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A STUDY OF SUBHARMONIC SOLUTIONS OF SECOND ORDER EQUATIONS

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ABSTRACT

A STUDY OF SUBHARMONIC SOLUTIONS OF
SECOND ORDER EQUATIONS

By

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Consider a second order differential equation

$$\ddot{x} + g(x) = -\lambda\dot{x} + \mu f(t)$$

with small damping and periodic forcing. We will investigate the condition on the parameters (λ,μ) to ensure the existence of subharmonic solutions of order k by deriving the bifurcation equation. We find in the (λ,μ)-plane there are two disjoint regions such that the equation has at least $2k$ k-periodic solutions in one region and none on the other region. The stability of these solutions is also discussed by computing the characteristic multipliers. Finally, some numerical experiments, such as locating those periodic solutions and increasing parameters to obtain period doubling phenomena, are performed on the forced pendulum problem.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

SECTION 1. INTRODUCTION

In [6], D'Humieres, Beasley, Huberman, and Libchaber presents a series of numerical experiments on the forced pendulum problem, which includes the period-doubling cascades which lead to chaotic motions. They also indicate that there is a non period-doubling case which also leads to chaotic states. A. Itō [9] also gave some evidence of successive subharmonic bifurcations which lead to chaos. Other experiments may be found in [7], [8]. However, in many cases it is not clear how these motions are created. Here we attempt to explain some of these phenomena.

Consider the 2nd order differential equation

$$\ddot{x} + g(x) = -\lambda\dot{x} + \mu f(t) \tag{1.1}$$

where $f(t)$ has least period 1, λ,μ are parameters and $g(x)$ is a function such that when $(\lambda,\mu) = (0,0)$, equation (1.1) has either a homoclinic orbit or heroclinic orbit Γ. Inside the orbit Γ, it is well-known that there exist periodic solutions with least periods tending to infinity as these periodic solutions tend to Γ.

For each periodic orbit Γ_k with least period k, we will investigate the condition on the parameters (λ,μ) to ensure the existence of subharmonic solutions of (1.1) of order k. To do this, we first invert a differential operator, whose inverse is denoted by G_k. Next, we derive the bifurcation equation for these subharmonic solutions in order to find the bifurcation diagram. We find that in the (λ,μ)-plane there are two disjoint regions such that (1.1) has at least $2k$ k-periodic solutions in one region, none on the other region, and exactly one on the curve. The above phenomena are called the saddle-node bifurcations, i.e., two periodic solutions coalesce and then disappear.

To prove that, we need to discuss the stability of the solutions by computing their characteristic multipliers. We also find a neighborhood of $(\lambda,\mu) = (0,0)$ such that the stability arguments holds uniformly for each k.

Chow, Hale, and Mallet-Paret [3] indicated that if there is a neighborhood U of $(\lambda,\mu) = (0,0)$, such that for $(\lambda,\mu) \in U$ there exist k-periodic solutions of any k, then there exist infinitely many periodic solutions which are derived from successive subharmonic bifurcation. Such neighborhood would exist if the operator G_k, as mentioned above, is uniformly bounded in k. We prove in Section 5 that G_k is uniformly bounded on the subspace of symmetric periodic functions.

Finally, we give a continuation method for finding the periodic orbits numerically. We will use the Runge-Kutta method to solve the initial value problem and Newton's method to locate the periodic orbit. Our numerical experiments are performed on the forced pendulum problem.

SECTION 2. BIFURCATION EQUATION

Consider the equation

$$\ddot{x} + g(x) = -\lambda\dot{x} + \mu f(t) \tag{2.1}$$

where λ,μ are real parameters, $g(x)$ is 3-times continuously differentiable and $f(t)$ is periodic with period 1. For $\lambda = \mu = 0$, assume the system

$$\ddot{x} + g(x) = 0 \tag{2.2}$$

has a nontrivial periodic solution $p(t)$ with least period k, where k is an integer. Let $\Gamma = \{(p(t),\dot{p}(t); 0 \leq t < k\}$. The problem is to find periodic solutions of (2.1) with least period k in a sufficiently small neighborhood of Γ for small λ,μ. If such solutions exist, they are called subharmonic solutions of order k since their period is k times the period of $f(t)$.

For this discussion, it is convenient to use a different coordinate system near Γ. Let $G : R^4 \to R^2$ be defined by

$$G(\alpha,a,x,y) = (p(\alpha) + a\ddot{p}(\alpha) - x, \dot{p}(\alpha) - a\dot{p}(\alpha) - y)$$

Since $G(\alpha_0,0,p(\alpha_0),\dot{p}(\alpha_0)) = (0,0)$ and

$$\det \frac{\partial G}{\partial(\alpha,a)} (\alpha_0,0,p(\alpha_0),\dot{p}(\alpha_0)) = \dot{p}(\alpha_0)^2 + \ddot{p}(\alpha_0)^2 \neq 0 .$$

It follows from Implicit function theorem that there exists $\delta(\alpha_0) > 0$, $a(\alpha_0) > 0$ and two functions $\alpha^*(x,y)$, $a^*(x,y)$ with $\alpha^*(p(\alpha_0),\dot{p}(\alpha_0)) = \alpha_0$, $a^*(p(\alpha_0),\dot{p}(\alpha_0)) = 0$, such that $G(\alpha^*(x,y),a^*(x,y),x,y) = 0$ for $|x - p(\alpha_0)| < \delta(\alpha_0)$, $|y - \dot{p}(\alpha_0)| < \delta(\alpha_0)$ and these solutions are unique for $|\alpha - \alpha_0| < a(\alpha_0)$, $|a| < a(\alpha_0)$. This defines a diffeomorphism from a neighborhood of $(p(\alpha_0),\dot{p}(\alpha_0))$ to a neighborhood of $(\alpha_0,0)$. We can apply the above argument to every point of Γ, then by compactness of Γ, there exists $a_0 > 0$ and a diffeomorphism F from a neighborhood of Γ onto $[o,k) \times \{a : |a| < a_0\}$. In summary, for any (x,y) near Γ, there exists unique (α,a), where α shows the position on Γ and $ap(\alpha)$ indicates the distance between the orbit Γ and the point (x,y), such that

$$x = p(\alpha) + a\ddot{p}(\alpha)$$
$$y = \dot{p}(\alpha) - a\dot{p}(\alpha)$$

If $x(t)$ is a k-periodic solution of (2.1) in a small neighborhood of Γ, then there exists a unique (α,a) such that

$$x(0) = p(\alpha) + a\ddot{p}(\alpha)$$
$$\dot{x}(0) = \dot{p}(\alpha) - a\dot{p}(\alpha)$$

Therefore we can write $x(t)$ in the form

$$x(t) = p(t + \alpha) + z(t + \alpha) \tag{2.3}$$

where $z(t+\alpha)$ has small magnitude, $(z(\alpha),\dot{z}(\alpha))$ is orthogonal to $(\dot{p}(\alpha),\ddot{p}(\alpha))$ and α is determined by the initial condition.

Let (2.3) be applied to (2.1), we get

$$\ddot{z}(t+\alpha) + g'(p(t+\alpha))z(t+\alpha)$$
$$= -\lambda\dot{z}(t+\alpha) - \lambda\dot{p}(t+\alpha) + \mu f(t) + G(t+\alpha,z)$$

where $G(t+\alpha,z) = -g(p(t+\alpha)+z(t+\alpha)) + g'(p(t+\alpha))z(t+\alpha) + g(p(t+\alpha))$, therefore $G(\cdot,z) = O(|z|^2)$. Here we let "," denote the derivative with respect to x. Replace $t+\alpha$ by t, we obtain the following equation

$$\ddot{z} + g'(p)z = -\lambda\dot{z} - \lambda\dot{p} + \mu f_\alpha(t) + G(t,z) \tag{2.4}$$

where $f_\alpha(t) = f(t-\alpha)$. Hence the problem now is to find k-periodic solutions of (2.4) with $(z(\alpha),\dot{z}(\alpha))$ orthogonal to $(\dot{p}(\alpha),\ddot{p}(\alpha))$.

Without loss of generality, suppose $\dot{p}(0) = 0$ and assume that

(H1) Every k-periodic solution of the homogeneous equation

$$\ddot{z} + g'(p)z = 0 \tag{2.5}$$

is a constant multiple of $\dot{p}(t)$.

Note that equation (2.2) is Hamiltonian. Hence $p(t)$ must be embedded on a one-parameter family of periodic solutions of (2.2). Let the parameter be denoted by b and the corresponding period be $T(b)$. Assume that $T(b_0) = k$.

<u>Proposition 2.1</u>. Let $T(b)$ be the period function as above. Then hypothesis (H1) is equivalent to $T'(b_0) \neq 0$.

<u>Proof</u>. For simplicity, let's assume that b is the place where the periodic orbit crosses the positive x-axis, therefore $\dot{p}(0) = 0$, $p(0) = b_0$.

Let $p(t,b)$ be the periodic solution of (2.2) with period $T(b)$, then

$$p(t + T(b),b) = p(t,b) \tag{2.6}$$

and $p(t,b_0) = p(t)$. Let

$$q(t) = \frac{\partial}{\partial b} p(t,b)|_{b=b_0} \tag{2.7}$$

then q satisfies (2.5) with initial condition $q(0) = 1$, $\dot{q}(0) = 0$. Take the derivative with respect to b of both sides of (2.6) and set $b = b_0$. We obtain

$$\dot{p}(t)T'(b_0) + q(t + T(b_0)) = q(t) \tag{2.8}$$

Set $t = 0$, we get $q(k) = 1$. Take the derivative with respect to t of both sides of (2.8) and set $t = 0$, and we get $\dot{q}(k) = -\ddot{p}(0)T'(b_0)$. If $T'(b_0) = 0$, then $\dot{q}(k) = 0$ which implies $q(t)$ is a k-periodic solution of (2.5), therefore (H1) does not hold. If $T'(b_0) \neq 0$, then $q(t)$ is not a k-periodic solution . Let

$$r(t) = \frac{\dot{p}(t)}{\ddot{p}(0)} \tag{2.9}$$

then

$$X(t) = \begin{bmatrix} q(t) & r(t) \\ \dot{q}(t) & \dot{r}(t) \end{bmatrix} \tag{2.10}$$

is a fundamental matrix solution of (2.5), that is the solution of (2.5) is a linear combination of q and r. This shows that the only k-periodic solution of (2.5) is a constant multiple of $\dot{p}(t)$. Q.E.D.

We now apply the method of Liapunov-Schmidt to equation (2.4). Let P_k^r be the space of r-times continuously differentiable periodic function with period k with $|f|_r = \sup\{|f^{(i)}(t)| : i = 0,1,\ldots,r, t \in [o,k)\}$. For any $y \in P_k^2$ let

$$\begin{aligned} Ay &= \ddot{y} + g'(p)y \\ Ny &= -\lambda\dot{y} - \lambda\dot{p} + \mu f_\alpha(t) + G(t,y) \end{aligned} \tag{2.11}$$

where f_α and $G(t,y)$ are the same as in (2.4). Then A is a continuous linear operator from $P_k^2 \to P_k^0$ and N is a continuous operator from $P_k^2 \to P_k^0$. (H1) implies that the null space of A is one dimensional. Define $P : P_k^0 \to P_k^0$ by

$$Py = \eta\dot{p}\int_0^k \dot{p}y\,dt \tag{2.12}$$

where

$$\eta = \left(\int_0^k \dot{p}^2\,dt\right)^{-1} \tag{2.13}$$

Then P is a continuous projection.

<u>Lemma 2.2</u>. Assume (H1) holds. Let $X(t)$ be the fundamental matrix of (2.5). For any $\phi \in P_k{}^0$, define $\tilde{G} : (I - P)P_k{}^0 \to P_k{}^0$ by

$$(2.14) \quad \begin{bmatrix} \tilde{G}\phi(t) \\ \dot{\tilde{G}\phi}(t) \end{bmatrix} = X(t)\begin{bmatrix} w_0 \\ 0 \end{bmatrix} + X(t)\int_0^t X^{-1}(s)\begin{bmatrix} 0 \\ \phi(s) \end{bmatrix} ds$$

where

$$w_0 = \frac{1}{-\dot{q}(k)}\int_0^k q(s)\phi(s)ds$$

and $q(t)$ is given by (2.7). Then G is a continuous linear operator, and $G\phi(t)$ is a solution of

$$(2.15) \quad \begin{cases} \ddot{z} + g'(p)z = \phi(t) \\ z \quad \text{is} \quad k\text{-periodic} \\ \dot{z}(0) = 0 \end{cases}$$

<u>Proof</u>: It follows from the variation of constants formula that the solution of (2.15) can be written as

$$\begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} = X(t)\begin{bmatrix} w_0 \\ 0 \end{bmatrix} + X(t)\int_0^t X^{-1}(s)\begin{bmatrix} 0 \\ \phi(s) \end{bmatrix} ds$$

Then $z(t)$ is a k-periodic solution if and only if

$$X(0)\begin{bmatrix} w_0 \\ 0 \end{bmatrix} = X(k)\begin{bmatrix} w_0 \\ 0 \end{bmatrix} + X(k)\int_0^k \begin{bmatrix} -r(s)\phi(s)ds \\ q(s)\phi(s)ds \end{bmatrix}$$

i.e.,

$$(I - X(k)) \begin{bmatrix} w_0 \\ 0 \end{bmatrix} = X(k) \int_0^k \begin{bmatrix} -r(s)\phi(s)ds \\ q(s)\phi(s)ds \end{bmatrix}$$

i.e.,

$$\begin{bmatrix} 0 \\ -\dot{q}(k)w_0 \end{bmatrix} =$$

$$= \begin{bmatrix} \int_0^k -r(s)\phi(s)ds \\ -\dot{q}(k) \int_0^k r(s)\phi(s)ds + \int_0^k q(s)\phi(s)ds \end{bmatrix}$$

$$= \begin{bmatrix} -\int_0^k r(s)\phi(s)ds \\ \int_0^k q(s)\phi(s)ds \end{bmatrix} .$$

Therefore $\int_0^k r(s)\phi(s)ds = 0$, and

$$w_0 = \frac{1}{-\dot{q}(k)} \int_0^k q(s)\phi(s)ds$$

Q.E.D.

Note that since both G and P are continuous linear operators, therefore the operator $G(I - P)$ has continuous second Fréchet derivative.

We obtain the following Lemma by adding an appropriate constant multiple of $\dot{p}(t)$.

<u>Lemma 2.3</u>. Let $\phi \in P_k{}^0$. Then

$$\begin{cases} \ddot{z} + g'(p)z = \phi(t) - P\phi(t) \\ z \quad \text{is} \quad k\text{-periodic} \end{cases}$$

has a unique solution $G(I - P)\phi$ which has continuous second Frèchet derivative with respect to ϕ and $P(G(I - P)\phi) = 0$.

