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THE ESTIMATION AND HYPOTHESIS TESTING OF TREATMENT EFFECTS IN NONEQUIVALENT CONTROL GROUP DESIGNS WHEN CONTINUOUS GROWTH MODELS ARE ASSUMED

By

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ABSTRACT

THE ESTIMATION AND HYPOTHESIS TESTING OF TREATMENT EFFECTS IN NONEQUIVALENT CONTROL GROUP DESIGNS WHEN CONTINUOUS GROWTH MODELS ARE ASSUMED

By

CAROL JOYCE BLUMBERG

The class of continuous growth models, where it is assumed there is a correlation of +1 between true scores at any two time points within each group, serves as the basis of discussion for the dissertation. This class can be expressed symbolically

$$Y^{*}_{ij}(t_2) = g_j(t_2) \cdot Y^{*}_{ij}(t_1) + h_j(t_2) + \alpha_j(t_2)$$

and

$$Y_{ij}(t) = Y^{*}_{ij}(t) + e_{ij}(t),$$

where t_1 and t_2 are any two time points;

$Y^{*}_{ij}(t)$ and $Y_{ij}(t)$ represent the true and observed scores, respectively, on the measure of interest, for the i<u>th</u> individual in the j<u>th</u> group;

$e_{ij}(t)$ represents the errors of measurement;

$g_j(t)$ and $h_j(t)$ are continuous functions;

and $\alpha_j(t)$ represents the population treatment effect.

The set of designs considered is that where there is one or more treatment groups with or without the presence of one or more control groups.

The various approaches that have been suggested for data analysis for these designs were examined under the given class of growth models. Appropriate procedures for the estimation and hypothesis testing of differences in treatment effects exist under these approaches when

(i) $g_j(t)$ is identical for all j and $h_j(t)$ is identical for all j;

(ii) no errors of measurement are present and $h_j(t) \equiv 0$;

or (iii) $g_j(t) = b_j \cdot (t - t_1) + 1$ and $h_j(t) = c_j \cdot (t - t_1)$ for b_j, c_j real-valued constants.

New methods of data analysis are developed which provide consistent estimates of treatment effects and differences in effects and appropriate procedures for the hypothesis testing of nonzero treatment effects and nonzero differences in effects under the entire class of growth models. These methods require only that either the functional forms of the $h_j(t)$'s are known or that $h_j(t)$ is the same, but unknown, for all j. When no errors of measurement exist, algebraic and numerical analysis techniques are used. When errors of measurement exist, the methods developed are several-stage procedures using numerical analysis techniques and the statistical techniques of maximum likelihood estimation and jackknifing. The advantage of these new methods is that they are applicable under a wider class of growth models than are the existing approaches.

ACKNOWLEDGEMENTS

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CHAPTER 1

INTRODUCTION

In many educational settings, a true experimental design is not possible when a researcher wants to evaluate the effects of different treatments. Thus, quasi-experimental designs are employed. One of the more commonly used quasi-experimental designs is the nonequivalent control group design (Campbell & Stanley, 1966; Campbell, 1969). Campbell and Stanely define this design as having two groups, a control group and a treatment group, which are formed by some method other than random assignment. They then require that both a preobservation and a postobservation be made on each individual on some measure of interest. Campbell (1969) extends the definition of a nonequivalent control group design to include any number of observations occurring at times before the treatment begins, during the treatment and after the treatment. More generally a nonequivalent control group design can include multiple-group designs, with or without the presence of one or more control groups, and can involve the investigation of interactions through the use of crossed factors.

There has been much discussion in the literature of the analysis strategies that are appropriate for use in connection with nonequivalent control group designs. The basic problem is to identify analysis strategies which will provide unbiased estimates of the treatment effects. This problem has come to be known as the problem of measuring change. Both Lord (1967) and Cronbach and Furby (1970) have argued that unless some assumptions are made, there is no way of knowing which analysis strategy is appropriate for use with any particular application of a nonequivalent control group design.

While the literature on the problem of measuring change is voluminous, considerable confusion remains as to appropriate solutions. One of the reasons for this confusion is that different authors have made different assumptions when making recommendations for the methods of analysis to be used with data arising from nonequivalent control group designs. Some of the more popular assumptions will now be discussed.

Kenny and Cohen (Kenny, 1975; Kenny & Cohen, 1980) and others (e.g., Cochran & Rubin, 1973) have taken as their assumption that the method of selection of the individuals into the treatment and control groups is known. They discuss various methods of selection and based on each method of selection they recommend a specific method of data analysis. Another popular assumption is the fan spread hypothesis (Bryk & Weisberg, 1977; Campbell & Erlebacher, 1970; Kenny, 1975; Olejnik, 1977). The fan spread hypothesis states that the ratio of the differences of population means to the standard deviation common to the populations of interest is constant over time. The fan spread hypothesis will be discussed further in Chapter 3. There it is described under what conditions data collected using a nonequivalent control group design will conform to the fan spread hypothesis. Campbell and Boruch (1975), Kenny (1975), and Olejnik and Porter (1981) have given examples of data sets which seem to conform to the fan spread hypothesis. Hence, evidence exists that the fan spread hypothesis may occur for some data sets arising from educational research settings.

A third group of assumptions are stated in terms of continuous growth models. The idea of continuous growth models dates back to at least 1964 (Potthoff & Roy, 1964). But, it was not until 1976 that the idea of applying continuous growth models to aid in the analysis of data from nonequivalent control group designs was born (Bryk & Weisberg, 1976).

Continuous Growth Models

Let J represent the number of groups in a particular design. Let $q_j + 1$ represent the number of time points at which observations for group j are made on the measure of interest. Let $Y_{ij}(t_{1_j})$, $Y_{ij}(t_{2_j})$, ... , $Y_{ij}(t_{q_j})$, and $Y_{ij}(t_{q_j+1})$ represent the observations for the i*th* individual in the j<u>th</u> group, where t_{1_j}, t_{2_j}, ... , t_{q_j+1} represent the $q_j + 1$ time points at which the observations are made. Let $Y^*_{ij}(t_{1_j})$, $Y^*_{ij}(t_{2_j})$, ... , $Y^*_{ij}(t_{q_j})$, $Y^*_{ij}(t_{q_j+1})$ and $e_{ij}(t_{1_j})$, $e_{ij}(t_{2_j})$, ... , $e_{ij}(t_{q_j})$, $e_{ij}(t_{q_j+1})$ represent the true scores and errors of measurement, respectively, for the ith individual in the jth treatment group at the $q_j + 1$ time points. The most general form for a growth model that will be considered here is that where classical measurement theory assumptions are made and where the functional relationships

$$Y^*_{ij}(t_{q_j+1}) = f_{ij}(Y^*_{ij}(t_{1_j}), Y^*_{ij}(t_{2_j}), \ldots, Y^*_{ij}(t_{q_j}), t_{1_j}, t_{2_j}, \ldots, t_{q_j+1}) \quad j = 1,2, \ldots, J; \tag{1-1}$$

hold, where the f_{ij}'s are continuous functions, which may be different for each individual in each of the J groups. The classical measurement theory assumptions of particular interest are, for each time t and for each j;

(i) $Y_{ij}(t) = Y^*_{ij}(t) + e_{ij}(t)$;

(ii) $E(e_{ij}(t)) = 0$;

(iii) $Cov(Y^*_{ij}(t), e_{ij}(t)) = 0$;

and (iv) $Cov(e_{ij}(t), e_{ij}(t')) = 0$ for any time $t' \neq t$.

The models considered by Bryk, Strenio, and Weisberg (1980), Strenio, Weisberg, and Bryk (in press), Olejnik (1977), and the models that will serve as the basis of the remainder of this dissertation can all be seen as special cases of the above general model. The most general model considered by Bryk, Strenio, and Weisberg (1980) is when equation (1-1) reduces to

$$Y^*_{ij}(t_{2_j}) = Y^*_{ij}(t_{1_j}) + b_{ij}(t_{2_j} - t_{1_j}) + \alpha_j(t_{2_j}) \quad (1\text{-}2)$$

where b_{ij} is some constant for the i<u>th</u> individual in the j<u>th</u> group and $\alpha_j(t)$ is the treatment effect for group j at time t ;

and $Y^*_{ij}(t_{1_j}) + b_{ij}(t_{2_j} - t_{1_j})$ represents natural growth (i.e., growth which occurs in the absence of any treatment effects or errors of measurement).

For this model and for all other continuous growth models to be discussed, the assumption is made that treatment effects are additive. That is, the treatment causes an increase or decrease of exactly the same amount for all individuals in the same group over the amount of growth accounted for by natural growth. The assumption of additive treatment effects is standard in the experimental design literature (e.g., Cox, 1958). Notice that this model states that each person's true growth over time is linear, albeit possibly a different line for each person.

Strenio, Weisberg, and Bryk (in press) describe a more general model of natural growth than that considered in equation (1-2). In this more recent paper they extend their model of natural growth to

$$Y^*_{ij}(t_{2_j}) = \sum_{\ell=1}^{L_j} \pi_{ij\ell} \cdot (t_{2_j} - t_{1_j})^{\ell} + Y^*_{ij}(t_{1_j}), \qquad (1\text{-}3)$$

where the L_j's are predetermined integers and the $\pi_{ij\ell}$'s are undetermined constants. The reason for the absence of any treatment effects in the model given by equation (1-3) is that in the paper they were only concerned with the estimation of natural growth curves. Notice that this model allows for natural growth over time which is a polynomial of any degree. Additionally, the coefficients of the various terms in the polynomial need not be the same for each person.

Olejnik (1977) discusses two different models of natural growth. One of his models assumes that each individual's natural growth over time is linear, that there is a correlation of +1 between true scores at any two points in time and that the fan spread hypothesis holds. Although he never explicitly expresses this model in terms of individual growth curves, the model can be expressed as

$$Y^*_{ij}(t) = [b(t - t_{1_j}) + 1]\cdot Y^*_{ij}(t_{1_j}) + c(t - t_{1_j}) + \alpha_j(t),$$

where b and c are positive constants and t is any point in time. Also, this model requires that the within groups standard deviation is the same for all J groups at any time point. Even though the expression given here is in terms of any number of groups, it should be pointed out that Olejnik only deals with two-group designs in his dissertation. Olejnik's other model will be discussed briefly in Chapter 3.

The growth models to be considered here are an extension of the model of Olejnik (1977) just described. As will be discussed later, this extension is both more and less general than the model considered by Strenio, Weisberg, and Bryk (in press). These models assume only three things:

(1) Classical measurement theory holds.

(2) The correlation within each group between true scores at any two points in time is +1.

and

(3) Treatment effects are additive.

These growth models can be expressed symbolically as

$$Y^*_{ij}(t) = g_j(t) \cdot Y^*_{ij}(t_{1_j}) + h_j(t) + \alpha_j(t)$$

and (1-4)

$$Y_{ij}(t) = Y^*_{ij}(t) + e_{ij}(t) \qquad \text{for all } t,$$

where $g_j(t)$ and $h_j(t)$ are continuous functions;

$g_j(t) \cdot Y^*_{ij}(t_{1_j}) + h_j(t)$ represents natural growth;

and $\alpha_j(t)$ represents the population treatment effect.

Let T be the time at which the treatments are initiated. Notice that $\alpha_j(t) \equiv 0$ for all $t \leq T$. Further, notice that $g_j(t) > 0$ for all t, since a correlation of +1 is being assumed.

There are several reasons for considering the class of growth models represented by equation (1-4). First, as will be shown in Chapter 3 these models are more general than the fan spread hypothesis. As mentioned earlier, Campbell and Boruch (1975) and others have stated their belief that in many actual situations, data conform to the fan spread hypothesis. Hence, they and others (e.g., Kenny, 1975; Olejnik, 1977) have mainly restricted their attention to situations where the fan spread hypothesis holds when discussing the problem of measuring change. Since the growth models being assumed for this dissertation are more general than the ones considered by others who have worked on the problem of measuring change, they are an important class of models to study. Second, as will be shown in Chapter 2, these growth models allow for a variety of different functional forms for individual growth and hence for group growth. They are not restricted to linear growth or even to polynomial growth as has been done in the past literature on the problem of measuring change (e.g., Kenny, 1975; Olejnik, 1977; Strenio, Weisberg & Bryk, in press). Finally, these models of natural growth have led to the development of very different approaches to data analysis than heretofore considered. Nevertheless, it is recognized that the models under study in this dissertation are only a small subset of all possible growth models.

Overview

This dissertation has three purposes. The first purpose is to explicitly show what types of individual growth are possible under the growth model given by equation (1-4). This will comprise Chapter 2. The second purpose is to explore the relationships between the growth models of equation (1-4), the fan spread hypothesis, and the various methods of data analysis that have been suggested in the past with respect to the nonequivalent control group design. This will comprise Chapter 3. Consistent with the models described by equation (1-4), the discussion of methods of data analysis will be restricted to those methods which do not require the use of additional information such as background variables, covariates other than the measure itself, nor replicates of the measure of interest within a point in time. As will be seen, currently proposed methods of data analysis are not sufficiently general as to apply to the full set of growth models represented by equation (1-4). Hence, new methods of data analysis must be found. The third and main purpose of this dissertation is to describe the data analysis procedures which have been developed. These data analysis procedures will be described in Chapters 4 through 6. The basic idea behind these new methods is to collect data at enough pretest time points so that one is able to estimate the natural growth patterns. Next, the estimated natural growth patterns are projected into the future. Point estimation and hypothesis testing methods are then developed which are based on the projected natural growth and the observed posttest data. Chapter 7 will provide a summary of the results from Chapters 3 through 6 and will also contain a discussion of some directions for further research.

CHAPTER 2

EXAMPLES OF NATURAL GROWTH

In this dissertation the term natural growth will be used to denote the relationship between true scores at time t_{1_j} and at any other time, t, that would have occurred if no treatments were applied. There are an infinity of different continuous natural growth curves represented by the models given in equation (1-4). The purpose of this chapter is to illustrate the flexibility of the model by discussing several specific examples. These examples were chosen because they model growth curves found in the educational and behavioral science research literature.

For simplicity of presentation, initial discussion is limited to a single group and hence, the j subscript will be dropped temporarily. Further, without loss of generality, it can be assumed that $t_{1_j} = 0$. Hence, natural growth under equation (1-4) can be expressed for a one-group design as

$$Y_i^*(t) = g(t) \cdot Y_i^*(0) + h(t) \ . \qquad (2\text{-}1)$$

Some specific examples of natural growth are:

(1) Parallel growth. Parallel growth is defined by $g(t) \equiv 1$, so that, $Y_i^*(t) = Y_i^*(0) + h(t)$, where h(t) is any

continuous function. Figure 1 provides a pictorial representation of an example of parallel growth. For ease of illustration, Figure 1 and all remaining figures will only show the growth curves for three individuals in the group.

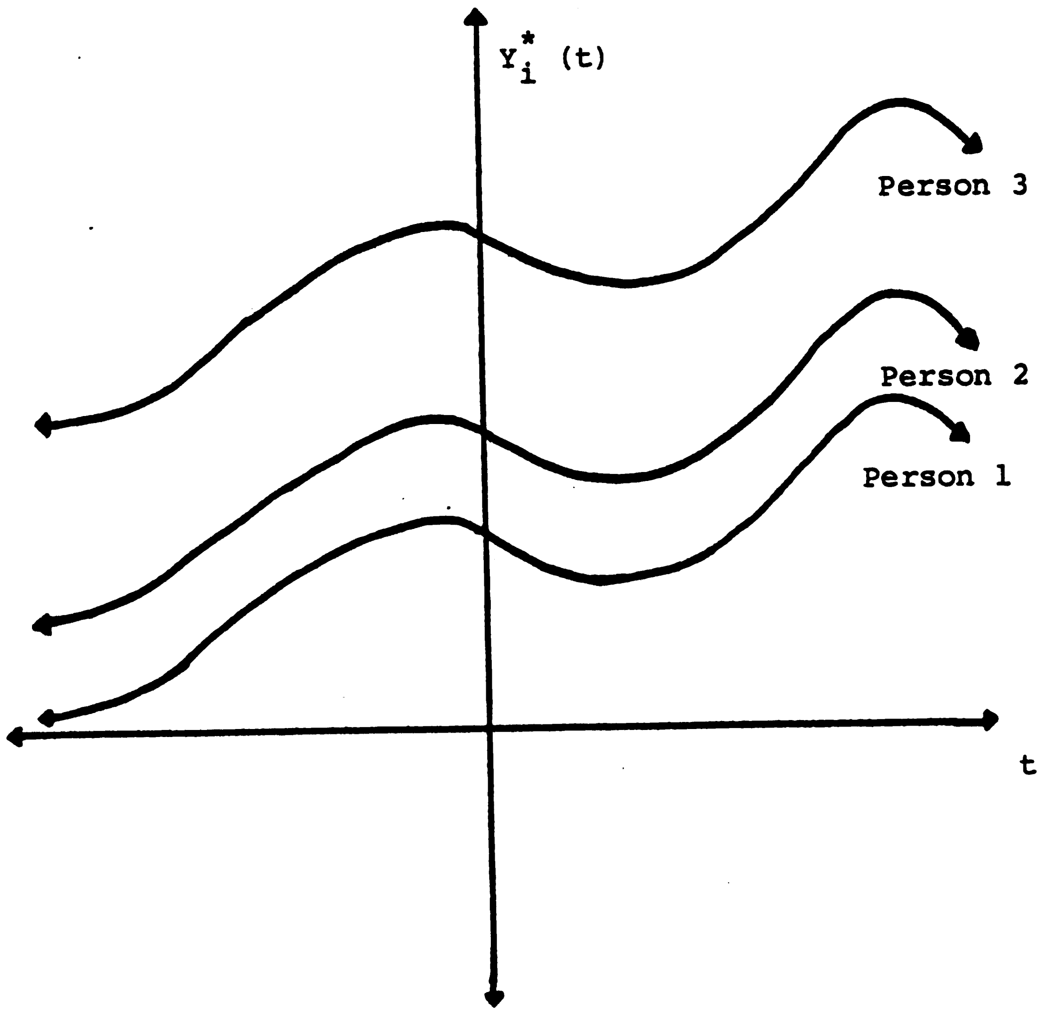

Figure 1. An example of parallel growth.

(2) Differential linear growth. Differential linear growth is defined by $Y_i^*(t) = (b \cdot t + 1) \cdot Y_i^*(0) + c \cdot t$, where b and c are real-valued constants with $b \neq 0$. Figures 2 through 4 provide pictorial representations of examples of differential linear growth.

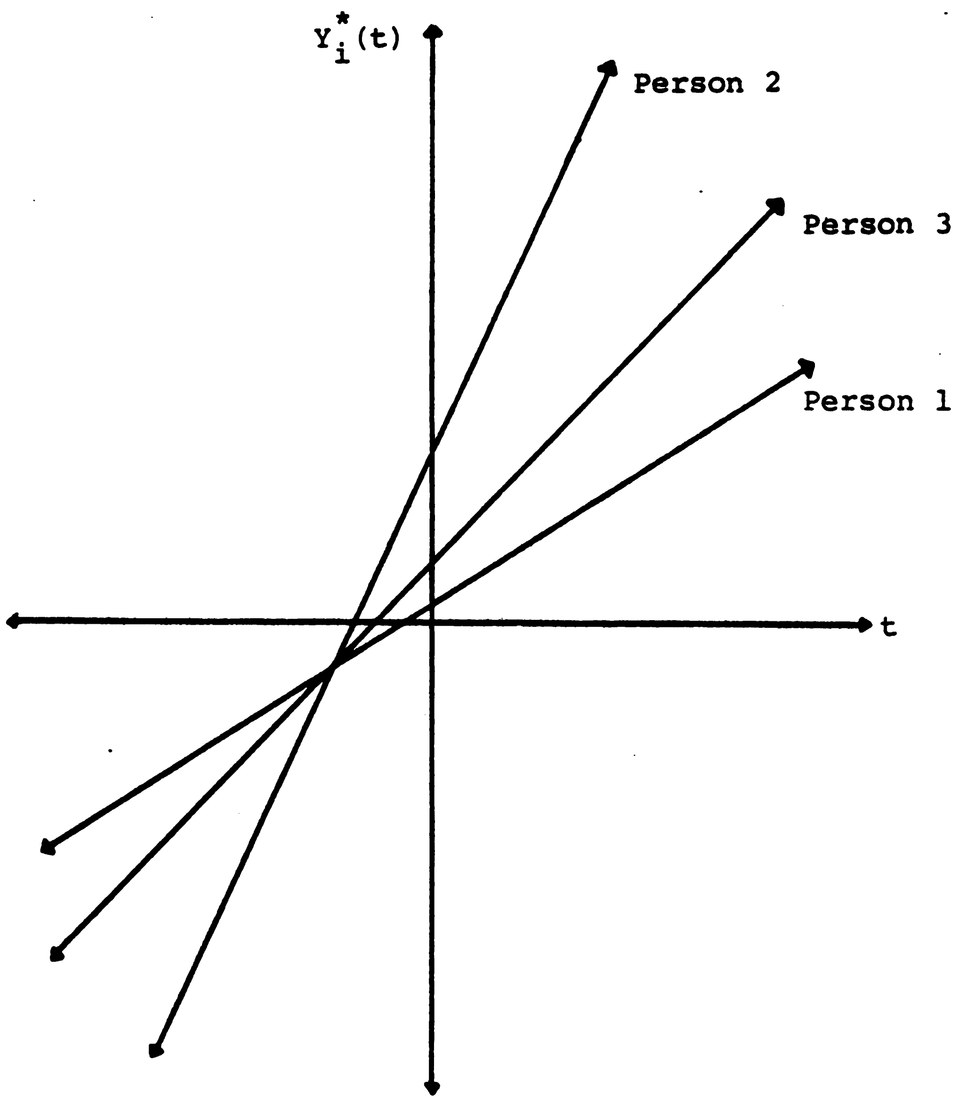

Figure 2. Differential linear growth when $b < 0$ and $c \geq 0$.

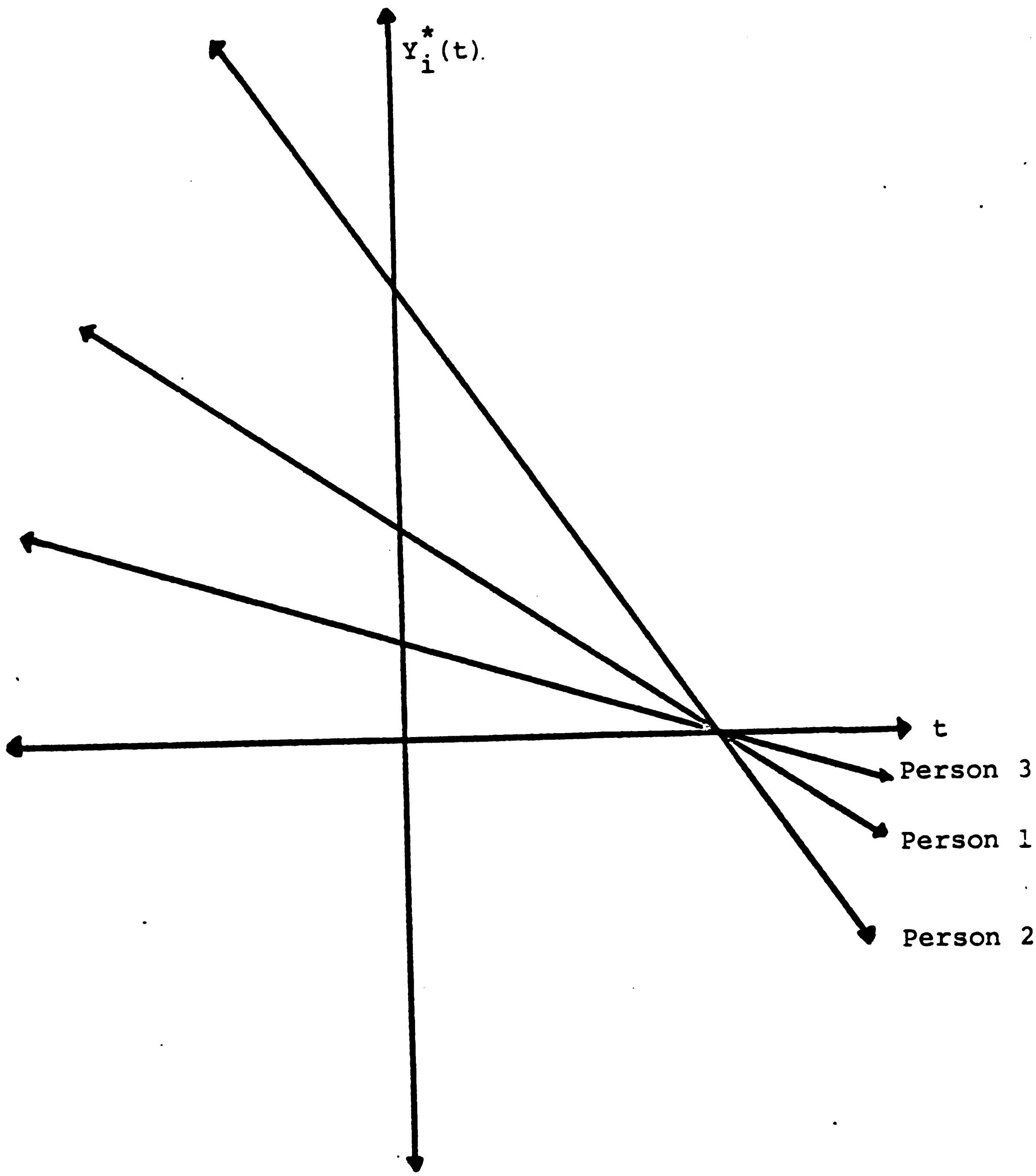

Figure 3. Differential linear growth when $b < 0$ and $c = 0$.

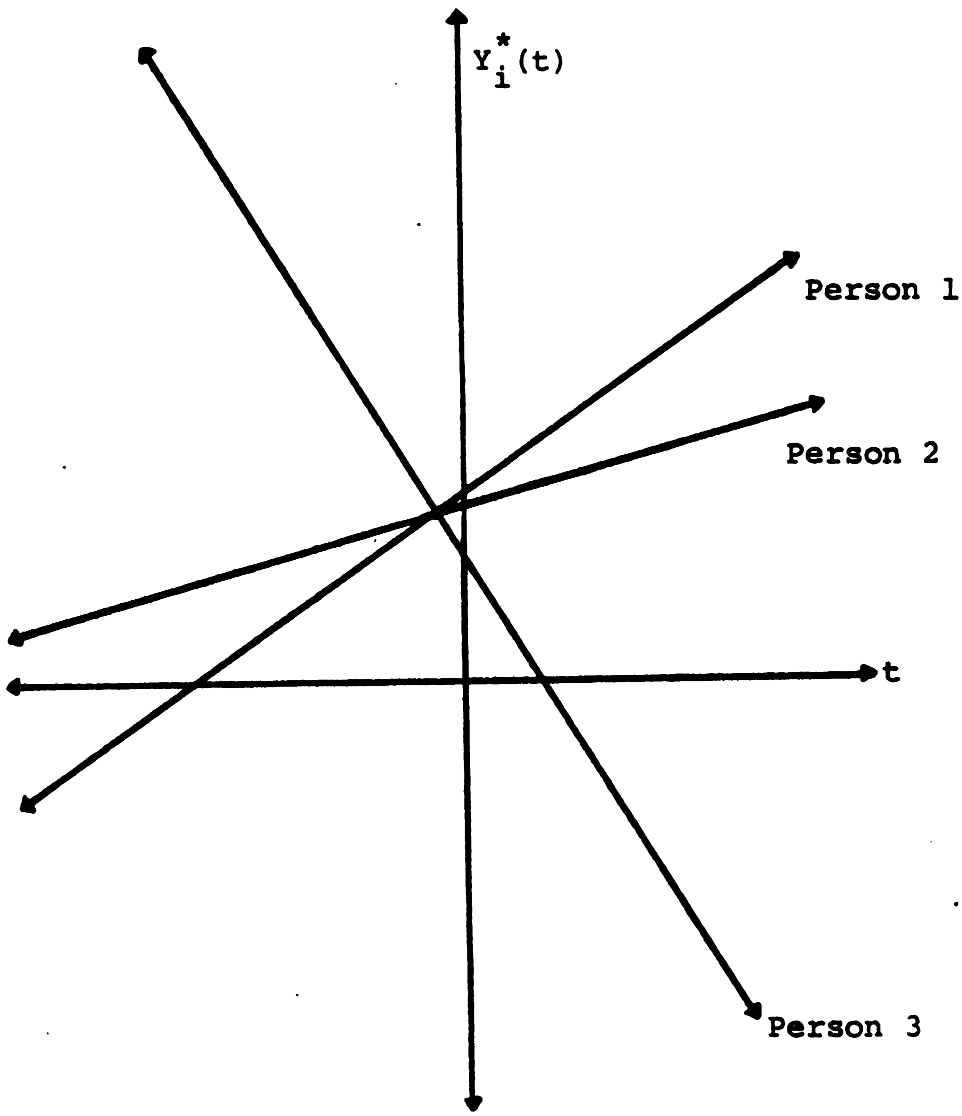

Figure 4. Differential linear growth when b > 0 and c < 0.

(3) Exponential growth. Exponential growth is defined by $Y_i^*(t) = [b \cdot c^t + (1 - b)] \cdot Y_i^*(0)$, where b and c are real-valued constants with $c > 0$. (Since measures of growth take on only real values, values of $c < 0$, which

yield complex values for $Y_i^*(t)$ are not allowed.) Figures 5 through 8 provide pictorial representations of examples of exponential growth.

