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CONSUMER BEHAVIOR AND WELFARE MEASUREMENT UNDER UNCERTAINTY: THEORY AND EMPIRICAL EVIDENCE FROM SENEGAL

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ABSTRACT

CONSUMER BEHAVIOR AND WELFARE MEASUREMENT UNDER UNCERTAINTY: THEORY AND EMPIRICAL EVIDENCE FROM SENEGAL

By

Aliou Diagne

In this dissertation we analyze theoretically and empirically how price uncertainty affects substitution among commodities by working within a general multivariate framework of choice under uncertainty. This framework can handle multiple sources of uncertainty as well as "non-expected" utility preferences, in contrast to the prevalent univariate expected utility framework.

We have extended the main results of deterministic demand theory to uncertainty, including Roy's identity, Shephard's lemma, the Slutsky equation, and a comparative statics result that generalizes the symmetry and negativity conditions of the Slutsky matrix. We have also derived elasticity formulas which generalize the ones for deterministic models. In both the comparative statics results and the elasticity formulas, we have disentangled the effects due to attitudes toward risk from the usual ones that are due to taste substitution and income, and which obtain in the absence of uncertainty. We have also extended the standard measures of welfare changes (equivalent variation, living standard index, and cost-of-living index) to uncertainty, and have derived new measures of risk premium and cost of risk.

We also show how to generate "flexible" integrable demand systems in the context of price uncertainty. An extension to uncertainty of the Almost Ideal Demand System (AIDS) was estimated using food consumption data from a two-year survey in Senegal collected by

the Institut Sénégalais de Recherches Agricoles and the International Food Policy Research Institute. For each household in the sample we estimated the risk and nonrisk elasticity matrices, and the consumption, welfare, and risk impacts of the recent devaluation of the CFA franc, and discussed their policy implications. A major finding was that higher food prices would have a strong (negative) income effect outweighing the substitution effect. Thus, the government's policy of raising the rice price is unlikely to significantly increase the demand for the locally produced cereals. Although their magnitudes were negligible, the price-risk effects were also found to dampen the overall substitution effect. Finally, the welfare impact of the devaluation was found to be positive (negative) for all rural (urban) income quartiles. However, the lowest urban income quartile was the hardest hit.

Dedicated

To

The Memory of Cheikh Anta Diop.

My Eternal Teacher in Life and Source of Inspiration.

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ABBREVIATIONS

BCEAO	Banque Centrale des Etats de l'Afrique de l'Ouest
CFA	Communauté Financière Africaine
CSA	Commissariat à la Sécurité Alimentaire
CPSP	Caisse de Péréquation et de Stabilization des Prix
EP	Enquêtes sur les Priorités
IFPRI	International Food Policy Research Institute
ISRA	Institut Sénégalais de Recherches Agricoles
MSU	Michigan State University
UMOA	Union Monétaire Ouest Africain
USAID	United States Agency for International Development

CHAPTER 1

INTRODUCTION

The failure of Senegal's agricultural policy under the decade-long structural adjustment program to decrease the consumption of imported rice and boost the domestic production of the coarse grains has raised the following three questions: **1)** Are imported rice and the coarse grains really substitutes in consumption? **2)** Are peanuts and coarse grains substitutes in production? **3)** Why do consumers in general and urban ones in particular not seem to respond to price incentives? While the first question has been raised since the beginning of the adjustment process, when early empirical evidence from Ross (1980) suggested that rice consumption was mainly driven by "non-price factors" such as convenience in preparation, the issue has not been settled yet and is still being debated in research and policy circles.[1] For the second and third questions, Martin (1988), Kelly (1988), Goetz (1990), and Ndoye (1992) have provided some empirical answers. The first answer is that growing peanuts is far more profitable for farmers compared to coarse grains. Thus, it would require a large increase in the producer prices of the coarse grains before they can be financially competitive with peanuts (Martin, 1988; and Kelly, 1988). The second answer is that the coarse grains are very demanding in soil nutrients, and agronomic imperatives require them to be rotated with peanuts (Kelly, 1988). The third answer is that for socio-economic reasons the coarse grains are mostly grown by household heads and primarily for the

[1] See Diagne (1990), and Kite (1993) for a more recent critical review of the empirical evidence.

household's food consumption needs. Meanwhile, peanut seed is used by the household's head to attract and keep dependent adult males who tend to grow exclusively cash crops (Goetz, 1990; and Ndoye, 1992).

All these answers abstract more or less from the fundamental fact that the government has always guaranteed known and fixed prices for both peanut and imported rice, while the prices for coarse grains, which are subject to market forces, can fluctuate from one year to another depending on weather conditions. Indeed, even without the above explanations by Martin, Kelly, Goetz, and Ndoye, price uncertainty alone may explain why some risk averse farmers might always prefer to grow peanuts and buy imported rice for their food consumption. Hence, our original research objective was to answer the following question: How does **a)** the relative stability of the price of rice controlled by the government, and **b)** the fact that prices of cash crops (peanut and cotton) are known and guaranteed before production decisions are made while the prices of food crops (millet, sorghum and maize) are uncertain and determined by domestic market forces, influence farmers' food security strategies? Also, how would a liberalization of the rice market affect this strategy?

The dissertation is dealing only with the consumption part of these questions. More precisely, the questions answered are **1)** to what extent are the agricultural household's levels of consumption of the various food commodities driven by differences in the levels of uncertainty in their respective prices, and **2)** how would a liberalization of the rice market (by letting the price of rice fluctuate randomly) affect the levels of consumption of the various food commodities, as well as the welfare of consumers? In order to answer the first question, one needs to define first the meaning of the word "change" for prices that are random. One also needs to separate out from the overall consumer's response to "changes" in random prices, the part that is strictly due to the variability of prices (i.e., the risk effect or response to uncertainty), from the usual part that is due to relative change in the level of prices (i.e., the "taste" substitution effect) which obtains in the absence of uncertainty. To answer the second question, one needs a general comparative statics method which allows one to quantify the impact of a change in prices that involves moving from certainty to uncertainty.

Besides their relevance from a disciplinary point of view, the two questions have important policy implications with respect to some aspects of the Senegalese government's Cereal Policy. The first policy implication is related to the following question: Given the fact that the Cereal Policy is presently designed to encourage consumers to switch from imported rice to the locally produced coarse grains, is it not the case that the policy of maintaining a fixed and stable price for rice while consumers face random prices for the coarse grains runs against that objective? The second policy is related to the debated issue of whether or not the Senegalese government should intervene in the market for the coarse grains to stabilize their prices or at least reduce their fluctuations. Although past government attempts to enforce officially set ceiling and floor prices were not successful, there are some policy observers and researchers who are still advocating the need for some coarse grains price stabilization in order to reduce the welfare cost to food insecure farmers resulting from the fluctuation in coarse grains prices. The policy question here is whether or not the overall cost of an eventual government price stabilization program is worth the potential gain in welfare for agricultural households.

In short, the general question we are asking in this dissertation is how does price uncertainty affect substitution among commodities and the welfare of consumers. In answering this question we found the standard univariate expected utility (E-U) framework inappropriate. This is because a satisfactory answer to the question requires a multivariate framework that can handle multiple sources of uncertainty, and that also has the ability to disentangle comparative statics effects of taste substitution from the effects of risk attitudes and time preference. In particular, intertemporal E-U models generally cannot separate out the behavioral consequences of (1) intertemporal substitution in consumption, (2) risk aversion regarding uncertain future consumption and (3) preference on the timing of the resolution of the uncertainty on future consumption (i.e., early resolution versus late resolution). These three aspects of preference, which are conceptually different, are indistinguishable in the expected utility framework. In univariate E-U models, the intertemporal elasticity of substitution in consumption is equal to the reciprocal of the coefficient of relative risk aversion. E-U models also imply that the decision maker is indifferent to the time when the uncertainty in future consumption is resolved (Spence and Zeckhauser, 1972; Kreps and Porteus, 1978, 1979a, and 1979b; Machina, 1984 and 1989; and Epstein and Zin, 1989). This problem has led to the development of alternative dynamic univariate non-expected utility models that separate these aspects of preference (Kreps and Porteus, 1978 and 1979; Machina, 1984, 1989; Epstein and Zin, 1989). However, as Khilstrom and Mirman (1974) first noticed, in a multivariate context one needs still to distinguish the effect of risk attitudes from that of taste substitution in order to make meaningful comparative risk statements. Moreover, the part of the question which is related to the liberalization of the rice market involves a situation where one moves from certainty to uncertainty. This type of comparative statics can hardly be handled with the standard comparative risk methodology.

Hence, in order to answer our basic policy question, we were led to some new theoretical developments in the areas of decision making under uncertainty. The dissertation also provides detailed empirical information on the consumption behavior and welfare of the agricultural household in Senegal. Most importantly, it provides the much-needed demand elasticities for the major food staples in Senegal. The thesis also rigorously evaluates the consumption and welfare impacts of the price changes following the January 1994 devaluation of the CFA franc on different groups of rural households and urban consumers. The estimation of the elasticities and the evaluation of the welfare impacts were both done assuming price uncertainty, with the taste substitution and pure income effects being both theoretically and empirically disentangled from the effects due to attitudes toward risk.

The remainder of this introduction gives a summary of the theoretical contributions of the thesis, and its organization.

1.1 Theoretical contributions of the thesis

The approach in this research has been to view the problem of choice under uncertainty as an infinite dimensional choice space problem, and to use functional analysis methods to tackle the problem systematically. Infinite dimensional choice spaces arise naturally in economics. Three typical examples have long been recognized: intertemporal choice with continuous time or infinite horizon discrete time, uncertainty with an infinite number of states, and differentiation of commodities. Some of the first systematic studies of infinite dimensional choice spaces were those by Berger and Meyer (1966) and Berger (1971) who studied the integrability problem in a infinite dimensional Banach space, using functional analysis methods. More recently, most of the classical general equilibrium results have been extended to arbitrary infinite dimensional choice spaces. In fact, the mathematical setting of our approach to choice under uncertainty follows this literature closely.[2]

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[2] See, for example, Back, 1988; Duffie, 1986a and 1986b; John, 1984 and 1986; Mas- Collel, 1986 and 1975; and Mas-Collel and Richard, 1991. However, much of this literature looks for conditions of price supportability, Pareto (weak) optima allocations, and existence of Walrasian equilibria. These are not a concern in our decision problem of one consumer with given income and facing a given set of prices. Thus, some of the difficulties discussed in this literature do not arise in our decision problem.

In the risk literature, functional analysis methods were used by Machina (1982 and 1984) who shows that most of the results of expected utility analysis remain locally valid in a non-expected utility framework, which does not rely on the controversial independence axiom. Machina's choice space is the space of distribution functions which is an infinite dimensional choice space. However, it is mathematically more convenient to take directly the space of random variables as the choice space. This choice is justified by the conceptual framework of choice under uncertainty laid out in Appendix 4. The conceptual framework is a slightly modified version of Savage's (1954) conceptual framework of choice under uncertainty, which is based on the concepts of *acts, consequences,* and *beliefs*.

Briefly, Savage defines an *act* as a mapping from the set of *states of nature* to the set of *consequences* (or *outcomes*). The interpretation is that in a choice situation involving uncertainty, you *act* and nature determines the *consequence* (*outcome*) of your *act*. *Preference* is then among alternative feasible *acts*. We found it intuitively and mathematically more convenient to make the distinction between *ex-ante consequence* that we simply call *consequence*, and *ex-post consequence* that we call *outcome*. We also reinterpret the meaning of Savage's *act* and introduce the additional primitive concept of *options* which has no counterpart in Savage's framework. We take an *act* to be a mapping from the set of *options* to the set of *consequences*, while a *consequence* is a mapping from the set of *states of nature* to the set of *outcomes*. These changes lead to the following alternative interpretation: in a choice situation you have a set of *options*, and you *act* by taking an *option*; the consequence of your *act* is ex-ante random and nature will ex-post determine the outcome of your *act*. *Preference* is among alternative *consequences* of your *acts*. This conceptual framework leads to a choice space which can be mathematically identified with a space of random variables, with the random variables being the *consequences* of *acts* and their realizations being the *outcomes*. In particular, in the static consumer choice under uncertainty context, consumer preferences among random consumption bundles are defined in $L_2(\Omega, \mathbb{R}^n)_+$, the positive orthant of the space of vector valued random variables with finite variances. Our mathematical framework is general and covers both expected and non-expected utility preferences. We have also identified a third psychological concept distinct from the ones of *beliefs* and *preference*, and which is called *perception*. We argue that many aspects of behavior that are usually attributed to either *beliefs* or *preference*, are in fact the results of the decision maker's *perception* of the *intrinsic* properties of the objects of choices, as well as the *intrinsic* relationships existing among them. Moreover, the disentanglement of the concept of *perception* from *preference* and *beliefs* simplifies the mathematical formalization analysis of choice under uncertainty and allows an intuitive interpretations of the mathematical concepts.

In the economic literature, there have been two axiomatic approaches to choice under uncertainty. One approach takes the space of probability distributions (or mixture sets) as the choice space and formulates behavioral axioms within this space. This approach is the most prevalent one in the literature dealing purely with risk. The other approach formulates the behavioral axioms within the space of random variables, considered as the choice space. This approach is the one generally used in the general equilibrium and mathematical finance literatures, and in models of intertemporal choice under uncertainty. Conceptually, the two approaches can be equivalent if one identifies a random variable with its probability distribution as in Yaari (1985 and 1986) with his "dual theory of choice under risk".[3] But, there are differences resulting from differences in both the behavioral foundations and the

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[3] It should be noted that in his dual theory, Yaari (1985 and 1986) replaces the independence axiom which leads to the expected utility model with an axiom he calls *dual independence* or *comonotonicity*. Neither of the two assumptions will be made here. Instead, we will use the most commonly used assumption of convexity. In the one commodity case, convex preferences have been associated with risk aversion (see, for example, Arrow (1964) and Yaari (1969)).

mathematical structure of the two choice spaces. Viewing decision makers as choosing among probability distributions is appealing. However, as Machina and Schmeidler (1992, p. 62) put it:

> ...real world uncertainty seldom presents itself in terms of exogenously specified probabilities, but rather, as alternative "events" or "states of nature," so that instead of well defined objective probability distributions, the objects of choice are typically "bets" or "acts" which assign outcomes to the alternative possible events or states.

Moreover, the examples in chapter 3 show the inherent contradictions of assuming that all decision makers face exogenously specified probabilities.

On the other hand, taking the space of random variables as the choice space will lead to the conclusion that, from a functional analysis perspective, choice under uncertainty (especially consumer and producer theory with many commodities) is not much different mathematically from choice in the absence of uncertainty (see the summary table contrasting the two cases in Appendix 1). In particular, all the standard results of deterministic demand theory are extended to the case of uncertainty. There is only a slight difference in the way numerical elasticities are computed and interpreted, which is due to the loss of the simplicity of linear functions defined in the finite dimensional Euclidean space. They in fact simplify to the usual ones when one assumes no uncertainty. One new problem created by the introduction of uncertainty is the need to separate out the effects due to attitude toward risk from the usual effects of taste substitution and income effects which obtain in the absence of uncertainty.

Our mathematical approach is similar to Machina's (1982 and 1984) in the sense that we both use functional analysis and differential calculus methods, with the difference that Machina works with distribution functions in place of random variables.[4] However, our

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[4] Machina's (1982 and 1984) analysis was limited only to the univariate case.

approach has several advantages compared to Machina's: **(1)** It is mathematically easier to work with a space of random variables than with a space of distribution functions. **(2)** In a multivariate context, it is very difficult and cumbersome to formulate constraints of optimization problems using distribution functions, while with our approach, the formulation of constraints is very similar to the way it is done in deterministic choice models. **(3)** With our approach, we can easily extend all the results of deterministic consumer and producer theories (Roy's identity, Shephard's lemma, Slutsky equations, etc...) to the case of uncertainty. **(4)** The results derived within our framework lend themselves to direct econometric estimation and testing using standard econometric methods, since they are expressed in terms of random variables instead of distribution functions.

As explained above, the mathematical approach is not certainly new because it is the standard approach in the general equilibrium and mathematical finance literatures, and was earlier used by Berger and Meyer (1966). What is new in the dissertation are the versions of Roy's Identity, Shephard's lemma, Slutsky equation and other comparative statics results we obtain in the context of multivariate risk. Another new result in the thesis is the derivation of a very general comparative statics result which does not require differentiability assumptions on either the utility function or the demand function.[5] The comparative statics can be stated as follows: *The compensated or Hicksian demand is a potential monotone operator with respect to prices*. The qualifications potential and monotone generalize respectively the usual symmetry and negative semi-definiteness of the Slutsky substitution matrix. They become respectively equivalent when demands are differentiable. Hence, this result exhausts all the implications of demand theory (given the standard assumptions on preferences), because the

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[5] The only assumptions required by this result are those that are usually made on preference (reflexivity, completeness, transitivity, continuity, convexity, and monotonicity).

symmetry and negative semi-definiteness of the Slutsky substitution matrix have been shown to be necessary and sufficient for the integrability of demand systems (see Hurwicz and Uzawa, 1971; and Berger, 1971). The result is general and encompasses both the case of uncertainty and the case of certainty.[6] Moreover, it is shown as a corollary of the comparative static result that the *risk premium*, defined as what the consumer is willing to pay for not having to face uncertainty, is always positive. In other words, everything else being equal, uncertainty cannot improve welfare.

The measures of price and income elasticities, and the monetary measures of welfare changes that we derive in the context of uncertainty are also new. The separation (in the comparative statics and in the elasticities) of the effects due to response to risk from the usual taste substitution and income effects which obtain in the absence of uncertainty is also new. Moreover, following Kimball (1990) and Drèze and Modigliani (1972), we separate the effects of risk aversion and the effects of what Kimball (1990) calls *precaution* or *prudence*. Risk aversion relates to the sensitivity of welfare to uncertainty, while *precaution* relates to the sensitivity of *ex-ante* optimal choices to uncertainty. The rationale behind this distinction is simple. Intuitively risk aversion is about "dislike" of uncertainty, and one "dislikes" uncertainty because of its possible negative effects on one's *ex-ante* welfare. Hence, risk aversion effects must be measuring how *ex-ante* welfare changes as a result of uncertainty. This risk aversion-related change in welfare might affect *ex-ante* optimal choices. But, the ability of decision makers to substitute across possible states of nature also affects *ex-ante* optimal choices. Hence, in the comparative static results we separate these two *precautionary*

[6] Although it is a direct application of a standard nonlinear functional analysis theorem, the statement of the general comparative statics is new in the best of our knowledge. The theory of potential monotone operator is well developed and has found wide applications in areas of optimization, numerical analysis, and mathematical physics (see Zeidler, 1984, 1986, and 1990).

risk effects in the same way that one separates the taste substitution and the pure income effect in deterministic consumer theory. Similarly, the equivalent variation, which is derived here under uncertainty, is decomposed into two components: one being the risk premium measuring in monetary units the sensitivity of welfare to risk aversion (i.e., "dislike" of uncertainty), and the other being the change in welfare that would obtain if the consumer were welfare-indifferent to uncertainty. The derived living standard and cost-of-living indices are also decomposed in a similar way.

