

AN INTRODUCTION TO THE HISTORY  
OF FOURIER SERIES AND THE THEORY  
OF INTEGRATION

Thesis for the Degree of Ph. D.

MICHIGAN STATE UNIVERSITY

DEAN WEBB

1971



This is to certify that the

thesis entitled

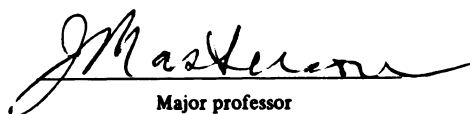
AN INTRODUCTION TO THE HISTORY OF  
FOURIER SERIES AND THE THEORY OF INTEGRATION

presented by

Dean Webb

has been accepted towards fulfillment  
of the requirements for

Ph.D. degree in Mathematics

A handwritten signature in cursive script, appearing to read "Masterman", written over a horizontal line.

Major professor

Date June 1, 1971

MAR 19 2011  
03 05 11

Consideration

(1) suggests many

of analysis. Exampl

establishing the ex

representation by Fo

that a function be r

Study of the

of functions has pr

evolution of analys

influence is partic

theory of functions

This paper is

the development of

integral from Cauch

theory of Fourier se

of Fourier series re

search for general s

The study be

the problem of

Allexbert, Euler, I

are described and I

## ABSTRACT

### AN INTRODUCTION TO THE HISTORY OF FOURIER SERIES AND THE THEORY OF INTEGRATION

By

Dean Webb

Consideration of the Fourier series corresponding to a function  $f(x)$  suggests many interesting questions which relate to basic concepts of analysis. Examples are the generality of the integrals admitted in establishing the existence of the Fourier coefficients, uniqueness of representation by Fourier series, and general forms of convergence such that a function be represented by its corresponding Fourier series.

Study of these and other aspects of Fourier series representation of functions has profoundly influenced the nature and direction of the evolution of analysis and, in turn, has been influenced by it. This influence is particularly apparent in the history of development of the theory of functions of a real variable.

This paper is an exposition of the history of relationships between the development of successively more general conceptions of definite integral from Cauchy to Lebesgue and the concomitant evolution of a theory of Fourier series. The unifying theme of the paper is the study of Fourier series representation of functions and in particular, the search for general sufficient conditions for such representation.

The study begins with the work of the first mathematicians to consider the problem of trigonometric series representation of functions, d'Alembert, Euler, D. Bernoulli and Lagrange. Fourier's contributions are described and I conjecture a relationship between Fourier's work and

Cantor's definition  
um for convergence  
are presented, as is  
given by Riemann in  
ation of necessary  
series.

The theories  
Jordan, Borel and Le  
uniqueness of repres  
initiated Cantor's co  
Integration is descri  
his conception of de  
sufficient condition  
is presented as well  
discovery of the Riesz  
getting convergence a  
to a function  $f \in L^2$   
Riesz's assertion  
condition for converg

Both the histo  
ries which motivated  
identified, and both  
the men who helped cr  
shaping passages f

Cauchy's definition of definite integral. Dirichlet's sufficient condition for convergence of Fourier series and modern conception of function are presented, as is the more general definition of definite integral given by Riemann in his Habilitationsschrift, which is devoted to investigation of necessary and sufficient conditions for convergence of trigonometric series.

The theories of measure created by Stolz, Cantor, Harnack, Peano, Jordan, Borel and Lebesgue are studied; efforts to solve the problem of uniqueness of representation by trigonometric series is shown to have initiated Cantor's creation of his theory of sets. Lebesgue's theory of integration is described, as are the first applications by Lebesgue of his conception of definite integral to study of Fourier series. Fatou's sufficient condition for convergence almost everywhere of Fourier series is presented as well as the work of Riesz and Fischer which led to the discovery of the Riesz-Fischer theorem. Finally, Lusin's conjecture regarding convergence almost everywhere of the Fourier series corresponding to a function  $f \in L^2[0, 2\pi]$  is studied, and the paper concludes with Carleson's assertion of the validity of Lusin's very general sufficient condition for convergence of Fourier series.

Both the historical sequence of events and the initiating influences which motivated the work of contributing mathematicians are identified, and both influences and events are expressed in the words of the men who helped create the theory. The latter is accomplished by excerpting passages from the original memoirs.

Several insin-  
uations which the problem  
has successively gen-  
erated, deficiency  
attributed to the crea-  
tion, clearly secondary to  
persistent controver-  
sions of a real

Several insights derive from such study. These include the manner in which the problem of Fourier series representation of functions has been successively generalized in order to render it capable of partial solution, deficiencies in the work of many of the mathematicians who contributed to the creation of the theory (deficiencies which are, of course, clearly secondary to the accomplishments of these men), and finally, the persistent controversy which accompanied the evolution of the theory of functions of a real variable.

AN INTRODUCTION TO THE HISTORY  
OF FOURIER SERIES AND THE THEORY  
OF INTEGRATION

by

*Bessie*  
Dean Webb

A THESIS

Submitted to

Michigan State University

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1971

To Marilyn

An

Se

A series  $c_n$   
and  $c$  are constant  
function whose domain  
the integrals

$$1) \quad a_n =$$

and

$$2) \quad b_n =$$

exist, then the Fourier  
series

$$3) \quad \frac{a_0}{2} +$$

The constants  $a_n$  and  $b_n$   
definition require  
a Fourier series  $c_n$   
series 3) converge  
series 3) should converge  
at a point  $x$  that  
at  $x$ . Since the  
general, represent  
dependence between

$f(x)$

# An Introduction to the History of Fourier Series and the Theory of Integration

## Preface

A series of the form  $c + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$  where the  $a_n, b_n$ , and  $c$  are constants is called a trigonometric series. If  $f(x)$  is a function whose domain of definition is the interval  $-\pi \leq x \leq \pi$  and if the integrals

$$1) \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx, \quad n = 0, 1, 2, \dots$$

and

$$2) \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx, \quad n = 1, 2, 3, \dots$$

exist, then the Fourier series corresponding to  $f(x)$  is the trigonometric series

$$3) \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

The constants  $a_n$  and  $b_n$  are the Fourier coefficients of  $f(x)$ . This definition requires only that the integrals 1) and 2) exist in order that a Fourier series correspond to  $f(x)$ ; there is no requirement that the series 3) converge in any sense. Indeed it is not apparent that the series 3) should converge in the interval  $[-\pi, \pi]$  nor if it does converge at a point  $x$  that its sum should be  $f(x)$ , i.e., that it should represent  $f$  at  $x$ . Since the Fourier series corresponding to  $f(x)$  does not, in general, represent  $f(x)$ , we will use the symbol " $\sim$ " to denote the correspondence between  $f$  and its Fourier series,

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Considerat

$f(x)$  suggests many  
of analysis. Exam  
establishing the  
forms of converg  
by its correspond  
and continuity, d  
series in interval

If a trigo  
then its sum  $f(x)$   
periodic with pe  
study of a great  
phenomena such a  
function satisfy  
trigonometric s  
problems are Fo  
and applied int

The pure  
viewed the work  
this and relat  
series. The c  
nature and dir  
influenced by  
development of

Consideration of the Fourier series corresponding to a function  $f(x)$  suggests many interesting problems which relate to basic concepts of analysis. Examples are the generality of the integrals admitted in establishing the existence of the Fourier coefficients 1) and 2), general forms of convergence of the series 3) such that a function be represented by its corresponding Fourier series, uniqueness of series representation, and continuity, differentiability and integrability properties of Fourier series in intervals of convergence.

If a trigonometric series converges (in the usual sense) for all  $x$ , then its sum  $f(x)$  has the property  $f(x + 2\pi) = f(x)$ , that is,  $f(x)$  is periodic with period  $2\pi$ . Periodic functions are encountered in the study of a great variety of physical problems which feature periodic phenomena such as vibration or wave motion. As is known, every periodic function satisfying certain general conditions can be represented by a trigonometric series<sup>1</sup> and most trigonometric series encountered in applied problems are Fourier series. Thus Fourier series are of both theoretical and applied interest.

The pure and applied aspects of study of Fourier series have motivated the work of several generations of mathematicians whose efforts in this and related areas have led to the creation of a theory of Fourier series. The creation of this theory has profoundly influenced the nature and direction of the evolution of analysis and, in turn, been influenced by it. These influences are particularly apparent in the development of a theory of functions of a real variable.

---

<sup>1</sup>See Appendix A for one form of this assertion.

The theory  
analysts dur  
years to con  
generally ac  
of Dirichlet  
or that the  
appeared in  
series; or t  
velopments o  
in his atten  
trigonometri  
was developed  
series....<sup>2</sup>

Indeed, its util

tributing factor

This paper is

tageous relations

of definite inte

of a theory of F

question of Four

be seen, much of

response to dema

paper begins wit

question, d'Alemb

the search for s

the theorems of

In writing t

al sequence of

work of contribu

ments in the wo

I have frequentl

[The theory of Fourier series] has been a source of new ideas for analysts during the last two centuries, and is likely to be so in years to come.... It is not accidental that the notion of function generally accepted now was first formulated in the celebrated memoir of Dirichlet (1837) dealing with the convergence of Fourier series; or that the definition of Riemann's integral in its general form appeared in Riemann's Habilitationsschrift devoted to trigonometric series; or that the theory of sets, one of the most important developments of nineteenth-century mathematics, was created by Cantor in his attempts to solve the problems of the sets of uniqueness for trigonometric series. In more recent times, the integral of Lebesgue was developed in close connection with the theory of Fourier series....<sup>2</sup>

Indeed, its utility in the study of Fourier series was an important contributing factor in the acceptance of Lebesgue's definition of integral.

This paper is an exposition of the history of such mutually advantageous relationships between the successively more general conceptions of definite integral from Cauchy to Lebesgue and the concomitant evolution of a theory of Fourier series. The unifying theme of the paper is the question of Fourier series representation of functions since, as will be seen, much of this part of the theory of integration developed in response to demands placed upon it by such series representations. The paper begins with the work of the first mathematicians to consider this question, d'Alembert, Euler, D. Bernoulli, and Lagrange, and following the search for sufficient conditions for such representation, ends with the theorems of Carleson and Hunt.

In writing the paper, I have attempted to identify both the historical sequence of events and the initiating influences which motivated the work of contributing mathematicians, and to express both influences and events in the words of the men who helped create the theory. Therefore I have frequently excerpted passages from the original memoirs.

---

<sup>2</sup>A. Zygmund, Trigonometric Series, Volume I, xi.

I have ass

nd have used st

not been given s

My purpose

context within w

ight assume gre

treatment of the

My hope is that

I should l

Wasterson, Clif

Ray and partic

and encourage

I have assumed the reader has a knowledge of undergraduate analysis, and have used standard notations and terminology; proofs of theorems have not been given since they can be found in the references cited.

My purpose in writing this paper has been to identify an historical context within which a study of the theory of functions of a real variable might assume greater meaning. Many texts provide a rigorous mathematical treatment of the theory, but, of necessity, little insight into its origins. My hope is that this paper will complement such texts.

I should like to thank Charles Wells, John Wagner, John Kinney, John Masterson, Clifford Weil, Francis Hildebrand, Albert Froderberg, Neil Gray and particularly Gottfried Adam and Arthur Kimmel for assistance and encouragement in the course of this work.

Preface . . . .

Chapter 1 On the

Contribut  
Lagrange

Fourier

Cauchy . .

Dirichlet

Riemann .

Chapter 2 On the

Heine, Ca

Cantor, H

Peano and

Borel .

Chapter 3 On the

Lebesgue

Trigonome

The Riesz

Lusin and

Epilogue . . . .

Appendices

A . . . .

B . . . .

C . . . .

D . . . .

References . .

## CONTENTS

|                                                                            |     |
|----------------------------------------------------------------------------|-----|
| Preface . . . . .                                                          | 1   |
| Chapter 1 On the Cauchy and Riemann Integrals. . . . .                     | 1   |
| Contributions of d'Alembert, Euler, D. Bernoulli and<br>Lagrange . . . . . | 1   |
| Fourier . . . . .                                                          | 11  |
| Cauchy . . . . .                                                           | 26  |
| Dirichlet . . . . .                                                        | 37  |
| Riemann . . . . .                                                          | 43  |
| Chapter 2 On the Creation of Theory of Measure . . . . .                   | 55  |
| Heine, Cantor, Hankel and Smith . . . . .                                  | 55  |
| Cantor, Harnack and Stolz . . . . .                                        | 71  |
| Peano and Jordan . . . . .                                                 | 78  |
| Borel . . . . .                                                            | 94  |
| Chapter 3 On the Lebesgue Integral . . . . .                               | 104 |
| Lebesgue Measure and Integral . . . . .                                    | 104 |
| Trigonometric Series . . . . .                                             | 142 |
| The Riesz-Fischer Theorem . . . . .                                        | 154 |
| Lusin and Carleson . . . . .                                               | 169 |
| Epilogue . . . . .                                                         | 184 |
| Appendices                                                                 |     |
| A . . . . .                                                                | 187 |
| B . . . . .                                                                | 191 |
| C . . . . .                                                                | 193 |
| D . . . . .                                                                | 198 |
| References . . . . .                                                       | 203 |

On

Co

One of the  
vibration of string  
the motion of a string  
differential equation

2)

where  $a$  is a constant  
is to find a solution  
for string, that is  
which determine  
 $F(x,t)$  uniquely

Consider,  
plane whose equation  
whose endpoints  
is deformed from  
in the  $(x,y)$  plane  
velocity  $g(x)$   
then the string  
its equilibrium  
the motion of  
 $F(0,t) = F(1,t)$

1  
A derivation  
Pren

## Chapter 1

### On the Cauchy and Riemann Integrals

### Contributions of d'Alembert, Euler,

### D. Bernoulli and Lagrange

One of the origins of the theory of Fourier series was study of the vibration of stretched strings. Given certain simplifying assumptions the motion of a stretched string is described by a solution of the partial differential equation

$$1) \quad \frac{\partial^2 F}{\partial t^2} = a^2 \frac{\partial^2 F}{\partial x^2},$$

where  $a$  is a constant.<sup>1</sup> This equation has many solutions and the problem is to find a solution  $F(x,t)$  of 1) which describes the motion of a particular string, that is, which satisfies given initial and boundary conditions which determine the motion of the string and thus determine the solution  $F(x,t)$  uniquely.

Consider, for example, a stretched string of unit length in the  $(x,y)$  plane whose equilibrium position is on the  $x$ -axis,  $0 \leq x \leq 1$ ,  $y = 0$ , and whose endpoints are fixed in this position through time. Assume the string is deformed from its equilibrium position at time  $t = 0$  by a force acting in the  $(x,y)$  plane to an initial position  $0 \leq x \leq 1$ ,  $y = f(x)$ . Let an initial velocity  $g(x)$  in the  $y$ -direction be imparted to the string at time  $t = 0$ . Then the string responds to the forces acting upon it by vibrating about its equilibrium position. If  $F(x,t)$  is the solution of 1) which describes the motion of this string then  $F(x,t)$  satisfies the boundary conditions  $F(0,t) = F(1,t) = 0$  and the initial conditions  $F(x,0) = f(x)$  and

---

<sup>1</sup>A derivation of this equation is given in D. Widder's Advanced Calculus, Prentice-Hall, 1947, 344.

$\frac{\partial F}{\partial t}(x,0) = g(x)$ . This equation describes the motion of the string in the sense that the form of the displaced string is given by the curve  $y = F(x, t_0)$  at a fixed instant in time  $t_0 \geq 0$ .

The reader will verify that the functions

$$F_n(x,t) = \sin n\pi x \cos n\pi at,$$

where  $n$  is a positive integer, are solutions of 1) which satisfy the boundary conditions  $F_n(0,t) = F_n(1,t) = 0$  and initial conditions  $F_n(x,0) = f_n(x) = \sin n\pi x$  and  $\frac{\partial F_n}{\partial t}(x,0) = g_n(x) = 0$ . Moreover, if the series

$$2) \quad F(x,t) = \sum_{n=1}^{\infty} a_n F_n(x,t) = \sum_{n=1}^{\infty} a_n \sin n\pi x \cos n\pi at,$$

where the  $a_n$  are constants, and the series obtained from it by differentiating term-by-term are convergent, and if term-by-term differentiation is justified, then 2) is a solution of 1) satisfying the boundary conditions  $F(0,t) = F(1,t) = 0$  and initial conditions  $F(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin n\pi x$  and  $\frac{\partial F}{\partial t}(x,0) = g(x) = 0$ . Thus  $F(x,t)$  describes the motion of a string stretched between the points  $(0,0)$  and  $(1,0)$ , fixed at these points, displaced in the form  $f(x) = \sum_{n=1}^{\infty} a_n \sin n\pi x$ ,  $0 \leq x \leq 1$ , and at rest at the instant of release  $t = 0$ .

The function  $F(x,t)$  is the most general solution of this particular case of the vibrating string provided it can be made to satisfy arbitrarily given initial conditions. This is the criterion for generality of solution. A solution, therefore, satisfying a particular initial condition  $F(x,0) = s(x)$  where of physical necessity  $s(x)$  is continuous on  $0 \leq x \leq 1$  and, in the context of this case of the vibrating string  $\frac{\partial F}{\partial t}(x,0) = g(x) = 0$ , is subsumed in the general solution. Again, a solution is obtained for each initial position of the stretched string; the general solution encompasses all of these solutions

simultaneously since  
conditions. Of course  
general solution of  
the question.

If  $F(x,t)$  is  
defined on  $[0, 1]$   
as a sine series on

allows since  $k(x)$  c  
string. Thus the at  
vibrating string lea  
representation of an  
expanded by the q  
solved by the cre

D'Alembert  
= 1 but I follow  
waves formed by a  
vibrating cord of l  
at the instant of r  
periodic function  
sequences to be  
after ironic under  
vibration of his

In his anal  
to be continu  
crossed by a sing

D'Alembert.  
"vibration," 214

simultaneously since by definition it satisfies arbitrarily prescribed initial conditions. Of course whether or not  $F(x,t) = \sum_{n=1}^{\infty} a_n \sin n\pi x \cos n\pi at$  is the general solution of this particular case of the vibrating string remains open to question.

If  $F(x,t)$  is the general solution, then an arbitrary continuous function  $k(x)$  defined on  $[0,1]$  and such that  $k(0) = k(1) = 0$  must admit representation by a sine series on  $[0,1]$ , that is,  $k(x) = \sum_{n=1}^{\infty} a_n \sin n\pi x$  for  $0 \leq x \leq 1$ . This follows since  $k(x)$  can be interpreted as an initial position of the stretched string. Thus the attempt to find the general solution for this case of the vibrating string leads to the very interesting question of trigonometric series representation of an arbitrary function. Historically, the controversy engendered by the question of such representation helped initiate and was resolved by the creation of the theory of Fourier series.

D'Alembert published a solution of the equation  $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$  (He assumed  $a = 1$  but I follow the usual convention.) in his 1747 memoir "Research on curves formed by a stretched vibrating cord."<sup>2</sup> He gave his solution for a vibrating cord of length  $\ell$ , fixed at the points  $(0,0)$  and  $(\ell,0)$  and at rest at the instant of release, in the form  $y = f(at + x) - f(at - x)$  where  $f$  denotes a periodic function with period  $2\ell$ . D'Alembert remarked, "There are many. . . consequences to be drawn from the general solution we have just given," a rather ironic understatement in light of the historical sequence initiated by publication of his solution général.

In his analysis d'Alembert took the initial position of the stretched cord to be continuous or regular, that is, a curve whose form could be expressed by a single definite function of the independent variable. Such a

---

<sup>2</sup>D'Alembert, "Recherches sur la courbe que forme un corde tendue mise en vibration," 214.

curve stood in con-  
stant to be composed  
of a single law the  
for their express

Euler argued  
of the stretched  
ords,"<sup>3</sup> published  
with considerable

communicated to ou  
often draws very  
solutions of the  
found on this que

Euler expr  
 $f(x + at) = f(x) + at$   
 $f(x + at) + f(x - at)$   
curve which "be s  
represent the  
it regular, c  
mechanical, its  
and for the sol

$f(x)$   
is a special cas  
continuous curve  
of its nature

---

<sup>3</sup> Euler, "

curve stood in contrast to discontinuous or irregular curves which were understood to be composed of portions of continuous curves and thus, not conforming to a single law throughout their course, to require several different functions for their expression.

Euler argued against d'Alembert's restriction of the initial position of the stretched cord to a continuous curve. In his "On the vibration of cords,"<sup>3</sup> published in 1748, he wrote "Mr. d'Alembert was the first to attack with considerable success the examination of this problem. . . and he has communicated to our Academy a very good solution of it. But as. . . one often draws very considerable profit from the comparison of several different solutions of the same problem I do not hesitate to propose the one I have found on this question."

Euler expressed his solution of the differential equation 1) in the form  $y = f(x + at) + f(x - at)$  where for every  $t$ ,  $f(at) + f(-at) = 0$  and  $f(l + at) + f(l - at) = 0$ , and concluded from these equations that every curve which "be situated alternatively above and below [the axis] is proper to represent the nature of the. . . function  $f$ . . . ." Thus, "a. . . curve, be it regular, contained in a certain equation, or be it irregular or mechanical, its arbitrary [ordinate] will furnish the functions which we need for the solution of the problem." He then gave the equation

$$f(x) = \alpha \sin \frac{\pi x}{l} + \beta \sin \frac{2\pi x}{l} + \gamma \sin \frac{3\pi x}{l} + . . .$$

as a special case of the "general solution" in which the function  $f(x)$  "is a continuous curve whose parts be bound in virtue of the law of continuity so that its nature can be understood by an equation."

---

<sup>3</sup>Euler, "Sur la vibration des cordes," 69.

Euler recog  
in time  $t = 0$  are  
the initial veloci  
to this rule [of r  
sees by this how t  
and is determined  
and go give it wha  
initial figure of  
have the greatest  
sufficiently gener  
and, Euler mainta  
D'Alembert, whose  
regular curve, did  
form of the cord f

D'Alembert  
is treated in the  
entirely similar t  
seems to me, a li  
office to transpo  
is; it is neces  
have expressed  
capable of res  
surpass the f  
ness y analyti

4  
D'Alembert  
et, mise en v.

Euler recognized that the vibrations of the stretched cord subsequent to time  $t = 0$  are completely determined by the initial form of the cord and the initial velocities of its points. ". . .if a single vibration conforms to this rule [of regularity], all the following must observe it also. One sees by this how the state of following vibrations depends on the preceding and is determined by them. . ." Furthermore "one can before letting the cord go give it whatever figure one wishes" and therefore, "so that the initial figure of the cord can be [given] arbitrarily, the solution must have the greatest extent possible." By obtaining a solution asserted to be sufficiently general to comprehend such initial positions of the stretched cord, Euler maintained that his solution was more general than that of d'Alembert, whose solution, by assuming the initial form of the cord to be a regular curve, did not encompass, for example, the case in which the initial form of the cord is polygonal.

D'Alembert rejoined Euler in a paper published in 1750.<sup>4</sup> "Mr. Euler has treated in the Memoirs of 1748 the problem of vibrating cords by a method entirely similar to mine as to the essential part of the problem and only, it seems to me, a little longer." He cautioned his readers that "it does not suffice to transport the initial curve alternatively above and below the axis; it is necessary in addition that the curve satisfy the conditions that I have expressed in my memoir. . . . In any other case the problem will not be capable of resolution, at least by my method, and I do not know if it will not surpass the force of continuous analysis. One cannot, it seems to me, express  $y$  analytically in a more general manner than by supposing it a

---

<sup>4</sup>D'Alembert, "Addition au memoire sur la courbe que forme un corde tendue, mise en vibration," 355.

function of  $x$  and  
only for the  
can be enclosed in  
is impossible to

Daniel Bern  
publication of his  
ords given by the  
ception to d'Alembert  
approach to the pro  
listen to abstract  
question is more  
that one need only  
in order to foresee  
geometricians four

Bernoulli  
ments upon Ta  
and, as a son of  
Bernoulli, admit  
the fundamen  
dicial position

Y  
Bernoulli wrote  
relation an

5 Bernoulli  
aordes ex

function of  $x$  and  $t$ . But in this supposition one finds the solution of the problem only for the cases where the different figures of the vibrating cord can be enclosed in a single and same equation. In all other cases it seems to me impossible to give to  $y$  a general form."

Daniel Bernoulli became a third party to this controversy in 1753 by publication of his "Reflections and enlightenments on the new vibrations of cords given by the Memoirs of the Academy of 1747 and 1748."<sup>5</sup> Bernoulli took exception to d'Alembert and Euler's reliance upon a strictly mathematical approach to the problem. He contended such reliance demonstrates that "to listen to abstract analysis without any synthetic examination of the proposed question is more likely to surprise rather than enlighten us. It seems to me that one need only give attention to the nature of simple vibrations of cords in order to foresee without any calculations all that these two great geometricians found by the most difficult and abstract analysis. . . ."

Bernoulli apprehended the problem in physical terms. Basing his arguments upon Taylor's De Methodo Incrementorum, he asserted the vibrating cord, as a sonorous body whose vibrations consist of a fundamental and its overtones, admits expression mathematically as the sum of terms corresponding to the fundamental and its harmonics. This led him to conclude that any initial position of the stretched cord admits representation in the form

$$y = \alpha \sin \frac{\pi x}{l} + \beta \sin \frac{2\pi x}{l} + \gamma \sin \frac{3\pi x}{l} + . . . .$$

Bernoulli wrote "Here is therefore an infinity of curves found without any calculation and our equation is the same as that of Mr. Euler. . . . It is

---

<sup>5</sup> Bernoulli, "Réfléxions et éclaircissemens sur les nouvelles vibrations des cordes exposées dans les Mémoires de l'Académie de 1747 et 1748," 147.

true that Mr. Euler

general and that h

are still other cu

Bernoulli's

function admits re

ider.<sup>6</sup> "Mr. Bernoulli

the solution of Tay

and is susceptible

$\frac{2^x}{\sin \frac{x}{2}} + \dots$ , be

general that it em

"All the curves ex

we augmented, have

other curves. If

comes also negat

abscissa x; and in

1-1 is negative

Therefore, assert

not a function w

other, cannot rep

solution to be mo

We should

possibility of

functions even i

is a violation o

<sup>6</sup> Euler, "

true that Mr. Euler does not treat this infinite multitude [of curves] as general and that he gives it. . . only for particular cases, but. . . if there are still other curves, I do not understand in what sense one can admit them."

Bernoulli's contention that every curve, and hence an arbitrary function admits representation by a sine series was immediately disputed by Euler.<sup>6</sup> "Mr. Bernoulli. . . sustains against Mr. d'Alembert and myself that the solution of Taylor is sufficient to explain all the movements of which a cord is susceptible. . . ." Any argument "that the equation  $y = \alpha \sin \frac{\pi x}{l} + \beta \sin \frac{2\pi x}{l} + . . .$ , because of the infinity of undetermined coefficients, is so general that it embraces all possible curves" must fail, Euler insisted, for "all the curves expressed by this equation, no matter how the number of terms be augmented, have certain characteristics which distinguish them from all other curves. If one takes the abscissa  $x$  negative, then the ordinate becomes also negative and equal to that which corresponds to the positive abscissa  $x$ ; and in the same way, the ordinate which corresponds to the abscissa  $x + l$  is negative and equal to that which corresponds to the abscissa  $x$ ." Therefore, asserted Euler, a sine series, being odd and periodic, cannot represent a function which does not possess both of these properties, and in particular, cannot represent an algebraic function. Thus Euler held Bernoulli's solution to be more restricted than the solution of d'Alembert.

We should observe that neither Euler nor his contemporaries could concede the possibility of trigonometric series representation of non-periodic continuous functions even in an interval. Indeed such a notion would have been dismissed as a violation of the concept of continuous function. Coeval opinion held that

---

<sup>6</sup>Euler, "Remarques sur les mémoires précédens de M. Bernoulli," 196.

the definition of  
everywhere in its  
beyond an interval  
functions were eq  
circumstance which  
tion of a non-per

Lagrange e  
"Researches on th  
of the first thr  
Bernoulli's solu  
necessary that  
enclose all the  
the possible c  
to distinguish  
we can imagin  
of an arbitra  
range."

Having  
of that of d  
Euler is  
later alway  
to the initia  
enclosure in  
function b  
of cont

the definition of a continuous function in an interval implied its definition everywhere in its domain, i.e., that a continuous function could be extended beyond an interval of definition in only one way. Thus, if two continuous functions were equal on an interval they were held to be equal everywhere, a circumstance which made the impossibility of a trigonometric series representation of a non-periodic function appear obvious.

Lagrange entered the controversy in 1759 with publication of his "Researches on the nature and propagation of sound."<sup>7</sup> In reviewing the methods of the first three protagonists, Lagrange reiterated Euler's objection to Bernoulli's solution, a criticism with which he concurred: "It would be necessary that the equation  $[y = \alpha \sin \frac{\pi x}{l} + \beta \sin \frac{2\pi x}{l} + \gamma \sin \frac{3\pi x}{l} + \dots]$  enclose all the figures that one can give to a stretched cord, that is, all the possible curves. This cannot be because of certain properties which seem to distinguish cords comprised in this equation from all the other curves that one can imagine; . . . in other words, in augmenting or diminishing the abscissa of an arbitrary multiple of the axis, the value of the ordinate  $y$  does not change."

Having dispatched Bernoulli's solution, Lagrange rejected the generality of that of d'Alembert, again following Euler's lead. "The construction of Mr. Euler is evidently much more general than that of Mr. d'Alembert, for the latter always supposed that the generating curve [i.e., the curve corresponding to the initial position of the stretched cord] be regular and susceptible to enclosure in a continuous equation. . . [and] . . . believed that such construction became insufficient whenever the generating curve did not follow the law of continuity. . . ."

---

<sup>7</sup> Lagrange, "Recherches sur la nature et la propagation du son," 39.

Lagrange a  
of demonstration.  
inferred by appli  
illegitimate in a  
and. It follows  
by its very nature  
from the integrati

Lagrange pr  
of the vibrating c  
the cord, each in  
from all criticism  
case of a weightle  
obtaining the sol  
solution as the n  
intermediate step

$F(x)$   
for a string of u  
such that the ini  
of particular int  
operations of sum  
other than the i  
representation of  
integrals.

Lagrange u  
of Bernoulli

Lagrange accepted Euler's solution as general but objected to its manner of demonstration. ". . .it seems undeniable that the consequences that are inferred by application of the rules of differential and integral calculus are illegitimate in all cases where this law [of continuity] is not assumed to hold. It follows from this that the construction of Mr. Euler is applicable by its very nature only to continuous curves since it is deduced immediately from the integration of the given differential equation. . . ."

Lagrange proffered a demonstration of Euler's solution of the problem of the vibrating cord "in which one considers the movements of the points of the cord, each in particular, . . .to arrive at a conclusion which be sheltered from all criticism." His argument consists of finding the solution for the case of a weightless cord composed of a finite number of particles and then obtaining the solution of the continuous cord as the limit of the first solution as the number of particles is increased without bound. In an intermediate step in his analysis, Lagrange expressed this limit in the form

$$F(x,t) = 2 \int \sum (\sin n\pi s \sin n\pi x \cos n\pi at) f(s) ds$$

for a string of unit length, whose initial position is given by  $f(x)$ , and such that the initial velocity of each of its points is zero. This form is of particular interest since it is only necessary to interchange in it the operations of summation and integration, that is, to write a sum of integrals rather than the integral of a sum, and let  $t = 0$  to obtain a sine series representation of  $f(x)$  in which the coefficients are determined as definite integrals.

Lagrange undoubtedly recognized the relationship between this form and that of Bernoulli but did not anticipate Fourier's conclusions with respect

to such expansion;  
most importantly,  
did not admit the  
function which con-  
equation. Second  
finite sum: "the  
series," and only  
particles composi-  
the summation to  
some criticism wh-  
arbitrary (perhap  
series and expres  
solution, Lagrang  
theory of this g  
being establishe  
way on the law o  
it happens that  
of W. Bernoulli  
mobile bodies is  
theory] when the  
advance in hi  
strict adher  
alternative furn

to such expansions. There are several reasons for this. First, and perhaps most importantly, Lagrange was constrained by his concept of function. He did not admit the possibility of such an expansion for any other than a periodic function which could be given analytically, i.e., expressed by a single equation. Second, Lagrange understood the integration symbol to denote a finite sum: "the integral sign  $\int$  is used to express the sum of all these series," and only after summing the series did Lagrange let the number of particles composing the cord tend to infinity. Indeed, had Lagrange considered the summation to be an integral, his demonstration would have been open to the same criticism which he directed toward that of Euler, i.e., integration of the arbitrary (perhaps discontinuous) function  $f(x)$ . Finally, having summed the series and expressed the limit of the sum in Euler's functional form of solution, Lagrange held his demonstration to be complete. "Here then is the theory of this great geometrician [Euler] placed beyond the reach of all criticism, being established on clear and direct principles which do not depend in any way on the law of continuity required by Mr. d'Alembert. Again, here is how it happens that the same formula serves to support and demonstrate the theory of Mr. Bernoulli on the mixture of isochronic vibrations when the number of mobile bodies is finite and reveals to us the insufficiency of [Bernoulli's theory] when the number of bodies becomes infinite." Lagrange "had formed in advance in his mind a definite conception of the path to be taken"<sup>8</sup> and in his strict adherence to this conception and consequent disregard of any alternative furnishes "an instructive example of the ease with which an author

---

<sup>8</sup>Riemann, Mathematische Werke, 220.

or fail to draw

another direction.

Lagrange w

Fourier's conclus

reciprocal of the

3)  $\frac{1}{\epsilon}(\epsilon) = A$

where  $\epsilon$  is the an

St. Petersburg Aca

series representat

$$A = \frac{1}{\pi}$$

result obtained

Mr. Euler only

representation wh

aps for this rea

series representa

Years pass

representation re

Bernoulli's conte

ier, and Lagran

Fourier wa

lections in the

9 Burkhardt

tion der Differ

10 Euler, "Di

anguli

can fail to draw an almost obvious conclusion if his attention is fixed in another direction."<sup>9</sup>

Lagrange was not the only mathematician to approach, but not attain, Fourier's conclusions. In discussing the problem of representation of the reciprocal of the distance between two planets by a cosine series

$$3) \quad \phi(\theta) = A + B \cos \theta + C \cos 2\theta + \dots,$$

where  $\theta$  is the angle between the radii, Euler, in a memoir presented to the St. Petersburg Academy in 1777,<sup>10</sup> asserted that if a function  $\phi(\theta)$  admits a series representation of the form 3), then

$$A = \frac{1}{\pi} \int_0^\pi \phi(\theta) d\theta, \quad B = \frac{2}{\pi} \int_0^\pi \phi(\theta) \cos \theta d\theta$$

a result obtained by multiplying the series by  $\cos n\theta$  and integrating term by term. Euler only used this argument to determine the coefficients of a series representation whose existence was verified by other means, however, and perhaps for this reason his paper had no effect upon the question of trigonometric series representation of an arbitrary function.

Years passed and the controversy regarding the possibility of series representation remained without conclusion. It was left to Fourier to carry Bernoulli's contention against the arguments and authority of d'Alembert, Euler, and Lagrange.

#### Fourier

Fourier was led to consider trigonometric series representation of functions in the course of his attempt to create a mathematical theory of the

---

<sup>9</sup> Burkhardt, "Entwicklungen nach oscillirenden Functionen und Integration der Differentialgleichungen der mathematischen Physik," 32.

<sup>10</sup> Euler, "Disquisito ulterior super seriebus secundum multipla cuiusdam anguli progredientibus," 114.

conduction of  
which leads to  
Consider a rec  
whose base A i  
the x-axis and  
each of its po  
maintained at  
the source A in  
longitudinal di  
towards the coo  
temperatures o  
to surpass nor  
different poin  
constant state

Fourier

$(x,y)$  of the p

4)

and the bounde  
where, of cour  
analysis the f

5)

conduction of heat. An example of the type of problem studied by Fourier which leads to the question of such series representation is the following. Consider a rectangular plate whose sides B and C are of infinite length and whose base A is of length  $\pi$ . Place the base A on the interval  $(-\frac{\pi}{2}, \frac{\pi}{2})$  on the x-axis and heat it in such a way that unit temperature is maintained at each of its points. Assume that each point of the sides B and C is maintained at zero temperature. Then "heat will pass continually from the source A into the solid BAC, and will be propagated there in the longitudinal direction, which is infinite, and at the same time will turn towards the cool masses B and C, which will absorb a great part of it. The temperatures of the solid BAC will be raised gradually but will not be able to surpass nor even attain a maximum of temperature, which is different for different points of the mass. It is required to determine the final and constant state to which the variable state continually approaches."<sup>11</sup>

Fourier showed that the steady state temperature  $T(x,y)$  at a point  $(x,y)$  of the plate must satisfy the partial differential equation

$$4) \quad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0,$$

and the boundary and initial conditions  $T(-\frac{\pi}{2}, y) = T(\frac{\pi}{2}, y) = 0$  and  $T(x,0) = 1$  where, of course,  $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ . He obtained as an intermediate step in his analysis the form

$$5) \quad T(x,y) = a e^{-y} \cos x + b e^{-3y} \cos 3x + c e^{-5y} \cos 5x + \dots$$

---

<sup>11</sup>Fourier, The Analytical Theory of Heat, §164.

It is evident that

the condition  $T(1)$

which is expressed

subject to the condition

$$1 = a \cos$$

The coefficient

by means of this

Thus it becomes

a cosine series in

the interval  $(-\frac{\pi}{2}, \frac{\pi}{2})$

been heated in such

function  $f(x)$ , then

of a cosine series

trigonometric series

initial problems of

this form [developed

the functions which

Fourier problem

reference on December

$f(x)$  is represented

$$f(x) =$$

$$a_n = \frac{1}{i}$$

"It is evident that the function. . .  $T(x,y)$  satisfies [equation 4] and the condition  $T(\pm \frac{\pi}{2}, y) = 0$ . A third condition remains to be fulfilled, which is expressed thus,  $T(x, 0) = 1$ . . . Equation 5) must therefore be subject to the following condition:

$$1 = a \cos x + b \cos 3x + c \cos 5x + . . . .$$

The coefficients  $a, b, c, \dots$  whose number is infinite are determined by means of this equation."

Thus it became important to Fourier to determine the coefficients of a cosine series representation of the function which is identically one on the interval  $(-\frac{\pi}{2}, \frac{\pi}{2})$ . Similarly, had the base A of the rectangular plate been heated in such a way that the temperature at a point  $x$  was given by the function  $f(x)$ , then it would have been required to represent  $f(x)$  in the form of a cosine series. This begins to explain why Fourier was interested in trigonometric series representation of "arbitrary" functions. "The fundamental problems of the theory of heat cannot be solved without reducing to this form [development in a series of sines and cosines of multiple arcs] the functions which represent the initial state of the temperatures."

Fourier presented the first of his memoirs to the French Academy of Science on December 21, 1807. His assertion that an "arbitrary" function  $f(x)$  is represented on the interval  $(-\ell, \ell)$  by a series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell}) \text{ where}$$

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell} dx \text{ and } b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell} dx$$

was met with disti-  
tationsschrift, w  
Lagrange that he  
that there is still  
this controversy,  
communicated to R  
archives without  
doubt withdrawn by

Perhaps as  
the competition fo  
to give the mathe  
to compare the res  
1811, Fourier subm  
slightly extended  
judges of the comp  
while they awarded  
superior of his analy  
tions of the trans  
with its importanc  
that the manner in  
for difficulty, a  
something to be de

was met with disbelief. Riemann, in the historical section of his Habilitationsschrift, wrote that Fourier's contention "was so unexpected by Mr. Lagrange that he contradicted it in the most decisive manner. It is said that there is still a document in the archives of the Paris Academy regarding this controversy," the statement of existence of such a document having been communicated to Riemann by Dirichlet. Fourier's memoir was deposited in the archives without being published and, according to Darboux, was "without doubt withdrawn by Fourier in 1810."<sup>12</sup>

Perhaps as a consequence of Fourier's work, however, the Academy set the competition for the grand prix de mathematiques for 1812 with the question: "To give the mathematical theory of the laws of the propagation of heat and to compare the results of this theory to exact experiments." On September 28, 1811, Fourier submitted to the Academy a work which essentially included and slightly extended his original memoir. Fourier's paper was referred to the judges of the competition, Lagrange, Laplace, Malus, Haüy and Legendre, and while they awarded him the prize, they were critical of the generality and rigor of his analysis. "This piece. . . contains the true differential equations of the transmission of heat. . . . The newness of the subject together with its importance has led the jury to crown this work, observing however that the manner in which the author arrives at his equations is not exempt from difficulty, and that his analysis in integrating his equations leaves something to be desired relative to generality and even with respect to rigor."

---

<sup>12</sup>Darboux, Oeuvres de Fourier, vii.

Fourier's  
Academy without  
the first part of  
of Heat, published  
Academy after the  
printed in its  
be published in  
his work "Fourier  
priority in an in

Fourier's m  
all of it justifi  
solutions to appl  
phenomena as a gu  
most fertile sour  
to offering a det  
cluding vague que  
method of forming  
concerns us to kn  
Richful to thi  
derated his pos  
The integrals w  
which satisfy the

Fourier's manuscript was again deposited in the archives of the Academy without being published. Resentful of this treatment, he incorporated the first part of this memoir almost without change in his Analytical Theory of Heat, published in 1822, and, having become Perpetual Secretary of the Academy after the death of Delambre, caused this part of the memoir to be printed in its original form in the Mémoires in 1824, and the second part to be published in the Mémoires in 1826. In light of the interest excited by his work "Fourier desired without doubt to thus establish his rights of priority in an incontestable manner. . . ."13

Fourier's methods and results have continued to receive criticism, not all of it justified. It is true that Fourier was a physicist interested in solutions to applied problems and in his search for such solutions utilized natural phenomena as a guide to mathematical theory. "Profound study of nature is the most fertile source of mathematical discoveries. Not only has this study, in offering a determinate object to investigation, the advantage of excluding vague questions and calculations without issue; it is besides a sure method of forming analysis itself, and of discovering the elements which it concerns us to know, and which natural science ought always to preserve. . . ."14 Faithful to this conception throughout the course of his work, Fourier reiterated his position in a summary statement toward the end of his treatise. "The integrals which we have obtained are not only general expressions which satisfy the differential equations; they represent in the most distinct

---

<sup>13</sup>Darboux, op. cit., viii.

