 1983).

The acquisition of similarity concepts in a school setting has been rarely investigated. One of the goals of this study was to determine patterns in the performance of middle-grades students in geometrical similarity tasks both before, and after instruction on similarity via the MGMP unit.

A second goal of this study was to determine the effect, if any, of instruction with the MGMP Similarity Unit on the spatial visualization of middle-grades students, and conversely, to determine whether instruction with the MGMP Spatial Visualization Unit effects the performance of middle-grades students in similarity tasks.

Design of the Study

The effect of 10-15 days of instruction with the MGMP Similarity Unit was determined by comparing pre, and post-instructional performance on a paper-and-pencil 25-item Similarity Test (Appendix A). Pre-instructional performance on the Similarity Test was analyzed to investigate possible grade level and sex differences in the performance of middle-grade students in similarity tasks.

The study followed three directions:

1. Investigation of student performance on four similarity-related areas: basic properties of similar shapes, proportional reasoning as expressed in similarity tasks, the area relationship of similar shapes, and applications of similarity as expressed in scaled drawings and indirect measurement.

2. Comparison of performance in similarity tasks presented in a visual presentation mode with performance on equivalent tasks presented verbally (i.e., without any illustrative aids).

3. Investigation of the relationship between performance on similarity tasks and performance in spatial visualization. This was accomplished by analyzing student performance on a 15-item subtest of the MGMP

Spatial Visualization Test, in addition to the achievement on the MGMP Similarity Test.

The sample included middle-grades students from five schools with a suburban, middle-class, midwestern, predominantly white population and whose teachers volunteered to the MGMP Similarity Unit. The sample was subdivided in two groups: Sample A consisted of students from four schools (N=514) that had no instruction in spatial visualization, and Sample B (N=161) consisted of students from a fifth school containing both students who had instruction in spatial visualization the year before this study, and students who did not have this kind of instruction.

The data obtained from Sample A was analyzed to determine trends in performance of middle-grades students in similarity tasks and to detect any influence of instruction in similarity on spatial visualization. The data from Sample B was used to analyze the influence of instruction in spatial visualization on performance in similarity tasks.

Before instruction, the teachers participated in a two-hour workshop on the mathematical content and instructional strategies of the unit, that was conducted by the MGMP staff, then taught and tested the students themselves.

Analysis of Variance and Analysis of Repeated Measures served as the main statistical tools to test the hypotheses that were derived from the research questions. Group means and correlation coefficients were additionally examined in order to provide further interpretation of the data. The analyses were performed by using SPSS programs on a

CYBER 170, Model 750 computer at Michigan State University.

Main Findings and Conclusions

Performance and Gains in Similarity Tasks

Two research questions were asked in this study on the performance of sixth, seventh, and eighth graders in similarity tasks. The first (Research Question 1.1) was related to grade level and sex differences in the pre-instructional performance in tasks that require the use of: (1) basic properties of similar shapes, (2) proportional reasoning related to similarity, (3) the principle of area growth, and (4) applications of the similarity concept in scaled drawings and in indirect measurement. The second question (Research Question 1.2) referred to the effect of instruction (i.e., the existence of gains) on achievement in these four areas.

The data obtained from Sample A on pre-instructional achievement was analyzed by Multivariate Analysis of Variance. At a significance level of .05, the test did not show significant sex differences in performance on any of the four topics, but indicated significant grade differences: on three topics the level of performance increased significantly with the grade level. The performance on the topic of area growth indicated a somewhat different pattern: it was only loosely correlated (correlation coefficients of less than .25) with the other three topics, and it was almost uniformly low (average rate of success of 18-22 percent) for all grade levels.

The general pattern of increasing achievement as a function of grade level (i.e., of age) indicates that cognitive development plays an important role in the acquistion of the concept of geometrical similarity. This result confirms the findings of other cognitive studies: Although researchers may differ on their explanations of factors that influence the development of the similarity concept or of proportional reasoning, they generally agree that starting from about the age of 11 there is a gradual improvement in the performance of children in similarity tasks. At earlier ages, perceptual judgments are possible (Piaget, 1967), but a more general and conceptual use of proportions could not be detected.

The sequence of topics arranged in an increasing order of difficulty as indicated by pre-instructional achievement is: (1) basic properties of similar shapes, (2) applications, (3) proportional reasoning, and (4) area growth, with overall sample averages of 37.2, 35.0, 33.7, and 19.8 percent respectively. More detailed data is presented in Table E.1 (Appendix E) and Figure 4.1. The sequence confirms the results obtained from the second mathematics assessment of NAEP (Carpenter et al., 1981) that indicate that children exhibit a better understanding of the basic geometrical concepts than of their applications. The lower performance in proportional reasoning supports the findings in cognitive research (e.g., in the Karplus studies) that a systematic use of this ability is possible about two years later than the first conceptual judgments in similarity tasks.

The very low achievement on tasks that involve the area relationship of similar shapes also corresponds to findings of other studies. This phenomenon can be explained in different ways: (1) at this age, only about half of the children know how to compute the area of a rectangle (Schnur & Callahan, 1973; Carpenter et al., 1981); (2) children often confound area with perimeter (e.g., Piaget et al., 1960) confusing accordingly area growth with linear growth; (3) the separation of the growth of sides and growth of area in similar shapes requires a formal reasoning ability that is rarely present at this transitional age (Montangero, 1976; Lunzer, 1968).

This study indicates a lack of significant sex differences in any of the four similarity-related topics. Other studies showed the same results for similarity (Cameron, 1925), for informal geometry (Olson, 1970; G.D. Peterson, 1973; Thomas, 1977), and for formal Euclidean geometry (Werdelin, 1961; Senk & Usiskin, 1982). On the other hand, some studies (Shonberger, 1976; Ben-Haim, 1983) detected a male superiority in informal geometry tasks and in spatial visualization. These sex differences were explained by the boys' richer extra-curricular experience related to visualization.

Other studies also indicate the existence of a clear male superiority in proportional reasoning (Chapman, 1975; Brendzel, 1978; Keating & Schaefer, 1975; Stage et al., 1980). However, the proportionality tasks employed in these studies were not related to geometrical similarity. The lack of sex differences in similarity-related proportional reasoning indicated by this study contradicts the findings of other proportionality studies. In view of more recent studies on sex differences in mathematics (e.g., Fennema & Sherman, 1978), it may be argued that the experience required in order to perform in geometrical proportionality tasks is common to boys and girls, and probably unrelated to extra-curricular activities.

The data on pre, and post-instructional achievement on the four similarity-related topics allowed for the analysis of gain scores for Sample A. The gains could be attributed to the instruction on similarity, and the possibility of any other confounding influences was considered low: due to the short period of instruction and to the fact that children probably did not encounter relevant extra-curricular experiences, the possibility that confounding events occured between the two tests may be considered remote. An effect of cognitive maturation may be also discarded for similar reasons. In order to avoid a test-retest effect, teachers were instructed not to discuss the tests after pretesting, and student results were available to both teachers and students only after posttesting.

At a significance level of .05, the Analysis of Repeated Measures again did not show significant sex differences, but indicated significant grade differences in gains. The seventh graders seem to have profited the most from instruction: their gains were similar to the eighth graders' and higher than the gains of the sixth graders on the topics of area growth and applications; on the other hand, they gained as much as the sixth graders and more than the eighth graders on the topics of basic properties of similar shapes and proportional reasoning (Figure 5.1).