Furthermore, the fact that the theoretical results derived in the thesis lend themselves to direct empirical estimation and testing using traditional econometric methods distinguishes the our approach from the other theoretical treatments of risk. Indeed the ISRA/IFPRI data set is used to estimate an AIDS model under uncertainty derived using the theoretical framework.

Other new analytical results are derived in the thesis. The optimality condition for the special case of a consumer maximizing a multivariate expected utility under a budget constraint is derived. Several examples illustrating how the approach and methodology are implemented in practice are worked out in detail. These are multivariate and univariate expected and "non-expected" utility examples. Moreover, the examples are not mere examples for they give insights into the nature of choice under uncertainty. Such insights are not perceptible from the general theory presented in chapter 2. The examples show clearly the qualitative differences between choice under uncertainty and choice in the absence of uncertainty. Readers used to the analysis of choice under uncertainty based on exogenously specified *objective* probability distributions will be uneasy with the implications of some of the examples. This is because they imply that the concept of an exogenous *objective* probability distribution facing *all* decision makers leads necessarily to contradictions. On the other hand, the implications will not be a surprise for readers familiar with the Bayesian approach to decision making under uncertainty in which only individual *subjective* probability distributions are considered to make sense.

The experimental evidence against expected utility is well documented. Each one of the expected utility examples given in chapter 3 sheds a different light on the restrictiveness and "instability" in terms of behavior, beliefs and expectations implied by the expected utility hypothesis. For example, for a consumer maximizing an expected utility with a multivariate Von Neumann-Morgenstern Cobb-Douglas utility that is homogenous of degree one, under price uncertainty and budget constraint, both the indirect utility and demands are independent of the probability distribution of prices. This also implies that some nonlinear function of the random prices must be independent of the state of nature (i.e., must have a degenerate probability distribution). In the univariate case, this restriction on the probability distribution of prices leads to the striking result that (Arrow-Pratt) risk neutrality is inconsistent with uncertainty, in the sense that for a maximizing consumer with a linear Von Neumann-Morgenstern univariate utility, price must be nonrandom.[7]

Of course, all these restrictions on beliefs are inconsistent with the concept of an "objective" probability distribution of uncertain events facing all consumers. That is why Appendix 4 was written to place our mathematical results within the Bayesian or

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[7] The restriction does seem to be implied by the independence axiom ,which leads to the expected utility form, in conjunction with the assumption that the Von Neumann-Morgenstern utility is homogenous of degree one. Indeed, both the Cobb-Douglas and constant elasticity of substitution (CES) expected utility functional forms lead to the same type of restrictions on "beliefs" and choices if they are homogenous of degree one; while there is no restriction on beliefs and choices if their degree of homogeneity is less than one. The log-linear expected utility does not lead to degenerate beliefs but the corresponding Marshallian demands are independent of "beliefs". "Non-expected" utility versions of Cobb-Douglas and CES functional forms that are homogenous of degree one do not lead to either degenerate "beliefs" or independence of demands with respect to beliefs.

"subjectivistic" framework of choice under uncertainty as laid down by Ramsey (1931), De Fenetti (1937) and Savage (1954), among others.[8]

Before we move to the organization of the dissertation, we will discuss one conceptual difficulty in the theoretical results derived in the thesis in order to put them into a proper perspective. The conceptual difficulty is related to the question of whether uncertainty is possible without the passage of time. All the theoretical results in the thesis are derived in the context of a static decision-making process. This follows the well-established tradition of analyzing choice under uncertainty in a static (atemporal) framework. The static framework used in chapters 2, 3, and 5 is in particular in line with the static analysis of consumption choices under uncertainty as in Khilstrom and Mirman (1974), Hanoch (1977), Duncan (1977), Karni (1979), and Yaari (1986) among others. Yaari (1986) in his "dual" framework considered explicitly the problem of a consumer optimally allocating a fixed budget on *n* commodities under price uncertainty.

However, there is the conceptual problem of whether in fact uncertainty can occur in a *timeless* world. That is, whether it is possible to imagine a realistic situation of choice under uncertainty where the resolution of uncertainty does not involve the passage of time (Kreps and Porteus, 1979 and Machina, 1984). To resolve this conceptual difficulty without a full-fledged intertemporal model, Machina (1984) uses the Drèze and Modigliani (1972) conceptualization to distinguish two alternative cases of static choice under uncertainty both involving the passage of time, but differentiated by whether or not the decision maker must make "auxiliary" decisions before the uncertainty is resolved. The situation where "auxiliary"

[8] The restrictions on beliefs also clearly indicate that the question of economic "rationality", as long as individual decision makers are concerned, is a personal matter. That is, the "rationality "of someone's choices must be judged in relation to his or her preference <u>and</u> beliefs. This means that the best way to characterize or evaluate individual choices is not in term of "rationality", but rather in terms of *good judgment* or *poor judgment*. All these issues are discussed in more detail in Appendix 4.

decisions must be made before the resolution of uncertainty is called "temporal" risk, while the alternative situation is called "timeless risk". Static models of production under output price uncertainty fall in the category of "temporal risk" because input choices must be made before the resolution of the uncertainty. On the other hand, most people would be inclined to describe static models of consumption choices under price uncertainty as a case of "timeless risk". But, then one would have to answer the question of why the consumer cannot wait until the price uncertainty is resolved before making his or her consumption choices? Thus, one must invoke some kind of "auxiliary" decisions that must be taken by the consumer so that the situation of "temporal risk" is justified in this static consumption context.[9] However, unlike the production case, these "auxiliary" decisions need not be explicitly modelled in order to solve the static consumption choice problems. For example, one can view the static consumption choice under price uncertainty as a two-period consumption choice problem where the primary choice problem is the allocation of second period total expenditure among various commodities, with the first period consumption and saving choices being the "auxiliary" decisions that must be made before the price uncertainty in the second period is resolved (see, for example, Drèze and Modigliani, 1972). As long as one is not interested in substitution across time but only in risk aversion, and substitution across possible states of nature and across commodities, the "auxiliary" first period consumption and saving decisions need not be explicitly modelled, and the static framework is adequate.[10]

In any case, from a functional analysis perspective all the theoretical results derived in the static framework will remain valid in a intertemporal context provided the consumption

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[9] Yaari (1986) suggests "to view the consumer as deciding not how much to *buy*, but rather how much to *spend*...on the various commodities", and to view the chosen quantities of the random commodities as what will be available for consumption when the price uncertainty is resolved.

[10] The variables related to the auxiliary decisions will act as parameters in the model.

choice variables, prices, and total expenditure are appropriately reinterpreted and an appropriate "topology" is chosen (see chapter 4).[11] The benefit of working within the simplified static framework is to get most of the new results on choice under uncertainty that are made possible by the application of functional analysis and differential calculus in general linear spaces methods, while avoiding the lengthy mathematical developments that are required in order to appropriately address information revelation issues, and the Hindy, Huang, and Kreps (1992) criticism of the standard intertemporal consumption choice models (see the mathematical framework outlined in chapter 4) .

1.2 Organization of the Dissertation

The thesis is organized as follows. Chapter 2 develops the mathematical framework of static consumer behavior under uncertainty and contains the main theoretical results of the thesis. It also contains the theoretical answers to our basic question of how uncertainty affects substitution across commodities and the welfare of consumers. This chapter may be hard to read for a non-mathematically oriented reader. But, its reading is essential for a good understanding of the rationale behind the interpretations of the empirical results and how they relate to the theory developed in this thesis, and the way we have conducted the empirical analysis. Hence, the reader interested in the empirical analysis only is advised to at least skim through it and pay attention to the intuitive explanations and interpretations. The intuitive explanations and interpretations are done at the cost of lengthening the chapter and boring the reader who is familiar with the mathematical framework and its concepts. The chapter begins with an introduction that discusses in detail the mathematical differences

[11] The two theorems **A2.1** and **A2.2** in Appendix 2 on which the theoretical results are based are formulated in very general terms, and can accommodate very general choice spaces (as long as they are *uniformly convex Banach spaces*).

between finite and infinite dimensional choice problems. It then states formally the behavioral and mathematical assumptions underlying the theoretical analysis. In particular, the metric we use for the static analysis of choice under uncertainty, and which governs the meaning of the word "change" and risk in a random variable, is defined. The chapter then proceeds to derive the demand, indirect utility, and expenditure functionals along with their respective properties. Next, the comparative statics results, the *precautionary* risk effects, and the different measures of price and income elasticities under uncertainty are derived. The final section of the chapter derives and discusses the various monetary measures of welfare changes under uncertainty, including the equivalent variation, living standard index, cost-of-living index, risk premiums, risk premium index, and cost-of-risk index.

Chapter 3 illustrates the general theory of chapter 2 by working in detail through examples of utility and expenditure functionals and deriving the associated Marshallian and Hicksian demand functionals along with the preference parameters (demand elasticities, measures of attitude toward risk and welfare changes). The chapter derives the optimality condition for a consumer maximizing a multivariate expected utility under a budget constraint. It then proceeds to derive the demand, indirect utility, and expenditure functionals for the popular univariate expected utility models (CARA, CRRA, and risk neutral). The multivariate generalizations of these univariate functional forms are then fully treated with all the elasticities and welfare indicators derived. The final part of the chapter shows how to generate and solve for "non-expected" utility functional forms. The Cobb-Douglas "non-expected" utility is thus treated as an illustration with all the functional, elasticities, and welfare indicators derived.

Chapter 4 gives a detailed outline of how the static results of chapters 2 and 3 are extended to intertemporal consumption choices under uncertainty. The chapter summarizes

Hindy, Huang, and Kreps (1992) criticism of the standard intertemporal consumption choice models and discusses in detail the mathematical framework that enables at the same time to extend the static results in chapters 2 and 3 and incorporate the more realistic types of intertemporal substitution in consumption discussed by Hindy, Huang, and Kreps (1992), and which are ruled out by the standard models. Most of the discussion in the chapter is based on advanced mathematical concepts which are not used in the simpler static framework of chapters 2 and 3. The chapter is not essential for the rest of the thesis.

Chapter 5 deals with the generation of "flexible" static demand systems in the context of uncertainty as well as their econometric estimations. The chapter shows how the duality theory of chapter 2 is used to derive non-restrictive ("flexible") functional forms under uncertainty in a way similar to deterministic demand theory. The chapter starts by deriving the "expected" and "non-expected" versions of the QUAIDS model under uncertainty. The QUAIDS model is the integrable quadratic extension of the popular AIDS (Almost Ideal Demand System) functional form of Deaton and Muellbauer (1980) recently proposed by Banks, Blundell, and Lewbel (1994). The last section discusses in detail the statistical issues involved in the estimation of demand systems derived under uncertainty and which can be estimated with the Iterated Moment Estimator of Blundell and Robin (1993).

Chapter 6 begins the empirical part of the dissertation. The chapter presents first a very brief description of the macroeconomic environment within which the Senegalese agricultural household operates. The chapter then proceeds to give a description of the ISRA/IFPRI survey and the resulting data set which is used in our empirical analysis. A tabular/descriptive analysis is used to describe very briefly the consumption, production, and transaction patterns of the Senegalese agricultural households as embodied in the ISRA/IFPRI data set. This brief description is intended as an introduction to the following econometric analysis which uses the food consumption data in the ISRA\IFPRI data set to implement empirically the theoretical results in chapter 5. The econometric analysis proceeds by estimating the deterministic and the "expected" AIDS demand systems for the food commodities. The more general "expected" and "non-expected" QUAIDS derived in chapter 5 were ultimately not estimated because of some computational problems we had in our early attempts to estimate them. The estimated coefficients of the "expected" and deterministic AIDS models are compared and model specification issues are discussed. The chapter then uses the estimated coefficients and the pre-devaluation prices to compare the elasticities for the median households calculated from the deterministic and "expected" AIDS for each household. The uncompensated and compensated elasticities for the "expected" AIDS, decomposed into their *precautionary* risk and nonrisk components, are then interpreted and the policy implications of their relative magnitudes are assessed. Descriptive and tabular analysis are also used to analyze the variations of the elasticities across agro-ecological zones and income groups. The last section concludes the chapter by summarizing the empirical findings on the consumption behavior of Senegalese households.

Chapter 7 uses the coefficients estimated in chapter 6 and the post-devaluation prices to evaluate the consumption and welfare impacts of the price changes following the January 1994 CFA devaluation. Both the post-devaluation elasticities and the predicted percentage changes in the shares and quantities demanded for the food commodities are calculated, and the post-devaluation elasticities are compared with the pre-devaluation ones calculated in chapter 6. Equivalent variation, compensating variation, living standard index, cost-of-living index, risk premium, risk premium index, and cost-of-risk index are also calculated for each household. The chapter then compares the welfare impacts of the devaluation across agro-ecological zones and income groups. The chapter concludes by summarizing the empirical findings on the consumption and welfare impacts of the change in prices following the CFA devaluation.

Chapter 8 concludes the thesis with a summary of the theoretical results, and empirical findings and their policy implications. The chapter concludes with some remarks on the limitations of the empirical analysis and directions for future research.

Except for chapter 7 and the concluding chapter, each chapter has an appendix that contains the proofs of the theorems and propositions stated in the chapter and/or additional tables and materials relevant to the chapter. Appendix 4 in particular develops the conceptual framework underlying the mathematical analysis in the dissertation. The objective of the appendix is threefold. First, it provides the conceptual basis for giving intuitive behavioral interpretations to the mathematical concepts and theoretical results which are in chapter 2. Second, it rationalizes the mathematical results derived in chapter 3 by putting them into the perspective of the Bayesian framework of decision making under uncertainty where only the concept of *subjective* probability distribution makes sense. Third, it provides the conceptual framework for extending the static analysis of the thesis to the intertemporal context. Readers familiar with functional analysis methods (and to a lesser extent martingale theory) can already see from this appendix how the static results of chapters 2, 3, and 5 extends to intertemporal choices. Chapter 4 contains the details of this outline.

Finally, an appendix to this introduction (Appendix 1) summarizes the theoretical results of the thesis and contrasts them with the deterministic case. The reader may want to start from this summary to get a global picture of the theoretical results before proceeding to the details of the derivations.

CHAPTER 2

STATIC CONSUMER BEHAVIOR UNDER PRICE UNCERTAINTY

2.1 Introduction

What this chapter shows is that mathematical concepts and technicalities aside, the only difference between choice under uncertainty and choice without uncertainty, is that the former is richer in the content of the analysis and behavioral concepts discussed.[12] The mathematical technicalities and concepts refer to the choice of an appropriate topology or metric which is a crucial step in infinite dimensional spaces. Since most of the mathematical technicalities that differentiate finite and infinite dimensional choice problems turn around the mathematical concept of topology, we need to say more about it and explain why, despite its abstractness, it is relevant to discuss in a economic theory context.

2.1.1 Mathematical differences between finite and infinite dimensional choice problems

The concept of topology is merely a mathematical formalization of the intuitive notion of "similarity" or "closeness" among elements belonging to the same set. Any formal analysis which goes beyond using the set theoretic, order, and algebraic structures of a set, needs to formalize and incorporate this notion (explicitly or not). A metric is a special kind of topology and a norm is a metric in a linear space with some special features. When one moves beyond the convenience of the Euclidean space, how to formalize and study the consequences of the fact that: *two economic situations are "similar" or "close";* or the fact

[12] As it will be clear in the next chapter, there are some qualitative differences that are related to beliefs and expectations.

that: *two objects have "similar" or "dissimilar" economically relevant characteristics*, becomes an important issue which determines the type of results you get from your analysis. Furthermore, if the issue is not adequately dealt with, it can lead to economically nonsensical results. For example, recently Hindy Huang and Kreps (1992) and Hindy and Huang (1992) have shown that the standard continuous time intertemporal model of consumption that is widely used in economics implies that consumptions at nearby dates are perfect nonsubstitutes. In other words, the model implies that consumers are perfectly unwilling to delay consumption for even one second, even when they are given the economic incentive for delaying. This conterintuitive result implied by this widely used model is due to the topology used on consumption "rates".[13]

The problem of the choice of an appropriate topology is implicitly assumed away in standard consumer and producer theories by the selection of the finite dimensional Euclidean space as the commodity space. The particularity of the Euclidean space, makes that all metrics defined in this space are equivalent, so that the choice of a topology is not an issue here.

Another mathematical concept that differentiates finite and infinite dimensional choice problems, is the one of differentiability. The concept of a differentiable function and its interpretation in terms of linear local approximations are familiar ones when one is working with the finite dimensional Euclidean space. But, in infinite dimensional linear spaces, one needs to distinguish several concepts of differentiability. There are at least three main concepts of differentiability that have been introduced: Fréchet differentiability, Gâteaux differentiability, and Hadamard or compact differentiability. The Fréchet concept is the

[13] Araujo (1988), also gives several troublesome nonexistence results related to demand functions, for some topologies that have been used in the economic literatures dealing with risk and intertemporal choice.

strongest one and is the direct generalization of the familiar concept of differentiability, Hadamard's one is the next strongest, and Gâteaux is the weakest one.[14] Depending on the topology used, many functionals defined in infinite dimensional spaces are not Fréchet differentiable. But they are usually "smooth" enough to be Gâteaux differentiable, so that solutions of optimization problems can be characterized with the usual first and second order conditions, and local comparative statics results be derived. All the three concepts of differentiability are equivalent in the Euclidean space, and there is no need to distinguish among them. Moreover, while in the Euclidean space the derivative of a function at a point can be identified with a vector of real numbers, in the general case this simplification is not in general possible and one needs to get used to a derivative of a function at a point being a continuous linear function.[15] This explains why the comparative statics results and the Slutsky equation derived in this chapter are stated in a slightly different way in this uncertainty case compared to the deterministic case. This also explains the formula for computing elasticities.

When extending the results of standard consumer and producer theory to infinite dimensional choice spaces, one has to answer four types of questions. (1) Under what conditions does a preference ordering defined in a infinite dimensional choice space admit a numerical utility representation? (2) Given a utility representation of preference, when does the consumer problem of maximizing utility subject to budget constraint have a solution? In particular, under what conditions do continuous and/or smooth demand functions exist? (3)

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[14] See Yamamuro (1974) for a general concept of derivative in a topological linear space. There are also other more general concepts of derivatives used in convex analysis (see, for example Aubin, 1979). Yet another more general concept of derivative, is the *generalized or distributional* derivative used in the theory of Sobolev spaces (see for example, Rudin, 1991 or Zeidler, 1990. In chapter 4 we argue that for both mathematical and economic reasons, the Sobolev spaces are the most appropriate (outcome) spaces for analyzing intertemporal choices.