<sup>14</sup>Fourier, op. cit., Preliminary Discourse.

manner the nat

chief condition

results of inve

When this cond

equation of the

of it, in the s

surface makes

In holdin

intuition to de

of solution of

Having determin

$1 = a \cos x + b$

$c = \frac{4}{5}, d = \frac{-4}{7}$

The complete s

form

6,

T(

Fourier then arg

temperature dist

not be determin

another solution

may be given to t

restricted withou

assumed uniqueness

15 Fourier, 2

16 Fourier, 2

manner the natural effect which is the object of the problem. This is the chief condition which we have always had in view, and without which the results of investigation would appear to us to be only useless transformations. When this condition is fulfilled, the integral is, properly speaking, the equation of the phenomenon; it expresses clearly the character and progress of it, in the same manner as the finite equation of a line or curved surface makes known all the properties of those forms."<sup>15</sup>

In holding to this view, Fourier occasionally allowed his physical intuition to dominate his mathematical analysis. His discussion of uniqueness of solution of the problem described above is an example of such domination. Having determined the coefficients  $a, b, c, \dots$  of the equation  $1 = a \cos x + b \cos 3x + c \cos 5x + \dots$  to be, respectively,  $a = \frac{4}{\pi}$ ,  $b = \frac{-4}{3\pi}$ ,  $c = \frac{4}{5\pi}$ ,  $d = \frac{-4}{7\pi}, \dots$  for values of  $x$  such that  $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ , Fourier asserted "the complete solution of the problem which we have proposed" to be of the form

$$6) \quad T(x, y) = \frac{4}{\pi} e^y \cos x - \frac{4}{3\pi} e^{-3y} \cos 3x + \frac{4}{5\pi} e^{-5y} \cos 5x - \dots$$

Fourier then argued, by means of physical considerations, that the final temperature distribution is unique; therefore "since the final state which must be determined is unique, it follows that the proposed problem admits no other solution than that which results from equation 6. Another form may be given to this result, but the solution can be neither extended nor restricted without rendering it inexact."<sup>16</sup> Thus Fourier essentially assumed uniqueness of solution.

---

<sup>15</sup> Fourier, op. cit., §428.

<sup>16</sup> Fourier, op. cit., §204.

This does  
sound mathemati  
demonstrate his  
physical or geome  
Euler or Lagrange  
convergence of s  
notions in his  
convergence of  
vergent. . . .  
of the terms dim  
establish the co  
which we arrive  
approach more an  
quantity which b  
value of the ser

$$\frac{x}{2}$$

which was given  
sinuum," publishe  
always convergen  
at less than  $\pi$ .  
the contrary val  
for a long time  
elucidate why the

This does not imply, however, that Fourier was disinterested in a sound mathematical development of his theory. Indeed, his efforts to demonstrate his propositions rigorously, while sometimes frustrated by his physical or geometrical intuition, were often as successful as those of Euler or Lagrange. For example, Fourier held the modern conception of convergence of series and this, of course, before Cauchy formalized such notions in his Cours d'analyse. He wrote, with respect to the question of convergence of (Fourier) series, "it is. . . easy to prove they are convergent. . . . This does not result solely from the fact that the values of the terms diminish continually, for this condition is not sufficient to establish the convergence of a series. It is necessary that the values at which we arrive on increasing continually the number of terms should approach more and more a fixed limit, and should differ from it only by a quantity which becomes less than any given magnitude; this limit is the value of the series."<sup>17</sup> Again, when discussing the series

$$\frac{x}{2} = \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - . . .$$

which was given without restrictions by Euler in his paper "Subsidium Calculi Sinuum," published in 1754, Fourier remarked "This infinite series, which is always convergent, has the value  $\frac{x}{2}$  so long as the arc  $x$  is greater than 0 and less than  $\pi$ . But it is not equal to  $\frac{x}{2}$  if the arc exceeds  $\pi$ ; it has on the contrary values very different from  $\frac{x}{2}$ . . . This series has been known for a long time but the analysis which served to discover it did not indicate why the result ceases to hold when the variable exceeds  $\pi$ .

---

<sup>17</sup> Fourier, op. cit., §228.

The method which  
and the origin  
is subject must  
undertake.

These remarks  
regarding conver-  
that the Fourier  
A partial answer  
series. Fourier

by considering  
expressed this

Seeking the limit  
integrated by part  
enter into the

Fourier concludes  
argument by demon-  
strated above.

Fourier and  
his work is of p

The method which we are about to employ must therefore be examined attentively and the origin of the limitation to which each of the trigonometrical series is subject must be sought,"<sup>18</sup> an examination which Fourier then proceeded to undertake.

These remarks are by way of prelude to a more important question regarding convergence: how adequately did Fourier demonstrate his assertion that the Fourier series corresponding to a function converges to the function. A partial answer is suggested by Fourier's work with particular Fourier series. Fourier initiated his discussion of the convergence of the series

$$y = \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \dots$$

by considering the sum of the first  $m$  terms of the series,  $m$  even. He expressed this finite sum  $S_m$  in the form

$$S_m = \frac{x}{2} - \int \frac{\cos (mx + \frac{x}{2})}{2 \cos \frac{x}{2}} dx.$$

Seeking the limit of this integral for increasing  $m$ , Fourier repeatedly integrated by parts to obtain "a series in which the powers of  $(m + \frac{1}{2})$  enter into the denominators," Thus  $\lim_{m \rightarrow \infty} \int \frac{\cos (mx + \frac{x}{2})}{2 \cos \frac{x}{2}} dx = 0$ , from which

Fourier concluded  $y = \lim_{m \rightarrow \infty} S_m = \frac{x}{2}$ . Having shown this, Fourier completed his argument by demonstrating the necessity of the interval of convergence noted above.

Fourier argued in this manner with respect to several explicit series. His work is of particular interest and importance since it is this type of

---

<sup>18</sup> Fourier, op. cit., §184.

argument, i.e., t  
the series and th  
finitly, with whic  
of convergence of

This remark  
established one f  
"arbitrary" funct

$f(x)$

where the interval  
 $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ . Fouri  
sumation and, to  
Be thus obtained t

$f(x)$

Fourier next wrote  
 $\cos j r + \sin j r$   
equation by  $f(x)$  e  
as the variable of  
In so doing he obt

$4f(x)$

rich concludes the

argument, i.e., to express by an integral the sum of the first  $m$  terms of the series and then to seek the limit of this integral as  $m$  tends to infinity, with which Lejeune-Dirichlet obtained the first rigorous theory of convergence of Fourier series. Thus Fourier anticipated Dirichlet.

This remark is general. For consider the manner in which Fourier established one form of his assertion regarding the representation of an "arbitrary" function  $f(x)$  by its corresponding Fourier series, to wit,

$$f(x) = \frac{1}{x} \sum_{i=-\infty}^{i=+\infty} \int_{-\frac{x}{2}}^{+\frac{x}{2}} [f(\alpha) \cos \frac{2i}{x} (\alpha - x)] d\alpha,$$

where the interval of convergence is the set of all  $x$  such that  $x \in (-\frac{x}{2}, \frac{x}{2})$ . Fourier first interchanged the order of integration and summation and, to simplify the work, let  $x = 2\pi$  and denoted  $\alpha - x$  by  $r$ . He thus obtained the form

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} [f(\alpha) \sum_{i=-\infty}^{i=+\infty} \cos i r] d\alpha.$$

Fourier next wrote the finite sum  $\sum_{i=-j}^{i=+j} \cos i r$  in the form  $\sum_{i=-j}^{i=+j} \cos i r = \cos j r + \sin j r \frac{\sin r}{1 - \cos r}$ . He then multiplied the second member of this equation by  $f(\alpha)$  expressed the product as an integral from  $-\pi$  to  $+\pi$  with  $\alpha$  as the variable of integration, and sought the form of the limit as  $j \rightarrow \infty$ . In so doing he obtained the expression

$$4f(x) \int_0^{+\infty} \frac{\sin u}{u} du = 2\pi f(x)$$

which concludes the proof.<sup>19</sup>

---

<sup>19</sup> Fourier's argument is given in greater detail in Appendix B.

Fourier

case in which

greater than  $2$

the order of  $1$

$f(x)$  such that

Fourier's reas

that Dirichlet

respect to tri

and conclusions

the criticism d

It was wi

Oeuvres de Four

cisely the way w

an extreme preci

This is not to d

memoirs of Dirich

had not been crow

that Fourier, wit

philosophy. . . ha

steps, the path a

stration of these

Part of Four

conception of func

from the tradition

the functions which

note "above all, i

Fourier also considered the form of the integral when  $x = \pm\pi$  and the case in which the limits of integration encompass an interval of length greater than  $2\pi$ . Of course Fourier's work is flawed because he interchanged the order of integration and summation and did not restrict the function  $f(x)$  such that the integrals exist. It is true, however, that the form of Fourier's reasoning was essentially correct and that it was by this means that Dirichlet later rigorously established Fourier's contentions with respect to trigonometric series representations. Thus Fourier's methods and conclusions have received less than justice with respect to much of the criticism directed toward them and concomitantly to Fourier himself.

It was with these considerations in mind that Darboux, in his Oeuvres de Fourier, commented that "Lejeune-Dirichlet. . . followed precisely the way which was indicated by Fourier, but brought into the work an extreme precision which is necessary to such an important question. This is not to deny the real and considerable progress brought by the memoirs of Dirichlet to a subject where the efforts of Poisson and Cauchy had not been crowned with complete success. But it seems just to remark that Fourier, with his very profound sense of the questions of natural philosophy. . . had indicated and even gone over, though with uncertain steps, the path along which one should go to find the first exact demonstration of these fundamental results. . . ."

Part of Fourier's success can be attributed to his more general conception of function. In this respect he represents a distinct break from the traditions of his eighteenth century colleagues. When describing the functions which admit trigonometric series representation, Fourier wrote "above all, it must be remarked that the function  $f(x)$ . . . is entirely

arbitrary and not  
 $f(x)$  represents a  
arbitrary. An in  
an equal number of  
or negative or ze  
common law; they s  
then is given as f  
are general than  
with his geometri  
equivalent to any  
(unique) dependen  
he makes referenc  
it has no existin  
given limits a  
that the curve h  
 $r = b$ , and coinci  
again, in anothe  
condition, and t  
for example, tha  
curved lines."  
continuous and h

arbitrary and not subject to a continuous law. . . .In general the function  $f(x)$  represents a succession of values or ordinates each of which is arbitrary. An infinity of values being given to the abscissa  $x$ , there are an equal number of ordinates  $f(x)$ . All have actual values, either positive or negative or zero. We do not suppose these ordinates to be subject to a common law; they succeed each other in any manner whatever, and each of them is given as if it were a single quantity."<sup>20</sup> This description is more general than Fourier intended. It is clear that Fourier, consistent with his geometric methodology, considered the notion of function to be equivalent to any relationship between the independent variable  $x$  and the (unique) dependent variable  $f(x)$  which could be given graphically.<sup>21</sup> Thus he makes reference to a function "such that the ordinate which represents it has no existing value except when the abscissa is included between two given limits  $a$  and  $b$ , all the other ordinates being supposed zero; so that the curve has no form or trace except above the interval  $x = a$  to  $x = b$ , and coincides with the  $x$ -axis in all other parts of its course." Again, in another context, Fourier wrote of a function "subject to no condition, and the line whose ordinate it represents may have any form; for example, that of a contour formed of a series of straight lines and curved lines." Thus the functions considered by Fourier were piecewise continuous and had at most finitely many discontinuities in the modern sense.

---

<sup>20</sup>Fourier, op. cit., §417.

<sup>21</sup>See Riemann, Mathematische Werke, 218.

In addition  
analysis, Fourier  
an interval of de  
tained the three

$$\frac{x}{2} =$$

$$\frac{x}{2} =$$

and  $\frac{x}{2} =$

and remarked that  
equal with referenc  
ments have a common  
and  $\frac{\pi}{2}$ . The constru  
comparison of the l  
the alternate coinci  
In general, Fourier  
vergent series. . .  
continuous law. We  
which have equal va  
included between tw  
two functions. . . a  
no substitutions a  
be represented by  
of their course. .

---

22 Fourier, op

23 Fourier, op

In addition to subjecting discontinuous functions to the methods of analysis, Fourier demonstrated that a function could be extended beyond an interval of definition in more than one way. For example Fourier obtained the three analytic expressions

$$\frac{x}{2} = \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - . . . ,$$

$$\frac{x}{2} = \frac{2}{\pi} \sin x - \frac{2}{3^2 \pi} \sin 3x + \frac{2}{5^2 \pi} \sin 5x - . . . ,$$

$$\text{and } \frac{x}{2} = \frac{\pi}{4} - \frac{2}{\pi} \cos x - \frac{2}{3^2 \pi} \cos 3x - . . . ,$$

and remarked that "these three values of  $\frac{x}{2}$  ought not to be considered as equal with reference to all possible values of  $x$ ; the three preceding developments have a common value only when the variable  $x$  is included between 0 and  $\frac{\pi}{2}$ . The construction of the values of these three series, and the comparison of the lines whose ordinates are expressed by them, render sensible the alternate coincidence and divergence of the values of these functions."<sup>22</sup> In general, Fourier wrote, "It is remarkable that we can express by convergent series. . .the ordinates of lines. . .which are not subject to a continuous law. We see by this that we must admit into analysis functions which have equal values whenever the variable receives any values whatever included between two given limits, even though on substituting in these two functions. . .a number included in another interval, the results of the two substitutions are not the same. The functions which enjoy this property are represented by different lines which coincide only in a definite portion of their course. . . ."<sup>23</sup> Thus Fourier's conception of the notion of

---

<sup>22</sup>Fourier, op. cit., §225.

<sup>23</sup>Fourier, op. cit., §230.

function was con

It is of i  
discontinuous fu  
pending trigonom  
argued geometric  
enter into the e  
are the values o  
 $\int_0(x) \sin n x dx$   
required. . . .  
of any curve wha  
the same part of  
 $y = \sin n x$  be c  
integral term.  
one value of  $\phi(x)$   
first and at the  
duct  $\phi(x) \sin n$   
those ordinates  
to the ordinates  
the area of the  
value of the co  
which correspond  
equation, or whe  
always serves to  
the reduced curv  
value of the coe

function was considerably more general than, say, that of d'Alembert.

It is of interest to consider how Fourier viewed the integrals of discontinuous function when determining the coefficients of the corresponding trigonometric series expansions. As might be expected, Fourier argued geometrically: "We see that the coefficients  $a, b, c, \dots$  which enter into the equation  $\frac{\pi}{2} \phi(x) = a \sin x + b \sin 2x + c \sin 3x + \dots$  are the values of definite integrals expressed by the general term  $\int \phi(x) \sin n x \, dx$ ,  $n$  being the number of the term whose coefficient is required.  $\dots$  if the function  $\phi(x)$  be represented by the variable ordinate of any curve whatever whose abscissa extends from  $x = 0$  to  $x = \pi$ , and if on the same part of the axis the known trigonometric curve, whose ordinate is  $y = \sin n x$  be constructed, it is easy to represent the value of any integral term. We must suppose that for each abscissa  $x$  to which corresponds one value of  $\phi(x)$  and one value of  $\sin n x$ , we multiply the latter by the first and at the same point of the axis raise an ordinate equal to the product  $\phi(x) \sin n x$ . By this continuous operation a third curve is formed whose ordinates are those of the trigonometric curve, reduced in proportion to the ordinates of the arbitrary curve represented by  $\phi(x)$ . This done, the area of the reduced curve taken from  $x = 0$  to  $x = \pi$  gives the exact value of the coefficient of  $\sin nx$ , and whatever the given curve may be which corresponds to  $\phi(x)$ , whether we can assign to it an analytical equation, or whether it depends on no regular law, it is evident that it always serves to reduce.  $\dots$  the trigonometric curve, so that the area of the reduced curve has, in all possible cases, a definite value which is the value of the coefficient of  $\sin nx$  in the development of the function."<sup>24</sup>

---

<sup>24</sup>Fourier, op. cit., §220.

In another place  
form of the curve  
The values of the  
included between  
evident that all  
figure of the body  
arbitrary form."<sup>25</sup>

This concept  
derived, of course  
of its primitive.  
conception in defe  
that he conceived  
below. In any case  
the modern standpo  
which was itself a  
purpose. That is,  
Fourier could inte

Fourier reco  
was aware of and r  
of d'Alembert, Eul  
principles to the  
the difficulties wh  
The solution given  
always be devel

In another place Fourier remarked "Whatever be the function  $\phi(x)$ , or the form of the curve which it represents, the integral has a definite value. . . . The values of these integrals are analogous to that of the whole area. . . included between the curve and the axis in a given interval. . . . It is evident that all these quantities have assignable values, whether the figure of the bodies be regular, or whether we give to them an entirely arbitrary form."<sup>25</sup>

This conception represents a break from the Eulerian tradition, derived, of course, from Newton, of defining a definite integral in terms of its primitive. Fourier may have chosen to abandon the traditional conception in deference to his geometrical viewpoint; it is more likely that he conceived of the definite integral as an area for reasons given below. In any case it should be remarked that, while unsatisfactory from the modern standpoint, the notion of the definite integral as an area, which was itself an essentially undefined concept, was adequate for Fourier's purpose. That is, given the prevailing intuitive conception of area, Fourier could integrate the functions he had in mind.

Fourier recognized that he had made a significant contribution. He was aware of and recognized the relationship of his work to the researches of d'Alembert, Euler, Daniel Bernoulli, and Lagrange. "If we apply these principles to the problem of the motion of vibrating strings, we can solve the difficulties which first appeared in the researches of Daniel Bernoulli. The solution given by this geometrician assumes that any function whatever may always be developed in a series of sines or cosines of multiple arcs

---

<sup>25</sup>Fourier, op. cit., §229.

Now the most common  
consists in actual  
determined coefficients  
had contributed to  
tation of function  
analysis. "It has  
in a series of s  
trigonometric series  
of the variable  
the function be  
it remains inco  
trigonometric s  
have insisted o  
the present ti  
isolated probl  
with the most  
that Darboux  
handsome wor  
all scientif

Now the most complete of all the proofs of this proposition is that which consists in actually resolving a given function into such a series with determined coefficients,"<sup>26</sup> a work which Fourier had accomplished. He had contributed to the solution of a long-standing problem in the representation of functions in such a way as to have important implications for analysis. "It had always been regarded as manifestly impossible to express in a series of sines of multiple arcs, or at least in a convergent trigonometric series, a function which has no existing values unless the values of the variable are included between certain limits, all the other values of the function being zero. But this point of analysis is fully cleared up and it remains incontestable that [such functions] are exactly expressed by trigonometric series which are convergent or by definite integrals. We have insisted on this consequence from the origin of our researches up to the present time, since we are not concerned here with an abstract and isolated problem, but with a primary consideration intimately connected with the most useful and extensive applications."<sup>27</sup> It is little wonder that Darboux could refer to Fourier's Analytical Theory of Heat as a "handsome work that one can place with justice beside the most perfect of all scientific writings."

---

<sup>26</sup> Fourier, op. cit., §230.

<sup>27</sup> Fourier, op. cit., §428.

Before Cauchy  
definite integrals  
knowledge of the  
especially by Newton  
had been obtained  
tenuous, however  
limiting process

Integration  
Leibnitz. Newton  
i.e., as the operation  
as a "calculus"  
familiar with the  
theorem of calculus  
integral calculus  
the course of the  
Institutiones calculi  
calculus as the  
the relationship  
a limiting summa

Of course,  
arithmetic expressions  
above, the concept

## Cauchy

Before Cauchy there was no rigorous definition of the concept of definite integral. Integration had been known from the time of Archimedes; knowledge of the integral calculus had been strengthened and extended, especially by Newton and Leibnitz, and results of fundamental importance had been obtained by its use. The basic concept had remained vague and tenuous, however, due to inadequate statements of definition of limits and limiting processes.<sup>28</sup>

Integration theory had followed complementary courses from Newton and Leibnitz. Newton regarded integration as "the inverse method of fluxions," i.e., as the operation inverse to differentiation; Leibnitz conceived integration as a "calculus summatorius," or limiting summation. Both men were familiar with the relationship between the two concepts, the fundamental theorem of calculus. While both concepts were known, the view of the integral calculus as the inverse of differentiation became predominant in the course of the formal development. Euler, for example, in his Institutiones calculi integralis, published in 1768, defined integral calculus as the method of finding the relationship between functions given the relationship between their differentials, and only used the concept of a limiting summation for the approximate evaluation of integrals.

Of course, one of the interpretations of the definite integral was the arithmetic expression of the geometric concept of area. Thus, as remarked above, the concept of area was essentially undefined. Moreover, in the

---

<sup>28</sup> See Bell, The Development of Mathematics, for these aspects of the history.

geometric-intui

definitions, and

converse was al

fined by means

work. With Fou

added or subtra

Cauchy w

and formalism.

of d'Alembert,

metaphysics of

definition of

continuity, de

Cauchy revived

of the integra

class of funct

members of thi

was first to c

functions of a

Cauchy e

d'analyse, put

analysis all

never have re

of algebra.

all in the pa

geometric-intuitive morass which prevailed in the absence of precise definitions, and in the confusion between definition and interpretation, the converse was also held to be true. That is, the definite integral was defined by means of the concept of area. We have seen this in Fourier's work. With Fourier, mathematicians "merely said which areas had to be added or subtracted in order to obtain the integral  $\int_a^b f(x)dx$ ." <sup>29</sup>

Cauchy wished to extricate analysis from the quagmires of intuitionism and formalism. To this end he successfully carried out the implied program of d'Alembert, who in 1754 had stated that "The theory of limits is the true metaphysics of the differential calculus." Thus, Cauchy first stated a definition of the concept of a limit and then defined the notions of continuity, derivative, and integral in terms of the limit concept. Cauchy revived the concept of limiting summation as the fundamental notion of the integral calculus. He defined the definite integral for an explicit class of functions, and demonstrated the existence of the integral for members of this class. He proved the fundamental theorem of calculus; he was first to create a theory of integration, and indeed, a theory of functions of a real variable.

Cauchy expressed his purpose in the introduction to his Cours d'analyse, published in 1821. "I have sought to give to the methods of analysis all the rigor which is demanded in geometry, in such a way as to never have recourse to reasons drawn from the generality [i.e., formalism] of algebra. Reasons of this type, although rather commonly admitted, above all in the passage from converging series to diverging series, and from real

---

<sup>29</sup> Lebesgue, "Sur le développement de la notion d'intégrale," 149.

quantities to im  
re, as induction  
relationship to  
sciences. One o  
algebraic formul  
formulas hold on  
quantities which  
values, and in f  
I use, I shall d  
"indefinite vali  
application of t

-1

Cauchy was  
tradition and a  
remain constant  
several proposi  
glance. For ex  
sm . . . . . Bu  
positions of t  
more precision  
which are too  
subjects of re

Cauchy's  
Polytechnical

quantities to imaginary expressions, can only be considered, it seems to me, as inductions which sometimes suggest the truth, but which bear little relationship to the exactitude which is so prized by the mathematical sciences. One ought even to observe that they tend to . . . attribute to algebraic formulas an indefinite validity, while in reality most of these formulas hold only under certain conditions, and for certain values of the quantities which they contain. In determining these conditions and these values, and in fixing in a precise manner the meaning of the notations which I use, I shall dispel all incertitude." By attributing to formulas an "indefinite validity" Cauchy meant, for example, an uncritical (formal) application of the binomial theorem to obtain

$$-1 = (1 - 2)^{-1} = 1 + 2 + 4 + 8 + 16 + . . . .$$

Cauchy was almost apologetic in breaking with the micawberian tradition and attempted to justify his work. "It is true that in order to remain constantly faithful to these principles I have been forced to admit several propositions which will perhaps appear a little hard on first glance. For example, I state in Chapter 6 that a divergent series has no sum . . . . But those who read my book will recognize, I hope, that propositions of this nature, which bring forth the happy necessity of placing more precision in theories and of bringing useful restrictions to assertions which are too general, turn to the profit of analysis and furnish several subjects of research which are not without importance."<sup>30</sup>

Cauchy's Course of Analysis, which he had given at the Royal Polytechnical School, and had been encouraged to publish by Laplace and

---

<sup>30</sup>Cauchy, Cours d'analyse, iv.

Prison, included

Let  $f(x)$  be a fun

value of  $x$  interme

admits a unique an

between these lim

increment  $\alpha$ , the

which will depend

been stated, the

tinuous function

between these li

decreases indef

continuous with

limits, an infi

infinitely smal

This def

Cauchy probabl

as his stateme

that Cauchy ha

even though i

function. Se

interval rath

defined. ". .

Poisson, included this definition of the concept of continuous function.

"Let  $f(x)$  be a function of the variable  $x$  and let us suppose that for each value of  $x$  intermediate between two given limits, the function always admits a unique and finite value. If, starting from a value of  $x$  included between these limits, one attributes to the variable  $x$  an infinitely small increment  $\alpha$ , the function itself will receive the increment  $f(x + \alpha) - f(x)$ , which will depend on the new variable  $\alpha$  and the value of  $x$ . This having been stated, the function  $f$  will be, between the two given limits, a continuous function of the variable  $x$  if, for each value of  $x$  intermediate between these limits, the numerical value of the difference  $f(x + \alpha) - f(x)$  decreases indefinitely with  $\alpha$ . In other terms, the function  $f(x)$  will be continuous with respect to  $x$  between the limits given if, between these limits, an infinitely small increment of the variable always produces an infinitely small increment of the function itself."<sup>31</sup>

This definition is interesting in a number of aspects. First, Cauchy probably did not intend his concept of function to be as general as his statement might be interpreted to imply. That is, it is unlikely that Cauchy had in mind a function like the Dirichlet function,

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1 \text{ and } x \text{ is irrational} \\ 1 & \text{if } 0 \leq x \leq 1 \text{ and } x \text{ is rational} \end{cases},$$

even though it is admissible in terms of his definition of (single-valued) function. Second, Cauchy defined the continuity of a function on an interval rather than at a point. Discontinuity of a function was similarly defined. ". . .when a function ceases to be continuous in a neighborhood of a

---

<sup>31</sup>Cauchy, op. cit., 34.

particular value of  
and that there is f  
 $x=0$  is a solution  
that Cauchy's defin  
required that the

We turn to  
Technical School  
such a definition  
limits. Cauchy r  
has been to recor  
Course of Analysis  
of infinitely sm

The first  
limites et des  
quantity is tha  
a constant quan  
a fixed and de  
to the same va  
it by as littl  
of all the for  
limit toward  
as the number  
discussed th  
concluded th

particular value of the variable  $x$ , then one says that it is discontinuous, and that there is for this particular value a solution of continuity. Thus,  $x = 0$  is a solution of continuity for the function  $\frac{1}{x}$ ." Finally, in order that Cauchy's definition of continuity of a function be meaningful, it is required that the concept of the infinitely small be defined.

We turn to Cauchy's Resume of the Lessons Given at the Royal Polytechnical School on the Infinitesimal Calculus, published in 1823, for such a definition. Indeed, it is here that Cauchy developed his theory of limits. Cauchy remarked in the preface to this work "My principal goal has been to reconcile the rigor which has been my guiding principle in my Course of Analysis with the simplicity which follows from the consideration of infinitely small quantities."

The first lesson of the text was devoted to "des variables, de leurs limites et des quantites infiniment petites." Cauchy wrote ". . . a variable quantity is that which successively receives several different values. . . , a constant quantity, on the other hand, is any quantity which receives a fixed and determined value. When the values attributed successively to the same variable approach a fixed value, so as to . . . differ from it by as little as one would wish, then the latter is called the limit of all the former. Thus, for example, the perimeter of a circle is the limit toward which the perimeters of regular inscribed polygons converge as the number of their sides is indefinitely increased."<sup>32</sup> Cauchy then discussed the limits of  $\frac{\sin \alpha}{\alpha}$  and  $(1 + \alpha)^{\frac{1}{\alpha}}$  as  $\alpha$  tends to zero. He concluded the lesson by defining the concept of the infinitely small.

---

<sup>32</sup>Cauchy, Résumé des leçons données à l'Ecole Royale Polytechnique sur le calcul infinitésimal, 13.

When the success  
that is, in such  
number, then this  
A variable of this

In a subse  
function as he ha  
to define in the  
function over a c  
Cauchy was aware  
had singled it ou  
calculus it has  
existence of int  
diverse properti  
necessary to est  
or definite int

Cauchy's  
suppose that th  
variable  $x$  betw  
by  $x_1, x_2, \dots$   
which always i  
These values a

which will be

"When the successive values of the same variable decrease indefinitely, that is, in such a manner as to go below [i.e., be less than] any given number, then this variable becomes what one calls infinitely small . . . A variable of this type has zero as its limit."<sup>33</sup>

In a subsequent lesson Cauchy defined the concept of continuous function as he had in his Cours d'analyse. Given this he was prepared to define in the 25th lesson, the definite integral of a continuous function over a closed (finite) interval as the limit of a set of sums. Cauchy was aware of the importance of his work in integration theory and had singled it out for special attention in his preface. "In integral calculus it has seemed necessary to me to demonstrate in general the existence of integrals or primitive functions before developing their diverse properties. In order to accomplish this it has first been necessary to establish the notion of integrals taken between given limits or definite integrals . . ."

Cauchy's definition of definite integral is as follows. "Let us suppose that the function  $y = f(x)$  is continuous with respect to the variable  $x$  between two finite limits  $x = x_0$  and  $x = X$ . We will designate by  $x_1, x_2, \dots, x_{n-1}$  new values of  $x$  interposed between these limits, which always increase or decrease from the first limit to the second. These values are used to divide the difference  $X - x_0$  into elements

$$x_1 - x_0, x_2 - x_1, \dots, X - x_{n-1}$$

which will be of the same sign. This stated, let us multiply each element

---

<sup>33</sup>Cauchy, op. cit., 16.

by the value

to say the el

and finally th

$S = ($

be the sum of

depend on 1) th

have been divid

the mode of div

numerical value

large, then the

influence on the

Cauchy th

the difference b

is small when th

tion that Cauchy

be then continue

of the difference

have very small r

third chosen in s

the first two mod

will find that t

first or second

difference  $X - x$

only a negligible

ical value of the

by the value of  $f(x)$  corresponding to the origin of the element, that is to say the element  $x_1 - x_0$  by  $f(x_0)$ , the element  $x_2 - x_1$  by  $f(x_1)$ , . . . , and finally the element  $X - x_{n-1}$  by  $f(x_{n-1})$ , and let

$$S = (x_1 - x_0)f(x_0) + \dots + (X - x_{n-1})f(x_{n-1})$$

be the sum of the products thus obtained. The quantity  $S$  will evidently depend on 1) the number  $n$  of elements in which the difference  $X - x_0$  will have been divided, and 2) the values of these elements and, in consequence, the mode of division used. Now it is important to remark that if the numerical values of the elements become very small and the number  $n$  very large, then the mode of division will no longer have any but a very small influence on the value of  $S$ ."

Cauchy then proceeded to demonstrate this assertion, that is, that the difference between sums  $S$  and  $S'$  corresponding to partitions  $P$  and  $P'$  is small when the norms of  $P$  and  $P'$  are small. It was in this demonstration that Cauchy appealed to the hypothesis of the (uniform) continuity of  $f(x)$ . He then continued, "Let us consider simultaneously of two modes of division of the difference  $X - x_0$ , in each of which the elements of the difference have very small numerical values. We will compare these two modes to a third chosen in such a way that . . . all the values of  $x$  interposed in the first two modes between the limits  $x_0$  and  $X$  be used in the third, and will find that the value of  $S$  is charged very little by passing from the first or second mode to the third . . . . Thus, when the elements of the difference  $X - x_0$  become infinitely small, the mode of division has on  $S$  only a negligible influence, and if one decreases indefinitely the numerical value of these elements while augmenting their number, the value of  $S$

will finish by  
by attaining a  
function  $f(x)$  and  
This limit is wh

Given thi  
properties" of t  
noted, the funda  
tion of definite  
Cauchy integrate  
subintervals in v  
theory of integra  
he had in mind, i  
Stated differentl  
purposes.

It remains  
vailing concepio  
is an important c  
definite integral  
Integration theo  
matter is furnis  
published in 182  
of differential e

---

<sup>34</sup> Cauchy,

<sup>35</sup> See Haw

will finish by being sensibly constant, or, in other terms, it will finish by attaining a certain limit which will depend uniquely on the form of the function  $f(x)$  and the extreme values  $x_0$  and  $X$  attributed to the variable  $x$ . This limit is what one calls a definite integral."<sup>34</sup>

Given this definition, Cauchy then went on to develop the "diverse properties" of the integral in succeeding lessons, including, as has been noted, the fundamental theorem of calculus. He also generalized the definition of definite integral to unbounded functions. In similar fashion, Cauchy integrated piecewise continuous functions by integrating over those subintervals in which there was no point of discontinuity. Thus, Cauchy's theory of integration was sufficiently general to integrate the functions he had in mind, i.e., continuous and piecewise continuous functions.<sup>35</sup> Stated differently, Cauchy's theory of integration was adequate for his purposes.

It remains to attempt to determine why Cauchy abandoned the prevailing conception of integration as the inverse of differentiation. This is an important question, it seems to me, since Cauchy's definition of definite integral as a limit of sums decisively influenced the course of integration theory in its subsequent development. Some insight into this matter is furnished by Cauchy's "General observations and additions," published in 1823 as an addendum to an earlier memoir on the integration of differential equations. ". . . we will consider each definite integral,

---

<sup>34</sup>Cauchy, op. cit., 122.

<sup>35</sup>See Hawkins, Lebesgués Theory of Integration, 12.

taken between two  
small values of  $t$   
correspond to the  
limits. When one  
shows easily that  
the two limits of  
the  $\int$  sign is fin  
later, and more i  
naturally led by  
regal, taken bet  
but, it seems to  
be adopted by pre  
cases, even those  
placed under the  
vantage of always  
to real functions  
equation into two  
were to consider  
necessarily equiv  
tive or even disc  
limit to pass to  
two cases, one wo  
those given by Mr

---

36 Cauchy.

37 Cauchy.

taken between two limits, as being nothing else than the sum of the infinitely small values of the differential expression placed under the sign  $\int$  which correspond to the diverse values of the variable contained within the given limits. When one adapts this manner of viewing definite integrals, one shows easily that such an integral has a unique and finite value, whenever the two limits of the variable are finite quantities, and the function under the  $\int$  sign is finite and continuous in the interval between the limits."<sup>36</sup>

Later, and more importantly, Cauchy wrote in a post scriptum "One is naturally led by the theory of quadrature to consider each definite integral, taken between two real limits as being [the limit of a set of sums]. But, it seems to me that this manner of viewing a definite integral should be adopted by preference, as we have done, because it suits equally all cases, even those in which one cannot generally pass from the function placed under the sign  $\int$  to the primitive. It has, in addition, the advantage of always furnishing real values for the integrals which correspond to real functions. Finally, it permits us to separate each imaginary equation into two real equations. All this would not take place if one were to consider the definite integral, taken between two real limits, as necessarily equivalent to the difference of the extreme values of a primitive or even discontinuous function, or if one caused the variable of one limit to pass to another by a series of imaginary values. In these last two cases, one would often obtain imaginary values for the integral, like those given by Mr. Poisson."<sup>37</sup>

---

<sup>36</sup>Cauchy, "Observations générales et additions" 571.

<sup>37</sup>Cauchy, op cit., 590.

Recall that  $f$  is Riemann integrable on  $[a, b]$  if and only if the Riemann sum  $\sum_{k=1}^n f(t_k) \Delta t_k$  converges to a limit as  $n \rightarrow \infty$  and  $\Delta t_k \rightarrow 0$ . This is  $\int_a^b f(t) dt + C$ . This

$f(t)$

Let  $f(x)$  be a function which is Cauchy integrable on  $[-1, 1]$ . This is a primitive on  $[-1, 1]$ .

$$F(x) = \int_{-1}^x f(t) dt$$

However,  $F'(x)$  does not exist for all  $x \in [-1, 1]$ . This is a mistake when he is thinking when he is passing from the function  $f$  to the function  $F$ .

Cauchy's perception of integration as the integration of an arbitrary function with coefficients are not the integration of such a function. Fourier's recognition of the equivalence of the two concepts which caused him

Recall that if  $f(x)$  is defined on  $[a,b]$  then  $F(x)$  is a primitive of  $f(x)$  on  $[a,b]$  if and only if  $F'(x) = f(x)$  for every  $x \in (a,b)$ . Also recall a theorem of Riemann's theory of integration of the form; if  $f(x)$  is integrable on  $[a,b]$  and if  $f(x)$  has a primitive on  $[a,b]$ , then a primitive is  $\int_a^x f(t)dt + C$ . This given, consider the function

$$f(x) = \begin{cases} 1 & \text{if } -1 \leq x \leq 0 \\ 2 & \text{if } 0 < x \leq 1. \end{cases}$$

Then  $f(x)$  is Cauchy integrable on  $[-1,1]$  and  $\int_{-1}^1 f(x)dx = 3$ . Now if  $f(x)$  has a primitive on  $[-1,1]$  it must be

$$F(x) = \int_{-1}^x f(t)dt + C = \begin{cases} x+C & \text{if } -1 \leq x \leq 0 \\ 2x+C & \text{if } 0 < x \leq 1. \end{cases}$$

However,  $F'(x)$  does not exist at  $x = 0$  and therefore  $f(x)$  has no primitive on  $[-1,1]$ . This is the kind of function of which Cauchy was undoubtedly thinking when he referred to those cases "in which one cannot generally pass from the function placed under the sign  $\int$  to the primitive."

Cauchy perceived that the integral of functions such as the one defined above cannot be obtained by direct appeal to the conception of integration as the inverse of differentiation. Fourier's assertion that an arbitrary function admits representation by a trigonometric series whose coefficients are given as definite integrals necessitated, however, the integration of such "arbitrary" functions. Indeed, it was probably Fourier's recognition that such functions cannot be considered "as necessarily equivalent to the difference of the extreme values of a primitive" which caused him to abandon the traditional conception of the definite integral

and to hold

I sho

ant influenc

integral. T

Fourier's as

conception w

by returning

sms.

Cauchy

after August,

"Memoir on de

"reciprocal"

these function

a great numbe

Bulletin de l

to remark tha

in which the

of Mr. Poisso

municated to

1807 and 1811

and I hastene

printed in De

with respect

work or comm

In sup

and to hold in its stead the notion of definite integral as an area.

I should like to conjecture that Fourier's work was an important influence in Cauchy's reformulation of the concept of definite integral. That is, Cauchy recognized the necessity, derived from Fourier's assertions, to integrate functions for which the traditional conception was inadequate, and yet for which an integral could be defined by returning to Leibnitz's conception of the integral as a limit of sums.

Cauchy stated that he was not aware of the work of Fourier until after August, 1817. In a note added in 1825 to his 1814 publication "Memoir on definite integrals," Cauchy wrote, with respect to certain "reciprocal" properties of functions, "The remarkable properties of these functions and the advantages offered by them in the solution of a great number of problems furnished me the subject of a note in the Bulletin de la Société philomathique of August, 1817. It is necessary to remark that when I wrote this note I did not yet know of any memoirs in which the . . . reciprocity is deduced or used, other than those of Mr. Poisson and myself . . . . Since that time, Mr. Fourier communicated to me his researches on heat, presented to the Institute in 1807 and 1811 and only published in 1819. I there saw the same formula and I hastened to render . . . the justice due to him in a second note printed in December, 1818."<sup>38</sup> Of course other obvious possibilities with respect to Cauchy's recognition of such necessity are independent work or communication through an intermediary.

In support of my conjecture I should remark that there is no

---

<sup>38</sup>Cauchy, "Mémoire sur les intégrales définies," 300.

evidence for  
 definite inte  
 his "Memoir o  
 in this monog  
 even implicit  
 contend to so  
 example, Cauc  
 Sciences on C

possible case  
 $\int_a^b f(x) dx$  whi

The q  
 influenced b  
 ing of addi  
 ever, it is  
 mathematica

The  
 converges  
 Lejeune-Di  
 convergenc  
 between gi

evidence for the conventional assertion that Cauchy conceived of the definite integral as a limit of sums as early as 1814, at least in his "Memoir on definite integrals." Cauchy did many wonderful things in this monograph, but to write a new definition of definite integral, even implicitly, was not one of them. Indeed, Cauchy himself did not contend to so early a priority. In a subsequent paper, for example, Cauchy began "In a memoir presented to the Academy of Sciences on October 28, 1822, . . . I showed how one can fix, in all possible cases, the sense that ought to be attached to the notation  $\int_a^b f(x)dx$  which represents a definite integral . . . ." <sup>39</sup>

The question of Cauchy's conception of definite integral being influenced by Fourier series is interesting and, in my opinion, deserving of additional research. Whatever the fate of my conjecture, however, it is clear that Cauchy began the long work of carrying out the mathematical development implied by the conceptions of Fourier.

#### Dirichlet

The first rigorous proof that the Fourier series of a function converges to the function was given for a general class of functions by Lejeune-Dirichlet. Dirichlet presented his proof in his memoir "On the convergence of trigonometric series which represent arbitrary functions between given limits," published in 1829. In his introductory remarks

---

<sup>39</sup>Cauchy, "Mémoire sur les intégrales définies prises entre des limites imaginaires," 265.

to this mem  
of Fourier  
opened a new  
ducing the  
are dealing  
researches  
present time  
dently unawa  
a demonstrat  
by Cauchy,<sup>41</sup>  
work credite  
of Sciences  
familiar wit  
series; inde  
had excited

After  
no doubt to  
by examining  
reduced."<sup>42</sup>

---

40 Di  
servent à re

41 Se  
fonctions en

42 A  
The Theory of  
Series, vol.

to this memoir Dirichlet observed that the property of convergence of Fourier series "had not escaped the illustrious geometer who had opened a new field of study for the application of analysis by introducing the manner of expressing the arbitrary functions with which we are dealing; this is found in the monograph which contains his first researches on heat. But to my knowledge, no one has given up to the present time a general demonstration of this."<sup>40</sup> Dirichlet was evidently unaware of several earlier unsuccessful attempts to give such a demonstration, most notably by Poisson in 1823 and in and after 1826 by Cauchy,<sup>41</sup> for he continued "I know nothing on this subject except a work credited to Mr. Cauchy which is part of the Memoirs of the Academy of Sciences of Paris for the year 1823." Of course Dirichlet was familiar with Fourier's attempts to demonstrate convergence of such series; indeed it was his acquaintance with Fourier in Paris which had excited his interest in the theory of Fourier series.

After giving a critique of Cauchy's work which seemed "to leave no doubt to its insufficiency," Dirichlet began his own demonstration by examining "the most simple case, to which all others can be reduced."<sup>42</sup> First he proved the following lemma: "If  $0 \leq g < h \leq \frac{\pi}{2}$ ,

---

<sup>40</sup>Dirichlet, "Sur la convergence des series trigonometriques qui servent à représenter une fonction arbitraire entre des limites données," 157.

<sup>41</sup>See, for example, Cauchy, "Mémoire sur les développements des fonctions en series periodiques," 196.

<sup>42</sup>A detailed account of Dirichlet's proof is given by Hobson, The Theory of Functions of a Real Variable and the Theory of Fourier's Series, vol. 2, §323 and 328.

then the integr

in which the fu

monotone decrea

limit as  $\epsilon$  tend

$\frac{1}{2} f(0)$  when  $g =$

Given th

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

of  $f(x)$ , a func

value of  $x$  incl

of the first  $2\pi$

$$7) \quad \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

Writing this in

then the integral

$$\int_g^h f(\beta) \frac{\sin i\beta}{\sin \beta} d\beta,$$

in which the function  $f(\beta)$  is continuous and monotone increasing or monotone decreasing over the interval of integration, converges to a limit as  $i$  tends to infinity. This limit is zero when  $g \neq 0$  and is  $\frac{\pi}{2} f(0)$  when  $g = 0$ ."