It follows from Lemma 2.3 that $AG = I$ on $(I - P)P_k^0$ and $GA = I - U$ on P_k^2 where

$$U = P|_{P_k^2} .$$

For this definition of G, (2.1) will have a solution of the form given in (2.3) if and only if the following two equations are satisfied.

$$(2.16) \qquad w = G(I - P)N(z)$$
$$= G(I - P)[-\lambda\dot{z} - \lambda\dot{p} + \mu f_\alpha + G(\ ,z)]$$

$$(2.17) \qquad PN(z) = p[-\lambda\dot{z} - \lambda\dot{p} + \mu f_\alpha + G(\ ,z)] = 0$$

where $z = w + a\dot{p}$ and $Pw = 0$. Let $B : P_k^2 \times R \times [0,k) \times R \times R \rightarrow P_k^0$ be defined by

$$B(w,a,\alpha,\lambda,\mu) = w - G(I - P)[-\lambda\dot{z} - \lambda\dot{p} + \mu f_\alpha + G(\ ,z)]$$

Since $B(0,0,\alpha,0,0) = 0$ and

$$\frac{\partial B}{\partial w}(0,0,\alpha,0,0) = I$$

on P_k^2; by the Implicit function theorem there is a neighborhood $U \subseteq P_k^2$ of zero and a neighborhood $V \subseteq R^3$ of $(a,\lambda,\mu) = (0,0,0)$, such that (2.16) has a solution $w^*(a,\alpha,\lambda,\mu)$ for $(a,\lambda,\mu) \in V$, $0 \leq \alpha < k$. This solution is unique and has continuous second derivative with respect to a,α,λ,μ and $w^*(a,\alpha,0,0) = 0$ for all α. Therefore (2.1) will have k-periodic solutions if and only if (a,α,λ,μ) satisfies equation (2.17) with z replaced by $w^*(a,\alpha,\lambda,\mu) + a\dot{p}$.

Proposition 2.4. Let

$$
\begin{aligned}
(2.18)\qquad C(a,\alpha,\lambda,\mu) &= \eta \int_0^k \dot{p}[-\lambda w^*(a,\alpha,\lambda,\mu) - \lambda\dot{p} + \mu f_\alpha(t) \\
&\qquad + G(t,w^*(a,\alpha,\lambda,\mu))]dt \\
&= -\lambda + h(\alpha)\mu + \text{h.o.t.}
\end{aligned}
$$

where

$$
h(\alpha) = \eta \int_0^k \dot{p}(t)f(t-\alpha)dt \quad \text{and}
$$

$$
\text{h.o.t.} = O(|\lambda|^2 + |\mu|^2 + |a|) \quad \text{as} \quad \lambda,\mu,a \to 0
$$

Then (2.1) has a k-periodic solution for some small $\tilde{\lambda}$, $\tilde{\mu}$, $\tilde{a}$ if and only if there exists an $\tilde{\alpha}$ such that

$$
(2.19)\qquad C(\tilde{a},\tilde{\alpha},\tilde{\lambda},\tilde{\mu}) = 0
$$

Remark 2.5. Equation (2.19) is often called the bifurcation equation.

Remark 2.6. Since $w^*(a,\alpha,\lambda,\mu)$ is a solution of (2.16) and $f(t)$ is 1-periodic, it follows that $C(a,\alpha+1,\lambda,\mu) = C(a,\alpha,\lambda,\mu)$, that is C is 1-periodic in α with λ μ fixed.

SECTION 3. EXISTENCE

To obtain the complete picture of the existence of k-periodic subharmonic solutions of (2.1), it remains to analyse the bifurcation equation (2.19). The procedure is similar to Chow and Hale [3]. Since a in the definition of the bifurcation equation is not important, we drop its dependence in our analysis.

Let $h(\alpha)$ be defined by (2.18) which has period 1. Assume that

(H2) There are only finitely many numbers $\alpha_i \in [0,1]$ $i = 1,\ldots,N$, such that $h'(\alpha_i) = 0$ and $h''(\alpha_i) \neq 0$.

For any $\mu \neq 0$, (2.19) is equivalent to

$$h(\alpha) - \frac{\lambda}{\mu} + G(\alpha,\frac{\lambda}{\mu},\lambda,\mu) = 0$$

where

$$\mu \cdot G(\alpha,\frac{\lambda}{\mu},\lambda,\mu) = \lambda - h(\alpha)\mu - C(\alpha,\lambda,\mu)$$

and

$$G(\alpha,\beta,0,0) = 0, \quad \beta = \frac{\lambda}{\mu} .$$

Hence to finding all possible solutions of (2.19) for small λ, μ, is equivalent to finding all possible solutions of

$$(3.1) \qquad F(\alpha,\beta,\mu) = h(\alpha) - \beta + G(\alpha,\beta,\beta\mu,\mu) = 0$$

for all $0 \le \alpha < k$, $\beta \in R$ and μ near 0. It follows from Remark 2.6 that $F(\alpha,\beta,\mu)$ is 1-periodic in α with λ,μ fixed. For $0 \le \alpha < 1$ and $\mu = 0$, the only solution of (3.1) is (α_0,β_0), which satisfies $h(\alpha_0) = \beta_0$. If $h'(\alpha_0) \neq 0$, then

$$\frac{\partial F}{\partial \alpha}(\alpha_0,\beta_0,0) \neq 0 \ .$$

The Implicit function theorem implies that there exists $\delta(\alpha_0,\beta_0) > 0$ and a unique solution $\alpha^*(\beta,\mu)$ such that $F(\alpha^*(\beta,\mu),\beta,\mu) = 0$ for $|\beta - \beta_0| < \delta(\alpha_0,\beta_0)$, $|\mu| < \delta(\alpha_0,\beta_0)$ and $\alpha^*(\beta_0,0) = \alpha_0$.

If $h'(\alpha_0) = 0$ then (H2) implies

$$\frac{\partial^2 F}{\partial \alpha^2}(\alpha_0,\beta_0,0) = h''(\alpha_0) \neq 0 \ .$$

By the Implicit function theorem there exists $\delta(\alpha_0,\beta_0) > 0$ and a unique solution $\alpha^*(\beta,\mu)$ such that

$$\frac{\partial F}{\partial \alpha}(\alpha^*(\beta,\mu),\beta,\mu) = 0$$

for $|\beta - \beta_0| < \delta(\alpha_0,\beta_0), |\mu| < \delta(\alpha_0,\beta_0)$. Hence $F(\alpha^*(\beta,\mu),\beta,\mu)$ is a maximum or minimum of $F(\alpha,\beta,\mu)$ with respect to α for β, μ fixed. For fixed β,μ, let $g(\alpha) = F(\alpha,\beta,\mu)$ then

$$g''(\alpha) = \frac{\partial^2 F}{\partial \alpha^2}(\alpha,\beta,\mu) \ .$$

In particular

$$g''(\alpha^*(\beta_0,0)) = \frac{\partial^2 F}{\partial \alpha^2}(\alpha_0,\beta_0,0) = h''(\alpha_0) \neq 0 .$$

If $h''(\alpha_0) > 0$ then for (β,μ) near $(\beta_0,0)$, $g''(\alpha^*(\beta,\mu)) > 0$ therefore $F(\alpha^*(\beta,\mu),\beta,\mu)$ is a minimum. If $h''(\alpha_0) < 0$ then by the same argument $F(\alpha^*(\beta,\mu),\beta,\mu)$ is a maximum. The number of solutions of (3.1) will depend on the sign of $F(\alpha^*(\beta,\mu),\beta,\mu)$. Let

(3.2) $$\gamma(\beta,\mu) = \text{sign } h''(\alpha_0) \cdot F(\alpha^*(\beta,\mu),\beta,\mu)$$

Then the following holds

(1) $\gamma(\beta,\mu) > 0 \Rightarrow$ there are no solutions of (3.1)

(2) $\gamma(\beta,\mu) = 0 \Rightarrow$ there is only one solution of (3.1)

(3) $\gamma(\beta,\mu) < 0 \Rightarrow$ there are exactly two solutions of (3.1)

Let $H(\beta,\mu) = F(\alpha^*(\beta,\mu),\beta,\mu) = 0$. Since

$$\frac{\partial H}{\partial \beta}(\beta_0,0) = \left(\frac{\partial F}{\partial \alpha}\frac{\partial \alpha}{\partial \beta} + \frac{\partial F}{\partial \beta}\right)(\alpha_0,\beta_0,0) = -1$$

and $H(\beta_0,0) = 0$, it follows from the Implicit function theorem that there exists $\delta(\beta_0) > 0$ and a unique solution $\beta^*(\mu)$, such that $H(\beta^*(\mu),\mu) = 0$ for $|\mu| < \delta(\beta_0)$. Therefore $F(\alpha^*(\beta^*(\mu),\mu),\beta^*(\mu),\mu) = 0$ or $\gamma^{-1}(0) = \{(\beta,\mu) : \beta = \beta^*(\mu), |\mu| < \delta(\beta_0)\}$. We conclude that there are two solutions of (3.1) near α_0 on one side of the curve $\beta = \beta^*(\mu)$ and none on the other side. In terms of the

coordinates (λ,μ), the curve becomes $\lambda = \beta^*(\mu)\mu$ which is tangent to $\lambda = h(\alpha_0)\mu$ at $\lambda = \mu = 0$.

The above argument can be applied to each $\alpha_0 + j$, $j = 1,\ldots,k-1$. Since $F(\alpha + 1,\lambda,\mu) = F(\alpha,\lambda,\mu)$, hence the curve we obtain will be the same for each $\alpha_0 + j$. This shows that altogether there are $2k$ solutions on one side of $\lambda = \beta^*(\mu)\mu$ and none on the other side. Let

$$h(\alpha^*) = \max_{\alpha\epsilon[0,1)} h(\alpha) \quad \text{and} \quad h(\alpha_*) = \min_{\alpha\epsilon[0,1)} h(\alpha)$$

and two curves $\lambda = C^*(\mu)$ $\lambda = C_*(\mu)$ which are respectively tangent to $\lambda = h(\alpha^*)\mu$, $\lambda = h(\alpha_*)\mu$ at $\lambda = \mu = 0$. We obtain the following theorem.1

<u>Theorem 3.1</u>. If hypotheses (H1) and (H2) are satisfied, then there are neighborhoods U of Γ, V of $\lambda = \mu = 0$ and a finite number of curves $C_j \epsilon V$ defined by $\lambda = C_j(\mu)$ which is tangent to the straight line $\lambda = h(\alpha_j)\mu$ at $\lambda = \mu = 0$, $j = 1,\ldots,N$. The number of k-periodic subharmonic solutions of (2.1) in U changes by $2k$ as each curve C_j is crossed. Moreover if

$$S = \{(\lambda,\mu) \epsilon V : C^*(\mu) < \lambda < C_*(\mu)\}$$

then there are no solutions of (2.1) in U for $(\lambda,\mu) \notin S$ and at least $2k$ in S

<u>Proof</u>. It remains to find the neighborhood V of $\lambda = \mu = 0$. For each α_j in Hypothesis (H2), by the same argument as before, there exists $\delta(\beta_j), \epsilon(\alpha_j)$ and bifurcation curve $\lambda = \beta_j^*(\mu)\mu$ such that there are two solutions of (3.1) on one side of $\lambda = \beta_j^*(\mu)\mu$ and none on the other side for $|\lambda|$, $|\mu| < \delta(\beta_j)$ and $|\alpha - \alpha_j| < \epsilon(\alpha_j)$. Let B be the complement of the union of $\{\alpha; |\alpha - \alpha_j| < \epsilon(\alpha_j)\}$, $j = 1,2,\cdots,N$ then B is compact in $[0,1]$ and $h'(\alpha) \neq 0$ on B. Therefore no further bifurcation will take place. By the same argument as before, for each $\alpha_0 \in B$, there exists $\delta(\alpha_0,\beta_0) > 0$, $\epsilon(\alpha_0,\beta_0) > 0$ such that equation (3.1) has exactly one solution for $|\lambda|,|\mu| < \delta(\alpha_0,\beta_0)$ and $|\alpha - \alpha_0| < \epsilon(\alpha_0,\beta_0)$. The sets $\{\alpha: |\alpha - \alpha_0| < \epsilon(\alpha_0,\beta_0)\}$ as α_0 varies over B, serves as an open covering of B. By the compactness of B, there exists a finite covering, $\{\alpha; |\alpha - \alpha_{0i}| < \epsilon(\alpha_{0i},\beta_{0i})\}$ $i = 1,2,\cdots,M$, of B. Let

$$\delta = \min_{\substack{i=1,\cdots,M \\ j=1,\cdots,N}} \{\delta(\alpha_{0i},\beta_{0i}),\delta(\beta_j)\} ,$$

then $V = \{(\lambda,\mu); |\lambda|,|\mu| < \delta\}$ will be the required neighborhood.

<u>Remark 3.2</u>. The above result can be generalized to a two dimensional systems

$$\dot{x} = g(x) + f(t,x,\mu)$$

where $g : R^2 \to R^2$, $f(t,x,\mu) : R \times R^2 \times R^2 \to R^2$ are r-times continuously differentiable and $f(t + 1,x,\mu) = f(t,x,\mu)$. Assume

$$\dot{x} = g(x)$$

has a periodic solution $p(t)$ of least period k. Let $q(t)$ be a nontrivial k-periodic solution of the equation

$$\dot{y} = -y \cdot A(t)$$

where

$$A(t) = \frac{\partial}{\partial x} g(p(t)) \ .$$

Then the bifurcation equation becomes

$$C(\alpha,\mu) = \int_0^k q(t) \cdot F(t,z^*(\alpha,\mu)(t),\mu,\alpha)dt = 0 \tag{3.2}$$

where $F(t,z,\mu,\alpha) = f(p(t) + z) - f(p(t)) - A(t)z + g(t - \alpha,p(t) + z,\mu)$ and "$\cdot$" is the inner product. Finding the solutions of (3.2) is equivalent to finding the solutions of

$$B(\alpha,\beta,\zeta) = \beta \cdot h(\alpha) + B_0(\alpha,\beta,\zeta) \tag{3.3}$$

where $\mu = \beta\zeta$, $\beta \in R^2$, $|\beta| = 1$, $\zeta \in R$ and

$$h(\alpha) = \int_0^k q(t) \cdot [\partial f(t - \alpha,p(t),0)/\partial\mu]dt$$

Apply the proof of Theorem 3.1 to (3.3). We obtain a result similar to that of Theorem 3.1.

<u>Remark 3.3</u>. For those (α,λ,μ) such that

$$C(\alpha,\lambda,\mu) = \lambda - h(\alpha)\mu + \text{h.o.t} = 0 ,$$

since

$$\frac{\partial C}{\partial \lambda}(\alpha_0,0,0) = 1 \neq 0 ,$$

the Implicit function theorem implies there exists $\delta > 0$ and a unique function $\lambda^*(\alpha,\mu)$ such that if $|\mu|$, $|\alpha - \alpha_0| < \delta$ then $C(\alpha,\lambda^*(\alpha,\mu),\mu) = 0$.

SECTION 4. STABILITY

Assume (H1) holds, we will discuss the stability of the subharmonic solution of (2.1) by computing the characteristic multipliers of the linearized equation.