[15] The simplification results from the fact that in the Euclidean space linear continuous functions can be represented by a vector or matrix of real numbers.

If an optimal solution to the consumer maximization problem exists, can we characterize it? In particular, can we have the sort of first order necessary conditions which allows one to solve explicitly for the optimal solution, when the utility function is given explicitly? **(4)** Finally, under what conditions can we do comparative statics in a way similar to the finite dimensional choice problem?

The first question was answered by Debreu (1954) for the case of perfectly separable topological spaces. In the context of topological vector spaces, Mas-Colell (1986) and Richard and Zame (1986) provide other answers that do not rely on the separability assumption, but instead exploit the linear and order structure of the space. To answer the other questions, one has to overcome two mathematical difficulties that are particular to infinite dimensional spaces. The first difficulty stems from the fact that infinite dimensional topological vector spaces are not well furnished with compact sets. The second difficulty results from the often emptiness of the interior of the positive cone (set of non negative elements) of infinite dimensional spaces, together with the economic requirement that consumption sets be part of the positive cone.

The key to the existence issue in **(2)** is some sort of generalization of the classical Weierstrass theorem which says that a real valued continuous function defined on a closed and bounded interval of the real line has a minimum and a maximum. The generalized Weierstrass theorem reads as follows: an upper (resp lower) semicompact function $\mathbf{f}: \mathbf{M} \subset \mathbf{X} \to \mathbf{R}$ defined on the non empty subset **M** of a topological vector space **X** has a maximum (resp minimum). A sufficient condition for **f** to be upper (resp lower) semicompact, is for **M** to be compact and **f** be upper (resp lower) semicontinuous. In applying this theorem in infinite dimensional topological vector spaces, one has to face two obstacles: **(i)** Unlike the finite dimensional case, closed and bounded subsets are not necessarily compact. Hence, in general there are fewer compacts to begin with. (ii) Compactness and continuity work against each other. That is, the stronger the topology of the space, the more functions are likely to be continuous, but the less sets are likely to be compact and vice versa.

To circumvent these two problems, one generally endows the space with a topology that is strong enough to allow sufficiently "regular" functions to be continuous, yet weak enough to generate enough compact sets. In reflexive Banach spaces (or more generally in topological vector spaces), one works usually with the weak (or the weak star) topology which is induced by the set of linear functionals defined on the space. The weak topology has the property that it is the weakest topology on the space that makes linear functionals continuous, yet it is just weak enough to let closed and bounded subsets to be compact. Hence, the weak topology seems to be about right for the generalized Weierstrass theorem to work in many optimization problems. Consequently, we will be using both the original norm topology and the weak topology in the Banach spaces we will be working with.

To answer the questions in (3) and (4), one need to overcome the mathematical difficulties stemming from the often emptiness of the positive cone of infinite dimensional vector spaces and the requirements that choice sets lie within the positive cone. This is so because of two things. First, economic optimization problems are almost always constrained optimization. In the finite dimensional Euclidean space, the most effective way of dealing with constrained optimization problems is the Lagrange-Kuhn-Tucker method. Hence, one would like to use some sort of generalized Lagrange-Kuhn-Tucker method that can handle an infinite number of constraints. However, in infinite dimensional spaces, the existence of Lagrange "multipliers" often relies on the so-called Slater condition which, in this context, requires the positive cone to have a non-empty interior. Second, if we want to characterize the optimal solution in a way similar to the finite dimensional case, we must be able to use differential calculus methods in our infinite dimensional choice spaces. However, the concepts of Fréchet and Gâteaux differentiability that we will be using in our analysis, require that the interior of the domain of definition of the function to be non empty. The justification of this requirement is obvious from the definition of Fréchet or Gâteaux differentiability.

Knowing that not all infinite Banach spaces have their positive cone with an empty interior, one might ask why not choose one, like L_∞, whose positive cone has a non empty interior? There are two problems with this space and the L_1 space (Machina's choice space). First, they are not reflexive so that Theorem **A2.1** in Appendix **2** cannot be used to guarantee the existence of Lagrange "multipliers" and the generalized Kuhn-Tucker conditions. Second, these spaces are not "smooth", which implies that functions defined on them are rarely differentiable. In particular, Theorem **A2.2** in Appendix **2**, which gives the general comparative statics result cited above, is not applicable in these spaces.[16] In fact, these two spaces are the standard pathological examples for most interesting properties in functional analysis.

The choice space, L_2,that we will be working with has a positive cone with an empty interior. But, we deal with this difficulty by doing two things. First, to guarantee the existence of Lagrange "multipliers" and the generalized Kuhn-Tucker conditions, we note that the Slater condition that the theorem seems to require is used only to obtain some type of weak coerciveness of the function defining the constraints. However, the weak coerciveness is satisfied in most cases, even if the Slater condition is impossible to meet.[17]

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[16] For L_1, since it is separable, one can possibly remove the assumption of reflexivity that Theorem **A2.1** requires, and instead of the weak topology, work with the weak* topology. However, the "smoothness" assumption of Theorem **A2.2** cannot be dispensed with.

[17] In some cases, the function defining the constraints takes its values in the Euclidean space so that emptiness of the interior of the positive cone is not an issue. However, in many cases, we would like to impose explicitly non-negativity constraints for the choice variables.

Second, we solve the differentiability problem by extending our functions to an open set containing their respective domains of definition. In most cases, this open set will be the whole space. The extended function will be used only for the purpose of deriving mathematical properties that require differentiability, and it needs not have economic meaning. Only its restriction to the positive cone which, by construction, coincides with our original function will have an economic meaning. For example, if we have a utility functional **U**, defined on a subset **M** of the positive cone, that we extend by a function $\mathbf{U}^*$ defined on the whole space, $\mathbf{U}^*$ is not anymore what we would call in economics a utility functional. However, for those mathematical properties that are defined for **U**, $\mathbf{U}^*$ restricted to **M** behaves in the same way as **U**. Thus, for example, if $\mathbf{U}^*$ has a local maximum on a point in **M**, that same point will be a maximum for **U** on **M**. Hence, while it is meaningless to speak of the Fréchet or Gâteaux differentiability of **U**, if the extended $\mathbf{U}^*$ is Fréchet or Gâteaux differentiable on **M** we can use some sort of necessary first order condition for $\mathbf{U}^*$ (the generalized Euler equation) to find or characterize the maximum of **U** on **M**. We can even set, as a definition, the Fréchet (resp Gâteaux) derivative of **U** at any point of **M** to be the Fréchet (resp Gâteaux) derivative of $\mathbf{U}^*$ at that point. In this way, the concepts of Fréchet and Gâteaux differentiability are extended to functions whose domains of definition may have empty interiors. The formal definitions are given in section 2.3 of this chapter and in Appendix 2. Hence, although the indirect utility, expenditure, and demand functional are all defined only for the strictly positive elements of the positive cone of the dual space (which also have a empty interior), Their Fréchet or Gâteaux derivatives will be defined using this extended definition of Fréchet and Gâteaux differentiability. For all the cases we will be dealing with (including the examples in chapter 3, the extension of functional defined in the positive cone by Fréchet or Gâteaux differentiable functional defined in the whole space is done naturally.

To summarize, our approach for dealing with the differentiability issue consists of two stages: (i) Extend the functional (utility, expenditure, indirect utility, and demand) by Fréchet or Gâteaux differentiable functional defined in a open set containing the positive cone, then derive the needed mathematical properties. (ii) Restrict the mathematical properties derived in stage (i) to apply only for elements contained in the positive cone.

The introduction and discussion of the above mathematical concepts are unavoidable when one loses the special features of the Euclidean space. But, in this chapter, we managed to introduce purely mathematical concepts and results only when they are strictly needed and we cannot do otherwise. Even when they are introduced, we do our best to explain intuitively their meanings. The chapter could have been shorter if we did adopt the mathematically more elegant formal language of convex analysis (using the concepts of conjugate and support functions, subdifferential, etc...). But, we preferred the more intuitive Varian type treatment which allows to get most duality results with the minimum of mathematics at the cost of length.[18] Moreover, since most of the proofs of the few propositions given in this chapter are either identical to the deterministic case or simple applications of existing convex analysis theorems, we have referred to Varian (1984) for some of the proofs, the others are put in Appendix 2 for the interested reader. For readers not familiar with the vocabulary of functional and convex analysis, the definitions of the most important concepts and facts used in the chapter are summarized in Appendix 2. To facilitate comparison with the standard

[18] Moreover, with the use of subgradient and the theory of monotone set-valued mapping, more general results under less restrictive assumptions (on preference) could have been derived. However, at this point, these generalization with the use of set-valued mappings are mostly of theoretical interest.

consumer theory, we have followed very closely Varian (1984, chapter 3) in our exposition, terminology, and notations. Also, it may be useful to read first the summary tables in Appendix 1 which contrast the two cases.

2.1.2 Organization of the chapter

The chapter is organized as follows: In section **2.2** we lay down the mathematical framework along with the behavioral and mathematical concepts and assumptions underlying the analysis. In section **2.3** we derive the demand, indirect utility, and expenditure functional, and study their properties. In section **2.4** we derive the comparative statics results. In section **2.5** the consumer's "precautionary" responses to price-risk are identified and separated from the usual taste substitution and income effects which obtain in the absence of uncertainty. In section **2.6** we derive the different measures of price and income elasticities under uncertainty. In section **2.7** we derive and discuss the various monetary measures of welfare changes under uncertainty including the measures of risk aversion, risk premium, and cost of risk. Finally, section **2.8** is a brief conclusion where we summarize the results obtained in this chapter.

2.2 The Conceptual and Mathematical Framework

In what follows, we summarize the conceptual framework of choice under uncertainty adopted in Appendix 4 as it applies to the analysis of static consumption choices under uncertainty, and lay out the mathematical framework within which the analysis is conducted.[19]

2.2.1 The Consumer's uncertain environment, "beliefs", and "expectations"

It is assumed that the *uncertain* environment of the decision maker (consumer) can be described by a measurable space $(\Omega, \mathscr{F})$, where Ω is the set of all relevant possible *states of nature* and $\mathscr{F}$ is the set of *events*. A generic element of Ω, noted ω, is one possible *state of nature* that is supposed to describe all relevant exogenous sources of uncertainty (including the actions taken by other economic agents). *Uncertainty* arises only because of the fact that among all the *possible states* the consumer does not *know exactly* which one is the *true state of nature*. An event is the *possibility* that the *true state of nature* belongs to a given subset A of Ω. A is then mathematically identified with that *event*. The set of all *events*, $\mathscr{F}$, is assumed to satisfy some stability property that makes it what is formally called a σ-algebra.[20] At any time t the consumer is assumed to be endowed with a *set* of *information* concerning the uncertain environment. An *information set* at a time t is the *fact known with certainty* by time t that the *true state of nature* belongs to one of the subsets of Ω that are in a given sub σ-algebra $\Im_t$ of $\mathscr{F}$ (i.e., $\Im_t$ is a σ-algebra contained in). $\Im_t$ is then mathematically identified with that *information set*.

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[19] The following mathematical set up is standard in decision making with general choice space (see for example, Debreu (1983, chapter 5), Berger and Meyers (1966), Berger (1971), Duffie (1986a and 1986b), Mas-Colell (1986). The purely mathematical facts stated here can be found in textbooks on measure and/or probability theories and functional analysis (see for example, Dudley (1989) and Dunford and Schwartz 1958).

[20] For a formal definition of a σ-algebra, see, for example, Billingsley (1986) or Dudley (1989).

It is important to note both the conceptual and mathematical differences between an *event* and an *information set*. An *event* is a possible *fact* the truthfulness of which is not known with certainty. On the other hand, an *information set* is a *fact* that will be known with certainty by the specified time. Mathematically, an *event* is identified with a subset Ω, while an *information set* is identified with a collection of subsets of Ω. The fact that an *information set* is identified with a collection of subsets of Ω instead simply with a subset Ω has to do with some mathematical technicalities related to the definitions of conditional probability and expectation.[21] However, the adopted definition has some intuitive appeal, because it captures the process embodied in the *partial* resolution of uncertainty through time. That is, the intuition that as time evolves additional "information" is obtained through the process of knowing with certainty by a specified time the truthfulness or not of all the *events* (possible *facts*) in the *information set*.[22]

When decisions have to be made, the consumer is assumed to have a set of *beliefs* about the relative likelihoods of occurrence of the various *events*. We assume that the consumer's set of *beliefs* can be mathematically identified with a binary relation (likelihood ordering) on the set of events. We also assume that this binary relation satisfy the regularity conditions in Chateauneuf (1985) so that it can be represented by a *subjective* probability measure $\mathcal{P}$ defined on (Ω,) and making (Ω, ,) a probability space. Hence, in this way for

[21] For example, why not consider "the *fact known with certainty* by time *t* that the *true state of nature* belongs to a given subset A of Ω" as a basis for a definition of an *information set*?

[22] At this point, it is perhaps worth recalling that from the conceptual framework of Appendix 4 it is assumed that the consumer *fully perceives* the set Ω of all possible *states of nature* and its partitioning into distinct subsets. Hence, if A is an *event*, for any given possible *state of nature* ω the consumer *knows exactly* whether or not it belongs to A. What he or she does not know is whether or not that particular ω is the *true state of nature*. Moreover, even if A is in the *information set* the *uncertainty* is still not completely resolved because **a)** he or she may not know exactly which element of the *information set* contains the *true state of nature*, until the time *t* has arrived and **b)** even if he or she knows that A contains it along with ω, ω may not be the *true state of nature* (unless A is a singleton).

each possible event A $\in \mathcal{F}$, the decision maker is able to assign a *subjective* probability (A) measuring the probability of occurrence of the event A (i.e., the *believed* chances that A contains the *true state of nature*). Here, we are not making explicit the way the consumer uses its *information set* $\Im_t$ in conjunction with its *subjective* probability $\mathcal{P}$ to *subjectively assess* the relative likelihoods of occurrence of the various *events*.[23] Strictly speaking, $\mathcal{P}$ should be a conditional *subjective* probability and noted $\mathcal{P}(. \mid \Im_t)$ so that for each event A $\in$, (A $\mid \Im_t$) is interpreted as the consumer's *subjective assessment* of the probability of occurrence of the event A given the *information* $\Im_t$ available to him or her at the time t where he or she is making his or her consumption choices.[24] However, although making the *information set* explicit in the notation would clarify some of the conceptual issues involved in the interpretation of the theoretical and empirical results, it only complicates the notation without adding any thing new in the results derived within the static framework of this chapter. Hence, reference to the *information set* will be subsequently omitted unless it clarifies the discussion.

Because we will be making reference to "expectations" in this chapter and in the next, we need to be more precise with its meaning in order to avoid potential confusion when it is used in the interpretations of the theoretical and empirical results. From the discussion in Appendix 4, given the measurable space $(\Omega, \mathcal{F})$, the consumer's *subjective* probability and its *information set* uniquely determine his or her "expectations" regarding the various real valued

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[23] See Appendix 4 for more on how the *information set* is used in the decision process.

[24] See, for example, Billingsley (1986) or Dudley (1989) for a formal definition of a conditional probability when the conditioning argument is a σ-algebra. One would note in particular that it is a random variable taking its value in [0, 1]. Roughly speaking, in this context, the randomness arises because the consumer may not know exactly which element of the information set contains the *true state of nature* until the time t.

random variables defined in the measurable space $(\Omega, \mathscr{F})$.[25] Here, by "expectations" we mean the process by which the consumer uses his or her *subjective* probability $\mathcal{P}$ and *information set* to *predict* the *most likely outcome* of a random phenomenon. That is, the process by which for each real valued random variable **x** the consumer associates what he or she *believes* is its *most likely* value, or (simply put) its *expected* value, $\mathbf{x}^e$. The consumer's "expectation" process can take many forms. But, throughout the thesis we formally identify it with the mathematical expectation operator E: $L_0(\Omega, \mathbb{R}) \to \mathbb{R}$ defined by $E\mathbf{x} = \int_\Omega \mathbf{x}(\omega)d\mathcal{P}(\omega)$ for all $\mathbf{x} \in L_0(\Omega, \mathbb{R})$; where $L_0(\Omega, \mathbb{R})$ is the space of real valued random variables.[26, 27]

2.2.2 The consumer's preference among random consumption bundles

We have assumed that in any static choice situation that involves uncertainty, the decision maker has a set of available *options* and his or her *decision* or *act* consists of selecting an option. This process generates a set which is the set of all *possible actions* that can be taken. It is assumed that the *consequences* of each act are not known with certainty at the time decision has to be made and can yield many *possible outcomes* depending on which

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[25] As usual, the real line $\mathbb{R}$ is equipped with its Borel σ-algebra.

[26] The right hand side of the identity defining the Ex is supposed to exist (the existence assumption will be made more precise later). Again, strictly speaking the mathematical identification should be with the conditional expectation operator $E[.\,|\,\Im_t]$ defined by $E[\mathbf{x}\,|\,\Im_t] = \int_\Omega \mathbf{x}(\omega)d\mathcal{P}[\omega\,|\,\Im_t]$ for all $\mathbf{x} \in L_0(\Omega, \mathbb{R})$. $E[\mathbf{x}\,|\,\Im_t]$ is interpreted as the consumer's *expected* value for the random variable **x** given the *information* $\Im_t$ available to him or her at time t. For the same reasons as for the conditional probability, $E[\mathbf{x}\,|\,\Im_t]$ is a real valued random variable (See, for example, Billingsley (1986) or Dudley (1989) for some of the technicalities involved in the definition of a conditional expectation).

[27] We note that by identifying the consumer's "expectation" *process* with the mathematical expectation operator, we are putting some restrictions on the way he or she obtained the "expected" values of the various random variables (linearity restrictions with respect to both the random variable **x** and the *belief* $\mathcal{P}$ in particular). This excludes for example the possibility that the consumer *predicts* the *location* or *most likely value* of a random variable by taking the median or mode of all its possible values.

state of nature get realized. Thus, the *(ex-ante) consequences* of the decision maker's actions are random, or more formally, they are measurable functions defined on the measurable space $(\Omega, \mathcal{F})$. In restricting ourself to *acts* affecting consumption, we have assumed that for each act corresponds, as an *(ex-ante) consequence* of the act, a random consumption bundle which yields several possible bundles of *consumption outcomes* depending on which state of nature gets realized.[28] Thus, we get a correspondence between the set of *acts* and the consumption set. We have further assumed that the decision maker care only about the *consequences* of *acts*, in other words, any *preference* among *acts* is derived from a *preference* among the different *consequences* they lead to.[29] This makes any explicit reference to the set of *options* and/or *acts* irrelevant for the mathematical analysis of consumption decisions.[30]

2.2.3 Structure of the consumption space

We assume that there are **n** consumption goods and that the realization of each n-dimensional bundle of consumption goods depend on the state of nature. In other words, given the measurable space $(\Omega, \mathcal{F})$, each consumption bundle is a vector valued measurable function $\mathbf{x}$ from Ω into the Euclidean space $\mathbf{R}^n$ which associates to each possible state of nature

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[28] We note that *(ex-ante) consequence* may be an independent function of *act* and/or the state of nature (i.e., possible outcomes may not depend on *acts* and/or the state of nature).