Given this, Dirichlet considered the Fourier series

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(\alpha) d\alpha + \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(\alpha) \cos \alpha d\alpha \cos x + \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(\alpha) \sin \alpha d\alpha \sin x + \dots$$

of  $\phi(x)$ , a function having "a finite and determined value for each value of  $x$  included between  $-\pi$  and  $\pi$ ." Dirichlet expressed the sum of the first  $2n+1$  terms of this series in the closed form

$$7) \quad \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(\alpha) \frac{\sin(n + \frac{1}{2})(\alpha - x)}{2 \sin \frac{1}{2}(\alpha - x)} d\alpha.$$

Writing this integral as the sum

$$\frac{1}{2} \frac{\sin(2n+1)x}{\sin x}$$

0

Dirichlet obt

use of the le

those function

and . . . hav

Dirichlet com

that the [Fou

for each valu

values  $\pi$  and

"all of whose

finite number

the limits  $-\pi$

convergence o

of Fourier and

It is f

of continuity

does not requi

that it be mon

Given Cauchy's

43 Recal

44 Dirich

$$\int_0^{\frac{\pi+x}{2}} \frac{\sin(2n+1)\beta}{\sin \beta} \phi(x-2\beta) d\beta + \int_0^{\frac{\pi-x}{2}} \frac{\sin(2n+1)\beta}{\sin \beta} \phi(x+2\beta) d\beta,$$

Dirichlet obtained the limits of these integrals with increasing  $n$  by use of the lemma, appealing to the lemma a finite number of times for those functions which "present several solutions of continuity . . . and . . . have several maxima and minima" in the interval of integration. Dirichlet concluded that "the [convergence of the] integral (7) . . . proves that the [Fourier] series is convergent . . . and equal to  $\frac{1}{2}(\phi(x+0) + \phi(x-0))$  for each value of  $x$  included between  $-\pi$  and  $\pi$ , and for each of the extreme values  $\pi$  and  $-\pi$  is equal to  $\frac{1}{2}(\phi(\pi-0) + \phi(-\pi+0))$ "<sup>43</sup> for those functions "all of whose values are finite and determined, and which have only a finite number of solutions of continuity and maxima and minima between the limits  $-\pi$  and  $\pi$ ."<sup>44</sup> Thus Dirichlet completely solved the problem of convergence of Fourier series representation of the "arbitrary" functions of Fourier and Cauchy.

It is instructive to study Dirichlet's use of Cauchy's conception of continuity in the proof of his lemma. Such study shows that the proof does not require continuity of the integrable function  $f(x)$  but only that it be monotone in an interval about  $x$  where, of course,  $x \in [0, \frac{\pi}{2}]$ . Given Cauchy's conception of definite integral of a piecewise continuous

---

<sup>43</sup> Recall  $\phi(x+0) = \lim_{k \rightarrow 0} \phi(x+k)$  and  $\phi(x-0) = \lim_{k \rightarrow 0} \phi(x-k)$ .  
 $k > 0$   $k > 0$

<sup>44</sup> Dirichlet, op. cit., 168.

function,  
continuity,  
and  $f(x)$ s  
in order  
he worked

Di  
for conve  
the quest  
could be  
sider the  
the numbe  
hold. Th  
ered. In  
of contin  
function  
arbitrary  
between a  
together  
r to s.  
ing that  
going bac  
the integ  
function  
tended, t  
function  
tegrabili  
should be

function, Dirichlet required a condition to ensure the piecewise continuity and hence integrability of  $f(x)$  and the products  $f(x) \cos mx$  and  $f(x) \sin mx$ . Thus Dirichlet stated his hypothesis of continuity in order to assure the existence of the definite integrals with which he worked.

Dirichlet wished to obtain more general sufficient conditions for convergence, i.e., to relax the restrictions on  $f(x)$ . He believed the question of convergence of Fourier series of more general functions could be reduced to the above "most simple case." "It remains to consider the case where the assumptions that we have made with respect to the number of solutions of continuity and maxima and minima cease to hold. These singular cases can be reduced to the ones we have considered. In order that the integral (7) have a meaning when the solutions of continuity are infinite in number it is only necessary that the function . . .  $\phi(x)$  be such that if one designates by  $a$  and  $b$  two arbitrary quantities included between  $-\pi$  and  $\pi$ , one can always place between  $a$  and  $b$  other quantities  $r$  and  $s$  which are sufficiently close together that the function remains continuous in the interval from  $r$  to  $s$ . The necessity of this restriction is easily seen by considering that the different terms of the series are definite integrals and going back to the fundamental notion of integral. Thus one sees that the integral of a function signifies nothing except inasmuch as the function satisfies the preceding stated condition." Dirichlet concluded, therefore, that the only requirement to be satisfied by the function is that it be integrable, and a sufficient condition for integrability is that the set of points of discontinuity of the function should be nowhere dense.

Dirichlet  
fulfill this  
stant  $c$  when  
constant  $d$  with  
thus defined  
however, it is  
integrals which  
Dirichlet's idea  
integrable.  
tion theory,  
function was  
such pathological  
terest,"<sup>45</sup> the

Dirichlet  
that I have found  
restrictions  
demonstrate that  
that one could  
mental principles  
another note  
able properties  
Dirichlet  
tinued to explain  
not only by

Dirichlet continued, "An example of a function which does not fulfill this condition is [the function]  $\phi(x)$  equal to a fixed constant  $c$  when the variable  $x$  is a rational number and equal to another constant  $d$  when the variable is an irrational number. The function thus defined has a finite and determined value for every value of  $x$ ; however, it cannot be substituted in the series, for the different integrals which enter into the series lose all meaning . . . ." Thus Dirichlet identified a function so discontinuous as to fail to be integrable. Dirichlet's example had little immediate impact on integration theory, however, for the concept of an everywhere discontinuous function was not yet taken seriously. Indeed, this was "an age in which such pathological functions appeared to be completely devoid of interest,"<sup>45</sup> there being no necessity to integrate such functions.

Dirichlet concluded his memoir by remarking "the restriction that I have just made and that of not becoming infinite are the only restrictions to which the function  $\phi(x)$  is subject . . . . But to demonstrate this, in order that the work be done with all the clarity that one could desire, requires several details related to the fundamental principles of infinitesimal analysis which will be presented in another note and in which I will also pursue some other rather remarkable properties of [Fourier] series."

Dirichlet never published the proposed subsequent note. He continued to exert an influence upon the ensuing development, however, not only by the implications for the theory of trigonometric series

---

<sup>45</sup> Bourbaki, Éléments d'histoire des mathématiques, 247.

expres

but al

excite

his st

Götting

tion of

necessa

tionssc

series.

complet

during

Dedekin

was pre

Univers

tended

origina

matter,

of the

differe

durch S.

expressed in his 1829 memoir, and again in a memoir published in 1837,<sup>46</sup> but also in his capacity as a teacher. In particular, Dirichlet excited an interest in this theory in Riemann while the latter was his student in Berlin, an interest which was expressed in Riemann's Göttingen inaugural dissertation, to which we now turn.

### Riemann

Dirichlet determined sufficient conditions for the representation of a function by a Fourier series. Riemann sought to establish necessary conditions for such representation in 1854 in his Habilitationsschrift "On the representation of a function by a trigonometric series."<sup>47</sup> As will be seen, Riemann's work was in some respects incomplete, and perhaps for this reason the memoir remained unpublished during his lifetime. It was printed in 1867 on the authority of Dedekind. Dedekind wrote in an introductory footnote, "This treatise was presented in 1854 by the author to obtain appointment at the University of Göttingen ... Although the author seems not to have intended to publish it, the present edition of this treatise in its original form seems to be justified by the high interest of the subject matter, and by the form of treatment of the most important principles of the infinitesimal calculus ..."

Riemann viewed his paper as consisting of "two essentially different parts. The first part is a history of the investigation ...

---

<sup>46</sup>Dirichlet, "Ueber die Darstellung ganz will-kürlicher Functionen durch Sinus - und Cosinusreihen," 152.

<sup>47</sup>Riemann, Mathematische Werke, 213.

of ar

by tr

a few

found

regar

which

this

conce

cepti

impos

funct

whom

histo

notion

d'Ale

a stat

the f

minima

the av

discon

remark

three

mains

counter

---

of arbitrary (graphically given) functions and their representation by trigonometric series. In its composition I was permitted to utilize a few hints of the famous mathematician to whom is owed the first profound work on this subject. In the second part I will present research regarding the representability of a function by trigonometric series which includes those cases which up to now have not been solved. [In this respect] it was necessary to introduce a short discussion of the concept of definite integral ..." Thus Riemann's more general conception of definite integral was derived in a context of necessity imposed by the problem of trigonometric series representation of functions. The "famous mathematician" referred to was Dirichlet, to whom Riemann had turned for information in the preparation of the historical section of his *Habilitationsschrift*.

In this first part, Riemann reviewed the development of the notion of trigonometric series representation of functions from d'Alembert to Dirichlet. He concluded his survey of the history with a statement of Dirichlet's sufficient conditions for such representation: the function must be integrable, have only finitely many maxima and minima, and "where the function suddenly changes values assumes the average of the mutual limit values." That is, if  $x_0$  is a point of discontinuity of  $f(x)$ , then  $f(x_0) = \frac{1}{2} (f(x_0 + 0) + f(x_0 - 0))$ . Riemann remarked "whether and when a function which does not fulfill these three conditions can be represented by a trigonometrical series remains undecided by these researches."<sup>48</sup>

Given the success of Dirichlet's work with respect to functions encountered in applied problems, and which therefore had been of primary

---

<sup>48</sup>Riemann, op.cit., 223.

into

tion

its

ing

and

the

to e

by D

Diri

matt.

infin

great

ject

Fouri

appli

numbe

trig

great

to re

prop

which

defin

ing t

interest, Riemann attempted to justify his own more general investigation of trigonometric series representation of functions in terms of its relationship to pure mathematics. "For all our ignorance concerning the manner in which the forces and states of matter vary with time and place in the infinitely small, we are able to hold as certain that the functions to which the researches of Dirichlet do not apply fail to express physical processes. Nevertheless, those cases not considered by Dirichlet seem to merit attention for two reasons. First, as Dirichlet himself remarked at the end of his treatise, this subject matter stands in the closest relationship with the principles of the infinitesimal calculus and it may serve to bring to these principles a greater clarity and certitude. In this respect the study of this subject matter has an immediate interest. Second, the application of Fourier series is not restricted to physical researches; it is also applied with success in a field of pure mathematics, the theory of numbers, and here precisely those functions whose representation by trigonometrical series was not examined by Dirichlet seem to be of the greatest importance. Toward the end of his treatise Dirichlet promised to return to these cases but this promise has remained unfilled ..."<sup>49</sup>

As a preliminary to his work with trigonometric series, Riemann proposed a more general concept of definite integral. "The uncertainty which still exists in a few fundamental points of the theory of the definite integral forces us to make some introductory statements regarding this concept ... Thus, what do we understand by  $\int_a^b f(x)dx$  ?"

Riemann answered this question as follows

---

<sup>49</sup>Riemann, op.cit., 224.

Let  $a, x_1, x_2, \dots$

$x_1 - a$  by  $\delta_1$

achter Bru

the sum

$S =$

will depend

the sum ...

then this v

contrary, t

defined." 50

Give

Riemann turn

"Let us show

cept: in wh

function no

Riem

integrabili

the sum  $S$  c

designate t

that is, th

interval, b

by  $D_n$ . The

Let  $a, x_1, x_2, \dots, x_{n-1}, b$  be a partition of the interval  $[a, b]$  and denote  $x_1 - a$  by  $\delta_1$ ,  $x_2 - x_1$  by  $\delta_2, \dots$ ,  $b - x_{n-1}$  by  $\delta_n$  and by  $\epsilon_n$  "einen positiven Mähter Bruch," i.e., a given position fraction. Then "the value of the sum

$$S = \delta_1 f(a + \epsilon_1 \delta_1) + \delta_2 f(x_1 + \epsilon_2 \delta_2) + \dots + \delta_n f(x_{n-1} + \epsilon_n \delta_n)$$

will depend on the choice of the interval  $\delta$  and the magnitude  $\epsilon$ . If the sum ... approaches a limit  $A$  when the  $\delta$ 's become infinitely small, then this value is called the definite integral  $\int_a^b f(x) dx$ . If, on the contrary, the sum does not have this property then the integral is not defined."<sup>50</sup>

Given this definition of the concept of definite integral, Riemann turned to the unresolved question of integrability of Dirichlet. "Let us shed light secondly as to the extent of validity of this concept: in which cases does a function admit and in which cases does a function not admit integration?"

Riemann established necessary and sufficient conditions for integrability by arguing in this way. "Let us ... first suppose that the sum  $S$  converges if all  $\delta$ 's become infinitely small. We will designate the greatest divergence of the function between  $a$  and  $x_1$ , that is, the difference between its maximal and minimal values in this interval, by  $D_1$ ; between  $x_1$  and  $x_2$  by  $D_2, \dots$ , and between  $x_{n-1}$  and  $b$  by  $D_n$ . Then

$$\delta_1 D_1 + \delta_2 D_2 + \dots + \delta_n D_n$$

---

<sup>50</sup>Riemann, op.cit., 225.

must become infinitely small with the magnitudes  $\delta$ . In addition, we assume that as long as all  $\delta$ 's remain smaller than  $d$ , the greatest value this sum may attain is  $\Delta$ ; therefore  $\Delta$  is a function of  $d$  which decreases with  $d$  and becomes infinitely small with this value. Now let the total length of the intervals in which the variation [of the function] is greater than  $\sigma$  be  $s$ . Then the sum  $\delta_1 D_1 + \delta_2 D_2 + \dots + \delta_n D_n$  is greater than or equal to  $\sigma s$ . Consequently

$\sigma s \leq \delta_1 D_1 + \delta_2 D_2 + \dots + \delta_n D_n \leq \Delta$ , and hence  $s \leq \frac{\Delta}{\sigma}$ . Now  $\frac{\Delta}{\sigma}$  can, if  $\sigma$  is given, be made arbitrarily small by a suitable choice of the magnitude  $d$ ; the same is true for  $s$ , and therefore we obtain the following:

"In order for the sums  $S$  to converge as the  $\delta$ 's become infinitely small, it is required, in addition to the finiteness of the function, that the total length of the intervals in which the amplitudes are greater than  $\sigma$ , no matter what  $\sigma$  may be, can be made arbitrarily small by a suitable choice of the magnitude  $d$ ."

Riemann also held the converse to be true. "If the function  $f(x)$  is always finite and if with infinitely decreasing magnitudes  $\delta$ , the total length  $s$  of all the intervals in which the amplitude of  $f(x)$  is greater than the given magnitude  $\sigma$  becomes infinitely small, then the sums  $S$  converge if all magnitudes  $\delta$  become infinitely small. For those intervals in which the variations of  $f(x)$  are greater than  $\sigma$  contribute less to the sum  $\delta_1 D_1 + \delta_2 D_2 + \dots + \delta_n D_n$  than  $s$  multiplied by the greatest variation in the function between  $a$  and  $b$ , which is finite; the other intervals contribute less than  $\delta(b-a)$ . Obviously we can

assume  $\sigma$  at first arbitrarily small, and then define the length of the interval so that  $s$  becomes arbitrarily small, by which the sum  $\delta_1 D_1 + \delta_2 D_2 + \dots + \delta_n D_n$  is given any chosen smallness, and consequently the value of the sum can be enclosed in arbitrarily chosen narrow limits.

"Thus we have found conditions which are necessary and sufficient for the sums  $S$  to converge with infinitely decreasing magnitudes  $\delta$ , and therefore, for the existence of the integral of a function  $f(x)$  from  $a$  to  $b$ ."<sup>51</sup>

In modern terms, a function  $f(x)$  is Riemann integrable on  $[a, b]$  if and only if the sum

$$\sum_{i=1}^n \delta_i D_i$$

tends to zero as the norm of the partition  $P$  tends to zero, where the  $\delta_i$  denote the lengths of the subintervals of  $[a, b]$  determined by  $P$ , and each  $D_i$  designates the oscillation of  $f(x)$  on the respective subinterval of length  $\delta_i$ .

We should observe that Riemann asserted necessary and sufficient conditions for integrability of a function on an interval without requiring that the function satisfy a condition with respect to continuity. Indeed, Riemann immediately gave an example of a discontinuous function not integrable in the sense of Cauchy but which admits a Riemann integral.

"Having examined the ... definite integral in general, i.e., without special assumptions as to the nature of the function to be integrated, we should apply this investigation in special cases ... and,

---

<sup>51</sup>Riemann, op.cit., 227.

at

bet

hav

spe

whe

the

int

for

to

and

If

poi

a d

Rie

osc

only

inte

tion

part

be a

at first, to functions which are infinitely often discontinuous between any two limits however close together. Since these functions have not been considered anywhere it will be good to start with a specific example." Riemann's example is the function

$$f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2},$$

where  $(nx)$  denotes the positive or negative difference between  $nx$  and the nearest integer, or is zero if  $nx$  is the midpoint of consecutive integers. This function converges for every  $x$ , and is discontinuous for every  $x$  of the form  $x = \frac{m}{2n}$  where  $m$  is an integer relatively prime to  $2n$ . This follows since, as Riemann wrote, if  $x = \frac{m}{2n}$ , then

$$f(x+0) = f(x) - \frac{1}{2n^2} \left(1 + \frac{1}{9} + \frac{1}{25} + \dots\right) = f(x) - \frac{\pi^2}{16n^2}$$

$$\text{and } f(x-0) = f(x) + \frac{1}{2n^2} \left(1 + \frac{1}{9} + \frac{1}{25} + \dots\right) = f(x) + \frac{\pi^2}{16n^2}.$$

If  $x$  is not of this form, then  $f(x+0) = f(x-0) = f(x)$ , i.e.,  $x$  is a point of continuity of  $f$ . Thus the points of discontinuity of  $f$  form a dense subset of the points on the real line. The function  $f$  is Riemann integrable on any finite interval  $[a, b]$ , however, since the oscillation of  $f$  at any point of discontinuity is  $\frac{\pi^2}{8n^2}$  and there are only finitely many values  $n$  such that  $\frac{\pi^2}{8n^2} > \delta$ . Thus, in the interval  $[a, b]$  there are only finitely many points at which the oscillation of  $f$  is greater than  $\delta$ , from which it follows that for a partition  $P$  of  $[a, b]$  of sufficiently small norm, the sum  $\sum \delta_i D_i$  will be arbitrarily small. Riemann's example is obviously not Cauchy

integrable; thus the class of Riemann integrable functions is more extensive than the class of Cauchy integrable functions, and encompasses the latter as a proper subset. Notice, however, that the Dirichlet function is not Riemann integrable.

Having completed his definition and discussion of a more general conception of definite integral, Riemann returned to the objective of his Habilitationsschrift, trigonometric series representation of functions. In introducing his "Research on the representability of a function by a trigonometrical series without special assumptions regarding the nature of the function," Riemann stated his intention in this way. "Up to now works on this subject have had the purpose of demonstrating the validity of Fourier series representation of functions which arise in applied work. Thus the proof could begin for an entirely arbitrarily assumed function and later, the course of the function could be subjected to . . . restrictions, provided these would not impair the first purpose, i.e., to demonstrate representability by means of Fourier series. The former works demonstrate that if a function has this and that property then it is describable by the series of Fourier. We will consider the converse question; if a function is represented by a trigonometrical series, then what follows out of this as to its course, that is, to changes in its values corresponding to changes in its argument?"<sup>52</sup>

There are several aspects of this work which are of particular interest.<sup>53</sup> First, Riemann investigated trigonometric series

---

<sup>52</sup>Riemann, op.cit., 230.

<sup>53</sup>See Hobson, The Theory of Functions of a Real Variable and the Theory of Fourier Series, vol. 2, §420-426 for statements and proofs of Riemann's theorems.

representation of functions in general rather than restricting his attention to the special case of Fourier series. That is, in considering the representation of a function  $f(x)$  in the form

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots,$$

Riemann made no a priori assumption regarding the coefficients  $a_n$  and  $b_n$ ; they might or might not be the Fourier coefficients of  $f$ . Riemann was first to study the general case, and demonstrated the distinction between the two types of series by giving several examples of trigonometric series which are not Fourier series.

Riemann utilized in a number of his proofs a new method of summing a trigonometric series. Thus, Riemann considered the sum of the series

$$7) \quad A_0 + A_1 + A_2 + \dots$$

where  $A_0 = \frac{a_0}{2}$ ,  $A_1 = a_1 \cos x + b_1 \sin x$ , ... , and  $\lim_{n \rightarrow \infty} A_n = 0$ , in the form

$$\lim_{h \rightarrow 0} \lim_{n \rightarrow \infty} \left[ A_0 + A_1 \left( \frac{\sin h}{n} \right)^2 + A_2 \left( \frac{\sin 2h}{2n} \right)^2 + \dots + A_n \left( \frac{\sin nh}{nh} \right)^2 \right]$$

This repeated limit is equal to the sum obtained at any point at which the series 7) converges in the usual manner, and may exist at points at which the series 7) fails to converge in the conventional sense. Riemann introduced, therefore, a more general conception of convergence of a trigonometric series.

Riemann was successful in obtaining necessary and sufficient

conditions such that a function be represented by a trigonometric series. In particular, Riemann proved that a function  $f(x)$  of period  $2\pi$  is represented (in the sense above) by a series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = 0$  if and only if

1) there exists a continuous function  $F(x)$  such that

$$\lim_{\alpha, \beta \rightarrow 0} \frac{F(x+\alpha+\beta) + F(x-\alpha-\beta) - F(x+\alpha-\beta) - F(x-\alpha+\beta)}{4\alpha\beta} = f(x)$$

and 2) for arbitrary constants  $b$  and  $c$ ,

$$\lim_{\mu \rightarrow \infty} \mu^2 \int_b^c F(x) \cos \mu(x-a) \lambda(x) dx = 0$$

where  $\lambda(x)$  is a function such that  $\lambda'(x)$  exists in  $(b, c)$  and vanishes at  $b$  and  $c$ , and  $\lambda''(x)$  has only a finite number of maxima and minima.

Of course this trigonometric series need not be a Fourier series.

Finally, Riemann ended his treatise by exhibiting a number of remarkable examples of functions with special properties. One of these is the function  $f(x)$  defined by

$$f(x) = \frac{d(x^v \cos \frac{1}{x})}{dx} \quad \text{where } 0 < v < \frac{1}{2}.$$

This function has infinitely many maxima and minima, is Riemann integrable, and yet, because the series does not converge, does not admit representation by a Fourier series. The last of Riemann's examples we will

consider is a function which is not integrable and yet is represented by a trigonometric series. The function is the series

$$f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n}$$

which exists for every rational  $x$  and is represented on this domain by the trigonometric series

$$\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sum_{\theta} (-1^{\theta})}{n} \sin 2n x,$$

where the summation with respect to  $\theta$  is over all divisors  $\theta$  of  $n$ .

The function  $f(x)$  is not integrable since it is unbounded on any interval, however small.

Riemann did not achieve his objective of finding necessary conditions for the convergence of a Fourier series. This fact, together with the many avenues opened but not explored may, as suggested above, have influenced Riemann to withhold publication of his Habilitationsschrift. Nevertheless, the work is a cornerstone, both with respect to Riemann's accomplishment and its implications for the subsequent development of real analysis. In addition to defining a more general integral and identifying a new field of investigation in the study of trigonometric series, Riemann initiated the creation of a theory of discontinuous functions. Dirichlet's definition of an everywhere discontinuous function had received little attention; Riemann's examples of integrable functions whose discontinuities are dense in every interval could not be ignored. Perhaps Dedekind envisaged the influence to be worked by "the high interest of the subject matter and . . . form of

treatment of . . . principles of the infinitesimal calculus . . ." of Riemann's treatise when he caused its publication. Riemann's work was an important contribution to the stream of ideas in which Dedekind had an intense interest, the study of the continuum of real numbers and the "arithmetization of analysis." As one aspect of this study, the development of a theory of measure of sets of real numbers had special significance for the evolution of integration theory, and it is this development to which we turn our attention in the next chapter.

"As has been said of the poet Coleridge, so it could be said of Riemann, he wrote but little, but that little should be bound in gold."<sup>54</sup>

---

<sup>54</sup>Van Vleck, "The influence of Fourier's series upon the development of mathematics," 116.

## Chapter 2

## On the Creation of a Theory of Measure

Heine, Cantor, Hankel and Smith

Many influences contributed to the creation of a theory of measure of sets of real numbers. As might be expected, the theory of Fourier series played a central role at the inception of this development.

Dirichlet had contended that the single requirement to be satisfied by a function in order that it admit representation by a Fourier series is that the function possess a definite integral, and had stated that the set of points of discontinuity of the function should be nowhere dense as a sufficient condition for integrability. In his "De explicatione per series trigonometricas...", published in 1864, Lipschitz attempted to extend Dirichlet's results with respect to Fourier series representation of functions. "The series which proceed according to the sines and cosines of multiples of an angle, by the aid of which one develops an arbitrary function of a single variable, have for a long time been used in different branches of mathematics by all those who are occupied with these questions. This is why it belongs to the mathematician to study in depth the extended field of these functions and to trace the limit between those which can be developed in trigonometric series and those which do not possess this property. The eminent Dirichlet has acquitted himself of this charge, without speaking of the efforts made before by others, in his celebrated memoir 'On the convergence of trigonometric series. . .'" Lipschitz noted that Dirichlet



had left consideration of the case of functions with infinitely many discontinuities and maxima and minima to a later paper and remarked, "As it is entirely regrettable for [our science] that this work has never seen the day, I propose to rediscover the traces of these researches..."<sup>1</sup>

Lipschitz argued that "the functions  $\phi(x)$  which do not satisfy [Dirichlet's] conditions can be distributed into the three following classes; in the interval  $(-\pi, \pi)$  they take infinite values, or possess an infinite number of discontinuities, or an infinite number of maxima and minima. It suffices to examine each of these three cases, so to speak, in itself, the combination of two or three of these cases modifying the form rather than the nature and the properties of the series [of Fourier]."

In considering "the second case, in which the function  $\phi(x)$  possesses in the finite interval  $(-\pi, \pi)$  an infinity of discontinuities," Lipschitz asserted "it is necessary that, if one designates by  $a$  and  $b$  two numbers placed between  $-\pi$  and  $\pi$ , one can always find between  $a$  and  $b$  numbers  $r$  and  $s$  such that the function remains finite and continuous on the interval  $(r, s)$ . One deduces from this, by an appropriate reasoning, that one can partition the interval  $(-\pi, \pi)$  into a finite number of partial intervals of which the two kinds are analogous to the two kinds of the first case." Lipschitz's "appropriate reasoning" was evidently

---

<sup>1</sup>Lipschitz, *Recherches sur le développement en series trigonométriques des fonctions arbitraires d'une variable et principalement de celles qui, in un intervalle fini, admettent une infinité de maxima et de minima* (Translation from the Latin by Paul Montel in Paris), 283.

intended to demonstrate that the nowhere dense set of points of discontinuity of  $\phi(x)$  admits only a finite number of limit points  $x_1, \dots, x_n$ . Hence Lipschitz delineated two kinds of partial intervals, those of a "first kind" which included the limit points  $x_1, \dots, x_n$ , and therefore all but a finite number of points of discontinuity of  $\phi(x)$ , and those of "the space remaining, . . . composed of a finite number of intervals of a second kind, in each of which the function  $\phi(x)$  satisfies the conditions imposed on the function  $f(\theta)$  in [Dirichlet's] theorem."

The integral of  $\phi(x)$  can be defined on the intervals of second kind as an improper integral in the Cauchy sense (Lipschitz was probably not aware of Riemann's generalization of Cauchy's conception of definite integral); given this Lipschitz defined the integral of  $\phi(x)$  in the form

$$\int_{-\pi}^{\pi} \phi(x) dx = \lim_{\alpha_1, \beta_1 \rightarrow 0} \sum_{i=1}^n \int_{x_{i-1} + \alpha_i}^{x_i - \beta_i} \phi(x) dx$$

where the  $(x_{i-1} + \alpha_i, x_i - \beta_i)$  are the intervals of second kind. Thus Lipschitz contended to have defined a definite integral for functions  $\phi(x)$  of this form, and therefore to satisfy Dirichlet's requirement of integrability in order that the functions  $\phi(x)$  admit representation by Fourier series.

Lipschitz's work is of interest here in that it called attention to the set of limit points of a nowhere dense set. Only later was it discovered that such a set might contain infinitely many elements.

Another influence which contributed to the recognition of more general point sets was the discovery of the importance of uniform convergence in the interchange of integration and summation of infinite

series. Dirichlet had shown that under certain conditions a function admits representation by a trigonometric series whose coefficients are the Fourier coefficients. The uniqueness of such representation was demonstrated by assuming that if the series

$$(1) \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

converges to zero for every  $x \in (-\pi, \pi)$ , then the coefficients  $a_n$  and  $b_n$  are identically zero for every  $n$ . This conclusion was obtained by multiplying the series by  $\cos nx$  or  $\sin nx$  and integrating term by term between  $-\pi$  and  $\pi$ . However, the validity of this argument is called into question if the convergence of the series is not uniform.

Heine considered the problem of uniqueness of representation of trigonometric series corresponding to functions in his "On trigonometrical series," published in 1870. "Until the most recent time it was believed that the integral of a convergent series, whose members remain finite between finite limits of integration, equals the sum of the individual members, and only Mr. Weierstrass has noticed the proof of this theorem requires not only that the series converges in the interval of integration, but that it converges uniformly. Thereby the theorem [that] a finite function given between  $-\pi$  and  $\pi$  can be developed in at most one way in a trigonometrical series ... has become untenable...."

Heine was successful in demonstrating the uniform convergence "in general" of the Fourier series of a function satisfying Dirichlet's conditions. He stated his work as follows: "the

Four

of ma

conve

clust

conve

ing 1

The e

the p

trigon

tion 1

in ger

an equ

unifor

ist v

possib

series

was su

Fourier series of a finite function  $f(x)$ , which has a finite number of maxima and minima, that is, those series of the form (1) in which

$$\pi a_m = \int_{-\pi}^{\pi} f(x) \cos mx \, dx \qquad \pi b_m = \int_{-\pi}^{\pi} f(x) \sin mx \, dx$$

converges uniformly whenever  $f(x)$  is continuous between  $-\pi$  and  $\pi$  inclusive, and  $f(\pi) = f(-\pi)$ ; in all other cases it is only uniformly convergent in general.

"The expression 'in general' will be used here and in the following if the exception is supposed to refer to a finite number of points. The exceptional places in the above theorem . . . are neighborhoods of the points of discontinuity . . . ." <sup>2</sup>

Heine returned to the problem of uniqueness of representation of trigonometric series in his second theorem. He proved that if a function is represented by a trigonometric series which converges uniformly in general, then this series is unique. He expressed this theorem in an equivalent form. "If a trigonometric series converges to zero uniformly in general from  $-\pi$  to  $\pi$ , . . . then all coefficients  $a$  and  $b$  must vanish, and the series therefore represents zero everywhere." The possibility of demonstrating this theorem for the case in which the series does not represent the function at a finite number of points was suggested to Heine by Cantor, one of Heine's colleagues at the

---

<sup>2</sup> Heine, Über trigonometrische Reihen, 3.

Unive

I had

in wh

mande

sente

be re

prope

might

every

metri

ficul

repre

subje

the p

so d

but

theo

onoma

theo

$t_1$ ,  $t$

give

incre

Canto

University of Halle. "I was encouraged by Mr. Cantor in Halle, whom I made acquainted with my research, to extend [the theorem] to the case in which coincidence in the points of discontinuity is no longer demanded...."

Of course it was still possible that a function uniquely represented by an in general uniformly convergent trigonometric series might be represented by another trigonometric series not possessing this property. That is, with reference to Heine's second theorem, there might exist a non-uniformly convergent trigonometric series converging everywhere to zero. Thus the uniqueness of representation of trigonometric series still could not be inferred. Heine recognized this difficulty and commented upon it. "...it is not yet known if a series which represents a continuous function must converge uniformly.... This subject is not enlightened even in the following...."

Stimulated by Heine's research, Cantor continued the study of the problem of uniqueness of representation by trigonometric series. In so doing, Cantor not only completely solved the uniqueness problem, but initiated the study which eventually led to his creation of the theory of sets. Cantor's first paper, "On a theorem concerning trigonometrical series," and dated March 20, 1870, contained a proof of the theorem "If two infinite sequences of constants  $a_1, a_2, a_3, \dots$  and  $b_1, b_2, b_3, \dots$  are of such a nature that for every value of  $x$  in a given interval  $(a, b)$  the limit of  $(a_n \cos nx + b_n \sin nx)$  is zero with increasing  $n$ , then  $a_n$  and  $b_n$  converge with increasing  $n$  to limit zero." Cantor remarked "if this theorem is applied to trigonometrical series

the

+ a

into

inf

of

uni

In

eve

in

"If

ser

whi

if

eve

the

va.

fr

a

ot

Let

then it yields the result that if the series  $\frac{b_0}{2} + a_1 \sin x + b_1 \cos x + \dots + a_n \sin nx + b_n \cos nx + \dots$  is convergent for each value of  $x$  in an interval  $(a < x < b)$  of the real line, then the coefficients  $a_n$  and  $b_n$  become infinitely small with increasing  $n$ ,"<sup>3</sup> and this without any requirement of uniform convergence.

Having demonstrated this, Cantor turned his attention to the uniqueness of representation of a function by a trigonometric series. In a memoir titled "The proof that a function  $f(x)$  which is given for every real value of  $x$  by a trigonometrical series can be represented in this form in one way only," and dated April 6, 1870, Cantor wrote, "If a function  $f(x)$  of a real variable  $x$  is given by a trigonometrical series

$$f(x) = \frac{b_0}{2} + (a_1 \sin x + b_1 \cos x) + \dots + (a_n \sin nx + b_n \cos nx) + \dots,$$

which is convergent for every value of  $x$ , then it is important to know if there are other series of the same form which also converge for every value of  $x$  and which represent the function  $f(x)$ ." Cantor showed that if  $\frac{b_0}{2} + (a_1 \sin x + b_1 \cos x) + \dots$  converges to zero for every value of  $x$  in an interval, then for every  $n$ ,  $a_n = b_n = 0$ . "It follows from this that ... if a function  $f(x)$  of a real variable  $x$  is given by a convergent trigonometric series for each value of  $x$ , then there is no other series of the same form which converges for each value of  $x$  and

---

<sup>3</sup>Cantor, "Ueber einen die trigonometrischen Reihen betreffenden Lehrsatz," 130.

re

in

se

un

ta

th

re

fo

ge

fe

se

of

th

th

.

w

c

e

e

f

F

l

represents the given function  $f(x)$ ." <sup>4</sup> Thus Cantor succeeded in establishing the uniqueness of representation of a function by a trigonometric series on an interval without the requirement that the series converge uniformly in general.

Next Cantor considered the possibility of uniqueness of representation of a function by a trigonometric series which fails to represent the function on a finite or infinite subset of its domain. Cantor soon realized that the requirement of convergence of the trigonometric series for all values of the independent variable in an interval is too stringent and could be remitted for some points of the interval without affecting the validity of his conclusion regarding uniqueness of representation. In a paper "On the extension of a theorem from the theory of trigonometrical series," dated November 8, 1871, Cantor wrote, "In the following I want to treat a certain generalization of the theorem that trigonometrical series representations are unique.

"I have attempted to prove that two trigonometrical series

$$\frac{b_0}{2} + \dots + (a_n \sin nx + b_n \cos nx) + \dots \text{ and } \frac{b'_0}{2} + \dots + (a'_n \sin nx + b'_n \cos nx) + \dots$$

which converge to the same sum for every value of  $x$  correspond in their coefficients ..., and in a note which refers to this work <sup>5</sup> I have ...

---

<sup>4</sup>Cantor, "Beweis, dass eine für jeden reellen Werth von  $x$  durch eine trigonometrische Reihe gegebene Function  $f(x)$  sich nur auf eine einzige Weise in dieser Form darstellen lässt," 142.

<sup>5</sup>The note referred to, "Notiz zudem Aufsatz: Beweis, dass eine für jeden reellen Werth von  $x$  durch eine trigonometrische Reihe gegebene Function  $f(x)$  sich nur auf eine einzige Weise in dieser Form darstellen lässt," was occasioned in part by Kronecker's suggestion of a simplified

den

num

ser

inf

of

of

con

lin

sus

fi

re

to

po

wi

no

pr

di

li

de

demonstrated that this theorem remains valid if one remits for a finite number of values of  $x$  either the convergence or the coincidence of the series sums.

"The extension intended here consists in the fact that for an infinite number of values of  $x$  in the interval  $(0, 2\pi)$ , the convergence of the sums of the series is renounced without invalidating the truth of the theorem.<sup>6</sup>

"For this purpose I feel compelled to mention a few preliminary considerations ..., " two of which were definitions of the concepts of limit point and derived set.

"By a limit point of a point set  $P$  I mean a point on the line in such a position that in any neighborhood of this point there exist infinitely many points of  $P$  .... By the term neighborhood of a point I mean any interval which has the point in its interior.... It is easy to prove that a (bounded) point set consisting of an infinite number of points has at least one limit point.

"It is now a definite relationship of any point of the line with respect to a given point set  $P$  either to be a limit point of  $P$  or not to be any such limit point. Therefore with the point set  $P$  is given

---

proof of Cantor's uniqueness of representation theorem, a proof which did not, as had Cantor's original proof, appeal to the theorem established in Cantor's first paper.

<sup>6</sup>Cantor, "Ueber die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen," 123.

the

the

de

sh

of

(a

the

con

tit

res

pro

to

cer

set

ser

wor

adm

the

the set of its limit points, which I shall designate by  $P'$  and call the first derived set of  $P$ ." <sup>7</sup> Cantor defined second, third, and so on derived sets  $P''$ ,  $P'''$ , ... in the obvious manner.

Then, returning to the study of trigonometric series, Cantor showed that the requirement of convergence can be remitted for any set of points whose  $n$ th derived set contains only finitely many points (and therefore whose  $(n+1)$ st derived set is empty) without invalidating the conclusion of uniqueness of representation.

These early papers of Cantor demonstrate an initial stage in his construction of a theory of sets. In a memoir published in 1879 and titled "On linear and infinite sets of points," Cantor remarked with respect to sets of points on the real line "... it does not seem improper that we study and seek to classify them; this is what we propose to do here ... we are lead to classify the sets of linear points into certain catagories. To begin ... we recall the notion of the derived set of a given set of points  $P$  as is given in a work on trigonometrical series...." <sup>8</sup>

Thus the work of Heine and Cantor was an important means by which more general point sets were called to the attention of mathematicians.

The conditions to be satisfied by a function in order that it admit Riemann integration were another impetus to the development of a theory of measure. Riemann had stated

$$\lim_{||P|| \rightarrow 0} \sum_{i=1}^n \delta_i D_i = 0$$

---

<sup>7</sup>Cantor, op. cit., 129.

<sup>8</sup>Cantor, "Ueber unendliche, lineare Punktmannichfaltigkeiten," 1.

as

a

la

Ri

of

co

".

in

be

cu

po

th

a

co

re

Ma

se

re

si

ar

ti

la

pr

as a necessary and sufficient condition for Riemann integrability of a bounded function; Hankel, in his "Research on infinitely often oscillating and discontinuous functions," published in 1870, expressed Riemann's conditions in a form which suggests the concept of measure of the set of points of discontinuity of a function in an interval.

Given the prevailing mood of indifference to the study of discontinuous functions, Hankel began by attempting to justify his work. "... the question of the existence of a differential quotient [of an integrable function] presents great difficulties. This has only rarely been discussed because the existence of a tangent at each point of a curve has been considered as certain from an immediate geometric viewpoint and one has taken this for granted as a self-evidence result of the *lex continuitatis* [law of continuity] which has been respected as a necessity in the field of mathematics. However, even if this law of continuity in fact governs all movements in nature, yet it should restrict the discipline of pure mathematics in no way....

"... I owe the stimulation to these studies essentially to Riemann's writings, especially to his excellent work on trigonometrical series, after whose publication there is no need to excuse research regarding these problems....If my essay fails in part over such a slippery and rarely trodden path, and if the results here and there are somewhat lacking ..., then I hope to find some benevolent consideration, for it was my only intention in publishing these ideas to stimulate other scholars, and to draw their interest to these fundamental problems of our science, which modern function theory does not permit

us to reject any longer."<sup>9</sup>

Hankel defined the terms discontinuity, jump and oscillation of a function at a point. "One says that a function  $f(x)$  is discontinuous at  $x=a$  when, among the values taken by the difference  $f(a+\delta) - f(a)$ , for all positive and negative  $\delta$  that are less than  $\epsilon$ , there are always values greater than a given finite quantity  $\sigma$ , no matter how small one chooses  $\epsilon$ . I will say such a function makes a jump at the point  $x=a$  which is greater than  $\sigma$  ... and will mean by the oscillation of a function in a given interval ... the difference between the largest and smallest values which the function takes in this interval."<sup>10</sup>

Hankel then classified functions into two categories, simply continuous functions and linearly discontinuous functions. "The simply continuous functions make jumps in individual points, which are allowed to exist only in a finite number on a given line .... Linear discontinuous functions are functions which are discontinuous at infinitely many points on a finite segment of a line."

Having observed that "the jumps which [a linearly discontinuous] function makes at these infinitely many points ... may show a very different nature since either the number of points at which jumps occur which are larger than a fixed magnitude  $\sigma$  is infinite, or whose number increases infinitely only if  $\sigma$  decreases infinitely," Hankel proposed to demonstrate "the great variety of particularities which the linear

---

<sup>9</sup>Hankel, "Untersuchungen über die unendlich oft oscillirenden und unstetigen Functionen," 70.

<sup>10</sup>Hankel, op. cit., 84.

discontinuous functions may show ... by a number of examples in which we let the variable  $x$  run through the interval from  $x=0$  to  $x=1$ ."

Hankel's first example is the Dirichlet function. Two of his other examples are "Define the function  $f(x)$  to have the value  $f(x) = 1$  for  $x = 0$  to  $x = 1$  with the exception of infinitely many intervals of length  $\delta^n$  which have their centers in the points  $x = (\frac{1}{2})^n$ ; in each of these intervals let the function ... have value zero for all rational values and value one for all irrational values of the argument. The total length of the intervals in which the functional oscillation is everywhere one is

$$s = \delta + \delta^2 + \delta^3 + \dots = \frac{\delta}{1-\delta} \dots "$$

Again, "Define the function  $f(x)$  to have the value  $f(x) = 1$  for  $x = 1$  to  $x > \frac{1}{2}$ , value  $f(x) = \frac{1}{2}$  for  $x = \frac{1}{2}$  to  $x > \frac{1}{4}$ , and generally the value  $f(x) = (\frac{1}{2})^n$  for  $x = (\frac{1}{2})^n$  to  $x > (\frac{1}{2})^{n+1}$ . This function, which is discontinuous only at the points  $x = (\frac{1}{2})^n$  and at these points jumps by  $(\frac{1}{2})^n$ , does not have an infinite number of points at which the jumps surpass a fixed finite magnitude as do the examples we have just considered. To be sure, the number of points at which the jumps are greater than  $\sigma$  is finite; with decreasing  $\sigma$  however their numbers increase constantly and without limit. The total length of the intervals in which jumps greater than  $\sigma$  occur obviously can be made arbitrarily small for every  $\sigma$  since these have to surround only the finitely many points of discontinuity."