It follows from Lemma 2.3 that for small λ, μ and $0 \leq \alpha < k$ there exists a unique solution z of

$$
(4.1) \quad \begin{cases} \ddot{z} + g'(p)z = -\lambda\dot{z} - \lambda\dot{p} + \mu f(t-\alpha) + G(t,z) - C(\alpha,\lambda,\mu)\dot{p} \\ z \text{ is k-periodic} \\ z(\alpha)\dot{p}(\alpha) + \dot{z}(\alpha)\ddot{p}(\alpha) = 0 \end{cases}
$$

where G is given by (2.4) and $C(\alpha,\lambda,\mu)$ is the expression given by (2.18). Note that the solution $z(t,\alpha,\lambda,\mu)$ has continuous second derivatives with respect to α,λ,μ. Let $\phi(t,\alpha,\lambda,\mu) = p(t) + z(t,\alpha,\lambda,\mu)$. Then $\phi(t,\alpha,\lambda,\mu)$ is a k-periodic subharmonic solution of

$$\ddot{x} + g(x) = -\lambda\dot{x} + \mu f(t-\alpha) - C(\alpha,\lambda,\mu)\dot{p}$$

Note that $\phi(t,\alpha,0,0) = p(t)$. We will find the information needed to decide the stability of $\phi(t,\alpha,\lambda,\mu)$.

Consider the linearized equation around $\phi(t,\alpha,\lambda,\mu)$

$$(4.2) \qquad \ddot{x} + g'(\phi)x + \lambda\dot{x} = 0$$

which can be rewritten as

$$\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -g'(\phi) & -\lambda \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

(4.3) $$= \tilde{A}(t) \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

Let

(4.4) $$Y(t,\alpha,\lambda,\mu) = \begin{bmatrix} y_1(t,\alpha,\lambda,\mu) & y_2(t,\alpha,\lambda,\mu) \\ \dot{y}_1(t,\alpha,\lambda,\mu) & \dot{y}_2(t,\alpha,\lambda,\mu) \end{bmatrix},$$

$$Y(0,\alpha,\lambda,\mu) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

be the fundamental matrix of (4.3). Note that

$$Y(t,\alpha,0,0) = \begin{bmatrix} q(t) & r(t) \\ \dot{q}(t) & \dot{r}(t) \end{bmatrix}$$

where q and r are given by (2.7) and (2.9). The characteristic multipliers of the linearized equation are the eigenvalues of $Y(k,\alpha,\lambda,\mu)$. Therefore, the characteristic multipliers satisfy

$$\sigma^2 - A(\alpha,\lambda,\mu)\sigma + D(\alpha,\lambda,\mu) = 0$$

where $A(\alpha,\lambda,\mu) = \operatorname{tr} Y(k,\alpha,\lambda,\mu)$, $D(\alpha,\lambda,\mu) = \det Y(k,\alpha,\lambda,\mu)$.

<u>Lemma 4.1</u>. $D(\alpha,\lambda,\mu) = \exp(-\lambda k)$.

<u>Proof</u>: $$\begin{aligned} D(\alpha,\lambda,\mu) &= \det Y(k,\alpha,\lambda,\mu) \\ &= \det Y(0,\alpha,\lambda,\mu) \cdot \exp \int_0^k \operatorname{tr} \tilde{A}(s)ds \\ &= \exp(-\lambda k). \end{aligned}$$

Lemma 4.2. If (H1) holds, then

$$(4.5) \qquad A(\alpha,\lambda,\mu) = a_1(\alpha) + a_2(\alpha)\lambda + a_3(\alpha)\mu + \text{h.o.t.}$$

where

$$a_1(\alpha) = 2,\ a_2(\alpha) = -k,\ a_3(\alpha) = -\zeta h'(\alpha),\ \zeta = \frac{\dot{q}(k)}{\ddot{p}(0)^2}$$

$$\text{h.o.t.} = O(|\lambda|^2 + |\mu|^2), \quad \text{as} \quad \lambda,\mu \to 0$$

$\dot{q}(k)$ is given by (2.7) and

$$\eta = \left(\int_0^k \dot{p}^2\, dt\right)^{-1}, \quad h'(\alpha) = \eta \int_0^k \ddot{p}(t) f(t-\alpha)\,dt.$$

Proof: Since both y_1 and y_2 are solutions of (4.4) and $\phi(t,\alpha,\lambda,\mu)$ is twice continuously differentiable with respect to α,λ,μ, $A(\alpha,\lambda,\mu) = y_1(k,\alpha,\lambda,\mu) + \dot{y}_2(k,\alpha,\lambda,\mu)$ is twice continuous differentiable with respect to α,λ,μ. Hence $A(\alpha,\lambda,\mu)$ has Taylor series expansion as in (4.5) with λ,μ in a neighborhood of $\lambda = \mu = 0$.

$$\begin{aligned} a_1(\alpha) = A(\alpha,0,0) &= y_1(k,\alpha,0,0) + \dot{y}_2(k,\alpha,0,0) \\ &= q(k) + \dot{r}(k) = 1 + 1 \\ &= 2. \end{aligned}$$

Let $b_1(t) = \frac{\partial\phi}{\partial\lambda}(t,\alpha,0,0)$ then $b_1(t)$ is a solution of the problem

$$\begin{cases} \ddot{z} + g'(p)z = 0 \\ z(\alpha) = \dot{z}(\alpha) = 0 \end{cases}$$

hence $b_1(t) \equiv 0$.

Let $b_2(t) = \frac{\partial}{\partial\mu}\phi(t,\alpha,0,0)$ then $b_2(t)$ satisfies

$$\begin{cases} \ddot{z} + g'(p)z = f(t-\alpha) - h(\alpha)\dot{p} \\ z(\alpha)\dot{p}(\alpha) + \dot{z}(\alpha)\ddot{p}(\alpha) = 0 \end{cases}$$

where $h(\alpha)$ is given by (2.18). Let P be the projection operator as in (2.12), since $P(f(t-\alpha) - h(\alpha)\dot{p}) = 0$, it follows from Lemma 2.2, that $b_2(t)$ is a k-periodic solution.

Let $b_3(t) = \frac{\partial}{\partial\lambda} y_1(t,\alpha,0,0)$, then b_3 is a solution of

$$\begin{cases} \ddot{z} + g'(p)z = -\dot{q} \\ z(0) = \dot{z}(0) = 0 \end{cases}$$

The variation of constants formula implies

$$b_3(t) = q(t)\int_0^t r(s)\dot{q}(s)ds - r(t)\int_0^t q(s)\dot{q}(s)ds$$

Let $b_4(t) = \frac{\partial}{\partial\lambda} y_2(t,\alpha,0,0)$ then b_4 satisfies

$$\begin{cases} \ddot{z} + q'(p)z = -\dot{r} \\ z(0) = \dot{z}(0) = 0 \end{cases}$$

Since $\int_0^k \dot{p}(-\dot{r})ds = 0$, b_4 is a k-periodic solution.

Again by the variation of constants formula

$$b_4(t) = \dot{q}(t)\int_0^t r(s)\dot{r}(s)ds - \dot{r}(t)\int_0^t q(s)\dot{r}(s)ds$$

Therefore

$$\begin{aligned} a_2(\alpha) &= \frac{\partial A}{\partial\lambda}(\alpha,0,0) \\ &= \frac{\partial y_1}{\partial\lambda}(k,\alpha,0,0) + \frac{\partial \dot{y}_2}{\partial\lambda}(k,\alpha,0,0) \end{aligned}$$

$$= b_3(k) + \dot{b}_4(k)$$

$$= \int_0^k r\dot{q}ds - \int_0^k q\dot{r}ds$$

$$= - \int_0^k (q\dot{r} - r\dot{q})ds$$

$$= -k$$

Let $b_5(t) = \frac{\partial y_1}{\partial \mu}(t,\alpha,0,0)$, then b_5 is a solution of

$$\begin{cases} \ddot{z} + g'(p)z = -g''(p)b_2q \\ z(0) = \dot{z}(o) = 0 \end{cases}$$

Applying the variation of constants formula, we have

$$b_5(t) = q(t) \int_0^t rg''(p)b_2qds - r(t) \int_0^t qg''(p)b_2qds$$

Let $b_6(t) = \frac{\partial y_2}{\partial \mu}(t,\alpha,0,0)$, then b_6 satisfies

$$\begin{cases} \ddot{z} + g'(p)z = -g''(p)b_2r \\ z(0) = \dot{z}(0) = 0 \end{cases}$$

by the variation of constants formula

$$b_6(t) = \dot{q}(t) \int_0^t rg''(p)b_2rds - \dot{r}(t) \int_0^t qg''(p)b_2rds$$

Therefore

$$\begin{aligned} a_3(\alpha) &= \frac{\partial A}{\partial \mu}(\alpha,0,0) \\ &= \frac{\partial y_1}{\partial \mu}(k,\alpha,0,0) + \frac{\partial \dot{y}_2}{\partial \mu}(k,\alpha,0,0) \end{aligned}$$

$$= b_5(k) + \dot{b}_6(k)$$

$$= \int_0^k rg''(p)b_2 q\,ds + \dot{q}(k) \int_0^k rg''(p)b_2 r\,ds$$

$$- \int_0^k qg''(p)b_2 r\,ds$$

$$= \dot{q}(k) \int_0^k rg''(p)b_2 r\,ds$$

$$= \frac{\dot{q}(k)}{\ddot{p}(0)^2} \int_0^k g''(p)\dot{p}^2 b_2\,ds$$

$$= \zeta \cdot \eta[g'(p)\dot{p}b_2 \Big|_0^k - \int_0^k g'(p)\ddot{p}b_2\,ds - \int_0^k g'(p)\dot{p}\dot{b}_2\,ds]$$

$$= -\zeta \cdot \eta[\int_0^k g'(p)\ddot{p}b_2\,ds + \int_0^k g'(p)\dot{p}\dot{b}_2\,ds]$$

$$= -\zeta \cdot \eta[\int_0^k g'(p)\ddot{p}b_2\,ds + g(p)\dot{b}_2 \Big|_0^k - \int_0^k g(p)\ddot{b}_2\,ds]$$

$$= -\zeta \cdot \eta[\int_0^k g'(p)\ddot{p}b_2\,ds + \int_0^k g(p)[g'(p)b_2 - f_\alpha$$

$$+ h(\alpha)\dot{p}]ds$$

$$= -\zeta \cdot \eta[\int_0^k g'(p)b_2[\ddot{p} + g(p)]ds$$

$$- \int_0^k g(p)f_\alpha\,ds + \int_0^k h(\alpha)g(p)\dot{p}\,ds]$$

$$= -\zeta \cdot \eta[\int_0^k \ddot{p}f_\alpha\,ds - h(\alpha) \int_0^k \ddot{p}\dot{p}\,ds]$$

$$= -\zeta \cdot \eta[\int_0^k \ddot{p}f_\alpha\,ds]$$

$$= -\zeta \cdot h'(\alpha)$$

Since

$$h(\alpha) = \eta \int_0^k \dot{p}(t)f(t-\alpha)dt$$

$$= \eta \int_{-\alpha}^{k-\alpha} \dot{p}(s+\alpha)f(s)ds$$

it follows that

$$h'(\alpha) = \eta\ \dot{p}(k)f(k-\alpha) - \eta\ \dot{p}(0)f(-\alpha) + \int_{-\alpha}^{k-\alpha} \ddot{p}(s+\alpha)f(s)ds$$

$$= \int_{-\alpha}^{k-\alpha} \ddot{p}(s+\alpha)f(s)ds$$

$$= \int_0^k \ddot{p}(t)f(t-\alpha)dt$$

$$= \int_0^k \ddot{p}(t)f_\alpha(t)dt$$

Q.E.D.

<u>Lemma 4.3</u>. If $h'(\alpha_0) = 0$, then for every small λ_0, μ_0 that satisfy the bifurcation equation, we have that 1 is a characteristic multiplier of the linearized equation of $\phi(t,\alpha_0,\lambda_0,\mu_0)$.

<u>Proof</u>: Suppose $\phi(t,\alpha,\lambda,\mu) = \phi(t,\alpha,\lambda,\mu,x,y)$ and $0 \leq \alpha < 1$, where (x,y) is the initial condition of $\phi(t,\alpha,\lambda,\mu)$, and $\phi(t,\alpha,\lambda,\mu)$ is a solution of (4.1).

Let $w_1(t) = \frac{\partial\phi}{\partial x}(t,\alpha_0,\lambda_0,\mu_0,x_0,y_0)$, then $w_1(t)$ is the solution of

$$\begin{cases} \ddot{x} + g'(\phi)x + \lambda\dot{x} = 0 \\ x(0) = 1, \quad \dot{x}(0) = 0 \end{cases}$$

Let $w_2(t) = \frac{\partial\phi}{\partial y}(t,\alpha_0,\lambda_0,\mu_0,x_0,y_0)$, then $w_2(t)$ satisfies

$$\begin{cases} \ddot{x} + g'(\phi)x + \lambda\dot{x} = 0 \\ x(0) = 1, \quad \dot{x}(0) = 1 \end{cases}$$

Let

$$W(t) = \begin{bmatrix} w_1(t) & w_2(t) \\ \dot{w}_1(t) & \dot{w}_2(t) \end{bmatrix}$$

then $W(t)$ is the fundamental matrix of

$$\ddot{x} + \lambda\dot{x} + g'(\phi)x = 0$$

If none of the characteristic multipliers of ϕ is one, then

$$\det(W(k) - I) \neq 0$$

Let $H(x,y,\alpha,\lambda,\mu) = (\phi(k,\alpha,\lambda,\mu,x,y) - x,$ $\phi(k,\alpha,\lambda,\mu,x,y,) - y)$. Since $H(x_0,y_0,\alpha_0,\lambda_0,\mu_0) = 0$ and

$$\det\left(\frac{\partial H}{\partial(x,y)}(x_0,y_0,\alpha_0,\lambda_0,\mu_0)\right) = \det(W(k) - I) \neq 0 ,$$

it follows from the Implicit function theorem that there exists $\delta > 0$ and three unique solutions $\lambda^*(x,y)$, $\mu^*(x,y)$, and $\alpha^*(x,y)$, such that $H(x,y, \alpha^*(x,y), \lambda^*(x,y), \mu^*(x,y)) = 0$ for $|x - x_0| < \delta$, $|y - y_0| < \delta$. Therefore, for (α,λ,μ) near $(\alpha_0,\lambda_0,\mu_0)$, there is a unique k-periodic solution of (2.1), which contradicts the result we obtained above which says that near $(\alpha_0,\lambda_0,\mu_0)$ either there are at least two solutions or no solutions.

Therefore, 1 must be the characteristic multiplier of the linearized equation. Q.E.D.

Without loss of generality, assume $\dot{q}(k) > 0$, then $\zeta > 0$. So we have the following theorem.

<u>Theorem 4.4</u>. Let λ and μ be small. Let $\phi(t,\alpha,\lambda,\mu)$ be the k-periodic solution of (2.1) from Theorem (3.1) and $|\sigma_1| \leq |\sigma_2|$ be its characteristic multipliers. We have,

(I) If $h'(\alpha)\mu > 0$, then

(i) if $\lambda > 0$, then either $0 < \sigma_1 < \sigma_2 < 1$ (stable node) or $|\sigma_1| = |\sigma_2| < 1$ (stable focus)

(ii) if $\lambda < 0$, then either $1 < \sigma_1 < \sigma_2$ (unstable node) or $|\sigma_1| = |\sigma_2| > 1$ (unstable focus)

(iii) if $\lambda = 0$, then both characteristic multipliers are complex and simple and have modulus 1.

(II) If $h'(\alpha)\mu < 0$, then $0 < \sigma_1 < 1 < \sigma_2$ (saddle)

(III) If $h'(\alpha) = 0$, then the characteristic multipliers are 1 and $e^{-\lambda k}$.