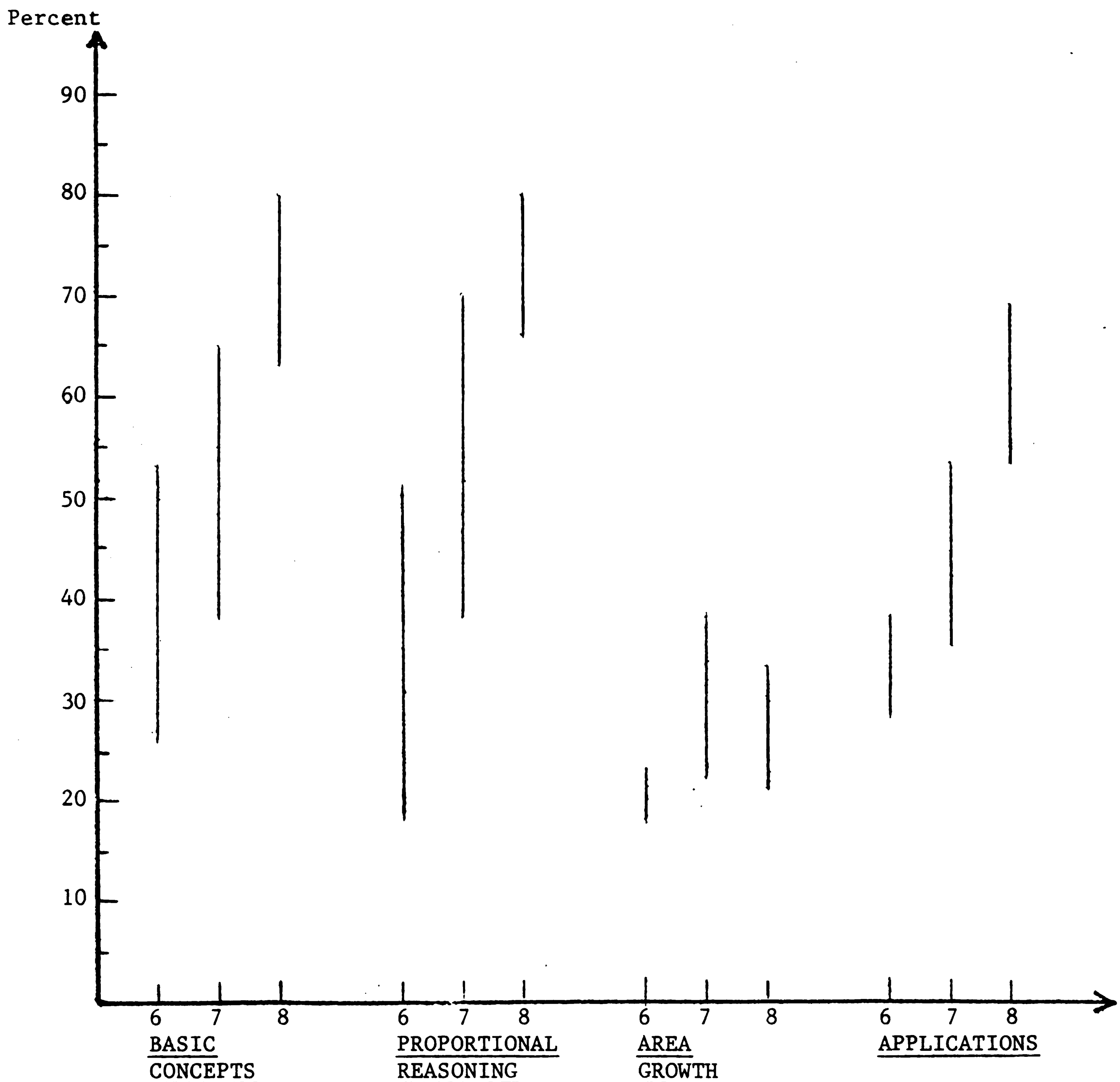


Figure 5.1 Gains in four similarity-related topics--Sample A.

Particularly high gains (27-32 percent) were detected in the sixth, and seventh graders' scores on the topics of basic properties of similar shapes and of similarity-related proportional reasoning. The relatively smaller gains (15-17 percent) of the eighth graders in the same topics may be attributed to a ceiling effect -- their average posttest score reached a mastery level of 80 percent. Although there are no other studies on similarity to which these results can be compared, these results are comparable to a study conducted by Kurz and Karplus (1977): after a three-week-long instruction in proportional reasoning the scores of the participating high-school prealgebra students on a proportional reasoning test rose to 62-70 percent. The same results were obtained by the subjects of this study at the sixth and the seventh grade level in proportionality tasks related to geometrical similarity.

Lower gains (5-20 percent) were found for the topics of applications and of area growth for all grade levels. These results may be attributed to the cognitive difficulties involving the transfer of the similarity concept to practical situations as required in the application tasks, and the differentiation between linear growth and area growth of shape as required in the area growth tasks. The particularly low level of post-instructional performance in area growth tasks for the sixth graders reported in this study (23 percent) and also in Fitzgerald and Shroyer (1979) may suggest that the cognitive difficulties related to area growth tasks cannot be overcome by instruction at this age. The fact that the MGMP Similarity Unit's treatment of area growth is less intensive

than the other topics cannot be the only reason for the low level of performance in this topic: the instruction in the study conducted by Fitzgerald and Shroyer concentrated mainly on the growth and shrinking of figures and objects, but did not produce better results.

The detection of generally significant gains for all similarity-related topics may be considered an indicator of the effectiveness of an instruction that employs the MGMP Similarity Unit and its corresponding instructional strategies. Although this study does not compare alternative instructional interventions, it may be concluded that teaching the concept of geometrical similarity through informal, exploratory activities that employ concrete materials at the middle grades level induced significant improvement in student performance in similarity tasks. This conclusion adds another empirical basis to recommendations made by math educators who stress the importance of teaching informal geometry (e.g., J. C. Peterson, 1973; Wirszup, 1976), using concrete materials (Fennema, 1972; Wollman & Lawson, 1978) through a problem-solving approach (Scott et al., 1971; Hempel, 1981).

Visual versus Verbal Presentation Mode

In order to compare student performance on SIMT items accompanied by drawings with achievement on items presented in a purely verbal form, four pairs of equivalent questions were selected to form a visual, and a corresponding verbal subtest. The obviously visual character of the instructional unit and pedagogical common sense seem to favor the visual presentation mode. In other words, it may be expected that students would have a higher performance and would gain more on geometry-related questions that use also a graphical representation of the concepts involved.

In this study, two research questions were used to investigate pre-instructional performance and gains on the verbal and the visual subtest: The first (Research Question 2.1) referred to grade level and sex differences in pre-instructional achievement on the items presented in the two modes, whereas the second question (Research Question 2.2) was related to the effect of instruction on achievement in the verbal and the visual subtests.

The data obtained from Sample A on pre-instructional achievement was analyzed by Multivariate Analysis of Variance. At a significance level of .05, no effects due to presentation mode were detected: all grade levels and both sexes performed at about the same level on verbal and on visual items. Furthermore, an examination of the mean scores (Table E.3, Appendix E) reveals a one-to-two percent higher performance in items presented in a verbal mode.

An item-analysis reveals a different picture. Two of the four items in each subtest dealt with area growth--a topic that, as indicated in the preceeding section, involves many cognitive difficulties. The achievement on these items is low (3-36 percent) and does not favor a particular presentation mode. On the item related to indirect measurement (application of similarity) the performance in the visual item was by 12-16 percent higher than in its verbal counterpart. The fourth item required the recognition of similar rectangles by drawings in the visual mode, and by the numerical ratio of their sides in the verbal form. In this case, the performance in the verbal mode was superior to the visual presentation by 15-47 percent.

Thus, although no overall difference for a particular presentation mode could be indicated, for different kinds of similarity-related tasks different presentation modes seem to be favored.

The fact that visual representations should not be always favored is also indicated by Karplus et al. (1974): in this study, performance on the Mr. Tall/Mr. Short proportionality task improved considerably, when their subjects were deprived of visual cues (i.e., the drawing of Mr. Tall) being thus forced to use the numerical data.