These examples suggested classes of points on the line. "If a



set of points on a line has a certain property [i.e., at each point of the set the jump of the function is greater than  $\sigma$ ] then I will say that these points fill up the line segment if in this segment, no interval, no matter how small, can be given which does not contain at least one point from this set; on the other hand, this set of points does not fill the line segment, but that the points are scattered in it, if between any two arbitrarily close points on the line segment, there can be given an interval which contains no point of this set."<sup>11</sup> Thus the set of points of discontinuity of a function at which the jump is greater than  $\sigma$  either "fills up" or is "scattered" on the line providing it is dense or nowhere dense, respectively, on an interval.

Hankel believed that the length  $s$  of the intervals in which the oscillation of a linear discontinuous function  $f(x)$  is greater than  $\sigma$  can be made arbitrarily small if and only if the points of discontinuity of  $f$  are nowhere dense.

This belief led Hankel to classify linear discontinuous functions into two types: 1) those that are totally discontinuous, i.e., whose points of discontinuity at which the jumps greater than  $\sigma$  are dense in an interval, and 2) functions which are pointwise discontinuous, that is, are such that for each  $\sigma > 0$ , the set of points at which the jump of the function is greater than  $\sigma$  is nowhere dense.

Finally, Hankel concluded that this classification separated the integrable and non-integrable functions. Thus, Hankel believed that a function is Riemann integrable if and only if it is a simply continuous or pointwise discontinuous function.

---

<sup>11</sup>Hankel, op. cit., 87.

Hankel's conclusion is only partially correct as will be shown by an example below. Nevertheless, his work was a significant contribution to an emerging theory of the mathematically discontinuous. Originally published as part of the Gratulationsprogramm of Tübinger University on March 6, 1870, Hankel's paper was published again in the Mathematische Annalen in 1882. The editor of the Annals remarked in a footnote, "...the work of Herman Hankel...has up to now been difficult to find due to its place of publication; since this paper is mentioned in almost all modern investigations of the concept of functions...we have made the transcription [of the Gratulationsprogramm] literal, and thus have considered as negligible any incorrect points which the work may contain."

In 1875 an English mathematician, H. J. S. Smith, published an example of a linear discontinuous function which is not Riemann integrable. He gave his example as follows: "Let  $m$  be any given integer greater than 2...now...divide the interval from 0 to 1 into  $m$  equal parts, exempting the last segment from any further division; let us divide each of the remaining  $m-1$  segments by  $m^2$ , exempting the last segment of each segment; let us again divide each of the remaining  $(m-1)(m^2-1)$  segments by  $m^3$ , exempting the last segment of each segment; and so on continually. After  $k-1$  operations we shall have

$$N = 1 + (m-1) + (m-1)(m^2-1) + \dots + (m-1)(m^2-1) \dots (m^{k-2}-1)$$

exempted segments, of which the sum will be

$$1 - (1 - \frac{1}{m}) (1 - \frac{1}{m^2}) \dots (1 - \frac{1}{m^{k-1}})$$

This sum, when  $k$  is increased without limit, approximates to the finite limit  $1 - E\left(\frac{1}{m}\right)$  where  $E\left(\frac{1}{n}\right)$  is the Eulerian product  $\prod_{k=1}^{\infty} \left(1 - \frac{1}{m^k}\right)$ , and is certainly different from zero."<sup>12</sup>

Smith went on to show that the points of division  $Q$  are nowhere dense on  $[0,1]$  and yet "a function having finite discontinuities at the points  $Q$  would be incapable of integration." Thus Smith gave the first example of a nowhere dense set with positive outer content.

Smith also expressed Riemann's integrability conditions in a form which strongly suggests the notion of "length" of the set of points of discontinuity of an integrable function. "Let  $\sigma$  be any given quantity, however small; if, in every [partition] of norm  $d$ , the sum [of lengths] of the segments, of which the ordinate differences surpass  $\sigma$ , diminishes without limit, as  $d$  diminishes without limit, the function admits of integration; and vice versa, if the function admits of integration, the sum [of lengths] of these segments diminishes without limit with  $d$ ."<sup>13</sup> The ordinate difference of a segment had been previously defined to be the difference between the greatest and least ordinates of  $f(x)$  on the segment.

Smith had undertaken this work in order to further discuss Riemann's theorem stating necessary and sufficient conditions for integrability of a function "partly because, in one particular at least, Riemann's demonstration is wanting in formal accuracy, and partly because the theorem ... appears ... to have been made the basis of erroneous inferences." Smith considered the sufficiency part of

---

<sup>12</sup>Smith, "On the integration of Discontinuous Functions," 148.

<sup>13</sup>Smith, op. cit., 142.



Riemann's proof to be incomplete; the latter remark refers of course to Hankel's work.

Thus the way was prepared to lead to the definition of a generalized conception of length, that is, a notion which would "measure" general point sets such as the set of points of discontinuity of a function, and reduce to the familiar concept of length when used to "measure" an interval.

#### Cantor, Harnack and Stolz

The first definitions of the measure of arbitrary point sets were given by Cantor, Harnack and Stolz. These measures were defined as a limiting form of a finite covering of the given set by elementary figures whose measures were known. The adherence to finite covers was undoubtedly due to the relationship between such covers and the summation, given a partition  $P$  of the interval of integration of a Riemann integrable function  $f$ , of the lengths of the (finitely many) intervals of  $P$  in which the oscillation of  $f$  is greater than  $\sigma > 0$ .

Cantor, in the fifth installment of his "On infinite linear point sets," published in 1884, defined the content of a bounded set  $P$  of points in  $n$ -dimensional space  $G_n$ . "If there is given a [bounded] point set  $P$  of  $G_n$ , then form around each point  $p$  of the ... set  $P + P'$  [the addition symbol denotes set union and  $P'$  designates the set of limit points of  $P$ ] an  $n$ -dimensional ball with center  $p$  and radius  $\rho$ , which with all its interior and boundary points will be designated by  $K(p, \rho)$ .

"The ... full balls which are obtained by letting  $p$  run through

all points of  $P + P'$  have one least common multiple [set union]

$$\sum_p K(p, \rho) ,$$

which point set is designated according to circumstances by  $\Pi(\rho, P \text{ in } G_n)$ , or more simply by  $\Pi(\rho, P)$  or  $\Pi(\rho)$ .

"Now this pointset  $\Pi(\rho)$  of  $G_n$  always consists of a finite number of regions, since  $P$  is assumed bounded, each of which is an  $n$ -dimensional continuum with its boundary. Consequently the  $n$ -times integral

$$\int dx_1 dx_2 \dots dx_n ,$$

taken over all regions of  $\Pi(\rho)$ , has a specific value which depends upon  $\rho$ ; we call this value  $F(\rho)$ ...

"... we define the content or volume of the set  $P$  [denoted  $I(P)$  after Inhalt] to be the limit value  $\lim_{\rho \rightarrow 0} F(\rho)$ ... and obtain therefore

$$I(P) = \lim_{\rho \rightarrow 0} F(\rho).^{14}$$

Cantor observed an additive property of the set function  $I(P)$ ; if  $P$  and  $Q$  are (bounded) sets which are "completely separated" then  $I(P+Q) = I(P) + I(Q)$ .

Cantor remarked further with respect to his definition of content

---

<sup>14</sup>Cantor, "Ueber unendliche, lineare Punktmannichfaltigkeiten," 474.

in a letter addressed to the editor of Acta Mathematica in November, 1883. "I wish to explicitly state that this ... volume or extent of an arbitrary set  $P$  in a ... space  $G_n$  of  $n$ -dimensions is absolutely dependent upon  $G_n$  ..." Thus "a square each of whose sides is equal to one, has its extent equal to zero when it is considered as a constituent part of a space of three dimensions, but it has extent equal to one when it is regarded as part of a plane of two dimensions. This general notion of volume or extent is indispensable to me in my researches on the dimensions of continuous sets ..."

Harnack, in 1883, in his paper "On the content of point sets" wrote, "I am going to develop in a series of explanations and theorems the general definition which I have given for discrete point sets within a closed linear interval. These theorems partially supplement the theorems which Mr. Cantor has in the meantime published on the same subject on the basis of a somewhat different definition ..."

"A point set within a linear interval of finite length is called discrete (with content zero) if all points [of the set] can be included in a finite number of intervals whose sum [of lengths] can be made arbitrarily small even if thereby the number of intervals may grow beyond any limit. The latter always occurs if one has to do with an infinite number of points."

Harnack defined the content of an arbitrary subset of an interval of length  $\ell$  in the following way: "If the point set is not everywhere dense in the entire interval (thereby the limit would be the magnitude  $\ell$ )

then one should fix a length ...  $\frac{\ell}{2}$  and construct the intervals [whose lengths] are greater than or equal to  $\frac{\ell}{2}$ , and which do not contain a point of the set in their interior. If such an interval exists and one has [deleted] it, then one should exclude from the residue parts of the interval those [whose lengths] are greater than or equal to  $\frac{\ell}{3}$  and contains no point of the set in their interior. One recognizes generally that there is always only a finite number of intervals [whose lengths] are  $\geq \frac{\ell}{n}$  and contain no point of the given set in their interior whenever  $n$  is any positive integer. The total length of the intervals excluded in this fashion which are  $\geq \frac{\ell}{n}$  may be [designated] by  $N$ . Then, except for the finite number of points which coincide perhaps with the endpoints of two adjoining excluded intervals and are isolated points, the points of the set lie in the interior or as endpoints of a finite number of intervals whose total length is  $\ell - N$ ... One needs only to arbitrarily diminish [the lengths of] the intervals which are taken out... The limit value of  $\ell - N$  for  $n = \infty$  is the limit for which one is looking. The point set is consequently discrete if  $\lim N = \ell$ .<sup>15</sup>

Harnack remarked that the union of a finite number of discrete sets is discrete, and continued "to the contrary the theorem is no longer valid if one has an infinite sequence of discrete sets, as is taught by the example of the sequence of rational numbers from 0 to 1, which can be composed by the [infinite sequence of discrete sets]

$$P_1 = \{\frac{1}{2}\}, P_2 = \{\frac{1}{3}, \frac{2}{3}\}, P_3 = \{\frac{1}{4}, \frac{3}{4}\}, P_4 = \{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}, \dots . "$$

Harnack considered a countable as opposed to a finite cover in a passage which is remarkable in its anticipation of Borel. "... in a certain sense any point set which can be counted has the property that

---

<sup>15</sup>Harnack, "Ueber den Inhalt von Punktmengen," 241.

that all its points may be included in intervals whose sum [of lengths] is arbitrarily small. Thus, for example, although they are everywhere dense on the unit interval, one is able to surround all the rational numbers between 0 and 1 with intervals whose sum [of lengths] is arbitrarily small. For, if one has a countable point set  $a_1, a_2, \dots$  then one surrounds the points with intervals of lengths  $\epsilon_1, \epsilon_2, \dots$  and chooses these magnitudes in such a way that  $\epsilon_1 + \epsilon_2 + \dots$  is less than an arbitrarily small magnitude  $\delta$ . If one carries out this process with the above mentioned rational numbers... then... the points not covered by these intervals... constitute a point set whose content is greater than  $1 - \delta$ .<sup>16</sup>

Harnack did not attempt to develop this observation; the notion of an everywhere dense set with arbitrarily small content must have seemed paradoxical. Indeed he believed to have obtained a "remarkable paradox" by consideration of a countable covering of intervals. "... if from an... interval of length  $a$ , one deletes an infinite number of subintervals  $a_1, a_2, \dots$  whose sum of lengths  $b$  is less than  $a$ , then there will remain infinitely many subintervals whose sum of lengths is never greater than  $b - a$ ; it may, however, be even smaller."<sup>17</sup> Harnack conceived of the latter occurrence in the case in which the endpoints of the intervals and their limit points do not form a discrete set. The difficulty in such a case is that content is not additive.

In his "On a limit value corresponding to an infinite point set," dated July, 1883, Stolz defined the measure of a point set in the following way: Let  $x'$  denote an arbitrary set of points in a finite

---

<sup>16</sup>Harnack, op. cit., 243.

<sup>17</sup>Harnack, op. cit., 244.



interval  $(a,b)$ , and consider a sequence of partitions  $t_1, t_2, t_3, \dots$  of  $(a,b)$  such that the norm  $||t_n||$  of  $t_n$  satisfies the inequality  $||t_n|| < \delta_n$  where  $\lim_{n \rightarrow \infty} \delta_n = 0$ . "Now if one carries out within the interval  $(a,b)$  a system of infinitely many such partitions  $t_1, t_2, t_3, \dots$ , and if for each partition one adds the lengths of those intervals containing points of the given set  $x'$ , then one obtains a sequence of sums  $S_1, S_2, S_3, \dots$  such that  $S_1 \geq S_2 \geq S_3 \geq \dots$ . Thus there exists a finite limit  $\lim_{n \rightarrow \infty} S_n = L$  where  $L \geq 0$ . The...number  $L$  is independent of the considered system of partitions  $t_m$  so that to each point set  $x'$  in the interval  $(a,b)$  there corresponds a unique limit  $L \geq 0$  which is called the interval limit [that is, content of the given set]."<sup>18</sup>

Later in his paper Stolz brought his work into relationship with Harnack's. "The point sets for which the limit value  $L=0$  coincide with the sets which have been called discrete by Mr. Harnack. According to his definition the points of a discrete set may be included in a finite number of intervals whose sum  $S$  is smaller than  $\epsilon$ . Consequently  $L < \epsilon$ , i.e.,  $L=0$ ."<sup>19</sup>

Other definitions equivalent to those of Harnack and Stolz were proposed. Pasch, in his article "On some topics in the theory of functions," dated March, 1887, first defined a function  $s[(x,z)]$  to be 1 if the interval  $(x,z)$  contains a point of a subset  $E$  of an interval  $(a,b)$ , and 0 otherwise. Then he continued "We partition the interval  $(a,b)$  in an arbitrary manner into a finite number of subintervals  $(a, a_1), (a_1, a_2), \dots, (a_{n-1}, b)$  and form the sum

---

<sup>18</sup>Stolz, "Ueber einen zu einer unendlichen Punktmenge gehörigen Grenzwert," 152.

<sup>19</sup>Stolz, op. cit., 154.

$$S = (a_1 - a)s[(a, a_1)] + (a_2 - a_1)s[(a_1, a_2)] + \dots (b - a_{n-1})s[(a_{n-1}, b)] \leq b - a,$$

that is, the sum of the lengths of subintervals which contain points of  $E$ . The lower limit [i.e., infimum] of all values  $S$  is a finite number  $\Sigma \geq 0$ ; it, so to say, represents the length covered by the point set  $E$  and is to be designated as the content of the set since, as will become clear from the following, it corresponds to the definition given by Mr. Cantor..."<sup>20</sup>

Pasch later commented, "If one calls  $\Sigma$  the extent of the point set then one can designate the point set as being extended if  $\Sigma > 0$  and as being unextended if  $\Sigma = 0$ . The first attempt to classify point sets in such a way for the purpose of the theory of functions can be found with Mr. Hankel. Mr. Harnack calls the point set linear if  $\Sigma > 0$  and discrete if  $\Sigma = 0$ ...deviating from the meaning of linear point set according to Mr. Cantor.... Mr. du Bois-Reymond has called the point set integrable if  $\Sigma = 0$ ...." These remarks give evidence of the influence of the new concept of measure upon the theory of functions.

The papers of Cantor, Harnack and Stolz were an important first step toward the creation of a theory of measure. The notion of content failed to satisfy one of the first expectations of a measure, however, that of additivity. Thus a generalized conception of length might be expected to possess the property that the measure of the union of disjoint sets be the sum of the measures of the sets. This expectation was fulfilled for finite collections of disjoint sets several years later in the definitions of measure proposed by Peano and Jordan.

---

<sup>20</sup>Pasch, "Ueber einige Punkte der Functionentheorie," 142.

# Peano and Jordan

Peano was led to develop a theory of measure by his attempt to find a simple condition for Riemann integrability of a function. At the time he published his paper "On the integrability of functions," in 1883, the statement of existence of the Riemann integral was the following: a function  $f$  is Riemann integrable on  $[a,b]$  if and only if for arbitrary positive  $\sigma$  and  $\delta$ , the content (in the sense of Stolz or Harnack) of the set of points in  $[a,b]$  at which the oscillation of  $f$  is greater than  $\sigma$  is less than  $\delta$ . Peano sought to express the integrability of  $f$  in terms of the measurability of the ordinate set of  $f$  on  $[a,b]$ , that is, the set  $\bigcup_{x_0 \in [a,b]} \{(x_0, y) \mid 0 \leq y \leq f(x_0)\}$  if  $f$  is non negative.

"The existence of the integral of functions of a single variable is not always demonstrated with the rigor and simplicity desirable in such questions. The method of reasoning of principal writers by recourse to geometric considerations is not satisfactory. The analytical demonstrations are generally long and complicated and conditions are introduced which are too restrictive or partly useless. In the present study I propose to demonstrate the existence of the integral by introducing a very simple condition of integrability. The reasoning will be analytical but can be interpreted geometrically in any of its parts."<sup>21</sup>

Peano began by stating Riemann's definition of integral: let  $f(x)$  be a function defined on  $[a,b]$  and be bounded above and below by

---

<sup>21</sup>Peano, "Sulla integrabilità delle funzioni," 439.

A and B respectively. Partition the interval  $[a,b]$  into subintervals  $h_1, h_2, \dots, h_n$  and let  $y_s = f(x)$  where  $x$  is an arbitrary element of  $h_s$ . Then if the sum  $u = h_1 y_1 + h_2 y_2 + \dots + h_n y_n$ , as  $h$  is "infinitely diminished," tends to a limit  $S$ , then the function  $f$  is integrable on  $[a,b]$  and conversely, and its integral  $\int_a^b f(x)dx$  on  $[a,b]$  is  $S$ .

Peano continued in this way. Let  $p_s$  and  $g_s$  be supremum (limiti superiore) and infimum (limiti inferiore), respectively, of the  $y_s$  on the subinterval  $h_s$ . Let  $P = \sum h_s p_s$  and  $Q = \sum h_s g_s$ ; then

$$A(b-a) \geq P \geq u \geq Q \geq B(b-a).$$

Hence the numbers  $P$  corresponding to all partitions of  $[a,b]$  admit a greatest lower bound  $M$ , and the numbers  $Q$  corresponding to all partitions of  $[a,b]$  admit a least upper bound  $N$ . Furthermore, Peano showed  $P \geq M \geq N \geq Q$ . Finally, if  $f(x)$  is an integrable function on  $[a,b]$ , then  $M = N = S$ . Thus "if the function  $f(x)$  is integrable then the quantities  $M$  and  $N$  are equal and their common value is equal to the value of the integral."<sup>22</sup>

---

<sup>22</sup>Peano, op. cit., 441. It is of interest to note that Darboux had asserted, without proof, the necessity and sufficiency of this condition for the existence of the Riemann integral in 1875. Darboux's purpose was to investigate several of Hankel's propositions which had been criticized by Gilbert, Schwarz, Klein and others. "I have imposed upon myself the duty of going back to several [of Hankel's assertions] and...to express them in a form such that they be sheltered from all criticism..." In particular, Darboux was interested in identifying a class of continuous functions which are not differentiable for infinitely many values of the independent variable. Thus Darboux did not develop the consequences of his definition of the Riemann integral with respect to the concept of area. See Darboux, "Memoire sus les fonctions discontinues," 72.



Peano also proved the converse of this theorem in the form:  
if for an arbitrary positive  $\epsilon$  there exists a partition of  $[a,b]$  with the property that the corresponding numbers  $P$  and  $Q$  are such that  $P-Q < \epsilon$ , then  $f(x)$  is an integrable function on  $[a,b]$ .

Thus Peano obtained a necessary and sufficient condition for the existence of the Riemann integral of a function on an interval. This condition conveys an immediate geometric intuition, the area of the ordinate set of  $f$ , and Peano turned his attention to this aspect of the work in his concluding remarks. "Many authors demonstrate the existence of the integral by means of geometric consideration, but... the reasonings are not at all adequate. In truth it is customary to consider in such a case the area of a figure without defining it... our minds conceive clearly, or believe to conceive clearly [notions such as area or] the length of the arc of a curve, and so on, but these need to be carefully defined before being introduced into analysis; and among these more importantly for area, because one is accustomed in elementary treatises to base other demonstrations upon this concept.

"Now if one takes a figure of simple form, the most natural method to conceive of its area is to imagine polygons which enclose within their interior the given figure, and polygons contained in the interior of the given figure. The areas of the first admit an inferior limit, and the areas of the second a superior limit. If these limits coincide their common value is the area of the given figure, a well defined quantity which can be calculated to any desired [degree of accuracy]; if instead these two limits are not equal, then the concept of area is excluded... Thus, in order to speak of the area of a figure it is necessary first to verify the equality of these two limits, which

is none other than the preceding condition of integrability."<sup>23</sup>

Peano developed these remarks in detail in his book Applications of Geometry to Infinitesimal Calculus, published in 1887. He began the section titled Area Plane by stating definitions of interior, exterior and boundary point of a set. "Let us say that a point P is internal to a plane field [i.e., set] A if it is possible to determine a length  $\rho$  in such a way that all the points of the plane which are separated from P by less than  $\rho$  belong to A; a point is called external to field A if it is internal to the field formed by points not belonging to A. A point neither internal nor external is called a limit point of A. The field formed of limit points of A is called the limit field or contour of A."

Having completed the statement of preliminary definitions, Peano stated his definition of area. "Let us take any plane field. We can in general imagine plane areas bounded by straight lines which contain in their interior plane field A, and plane areas also limited by straight lines contained in the interior of the given field. If as happens in the most common cases, the inferior limit of the first areas coincides with the superior limit of the second, then we give the name area of the given plane field to the common value of these two limits.

"But it can happen that these limits are not equal; in this case we will call the inferior limit of the polygonal areas which contain in their interior the given figure the external area, and the superior limit of the polygonal areas contained in the interior the internal area of the given figure."<sup>24</sup>

---

<sup>23</sup> Peano, op. cit., 445.

<sup>24</sup> Peano, Applicazioni geometriche del calcolo infinitesimale, 156.

Peano then gave necessary and sufficient conditions such that a point set in the plane have an area. "If from a field limited by straight lines, containing in its interior field A, we take a field none-the-less limited by straight lines and contained in A, we thus obtain a field limited by straight lines and which contains in its interior the limit field of A. The area of this field is the difference between the areas of the two first. Thus we can be assured that the inferior limit of the first area coincides with the superior limit of the second area if their difference can be made as small as one would wish. This is equivalent to saying that a plane field has an area...if and only if there can be formed a plane field limited by lines containing in its interior all the limit points of the given field and whose area be as small as one wishes. It can also happen that there exists no polygon of finite area containing in its interior the given field and then we say that the external area of the field is infinite. If there exists no polygon contained in the interior of the given field then we will say that its internal area is zero."

Peano, in his discussion of length, had remarked that whether or not a linear point set has a length, "it is seen that the difference between the external and internal length of the set A is equal to the external limit of the limit field of A," a statement which obviously carries over to plane sets. In modern notation, if we denote the interior area, exterior area, and "contour" or boundary of planar set E by  $\underline{A}(E)$ ,  $\overline{A}(E)$ , and  $B(A)$ , respectively, then Peano's statement is of the form  $\overline{A}(E) - \underline{A}(E) = \overline{A}(B(E))$ . Thus E has area if and only if  $\overline{A}(B(E)) = 0$ .

Peano recognized that the set function thus defined is finitely additive, that is, if we denote the area of a set  $E$  that has area by  $A(E)$ , and if  $E_1, E_2, \dots, E_n$  are nonoverlapping sets in the plane each of which has area, then  $\bigcup_{i=1}^n E_i$  has area and

$$A\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n A(E_i).$$

Peano used this "distributive function" (*funzione distributiva*) to develop a surprisingly sophisticated theory of integration and differentiation. Thus Peano stated and proved that a bounded nonnegative function  $f(x)$  defined on an interval  $[a, b]$  is Riemann integrable on  $[a, b]$  if and only if the set of ordinates of  $f(x)$  on  $[a, b]$  has area, and in this case,

$$\int_a^b f(x) dx = A\left(\bigcup_{x_0 \in [a, b]} \{(x_0, y) \mid 0 \leq y \leq f(x_0)\}\right).$$

This form of statement of integrability and characterization of the integral as the measure of a point set opened the way to generalize the conception of Riemann integration. It suggests that a more general conception of measurability and measure of the set of ordinates implies a more general conception of integrability and definite integral of functions. Indeed Lebesgue followed this course several years later in his geometric definition of integral.

Jordan was led to consider a theory of measure in connection with his investigations of the role played by the set over which a function is defined, in influencing the existence and properties of the integral. In his "Remarks on definite integrals," published in

1892, Jordan wrote "the definite integral, simple or multiple, of a function  $f$  in a set  $E$  is obtained as one is aware...in the following manner:

"One decomposes the set into infinitely small elements in all directions; one multiplies the extent  $d\sigma$  of each of these elements by the value of  $f$  chosen at will in the element, and one searches for the limit of the sums  $\sum f d\sigma$  thus formed.

"One knows that this limit has a well determined value when the function is continuous. This property subsists even for a more general class of functions, defined in a precise manner by a well known theorem of Riemann.

"Finally, Mr. Darboux has shown that, whatever be the bounded function  $f$ , each of the two sums  $\sum M d\sigma$  and  $\sum m d\sigma$ , where  $M$  and  $m$  represent the maximum and minimum of  $f$  in the element  $d\sigma$ , always has a perfectly determined limit.

"These results are very clear cut and enlighten completely the role played by the function in the integral.

"The influence of the nature of the set does not seem to have been studied with the same care. All demonstrations rest upon this double postulatam, that each set  $E$  has a determined extent, and that, if one decomposes it into several parts  $E_1, E_2, \dots$ , the sum of the extents of these parts is equal to the total extent of  $E$ . Now these propositions are far from being evident if one leaves the conception of set to all its generalities."<sup>25</sup>

Jordan proposed to define, corresponding to a set  $E$ , an interior

---

<sup>25</sup>Jordan, "Remarques sur les intégrales définies," 69.

extent  $E'$  and an exterior extent  $E''$  and, in the case of equality of these limits, to call the set  $E$  measurable with extent the common value  $E'=E''$ . Jordan further proposed to show that a function  $f$  bounded on a set  $E$  admits an upper and lower integral on  $E$ , and to define the integral of  $f$  over  $E$  to be the common value of the upper and lower integrals in the case of equality. Finally, by utilizing the previously defined concepts of measurability and integrability, Jordan set himself the task of demonstrating that the multiple integral of a function of several variables can be reduced to repeated simple integrals if the set over which the function is to be integrated is measurable.

In carrying out this program, Jordan defined the extent of a set in this way. "We seek...to make precise the notion of the extent of a set.

"This extent will be a length, an area, a volume,..., if the number of dimensions of the set is 1,2,3,... . We suppose, in order to fix the ideas, that this number is equal to 2. Each point  $(u,v)$  of  $E$  is represented geometrically on a plane by the point whose rectangular coordinates are  $u,v$ .

"Decompose the plane by parallels to the axes into squares of side  $r$ . The set of those squares all of whose points are interior to  $E$  form a domain  $S$  interior to  $E$ ; the set of those which are interior to  $E$  or which contain a point of its frontier form a new domain  $S+S'$ , to which  $E$  is interior. These domains, being formed by the union of squares, have determined areas which we will represent by  $S$  and  $S+S'$ .

"We will vary the decomposition of squares in such a way that  $r$  tends toward zero: the areas  $S$  and  $S+S'$  tend toward fixed limits."

Jordan proved this assertion, designating the limit of the areas  $S$  by  $A$ , and the limit of the areas  $S+S'$  by  $a$ . He continued, "As one has always  $S+S' \geq S$ ,  $a$  is at least equal to  $A$ .

"We will call  $A$  the interior area of  $E$ ,  $a$  its exterior area. If  $S'$  has limit zero, then we will say that  $E$  is measurable, and has as its area the quantity  $a = A$ ."

Jordan stated the (finite) additivity of the measure as follows: "Suppose  $E$  is formed by the union of several partial sets  $E_1, E_2, \dots$ , and consider an arbitrary decomposition of the plane into squares. Let, respectively,  $S, S_1, S_2, \dots$  be the sums [of areas] of the squares interior to  $E, E_1, E_2, \dots$ ; and  $S', S'_1, S'_2, \dots$  the ones of squares which meet their frontiers. Each square interior to one of the sets  $E_1, E_2, \dots$  is interior to  $E$ , and, on the other hand, each square not exterior to  $E$  is not exterior to at least one of the sets  $E_1, E_2, \dots$ ; one has therefore

$$S \geq S_1 + S_2 + \dots, \quad S + S' \leq S_1 + S'_1 + S_2 + S'_2 + \dots,$$

and in the limit

$$A \geq A_1 + A_2 + \dots, \quad a \leq a_1 + a_2 + \dots.$$

$A, A_1, A_2, \dots$  and  $a, a_1, a_2, \dots$  represent the interior and exterior areas of the sets  $E, E_1, E_2, \dots$ . These inequalities are changed, moreover, into equalities if the sets are measurable."<sup>26</sup>

---

<sup>26</sup>Jordan, op. cit., 76-78.

Jordan gave his definition of definite integral in the section of the same title. "Let  $f(x,y,...)$  be a bounded function in the interior of a domain  $E$  supposed measurable.

"Decompose  $E$  into elementary measurable domains  $e_1, e_2, \dots$ . Designate by  $M$  and  $m$  the maximum and minimum of the function  $f$  in  $E$ ; and by  $M_k$  and  $m_k$  its maximum and minimum in  $e_k$ , and form the sums

$$S = \sum M_k e_k, \quad s = \sum m_k e_k."$$

Of course the symbol  $e_k$  represents the extent of the "measurable domain"  $e_k$  in each of these summations. Jordan showed that the sum  $S$  and  $s$ , corresponding to given decompositions of  $E$  tend toward limits fixes as the diameters of the elements of the decompositions tend to zero. Designating these limits by  $T$  and  $t$ , respectively, Jordan wrote "This fixed number  $T = \lim S$  is called the upper integral (integrale par excès) of the function  $f(x,y,...)$  in the interior of  $E$ .

"... the sums  $s$  tend toward their maximum  $t$ , which will be the lower integral (intégrale par défaut) of  $f(x,y,...)$ .

"One evidently has  $T \geq t$ . If  $T = t$ , the function will be integrable and  $T = t$  will be its integral, and will be represented by the notation  $S_E f(x,y,...)$ ."<sup>27</sup>

Jordan extended this definition of integrability to functions defined on a nonmeasurable set. "We have assumed until now that the domain  $E$  is measurable. We are now able to suppress this restriction... Consider the limit of a sequence of measurable domains  $E_1, E_2, \dots, E_n, \dots$  [each of which is interior to the following and to  $E$ ]<sup>28</sup> and whose extents converge toward a limit which, by definition, is the interior extent of

<sup>27</sup>Jordan, op. cit., 81-84.

<sup>28</sup>Jordan had imposed this condition in the preceding paragraph and obviously intended it to hold here.

$E$ . Then the integral, upper or lower, taken in  $E_n$  tends toward a limit" for the difference between the integrals taken in  $E_n$  and  $E_{n+p}$  is

$$\left| \int_{E_{n+p}} f - \int_{E_n} f \right| = \left| \int_{E_{n+p} - E_n} f \right| \leq D \left| A(E_{n+p}) - A(E_n) \right|$$

$$\leq D \left| \underline{A}(E) - A(E_n) \right|$$

where  $|f| \leq D$ . "We consider this limit of the integrals taken in  $E_n$  as representing the value of the integral in  $E$ ." Thus, in modern notation

$$\lim_n \int_{E_n} f = \int_E f.$$

Jordan then turned to the theorem which was the primary object of his paper. "If a function  $f(x,y,\dots)$  of  $n$  variables is integrable in a domain  $E$  of measurable extent, then the calculation of the multiple integral  $I = S_E f(x,y,\dots)$  is reduced to the calculation of  $n$  successive simple integrals."

Jordan proved this theorem in the following way. "For greatest simplicity we will suppose  $n=2$  in the demonstration. The set  $E$  will be represented geometrically by a set of points  $(x,y)$  situated in a plane.

"The values of  $y$  to which correspond points of  $E$  form a bounded set  $F$ . Let one of them be  $\eta$ ; the values of  $x$  which, associated with  $\eta$  give points of  $E$ , form a bounded set  $G_\eta$ . We are not able to affirm that  $G_\eta$  has a measurable length, nor that the function  $f(x,\eta)$  is

integrable there; but this function being bounded, always has determined in the interior of  $G_\eta$  its upper and lower integrals. These are functions of  $\eta$ , that we designate by  $J(\eta)$  and  $j(\eta)$ , and which are bounded in the domain  $F$ . We are able therefore to determine in the interior of  $F$ : (1) the upper integral of  $J(\eta)$  which we designate by  $K$ , and (2) the lower integral of  $j(\eta)$  which we designate by  $k$ ."

Observing that  $K \geq k$ , Jordan succeeded in showing, using the notation of extent, "that the integral  $K$  is at most equal to the upper double integral  $\overline{S}_E f(x,y)$ ."

"One shows, by a similar reasoning, that the integral  $k$  is at least equal to the lower double integral."

Thus, if we denote the upper and lower integrals of  $f(x,y)$  on  $E$  by  $\overline{S}_E f(x,y)$  and  $\underline{S}_E f(x,y)$ , respectively, Jordan demonstrated the inequalities

$$\overline{S}_E f(x,y) \geq K \geq k \geq \underline{S}_E f(x,y).$$

He concluded the proof by remarking "Until now we have not made use of the hypothesis that the function  $f(x,y)$  is integrable. If this is assumed, then the upper and lower double integrals are equal and, therefore, equal the integrals  $K$  and  $k$ . Now each of the latter can be calculated by a simple integration successively effected."<sup>29</sup>

In a footnote following the proof Jordan emphasized the necessity of the hypothesis of measurability of  $E$ . "The demonstration above [requires] that the domain  $E$  be measurable. If it is not, then the proposition will be found wanting. Suppose, for example, that  $E$  consists

---

<sup>29</sup>Jordan, op. cit., 85-87.

of the points  $0 \leq y \leq 1$  and  $0 \leq x \leq 1$  if  $y$  is rational or  $-1 \leq x \leq 0$  if  $y$  is irrational, and define the function to be an integral constant  $c$  [where  $c \neq 0$ ]. Then the double integral  $S_E c$  [in the extended sense previously defined] is zero, for the interior area of  $E$  is zero. But, on the other hand, the domain  $G_\eta$  and  $F$ , having a length equal to 1, one will have

$$\int_F d_\eta \int_{G_\eta} c \, dx = \int_F c d\eta = c. "$$

Thus Jordan called attention to the relationship between integration theory and the theory of measure of sets.

It might be of interest to observe that Jordan's work was an important contribution to the viability of set theory itself. We have seen Jordan's appeal to Cantor's  $n$ -dimensional set theoretic conceptions in his study of the theory of measure of sets and the theory of integration; he also made use of Cantor's set theory in his investigations of the rectification of curves, trigonometric series and analysis situs. "In daring to incorporate certain parts of the theory of sets into his course at the Polytechnical School,<sup>30</sup> Jordan rehabilitated... this theory; he affirmed that it is a useful branch of mathematics."<sup>31</sup> Jordan's authority, arguments and results encouraged an acceptance of Cantor's theory of sets which had been severely criticized by Kronecker.

---

<sup>30</sup>The integration theory described above appears in the second edition of Jordan's Cours D'Analyse, published in 1893.

<sup>31</sup>Lebesgue, Notice sur les travaux scientifiques de M. Henri Lebesgue, 16.

In this respect, the evolution of a theory of functions of a real variable was hardly greeted with equanimity by some mathematicians. The study of functions possessing strange and unexpected properties was sometimes viewed with disdain or even hostility. "The distrust with which this new field of investigation was regarded is typified by the attitude of H. Poincaré who wrote, 'in the past new functions were invented with some practical purpose in mind; today they are invented intentionally in order to baffle the reasonings of our fathers, and one cannot deduce anything from them but that'....Researches dealing with...functions violating laws which one hoped were universal, were regarded almost as the propagation of anarchy and chaos where past generations had sought order and harmony. Even the first attempts to establish a positive theory were rather skeptically received; it was feared that an excessively pedantic exactitude in formulating hypotheses would spoil the elegance of classical methods, and that discussions of details would end by obscuring the main ideas of analysis."<sup>32</sup>

The concept of exterior extent defined by Jordan and Peano is equivalent to Cantor, Stolz, and Harnack's conception of content. The measure-theoretic accomplishment of Peano and Jordan was the introduction of the concepts of interior extent and measurability of a set. Not all writers are in agreement, however, regarding the form of definition given by Peano and Jordan to the concept of interior extent. Thus Loève, in his *Encyclopaedia Britannica* article "Integration and measure," states "this requirement [of additivity] was recognized by G. Peano and C. Jordan who introduced approximations both from above and below.

---

<sup>32</sup> Saks, Theory of the Integral, viii.

To a set  $S$  contained in a bounded interval  $I$  they assigned also its inner content (the difference between the ordinary measure of  $I$  and the outer content of  $S$ )."<sup>33</sup> This statement of definition is in error and should read "the difference between the ordinary measure of  $I$  and the outer content of  $I-S$ ." The second source in which I have been able to find a definition of this form attributed to Peano and Jordan is in Bourbaki's Intégration (and the same passage in Bourbaki's Eléments d'histoire des mathématiques). Bourbaki, after observing that the concept of extent is not additive, writes "without doubt to alleviate this last difficulty, Peano and Jordan, several years later, introduced beside the 'measure' of Cantor  $\mu(A)$  of a set  $A$  contained in an interval  $I$ , its 'interior measure'  $\mu(I)-\mu(I-A)$ ..."<sup>34</sup>

I have been able to find nothing in the work of either Peano or Jordan to support these assertions. Such contentions are of interest, however, for if true, they may have guided the form given by Lebesgue to the definition of inner measure of a set contained in an interval. I have not been successful in corresponding with any members of Bourbaki with respect to this question; Loeve responded to my query by writing "I have no recollection...of the source (or sources) of my statement...My feeling is that Lebesgue's use of difference is his own. For he writes of Jordan content (étendue) in terms of outer and inner contents both [defined by] a limiting process..."

However these differing statements of definition might be reconciled, the measure theoretic concepts introduced by Peano and Jordan

---

<sup>33</sup>Loève, "Integration and measure," 343.

<sup>34</sup>Bourbaki, op. cit., 249.

imply a rather comprehensive theory of measure. Thus the collection of Peano Jordan measurable sets  $R$  forms a ring, i.e.,  $R$  is a ring in the usual algebraic sense with respect to the operations of "addition" and "multiplication" defined by the symmetric difference  $E \Delta F = (E - F) \cup (F - E)$  and intersection  $E \cap F$ , respectively. These operations are meaningful since  $R$  is closed under set theoretic union and difference. (We might note  $E \cap F = E - (E - F)$ ). Furthermore, the set function  $A$  defined on  $R$  is finitely additive on disjoint elements of  $R$ . That is, if  $E_i \in R$  for  $1 \leq i \leq n$  and  $E_i \cap E_j = \emptyset$  if  $i \neq j$ , then

$$A\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n A(E_i).$$

The example of the rational numbers in the unit interval demonstrates, however, that the countable union of sets in  $R$  need not be in  $R$ , and the set function  $A$  does not possess the property of countable additivity. These facts might call into question the usefulness of this class of sets and its associated set function. Analysis is essentially a study of limiting processes; thus a theory of measure adequate to the requirements of analysis might be expected to possess general properties with respect to certain limiting operations. In particular, the ring  $R$  of measurable sets might be expected to be closed under countable unions, and its associated set function  $m$  might be expected to be countably additive on sequences of disjoint elements of  $R$ . The necessity of these analytic properties, as opposed to the purely algebraic properties of the ring of Peano Jordan measurable sets, was not immediately recognized. It remained for Borel to introduce such

conceptions into the stream of measure theoretic ideas only after having discovered their efficacy in a context removed from integration theory.

### Borel

Borel developed a theory of measure as a consequence of his investigation of analytic continuation of functions. In a memoir published in 1895 and titled "On several points of the theory of functions," Borel proposed this question: "Being given two functions of a complex variable, the first defined when the variable is in a certain domain, the other defined when the variable is in a different domain, in which cases can one say that they are the same function?"<sup>35</sup> A particular case of the problem considered by Borel is the following: given a circle  $K$  in the complex plane of which the set  $\{a_n\}$  of points is a dense subset, and the function  $f(z)$  defined by  $f(z) = \sum \frac{A_n}{(z-a_n)^{m_n}}$  where  $\sum u_n$  is a convergent series of positive terms such that  $\sum \frac{|A_n|}{u_n^{m_n}}$  is convergent, and "the integral exponents  $m_n$  are at most equal to a fixed number  $m$ ," is it possible to define an analytic continuation of  $f(z)$  across  $K$ ?

Borel argued in this way. Let  $P$  and  $Q$  be points inside and outside the circle  $K$ , respectively. "We will consider the circles  $C$  passing through  $P$  and  $Q$  whose centers  $O$  are situated in a determined segment  $AB$ ...perpendicular to the middle of  $PQ$ ." Assume that for each  $n$ ,

---

<sup>35</sup>Borel, "Sur quelques points de la théorie des fonctions," 9.

the circle determined by  $P, Q$  and  $a_n$  has center  $O_n$  on the line containing  $AB$ . "Designate by  $\ell$  the length of  $AB$ ; the series  $\sum u_n$  being convergent, we are able to choose an  $n$  such that

$$2 \sum_{n+1}^{\infty} u_i < \ell.$$

"Now for  $i > n$ , cover the center

$O_i$  by an interval  $A_i B_i$  of length  $2u_i$ ; "the sum [of lengths] of all the segments, infinite in number,  $A_i B_i$ , situated on the segment  $AB$  or on its extension, is less than the length  $\ell$  of  $AB$ ; therefore there exists on  $AB$  a nondenumerable infinity of points belonging to none of these segments. Let  $\omega$  be one of these points which does not coincide with any of the points  $O_i$  for  $i \leq n$ , and let  $\Gamma$  be the circle of center  $\omega$  passing through the points  $P$  and  $Q$ ; I say that this circle  $\Gamma$  has the required property, that is to say that the series  $\sum \frac{A_n}{(z-a_n)^m}$  is uniformly convergent on this circle."<sup>36</sup> Thus Borel determined a sense in which  $f(x)$  can be analytically continued across  $T$ .

In a note at the end of his paper, Borel commented further on the existence of a nondenumerable infinity of points not belonging to the union of the intervals  $A_i B_i$ . "...if one has an infinity of partial intervals given on a line whose sum [of lengths] is less than the length of a given interval, then there exists at least one point of the interval contained in none of the partial intervals. It is clear that, if there is such a point, then there exists a nondenumerable infinity of them, for, if there is a denumerable infinity, then one is able to enclose them in intervals whose sum [of lengths] is as small as one wants, and chosen in such a manner that, in adjoining these intervals

---

<sup>36</sup>Borel, op. cit., 25-26.

to those which are already given, one has a sum less than the length of the interval; it suffices therefore [to demonstrate the existence of] a point on the line belonging to none of these intervals."<sup>37</sup>

Borel succeeded in demonstrating the existence of such a point by means of what has since become known as the Heine-Borel theorem.