<u>Proof</u>: Let σ_1, σ_2 be the characteristic multipliers of the linearized equation, then by Lemma 4.1 and 4.2, they are the solutions of the following equation.

$$\sigma^2 - A(\alpha,\lambda,\mu)\sigma + \exp(-\lambda k) = 0.$$

Therefore $\sigma_1, \sigma_2 = \dfrac{A(\alpha,\lambda,\mu) \pm \sqrt{(A(\alpha,\lambda,\mu))^2 - 4\cdot\exp(-\lambda k)}}{2}$

Case I. If $h'(\alpha_0)\mu > 0$, then there exists a $\delta(\alpha_0) > 0$, such that if $|\lambda|, |\mu| < \delta(\alpha_0)$, we have either $2\exp(\frac{-\lambda k}{2}) < A(\alpha,\lambda,\mu) < 1 + \exp(-\lambda k)$ or $A(\alpha,\lambda,\mu) < 2\exp(\frac{-\lambda k}{2})$ for $|\alpha - \alpha_0| < \epsilon(\alpha_0)$. If $2\exp(\frac{-\lambda k}{2}) < A(\alpha,\lambda,\mu) < 1 + \exp(-\lambda k)$, then $\sigma_1 < \sigma_2 < 1$ for $\lambda > 0$ and $1 < \sigma_1 < \sigma_2$ for $\lambda < 0$. If $A(\alpha,\lambda,\mu) < 2\exp(\frac{-\lambda k}{2})$, then σ_1, σ_2 are complex conjugate with modulus greater than one or less than one according to $\lambda < 0$ or $\lambda > 0$. The above argument holds for $|\alpha - \alpha_0| < \epsilon(\alpha_0)$.

Case II. If $h'(\alpha_0)\mu < 0$, then there exists a $\delta(\alpha_0) > 0$ such that if $|\lambda|, |\mu| < \delta(\alpha_0)$ we have $A(\alpha,\lambda,\mu) > 1 + \exp(-\lambda k)$ for $|\alpha - \alpha_0| < \epsilon(\alpha_0)$. Then $\sigma_1 < 1 < \sigma_2$.

Case III. If $h'(\alpha) = 0$, Lemma 4.3 shows that $\sigma_1, \sigma_2 = 1, \quad \exp(-\lambda k)$.

The way to find a neighborhood V_0 of $\lambda = \mu = 0$ such that the stability arguments hold uniformly is similar to the proof of Theorem 3.1. It follows from Remark 3.3. that for (α,λ,μ) in the region of existence, λ can be written as a function of μ,α, say $\lambda^*(\mu,\alpha)$. Let

$$H(\alpha,\mu) = A(\alpha,\lambda^*(\alpha,\mu),\mu) - (1 + \exp(-\lambda^*(\alpha,\mu)k))$$

$$= -\zeta\ h'(\alpha)\mu + O(|\mu|^2)$$

$$= \mu\cdot(-\zeta\ h'(\alpha) + O(|\mu|)).$$

Let $G(\alpha,\mu) = -\zeta\ h'(\alpha) + O(|\mu|)$. For those α_j's in Hypothesis (H2) we have $G(\alpha_j,\mu) = 0$. Since

$$\frac{\partial G}{\partial \alpha}(\alpha,\mu) = -\zeta\ h''(\alpha) + O(|\mu|)$$

and $h''(\alpha_j) \neq 0$. Choose $\epsilon(\alpha_j) > 0$, such that $h''(\alpha)$ is bounded away from zero for $|\alpha - \alpha_j| < \epsilon(\alpha_j)$, then there exists a $\delta(\alpha_j) > 0$ such that if $|\mu| < \delta(\alpha_j)$ then

$$\frac{\partial G}{\partial \alpha}(\alpha,\mu) \neq 0,$$

in particular

$$\frac{\partial G}{\partial \alpha}(\alpha_j,\mu) \neq 0.$$

Which shows that $G(\alpha,\mu)$ changes sign as α varies from one side of α_j to the other side of α_j. That is $A(\alpha,\lambda,\mu) < 1 + \exp(-\lambda k)$ (=> node) on one side of α_j and $A(\alpha,\lambda,\mu) > 1 + \exp(-\lambda k)$ (=> saddle) on the other side of α_j. Let

$$U = [0,1] \sim \bigcup_{j=1}^{N} \{\alpha;\ |\alpha - \alpha_j| < \epsilon(\alpha_j)\}$$

then U is compact and $h'(\alpha) \neq 0$ in U. Apply the same argument as in Theorem 3.1. There exists $\delta_0 > 0$ such that the stability result holds in

$$V_0 = \{(\lambda,\mu)\ ;\ |\lambda|,|\mu| < \delta_0\}.$$

Q.E.D.

Remark 4.5. By continuity of the eigenvalues of the matrix $Y(k,\alpha,\lambda,\mu)$, we can see that when $h'(\alpha)$ is near 0, we have a node and when $h(\alpha)$ is near 0 we have a focus.

To illustrate the theorem, consider the following example.

Example: Suppose $h(\alpha)$ has one maximum and one minimum on $[0,1]$ which occurs at α_M and α_m respectively. Since $h(\alpha)$ has period one, that identify 1 with 0. It follows from Theorem 3.1, there are two curves $\lambda = C^*(\mu)$, $\lambda = C_*(\mu)$ which are respectively tangent to $\lambda = h(\alpha_M)\mu$, $\lambda = h(\alpha_m)\mu$ at $\lambda = \mu = 0$, and which divide a neighborhood of $\lambda = \mu = 0$ into two disjoint open sets S_1 and S_2 (see Fig. 1), such that (2.1) has two solutions if $(\lambda,\mu) \in S_1$, no solution if $(\lambda,\mu) \in S_2$ and one solution if (λ,μ) is on either curve

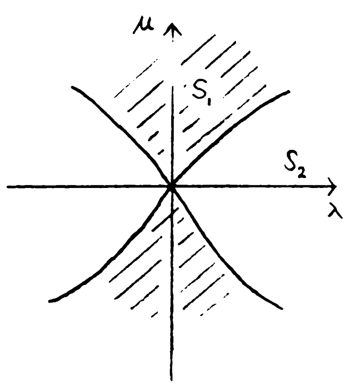


Fig. 1

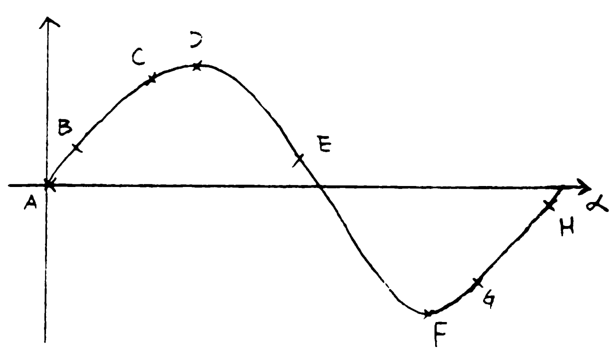
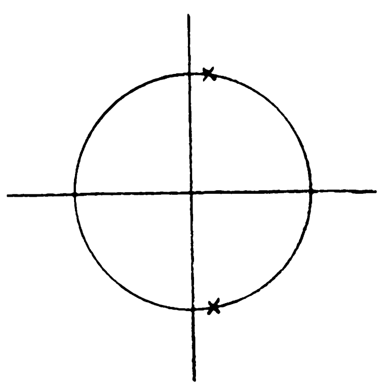


Fig. 2

Fix $\mu > 0$, let λ vary from greater than $C^*(\mu)$ to less than $C_*(\mu)$ then the number of solutions of (2.1) varies from 0,1 then 2 and back to 1 then 0. Let $y = \beta$ (we used $\lambda = \beta\mu$ in section 2.3) be a horizontal line in the parameter plane, as β varies from greater than $h(\alpha_M)$ to less than $h(\alpha_m)$, the number of intersections of the line $y = \beta$ and $h(\alpha)$ changes again from 0 to 1 then 2 and back to 1 then 0. We can see how the two solutions of (2.1) change by looking at when β changes from $h(\alpha_M)$ to less than $h(\alpha_m)$.

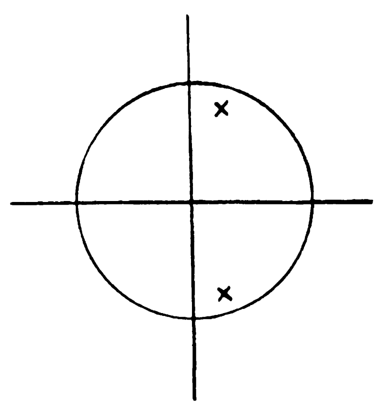
To see how the characteristic multipliers of the linearized equation move when α moves along $h(\alpha)$. Let $|\sigma_1| \leq |\sigma_2|$ be the characteristic multipliers and label some points on $h(\alpha)$, see Fig. 2. Then we obtain the corresponding σ_1, σ_2 situated near the unit circle in the complex plane (see Fig. 3).

(A)



$|\sigma_1| = |\sigma_2| = 1$

(B)



$|\sigma_1| = |\sigma_2| < 1$

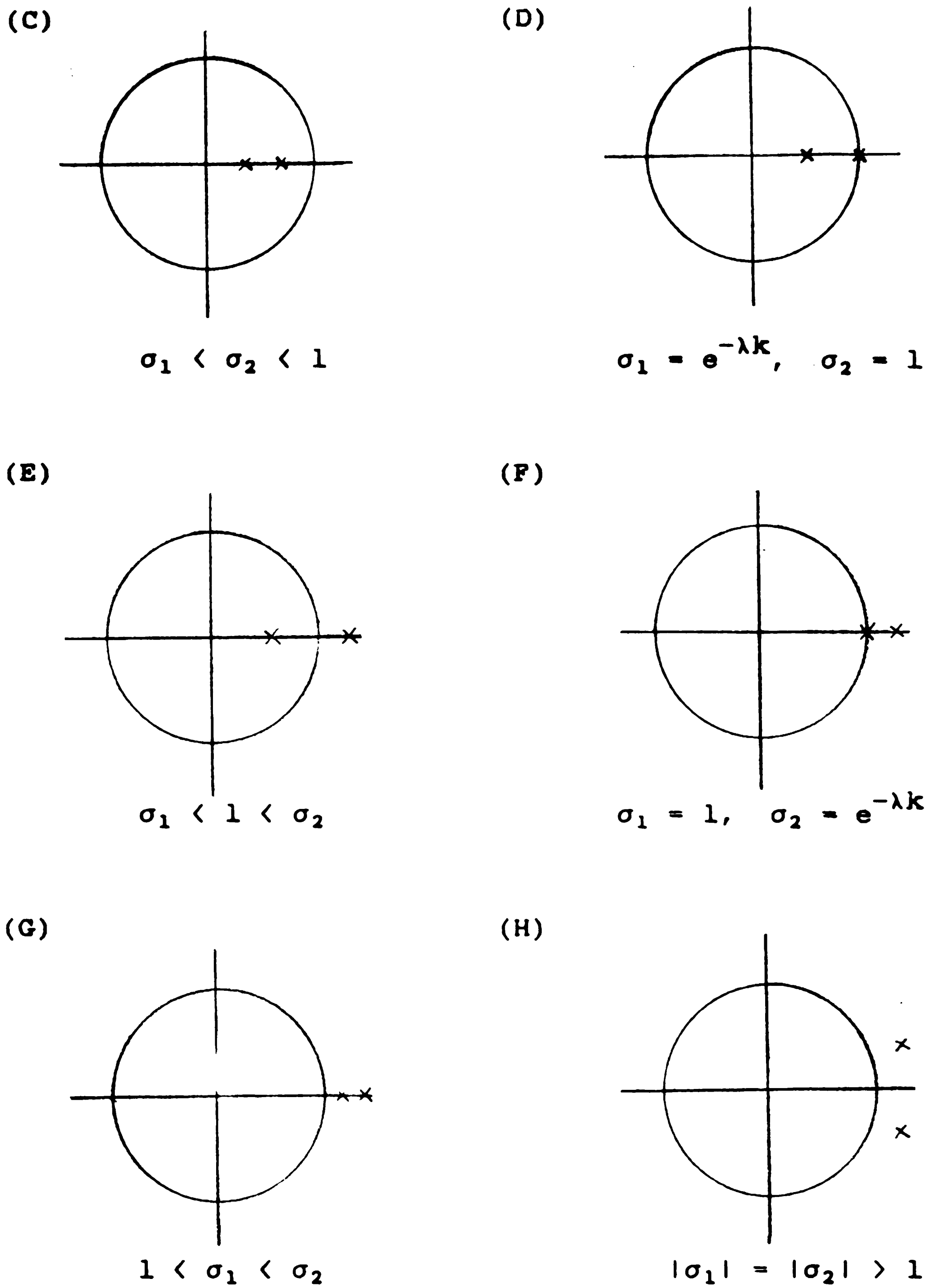


Fig. 3

<u>Remark 4.6</u>. Note that the above results hold only when λ, μ are small. If λ, μ are not small, then some

interesting phenomena might occur which will be illustrated later by numerical experimentation.

Remark 4.7. We have analyzed the stability of the solution of $\ddot{x} + g(x) = -\lambda\dot{x} + \mu f_\alpha(t)$. To discuss the stability of the solution of $\ddot{x} + g(x) = -\lambda\dot{x} + \mu f(t)$, let's consider the following two systems

(1) $\dot{x} = A(t)x$

(2) $\dot{y} = A(t+\alpha)y$

where $A(t)$ is periodic with least period k. Let $X(t,s)$ be the matrix solution of (1) with $X(0,0) = I$, and let $Y(t) = X(t+\alpha,0)$. Then $Y(t)$ is a matrix solution of (2) with $Y(0) = X(\alpha,0)$. Define $Z(t) = Y(t)X(0,\alpha)$. Then $Z(t)$ is a matrix solution of (2) with $Z(0) = I$.

First, we claim that $X(k+\alpha,k) = X(\alpha,0)$. Let $W(t,s) = X(t+k,s+k)$. Then $W(0,0) = I$ and $\dot{W} = A(t+k)W = A(t)W$. Therefore $W(t,s) = X(t,s)$ for any t,s, in particular $X(\alpha+k,k) = X(\alpha,0)$.

$$
\begin{aligned}
\text{Since}\quad Z(T) &= Y(k)X(0,\alpha) \\
&= X(k+\alpha,0)X(0,\alpha) \\
&= X(k+\alpha,k)X(k,0)X(0,\alpha) \\
&= X(\alpha,0)X(k,0)X(0,\alpha)
\end{aligned}
$$

and $X(\alpha,0) = X(0,\alpha)^{-1}$, it follows that $Z(k)$ is similar to $X(k,0)$. In particular they have the same eigenvalues. Thus the stability result of (1) can be applied to (2).