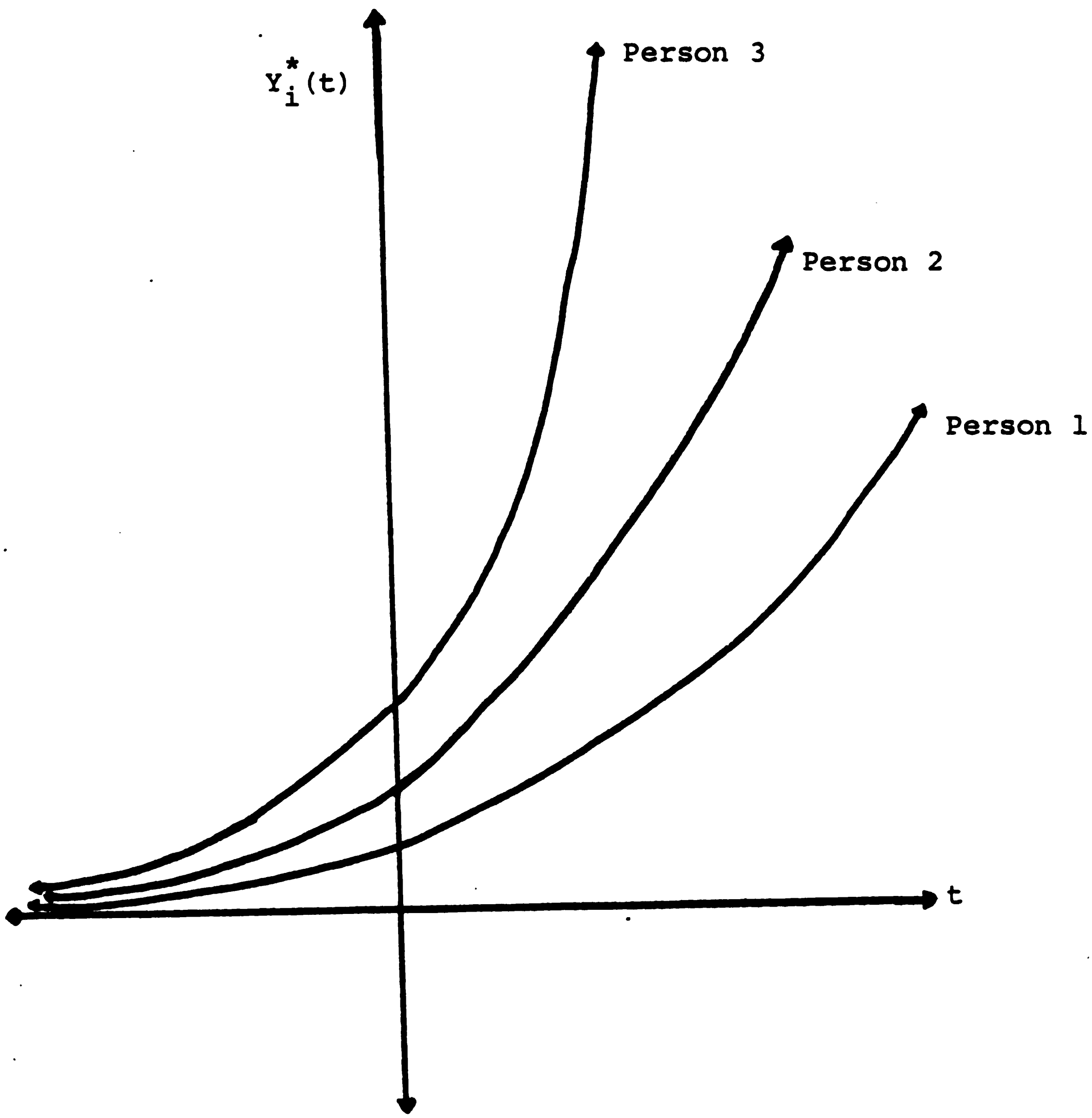

Figure 5. Exponential growth when b = 1 and c > 1.

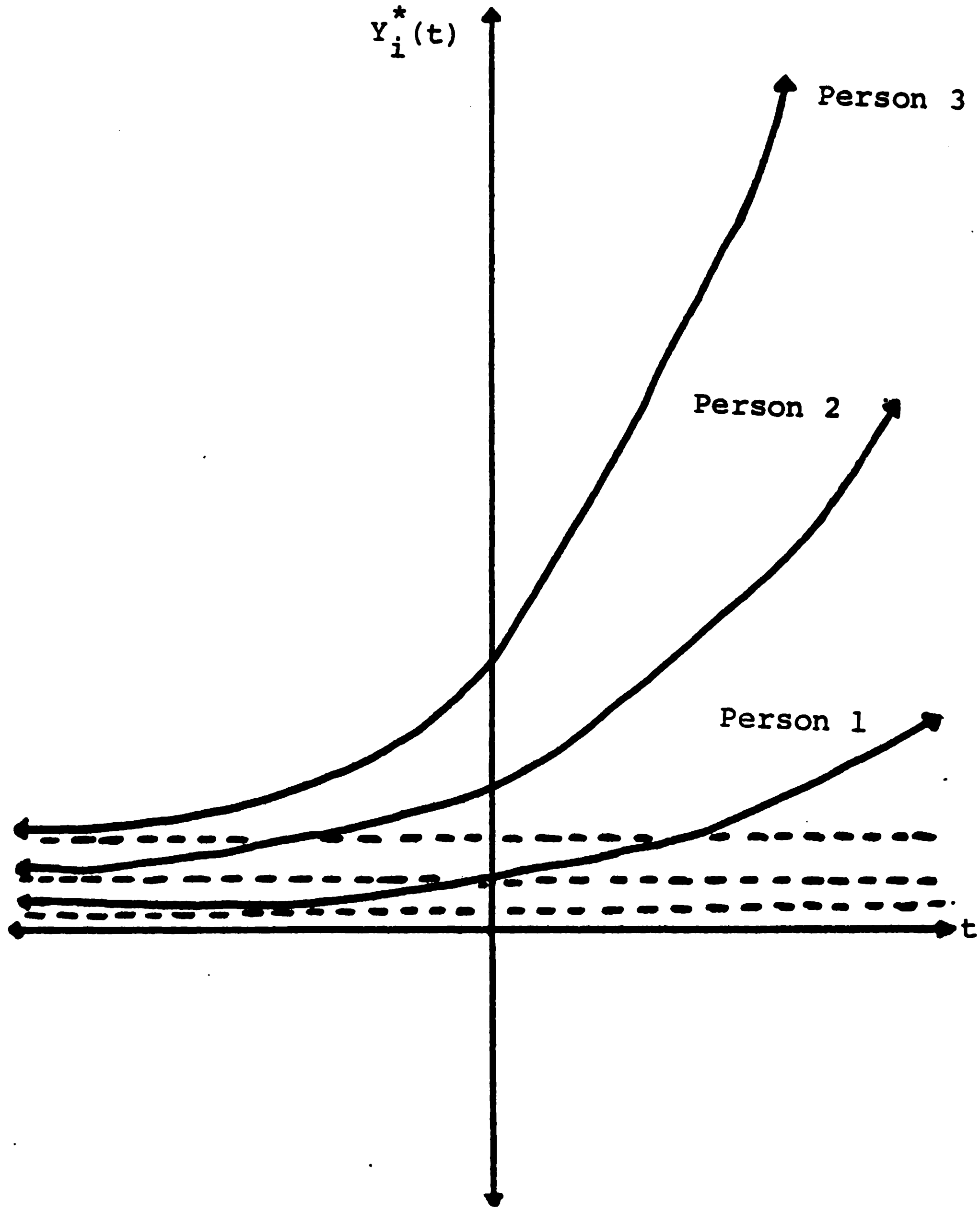

Figure 6. Exponential growth when $0 < b < 1$ and $c > 1$.

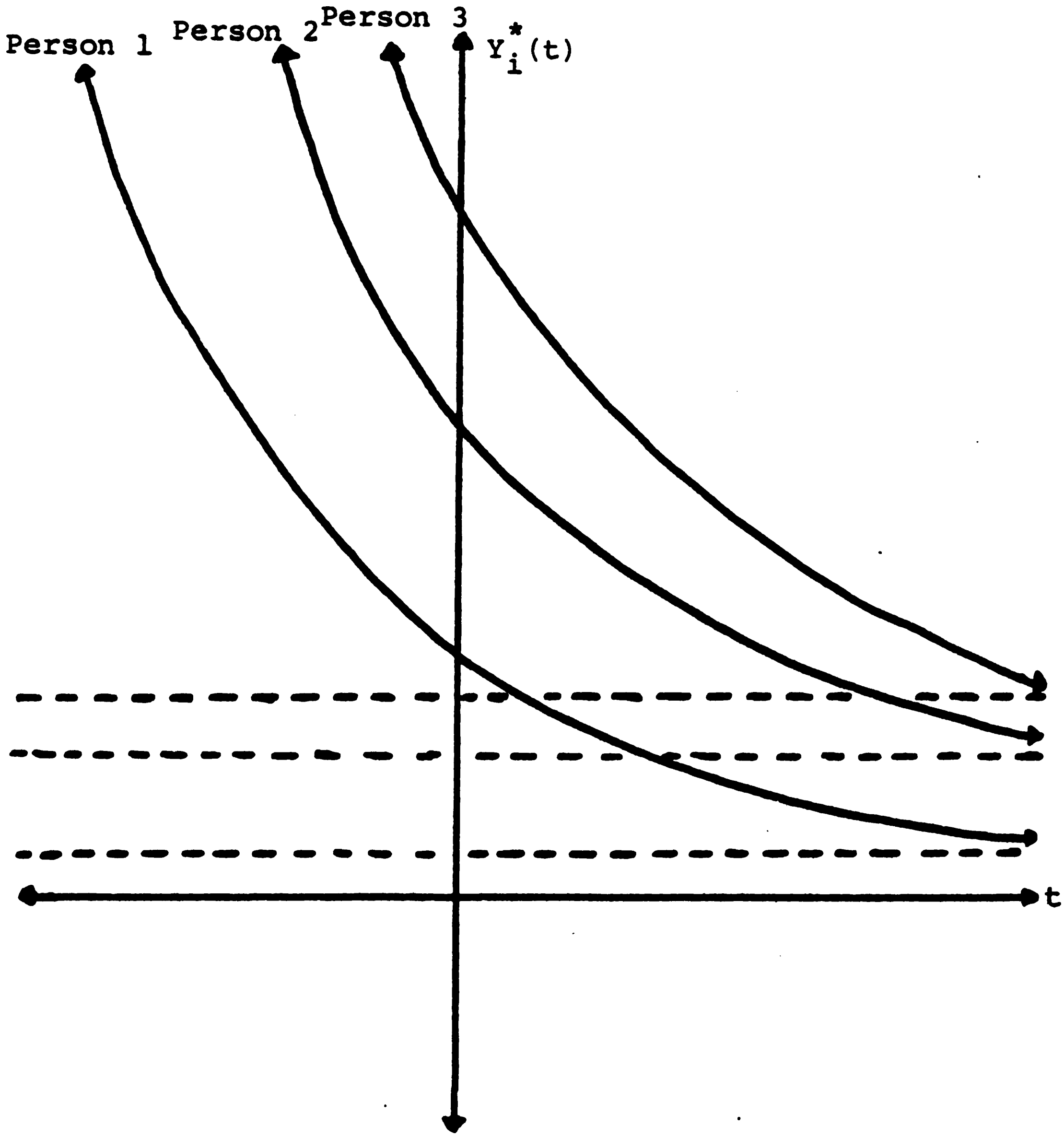

Figure 7. Exponential growth when

$0 < b < 1$ and $0 < c < 1$.

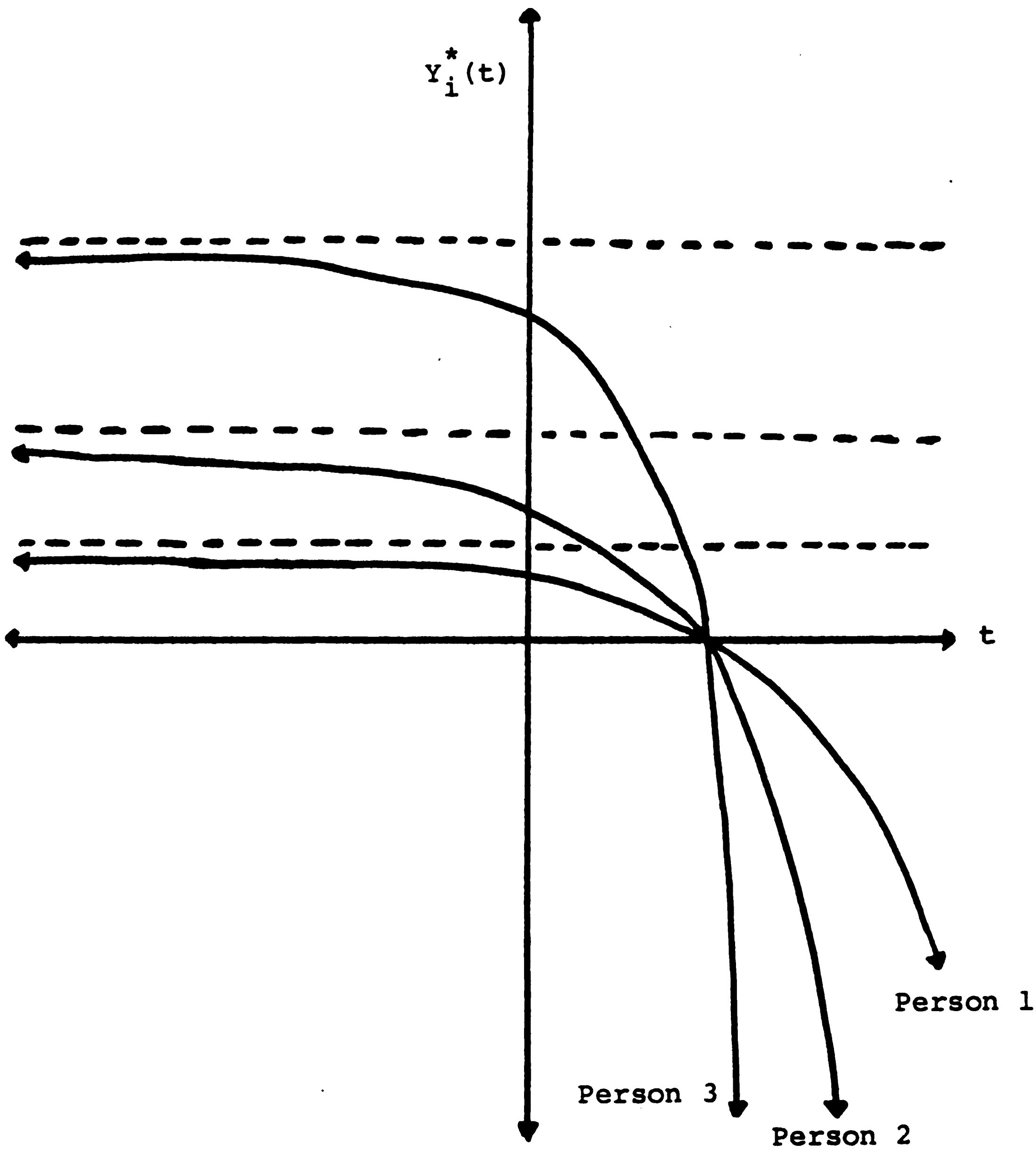

Figure 8. Exponential growth when

$b < 0$ and $c > 1$.

The inclusion of monotone decreasing functions (see e.g., Figures 3, 7, and 8) as representatives of natural growth was motivated by learning theorists interest in forgetting curves. Other forms of natural growth included under equation (2-1) are:

(4) Logarithmic growth. Logarithmic growth is defined by

$$Y_i^*(t) = [\log_c(b \cdot t + c)] \cdot Y_i^*(0),$$

where b and c are real-valued constants with $c > 0$ and $c \neq 1$.

(5) Cumulative normal (Normal Ogive) growth. Cumulative normal (Normal Ogive) growth is defined by

$$Y_i^*(t) = 2 \cdot [\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \cdot \exp(-\tfrac{1}{2}v^2)\ dv] \cdot Y_i^*(0) .$$

(6) Logistic growth. Logistic growth (Lord & Novick, 1968) is defined by

$$Y_i^*(t) = \frac{(1 + c) \cdot d^t}{1 + c \cdot d^t} \cdot Y_i^*(0),$$

where c and d are real-valued constants with $c > 0$ and $d > 1$.

(7) Polynomial growth. Polynomial growth is defined by

$$Y_i^*(t) = (1 + \sum_{n=1}^{p} c_n \cdot t^n) \cdot Y_i^*(0) + \sum_{k=1}^{q} d_k \cdot t^k,$$

where the c_n's; $n = 1,2, \ldots , p$ and the d_k's; $k = 1,2, \ldots, q$ are real-valued constants with $c_p \neq 0$ and $d_q \neq 0$.

Figures 9 through 12 provide pictorial representations of logarithmic growth. Figure 13 provides a pictorial representation of Cumulative normal and Logistic growth. Figure 14 provides a pictorial representation of polynomial growth when $p = 2$, $q = 1$, $c_2 > 0$, $c_1 > 0$, and $d_1 < 0$.

As stated previously the examples given here represent only a small fraction of the types of natural growth allowed under equation (2-1). Continuous functions of any form are allowed for g(t) and h(t), with the only restrictions being that $g(t) > 0$ (because, it is being assumed that the correlation of true scores at any two points in time is +1) and that $g(0) = 1$ and $h(0) = 0$. The reason for the restrictions $g(0) = 1$ and $h(0) = 0$ is consistency. For equation (2-1) to hold at time $t = 0$, it is necessary to have $Y_i^*(0) = g(0) \cdot Y_i^*(0) + h(0)$ for each individual. Consequently, $g(0) = 1$ and $h(0) = 0$.

All of the types of growth possible for single-group designs are also possible for multi-group designs. For multi-group designs, the natural growth curves may be

1) exactly the same for all the groups

2) of the same form for all the groups, but with different constants specifying the functions. For example, a three-group design where all three groups follow exponential growth patterns would be expressed as

$$Y^*_{i1}(t) = (b_1 \cdot c_1^t + (1 - b_1)) \cdot Y^*_{i1}(0)$$

$$Y^*_{i2}(t) = (b_2 \cdot c_2^t + (1 - b_2)) \cdot Y^*_{i2}(0)$$

and

$$Y^*_{i3}(t) = (b_3 \cdot c_3^t + (1 - b_3)) \cdot Y^*_{i3}(0),$$

where b_1, b_2, and b_3 take on possibly distinct values and c_1, c_2, and c_3 take on possibly distinct values.

or 3) of different forms for each group (e.g., group 1 follows logarithmic growth, group 2 follows exponential growth, group 3 follows polynomial growth, etc.).

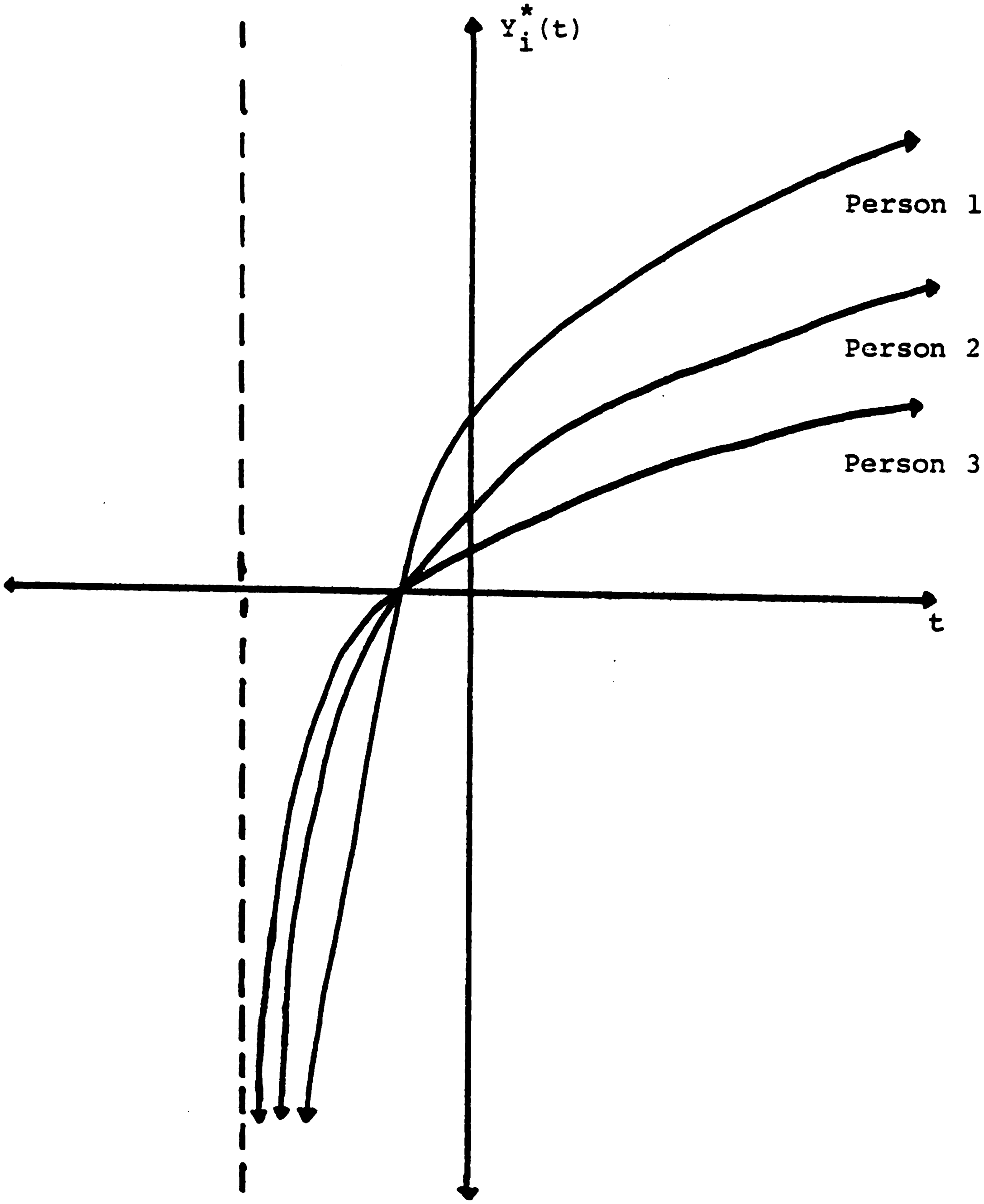

Figure 9. Logarithmic growth when
$c > 1$ and $b > 0$.

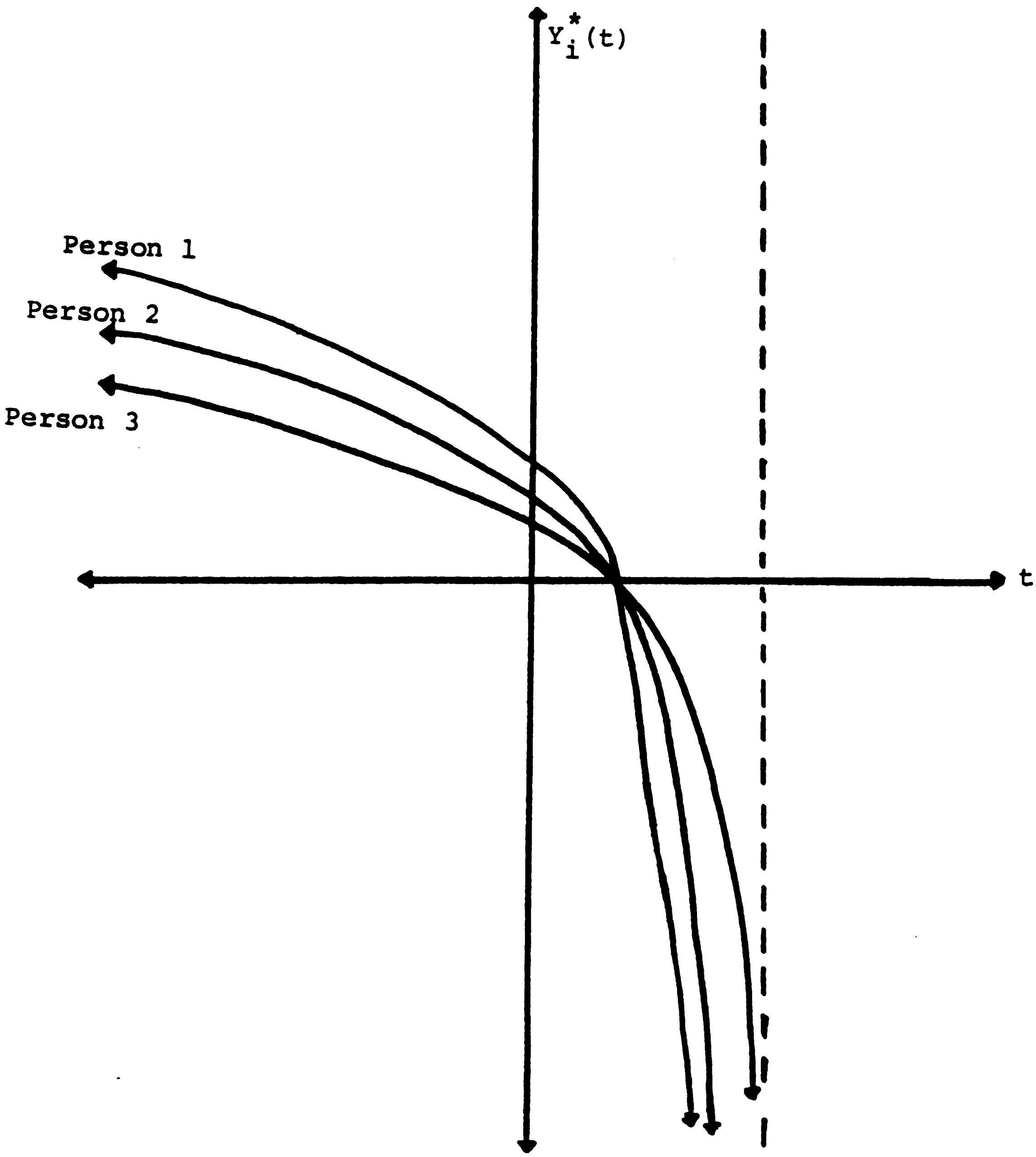

Figure 10. Logarithmic growth when $c > 1$ and $b < 0$.

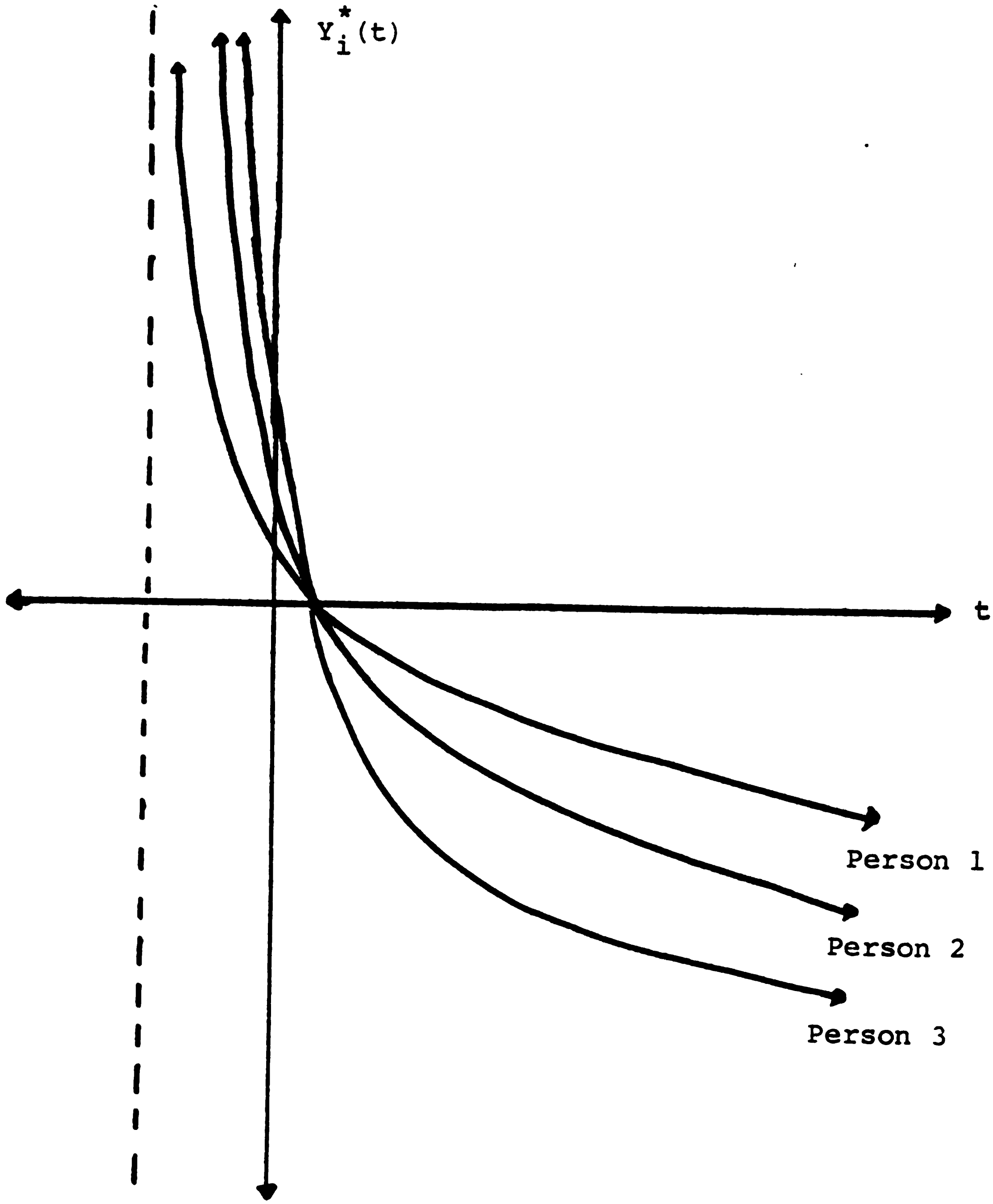

Figure 11. Logarithmic growth when $0 < c < 1$ and $b > 0$.