[29] We recall that "*do-nothing*" is considered as an *act* (see Appendix 4).

[30] As explained in the introductory chapter, these acts constitute Machina's (1984) "auxiliary" decisions that must be made before the uncertainty is resolved (labor supply and saving decisions for examples). Conceptually, the random consumption bundles can be thought as plan future consumptions. Hence, in the classification of Machina (1984) our static framework of choice under uncertainty constitute a situation of "temporal risk" as opposed to the alternative situation of "timeless risk" where no "auxiliary" decisions need to be taken. However, unlike in the case of a static analysis of production decisions under uncertainty where the "auxiliary" decisions (which are made of the choice of inputs) need to be explicitly modelled, the explicit modelling of these "auxiliary" decisions is unnecessary here (they can be taken as parameters in the model).

ω a realized consumption bundle $\mathbf{x}(\omega)$.[31] For economic reasons, the consumption set is restricted to contain only positive elements; i.e, random consumption bundles take their values in $\mathbb{R}^n_+$, the positive orthant of $\mathbb{R}^n$. Formally, the consumption set is defined to be the positive cone $L_2(\Omega, \mathbb{R}^n)_+ \equiv \{ \mathbf{x} \in L_2(\Omega, \mathbb{R}^n)\colon \mathbf{x}(\omega) \geq 0 \text{ for all } \omega \in \Omega \}$. Where $L_2(\Omega, \mathbb{R}^n)$ is the space of vector valued random variables $\mathbf{x}$ defined on Ω, taking values in $\mathbb{R}^n$, and satisfying $\int_\Omega \|\mathbf{x}(\omega)\|^2 dP(\omega) < \infty$ with $\| \ \|$ being the Euclidean norm of $\mathbb{R}^n$.[32]

It is important to note that we have restricted the consumption set to include only positive random consumption vectors <u>with finite second moments</u>. This restriction, which is mainly for mathematical reasons, rules out from consideration random consumption bundles with finite expected values but infinite variances. Ruling out random consumption bundles with finite expected values but infinite variances may appear restrictive. But, this was done because most of our results depend on the reflexivity and "smoothness" of the choice space. Moreover, Araujo (1988) has proved the general non-existence of a quasi-concave continuous utility functional defined on non-reflexive Banach spaces like L_1 or L_∞ and which would give rise to well defined demand functionals. [33] For $1<p<\infty$ $L_p(\Omega, \mathbb{R}^n)$ is a reflexive Banach space, and $L_2(\Omega, \mathbb{R}^n)$ is an Hilbert space (meaning that its norm can be defined from a inner

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[31] To be more precise we should have written $\mathbf{x}(a, \omega)$ instead of $\mathbf{x}(\omega)$ to express the dependence of the consumption bundle (*outcome)* on the *act* taken by the consumer (see Appendix 4). But, we have suppress this explicit dependence to simplify the notation since it is relevant only at the conceptual level. Moreover, to shorten the discussions we will stop referring to $\mathbf{x}$ and $\mathbf{x}(\omega)$ as (*ex-ante*) *consequence* and *outcomes* respectively as dictated by the conceptual framework presented in Appendix 4, unless such references add clarity to the interpretations of the results.

[32] As usual, $\mathbb{R}^n$ is equipped with the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$ defined by the Euclidean norm.

[33] Another result of Araujo (1988) is that with L_p, $p \neq 2$ as a consumption space, there is no strictly quasi-concave and twice continuously differentiable utility functional attaining its maximum on the interior of the consumption set and giving rise to a continuously differentiable demand functional defined on its dual. However, even if we use $p \neq 2$, this result will not apply in our case because we are not going to make any differentiability assumption on the derived functional utility representation of preference.

product).[34] Since Hilbert spaces are natural generalizations of the finite dimensional Euclidean space $\mathbf{R}^n$ (the standard commodity space), in the sense that most of the nice properties of the Euclidean space extend into these spaces, we chose $L_2(\Omega, \mathbf{R}^n)$ as the consumption space. However, most of the results derived below hold when we consider a generic L_p space with $1<p<\infty$. [35]

In what follows, $L_2(\Omega, \mathbf{R}^n)$ is endowed with the norms $\| \ \|_2$ defined by $\|\mathbf{x}\|_2 \equiv \left[\int_\Omega \|\mathbf{x}(\omega)\|^2 dP(\omega)\right]^{\frac{1}{2}}$ for $\mathbf{x} \in L_2(\Omega, \mathbf{R}^n)$, and with its linear natural order $\geq$ defined by: for all $\mathbf{x}, \mathbf{y} \in L_2(\Omega, \mathbf{R}^n)$, $\mathbf{x} \geq \mathbf{y}$ if and only if $\mathbf{x}(\omega) \geq \mathbf{y}(\omega)$ for all $\omega \in \Omega$. By $\mathbf{x} > \mathbf{y}$ we will mean $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$.[36]

2.2.4 Choice of price functionals

We take the price space to be the topological dual of $L_2(\Omega, \mathbf{R}^n)$, that is, the space of continuous linear functionals defined on $L_2(\Omega, \mathbf{R}^n)$ endowed with the norm defined above. Again, for economic reasons prices are restricted to be strictly positive linear functionals. Formally, prices are strictly positive elements of the positive cone of $L_2(\Omega, \mathbf{R}^n)^*$ defined as $L_2(\Omega, \mathbf{R}^n)_{++}^* \equiv \{\mathbf{p} \in L_2(\Omega, \mathbf{R}^n)^* : \mathbf{p}(\mathbf{x}) > 0 \text{ for all } \mathbf{x} \in L_2(\Omega, \mathbf{R}^n)_+, \mathbf{x} \neq 0\}$. This definition of price is the same as in the Arrow-Debreu framework where the consumption space is $\mathbf{R}^n$. It just happens to be the case that any linear form defined on $\mathbf{R}^n$ can be identified with some

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[34] The reflexivity of $L_p(\Omega, \mathbf{R}^n)$ means that it is identical to the dual of its dual, i.e., $L_p(\Omega, \mathbf{R}^n)^{**} \equiv L_q(\Omega, \mathbf{R}^n)^* = L_p(\Omega, \mathbf{R}^n)$. In particular, we have $L_2(\Omega, \mathbf{R}^n)^* = L_2(\Omega, \mathbf{R}^n)$.

[35] We did initially derive our results by considering a generic L_p, $1<p<\infty$; the difference with the case of $p=2$, is that the duality mapping of L_p would appear in some of the results and examples worked out in chapter 3; otherwise, all the expressions are the same. This should not be a surprise since in the case of an Hilbert space, the duality mapping is equal to the identity operator.

[36] From now on, for a linear ordered space A with order $\geq$, A_+ and A_{++} will respectively designate the sets $\{ \mathbf{x} \in A: \mathbf{x} \geq \mathbf{0} \}$ and $\{ \mathbf{x} \in A: \mathbf{x} \geq \mathbf{0} \text{ and } \mathbf{x} \neq \mathbf{0} \}$.

vector in the same space $\mathbb{R}^n$. The same is true for infinite dimensional spaces identified with their biduals. But, in general a price **p** is taken to be a positive linear functional taking value in $\mathbf{R}_+$ and which assigns to each random consumption bundle **x** its market (or shadow) value **p**(**x**) (a positive real, non random number). The value of the linear functional **p** at **x** is by convention noted as $\mathbf{p}(\mathbf{x}) \equiv \langle \mathbf{p}, \mathbf{x} \rangle$ or sometimes simply as $\mathbf{p} \cdot \mathbf{x}$.

Since by the duality result cited above, we have $L_2(\Omega, \mathbb{R}^n)^* = L_2(\Omega, \mathbb{R}^n)$, we can identify a price **p** with an element of $L_2(\Omega, \mathbb{R}^n)$. That is, **p** is identified with a vector valued random variable defined on Ω and taking values in $\mathbb{R}^n_+$, and satisfying $\|\mathbf{p}\|_2 \equiv \left[\int_\Omega \|\mathbf{p}(\omega)\|^2 d\mathcal{P}(\omega)\right]^{\frac{1}{2}} < \infty$. We can then write the value of **p** at **x** as $\mathbf{p} \cdot \mathbf{x} = \langle \mathbf{p}, \mathbf{x} \rangle = \int_\Omega \mathbf{p}(\omega) \cdot \mathbf{x}(\omega) d\mathcal{P}(\omega)$. Where the dot appearing in the first expression is the inner product of the Hilbert space $L_2(\Omega, \mathbb{R}^n)$ and the one inside the integral is the usual inner product of the Euclidean space $\mathbb{R}^n$.

2.2.5 "Changes" and *expected* changes in random consumption bundles and prices

The topology induced by $\| \ \|_2$ formalizes the consumer's *perception* of "similarities" and "dissimilarities" among the random consumption bundles. Hence, the consumer's *belief* as represented by the subjective probability distribution $\mathcal{P}$ is implicit in this formalization of his or her *perception* of "similarity" or "closeness", since the norm $\| \ \|_2$ depends on it. As explained in Appendix 4, before the consumer uses his or her *preference* to rank the various random consumption bundles in his or her choice set, its *belief* $\mathcal{P}$ is used to "adjust" his or her *perception* of the *intrinsic* "similarities" and dissimilarities existing among the possible bundles of consumption *outcomes* (i.e., for example, his or her *perception* of how the pair constituted by 1 kg of rice and 1/2 kg of millet is "close" to the pair constituted by 1/2 kg of rice and 2 kg of millet) to arrive at a "*belief-adjusted*" *perception* of "similarities" and "dissimilarities" among the respective corresponding random consumption bundles. Under the norm $\| \ \|_2$ this "adjustment" is done by "weighting" the Euclidean norms of all the possible bundles of consumption *outcomes* corresponding to each random consumption bundle **x** with the respective *believed* relative likelihoods of occurrence of the states of nature where they may be respectively realized; so as to arrive at a *"belief-weighted" perception* of how **x** is "close" to the degenerate random consumption bundle consisting of zero consumption in all possible states of nature.

The norm $\| \ \|_2$ also governs the meaning of the word "change" when it is used in this thesis and is attached to a random variable. In particular, when we take "derivative" of functionals with respect to random consumption bundles or random prices, the meaning of the implied "changes in the random variables" is governed by the metric $\| \ \|_2$. Similarly, in the comparative static statements as well as in the elasticities and measures of welfare change formula derived later, the meaning of the "changes in the random variables" is governed by the metric $\| \ \|_2$. Moreover, given our mathematical identification of the consumer's *expectation process* with the expectation operator E, under this metric we can interpret a "change in a random variable" as an *expected* change in its level. Hence, a "change in a random price vector" is interpreted as an *expected change* in prices, and a "change in a random consumption bundle" is interpreted as an *expected change* in its consumption. To see the rationale behind this interpretation, we note that if, for example, a random price vector "change" from **p** to **q** then under a given possible state of nature ω the actual change in price is as usual measured by the Euclidean distance between $\mathbf{p}(\omega)$ and $\mathbf{q}(\omega)$. That is, by the quantity $\| \mathbf{p}(\omega) - \mathbf{q}(\omega) \|$. Hence, to get the *expected* change in prices one would simply take a "weighted average" across all states of nature to find $E(\| \mathbf{p} - \mathbf{q} \|)$, the expected value of the real valued random variable $\| \mathbf{p} - \mathbf{q} \|$. But, under the metric $\| \ \|_2$ (which governs the meaning of "change") the "averaging" across states of nature is made "smooth" by taking instead, the squared root of the expected value of the real valued random variable $\|\mathbf{p} - \mathbf{q}\|^2$. That is, by getting $(E\|\mathbf{p}-\mathbf{q}\|^2)^{\frac{1}{2}} = \|\mathbf{p}-\mathbf{q}\|_2$ which is by definition the distance between the two random variables **p** and **q** under the metric $\| \ \|_2$.[37] Hence, under this metric a "change" in the random vector of prices can be interpreted as an *expected change* in prices.[38, 39]

Finally, for more clarification on the meaning of this interpretation, we call attention on the difference between an *expected change in prices* and a *change in expected prices* from both behavioral and mathematical perspectives. From the behavioral perspective, the latter change is about the consumer changing his or her *predictions* of the "location" of the random prices and this may occur even if there is no "change" in the random prices (for example, the consumer may obtain new information on the random prices which leads him or her to "revise" his or her "expectation"). From the mathematical perspective, the later change is a change in the expected values and hence involves only the Euclidean distance.[40]

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[37] This "smooth averaging" is what makes $L_2(\Omega, \mathbf{R}^n)$ a *uniformly convex* space, thus making it possible to derive the theoretical results.

[38] Along the same lines, we note that with the assumption that $L_2(\Omega, \mathbf{R}^n)$ is endowed with its natural linear order, an "increase" in the random price vector from **p** to **q** (i.e., the vector of prices change from **p** to **q** with $\mathbf{q} \geq \mathbf{p}$) means that in any given possible state of nature the value taken by the random vector **q** is higher than the one taken by **p**. In this context, because of the positivity of the expectation operator, an "increase" in random prices always imply an *expected* increase in prices. But, the converse is not true.

[39] We note that from a functional analysis perspective the (*unconditional*) expectation operator E is just an element of the dual of $L_2(\Omega, \mathbf{R})$, that is an element of $L_2(\Omega, \mathbf{R})$ itself since this is an Hilbert space. On the other hand, the *unconditional* expectation operator $E[\,.\,|\mathfrak{F}_t]$ is an continuous linear operator from $L_2(\Omega, \mathbf{R})$ to $L_2(\Omega, \mathbf{R})$. That is, it is an element of the space $\mathcal{L}(L_2(\Omega, \mathbf{R}); L_2(\Omega, \mathbf{R}))$. Thus, the mathematical identification of the consumer's "expectation" *process* with the *conditional* or *conditional* expectation operator opens the door for quantifying the impact of "changes" in the consumer's "expectation" *process* on behavior and welfare.

[40] The metric $\| \ \|_2$ will be involved when conditional expectation is used.

2.2.6 Measures of the consumer's perception of the risk in its consumption choices

Given the consumer's *subjective* probability distribution $\mathcal{P}$, the "riskiness" involved in its consumption choices under price uncertainty depends on how large the set of possible states of nature is *perceived* to be, how each possible state of nature jointly determine the respective values of the random variables, and how consumption decisions based on his or her *predictions* or *forecasts* of the *most likely* values (or *location*) of the random variables affect his or her welfare.[41] However, for each random variable several possible states of nature can lead to the same value or to values that are "close". These values may also be related in some way to values taken by the other random variables. Hence, the consumer's *perception* of the degree of uncertainty or risk in his or her consumption choices is determined by **a)** his or her *subjective assessment* of the relative likelihood of each possible state of nature, **b)** the degrees of variability of the random variables around their respective *predicted* or "expected" values, and **c)** the degrees of covariability of the random variables as they vary around their respective *predicted* or "expected" values. Therefore, to have an acceptable measure of the consumer's *perception* of the degree of risk in the random prices in particular, one need to find a way to appropriately "aggregate" the degree of variability of each random price around its "expected" value and the degree of covariability of each pair of random prices (when they vary around their "expected" values) with his or her *subjective assessment* of the relative likelihood of each possible state of nature so as to arrive at a "belief-adjusted" numerical measures of the *perceived* degree of "variability" of each random price and degree of "covariability" of each pair of random prices.

[41] The random variables related to consumption choices include prices, the commodities, and (future) total expenditure.

Given our mathematical framework and identification of the consumer's "expectation" process with the expectation operator E, it is natural to measure the consumer's *perceived* degree of "variability" of a real-valued random variable $\mathbf{x}$ by the distance between $\mathbf{x}$ and its expected value under the metric $\| \ \|_2$. That is by $\| \mathbf{x} - E\mathbf{x} \|_2 = \left(E(\mathbf{x} - E\mathbf{x})^2\right)^{\frac{1}{2}}$, which is just its standard deviation.[42] Similarly, it is natural to measure the consumer's *perceived* degree of "covariability" of two real-valued random variable $\mathbf{x}$ and $\mathbf{y}$ by the scalar (or dot) product of their deviations from their respective expected values under the scalar product associated with the metric $\| \ \|_2$. That is by $\langle \mathbf{x} - E\mathbf{x}, \mathbf{y} - E\mathbf{y} \rangle_2 = E\left((\mathbf{x} - E\mathbf{x})(\mathbf{y} - E\mathbf{y})\right)$, which is just their covariance.[43]

Before concluding this section on the consumer's *perception* of price-risk, we note two implications that will be important in the interpretations of comparative static results and elasticities measuring the effects of price-risk. First, with this measure of the variability of a random variable, it is easy to see that not all "changes" or *expected changes* in a random price will necessarily lead to a change in its variability. However, a deterministic homothetic *expected change* in prices (i.e., a multiplicative shift in the random price) will lead to change in its variability. Such *expected change* in prices characterizes deterministic changes that are expressed in percentage terms (e.g., sale tax, devaluation, etc...). On the other hand,

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[42] This is equivalent to "weighing" the Euclidean distances of the random variable's possible values to its "expected" value by the respective consumer's *subjective assessment* of relative the likelihoods of the possible states of nature and "appropriately" aggregating them. Note that if instead of L_2 we have worked with a generic L_p, $1 < p < \infty$ then the measure of "variability" would correspond to the *p*th root of the *p*th centered moment. Hence, "variability" measures based on higher moments are accomodated by working with L_p $2 \leq p < \infty$.

[43] For comparability across pairs of random variables, one can normalize either by dividing by the products of the norms of the random variables or by the products of their measures of "variability" (i.e., their standard errors). If the latter normalization is chosen, then the measure of covariability reduces to the familiar coefficient of correlation. For convenience, in the derivation of elasticity estimates, we choose the first normalization method. Note that with a generic L_p, $1 < p < \infty$ and $p \neq 2$, the random variables $\mathbf{x}$ and $\mathbf{y}$ do not belong to the same space and the operator $\langle \ \rangle$ is not a product scalar. However, in this general case the interpretation in terms of covariance is still valid.

deterministic additive shift in a random price does not change its variability. Second, because of the consumer's *perceived* covariability among the random prices, an *expected* change in one price may possibly imply *expected* changes in other prices. This contrasts with the deterministic multivariate case where the statement "change in one price holding the other prices constant" has always a well defined meaning. This *perceived* covariability among random prices will make difficult the predictions of the signs of the own-price and substitution effects induced by price-risk.