SECTION 5. UNIFORMITY

By Theorem 3.1, for each integer $k > 1$, we obtain a neighborhood V_k of $\lambda = \mu = 0$ such that if $(\lambda, \mu) \in V_k$, there exists at least $2k$ k-periodic solutions of

$$(5.1) \qquad \ddot{x} + g(x) = -\lambda\dot{x} + \mu f(t)$$

where $g(x)$ is defined as before and $f(t)$ has least period 1. It is interesting to know whether there exists a neighborhood $V \subseteq \cap V_k$ such that the existence theorem holds. In other words, is there a neighborhood V such that if $(\lambda, \mu) \in V$, there exists k-periodic solution of (5.1) for every k? If the operator G as defined in Lemma 2.3 is uniformly bounded for every k, then such neighborhood exists. We will show that sometimes this is true (Theorem 5.4) and in some other cases, it is not true (Theorem 5.1). To discuss this, we first consider the following equation

$$(5.2) \qquad \ddot{x} = \begin{cases} x, & x \leq 1 \\ x - 2, & x \geq 1 \end{cases}$$

Let $\tilde{G}_k$ be the operator as in Lemma 2.2 corresponds to $p_k(t)$, where $p_k(t)$ is the k-periodic solution of (5.2).

Theorem 5.1. The Green's function G_k for equation (5.2) satisfies

$$\|G_k\| \geq \cosh(\tfrac{k}{4}) - 1 \tag{5.3}$$

for all $k = 1,2,\cdots$.

Proof: Let $\phi(t)$ be any k-periodic function, it follows from Lemma 2.2 that $\tilde{G}_k\phi$ is the solution of

$$\begin{cases} \ddot{x} - x = \phi(t) \\ x \quad \text{is} \quad \text{k-periodic} \\ \dot{x}(0) = 0 \end{cases}$$

Since $\ddot{x} - x = 0$ has $\cosh t$ and $\sinh t$ as linearly independent solutions, we obtain (see (2.7) and (2.9)),

$$r(t) = \begin{cases} \sinh t & 0 \leq t \leq \frac{k}{4} \\ -\sinh(t - \frac{k}{2}) & \frac{k}{4} \leq t \leq \frac{3k}{4} \\ \sinh(t - k) & \frac{3k}{4} \leq t \leq k \end{cases}$$

and

$$q(t) = \begin{cases} \cosh t & & 0 \leq t \leq \frac{k}{4} \\ -\cosh(t - \frac{k}{2}) - & c_1 \ \sinh(t - \frac{k}{2}) & \frac{k}{4} \leq t \leq \frac{3k}{4} \\ \cosh(t - k) + & c_2 \ \sinh(t - k) & \frac{3k}{4} \leq t \leq k \end{cases}$$

where

$$c_1 = \frac{2\cosh\frac{k}{4}}{\sinh\frac{k}{4}}, \quad c_2 = \frac{4\cosh\frac{k}{4}}{\sinh\frac{k}{4}}$$

Take $\phi = 1$ and $0 \le t \le \frac{k}{4}$. Then

$$\tilde{G}\phi(t) = \frac{\cosh t}{-c_2}\Big(\int_0^{\frac{k}{4}} \cosh ds$$

$$+ \int_{\frac{k}{4}}^{\frac{3k}{4}} (-\cosh(s - \tfrac{k}{2}) - c_1 \sinh(s - \tfrac{k}{4}))ds$$

$$+ \int_{\frac{3k}{4}}^{k} (\cosh(s - k) + c_2 \sinh(s - k))ds\Big)$$

$$- \cosh t \int_0^t (\sinh s\, ds + \sinh t \int_0^t \cosh s\, ds)$$

$$= \frac{\cosh t}{-c_2}(\sinh\tfrac{k}{4} - 0 - \sinh(\tfrac{k}{4}) + \sinh(-\tfrac{k}{4}) - c_1 \cosh(\tfrac{k}{4})$$

$$+ c_1 \cosh(-\tfrac{k}{4}) + 0 - \sinh(-\tfrac{k}{4}) + c_2 - c_2 \cosh\tfrac{k}{4})$$

$$- \cosh t(\cosh t - 1) + \sinh t \cdot \sinh t$$

$$= -\cosh t(1 - \cosh\tfrac{k}{4}) + \cosh t - 1$$

$$= \cosh t \cosh\tfrac{k}{4} - 1$$

Set $t = 0$, then $\|\tilde{G}_K\phi\| \ge \cosh\frac{k}{4} - 1$, therefore

$$\|\tilde{G}_k\| \geq \cosh \frac{k}{4} - 1$$

Next, it follows from Lemma 2.3 that

$$G_k\phi = \tilde{G}_k\phi + a_k(\phi)\dot{p}_k$$

where

$$a_k(\phi) = -\eta \int_0^k \dot{p}_k \tilde{G}\phi \; dt \quad \text{and} \quad \eta = (\int_0^k p_k^2 \; dt)^{-1} .$$

Set $t = 0$, we obtain $G_k\phi(0) = \tilde{G}_k\phi(0) + a_k(\phi)\dot{p}_k(0) = \tilde{G}_k\phi(0)$, therefore G_k satisfies (5.3). Q.E.D.

On the otherhand, consider the following equation

$$\text{(5.4)} \qquad \ddot{x} = \begin{cases} -x & x \leq 1 \\ x - 2 & x \geq 1 \end{cases}$$

For equation (5.4), the equilibrium point $(0,0)$ is a saddle and the other equilibrium point $(2,0)$ is a center. Also, the global stable and unstable manifolds of $(0,0)$ coincide. That is equation (5.4) has a homoclinic orbit which crosses the x-axis at $(0,0)$ and $(4,0)$.

Let $p_k(t)$ be the k-periodic solution of (5.4) with $p_k(0) = b_k$ and $p_k(k/2) = c_k$. Since equation (5.4) admits the first integral

$$E = \tfrac{1}{2}\dot{x}^2 + \int^{x} g(s)ds$$

$$= \begin{cases} \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}x^2 & \text{if} \quad x \le 1 \\ \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}x^2 - 2x & \text{if} \quad x \ge 1 \end{cases}$$

Therefore the period of the periodic orbit $p_k(t)$ is given by the formula

$$T(b_k) = 2\left(\int_{b_k}^{1} \frac{dx}{\sqrt{x^2 - b_k^2}} + \int_{1}^{c_k} \frac{dx}{\sqrt{-(x-2)^2 + 4 - b_k^2}}\right)$$

where $c_k = 2 + \sqrt{4 - b_k^2}$, therefore

$$(5.5) \quad T(b_k) = 2\ln\frac{1 + \sqrt{1 - b_k^2}}{b_k} + \pi - 2\sin^{-1}\left(\frac{-1}{\sqrt{4 - b_k^2}}\right)$$

$$= 2\tau_k + 2\sigma_k$$

$$= k$$

<u>Lemma 5.2</u>. $a = \dot{q}(k) \to 2$, as $k \to \infty$.

Proof: It follows from Proposition 2.1 that $q(k) = T'(b_k)\cdot g(b_k)$. Hence

$$\dot{q}(k) = \left(\frac{-2b_k}{\sqrt{1 - b_k^2}\,(1 + \sqrt{1 - b_k^2})} - \frac{2}{b_k} - \frac{4b_k}{\sqrt{4 - b_k^2}\,\sqrt{3 - b_k^2}}\right)(-b_k)$$

$$= b_k^2\left(\frac{2}{\sqrt{1 - b_k^2}\,(1 + \sqrt{1 - b_k^2})} + \frac{4}{\sqrt{4 - b_k^2}\,\sqrt{3 - b_k^2}}\right) + 2 .$$

As $b_k \to 0$, $\dot{q}(k) \to 2$. Q.E.D.

Note that $p_k(t)$ is given precisely by the following

(5.6)

$$p_k(t) = \begin{cases} b_k \cosh t & 0 \le t \le \tau \\ \dfrac{\sinh \tau}{\cosh \tau \sin \sigma}\cos(t - \frac{k}{2}) + 2 & \tau \le t \le k - \tau = \tau + 2\sigma \\ b_k \cosh(t - k) & k - \tau \le t \le k \end{cases}$$

where, $b_k \cosh \tau = 1$.

Lemma 5.3. $\tilde{G}_k\phi$ is uniformly bounded in k, where $\phi \in \tilde{P} = \{\phi$ is a k-periodic characteristic function and is symmetric with respect to $k/2\}$.

Proof: For simplicity, we will drop the subscript k. First choose k large enough, such that $3\pi/4 < \sigma < 5\pi/6$, where σ is given by (5.5). Let $r(t)$, $q(t)$ be the solutions of the linearlized equation (see also (2.7) and (2.9)):

$$\begin{cases} \ddot{x} - x = 0 & 0 \le t \le \tau \\ \ddot{x} + x = 0 & \tau \le t \le k - \tau \\ \ddot{x} - x = 0 & k - \tau \le t \le k \end{cases}$$

It follows from (5.6) and $p(t) = br(t)$ that

$$(5.7) \qquad r(t) = \begin{cases} \sinh t & 0 \le t \le \tau \\ F(t) & \tau \le t \le k - \tau \\ \sinh(t-k) & k - \tau \le t \le k \end{cases}$$

where $F(t) = \frac{\sinh \tau}{-\sin \sigma} \sin(t - \frac{k}{2})$ and

$$(5.8) \quad q(t) = \begin{cases} \cosh t & 0 \le t \le \tau \\ Q(t) & \tau \le t \le k - \tau \\ \cosh(t-k) + a \sin h(t-k) & k - \tau \le t \le k \end{cases}$$

where $a = \dot{q}(k)$ and

$$Q(t) = \frac{2 \cosh \tau - a \sinh \tau}{2 \cos \sigma} \cos(t - \frac{k}{2}) - \frac{a \sinh \tau}{\sin \sigma} \sin(t - \frac{k}{2})$$

It follows from Lemma 2.2, that

$$(5.9) \qquad \tilde{G}\phi(t) = \frac{q(t)}{-a} \int_0^k q(s)\phi(s)ds - q(t) \int_0^t r(s)\phi(s)ds$$
$$+ r(t) \int_0^t q(s)\phi(s)ds \ .$$

For $\phi \in \tilde{P}$, there exist constants $\beta_1, \beta_2, \beta_3$ such that

$$\phi(t) = \begin{cases} 1 & \beta_1 \le t \le \beta_2 \quad , \qquad k - \beta_2 \le t \le k - \beta_1 \\ 1 & \beta_3 \le t \le k - \beta_3 \\ 0 & \text{otherwise} \end{cases}$$

Therefore we only have to consider the following forms of ϕ

$$(5.10)\quad \phi(t) = \begin{cases} 1 & \beta_1 \le t \le \beta_2, \quad k - \beta_2 \le t \le k - \beta_1 \\ \\ 0 & \text{otherwise} \end{cases}$$

where $\beta_1 < \beta_2 \le \tau$, and

$$(5.11)\qquad \phi(t) = \begin{cases} 1 & \beta_3 \le t \le k - \beta_3 \\ \\ 0 & \text{otherwise} \end{cases}$$

where $\tau \le \beta_3 \le \frac{k}{2}$.

Substitute (5.7), (5.8) and (5.10) into (5.9), we obtain the following, for $0 \le t \le \beta_1$

$$\tilde{G}\phi(t) = \frac{\cosh t}{-a} [\int_{\beta_1}^{\beta_2} \cosh s \, ds$$

$$+ \int_{k-\beta_2}^{k-\beta_1} (\cosh(s - k) + a \sinh(s - k))ds$$

$$= \frac{\cosh t}{-a} [(2 \sinh \beta_2) - a \cosh \beta_2 - 2 \sinh \beta_1$$

$$+ \cosh \beta_1]$$

$$= \frac{\cosh t}{-a} [(2 - a)e^{\beta_2} - (a + 2)e^{-\beta_2} + (a - 2)e^{\beta_1}$$

$$+ (a + 2)e^{-\beta_1})$$

It follows from Lemma (5.2) that $a - 2 = O(e^{-2\tau})$ and $t \le \beta_1 < \beta_2 \le \tau$. Hence, there exists $M_1 > 0$ such that $|\tilde{G}\phi(t)| \le M_1$ for $0 \le t \le \beta_1$. Next, for $\beta_1 \le t \le \beta_2$, we have

$$\tilde{G}\phi(t) = \frac{\cosh t}{-a}[2\sinh\beta_2 - a\cosh\beta_2 - 2\sinh\beta_1$$

$$+ a\cosh\beta_1] - \cosh t(\cosh t - \cosh\beta_1)$$

$$+ \sinh t(\sinh t - \sinh\beta_1)$$

$$= \frac{1}{-a}[(2 - a)e^{t+\beta_2} + (a - 2)e^{t+\beta_1} - (2 + a)e^{t-\beta_2}$$

$$+ (2 - a)e^{\beta_2 - t}$$

$$+ (a - 2)e^{\beta_1 - t} - (a + 2)e^{-(t+\beta_2)}$$

$$+ (a + 2)e^{-(t+\beta_1)}]$$

$$+ (\frac{2}{-a} + 1)e^{t-\beta_1} + 2e^{\beta_1 - t} .$$

Since $a - 2 = O(e^{-2\tau})$ and $\beta_1 \le t \le \beta_2 \le \tau$, there exists $M_2 > 0$, such that $|\tilde{G}\phi(t)| \le M_2$ for $\beta_1 \le t \le \beta_2$.

Consider now the interval $\beta_2 \le t \le k - \beta_2$. We have

$$\tilde{G}\phi(t) = Q(t)\ [\frac{2}{-a}(\sinh\beta_2 - \sinh\beta_1) + (\cosh\beta_2 - \cosh\beta_1)]$$

$$- Q(t)(\cosh\beta_2 - \cosh\beta_1)$$

$$+ F(t)(\sinh\beta_2 - \sinh\beta_1)$$

$$= [(2\cosh\tau - a\sinh\tau)\ \frac{\cos(t - \frac{k}{2})}{-a\cos\sigma}$$

$$- (2\sinh\tau - a\sinh\tau)\ \frac{\sin(t - \frac{k}{2})}{\sin\sigma}]$$

$$\cdot (\sinh\beta_2 - \sinh\beta_1)$$

since $a - 2 = O(e^{-2\tau})$, $3\pi/4 \le \sigma \le 5\pi/6$ and $-\sigma \le t - k/2 \le \sigma$, there exists $M_3 \ge 0$, such that $|\tilde{G}\phi(t)| \le M_3$ for $\beta_2 \le t \le k - \beta_2$.

For $k - \beta_2 \le t \le k - \beta_1$ and $k - \beta_1 \le t \le k$, the computations are similar to the first two cases by replacing $\cosh t$ by $\cosh(t - k) + a \sinh(t - k)$ and $\sinh t$ by $\sinh(t - k)$. We obtain for some $M_4 \ge 0$

$$|\tilde{G}\phi(t)| \le M_4 , \quad k - \beta_2 \le t \le k .$$

One can see that the constants M_1, M, M_3 and M_4 can be chosen independent of k.