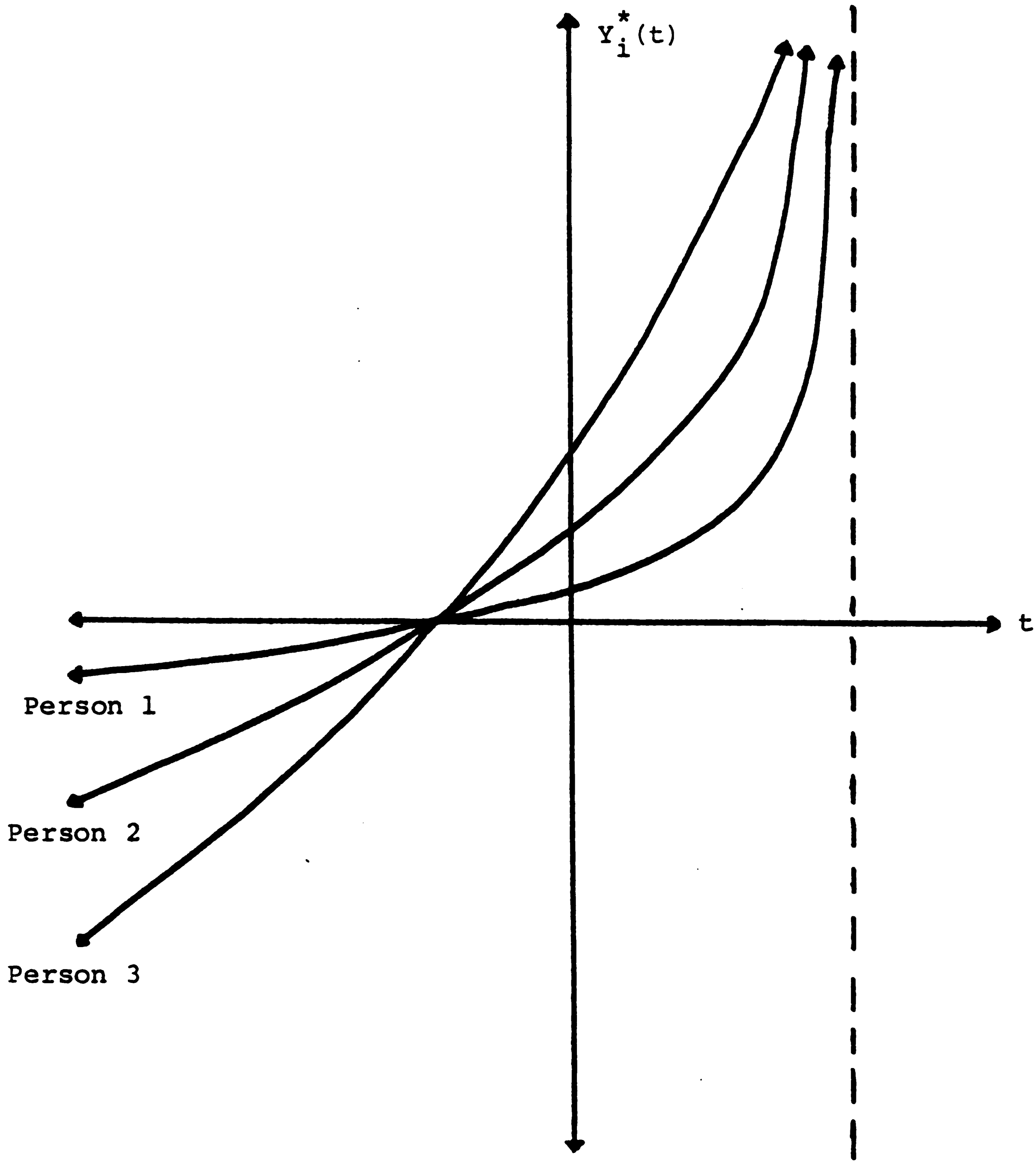

Figure 12. Logarithmic growth when $0 < c < 1$ and $b < 0$.

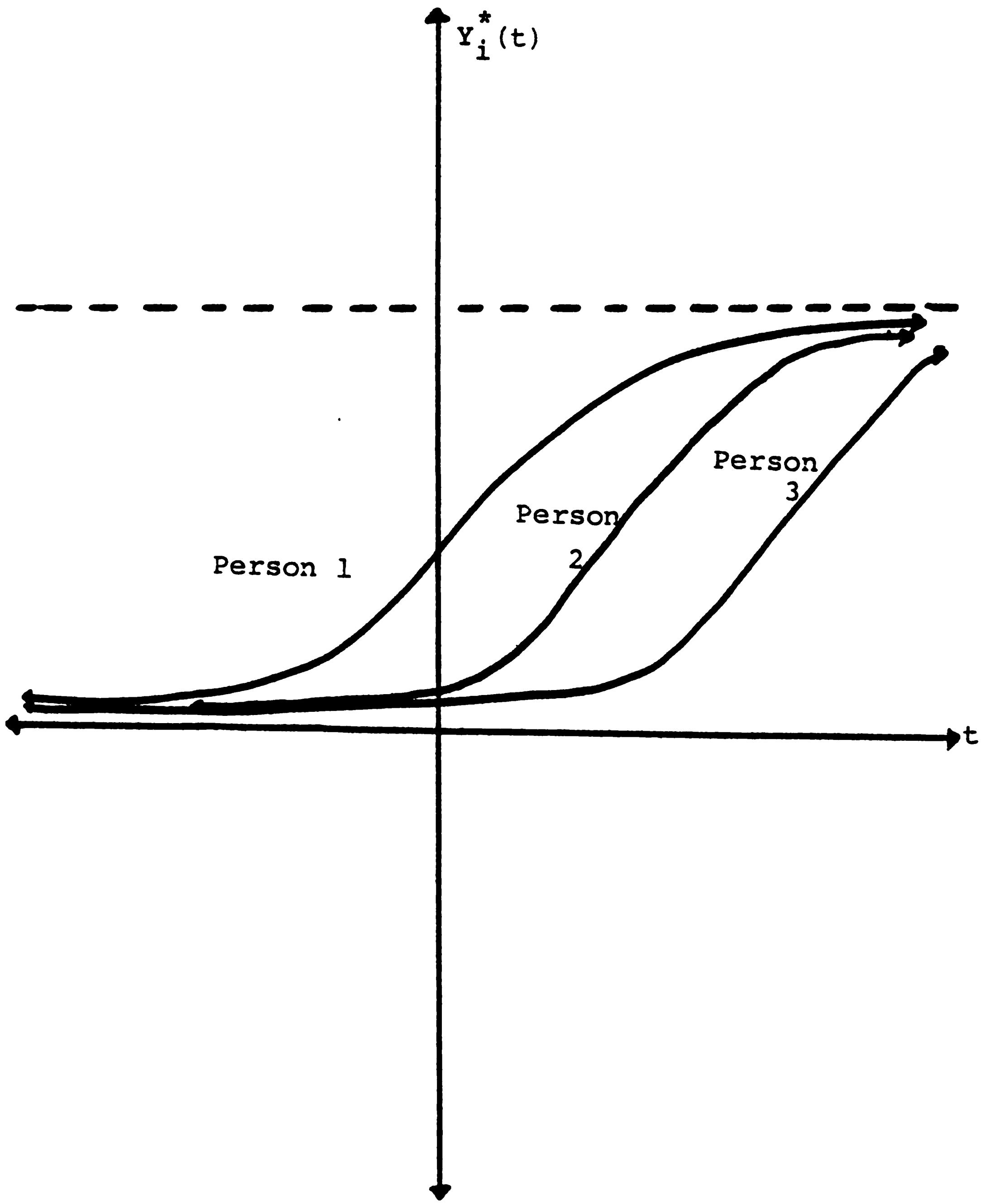

Figure 13. Cumulative Normal or Logistic growth.

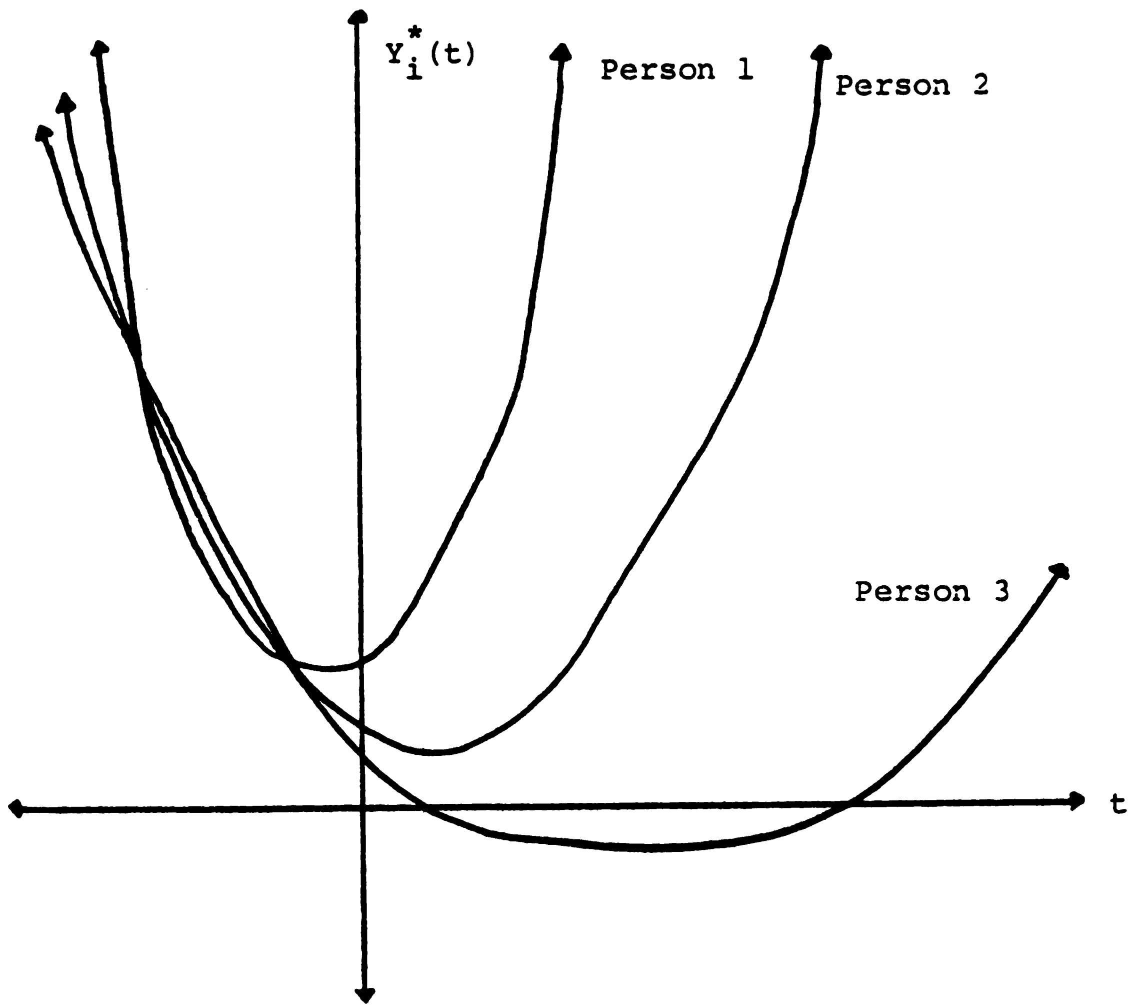

Figure 14. An example of polynomial growth when

$Y_i^*(t) = (c_2 \cdot t^2 + c_1 \cdot t + 1) \cdot Y_i^*(0) + d_1 \cdot t$

with $c_2 > 0$, $c_1 > 0$, and $d_1 < 0$.

CHAPTER 3

REVIEW OF THE LITERATURE

In this chapter the various methods that have been suggested for the analysis of data arising from the application of nonequivalent control group designs are discussed. Following the lead of Campbell and Stanley (1966) most of the literature on the problem of measuring change has restricted its attention to designs having only two groups--a treatment group and a control group. This sole attention to two group designs is unfortunate. Methods of data analysis should be discussed in the context of designs with any number of treatment groups and with or without the presence of a control group. For the remainder of this dissertation the discussion will usually be in the context of both one-group and multi-group designs where random assignment has not taken place. For multi-group designs, the presence of a control group will not be assumed. The discussion here will be restricted to two-group designs only when absolutely necessary.

The literature on the problem of measuring change is very confusing, in that each author(s) makes different assumptions and oftentimes the assumptions are implicit, rather than explicit. The attempt here is to clarify the literature by separately discussing many of the methods of data analysis that have been suggested in the literature.

The discussion for each analysis method will include an explicit statement of the assumptions being made and a short description of the method.

Since the fan spread hypothesis is the most widely made assumption in the past literature on the problem of measuring change (e.g., Bryk & Weisberg, 1977; Campbell, 1971; Kenny, 1975; Kenny & Cohen, 1980; Olejnik, 1977), a discussion of the relationship between the fan spread hypothesis and the growth models defined by equation (1-4) is offered first. This discussion will help to clarify the motivation for and the appropriateness of some of the analysis strategies to be discussed later in this chapter.

Relationship Between the Fan Spread Hypothesis and Natural Growth Models

In the previous literature the fan spread hypothesis has only been discussed in the context of two-group designs. For those designs the fan spread hypothesis states that at the population level the ratio of the difference of the group means to the standard deviation common to the populations is constant over time when there are no treatment effects (Kenny, 1975). In the past literature it is not made clear whether the standard deviation is that for the true scores or that for the observed scores. The discussion here will be in terms of both true scores and observed scores.

Symbolically, the fan spread hypothesis on the true scores can be expressed

$$\frac{\mu^*_{Y_1}(t_1) - \mu^*_{Y_2}(t_1)}{\sigma^*_Y(t_1)} = \frac{\mu^*_{Y_1}(t_2) - \mu^*_{Y_2}(t_2)}{\sigma^*_Y(t_2)},$$

where t_1 and t_2 are any two points in time;

$\mu^*_{Y_j}(t_k)$ is the population mean for group j on the measure of interest at time t_k; j=1,2; k=1,2;

and $\sigma^*_Y(t_k)$ is the standard deviation common to both populations for the true scores on the measure of interest at time t_k; k = 1,2 .

An extension of the fan spread hypothesis to multi-group designs is straight-forward by assuming that at the population level the fan spread hypothesis holds for every pair of two groups chosen without replacement from the J groups. Hence, the extended fan spread hypothesis for true scores will be defined as

$$\frac{\mu^*_{Y_j}(t_1) - \mu^*_{Y_{j'}}(t_1)}{\sigma^*_Y(t_1)} = \frac{\mu^*_{Y_j}(t_2) - \mu^*_{Y_{j'}}(t_2)}{\sigma^*_Y(t_2)}, \qquad (3\text{-}1)$$

where j, j' represent any two of the J groups.

Recall that natural growth on latent variables was defined in the system of equations (1-4) by

$$Y^*_{ij}(t) = g_j(t) \cdot Y^*_{ij}(0) + h_j(t) \ , \qquad (2\text{-}1)$$

By deriving the means and variances for these variables, it can be seen under what conditions natural growth satisfying equation (2-1) conforms to the fan spread hypothesis.

First, taking the variance of both sides of equation (2-1) yields

$$[\sigma^*_{Y_j}(t)]^2 = [g_j(t)]^2 \cdot [\sigma^*_{Y_j}(0)]^2 \ . \qquad (3\text{-}3)$$

But as stated, the fan spread hypothesis requires that the variances for any time, t, be the same for all groups. That is, it is required to have

$$\sigma^*_{Y_j}(t) = \sigma^*_Y(t) \qquad \text{for all } j.$$

Hence, by equation (3-3) it is necessary to have $g_j(t) \equiv g(t)$, for all j at each time t, and to have $\sigma^*_{Y_j}(0) \equiv \sigma^*_Y(0)$ for all j in order for the fan spread hypothesis on true scores to hold.

Consequently, equation (3-3) can be rewritten as

$$[\sigma^*_Y(t)]^2 = [g(t)]^2 \cdot [\sigma^*_Y(0)]^2 \ . \qquad (3\text{-}4)$$

Second, by taking the means on both sides of equation (2-1),

$$\mu^*_{Y_j}(t) = g(t)\mu^*_{Y_j}(0) + h_j(t) \qquad (3\text{-}5)$$

and

$$\mu^*_{Y_{j'}}(t) = g(t)\mu^*_{Y_{j'}}(0) + h_{j'}(t) \ . \qquad (3\text{-}6)$$

Substituting equations (3-4), (3-5), and (3-6) into equation (3-1) shows that for the fan spread hypothesis for true scores to hold under the growth model being considered it is necessary to have

$$\frac{[g(t_1)\mu^*_{Y_j}(0) + h_j(t_1)] - [g(t_1)\mu^*_{Y_{j'}}(0) + h_{j'}(t_1)]}{g(t_1)\sigma^*_Y(0)}$$

$$= \frac{[g(t_2)\mu^*_{Y_j}(0) + h_j(t_2)] - [g(t_2)\mu^*_{Y_{j'}}(0) + h_{j'}(t_2)]}{g(t_2)\sigma^*_Y(0)} \ . \qquad (3\text{-}7)$$

By simple algebra, equation (3-7) reduces to

$$\frac{h_j(t_1) - h_{j'}(t_1)}{g(t_1)} = \frac{h_j(t_2) - h_{j'}(t_2)}{g(t_2)}$$

for any two times, t_1 and t_2. In particular, for times $t_1 = 0$ and $t_2 = t$,

$$\frac{h_j(t) - h_{j'}(t)}{g(t)} = \frac{h_j(0) - h_{j'}(0)}{g(0)} . \qquad (3-8)$$

But, by definition, $h_j(0) = 0$, $h_{j'}(0) = 0$, and $g(0) = 1$.

Consequently by equation (3-8) $\dfrac{h_j(t) - h_{j'}(t)}{g(t)} = 0$.

Therefore, $h_j(t) \equiv h_{j'}(t)$ for each t. Hence the fan spread hypothesis for true scores holds when natural growth under equation (1-4) reduces to

$$Y^*_{ij}(t) = g(t)Y^*_{ij}(0) + h(t)$$

and when

$$\sigma^*_{Y_j}(0) = \sigma^*_Y(0) .$$

The definition of the fan spread hypothesis for observed scores is obtained by simply replacing the common to all groups standard deviation for true scores by the common to all groups standard deviation for observed scores. The numerators in equation (3-1) need not be changed since under classical measurement theory $\mu_{Y_j}(t) = \mu^*_{Y_j}(t)$ for all j and for each t. Also, under classical measurement theory

$$\sigma_Y{}^2(t) = [\sigma_{Y_j}{}^*(t)]^2 + \sigma_{e_j}{}^2(t) .$$

Hence, if $\sigma^*_{Y_j}(t) = \sigma^*_Y(t)$ and $\sigma_{e_j}(t) = \sigma_e(t)$, at each time t, then $\sigma^2_{Y_j}(t) = \sigma^2_Y(t)$ for all j, where $\sigma_e(t)$ denotes the common to all groups standard deviation of the errors of measurement and $\sigma_Y(t)$ denotes the common to all groups standard deviation of the observed scores. But for $\sigma^*_{Y_j}(t)$ to be equal to $\sigma^*_Y(t)$, it was previously shown that $g_j(t)$ must be equal to $g(t)$. Further, by equation (3-4), $\sigma^*_Y(t) = g(t)\cdot\sigma^*_Y(0)$. Hence,

$$\sigma^2_Y(t) = [g(t)]^2\cdot[\sigma^*_Y(0)]^2 + \sigma^2_e(t) \quad . \qquad (3\text{-}9)$$

The fan spread hypothesis on observed scores can be expressed as

$$\frac{\mu_{Y_j}(t_1) - \mu_{Y_{j'}}(t_1)}{\sigma_Y(t_1)} = \frac{\mu_{Y_j}(t_2) - \mu_{Y_{j'}}(t_2)}{\sigma_Y(t_2)} \quad . \qquad (3\text{-}10)$$

Substituting equations (3-4), (3-5), and (3-9) into equation (3-10) yields

$$\frac{[g(t_1)\mu_{Y_j}(0) + h_j(t_1)] - [g(t_1)\mu_{Y_{j'}}(0) + h_{j'}(t_1)]}{\sqrt{g^2(t_1)[\sigma_Y^*(0)]^2 + \sigma_e^2(t_1)}}$$

$$= \frac{[g(t_2)\mu_{Y_j}(0) + h_j(t_2)] - [g(t_2)\mu_{Y_{j'}}(0) + h_{j'}(t_2)]}{\sqrt{g^2(t_2)[\sigma_Y^*(0)]^2 + \sigma_e^2(t_2)}} . \quad (3\text{-}11)$$

If, in addition $\sigma_e^2(t) = [g(t)]^2 \cdot \sigma_e^2(0)$ for each t, then equation (3-11) simplifies to equation (3-8). Hence, $h_j(t) \equiv h_{j'}(t)$. Therefore, the fan spread hypothesis for observed scores holds when natural growth under equation (1-4) reduces to

$$Y_{ij}^*(t) = g(t) \cdot Y_{ij}^*(0) + h(t)$$

and when

$$\sigma_{Y_j}^*(0) = \sigma_Y^*(0) \quad \text{and} \quad \sigma_{e_j}(t) = g(t) \cdot \sigma_e(0).$$

The statement that $\sigma_{e_j}(t) = g(t)\sigma_e(0)$ is equivalent to requiring that the reliability of Y be constant over time and across groups.

Olejnik (1977) has asserted that the fan spread hypothesis for either true scores or observed scores can hold even when there is not a correlation of +1 within each group between true scores at any two points in time.

Although he gives no mathematical formulation of individual growth in this case, Figure 15 shows the example he gave in his dissertation. For this example, he has assumed that at the population mean level both groups (he deals only with two-group designs) follow the same differential linear growth pattern, and hence, he has only drawn growth curves for individuals from one group. It should be noted that Olejnik's restriction to differential linear growth is not necessary. Figure 16 gives a sketch of an example where the fan spread hypothesis holds for an exponential type model for group mean growth and where a correlation of +1 between true scores within each group is not assumed. For simplicity, only two groups have been drawn. In Figures 15 and 16 the solid lines represent population mean growth and the dotted lines indicate the growth curves for some selected individuals.

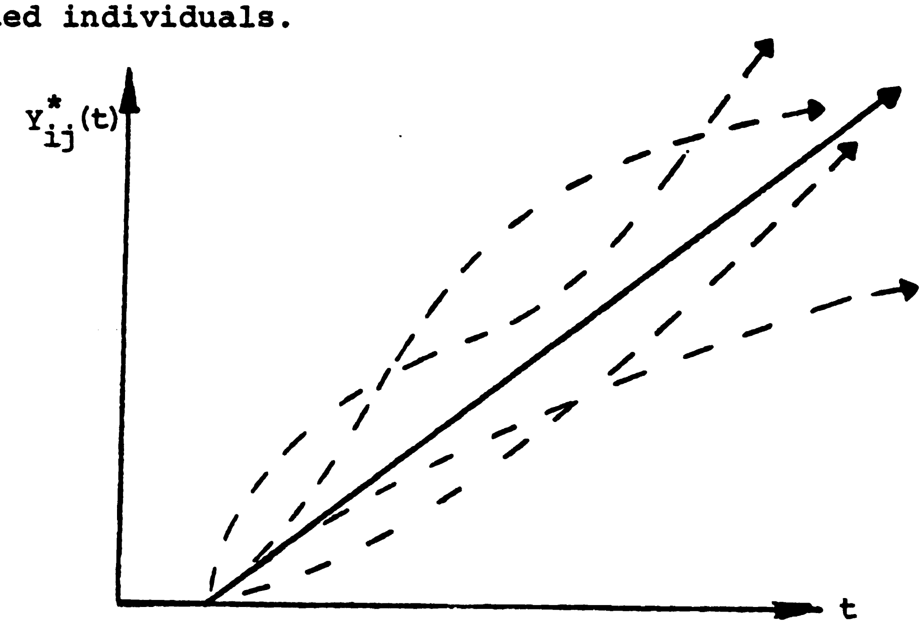

Figure 15. Olejnik's example.

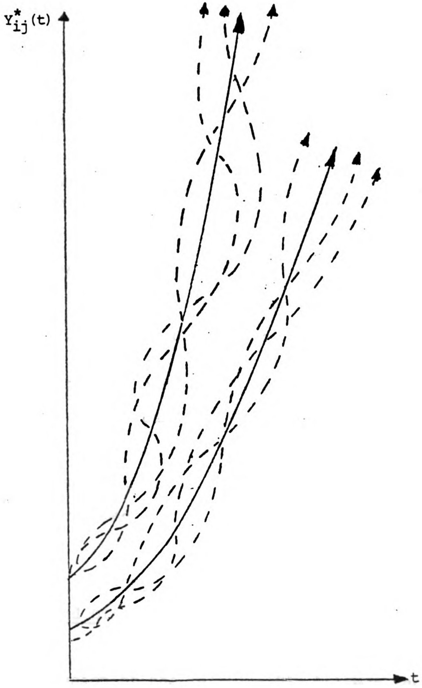

Figure 16. Exponential group growth under the fan spread hypothesis.

In summary, the fan spread hypothesis for true scores holds when equation (1-4) reduces to

$$Y^*_{ij}(t) = g(t)Y^*_{ij}(0) + h(t) + \alpha_j(t)$$

$$Y_{ij}(t) = Y^*_{ij}(t) + e_{ij}(t)$$

and (3-12)

$$\sigma^*_{Y_j}(t) = \sigma^*_Y(t) \quad .$$

The fan spread hypothesis for observed scores holds for those growth models where the system of equations (3-12) are fulfilled and, in addition, $\sigma_{e_j}(t) = g(t) \cdot \sigma_e(0)$.

Further, both the fan spread hypothesis for true scores and for observed scores can hold for some models of individual growth where there is not a correlation of +1 within each group. No mathematical formulation of these models has yet to be developed.

Relationship Between the Fan Spread Hypothesis and Differential Linear Growth

A discussion of the relationship between the fan spread hypothesis and differential linear growth is needed since in the past literature the distinction between these two concepts has been blurred. The concepts of differential linear growth and of the fan spread hypothesis are, however, distinct concepts. In the previous two sections it was shown that the fan spread hypothesis holds for many forms of natural growth other than differential linear growth. Hence, linear growth that conforms to the fan spread hypothesis is a subset of all natural growth that conforms to the fan spread hypothesis. Further, differential linear growth conforms to the fan spread hypothesis only in rare cases. Differential linear growth is represented in general by

$$Y^*_{ij}(t) = (b_j \cdot t + 1) \cdot Y^*_{ij}(0) + c_j \cdot t \ ,$$

That is, when $g_j(t) = b_j \cdot t + 1$ and $h_j(t) = c_j \cdot t$. But, for the fan spread hypothesis (either for true scores or observed scores) to hold it is necessary to have $g_j(t) \equiv g(t)$ and $h_j(t) \equiv h(t)$. Hence, the fan spread hypothesis does not hold under differential linear growth unless all of the b_j's are equal to some common value and all of the c_j's are equal to some common value.

Analysis Strategies

For the remainder of this dissertation the only continuous growth models that will be considered are those represented by the system of equations (1-4)

$$Y^*_{ij}(t) = g_j(t) \cdot Y^*_{ij}(t_{1_j}) + h_j(t) + \alpha_j(t)$$

and (1-4)

$$Y_{ij}(t) = Y^*_{ij}(t) + e_{ij}(t) \ .$$

Recall that $\alpha_j(t)$ represents the amount of growth for group j, over that which would have occurred under natural growth. Let $\mu_\alpha(t)$ represent the mean of the $\alpha_j(t)$'s. Then, $\gamma_j(t)$, as defined by $\gamma_j(t) = \alpha_j(t) - \mu_\alpha(t)$, represents what has traditionally been called a treatment effect under comparative experiments.

The past literature on the problem of measuring change has been divided into two groups with respect to the discussion of treatment effects. The majority of the literature has been concerned with treatment effects as defined by the $\gamma_j(t)$'s. A small subset of the literature has been concerned with treatment effects as defined by the $\alpha_j(t)$'s. Those analysis strategies which deal with treatment effects as defined by the $\gamma_j(t)$'s will be discussed first and with respect to three criteria:

(1) Under what additional conditions over those of equation (1-4) does the analysis strategy provide unbiased point estimates of the differences between treatment effects?

(2) If under certain conditions the analysis strategy provides unbiased point estimates of the differences between

treatment effects, then how are interval estimates of the differences constructed?

and (3) How is the hypothesis testing of

$$H_o: \sum_{j=1}^{J} (\gamma_j(t))^2 = 0$$

(i.e., H_o: All of the $\gamma_j(t)$'s are equal)

versus

$$H_1: \sum_{j=1}^{J} (\gamma_j(t))^2 \neq 0$$

(i.e., H_1: All of the $\gamma_j(t)$'s are not equal)

accomplished?

Before describing the analysis strategies it should be pointed out that, by definition,

$$\gamma_j(t) - \gamma_{j'}(t) = [\alpha_j(t) - \mu_\alpha(t)] - [\alpha_{j'}(t) - \mu_\alpha(t)]$$

Hence, $\gamma_j(t) - \gamma_{j'}(t) = \alpha_j(t) - \alpha_{j'}(t)$. Since the growth models described by the system of equations (1-4) are in terms of the $\alpha_j(t)$'s rather than the $\gamma_j(t)$'s, the discussion of the three criteria will be in terms of the $\alpha_j(t) - \alpha_{j'}(t)$'s.

ANOVA of Index of Response

Analysis of Variance (ANOVA) of Index of Response refers to a group of analysis strategies, rather than a single method. Much of the literature on the problem of measuring change has considered various forms of ANOVA of Index of Response as approaches to data analysis for nonequivalent control group designs. A general discussion of ANOVA of Index of Response will be given first. Then, specific forms of ANOVA of Index of Response will be discussed. A description of ANOVA of Index of Response is also being included here since the methods of data analysis to be developed in Chapters 4 through 6 can be thought of as generalizations of this group of approaches.

To employ ANOVA of Index of Response it is only necessary to have scores on the measure of interest at two points in time for two or more groups. Taking data at the first point in time as a pretest, without loss of generality, it can be assumed that the pretest observations are taken at time $t_{1_j} = 0$ for all j. The second time point is considered to be a posttest, given at some time, t, past when the treatments have been initiated. It is assumed here and throughout the remainder of the dissertation that all individuals in all of the treatment groups start receiving the treatment at the same time. A new score is formed, defined by $Z_{ij}(t) = Y_{ij}(t) - K \cdot Y_{ij}(0)$, where K is some known constant (Cox, 1958; Porter, 1973). The method of data analysis then used is an ANOVA with the $Z_{ij}(t)$'s as the dependent variable. Notice that ANOVA of Index of Response with K = 0 reduces to an ANOVA of the $Y_{ij}(t)$'s.

The linear model for ANOVA of Index of Response is as for ANOVA,

$$Z_{ij}(t) = \mu_Z(t) + (\tau_{IR})_j + (f_{IR})_{ij},$$

where $\mu_Z(t) = \mu_Y(t) - K\cdot\mu_Y(0)$ (3-13)

is the population grand mean for $Z(t)$;

$\mu_Y(t)$ is the population grand mean for $Y(t)$;

$(\tau_{IR})_j = \mu_{Z_j}(t) - \mu_Z(t)$; (3-14)

IR denotes Index of Response;

and $(f_{IR})_{ij}$ is the error term for an individual.