2.2.7 The Budget constraint under price uncertainty

With the above notations, for a given price $\mathbf{p}$ and *expenditure* $m \in \mathbf{R}_{++}$, the budget set is given by: $\mathbf{B}(\mathbf{p}, m) \equiv \{\mathbf{x} \in L_2(\Omega, \mathbf{R}^n)_{++} : \mathbf{p}\cdot\mathbf{x} \leq m\}$. We will assume that for any given price and income, the budget set is non-empty (minimum subsistence level). Formally, $\mathbf{B}(\mathbf{p},\mathrm{m}) \neq \varnothing$ for all $(\mathbf{p}, m) \in L_2(\Omega, \mathbf{R}^n)_{++} \times \mathbf{R}_{++}$.

To be more precise, m should be called *expected* total expenditure. But for shortness we will keep the term total expenditure. The important point here is that m is a predetermined nonrandom scalar; and that results such as the Slutsky equation apply only to *expected* total expenditure, not to total expenditure which is a real valued random variable $\mathbf{m} \in L_2(\Omega, \mathbf{R}_{++})$. Nevertheless, we can still do our analysis in terms of total expenditure $\mathbf{m}$ as a random variable by taking *expected* total expenditure as $m \equiv \langle \mathbf{1}, \mathbf{m} \rangle$, where $\mathbf{1}$ is the (constant) random variable that is equal to 1 for all states of nature, with the assumption that total expenditure is expressed in terms of the numeraire. Since $\langle \mathbf{1}, \mathbf{m} \rangle = \int_{\Omega} \mathbf{m}(\omega) d\mathbf{P}(\omega) = E\mathbf{m}$, with E being the expectation operator, all the results derived below and that do not involve differentiation can be applied to random total expenditure by just replacing m by $E\mathbf{m}$.[44] For results that involve differentiation (such the Slutsky equation and the expenditure elasticities), one can derive them in terms of $\mathbf{m}$ by using the chain rule since the expectation operator is Fréchet differentiable at all points with a constant derivative equals to the same expectation operator. However, since these translations are straightforward, and the chapter is already very long, we will not do them here (see footnote 72). As in deterministic consumer theory, we will also maintain a strict difference between *expected total expenditure* and *expected total income*. The economic reason for making the difference, results from the fact that with the assumption of non satiation, the consumer always spend all of its predetermined *expected* total expenditure, while this need not be the case for *expected* total income, part of which may be saved.

2.2.8 Behavioral assumptions and utility representation of preference

We will restrict further the consumption set as being a weakly compact subset, $\mathbf{X}$, in $L_2(\Omega, \mathbf{R}^n)_+$. This "boundness" condition which is for technical reasons can be hardly considered as restrictive since in practice, random consumptions goods can be considered to always lie in a given closed and bounded subset of $\mathbf{R}^n_+$

Thus, the consumer's preference ordering $\succeq$ is defined now on $\mathbf{X}$. For notational conveniences, we introduce the following sets:

[44] From a behavioral perspective, this assumes that consumers are making their consumption choices based on the expenditure level they don't "expect" to exceed. Hence, given our identification of the consumer's "expectation" process with the mathematical expectation operator, this is equivalent to basing the consumption choices on their maximum *expected* level of expenditure. Note that the issues involved in the formulation of the budget constraint in terms of maximum *expected* expenditure level and the identification of the expectation process with the mathematical expectation operator are unrelated to issues concerning the properties of the consumer's *preference* ordering which are discussed in the next section (see footnote 27 and section 2.2.6).

WP(x) $\equiv$ {y $\in$ **X**: y $\succeq$ **x** } (set of random consumption bundles weakly preferred to **x**).

NSP(x) $\equiv$ {y $\in$ **X**: **x** $\succeq$ y } (set of random consumption bundles not strictly preferred to **x**).

We assume that $\succeq$ satisfies the following standard axioms on preference:

A1: **Reflexivity**: $\succeq$ is reflexive i.e, **x** $\succeq$ **x** for all **x** $\in$ **X** (every random consumption bundle is preferred to itself).

A2: **Completeness**: $\succeq$ is complete i.e, for all **x, y** $\in$ **X** we have either **x** $\succeq$ **y**, **y** $\succeq$ **x** or **x** $\sim$ **y** (all random consumption bundles in the consumption set are comparable).

A3: **Transitivity**: $\succeq$ is transitive i.e, for all **x, y, z** $\in$ **X**, **x** $\succeq$ **y** and **y** $\succeq$ **z** implies **x** $\succeq$ **z** (consistency of the preference ordering).

A4: **Continuity**: $\succeq$ is continuous i.e, for all **x** $\in$ **X** the sets **WP(x)** and **NSP(x)** are closed subsets of $L_2(\Omega, \mathbf{R}^n)_+$ endowed with the topology induced by $\| \ \|_2$ (for all **x, y** $\in$ **X** if **x** $\succ$ **y** then for any **z** "close" enough to **x** we have **z** $\succ$ **y**)[45]

A5: **Strict convexity**: $\succeq$ is strictly convex i.e, for all **x** $\in$ **X** the set **WP(x)** is a strictly convex subset of **X** meaning that if **y, z** $\in$ **WP(x)** and **x** $\neq$ **y** then $\alpha \cdot$**y** + $(1-\alpha)$**z** $\in$ **WP(x)** for all $0<\alpha<1$ (diversification or preference for averages over extremes).

A6: **Monotonicity**: $\succeq$ is monotone i.e, for all **x, y** $\in$ **X** , **x** $\geq$ **y** implies **x** $\succeq$ **y** (if in every state of nature the realized level of consumption good **x** is greater than the one of **y**, then **x** is preferred to **y**)[46].

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[45] We recall that the topology induced by $\| \ \|_2$ formally defines what we mean by the statement : a random consumption **x** is "close" or "similar" to another random consumption **y**.

[46] In the risk literature, this assumption is also called "first order stochastic dominance". As usual, this assumption can be replaced by the weaker local non-satiation assumption.

We note in passing that strict convexity and monotonicity together implies Strict monotonicity. That is, if $\mathbf{x} > \mathbf{y}$ then $\mathbf{x} \succ \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. [47]

Given these assumptions, one can establish the numerical representation of the preference ordering $\succeq$ by a utility functional. This utility representation will be used for the rest of our analysis.

Proposition 2.2.1: *Let the preference ordering $\succeq$ satisfies axioms A1-A6, then there is a continuous, strictly quasi-concave, and bounded utility functional* $\mathbf{U}$: $\mathbf{X} \rightarrow \mathbf{R}_+$ *such that* $\mathbf{x} \succeq \mathbf{y}$ *if and only if* $\mathbf{U}(\mathbf{x}) \geq \mathbf{U}(\mathbf{y})$.

2.3 Demand Functionals in $L_2(\Omega, \mathbf{R}^n)_{++}$ and Duality

As in the finite dimensional case, the assumptions of strict convexity and monotonicity imply that if the solution to the consumer's preference maximization exists, then it is unique and must exactly satisfy the budget constraint. More formally:

Lemma 2.3.1 (adding-up property): *If the set* $\mathbf{S}^* \equiv \{ \mathbf{x} \in \mathbf{B}(\mathbf{p}, m)$: $\mathbf{x} \succeq \mathbf{y}$ for all $\mathbf{y} \in \mathbf{B}(\mathbf{p}, m) \}$ *is non empty, then it is reduced to a singleton* $\{\mathbf{x}^*\}$ *with:* $\mathbf{p}\cdot\mathbf{x}^* = m$.

With the utility representation $\mathbf{U}$, the consumer optimizing behavior or preference maximization assumption translates to utility maximization subject to the budget constraint. Hence, we can write the consumer problem as:

[47] To see that, note that $\mathbf{x} > \mathbf{y}$ implies $2\mathbf{x} - \mathbf{y} > \mathbf{y}$ which, by monotonicity implies $2\mathbf{x} - \mathbf{y} \succeq \mathbf{y}$ which in turn by strict convexity implies $\frac{1}{2}(2\mathbf{x}-\mathbf{y}) + \frac{1}{2}\mathbf{y} \succ \mathbf{y}$ or $\mathbf{x} \succ \mathbf{y}$.

$$\max_{s.t.\ \mathbf{p}\cdot\mathbf{x} \leq m} \mathrm{U}(\mathbf{x}) \qquad (\mathcal{P}1)$$

Moreover, by Lemma **2.3.1** for given price **p** and expenditure *m* we can restrict our search for an optimum to the subset of the budget set defined by $\mathbf{B}^*(\mathbf{p}, m) \equiv \{\mathbf{x} \in \mathbf{X}\colon\ \mathbf{p}\cdot\mathbf{x} = m\}$.

2.3.1 Existence of Continuous Demand Functionals on $L_2(\Omega, \mathbf{R}^n)_{++} \times \mathbf{R}_{++}$

Since the consumption set **X** is a weakly compact subset of the Hilbert space $L_2(\Omega, \mathbf{R}^n)$ and **U** is quasi-concave and continuous; for a given $(\mathbf{p}, m) \in L_2(\Omega, \mathbf{R}^n)_{++} \times \mathbf{R}_{++}$ such that the budget set $\mathbf{B}(\mathbf{p}, m)$ is non-empty there exists a solution $\mathbf{x}^*$ to the consumer maximization problem defined as $\mathbf{x}^*(\mathbf{p}, m) \equiv \mathrm{argmax}\{ \mathrm{U}(\mathbf{x}) : \mathbf{x} \in \mathbf{B}(\mathbf{p}, m)\}$.[48]

By Lemma **2.3.1** the solution $\mathbf{x}^*$ is unique. Thus, we can define a demand functional $\mathbf{x}\colon L_2(\Omega, \mathbf{R}^n)_{++} \times \mathbf{R}_{++} \to L_2(\Omega, \mathbf{R}^n)_{++}$ which associates to each $(\mathbf{p}, m) \in L_2(\Omega, \mathbf{R}^n)_{++} \times \mathbf{R}_{++}$ the unique vector $\mathbf{x}(\mathbf{p}, m) \equiv \mathrm{argmax}\{ \mathrm{U}(\mathbf{x}) : \mathbf{x} \in \mathbf{B}(\mathbf{p}, m)\}$ element of $L_2(\Omega, \mathbf{R}^n)_{++}$. Moreover, because of the canonical isomorphism between $L_2(\Omega, \mathbf{R}^n)$ and the n-cartesian product of $L_2(\Omega, \mathbf{R})$, $\prod_{i=1}^{n} L_2(\Omega, \mathbf{R}) \equiv L_2(\Omega, \mathbf{R})^n$, the demand functional for the i^{th} consumption good, $\mathbf{x}_i$, is given by $\mathbf{x}_i \equiv \pi_i \circ \mathbf{x} : L_2(\Omega, \mathbf{R}^n)_{++} \times \mathbf{R}_{++} \to L_2(\Omega, \mathbf{R})_{++}$ where π_i is the i^{th} canonical projection of $L_2(\Omega, \mathbf{R})^n$.

The demand functional is continuous in both of its arguments.[49] It is also homogenous of degree 0 since multiplying both price and expenditure by a positive scalar does not change the budget set.

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[48] This is a well known result which follows from the weak compactness of **X** and the upper semicontinuity of the quasi-concave, continuous and bounded utility function (see, for example, Aubin 1979b, p. 45, Theorem 1; or Zeidler, 1985, p. 144. proposition **38.12**).

[49] See, for example, Aubin (1979b, page 70, Theorem 3) or Theorem A2 in Appendix 2. Note that upper semicontinuity (of correspondence) reduces to continuity when the correspondence is single valued.

For subsequent analysis we need to express the consumer maximization problem in Lagrangian form. The existence of Lagrange multipliers for constrained optimization problems in infinite dimensional spaces is not a trivial question (see Aubin 1979b, chapter **14**; or Zeidler, 1986 chapters **47-50**). The topologies of the spaces involved play crucial roles in guaranteeing existence of Lagrange multipliers. The existence of Lagrange multipliers for our optimization problems will be based in a existence theorem in reflexive Banach spaces given in Zeidler (1986, p. 480, Theorem **50.A**).

Proposition 2.3.1: *There exists a Lagrange multiplier* $\lambda \in \mathbf{R}_+$ *such that the consumer problem:*

$$\mathbf{V}(\mathbf{p}, m) = \max_{s.t.\ \mathbf{p}\cdot\mathbf{x} = m} \mathbf{U}(\mathbf{x}) \qquad (\mathcal{P}1)$$

is equivalent to the unconstrained maximization problem:

$$\mathbf{V}(\mathbf{p}, m) = \max_{\mathbf{x}}\{\mathbf{U}(\mathbf{x}) - \lambda(\mathbf{p}\cdot\mathbf{x} - m)\} \qquad (\mathcal{P}1')$$

$$\textit{with} \quad \lambda(\mathbf{p}\cdot\mathbf{x}^* - m) = 0 \quad \textit{for any optimal solution } \mathbf{x}^* \in X$$

2.3.2 Properties of the Indirect Utility functional and Roy's Identity

As in the finite dimensional case, if we define an indirect utility functional **V**: $L_2(\Omega, \mathbf{R}^n)_{++} \times \mathbf{R}_{++} \rightarrow \mathbf{R}_+$ with $\mathbf{V}(\mathbf{p}, m) = \max_{s.t.\ \mathbf{p}\cdot\mathbf{x} = m} \mathbf{U}(\mathbf{x})$ for all $(\mathbf{p}, m) \in L_2(\Omega, \mathbf{R}^n)_{++} \times \mathbf{R}_{++}$ then **V** has the following properties:[50]

Proposition 2.3.2

(1) **V** *is continuous at all* $(\mathbf{p}, m) \in L_2(\Omega, \mathbf{R}^n)_{++} \times \mathbf{R}_{++}$.

[50] The proofs are omitted because they are identical to the finite dimensional case which can be found in Varian (1984, p. 121).

(2) *for given* m, $\mathbf{V}(., m)$ *is a monotonic nonincreasing functional; i.e,* $\mathbf{p} \geq \mathbf{p}'$ *implies* $\mathbf{V}(\mathbf{p}, m) \leq \mathbf{V}(\mathbf{p}', m)$.

(3) *for given* $\mathbf{p}$, $\mathbf{V}(\mathbf{p}, .)$ *is a strictly increasing function; i.e,* $m > m'$ *implies* $\mathbf{V}(\mathbf{p}, m) > \mathbf{V}(\mathbf{p}, m')$.

(4) *for given* m, $\mathbf{V}(., m)$ *is a quasi-convex functional; i.e, the set* $\{\mathbf{p} \in L_2(\Omega, \mathbb{R}^n)_{++}: \mathbf{V}(\mathbf{p}, m) \leq k\}$ *is convex for all* $k \in \mathbb{R}$.

(5) $\mathbf{V}$ *is homogenous of degree 0 in both of its arguments.*

In fact, as we will see below, the indirect utility functional satisfies stronger properties.

We now move to the characterization of the optimal solution which is essential for comparative statics. However, before that, as already indicated in the introduction we need to extend the concept of Fréchet differentiability to functions whose domains of definition may have an empty interior.

Definition 2.3.1 Let **X** and **Y** be two real Banach spaces, **M** a subset of **X** which is not necessarily open, and **f** a function $\mathbf{f}: \mathbf{M} \subseteq \mathbf{X} \rightarrow \mathbf{Y}$. **f** is said to be *Fréchet (resp Gâteaux) differentiable* at $\mathbf{x}_0 \in \mathbf{M}$ if and only if there exists a neighborhood $\mathbf{O}(\mathbf{x}_0)$ of $\mathbf{x}_0$ that contains **M**, and a function $\mathbf{f}^*: \mathbf{O}(\mathbf{x}_0) \subseteq \mathbf{X} \rightarrow \mathbf{Y}$ which is Fréchet (resp Gâteaux) differentiable at $\mathbf{x}_0$, such that $\mathbf{f}^*(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{M}$. The Fréchet (resp Gâteaux) derivative of **f** at $\mathbf{x}_0$ is then defined as: $\mathbf{f}'(\mathbf{x}_0) = \mathbf{f}^{*\prime}(\mathbf{x}_0)$. Higher derivatives of **f** are also defined similarly (see Definition **A21** in Appendix **2**).[51]

[51] It is clear that for a given $\mathbf{x}_0$ $\mathbf{f}'(\mathbf{x}_0)$ is well defined since if there are two pairs of neighborhood of $\mathbf{x}_0$ and function $(\mathbf{O}_1(\mathbf{x}_0), \mathbf{f}_1^*)$ and $(\mathbf{O}_2(\mathbf{x}_0), \mathbf{f}_2^*)$ satisfying the definition then it immediately follows that $\mathbf{f}_1^{*\prime}(\mathbf{x}_0) = \mathbf{f}_2^{*\prime}(\mathbf{x}_0)$, and $\mathbf{f}^*_1 = \mathbf{f}_2^*$ on **M**. It is also clear that when **M** is an open set the definition reduces to the usual definition of Fréchet and Gâteaux differentiability (since it suffices to take $\mathbf{O}(\mathbf{x}) = \mathbf{M}$, and $\mathbf{f}^* = \mathbf{f}$ for all $\mathbf{x} \in \mathbf{M}$).

Hence, in what follows, whenever we speak of the Fréchet or Gâteaux differentiability of a function $\mathbf{f}$: $\mathbf{M} \subseteq \mathbf{X} \to \mathbf{Y}$, and $\mathbf{M}$ is not open or has an empty interior, it must be understood in the sense of Definition **2.3.1**, even if the open neighborhood and the extended function are not explicitly given.[52]

It is frequent, especially in economics, to characterize optimal solutions by the first and/or second order condition. But this would requires the utility functional to be differentiable in some appropriate sense. However, the assumption of differentiability rules out a class of non-smooth preferences such as the Leontief utility which admits infinitely continuously differentiable demand functionals. Nevertheless, we can characterize the optimal solution and obtain all the comparative static results of economic interest without using the first order condition. This will be achieved by using a mathematical result of Ekeland and Lebourg (1976) which gives a characterization of optimal solutions of perturbed optimization problems, which does not require the differentiability of the objective function with respect to the choice variable. In fact, the characterization can be seen as a generalization of the well known envelop theorem.