Thus Borel was led to consider the union of a countable collection of intervals whose measure is defined to be the sum of lengths of the constituent intervals. This measure was appropriate for Borel's purpose in that it could distinguish between the measure of a countably dense set and the measure of its closure, a property not enjoyed by the measure of Peano Jordan. Borel was probably influenced in his statement of definition of measure by Cantor and Harnack. Cantor, in the fourth installment of his "Ueber unendliche, lineare Punktmannichfaltigkeiten," had characterized each open set of real numbers as a countable union of disjoint open intervals; an aspect of Harnack's work in this respect has previously been described.

Borel developed the measure theoretic implications of his 1895 paper in his treatise Lessons on the Theory of Functions, published in 1898. "We now define...a notion which will be very useful to us, the notion of a measurable set.

"All of the sets we consider are formed of points included between 0 and 1. When a set is formed of all the points included in a denumerable infinity of intervals which are disjoint and have a total length  $s$ , we will say that the set has measure  $s$ . When two sets have

---

<sup>37</sup>Borel, op. cit., 51.

no common points and their measures are  $s$  and  $s'$  then their union has measure  $s+s'$ . Moreover, it is of little importance in the definition of measure of a set, or that of the union of two sets, if we neglect . . . the denumerable infinity of endpoints of intervals.

"More generally, if we have a denumerable infinity of sets which are mutually disjoint and have measures respectively  $s_1, s_2, \dots, s_n, \dots$ , then their union has measure  $s_1 + s_2 + \dots + s_n + \dots$  .

"...if a set  $E$  has measure  $s$ , and contains all the points of a set  $E'$  whose measure is  $s'$ , then the set  $E-E'$  formed of points of  $E$  which do not belong to  $E'$  will be said to have measure  $s-s'$ ...."<sup>38</sup>

Borel thus imagined sets formed by countable unions and set theoretic difference, and their associated measure given by infinite series and arithmetic difference. A nonempty class of sets closed under the set operations of difference and countable union is called a  $\sigma$ -ring. The ring  $B$  of Borel measurable sets is, therefore, a  $\sigma$ -ring. The  $\sigma$ -ring  $B$  of Borel measurable sets contains all open sets, closed sets and countable sets; sets of particular importance in analysis. None of these sets is, in general, measurable in the sense of Peano and Jordan. Thus the  $\sigma$ -ring of Borel measurable sets  $B$  might be expected to be more useful in analysis than the ring of Peano-Jordan measurable sets  $R$ . The  $\sigma$ -ring  $B$  does exhibit a deficiency: it is not complete. That is, it is not true that if  $E$  is a Borel measurable set with Borel measure zero, and if  $F$  is any subset of  $E$ , then  $F$  is a member of  $B$ . This assertion is proved by means of a cardinality argument described

---

<sup>38</sup> Borel, Lecons sur la Theorie des Fonctions, 46.

below; the importance of the property of completeness of the  $\sigma$ -ring of measurable sets will become evident in the next chapter.

Borel undoubtedly had the Peano-Jordan conception of measure in mind when he wrote "the sets whose measures one can define by the preceding definitions are called by us measurable, without necessarily implying by this that it is not possible to give a definition of the measure of other sets, but such a definition would be useless to us, it could even hinder us if it did not leave to measure the fundamental properties that we have attributed to it in the definitions that we have given."

Borel stated the "essential properties" of a measure to be the following: "the measure of the union of a denumerable infinity of sets is equal to the sum of their measures; the measure of the difference of two sets is equal to the difference of their measures; measure is never negative; each set whose measure is not zero is nondenumerable. It is above all this last property that we will use."<sup>39</sup>

The "essential properties" of a measure were referred to again in a footnote. "The procedure that we have used comes back to this: we have recognized that a definition of measure can be useful only if it has certain fundamental properties; we have posed a priori these properties and it is these which have served us in defining the class of sets that we regard as measurable....In all cases, it proceeds from the same fundamental idea: define the new elements which one introduces with the aid of their essential properties, that is to say, of

---

<sup>39</sup>Borel, op. cit., 48.

those which are strictly indispensable for the reasonings which must follow." Thus Borel attempted a postulational approach to the theory of measure. This was a significant contribution in itself for it made explicit important properties a measure might be expected to possess.

Borel did not endeavor to apply his concept of measure to the theory of integration. There are two reasons for this. First, the problems studied by Borel in the theory of functions were presumably unrelated to integration theory. Second, the theory of measure proposed by Borel is in a sense not as general as the concepts of measure defined by Harnack and Stolz, and Peano and Jordan. Thus Borel's work was ostensibly not as appropriate to the theory of integration. Borel wrote, "One will compare fruitfully the definitions that we are going to give with the more general definitions given by Mr. Jordan in his Course of Analysis. The problem that we are studying here is besides, completely different from the one resolved by Mr. Jordan." Lebesgue later attributed this remark to Borel with respect to Borel measurable sets: "By renouncing the definition of measure for an arbitrary set one founds a less general theory; that is to say, it applies to fewer cases, but more precisely in the cases in which it is applied."<sup>40</sup>

The sense in which Borel's theory of measure is "less general" is that the cardinal number of Borel measurable sets is less than the cardinal number of measurable sets in the theories of Cantor, Harnack and Stolz, and Peano and Jordan. Any bounded set of real numbers is

---

<sup>40</sup>Lebesgue, Notice sur les travaux scientifiques de M. Henri Lebesgue, 33.

measurable in the sense of Cantor, Harmack and Stolz; thus the cardinality of the collection of measurable sets in this theory of measure is  $2^c$  where  $c$  is the cardinality of the continuum. The Cantor ternary set is measurable in the sense of Peano and Jordan, with extent zero. It follows that all subsets of this set have extent zero and there are, therefore, at least  $2^c$  Peano-Jordan measurable sets. (There are also at most  $2^c$  sets of real numbers measurable in the sense of Peano-Jordan. This does not mean, of course, that every subset of real numbers is Peano-Jordan measurable). There are, however, only  $c$  Borel measurable sets. This assertion follows from the propositions that there are  $c$  open sets, that the  $\sigma$ -ring of Borel measurable sets is generated by the collection of open sets, that is, the Borel measurable sets are the elements of the smallest  $\sigma$ -ring of sets containing the open sets, and, if  $M$  is a class of sets and the cardinality of  $M$  is less than or equal to  $c$ , then the cardinality of the  $\sigma$ -ring generated by  $M$  is also less than or equal to  $c$ .<sup>41</sup> Thus there are "fewer" Borel measurable sets than sets with content or extent.

It was probably with this in mind that Borel appended the following statement to his definition of measure. "It is expressly understood that we will be speaking of measure only with respect to sets that we have called measurable.

"However, if a set  $E$  contains all the elements of a measurable set  $E_1$ , of measure  $\alpha$ , we will say that the measure of  $E$  is greater than  $\alpha$  without inquiring whether  $E$  is measurable or not. Inversely, if  $E_1$  contains all the elements of  $E$ , we will say that the measure of  $E$  is

---

<sup>41</sup>See Halmos, Measure Theory, 26.

less than  $\alpha$ . The words greater than and less than do not, moreover, exclude equality." This statement contributed to the subsequent history of measure theory; in particular it became one of the issues of controversy in the polemic between Borel and Lebesgue.

Arthur Schoenflies promulgated a critique of Borel's theory of measure in his treatise The Development of the Theory of Point Sets. Schoenflies' work is of interest in two respects: first, it was published in 1900 and hence after Borel had published his theory of measure but before Lebesgue had published his work on this subject, and second, Schoenflies was at the time a well-known and respected mathematician, and his treatise was a standard reference, being frequently cited in the literature. Indeed, Schoenflies' text was an outgrowth of a report on the topic "curves and point sets," commissioned two years before by the German Mathematical Association.<sup>42</sup> The work might be expected to reflect a conservative point of view and therefore to illustrate the skepticism with which contributions to the emerging theory of functions of a real variable continued to be received.

Schoenflies began his discussion of "the content of point sets" by remarking on the existence of three essentially different theories of measure. "The consideration of the content of point sets constitutes a subject from which various controversies have emanated. On one hand, we utilize results that could appear to be paradoxical; on the other hand, the definition of content, like every mathematical definition, has above all a certain subjective character, and the consequences that proceed from it vary, if it has been chosen in accordance

---

<sup>42</sup>Schoenflies, Die Entwicklung der Lehre von den Punktmannigfaltigkeiten, iii.

with the purpose to be accomplished."

Schoenflies reviewed the conceptions of Hankel, Cantor and Harnack, and Peano and Jordan, and continued, "An essentially different position has been taken recently by E. Borel. Borel does not add the limit points to the point set  $P$ , and dismisses the requirement that a finite number of regions contain all the points of the set. He imagines that every point of  $P$  is surrounded by an arbitrary domain, and considers the areas of these and their limits respectively. A consequence of this definition is that all countable point sets have content zero...."

Schoenflies was critical of Borel's form of definition of measure. "...Borel has taken [additivity] as the basis of his definition of content. He considers this as the essential property of the concept of content....If the continuum  $C$  on a line is divided into two point sets  $P$  and  $P_1$ , then it must follow that

$$J(C) = J(P) + J(P_1).$$

To this Borel adds the second requirement that this equation must hold for any countable collection of point sets....Now [this] second requirement of Borel has of course by no means the same character as the first. It has above all only the character of a postulate; the question of extending a property of finite sums to the sum of infinitely many terms cannot be settled by assertion but requires investigation...."<sup>43</sup>

Borel's postulates describe the properties he wishes the concept of measure to possess. But where, Schoenflies seems to ask, are Borel's

---

<sup>43</sup>Schoenflies, op. cit., 93.

proofs of existence and uniqueness of a measure with these properties? Of course these objections are sound; proof of Borel's assertions came only with Lebesgue.

Schoenflies ended by rejecting the theories of both Borel and Peano and Jordan. "Since in applications it is always only a question of the outer content, I will from now on call it the content of  $T$ , designated by  $J(T)$ ." Thus Schoenflies conceived the measure of Cantor and Harnack to be adequate for the theory of integration.

Not all mathematicians dismissed Borel's conception of measure. Even as Schoenflies was writing his monograph, Lebesgue was developing his theory of measure as a completion of that of Borel. "[Borel's] definition was to inaugurate a new era in analysis: in connection with the contemporaneous work of Baire, it formed the point of departure of a whole series of researches of a topological nature on the classification of sets of points; above all, it went to serve as a basis for the extension of the notion of integral, realized by Lebesgue in the first years of the 20th century."<sup>44</sup> This is the extension to which we turn in the next chapter.

---

<sup>44</sup>Bourbaki, op. cit., 250.

## Chapter 3

## On the Lebesgue Integral

## Lebesgue measure and integral

Lebesgue undertook his study of measure and integral in an attempt to free classical results in analysis from restrictive continuity hypotheses. Thus, the Riemann integral solved the problems of determining the primitive of a continuous derivative and the length of an arc with a continuously turning tangent line in the forms

$$\int f'(x) dx = f(x) + c$$

and

$$\int_a^b (1 + (f'(x))^2)^{1/2} dx,$$

respectively. Volterra's example of a bounded derivative which is not Riemann integrable, and Scheeffer's example of a continuous increasing function whose derivative is unbounded in any interval<sup>1</sup> had demonstrated, however, the failure of these classical forms in the absence of the continuity requirement necessary for the existence of the Riemann integral. Lebesgue's investigation of these and other classical theorems led to his generalization of the concept of integral.

The succession of ideas which resulted in Lebesgue's definition of integral, and which formed the basis of his thesis, can be found in a sequence of papers published by Lebesgue between June, 1899 and April, 1901. In the first of these, "On several non ruled surfaces applicable on the plane," Lebesgue proposed "to seek to determine if there exist

---

<sup>1</sup>These examples are given in Appendix C.

surfaces applicable on the plane other than developable surfaces."<sup>2</sup>

Bonnet had proved that a surface is applicable on the plane if and only if it is developable,<sup>3</sup> a classical theorem whose generalization requires a less restrictive hypothesis than that of a continuously turning tangent plane. Lebesgue gave a procedure for identifying non developable surfaces applicable on the plane, basing his procedure on the exemplar of a crumpled handkerchief or sheet of paper. He concluded, "These examples demonstrate that the question of finding all the surfaces applicable on the plane is not completely resolved by the theorem of Ossian Bonnet."

Montel wrote, with respect to Lebesgue's violation of the conventional bounds of classical differential geometry, "This observation [of non developable surfaces applicable on the plane], in conjunction with the construction of polyhedrons by means of cutout cardboard, was the origin of the great discovery to which his name remains attached, this integral of Lebesgue, which for the study of discontinuous functions is the principal algorithm created since the series of Fourier."<sup>4</sup>

---

<sup>2</sup>Lebesgue, "Sur quelques surfaces non réglées applicables sur le plan," 1503.

<sup>3</sup>Roughly speaking, surfaces which can be continuously deformed into each other in such a way that length of every arc on either of the surfaces is preserved, are called applicable. A ruled surface is a surface which can be generated by the continuous motion of a line in space. The instantaneous positions of the line are called generators of the ruled surface. A developable surface is a ruled surface with the property that it has the same tangent plane at all points of a given generator.

<sup>4</sup>Montel, "Notice nécrologique sur M. Henri Lebesgue," 198.

In his second research note, "On the definition of the area of a surface," Lebesgue stated, "the problem of measure of plane surfaces" in the following way: "To make correspond to each surface a number called its area, in such a way that two equal surfaces have equal areas, and that the surface formed by the union of a finite or infinite number of [non overlapping] surfaces, has area the sum of the areas of the composing surfaces."<sup>5</sup> Thus Lebesgue had adopted Borel's conception of a countably additive measure. Lebesgue's third and fourth notes are a continuation of his investigations of the concept of surface area.

The fifth note in this sequence, "On a generalization of the definite integral," includes statements of definition of Lebesgue measure and integral. Lebesgue began by reviewing properties of the Riemann integral with respect to existence of a primitive. "In the case of continuous functions there is an identity between the notions of integral and primitive function. Riemann defined the integral of certain discontinuous functions, but not all derived functions are integrable in the sense of Riemann. The problem of the research of primitive functions is therefore not resolved by [Riemann] integration, and one can seek a definition of integral which includes the Riemann integral as a particular case, and which permits the resolution of the problem of primitive functions."<sup>6</sup>

After reviewing Riemann's definition of integral, Lebesgue stated

---

<sup>5</sup>Lebesgue, "Sur la definition de l'aire d'une surface," 870.

<sup>6</sup>Lebesgue, "Sur une généralisation de l'intégrale définie," 1025.



his own definition: "Let  $y$  be a function [bounded below and above respectively by]  $m$  and  $M$ . We give

$$m = m_0 < m_1 < m_2 < \cdots < m_{p-1} < M = m_p ;$$

$y = m$  when  $x$  is a member of a set  $E_0$ ;  $m_{i-1} < y \leq m_i$  when  $x$  is a member of a set  $E_i$ .

"We will define the measures  $\lambda_0, \lambda_i$  of these sets below. Consider one or the other of the two sums

$$m_0 \lambda_0 + \sum m_i \lambda_i \quad ; \quad m_0 \lambda_0 + \sum m_{i-1} \lambda_i \quad ;$$

if, when the maximum difference between two consecutive  $m_i$  tend toward zero, these sums tend toward the same limit independently of the choice of the  $m_i$ , then this limit will be by definition the integral of  $y$ , which will be called integrable."<sup>7</sup>

Lebesgue immediately defined his conception of the measure of a set. "Consider a set of points of  $(a,b)$ ; one can in infinitely many ways enclose these points in a denumerable infinity of intervals; the greatest lower bound of the sum of lengths of these intervals is the measure of the set. A set  $E$  is said to be measurable if its measure plus the one of the set of points not contained in  $E$  gives the measure of  $(a,b)$ ." Lebesgue continued by remarking that the countable union or intersection of measurable sets is measurable, and measure is countably additive on disjoint sequences of measurable sets.

"It is natural to consider...functions such that the sets which figure in the definition of the integral are measurable. One finds that

---

<sup>7</sup>Lebesgue, op. cit., 1026.

if a function bounded in absolute value is such that, for every  $A$  and  $B$ , the set of values of  $x$  for which  $A < y \leq B$  is measurable, then it is integrable....Such a function will be called summable." Thus Lebesgue defined what he later called a measurable function.

Lebesgue asserted that the integral of a summable function is between the intégrale par défaut and the intégrale par excès. Therefore, "if a function integrable in the sense of Riemann is summable, the integral is the same with the two definitions. Now, each function integrable in the sense of Riemann is summable, for the set of points of discontinuity is of measure zero..."

Lebesgue gave the Dirichlet function as an example of a summable function not integrable in the sense of Riemann. He stated that "the set of summable functions has a cardinality greater than that of the continuum," and that "if  $f$  and  $\phi$  are summable,  $f+\phi$  and  $f\phi$  are also, and the integral of  $f+\phi$  is the sum of the integrals of  $f$  and of  $\phi$ ."

Lebesgue contended that "if a sequence of summable functions has a limit, it is a summable function," and therefore, by these last two properties "The set of summable functions...contains all continuous functions, all the limits of continuous functions, that is to say the functions of the first class [of Baire], it contains all those of the second class, and so on.

"In particular, each derived function, bounded in absolute value, being of the first class, is summable, and one can demonstrate that its integral, considered as a function of its upper limit, is one of its primitive functions."

Lebesgue concluded this remarkable outpouring by asserting an integral form of a concept which had occupied his attention in a prior note, the length of a curve. "Here is an application in geometry: if  $|f'|$ ,  $|\phi'|$ ,  $|\psi'|$  are bounded, then the curve

$$x=f(t), y=\phi(t), z=\psi(t)$$

has its length given by the integral of  $(f'^2 + \phi'^2 + \psi'^2)^{1/2}$ ." Thus, in this introductory note, Lebesgue demonstrated the usefulness of his more general conception of integral by applying it to previously unsolved problems in analysis. That is, Lebesgue reinstated for a more general class of functions the classical integral form of the length of a curve, and of the relationship between a derivative and its primitive.

Lebesgue developed and extended the ideas introduced in his research notes in his thesis "Integral, Length, and Area," published in 1902. Lebesgue attempted to justify and describe his work in his Introduction. "In this work I try to give as precise and general definitions as possible to some of the numbers that are considered in analysis: definite integral, length of a curve, area of a surface.

"Mr. Jordan, in the second edition of his course of analysis has studied these numbers in depth. It seems useful to me, however, to take up this study again and here is why. It is known that there exist derived functions which are not integrable, when one adopts, as Mr. Jordan did, the definition of the integral given by Riemann. Thus integration, as Riemann defined it, does not permit the resolution in all cases of the fundamental problem of the integral calculus:

"To find a function knowing its derivative.

"It seems natural therefore to search for another definition of the integral such that in more extended cases, integration is the inverse operation of differentiation.

"On the other hand, as Mr. Jordan remarked, the area of a surface not having tangent planes varying in a continuous fashion is not defined; and the things...which one would be tempted to admit as analogous to the definition of the length of a curve cannot be adopted. There is therefore reason to seek a definition of area and perhaps also to modify that of length so that the two definitions are as analogous as possible.

"In the study of questions relative to the theory of functions of real variables it is often recognized that it would be useful to be able to attach to sets of points numbers enjoying certain of the properties of lengths of segments or areas of polygons.

"In the first chapter I define with Mr. Borel the measure of a set by its essential properties. After having completed and making more precise the rather sketchy arguments given by Mr. Borel, I indicate what relations there are between measure thus defined and the measure of Mr. Jordan....

"[the method] of defining the integral of a continuous function as the area of a plane domain...has the advantage of leading to a definition of the integral of a bounded discontinuous function as the measure of a certain set of points. I adopt this geometric definition in the second chapter; one can replace it by an analytical definition, the integral then being presented as the limit of a series of sums rather analogous to those that are considered in the definition of Riemann....

"The integral of a bounded derivative, considered as a function

of the upper limit of integration, is a primitive function of the given derivative; the fundamental problem of integral calculus is therefore theoretically resolved whenever the given derived function is bounded.

"In order to obtain more general results it is necessary to give a definition of integral applying to unbounded functions. It is easy to find such a definition but that which seemed the most simple and natural to me does not apply to all unbounded derived functions, so that for unbounded functions the problem of research of primitive functions is not resolved....

"The effective calculation of an integral depends essentially on the form in which the function to be integrated is given. In the case in which the function is defined with the aid of series, one can make use of this property, of which a particular case was obtained by Mr. Osgood: a series whose terms have integrals and whose remainders are in absolute value less than a fixed number is integrable term by term.

"In the third chapter...I adopt the following definition: the length of a curve  $C$  is the infimum of the lengths of polygonal lines which tend uniformly toward  $C$ . This definition is exactly equivalent to the classical definition.

"The search for an expression of the length of a curve having tangents leads to a new application of the definite integral....If  $f'$ ,  $\phi'$ ,  $\psi'$  exist then the necessary and sufficient condition that the curve

$$x=f(t), y=\phi(t), z=\psi(t)$$



[have finite length] is that the integral of  $(f^2 + \phi^2 + \psi^2)^{1/2}$  exists.

This integral represents the length of the curve whenever it exists.

The definition...is therefore a particular case of the classical definition....

"In the fourth chapter I call the area of a surface  $L$  the infimum of the areas of polyhedral surfaces which tend uniformly toward  $L$ .

"The study of the representation of area with the aid of a double integral is approached only in the very particular case in which the surface admits plane tangents varying in a continuous manner; one recovers the classical integral

$$\iint (EG - F^2)^{1/2} \, du \, dv."$$
<sup>8</sup>

Lebesgue devoted the last two chapters of his thesis to the study of surfaces applicable on the plane and surfaces of minimum area bounded by a given contour, respectively. The latter had been a topic of investigation in his fourth research note.

In the first chapter, Mesure de Ensembles, Lebesgue began by proposing "to attach to each bounded set a nonnegative number that we will call its measure and which satisfies the following conditions:

1. There exist sets whose measure is not zero.
2. Two equal [i.e., congruent] sets have the same measure.
3. The measure of the union of a finite or denumerable infinity of [mutually disjoint] sets is the sum of the measures of these sets.

---

<sup>8</sup>Lebesgue, "Intégrale, Longueur, Aire," 231 ff.

"We will resolve the problem of measure only for the sets that we will call measurable."

Having identified the essential properties of a measure in the manner of Borel, Lebesgue stated his definition of a measurable set.

"A set  $E$  being given, one can in infinitely many ways enclose its points in a finite or denumerable infinity of intervals. The set  $E_1$  of points of these intervals contains  $E$ ; therefore the measure  $m(E)$  of  $E$  is at most equal to the measure  $m(E_1)$  of  $E_1$ , that is to say at most equal to the sum of the lengths of the intervals considered. The greatest lower bound of this sum is an upper bound of  $m(E)$ , we call it the exterior measure of  $E$ ,  $m_e(E)$ .

"Suppose that all the points of  $E$  belong to a segment  $AB$ . We will call the complement of  $E$  with respect to  $AB$ ,  $C_{AB}(E)$ , the set  $AB-E$ . Since the measure of  $C_{AB}(E)$  is at most  $m_e(C_{AB}(E))$ , that of  $E$  is at least  $m(AB)-m_e(C_{AB}(E))$ . This number is independent of the chosen segments  $AB$  containing  $E$ ; we call it the interior measure of  $E$ ,  $m_i(E)$ .

"We will call measurable those sets whose exterior and interior measures are equal, the common value of these two numbers will be the measure of the set....From the properties which follow...the number  $m(E)$  thus defined satisfies very well the conditions of the problem of measure...."<sup>9</sup>

Lebesgue demonstrated that the union of a denumerable infinity of measurable sets is measurable. He continued, "If  $E_1, E_2, \dots$  are mutually disjoint then the points of  $E_1$  are interior to intervals  $\alpha_1$  in such a manner that  $m(\alpha_1)-m(E_1)$  is at most equal to  $\epsilon_1$ . Now  $m(E)$

---

<sup>9</sup>Lebesgue, op. cit., 236ff.

differs from

$$m(\alpha_1) + m(\alpha_2) + m(\alpha_3) + \dots$$

by less than

$$\epsilon_1 + \epsilon_2 + \epsilon_3 + \dots,$$

therefore one has

$$m(E) = m(E_1) + m(E_2) + \dots \dots^{10}$$

Thus Lebesgue measure is countably additive on disjoint sequences of Lebesgue measurable sets.

Lebesgue obtained the measure of the difference of two measurable sets. "Let  $E_1$  contain  $E_2$ ,  $E_1 - E_2$  is the set of points common to  $E_1$  and  $C(E_2)$ , therefore if  $E_1$  and  $E_2$  are measurable,  $E_1 - E_2$  is measurable. Moreover, since one has  $E_1 = (E_1 - E_2) + E_2$ ,  $m(E_1 - E_2) = m(E_1) - m(E_2)$ ."

These two properties of Lebesgue measure, countable additivity and the form of the measure of the difference of two sets, imply that any Borel measurable set is measurable in the sense of Lebesgue and the measures are equal. In particular, the collection of Borel measurable sets is a subset of the class of Lebesgue measurable sets. Lebesgue commented, "Mr. Borel called measurable the sets that one obtains by [denumerable union and difference]; I will call such sets measurable (B)....[The set of Borel measurable sets] has the cardinality of the continuum. Among these sets it is necessary to cite those which are unions of intervals and the closed sets, that is to say which contain

---

<sup>10</sup>Lebesgue, op. cit., 239.

their derivative, whose complements are unions of intervals." Thus every open and every closed set is Lebesgue measurable; the collection of Lebesgue measurable sets is closed under countable union and difference, and therefore is a  $\sigma$ -ring.

The deficiency exhibited by the collection of Borel measurable sets, that the measure is not complete, is obviated in Lebesgue's theory of measure. Having shown that the Lebesgue measure of the Cantor set  $E$  is zero, Lebesgue remarked, " $E$  has the cardinality of the continuum, therefore one can form with the points of  $E$  an infinity of sets all of which, having exterior measure zero, are measurable. The cardinality of the set of these sets is that of the set of sets of points [i.e., cardinality  $2^c$  of the set of subsets of real numbers]; there exist therefore measurable sets which are not measurable (B), and the cardinality of the set of measurable sets is  $[2^c]$ ."<sup>11</sup>

Lebesgue proved that "...each measurable set is contained in a set  $E_1$  and contains a set  $E_2$ ,  $E_1$  and  $E_2$  being measurable (B) and having the same measure." In modern terms, if  $E$  is Lebesgue measurable, then there exist Borel measurable sets  $E_1$  and  $E_2$  such that  $E_1 \supset E \supset E_2$  and  $m(E_1) = m(E) = m(E_2)$ . We note that since  $E_2 \subset E$  and  $E_2$  and  $E$  are Lebesgue measurable,  $m(E - E_2) = 0$ . Since  $E = E_2 \cup (E - E_2)$ , it follows that every Lebesgue measurable set is the disjoint union of a Borel measurable set and a set of Lebesgue measure zero.

Lebesgue also expressed a relationship between Lebesgue measurable sets and sets measurable in the sense of Peano-Jordan. "From this definition [of Peano-Jordan extent] it follows that the exterior extent is at least equal to the exterior measure and that the interior extent is at

---

<sup>11</sup>Lebesgue, op. cit., 241.

most equal to the interior measure." That is,  $\overline{A}(E) \geq m_e(E) \geq m_i(E) \geq \underline{A}(E)$ .

"Mr. Jordan calls measurable the sets whose interior and exterior extents are equal; these sets which we call measurable (J) are therefore measurable in the sense we have adopted...."

Lebesgue completed his discussion of the Lebesgue measure of sets on the real line by demonstrating that the measure of the set "of points common to all the measurable sets  $E_1, E_2, \dots$  which are such that each contains all those that follow it, is the infimum of the sequence  $m(E_1), m(E_2), \dots$ ." Thus, if  $E_1 \supset E_2 \supset \dots$ , then  $m(\bigcap_{i=1}^{\infty} E_i) = \lim_{n \rightarrow \infty} m(E_n)$ .

Lebesgue then extended his definition of measure to point sets in "a space of several dimensions," in particular to sets of points in the plane.

It is of interest to attempt to understand how Lebesgue was led to his definition of interior measure. He commented in this respect in his memoir "On the development of the concept of integral," published in 1927. Lebesgue remarked that the measure in (E) of a set E formed of an infinity of disjoint intervals is defined to be the sum of the lengths of the intervals. In the general case this "leads us to proceed as follows. Enclose E in a finite or denumerable infinity number of intervals and let  $\ell_1, \ell_2, \dots$  be the length of these intervals. We evidently wish to have

$$m(E) \leq \ell_1 + \ell_2 + \dots.$$

"If we seek the greatest lower bound of the second member for all possible systems of intervals that cover E, then this bound is an upper bound of  $m(E)$ . For this reason we represent it by  $\overline{m(E)}$ , and we have

$$1) \quad m(E) \leq \overline{m(E)}.$$

"If C is the set of points of the interval (a,b) that do not

belong to E, we have similarly

$$m(C) \leq \overline{m(C)}.$$

"Now we manifestly wish to have

$$m(E) + m(C) = m([a,b]) = b-a ;$$

and therefore we must have

$$2) \quad m(E) \geq (b-a) - m(C).$$

"The inequalities 1) and 2) give upper and lower bounds for  $m(E)$ ....When the upper and lower bounds for  $m(E)$  are equal,  $m(E)$  is defined, and we then say that E is measurable."<sup>12</sup>

In the second chapter of his thesis, Integrale, Lebesgue utilized his theory of measure to create a more general theory of integration. His first definition of integral was expressed in a geometric form, i.e., the integral of a function as the measure of its set of ordinates. "From the geometric point of view the problem of integration can be expressed as follows:

"Being given a curve C by its equation  $y=f(x)$  (where f is a continuous positive function...), find the area of the domain bounded by an arc of C, a segment of the x-axis, and two parallels to the y-axis whose abscissas are a and b where  $a < b$ .

"This area is called the definite integral of f taken between the limits a and b and is represented by  $\int_a^b f(x)dx$ ."

Lebesgue reviewed the geometric definition of integral given by

---

<sup>12</sup>Lebesgue, 'Sur le développement de la notion d'integrale,' 153-154.

Peano and Jordan. "In order that the function  $f$  be integrable it is necessary and sufficient that [the set of ordinates]  $E$  be measurable (J); the measure of  $E$  is the integral."

If the function  $f$  is of arbitrary sign, that is, not necessarily positive, then the set  $E$  is the union of the set  $E_1$  of positive ordinates and the set  $E_2$  of negative ordinates. "The intégrale par défaut is the interior extent of  $E_1$  minus the exterior extent of  $E_2$ ; the intégrale par excès is the exterior extent of  $E_1$  minus the interior extent of  $E_2$ . If  $E$  is measurable (J), in which case  $E_1$  and  $E_2$  are also, then the function is integrable, with integral  $A(E_1) - A(E_2)$ ."

"These results immediately suggest the following generalization: if the set  $E$  is measurable, in which case  $E_1$  and  $E_2$  are also, we will call the definite integral of  $f$ , taken between  $a$  and  $b$ , the quantity

$$m(E_1) - m(E_2) .$$

"The corresponding functions  $f$  are called summable."

It may bear repeating that Lebesgue later called such functions "measurable." In order to avoid confusion (since the term "summable" is a member of the technical vocabulary as will be seen below), I will conform to Lebesgue's later usage by referring to such functions as "measurable functions."

Lebesgue continued, "Relative to non-measurable functions, if such exist, I will define the inferior and superior integrals as equal to

$$m_i(E_1) - m_e(E_2) \quad m_e(E_1) - m_i(E_2) .$$

"These two numbers are included between the intégrale par défaut and the intégrale par excès."<sup>13</sup>

Thus if a function is Riemann integrable it is Lebesgue integrable, and since the Dirichlet function is integrable in the sense of Lebesgue, Lebesgue's definition of integral is a generalization of that of Riemann.

Lebesgue next turned his attention to an analytic definition of integral. His first result in this direction was an analytic characterization of measurable functions. Arguing by means of the definition of measurable function, the relationship between Borel measurable and Lebesgue measurable sets identified above, and the fact that the intersection of a planar Borel measurable set by a line parallel to a coordinate axis is a linear Borel measurable set, Lebesgue succeeded in establishing this analytic form of measurability: a bounded function  $f(x)$  is measurable if and only if for arbitrary real numbers  $a$  and  $b$ ,  $a > b$ , the set  $\{x | a > f(x) > b\}$  is measurable.

Lebesgue immediately utilized this characterization of measurability in his analytic definition of integral. As a prelude to his definition he wrote, "Let  $f(x)$  be a continuous increasing function defined between  $\alpha$  and  $\beta$  ( $\alpha < \beta$ ) and varying between  $a$  and  $b$  ( $a < b$ ). Arbitrarily choose values of  $x$  such that

$$x_0 = \alpha < x_1 < x_2 < \cdots < x_n = \beta$$

to which correspond the values of  $f(x)$

$$a_0 = a < a_1 < a_2 < \cdots < a_n = b.$$

"The definite integral, in the usual sense of the word, is the

---

<sup>13</sup>Lebesgue, "Intégrale, Longueur, Aire," 250.



common limit of the two sums

$$\sum_{i=1}^n (x_i - x_{i-1}) a_{i-1} \qquad \sum_{i=1}^n (x_i - x_{i-1}) a_i$$

when the maximum of  $x_i - x_{i-1}$  tends toward zero.

"But  $x_i$  is given if  $a_i$  is, and  $x_i - x_{i-1}$  tends toward zero if  $a_i - a_{i-1}$  tends toward zero. Therefore, in order to define the integral of a continuous increasing function  $f(x)$  one can give the  $a_i$ , that is to say, the division of the interval of variation of  $f(x)$  in place of giving the  $x_i$ , that is to say, the division of the interval of variation of  $x$ ."

This observation, in conjunction with the form of measurability of a function and therefore the properties of the measure of sets on the real line, led Lebesgue to state an analytic definition of integral in which the range of the function to be integrated is partitioned rather than its domain. "Let  $f(x)$  be a continuous function defined in  $(\alpha, \beta)$  and varying between  $a$  and  $b$ , ( $a < b$ ). Arbitrarily choose

$$a = a_0 < a_1 < a_2 < \dots < a_n = b ;$$

$f(x) = a_i$  for the points of a closed set  $e_i$ , ( $i=0,1,\dots,n$ );  $a_i < f(x) < a_{i+1}$  for the points of an [open] set  $e_i'$ , ( $i = 0,1,2,\dots,n-1$ ); the sets  $e_i$  and  $e_i'$  are measurable.

The two quantities

$$\sigma = \sum_0^n a_i m(e_i) + \sum_1^n a_i m(e_i') \qquad \gamma = \sum_0^n a_i m(e_i) + \sum_1^n a_{i+1} m(e_i')$$

tend toward  $\int_a^b f(x) dx$  when the number of  $a_i$ 's is augmented in such a way that the maximum of  $a_i - a_{i-1}$  tends toward zero.

"This property obtained, one can take it for the definition of the integral of  $f(x)$ . But the two quantities  $\sigma$  and  $\gamma$  have meaning for functions other than continuous functions, they [have a sense] for measurable functions. ...for these functions  $\sigma$  and  $\gamma$  have the same limit independently of the choice of the  $a_i$ ; this limit is by definition the integral of  $f(x)$  taken between  $\alpha$  and  $\beta$ ."

This form of definition had important consequences in the ensuing development of the properties of the integral. It enabled Lebesgue and subsequent mathematicians to exploit the properties of measurable functions and measurable sets both as a guide and as a means to develop the theory. Jeffery commented that the form given by Lebesgue to his definition of integral "...proved fortunate. It influenced [the form of results] in real variable theory, and gave them an elegance and simplicity which they might not otherwise have obtained."<sup>14</sup>

Lebesgue described this form of definition in a later paper in this way. "It is evident that partitioning the interval  $(\alpha, \beta)$  into smaller and smaller intervals will make the differences  $\bar{f}_i - \underline{f}_i$  [where  $\bar{f}_i$  and  $\underline{f}_i$  denote the supremum and infimum of  $f(x)$  in  $[x_{i-1}, x_i]$ , respectively] smaller and smaller if  $f(x)$  is continuous, and continued refinement of the partition will make  $\bar{S} - \underline{S}$  [upper and lower Riemann sums, respectively] tend toward zero if there are only a few points of discontinuity. But one has no reason to hope that this will happen for an everywhere discontinuous function. Then, to take smaller and smaller intervals  $(x_i, x_{i+1})$ ,

---

<sup>14</sup>Jeffery, The Theory of Functions of a Real Variable, 74.

that is to say values of  $f(x)$  corresponding to values of  $x$  closer and closer together, guarantees in no way that one takes values of  $f(x)$  whose differences become smaller.

"Let us be guided by the goal to be attained: to group approximately equal values of  $f(x)$ . It is clear then that we must partition, not  $(\alpha, \beta)$ , but the interval  $(\underline{f}, \overline{f})$  bounded by the lower and upper bounds of  $f(x)$  in  $(\alpha, \beta)$ . Let us do this with the aid of numbers  $y_i$  differing between themselves by less than  $\epsilon$ . We are thus led to consider the values of  $f(x)$  defined by

$$y_i \leq f(x) \leq y_{i+1}.$$

"The corresponding values of  $x$  form a set  $E_i$ ... which plays a role analogous to the interval  $(x_i, x_{i+1})$  in the definition of the integral of continuous functions, since it identifies the values of  $x$  which give to  $f(x)$  approximately equal values."<sup>15</sup>

Lebesgue observed that if  $\eta_i$  is chosen such that  $y_i \leq \eta_i \leq y_{i+1}$ , then  $\eta_i$  plays the role of  $f(\xi_i)$  in the usual summation formula  $\sum f(\xi_i)(x_{i+1} - x_i)$  and the measure  $m(E_i)$  of  $E_i$  plays the role of the length of the interval  $(x_i, x_{i+1})$ . Thus, corresponding to the ordinary sum we are led to consider the sum

$$S = \sum \eta_i m(E_i),$$

whose limit as the maximum of the  $y_{i+1} - y_i$  tends to zero is the integral of  $f$  from  $\alpha$  to  $\beta$ .

If the integral is thought of as the sum of an infinity of

---

<sup>15</sup>Lebesgue, "Sur le développement de la notion d'intégrale," 151-152.

indivisibles, each of which is a positive or negative ordinate of  $f$ , then "one could say that according to Riemann's procedure, one attempts to add the indivisibles by taking them in the order in which they are furnished by the variation in  $x$ , operating therefore like a merchant without method who counts coins and bills at random in the order in which they come to hand...", while the procedure of Lebesgue is to separate the coins and bills into collections of like denominations, to find the value of each of these collections and then to find the total of the values thus obtained. Lebesgue continued, "The two procedures certainly lead the merchant to the same result because no matter how much money he has there are only finitely many coins or bills to count. But for us who must add an infinity of indivisibles, the difference between the two methods is of capital importance."

Returning to Lebesgue's thesis, "A very important proposition is the following: if a bounded function  $f$  is the limit of a sequence  $\{f_i\}$  of measurable functions, then  $f$  is measurable.

"For let  $e_i$  be the set of values for which  $f_i$  is included between  $a$  and  $b$ . The set  $e$  of points common to all the  $e_i$ , at least beginning with a certain value of  $i$ , is the set of values of  $x$  for which  $f$  is included between  $a$  and  $b$ . Now the  $e_i$  being measurable imply  $e$  is measurable; therefore  $f$  is measurable." If  $\{E_n\}$  is a sequence of sets of points, then the set  $E$  of points which belong to  $E_n$  for all but a finite number of values of  $n$  is called the limit inferior of the sequence and is denoted by  $E = \liminf_n E_n$ . It follows that

$$\liminf_n E_n = \bigcup_n \bigcap_{i \geq n} E_i .$$

Thus, if we designate  $e = \{x | a \leq f(x) \leq b\}$  and  $e_n = \{x | a \leq f_n(x) \leq b\}$ , Lebesgue's conclusion follows from the equality

$$e = \bigcup_n \bigcap_{i \geq n} e_i$$

and the measurability of the limit inferior of the sequence  $\{e_n\}$ . The "great importance" of this proposition will become apparent below.

Lebesgue stated a definition of integral for a bounded measurable function defined only for the points of a (measurable) set  $E$ . "Let  $AB$  be a segment containing  $E$  and define a function  $\phi$  to be equal to  $f$  for the points of  $E$  and equal to zero for the points of  $C_{AB}(E)$ . The integral of  $f$  taken in  $E$  is, by definition, the integral of  $\phi$  taken in  $AB$ ." Thus Lebesgue could integrate functions over more general sets than intervals. In particular, "If  $E$  is the union of  $E_1, E_2, \dots$ , all these sets being measurable and [mutually disjoint], and if the function  $f$  is measurable in  $E$ , then one has

$$\int_E f(x) dx = \sum \int_{E_i} f(x) dx."$$

Lebesgue extended his definition of integral to unbounded functions in the following way. "The geometric method which was so useful to us at the beginning of this chapter, being based on the concept of the measure of a bounded set, applies only to bounded functions.<sup>16</sup> To the contrary, the analytic method [of definition] can be applied almost without modification to [unbounded] functions.

"A function is called measurable if, for arbitrary  $a$  and  $b$ ,

---

<sup>16</sup>Lebesgue observed in a footnote, however, that "there is no difficulty in stating the problem of the measure of points for all sets, bounded or not."

the set of values of  $x$  for which

$$a < f(x) < b$$

is measurable.

"Let  $f(x)$  be a summable function. Choose the numbers

$\dots < m_{-2} < m_{-1} < m_0 < m_1 < m_2 < \dots$  varying between  $-\infty$  and  $+\infty$  and such that  $m_i - m_{i-1}$  is bounded in absolute value. Let  $e_i$  be the set of values of  $x$  for which  $f(x)$  is equal to  $m_i$  and  $e'_i$  be the set of values for which

$$m_i < f(x) < m_{i+1}.$$

"Consider the two sums

$$\sigma = \sum m_i m(e_i) + \sum m_i m(e'_i) \qquad \gamma = \sum m_i m(e_i) + \sum m_{i+1} m(e'_i)$$

in which the symbols  $\sum$  represent the sums of two series, one of positive terms, the other of negative terms. [That is, as the index  $i$  runs through negative and nonnegative integral values]. These series can be convergent or divergent; if the series which figure in  $\sigma$  are convergent, that is to say, if  $\sigma$  has a meaning, then  $\gamma$  has a meaning and conversely, and this holds however the  $m_i$  are chosen.

"By reasoning as before, one will see that the two sums of  $\sigma$  and  $\gamma$  tend toward the same limit, independently of the choice of the  $m_i$ , when the number of  $m_i$  is augmented in such a way that the maximum of  $m_i - m_{i-1}$  tends toward zero. This limit is the integral."

If the limit exists as a finite real number, then  $f$  is called integrable. The term summable, which conveys an immediate intuition in the context of Lebesgue's definition, is used by some authors as a synonym for "integrable."

The result of greatest interest for our purposes is the theorem Lebesgue stated and proved regarding the interchange of the operations of limit and integral and its consequent, term by term integration of infinite series. "Most of the discontinuous functions that have been considered up to now in analysis have been defined by means of infinite series; it is therefore of interest to know the following theorem.