Now, repeat the above procedure for ϕ defined by (5.11), for $0 \le t \le \beta_3$ and $\beta_3 \ge \tau$, we have

$$\tilde{G}\phi(t) = \frac{q(t)}{-a} \left(\int_{\beta_3}^{k-\beta_3} Q(s)ds\right)$$

$$= \begin{cases} \cosh t\left(\dfrac{2 \cosh \tau - a \sinh \tau}{-2a \cos \sigma}(-2 \cos \sigma)\right) & t \le \tau \\ Q(t) \left(\dfrac{2 \cosh \tau - a \sinh \tau}{a}\right) & \tau \le t \le \beta_3 \end{cases}$$

Again, since $a - 2 = O(e^{-2\tau})$, there exists $N_1 > 0$, such that

$$|\tilde{G}\phi(t)| \le N_1 , \quad \text{for} \quad 0 \le t \le \beta_3 .$$

For $\beta_3 \le t \le k - \beta_3$, we obtain

$$\tilde{G}\phi(t) = Q(t)\left(\frac{2 \cosh \tau - a \sinh \tau}{a}\right) - Q(t) \int_{\beta_3}^{t} F(s)ds$$
$$+ F(t) \int_{\beta_3}^{t} Q(s)ds$$

$$= Q(t)\left(\frac{2\cosh\tau - a\sinh\tau}{a}\right) + \left(\frac{2\cosh\tau - a\sinh\tau}{2\cos\sigma}\right)$$

$$\left(\frac{\sinh\tau}{\sin\sigma}\right)\left[-1 + \cos\left(t - \frac{k}{2}\right)\cos\left(\beta_3 - \frac{k}{2}\right)\right.$$

$$\left. + \sin\left(t - \frac{k}{2}\right)\sin\left(\beta_3 - \frac{k}{2}\right)\right]$$

Since $a - 2 = 0(e^{-2\tau})$, $-\sigma \le t - k/2 \le \sigma$ and $3\pi/4 \le \sigma \le 5\pi/6$, there exists $N_2 > 0$, such that $|\tilde{G}\phi(t)| \le N_2$ for $\beta_3 \le t \le k - \beta_3$. For $k - \beta_3 \le t \le k$, by the similar arguments, we can choose $N_3 \ge 0$ which is independent of k such that $G\phi(t)$ is bounded by N_3.

It follows that there exists $M \ge 0$ independent of k such that

$$\max_{0\le t\le k} |\tilde{G}\phi(t)| \le M .$$

Since

$$\tilde{G}\dot{\phi}(t) = \dot{q}(t)\left[\frac{1}{-a}\int_0^k q(s)\phi(s)ds\right] - \dot{q}(t)\int_0^t r(s)\phi(s)ds$$

$$+ \dot{r}(t)\int_0^t q(s)\phi(s)ds ,$$

one can see by similar arguments that there exists $N \ge 0$ which is independent of k such that

$$\max_{0\le t\le k} |\tilde{G}\dot{\phi}(t)| \le N$$

Since

$$\ddot{\tilde{G}}\phi = -g'(p)\tilde{G}\phi + \phi$$

we have

$$\max_{0\le t\le k} |\ddot{\tilde{G}}\phi| \le \max_{0\le t\le k} |g'(p)| \max_{0\le t\le k} |\tilde{G}\phi| + 1$$

$$\le JM + 1$$

where $J = \max_{0\le t\le k} |g'(p)|$. Since $p(t)$ is uniformly bounded, therefore J can be chosen independent of k.

We have shown that

$$\|\tilde{G}\phi\| \le \max(N,M,jM + 1) = K_0$$

Q.E.D.

Theorem 5.4. The operator G_k is uniformly bounded in P_s, where G_k is defined by Lemma 2.3 and $P_s = \{\phi$ is a continuous k-periodic function and is symmetric with respect to $k/2\}$.

Proof: We first show that $\tilde{G}_k$ is uniformly bounded in P_s. For simplicity, we will drop the subscript k. For any $\phi \in P_s$, there exist $\phi_i \in \tilde{P}$, where $\tilde{P}$ is defined by Lemma 5.3, such that

$$\sum_{i=1}^{N} \alpha_i\phi_i(x) \to \phi(x) \quad \text{uniformly in } [0,k] .$$

Let $E_i = \{x : \phi_i(x) = 1\}$. Then $\phi_i(x) = \chi_{E_i}$, the characteristic function of E_i. Let $x \in E_i$ be fixed. We have

$$\tilde{G}\phi(x) - \tilde{G}\sum_{i=1}^{N} \alpha_i \phi_i(x) = \tilde{G}\phi(x) - \alpha_i \tilde{G}\phi_i(x)$$

It follows from Lemma 5.3, that

$$\|\tilde{G}\phi\| \le K_0 \|\phi\| ,$$

where K_0 is independent of k.

For any $\phi \in P_S$, since ϕ is symmetric with respect to $k/2$, $P\phi = 0$. It follows from Lemma 2.3, that

$$G\phi = \tilde{G}\phi + a\dot{p} ,$$

such that $P(\tilde{G}\phi + a\dot{p}) = 0$. Therefore

$$\begin{aligned} a(\phi) &= -P(\tilde{G}\phi) \\ &= -\eta\dot{p} \int_0^k \dot{p}(t)\tilde{G}\phi(t)dt \end{aligned}$$

where $\eta = (\int_0^k \dot{p}^2 dt)^{-1}$. Since, $\dot{p}, p, p$ are uniformly bounded and is near 0 for very long time, hence η is uniformly bounded too. Similarly

$$\int_0^k |\dot{p}(t)|dt ,$$

is bounded. Therefore

$$\|a(\phi)\| \le K_1\|\tilde{G}\phi\|$$

$$\le K_1K_0\|\phi\|$$

hence $a(\phi)$ is a uniformly bounded operator. This shows that G is a uniformly bounded operator. Q.E.D.

SECTION 6. NUMERICAL STUDY

In this section, we will give a numerical scheme to find the periodic solutions of (2.1). Let $\phi(t,x,y)$ be the solution of (2.1) with initial condition (x,y) at $t = 0$. In order for $\phi(t,x,y)$ to be a k-periodic solution of (2.1), the following equations have to be satisfied.

$$\phi(k,x,y) = x$$

$$\dot{\phi}(k,x,y) = y \ .$$

Let

$$F(x,y) = \begin{bmatrix} \phi(k,x,y) \\ \dot{\phi}(k,x,y) \end{bmatrix} \tag{6.1}$$

The problem is reduced to finding the fixed points of $F(x,y)$. We apply the Newton's method to find the zeros of

$$G(x,y) = F(x,y) - \begin{bmatrix} x \\ y \end{bmatrix} \ . \tag{6.2}$$

We obtain the following scheme:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - DG(x_n, y_n)^{-1} G(x_n, y_n) \tag{6.3}$$

$$= \begin{bmatrix} x_n \\ y_n \end{bmatrix} - (DF(x_n, y_n) - I)^{-1} G(x_n, y_n)$$

where

$$DF(x,y) = \begin{bmatrix} \frac{\partial \phi}{\partial x}(k,x,y) & \frac{\partial \phi}{\partial y}(k,x,y) \\ \frac{\partial \dot{\phi}}{\partial x}(k,x,y) & \frac{\partial \dot{\phi}}{\partial y}(k,x,y) \end{bmatrix}, \tag{6.4}$$

and $\partial\phi(t,x,y)/\partial x$, $\partial\phi(t,x,y)/\partial y$ satisfy respectively

$$\begin{cases} \ddot{x} + g'(\phi)x = 0 \\ x(0) = 1, \quad \dot{x}(0) = 1 \end{cases} \tag{6.5}$$

and

$$\begin{cases} \ddot{x} + g'(\phi)x = 0 \\ x(0) = 0, \quad \dot{x}(0) = 1 \end{cases}. \tag{6.6}$$

Let

$$Z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \\ z_5(t) \\ z_6(t) \end{bmatrix} = \begin{bmatrix} \phi(t,x,y) \\ \dot{\phi}(t,x,y) \\ \frac{\partial \phi}{\partial x}(t,x,y) \\ \frac{\partial \dot{\phi}}{\partial x}(t,x,y) \\ \frac{\partial \phi}{\partial y}(t,x,y) \\ \frac{\partial \dot{\phi}}{\partial y}(t,x,y) \end{bmatrix},$$

then it follows from (6.3), (6.4), (6.5) that

(6.7)

$$\dot{z}(t) = \begin{bmatrix} z_2(t) \\ -g(z_1(t)) - \lambda \dot{z}_2(t) + \mu f(t) \\ z_4(t) \\ -g'(z_1(t))z_3(t) - \lambda z_4(t) \\ z_5(t) \\ -g'(z_1(t))z_5(t) - \lambda z_6(t) \end{bmatrix} \quad \text{with} \quad z(0) = \begin{bmatrix} x_0 \\ y_0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Now we use Runge-Kutta method to solve the above initial value problem. We obtain

(6.8)

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \begin{bmatrix} z_3(k) - 1 & z_5(k) \\ z_4(k) & z_6(k) - 1 \end{bmatrix}^{-1} \begin{bmatrix} z_1(k) - x_n \\ z_2(k) - y_n \end{bmatrix}$$

Note that if $\phi(t,\tilde{x},\tilde{y})$ is the k-periodic solution of (2.1), then the characteristic multipliers of ϕ are the eigenvalues of the Jacobian matrix $DF(\tilde{x},\tilde{y})$. Hence, if both eigenvalues of $DF(\tilde{x},\tilde{y})$ have modulus less than 1, then the fixed point $(\tilde{x},\tilde{y})$ or the periodic solution through $(\tilde{x},\tilde{y})$ is stable and if one of the eigenvalues has modulus greater than 1 then it becomes unstable. Therefore, when we compute the fixed points, we determine the characterist multipliers simultaneously.

Our experiments will be performed on the forced pendulum problem. Let $p_k(t)$ be the k-periodic solution of

$$(6.9) \qquad \ddot{x} + \sin x = 0 .$$

Consider the perturbed system:

$$(6.10) \qquad \ddot{x} + \sin x = -\lambda\dot{x} + \mu f(t)$$

where $f(t + 1) = f(t)$. Since the initial conditions of a k-subharmonic solution of (6.10) are near $(p(\alpha),\dot{p}(\alpha))$ and $\lambda \approx h(\alpha)\mu$ for small λ and μ (see Section 2), we can choose our initial guess to be $(p(\alpha),p(\alpha))$ with $h(\alpha) = 0$ and $\lambda = 0$.

In Figures 6.1 and 6.2, we show the Poincare map under iterations for (6.10) with $f(t) = \sin 2\pi t$, $\lambda = 0$, $\mu = .2$. Note that in Figure 6.2, we easily observe the subharmonic motions. Figure 6.3 is the magnification of the square box in Figure 6.2. We observe that there are saddle connections and the stable periodic solutions are enclosed by invariant tori.

For fixed $\lambda = 0$ and $f(t) = \sin 2\pi t$, we increase μ and follow numerically the subharmonics of order 9. We do not observe bifurcations of these solutions. See Table 6.1.

In order for us to observe bifurcation phenomena for these solutions, we let $f(t) = \cos(.67t)$ and look for subharmonic solution of order 1. For this $f(t) = \cos(.67t)$, we have $\max|h(\alpha)| = .61681$. It follows from Theorem 3.1, we can only have periodic solutions if λ lies between $-.61681$ and $.61681$. Table 6.2 shows how the characteristic multiplies of the periodic solution vary as λ goes from $-.61681$ to $.61681$ and $\mu = .2$. In Figure 6.4, we fixed $\mu = .2$ and plot λ against r, where $r = \sqrt{x_0{}^2+y_0{}^2}$ and (x_0, y_0) is the initial condition of the periodic solution.

From now on we would like to consider the Poincare map $F(x,y)$ defined in (6.1). First, we will show how the stable fixed point of F loses its stability through period doubling. Again we fixed $\lambda = 0$ and vary μ. At each stage, we apply (6.7) and (6.8) to obtain the fixed point for the next μ by using the present fixed point as the initial guess. We continue this process - until one of the characteristic multipliers of the fixed point passed through -1. Let's assume it occurs at $\mu = \mu_0$, and the fixed point is $(x_{\mu 0}, y_{\mu 0})$. Let (x_μ, y_μ) be the fixed point of $F(x,y)$ for μ slightly larger than μ_0, then (x_μ, y_μ) is a unstable fixed point of $F^2(x,y)$. There are also two stable fixed points of $F^2(x,y)$ which are not the fixed point of $F(x,y)$.

We find numerically the unstable fixed points (x_μ, y_μ) by our scheme (6.7), (6.8). To find the stable fixed points, we first locate numerically the saddle connections and choose an initial guess in the interior of the connections. The method allows us to continue and to detect the bifurcation points. In Figure 6.5 we fixed $\lambda = 0$ and increase μ, one can see how the periodic solution changes from one loop to two loops then 4 loops in the sinx x-plane. This shows the period of the solution doubles and doubles again. Figure 6.6 shows the same pheonomena with graph μ against r, where $r = \sqrt{x_0{}^2+y_0{}^2}$ and (x_0, y_0) is the initial data.

Now we start with the fixed point $(x_{\mu 0}, y_{\mu 0})$ and increase λ to obtain the period-doubling curve, see Figure 6.7. That is if (λ, μ) goes through the curve with increasing μ then the stable periodic solution loses its stability and another stable periodic solution is created, with least period 2 times the original one.

μ	x_0	y_0	σ_1, σ_2
0	2.18106292	-.28080103	.99483581 ± .10149725i
1	2.18093417	-.35920429	.98968713 ± .14324583i
2	2.18036890	-.51601111	.97947406 ± .20157020i
3	2.17940921	-.67282215	.96943533 ± .24534694i
4	2.17805323	-.82964198	.95964335 ± .28121990i
5	2.17629843	-.98647512	.95016770 ± .31173919i
6	2.17414159	-1.14332606	.94107458 ± .33819909i
7	2.17157875	-1.30019926	.93242616 ± .36136054i
8	2.16860526	-1.45709914	.92428013 ± .38171485i
9	2.16521572	-1.61403012	.91668919 ± .39960094i
10	2.16140393	-1.77099662	.90970662 ± .41526461i
11	2.15603471	-1.96726143	.90187488 ± .43199732i
12	2.15248489	-2.08505389	.89769104 ± .44062544i
13	2.14736114	-2.24215356	.89273472 ± .45058263i
14	2.14178208	-2.39930658	.88850988 ± .45885745i
15	2.13573716	-2.55651749	.88503265 ± .46552894i
16	2.12921478	-2.71379091	.88231246 ± .47066412i
17	2.12220229	-2.87113151	.88035202 ± .47432089i

TABLE 6.1

λ	x_0	y_0	σ_1,	σ_2
.12336274	.28465940	1.77898547	.99789666,	.31512796
.1233624	.27953712	1.78025094	.98806257,	.31802605
.123361	.27177950	1.78213926	.97491078,	.32256313
.123330	.22472892	1.79286449	.88629559,	.35491738
.12320	.14854213	1.80758616	.70526800,	.44656136
.12315	.12855802	1.81090718	.62271744,	.50599698
.12314	.12484698	1.81149919	.59330936,	.53112713
.12313	.12121602	1.81207094	.56012761 ±	.03754078i
.123	.07952532	1.81810551	.53627383 ±	.16717376i
.12	-.36144268	1.82265190	.29791170 ±	.48558065i
.11	-1.01698498	1.63434569	-.02793731 ±	.59637841i
.10	-1.40960328	1.41440422	-.21776604 ±	.58657520i
.05	-2.30221653	.56463577	-.69586196 ±	.37612466i
.0	-2.50801521	-.00000000	-.95303355 ±	.30286473i
-.05	-2.30221653	-.56463577	-1.11214486 ±	.60113230i
-.10	-1.40960328	-1.41440422	-.55624572 ±	1.49830499i
-.11	-1.01698498	-1.63434569	-.07837704 ±	1.67311615i
-.115	-.74595905	-1.73982553	.30663149 ±	1.68704385i
-.12	-.36144268	-1.82265190	.91795078 ±	1.49621223i
-.123	.07952532	-1.81810550	1.69956093 ±	.52980762i
-.1231	.11075522	-1.81367677	1.75775532 ±	.28718137i
-.1232	.14854215	-1.80758616	2.39334066,	1.41790061
-.1233	.20103788	-1.79779243	2.66166160,	1.19404065
-.123361	.27177964	-1.78213923	3.10016907,	1.02573461
-.1233626	.28169078	-1.77972067	3.15658155,	1.00741843