Proposition 2.3.3: *Let the indirect utility functional* $\mathbf{V}$ *be defined as* $\mathbf{V}(\mathbf{p},m) \equiv \max_{\mathbf{x}}\{\mathbf{U}(\mathbf{x}) - \lambda(\mathbf{p}\mathbf{x} - m)\}$ *for all* $(\mathbf{p}, m) \in L_2(\Omega, \mathbf{R}^n)_{++} \times \mathbf{R}_{++}$, *then:*
$\mathbf{V}$ *is locally lipschitzian on* $L_2(\Omega, \mathbf{R}^n)_{++} \times \mathbf{R}_{++}$ *and is "generically" Fréchet-differentiable i.e, there exists a dense* G_δ *subset* $\mathbf{S}_1 \times \mathbf{S}_2$ *of* $L_2(\Omega, \mathbf{R}^n)_{++} \times \mathbf{R}_+$ *such that* $\mathbf{V}$ *is Fréchet-differentiable at every* $(\mathbf{p}, m) \in \mathbf{S}_1 \times \mathbf{S}_2$ *with:*

[52] In general, the open neighborhood and the differentiable function extending **f** is the same for all elements of **M**. In the propositions that follows, and in all the examples worked out in chapter **3** where **M** has an empty interior, the open neighborhood is the whole space and **f** is naturally extended to the whole space.

(i)

$$\frac{\partial \mathbf{V}(\mathbf{p},m)}{\partial \mathbf{p}_i} = -\lambda \mathbf{x}_i(\mathbf{p},m) \qquad i=1,\ldots,n; \qquad \textit{and} \qquad \frac{\partial \mathbf{V}(\mathbf{p},m)}{\partial m} = \lambda \tag{2.3.1}$$

(ii) *The restriction to* $\mathbf{S}_1 \times \mathbf{S}_2$ *of the derivative* $\mathbf{V}': L_2(\Omega, \mathbb{R}^n)_{++} \times \mathbb{R}_{++} \rightarrow L_2(\Omega, \mathbb{R}^n) \times \mathbb{R}$ *is continuous.*[53]

As an immediate consequence of (i) we get Roy's Identity which links the **n** Marshallian demand functionals to the indirect utility functional, i.e, generically we have:[54]

$$\mathbf{x}_i(\mathbf{p},m) = -\frac{\dfrac{\partial \mathbf{V}(\mathbf{p},m)}{\partial \mathbf{p}_i}}{\dfrac{\partial \mathbf{V}(\mathbf{p},m)}{\partial m}} \qquad i=1,\ldots,n \tag{2.3.2}$$

2.3.3 Properties of the Expenditure Functional and Shephard's Lemma

Since for given **p**, **V(p, .)** is a strictly increasing function from $\mathbf{R}_{++}$ into $\mathbf{R}_{++}$, we can invert it and define the expenditure function **E(p,)**: $\tilde{\mathbf{R}} \rightarrow \mathbf{R}_{++}$ with $\mathbf{V}(\mathbf{p}, \mathbf{E}(\mathbf{p}, u)) = u$ and $\mathbf{E}(\mathbf{p}, \mathbf{V}(\mathbf{p}, m)) = m$ for all $u \in \tilde{\mathbf{R}}$ and $m \in \mathbf{R}_{++}$, where $\tilde{\mathbf{R}} \equiv \mathbf{U}(\mathbf{X})$. Thus defined, **E(p,** u**)**

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[53] A $\mathbf{G}_\delta$ subset is defined as the intersection of a countable family of open dense subsets (see Appendix 2). Genericity is an important concept of modern mathematics. Roughly speaking, a generic property is a property which is true in a "big set", or almost everywhere. In other words, a generic property is always true except in "very rare" pathological cases. It is a topological property and should not be confused with the more familiar measure theoretic concept of "almost everywhere" usually associated with the Lebesgue or probability measures. They transmit the same idea but genericity is the appropriate concept in infinite dimensional spaces.

[54] Let us recall that for a function **f**: **O** → **B** where **B** is a Banach space and **O** is an open subset of another Banach space **A**, its Fréchet derivative **f′** is, by definition, a continuous linear functional from **A** to **B**. Thus, here we have $\frac{\partial \mathbf{V}(\mathbf{p},m)}{\partial \mathbf{p}_i} \in L(\Omega, \mathbf{R})^* = L(\Omega, \mathbf{R})$ which is consistent with the fact that $\mathbf{x}_i(\mathbf{p},m) \in L(\Omega, \mathbf{R})$.

is the minimal expenditure necessary to achieve utility u at price $\mathbf{p}$.[55] Since $\mathbf{E}(\mathbf{p}, u)$ is defined for all price $\mathbf{p} \in L_2(\Omega, \mathbb{R}^n)_{++}$, the expenditure functional can be constructed as the functional $\mathbf{E}$: $L_2(\Omega, \mathbb{R}^n)_{++} \times \tilde{\mathbb{R}} \to \mathbb{R}_{++}$ which associates to each $(\mathbf{p}, u)$ $\mathbf{E}(\mathbf{p}, u)$ the minimal expenditure necessary to achieve utility u at price $\mathbf{p}$.

Alternatively, we can define the expenditure functional through the following minimization problem:

$$\mathbf{E}(\mathbf{p}, u) = \min_{\text{s.t. } \mathbf{U}(\mathbf{x}) \geq u} \mathbf{p}\cdot\mathbf{x} \qquad (\mathcal{P}2)$$

where u is a fixed utility level in the range of $\mathbf{U}$, $\tilde{\mathbb{R}}$.[56]

The functional $\mathbf{h}$: $L_2(\Omega, \mathbb{R}^n)_{++} \times \tilde{\mathbb{R}} \to L_2(\Omega, \mathbb{R}^n)_{++}$ which associates to each $(\mathbf{p}, u)$ $\mathbf{h}(\mathbf{p}, u) \equiv \operatorname{argmin}\{ \mathbf{p}\cdot\mathbf{x} : \mathbf{U}(\mathbf{x}) \geq u \}$ is the Hicksian or compensating demand functional; and $\mathbf{h}_i \equiv \pi_i \circ \mathbf{h}$ is the i^{th} compensated or Hicksian demand functional.

Given our assumptions, like in the finite dimensional case, the solution to the maximization problem $(\mathcal{P}1)$ is identical to the one of the above minimization problem where we take u as the maximum utility attained in $(\mathcal{P}1)$; more formally:[57]

Proposition 2.3.4: *if an element* $\mathbf{x}^* \in \mathbf{X}$ *solves the maximization problem* $(\mathcal{P}1)$ *then* $\mathbf{x}^*$ *solves the minimization problem:*

---

[55] An easy way to see that is to recall the definition of the generalized inverse, g^{-1}, of a real valued function g on the real line as being $g^{-1}(t) = \inf\{ y \in \mathbb{R} : g(y) \geq t\}$.

[56] It is important that u be in the range of $\mathbf{U}$, otherwise $\mathbf{E}$ may not be defined unless one allows $\mathbf{E}$ to take the value $+\infty$ and sets, for given $\mathbf{p}$, $\mathbf{E}(\mathbf{p}, u) = +\infty$ if $u \notin \mathbf{U}(\mathbf{X})$. Also, we do not need to use inf in the definition of $\mathbf{E}$ since $\mathbf{X}$ is weakly compact.

[57] For a proof, see Varian (1984, Appendix 1 page 162). The topological arguments used in his proof are also valid in the infinite dimensional case.

$$\min_{s.t.\ U(x) \geq U(x^*)} \mathbf{p}\cdot\mathbf{x} \qquad (2.3.3)$$

conversely, if $\mathbf{x}^*$ *solves the minimization problem* ($\mathcal{P}$**2**), *then* $\mathbf{x}^*$ *solves the maximization problem:*

$$\max_{s.t.\ \mathbf{p}\cdot\mathbf{x} \leq \mathbf{p}\cdot\mathbf{x}^*} U(x) \qquad (2.3.4)$$

Proposition **2.3.4** is often stated by saying that the Marshallian demand at price **p** and expenditure m is equal to the Hicksian demand at price **p** and utility **V**(**p**, m) and vice versa, the Hicksian demand at price **p** and utility u is equal to the Marshallian demand at price **p** and expenditure **E**(**p**, u). That is :

$$\mathbf{x}_i(\mathbf{p},m) = \mathbf{h}_i(\mathbf{p},\ \mathbf{V}(\mathbf{p},m)), \quad \text{and} \quad \mathbf{h}_i(\mathbf{p},u) = \mathbf{x}_i(\mathbf{p},\ \mathbf{E}(\mathbf{p},u)) \quad i = 1,\ldots,n \qquad (2.3.5)$$

Since the optimal solution of the minimization is the same as the corresponding maximization problem, it is easy to see that the two definitions of the expenditure functional, given above, are equivalent. Thus we have the following properties for the expenditure functional:[58]

Proposition 2.3.5:

(1) **E** *is continuous at all* (**p**, u) $\in L_2(\Omega, \mathbf{R}^n)_{++} \times \bar{\mathbf{R}}$

(2) *for given* u, **E**(., u) *is a monotone nondecreasing functional; i.e,* $\mathbf{p} \geq \mathbf{p}'$ *implies* **E**(**p**, m) $\geq$ **E**(**p**′, m).

(3) *for given* **p**, **E**(**p**, .) *is a strictly increasing function; i.e,* $u > u'$ *implies* **E**(**p**, u) $>$ **E**(**p**, u').

(4) *for given* u, **E**(., u) *is a strictly concave functional.*

---

[58] The proofs are the same as in the finite dimensional case and are omitted (they can be found in Varian (1984, chapter 1)). The strict concavity follows directly from the strict convexity assumption (via Proposition **2.3.4** and the unicity of the solution of the associated maximization problem). For the continuity property, see, for example, Aubin (1979, p. 70, Theorem 3).

(5) *for given u, $\mathbf{E}(., u)$ is homogenous of degree 1.*

To get the differential properties analogous to those of the indirect utility functional, we need to express the minimization problem in Lagrangian form.

Proposition 2.3.6: *For all $(\mathbf{p}, u) \in L_2(\Omega, \mathbf{R}^n)_{++} \times \tilde{\mathbf{R}}$ such that $u \in \operatorname{int}\tilde{\mathbf{R}}$ there exists a Lagrange multiplier $\lambda \in \mathbf{R}_+$ such that the consumer problem:*

$$\mathbf{E}(\mathbf{p}, u) = \min_{\text{s.t. } \mathbf{U}(\mathbf{x}) \geq u} \mathbf{p}\mathbf{x} \qquad (\mathcal{P}2)$$

is equivalent to the unconstrained minimization problem:

$$\mathbf{E}(\mathbf{p}, u) = \min_{\mathbf{x}}\{\mathbf{p}\mathbf{x} - \lambda(\mathbf{U}(\mathbf{x}) - u)\} \qquad (\mathcal{P}2')$$

with $\lambda(\mathbf{U}(\mathbf{x}^*) - u) = 0$ *for any optimal solution* $\mathbf{x}^* \in X$

We can now use Ekeland and Lebourg's result to characterize the optimal solution of the cost minimization problem.

Proposition 2.3.7: *Let the expenditure functional $\mathbf{E}$ be defined as*

$$\mathbf{E}(\mathbf{p}, u) = \min_{\mathbf{x}}\{\mathbf{p}\mathbf{x} - \lambda(\mathbf{U}(\mathbf{x}) - u)\} \qquad \textbf{(2.3.6)}$$

Then, $\mathbf{E}$ is locally lipschitzian on $L_2(\Omega, \mathbf{R}^n)_{++} \times \tilde{\mathbf{R}}$ and is "generically" Fréchet-differentiable i.e, there exists a dense G_δ subset $\mathbf{Q}_1 \times \mathbf{Q}_2$ of $L_2(\Omega, \mathbf{R}^n)_{++} \times \tilde{\mathbf{R}}$ such that $\mathbf{E}$ is Fréchet-differentiable at every $(\mathbf{p}, m) \in \mathbf{Q}_1 \times \mathbf{Q}_2$ with:

(i)

$$\frac{\partial \mathbf{E}(\mathbf{p}, u)}{\partial \mathbf{p}_i} = \mathbf{h}_i(\mathbf{p}, u) \quad i = 1, \ldots, n, \quad \text{and} \quad \frac{\partial \mathbf{E}(\mathbf{p}, u)}{\partial u} = \lambda \qquad \textbf{(2.3.7)}$$

(ii) The restriction to $\mathbf{Q}_1 \times \mathbf{Q}_2$ of the derivative $\mathbf{E}'$: $L_2(\Omega, \mathbf{R}^n)_{++} \times \tilde{\mathbf{R}} \rightarrow L_2(\Omega, \mathbf{R}^n) \times \mathbf{R}$ is continuous.

In particular, (i) yields Shephard's lemma which gives the compensated demand functionals as the partial derivatives of the expenditure functional. That is, generically we have:

$$\mathbf{h}_i(\mathbf{p},u) = \frac{\partial \mathbf{E}(\mathbf{p},u)}{\partial \mathbf{p}_i} \qquad i = 1,\ldots,n \tag{2.3.8}$$

2.4 Comparative Statics: The Slutsky price-substitution matrix under uncertainty

For the derivation and correct statement of the comparative static results in the context of uncertainty, we need to introduce few mathematical concepts and results from nonlinear functional analysis. The main ones are the monotonicity and potentiality concepts. In this economic context, the monotonicity property is merely an abstract way of saying that the compensated demands are decreasing functions of prices. It thus generalizes the negative semi-definiteness of the Slutsky matrix. The potential property is a characterization of the integrability (also called path independence) of demands, hence generalizing the Slutsky symmetry condition. The formal definitions and theorems related to them are given in Appendix **2**.

2.4.1 Global comparative statics

As a corollary of Proposition **2.3.6** (strict concavity and continuity of $\mathbf{E}(., u)$) and Proposition **2.3.7** (Shephard's lemma), Theorems **A2.3-A2.4** in Appendix **2** give us the main result of demand theory: The monotonicity and potential property with respect to price of the compensated or Hicksian demand. The result is global in the sense that unlike the comparative statics results based on differentiability assumptions, it is also valid for changes that are not infinitesimally small.[59]

Proposition 2.4.1: *There exists a dense* G_δ *subset* $\mathbf{Q}_1 \times \mathbf{Q}_2$ *of* $L_2(\Omega, \mathbb{R}^n)_{++} \times \tilde{\mathbb{R}}$ *such that for fixed* $u \in \tilde{\mathbb{R}}$, *the compensated demand operator* $\mathbf{h}(., u)$ *is a strictly monotone potential operator from* $\mathbf{Q}_1$ *to* $L_2(\Omega, \mathbb{R}^n)_{++}$. *In particular,* $\mathbf{h}(., u)$ *is locally bounded and satisfies the following properties:*

(i) ***integrability***

$$\int_0^1 \langle \mathbf{h}(t\mathbf{p}, u),\ \mathbf{p}\rangle dt - \int_0^1 \langle \mathbf{h}(t\mathbf{q}, u),\ \mathbf{q}\rangle dt = \int_0^1 \langle \mathbf{h}(\mathbf{q} + t(\mathbf{p} - \mathbf{q}), u),\ \mathbf{p} - \mathbf{q}\rangle dt \tag{2.4.1}$$
$$\text{for all}\quad \mathbf{p},\ \mathbf{q},\ u \in \mathbf{Q}_1 \times \mathbf{Q}_2$$

(ii) ***strict monotonicity***

$$\langle \mathbf{h}(\mathbf{q}, u) - \mathbf{h}(\mathbf{p}, u), \mathbf{q} - \mathbf{p}\rangle < 0 \tag{2.4.2}$$
$$\text{for all}\ \mathbf{p}, \mathbf{q} \in \mathbf{Q}_1\ \text{and}\ u \in \mathbf{Q}_2\quad \text{with equality if}\ \mathbf{p} = \mathbf{q}.$$

By taking in (ii) $u \equiv \mathbf{V}(\mathbf{p}, m)$, the indirect utility corresponding to the demanded bundle $\mathbf{x}(\mathbf{p}, m)$, we obtain:

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[59] The result is also general in that it applies to any choice situation where the preference is convex and monotonic, and the choice set is a uniformly convex Banach space. Most choice spaces used in economics satisfy this "smoothness" condition. Machina's (1982 and 1984) choice space of cumulative distribution functions does not qualify because he uses the L_1 topology which makes the space not uniformly convex.

$$\langle \mathbf{x}(\mathbf{q},\ \mu(\mathbf{q},\mathbf{p},m)) - \mathbf{x}(\mathbf{p},m),\ \mathbf{q}-\mathbf{p}\rangle < 0 \quad \text{for all } \mathbf{p},\mathbf{q} \in \mathbf{Q}_1 \text{ and } m>0 \text{ with equality if } \mathbf{p}=\mathbf{q}. \tag{2.4.3}$$

where $\mu(\mathbf{q}, \mathbf{p}, m) \equiv \mathbf{E}(\mathbf{q}, \mathbf{V}(\mathbf{p}, m))$ is the compensated expenditure which maintains the level of utility at the same level as when the bundle $\mathbf{x}(\mathbf{p}, m)$ is consumed. The compensated functional is defined and discussed further in section **2.7** below.

If instead of $\mathbf{V}(\mathbf{p}, m)$ we take $u \equiv \mathbf{V}(\mathbf{q}, m')$, the indirect utility corresponding to the demanded bundle $\mathbf{x}(\mathbf{q}, m')$, we obtain a similar inequality:

$$\langle \mathbf{x}(\mathbf{q},m') - \mathbf{x}(\mathbf{p},\ \mu(\mathbf{p},\mathbf{q},m')),\ \mathbf{q}-\mathbf{p}\rangle < 0 \quad \text{for all } \mathbf{p},\mathbf{q} \in \mathbf{Q}_1 \text{ and } m'>0 \text{ with equality if } \mathbf{p}=\mathbf{q}. \tag{2.4.4}$$

The two inequalities (2.4.3) and (2.4.4) essentially state that for any price change the pure price effects in changes in the ordinary or Marshallian demands are negative. Moreover, we can decompose the change in the ordinary demand induced by a price/expenditure change from $(\mathbf{p}, m)$ to $(\mathbf{q}, m')$ into a pure price effect and a expenditure effect:

$$\begin{aligned}\langle \mathbf{x}(\mathbf{q},m') - \mathbf{x}(\mathbf{p},m),\ \mathbf{q}-\mathbf{p}\rangle &= \langle \mathbf{x}(\mathbf{q},m') - \mathbf{x}(\mathbf{p},\ \mu(\mathbf{p},\mathbf{q},m')),\ \mathbf{q}-\mathbf{p}\rangle \\ &+ \langle \mathbf{x}(\mathbf{p},\ \mu(\mathbf{p},\mathbf{q},m')) - \mathbf{x}(\mathbf{p},m),\ \mathbf{q}-\mathbf{p}\rangle\end{aligned} \tag{2.4.5}$$

The first part of the right hand side of the equality is the sum of the *own price effects* of the change in the individual Marshallian demands, which was shown to be negative, and the second part is the *income effect* (or more precisely *expenditure effect*). The income effect can further be decomposed into two components:

$$\begin{aligned}\langle \mathbf{x}(\mathbf{p},\ \mu(\mathbf{p},\mathbf{q},m')) - \mathbf{x}(\mathbf{p},m),\ \mathbf{q}-\mathbf{p}\rangle &= \langle \mathbf{x}(\mathbf{p},\ \mu(\mathbf{p},\mathbf{q},m')) - \mathbf{x}(\mathbf{p},m),\ \mathbf{q}\rangle \\ &- \langle \mathbf{x}(\mathbf{p},\ \mu(\mathbf{p},\mathbf{q},m')) - \mathbf{x}(\mathbf{p},m),\ \mathbf{p}\rangle\end{aligned} \tag{2.4.6}$$

The second component will be shown below, in section **2.7**, to be the *equivalent variation*, a monetary measure of the change in welfare induced by a change in price and expenditure from ($\mathbf{p}$, m) to ($\mathbf{q}$, m').[60] The first component could not, at this point, be given an intuitive economic interpretation.