First, notice that

$$\mu_{Z_j}(t) = \mu_{Y_j}(t) - K\cdot\mu_{Y_j}(0) \ . \qquad (3\text{-}15)$$

Second, by substituting equations (3-13) and (3-15) into equation (3-14)

$$(\tau_{IR})_j = \mu_{Y_j}(t) - \mu_Y(t) - K(\mu_{Y_j}(0) - \mu_Y(0)) \ . \qquad (3\text{-}16)$$

Consequently, for any two groups j and j',

$$(\tau_{IR})_j - (\tau_{IR})_{j'} = \mu_{Y_j}(t) - \mu_{Y_{j'}}(t) - K(\mu_{Y_j}(0) - \mu_{Y_{j'}}(0)) \ . \qquad (3\text{-}17)$$

Next, taking the means on both sides of the system of equations (1-4) yields

$$\mu_{Y_j}(t) = g_j(t)\mu_{Y_j}(0) + h_j(t) + \alpha_j(t) .$$

Solving for $\alpha_j(t)$ gives

$$\alpha_j(t) = \mu_{Y_j}(t) - [g_j(t)\mu_{Y_j}(0) + h_j(t)] . \quad (3\text{-}18)$$

So,

$$\alpha_j(t) - \alpha_{j'}(t) = \{\mu_{Y_j}(t) - [g_j(t)\mu_{Y_j}(0) + h_j(t)]\} - \{\mu_{Y_{j'}}(t) - [g_{j'}(t)\mu_{Y_{j'}}(0) + h_{j'}(t)]\} .$$

Therefore,

$$\alpha_j(t) - \alpha_{j'}(t) = \mu_{Y_j}(t) - \mu_{Y_{j'}}(t) - \{[g_j(t)\mu_{Y_j}(0) + h_j(t)] - [g_{j'}(t)\mu_{Y_{j'}}(0) + h_{j'}(t)]\} . \quad (3\text{-}19)$$

Hence, by comparing equations (3-17) and (3-19), ANOVA of Index of Response theoretically provides correctly defined differences in treatment effects between groups j and j' if and only if

$$[g_j(t)\mu_{Y_j}(0) + h_j(t)] - [g_{j'}(t)\mu_{Y_{j'}}(0) + h_{j'}(t)]$$
$$= K(\mu_{Y_j}(0) - \mu_{Y_{j'}}(0)).$$

That is, if and only if

$$K = \frac{[g_j(t)\mu_{Y_j}(0) + h_j(t)] - [g_{j'}(t)\mu_{Y_{j'}}(0) + h_{j'}(t)]}{\mu_{Y_j}(0) - \mu_{Y_{j'}}(0)} . \quad (3\text{-}20)$$

Hence for two-group designs ANOVA of Index of Response with

$$K = \frac{[g_1(t)\mu_{Y_1}(0) + h_1(t)] - [g_2(t)\mu_{Y_2}(0) + h_2(t)]}{\mu_{Y_1}(0) - \mu_{Y_2}(0)}$$

provides correctly defined differences in treatment effects. For designs with more than two groups, it is required that equation (3-20) hold simultaneously for all possible pairs of groups if ANOVA of Index of Response is to provide correctly defined treatment effects. Equation (3-20) will hold simultaneously for all possible pairs of groups when $g_j(t) \equiv g(t)$ and $h_j(t) \equiv h(t)$, which previously was shown to be equivalent to the fan spread hypothesis on true scores. Also, the fan spread hypothesis on observed scores will result in this being satisfied, but recall that the fan spread hypothesis on observed scores requires even more than this. When $g_j(t) \equiv g(t)$ and $h_j(t) \equiv h(t)$, equation (3-20) simplifies to $K = g(t)$. Further, there are, however, rare cases where equation (3-20) holds simultaneously for all possible pairs and $g_j(t) \not\equiv g(t)$ and/or $h_j(t) \not\equiv h(t)$.

No mathematical formulation of these rare cases has yet to be developed.

In summary, ANOVA of Index of Response always provides correctly defined differences in treatment effects for two-group designs by setting

$$K = \frac{[g_1(t)\mu_{Y_1}(0) + h_1(t)] - [g_2(t)\mu_{Y_2}(0) + h_2(t)]}{\mu_{Y_1}(0) - \mu_{Y_2}(0)} .$$

Further, ANOVA of Index of Response with $K = g(t)$ provides correctly defined differences in treatment effects for designs with two or more groups when the system of equations (1-4) reduces to

$$Y^*_{ij}(t) = g(t) \cdot Y^*_{ij}(0) + h(t) + \alpha_j(t)$$

and

$$Y_{ij}(t) = Y^*_{ij}(t) + e_{ij}(t) . \qquad (3\text{-}21)$$

Under ANOVA of Index of Response, estimates of differences in treatment effects are given by

$$\widehat{\alpha_j(t) - \alpha_{j'}(t)} = \overline{Z_j(t)} - \overline{Z_{j'}(t)}$$

$$= \overline{Y_j(t)} - \overline{Y_{j'}(t)} - K[\overline{Y_j(0)} - \overline{Y_{j'}(0)}] .$$

By elementary algebra and statistics it can be shown that under a variety of statistical distributions

$$E(\widehat{\alpha_j(t) - \alpha_{j'}(t)}) = \alpha_j(t) - \alpha_{j'}(t) \ .$$

The interval estimation and hypothesis testing procedures are those of a traditional analysis of variance, but now performed on the $Z_{ij}(t)$'s.

ANOVA of Gain Scores

ANOVA of Gain Scores (Bereiter, 1963) is the special case of ANOVA of Index of Response where K = 1. Hence, by applying the results of the previous section, ANOVA of Gain Scores yields correctly defined (i.e., unbiased) differences in treatment effects and a correct test of the null hypothesis of nonzero differences in treatment effects when $Y^*_{ij}(t) = Y^*_{ij}(0) + h(t) + \alpha_j(t)$. That is, ANOVA of Gain Scores provides correctly defined differences in treatment effects and correct hypothesis testing procedures for designs where random assignment has not occurred when natural growth takes the form of parallel growth. This same result was proved by Porter (1973) and Kenny (1975) using different methods of proof.

ANOVA of Standardized Change Scores

ANOVA of Standardized Change Scores was introduced by Kenny (1975) to provide a method of data analysis which will correctly test for differences in treatment effects in nonequivalent control group designs where the fan spread hypothesis holds. Although Kenny only discussed ANOVA of Standardized Change Scores in terms of two-group designs, the discussion here will be in terms of designs with any number of groups. Kenny does not clearly distinguish between those cases where errors of measurement are present and not present. Further, he does not distinguish between the sample and population values for the variances. Hence, ANOVA of Standardized Change Scores can really be considered as three separate variations of an ANOVA of Index of Response:

(1) $$K = \frac{\sigma^*_Y(t)}{\sigma^*_Y(0)},$$

(2) $$K = \frac{\sigma_Y(t)}{\sigma_Y(0)},$$

and

(3) $$K = \frac{S_Y(t)}{S_Y(0)},$$

where $S_Y^2(t)$ is the pooled-within groups variance of Y(t) for t $\geq$ 0. The first two of these, which use population values, will be discussed presently. The third one, which uses sample values, will be discussed later in this chapter.

When $K = \frac{\sigma_Y^*(t)}{\sigma_Y^*(0)}$ the analysis strategy is called ANOVA of Standardized Change Scores with reliability correction (Kenny & Cohen, 1980). When $K = \frac{\sigma_Y(t)}{\sigma_Y(0)}$ the analysis strategy will be called ANOVA of Observed Standardized Change Scores and when $K = \frac{S_Y(t)}{S_Y(0)}$ the analysis strategy will be called ANOVA of Estimated Standardized Change Scores.

<u>ANOVA of Standardized Change Scores with reliability correction</u>

Since ANOVA of Standardized Change Scores with reliability correction uses $K = \frac{\sigma_Y^*(t)}{\sigma_Y^*(0)}$, the assumption is being made that there is a common variance for true scores at any point in time for all J groups. In the section on the fan spread hypothesis it was shown that the J groups would have a common true scores variance, $[\sigma_Y^*(t)]^2$, at time t, when

$$Y_{ij}^*(t) = g(t) \cdot Y_{ij}^*(0) + h(t) + \alpha_j(t)$$

and

$$\sigma_{Y_j}^*(0) = \sigma_Y^*(0) \text{ for all } j .$$

Further, it was shown in that section that

$$[\sigma_Y^*(t)]^2 = [g(t)]^2 \cdot [\sigma_Y^*(0)]^2 . \qquad (3\text{-}22)$$

Hence,

$$K = \frac{\sigma_Y^*(t)}{\sigma_Y^*(0)} = \frac{g(t) \cdot \sigma_Y^*(t)}{\sigma_Y^*(t)} = g(t) .$$

So ANOVA of Standardized Change Scores with reliability correction is the special case of ANOVA of Index of Response with $K = g(t)$ and where there is a common variance known to exist at each time t. Consequently, ANOVA of Standardized Change Scores with reliability correction provides correctly defined differences in treatment effects and tests for non-zero differences in treatment effects when

$$Y^*_{ij}(t) = g(t)\cdot Y^*_{ij}(0) + h(t) + \alpha_j(t)$$

and

$$\sigma^*_{Y_j}(0) = \sigma^*_Y(0) \qquad \text{for all } j \ .$$

ANOVA of Observed Standardized Change Scores

ANOVA of Observed Standardized Change Scores is the special case of ANOVA of Index of Response where $K = \dfrac{\sigma_Y(t)}{\sigma_Y(0)}$. Recall that ANOVA of Index of Response yields correctly defined differences in treatment effects and correctly tests for nonzero differences in treatment effects when $K = g(t)$. Hence, ANOVA of Observed Standardized Change Scores yields correctly defined differences in treatment effects and correctly tests for nonzero differences when $g(t) = \dfrac{\sigma_Y(t)}{\sigma_Y(0)}$. In the section on the fan spread hypothesis it was shown that $g(t) = \dfrac{\sigma_Y(t)}{\sigma_Y(0)}$ when

$$Y^*_{ij}(t) = g(t)\cdot Y^*_{ij}(0) + h(t) + \alpha_j(t) \tag{3-23}$$

$$Y_{ij}(t) = Y^*_{ij}(t) + e_{ij}(t) \tag{3-24}$$

$$\sigma_{e_j}(t) = g(t)\cdot\sigma_{e_j}(0) \qquad (3\text{-}25)$$

and

$$\sigma_{Y_j}(t) = \sigma_Y(t) \quad . \qquad (3\text{-}26)$$

Hence, ANOVA of Observed Standardized Change Scores yields correctly defined differences in treatment effects and tests for nonzero differences when equations (3-23) to (3-26) hold. It should be noted that there are conditions where the system of equations (1-4) holds and under which ANOVA of Observed Standardized Change Scores yields correctly defined differences in treatment effects and correctly tests for nonzero differences that have not been included in the set of conditions delineated by equations (3-23) to (3-26). No mathematical formulation of these other conditions has yet to be developed.

ANOVA of Residual Gain Scores

ANOVA of Residual Gain Scores has been widely suggested in the past literature on the problem of measuring change as being a possible method of data analysis for nonequivalent control group designs (Cronbach & Furby, 1970; Linn & Slinde, 1977; Porter & Chibucos, 1974). ANOVA of Residual Gain Scores actually represents three different methods. The first, which is called ANOVA of True Residual Gain Scores, is defined by $K = \beta_{Y^*(t)\cdot Y^*(0)}$, where $\beta_{Y^*(t)\cdot Y^*(0)}$

is the slope of the $Y^*(t)$ on $Y^*(0)$ regression line (Cronbach & Furby, 1970). The second, which is called ANOVA of Raw Residual Gain Scores, is defined by $K = \beta_{Y(t)\cdot Y(0)}$, where $\beta_{Y(t)\cdot Y(0)}$ is the slope of the Y(t) on Y(0) regression line (Cronbach & Furby, 1970). The third, which will be called ANOVA of Estimated Residual Gain Scores, is defined by $K = \hat{\beta}_{Y(t)\cdot Y(0)}$, where $\hat{\beta}_{Y(t)\cdot Y(0)}$ is the least squares estimate of the slope of the Y(t) on Y(0) regression line or by $K = \dfrac{\hat{\beta}_{Y(t)\cdot Y(0)}}{\hat{\rho}_{Y(0)Y(0)}}$, where $\hat{\rho}_{Y(0)Y(0)}$ is some estimator of the reliability of Y(0). ANOVA of True Residual Gain Scores and ANOVA of Raw Residual Gain Scores will be discussed presently. The discussion of ANOVA of Estimated Residual Gain Scores will be postponed until later in this chapter.

ANOVA of True Residual Gain Scores

Recall that ANOVA of True Residual Gain Scores is defined by $K = \beta_{Y^*(t)\cdot Y^*(0)}$. By definition, $\beta_{Y^*(t)\cdot Y^*(0)}$ $= \rho_{Y^*(t)\cdot Y^*(0)} \cdot \dfrac{\sigma^*_Y(t)}{\sigma^*_Y(0)}$. But, for the growth models considered here, $\rho_{Y^*(t)Y^*(0)} = 1$. Hence, $\beta_{Y^*(t)\cdot Y^*(0)}$

$$= \frac{\sigma_Y^*(t)}{\sigma_Y^*(0)}.$$ Consequently, for the class of growth models under consideration ANOVA of True Residual Gain Scores is identical to ANOVA of True Standardized Change Scores.

ANOVA of Raw Residual Gain Scores

ANOVA of Raw Residual Gain Scores is defined by $K = \beta_{Y(t)\cdot Y(0)}$. It is well known from classical measurement theory that $\beta_{Y(t)\cdot Y(0)}$ is equal to $\rho_{Y(0)Y(0)}\cdot\beta_{Y^*(t)\cdot Y^*(0)}$, where $\rho_{Y(0)Y(0)}$ is the reliability of Y at time 0. Further, by the previous section, $\beta_{Y^*(t)\cdot Y^*(0)} = \frac{\sigma_Y^*(t)}{\sigma_Y^*(0)}$. But, by equation (3-3), $\sigma_Y^*(t) = g(t)\cdot\sigma_Y^*(0)$. Hence,

$$\beta_{Y^*(t)\cdot Y^*(0)} = g(t). \qquad (3\text{-}27)$$

Whence, $\beta_{Y(t)\cdot Y(0)} = \rho_{Y(0)Y(0)}\cdot g(t)$. Consequently, ANOVA of Raw Residual Gain Scores only provides correctly defined differences and tests of nonzero differences when there are no errors of measurement present (i.e., when $\rho_{Y(0)Y(0)} = 1$) and for each time t, $g_j(t) \equiv g(t)$ and $h_j(t) \equiv h(t)$ for all j.

True Difference Scores

In their 1966 paper, Tucker, Damarin, and Messick (1966) introduced the concept of a true difference score. A true difference score, $D_{ij}(t)$, is defined by

$$D_{ij}(t) = Y_{ij}(t) - \frac{\rho_{Y(t)Y(0)} \cdot \sigma_Y(t)}{\rho_{Y(0)Y(0)} \cdot \sigma_Y(0)} \cdot Y_{ij}(0),$$

which is in the form on an index of response. But,

$$\frac{\rho_{Y(t)Y(0)} \cdot \sigma_Y(t)}{\rho_{Y(0)Y(0)} \cdot \sigma_Y(0)} = \frac{\beta_{Y(t) \cdot Y(0)}}{\rho_{Y(0)Y(0)}} = \beta_{Y^*(t) \cdot Y^*(0)} \; .$$ Further,

by equation (3-27), $\beta_{Y^*(t) \cdot Y^*(0)}$ is equal to $g(t)$. Therefore, the $D_{ij}(t)$'s are the same as the $Z_{ij}(t)$'s as defined in the ANOVA of Index of Response with $K = g(t)$. Thus, True Difference Scores provides correctly defined differences in treatment effects and correctly tests for nonzero differences in treatment effects when

$$Y^*_{ij}(t) = g(t) \cdot Y^*_{ij}(0) + h_j(t) + \alpha_j(t)$$

and

$$Y_{ij}(t) = Y^*_{ij}(t) + e_{ij}(t) \; .$$

ANOVA of Estimated Standardized Change Scores

ANOVA of Estimated Standardized Change Scores (Kenny, 1975; Olejnik, 1977) is the analysis strategy in which an ANOVA is performed on the $Z_{ij}(t)$'s as defined by

$$Z_{ij}(t) = Y_{ij}(t) - \frac{S_Y(t)}{S_Y(0)} \cdot Y_{ij}(0) \ .$$

ANOVA of Estimated Standardized Change Scores is an attempt to develop a method of data analysis when $\sigma_Y(0)$ and $\sigma_Y(t)$ are not known, which is analogous to ANOVA of Observed Standardized Scores. There are, however, several problems which arise when trying to go from observed standardized change scores to estimated standardized change scores.

The first problem is that the distribution of the $Z_{ij}(t)$'s may not be a normal distribution, even if the vector $(Y(0),Y(t))$ has a bivariate normal distribution. The second problem is that in order to decide whether ANOVA of Estimated Standardized Change Scores yields correctly defined differences in treatment effects at the population level, it is first necessary to derive an expression for $E(Z_{ij}(t))$. Unfortunately, the problem of finding $E(Z_{ij}(t))$ is an unsolved problem, even for the case when it can be assumed that the vector $(Y(0),Y(t))$ has a bivariate normal distribution. So, one can not determine theoretically under what conditions the $E(Z_{ij}(t))$'s lead to correctly defined differences in treatment effects.

For large enough sample sizes (i.e., $N \geq 30$) it has been shown that ANOVA is robust with respect to the violation of the assumption of a normal distribution for the $Z_{ij}(t)$'s (Glass, Peckham, & Sanders, 1972). Further, asymptotically

$$\mu_{Z_j}(t) = \mu_{Y_j}(t) - \frac{\sigma_Y(t)}{\sigma_Y(0)} \cdot \mu_{Y_j}(0) .$$

Hence, asymptotically, ANOVA of Estimated Standardized Change Scores provides correctly defined differences in treatment effects and correctly tests the hypothesis of nonzero differences in treatment effects under the same conditions as does ANOVA of Observed Standardized Change Scores. Until the problem of finding the expected value of the $Z_{ij}(t)$'s has been solved and a test statistic has been defined with known distribution, it is recommended that ANOVA of Estimated Standardized Change Scores not be used as a method of data analysis, even for true experiments, when small sample sizes are present.

ANOVA of Estimated Residual Gain Scores

ANOVA of Estimated Residual Gain Scores (Manning & Dubois, 1962; Olejnik, 1977; Porter & Chibucos, 1974) is an analysis strategy in which an ANOVA is performed on either the $V_{ij}(t)$'s or $W_{ij}(t)$'s (depending on the particular reference) as defined by

$$V_{ij}(t) = Y_{ij}(t) - \hat{\beta}_{Y(t)\cdot Y(0)} \cdot Y_{ij}(0)$$

and

$$W_{ij}(t) = Y_{ij}(t) - \frac{\hat{\beta}_{Y(t)\cdot Y(0)}}{\hat{\rho}_{Y(0)Y(0)}} \cdot Y_{ij}(0),$$

where $\hat{\beta}_{Y(t)\cdot Y(0)}$ is the least squares estimate of the slope of the Y(t) on Y(0) regression line and where $\hat{\rho}_{Y(0)Y(0)}$ is some estimator of the reliability of Y(0).

Two problems arise when one performs the ANOVA procedures on the $V_{ij}(t)$'s or $W_{ij}(t)$'s. These problems parallel the problems discussed in the section on ANOVA of Estimated Standardized Change Scores. The first problem is that neither $E(V_{ij}(t))$ nor $E(W_{ij}(t))$ is known (Draper & Smith, 1981). The second problem is that the distributions of the $V_{ij}(t)$'s and $W_{ij}(t)$'s may not be normal, even if the vector $(Y(0),Y(t))$ has a bivariate normal distribution. Asymptotically, however,

$$\mu_{V_j}(t) = \mu_{Y_j}(t) - \beta_{Y(t)\cdot Y(0)} \cdot \mu_Y(0)$$

and

$$\mu_{W_j}(t) = \mu_{Y_j}(t) - \frac{\beta_{Y(t)\cdot Y(0)}}{\rho_{Y(0)Y(0)}} \cdot \mu_Y(0) \ .$$

Hence, asymptotically, ANOVA of Estimated Residual Gain Scores using the $V_{ij}(t)$'s provides correctly defined treatment effects and correctly tests the hypothesis of nonzero differences in treatment effects under the same conditions as does ANOVA of Raw Residual Gain Scores. Recalling that

$$\beta_{Y^*(t)\cdot Y^*(0)} = \frac{\beta_{Y(t)\cdot Y(0)}}{\rho_{Y(0)Y(0)}},$$ then asymptotically, ANOVA of Estimated Residual Gain Scores using the $W_{ij}(t)$'s provides correctly defined treatment effects and correctly tests the hypothesis of nonzero differences under the same conditions as does ANOVA of True Residual Gain Scores. As with ANOVA of Estimated Standardized Change Scores, it is recommended that ANOVA of Estimated Residual Gain Scores not be used when small sample sizes are present.

Analysis of Covariance

Analysis of Covariance is a method of data analysis where the slope of the Y(t) on Y(0) regression line is estimated, as in ANOVA of Estimated Residual Gain Scores, but where the statistical difficulties involved in doing an ANOVA of Estimated Residual Gain Scores are eliminated. The methodology involved in performing an Analysis of Covariance (ANCOVA) is well-known (see e.g., Glass & Stanley, 1970; Seber, 1977; or Winer, 1971) and will not be repeated here.

As with ANOVA of Index of Response, ANCOVA requires that the observations be taken at the same time points for all J groups. Without loss of generality, assume that there are a pretest and a posttest, given at times 0 and t, respectively. For the models under consideration here Y(0) is the covariate and Y(t) is the dependent variable for the ANCOVA. In general, when deriving the Sum of Squares to be used for an ANCOVA, the assumption is made that the covariate is fixed. In many settings such as those assumed in this dissertation it is unreasonable to assume that the covariate is fixed. In these settings, the covariate must be considered as a random variable. DeGracie and Fuller (1972) have shown, however, that ANCOVA still works when the covariate is taken as a random variable. A second assumption of ANCOVA is that the covariate is measured without error. Since the covariate and the dependent variable here are the same measure taken at two different time points, it will be assumed here that both Y(0) and Y(t) are measured without error. A further assumption made for ANCOVA is that there is a linear relationship between $Y^*(t)$ and $Y^*(0)$ when all of the subjects are considered as coming from one population. That is, ANCOVA can only be used for the models under consideration in this dissertation when

$$Y^*_{ij}(t) = g(t) \cdot Y^*_{ij}(0) + h(t) + \alpha_j(t) \ .$$

The linear model for ANCOVA with a random covariate is

$$Y_{ij}(t) = \mu_Y(t) + (\tau_{AC})_j + \beta_{Y(t)\cdot Y(0)}(Y_{ij}(0) - \mu_Y(0)) + (f_{AC})_{ij}$$

where AC denotes Analysis of Covariance;

$$(\tau_{AC})_{ij} = [\mu_{Y_j}(t) - \mu_Y(t)] - \beta_{Y(t)\cdot Y(0)}[\mu_{Y_j}(0) - \mu_Y(0)];$$

and $(f_{AC})_{ij}$ is the error term for an individual.

In the section on ANOVA of Raw Residual Gain Scores it was shown that $\beta_{Y(t)\cdot Y(0)}$ is well-defined only when, for each time t, $\sigma_{Y_j}(t) = \sigma_Y(t)$ for all j. Hence, ANCOVA provides correctly defined differences in treatment effects and correctly tests for nonzero differences in treatment effects if and only if, for each time t,

(i) no errors of measurement are present in the data;

(ii) $Y^*_{ij}(t) = g(t) \cdot Y^*_{ij}(0) + h(t) + \alpha_j(t)$;

and (iii) $\sigma^*_{Y_j}(t) = \sigma^*_Y(t)$.

The estimates of the treatment effects under ANCOVA are given by

$$\widehat{(\tau_{AC})}_j = \overline{Y_j(t)} - \overline{Y(0)} - \hat{\beta}_{Y(t) \cdot Y(0)} \cdot (\overline{Y_j(0)} - \overline{Y(0)}) \ .$$

It has been shown that $E\widehat{(\tau_{AC})}_j = (\tau_{AC})_j$ (Seber, 1977). Further, by simple algebra it can be shown that $E[\widehat{(\tau_{AC})}_j - \widehat{(\tau_{AC})}_{j'}] = \alpha_j(t) - \alpha_{j'}(t)$. Consequently, $\widehat{(\tau_{AC})}_j - \widehat{(\tau_{AC})}_{j'}$ provides an unbiased estimator of $\alpha_j(t) - \alpha_{j'}(t)$. The procedures for interval estimation and for the testing for nonzero differences in treatment effects are given in Glass & Stanley (1970) and Winer (1971).

Estimated True Scores Analysis of Covariance

Estimated True Scores Analysis of Covariance was developed by Porter (Porter, 1967; Porter & Chibucos, 1974) as an extension of Analysis of Covariance techniques to situations where a random covariate is used which contains errors of measurement. Estimated true scores, $\hat{T}_{ij}$'s, are defined by

$$\hat{T}_{ij} = \overline{Y_j(0)} + \rho_{Y(0)Y(0)} \cdot (Y_{ij}(0) - \overline{Y_j(0)}),$$

where $\rho_{Y(0)Y(0)}$ is the reliability of the measure, $Y(0)$, which is assumed the same for all J groups. An Analysis of Covariance is then performed using the $\hat{T}_{ij}$'s as the covariate and the $Y_{ij}(t)$'s as the dependent variable. In addition to assuming that the reliability of $Y(0)$ is the same for all J groups, Estimated True Scores ANCOVA makes all of the usual ANCOVA assumptions. The assumption of equal reliability, combined with the usual ANCOVA assumption that $\sigma^*_{Y_j}(0) = \sigma^*_Y(0)$ for all j, implies that $\sigma_{Y_j}(0) = \sigma_Y(0)$ for all j. Hence, Estimated True Scores ANCOVA can be used for the growth models under consideration in this dissertation only when

$$Y^*_{ij}(t) = g(t) \cdot Y^*_{ij}(0) + h(t) + \alpha_j(t)$$

$$Y_{ij}(t) = Y^*_{ij}(t) + e_{ij}(t)$$

$$\sigma_{Y_j}(0) = \sigma_Y(0)$$

and

$$\sigma^*_{Y_j}(0) = \sigma^*_Y(0) \ .$$

The linear model for Estimated True Scores ANCOVA is given by

$$Y_{ij}(t) = \mu_Y(t) + (\tau_{EAC})_j + \beta_{Y(t)\cdot\hat{T}}(\hat{\tau}_{ij} - \mu_{\hat{T}}) + (f_{EAC})_{ij} \ ;$$

where EAC denotes Estimated True Scores ANCOVA;

$(\tau_{EAC})_j = \mu_{Y_j}(t) - \mu_Y(t) - \beta_{Y(t)\cdot\hat{T}}(\mu_{\hat{T}_j} - \mu_{\hat{T}})$;

$\mu_{\hat{T}_j}$ represents the population mean for the $\hat{T}_{ij}$'s for group j ;

$\mu_{\hat{T}}$ represents the population grand mean for the $\hat{T}_{ij}$'s ;

$\beta_{Y(t)\cdot\hat{T}}$ is the slope of the Y(t) on $\hat{T}$ regression line;

and

$(f_{EAC})_{ij}$ is the error terms for an individual .

Treatment effects are estimated using

$$(\hat{\tau}_{EAC})_j = (\overline{Y_j(t)} - \overline{Y(t)}) - \hat{\beta}_{Y(t)\cdot\hat{T}}(\overline{Y_j(0)} - \overline{Y(0)}) \ ,$$

where $\hat{\beta}_{Y(t)\cdot\hat{T}} = \hat{\beta}_{Y(t)\cdot Y(0)}/\rho_{Y(0)Y(0)}$. Using Monte Carlo simulations Porter (1967) showed that $(\hat{\tau}_{EAC})_j - (\hat{\tau}_{EAC})_{j'}$ provides an estimator of $\alpha_j(t) - \alpha_{j'}(t)$, for any two groups j and j', which contains no identifiable bias. He further showed that the resulting test statistic from an ANCOVA on the $Y_{ij}(t)$'s using the $\hat{T}_{ij}$'s as a covariate is approximately distributed as an F statistic with $J - 1$ and $N - J - 1$ degrees of freedom when there is an equal number of individuals in each group. The properties of the test statistic have not been studied for those situations where there is an unequal number of individuals in the groups.

The remaining analysis strategies are concerned with the direct assessment of treatment effects (i.e., the $\alpha_j(t)$'s) as well as differences in treatment effects. These remaining analysis strategies will be discussed with respect to the estimation of treatment effects and the testing of the hypothesis of nonzero treatment effects, as well as testing differences in treatment effects.

Rogosa's Method

Rogosa (1980) has developed a method for estimating treatment effects for two-group designs in those situations where no errors of measurement are present, $h_j(t) \equiv 0$, and the observations are taken at the same time points for both groups. Rogosa's method is included here because the methods to be developed in Chapter 4 can be thought of as generalizations of this method. Under Rogosa's assumptions the system of equations (1-4) reduces to

$$Y_{ij}(t) = g_j(t) \cdot Y_{ij}(0) + \alpha_j(t); \; j = 1,2. \qquad (3\text{-}28)$$

Rogosa rewrites the set of equations (3-28) as

$$Y_{ij}(t) = \gamma_1 + \gamma_2 \cdot T_{ij} + \gamma_3 \cdot Y_{ij}(0) + \gamma_4 \cdot Y_{ij}(0) \cdot T_{ij} \; ,$$

where

$$T_{ij} = \begin{cases} 1 & \text{if } j = 1 \\ 0 & \text{if } j = 2 \end{cases} ;$$

$$\gamma_1 = \alpha_2(t) \; ;$$

$$\gamma_2 = \alpha_1(t) - \alpha_2(t) \; ;$$

$$\gamma_3 = g_2(t) \; ;$$

and

$$\gamma_4 = g_1(t) - g_2(t) \; .$$

Next, he computes the least squares estimates of γ_1, γ_2, γ_3, and γ_4. Call these least squares estimates $\hat{\gamma}_1$, $\hat{\gamma}_2$, $\hat{\gamma}_3$, and $\hat{\gamma}_4$.

Rogosa assumes, as is done in traditional regression analysis, that the only values of $Y(0)$ which are of interest are the observed values from the sample used. Hence, the $\hat{\gamma}_i$'s are unbiased estimators of the γ_i's. But, the derivations given in DeGracie and Fuller (1972) can be applied here to show that the $\hat{\gamma}_i$'s are unbiased estimators of the γ_i's even when Y(0) is taken to be a random variable. Hence, $\hat{\gamma}_1 + \hat{\gamma}_2$ provides an unbiased estimate of $\alpha_1(t)$, $\hat{\gamma}_1$ provides an unbiased estimate of $\alpha_2(t)$, and $\hat{\gamma}_2$ provides an unbiased estimate of $\alpha_1(t) - \alpha_2(t)$. Rogosa does not discuss how to test whether the difference in treatment effects is nonzero or how to test whether the treatment effects themselves are nonzero. Notice that Rogosa's method can provide unbiased estimates of treatment effects and differences in treatment effects for designs with any number of groups, as long as the assumptions detailed in the first paragraph of this section hold, by repeating his procedure for each combination of two groups chosen from the J groups.

Adjusted Gain Scores

Olejnik (1977) developed a method of data analysis for those two-group designs where it is assumed that mean group growth at the population level is linear over time when no treatment effects are present. The reason for including Adjusted Gain Scores here is that the methods to be developed in Chapters 4 and 5 can be thought of as generalizations of this method. Olejnik requires that two pretest observations and one posttest observation are taken on the measure of interest. The pretest observations are taken at times 0 and t_2 and the posttest observations are taken at time t_3. Figure 17 provides a pictorial representation of Olejnik's model.