TABLE 6.2

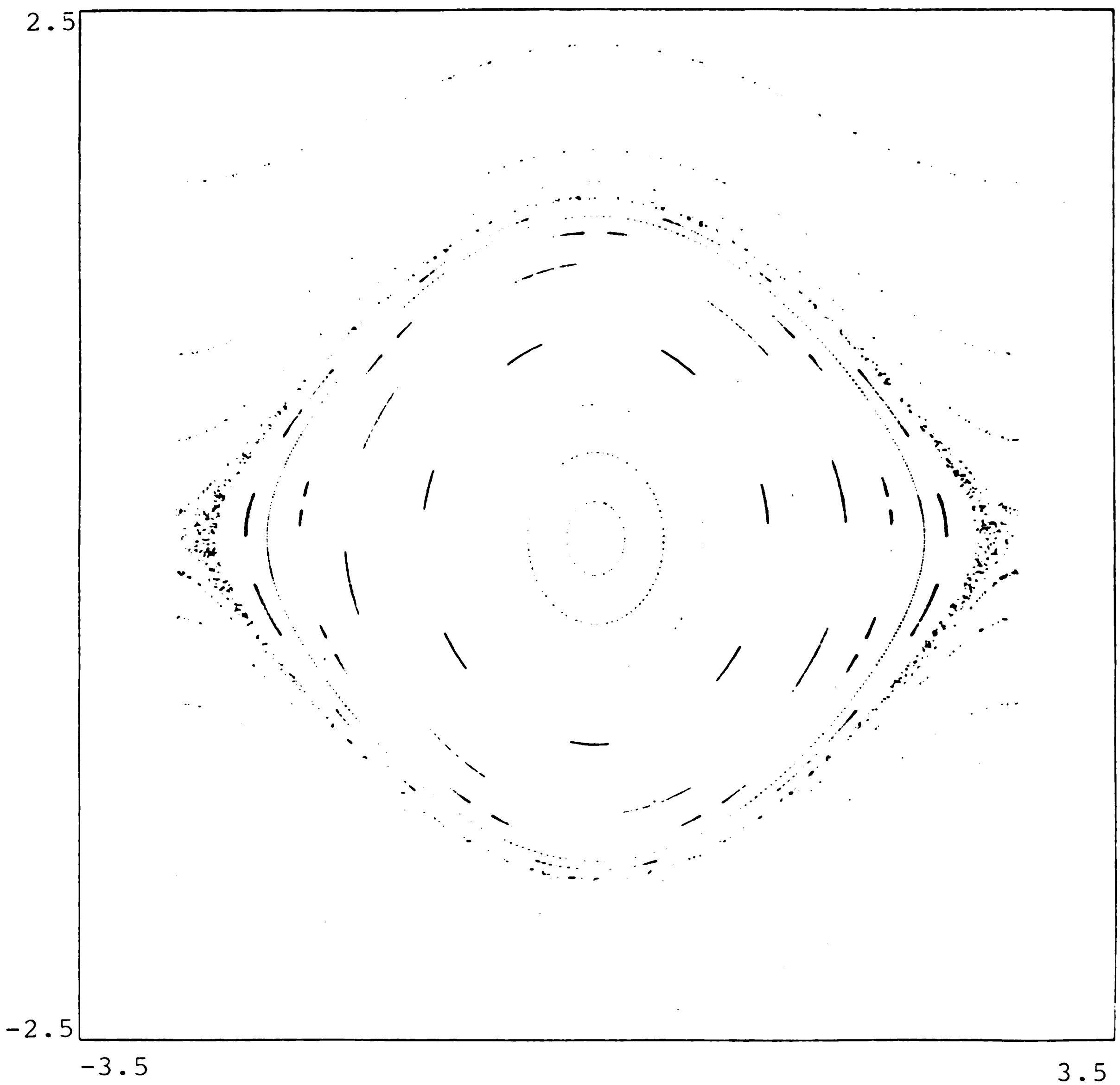
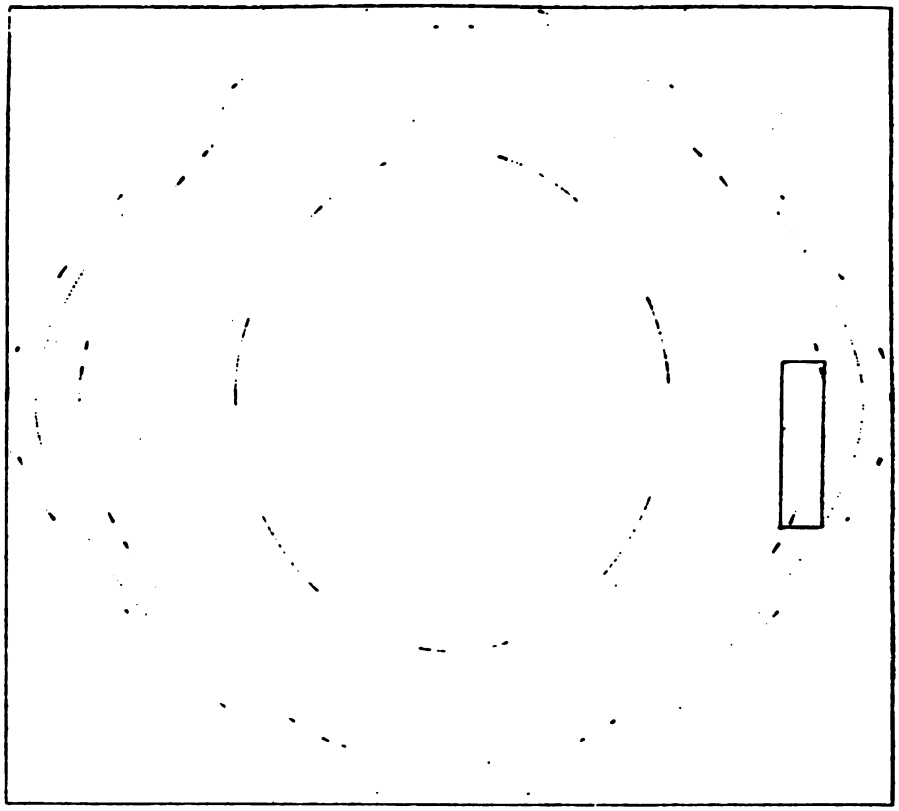