"If a sequence of integrable [and therefore measurable] functions  $f_1, f_2, \dots$  has a limit  $f$ , and if  $|f - f_n|$  remains, for arbitrary  $n$ , less than a fixed number  $M$ , then  $f$  has an integral which is the limit of the integrals of  $f_n$ ."

Lebesgue's method of proof of this theorem demonstrates the way he utilized the properties of measurable functions and measurable sets to obtain his results. First, the measurability of the limit function  $f$  follows from the measurability of the functions  $f_n$ . Let  $E$  be a measurable set. "Choose an arbitrary positive number  $\epsilon$ . Let  $e_n$  be the set of values of  $x \in E$  for which  $|f - f_{n+p}| < \epsilon$  fails for all nonnegative values of  $p$ ; then  $e_n$  is measurable." Thus,  $e_n = \{x \in E \mid |f - f_{n+p}| \geq \epsilon \text{ for all } p \geq 0\}$ . Now  $E - e_n = \{x \in E \mid |f - f_{n+p}| < \epsilon \text{ for all } p \geq 0\}$  for all  $p \geq 0$  and  $E - e_n$  is a measurable set. Since, as usual,

$$\left| \int_E f - \int_E f_n \right| = \left| \int_E f - f_n \right| \leq \int_E |f - f_n|,$$

and  $E - e_n$  and  $e_n$  are disjoint measurable sets whose union is  $E$ ,

$$\int_E |f - f_n| = \int_{e_n} |f - f_n| + \int_{E - e_n} |f - f_n|.$$

Furthermore

$$\int_{e_n} |f - f_n| \leq Mm(e_n) \text{ and } \int_{E - e_n} |f - f_n| \leq \epsilon m(E - e_n),$$

from which it follows that

$$\left| \int_E f - \int_E f_n \right| \leq Mm(e_n) + \epsilon m(E - e_n).$$

"Now each set  $e_n$  contains all those of greater indices and there exists no point common to all the  $e_n$ . Therefore  $m(e_n)$  tends to zero with  $\frac{1}{n}$  and consequently so does

$$\left| \int_E f - \int_E f_n \right|."$$

That is, since  $e_1 \supset e_2 \supset \dots$  and  $\bigcap_{n=1}^{\infty} e_n = \phi$ ,  $m(\bigcap_{n=1}^{\infty} e_n) = \lim_n m(e_n) = 0$ .

Another way of expressing the conclusion of the theorem is

$$\int_E \lim_n f_n = \lim_n \int_E f_n,$$

i.e., in these circumstances, the limiting operations can be interchanged.

This proposition is called Lebesgue's bounded convergence theorem.

Lebesgue continued, "When  $f$  is bounded the proposition can be

stated as follows: If a sequence  $f_1, f_2, \dots$  of measurable functions, bounded above in absolute value..., has a limit  $f$ , then the integral of  $f$  is the limit of the integrals of the functions  $f_n$ .

"Here is another form of statement of the theorem in the general case:

"If the set of remainders of a convergent series of integrable functions is bounded above in absolute value, then the series is integrable term by term."

We observe with respect to the statement of this last theorem that the limit of a convergent series of measurable functions is a measurable function.

It is of interest to compare these theorems with the strongest corresponding propositions which can be obtained in Riemann's theory of integration. These are the theorems of Arzelà<sup>17</sup> and independently of Osgood,<sup>18</sup> published in 1885 and 1897, respectively, which may be stated as follows: If  $\{f_n\}$  is a sequence of Riemann integrable functions such that  $|f_n| \leq M$  and  $\lim_{n \rightarrow \infty} f_n = f$ , and if  $f$  is Riemann integrable, then

$$\int_a^b \lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} \int_a^b f_n.$$

It follows from this that if  $\{f_n\}$  is a sequence of Riemann integrable functions such that  $f = \sum_{n=1}^{\infty} f_n$  where  $f$  is Riemann integrable, then

---

<sup>17</sup>Arzelà, "Sulla integrazione per series," 532-537, 566-569.

<sup>18</sup>Osgood, "Non-uniform Convergence and the Integration of Series Term by Term," 182.

$$\int_a^b \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_a^b f_n. \quad ^{19}$$

An essential difference between these propositions is that in Lebesgue's theory the measurability of the limit function is implied by the measurability of the  $f_n$ , while in the theory of Riemann the integrability of the limit function must be made the consequence of a special hypothesis: the limit of a sequence of Riemann integrable functions need not be Riemann integrable. This distinction, in concurrence with the more general class of Lebesgue integrable functions, means that convergence theorems relating to the interchange of limit and integral are valid in Lebesgue's theory of integration under considerably less restrictive hypotheses than the corresponding propositions in the integration theory of Riemann. Such interchanges are frequently encountered in analysis since a function is often defined by a limit process. Thus the Lebesgue integral is better adapted to analytical processes, and is therefore of greater usefulness in analysis than the integral of Riemann. A classic example of this fact is the theorem of Riesz-Fischer, which will be described below.

One way to generalize Lebesgue's bounded convergence theorem is to relax the requirement that the sequence  $\{f_n\}$  converge to  $f$  for every  $x \in E$ . It might be required instead that  $\{f_n\}$  converge to  $f$  for every  $x \in E - A$  where  $A$  is a set of measure zero. If the limit function  $f$

---

<sup>19</sup>See Kestelman, "Riemann Integration of Limit Functions," 182, for elementary proofs of these theorems.

remains measurable then the proof given by Lebesgue is clearly valid in this more general case.

If a proposition with respect to a measurable set  $E$  is true for every element of  $E$  with the exception of at most a measurable subset of  $E$  of measure zero, then the proposition is said to be true almost everywhere in  $E$ . The phrase "almost everywhere" is abbreviated by a.e. Thus, in the circumstance above, it is required that  $\lim_n f_n = f$  a.e. in  $E$ .

Now if the measure is complete and if  $f$  is a measurable function, then  $f = g$  a.e. implies  $g$  is a measurable function. This assertion fails in general if the measure is not complete. For consider the  $\sigma$ -ring of Borel measurable sets and define, for purposes of argument, a function to be Borel measurable if and only if for every  $a$  and  $b$ ,  $a > b$ , the set  $\{x | a > f(x) > b\}$  is measurable (B). Let  $C$  be the Cantor set on  $[0,1]$  and  $E$  be a subset of  $C$  which is not Borel measurable. Define  $f(x) = 0$  for every  $x \in [0,1]$  and

$$g(x) = \begin{cases} 0 & \text{if } x \in [0,1] - C \\ 1 & \text{if } x \in C - E \\ -1 & \text{if } x \in E. \end{cases}$$

Then  $f$  is Borel measurable,  $f = g$  a.e., and yet  $g$  is not Borel measurable. Thus, in this circumstance, the measurability of a function may be lost by changing its values on a set of measure zero. Lebesgue's measure is complete and therefore if  $f$  is an integrable function and if  $f = g$  a.e. then  $g$  is integrable; furthermore

$$\int_E f = \int_E g.$$

This property of completeness is also expressed with respect to the measurability of the limit a.e. of a sequence of measurable functions. Define, for example, the sequence  $\{f_n\}$  of Borel measurable functions by

$$f_n = \begin{cases} \frac{1}{n} & \text{if } x \in [0,1] - C \\ (-1)^n & \text{if } x \in C. \end{cases}$$

Then  $\liminf_n f_n = g$  a.e. where  $g$  is the function defined above. Thus, if Lebesgue's bounded convergence theorem were expressed in this more general form, i.e.,  $\liminf_n f_n = f$  a.e., for Borel measurable functions, then an additional hypothesis would be required to ensure the measurability of the limit function  $f$ . Of course the limit a.e. of a sequence of Lebesgue measurable functions is Lebesgue measurable. The validity of Lebesgue's bounded convergence in this more general case may help to explain why a measure which is complete is preferred to one which is not.

Lebesgue gave the Cours Peccot on integration theory at the College of France during the academic year 1902-1903. His lectures were collected and published in 1904 as one of the Borel monographs on the theory of functions under the title, "Lessons on integration and research on primitive functions." Lebesgue devoted six of the seven chapters of his text to the history of the development of the concept of integral. "A complete history could not be given in twenty lessons; thus leaving aside many important results . . . [and] numerous definitions which have been successively proposed for the integral of real valued functions of one real variable, I have retained only those that

in my opinion are indispensable to know in order to understand well all the transformations the problem of integration has undergone and in order to comprehend the relationships that exist between the notion of area, so simple in appearance, and certain analytical definitions of the integral with very complicated aspects."<sup>20</sup>

Again, Lebesgue attempted to justify his work. "One may ask, it is true, if there is any interest in considering such complications and if it would not be better to limit oneself to the study of functions which only necessitate simple definitions. This has hardly any advantage . . . [for] as one will see in these lessons, if one wishes to limit oneself to consideration of these simple functions, it will be necessary to renounce the resolution of many problems with simple statements which have been asked for a long time. It is for the resolution of these problems, and not through love for complications, that I have introduced in this book [my] definition of the integral . . . ."

Lebesgue expostulated with those who objected to the study of discontinuous functions. "Those who read me carefully, while regretting perhaps that things are not simple, will agree with me, I think, that this definition is necessary and natural. I dare to say that it is in a certain sense simpler than that of Riemann, as easy to comprehend as his, and that only habits of mind which have been acquired earlier can make it seem more complicated. It is more simple because it places in evidence the most important properties of the integral . . . ."

Lebesgue noted that his researches on primitive functions and

---

<sup>20</sup> Lebesgue, Lecons sur l'intégration et la recherche des fonctions primitives, v.

rectification of curves were given in his course as applications of his definition of integral. "To these two applications I wished to join another which is very important: the study of the trigonometric development of functions, but in my course I have given this subject such scant attention that I have decided not to reproduce it here." Lebesgue had pursued research on trigonometric series representation of Lebesgue integrable functions during his Cours Peccot tenure, the results of which were published as separate papers.

Having described the theories of integration of Cauchy and Riemann, Lebesgue began his investigation of the "problem of integration" by stating conditions to be satisfied by an integral "if one wants that there be some analogy between [it] and the integral of continuous functions." Lebesgue proposed "to attach to each bounded function  $f(x)$ , defined in a finite interval  $(a,b)$ , a finite number, positive, negative, or zero,  $\int_a^b f(x)dx$ , that we call the integral of  $f(x)$  in  $(a,b)$  and which satisfies the following conditions:

1. For arbitrary,  $a, b, h$ , one has

$$\int_a^b f(x)dx = \int_{a+h}^{b+h} f(x-h)dx.$$

2. For arbitrary  $a, b, c$ , one has

$$\int_a^b f(x)dx + \int_b^c f(x)dx + \int_c^a f(x)dx = 0.$$

$$3. \quad \int_a^b (f(x) + \phi(x)) dx = \int_a^b f(x) dx + \int_a^b \phi(x) dx.$$

4. If  $f \geq 0$  and  $b > a$  then

$$\int_a^b f(x) dx \geq 0.$$

5. One has

$$\int_0^1 1 dx = 1$$

6. If  $f_n(x)$  tends increasingly toward  $f(x)$ , the integral of  $f_n(x)$  tends toward the integral of  $f(x)$ .<sup>21</sup>

"In enunciating the six conditions of the problem of integration, we define the integral. This definition belongs to the class of definitions that one can call descriptive; in these definitions one states the characteristic properties of the object that one wants to define. In constructive definitions, one states which operations it is necessary to complete in order to obtain the object that is defined.

"When one states a constructive definition it must be shown that the indicated operations are possible; a descriptive definition is also subject to certain conditions: it is necessary that the conditions stated be compatible [i.e., non contradictory] . . . It is also necessary to study the possibly ambiguous nature of the objects that one wants to define. Suppose, for example, that one has demonstrated the impossibility of the existence of two different classes of objects satisfying the conditions indicated and that, in addition, one has

---

<sup>21</sup>Lebesgue, op. cit., 98.

demonstrated the compatibility of these conditions by choosing a class of objects satisfying them. This class of objects is then uniquely defined, so that the constructive definition which has served to effect the choice is exactly equivalent to the descriptive definition.

"We seek a constructive definition equivalent to the descriptive definition of the integral."

Lebesgue stated in this respect the primary purpose of his work. ". . . one can say that the investigations reported in this treatise have as their principal goal the discovery of a constructive definition equivalent to the descriptive definition of primitive functions."<sup>22</sup>

Lebesgue succeeded in demonstrating, for the class of Lebesgue integrable functions, that there is exactly one constructive definition of integral which satisfies the descriptive definition, and that is the definition of the Lebesgue integral. Thus "the reasonings employed show that the problem of integration is possible and in only one way, if it is posed for summable functions."

Lebesgue resolved a question which had remained open in his thesis by proving this theorem: ". . . the indefinite integral of a summable function admits this function as its derivative except at the points of a set of measure zero." That is, if  $f(x)$  is integrable on  $[a,b]$ , then

$$\left( \int_a^x f \right)' = f(x) \text{ a.e.}$$

---

<sup>22</sup>Lebesgue, op. cit., 100.

This remarkable theorem is only one of many stated and proved by Lebesgue with respect to "the research of primitive functions," which is of course the focal point of the Lecons.<sup>23</sup>

---

<sup>23</sup>It is necessary to mention the work of W. H. Young in his attempt to generalize the Riemann integral as the natural generalization of the definition given by Darboux. In a paper titled "On the General theory of Integration" published in 1904, Young recalled Darboux's definition of Riemann integral, and in particular, that the interval of integration is partitioned into a finite number of subintervals. "The progress of the modern theory of sets of points . . . due, as is well known, chiefly to G. Cantor, though taking its origin in Riemann's paper 'Ueber die Darstellbarkeit . . . ' naturally leads us to put the question how far these definitions can be generalized. This theory has in fact taught us on the one hand that many of the theorems hitherto stated for finite numbers are true with or without modification for a countably infinite number, and on the other hand that closed sets of points possess many of the properties of intervals.

"What would be the effect on the Riemann and Darboux definitions, if in those definitions, the word 'finite' were replaced by 'countably infinite,' and the word 'interval' by set of points? A further question suggests itself: are we at liberty to replace the segment  $(a,b)$  itself by a closed set of points, and so define integration with respect to any closed set of points?"

In the course of his investigations of these problems, Young created a definition of measure and integral equivalent to that of Lebesgue. The accomplishments of the two men, however, are reflected in their intentions. Young wished to generalize the definition of integral and succeeded in this. Lebesgue wished to resolve fundamental problems in analysis and created his theory of measure and integral as a means to this end. Lebesgue's attempts to reinstate the classical forms relating integral and derivative, for example, led to his bounded convergence theorem and its consequent, integration of any bounded

It is of interest for our purpose to call attention to Lebesgue's publication of a convergence theorem of which the bounded convergence theorem is a special case. In "On the method of Mr. Goursat for the resolution of the equation of Fredholm," printed in January, 1908, Lebesgue stated and proved this theorem: "A convergent sequence of summable functions  $f_i$  is integrable term by term when there exists a summable function  $F$  such that, whatever be  $i$  and the variable  $x$ ,  $|f_i| \leq |F|$ ."<sup>24</sup> In modern terms, and taking advantage of the completeness of Lebesgue measure, if  $\{f_n\}$  is a sequence of summable functions and if  $h$  is a summable function such that  $\lim_{n \rightarrow \infty} f_n = f$  a.e. and  $|f_n| \leq h$  for every  $n$ , then  $f$  is summable and

$$\int f = \int \lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} \int f_n .$$

This proposition is called Lebesgue's dominated convergence theorem.

It is to be expected that Lebesgue's work was rejected by some members of the mathematical establishment. Denjoy, in a memoir "Henri Lebesgue, the scholar, the teacher, the man," published in 1957,

---

derivative. Young did not discover these results because he was concerned with the question of definition rather than application of his integral. See Pesin, Classical and Modern Integration Theories for a description of Young's work and its relationship with that of Lebesgue.

<sup>24</sup>Lebesgue, "Sur la méthode de M. Goursat pour la résolution de l'équation de Fredholm," 12.

observed that Lebesgue "presented in his thesis . . . to mathematical service a tool of extraordinary power." Yet "The acceptance of the works of Lebesgue by the masters of the time was rather reserved. Many feared to see installed a teratology of functions. Darboux, whom one might have thought favorable because of his memoir of 1875 on discontinuous functions, was hostile to him. Boussinesq was supposed to have said, 'But a function has every interest in having a derivative!' He was speaking of the interest of he who uses it. Only Picard defended the research of Lebesgue and appreciated its qualities."<sup>25</sup>

Lebesgue reflected on these and other aspects of the objections which had been raised against the study of the functions of real variables in the Introduction of his "Notice on the scientific works of Henri Lebesgue," published in 1922. "In order to demonstrate the state of mind at the time when I began my research, I will indicate certain resistance which I encountered; all those who have taken up the same type of studies have met analogous resistance. I can do this without hesitation, for it has never been anything but conflicts in ideas, and I have always found the greatest personal goodwill in the case of those very people to whom my works were the least agreeable."<sup>26</sup>

"In 1899 I remitted to Mr. Picard a note on non ruled surfaces applicable on the plane; Hermite wished for a moment to oppose its insertion in the Comptes Rendus of the Academy; Mr. Picard had to defend my note. One knows how much, however, Hermite was filled with

---

<sup>25</sup>Denjoy, "Henri Lebesgue, Le Savant, Le Professeur, L'Homme," 15.

<sup>26</sup>Lebesgue, Notice sur les travaux scientifiques de M. Henri Lebesgue, 13.

goodwill and praise, but this was near the time he was writing to Stieltjès, 'I turn away with fright and horror from this lamentable plague of functions which have no derivatives,' and he wished to see excluded from the domain of mathematics all research in which these horrifying functions intervene."

Lebesgue observed that he became, for many mathematicians, the man of functions without derivatives, a charge which Lebesgue denied. Nevertheless, "as the horror manifested by Hermite was felt by nearly all, as soon as I tried to take part in a mathematical conversation there was always an analyst to tell me 'this cannot interest you; we are talking about functions with derivatives," and a geometer to repeat in his language 'We are taking up surfaces having a tangent plane.'"

Lebesgue expressed doubt that Darboux ever entirely pardoned his memoir on applicable surfaces and noted that "for a long time he was hardly interested in my memoir on integration . . . .It is said that in 1875 Darboux was somewhat criticized for having allowed himself to study such questions; whether because of these remonstrances, or whether because of the beauty and importance of the problems he took up after that, Darboux made no other incursion into the domain of non analytical functions."

Lebesgue believed that Borel was the first "to think that my work would have practical utility in some way. He did, in any case, think it before I did. I saw myself still hesitant before deciding to present as a doctoral thesis the memoir where I took up nearly all the research that I have since developed . . . .A little later, in 1903, I insisted on the necessity of these studies in the preface of my Lessons

on Integration. In an analysis of the book, Mr. Picard, while encouraging me as he has always done . . ., allowed some uneasiness to show through on the subject of possible exaggerations of the tendency that I represented."

Given widespread resistance to the study of functions of real variables, Lebesgue asked if these objections might have merit. He responded to this question by observing that most of the prior works on real functions, except for those relating to trigonometric series, were without comprehension in the sense that one could attribute to them a coherent body of theory. Many statements were negative; positive statements were sought after but seldom achieved. ". . .if one searched to generalize [a property or definition], then one would too often end up with a notion certainly new but serving nothing other than being defined. . . .If one sought for the most general functions possessing a certain property or to which is applied a certain definition, then one would end up with a class of functions variable with the property or definition envisaged, and which by consequent, could not naturally be put into any research; such had been the case for the class of functions integrable in the sense of Riemann."

Lebesgue argued that since the work was essentially an exploration of a disordered mass of functions, properties, and definitions, without knowledge of interest or application, and no criteria with which to judge such questions, mathematicians could be led to think that researches with respect to real functions could be suspended until the necessity of such researches became more apparent. Yet, "in spite

of the indifference and sometimes the opposition manifested with regard to the theory of functions of real variables, . . . it happened, as in the past with trigonometric series, that one encountered functions whose analyticity did not have to be assumed. It was thus for example in the study of solutions of differential equations by the method of Cauchy-Lipschitz, or by that of successive approximations by Mr. Picard. . . . Sometimes certain of the data or solutions could be or even necessarily were discontinuous functions, . . . at other times, as in the questions studied by Mr. Borel, the solution is continuous but non analytic. Thus one could become familiar with the idea that a discontinuity or a singularity is not necessarily a monstrosity."<sup>27</sup>

The reviews Lebesgue's work received in the Jahrbuch über die Fortschritte der Mathematik were mixed. Haussner wrote a three line review of Lebesgue's memoir "Sur une généralisation de l'intégrale définie" as follows: "The author gives a generalization of the definite integral which encompasses Riemann's definition as a special case and simultaneously permits the solution of the problem of primitive functions." On the other hand, Lebesgue's "Intégrale, longueur aire" received a very complete and unbiased review from Kowalewski. Again, Stäckel, reviewer of Leçons sur l'intégration et la recherche de fonctions primitives, was moved to write "The definition of Lebesgue [integral] . . . corresponds to a need which obviously cannot be contested. However, if one goes so far to say, as does the author, that in a certain sense it is more simple than Riemann's theory, and is just as easy to comprehend, since only certain methods of thought which had been assumed previously

---

<sup>27</sup>Lebesgue, op. cit., 16.

let them appear complicated, then all these things are debatable to the reviewer. At any rate it will have to be expressed in a set theoretic context and will hardly be amenable to elementary lectures, for which purpose however the definition of Riemann is very suitable."

Of course Lebesgue was not without support. Picard's advocacy has been noted. Lebesgue received his first university appointment in Rennes in 1902; he was chosen to give the Cours Peccot in 1902-1903. Indeed, in his inaugural lecture at the Collège de France, Lebesgue remarked how "the great authority of Camille Jordan gave to the new school a valuable encouragement which amply compensated for the few reproofs it had to suffer."

General acceptance of Lebesgue's ideas was another matter, however, and was achieved only as Lebesgue and others exploited the remarkable properties of his integral. One of the areas in which Lebesgue's conceptions were first applied was the study of trigonometric series. "Of all branches of analysis in which the use of an integral more powerful than Riemann's offered a rich reward, none was so promising as the theory of trigonometrical series,"<sup>28</sup> and it was to the investigation of such series that Lebesgue turned.

### Trigonometric Series

The theory of trigonometric series representation of functions had continued to develop in the period in which there was created a theory of measure and a more general theory of integration.

The reader will recall the memoirs of Heine and Cantor in which the uniqueness of representation of trigonometric series of functions was

---

<sup>28</sup>Burkill, "Henri Lebesgue," 58.

demonstrated for a particular class of functions, after the uniqueness of such representation had been called into question by the discovery of the relationship between uniform convergence and term by term integration of infinite series. The propositions of Heine and Cantor reinstated the uniqueness of representation of trigonometric series under certain conditions but did not speak to the question of the form of coefficients of such representations.

Ascoli considered this aspect of the problem in his memoir "On trigonometrical series," dated April, 1872.

"In a treatise on trigonometrical series...Mr. Heine has verified the following theorem:

"A function which is in general continuous but not necessarily finite can be developed in at most one way in a trigonometrical series of the form

$$3) \quad \sum_0 (a_n \sin nx + b_n \cos nx)$$

if the series is required to converge uniformly in general.

"Shortly after that Mr. Cantor...demonstrated how a function given by a trigonometrical series which is convergent in general for every value of  $x$  cannot be represented by another series of the same form. From this it seems to follow that the preconditions which have been made in Heine's theorem on the continuity of the function and on the type of convergence of the series are unnecessary.

"Also it seems to me that if a periodically repeating function in the interval  $[0, 2\pi]$  which is continuous in general...is representable by a trigonometric series of the form 3), then the development is not only unique but must be the development of Fourier."<sup>29</sup>

---

<sup>29</sup> Ascoli, "Über trigonometrische Reihen," 231.

Thus, Ascoli asserted, the coefficients of the unique trigonometric series representation of a function continuous except at a finite number of points are the Fourier coefficients. Dini published a paper, "On the series of Fourier," also dated April, 1872, which extended Ascoli's result to functions whose points of discontinuity have at most a finite number of limit points.<sup>30</sup>

These propositions were generalized in turn by du Bois Reymond.

In his "Proof that the coefficients of the trigonometric series  $f(x) =$

$$\sum_{p=0}^{p=\infty} (a_p \cos px + b_p \sin px) \text{ have the values } a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(\alpha) d\alpha, a_p = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(\alpha) \cos p\alpha d\alpha, b_p = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(\alpha) \sin p\alpha d\alpha, \text{ whenever these integrals are}$$

finite and determinate," published in 1875, du Bois Reymond argued that the coefficients of the trigonometric series corresponding to a Riemann integrable function are the coefficients of Fourier. Du Bois Reymond noted that "the main theorems of the theory of trigonometric series are called into question by introduction of the concept of uniform convergence," these being the theorems "if a trigonometrical development of the form  $f(x) = \sum_{p=0}^{p=\infty} (a_p \cos px + b_p \sin px)$  is given then there is no second of the same form," and "the coefficients of the developments can only be expressed [in the form of the Fourier coefficients].

"Both these theorems were proven by term by term integration of the series  $f(x)$  and both were suddenly left without meaning."<sup>31</sup>

<sup>30</sup>Dini, "Sopra la serie di Fourier," 161.

<sup>31</sup>duBois Reymond, "Beweis, dass die Coefficienten der

In his section "History of the further evolution of the study of trigonometrical series. The first theorem is restored," duBois Reymond observed that the work of Heine and Cantor "not only restored the theorem of the unambiguity of trigonometric representations...but also gave to it a general applicability of which one had not thought before these events took place."

Turning to the purpose of his paper in the section "On the second main theorem. The author announces that he is able to restore it," du Bois Reymond wrote, "With regard to the first main theorem of the theory of trigonometric series, we can consider the research as being closed. There seems to be no publication on the subject of the second theorem, that under certain necessary conditions on  $f(x)$  the coefficients have the form discovered by Fourier, so nothing has been clarified here.

"I have known for some time that the second theorem can also be restored if one assumes  $f(x)$  to be continuous except at special points.

---

trigonometrischen Reihe

$$f(x) = \sum_{p=0}^{p=\infty} (a_p \cos px + b_p \sin px)$$

die Werthe

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(\alpha) d\alpha, a_p = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(\alpha) \cos p\alpha d\alpha, b_p = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(\alpha) \sin p\alpha d\alpha$$

haben, jedesmal wenn diese Integrale endlich und bestimmt sind,"121.

"This condition on  $f(x)$  is more restrictive, however, than integrability in general, which suffices for the existence of Fourier coefficients. The task which challenged my mathematical curiosity most was to find if the coefficients of the series of  $f(x)$  have the form of Fourier whenever  $f(x)$  is integrable....

"Now I believe I am able to solve the problem and in a most general fashion."<sup>32</sup>

Du Bois Reymond's proof of his theorem is extremely long and complicated.<sup>33</sup> Indeed the length and difficulty of the proof were such that du Bois Reymond was moved to comment in this respect in a concluding statement. "If the primeval saga is a juxtaposing of enjoyment without effort in paradise and the hard work of the just after the fall, then our science shows a similar anthesis. After the first analytical epoch, which ended approximately with Fourier and Poisson, and in which there were many new discoveries of formulas and theorems but little concern with their precise formulation and range of validity... we, having eaten from the tree of cognizance [of uniform convergence], have had to struggle with profound difficulties.., in order to gain again these results for science."<sup>34</sup>

Some years later, in 1881, in a paper titled "On the integration of trigonometric series," du Bois Reymond observed, "Heine's theorem shows that a Fourier series which converges [uniformly in general] to a function continuous at all but a finite or suitably grouped infinite set of points

---

<sup>32</sup> du Bois Reymond, op. cit., 123.

<sup>33</sup> See Gibson, "On the History of the Fourier Series," 163, for a partial synopsis.

<sup>34</sup> du Bois Reymond, op. cit., 160.

admits integration in its members, but nothing can be said on the basis of Heine's theorem of the term by term integration of a trigonometrical series, assuming only that the sum is integrable."<sup>35</sup> Du Bois Reymond was able to demonstrate that a trigonometric series which converges to a Riemann integrable function can be integrated term by term. Thus the significance of uniform convergence for such integration of a trigonometric series was shown to be less important than initially assumed. Of course the theorems of Argelà and Osgood are generalizations of this proposition of du Bois Reymond.

In the second section of the same volume of the Abhandlungen der Bayerischen Academie in which du Bois Reymond proved his theorem that if a trigonometric series converges in  $(-\pi, \pi)$  to a Riemann integrable function  $f(x)$ , then the series is the Fourier series of  $f(x)$ , du Bois Reymond published another long and arduous paper in which, by considering particular forms of  $f(\alpha)$  in the Dirichlet integral  $\int_0^b f(\alpha) \frac{\sin h \alpha d\alpha}{\alpha}$ , he demonstrated the existence of a continuous function whose Fourier series does not converge at a particular point, and more generally, a function continuous in  $(-\pi, \pi)$  whose Fourier series does not converge at the points of an everywhere dense set.<sup>36</sup> Thus the question of Fourier series representation at every point of every continuous function was decided in the negative.

Jordan obtained a simplified sufficient condition for convergence in his memoir "On the series of Fourier," published in 1881. The sufficient

---

<sup>35</sup> du Bois Reymond, "Ueber die Integration der trigonometrischen Reihe," 260.

<sup>36</sup> du Bois Reymond, "Untersuchungen über die Convergenz und Divergenz der Fourierschen Darstellungsformeln," 72.

conditions of Dirichlet wrote Jordan "depend upon the two following propositions:

$$1) \lim_{p \rightarrow \infty} \int_a^b f(x) \frac{\sin px}{x} dx = 0 \text{ if } 0 < a < b < \pi$$

and 2) This limit is equal to  $f(+0)$  if  $a = 0$  and  $0 < b < \pi$ .

"Dirichlet assumed for his demonstration that in the interval of integration  $f(x)$  has only a finite number of discontinuities and a finite number of maxima and minima....

"One sees easily that the first proposition holds under the single condition that  $f(x)$  is integrable from  $a$  to  $b$ . [Jordan attributed this remark to Darboux].

"With regard to the second proposition, its demonstration requires only that there exists, in a neighborhood of the point  $x = 0$ , a finite interval  $(0, \epsilon)$  in which  $f(x)$  is constantly non increasing or non decreasing.

"The theorem holds therefore whenever  $f(x)$  can be represented from 0 to  $\epsilon$  by  $h(x) - g(x)$  where  $h(x)$  and  $g(x)$  are two finite non decreasing functions."<sup>37</sup>

Jordan then introduced the concepts of negative, positive, total and bounded variation of a function in an interval, and showed that a function of bounded variation can be expressed as the difference of two monotone functions.

"The demonstration of Dirichlet is therefore applicable, without modification, for every function of bounded variation." Expressed

---

<sup>37</sup> Jordan, "Sur la série de Fourier," 228.



differently, Jordan's sufficient condition for the convergence of the Fourier series of  $f(x)$  at a point  $x$  to  $\frac{1}{2} (f(x+0) + f(x-0))$  is that  $f(x)$  be Riemann integrable and of bounded variation in a neighborhood of the point  $x$ . Dini<sup>38</sup> and Lipschitz<sup>39</sup> had previously published sufficient conditions for convergence of the Fourier series of an integrable function  $f(x)$  at a point  $x$  in the form  $\lim_{\delta \rightarrow 0} \log |\delta| (f(x+\delta) - f(x)) = 0$  and  $|f(x+\delta) - f(x)| < A \delta^\alpha$  where  $A$  and  $\alpha$  are constants, respectively. The inequality is now called a Lipschitz condition.

One more result from the theory of Fourier series which will be of interest in later work is Parseval's equality. Properties of the Fourier coefficients had been sought for some time and it was well known that if  $f$  and  $f^2$  are Riemann integrable on  $(-\pi, \pi)$  and if  $a_n$  and  $b_n$  are the Fourier coefficients of  $f(x)$ , then

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leq \frac{1}{\pi} \int_{-\pi}^{+\pi} f^2 .$$

In a paper "On several applications of the integral of Poisson," published in 1893, de la Vallée - Poussin was successful in demonstrating the inequality to be an equality,<sup>40</sup> a result obtained independently some years later by Hurwitz.<sup>41</sup> The equality has since become known

<sup>38</sup>Dini, op. cit., 161.

<sup>39</sup>Lipschitz, op. cit., 294.

<sup>40</sup>de la Vallée-Poussin, "Sur quelques applications de l' integral de Poisson," 18.

<sup>41</sup>Hurwitz, "Über die Fourierschen Konstanten integurbarer Funktionen," 425.

as Parseval's equality.

Lebesgue published one of the first applications of his conception of integral to the study of trigonometric series corresponding to functions in 1903 in a memoir titled, "On trigonometrical series." He wrote, "In my work with trigonometric series, my principal goal has been to demonstrate the utility that the notion of integral I introduced in my thesis can have in the study of discontinuous functions of a real variable."<sup>42</sup>

After a brief description of his conception of measure and integration, Lebesgue stated the means by which he sought to attain his purpose. "I am going to apply the concept of integral to the study of trigonometric development of functions non integrable in the sense of Riemann.

"Among the methods which have been used for the study of trigonometric series, the only one which can be applied to these functions is that of Riemann. But this method has led up to now only to two or three general properties, [one of which is] the theorem of Cantor on the impossibility of two developments of the same function...it results from the work of Dini and Ascoli that a continuous function can only be represented trigonometrically by means of a Fourier series. P. du Bois Reymond, with the aid of considerations which are not perhaps exempt from all criticism has extended the same theorem to Riemann integrable functions. The search for sufficient conditions for the possibility of the trigonometric development of non integrable functions, in the sense of Riemann, has not yet been taken up because of the ignorance in which one finds oneself regarding the form of the coefficients of the development

---

<sup>42</sup>Lebesgue, "Sur les series trigonometriques," 453.

of these functions."<sup>43</sup>

Lebesgue resolved this problem by demonstrating the theorem "If a bounded function admits a convergent trigonometric development for all values of the variable, except perhaps for values of a reducible set, then this trigonometric development is the series of Fourier." Thus, the coefficients of the trigonometric series corresponding to such functions are the Fourier coefficients.

As a second important result, Lebesgue succeeded in demonstrating a generalization of a theorem of Riemann: "The integrals of Fourier, corresponding to a function having an integral in the generalized sense of the word, tend toward zero when their index is indefinitely increased."<sup>44</sup> That is, if  $f(x)$  is an integrable function in the sense of Lebesgue, then

$$\lim_{t \rightarrow \infty} \int f(x) \cos t x dx = \lim_{t \rightarrow \infty} \int f(x) \sin t x dx = 0$$

Lebesgue then used this result to obtain a number of sufficient conditions for the convergence of a Fourier series. Given sufficient conditions for such convergence Lebesgue was able to demonstrate the existence of "a function non integrable in the sense of Riemann and representable by a [Fourier] series for all values of the variable." In this respect Lebesgue observed that "the calculation of the coefficients which was made at the beginning was therefore not without purpose."

Lebesgue ended his memoir with the statement and proof of another striking theorem. Lebesgue showed that if  $f(x)$  is a summable function

---

<sup>43</sup>Lebesgue, op. cit., 466.

<sup>44</sup>Lebesgue, op. cit., 474.

and if  $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$  is the Fourier series corresponding to  $f(x)$ , then

$$\int_0^x f(x) dx = \frac{a_0}{2}x + \sum_{n=1}^{\infty} \frac{1}{n} (a_n \sin nx + b_n (1 - \cos nx)) \text{ for every}$$

$x \in (0, 2\pi)$ . "The series of Fourier are therefore integrable term by term in each interval....[This] proposition...is the generalization of a theorem of du Bois Reymond."<sup>45</sup> One remarkable aspect of this theorem is that the conclusion follows whether or not the Fourier series of  $f(x)$  converges.

Lebesgue returned to the consideration of sufficient conditions for convergence of Fourier series in a subsequent paper, "Researches on the convergence of the series of Fourier," published in 1905. Lebesgue argued that "in order to study the convergence of the series of Fourier toward the corresponding function  $f(x)$ , it suffices...to study the convergence toward zero of one or the other of the integrals [when  $S_m$  denotes the sum of the first  $m$  terms of the Fourier series]

$$\pi(S_m - f(x)) = \int_0^{\frac{\pi}{2}} \frac{\phi(t)}{\sin t} \sin(2m+1)t dt = \int_0^{\frac{\pi}{2}} \psi(t) \sin(2m+1)t dt$$

where  $\phi(t) = \sin t$   $\psi(t) = f(x+2t) + f(x-2t) - 2f(x)$ ."<sup>46</sup> Of course this is an adoption of the method of Riemann. Lebesgue succeeded in demonstrating the theorem: "The series of Fourier converges to the function  $[f(x)]$  at the point  $x$  if the integral of  $|\phi(t)|$  has a derivative equal to zero at  $t = 0$ , and if the quantity

<sup>45</sup>Lebesgue, op. cit., 485.

<sup>46</sup>Lebesgue, "Recherches sur la convergence de series de Fourier," 252.

$$\int_{\delta}^{\alpha} |\psi(t+\delta) - \psi(t)| dt, \quad 0 < \alpha < \pi, \quad 0 < \delta,$$

tends toward zero with  $\delta$ .

"This is the statement that I wanted to obtain."<sup>47</sup> This is a more general sufficient condition for convergence for, as shown by Lebesgue, it includes as special cases the previously given sufficient conditions of Dirichlet, Lipschitz, Dini, and Jordan.

Lebesgue was chosen to give the Cours Peccot on trigonometric series at the Collège de France in the year 1904-1905. He published his collected lectures as another of the Borel monographs in 1906 under the title "Lessons on Trigonometric Series." As might be expected, the focus of this "petit livre" is the theory of the trigonometric series representation of functions. Lebesgue discussed a wide range of topics relating to such study, including applications to the Poisson integral and the Dirichlet problem, and a history of the development of the theory. Having described some of Lebesgue's earlier results, there is one proposition of particular interest in the Lecons, a generalization of Parseval's equality. Lebesgue demonstrated that "for each bounded summable function  $f$ , such that

$$f \sim \frac{1}{2}a_0 + \sum (a_p \cos px + b_p \sin px)$$

one has

$$\frac{1}{\pi} \int_0^{2\pi} f^2(\theta) d\theta = \frac{1}{2}a_0^2 + \sum_{p=1}^{\infty} (a_p^2 + b_p^2)."$$
<sup>48</sup>

---

<sup>47</sup>Lebesgue, op. cit., 263.

<sup>48</sup>Lebesgue, Lecons sur les Series Trigonometriques, 100.

Lebesgue published a number of papers on the topic of trigonometric series; these first propositions suffice to demonstrate the usefulness of his conception of integral as a means to obtain a more comprehensive theory. In Lebesgue's words, "...the generalization of the notion of integral permits me to give a much more extended meaning to [the classical theorems]." These results helped to stimulate interest in Lebesgue's work. It was left to Riesz and Fischer, however, to independently and almost simultaneously appeal to Lebesgue's ideas to discover a theorem sufficiently important to establish Lebesgue's conceptions as fundamental in the theory of real functions.

#### The Riesz-Fischer theorem

Fatou was one of the first after Lebesgue to apply Lebesgue's more general conception of integral to the study of trigonometric series. In a paper "Trigonometric series and series of Taylor," published in 1906, Fatou wrote "...the introduction of these notions of measure and generalized integral, which constitutes an important progress in the study of point sets and of functions of real variables, can equally serve to resolve problems which are posed in formerly cultivated branches of analysis.

"Already Mr. Lebesgue, in a memoir which appeared in the Annales de l'École normale supérieure, has applied his notion of integral to the study of trigonometric series, and demonstrated, among other things, that if a trigonometric series is convergent and represents a bounded function, then the coefficients of this series are given by the formulas of Euler-Fourier where the integrals are taken in the generalized sense of the

word. Now there exist bounded functions non integrable in the sense of Riemann which are representable at each point by a convergent trigonometric series; this result permits therefore...more unity and generality in the theory of Fourier series.

"In this work I demonstrate an analogous result relative to the integral of Poisson; if a harmonic function regular in the interior of a circle remains bounded there, then it can be expressed with the aid of a Poisson integral, the integral being taken in the sense of Mr. Lebesgue."<sup>49</sup> Fatou had undertaken this work, he remarked, "in large part for the purpose of demonstrating the advantage that one can obtain in these questions from the new notions of measure of sets and of generalized definite integral."

One of Fatou's results which is of independent interest is a proposition which is now called "Fatou's Lemma." It is "if a sequence  $f_1(x), f_2(x), \dots$  of positive bounded summable functions tends toward a function  $f(x)$  bounded or not, and if

$$\int_a^b f_n(x) dx$$

remains, for every  $n$ , less than a fixed number, then the function  $f(x)$  is integrable and one has

---

<sup>49</sup>Fatou, "Series trigonometriques et series de Taylor," 337.

$$\int_a^b f(x) dx \leq \liminf \int_a^b f_n(x) dx."^{50}$$

Fatou used this theorem to obtain a more general form of Parseval's equality. He demonstrated "if  $f(u)$  is a square summable function, then

$$\frac{1}{\pi} \int_{-\pi}^{+\pi} f^2(u) du = 2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2),$$

$a_n$  and  $b_n$  being the Euler-Fourier constants attached to  $f(u)$ ."<sup>51</sup> Thus

Fatou succeeded in demonstrating Parseval's equality for any square summable function, i.e., without restricting his argument to bounded functions as had Lebesgue in his proof of this theorem.

As will be seen, Fatou's lemma was important in another respect; it influenced Riesz to utilize Lebesgue's conception of integral in his attempt to extend Hilbert's results in the study of linear integral equations. Hilbert, in his investigation of the equation

<sup>50</sup>Fatou, op. cit., 375. We should observe that if  $\{a_n\}$  is a sequence of real numbers and if  $b_n = \inf_{i \geq n} \{a_i\}$ , then  $b_n = -\infty$  for every  $n$ , or  $\{b_n\}$  is a monotonically increasing sequence of real numbers. Therefore  $\lim_{n \rightarrow \infty} b_n$  exists as a finite real number or  $+\infty$ . This limit is defined to be the limit inferior of the sequence  $\{a_n\}$  and is denoted  $\liminf_{n \rightarrow \infty} a_n$ . If  $\liminf_{n \rightarrow \infty} a_n = a$  and  $b$  is any limit point of the sequence  $\{a_n\}$  then  $a \leq b$ , that is, the limit inferior of  $\{a_n\}$  is the least limit point of the sequence. This may help to explain the interest in Fatou's lemma.

<sup>51</sup>Fatou, op. cit., 379.

$$4) \quad f(x) + \int_a^b K(x,y)f(y)dy = u(x)$$

where the functions  $K(x,y)$  and  $u(x)$  are given and  $f(x)$  is unknown, had introduced the concept of a complete orthogonal system of functions  $\{\phi_n\}$ , and had, in the course of solution of the integral equation 4) sought the "generalized Fourier coefficients" of the unknown function  $f(x)$  with respect to  $\{\phi_n\}$ , that is,

$$c_n = \int_a^b f(x)\phi_n(x)dx. \quad 52$$

Riesz commented in this regard in the paper in which he published his form of the Riesz-Fischer theorem, "On orthogonal systems of functions," dated March 11, 1907. "Mr. Hilbert has introduced a general method for the resolution of certain functional equations of the type... of Fredholm. This method consists of relating the resolution of these functional equations to the resolution of an infinite system of linear equations in an infinite number of unknowns. Mr. Hilbert makes the connection between these two problems by using an orthogonal system of functions; the coefficients, like the unknowns of the latter equations, [are] integrals obtained from the given functions and the unknown functions of the problem, in a manner analogous to the coefficients of Fourier, with the aid of an orthogonal system of functions.