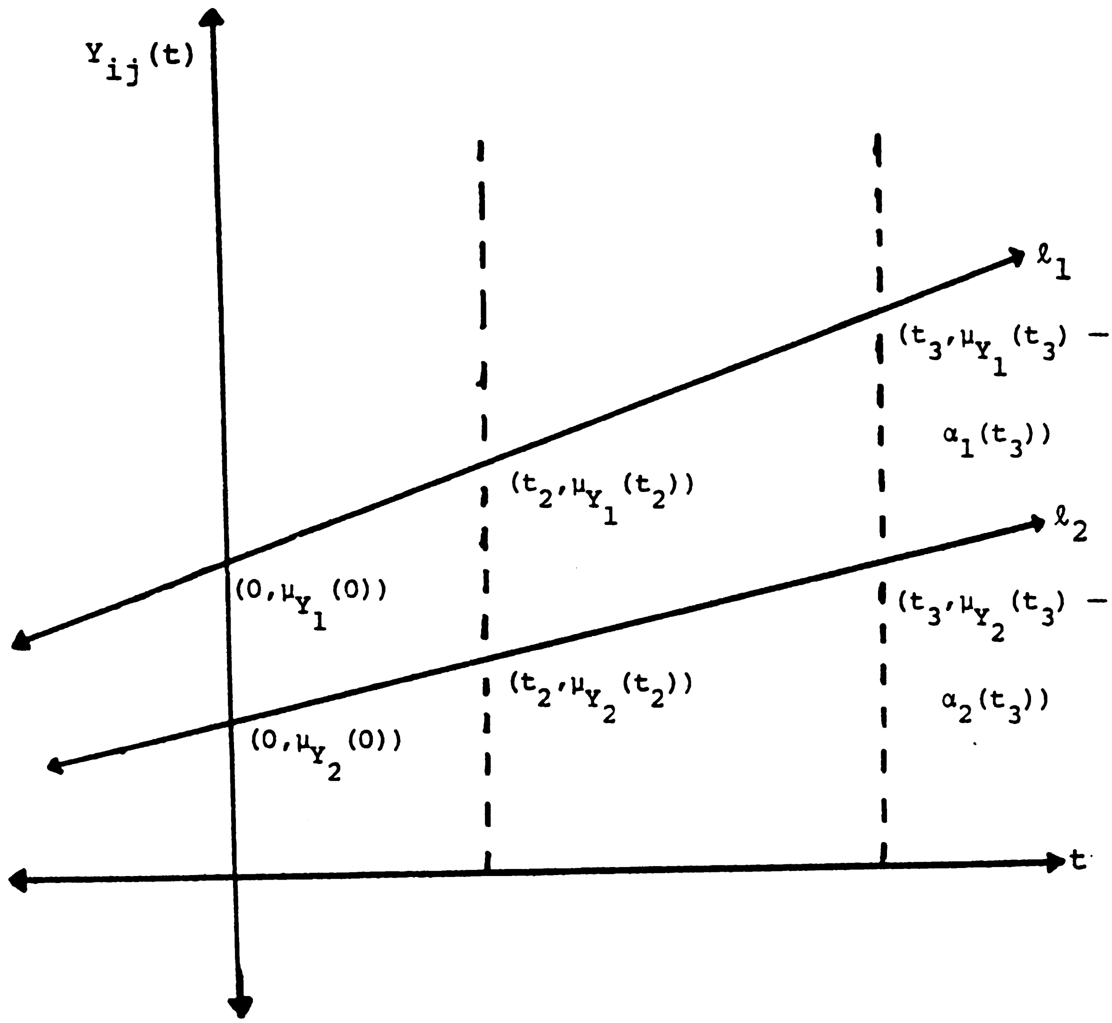

Figure 17. Olejnik's model.

Olejnik defines an Adjusted Gain Score by

$$W_{ij}(t_3) = Y_{ij}(t_3) - Y_{ij}(0) - (\overline{Y_j(t_2)} - \overline{Y_j(0)}) \cdot \frac{t_3}{t_2} .$$

He shows that the $\overline{W_j(t)}$'s provide unbiased estimates of the treatment effects (i.e., the $\alpha_j(t)$'s).

It should be noted here that the development of the estimates for treatment effects using Adjusted Gain Scores does not depend upon having exactly two groups in the design. Hence, unbiased estimates of treatment effects can be found using Adjusted Gain Scores for any growth situation for which

$$Y^*_{ij}(t) = (b_j \cdot t + 1) \cdot Y^*_{ij}(0) + c_j \cdot t + \alpha_j(t)$$

and

$$Y_{ij}(t) = Y^*_{ij}(t) + e_{ij}(t) ,$$

where the b_j's and c_j's are real-valued constants with $b_j \neq 0$. The hypothesis of nonzero treatment effects is tested by performing a one-sample t-test, separately, on the $W_{ij}(t)$'s for each of the J groups. The hypothesis of nonzero differences is tested by performing an ANOVA on the $W_{ij}(t)$'s.

Empirical Bayes Estimation

Strenio, Bryk, and Weisberg (Bryk, Strenio, & Weisberg, 1980; Strenio, Weisberg, and Bryk, in press) have applied the ideas of empirical Bayes estimation (Fearn, 1975) in order to estimate treatment effects when certain types of continuous growth models are assumed. The most general continuous growth model assumed by Strenio, Weisberg, and Bryk (in press) is

$$Y_{ij}^{*}(t) = (1 + \sum_{\ell=1}^{L_j} k_{ij\ell} \cdot (t - t_{1_j})^{\ell}) \cdot Y_{ij}^{*}(t_{1_j})$$

and (3-29)

$$Y_{ij}(t) = Y_{ij}^{*}(t) + e_{ij}(t),$$

where the $k_{ij\ell}$'s are unknown real-valued constants, the L_j's are predetermined positive integers, and t_{1_j} is the time of the first observations for the jth group. The reason for the absence of a treatment effect in equation (3-29) is that Strenio, Weisberg, and Bryk consider only natural growth in their paper. The model given by equation (3-29) and the models of growth given by the system of equations (1-4) overlap only in the case where, for each j, the $k_{ij\ell}$'s are equal to some common value (say $k_{j\ell}$) across individuals. That is, (3-29) and (1-4) overlap only when

$$g_j(t) = 1 + \sum_{\ell=1}^{L_j} k_{j\ell}\cdot(t - t_{1_j})^{\ell} \text{ and } h_j(t) \equiv 0 \text{ .} \quad (3\text{-}30)$$

The idea of empirical Bayes estimation, in general, is to obtain estimates for the $k_{ij\ell}$'s for each person by using a weighted sum of the information available for that person and the information available about the remainder of the individuals in the group. One of the requirements of the empirical Bayes method is that the variance-covariance matrix of the vector $\underline{Y}_j^* = (Y_j^*(t_{2_j}), Y_j^*(t_{3_j}), \ldots, Y_j^*(t_{p_j}))$ be nonsingular, where $t_{2_j}, t_{3_j}, \ldots, t_{p_j}$ are the times of the additional observations on the measure of interest. The variance-covariance matrix of $\underline{Y}_j^*$, $V(\underline{Y}_j^*)$, under (3-30) is

$$\begin{bmatrix} g_j(t_{2_j}) & g_j(t_{3_j}) & \cdots & g_j(t_{p_j}) \\ g_j(t_{2_j}) & g_j(t_{3_j}) & \cdots & g_j(t_{p_j}) \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ & & & \\ g_j(t_{2_j}) & g_j(t_{3_j}) & \cdots & g_j(t_{p_j}) \end{bmatrix} \cdot \begin{bmatrix} g_j(t_{2_j}) \\ g_j(t_{3_j}) \\ \cdot \\ \cdot \\ \cdot \\ g_j(t_{p_j-1}) \\ g_j(t_{p_j}) \end{bmatrix} \cdot [\sigma_{Y_j}^*(t_{1_j})]^2$$

Notice that, Rank $(V(\underline{Y}_{j}^{*})) = 1$. Hence, $V(\underline{Y}_{j}^{*})$ is nonsingular. Consequently, the method of empirical Bayes estimation cannot be used for the models being considered in this dissertation.

CHAPTER 4

RESULTS FOR CASES WHEN NO ERRORS OF MEASUREMENT ARE PRESENT

In Chapter 3 it was shown that presently available methods of data analysis can be used to estimate and test for differences in treatment effects for the growth models under consideration (see equation 1-4) when one of the following conditions hold:

(i) $Y^*_{ij}(t) = g(t) \cdot Y^*_{ij}(0) + h(t) + \alpha_j(t)$;

(ii) There are only two groups in the design and the values of $g_1(t)$, $g_2(t)$, $h_1(t)$, $h_2(t)$, $\mu_{Y_1}(0)$, and $\mu_{Y_2}(0)$ are known;

or (iii) $Y^*_{ij}(t) = (b_j \cdot t + 1) \cdot Y^*_{ij}(0) + c_j \cdot t + \alpha_j(t)$.

For designs where condition (i) holds, Classical ANCOVA can be used if there are no errors of measurement present and $\sigma_{Y_j}(t) = \sigma_Y(t)$ for all j. Estimated True Scores ANCOVA (Porter, 1967) can be used when errors of measurement are present if $\rho_{Y(0)Y(0)}$ is known, $\sigma^*_{Y_j}(t) = \sigma^*_Y(t)$ for all j, and $\sigma_{Y_j}(t) = \sigma_Y(t)$ for all j. If the values of g(t) are known, then ANOVA of Index of Response with K set equal to g(t) can be used. For designs where condition (ii) holds an ANOVA of Index of Response with

$$K = \frac{[g_1(t)\mu_{Y_1}(0) + h_1(t)] - [g_2(t)\mu_{Y_2}(0) + h_2(t)]}{\mu_{Y_1}(0) - \mu_{Y_2}(0)}$$

can be used. For designs where condition (iii) holds, Olejnik's (1977) Adjusted Gain Scores can be used. Further, in Chapter 3 it was shown that point estimates of treatment effects as defined in equation (1-4) can be found by presently available methods of data analysis only when

(iii) $Y^*_{ij}(t) = (b_j \cdot t + 1) \cdot Y^*_{ij}(0) + c_j \cdot t + \alpha_j(t)$;

or (iv) There are no errors of measurement present in the data and $h_j(t) \equiv 0$ for all t and for all j.

For designs where condition (iv) holds, Rogosa's (1980) method is appropriate. For designs where condition (iii) holds, Adjusted Gain Scores is appropriate. Finally, it was shown in Chapter 3 that an appropriate test of nonzero treatment effects (i.e., H_o: $\alpha_j(t) = 0$ versus H_1: $\alpha_j(t) \neq 0$) exists only when condition (iii) holds, by using Adjusted Gain Scores.

Hence, new methods of data analysis need to be developed which provide estimates and hypothesis tests for treatment effects and differences in treatment effects for data sets conforming to equation (1-4) and where conditions other than (i), (ii), (iii), and (iv) hold. The remainder of this dissertation is devoted to the discussion of new methods of data analysis developed by the author.

As has already been illustrated, the development of methods of data analysis can be facilitated through placing various constraints on the parameters in equation (1-4). First, since the $h_j(t)$ and $\alpha_j(t)$ terms are confounded, some information about the $h_j(t)$'s is necessary. Three possible types of available information are:

(a) the exact natures of the $h_j(t)$'s are known (e.g., $h_1(t) = 3 \cdot t$, $h_2(t) = 4 \cdot t^2 + 3 \cdot t^{\frac{1}{2}}$, ... , $h_J(t) = \log_{10}(5 \cdot t^3 + 1)$;

(b) the functional forms of the $h_j(t)$'s are known (e.g., $h_1(t) = k_1 \cdot t$, $h_2(t) = k_2 \cdot t^2 + k_3 \cdot t^{\frac{1}{2}}$, ... , $h_J(t) = \log_{10}(k_4 \cdot t^3 + 1)$, where k_1, k_2, k_3, and k_4 are unknown real-valued constants);

and

(c) for each time t, the $h_j(t)$'s are equal to some common value, say h(t) (i.e., for each t, $h_1(t) = h_2(t) = \ldots = h_J(t) = h(t)$).

In this chapter and in Chapters 5 and 6 methods of data analysis are developed under each of these types of information. The discussion of methods of data analysis under each type of information is further broken down into six cases according to whether the exact natures of the $g_j(t)$'s are known, the functional forms of the $g_j(t)$'s are known, or nothing is known about the $g_j(t)$'s and according to whether or not errors of measurement are present (see Figure 18).

Cases where errors of measurement are present in the data are more important for educational research and so will receive greater attention in this dissertation. Nevertheless, a discussion of those cases where errors of measurement are not present will also be included, since there are educational and behavioral research settings in which it can be assumed that there are no, or perhaps negligible, errors of measurement present (e.g., settings where elapsed time, weight, or height is the measure of interest). The results for those cases when no errors of measurement are present are discussed in the remainder of this chapter. The results for those cases when errors of measurement are present are discussed in Chapters 5 and 6.

Case 1

For Case 1 there are no errors of measurement present in the data and the exact natures of the $g_j(t)$'s and $h_j(t)$'s

	$g_j(t)$'s known		Functional forms of $g_j(t)$'s known		Nothing known about $g_j(t)$'s	
	No Errors of Measurement	Errors of Measurement Present	No Errors of Measurement	Errors of Measurement Present	No Errors of Measurement	Errors of Measurement Present
$h_j(t)$'s known	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
Functional forms of $h_j(t)$'s known	Case 7	Case 8	Case 9	Case 10	Case 11	Case 12
$h_j(t) \equiv h(t)$	Case 13	Case 14	Case 15	Case 16	Case 17	Case 18

Figure 18. Subcases of the general growth model.

are known. Notice that $Y^*_{ij}(t) = Y_{ij}(t)$ in this case and for the remaining cases where no errors of measurement are present. Hence, the system of equations (1-4) reduces to

$$Y_{ij}(t) = g_j(t) \cdot Y_{ij}(t_{1_j}) + h_j(t) + \alpha_j(t) \quad (4\text{-}1)$$

$$; \ i = 1,2,\ldots, N_j$$

$$j = 1,2,\ldots, J \ ,$$

where $g_j(t)$ and $h_j(t)$ are known continuous functions and N_j represents the number of individuals in the sample from group j. Notice that the pretests for the different groups need not occur at the same time. Solving equation (4-1) for $\alpha_j(t)$,

$$\alpha_j(t) = Y_{ij}(t) - [g_j(t) \cdot Y_{ij}(t_{1_j}) + h_j(t)]. \quad (4\text{-}2)$$

Hence, only a sample of size equal to one from each group is necessary in order to determine the $\alpha_j(t)$'s. No estimation or hypothesis testing procedures are necessary, since equation (4-2) determines the $\alpha_j(t)$'s exactly.

Cases 3 and 5

For Cases 3 and 5 there are no errors of measurement present in the data and the exact natures of the $h_j(t)$'s are known. In Case 3 the functional forms of the $g_j(t)$'s are known while in Case 5 nothing is known about the $g_j(t)$'s. Under equation (4-1) any two individuals from the j<u>th</u> group are represented by

$$Y_{1j}(t) = g_j(t) \cdot Y_{1j}(t_{1_j}) + h_j(t) + \alpha_j(t) \qquad (4\text{-}4)$$

and

$$Y_{2j}(t) = g_j(t) \cdot Y_{2j}(t_{1_j}) + h_j(t) + \alpha_j(t) \ . \qquad (4\text{-}5)$$

Subtracting equation (4-4) from equation (4-5) gives

$$Y_{2j}(t) - Y_{1j}(t) = g_j(t) \cdot [Y_{2j}(t_{1_j}) - Y_{1j}(t_{1_j})] \ .$$

Hence,

$$g_j(t) = \frac{Y_{2j}(t) - Y_{1j}(t)}{Y_{2j}(t_{1_j}) - Y_{1j}(t_{1_j})} \ . \qquad (4\text{-}6)$$

So, for Cases 3 and 5 the values of the $g_j(t)$'s are easily derived solely from knowledge of the pretest and posttest scores of two individuals. Once the values of the $g_j(t)$'s have been determined using equation (4-6), Cases 3 and 5 reduce to Case 1. Nothing is gained by knowing the functional forms of the $g_j(t)$'s. Further, if one assumes incorrect exact expressions for the $g_j(t)$'s, then the values computed for the $\alpha_j(t)$'s will, consequently, be incorrect. Hence, it is recommended that when no errors of measurement are present, no assumptions be made about the $g_j(t)$'s, but instead equation (4-6) should be used to determine the values of the $g_j(t)$'s at the time points at which observations have been collected. For those cases where errors of measurement are present, however, the distinction between those cases where the exact natures are known, the functional forms are known, or nothing is known about the $g_j(t)$'s will be important.

Cases 7, 9, and 11

For Cases 7, 9, and 11 there are no errors of measurement in the data and the functional forms of the $h_j(t)$'s are known. The general functional forms of the $h_j(t)$'s are given in terms of undetermined constants, the values of which must be calculated. The method used to calculate the estimates for these constants is a generalization of Olejnik's (1977) Adjusted Gain Scores to nonlinear growth. Following Olejnik's lead, the presence of multiple pretests is introduced. For group j, let $t_{1_j}, t_{2_j}, \ldots , t_{p_j,j}$ represent the times at which the p_j pretests are administered, where p_j denotes the number of pretests. Let $t_{p_j+1,j}$ represent the time of the posttest for the j*th* group. Again, the times of the pretests and posttest need not be the same for the different groups. For simplicity, the j subscript will be dropped because the $\alpha_j(t)$'s are computed separately for each group.

Consider any two individuals from the j<u>th</u> group, say individuals 1 and 2. By equation (4-1) [and dropping the j subscript]

$$Y_1(t_k) = g(t_k)\cdot Y_1(t_1) + h(t_k)$$

and (4-7)

$$Y_2(t_k) = g(t_k)\cdot Y_2(t_1) + h(t_k) \; ; \; k = 2,3,\ldots,p \; ,$$

since $\alpha(t) \equiv 0$ for all $t \leq t_p$. Solving the system of equations (4-7) for $g(t_k)$ and $h(t_k)$ in terms of $Y_1(t_1)$, $Y_2(t_1)$, $Y_1(t_k)$, and $Y_2(t_k)$ yields

$$g(t_k) = \frac{Y_1(t_k) - Y_2(t_k)}{Y_1(t_1) - Y_2(t_1)} \tag{4-8}$$

and

$$h(t_k) = \frac{Y_2(t_k)Y_1(t_1) - Y_2(t_1)Y_1(t_k)}{Y_1(t_1) - Y_2(t_1)} \; . \tag{4-9}$$

Hence, the values of $h(t_k)$ for $k = 2,3, \ldots , p$ can be determined from knowledge of the pretest scores for only two individuals.

Once the values of the function h are derived at the time points t_2, t_3, ... , and, t_p, the values of the unknown constants in the general expressions for h(t) can be determined. The method for determining the values of the constants is, however, dependent on the functional form of h(t). To illustrate the method for determining the values of the constants, the following two examples will be used:

a polynomial form: $$h(t) = 1 + \sum_{d=1}^{3} c_d \cdot (t - t_1)^d \qquad (4\text{-}10)$$

and

an exponential form: $$h(t) = b \cdot c^{(t-t_1)} + (1 - b) \quad . \qquad (4\text{-}11)$$

Polynomial Form

The object of the method to be described is to determine the values of c_1, c_2, and c_3 in the polynomial from knowledge of the $h(t_k)$'s; $k = 2,3, \ldots , p$. For simplicity, assume that $p = 5$, $t_1 = 0$, $t_2 = 1$, $t_3 = 4$, $t_4 = 5$, and $t_5 = 7$. The method to be described will work for any $p \geq 4$ and for any set of values for the t_k's. Substituting $t_1 = 0$, $t_2 = 1$, $t_3 = 4$, $t_4 = 5$, and $t_5 = 7$ into equation (4-10) yields

$$h(1) = 1 + c_1 + c_2 + c_3,$$

$$h(4) = 1 + 4c_1 + 16c_2 + 64c_3,$$

$$h(5) = 1 + 5c_1 + 25c_2 + 125c_3, \qquad (4\text{-}12)$$

and

$$h(7) = 1 + 7c_1 + 49c_2 + 343c_3.$$

The system of equations (4-12) is then solved for c_1, c_2, and c_3 in terms of h(1), h(4), h(5), and h(7). This can be done since the values of h(1), h(4), h(5), and h(7) have already been determined using equation (4-9).

Exponential Form

The object of the method to be described in this subsection is to determine the values of b and c in (4-11) from knowledge of the $h(t_k)$'s. The method to be described will work for any $p \geq 3$ and for any set of values for the t_k's. Recall that the values of $h(t_2)$, $h(t_3)$, $h(t_4)$, ... , $h(t_p)$ were already determined by using equation (4-9). Next, the values of the t_k's and $h(t_k)$'s; $k = 2,3, \ldots , p$ are substituted into equation (4-11) to yield a new system of equations analogous to (4-12). This new system is then solved for b and c. For the sake of illustration, assume that $p = 4$, $t_1 = 0$, $t_2 = 1$, $t_3 = 4$, and $t_4 = 5$.

Substituting these values into equation (4-11) yields the system

$$h(1) = b\cdot c + (1 - b), \qquad (4\text{-}13\ \text{i})$$

$$h(4) = b\cdot c^4 + (1 - b), \qquad (4\text{-}13\ \text{ii})$$

and

$$h(5) = b\cdot c^5 + (1 - b)\ . \qquad (4\text{-}13\ \text{iii})$$

Since these equations are not linear in the parameters, b and c, they cannot be solved algebraically. One must take the different possible subsystems of two equations each and see if the same solutions for b and c are found. If the solutions agree, then b and c have been found. If the solutions do not agree, then no general solution exists for the system (4-13). Contradictory solutions could have arisen for one of several reasons. A discussion of those situations where contradictory solutions occur will be postponed until the end of this chapter.

The method for obtaining the actual solutions for b and c will be described using equations (4-13 ii) and (4-13 iii). This same method can be applied if instead the pair of equations (4-13 i) and (4-13 ii) or the pair of equations (4-13 i) and (4-13 iii) is used. Solving equations (4-13 ii) and (4-13 iii) for c gives

$$c = \left[\frac{h(4) - (1-b)}{b}\right]^{1/4} \qquad (4\text{-}14)$$

and

$$c = \left[\frac{h(5) - (1-b)}{b}\right]^{1/5}, \text{ respectively.} \qquad (4\text{-}15)$$

Equating (4-14) and (4-15) gives

$$\left[\frac{h(4) - (1 - b)}{b}\right]^{1/4} = \left[\frac{h(5) - (1 - b)}{b}\right]^{1/5}, \qquad (4\text{-}16)$$

which must now be solved for b.

There is, however, no algebraic method for directly solving equation (4-16) for b. Hence, numerical analysis techniques must be used. (e.g., Newton-Raphson method [Froberg, 1965; Thomas & Finney, 1979]).

From the preceding discussion the general approach should be apparent. First, the values of the $h(t_k)$'s; $k = 2,3, \ldots, p$ are determined using equation (4-9). A system of equations analogous to (4-12) or (4-13) is then generated which relates the values of the $h(t_k)$'s to the parameters involved in the general expression for h(t). This system of equations is then solved for the parameters using algebraic techniques (when possible) or numerical analysis methods. The minimum number of pretest time points required in order to obtain the values of the unknown parameters is $m_h + 1$, where m_h is the number of unknown parameters in the functional form expression for h(t).

By substituting the computed values of the parameters into the functional form expressions for each of the $h_j(t)$'s, the exact natures of the $h_j(t)$'s are determined. For Case 9, where the functional form expressions of the $g_j(t)$'s are known, and for Case 11, where nothing is known about the $g_j(t)$'s, the values of the $g_j(t)$'s are determined using the method described for Cases 3 and 5. For Case 7, the $g_j(t)$'s are already known.

Knowing both the $g_j(t)$'s and $h_j(t)$'s, the $\alpha_j(t)$'s are determined by

$$\alpha_j(t) = Y_{ij}(t) - [g_j(t) \cdot Y_{ij}(0) + h_j(t)],$$

where i is any individual from the j<u>th</u> group. As with Cases 1, 3, and 5, since assumptions about the $g_j(t)$'s are unnecesary and may even lead to incorrect values for the $\alpha_j(t)$'s, it is recommended that for Cases 7 and 9 the assumptions on the $g_j(t)$'s be ignored and that these cases be treated as if they were part of Case 11.

Since the determination of the values of the treatment effects is done separately for each group, the groups in a design need not all belong to the same case. All that is necessary is sufficient information to place each of the J groups into one or the other of Cases 1, 3, 5, 7, 9, or 11.

Cases 13, 15, and 17

For Cases 13, 15, and 17, $h_j(t) \equiv h(t)$ with $h(t)$ unknown and no errors of measurement are present in the data. For these three cases only differences in treatment effects can be calculated since the $h(t)$ and $\alpha_j(t)$ terms are confounded. For Case 13 the exact natures of the $g_j(t)$'s are known, while for Case 15 only the functional form expressions for the $g_j(t)$'s are known and for Case 17 nothing is known about the $g_j(t)$'s. For all three cases the system of equations (1-4) can be rewritten as

$$Y_{ij}(t) = g_j(t) \cdot Y_{ij}(t_{1_j}) + h(t) + \alpha_j(t) \ . \qquad (4\text{-}17)$$

Case 13

Pick one person from group j, say person i, and one person from group j', say person i', where j and j' are two distinct treatment groups. Then by equation (4-17), for the i'th person from group j',

$$Y_{i'j'}(t) = g_{j'}(t) \cdot Y_{i'j'}(t_{1_{j'}}) + h(t) + \alpha_{j'}(t) \ . \qquad (4\text{-}18)$$

Subtracting equation (4-18) from equation (4-17) yields

$$Y_{ij}(t) - Y_{i'j'}(t)$$
$$= g_j(t)\cdot Y_{ij}(t_{1_j}) + \alpha_j(t) - g_{j'}(t)\cdot Y_{i'j'}(t_{1_{j'}}) - \alpha_{j'}(t) \ .$$

Hence,

$$\alpha_j(t) - \alpha_{j'}(t)$$
$$= Y_{ij}(t) - Y_{i'j'}(t) - [g_j(t)\cdot Y_{ij}(t_{1_j})$$
$$- g_{j'}(t)\cdot Y_{i'j'}(t_{1_{j'}})] .$$

So, the exact value of $\alpha_j(t) - \alpha_{j'}(t)$ can be determined by taking a subsample of size equal to one from each of the groups. Notice that no interval estimation or hypothesis testing procedures are needed since the exact values of the $\alpha_j(t) - \alpha_{j'}(t)$'s have been determined.

Cases 15 and 17

Define new functions, $H_j(t)$'s, by $H_j(t) = h(t) + \alpha_j(t)$. Then, for these two cases the system of equations (4-17) can be rewritten as

$$Y_{ij}(t) = g_j(t)\cdot Y_{ij}(t_{1_j}) + H_j(t) \ . \qquad (4\text{-}19)$$

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Consider two individuals from the jth group, say individuals 1 and 2. Then, by equation (4-19)

$$Y_{1j}(t) = g_j(t) \cdot Y_{1j}(t_{1_j}) + H_j(t)$$

and (4-20)

$$Y_{2j}(t) = g_j(t) \cdot Y_{2j}(t_{1_j}) + H_j(t) \ .$$

Solving the system of equations (4-20) for $g_j(t)$ and $H_j(t)$ in terms of $Y_{1j}(t_{1_j})$, $Y_{2j}(t_{1_j})$, $Y_{1j}(t)$, and $Y_{2j}(t)$ yields

$$g_j(t) = \frac{Y_{1j}(t) - Y_{2j}(t)}{Y_{1j}(t_{1_j}) - Y_{2j}(t_{1_j})}$$

and

$$H(t) = \frac{Y_{2j}(t) \cdot Y_{1j}(t_{1_j}) - Y_{2j}(t_{1_j}) \cdot Y_{1j}(t)}{Y_{1j}(t_{1_j}) - Y_{2j}(t_{1_j})} \ . \qquad (4\text{-}21)$$

Similarly, for group j', consider a subsample of size two from that group, say individuals 1' and 2'. By a derivation analogous to that for group j,

$$g_{j'}(t) = \frac{Y_{1'j'}(t) - Y_{2'j'}(t)}{Y_{1'j'}(t_{1_{j'}}) - Y_{2'j'}(t_{1_{j'}})}$$

and (4-22)

$$H_{j'}(t) = \frac{Y_{2'j'}(t)\cdot Y_{1'j'}(t_{1_{j'}}) - Y_{2'j'}(t_{1_{j'}})\cdot Y_{1'j'}(t)}{Y_{1'j'}(t_{1_{j'}}) - Y_{2'j'}(t_{1_{j'}})} .$$

Subtracting equation (4-22) from equation (4-21) yields

$$H_j(t) - H_{j'}(t)$$

$$= \frac{Y_{2j}(t)\cdot Y_{1j}(t_{1_j}) - Y_{2j}(t_{1_j})\cdot Y_{1j}(t)}{Y_{1j}(t_{1_j}) - Y_{2j}(t_{1_j})} \qquad (4\text{-}23)$$

$$- \frac{Y_{2'j'}(t)\cdot Y_{1'j'}(t_{1_{j'}}) - Y_{2'j'}(t_{1_{j'}})\cdot Y_{1'j'}(t)}{Y_{1'j'}(t_{1_{j'}}) - Y_{2'j'}(t_{1_{j'}})} .$$

But, by definition

$$H_j(t) - H_{j'}(t) = [h(t) + \alpha_j(t)] - [h(t) + \alpha_{j'}(t)] .$$

So,

$$H_j(t) - H_{j'}(t) = \alpha_j(t) - \alpha_{j'}(t) . \qquad (4\text{-}24)$$

Consequently, by equating equations (4-23) and (4-24)

$$\alpha_j(t) - \alpha_{j'}(t)$$
$$= \frac{Y_{2j}(t)\cdot Y_{1j}(t_{1_j}) - Y_{2j}(t_{1_j})\cdot Y_{1j}(t)}{Y_{1j}(t_{1_j}) - Y_{2j}(t_{1_j})}$$
$$- \frac{Y_{2'j'}(t)\cdot Y_{1'j'}(t_{1_{j'}}) - Y_{2'j'}(t_{1_{j'}})\cdot Y_{1'j'}(t)}{Y_{1'j'}(t_{1_{j'}}) - Y_{2'j'}(t_{1_{j'}})} .$$

Since this is the exact value of $\alpha_j(t) - \alpha_{j'}(t)$, no interval estimation or hypothesis testing procedures need be discussed. Notice that here, as with Cases 3 and 5, there is no advantage in having knowledge of the functional forms of the $g_j(t)$'s.