Figure 6.1



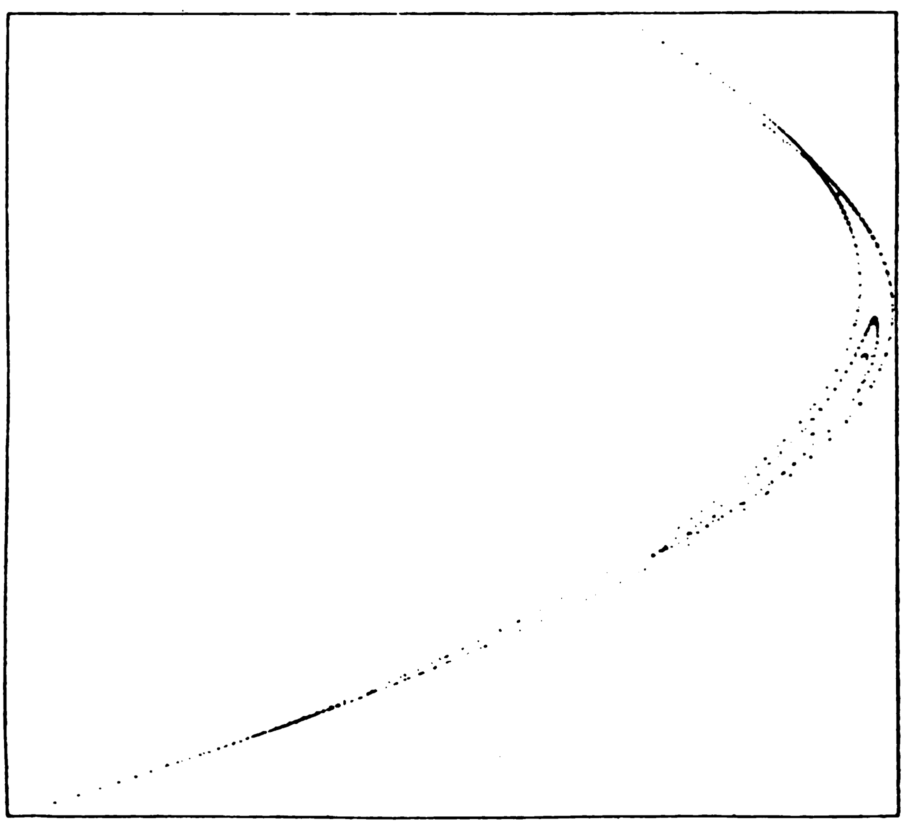


Figure 6.2,6.3

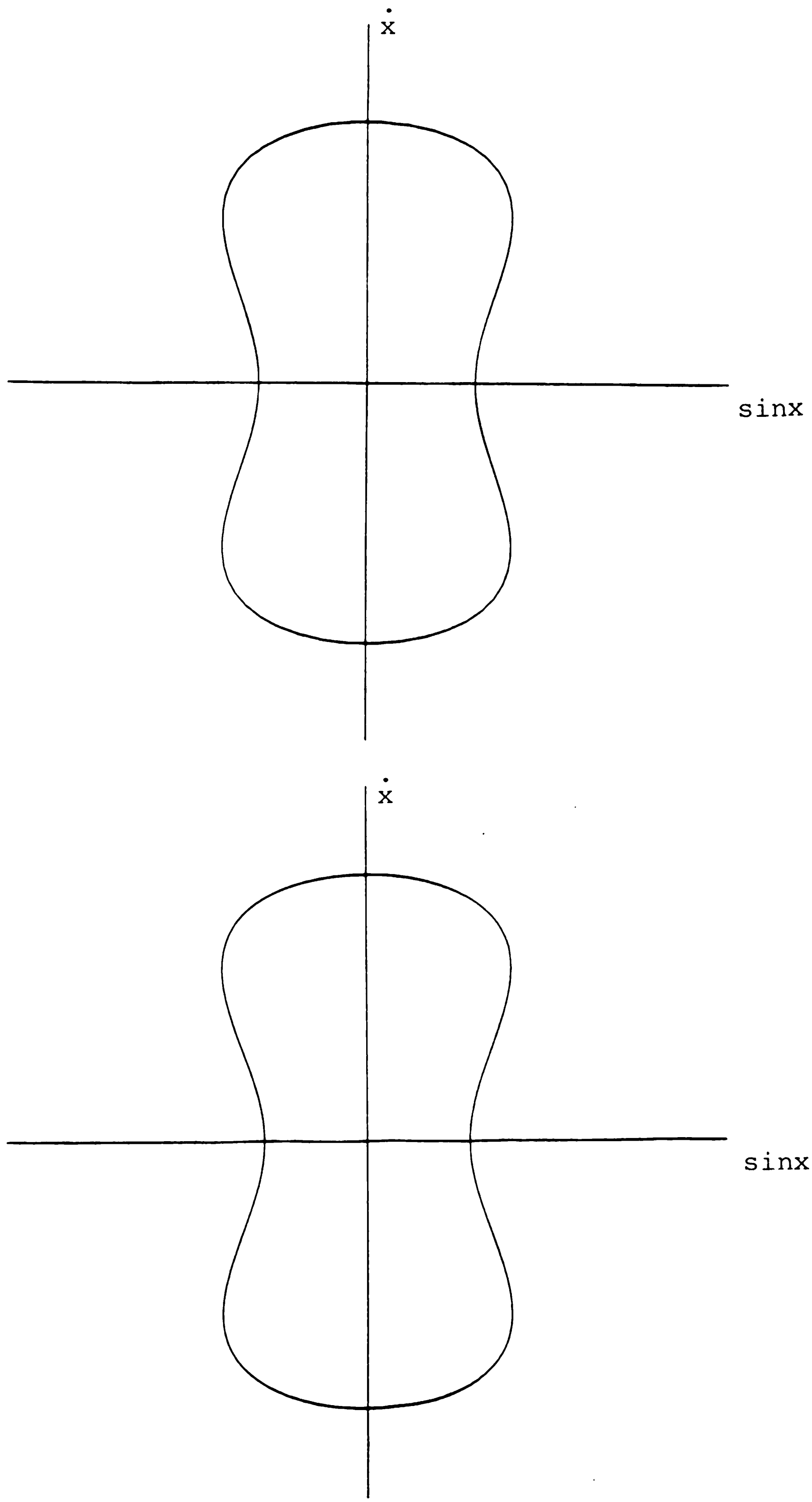


Figure 6.4-a

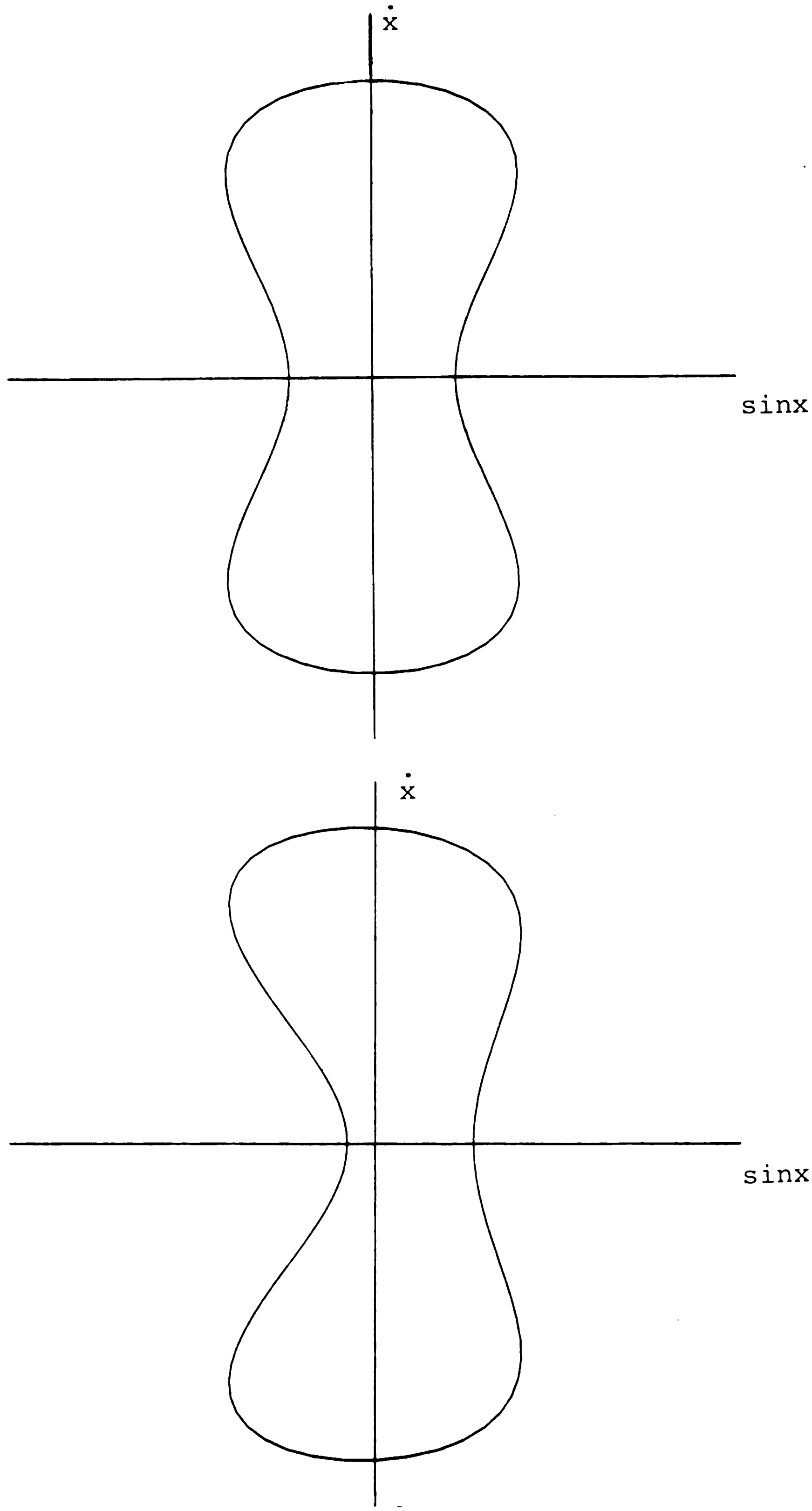


Figure 6.4-b

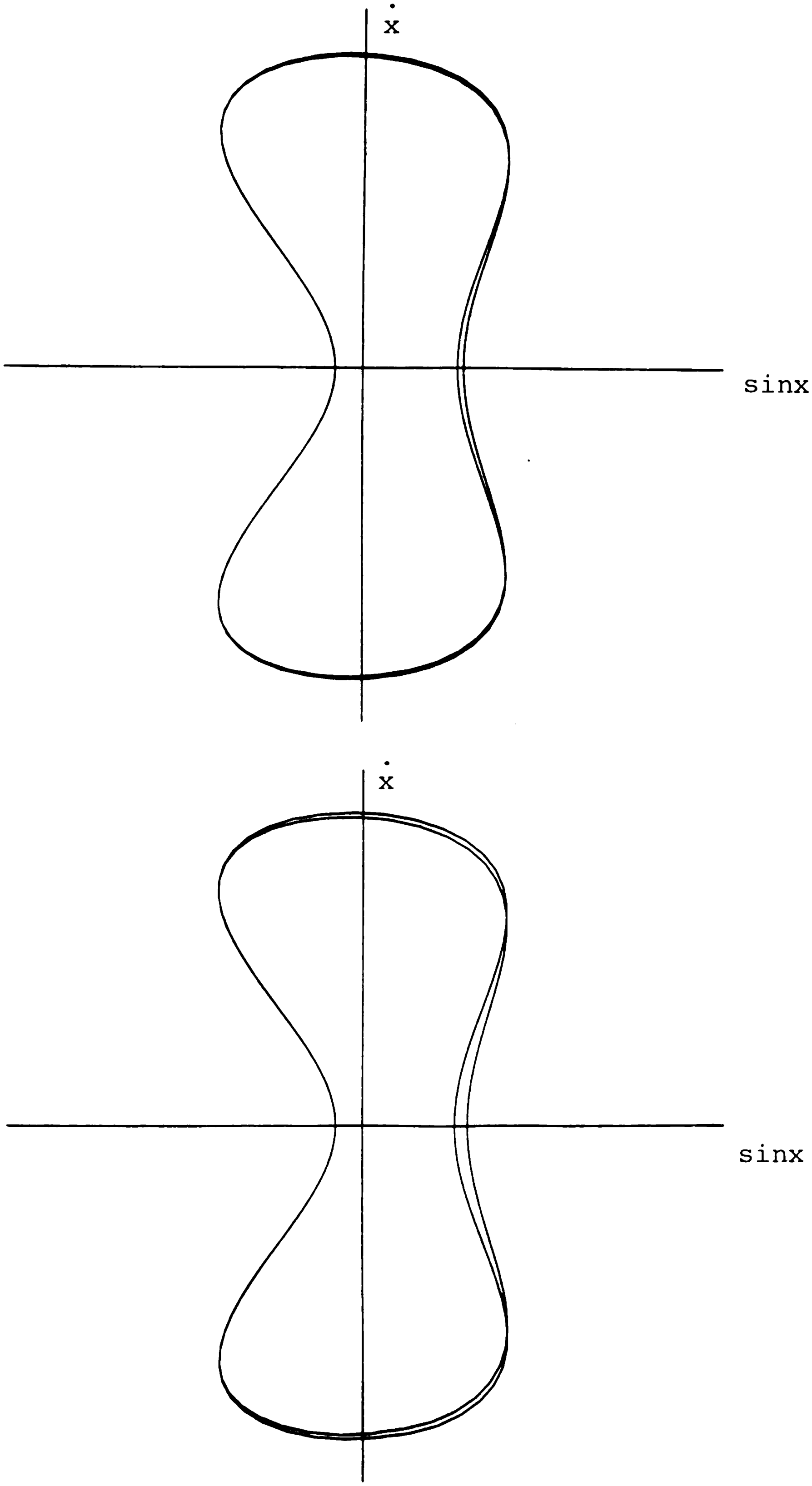
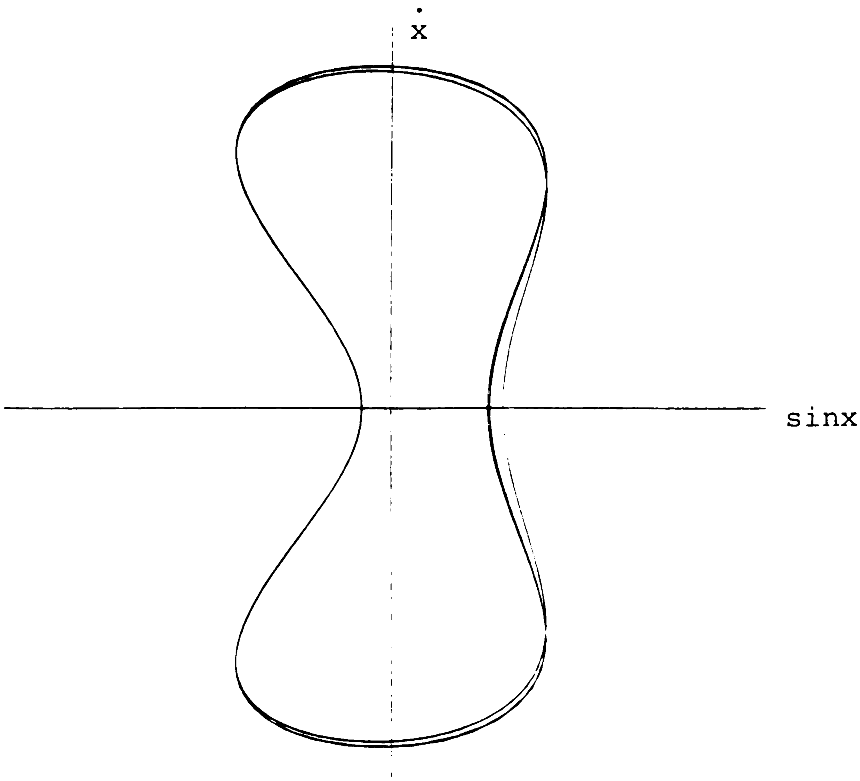


Figure 6.4-c



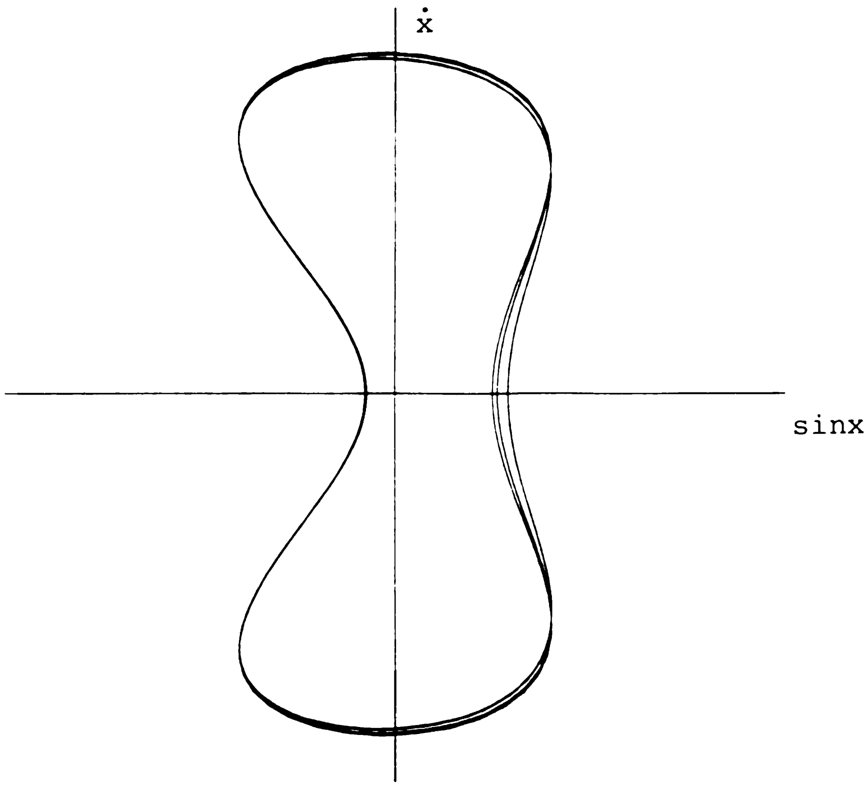


Figure 6.4-d

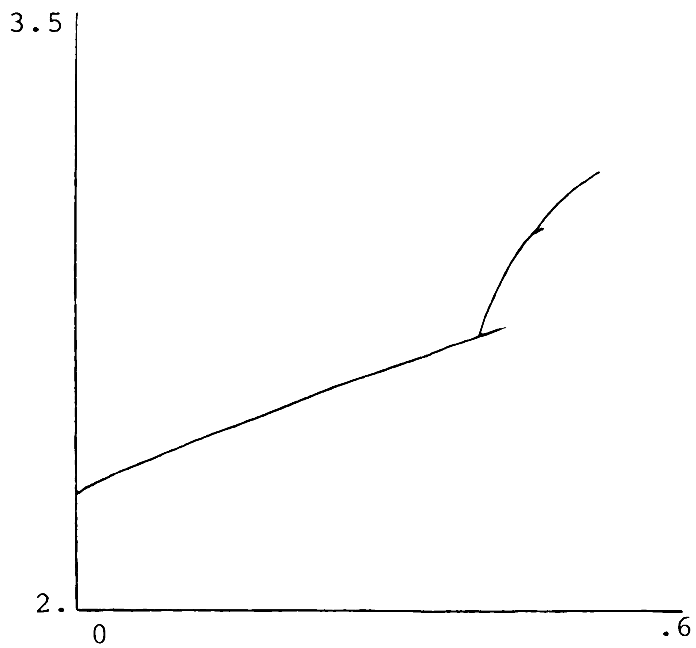


Figure 6.5

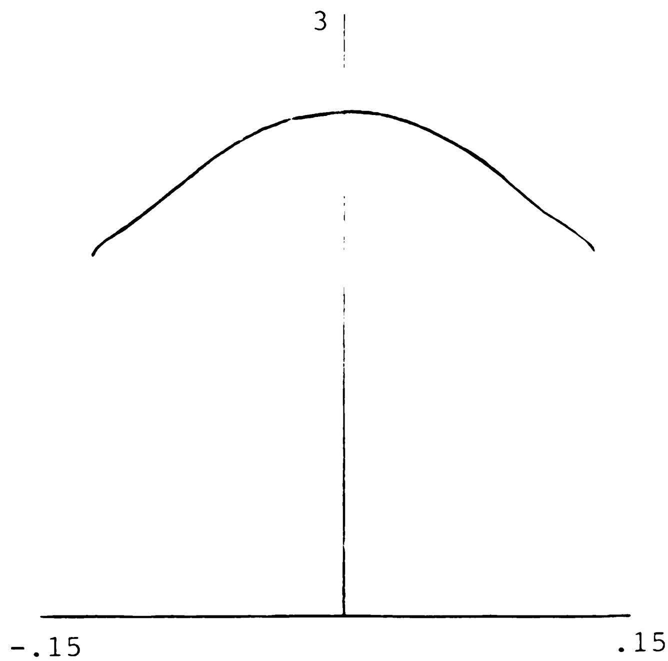


Figure 6.6

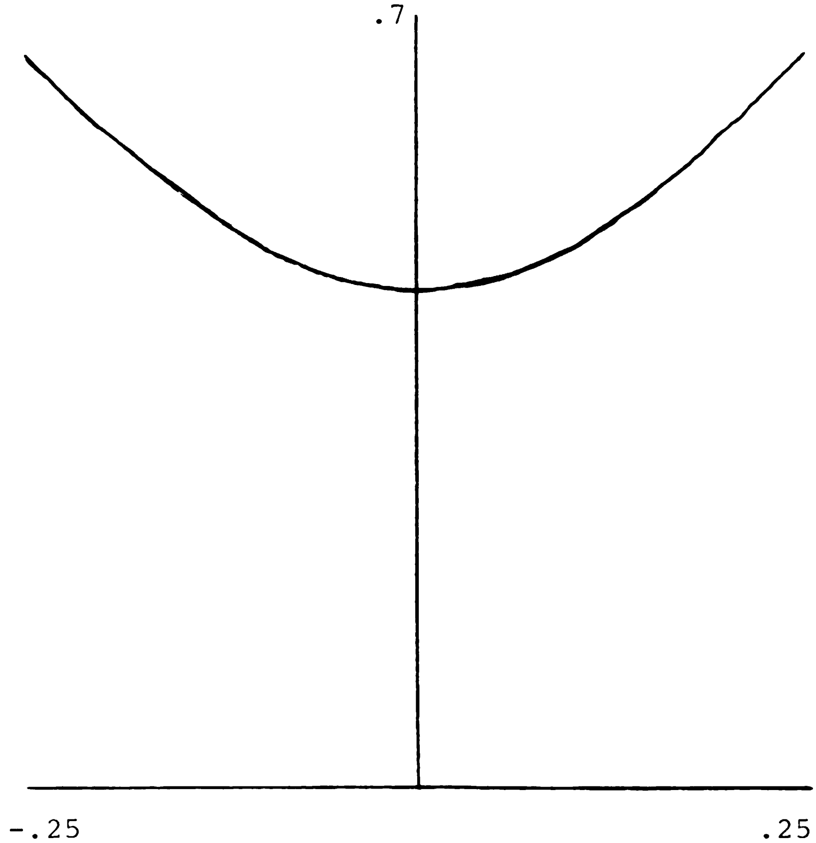


Figure 6.7

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