The gain scores obtained from the pre, and post-instructional scores of Sample A were analyzed by conducting an Analysis of Repeated Measures. At a significance level of .05, no significant sex differences in gain scores of the verbal and visual presentation were detected. However, the test indicated a significant difference in gain pattern as a function of grade level: seventh, and eighth graders gained more (by 2 and 5 percent respectively) in the visual score, whereas the gains of the sixth graders in the verbal items were 5 percent higher than in the visual subtest.

The results indicate that boys and girls have the same level of performance and the same gain patterns with regard to the two presentation modes. This finding agrees with the lack of sex differences in geometry achievement indicated by some studies. However other studies (e.g., Werdelin, 1961; Smith & Litman, 1979; Senk & Usiskin, 1982) indicate

a male superiority on geometry tasks that require visualization, but equal performance or female superiority on more formal or verbal geometrical tasks. Therefore, a sex by presentation mode interaction would be expected. The fact that this phenomenon did not occur in this study may be a result of unfamiliarity with similarity tasks for subjects of both sexes.

The MGMP Similarity Unit emphasizes the visual and the informal aspect of the concept of similarity. The fact that sixth graders gained less in their visual tasks than in the verbal ones may be another indicator of a lower effect of instruction for this group. It is also possible that during the period of instruction the sixth graders achieved a higher level in the mastery of the "theoretical" aspects of the similarity tasks (e.g., numerical ratios, the verbal rule related to area growth) but were less able to apply it when the data had to be deduced from drawings.

Relationship with Spatial Visualization

Studies that examined the possibility of a relationship between geometrical ability and spatial visualization did not show uniform results: some studies (Holzinger & Swineford, 1946; Blade & Watson, 1955; Werdelin, 1961) indicate the existence of such a relationship, others (Weiss, 1954; French, 1964; Bennet et al., 1966) claim that geometrical ability is independent of space factors, while additional studies (e.g., Hanson, 1972) show mixed results. However, there are strong cognitive and pedagogical reasons (Piaget, 1964; Hoffer, 1977) to assume a close relationship between informal geometry and spatial ability.

Two research questions were designed to refer to the possibility of a relationship between achievement in similarity tasks and spatial visualization. The first (Research Question 3.1) was related to the effect of instruction in similarity on spatial visualization, whereas the second (Research Question 3.2) dealt with the effect of instruction in spatial visualization on performance in similarity tasks.

In order to answer the first question, Sample A scores on the Similarity Test and on the Spatial Visualization Subtest (Appendices A and B) prior to, and after instruction in similarity were analyzed by an Analysis of Repeated Measures. At a significance level of .05, no grade or sex differences in spatial visualization gains could be detected. However, there were significant gains in achievement in spatial visualization (from 37 to 47 percent for the whole sample). Ben-Haim (1983) indicates that the MGMP Spatial Visualization Test shows no test-retest effect. This study also indicates a strong correlation between the performance on the similarity, and on the spatial visualization test ($r=.56$ prior to similarity instruction, and $r=.58$ after instruction). Therefore, it may be assumed that the gains in spatial visualization detected in this study are related to the instruction on similarity.

When similarity achievement was used as a covariate, the results of the Analysis of Variance for achievement in the MGMP Spatial Visualization Subtest remained basically unchanged. The fact that the overall gains in spatial visualization stayed significant even after controlling for gains in similarity, may indicate the existence of additional factors that caused the growth in spatial visualization.

Thus, it may be concluded that instruction in similarity had a positive (although possibly not exclusive) effect on spatial visualization. This result is consistent with the findings of other studies (Ekman, 1968; Brinkmann, 1966; Wolfe, 1970; Battista et al., 1982) that showed significant effects on spatial visualization due to training in informal geometry.

The data obtained from Sample B was examined to detect any effect of instruction with the MGMP Spatial Visualization Unit (conducted one year before this study) on achievement in similarity.

The Analysis of Repeated Measures for similarity test scores indicates that at a significance level of .05, there are no sex or grade level (seven versus eight) differences in similarity gains. Moreover, there was no significant instruction effect: children that underwent instruction in Spatial Visualization performed at the same level and gained in similarity scores about the same as children that did not study that instructional unit. When achievement on the Spatial Visualization Subtest was controlled, by using spatial visualization scores as covariates, the same pattern of similarity gains as in Sample A could be shown: no sex differences, but significantly superior gains of seventh graders were revealed. The restricted number of subjects, the use of subjects from a single school, and the post-hoc blocking of students according to level of instruction in spatial visualization posed serious limitations on the data obtained from Sample B.

Pedagogical Implication

The results of this study suggest that an informal and activity-oriented approach in teaching the concept of geometrical similarity, at the middle-grades level may be effective. Although age appears to be a considerable factor in children's understanding of similarity, three weeks of instruction with the MGMP Similarity Unit helped to improve the average performance on the Similarity Test by 19, 23, and 17 percent for sixth, seventh and eighth graders respectively.

Considerable gains were obtained in the two topics that were emphasized the most in the unit: basic properties of similar shapes (17-27 percent) and proportional reasoning (15-32 percent). In order to achieve better results in tasks that involve applications of the concept of similarity or the principle of area growth, these topics will have to be accorded an equal amount of attention. However, the teaching of area growth at the sixth grade level seems to be particularly unrewarding. Three possible solutions to this difficulty may be suggested: (1) to postpone the instruction on this topic to an age when children have the required cognitive skills, (2) to teach first the principle of area growth in an intensive way (e.g., by the use of the instructional unit developed by Shroyer and Fitzgerald, 1979) and to use the MGMP Similarity Unit as a follow-up on the topic, or (3) to consider the teaching of the Similarity Unit as a first step in teaching area growth that will have to be followed at a later stage by additional instruction. Since seventh graders were shown to profit most from instruction, this grade level seems to be particularly favorable for the instruction of the Similarity Unit. It is possible that sixth graders encounter more cognitive difficulties than their older peers, whereas the concrete-activity orientation of the unit appeals less to the eighth graders than to younger students.

The lack of sex differences in performance on similarity tasks found by this study should be a warning against generalizing findings on male superiority in some visualization tasks to all activities of this kind. As pointed out by others, the nature of the extra-curricular experiences required to complete a task seems to be the key to sex differences in mathematics: tasks that are at the same familiarity level for both sexes are likely to induce an equal level of performance for both sexes.

This study questions the assumption that performance in informal geometry tasks presented in a visual mode is always superior to verbal tasks. A careful task analysis seems to be required in each case. Some similarity, and probably other geometrical tasks seem to attract a higher number of wrong answers when accompanied by drawings. This finding should not imply that misleading representations should be avoided. It is recommended that (1) different approaches should be used for teaching the same topic, and (2) the misleading representation should be used as the final test for understanding a concept.

The fact that instruction in similarity was accompanied by an improvement in spatial visualization seems to indicate that if the MGMP Similarity Unit will be followed by instruction with the MGMP Spatial Visualization Unit a better final achievement in spatial visualization may be expected. The first unit will raise indirectly the achievement in spatial visualization, and thus will allow students to start the second unit at a higher level.

In this study, instruction in spatial visualization did not have a similar effect on achievement in similarity. This result seems to indicate that placing the two instructional units in a reversed sequence would not have the same favorable effect for understanding the concept of similarity. However, since the instruction in spatial visualization took place a year before this study, and the sample used in this case was restricted, further inquires are needed to verify this last assertion.

Recommendations for Further Research

It is recommended that attempts should be made to overcome some of the restrictions due to the amount of available time, conditions of sampling and structure of the evaluation tools. Several studies that further pursue the directions established in this study are possible:

1. This study may be replicated with samples differing in terms of student SES, site and ethnic background to verify the generalizability of the results.

2. A replication of this study that would also include a delayed testing would allow for an investigation of retention of the initial gains.