Contradictory Solutions

Each of the methods for computing the $\alpha_j(t)$'s considered thus far has required data for only two subjects per group, except for Cases 1 and 13, where only one subject per group was needed. If the conditions describing the cases hold exactly, then the solutions for the $\alpha_j(t)$'s will be invariant across pairs of subjects. Thus, if using different pairs of subjects yields contradictory solutions for $\alpha_j(t)$, then one or more of the conditions describing the cases must be false. That is, either

(i) Errors of measurement are present in the data ;

(ii) There is not a correlation of +1 between true scores at two points in time within group j ;

and/or (iii) For Cases 1, 3, 5, 7, 9, and 11, depending on the case, either the exact nature or the general functional form of $h_j(t)$ has been misspecified.

Also, when discussing the methods for Cases 7, 9, and 11 it was mentioned that different, and hence contradictory, solutions may arise when different combinations of time points are used for the pretest observations. The possible reasons for these different solutions are the same as those just listed.

The problem, however, is that one does not know which of the reasons caused the contradiction. If only reason (i) is the cause, then the results to be derived in Chapters 5 and 6 should be used. If only reason (iii) is the cause, then the correct exact nature or functional form expression for $h_j(t)$ must be determined. If both reasons (i) and (iii) are the case, then the results of Chapters 5 and 6 should be used, but first the correct expression for $h_j(t)$ must be determined. If reason (ii) is the case, then the methods developed in this dissertation are not applicable. It should be noted here that the methods of data analysis developed by Strenio, Bryk, and Weisberg (Bryk, Strenio, and Weisberg, 1981; Strenio, Weisberg, and Bryk, in press) seem very promising for some of the situations where there is not a correlation of +1 between true scores.

CHAPTER 5

POINT ESTIMATION WHEN ERRORS OF MEASUREMENT ARE PRESENT

In this chapter point estimation procedures for those cases where errors of measurement are present in the data are discussed. The interval estimation and hypothesis testing procedures for these cases will be given in Chapter 6.

Case 2 Estimators

For Case 2 the exact natures of both the $g_j(t)$'s and $h_j(t)$'s are known and errors of measurement are present. Recall that the general growth model is

$$Y_{ij}^*(t) = g_j(t) \cdot Y_{ij}^*(t_{1_j}) + h_j(t) + \alpha_j(t)$$

and (1-4)

$$Y_{ij}(t) = Y_{ij}^*(t) + e_{ij}(t) \ .$$

Taking the population mean on both sides of equation (1-4),

$$\mu_{Y_j}(t) = g_j(t) \cdot \mu_{Y_j}(t_{1_j}) + h_j(t) + \alpha_j(t) \ . \qquad (5\text{-}1)$$

Hence,

$$\alpha_j(t) = \mu_{Y_j}(t) - [g_j(t) \cdot \mu_{Y_j}(t_{1_j}) + h_j(t)] \ . \qquad (5-2)$$

If $\mu_{Y_j}(t_{1_j})$ and $\mu_{Y_j}(t)$ are known then equation (5-2) gives $\alpha_j(t)$ exactly. If, as is usually the case, the population means are not known, then for a wide variety of statistical distributions, a point estimator of $\alpha_j(t)$ is given by

$$\widehat{\alpha_j(t)} = \overline{Y_j(t)} - [g_j(t) \cdot \overline{Y_j(t_{1_j})} + h_j(t)] \ . \qquad (5-3)$$

Overview of the Procedures for the Remaining Cases

Each of the procedures to be discussed is a variation of a several stage method. At the first stage, for Cases 4, 8, 10, 12, 14, and 16, estimates are obtained for the unknown constants (i.e., parameters) in the functional form expressions for the $g_j(t)$'s and $h_j(t)$'s. The method used to obtain the estimates of the parameters in the functional form expressions is the same whether the parameters for the $g_j(t)$'s, the $h_j(t)$'s, or for both are being estimated. Hence, the discussion of the first stage will be done in general. The second stage concerns the estimation of treatment effects (i.e., the $\alpha_j(t)$'s in equation (1-4)) given the estimates obtained at the first stage and the particulars of the case of interest. Hence for the second stage, each of the cases must be discussed separately. For Cases 6 and 18, the first and second stages are replaced by a process which directly estimates the $\alpha_j(t)$'s. The third stage concerns methods for interval estimation and hypothesis testing of both treatment effects and differences in treatment effects. These third stage methods are the same for all the cases, once estimates of the $\alpha_j(t)$'s have been obtained, and are described in Chapter 6.

Stage 1: Estimation of the $g_j(t)$'s and $h_j(t)$'s

Stage 1 is divided into two substages. At the first substage, estimates of the $g_j(t_{k_j})$'s and $h_j(t_{k_j})$'s are obtained, where the t_{k_j}'s; $k = 1,2,3,\ldots,p_j$ are the times of the p_j pretests for the jth group. At the second substage, these estimates of the $g_j(t_{k_j})$'s and $h_j(t_{k_j})$'s are used to obtain estimates of the unknown constants in the functional form expressions of the $g_j(t)$'s and/or $h_j(t)$'s.

For simplicity, the j subscript will be dropped since the estimation of the unknown constants in the functional form expressions for the $g_j(t)$'s and $h_j(t)$'s is done separately for each of the J groups. Dropping the j subscript and considering only the pretest time points, equation (1-4) becomes

$$Y_i^*(t_k) = g(t_k) \cdot Y_i^*(t_1) + h(t_k) \tag{5-4}$$

and

$$Y_i(t_k) = Y_i^*(t_k) + e_i(t_k) \ ; \ k = 1,2,\ldots,p \ . \tag{5-5}$$

Some of the decisions made at substage 1 are dependent upon the context of what is to be done at substage 2. Since the second substage provides the motivation for the methods chosen to complete the first substage, it will be discussed first.

<u>Substage 2: Estimation of the unknown parameters in g(t) and h(t)</u>

Once the substage 1 estimates of the $g(t_k)$'s and $h(t_k)$'s are calculated, estimates of the unknown parameters in the general functional form expressions for g(t) and h(t) can be computed. For simplicity, the discussion here will be in terms of g(t). The procedures to be used for h(t) are analogous.

The second substage of Stage 1 begins by setting up a system of equations similar to the systems (4-12) and (4-13) which relates the estimates of the $g(t_k)$'s to the unknown parameters. For example, if $g(t) = \log_c[b\cdot(t - t_1) + c]$ then the system of equations would be

$$
\begin{aligned}
\widehat{g(t_2)} &= \log_c[b\cdot(t_2-t_1) + c] \ , \\
\widehat{g(t_3)} &= \log_c[b\cdot(t_3-t_1) + c] \ , \\
&\ \ \vdots \\
\widehat{g(t_p)} &= \log_c[b\cdot(t_p-t_1) + c] \ .
\end{aligned}
\tag{5-6}
$$

In general, define m_g to be the number of unknown parameters in the general expression for $g(t)$. If $p \leq m_g$ then estimates of the unknown parameters can not be found. If $p = m_g + 1$ then the system of equations is solved for the unknown parameters using the appropriate algebraic and numerical analysis techniques, as was done in Chapter 4. It should be remembered, however, that the resulting answers for the unknown parameters in the general expression of $g(t)$ are now estimates of the parameters rather than their exact values.

When $p > m_g+1$ the method of least squares can be used to provide estimates of the parameters. The methods developed in Chapter 4 for the situations where p was greater than m_g can not be used here because errors of estimation are present. Once errors of estimation are present, systems of equations such as (5-6) usually become contradictory simply because of the presence of these errors and not for the reasons discussed at the end of Chapter 4. The method of least squares resolves this contradiction by finding the estimates for the unknown parameters which minimize the inconsistency. For example, consider $g(t) = \log_c[b\cdot(t-t_1) + c]$ and let $p = 5$. Let $\widehat{g(t_2)}$, $\widehat{g(t_3)}$, $\widehat{g(t_4)}$, and $\widehat{g(t_5)}$ denote the estimates of the $g(t_k)$'s which were computed using one of the substage 1 methods to be described later in this section. The quantity,

$$L = \sum_{k=2}^{5} [\widehat{g(t_k)} - \log_c(b\cdot(t_k-t_1) + c)]^2$$

represents the inconsistency, in the least squares sense, inherent in the system of equations (5-6). The method of least squares finds the values of b and c which minimize L. These values are considered to be the estimators of b and c. In general, for any differentiable function $g(t)$, the estimates of the unknown parameters are found by finding the vector of values for the parameters for which the minimum value of the expression

$$L = \sum_{k=2}^{p} [\widehat{g(t_k)} - g(t_k)]^2$$

occurs.

Notice that the method of least squares can be used to find estimates of the unknown parameters whenever p is at least two greater than the number of unknown parameters. Further, in general, the precision of least squares estimates improves as the number of pieces of available information used increases. Here, the pieces of information are the $g(t_k)$'s; $k = 2,3,\ldots, p$. Hence, the greater the number of pretest time points, the better the precision of the estimators of the unknown parameters.

Substage 1: Estimation of the $g(t_k)$'s and $h(t_k)$'s

Equation (5-4) represents a linear structural (Madansky, 1959; Moran, 1971) or functional relation (DeGracie & Fuller, 1972; Lindley, 1947), where $g(t_k)$ and $h(t_k)$ are the slope and intercept of the $Y^*(t_k)$ on $Y^*(t_1)$ regression line. The general problem of how to estimate the slope and intercept of a linear structural relation is known as the errors-in-variables problem and has been widely discussed in the literature, especially in the area of econometrics (e.g., Johnston, 1972; Madansky, 1959; Moran, 1971; Sprent, 1966). The econometrics literature deals mostly with those situations where there is only one independent variable and one dependent variable.

The maximum likelihood structural equations approach developed most notably by Jöreskog (Goldberger & Duncan, 1973; Jöreskog, 1969; Jöreskog, 1977; Magidson, 1979; W. Schmidt, 1975), however, allows for any number of independent and dependent variables. Also, as has just been shown, in order to complete the second substage of the process for determining the unknown parameters, it is necessary to have at least one more pretest time point than the number of unknown parameters. So, if the functional form under consideration has two or more parameters, then the necessary number of pretests is three or greater. For these situations, only the maximum likelihood approach can be used. Further, since the more pretests used, the better the precision of the substage 2 estimators, it is recommended that pretest observations be collected at as many time points as possible. It is realized that the number of pretest time points is, however, constrained in educational settings by the amount of money and investigator time available. Also, in most settings, the number of tests is constrained because the subjects can only take a certain number of tests without either reactivity, fatigue, or attrition occurring. Finally, as will be shown later in this section, when only two pretest observations are available, additional assumptions must be made before estimates of the $g(t_k)$'s and $h(t_k)$'s can be determined. Hence, it is recommended that observations always be collected at three or more pretest time points, for those cases where Stage 1 is implemented, and that maximum likelihood estimation be used to estimate the $g(t_k)$'s and $h(t_k)$'s. Since the maximum likelihood approach is applicable in a wider variety of situations than the other approaches, it will be discussed first.

Maximum Likelihood Approach

For a design with p pretests, the system of equations represented by equations (5-4) and (5-5) can be described pictorially as in Figure 19.

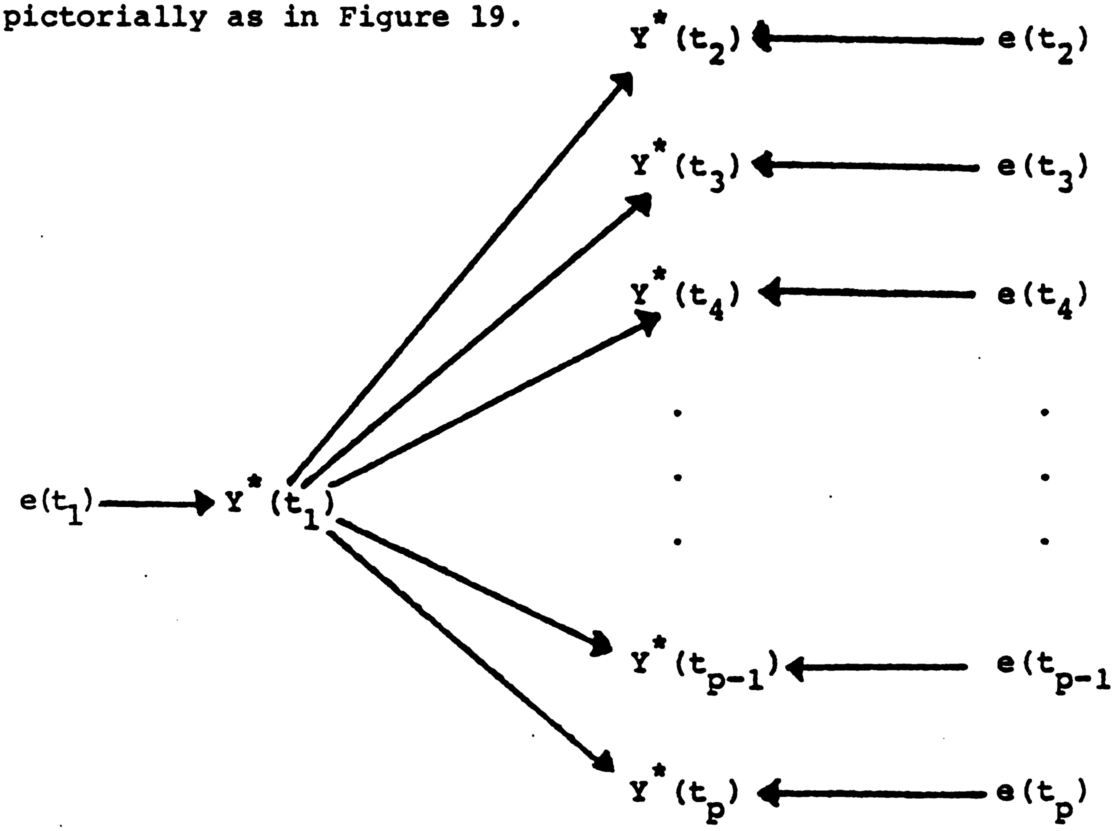

Figure 19. Pictorial representation of the structural relation.

The system can be written in vector form as

$$\underline{Y_i^*(t_k)} = \underline{g(t_k)} \cdot Y_i^*(t_1) + \underline{h(t_k)} \qquad (5\text{-}7)$$

and

$$\underline{Y_i(t_k)} = \underline{Y_i^*(t_k)} + \underline{e_i(t_k)} \; . \qquad (5\text{-}8)$$

Maximum likelihood requires expressions for the means, variances, and covariances of the observed variables under consideration. Taking the mean on both sides of equation (5-7)

$$\underline{\mu_Y(t_k)} = \underline{g(t_k)} \cdot \mu_Y(t_1) + \underline{h(t_k)} \; , \qquad (5\text{-}9)$$

where $\underline{\mu_Y(t_k)}$ is the vector of means. The variance of $Y(t_1)$ is given by

$$\sigma_Y^2(t_1) = [\sigma_Y^*(t_1)]^2 + \sigma_e^2(t_1) \; , \qquad (5\text{-}10)$$

where $\sigma_e^2(t)$ represents the variance of the errors of measurement at time t. The variance of $Y(t_k)$ is given by

$$\sigma_Y^2(t_k) = [g(t_k)]^2 \cdot [\sigma_Y^*(t_1)]^2 + \sigma_e^2(t_k) \; , \qquad (5\text{-}11)$$

for k=1,2, ... , p. The covariance of $Y(t_k)$ and $Y(t_1)$ is given by

$$\sigma_{Y(t_k)Y(t_1)} = g(t_k) \cdot [\sigma_Y^*(t_1)]^2 \; . \qquad (5\text{-}12)$$

The covariance of $Y(t_k)$ and $Y(t_{k'})$, where $k,k' = 2,3, \ldots ,$ p and $k \neq k'$, is given by

$$\sigma_{Y(t_k)Y(t_{k'})} = g(t_k) \cdot g(t_{k'}) \cdot [\sigma_Y^*(t_1)]^2 . \quad (5\text{-}13)$$

Consider the systems of equations (5-9), (5-10), (5-11), (5-12), and (5-13) as one large system of equations. The maximum likelihood approach requires that this large system be identifiable. A system of equations is said to be identifiable if and only if each of the unknown parameters on the right hand side of the system can be expressed in terms of the unknown parameters on the left hand side of the system. For identifiability only the existence of expressions is required; the expressions need not be unique. For the models of continuous growth being considered here, the unknown parameters on the left hand side are $\mu_Y(t_1)$, $\mu_Y(t_2), \ldots,$ $\mu_Y(t_p)$ and the $\frac{p(p+1)}{2}$ parameters in the variance-covariance matrix of $Y(t_1)$, $Y(t_2)$, $\ldots$, $Y(t_p)$. The unknown parameters on the right hand side are $g(t_2)$, $g(t_3)$, $\ldots$, $g(t_p)$, $h(t_2)$, $h(t_3)$, $\ldots$, $h(t_p)$, $[\sigma_Y^*(t_1)]^2$, $\sigma_e^2(t_1)$, $\sigma_e^2(t_2)$, $\ldots$, $\sigma_e^2(t_{p-1})$, and $\sigma_e^2(t_p)$. In Appendix A it is shown that the system of equations (5-9) to (5-13) is identifiable if and only if $p \geq 3$. Hence, for the remainder of the discussion of the maximum likelihood approach it will be assumed that $p \geq 3$. Those designs for which $p = 2$ are exactly those designs where the system of equations (5-7) reduces to a single equation with one independent variable, $Y^*(t_1)$, and one dependent variable, $Y^*(t_2)$. When $p = 2$, the general maximum likelihood approach must be modified slightly or methods other than maximum likelihood can be used.

Assume that the variables $Y^*(t_1)$, $Y^*(t_2)$, ... , $Y^*(t_p)$ have a multivariate normal distribution and that classical measurement theory assumptions hold. It follows that $Y(t_1)$, $Y(t_2)$, ... , $Y(t_p)$ have a multivariate normal distribution. From elementary statistics (Mood, Graybill and Boes, 1974) it can be shown that the maximum likelihood estimates of the means are given by $\overline{Y(t_1)}$, $\overline{Y(t_2)}$, ... , $\overline{Y(t_p)}$. The maximum likelihood of the variances, $\sigma_Y^2(t_1)$, $\sigma_Y^2(t_2)$, ... , $\sigma_Y^2(t_p)$, are given by $S_Y^2(t_1)$, $S_Y^2(t_2)$, ... , $S_Y^2(t_p)$ where

$$S_Y^2(t_k) = \frac{1}{N} \cdot \sum_{i=1}^{N} (Y_i(t_k) - \overline{Y_i(t_k)})^2 \qquad (5\text{-}14)$$

$$; k = 1,2,\ldots,p .$$

The maximum likelihood estimates of the covariances, the $\sigma_{Y(t_k)Y(t_{k'})}$'s, are given by the $S_{Y(t_k)Y(t_{k'})}$'s, where

$$S_{Y(t_k)Y(t_{k'})} =$$

$$\frac{1}{N} \cdot \sum_{i=1}^{N} [Y_i(t_k) - \overline{Y_i(t_k)}][Y_i(t_{k'}) - \overline{Y_i(t_{k'})}] \qquad (5\text{-}15)$$

$$; k,k' = 1,2, \ldots , p .$$

The goal of the maximum likelihood structural equations approach is to find the maximum likelihood estimates of the parameters on the right hand side of the system of equations (5-9) to (5-13) in terms of the maximum likelihood estimates of the means, variances, and covariances of the observed scores. Jöreskog (1969) has derived expressions for the first and second derivatives which need to be used in order to determine the maximum likelihood estimates for systems of structural equations and has written a computer program, LISREL (Jöreskog and Sörbom, 1978), which evaluates these expressions in order to find the estimates of the parameters of interest. The LISREL output gives the maximum likelihood estimates for the $g(t_k)$'s, $[\sigma^*_Y(t_1)]^2$, and the $\sigma^2_e(t_k)$'s. The estimates of the $h(t_k)$'s are computed using

$$\widehat{h(t_k)} = \overline{Y(t_k)} - \widehat{g(t_k)}\,\overline{Y(t_1)},$$

where $\widehat{g(t_k)}$ is the maximum likelihood estimate of $g(t_k)$.

The method just described requires $p \geq 3$ in order to estimate the $g(t_k)$'s and $h(t_k)$'s. Since pretests are expensive, the situation when p = 2 is important to consider, even though p = 2 can only be used if the number of unknown parameters in the functional form expression for h(t) is equal to 1. The reason why there need not be a restriction on the number of parameters in g(t) will be explained in the section covering Stage 2.

When p = 2, the system of equations (5-7) to (5-13) reduces to

$$Y_i^*(t_2) = g(t_2)\cdot Y_i^*(t_1) + h(t_2) \quad (5\text{-}16)$$

from (5-7);

$$Y_i(t_1) = Y_i^*(t_1) + e_i(t_1)$$

and

$$Y_i(t_2) = Y_i^*(t_2) + e_i(t_2) \quad \text{from (5-8);}$$

$$\mu_Y(t_1) = \mu_Y(t_1) \quad (5\text{-}17)$$

and

$$\mu_Y(t_2) = g(t_2)\mu_Y(t_1) + h(t_2) \quad (5\text{-}18)$$

from (5-9);

$$\sigma_Y^2(t_1) = [\sigma_Y^*(t_1)]^2 + \sigma_e^2(t_1) \quad (5\text{-}19)$$

from (5-10);

$$\sigma_Y^2(t_2) = [g(t_2)]^2[\sigma_Y^*(t_1)]^2 + \sigma_e^2(t_2) \quad (5\text{-}20)$$

from (5-11);

and

$$\sigma_{Y(t_1)Y(t_2)} = g(t_2)[\sigma_Y^*(t_1)]^2 \qquad (5\text{-}21)$$

from (5-12).

The maximum likelihood structural equations approach can still be used for the system (5-17) to (5-21), but an additional assumption must be made to insure identifiability. Some of the possible additional assumptions are:

Assumption 1: $\rho_{Y(t_1)Y(t_1)}$ is known

Assumption 2: $\sigma_e^2(t_1)$ is known

Assumption 3: $\sigma_e^2(t_2)$ is known

Assumption 4: The ratio $\dfrac{\sigma_e^2(t_2)}{\sigma_e^2(t_1)}$ is known

Assumption 5: $\rho_{Y(t_1)Y(t_1)} = \rho_{Y(t_2)Y(t_2)}$

and

Assumption 6: $\rho_{Y(t_2)Y(t_2)}$ is known .

Assumptions 1 to 4 are included because they are assumptions that have appeared in the errors-in-variables literature (Johnston, 1972; Moran, 1971). Assumption 5 is included because it is an assumption which is often made in educational settings. Assumption 6 is included because of its similarity to Assumption 1. The maximum likelihood estimates for Assumptions 1 to 3 are given in many places (see e.g., Johnston, 1972; Moran, 1971). The maximum likelihood estimates for $g(t_2)$ under Assumptions 1 to 3 are:

under Assumption 1,

$$\widehat{g(t_2)} = \frac{s_{Y(t_1)Y(t_2)}}{\rho_{Y(t_1)Y(t_1)} \cdot s_Y^2(t_1)} ;$$

under Assumption 2,

$$\widehat{g(t_2)} = \frac{s_{Y(t_1)Y(t_2)}}{s_Y^2(t_1) - \sigma_e^2(t_1)} ;$$

and under Assumption 3,

$$\widehat{g(t_2)} = \frac{s_Y^2(t_2) - \sigma_e^2(t_2)}{s_{Y(t_1)Y(t_2)}} .$$

Under all three assumptions,

$$\widehat{h(t_2)} = \overline{Y(t_2)} - \widehat{g(t_2)} \cdot \overline{Y(t_1)} .$$

Assumption 4 has been widely discussed in the literature on the errors-in-variables problem (Johnston, 1972; Madansky, 1959; Moran, 1971; Mullet & Murray, 1976). Johnston and Mullet and Murray give multiple estimates under this assumption and then describe ad-hoc methods for choosing between these estimators. But, both Madansky and Moran have pointed out that none of these estimates is a maximum likelihood estimate. When it can be assumed, however, that $g(t_2) > 0$, as is being assumed for the growth models under consideration here, then maximum likelihood estimates of $g(t_2)$ and $h(t_2)$ exist and are given by

$$\widehat{g(t_2)} = \frac{S_Y^2(t_2) - \lambda S_Y^2(t_1) + \sqrt{(S_Y^2(t_2) - \lambda S_Y^2(t_1))^2 + 4\lambda [S_{Y(t_1)Y(t_2)}]^2}}{2S_{Y(t_1)Y(t_2)}}$$

and

$$\widehat{h(t_2)} = \overline{Y(t_2)} - \widehat{g(t_2)} \cdot \overline{Y(t_1)} ,$$

where $\lambda = \dfrac{\sigma_e^2(t_2)}{\sigma_e^2(t_1)}$. Appendix B provides a derivation of $\widehat{g(t_2)}$ and $\widehat{h(t_2)}$ under Assumption 4. Appendix B also provides derivations of the maximum likelihood estimates under Assumptions 5 and 6, since a search of the literature did not reveal any place where the estimates under these assumptions are discussed.

The maximum likelihood estimate of $g(t_2)$ under Assumption 5 is

$$\widehat{g(t_2)} = \frac{s_Y(t_2)}{s_Y(t_1)} .$$

Under Assumption 6,

$$\widehat{g(t_2)} = \frac{s_Y^2(t_2)}{s_{Y(t_1)Y(t_2)}} \cdot \rho_{Y(t_2)Y(t_2)} .$$

Under both assumptions, $\widehat{h(t_2)} = \overline{Y(t_2)} - \widehat{g(t_2)} \cdot \overline{Y(t_1)}$.

Other Approaches

When $p = 2$, there are also approaches other than maximum likelihood available to estimate $g(t_2)$ and $h(t_2)$. When an independent estimates of $\sigma_e^2(t_1)$ is available DeGracie and Fuller (1972) have suggested using

$$\widehat{g(t_k)} = \frac{\tilde{S}_{Y(t_1)Y(t_k)}}{\left\{ \widehat{S_{Y^*(t_1)Y^*(t_1)}} + \frac{1}{N-1}\left(2\widehat{\sigma_e^2(t_1)} + \frac{2\widehat{\sigma_e^4(t_1)}}{\widehat{S_{Y^*(t_1)Y^*(t_1)}}} \right) + \frac{2\widehat{\sigma_e^4(t_1)}}{q \cdot \widehat{S_{Y^*(t_1)Y^*(t_1)}}} \right\}}$$

as the estimator for $g(t_k)$, where

$$\tilde{S}_{Y(t_1)Y(t_k)} = \frac{1}{N-1} \sum_{i=1}^{N} (Y_i(t_1) - \overline{Y(t_1)})(Y_i(t_k) - \overline{Y(t_k)}) \ ;$$

$\widehat{\sigma_e^2(t_1)}$ represents the independent estimate of $\sigma_e^2\ (t_1)$;

$$\tilde{S}_{Y(t_1)Y(t_1)} = \frac{1}{N-1} \sum_{i=1}^{N} (Y_i(t_1) - \overline{Y(t_1)})^2 \ ;$$

$$\widehat{S_{Y^*(t_1)Y^*(t_1)}} = \begin{cases} \tilde{S}_{Y(t_1)Y(t_1)} - \widehat{\sigma_e^2(t_1)} & \text{if } \tilde{S}_{Y(t_1)Y(t_1)} - \widehat{\sigma_e^2(t_1)} > \frac{1}{N-1}\widehat{\sigma_e^2(t_1)} \\ \frac{1}{N-1}\widehat{\sigma_e^2(t_1)} & \text{otherwise;} \end{cases}$$

and q is the number of degrees of freedom for the χ^2

distribution of which the distribution of $\widehat{\sigma_e^2(t_1)}$ is a multiple.

The estimator for $h(t_k)$ can then be defined as

$$\widehat{h(t_k)} = \overline{Y(t_k)} - \widehat{g(t_k)} \cdot \overline{Y(t_1)} \ .$$

Notice that DeGracie and Fuller's method is only concerned with two time points. Any number of pretests can be used, however, but the determinations of the estimates for $g(t_k)$ and $h(t_k)$ are done separately for each k; $k=2,3, \ldots , p$.

Another approach for estimating the $g(t_k)$'s is given by Spiegelman (1979). Spiegelman's approach requires a knowledge of real analysis on the part of the data analyst. Even though Spiegelman's technique is a possible method for estimating the $g(t_k)$'s, it will not be discussed here because of its complexity.

Several other approaches discussed in the literature are variations of the techniques of instrumental variables or method of grouping (Johnston, 1972; Madansky, 1959; Wald, 1940). The methodology of the instrumental variables and method of grouping techniques requires that the errors of measurement in the independent variable be statistically independent of the instrumental variable or grouping variable. As has been pointed out by P. Schmidt (1978) and Madansky (1959), unless there is information available besides the pretest and posttest scores, the errors of measurement will be correlated with any instrumental or grouping variable used. Hence, these methods should not be used in the data collection situations considered here.

The discussion so far in this chapter has been focused on single-group designs. For multi-group designs the methods developed here can be applied to each group separately to find the estimates of the unknown parameters in the expressions for the $g_j(t)$'s and $h_j(t)$'s. Treating each group separately will suffice except in those instances where it is assumed that $g_j(t) \equiv g_{j'}(t)$ and/or $h_j(t) \equiv h_{j'}(t)$ for two distinct groups, j and j'. If it is assumed that $g_j(t) \equiv g_{j'}(t)$, $h_j(t) \equiv h_{j'}(t)$, $p_j = p_{j'}$, and $t_{k_j} = t_{k_{j'}}$ for $k = 1,2, \ldots, p_j (=p_{j'})$, then the two groups j and j' should be combined and considered as one group for the purposes of this first stage. If $g_j(t) \equiv g_{j'}(t)$, but $h_j(t) \not\equiv h_{j'}(t)$, and still $p_j = p_{j'}$ and $t_{k_j} = t_{k_{j'}}$ for $k = 1,2,\ldots,p_j$, then the two groups should be combined when the $\widehat{g_j(t_k)}$'s (and hence, the $\widehat{g_{j'}(t_k)}$'s) are determined and when the estimates of the unknown parameters in the expression for $g_j(t)$ (and hence, $g_{j'}(t)$) are calculated. Call the common estimate of $g_j(t_k)$ and $g_{j'}(t_k)$ by $\widehat{g_{j,j'}(t_k)}$, for $k = 2,3,\ldots, p_j$. Then determine the values of $\widehat{h_j(t_k)}$ and $\widehat{h_{j'}(t_k)}$ separately using

$$\widehat{h_j(t_k)} = \overline{Y_j(t_k)} - \widehat{g_{j,j'}(t_k)}\cdot\overline{Y_j(t_1)} \qquad (5\text{-}22)$$

and

$$\widehat{h_{j'}(t_k)} = \overline{Y_{j'}(t_k)} - \widehat{g_{j,j'}(t_k)}\cdot\overline{Y_{j'}(t_1)} \ . \qquad (5\text{-}23)$$

The values of the unknown parameters in the expressions for $h_j(t)$ and $h_{j'}(t)$ are computed separately using the values of the estimates given by equations (5-22) and (5-23), respectively.