With the canonical isomorphism between $L_2(\Omega, \mathbb{R}^n)$ and $\prod_{i=1}^{n} L_2(\Omega, \mathbb{R}) \equiv L_2(\Omega, \mathbb{R})^n$, the pure price, and income effects decomposition can be written in terms of the individual demands. That is, the change in the i^{th} Marshallian demand $\mathbf{x}_i$, induced by a change in the j^{th} price from $\mathbf{p}_j$ to $\mathbf{q}_j$ can be decomposed into a *cross price* or *substitution* effect and an *income effect*:

$$\begin{aligned}\langle \mathbf{x}_i(\mathbf{q},m') - \mathbf{x}_i(\mathbf{p},m), \mathbf{q}_j - \mathbf{p}_j\rangle &= \langle \mathbf{x}_i(\mathbf{q},m') - \mathbf{x}_i(\mathbf{p}, \mu(\mathbf{p},\mathbf{q},m')), \mathbf{q}_j - \mathbf{p}_j\rangle \\ &+ \langle \mathbf{x}_i(\mathbf{p}, \mu(\mathbf{p},\mathbf{q},m')) - \mathbf{x}_i(\mathbf{p},m), \mathbf{q}_j - \mathbf{p}_j\rangle \quad i,j = 1,\ldots,n.\end{aligned} \tag{2.4.7}$$

Again, the strict monotonicity implies that for $i = j$ the *own price* effect is negative, but for $i \neq j$ it does not imply anything about the sign of *cross price or substitution* effects which may be positive or negative. In fact, the signs of the cross price and income effects are the basis for the usual classification of commodities as substitutes, complements, inferior and superior.

In summary, the strict monotonicity and potential properties of the compensated demand gives us the main comparative statics conclusions of demand theory. These conclusions are usually stated in terms of the Slutsky price-substitution matrix which requires the differentiability of the demand functional. But it should be noted that these conclusions do not require differentiability assumptions on the demand functionals or even on the utility functional U. However, as shown below, there is equivalence when the demand functional is differentiable. Besides the mostly technical assumptions on preference (reflexivity, transitivity, continuity etc...), the key behavioral assumptions required for this conclusion to

[60] This term would have been the *compensating variation* if the left hand side of (2.4.3) was used in the decomposition.

hold are strict convexity, monotonicity and preference maximization.[61] The symmetry and negative semi-definiteness of the Slutsky matrix has been shown to be a necessary and sufficient condition for the integrability of demand systems (see, for example, Hurwicz and Uzawa (1971) for the finite dimensional case, and, Berger and Meyers (1965) and Berger (1971) for the general case). Hence, Proposition **2.4.1** exhausts all the implications of demand theory (given the above assumptions on preference).

2.4.2 Differential or local comparative statics

Differential comparative statics (assessing the effects of infinitesimally small changes in prices and expenditure on demands) makes sense only if the demand functional is at least differentiable in the neighborhood of the point where it is evaluated. The differentiability of the demand functional is also important as a theoretical requirement for calculating the usual elasticities, which is the aim of applied demand analysis.

So far, nothing in the above results guarantees the differentiability of the demand functionals. The existence of differentiable demand functional in infinite dimensional choice spaces is very problematic. In particular Araujo (1988) shows that continuously differentiable demand functionals cannot be derived from a strongly concave and twice continuously

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[61] Some authors (e.g. Deaton and Muellbauer, 1980, pp. 36, 44-45, 50) attribute the negative semi-definiteness of the Slutsky price-substitution matrix to the assumption of preference maximization alone. This is because the property follows directly from the concavity of the expenditure function. But, it is important to see that the equivalence between preference maximization and expenditure minimization depends, as the proof of Proposition **2.3.4** shows, on both the convexity and monotonicity (or at least local non-satiation) assumptions. Hence, as far as the global nature of the property is concerned, preference maximization alone is not sufficient for having the <u>global</u> monotonicity of the compensated demand.

differentiable utility functional unless the choice space is a Hilbert space. As he puts it, "...in the case of infinitely many commodities, strong, non-existence results are the rule".[62]

The Slutsky price-substitution matrix under uncertainty

If the demand functionals are differentiable on $Q_1 \times Q_2$ so that the Fréchet derivative $D_p h(p, u)$ exists for all $p, u \in Q_1 \times Q_2$, then by Theorem **A2.5** in Appendix **2**, conditions **(i)** and **(ii)** of Proposition **2.4.1** are respectively equivalent to:

(iii) ***symmetry of*** $D_p h(p, u)$:

$$\langle D_p h(p,u)q, s \rangle = \langle D_p h(p,u)s, q \rangle \quad \text{for all } (p, u) \in Q_1 \times Q_2, \text{ and } q, s \in L_2(\Omega, R^n) \tag{2.4.8}$$

(iv) ***negativity of*** $D_p h(p, u)$:

$$\langle D_p h(p,u)q, q \rangle < 0 \quad \text{for all } (p, u) \in Q_1 \times Q_2, \text{ and } q \in L_2(\Omega, R^n) \tag{2.4.9}$$

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[62] It is important to address the differentiability of demand question if one wants to avoid making vacuous local comparative static statements or taking meaningless numbers as elasticities. The question we need to answer is whether the assumptions thus far made on the primitive concepts are enough to guarantee the differentiability of the demand functionals, and if not, what additional assumptions, consistent with previously made ones, on the primitive concepts are needed? In the finite dimensional case, Debreu (1983, chapter 15) shows that a sufficient condition for the differentiability of demand is the one of smooth preference. He also shows that this condition is equivalent to the condition that the utility function be twice continuously differentiable with non vanishing Heissian determinant.

We recall that $\mathbf{D}_p\mathbf{h}(\mathbf{p}, u)$ is a continuous linear operator from $L_2(\Omega, \mathbb{R}^n)$ to $L_2(\Omega, \mathbb{R}^n)$ and that by a standard result of differential calculus in Banach spaces (see, for example, Cartan 1977, p. 40), it is completely determined by the $n \times n$ matrix of partial derivatives

$$\left[\frac{\partial \mathbf{h}_i(\mathbf{p}, u)}{\partial \mathbf{p}_j}\right]_{\substack{1 \le i \le n \\ 1 \le j \le n}} \qquad (2.4.10)$$

with each $\frac{\partial \mathbf{h}_i(\mathbf{p}, u)}{\partial \mathbf{p}_j}$ being a continuous linear functional from $L_2(\Omega, \mathbb{R})$ to $L_2(\Omega, \mathbb{R})$. Hence the symmetry condition can be written in the usual form:

$$\frac{\partial \mathbf{h}_i(\mathbf{p}, u)}{\partial \mathbf{p}_j} = \frac{\partial \mathbf{h}_j(\mathbf{p}, u)}{\partial \mathbf{p}_i} \qquad i, j = 1, \ldots, n. \quad (2.4.11)$$

The negativity condition implies that the continuous linear operators $\frac{\partial \mathbf{h}_i(\mathbf{p}, u)}{\partial \mathbf{p}_i}$ $\quad i = 1, \ldots, n$ which measure the own price effects, are strictly negative i.e:

$$\langle \frac{\partial \mathbf{h}_i(\mathbf{p}, u)}{\partial \mathbf{p}_i} \mathbf{q}_i, \mathbf{q}_i \rangle < 0 \quad \text{for all } \mathbf{q}_i \in L(\Omega, \mathbb{R}) \setminus \{\mathbf{0}\}; \; i = 1, \ldots, n \qquad (2.4.12)$$

Furthermore, by differentiating both sides of the second equality in (2.3.5) with respect to the j^{th} price we get:

$$\frac{\partial \mathbf{h}_i(\mathbf{p}, u)}{\partial \mathbf{p}_j} = \frac{\partial \mathbf{x}_i(\mathbf{p}, \mathbf{E}(\mathbf{p}, u))}{\partial \mathbf{p}_j} + \frac{\partial \mathbf{x}_i(\mathbf{p}, \mathbf{E}(\mathbf{p}, u))}{\partial m} \circ \frac{\partial \mathbf{E}(\mathbf{p}, u)}{\partial \mathbf{p}_j} \qquad (2.4.13)$$

where the symbol $\circ$ represents the composition operation between the two linear functionals.

By taking $u \equiv \mathbf{V}(\mathbf{p}, m)$ and using Shephard's lemma along with the facts that $\mathbf{E}(\mathbf{p}, \mathbf{V}(\mathbf{p}, m)) = m$ and $\mathbf{h}(\mathbf{p}, \mathbf{V}(\mathbf{p}, m)) = \mathbf{x}(\mathbf{p}, m)$, we obtain the Slutsky equation:[63]

$$\frac{\partial \mathbf{h}_i(\mathbf{p}, \mathbf{V}(\mathbf{p},m))}{\partial \mathbf{p}_j} = \frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j} + \frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathrm{m}} \circ \mathbf{x}_j(\mathbf{p},m) \quad i,j = 1,\ldots,n. \tag{2.4.14}$$

Hence, in terms of the Marshallian demand operator, the potential and monotonicity properties of the compensated demand operator are respectively equivalent to the symmetry and negativity of the Slutsky matrix

$$\left[\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j} + \frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial m} \circ \mathbf{x}_j(\mathbf{p},m)\right]_{\substack{1 \le i \le n \\ 1 \le j \le n}} \tag{2.4.15}$$

The Slutsky equation also gives the decomposition of a differential change in the Marshallian demands into a substitution effect and a income effect:

$$\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j} = \frac{\partial \mathbf{h}_i(\mathbf{p}, \mathbf{V}(\mathbf{p},m))}{\partial \mathbf{p}_j} - \frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial m} \circ \mathbf{x}_j(\mathbf{p},m) \quad i,j = 1,\ldots,n. \tag{2.4.16}$$

Finally, to conclude this section, we point out that if the utility functional is Fréchet or Gâteaux differentiable (in the sense of Definition **2.3.1** of course), then Corollary **A2.1** in Appendix **2** gives a necessary first order condition that is similar to the finite dimensional case. Moreover, as in the finite dimensional case, if **U** is twice differentiable, one can use this first order condition along with a generalized implicit function theorem (see Zeidler, 1985, p. 150), to derive all the above differential comparative statics results.

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[63] For two functionals $\mathbf{A}: \mathbf{H} \to \mathbf{F}$ and $\mathbf{B}: \mathbf{G} \to \mathbf{H}$, $\mathbf{A} \circ \mathbf{B}: \mathbf{G} \to \mathbf{F}$ is defined by $\mathbf{A} \circ \mathbf{B}(\mathbf{p}) \equiv \mathbf{A}(\mathbf{B}(\mathbf{p}))$ for all $\mathbf{p} \in \mathbf{G}$. We recall that $\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial m}$ is linear functional from $\mathbf{R}$ to $L_2(\Omega, \mathbf{R})$; and because $L_2(\Omega, \mathbf{R})$ is equal to its dual, we can take $\mathbf{x}_j(\mathbf{p}, m)$ as a linear operator defined on $L_2(\Omega, \mathbf{R})$. Hence, $\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial m} \circ \mathbf{x}_j(\mathbf{p},m)$ is a linear functional from $L_2(\Omega, \mathbf{R})$ to $L_2(\Omega, \mathbf{R})$.

2.5 The consumer's "precautionary" response to price-risk

The comparative statics conclusions derived in the preceding section reflect both the effects dues to taste substitution, income, and the consumer's response to price-risk. Indeed, the consumer's preference ordering of the different bundles of consumption goods by $\succeq$ reflects both its taste (i.e., the relative strengths of the consumer's preference or desire for consuming one type of commodity over another) and attitude toward risk (i.e., the consumer's attitude toward the relative riskiness or different degrees of risk regarding the availability and affordability of the different commodities as reflected in their uncertain relative prices).[64] For example, the convexity assumption (i.e., preference for commodity bundles that include more types of good over those with fewer types) reflect diversification either for taste reason or for risk aversion reason, or for both reasons. Hence in choice under uncertainty with many commodities, there are two possible ways to justify the crucial convexity assumption.

The separation in comparative statics of the effects due solely to attitude toward risk from the usual taste substitution and income effects, is essential for a meaningful comparative risk analysis in the context of uncertainty with many commodities. This problem which does not arise in the static one-commodity case has been recognized and discussed since Khilstrom and Mirman (1974).[65] The same problem arises in the analysis of intertemporal consumption choice under uncertainty with one aggregated commodity (the consumption-saving decision for example). But in the one-commodity intertemporal case, the separation of the effects of

[64] Some authors refer to the taste aspect as *ordinal preference*. But we prefer the term *taste* because we view the ordinality of $\succeq$ as reflecting both taste and attitude toward risk.

[65] See for example, Khilstrom and Mirman (1974), Paroush (1974), Keeney (1975), Duncan (1977), Kerni (1979), Olson (1988). With one commodity only (usually taken to be money), the taste aspect is irrelevant. The same is true for the static analysis of portfolio allocation with many risky securities, because all returns are measured in monetary units so that any substitution will be due to attitude toward risk.

attitude toward risk is not from those of taste (which is irrelevant for one commodity only), but rather from the effects of *time preference* or *impatience*, which is another aspect of preference.[66] The separation problem in the multi-commodity static case led Khilstrom and Mirman to restrict their comparative risk analysis to preferences with same taste for the commodities. We do not need this restriction in order to assess the impact of uncertainty on the consumer's behavior and welfare. But, first we need to make the distinction between a measure of the consumer's "risk aversion" and a measure of his or her "optimal response to risk".

2.5.1 Risk aversion and the "precautionary" response to risk

Following Leland (1968), Sandmo (1970), and Drèze and Modigliani (1972), Kimball (1990), in his analysis of the precautionary motive of saving, makes the distinction between the consequence of risk aversion and what he call "the optimal response of decision variables to risk". The rationale behind this distinction is simple. Intuitively risk aversion is about "dislike" of uncertainty, and one "dislikes" uncertainty because its possible negative effects on one's welfare. Hence, risk aversion measures must be measuring how welfare changes as a result of the introduction of uncertainty or "change" in it. But, a measure of how welfare changes as a result of the introduction of uncertainty (or change in it) alone is not sufficient to infer a decision maker's response to risk (i.e., the sensitivity of *ex-ante* optimal choices to risk). Indeed, while this risk aversion-related change in welfare might affect *ex-ante* optimal choices, the ability of decision makers to substitute across possible states of nature does also affect *ex-ante* optimal choices. Hence, one needs measures that allow to infer and assess the

[66] See in particular, Leland (1968), Sandmo (1970), Drèze and Modigliani (1972), and Kimball (1989).

sensitivity of *ex-ante* optimal choices to risk. Kimball refers to such measures as measures of *prudence* or *precaution* to suggest, in his words, "the propensity to prepare and forearm oneself in the face of uncertainty". In the univariate expected utility framework, the degree of convexity of the first derivative of the von Neumann-Morgenstern utility function plays the same role for *prudence* or *precaution* that the von Neumann-Morgenstern utility function plays for *risk aversion*.

Kimball's distinction between *prudence* or *precaution* as concepts related to the sensitivity of *ex-ante* optimal choices to risk and *risk aversion* as a concept related to the sensitivity of welfare to risk is very intuitive and important if one wants to assess the ability of decision makers to make substitutions across the numerous possible states of nature depending on their *risk preference* (see the related discussions in Appendix 4). Indeed, a risk averse decision maker may be able to eliminate (or at least reduce) the negative welfare impact resulting from the introduction of uncertainty or ("change" in it) if he or she is able to make substitution across possible states of nature. The way that this substitution across possible states of nature affects welfare is similar to the way, in the multi-commodity deterministic case, the pure taste substitution effects affect the welfare change resulting from a change in relative prices. Moreover, the reduction in *ex-ante* welfare implied by risk aversion may have some *precautionary* effects on *ex-ante* optimal choices in way similar to how change in real income affects optimal choices in deterministic demand theory.

Hence, to summarize, in the analysis of static consumption choices under uncertainty with many commodities there are three issues involved in the comparative static and welfare impact assessment of an *expected* change in prices. First, one need to separate out the effect of the consumer's risk attitude on welfare from the welfare change that would have resulted in the absence of uncertainty. Second, one needs to disentangle the effects of price-risk or uncertainty on *ex-ante* optimal choices from the usual effects that obtain in the absence of uncertainty. Third, when analyzing the sensitivity of *ex-ante* optimal choice to uncertainty, one needs to distinguish the *precautionary* substitution across possible states of nature effect from the *precautionary* risk aversion (i.e., "dislike" of uncertainty) effect, in the same way one distinguishes the taste substitution and income effects in static deterministic consumer theory. Drèze and Modigliani (1972), in the context of a two period consumption-saving model with one aggregated commodity, call these two effects simply "substitution" and "income" effects respectively. But here, we will adopt the terms "*precautionary* risk-substitution" and "*precautionary* risk aversion" effects respectively, so as to distinguish them from the taste substitution and income effects that obtain in the absence of uncertainty. The sum of these two *precautionary* effects will be called the"uncompensated *precautionary* risk effect". In section **2.7** below where we deal with welfare measurement under uncertainty issues, we derive risk aversion, risk premium, and cost of risk measures for assessing the sensitivity of welfare to risk. In the remainder of this section we concentrate on the derivation of the measures of the *precautionary* risk substitution and risk aversion responses to price-risk, and their disentanglements from the usual taste substitution and income effects respectively.