3. A systematic comparison of student responses in similarity tasks presented in a verbal and a visual mode may be conducted to confirm the primary findings of this study. In this case, there is a need

for a test that would include several pairs of equivalent verbal, and visual items in each of the four similarity-related topics identified in this study.

4. In order to further investigate the relationship between the concept of similarity and spatial visualization and its pedagogical implications, a sample of students should be divided into three groups: one group will study the MGMP Similarity Unit first and the MGMP Spatial Visualization Unit next, another group will study the two instructional units in reversed order, and the third group will serve as control. The SIMT and the SVST will be administered at the beginning and at the end of each instructional unit in order to draw conclusions about an optimal sequence.

5. The concept of area growth may be investigated in a separate study. Results after instruction with the MGMP Similarity Unit, The Mouse and the Elephant (Fitzgerald & Shroyer, 1979), and possibly after other alternative approaches may be compared. It would be also interesting to follow the development of the concept of area growth throughout the high school, and even at the college level.

6. An in-depth study on the cognitive development of the concept of similarity may be conducted: by using the SIMT as a primary selection tool, a limited number of average students will be interviewed extensively both before, and after instruction with the Similarity Unit. Error-patterns and effect of instruction on students' reasoning

may be revealed and considered in a revision of the present instruction.

7. The relationship between the concept of similarity and proportional reasoning would be another topic for investigations: A test in proportional reasoning will allow the researcher to compare student performance on the two related topics and detect any transfer from proportional reasoning in similarity tasks to more general proportionality tasks.

The first four recommendations may extend the primary findings of this study one step further, whereas the other three may open new directions in the investigation of the development of children's geometrical understanding in general, and of the concept of similarity in particular.

APPENDICES

APPENDIX A

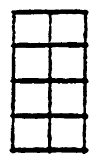
MGMP SIMILARITY TEST:

ITEMS INCLUDED IN THE FOUR SUBTESTS

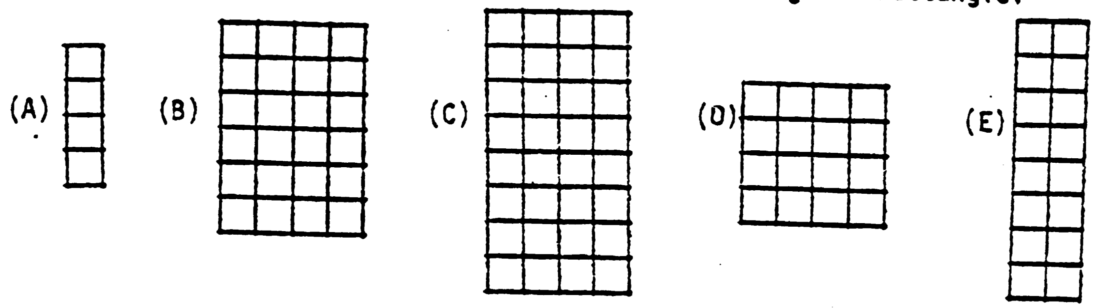
ITEMS INCLUDED IN THE TWO PRESENTATION MODES

BASIC PROPERTIES OF SIMILAR SHAPES

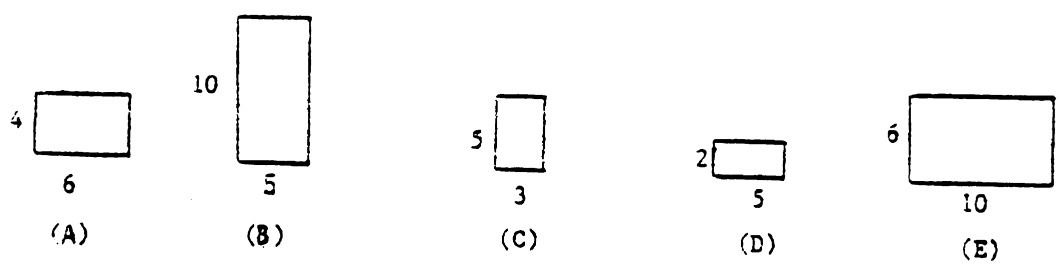
1. Given the rectangle



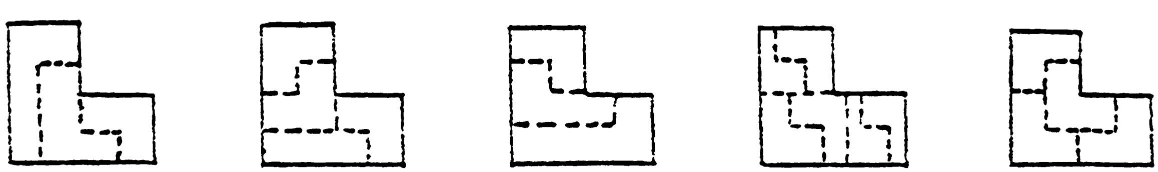
Which of the following rectangles is similar to the given rectangle?



13. Which of the following rectangles is similar to a 10 x 15 rectangle?



14. Which of these figures is divided into small shapes each similar to the big shape?

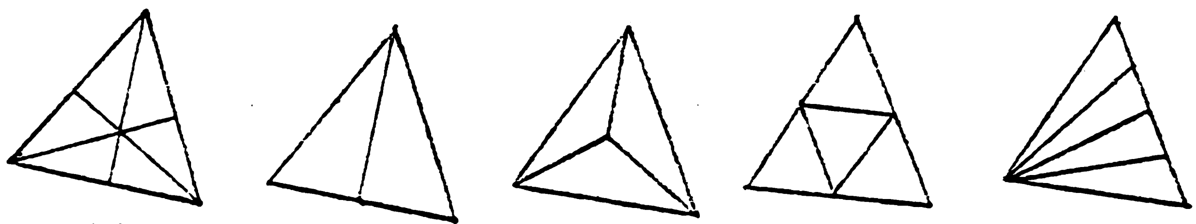


(A) (B) (C) (D) (E)

15. The ratio of the sides of a rectangle is $\frac{2}{3}$.
Which of the following could be the ratio of sides of a similar rectangle?

(A) $\frac{4}{9}$ (B) $\frac{4}{3}$ (C) $\frac{2}{6}$ (D) $\frac{4}{5}$ (E) $\frac{6}{9}$

17. Which of these figures shows the big triangle divided into smaller triangles each similar to the big triangle?



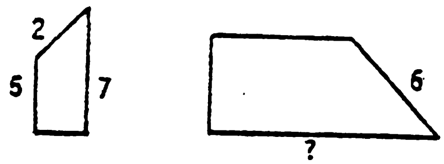
(A) (B) (C) (D) (E)

19. If two figures are similar, which of the following might be <u>different</u>?

(A) number of sides (B) lengths of corresponding sides (C) shape (D) size of angles (E) ratio of corresponding sides

PROPORTIONAL REASONING

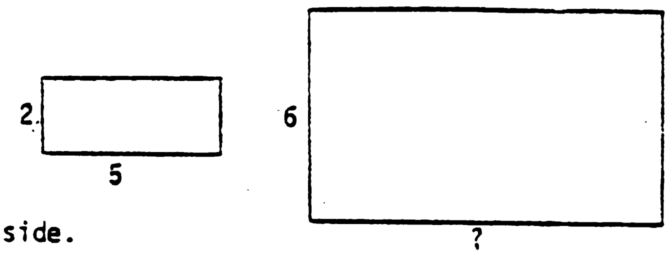
3. The given figures are similar.



Find the missing length.

(A) 11 (B) 14 (C) 15 (D) 18 (E) 21

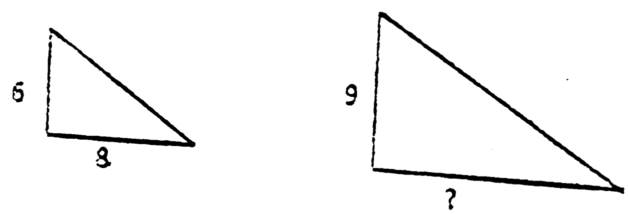
4. These rectangles are similar.