If $g_j(t) \equiv g_{j'}(t)$ and if for some k, $t_{k_j} \neq t_{k_{j'}}$ then the methods just described can not be used. Methods need to be developed which will insure that the estimates of the unknown parameters in the expressions for $g_j(t)$ and $g_{j'}(t)$ are the same in these situations. The development of these methods is left as an open question. Further, if $g_j(t) \not\equiv g_{j'}(t)$ and $h_j(t) \equiv h_{j'}(t)$, something should be done to insure that the estimates of the unknown parameters in $h_j(t)$ and $h_{j'}(t)$ are the same. How this should be accomplished is also left as an open question.

Stage 2: Point Estimation of Treatment Effects

The discussion of the determination of the point estimators of treatment effects will be discussed separately for each case, since the process is slightly different in each case.

Case 4

In Case 4 the exact natures of the $h_j(t)$'s are known but only the functional forms of the $g_j(t)$'s are known. For this case, there are two methods possible for determining the $\widehat{\alpha_j(t)}$'s. For the first method, recall that at Stage 1, estimates were obtained for the unknown constants in the functional form expressions for the $g_j(t)$'s. For the jth group, let $\hat{g}_j(t)$ denote the function formed by substituting these estimates of the constants into the general functional form expression for $g_j(t)$. For example, if $g_j(t) = \log_c[b\cdot(t - t_1) + c]$, $\hat{b} = 1.2$, and $\hat{c} = 4.17$, where $\hat{b}$ and $\hat{c}$ are the Stage 1 estimates of b and c, then $\hat{g}_j(t) = \log_{4.17}[1.2(t - t_1) + 4.17]$. Point estimators of treatment effects are then given by

$$\widehat{\alpha_j(t)} = Y_j(t) - [\hat{g}_j(t)\cdot Y_j(t_{1_j}) + h_j(t)] \ .$$

Define m_{g_j} to be the number of unknown parameters in the functional form expression for $g_j(t)$. The method just described will work only if $p_j > m_{g_j}$.

The choice of $Y_j(t_{1_j})$ as the pretest to be used as the exogenous (i.e., independent) variable for estimating $\alpha_j(t)$ was arbitrary. The general growth model could have been stated alternatively as

$$Y^*_{ij}(t) = g_{jk}(t) \cdot Y^*_{ij}(t_{k_j}) + h_{jk}(t) + \alpha_j(t) \; , \qquad (5\text{-}24)$$

where the $g_{jk}(t)$'s and $h_{jk}(t)$'s are some continuous functions and where k is set equal to either 1, 2, ... , p_{j-1}, or p_j. Notice that equation (1-4) is the special case of (5-24) when k = 1. For a fixed k, then, $Y^*_j(t_{k_j})$ can be thought of as the exogenous variable and the remaining pretests $Y^*_j(t_{1_j})$, $Y^*_j(t_{2_j})$, ... , $Y^*_j(t_{k-1_j})$, $Y^*_j(t_{k+1_j})$, ... , $Y^*_j(t_{p_j})$ can be thought of as the endogenous variables in a structural equations causal model. It is now possible to generate p_j different point estimates of $\alpha_j(t)$, one from each of the p_j different versions of equation (5-24), where each time a different pretest becomes the exogenous variable.

It is left as an open question as to how point estimators of the $\alpha_j(t)$'s can be developed, which are improvements in the sense of increased precision over the p_j separate estimates, by using some function, such as the mean or median, of the p_j separate estimates of $\alpha_j(t)$.

For the second method, define a new variable, $W_{ij}(t) = Y_{ij}(t) - h_j(t)$. The system of equations (1-4) can then be rewritten as

$$W^*_{ij}(t) = g_j(t) \cdot W^*_{ij}(t_{1_j}) + \alpha_j(t)$$

and (5-25)

$$W_{ij}(t) = W^*_{ij}(t) + e_{ij}(t) \quad .$$

Notice that the system of equations (5-25) is, for each j, a linear structural relation with a slope of $g_j(t)$ and an intercept of $\alpha_j(t)$. Hence, for any particular time t, estimates of $g_j(t)$ and $\alpha_j(t)$ can be obtained directly by using the techniques described for substage 1 of Stage 1, with $W_j(t_{1_j})$ as the independent variable and $W_j(t)$ as the dependent variable. It should be recalled here that in order to compute the estimates in situations where there is only one independent and one dependent variable, it is necessary to make one of the assumptions listed in substage 1 (or some other assumption that will make the system identified). These direct estimates of the $\alpha_j(t)$'s are considered to be the Stage 2 estimates and are labelled as the $\widehat{\alpha_j(t)}$'s. Notice that Stages 1 and 2 are combined for this method. This is possible because the exact natures of the $h_j(t)$'s are known.

As with the first method, the use of $W_j(t_{1_j})$ as the independent variable is arbitrary. Rewriting equation (5-25) in terms of $W_j^*(t_{k_j})$ yields

$$W_{ij}^*(t) = g_{jk}(t) \cdot W_{ij}^*(t_{k_j}) + \alpha_j(t) \ . \qquad (5\text{-}26)$$

Equation (5-26) is a linear structural relation with $W_j^*(t_{k_j})$ as the independent variable and $W_j^*(t)$ as the dependent variable. Hence, p_j different estimates of $\alpha_j(t)$ can be calculated, one for each of the p_j pretests, using the techniques described for substage 1 of Stage 1, once one of the additional assumptions is made. As with the first method, it is left as an open question as to how point estimators of the $\alpha_j(t)$'s can be developed, which are improvements, in the sense of increased precision, over the p_j estimates of $\alpha_j(t)$ generated separately using only one pretest at a time, by using some function, such as the mean or median, of these p_j different estimates.

Since two methods have been proposed when Case 4 holds, the question of when each method is appropriate needs to be discussed. The first method has the advantage over the second method in that no additional assumptions need be made in order to implement it. The disadvantages of the first method are that it is computationally more complex than the second method and that a minimum of $m_{g_j} + 1$ pretest time points is needed. Hence, when observations are available for at least $m_{g_j} + 1$ time points and the data analyst is not willing to make any additional assumptions, the first method must be used. Further, when m_{g_j} or fewer pretest time points are available then the second method must be used and one of the additional assumptions made in order to insure the identifiability of the system. When there are at least $m_{g_j} + 1$ pretest time points available and one of the additional assumptions seems reasonable, then a choice must be made between the two methods. The method of choice should be that method which leads to the smallest standard error of estimate (i.e., the better precision). It is left as an open question as to which method possesses better precision.

Case 6

In Case 6 the exact natures of the $h_j(t)$'s are known and nothing is known about the $g_j(t)$'s. For this case, the $\widehat{\alpha_j(t)}$'s are determined using the second method described for Case 4 situations.

Case 8

In Case 8 the exact natures of the $g_j(t)$'s are known but only the functional forms of the $h_j(t)$'s are known. For this case estimators of the $h_j(t_{k_j})$'s are given by

$\widehat{h_j(t_{k_j})} = \overline{Y_j(t_{k_j})} - g_j(t_{k_j}) \cdot \overline{Y_j(t_{1_j})}$. The $\widehat{h_j(t_{k_j})}$'s are then used, exactly as was described for the second substage of Stage 1, to provide estimates of the unknown parameters in the functional form expressions of the $h_j(t)$'s. A new function, called $\hat{h}_j(t)$, is formed by substituting the estimates of the unknown parameters that were obtained in Stage 1 into the functional form expression for $h_j(t)$. Recall that in order for this method to be implemented it is necessary to have at least $m_{h_j} + 1$ pretest time points, where m_{h_j} is the number of unknown parameters in $h_j(t)$.

Point estimators of treatment effects are then given by

$$\widehat{\alpha_j(t)} = \overline{Y_j(t)} - [g_j(t)\cdot\overline{Y_j(t_{1_j})} + \hat{h}_j(t)] .$$

Case 10

In Case 10 the functional forms of both the $g_j(t)$'s and $h_j(t)$'s are known. The point estimation method for this case begin by using substage 1 of Stage 1 to generate the $\widehat{h_j(t_k)}$'s. Substage 2 is then used to find the estimates of the unknown parameters in the functional form expressions for the $h_j(t)$'s. The $\hat{h}_j(t)$'s are then formed by substituting the estimates of the unknown parameters into the functional form expressions for the $h_j(t)$'s. The method then proceeds as in Case 4, except that the $h_j(t)$'s of Case 4 are replaced by the $\hat{h}_j(t)$'s.

Case 12

Case 12 occurs when only the functional forms of the $h_j(t)$'s are known and when nothing is known about the $g_j(t)$'s. In this case a new variable is formed by defining $U_{ij}(t) = Y_{ij}(t) - \hat{h}_j(t)$, where $\hat{h}_j(t)$ is defined as in Case

10. The method described for Case 6 is then used to obtain the $\widehat{\alpha_j(t)}$'s by replacing the $W_{ij}(t)$'s of Case 6 with the $U_{ij}(t)$'s.

In developing the point estimators of the treatment effects it was assumed that for any particular nonequivalent control group design, the known information about the growth curves for each of the J groups belonged to the same case. Since the determination of the point estimators is done separately for each group, insisting that all of the groups belong to the same case is overly restrictive. Hence, it should be assumed that for each of the J groups, the known information about the growth curves allows the data analyst to place each of the groups into one of the cases 2,4,6,8,10, or 12. Once this is done, the $\widehat{\alpha_j(t)}$'s are derived separately for each group by using the methods given in this chapter for the case to which the group's growth curves belong.

Case 14

For Case 14 the exact natures of the $g_j(t)$'s are known and $h_j(t) \equiv h(t)$, with $h(t)$ unknown. In this case, the system of equations (1-4) can be rewritten as

$$Y^*_{ij}(t) = g_j(t)\cdot Y^*_{ij}(t_{1_j}) + h(t) + \alpha_j(t)$$

and (5-27)

$$Y_{ij}(t) = Y^*_{ij}(t) + e_{ij}(t) \ .$$

For this case, and for Cases 16 and 18, only differences in treatment effects can be estimated since the h(t) and $\alpha_j(t)$ terms are confounded. Taking the population mean on both sides of equation (5-27) yields

$$\mu_{Y_j}(t) = g_j(t)\cdot\mu_{Y_j}(t_{1_j}) + h(t) + \alpha_j(t) \ . \quad (5\text{-}28)$$

Similarly, for any other group, j',

$$\mu_{Y_{j'}}(t) = g_{j'}(t)\cdot\mu_{Y_{j'}}(t_{1_{j'}}) + h(t) + \alpha_{j'}(t) \ . \quad (5\text{-}29)$$

Subtracting equation (5-29) from equation (5-28) yields

$$\mu_{Y_j}(t) - \mu_{Y_{j'}}(t) =$$
$$g_j(t)\mu_{Y_j}(t_{1_j}) - g_{j'}(t)\mu_{Y_{j'}}(t_{1_{j'}}) + \alpha_j(t) - \alpha_{j'}(t) \ .$$

Hence,

$$\alpha_j(t) - \alpha_{j'}(t) =$$
$$\mu_{Y_j}(t) - \mu_{Y_{j'}}(t) - [g_j(t)\mu_{Y_j}(t_{1_j}) -$$
$$g_{j'}(t)\mu_{Y_{j'}}(t_{1_{j'}})] \ . \quad (5\text{-}30)$$

If the values of $\mu_{Y_j}(t_{1_j})$, $\mu_{Y_{j'}}(t_{1_{j'}})$, $\mu_{Y_j}(t)$, and $\mu_{Y_{j'}}(t)$ are known, then equation (5-30) determines the value of $\alpha_j(t) - \alpha_{j'}(t)$ exactly, and no interval estimation or hypothesis testing procedures need be discussed. It is, however, rarely the case that $\mu_{Y_j}(t_{1_j})$, $\mu_{Y_{j'}}(t_{1_{j'}})$, $\mu_{Y_j}(t)$, and $\mu_{Y_{j'}}(t)$ are known. When these populations means are unknown, a point estimator of $\alpha_j(t) - \alpha_{j'}(t)$ is given by

$$\widehat{\alpha_j(t) - \alpha_{j'}(t)} = \overline{Y_j(t)} - \overline{Y_{j'}(t)} - [g_j(t)\cdot\overline{Y_j(t_{1_j})} - g_{j'}(t)\cdot\overline{Y_{j'}(t_{1_{j'}})}] \ .$$

<u>Case 16</u>

For Case 16 the functional forms of the $g_j(t)$'s are known and $h_j(t) \equiv h(t)$, with $h(t)$ unknown. Define $H_j(t)$ to be $h(t) + \alpha_j(t)$. The system of equations (5-27) can then be rewritten as

$$Y^*_{ij}(t) = g_j(t)\cdot Y^*_{ij}(t_{1_j}) + H_j(t)$$

and (5-31)

$$Y_{ij}(t) = Y^*_{ij}(t) + e_{ij}(t) \ .$$

The two methods described under Case 4 for estimating $\alpha_j(t)$ can now be used here to estimate $H_j(t)$, simply by replacing $\alpha_j(t)$ by $H_j(t)$ in the Case 4 discussion. Once the $\widehat{H_j(t)}$'s have been determined, point estimates of the $\alpha_j(t) - \alpha_{j'}(t)$'s, for any two groups j and j', are formed by using $\widehat{H_j(t)} - \widehat{H_{j'}(t)}$.

Case 18

For Case 18 nothing is known about the $g_j(t)$'s and $h_j(t) \equiv h(t)$. The second method for estimating $\alpha_j(t)$ under Case 4 can be used here to estimate each of the $H_j(t)$'s in the system of equations (5-31), simply by replacing the $\alpha_j(t)$'s by the $H_j(t)$'s in the Case 4 discussion. Once the $\widehat{H_j(t)}$'s have been determined the point estimates of the differences in treatment effects are determined as in Case 16.

Bias of the Point Estimators

The point estimation techniques introduced in this chapter lead almost always to biased estimates of treatment effects. The discussion of this bias will be broken down into three categories according to the information available about the $g_j(t)$'s. The cases where the exact natures of the $g_j(t)$'s are known will be discussed first, followed by those cases where the functional forms of the $g_j(t)$'s are known and then by those cases where nothing is known about the $g_j(t)$'s.

Cases 2, 8, and 14 comprise those cases where the exact natures of the $g_j(t)$'s are known. For Cases 2 and 14, the formulas given in the previous section provide unbiased estimates of treatment effects and differences in treatment effects, respectively, since for all t, $E(\overline{Y_j(t)}) = \mu_{Y_j}(t)$ under a wide variety of statistical distributions. For Case 8, however, the $\alpha_j(t)$'s almost always provide biased estimates of the $\alpha_j(t)$'s. The Case 8 estimation method begins by finding estimates for the $h_j(t_{k_j})$'s, where the t_{k_j}'s are the pretest time points. These estimates are given by $\widehat{h_j(t_{k_j})} = \overline{Y_j(t_{k_j})} - g_j(t_{k_j}) \cdot \overline{Y_j(t_{1_j})}$. Notice that these $\widehat{h_j(t_{k_j})}$'s are unbiased estimates of the $h_j(t_{k_j})$'s under a wide variety of statistical distributions. In order to find estimates for the parameters in the functional form expressions of the $h_j(t)$'s, a system of equations analogous to (5-6) is set up which relates the $h_j(t_{k_j})$'s to the unknown parameters. This system is then solved for the unknown parameters. Hence, expressions are obtained which are estimates of the unknown parameters in terms of the $h_j(t_{k_j})$'s. That is, estimates of the unknown parameters are given as functions of the $\overline{Y_j(t_{k_j})}$'s and $g_j(t_{k_j})$'s; $k = 1,2,\ldots,p_j$. Even though the $\overline{Y_j(t_{k_j})}$'s are unbiased estimators of the $\mu_{Y_j}(t_{k_j})$'s, it is rarely the case that functions of the $\overline{Y_j(t_{k_j})}$'s are unbiased estimators of the corresponding functions of the $\mu_{Y_j}(t_{k_j})$'s (Bickel & Doksum, 1977). For example, $E\left(\dfrac{\overline{Y_j(t_{1_j})}}{\overline{Y_j(t_{2_j})}}\right) \neq \dfrac{\mu_{Y_j}(t_{1_j})}{\mu_{Y_j}(t_{2_j})}$, when a bivariate normal distribution is assumed for $Y_j(t_{1_j})$ and $Y_j(t_{2_j})$ (Cochran, 1977). Hence, the estimates of the unknown parameters are usually biased. Therefore, the $\hat{h}_j(t)$'s, which were formed by substituting the estimates of the unknown parameters back into the functional form expressions of the $h_j(t)$'s, are biased estimates of the $h_j(t)$'s. Consequently, the $\widehat{\alpha_j(t)}$'s, as defined by,

$\widehat{\alpha_j(t)} = Y_j(t) - [g_j(t) \cdot Y_j(t_{1_j}) + \hat{h}_j(t)]$, are almost always biased estimators of the $\alpha_j(t)$'s.

Since the $\alpha_j(t)$'s are functions of the $\overline{Y_j(t_{k_j})}$'s, by taking limits as the sample sizes approach infinity, one can see that the $\alpha_j(t)$'s are, however, consistent estimators, provided that the limits exist. It should be pointed out here that it is an accepted technique in the applied statistics literature when a point estimator is desired of a quantity which is a function of several parameters, to first find unbiased, or even just consistent, point estimators of the parameters. These estimates of the parameters are then substituted into the original function to yield a useable point estimate of the desired quantity. For example, a point estimate of $\frac{\sigma_Y}{\sigma_X}$ is given by $\frac{S_Y^2}{S_X^2}$. Since in many instances in the past literature the techniques used for Case 8 lead to point estimators with an acceptable amount at bias, it is conjectured that the Case 8 techniques will also lead to point estimates with an acceptable amount of bias. Further study of this bias is needed.

For Cases 4, 10, and 16, where only the functional form expressions for the $g_j(t)$'s are known, maximum likelihood techniques are first used to generate estimates of the $g_j(t_{k_j})$'s. It is well known that maximum likelihood techniques often lead to biased estimation (Bickel & Doksum, 1977: Mood, Graybill, and Boes, 1974). Maximum likelihood estimators are, however, usually consistent (Patel, Kapadia, & Owen, 1976). Further, maximum likelihood estimation is probably the most widely used and accepted estimation technique, since it has been found in a wide variety of situations to produce estimators which have a negligible amount of bias and which are asymptotically efficient when compared to a large class of possible estimators (Zacks, 1971).

The maximum likelihood estimates of the $g_j(t_{k_j})$'s are then used to generate estimates for the unknown parameters in the functional form expressions of the $g_j(t)$'s, in a manner analogous to that just described for finding estimates of the unknown parameters in the expressions for the $h_j(t)$'s in Case 8. Further, for Case 10, maximum likelihood estimates of the $h_j(t_{k_j})$'s are generated using $\widehat{h_j(t_{k_j})} = Y_j(t_{k_j}) - \widehat{g_j(t_{k_j})} \cdot Y_j(t_{1_j})$, where the $\widehat{g_j(t_{k_j})}$'s are the maximum likelihood estimates of the $g_j(t_{k_j})$'s. These maximum likelihood estimates of the $h_j(t_{k_j})$'s are then used to replace the unbiased estimates of the $h_j(t_{k_j})$'s used in Case 8, and the $\hat{h}_j(t)$'s are then generated as in Case 8. Hence, for Cases 4, 10, and 16 maximum likelihood estimation along with the techniques from Case 8 are used to generate estimates of treatment effects and differences in treatment effects. Consequently, these estimators are usually consistent estimators. Since both the maximum likelihood estimation and the Case 8 techniques have been found to lead to acceptable levels of bias, it is conjectured that a combination of these two methods will still lead to estimators with an acceptable amount of bias. The nature of the bias needs to be studied further.

For Cases 6 and 18 maximum likelihood techniques are used to directly arrive at estimates of the treatment effects and/or differences in treatment effects. Case 12 combines the maximum likelihood techniques used in Case 6 with the techniques used in Case 10 to find the $\hat{h}_j(t)$'s. The $\hat{h}_j(t)$'s are then used as in Case 8, to find estimates of the treatment effects and differences in treatment effects. Hence in these three cases the estimators used are also almost always biased, although usually consistant, estimators. In these cases, also, the nature of the bias of the estimators still needs to be investigated. For these cases, as well as for Cases 4, 8, 10, and 16 computer simulation techniques appear to be the only feasible method for further study of the bias.

CHAPTER 6

INTERVAL ESTIMATION AND HYPOTHESIS TESTING PROCEDURES

In Chapter 5 methods were developed for the point estimation of treatment effects for those cases where errors of measurement were present and where either the exact natures or the functional forms of the $h_j(t)$'s were known. The beginning of this chapter is concerned with the interval estimation of treatment effects and with the testing of the hypotheses of nonzero effects (i.e., H_o: $\alpha_j(t) = 0$ versus H_1: $\alpha_j(t) \neq 0$). Next, procedures for the interval estimation and hypothesis testing of differences in treatment effects are developed.

In order to develop interval estimates and hypothesis testing procedures it is necessary to have estimates of the variance of the $\widehat{\alpha_j(t)}$'s. Since the probability distributions of the $\widehat{\alpha_j(t)}$'s are unknown, except in Case 2, traditional methods can not be used to estimate the variances of the $\widehat{\alpha_j(t)}$'s. Two techniques which have been suggested in the literature for estimating variances when nothing is known about the probability distributions are the δ-method (Bishop, Feinberg, and Holland, 1975) and jackknifing (Tukey, 1958). The δ-method quickly becomes computationally intractable in many cases. Moran (1971) has shown that this intractableness occurs for the growth models and designs being considered here when even only two pretest observations are available. Because of its computational difficulties, it was decided not to use the δ-method to find estimates of the variances of the $\widehat{\alpha_j(t)}$'s. The method of jackknifing, however, avoids the computational difficulties inherent in the δ-method. Further, jackknifing was originally designed as a bias reduction method (Quenouille, 1956). Hence, besides providing estimates of the variances of the $\widehat{\alpha_j(t)}$'s, jackknifing will also in most cases (Gray & Schucany, 1972) provide a reduction in the bias of the point estimators of the treatment effects and differences in effects.

The technique of jackknifing begins by drawing a random sample from a specified population. Let N denote the number of subjects in the sample. The N subjects are then divided into m disjoint subsets, each of size $\frac{N}{m}$. Let γ be the parameter of interest and $\hat{\gamma}$ be an estimator of γ. Further, let $\hat{\gamma}^T$ be the value of $\hat{\gamma}$ when all N subjects are used, and let $\hat{\gamma}^{(\ell)}$ be the value of $\hat{\gamma}$ when the subsample of size $N - \frac{N}{m}$, where the ℓ<u>th</u> subset has been deleted, is used.

Next define $J_{\ell}(\hat{\gamma})$ by

$$J_{\ell}(\hat{\gamma}) = m \cdot \hat{\gamma}^{T} - (m-1)\hat{\gamma}^{(\ell)} \qquad ; \ \ell = 1, 2, \ldots, m$$

and define $J(\hat{\gamma})$ by

$$J(\hat{\gamma}) = \frac{1}{m} \cdot \sum_{\ell=1}^{m} J_{\ell}(\hat{\gamma}) \ .$$

An estimate of the variance of $J(\hat{\gamma})$ is given by (Tukey, 1958)

$$S_J^2 = \frac{\sum_{\ell=1}^{m} [J_{\ell}(\hat{\gamma}) - J(\hat{\gamma})]^2}{m - 1} \ .$$

Think of m as being fixed. Gray and Schucany (1972) have shown that $\frac{\sqrt{m} \cdot [J(\hat{\gamma}) - \gamma]}{\sqrt{S_J^2}}$ is asymptotically distributed (as $\frac{N}{m} \to \infty$) as a t-random variable with m − 1 degrees of freedom. An open problem is to determine how m should be chosen so as to allow $\frac{\sqrt{m}[J(\hat{\gamma}) - \gamma]}{\sqrt{S_J^2}}$ to be distributed approximately as a Student's t random variable and at the same time allow for enough degrees of freedom so that the power of the test is not too low (Miller, 1974).

The estimators of interest here are the $\widehat{\alpha_j(t)}$'s from Cases 2, 4, 6, 8, 10, and 12. It should be kept in mind here that once the $\widehat{\alpha_j(t)}$'s have been determined it is no longer important to consider which of the cases the growth curves belong to. To apply the jackknifing technique, first divide the N_j subjects from the j<u>th</u> group into m_j disjoint subsets. Next define $\widehat{\alpha_j^{(\ell)}(t)}$ to be the value of the estimator, $\widehat{\alpha_j(t)}$, when the ℓ<u>th</u> subset is deleted from the sample for the j<u>th</u> group. Notice that the values of the estimates of the $g_j(t_k)$'s and $h_j(t_k)$'s change when the ℓ<u>th</u> subset is deleted, and hence also the values of the estimates of the unknown parameters in the expressions for $g_j(t)$ and $h_j(t)$ will change once the ℓ<u>th</u> subset is deleted. Hence, in order to compute all of the $\widehat{\alpha_j^{(\ell)}(t)}$'s, the entire two-stage process described in Chapter 5 must be repeated m_j times for the j<u>th</u> group. Once the $\widehat{\alpha_j^{(\ell)}(t)}$'s have been computed, new estimators of the $\alpha_j(t)$'s are given by

$$J(\widehat{\alpha_j(t)}) = \frac{1}{m_j} \sum_{\ell=1}^{m_j} J_\ell(\widehat{\alpha_j(t)}) ,$$

where (6-1)

$$J_\ell(\widehat{\alpha_j(t)}) = m_j \cdot \widehat{\alpha_j(t)} - (m_j - 1) \cdot \widehat{\alpha_j^{(\ell)}(t)} .$$

These new estimators, the $J(\widehat{\alpha_j(t)})$'s, have two advantages over the original estimators, the $\widehat{\alpha_j(t)}$'s. The first advantage is that in most situations the bias of the $J(\widehat{\alpha_j(t)})$'s as estimates of the $\alpha_j(t)$'s is less than the bias of the $\widehat{\alpha_j(t)}$'s. There are, however, situations in which the bias of $J(\widehat{\alpha_j(t)})$ is greater than the bias of $\widehat{\alpha_j(t)}$ (Gray & Schucany, 1972). Gray and Schucany (1972) discuss various conditions under which jackknifing increases, decreases, or does not affect the bias of the estimator of interest. The conditions, however, demand knowledge of some of the properties of the distribution of the estimator. But, for the estimators of interest here (i.e., the $\widehat{\alpha_j(t)}$'s) nothing is known about their distributions, so one can not know for sure whether jackknifing increases, decreases, or does not affect their biases.

The second advantage of $J(\widehat{\alpha_j(t)})$ over $\widehat{\alpha_j(t)}$ is that once the $J_\ell(\widehat{\alpha_j(t)})$'s and $J(\widehat{\alpha_j(t)})$'s are computed, interval estimation and hypothesis testing procedures are available by observing that

$$\frac{\sqrt{m_j}\cdot[J(\widehat{\alpha_j(t)}) - \alpha_j(t)]}{\sqrt{(S_J^2(t))_j}}$$

is asymptotically distributed as a random variable with a Student's t distribution with $m_j - 1$ degrees of freedom

$$\text{where} \quad (S_J^2(t))_j = \frac{\sum_{\ell=1}^{m_j}[J_\ell(\widehat{\alpha_j(t)}) - J(\widehat{\alpha_j(t)})]^2}{m_j - 1} .$$

An approximate $(1 - \alpha)\%$ confidence interval for $\alpha_j(t)$ (and hence, an α-level test for nonzero $\alpha_j(t)$) is then given by

$$J(\widehat{\alpha_j(t)}) \pm t_{m_j-1}(1 - \tfrac{\alpha}{2}) \cdot \frac{\sqrt{(S_J^2(t))_j}}{\sqrt{m_j}} .$$

The estimation and hypothesis testing of differences in treatment effects are also of interest. A point estimator of $\alpha_j(t) - \alpha_{j'}(t)$, for two groups j and j', can be given by $J(\widehat{\alpha_j(t)}) - J(\widehat{\alpha_{j'}(t)})$ where $J(\widehat{\alpha_j(t)})$ and $J(\widehat{\alpha_{j'}(t)})$ are defined by equation (6-1). An approximate $(1 - \alpha)\%$ test of the hypothesis of nonzero differences in treatment effects is accomplished by performing an ANOVA on the $J_\ell(\widehat{\alpha_j(t)})$'s. Notice that the unit of analysis for the ANOVA being performed here is the disjoint subsets formed in order to do the jackknifing and that the dependent variable is $J_\ell(\widehat{\alpha_j(t)})$. These procedures can be used as long as all of the growth curves under consideration belong to either Case 2, 4, 6, 8, 10, or 12.

For Cases 14, 16, and 18 jackknifing can be used to provide estimation and hypothesis testing procedures for differences in treatment effects. For these cases, the growth models under consideration are written as

$$Y^*_{ij}(t) = g_j(t) \cdot Y^*_{ij}(t_{1_j}) + H_j(t) , \qquad (5\text{-}31)$$

where $H_j(t) = h(t) + \alpha_j(t)$. For Case 14, where the exact natures of the $g_j(t)$'s are known, $\widehat{H_j(t)}$ is given by $\overline{Y_j(t)} - g_j(t) \cdot \overline{Y_j(t_{1_j})}$. For Cases 16 and 18, the procedures for obtaining the $\widehat{H_j(t)}$'s were described in Chapter 5. The estimation and hypothesis testing techniques discussed in the previous paragraph can be applied to the $\widehat{H_j(t)} - \widehat{H_{j'}(t)}$'s to provide point and interval estimates for the $\alpha_j(t) - \alpha_{j'}(t)$'s and tests of H_o: $\alpha_j(t) = \alpha_{j'}(t)$ versus H_1: $\alpha_j(t) \neq \alpha_{j'}(t)$, for any two groups j and j', since $\widehat{H_j(t) - H_{j'}(t)}$ is the point estimator of $\alpha_j(t) - \alpha_{j'}(t)$ derived in Chapter 5. The problem arises here that each of the differences between treatment effects for each pair of two groups is tested separately. But, what is needed is an ANOVA type procedure which tests the hypothesis H_o: $\sum_{j=1}^{J}(\alpha_j(t) - \mu_\alpha(t))^2 = 0$ versus H_1: $\sum_{j=1}^{J}(\alpha_j(t) - \mu_\alpha(t))^2 \neq 0$, where $\mu_\alpha(t)$ is the grand mean of the $\alpha_j(t)$'s. The development of this hypothesis testing procedure is left as an open question. Recall, however, that if $g_j(t) \equiv g(t)$ for each t and if $\rho_{Y(t_1)Y(t_1)}$ is known, then Estimated True Scores ANCOVA (Porter, 1967) provides a test of H_o: $\sum_{j=1}^{J}(\alpha_j(t) - \mu_\alpha(t))^2 = 0$ versus H_1: $\sum_{j=1}^{J}(\alpha_j(t) - \mu_\alpha(t))^2 \neq 0$. Notice that the use of Estimated True Scores ANCOVA requires that $t_{1_j} \equiv t_1$.