---

<sup>52</sup>The definition of a complete orthogonal system of functions and a description of Hilbert's method of solution of the linear integral equation 4) is given in Appendix D.

"For the method of Mr. Hilbert, the following question is of great importance:

"Being given an orthogonal system of functions [on] a determined interval, attribute to each function of the system a real number. Then under what conditions will there exist a function such that for each function of the system [of orthogonal functions], the integral of the product of this function and the function in question, taken on the interval, will be equal to the given number?" That is, if  $\{\phi_n\}$  is an orthogonal sequence of functions defined on  $[a,b]$  and if  $\{c_n\}$  is a given sequence of real numbers, then under what conditions does there exist a function  $f(x)$  such that

$$c_n = \int_a^b f(x)\phi_n(x)dx \quad \text{for } n = 1, 2, \dots?$$

Riesz continued, "For the class of summable functions, bounded or not, but whose square is summable, the theorem that I am going to give completely resolves the question."<sup>53</sup>

After observing that "an orthogonal system of functions of which none has integral zero, must be finite or denumerable,"<sup>54</sup> Riesz stated

<sup>53</sup>Riesz, "Sur les systèmes orthogonaux de fonctions," 616.

<sup>54</sup>As a generalization of a theorem of Schmidt, Riesz had proved this assertion for bounded functions in his memoir "Sur les ensembles de fonctions," published in November, 1906. Riesz observed that the theorem "can be extended without difficulty to all square summable functions," i.e., whether or not the functions are bounded.

his theorem. "Let  $\phi_1(x), \phi_2(x), \dots$  be a ... system of functions, defined on an interval  $[a, b]$ , orthogonal two by two, bounded or not, summable and square summable.... Attribute to each function  $\phi_1(x)$  of the system a number  $a_1$ . Then the convergence of  $\sum a_1^2$  is a necessary and sufficient condition for the existence of a function  $f(x)$  such that one has

$$\int_a^b f(x) \phi_1(x) dx = a_1$$

for each function  $\phi_1(x)$  and each number  $a_1$ ."

Riesz observed, "The necessity of the given condition follows immediately from the well known inequality of Bessel, given for continuous functions, but which remains true for arbitrary functions, summable and square summable." The necessity of this condition is also seen by appeal to Fatou's form of Parseval's equality (which is, of course, a strengthened form of Bessel's inequality in the presence of a complete orthonormal sequence). As Riesz stated, therefore, what must be shown is the sufficiency of the given condition. That is, if  $\{a_n\}$  is a square summable sequence, then does there exist a square summable function  $f$  such that for every  $n$

$$a_n = \int f \phi_n \quad ?$$

Riesz was successful in demonstrating the existence of such a function  $f$ . In the proof of existence of this function Riesz utilized in an essential way the convergence properties of the Lebesgue integral;

properties which fail for the Riemann integral. Thus a Riesz-Fischer theorem is not possible in the context of Riemann's theory of integration.

Fischer demonstrated an equivalent form of this theorem in his note "On convergence in mean," published April 27, 1907. He wrote "On the 11th of March, Mr. Riesz presented to the Academy a note on orthogonal systems of functions.... I had arrived at the same result and had demonstrated it at a conference of the mathematical society in Brunn on March 5th. Thus my independence is evident, but the priority of publication belongs to Mr. Riesz."<sup>55</sup>

As a prelude to the statement and proof of his form of the Riesz-Fischer theorem, Fischer introduced the notion of convergence in mean.

Let  $\Omega$  be the set of real [valued] functions of a real variable  $x$  such that  $f$  and  $f^2$  are summable...on a finite interval  $(a,b)$ . Then "a sequence  $f_1, f_2, \dots$  of functions belonging to  $\Omega$  is said to converge in mean if

$$\lim_{m,n \rightarrow \infty} \int_a^b (f_m - f_n)^2 dx = 0.$$

[The sequence] converges in mean toward a function  $f$  of  $\Omega$  if

$$\lim_{n \rightarrow \infty} \int_a^b (f - f_n)^2 dx = 0;$$

we will write then the 'equivalence'  $\lim_{n \rightarrow \infty} f_n \sim f$ . This does not imply the

---

<sup>55</sup>Fischer, "Sur la convergence en moyenne," 1023.

existence of a limit in the ordinary sense of the word." That is, the sequence  $f_n$  need not converge a.e. to  $f$ . Indeed, consider the following sequence of functions. For each natural number  $n$  define  $n$  functions.

$$f_1^n, f_2^n, \dots, f_n^n$$

on the interval  $(0,1]$  by

$$f_1^n(x) = \begin{cases} 1 & \text{if } \frac{1-1}{n} < x \leq \frac{1}{n} \\ 0 & \text{elsewhere.} \end{cases}^{56}$$

Then the sequence of functions  $f_1^1, f_1^2, f_2^2, f_1^3, f_2^3, f_3^3, \dots$  converges in mean to the function which is identically zero but does not converge at any point of  $(0,1]$ . There does exist, however, a subsequence of this sequence which converges a.e. For example, the subsequence of functions  $\{f_n^n\}$  converges a.e. to zero. This behavior is typical: if a sequence of functions  $\{f_n\}$  converges in mean, then there exists a subsequence  $\{f_{n_k}\}$  of  $\{f_n\}$  which converges a.e.

Given these preliminaries, Fischer stated his theorem. "If a sequence of functions belonging to  $\Omega$  converges in mean, then there exists in  $\Omega$  a function  $f$  toward which [the sequence] converges in mean." That is, if  $\{f_n\}$  is a sequence of functions, where  $f_n \in \Omega$ , is such that

$$\lim_{m,n \rightarrow \infty} \int_a^b (f_n - f_m)^2 = 0,$$

---

<sup>56</sup>This example is taken from Kolmogorov and Fomin, Elements of the Theory of Functions and Functional Analysis, Vol. 2.

then there exists an  $f \in \Omega$  such that

$$\lim_{n \rightarrow \infty} \int_a^b (f_n - f)^2 = 0.$$

Of course Fischer, as had Riesz, utilized in an indispensable manner properties of the Lebesgue integral in the proof of this theorem.

After demonstrating his proposition, Fischer brought his work into relationship with that of Riesz. "Let  $\phi_1(x), \phi_2(x), \dots$  be a denumerable set in  $\Omega$  such that

$$\int_a^b \phi_m \phi_n dx = 0 \text{ if } m \neq n,$$

and

$$\int_a^b \phi_n^2 dx = 1.$$

"If the series of constant non negative terms  $a_1^2 + a_2^2 + \dots$  converges, then  $a_1\phi_1 + a_2\phi_2 + \dots$  converges in mean, and the theorem demonstrated proves the existence of a function  $\phi$  of  $\Omega$  essentially determined [that is, with the possible exception of a set of measure zero] for which

$$\phi(x) \sim a_1\phi_1(x) + a_2\phi_2(x) + \dots .$$

Now ... one can calculate the  $a_n$  by the classical method,

$$a_n = \int_a^b \phi \phi_n dx ;$$

this demonstrates the theorem previously stated by Mr. Riesz." It should be observed that by " $a_1\phi_1 + a_2\phi_2 + \dots$  converges in mean," Fischer meant the sequence of partial sums  $f_n = \sum_{i=1}^n a_i\phi_i$  converges in mean.

Riesz reflected on the creation of the Riesz-Fischer theorem in a memoir titled "The evolution of the notion of integral since Lebesgue," published in 1949. "If I am not mistaken it was the book of Lebesgue on trigonometric series, in the Borel collection, which drew my attention to his notion of integral; after, in order to penetrate into the details, I studied also his thesis and his book on integration. However, the idea and the courage to try to apply this notion to the problems with which I was occupied came to me while reading in 1906 the excellent memoir of Fatou, printed in the Acta Mathematica and which the author presented also as a thesis. It is in particular, a very simple theorem generally called the lemma of Fatou...which helped me to demonstrate, in February, 1907, a few weeks after the reading of the thesis, the theorem also discovered independently and simultaneously by Mr. Ernest Fischer and which is cited under both our names. The theorem served first of all as a permanent ticket to go and return between the two spaces with infinitely many dimensions whose interest is attached to the study of integral equations, i.e., the space with an infinity of coordinates of Hilbert, and the set  $L^2$  of square summable functions.... This was perhaps the first application of the theory of Lebesgue after, naturally, those given by himself and Fatou, which drew the attention of mathematicians and which brought into the light the importance of his notion of integral."<sup>57</sup>

---

<sup>57</sup>Riesz "L'evolution de la notion d'intégrable depuis Lebesgue," 29.

The application referred to by Riesz might be described as follows:

Let  $L^2[a,b]$  denote the set of all square summable functions on  $[a,b]$ .

Identify functions  $f \in L^2[a,b]$  that are equal a.e., that is, consider the equivalence classes  $[f]$  of functions of  $L^2[a,b]$  defined by  $[f] = \{g \in L^2[a,b] \mid f = g \text{ a.e.}\}$ . Then every complete orthonormal sequence of functions  $\{\phi_n\}$  effects a one-to-one correspondence between the equivalence classes  $[f]$  of square summable functions defined on  $[a,b]$  and the square summable sequences  $\{c_n\}$  of real numbers, i.e., sequences such that  $\sum_{n=1}^{\infty} c_n^2$  converges. Indeed, the representation

$$f \sim \sum_{n=1}^{\infty} c_n \phi_n$$

shows that the equivalence classes  $[f]$  of  $L^2[a,b]$  can be regarded as the infinite dimensional Hilbert space of square summable sequences introduced by Hilbert and studied in detail by Schmidt.<sup>58</sup> This space is a generalization of the concept of finite dimensional Euclidean space. The point  $[f]$  has coordinates  $(c_1, c_2, \dots)$  and the functions  $\phi_n$  are the coordinate vectors. The distance  $d([f], [g])$  between  $[f]$  and  $[g]$  is defined by

$$d([f], [g]) = \left( \int_a^b (f-g)^2 \right)^{1/2}$$

which is a generalization of the finite dimensional Euclidean distance

$$d(x, y) = \left( \sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2}.$$

If  $f$  and  $g$  are in  $L^2[a,b]$ , then the product  $fg$  is summable. Thus the inner product  $([f], [g])$  of  $[f]$  and  $[g]$  is defined by

---

<sup>58</sup>See Berberian, Introduction to Hilbert Space for an elementary account of the theory.

$$([f],[g]) = \int_a^b fg,$$

which is a generalization of the finite dimension Euclidean inner product

$$(x,y) = \sum_{i=1}^n x_i y_i .$$

Finally, the existence of an inner product makes possible the definition of the norm of  $[f]$ ,

$$|| [f] || = \sqrt{([f],[f])} = \left( \int_a^b f^2 \right)^{1/2},$$

which is a generalization of the norm of a vector in a finite dimensional Euclidean space.

Now if  $[f]$  and  $[g]$  are two equivalence classes of functions of  $L^2[a,b]$ , then the distance between  $[f]$  and  $[g]$  is  $|| [f] - [g] ||$ . Thus a sequence of equivalence classes of functions  $\{[f_n]\}$  converges to an equivalence class of functions  $[f]$  if and only if  $\lim_n || [f] - [f_n] || = 0$ , that is, convergence in the space of equivalence classes of functions is convergence in mean. In this context the Riesz-Fischer theorem asserts that if  $\{[f_n]\}$  is a sequence of equivalence classes of functions of  $L^2[a,b]$ , then a necessary and sufficient condition that there exists on  $f \in L^2[a,b]$  such that  $\{[f_n]\}$  converges to  $[f]$  is that  $\lim_n || [f_n] - [f_m] || = 0$ . This theorem, which is analogous to the Cauchy convergence theorem, shows that any Cauchy sequence  $\{[f_n]\}$  converges, i.e., the space of equivalence classes of functions of  $L^2[a,b]$  is complete. It might be observed in this respect that the form of the Riesz-Fischer theorem proved by Riesz implies the form given by Fischer since Hilbert space is

complete in the inner product norm. Thus the two forms are equivalent which explains the name of the theorem.

Fischer emphasized the necessity of Lebesgue's conceptions for the validity of the Riesz-Fischer theorem in a note "Applications of a theorem on convergence in mean" published on May 27, 1907. "I shall prove that use of the notions of Mr. Lebesgue is necessary for our subject. Let  $\Pi$  be the set of continuous functions, and  $\pi_1, \pi_2, \dots$  [be an orthonormal sequence of continuous functions]. Then there exists in  $\Omega$ <sup>59</sup> a function  $\chi$  which is essentially different from all the functions of  $\Pi$ . Suppose

$$a_n = \int_a^b \chi \pi_n dx.$$

Then the series of continuous terms

$$a_1 \pi_1 + a_2 \pi_2 + \dots$$

converges in mean without converging in mean toward any continuous function; therefore for  $\Pi$ , the theorem fails in general."<sup>60</sup>

Riesz remarked with respect to geometric aspects of the Fourier coefficients of square summable functions on June 24, 1907, in a memoir titled "On a type of analytic geometry of systems of summable functions." He wrote "In a lecture given in Göttingen to the mathematical society, February 26th of this year, I set forth the results of my research on

---

<sup>59</sup> Recall that in his previous Comptes Rendus note Fischer had defined  $\Omega$  to be the set of all real valued functions of a real variable  $x$  such that  $f$  and  $f^2$  are summable on a finite interval  $(a,b)$ .

<sup>60</sup> Fischer, "Applications d'un theoreme sur la convergence en moyenne," 1150.

systems of summable functions. Afterwards I communicated the principal of these results in two notes published in Comptes Rendus....

"The goal of my research was to investigate the method of co-ordinates applied to the study of systems of summable functions. To whom goes the credit for having introduced the notion of coordinates into the theory of summable functions? It would be difficult to say. What is certain is that after the fundamental results relative to Fourier series...the idea of representing a function by its Fourier constants ought to have become very familiar. In this fashion, one arrived at representing the set of summable functions by a subset of the space of a denumerable infinity of dimensions. What is this subset? Until today no one could really say.

"Now, for a more special class, the system of square summable functions, the solution of the problem no longer carries with it so many difficulties. For this class there exists a more intimate bond between the function and its Fourier series.... For this class of functions one can define a notion of distance and can found upon this notion a geometric theory of systems of functions, a theory which resembles synthetic geometry. On the other hand, the notion of distance can also be defined in a simple manner for a subset of points of our space, the set of points whose sum of squares of coordinates converge. Now thanks to the theorem on the integration of the product of two functions represented by their Fourier constants, the bond between these two notions of distance is very intimate; it permits one to make a correspondence between the synthetic geometry of functions and an analytic geometry. This parallelism of the two theories becomes complete only through my theorem of existence which

assures that each point playing a role in this analytic geometry can be regarded as the image of a square summable function. Then, the whole geometry of our subset of points, a geometry which can be developed without difficulty, can be translated into a theory of systems of square summable functions....

"...it is the analytical theory I had in mind. On the contrary, in his two notes...Mr. Fischer developed, in a very elegant manner, the synthetic theory...."<sup>61</sup>

It remains to make explicit the relationship between the Riesz-Fischer theorem and the theory of Fourier series. This special case of the Riesz-Fischer theorem is given as follows: if  $\{a_n\}$  and  $\{b_n\}$  are sequences of real numbers such that

$$\frac{1}{4}a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) < \infty$$

then there exists a function  $f \in L^2 [0, 2\pi]$  such that

$$5) \quad \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is the Fourier series corresponding to  $f$ , and conversely.

Thus the Riesz-Fischer theorem implies a very general solution to the problem of existence of a function  $f$  such that a given sequence of constants  $\{c_n\}$  is the set of Fourier coefficients of  $f$ . It should be noted, however, that the Riesz-Fischer theorem does not speak to the questions of pointwise convergence or representation of a function by its corresponding Fourier series. The sequence of partial sums of 5) converges in

---

<sup>61</sup>Riesz, "Sur une espèce de Géométrie analytique des systèmes de fonctions sommables," 1409.

mean; this does not imply convergence of the series at a point  $x \in [0, 2\pi]$ , nor in case of convergence, convergence to  $f(x)$ .

It is natural to attempt to determine conditions such that the Fourier series corresponding to a square summable function represents the function. In 1915 Lusin conjectured that if a function is square summable then it is represented almost everywhere by its corresponding Fourier series. This sufficient condition for Fourier series representation remained open until 1966 when Carleson succeeded in affirming its validity.

#### Lusin and Carleson

The first sufficient condition for representation almost everywhere of a function  $f \in L^2[0, 2\pi]$  by its corresponding Fourier series was given by Fatou. "One can seek," Fatou remarked, "criteria for convergence [of trigonometric series], or, supposing that convergence takes place, seek the properties of the functions thus defined. These problems, which appear difficult, have been little studied." With respect to the former, criteria for convergence, Fatou obtained the following proposition as a consequence of Parseval's equality. "...let  $a_n$  and  $b_n$  be the Fourier coefficients of  $f(x)$ ; if the series  $\sum (a_n^2 + b_n^2)$  is convergent, then  $f(x)$  [is represented by its corresponding] Fourier series, except perhaps for a set of values of  $x$  of measure zero; practically this proposition does not seem very useful."<sup>62</sup>

After examining several examples of trigonometric series which diverge at certain points, Fatou proposed this problem. "Here is a

---

<sup>62</sup>Fatou, op. cit., 379.

question...which appears interesting to me and for which I have not been able to find a solution: consider a trigonometric series whose coefficients tend toward zero; we have seen that [such a trigonometric series] can have points of divergence in every interval, but the set of points for which we can demonstrate divergence... is always of measure zero. Can one give an example of a trigonometric series, whose coefficients tend toward zero, and which is divergent for all values for a set of non-zero measure of values of the argument?"<sup>63</sup>

Three years later, in 1909, Weyl obtained a generalization of Fatou's proposition regarding sufficient conditions for convergence almost everywhere of a trigonometric series. Weyl considered the convergence of the series

$$c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x) + \dots$$

where the orthonormal functions  $\phi_1(x), \phi_2(x), \dots$  are defined on the interval  $0 \leq x \leq 1$ . He succeeded in demonstrating, by a method due to Jerosch and developed by Weyl, that if the functions  $\phi_n(x)$  are such that  $|\phi_n(x)| < M$  for every  $n$  and  $x \in (0,1)$ , then the trigonometric series

$$\sum_{n=1}^{\infty} c_n \phi_n(x)$$

converges almost everywhere on  $(0,1)$  if

$$\sum_{n=1}^{\infty} c_n^2 \frac{1}{n^3}$$

is convergent.<sup>64</sup>

---

<sup>63</sup>Fatou, op. cit., 398.

<sup>64</sup>Weyl, "Über die Konvergenz von Reihen, die nach Orthogonalfunktionen fortschreiten," 241.

In the converse direction, Lusin gave an affirmative answer to Fatou's question regarding divergence of a trigonometric series on a set of positive measure by publishing in 1911 the first example of a trigonometric series whose coefficients tend to zero, and which diverges almost everywhere on  $[0, 2\pi]$ .<sup>65</sup>

Hobson contributed a generalization of Weyl's theorem in 1912. He demonstrated the following proposition. "If  $\phi_1(x), \phi_2(x), \dots$  [is] a sequence of [orthonormal] functions, and if the series  $1^k c_1^2 + 2^k c_2^2 + \dots + n^k c_n^2 + \dots$  converges for some value of  $k$  that is greater than zero, then the series  $c_1 \phi_1(x) + c_2 \phi_2(x) + \dots + c_n \phi_n(x) + \dots$  converges at all points of the interval for which the [orthonormal] functions are defined, with at most the exception of a set of points of ... measure ... zero."<sup>66</sup> Hobson observed that "the particular case of [this] theorem which arises when  $k$  has the value  $1/2$  was established by Weyl... Weyl also established the theorem for the case  $k = 1/3$ , on the assumption that the functions  $\phi_n(x)$  are less in absolute value than some fixed positive number, for all ... values of  $n$  and  $x$ ; this last restriction has been shown... to be unnecessary."

In a paper published in 1913 Plancherel wrote, "By modifying in its details the method given by Mr. Hobson... I have obtained the following theorem, of which the theorems of Weyl and Hobson are corollaries: If the functions  $\phi_n(x)$  ( $n = 1, 2, 3, \dots$ ) form an [orthonormal] system of functions in the interval  $(a, b)$  ..., and if, moreover, the real constants

---

<sup>65</sup>Lusin, "Über eine Potenzreihe," 386.

<sup>66</sup>Hobson, "On the convergence of series of orthogonal functions," 307.

$c_n$  are such that  $\sum_{n=1}^{\infty} c_n^2 (\log n)^3$  converges, then the series  $\sum_{n=1}^{\infty} c_n \phi_n(x)$  converges almost everywhere in the interval  $(a,b)$ ."67

Hardy contributed to the theory by proving in a paper also published in 1913 that the convergence of  $\sum_{n=1}^{\infty} c_n^2 (\log n)^2$  is a sufficient condition for convergence almost everywhere of the corresponding trigonometric series.68

Hardy's was the most general result known when Lusin wrote his thesis, Integral and Trigonometric Series, which was published in 1915. This work contains Lusin's famous conjecture: the Fourier series of a square summable function converges almost everywhere.

Lusin inferred the truth of his supposition from two lines of evidence. The first is related to the concept of the series conjugate to a given trigonometric series. The conjugate of the trigonometric series

$$6) \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is defined to be the series

$$7) \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} (-b_n \cos nx + a_n \sin nx).$$

We might observe in justifying the name "conjugate series" that these are the real and imaginary parts, respectively, of the series

$$\sum_{n=1}^{\infty} (a_n - ib_n) z^n$$

on the circle  $|z| = 1$ .

<sup>67</sup>Plancherel, "Sur la convergence des series de fonctions orthogonales," 540.

<sup>68</sup>Hardy, "On the summability of Fourier's series," 365.

If 6) is the Fourier series of a function  $f \in L^2[0, 2\pi]$ , then  $\sum_{n=1}^{\infty} (a_n^2 + b_n^2) < \infty$ ; therefore by the Riesz-Fischer theorem there exists a square summable function  $\bar{f}$  such that 7) is its Fourier series. The function  $\bar{f}$  is called the conjugate of  $f$ . Lusin showed that if  $f$  is square summable then  $\bar{f}$  can be expressed in terms of  $f$  independently of the Fourier series 6) and 7). He demonstrated that in this case,  $\bar{f}$  is defined almost everywhere by the integral

$$8) \quad -\frac{1}{\pi} \int_0^{\pi} \frac{f(x+t) - f(x-t)}{2 \tan(\frac{t}{2})} dt. \quad 69$$

The integral in 8) can be expressed in the form

$$9) \quad \int_0^{\pi} \frac{f(x+t) - f(x-t)}{t} dt,$$

for the integrals 8) and 9) exist or fail to exist simultaneously since

$$\frac{1}{2 \tan(\frac{t}{2})} - \frac{1}{t}$$

is a continuous function on  $[0, \pi]$ .

Lusin was fascinated by 9) and showed that it exists because of the mutual effect of the positive and negative quantities in the integral. Indeed, he considered this interference to be of fundamental importance in the convergence of Fourier series. "It is necessary to examine the interference of the positive and negative magnitudes of the expression

$$\frac{g(x+\alpha) - g(x-\alpha)}{\alpha}$$

---

<sup>69</sup>Lusin, Integral and Trigonometric Series, 200 ff. (1951 edition).

as the true source of the convergence of Fourier-Lebesgue series. All investigations which have been carried out up to this time concerning convergence of Fourier series have been based on an examination of the absolute value of only one or another expression. It is necessary therefore to consider these investigations as approximate and not actually entering into the [ cause of] convergence of Fourier series. Unfortunately, the fact of the existence of a limit value of the integral 9) is deeply hidden in the Riesz-Fischer theory...."<sup>70</sup>

Lusin demonstrated the existence of 9) in the course of his proof of the following theorem: the Fourier series of a square summable function  $f(x)$  is convergent a.e. in  $[-\pi, \pi]$  if and only if almost everywhere in  $[-\pi, \pi]$

$$10) \quad \lim_{n \rightarrow \infty} \int_0^\pi \frac{\bar{f}(x+t) - \bar{f}(x-t)}{t} \cos nt dt = 0.$$

Lusin held these considerations to be a first argument for the validity of his conjecture. For "having noticed that the integral in equation 10)... differs from the integral 9) only by the factor  $\cos nt$ , which acquires positive and negative values which are uniformly distributed over the interval  $[0, 2\pi]$  when  $n$  converges to  $+\infty$ , we are led to expect that the Fourier-Lebesgue series of any square summable function  $f(x)$  is always a series which converges almost everywhere in the interval  $[0, 2\pi]$ . All the results which have been obtained up to this time in the theory of trigonometric series confirm the probability of this hypothetical proposition."<sup>71</sup>

---

<sup>70</sup>Lusin, op. cit., 218.

<sup>71</sup>Lusin, op. cit., 219.

A second line of reasoning which led Lusin to his conjecture is related to the collection of papers cited above, i.e., the papers of Fatou, Weyl, Hobson, Plancherel and Hardy. Lusin argued in the following way. "It is possible, in general, to study the trigonometric series

$$11) \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

from two points of view. First, [it is possible] to study the question of convergence or divergence of 11) in direct dependence on the numerical character of the coefficients  $a_n, b_n$  ( $n=1,2,3,\dots$ ). It is, for example, possible to study the convergence or divergence [of 11) as it depends] on the character of the magnitude of  $a_n, b_n$  with increasing  $n$ . Classical analysis often applies this point of view. Second, it is possible in the study of these questions to completely exclude the numerical character of the coefficients of the trigonometric series, expressing these coefficients directly through the function; for example, determining them according to the formulas of Fourier.... From this point of view the questions of convergence or divergence of the trigonometric series or other analogous questions are no longer related to the properties of the coefficients, but to the properties of the function  $f(x)$  itself.... This point of view is primarily the point of view of the theory of functions.

"Both of these points of view almost coincide when we limit ourselves to the examination of square summable functions. In this case the natural necessary and sufficient numerical characteristic of the coefficients is the convergence of the series

$$\sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

Earlier we were at the second or the function theoretic point of view in the study of trigonometric series, and we saw a great probability that the Fourier-Lebesgue series of every square summable function  $f(x)$  converges almost everywhere in the interval  $[0, 2\pi]$ . The probability of the indicated hypothetical proposition becomes clear with an examination of this question from the first point of view."

Lusin then reviewed the sufficient conditions for convergence almost everywhere of Fourier series cited above, the tests of Fatou through Hardy. He continued, "Examining this table of tests for convergence, we see that their general type is the convergence of the series

$$12) \quad \sum_{n=1}^{\infty} W(n) (a_n^2 + b_n^2)$$

where  $W(n)$  is a positive increasing function. We shall call such a test for convergence Weyl's test since Weyl first attracted attention to tests for convergence of this type. The function  $W(n)$  is a positive non-decreasing function such that the convergence of the series 12) implies the convergence almost everywhere of the corresponding trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

We shall call such a function Weyl's function.

"The more slowly Weyl's function  $W(n)$  increases, the more extensive the class of trigonometric series which converge on the basis of this test, and consequently the greater the generality of this test of convergence. Hence the problem of convergence of the trigonometric series of Fourier-Lebesgue of a square summable function  $f(x)$  leads to the task

of seeking out the least increasing functions of Weyl.

"...It is easy to see that if each increasing function ... is a Weyl function, then the Fourier series of each square summable function converges almost everywhere."<sup>72</sup>

Lusin was thus led to a conjecture whose validity remained undecided for over fifty years after it was first expressed.

New ground was broken with respect to study of the divergence of Fourier series with Kolmogorov's publication in 1922 of the first "example of a summable function whose Fourier series diverges almost everywhere...."<sup>73</sup> Kolmogorov noted that the function constructed "is not square summable and I know nothing of the magnitude of the coefficients of its Fourier series." It might be observed that the Fourier series of the function constructed by Kolmogorov diverges almost everywhere because the sequence of partial sums of the Fourier series is almost everywhere unbounded. In a paper published in 1923, Menchov stated and proved the following generalization of Hardy's test for convergence almost everywhere of Fourier series: "If the functions  $\phi_n(x)$ , ( $n = 1, 2, \dots$ ) form an [orthonormal] system of functions in the interval  $(a, b)$ , and if the series

$$\sum_{n=1}^{\infty} a_n^2 (\log n)^2$$

converges, then the series

$$\sum_{n=1}^{\infty} a_n \phi_n(x)$$

---

<sup>72</sup>Lusin, op. cit., 227ff.

<sup>73</sup>Kolmogorov, "Une série de Fourier-Lebesgue divergente presque partout," 324.

converges almost everywhere in the interval  $(a,b)$ ." <sup>74</sup> (The reader will recall that Hardy proved this theorem for the case of trigonometric series). It may be of interest to note that Rademacher had stated this generalization of Hardy's theorem without proof in a paper published in 1822. <sup>75</sup> Menchov's work, however, was independent of that of Rademacher.

The principal result of Menchov's paper was original and striking. Having observed that the notation

$$W(n) = O[(\log n)^2]$$

means "the order of increase of  $W(n)$  is less than that of  $(\log n)^2$ ," Menchov stated and proved the following theorem: "If  $W(n)$  is an arbitrary positive function satisfying the condition  $W(n) = O[(\log n)^2]$ , then there exists an [orthonormal] system of functions  $\phi_n(x)$ ,  $n = 1, 2, 3, \dots$ , and a sequence of real constants  $a_n$  such that the series

$$\sum_{n=1}^{\infty} a_n \phi_n(x)$$

diverges everywhere in  $(0,1)$ , whereas the series

$$\sum_{n=1}^{\infty} a_n^2 W(n)$$

converges." <sup>76</sup> Thus Menchov's first theorem above is the best possible for arbitrary systems of orthonormal functions.

<sup>74</sup>Menchov, "Sur les series de fonctions orthogonales," 82.

<sup>75</sup>Rademacher, "Einige Satze uber Reihen von allgemeinen Orthogonal funktionen," 112.

<sup>76</sup>Menchov, op. cit., 89.

Menčov's result did not negate the possibility of replacing the Weyl function  $(\log n)^2$  by a function  $W(n)$  satisfying the condition  $W(n) = O[(\log n)^2]$  for the case of trigonometric series, and this opportunity was exploited by Kolmogorov and Seliverstov in a paper published in 1925. These men demonstrated that in the case of trigonometric series, the factor  $(\log n)^2$  in Weyl's test for convergence can be replaced by the factor  $(\log n)^{1+\delta}$  where  $\delta > 0$ .<sup>77</sup> This was accomplished by proving the theorem: "If the series

$$13) \quad \sum_{n=1}^{\infty} \tau(n) (a_n^2 + b_n^2)$$

and

$$14) \quad \sum_{n=1}^{\infty} \frac{1}{n^{\tau(n)}} \text{ where } \tau(n) < \tau(n+1)$$

converge, then the series

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

converges almost everywhere," i.e., the convergence of

$$\sum_{n=1}^{\infty} (\log n)^2 (a_n^2 + b_n^2)$$

can be replaced by the convergence of both of the series 13) and 14) where in the latter series,  $\tau(n) < \tau(n+1)$ .

In the same year, 1925, Plessner obtained a generalization of the result of Kolmogorov-Seliverstov. By utilizing the methods of Jerosch

---

<sup>77</sup>Kolmogorov and Seliverstov, "Sur la convergence des séries de Fourier," 303.

and Weyl, as had Kolmogorov and Seliverstov, Plessner proved that if

$$\sum_{n=1}^{\infty} (\log n) (a_n^2 + b_n^2)$$

is convergent, then

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is convergent almost everywhere.<sup>78</sup> This result remained the most general known for over forty years.

Turning again to the study of divergence of Fourier series, Kolmogorov published in 1926 an example of a summable function whose Fourier series diverges everywhere.<sup>79</sup> Even more significant for our purpose is the assertion, due to Kolmogorov, and published in 1927 in a paper authored by Kolmogorov and Menchov, that there exists a function  $f \in L^2[0, 2\pi]$  with the property that the terms of its Fourier series can be rearranged to form a series which diverges almost everywhere.<sup>80</sup>

Kolmogorov stated this proposition without proof. Assuming the validity of Kolmogorov's theorem, Ulyanov published a generalization of it in 1958. To understand Ulyanov's generalization requires that we define the concept of an  $L^p$  space. Let  $1 \leq p \leq \infty$ ; then  $L^p$  is the space of all measurable functions  $f$  such that  $|f|^p$  is summable. We have already encountered the two special cases  $p = 1$  and  $p = 2$ ; these are

<sup>78</sup>Plessner, "Über Konvergenz von trigonometrischen Reihen," 16.

<sup>79</sup>Kolmogorov, "Une série de Fourier-Lebesgue divergente partout," 1327.

<sup>80</sup>Kolmogorov and Menchov, "Sur la convergence des series de fonctions orthogonales," 433.

the spaces of summable and square summable functions, respectively. We might observe that the use of the term "space" is meaningful for, as in the case of  $L^2$ , it is possible to define a norm on the elements of  $L^p$  such that  $L^p$  is a normed linear (or vector) space.<sup>81</sup>

Given the definition of an  $L^p$  space, Ulyanov's generalization of Kolmogorov's theorem can be stated as follows: if  $p > 2$  then there exists a function  $f \in L^p$  such that the terms of the Fourier series of  $f$  can be rearranged to form a series which is divergent almost everywhere.<sup>82</sup>

The first construction of such a Fourier series was given in 1960 in a paper by Zahorski. Zahorski sketched a means of determining the coefficients of the Fourier series of a function  $f \in L^2$  and indicated how to permute the terms of the series in order that the rearranged series diverges almost everywhere.<sup>83</sup> Zahorski thus verified in print Kolmogorov's proposition and Ulyanov's generalization of it.

In the converse direction, and "[lending] some support to the conjecture of N. N. Lusin," Garsia proved the following two theorems in a paper published in 1963. "The Fourier series of every function in  $L^2(-\pi, \pi)$  can be so rearranged as to converge almost everywhere," and, "If successively and independently for each  $k$  we permute at random the terms of the Fourier series of  $f$  whose indices are comprised between  $m_k$  and  $m_{k+1}$  among themselves, then with probability one, the resulting rearranged series will converge almost everywhere," where  $f$  is a function

---

<sup>81</sup>See Royden, Real Analysis, 93.

<sup>82</sup>Ulyanov, "On unconditional convergence and summability," 828.

<sup>83</sup>Zahorski, "Une série de Fourier permutée d'une fonction de classe  $L^2$  divergent presque partout," 501.

in  $L^2(-\pi, \pi)$  and  $\{m_k\}$  is a sequence of integers such that

$$S_{m_k}(x, f) \rightarrow f(x).$$

Having noted that these theorems are valid for arbitrary orthonormal expansions, Garsia observed that "the construction of divergent orthonormal expansions is elaborate.... The corresponding form of [Garsia's second theorem] may explain this difficulty, since convergence almost everywhere appears to be more the rule than the exception."<sup>84</sup>

Such was the situation when Carleson published his paper in 1966. Carleson stated his purpose in this way. "In the present paper we shall introduce a new method to estimate partial sums of Fourier series. This will give quite precise results and will in particular enable us to solve the long open problem concerning convergence almost everywhere for functions in  $L^2$ ."<sup>85</sup> In particular, Carleson proved that if  $f \in L^2[0, 2\pi]$  then the sequence of partial sums of the Fourier series of  $f$  converges almost everywhere.

Kahane reviewed Carleson's paper in Mathematical Reviews: "The spectacular discovery contained in this article is the validity of Lusin's hypothesis.... The coherence [of results of previous workers] and [their] great difficulty... had made specialists think that they were probably the best possible; from which followed a skepticism, justifiable until a few months ago, with respect to proposed demonstrations of Lusin's hypothesis.

---

<sup>84</sup>Garsia, "Existence of almost everywhere convergent rearrangements for Fourier series of  $L^2$  functions," 623.

<sup>85</sup>Carleson, "On convergence and growth of partial sums of Fourier series," 135.

"[Carlson's] proof is very delicate and demands admiration....

The techniques used...are refined but classical (maximal Hilbert transforms, harmonic functions in a half plane, convolutions, and Young's inequalities on Fourier transforms).

"The article of the author is very difficult to read. It would be desirable to find either a more rapid demonstration by another method or several general theorems suggested by the author's method from which the theorems on convergence of Fourier series would follow...."

Kahane observed that Carleson's results "...are without doubt not the best possible; one can conjecture that...[the sequence of partial sums]  $s_n(x)$  is convergent a.e. when  $f \in L^p$ ,  $p > 1$ ." Hunt succeeded in demonstrating this generalization of Carleson's theorem in a paper published in 1968. By utilizing essentially Carleson's proof but modifying certain of Carleson's definitions and constructions, Hunt showed that if  $f \in L^p[0, 2\pi]$  where  $1 < p < \infty$ , then the Fourier series corresponding to  $f$  converges almost everywhere.<sup>86</sup>

Thus the theorem of Carleson and its generalization by Hunt provide a very general answer to the question of sufficient conditions for representation almost everywhere of a function by its corresponding Fourier series.

---

<sup>86</sup>Hunt, "On the convergence of Fourier series," 235.

## Epilogue

Lebesgue remarked that "... many branches of our science have died just at the time when general results seemed to guarantee them a new activity. I cite as examples the theory of forms and elliptic functions--so completely ignored since Weierstrass presented the general theorems about them. General theories reply to the questions asked of them. Unfortunately, they reply too easily, without requiring of us any effort, and since they give us the solution of problems before we have studied them, they weaken our curiosity and deny us the intimate knowledge which would have led to new problems."<sup>86</sup> We need not share Lebesgue's concern regarding continued study of Fourier series. Carleson's result is very general, yet many questions with respect to Fourier series representation of functions remain open. This has been and is one of the great problems of analysis. Investigation of it can be expected to continue and I predict the pursuance of such investigation will continue to contribute to the development of analysis.

The question of Fourier series representation of functions has been successively generalized. Such generalization, by making the problem more rather than less comprehensive, renders it capable of partial solution. Thus the question of representing a generalized Fourier series  $\sum a_n \phi_n$ , which may be convergent or divergent, by a function  $f$  derived from the series, i.e., which is related to the series by the equations

$$a_n = \int_a^b f \phi_n$$

---

<sup>86</sup>Lebesgue, "Humbert et Jordan, Roberval et Ramus," 192.

and agrees in value with the series at almost all points of  $(a,b)$  at which the series converges, becomes possible only with the introduction of a more general conception of integral and convergence of series. Saks, describing the "regularity and harmony, un hoped for by the older methods, concerning, for instance, the existence of a limit, a derivative, or a tangent," remarked that "... many branches of analysis... have lost none of their elegance where they have been inspired by methods of the theory of real functions. On the contrary, we have learned to admire in the arguments not only cleverness of calculation, but also the generality which, by an apparent abstraction, often enables us to grasp the real nature of the problem."<sup>87</sup> Indeed, the generalization of a problem in order to grasp its "real nature" and therefore to make it amenable to solution is characteristic of the evolution of the theory.

It is evident from reading the almost literal excerpts that there were many ambiguities and omissions in the statements of definitions and assertions of results in the original papers. Borel's definition of measure, for example, is in my opinion almost incomprehensive in its indefiniteness. Proofs were often wanting in accuracy; even Lebesgue slipped into occasional error in his Lecons sur l'Intégration et la Recherche des Fonctions Primitives. Such logical deficiencies are clearly secondary, however, to the very real accomplishments of the men who contributed to the creation of the theory: the explication of new ideas, lines of investigation, methods of attack, and fruitful generalizations.<sup>88</sup>

---

<sup>87</sup> Saks, op. cit., x.

<sup>88</sup> See Kline, "Logic versus Pedagogy," 264.

Finally, mention should be made of the controversy which accompanied the creation of the theory of functions of a real variable. There was polemic between Bernoulli and d'Alembert and Euler; Fourier was criticized by Lagrange. Cauchy's formulation of the notion of limit was criticized by Cournot and ignored by Poisson.<sup>89</sup> While I have no evidence to support such a contention, I suspect Riemann did not publish his Habilitationsschrift in part because of the criticism he knew would follow. Cantor was criticized by Kronecker, Borel by Schoenflies, and Lebesgue by Hermite. In all of this there is the question of the role of authority in the determination of what is admissible in mathematical science. It is apparent that mathematics was not the open and impartial discipline described by the stereotype. Indeed, evidence suggests that this is true of science in general even today;<sup>90</sup> hopefully, the possibility of such controversy no longer exists in the science of mathematics.

---

<sup>89</sup> Boyer, The History of the Calculus and its Conceptual Development, 283.

<sup>90</sup> See de Grazia, The Velikovsky Affair; The Warfare of Science and Scientism.

Appendix A

Let  $f$  be a real valued function of a single real variable such that  $f$  is periodic with period  $2\pi$ , i.e.,  $f(x + 2\pi) = f(x)$ ; and  $f'$  and  $f''$  exist and are continuous. First we will show that  $f$  can be represented by a Fourier series on  $[-\pi, \pi]$ .

Let

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos n t dt \text{ for } n = 0, 1, 2, \dots,$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin n t dt \text{ for } n = 1, 2, \dots$$

Then

$$\begin{aligned} |a_n| &= \left| \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos n t dt \right| \\ &= \frac{1}{\pi} \left| \int_{-\pi}^{\pi} f(t) d\left(\frac{\sin n t}{n}\right) \right| \\ 1) \quad &= \frac{1}{\pi} \left| f(t) \frac{\sin n t}{n} \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(t) \frac{\sin n t}{n} dt \right| \end{aligned}$$

by integration by parts.

Evaluate 1) to obtain

$$\begin{aligned} &\frac{1}{n\pi} \left| \int_{-\pi}^{\pi} f'(t) \sin n t dt \right| \\ &= \frac{1}{n\pi} \left| \int_{-\pi}^{\pi} f'(t) d\left(\frac{-\cos n t}{n}\right) \right| \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{n\pi} \left| -f'(t) \frac{\cos nt}{n} \right|_{\pi}^{\pi} + \left| \int_{-\pi}^{\pi} f''(t) \frac{\cos nt}{n} dt \right| \\
&\leq \frac{1}{n^2\pi} \left| f'(\pi) \right| + \left| f'(-\pi) \right| + 2\pi M
\end{aligned}$$

where  $M = \max \{ |f''(t)| \mid \pi \leq t \leq \pi \}.$

Thus there exists a constant  $M_1$  such that

$$|a_n| \leq \frac{M_1}{n^2\pi} \text{ for } n = 1, 2, \dots$$

Similarly, there exists a constant  $M_2$  such that

$$|b_n| \leq \frac{M_2}{n^2\pi} \text{ for } n = 1, 2, \dots$$

Therefore the series

$$2) \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

converges uniformly and absolutely on  $[-\pi, \pi]$  to a (continuous) function, say  $g(x)$ . That is,

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \text{ for every } x \in [-\pi, \pi].$$

Now let

$$a'_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos nx dx \text{ for } n = 0, 1, 2, \dots,$$

and

$$b'_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \sin nx dx \text{ for } n = 1, 2, \dots.$$

Since the series 2) converges uniformly,

$$\begin{aligned} a'_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \left\{ \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx) \right\} \cos nx dx \\ &= \frac{a_0}{2\pi} \int_{-\pi}^{\pi} \cos nx dx + \sum_{m=1}^{\infty} \left( \frac{a_m}{\pi} \int_{-\pi}^{\pi} \cos mx \cos nx dx + \frac{b_m}{\pi} \int_{-\pi}^{\pi} \sin mx \cos nx dx \right). \end{aligned}$$

Using the facts

$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

and

$$\int_{-\pi}^{\pi} \sin mx \cos nx dx = 0,$$

it follows that  $a'_n = a_n$  for every  $n$ . Similarly,  $b'_n = b_n$ .