Find the length of the missing side.

(A) 7 (B) 9 (C) 10 (D) 11 (E) 15

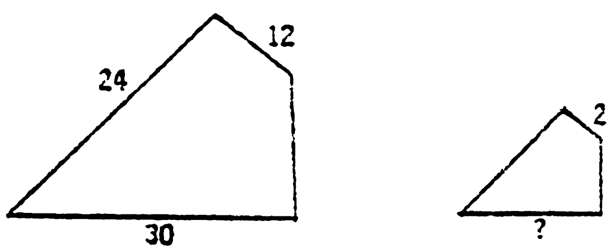
3. These triangles are similar:



Find the missing length.

(A) 10 (B) 11 (C) 12 (D) 13 (E) 14

18. The given figures are similar.



Find the missing length.

(A) 4 (B) 5 (C) 6 (D) 12 (E) 15

AREA GROWTH

2. A 1x5 rectangle grows into a 4x20 rectangle.
The area of the new rectangle is how many times larger than the area of the small rectangle?

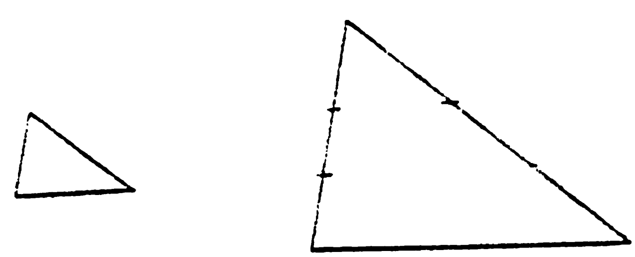
(A) 3 times (B) 4 times (C) 5 times (D) 15 times (E) 16 times

6. If the lengths of the sides of a triangle are each multiplied by 3, then the area of the new triangle is?

(A) 3 times larger (B) 6 times larger (C) 9 times larger

(D) 12 times larger (E) 15 times larger

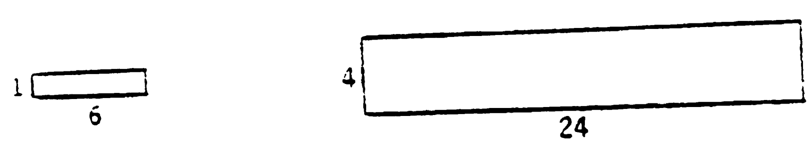
9. The two triangles below are similar and the lengths of the sides of the larger are 3 times that of the smaller.



How many of the smaller triangles will exactly fit into the larger one?

(A) 4 (B) 6 (C) 7 (D) 8 (E) 9

11. Given rectangles of dimensions 1x6 and 4x24.

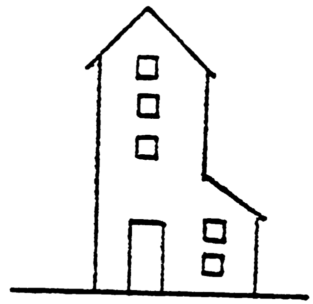


The area of the larger rectangle is how many times as big as the area of the smaller rectangle?

(A) 4 times (B) 6 times (C) 8 times (D) 16 times (E) 18 times

APPLICATIONS

5. A picture of a tall building is taken. The following measurements are known.



	Picture	Actual
Height of door	10mm	2m
Height of building	40mm	?

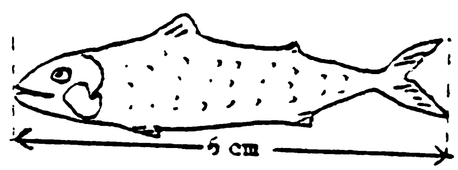
What is the actual height of the building?

(A) 8m (B) 6m (C) 10m (D) 60m (E) 80m

10. A man who is 6 feet tall has a shadow which is 8 feet long. At the same time a nearby tree has a shadow which is 32 feet long. How tall is the tree?

(A) 30 feet (B) 21 feet (C) 24 feet (D) 42 feet (E) 48 feet

12. Given a picture of a trout.

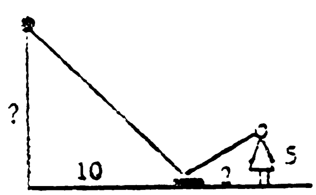


The scale of the picture to the real trout is 1 to 12.

What is the actual length of the real trout?

(A) 1/2 cm (B) 72 cm (C) 2 cm (D) 144 cm (E) 18 cm

21. Joan estimates the height of a flagpole by using a mirror.



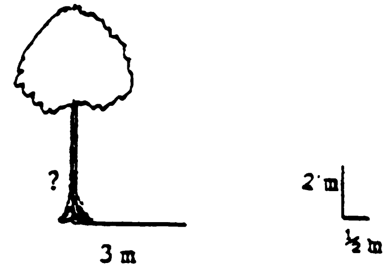
<u>Distances</u>

To eye level	5 ft.
Joan to mirror	2 ft.
Mirror to pole	10 ft.

How tall is the pole?

(A) 10 ft. (B) 13 ft. (C) 15 ft. (D) 25 ft. (E) 100 ft.

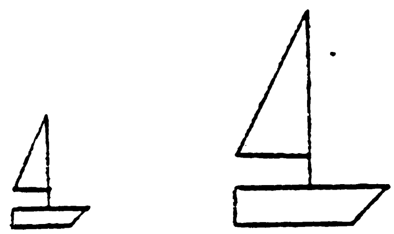
23. A 2 meter stick has a shadow of ½ m at the same time that a nearby tree has a shadow of 3m.



How tall is the tree?

(A) 6m (B) 12 m (C) 1½ m (D) 3 m (E) 15 m

25. What scale factor has been used to enlarge the small sailboat?



(A) 2 (B) 3 (C) 4 (D) 6 (E) 1/4

VERBAL PRESENTATION MODE

2. A 1x5 rectangle grows into a 4x20 rectangle.
The area of the new rectangle is how many times larger than the area of the small rectangle?

(A) 3 times (B) 4 times (C) 5 times (D) 15 times (E) 16 times

6. If the lengths of the sides of a triangle are each multiplied by 3, then the area of the new triangle is?

(A) 3 times larger (B) 6 times larger (C) 9 times larger

(D) 12 times larger (E) 15 times larger

10. A man who is 6 feet tall has a shadow which is 8 feet long. At the same time a nearby tree has a shadow which is 32 feet long. How tall is the tree?

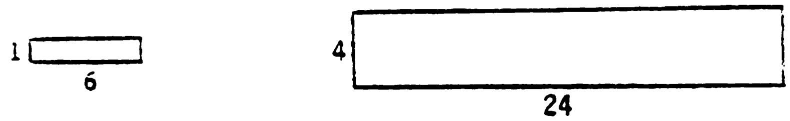
(A) 30 feet (B) 21 feet (C) 24 feet (D) 42 feet (E) 48 feet

15. The ratio of the sides of a rectangle is $\frac{2}{3}$.
Which of the following could be the ratio of sides of a similar rectangle?

(A) $\frac{4}{9}$ (B) $\frac{4}{3}$ (C) $\frac{2}{6}$ (D) $\frac{4}{5}$ (E) $\frac{6}{9}$

VISUAL PRESENTATION MODE

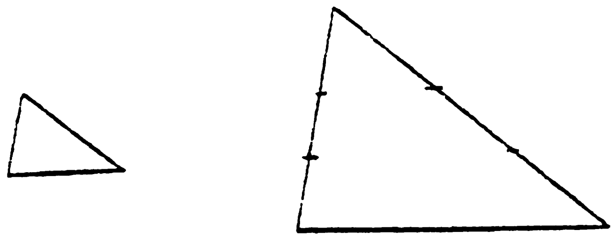
11. Given rectangles of dimensions 1x6 and 4x24.