Several generalizations of jackknifing have been proposed in the literature: the generalized jackknife (Gray & Schucany, 1972), the infinitesimal jackknife (Jaeckel, 1972), and bootstrapping (Efron, 1979). All three of these methods require information about the distributional properties of the estimator on which the jackknife is to be performed. For the estimators being considered here (i.e., the $\widehat{\alpha_j(t)}$'s and $\widehat{H_j(t)}$'s), the information about their distributions that is needed to use any of the three methods is not available.

Since one of the benefits of jackknifing is that usually the amount of bias in the estimators that have been jackknifed is less than that of the original estimators (Quenouille, 1956; Gray & Schucany, 1972), the nature of the bias of the $J(\widehat{\alpha_j(t)})$'s (the $J(\widehat{H_j(t)})$'s in Cases 14, 16, and 18) needs to be studied. As with the $\widehat{\alpha_j(t)}$'s, it appears that the only feasible method for studying this bias is through the use of computer simulation techniques. If it is found, as expected, that the bias of the $J(\widehat{\alpha_j(t)})$'s is less than the bias of the $\widehat{\alpha_j(t)}$'s, then the $J(\widehat{\alpha_j(t)})$'s should be used as the point estimators of treatment effects and the $J(\alpha_j(t))$'s, or $J(\widehat{H_j(t)})$'s, should be used to form the point estimators of differences in treatment effects, depending on the case. For those situations, if any, where it is found that jackknifing leads to an increase in bias, the original estimators, the $\widehat{\alpha_j(t)}$'s, should be used as the point estimators. Finally, since the jackknifed estimators are linear functions of the original estimators, it follows from the previous discussion of the $\widehat{\alpha_j(t)}$'s that the jack-knifed estimators will also usually be consistent point estimators.

CHAPTER 7

DISCUSSION

In this chapter a summary of the major results from Chapters 2 to 6 and a discussion of directions for further research are provided. The implications of this dissertation for the analysis of data collected using quasi-experimental designs are also discussed.

Summary

In this dissertation point estimation, interval estimation, and hypothesis testing procedures for the assessment of treatment effects and differences in treatment effects have been described for the class of growth models where a correlation within each group of +1 between true scores at any two points in time is assumed. In Chapter 2 it was shown that this class includes an infinite variety of types of growth over time, and not just linear growth. The set of designs allowed under this class of models includes single-group and multi-group designs with or without the presence of one or more control groups. The techniques described here can easily be extended to designs which involve the investigation of interactions through the use of crossed factors.

In Chapter 3 it was shown that for this set of designs the existing methods of data analysis rarely result in adequate estimation and hypothesis testing procedures under the class of growth models being considered. In Chapters 4 through 6 methcds of data analysis were developed which were consistent with the class of growth models. The only additional assumptions made in developing these new methods were:

(i) Classical measurement theory holds;

(ii) Treatment effects are additive;

and (iii) Either $h_j(t) \equiv h(t)$ or for each of the J groups, either the exact nature of or the functional form of $h_j(t)$ is known.

For those cases where no errors of measurement are present, methods were developed under which the exact values of the treatment effects (for those cases where either the exact natures of the $h_j(t)$'s or their functional forms were known) or differences in treatment effects (for those cases where $h_j(t) \equiv h(t)$) were computed. Even though these methods have the advantage that they provide the exact values of the treatment effects and differences in treatment effects, they have the disadvantage that they can rarely be applied to analyze data arising in educational settings, since it is usually the case that errors of measurement are present.

For the cases where errors of measurement are present and either the exact natures or the functional forms of the $h_j(t)$'s are known, point estimation, interval estimation, and hypothesis testing procedures were developed for both the assessment of treatment effects and differences in treatment effects. For those cases where $h_j(t) \equiv h(t)$ estimation and hypothesis testing procedures were developed for the assessment of differences in treatment effects. Except for the cases where the exact natures of the $g_j(t)$'s are known and either $h_j(t) \equiv h(t)$ or the exact natures of the $h_j(t)$'s are known, the point estimation procedures used almost always lead to biased estimates of treatment effects. As indicated in Chapter 5, the nature of the bias needs to be studied further. It should be remembered that the estimation methods used are methods which are frequently used by applied statisticians. It is well known that these methods lead to biased estimates, but whose bias is generally at an acceptable level. Despite being biased, these estimators are usually consistent. Thus, these estimators have the advantage, over the estimators generated using existing methods suggested in the past literature, that they are consistent whereas, except in rare cases, the existing estimation methods do not yield consistent estimators.

Directions For Further Research

In Chapters 5 and 6, when developing the methods for the estimation and hypothesis testing of both treatment effects and differences in treatment effects when errors of measurement are present, several directions for further research were mentioned. This section provides a synthesis of these directions plus additional directions for further research. As mentioned above, one direction for further research is the study of the bias and other statistical properties of the point estimators. Further, for those cases where the functional forms of the $g_j(t)$'s are known, two different methods were suggested for determining point estimators of treatment effects and differences in treatment effects. It is left as an open question for further study to determine which of the two methods provides better precision. Besides comparing these two methods with respect to precision, they should also be compared with respect to the amount of bias inherent in each method. Once more is known about the precision and bias of each of the two methods, then a decision can be made as to which method should be used for those cases where the functional forms of the $g_j(t)$'s are known.

The technique of jackknifing was used to generate interval estimation and hypothesis testing procedures. There are still several open questions which need to be explored with respect to jackknifing. First, it is known that the method of jackknifing can either reduce, increase, or not affect the amount of bias of an estimator. It is important to determine whether for the models of growth considered here there is an increase or a decrease in the amount of bias as a result of the jackknifing procedure. If the Stage 2 estimators have less bias then the jackknifed estimators, then they should be retained as the point estimators of treatment effects. But, as is suspected, if the jackknifed estimators have the lesser amount of bias, then they should be used as the point estimators of treatment effects. A second direction for further research is the determination of the best way to form the disjoint subsets which are used to perform the jackknifing. Finally, a test of H_o: $\sum_{j=1}^{J} (\alpha_j(t) - \mu_\alpha(t))^2 = 0$ versus H_1: $\sum_{j=1}^{J} (\alpha_j(t) - \mu_\alpha(t))^2 \neq 0$ still needs to be developed for Cases 14, 16, and 18.

Implications for Data Analysis and Collection

When confronted with a data set collected using the generalized nonequivalent control group designs considered here, the data analyst must be careful when applying one of the existing methods of data analysis since, except under rare conditions, these existing methods lead to incorrect estimates of treatment effects. For data sets which do not conform to these rare conditions, the methods developed in this dissertation should be seriously considered.

The methods of data analysis developed here depend heavily on having observations at several pretest time points. Hence, when planning and collecting data in future studies using nonequivalent control group designs, it is important to have available observations from as many time points as is possible given the constraints of the research setting. Further, the methods developed here depend on having some a priori knowledge of the nature of the natural growth curves. Hence, attention should be focused on collecting data in the absence of any treatment which could then be used to determine the functional forms of the $h_j(t)$'s or, in some situations, to convince the data analyst that it is reasonable to assume that for all j, $h_j(t) \equiv h(t)$.

Finally, it should be remembered that these new data analysis techniques were shown to be applicable only when a correlation within each group of +1 between true scores exists at any two points in time. Hence, these techniques should be used cautiously when one suspects that the actual correlation is not approximately equal to +1, until robustness studies have been carried out to determine the extent to which the assumption of a correlation of +1 can be violated. The development of methods when a correlation of less than +1 exists has been virtually ignored. The notable exception is the work of Strenio, Bryk, and Weisberg (Bryk, Strenio, & Weisberg, 1980; Strenio, Weisberg, & Bryk, in press). Hence, even though the techniques developed in this dissertation are applicable only in a limited number of data analysis situations, they are still potentially important techniques to consider since they are applicable in a wider variety of situations than the previously available techniques.

APPENDIX A

APPENDIX A

IDENTIFIABILITY

This appendix provides a demonstration that the system of equations (5-9), (5-10), (5-11), (5-12), and (5-13) is identifiable if and only if $p \geq 3$.

First, for k = 2, equation (5-12) can be written as

$$\sigma_{Y(t_2)Y(t_1)} = g(t_2)\cdot[\sigma_Y^*(t_1)]^2 \; . \qquad \text{(A-1)}$$

For k' = 2, equation (5-13) can be written as

$$\sigma_{Y(t_k)Y(t_2)} = g(t_k)\cdot g(t_2)\cdot[\sigma_Y^*(t_1)]^2 \; . \qquad \text{(A-2)}$$

Hence, by equations (A-1) and (A-2)

$$\frac{\sigma_{Y(t_k)Y(t_2)}}{\sigma_{Y(t_2)Y(t_1)}} = \frac{g(t_k)\cdot g(t_2)\cdot[\sigma_Y^*(t_1)]^2}{g(t_2)\cdot[\sigma_Y^*(t_1)]^2}$$

$$= g(t_k) \text{ for } k=3,4,\ldots,p \; .$$

That is,

$$g(t_3) = \frac{\sigma_{Y(t_3)Y(t_2)}}{\sigma_{Y(t_2)Y(t_1)}} ,$$

$$g(t_4) = \frac{\sigma_{Y(t_4)Y(t_2)}}{\sigma_{Y(t_2)Y(t_1)}} , \qquad \text{(A-3)}$$

$$\vdots$$

$$g(t_p) = \frac{\sigma_{Y(t_p)Y(t_2)}}{\sigma_{Y(t_2)Y(t_1)}} .$$

By interchanging the roles of t_2 and t_3, it can be seen that

$$g(t_2) = \frac{\sigma_{Y(t_3)Y(t_2)}}{\sigma_{Y(t_3)Y(t_1)}} . \qquad \text{(A-4)}$$

Substituting equation (A-4) into equation (A-1) gives

$$\sigma_{Y(t_2)Y(t_1)} = \frac{\sigma_{Y(t_3)Y(t_2)}}{\sigma_{Y}(t_3)Y(t_1)} \cdot [\sigma_Y^*(t_1)]^2 . \qquad \text{(A-5)}$$

Solving equation (A-5) for $[\sigma_Y^*(t_1)]^2$ yields

$$[\sigma_Y^*(t_1)]^2 = \frac{\sigma_{Y(t_2)Y(t_1)} \cdot \sigma_{Y(t_3)Y(t_1)}}{\sigma_{Y(t_3)Y(t_2)}} . \tag{A-6}$$

Next, solving the set of equations (5-9) for the <u>h(t_k)</u> yields

$$\begin{bmatrix} h(t_2) \\ h(t_3) \\ \cdot \\ \cdot \\ \cdot \\ h(t_p) \end{bmatrix} = \begin{bmatrix} \mu_Y(t_2) \\ \mu_Y(t_3) \\ \cdot \\ \cdot \\ \cdot \\ \mu_Y(t_p) \end{bmatrix} - \begin{bmatrix} g(t_2) \\ g(t_3) \\ \cdot \\ \cdot \\ \cdot \\ g(t_p) \end{bmatrix} \cdot \mu_Y(t_1) . \tag{A-7}$$

Substituting the system of equations (A-3) and equation (A-4) into the set of equations (A-7) yields

$$\begin{bmatrix} h(t_2) \\ h(t_3) \\ h(t_4) \\ \cdot \\ \cdot \\ \cdot \\ h(t_p) \end{bmatrix} = \begin{bmatrix} \mu_Y(t_2) \\ \mu_Y(t_3) \\ \mu_Y(t_4) \\ \cdot \\ \cdot \\ \cdot \\ \mu_Y(t_p) \end{bmatrix} - \begin{bmatrix} \dfrac{\sigma_{Y(t_3)Y(t_2)}}{\sigma_{Y(t_3)Y(t_1)}} \\ \dfrac{\sigma_{Y(t_3)Y(t_2)}}{\sigma_{Y(t_2)Y(t_1)}} \\ \dfrac{\sigma_{Y(t_4)Y(t_2)}}{\sigma_{Y(t_2)Y(t_1)}} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{\sigma_{Y(t_p)Y(t_2)}}{\sigma_{Y(t_2)Y(t_1)}} \end{bmatrix} \cdot \mu_Y(t_1) \ . \qquad \text{(A-8)}$$

Further, solving equation (5-10) for $\sigma_e^2(t_1)$ gives

$$\sigma_e^2(t_1) = \sigma_Y^2(t_1) - [\sigma_Y^*(t_1)]^2 \ . \qquad \text{(A-9)}$$

Substituting equation (A-6) into equation (A-9) yields

$$\sigma_e^2(t_1) = \sigma_Y^2(t_1) - \frac{\sigma_{Y(t_2)Y(t_1)} \cdot \sigma_{Y(t_3)Y(t_1)}}{\sigma_{Y(t_3)Y(t_2)}} \ . \qquad \text{(A-10)}$$

Finally, by rewriting equation (5-11),

$$\sigma_e^2(t_k) = \sigma_Y^2(t_k) - [g(t_k)]^2 \cdot [\sigma_Y^*(t_1)]^2 \ . \qquad \text{(A-11)}$$

Substituting equations (A-4) and (A-3) into the set of equations (A-11) yields

$$\begin{bmatrix} \sigma_e^2(t_2) \\ \sigma_e^2(t_3) \\ \sigma_e^2(t_4) \\ \cdot \\ \cdot \\ \cdot \\ \sigma_e^2(t_p) \end{bmatrix} = \begin{bmatrix} \sigma_Y^2(t_2) \\ \sigma_Y^2(t_3) \\ \sigma_Y^2(t_4) \\ \cdot \\ \cdot \\ \cdot \\ \sigma_Y^2(t_p) \end{bmatrix} - \begin{bmatrix} \left(\dfrac{\sigma_{Y(t_3)Y(t_2)}}{\sigma_{Y(t_3)Y(t_1)}}\right)^2 \\ \left(\dfrac{\sigma_{Y(t_3)Y(t_2)}}{\sigma_{Y(t_2)Y(t_1)}}\right)^2 \\ \left(\dfrac{\sigma_{Y(t_4)Y(t_2)}}{\sigma_{Y(t_2)Y(t_1)}}\right)^2 \\ \cdot \\ \cdot \\ \cdot \\ \left(\dfrac{\sigma_{Y(t_p)Y(t_2)}}{\sigma_{Y(t_2)Y(t_1)}}\right)^2 \end{bmatrix} \cdot \kappa$$

where $\kappa = \dfrac{\sigma_{Y(t_2)Y(t_1)}\sigma_{Y(t_3)Y(t_1)}}{\sigma_{Y(t_2)Y(t_3)}}$. Hence, by inspecting equations (A-3), (A-4), (A-6), (A-8), (A-10), and (A-12) it can be seen that the parameters on the right hand side of equations (5-9) through (5-13), namely $\mu_Y(t_1)$, $[\sigma_Y^*(t_1)]^2$, $g(t_2)$, $g(t_3)$, ... , $g(t_p)$, $h(t_2)$, $h(t_3)$, ... , $h(t_p)$, $\sigma_e^2(t_1)$, $\sigma_e^2(t_2)$, $\sigma_e^2(t_3)$, ... , and $\sigma_e^2(t_p)$ can be written in terms of the parameters on the left hand side, namely $\mu_Y(t_1)$, $\mu_Y(t_2)$, $\mu_Y(t_3)$, ... , $\mu_Y(t_p)$ and the $\dfrac{p(p + 1)}{2}$ parameters in the variance-covariance matrix of $Y(t_1)$, $Y(t_2)$, $Y(t_3)$, ... , and $Y(t_p)$. Consequently, the system of equations (5-9) through (5-13) is identifiable whenever $p \geq 3$.

When $p = 2$ there are 6 parameters on the right hand side of equations (5-9) through (5-13) [namely, $\mu_Y(t_1)$, $[\sigma_Y^*(t_1)]^2$, $g(t_2)$, $h(t_2)$, $\sigma_e^2(t_1)$, and $\sigma_e^2(t_2)$] and 5 parameters on the left hand side [namely, $\mu_Y(t_1)$, $\mu_Y(t_2)$, $\sigma_Y^2(t_1)$, $\sigma_Y^2(t_2)$, and $\sigma_{Y(t_1)Y(t_2)}$]. Since the number of parameters on the right hand side is greater than the number on the left hand side the system is automatically underidentified.

APPENDIX B

APPENDIX B

DERIVATION OF MAXIMUM LIKELIHOOD ESTIMATES UNDER ASSUMPTIONS 4, 5, AND 6

Assumption 4: The ratio of $\sigma_e^2(t_2)$ to $\sigma_e^2(t_1)$ is known

Let λ represent the ratio of $\sigma_e^2(t_2)$ to $\sigma_e^2(t_1)$. Hence equation (5-20) can be rewritten as

$$\sigma_Y^2(t_2) = [g(t_2)]^2 \cdot [\sigma_Y^*(t_1)]^2 + \lambda \cdot \sigma_e^2(t_1) \quad . \qquad \text{(B-1)}$$

Solving the system of equations (5-17), (5-18), (5-19), (B-1), and (5-21) for $g(t_2)$ in terms of $\mu_Y(t_1)$, $\mu_Y(t_2)$, $\sigma_Y^2(t_1)$, $\sigma_Y^2(t_2)$, and $\sigma_{Y(t_1)Y(t_2)}$ gives

$$g(t_2) = \frac{\left(\sigma_Y^2(t_2) - \lambda\sigma_Y^2(t_1) \pm \sqrt{(\sigma_Y^2(t_2) - \lambda\sigma_Y^2(t_1))^2 + 4\lambda[\sigma_{Y(t_1)Y(t_2)}]^2}\right)}{2\sigma_{Y(t_1)Y(t_2)}}$$

Since $g(t_2) > 0$ and $[\sigma_Y^*(t_1)]^2 > 0$, then by equation (5-21) $\sigma_{Y(t_1)Y(t_2)} > 0$. Consequently,

$$K_1 = \frac{\left(\sigma_Y^2(t_2) - \lambda\sigma_Y^2(t_1) + \sqrt{(\sigma_Y^2(t_2) - \lambda\sigma_Y^2(t_1))^2 + 4\lambda[\sigma_{Y(t_1)Y(t_2)}]^2} \right)}{2\sigma_{Y(t_1)Y(t_2)}} > 0$$

and

$$K_2 = \frac{\left(\sigma_Y^2(t_2) - \lambda\sigma_Y^2(t_1) - \sqrt{(\sigma_Y^2(t_2) - \lambda\sigma_Y^2(t_1))^2 + 4\lambda[\sigma_{Y(t_1)Y(t_2)}]^2} \right)}{2\sigma_{Y(t_1)Y(t_2)}} < 0 .$$

Hence, since $g(t_2) > 0$, K_2 can never be a solution for $g(t_2)$. Consequently, K_1 provides the unique solution for $g(t_2)$ in terms of $\mu_Y(t_1)$, $\mu_Y(t_2)$, $\sigma_Y^2(t_1)$, $\sigma_Y^2(t_2)$ and $\sigma_{Y(t_1)Y(t_2)}$. Therefore, a one-to-one transformation exists between the set of parameters $\mu_Y(t_1)$, $\mu_Y(t_2)$, $\sigma_Y^2(t_1)$, $\sigma_Y^2(t_2)$, and $\sigma_{Y(t_1)Y(t_2)}$ and the set of parameters $\mu_Y(t_1)$, $[\sigma_Y^*(t_1)]^2$, $g(t_2)$, $h(t_2)$ and $\sigma_e^2(t_1)$ as defined by

$$\mu_Y(t_1) = \mu_Y(t_1) \ ;$$

$$[\sigma_Y^*(t_1)]^2 = \frac{2[\sigma_{Y(t_1)Y(t_2)}]^2}{\left(\sigma_Y^2(t_2) - \lambda\sigma_Y^2(t_1) + \sqrt{(\sigma_Y^2(t_2) - \lambda\sigma_Y^2(t_1)^2 + 4\lambda[\sigma_{Y(t_1)Y(t_2)}]^2}\right)} ;$$

$$g(t_2) = \frac{\left(\sigma_Y^2(t_2) - \lambda\sigma_Y^2(t_1) + \sqrt{(\sigma_Y^2(t_2) - \lambda\sigma_Y^2(t_1)^2 + 4\lambda[\sigma_{Y(t_1)Y(t_2)}]^2}\right)}{2\sigma_{Y(t_1)Y(t_2)}} ;$$

(B-2)

$$h(t_2) = \mu_Y(t_2) - g(t_2)\mu_Y(t_1) ;$$

and

$$\sigma_e^2(t_1) = \sigma_Y^2(t_1) - \frac{2[\sigma_{Y(t_1)Y(t_2)}]^2}{\left(\sigma_Y^2(t_2) - \lambda\sigma_Y^2(t_1) + \sqrt{(\sigma_Y^2(t_2) - \lambda\sigma_Y^2(t_1))^2 + 4\lambda[\sigma_{Y(t_1)Y(t_2)}]^2}\right)} ;$$

One of the properties of maximum likelihood estimation is that if one set of parameters is related to another set of parameters by a one-to-one transformation, say f, so that $f(\underline{\theta}) = \underline{\Psi}$ where $\underline{\theta}$ and $\underline{\Psi}$ represent the first and second sets of parameters respectively, then $\hat{\underline{\Psi}} = f(\hat{\underline{\theta}})$ where $\hat{\underline{\theta}}$ and $\hat{\underline{\Psi}}$ represent the maximum likelihood estimates of $\underline{\theta}$ and $\underline{\Psi}$

(Mood, Graybill, & Boes, 1974). Under assumption 1, $\underline{\theta}$ = $(\mu_Y(t_1),\ \mu_Y(t_2),\ \sigma_Y^2(t_1),\ \sigma_Y^2(t_2),\ \sigma_{Y(t_1)Y(t_2)})$ and $\underline{\Psi}$ = $(\mu_Y(t_1),\ [\sigma_Y^*(t_1)]^2,\ g(t_2),\ h(t_2),\ \sigma_e^2(t_2))$. Assuming that the distribution of the vector $(Y(t_1),\ Y(t_2))$ is multivariate normal, the maximum likelihood estimator of $\underline{\theta}$ is given by $\hat{\underline{\theta}} = (\overline{Y(t_1)},\ \overline{Y(t_2)},\ S_Y^2(t_1),\ S_Y^2(t_2),\ S^2_{Y(t_1)Y(t_2)})$, where $S_Y^2(t_1)$, $S_Y^2(t_2)$, and $S_{Y(t_1)Y(t_2)}$ are given by equations (5-14) and (5-15). Substituting $\hat{\underline{\theta}}$ into the system of equations (B-2) gives the maximum likelihood estimates of $\underline{\Psi}$. That is,

$$\widehat{\mu_Y(t_1)} = \overline{Y(t_1)}\ ;$$

$$\widehat{[\sigma_Y^*(t_1)]^2} = \frac{2[S_{Y(t_1)Y(t_2)}]^2}{\left(S_Y^2(t_2) - \lambda S_Y^2(t_1) + \sqrt{(S_Y^2(t_2) - \lambda S_Y^2(t_1))^2 + 4\lambda[S_{Y(t_1)Y(t_2)}]^2}\right)}\ ;$$

$$\widehat{g(t_2)} = \frac{\left(S_Y^2(t_2) - \lambda S_Y^2(t_1) + \sqrt{(S_Y^2(t_2) - \lambda S_Y^2(t_1))^2 + 4\lambda[S_{Y(t_1)Y(t_2)}]^2}\right)}{2[S_{Y(t_1)Y(t_2)}]}\ ;$$

$$\widehat{h(t_2)} = \overline{Y(t_2)} - \widehat{g(t_2)}\overline{Y(t_1)} \ ;$$

and

$$\widehat{\sigma_e^2(t_1)} = s_Y^2(t_1) - \widehat{[\sigma_Y^*(t_1)]^2} \ .$$

Assumption 5: $\rho_{Y(t_1)Y(t_1)} = \rho_{Y(t_2)Y(t_2)}$

The assumption that $\rho_{Y(t_1)Y(t_1)} = \rho_{Y(t_2)Y(t_2)}$ can be rewritten as

$$\frac{[\sigma_Y^*(t_1)]^2}{[\sigma_Y^*(t_1)]^2 + \sigma_e^2(t_1)} = \frac{[\sigma_Y^*(t_2)]^2}{[\sigma_Y^*(t_2)]^2 + \sigma_e^2(t_2)} \ . \qquad \text{(B-3)}$$

But, recall that

$$[\sigma_Y^*(t_2)]^2 = [g(t_2)]^2 \cdot [\sigma_Y^*(t_1)]^2 \ . \qquad \text{(B-4)}$$

Hence, by substituting equation (B-4) into equation (B-3)

$$\frac{[\sigma_Y^*(t_1)]^2}{[\sigma_Y^*(t_1)]^2 + \sigma_e^2(t_1)} = \frac{g^2(t_2)\,[\sigma_Y^*(t_1)]^2}{g^2(t_2)\,[\sigma_Y^*(t_1)]^2 + \sigma_e^2(t_2)} \ . \qquad \text{(B-5)}$$

Solving equation (B-5) for $\sigma_e^2(t_2)$ gives

$$\sigma_e^2(t_2) = g^2(t_2)\sigma_e^2(t_1) \ . \qquad \text{(B-6)}$$

The system of equations (5-17), (5-18), (5-19), (5-20), (5-21), and (B-6) is then solved for $\mu_Y(t_1)$, $[\sigma_Y^*(t_1)]^2$, $g(t_2)$, $h(t_2)$, and $\sigma_e^2(t_1)$ in terms of $\mu_Y(t_1)$, $\mu_Y(t_2)$, $\sigma_Y^2(t_1)$, $\sigma_Y^2(t_2)$, and $\sigma_{Y(t_1)Y(t_2)}$. Using the property that one-to-one transformations of maximum likelihood estimates are themselves maximum likelihood, the maximum likelihood estimates of $\mu_Y(t_1)$, $[\sigma_Y^*(t_1)]^2$, $g(t_2)$, $h(t_2)$, and $\sigma_e^2(t_1)$ are then given by substituting $\overline{Y(t_1)}$, $\overline{Y(t_2)}$, $S_Y^2(t_1)$, $S_Y^2(t_2)$, and $S_{Y(t_1)Y(t_2)}$ in place of $\mu_Y(t_1)$, $\mu_Y(t_2)$, $\sigma_Y^2(t_1)$, $\sigma_Y^2(t_2)$, and $\sigma_{Y(t_1)Y(t_2)}$ respectively. These estimates are

$$\widehat{\mu_Y(t_1)} = \overline{Y(t_1)} \quad ;$$

$$\widehat{[\sigma_Y^*(t_1)]^2} = \frac{S_{Y(t_1)Y(t_2)} S_Y(t_1)}{S_Y(t_2)} \quad ;$$

$$\widehat{g(t_2)} = \frac{S_Y(t_2)}{S_Y(t_1)} \quad ;$$

$$\widehat{h(t_2)} = \overline{Y(t_2)} - \frac{S_Y(t_2)}{S_Y(t_1)}\overline{Y(t_1)} \quad ;$$

and

$$\widehat{\sigma_e^2(t_1)} = S_Y^2(t_1) - \frac{S_{Y(t_1)Y(t_2)} S_Y(t_1)}{S_Y(t_2)} .$$

<u>Assumption 6: $\rho_{Y(t_2)Y(t_2)}$ is known</u>

By definition,

$$\rho_{Y(t_2)Y(t_2)} = \frac{[\sigma_Y^*(t_2)]^2}{[\sigma_Y^*(t_2)]^2 + \sigma_e^2(t_2)} .$$

So, by equation (B-4),

$$\rho_{Y(t_2)Y(t_2)} = \frac{[g(t_2)]^2 \cdot [\sigma_Y^*(t_1)]^2}{[g(t_2)]^2 \cdot [\sigma_Y^*(t_1)]^2 + \sigma_e^2(t_2)} . \qquad \text{(B-7)}$$

Solving equation (B-7) for $\sigma_e^2(t_2)$ gives

$$\sigma_e^2(t_2) = \frac{[g(t_2)]^2 \cdot [\sigma_Y^*(t_1)]^2 \cdot [1 - \rho_{Y(t_2)Y(t_2)}]}{\rho_{Y(t_2)Y(t_2)}} . \qquad \text{(B-8)}$$

The system of equations (5-17), (5-18), (5-19), (5-20), (5-21), and (B-8) is then solved for $\mu_Y(t_1)$, $[\sigma_Y^*(t_1)]^2$, $g(t_2)$, $h(t_2)$, and $\sigma_e^2(t_1)$ in terms of $\mu_Y(t_1)$, $\mu_Y(t_2)$,

$\sigma^2_Y(t_1)$, $\sigma^2_Y(t_2)$, and $\sigma_{Y(t_1)Y(t_2)}$. Using the property that one-to-one transformations of maximum likelihood estimates are themselves maximum likelihood, the maximum likelihood estimates of $\mu_Y(t_1)$, $[\sigma^*_Y(t_1)]^2$, $g(t_2)$, $h(t_2)$, and $\sigma^2_e(t_1)$ are then given by substituting $\overline{Y(t_1)}$, $\overline{Y(t_2)}$, $S^2_Y(t_1)$, $S^2_Y(t_2)$, and $S_{Y(t_1)Y(t_2)}$ in place of $\mu_Y(t_1)$, $\mu_Y(t_2)$, $\sigma^2_Y(t_1)$, $\sigma^2_Y(t_2)$, and $\sigma_{Y(t_1)Y(t_2)}$ respectively. These estimates are

$$\widehat{\mu_Y(t_1)} = \overline{Y(t_1)} \;;$$

$$\widehat{[\sigma^*_Y(t_1)]^2} = \frac{[S_{Y(t_1)Y(t_2)}]^2}{S^2_Y(t_2)\cdot\rho_{Y(t_2)Y(t_2)}} \;;$$

$$\widehat{g(t_2)} = \frac{S^2_Y(t_2)}{S_{Y(t_1)Y(t_2)}}\cdot\rho_{Y(t_2)Y(t_2)} \;;$$

$$\widehat{h(t_2)} = \overline{Y(t_2)} - \frac{S^2_Y(t_2)}{S_{Y(0)Y(t_2)}}\rho_{Y(t_2)Y(t_2)}\overline{Y(t_1)} \;;$$

and

$$\widehat{\sigma^2_e(t_1)} = S^2_Y(t_1) - \frac{[S_{Y(t_1)Y(t_2)}]^2}{S^2_Y(t_2)\cdot\rho_{Y(t_2)Y(t_2)}} \,.$$

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