Thus  $f$  and  $g$  have exactly the same Fourier coefficients. It is easy to show that the function which is identically zero is the only continuous function with the property that all its Fourier coefficients are zero. Thus  $f = g$ , and

$$3) \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

for all  $x \in [-\pi, \pi]$ .

Now since  $f(x)$  has the property that  $f(x + 2\pi) = f(x)$  for every real number  $x$ ,  $f$  is represented for every real number by the Fourier series 3).

## Appendix B

The difficult part of the proof is to evaluate the integral

$\lim_{j \rightarrow \infty} \int_{-\pi}^{\pi} f(\alpha) \sin j r \frac{\sin r}{1 - \cos r} d\alpha$ . An analytic analog of Fourier's geometric argument might be as follows.

Express the integral in the form

$\int_{-\pi-x}^{\pi-x} f(x+r) \left( \sin j r \frac{\sin r}{1 - \cos r} \right) dr$  recalling, of course, the relationship  $r = \alpha - x$ . Now let  $u = jr$ . Then the integral is written

$$\int_{j(-\pi-x)}^{j(\pi-x)} f\left(x+\frac{u}{j}\right) \left( \sin u \right) \left( \frac{1}{j} \right) \left( \frac{\sin \frac{u}{j}}{1 - \cos \frac{u}{j}} \right) du.$$

Appealing to the familiar equalities  $\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots$  and  $\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots$ , we obtain the integral in the form

$$\int_{j(-\pi-x)}^{j(\pi-x)} f\left(x+\frac{u}{j}\right) \left( \sin u \right) \left( \frac{1}{j} \right) \left( \frac{\frac{u}{j} - \frac{u^3}{3!j^3} + \frac{u^5}{5!j^5} - \dots}{\frac{u^2}{2!j^2} - \frac{u^4}{4!j^4} + \frac{u^6}{6!j^6} - \dots} \right) du.$$

Now consider the limit of this integral as  $j \rightarrow \infty$ . The limit is of the form

$$f(x) \int_{-\infty}^{+\infty} \left( \sin u \right) \left( \frac{u}{2} \right) du.$$

Hence, completing the evaluation, we write

$$2f(x) \int_{-\infty}^{+\infty} \frac{\sin u}{u} du = 4f(x) \int_0^{+\infty} \frac{\sin u}{u} du = 2\pi f(x).$$

Thus, Fourier concluded

$$f(x) = \frac{1}{2\pi} \sum_{i=-\infty}^{i=+\infty} \int_{-\pi}^{\pi} [f(\alpha) \cos i(\alpha - x)] d\alpha.$$

It is of interest to observe that Fourier's proof is quite different from the argument given here. Fourier argued geometrically rather than analytically.

He obtained the integral

$$2f(x) \int_0^{+\infty} (\sin j r) \frac{r}{\left(\frac{r}{2}\right)} dr$$

by asserting that the area under the curve whose abscissas are  $\alpha$  and ordinates are  $f(\alpha) (\sin j r) \frac{\sin r}{1-\cos r}$  is zero "except for certain intervals infinitely small, namely, when the ordinate  $\frac{\sin r}{1-\cos r}$  becomes infinite. This will take place if  $r = \alpha - x$  is zero; and in the interval in which  $\alpha$  differs infinitely little from  $x$ , the value of  $f(\alpha)$  coincides with  $f(x)$ . Hence the integral

$$\int f(\alpha) (\sin j r) \frac{\sin r}{1-\cos r} d\alpha$$

becomes  $2f(x) \int_0^{+\infty} (\sin j r) \frac{r}{\left(\frac{r}{2}\right)} dr. . . ."$

It is enjoyable to attempt to visualize Fourier's assertion; such an exercise will also begin to reveal remarkable depth of insight and geometric intuition which Fourier almost invariably brought to bear upon such problems.

## Appendix C

Volterra's<sup>1</sup> example of a bounded derivative which is not Riemann integrable may be described as follows: In analogy with the construction of the Cantor ternary set, construct a nowhere dense perfect subset  $E$  of  $[0,1]$  of positive Lebesgue measure,<sup>2</sup> i.e., such that the sum of lengths of the open intervals  $(a,b)$  of  $[0,1]$  whose union is the complement of  $E$  is less than 1.

Recall that the function  $f$  defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable on  $[0,1]$ , where  $f'(0)$  and  $f'(1)$  are left and right derivatives respectively, and  $f'$  is bounded on  $[0,1]$  but is discontinuous at the origin. Volterra's example is a function  $g$  defined on  $[0,1]$  such that  $g'$  exists and is bounded on  $[0,1]$  and, on each interval contiguous to  $E$  behaves as the function  $f$  above behaves on  $(0,1)$ . Thus  $g'$  is discontinuous at every point of  $E$ .

To define  $g$ , proceed in this way. Let  $E$  be given as above and let  $(a,b)$  be any of the countably many open intervals whose union is the complement of  $E$ . Then on  $(a,b)$  the function

$$h(x,a) = (x-a)^2 \sin \frac{1}{x-a} \quad x \in (a,b)$$

is differentiable and the derivative  $h'(x,a)$  is given by

$$h'(x,a) = 2(x-a) \sin \frac{1}{x-a} - \cos \frac{1}{x-a} \quad x \in (a,b).$$

---

<sup>1</sup>Volterra, "Sui principii del calcolo integrale," 334.

<sup>2</sup>See Gelbaum and Olmsted, Counterexamples in Analysis, 88, for the construction of such a set.

Now there exists a sequence  $\{x_n\}$  of points in  $(a,b)$  converging to  $a$  for which  $h'(x_n, a) = +1$ , and there exists a sequence  $\{y_n\}$  of points in  $(a,b)$  converging to  $a$  for which  $h'(y_n, a) = -1$ . Indeed, there exists a fixed positive integer  $N$  such that  $a < a + \frac{1}{2N\pi} < b$ ; hence for every  $n > N$ ,

$a < a + \frac{1}{(2n+1)\pi} < a + \frac{1}{2n\pi} < b$ . Therefore for  $n > N$ ,  $\{a + \frac{1}{(2n+1)\pi}\}$  and  $\{a + \frac{1}{2n\pi}\}$  are sequences contained in  $(a,b)$  which converge to  $a$  and are such that  $h'(a + \frac{1}{(2n+1)\pi}, a) = 1$  and  $h'(a + \frac{1}{2n\pi}, a) = -1$ .

By the intermediate value property of derivatives, for every  $n > N$  there exists an  $x \in (a + \frac{1}{(2n+1)\pi}, a + \frac{1}{2n\pi})$  such that  $h'(x, a) = 0$ . Thus there are countably many zeros of  $h'(x, a)$  in the interval  $(a, \frac{a+b}{2}]$ . Let  $\gamma$  denote a fixed zero of  $h'(x, a)$  in this interval.

Now define  $g(x)$  by

$$g(x) = \begin{cases} h(x, a) & \text{if } a < x \leq \gamma \\ h(\gamma, a) & \text{if } \gamma \leq x \leq a + b - \gamma \\ h(a+b-x, a) = -f(x, b) & \text{if } a+b-\gamma \leq x < b. \end{cases}$$

Thus, on each of the intervals  $(a,b)$ ,  $g(x)$  is symmetric with respect to the line  $x = \frac{a+b}{2}$ . Finally, define

$$g(x) = 0 \text{ if } x \in E.$$

Therefore  $g(x)$  is defined on  $[0,1]$ .

Now  $g(x)$  is differentiable on  $[0,1]$ . For if  $x$  is a member of the complement of  $E$ , then there exists an  $(a,b)$  such that  $x \in (a,b)$ .

Hence  $g'(x) = 2(x-a) \sin \frac{1}{x-a} - \cos \frac{1}{x-a}$  if  $a < x \leq \gamma$ ,  $g'(x) = 0$  if

$\gamma \leq x \leq a+b-\gamma$ , and  $g'(x) = -2(b-x)\sin \frac{1}{b-x} + \cos \frac{1}{b-x}$  if  $a+b-\gamma \leq x < b$ . If  $x \in E$  then let  $\epsilon > 0$  and  $y \in [0,1]$  such that  $|y-x| < \epsilon$  and consider

$$\left| \frac{g(y) - g(x)}{y-x} \right|.$$

If  $y \in E$  then by definition of  $g$ ,  $g(y) = g(x) = 0$  and hence

$$\left| \frac{g(y) - g(x)}{y-x} \right| = 0 < \epsilon.$$

If  $y$  is a member of the complement of  $E$  then there exists an  $(a,b)$  such that  $y \in (a,b)$ . Then  $x \leq a$  or  $b \leq x$ . If  $x \leq a$  then  $|y-a| \leq |y-x|$ , and since  $|g(y)| \leq (y-a)^2$  on  $(a,b)$ ,

$$\left| \frac{g(y) - g(x)}{y-x} \right| = \left| \frac{g(y)}{y-x} \right| \leq \frac{|g(y)|}{|y-a|} \leq \frac{(y-a)^2}{|y-a|} = |y-a| \leq |y-x| < \epsilon.$$

Of course a similar argument can be written if  $b \leq x$  for then

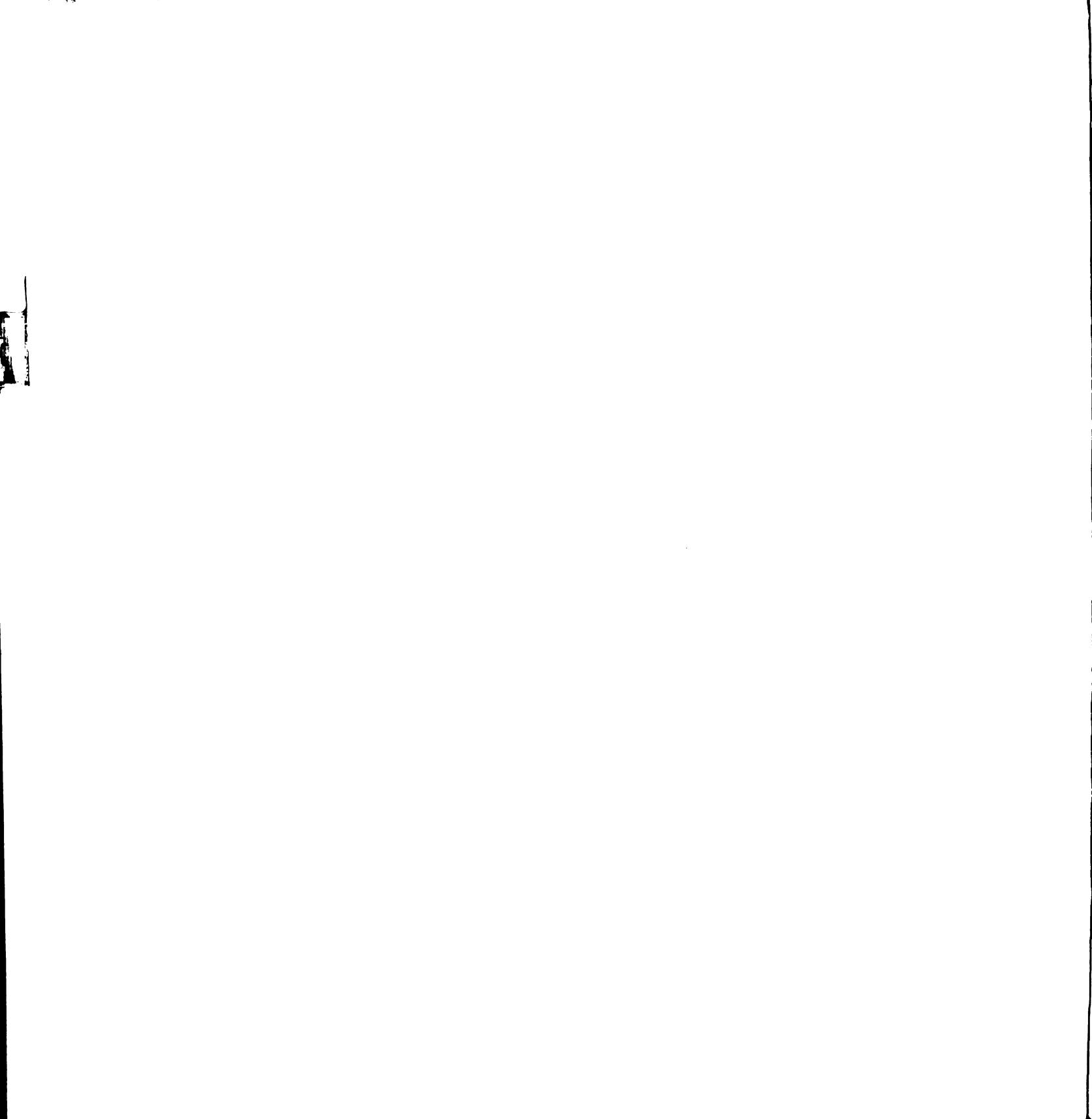
$|y-b| \leq |y-x|$  and  $|g(y)| \leq (y-b)^2$  on  $(a,b)$ . Therefore for every  $x \in E$ , if  $y \in [0,1]$  is such that  $|y-x| < \epsilon$ , then

$$\left| \frac{g(y) - g(x)}{y-x} \right| < \epsilon$$

which implies  $g'(x)$  exists and  $g'(x) = 0$ . Notice that  $g'(0)$  and  $g'(1)$  are left and right derivatives respectively.

Now  $g'$  is bounded on  $[0,1]$  for  $|g'(x)| \leq 3$ .

Finally,  $g'$  is discontinuous at every point of  $E$ . For since  $E$  is nowhere dense in  $[0,1]$ , for every  $x \in E$  and every open interval containing  $x$  there exists a subinterval of  $[0,1]$  containing no point



of  $E$ . Hence for every  $x \in E$ , every open interval containing  $x$  contains an endpoint of an interval complementary to  $E$ , and since  $g$  assumes values  $\pm 1$  in every neighborhood of such a point,  $g'$  is discontinuous at every point of  $E$ . Thus there exists a bounded derivative  $g$  on  $[0,1]$  which is not Riemann integrable there.

These remarks indicate that the assertion "integration is the inverse of differentiation" requires some qualification. Given a differentiable function  $f$  we differentiate to obtain  $f'$ . But given  $f'$  we cannot necessarily integrate, in the sense of Riemann, to recover  $f$  up to a constant. Thus for Riemann integrals the classical form

$$\int_a^x f' = f + c$$

fails in general.

An example of a function for which Scheefer asserted the classical integral form of the length of curve

$$\int_a^b (1+(f')^2)^{1/2}$$

has no meaning is constructed in the following way.

"Let  $w_1, w_2, \dots$  be the set of all rational numbers. With the sequence  $w_1, w_2, \dots$  associate a sequence of positive quantities  $c_1, c_2, \dots$  so that

$$f(x) = \sum_{r=1}^{\infty} c_r (x-w_r)^{1/3}$$

is uniformly convergent for all values of  $x$  from  $x_0$  to  $x_1 \dots$ . The function

$f(x)$  is then continuous and increasing with  $x$ ...it follows that the curve between any two points  $(x_0, y_0)$  and  $(x_1, y_1)$  has a determined length. The integral

$$1) \int_{x_0}^{x_1} (1 + f'(x)^2)^{1/2} dx$$

by which the length usually is expressed is in this case completely meaningless, since...at all points  $x = w_r$  a differential quotient exists and has the value  $+\infty$ ."<sup>3</sup> That is, for any  $x \in (x_0, x_1)$  the oscillation at  $f'(x)$  is  $+\infty$  and therefore the integral 1) does not exist.

---

<sup>3</sup>Scheeffe, "Allgemeine Untersuchungen über Rectification der Curven," 66.

## Appendix D

It should be remarked at the outset that Hilbert worked with Riemann integrals and therefore the integral symbol  $\int$  is understood to denote an integral in the Riemann sense.

We will begin by attempting to develop some intuition regarding how a linear integral equation

$$1) \quad f(s) + \int_a^b K(s,t) f(t) dt = u(s),$$

where the functions  $K$  and  $u$  are given and  $f$  is unknown, can be reduced to a system of infinitely many linear equations in infinitely many unknowns and why this might be of assistance in obtaining a solution  $f(x)$  of the integral equation.

Let the interval of integration  $[a,b]$  be partitioned into  $n$  subintervals of equal length by the points  $x_0, x_1, \dots, x_n$ . Then  $x_i = a + \frac{i(b-a)}{n}$  where  $i = 0, 1, 2, \dots, n$ . Designate the functional values  $f(x_i)$  by  $f_i$ ,  $K(x_i, x_j)$  by  $K_{ij}$  and  $u(x_i)$  by  $u_i$ . Then the integral equation 1) assumes the form

$$2) \quad f_i + \sum_{j=0}^n K_{ij} f_j = u_i, \quad i = 0, 1, 2, \dots, n,$$

that is, a system of  $n+1$  linear equations in the  $n+1$  unknowns  $f_i$ .

Written out, these equations are of the form

$$\begin{aligned}
(1 + K_{00})f_0 + K_{01}f_1 + K_{02}f_2 + \dots &+ K_{0n}f_n = u_0 \\
K_{10}f_0 + (1+K_{11})f_1 + K_{12}f_2 + \dots &+ K_{1n}f_n = u_1 \\
K_{20}f_0 + K_{21}f_1 + (1+K_{22})f_2 + \dots &+ K_{2n}f_n = u_2 \\
\vdots & \\
K_{n0}f_0 + K_{n1}f_1 + K_{n2}f_2 + \dots &+ (1+K_{nn})f_n = u_n .
\end{aligned}$$

For a given  $n$ , let  $f_0, f_1, \dots, f_n$  be a solution of 2) and consider the set of points

$$P_n = \{(x_i, f_i) \mid 0 \leq i \leq n\}.$$

Now if  $K$  and  $u$  are sufficiently well behaved, then as  $n$  tends to infinity, the systems of linear equations 2) tend to the integral equation 1), and the sets of points  $P_n$  tend to a solution function  $f(x)$  of the integral equation. This may help to explain why Hilbert reduced the equation 1) to a system of linear equations.

More specifically, Hilbert argued as follows. He first introduced the concept of a complete orthogonal sequence of functions  $\{\phi_n\}$ . Such a sequence of functions, each of which is defined and continuous on  $[a, b]$ , is orthogonal, that is,

$$\int_a^b \phi_i \phi_j = 0 \text{ if } i \neq j ,$$

and complete, i.e., if  $s(x)$  is any continuous function defined on  $[a, b]$  and if

$$s_n = \int s(x) \phi_n(x) dx$$

is defined to be the generalized  $n$ th Fourier coefficient of  $s(x)$  with respect to  $\{\phi_n\}$ , then  $\{\phi_n\}$  is complete if and only if

$$\int s^2(x) = \sum_{n=1}^{\infty} s_n^2 .$$

This is, of course, a more general form of Parseval's equality. Hilbert did not give this definition of completeness of an orthogonal set but he did demonstrate the equivalence of the above condition and his definition. We note that an orthogonal sequence of functions  $\{\phi_n\}$  is said to be orthonormal if and only if for every  $n$ ,

$$\int \phi_n^2 = 1 .$$

The sequence

$$\frac{1}{\sqrt{2\pi}} , \frac{\cos x}{\sqrt{\pi}} , \frac{\sin x}{\sqrt{\pi}} , \dots , \frac{\cos nx}{\sqrt{\pi}} , \frac{\sin nx}{\sqrt{\pi}} , \dots$$

is a complete orthonormal sequence of functions defined on  $[-\pi, \pi]$ ; this example may help to explain the origin of the conception of complete orthonormal sequence and generalized Fourier coefficients.

Hilbert then reduced the linear integral equation 1) to a system

of infinitely many linear equations in infinitely many unknowns

$$3) \quad x_i + \sum_{j=1}^{\infty} a_{ij} x_j = b_i \quad , i = 1, 2, 3, \dots,$$

where  $a_{ij}$  and  $b_i$  are the generalized Fourier coefficients of  $K$  and  $u$ , respectively, in reference to a complete orthogonal sequence of functions  $\{\phi_n\}$ . That is,

$$a_{ij} = \int_a^b \int_a^b K(s, t) \phi_i(s) \phi_j(t) ds dt ,$$

a "double" Fourier series, and

$$b_i = \int_a^b u(s) \phi_i(s) ds .$$

This enabled Hilbert to view the problem of obtaining a solution of 1) as the search for a solution  $(x_1, x_2, x_3, \dots)$  of 3) where  $x_i$  is the  $i$ th generalized Fourier coefficient of the unknown function  $f$  with respect to  $\{\phi_n\}$ .

Hilbert showed that the  $b_i$  are square summable, that is,  $\sum_{i=1}^{\infty} b_i^2 < \infty$ . Therefore, by a theorem previously demonstrated by Hilbert, there are two possibilities for 3): there exists a unique square summable solution or no unique solution exists. We will consider only the first of these two possibilities.

If  $(\alpha_1, \alpha_2, \dots)$  is the unique square summable solution of 3) then Hilbert proved,

$$\sum_{i=1}^{\infty} \alpha_i K_i(s) ,$$

where

$$K_i(s) = \int_a^b K(s,t) \phi_i(t) dt ,$$

converges uniformly on  $[a,b]$  to a (continuous) function,  $\alpha(s)$ .

Finally, Hilbert showed if  $f(x)$  is defined by  $f(x) = u(x) - \alpha(x)$  then

$$\int_a^b f(x) \phi_i(x) dx = \alpha_i$$

and hence, since  $\alpha_i$  is the  $i$ th generalized Fourier coefficient of the solution function of 1) with respect to  $\{\phi_n\}$ ,  $f(x)$  is a solution of the integral equation. Conversely, the conditions satisfied by  $(\alpha_1, \alpha_2, \dots)$  must be satisfied by the generalized Fourier coefficients with respect to  $\{\phi_n\}$  of any continuous solution of 1). Thus, Hilbert concluded, if 3) has a unique solution then there exists a unique continuous solution of 1).

This argument is given in Hilbert's Grundzüge Einer Allgemeinen Theorie der linearen Integralgleichungen, 174 ff., a collection of six papers of which the first five appeared in the Göttingen Nachrichten in 1904-1906; the collected works were published in 1912. For a more detailed account in English of Hilbert's work in integral equations the reader is directed to Bernkopf's "The Development of Function Spaces with Particular Reference to their Origins in Integral Equation Theory."

### Bibliography

- Arzelà, C. "Un teorema interno alle serie di funzioni," 262-267, "Sulla integrabilità di una serie di funzioni," 321-326, "Sulba integrazione serie," 532-537, "Sulla integrazione serie," 566-569, Atti della R. Accademia dei Lincei, Serie IV Vol. 1 (1884-85).
- Ascoli, G. "Über trigonometrische Reihen," Math. Ann. 6 (1873), 231-240.
- Bary, N. A Treatise on Trigonometric Series, Pergamon Press, 1964.
- Bell, E. Men of Mathematics. Simon and Schuster: New York, 1937.
- \_\_\_\_\_. The Development of Mathematics. New York: McGraw Hill, 1945.
- Berberian, S. Introduction to Hilbert Space. Oxford University Press: New York, 1961.
- Bernkopf, M. "The Development of Function Spaces with Particular Reference to their Origins in Integral Equation Theory," Arch. Hist. Exact Sci. 3 (1966), 1-96.
- Bernoulli, D. "Réflexions et éclaircissements sur les nouvelles vibrations des cordes exposées dans les Mémoires de L'Académie," Histoire de l'Académie Royale des Sciences de Berlin 9 (1753), 147-172.
- Borel, E. "Sur quelques points de la théorie des fonctions," Annales de L'École Normale Supérieure, 3 Serie, 12 (1895), 9-55.
- \_\_\_\_\_. Leçons sur la Théorie des Fonctions. Paris: Gauthier-Villars, 1898.
- Bourbaki, N. Éléments d'histoire des mathématiques. Hermann, 1960. Also appears in Bourbaki, N. Integration, Actualities Sci. Indust. 1244. Paris: Gauthier-Villars, 1956.
- Boyer, C. The History of the Calculus and its Conceptual Development. Dover: New York, 1959.
- Burkhardt, H. "Entwicklungen nach oscillirenden Functionen und Integration der Differentialgleichungen der mathematischen Physik," Jahresbericht der Deutschen Mathematiker-Vereinigung, Leipzig, 10 2, 1901.
- \_\_\_\_\_. "Trigonometrische Reihen und Integrale," Encyklopädie der Mathematischen Wissenschaften, Zweiter Band in drei teilen, Erster teil, zweite hälfte, 1904-1916, 825-1353.
- Burkill, J. "Henry Lebesgue," J. Lon. Math. Soc. 19 (1944), 56-64.

Cantor, G. "Über einen trigonometrischen Reihen betreffenden Lehrsatz," J. de Crelle LXXII (1870), 130-138.

\_\_\_\_\_. "Beweis, dass eine für jeden reellen Werth von  $x$  durch eine trigonometrische Reihe gegebene Function  $f(x)$  sich nur auf eine einzige Weise in dieser Form darstellen lässt," J. de Crelle LXXII (1870), 139-142.

\_\_\_\_\_. "Notiz zuden Aufsatz: Beweis, dass eine für jeden reellen Werth von  $x$  durch eine trigonometrische Reihe gegebene Function  $f(x)$  sich nur auf eine einzige Weise in dieser Form darstellen lässt," J. de Crelle LXXIII (1871), 294-296.

\_\_\_\_\_. "Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen," Math. Ann. 5 (1871), 123-132.

\_\_\_\_\_. "Über unendliche, lineare Punktmannichfaltigkeiten." Math. Ann. vol. 15 (1879), 1-7.

\_\_\_\_\_. "Über unendliche, lineare Punktmannigfaltigkeiten," Math. Ann. vol. 23 (1884), 453-488.

\_\_\_\_\_. "De la puissance des ensembles parfaits de points," Acta Mathematica 4 (1884), 381-392.

Carleson, L. "On Convergence and Growth of Partial Sums of Fourier Series," Acta Math. 116 (1966), 135-157.

Carslaw, H. S. Introduction to the Theory of Fourier's Series and Integrals. Cambridge University Press, 1929.

Cauchy, A. "Memoire sur les intégrales definies," Oeuvres complètes, (1) 1 (1882), 319-506.

\_\_\_\_\_. Cours d'Analyse de l'École Royale Polytechnique, Oeuvres complètes, (2) 3 (1897).

\_\_\_\_\_. Résumé des leçons données à l'école royale polytechnique sur le calcul infinitesimal. Oeuvres complètes, (2) 4 (1899).

\_\_\_\_\_. "Observations générales et additions." Journal de l'école polytechnique 19 (1823), 571-590.

\_\_\_\_\_. "Memoire sur les développements des fonctions en series periodiques," Oeuvres complètes, (1) 2 (1908), 12-19.

\_\_\_\_\_. "Mémoire sur les intégrales définies prises entre des limites imaginaires," Bull. Sci. Math. (1) 7 (1874), 265-304 and (1) 8 (1875), 43-55 and 148-159.

Clairaut, A. "Memoire sur l'orbite apparente de soleil autour de la terre," Acad. Sci. Paris Math. Physique (1754), 521-564.

d'Alembert, J. "Recherches sur la courbe que forme un corde tendue mise en vibration," Histoire de l'Academie Royale des Sciences de Berlin 3 (1747), 214-219.

\_\_\_\_\_. "Suite des recherches sur la courbe que forme une corde tendue en vibration," Histoire de l'Academie Royale des Sciences de Berlin 3 (1747), 220-249.

\_\_\_\_\_. "Addition au memoire sur la courbe que forme un corde tendue, mise en vibration," Histoire de l'Academie Royale des Sciences de Berlin 6 (1750), 355-378.

Darboux, G. Oeuvres de Fourier, Paris: Gauthier-Villars, 1888, Vol. 1.

Darboux, M. G. "Memoire sur les fonctions discontinues," Annales de l'Ecole Normale, 2<sup>e</sup> Serie Tome IV (1875), 57-112.

DeGrazia, A. (ed.) The Velikovsky Affair: the Warfare of Science and Scientism. New Hyde Park, N. Y.: University Books, 1966. See also the September, 1963 issue of the American Behavioral Scientist.

de la Vallee-Poussin, C. "Sur quelques applications de l'integral de Poisson," Ann. Soc. sc. Brux, 17B (1893), 18-34.

Denjoy, A., L. Felix, P. Montel. "Henri Lebesgue, le savant, le professeur, l'homme," L'Enseignement Mathematique (2) 3 (1957), 1-18.

Dini, U. "Sopra la serie di Fourier," Univ. Toscane Ann. Risa 14 (1874), 161-176.

Dirichlet, L. "Sur la convergence des séries trigonometriques qui servent à représenter une fonction arbitraire entre des limites données," J. reine und angew. Math 4 (1829), 157-169.

\_\_\_\_\_. "Über die Darstellung ganz willkürlicher Functionen durch Sinus und Cosinusreihen," Repertorium der Physik 1 (1837), 152-174.

du Bois Reymond, P. "Beweis, dass die coefficienten der trigonometrischen

$$\text{Reihe } f(x) = \sum_{p=0}^{\infty} (a_p \cos px + b_p \sin px) \text{ die Werthe } a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(\alpha) \cos p\alpha, \\ b_p = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(\alpha) \sin p\alpha \text{ haben, jedesmal wenn diese Integrale endlich}$$

und bestimmt sind," Abhandlungen der Bayerischen Akademie 12 (1876) 117-166.

du Bois Reymond, P. "Über die Integration der trigonometrischen Reihe," Math. Ann. 22 (1883), 260-268.

Encyklopädie der Mathematischen Wissenschaften, Band II, 3. Teil, 2. Hälfte. 1923-1927. Analysis: Integration und Differentiation, 1023-1049.

Encyklopädie der Mathematischen Wissenschaften, Band II, 3. Teil, 2. Hälfte. 1923-1927. Analysis: Der Inhalt der Punktmergen, 962-1001.

Euler, L. "Sur la vibration des cordes," Histoire de l'Académie Royale des Sciences de Berlin 4 (1748), 69-85.

\_\_\_\_\_. "De propagatione pulsuum per medium elasticum," Novi. Comm. Acad. Sci. Petrop. 1 (1750), 67-105.

\_\_\_\_\_. "Remarques sur les mémoires précédens de M. Bernoulli," Histoire de l'Académie Royale des Sciences de Berlin 9 (1753), 196-222.

\_\_\_\_\_. "Disquisitio ulterior super seriebus secundum multipla anguli progredientibus," Nova Acta Academiae Scientiarum Imperialis. Tome XI (1793), 114-132.

Fatou, P. "Series trigonométriques et series de Taylor," Acta Math. 30 (1906), 335-400.

Fischer, E. "Sur la convergence en moyenne," Compt. Rend. Acad. Sci. Paris 144 (1907), 1022-1024.

\_\_\_\_\_. "Applications d'un théorème sur la convergence en moyenne," Compt. Rend. Acad. Sci. Paris 144 (1907), 1148-1150.

Fourier, J. Théorie analytique de la chaleur. Paris, F. Didot père et fils, 1822. Translated as The Analytical Theory of Heat by A. Freeman, New York: G. E. Stechert & Co.

Garsia, A. "Existence of almost everywhere convergent rearrangements for Fourier series of  $L_2$  functions," Ann. of Math. (2) 79 (1964), 623-29.

Gelbaum, B. and Olmsted, J. Counterexamples in Analysis. Holden-Day: San Francisco, 1964.

Gibson, G. A. "On the history of the Fourier series," Edinburgh Math. Soc. Proc. 11 (1893), 137-170.

Grattan-Guinness, I. "Joseph Fourier and the revolution in mathematical physics," Jour. Inst. Maths. Applics. 5 (1969), 230-253.

\_\_\_\_\_. "Bolzano, Cauchy and the 'new analysis' of the early nineteenth century," Arch. Hist. Exact. Sci. 6 (1970), 372-400.

\_\_\_\_\_. The Development of the Foundations of Mathematical Analysis from Euler to Riemann. MIT Press, 1970.

- Halmos, P. Measure Theory. Princeton, N. J. Van Nostrand, 1950.
- Hankel, H. "Untersuchungen über die unendlich oft oscillirenden und unsteigen Functionen," Math. Ann. 20 (1882), 63-112. Originally published in 1870 in Ersch and Gruber's Allgemeine Encyklopädie.
- Hardy, G. "On the summability of Fourier series," Proc. Lon. Math. Soc. (2) 12 (1913), 365-372.
- Harnack, A. "Über den Inhalt von Punktmengen," Math. Ann. vol. 25 (1885), 241-250.
- Hawkins, T. Lebesgue's Theory of Integration. The University of Wisconsin Press, 1970.
- Heine, E. "Über trigonometrische Reihen," J. de Crelle LXXI (1870), 353-365.
- Hilbert, D. Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen. Teubner: Berlin and Leipzig, 1912.
- Hobson, E. "On the convergence of series of orthogonal functions," Proc. Lon. Math. Soc. (2) 12 (1913), 297-308.
- \_\_\_\_\_. The Theory of Functions of a Real Variable and The Theory of Fourier Series. Cambridge: Cambridge University Press, 1927. Vol. II.
- Hunt, R. "On the Convergence of Fourier Series," Orthogonal Expansions and their Continuous Analogues (Proc. Conf., Edwardsville, Ill., 1967), 235-255. Southern Illinois University Press, Carbondale, Ill., 1968.
- Hurwitz, A. "Über die Fourierschen Konstanten integrierbarer Funktionen," Math. Ann. 57 (1903), 425-446.
- Jarbuch über die Fortschritte der Mathematik, Haussner, 32 (1901), 299; Kowalewski, 33 (1902), 307-309; Stackel, 35 (1904), 377-378.
- Jeffery, R. The Theory of Functions of a Real Variable. Toronto: University of Toronto Press, 1951.
- Jordan, C. "Remarques sur les intégrales définies," J. Math. Pur. Appl. 4 Sér. 8 (1892), 69-99.
- \_\_\_\_\_. Cours d'Analyse de l'Ecole Polytechnique, 2<sup>e</sup> ed., Tome Premier. Paris: Gauthier-Villars, 1893.
- \_\_\_\_\_. "Sur la série de Fourier," Compt. Rend. Acad. Sci. Paris 92 (1881), 228-230.
- Jordain, P. "The Origin of Cauchy's Conceptions of a Definite Integral and of the Continuity of a Function," Isis 1 (1913), 661-703.

- Kestleman, H. "Riemann Integration of Limit Functions," American Mathematical Monthly 77 (2), February, 1970, 182-187.
- Kline, M. "Logic Versus Pedagogy," The American Mathematical Monthly, (77) 3 (1970), 264-281.
- Kolmogorov, A. "Une série de Fourier-Lebesgue divergente presque partout," Fund. Math. 4 (1923), 324-328.
- \_\_\_\_\_. "Une série de Fourier-Lebesgue divergent partout," Compt. Rend. Acad. Sci. Paris 183 (1926), 1327-1328.
- Kolmogorov, A. and D. Menshov. "Sur la convergence des séries de fonctions orthogonales," Math. Z. 26 (1927), 432-441.
- Kolmogorov, A. and G. Seliverstov. "Sur la convergence des séries de Fourier," Compt. Rend. Acad. Sci. Paris 178 (1925), 303-305.
- Kolmogorov, A., and Fomin, S. Elements of the Theory of Functions and Functional Analysis. Graylock Press, Albany, N. Y., 1961.
- Lagrange, J. L. "Recherches sur la nature et la propagation du son," Oeuvres 1, 39-332. (originally published in Miscellanea Taurinensia 1 (1759)).
- Lebesgue, H. "Sur quelques surfaces non réglées applicables sur le plan," Compt. Rend. Acad. Sci. Paris 128 (1899), 1502-1505.
- \_\_\_\_\_. "Sur la définition de l'aire d'une surface," Compt. Rend. Acad. Sci. Paris 129 (1899), 870-873.
- \_\_\_\_\_. "Sur une generalisation de l'intégrale définie," Compt. Rend. Acad. Sci. Paris 132 (1901), 1025-1028.
- \_\_\_\_\_. "Intégrale, longueur, aire," Ann. Mat. Pura. Appl. (3) 7 (1902), 231-359.
- \_\_\_\_\_. "Sur les series trigonometriques," Ann. Ec. Norm. Sup. (3) t XX (1903), 453-485.
- \_\_\_\_\_. Lecons sur l'Intégration et la Recherche des Fonctions Primitives. Paris: Gauthier-Villars, 1904.
- \_\_\_\_\_. "Humbert et Jordan Roberval et Ramus," L'Enseignement Mathématique (2) B (1957), 188-215.
- \_\_\_\_\_. "Recherche sur la convergence des séries de Fourier," Math. Ann. 61 (1905), 251-280.
- \_\_\_\_\_. Lecons sur les Series Trigonometriques. Paris, 1906.
- \_\_\_\_\_. "Sur la méthode de M. Goursat pour la résolution de l'équation de Fredholm," Bulletin de la Société Mathématique de France 36 (1908), 3-19.

Lebesgue, H. Notice sur les travaux scientifiques de M. Henri Lebesgue.  
Toulouse, 1922.

\_\_\_\_\_. "Sur le développement de la notion d'intégrale," Revue de  
Metaphysique et de Morale 34 (1927), 149-167. Translated as  
"Development of the Integral Concept," Part II of Measure and  
the Integral, Holden-Day, 1966.

Lipschitz, R. "Recherches sur le développement en series trigono-  
métriques des fonctions arbitraires d'une variable et principalement  
de celles qui, dans un intervalle fini, admettent une infinité de  
maxima et de minima." (Translation from the Latin by M. Paul  
Montel, in Paris). Acta Mathematica 35-36 (1912-12), 281-295.

Littlewood, J. and R. Paley. "Theorems on Fourier series and Power  
Series," Journal of the London Math. Soc. 6 (1931), 230-233.

Loéve, M. "Integration and measure," Encyclopaedia Britannica, 1966  
edition.

Lusin, N. "Über eine Potenzreihe," Rend. Circ. Mat. Palermo 32 (1911),  
386-390.

\_\_\_\_\_. Integral and Trigonometric Series (in Russian). M. L.  
Postekhzdat, Moscow, 1951.

Mathematical Reviews. Kahane, 33 (1967), 7774.

Menchov, D. "Sur les series des fonctions orthogonales," Fund. Math. 4  
(1923), 82-105.

Montel, P. "Notice nécrologique sur M. Henri Lebesgue," Compt. Rend.  
Acad. Sci. Paris 213 (1941), 197-200.

Osgood, W. "Nonuniform convergence and the integration of series term  
by term," Amer. J. Math. 19 (1897), 155-190.

Pasch, M. "Über einige Punkte der Functionentheorie," Math. Ann.  
vol. 30 (1887), 132-154.

Peano, G. "Sulla integrabilità delle funzioni," Torino Acc. Sci. Atti.  
18 (1883), 439-446.

\_\_\_\_\_. Applicazioni geometriche del calcolo infinitesimale. Torino,  
1887.

Pesin, I. Classical and Modern Integration Theories. Academic Press:  
New York, 1970.

Plancherel, M. "Sur la convergence des séries de fonctions orthogonales,"  
Compt. Rend. Acad. Sci. Paris 157 (1913), 539-541.

- Plessner, A. "Über Konvergenz von trigonometrischen Reihen," J. reine und angew. Math. 155 (1925), 15-25.
- Poisson, S. "Suite du mémoire sur les intégrales définies et sur la sommation des séries," Journal de l'École Royale Polytechnique, 19 (1823), 404-593.
- Rademacher, H. "Einige Sätze über Reihen von allgemeinen Orthogonal funktionen," Math. Ann. 87 (1922), 112-138.
- Ravetz, J. "Vibrating strings and arbitrary functions," Logic of Personal Knowledge: Essays Presented to M. Polanyi on his 70th Birthday (1961), 71-78.
- Riemann, B. Gesammelte mathematische Werke, 2nd ed. Leipzig: Teubner, 1892 (Dover, 1953).
- Riesz, F. "Sur les ensembles de fonctions," Compt. Rend. Acad. Sci. Paris 143 (1906), 738-741.
- \_\_\_\_\_. "Sur les systèmes orthogonaux des fonctions," Compt. Rend. Acad. Sci. Paris 144 (1907), 615-619.
- \_\_\_\_\_. "Sur une espèce de Géométrie analytique des systèmes de fonctions sommables," Compt. Rend. Acad. Sci. Paris 144 (1907), 1409-1411.
- \_\_\_\_\_. "L'évolution de la notion d'intégrale depuis Lebesgue," Ann. Inst. Fourier (Grenoble) 1 (1949), 29-42.
- Royden H. Real Analysis. Macmillan, New York, 1963.
- Saks, S. Theory of the Integral. New York: G. E. Stechert & Co., 1937.
- Scheeffer, L. "Allgemeine Untersuchungen über Rectification der Curven," Acta Math. 5 (1884), 49-82.
- Schoenflies, A. "Die Entwicklung der Lehre von den Punktmannigfaltigkeiten," Jahresbericht der Deutschen Mathematiker-vereinigung, Leipzig 8, (1900).
- Smith, H. J. S. "On the Integration of Discontinuous Functions," London Math. Soc. Proc. 6 (1875), 140-153.
- Stolz, O. "Über einen zu einer unendlichen Punktmenge gehörigen Grenzwert," Math. Ann. vol. 23 (1884), 152-156.
- Ulyanov, P. "On unconditional convergence and summability" (in Russian), Izv. Akad. Nauk SSSR Ser. Mat. 22 (1958), 811-840.
- Van Vleck, E. "The influence of Fourier's series upon the development of mathematics," Science, 39 (1914), 113-124.
- Volterra, V. "Sui principii del calcolo integrale," Giorn. Mat. 19 (1881), 333-372.

Weyl, H. "Über die Konvergenz von Reihen, die nach Orthogonalfunktionen fortschreiten," Math. Ann. 67 (1909), 225-245.

Widder, D. Advanced Calculus. Prentice-Hall, 1947.

Young, W. "On the general theory of integration," Philos. Trans. Roy. Soc. London 204 (1905), 221-252.

Zahorski, Z. "Une série de Fourier permutée d'une fonction de classe  $L_2$  divergente presque partout," Compt. Rend. Acad. Sci. Paris 251 (1960), 501-503.

Zygmund, A. Trigonometrical Series. Warszawa-Livów, 1935.

MICHIGAN STATE UNIVERSITY LIBRARIES



3 1293 50040 3476