The area of the larger rectangle is how many times as big as the area of the smaller rectangle?

(A) 4 times (B) 6 times (C) 8 times (D) 16 times (E) 18 times

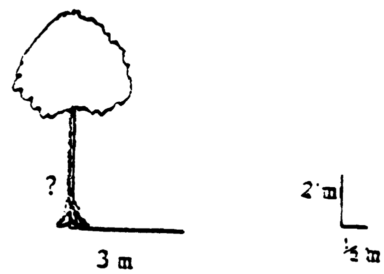
9. The two triangles below are similar and the lengths of the sides of the larger are 3 times that of the smaller.



How many of the smaller triangles will exactly fit into the larger one?

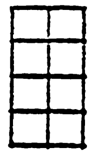
(A) 4 (B) 6 (C) 7 (D) 3 (E) 9

23. A 2 meter stick has a shadow of ½ m at the same time that a nearby tree has a shadow of 3m.

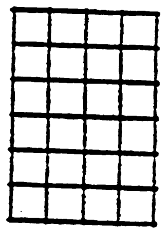
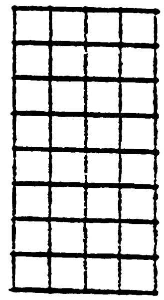
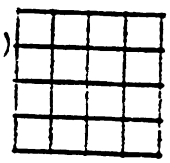
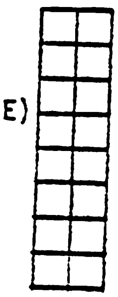


How tall is the tree?

(A) 6m (B) 12 m (C) 1½ m (D) 3 m (E) 15 m

1. Given the rectangle 

Which of the following rectangles is similar to the given rectangle?

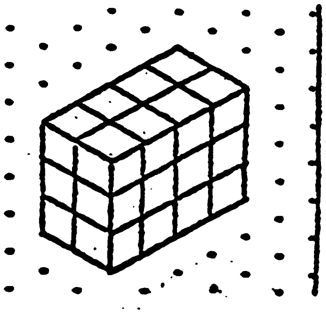
(A)  (B)  (C)  (D)  (E) 

APPENDIX B

MGMP SPATIAL VISUALIZATION SUBTEST:

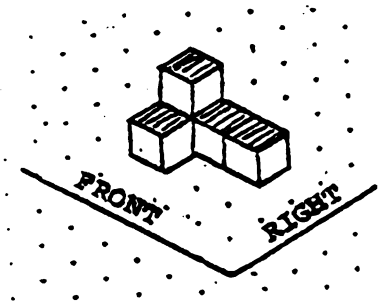
SAMPLE ITEMS

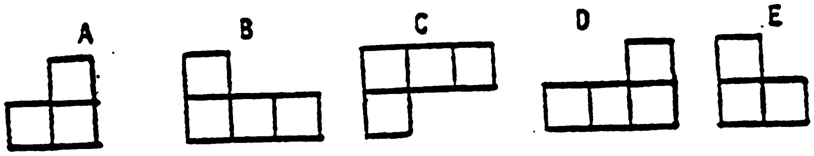
26. How many cubes are needed to build this rectangular solid?



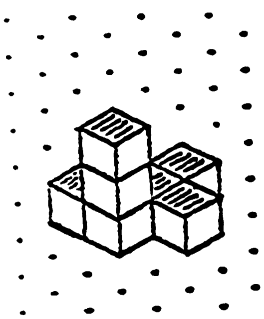
A	B	C	D	E
18	24	26	36	52

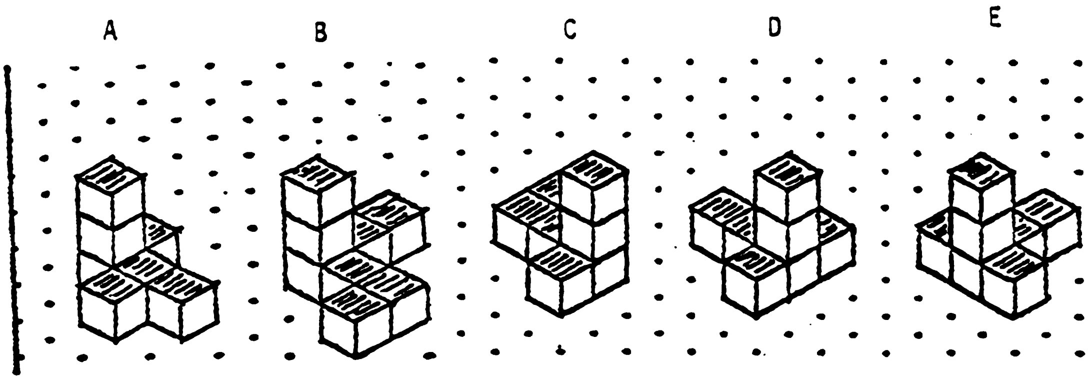
27. You are given a picture of a building drawn from the FRONT-RIGHT corner. Find the <u>RIGHT VIEW</u>.



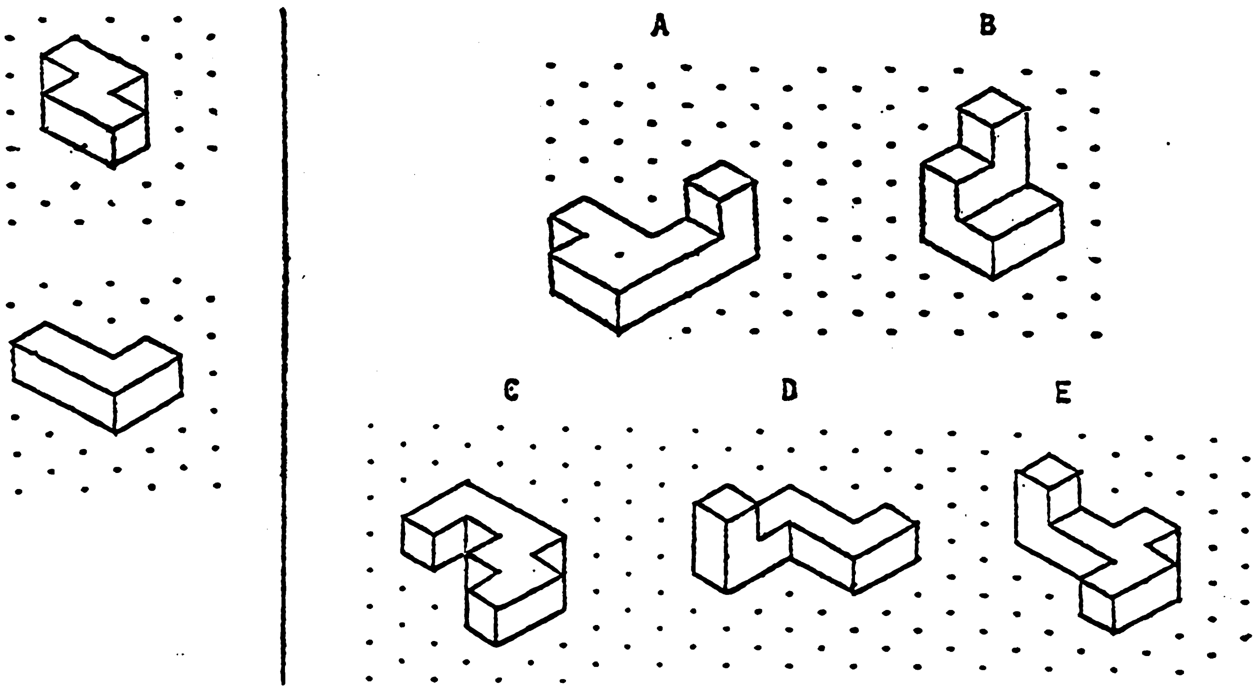


31. Find another view of the first building.

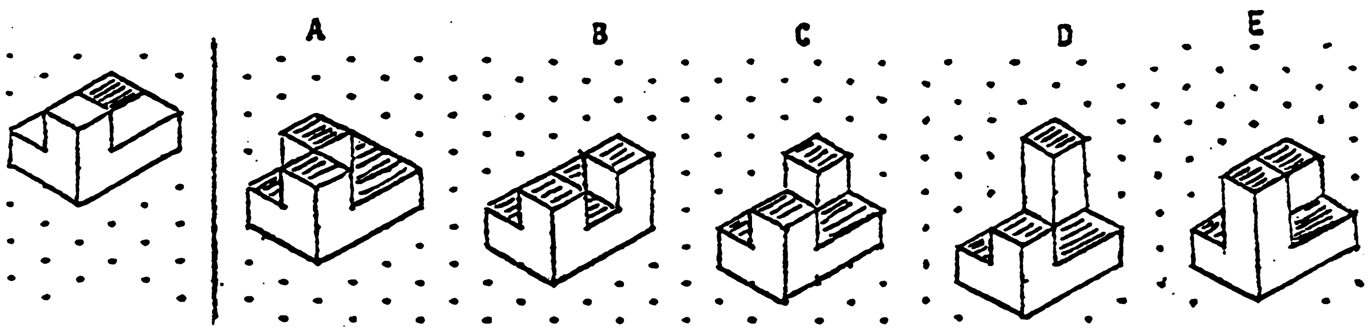




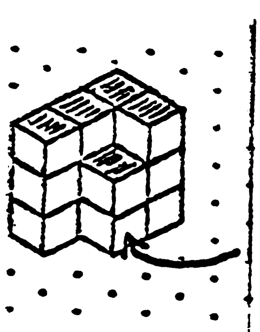
32. Which of these buildings can be made from the two pieces given?



33. If a cube were added to the shaded face of the given building, what would the new building look like?



38. How many cubes touch the indicated cube face to face?



A	B	C	D	E
1	2	3	4	5

APPENDIX C

THE MGMP SIMILARITY TEST AND

THE MGMP SPATIAL VISUALIZATION SUBTEST:

RELIABILITY COEFFICIENTS, AND

PRETEST - POST TEST CORRELATION COEFFICIENTS

Table C.1 RELIABILITY COEFFICIENTS--CRONBACH α FOR THE MGMP SIMILARITY TEST AND FOR THE MGMP SPATIAL VISUALIZATION SUBTEST BY TIME, BY GRADE, AND BY SEX.

	N	MGMP SIMT		MGMP SVST	
		Pretest α	Posttest α	Pretest α	Posttest α
Grade 6	248	.57	.76	.71	.69
Male	137	.59	.75	.76	.70
Female	111	.53	.76	.59	.65
Grade 7	240	.64	.77	.77	.78
Male	120	.68	.79	.79	.79
Female	120	.60	.74	.75	.76
Grade 8	207	.75	.82	.77	.79
Male	116	.76	.82	.80	.79
Female	91	.74	.82	.71	.77

Table C.2 PEARSON CORRELATION COEFFICIENTS BETWEEN PRETEST AND POST TEST SCORES ON THE MGMP SIMILARITY TEST AND ON THE MGMP SPATIAL VISUALIZATION SUBTEST BY GRADE AND BY SEX

	N	MGMP SIMT Pre - Post Correlation*	MGMP SVST Pre - Post Correlation*
Grade 6	248	.54	.66
Male	137	.49	.69
Female	111	.61	.60
Grade 7	240	.49	.71
Male	120	.52	.72
Female	120	.46	.70
Grade 8	207	.78	.78
Male	116	.76	.75
Female	91	.79	.80

*All significant ($P < .001$)

APPENDIX D

THE MGMP SIMILARITY UNIT: OVERVIEW

OVERVIEW

The unit begins with a very intuitive approach to enlargement using the rubber band stretcher. While the scale and the shape are not precise, the basic idea stands out-if we produce an enlargement with two rubber bands knotted together we get a figure whose sides are twice as long and whose area is four times as large as the original.

Enlargements and contractions are also the focus of the second activity. However, in this activity the use of coordinate systems allows more precise pictures. Tests for similarity can now be made by comparing angle sizes and corresponding distances. The area of Morris's nose provides a concrete example of the relationship that area grows as the square of the scale factor. For example if the sides are enlarged by a factor of three, the area is three squared or nine times as large. This is an important but subtle idea which will not be learned in this first exposure.

Activities three and four allow a close look at similarity of rectangles and develop two tests which can be used to determine whether rectangles are similar. In activity three, students test two rectangles by dividing the short side of each rectangle by its long side. If the resulting ratios are equal, the rectangles are similar.

In activity four, the students find that families of similar rectangles will have coinciding diagonals when the rectangles are nested in their lower left corners. In activity five, first the students apply what they learned in activities three and four to the idea as similar right triangles (which are simply halves of the rectangles), then to

any similar triangles.

The idea of repeating tiles (reptiles) in activity six brings together in one activity the ideas of similarity, congruence and infinity. The student will realize that the reptiling process can go on infinitely in both directions. The tiles can continue to grow as large as one might want and they can be continually subdivided as small as desired. Many children are fascinated by the ideas of infinity.

Activity seven provides a means of enlarging figures much more accrately than the stretcher in activity one. With an unmarked straight edge the students can construct point enlargements to various <u>scale factors</u>, both positive and negative. The effects of moving the point of enlargement determines <u>where</u> the image will lie, but not how large it will be. The scale factor completely determines size.

Activity eight provides a short respite from the more intensive activities by having the students pause and apply the ideas they have been studying. They can choose a picture to reproduce by any of the techniques they have learned so far. They can experiment to see which of the techniques provides the best results for their purposes.

If you have access to a pantagraph (a mechanical linkage device for enlarging figures), you might introduce it at this point.

Activity nine engages students in real life applications of similarity to finding distances or heights that cannot be measured directly.

APPENDIX E

MEAN SCORES, STANDARD DEVIATIONS,

AND CORRELATIONS

RELATED TO THE RESEARCH QUESTIONS

Table E.1 MEAN SCORES (IN PERCENT) AND STANDARD DEVIATIONS ON FOUR SIMILARITY-RELATED AREAS FOR SAMPLE A BY GRADE LEVEL AND BY SEX

	Basic Properties M (S.D.)		Proportional Reasoning M (S.D.)		Area Growth M (S.D.)		Applications M (S.D.)	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Grade 6	26.48 (19.61)	53.43 (23.82)	18.25 (26.47)	50.60 (34.41)	17.84 (19.66)	23.19 (24.47)	27.96 (20.07)	38.31 (21.83)
Males	29.81 (20.56)	53.89 (24.19)	19.53 (28.07)	54.20 (33.56)	18.43 (19.48)	23.91 (24.04)	29.08 (21.86)	40.15 (21.44)
Females	22.37 (17.62)	52.85 (23.45)	16.67 (24.39)	46.17 (35.06)	17.12 (19.94)	22.30 (25.08)	26.58 (17.61)	36.04 (22.20)
Grade 7	38.04 (24.68)	65.10 (22.15)	38.09 (35.17)	70.15 (31.16)	22.06 (20.86)	37.65 (29.82)	34.80 (20.74)	53.33 (23.89)
Males	37.93 (25.88)	65.90 (21.85)	37.64 (34.71)	70.69 (31.94)	25.00 (22.33)	37.65 (31.64)	33.91 (20.40)	55.71 (25.20)
Females	38.15 (23.50)	64.26 (22.56)	38.55 (35.85)	69.58 (30.51)	18.98 (18.96)	37.65 (27.98)	35.74 (21.17)	51.41 (22.42)
Grade 8	63.37 (20.89)	79.86 (16.74)	65.63 (31.47)	80.47 (22.13)	10.83 (24.78)	33.33 (30.49)	53.47 (23.06)	69.44 (22.25)
Males	63.58 (22.68)	79.01 (16.25)	66.67 (31.85)	80.56 (22.61)	18.06 (22.99)	36.11 (30.20)	56.17 (24.93)	73.46 (22.55)
Females	63.10 (18.60)	80.95 (17.49)	64.29 (31.30)	80.36 (21.76)	24.40 (26.76)	29.76 (30.86)	50.00 (20.16)	64.29 (21.01)

Table E.2 CORRELATIONS AMONG PRE-INSTRUCTIONAL PERFORMANCE ON FOUR SIMILARITY-RELATED AREAS FOR SAMPLE A BY GRADE[a]

		Basic Properties	Proportional Reasoning	Area Growth
Proportional Reasoning	Total	.516		
	Grade 6	.255		
	Grade 7	.434		
	Grade 8	.313		
Area Growth	Total	.117	.066*	
	Grade 6	.183	.062*	
	Grade 7	-.054*	-.028*	
	Grade 8	.236	.110*	
Applications	Total	.529	.489	.097
	Grade 6	.346	.265	.056*
	Grade 7	.443	.484	.027*
	Grade 8	.461	.348	.210

[a] N = 514, # Grade 6 = 248, # Grade 7 = 170, # Grade 8= 96.

*Not significant at $\alpha = .05$

Table E.3 MEAN SCORES (IN PERCENT) AND STANDARD DEVIATIONS ON TEST ITEMS PRESENTED IN A VERBAL AND A VISUAL MODE FOR SAMPLE A BY GRADE LEVEL AND BY SEX

	N	Verbal Mode M (S.D.) Pretest	Verbal Mode M (S.D.) Posttest	Visual Mode M (S.D.) Pretest	Visual Mode M (S.D.) Posttest
Grade 6	248	19.76 (21.17)	35.28 (24.65)	18.75 (20.89)	29.44 (25.75)
Males	137	21.72 (22.24)	35.77 (24.40)	19.53 (20.94)	31.20 (25.33)
Females	111	17.34 (19.60)	34.68 (25.04)	17.79 (20.89)	27.25 (26.23)
Grade 7	170	27.35 (20.76)	48.97 (28.57)	24.71 (21.58)	48.53 (27.56)
Males	87	27.59 (23.82)	48.56 (28.11)	26.15 (22.53)	51.15 (28.51)
Females	83	27.11 (17.11)	49.40 (29.21)	23.19 (20.58)	45.78 (26.43)
Grade 8	96	40.36 (22.17)	52.34 (24.62)	38.02 (26.65)	55.21 (26.64)
Males	54	39.35 (20.95)	54.63 (23.33)	38.43 (26.91)	57.87 (26.96)
Females	42	41.67 (23.86)	49.40 (26.18)	37.50 (26.62)	51.79 (26.13)

Table E.4 CORRELATIONS[a] AMONG PRE, AND POST-INSTRUCTIONAL PERFORMANCE ON SIMILARITY AND SPATIAL VISUALIZATION TASKS FOR SAMPLE A BY GRADE[b]

		SIMT Pre	SIMT Post	SVST Pre
SIMT Post	Total	.678		
	Grade 6	.542		
	Grade 7	.512		
	Grade 8	.733		
SVST Pre	Total	.562	.509	
	Grade 6	.297	.270	
	Grade 7	.437	.387	
	Grade 8	.551	.608	
SVST Post	Total	.552	.582	.741
	Grade 6	.339	.472	.663
	Grade 7	.452	.476	.687
	Grade 8	.577	.583	.740

[a] All correlation coefficients are significant ($P < .001$)

[b] N=514, # Grade 6 = 248, # Grade 7 = 170, # Grade 8 = 96.

Table E.5 MEAN SCORES (IN PERCENT) AND STANDARD DEVIATIONS ON SIMILARITY AND ON SPATIAL VISUALIZATION FOR SAMPLE A BY GRADE LEVEL AND BY SEX

	N	Similarity Test M (S.D.) Pre	Similarity Test M (S.D.) Post	Spatial Visualization Subtest M (S.D.) Pre	Spatial Visualization Subtest M (S.D.) Post
Grade 6	248	22.89 (11.87)	41.76 (17.28)	28.44 (18.85)	39.95 (19.54)
Males	137	24.15 (12.30)	43.36 (17.04)	30.51 (20.75)	43.41 (19.87)
Females	111	21.33 (11.16)	39.78 (17.45)	25.89 (15.92)	35.68 (18.33)
Grade 7	170	31.79 (14.67)	54.82 (17.52)	39.57 (22.37)	48.75 (22.27)
Males	87	31.82 (15.72)	55.45 (18.86)	40.15 (22.46)	49.89 (23.14)
Females	83	31.76 (13.58)	54.17 (16.08)	38.96 (22.39)	47.55 (21.41)
Grade 8	96	50.08 (15.70)	66.88 (15.03)	56.74 (20.84)	63.61 (21.58)
Males	54	50.00 (16.33)	68.44 (14.67)	56.91 (22.61)	64.32 (20.81)
Females	42	50.19 (15.05)	64.86 (15.42)	56.51 (18.58)	62.70 (22.77)

Table E.6 MEAN SCORES (IN PERCENT) AND STANDARD DEVIATIONS ON SIMILARITY AND ON SPATIAL VISUALIZATION FOR SAMPLE B BY LEVEL OF INSTRUCTION IN SPATIAL VISUALIZATION, BY GRADE, AND BY SEX

	N	Similarity Test M (S.D.) Pre	Similarity Test M (S.D.) Post	Spatial Visualization Subtest M (S.D.) Pre	Spatial Visualization Subtest M (S.D.) Post
SV Unit[a]	98	33.67 (12.96)	47.43 (16.13)	61.56 (21.49)	67.07 (21.76)
Grade 7	29	31.86 (11.98)	48.83 (15.06)	56.78 (22.51)	60.23 (22.71)
Males	15	34.93 (8.34)	52.00 (11.71)	60.44 (22.88)	68.89 (17.58)
Females	14	28.57 (14.54)	45.43 (17.79)	52.86 (22.26)	50.95 (24.48)
Grade 8	69	34.43 (13.36)	46.84 (16.64)	63.57 (20.88)	69.95 (20.85)
Males	43	36.28 (14.59)	47.35 (18.33)	68.68 (20.60)	74.88 (20.57)
Females	26	31.38 (10.59)	46.00 (13.68)	55.13 (18.81)	61.79 (19.00)
No SV Unit[b]	63	32.06 (14.34)	45.27 (17.42)	40.74 (20.88)	45.50 (20.27)
Grade 7	29	30.62 (13.66)	45.79 (18.16)	45.52 (19.93)	48.05 (23.19)
Males	13	33.85 (15.02)	50.77 (19.76)	55.90 (18.77)	60.00 (24.65)
Females	16	28.00 (12.31)	41.75 (16.26)	37.08 (17.03)	38.33 (17.13)

Table continued

Table E.6 (Cont'd.)

	N	Similarity Test M (S.D.) Pre	Post	Spatial Visualization Subtest M (S.D.) Pre	Post
Grade 8	34	33.29 (14.99)	44.82 (17.02)	36.67 (21.10)	43.33 (17.47)
Males	16	33.50 (17.34)	48.50 (15.79)	39.17 (25.05)	46.25 (18.77)
Females	18	33.11 (13.07)	41.56 (17.84)	34.44 (17.30)	40.74 (16.31)

[a] Studied the MGMP Spatial Visualization Unit.

[b] Did not study the MGMP Spatial Visualization Unit.

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