2.5.2 Measures of the "precautionary" responses to price-risk

In section **2.2.6** we have identified the degree of risk in a random price by the consumer's "belief-adjusted" *perception* of its degree of variability which we have measured by its distance to the consumer's *expected* value for it under the metric $\| \ \|_2$. Hence, to measure the consumer's overall *precautionary* response to the risk in a random price we need to measure how changes in its *ex-ante* optimal consumption choices, resulting from an *expected* change in a random price, is related to the variability of the random price in question. In other words, for uncompensated risk effects for example, we are looking for numerical measures of the relations between changes in the uncompensated demand of each individual commodity, $\mathbf{x}_i(\mathbf{p}, m) - \mathbf{x}_i(\mathbf{q}, m)$, and the variability in changes in the random variable $\mathbf{p}_j - \mathbf{q}_j$ associated with the j^{th} price which is *expected* to change from $\mathbf{p}_j$ to $\mathbf{q}_j$. In the case of differentiable demands and a marginal "change" in $\mathbf{p}_j$, we are looking for numerical measures of the relations between each random variable $\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j} \cdot \mathbf{p}_j \quad i=1,\dots n$ and the variability of the price $\mathbf{p}_j$. That is to say in both cases we are looking for numerical measures of the respective degrees of "covariability" between the $\mathbf{x}_i(\mathbf{p}, m) - \mathbf{x}_i(\mathbf{q}, m)$ $j=1,..,n$ and $\mathbf{p}_j - \mathbf{q}_j$, and between the random variables $\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j} \cdot \mathbf{p}_j \quad i=1,\dots n$ and $\mathbf{p}_j$ respectively. But, from section **2.2.6**, under the scalar product $\langle . \, , . \rangle$ associated with the metric $\| \;\|_2$ these measures of covariability are given respectively by

$$\begin{aligned} \tilde{\mathbf{r}}_{ij} &\equiv \langle \mathbf{x}_i(\mathbf{p},m) - \mathbf{x}_i(\mathbf{q},m) - E(\mathbf{x}_i(\mathbf{p},m) - \mathbf{x}_i(\mathbf{q},m)) \, , \, \mathbf{p}_j - \mathbf{q}_j - E(\mathbf{p}_j - \mathbf{q}_j) \rangle_2 \\ &= cov\big(\mathbf{x}_i(\mathbf{p},m) - \mathbf{x}_i(\mathbf{q},m) \, , \, \mathbf{p}_j - \mathbf{q}_j\big) \end{aligned} \tag{2.5.1}$$

and

$$\mathbf{r}_{ij} \equiv \langle \frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j} \cdot \mathbf{p}_j - E(\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j} \cdot \mathbf{p}_j) \, , \, \mathbf{p}_j - E\mathbf{p}_j \rangle_2 = cov\left[\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j} \cdot \mathbf{p}_j \, , \, \mathbf{p}_j\right] \tag{2.5.2}$$

If in (2.5.1) and (2.5.2) we replace the uncompensated demands with the compensated or Hicksian demands which hold the consumer's utility constant, then we would obtain numerical measures of the consumer's *precautionary* risk substitution effects:

$$\begin{aligned} \tilde{\mathbf{r}}^s_{ij} &\equiv \langle \mathbf{h}_i(\mathbf{p},u) - \mathbf{h}_i(\mathbf{q},u) - E(\mathbf{h}_i(\mathbf{p},u) - \mathbf{h}_i(\mathbf{q},u)) \, , \, \mathbf{p}_j - \mathbf{q}_j - E(\mathbf{p}_j - \mathbf{q}_j) \rangle_2 \\ &= cov\big(\mathbf{h}_i(\mathbf{p},u) - \mathbf{h}_i(\mathbf{q},u) \, , \, \mathbf{p}_j - \mathbf{q}_j\big) \end{aligned} \tag{2.5.3}$$

and

$$\mathbf{r}^s_{ij} \equiv \langle \frac{\partial \mathbf{h}_i(\mathbf{p},u)}{\partial \mathbf{p}_j} \cdot \mathbf{p}_j - E(\frac{\partial \mathbf{h}_i(\mathbf{p},u)}{\partial \mathbf{p}_j} \cdot \mathbf{p}_j) \,,\, \mathbf{p}_j - E\mathbf{p}_j \rangle_2 = cov\left[\frac{\partial \mathbf{h}_i(\mathbf{p},u)}{\partial \mathbf{p}_j} \cdot \mathbf{p}_j \,,\, \mathbf{p}_j\right] \tag{2.5.4}$$

since it is his or her responses to price-risk when his her *ex-ante* welfare remain constant and is thus not affected by risk aversion (i.e., "dislike" of uncertainty).

The respective differences between the total *precautionary* effects and the *precautionary* risk substitution effects give by definition the numerical measures of the *precautionary* risk aversion effects which measure parts of the changes in demands resulting from the loss in *ex-ante* welfare associated with risk aversion (i.e., "dislike" of uncertainty):

$$\tilde{\mathbf{r}}^m_{ij} \equiv \tilde{\mathbf{r}}_{ij} - \tilde{\mathbf{r}}^s_{ij} \qquad i,j = 1,\dots,n. \tag{2.5.5}$$

for the global case, and

$$\mathbf{r}^m_{ij} \equiv \mathbf{r}_{ij} - \mathbf{r}^s_{ij} \qquad i,j = 1,\dots,n. \tag{2.5.6}$$

for the differential case.

In the next sections, we turn to the disentanglement of these three *precautionary* effects from the overall consumer's response to an *expected* change in a random price.

2.5.3 Disentanglements of the "precautionary" risk effects from the nonrisk effects

Because of the bilinearity of the scalar product $\langle . \,,\, . \rangle$, for all $\mathbf{x}$, $\mathbf{y} \in L_2(\Omega, \mathbf{R}^n)$ we have:

$$\begin{aligned} \langle \mathbf{x},\mathbf{y} \rangle &= E\big(\mathbf{x} - E(\mathbf{x})\big)\big(\mathbf{y} - E(\mathbf{y})\big) + E\mathbf{x} \cdot E\mathbf{y} \\ &= \operatorname{tr} cov[\mathbf{x},\mathbf{y}] + E\mathbf{x} \cdot E\mathbf{y} \end{aligned} \tag{2.5.7}$$

where $cov[\mathbf{x}, \mathbf{y}]$ is the $n \times n$ covariance matrix of the two random vectors $\mathbf{x}$ and $\mathbf{y}$, and tr stands for its trace, which is by definition the sum of its diagonal elements. Hence, in both the global and marginal cases we can always decompose the total "changes" in the demand of the individual commodities resulting from an *expected* change in $\mathbf{p}_j$, and thus isolate the *precautionary* risk part of them.

2.5.3.1 Disentanglement of the uncompensated "precautionary" risk effects

Substituting the total "change" in the uncompensated or Marshallian demands resulting from an *expected* change in $\mathbf{p}_j$ in (2.5.7) we obtain their respective decomposition into two components constituted by the uncompensated *precautionary* risk effects and the uncompensated nonrisk effects (noted $\bar{\mathbf{t}}_{ij}$ for the global case and $\mathbf{t}_{ij}$ for the differential case) which obtain if the consumer's behavior were not affected by uncertainty. Thus, for the global case we have:

$$\begin{aligned}\langle \mathbf{x}_i(\mathbf{q},m') - \mathbf{x}_i(\mathbf{p},m)\,,\, \mathbf{q}_j - \mathbf{p}_j\rangle &= cov[\mathbf{x}_i(\mathbf{q},m') - \mathbf{x}_i(\mathbf{p},m)\,,\, \mathbf{q}_j - \mathbf{p}_j] \\ &\quad + E(\mathbf{x}_i(\mathbf{q},m') - \mathbf{x}_i(\mathbf{p}m))E(\mathbf{q}_j - \mathbf{p}_j) \\ &= \bar{r}_{ij} + \bar{t}_{ij}\end{aligned} \tag{2.5.8}$$

and for the case of differentiable demands and marginal change in $\mathbf{p}_j$ we have:

$$\begin{aligned}\langle \frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j}\,\mathbf{\dot{p}}_j\,,\,\mathbf{p}_j\rangle &= cov[\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j}\,\mathbf{\dot{p}}_j\,,\,\mathbf{p}_j] + E\left[\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j}\,\mathbf{\dot{p}}_j\right]E\mathbf{p}_j \\ &= r_{ij} + t_{ij}\end{aligned} \tag{2.5.9}$$

2.5.3.2 Disentanglement of the "precautionary" risk substitution effects

A similar decomposition for the compensated or Hicksian demands based on (2.5.7) gives a separation of the *precautionary* risk substitution effects from the taste substitution effects (noted $\bar{\mathbf{t}}^{s}_{ij}$ for the global case and $\mathbf{t}^{s}_{ij}$ for the differential case) which obtain when the consumer's substitution across commodities is not affected by uncertainty. For the global case we have:

$$\begin{aligned}\langle \mathbf{h}_i(\mathbf{q},u) - \mathbf{h}_i(\mathbf{p},u)\,,\,\mathbf{q}_j - \mathbf{p}_j\rangle &= cov[\mathbf{h}_i(\mathbf{q},u') - \mathbf{h}_i(\mathbf{p},u)\,,\,\mathbf{q}_j - \mathbf{p}_j] \\ &\quad + E(\mathbf{h}_i(\mathbf{q},u) - \mathbf{h}_i(\mathbf{p},u))E(\mathbf{q}_j - \mathbf{p}_j) \\ &= \bar{r}_{ij}^s + \bar{t}_{ij}^s\end{aligned} \tag{2.5.10}$$

and for the differentiable case we have:

$$\begin{aligned}\langle \frac{\partial \mathbf{h}_i(\mathbf{p},u)}{\partial \mathbf{p}_j}\mathbf{p}_j\,,\,\mathbf{p}_j\rangle &= cov[\frac{\partial \mathbf{h}_i(\mathbf{p},u)}{\partial \mathbf{p}_j}\mathbf{p}_j\,,\,\mathbf{p}_j] + E\left[\frac{\partial \mathbf{h}_i(\mathbf{p},u)}{\partial \mathbf{p}_j}\mathbf{p}_j\right]E\mathbf{p}_j \\ &= r_{ij}^s + t_{ij}^s\end{aligned} \tag{2.5.11}$$

Since the $\mathbf{t}_{ij}^s$ $i,j = 1,..,n$ measure the price substitution effects due to taste preference, they are the basis for classifying commodities into complements and substitutes (i.e in taste), with the usual understanding. It is tempting to use the signs of the *precautionary* risk substitution effects to classify the commodities into *risk substitutes* and *risk complements*. The logic of this classification would be justified by the intuitive perception that, taste preference and income aside, if the prices of two different commodities are negatively correlated, then a risk averse consumer would *ex-ante* demand both of the commodities. However, unlike the total substitution matrix of the $\mathbf{s}_{ij} \equiv \mathbf{t}_{ij}^s + \mathbf{r}_{ij}^s$ the $n \times n$ matrix of the $\mathbf{r}_{ij}^s$ is not a symmetric matrix, thus making such a classification less appealing. Also, although the monotonicity of the compensated demand functional implies that $s_{ii} \equiv t_{ii} + r^s_{ii} < 0$, this latter condition does not require both terms to be negative.[67]

2.5.3.3 Disentanglement of the "precautionary" risk aversion effects

Since, by definition, for each commodity the total income effect is the difference between the total uncompensated and compensated effects, its decomposition based on (2.5.7) gives exactly the sum of the *precautionary* risk aversion effect and the pure income effect (noted $\bar{t}^m_{ij}$ for the global case and t^m_{ij} for the differential case) which obtain when the consumer's behavior is not affected by the sensitivity of *ex-ante* welfare to risk aversion. Hence for the global case we have:

$$\begin{aligned}\langle x_i(p,\ \mu(p,q,m')) - x_i(p,m),\ q_j - p_j\rangle &= cov[x_i(p,\ \mu(p,q,m')) - x_i(p,m)\ ,\ q_j - p_j] \\ &+ E\big(x_i(p,\ \mu(p,q,m')) - x_i(p,m)\big)\cdot E(q_j - p_j) \qquad \textbf{(2.5.12)} \\ &= \bar{r}^m_{ij} + \bar{t}^m_{ij}\end{aligned}$$

and for the differential case we have:[68]

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[67] We know that at least one of the two terms must be negative, but which one? In the two-period expected utility consumption-saving context with one commodity and income risk, Drèze and Modigliani shows that the equivalent of r^s_{ii} is positive, equal to zero or negative when the Arrow-Pratt absolute risk aversion coefficient with respect to second period consumption is decreasing, constant, or increasing. But so far, we have been unable to derive a condition on preference that would allow to sign r^s_{ii}. The difficulty seems to be related to two facts: 1) the univariate Arrow- Pratt measure of risk aversion is irrelevant in a multivariate and/or non-expected utility context, and 2) the possible correlation among the random prices means that an *expected* change in one price may implies *expected* changes in other prices in the same or opposite direction, thus complicating the consumer's substitution process.

[68] We recall that for given *i,j* and (p, *m*), both $\dfrac{\partial x_i(p,m)}{\partial p_j}$ and $\dfrac{\partial x_i(p,m)}{\partial m} \circ x_j(p,m)$ are continuous linear operators from $L_2(\Omega,\ \mathbb{R})$ to $L_2(\Omega,\ \mathbb{R})$. Hence, for any random price p_j $\dfrac{\partial x_i(p,m)}{\partial p_j}\cdot p_j$ and $\dfrac{\partial x_i(p,m)}{\partial m} \circ x_j(p,m)\cdot p_j$ are random variables, elements of $L_2(\Omega,\ \mathbb{R})$.

$$\langle \frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j} \cdot \mathbf{p}_j + \frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial m} \circ \mathbf{x}_j(\mathbf{p},m) \cdot \mathbf{p}_j \,,\, \mathbf{p}_j \rangle = cov\left[\left[\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j} \cdot \mathbf{p}_j + \frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial m} \circ \mathbf{x}_j(\mathbf{p},m) \cdot \mathbf{p}_j \right] \,,\, \mathbf{p}_j \right]$$

$$+ E\left[\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j} \cdot \mathbf{p}_j + \frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial m} \circ \mathbf{x}_j(\mathbf{p},m) \cdot \mathbf{p}_j \right] E\mathbf{p}_j$$

$$= r_{ij}^m + t_{ij}^m \qquad \textbf{(2.5.13)}$$

$\mathbf{r}_{ij}^m$ measures the impact on the demand of commodity i of the consumer's risk aversion toward the price of commodity j when it is *expected* to change. The signs of $\mathbf{r}_{ij}^m$ can be used to classify the commodities according to how their demands are sensitive to risk aversion (i.e.,the consumer's "dislike" of uncertainty), in the same way that the signs of the pure income effects, the $\mathbf{t}_{ij}^m$, are used to classify the commodities into normal and inferior commodities.[69]

2.5.3.4 Disentanglement of the "precautionary" risk effects in the comparative statics

To conclude the analysis of the consumer's *precautionary* response to price-risk, we will use the above decompositions to rewrite the comparative statics equations (2.4.7) and (2.4.16) in a way that separates the *precautionary* risk effects from the nonrisk effects which obtain when the consumer's behavior is not affected by uncertainty. Indeed, equations (2.5.8), (2.5.10), and (2.5.12) allow to decompose the global comparative static equations (2.4.7) as

[69] In their one commodity two-period expected utility model, Drèze and Modigliani (1972) showed that if both first and second period consumptions are not inferior commodities, then the analogue of $\mathbf{r}_{ij}^m$ in their model is always positive under risk aversion. But, so far, we have not been able to find conditions that would allow us to sign the $\mathbf{r}_{ij}^m$.

$$\tilde{\mathbf{r}}_{ij} \equiv \tilde{\mathbf{r}}^{s}_{ij} + \tilde{\mathbf{r}}^{m}_{ij} \qquad i,j = 1,\ldots,n. \tag{2.5.14}$$

$$\tilde{\mathbf{t}}_{ij} \equiv \tilde{\mathbf{t}}^{s}_{ij} + \tilde{\mathbf{t}}^{m}_{ij} \qquad i,j = 1,\ldots,n. \tag{2.5.15}$$

and

$$\tilde{\mathbf{r}}_{ij} + \tilde{\mathbf{t}}_{ij} = \left(\tilde{\mathbf{r}}^{s}_{ij} + \tilde{\mathbf{t}}^{s}_{ij}\right) + \left(\tilde{\mathbf{r}}^{m}_{ij} + \tilde{\mathbf{t}}^{m}_{ij}\right) = \left(\tilde{\mathbf{r}}^{s}_{ij} + \tilde{\mathbf{r}}^{m}_{ij}\right) + \left(\tilde{\mathbf{t}}^{s}_{ij} + \tilde{\mathbf{t}}^{m}_{ij}\right) \qquad i,j = 1,\ldots,n. \tag{2.5.16}$$

Of course, we have a similar decomposition for the differential comparative statics equations (2.4.16).

The equations in (2.5.14) isolate from the general comparative statics result the *precautionary* risk effects and decompose them into their respective *precautionary* risk substitution and *precautionary* risk aversion effects. On the other hand, the equations in (2.5.15) isolate the nonrisk components and decompose them into their respective taste substitution and pure income effects. The equations in (2.5.16) provide two different ways of grouping these four distinct effects which constitute the total uncompensated consumer's response to an *expected* change in a random price: either along the substitution/income dimension, or along the risk/nonrisk dimension. Naturally, the risk terms disappear in the absence of uncertainty and the comparative statics reduces to the ones found in deterministic consumer theory. The consumer's indifference to price uncertainty (i.e., insensitivity of optimal consumption choices to price uncertainty) can occur in two situations **a)** when the *precautionary* risk substitution and risk aversion effects are all identically equal to zero, and **b)** when the *precautionary* risk substitution and risk aversion effects cancel each other respectively. In both cases, the uncompensated *precautionary* risk effects will be equal to zero .

2.6 Measures of price, expenditure, and precautionary risk elasticities

At this level of generality, except for the negativity of the own price substitution effects, the comparative statics results derived above cannot theoretically determine the signs of the various effects without further restrictions on preference. Furthermore, the information contained in these results is only of qualitative value, in the sense that only the direction of change can possibly be inferred from the results. To go beyond this qualitative assessment, one needs to numerically calculate and estimate the magnitudes of these various effects.

In the deterministic consumer theory, the quantitative assessment of these various effects are usually and conveniently presented in terms of elasticities which measure the percentage changes in demands corresponding to one percent change in prices or expenditure. The question that arises then is, how would we numerically calculate and interpret elasticities in this uncertainty context? But, from the previous section we already have numerical measures of the various risk and nonrisk effects resulting from a "change" in a random price. Hence, if we appropriately "normalize" these numerical measures we have elasticity measures in a way similar to the deterministic case: the percentage change in demand corresponding to one percent change in price. The only difference is the meaning attached to the word "percentage change". While in the deterministic context the meaning is governed by the Euclidean distance, in this context the meaning of "percentage change" is governed by the metric $\| \|_2$ defined in $L_2(\Omega, \mathbf{R})$. Moreover, as discussed in section **2.2.5**, under this metric and given our identification of consumer's "expectation" process with the expectation operator E, we can always interpret a "percentage change in a random variable" as a percentage *expected* change in the random variable. Hence, in this framework elasticities are interpreted as: the percentage *expected* change in demands resulting from one percent *expected* change in a price.[70]

2.6.1 Elasticity measures for marginal change in random prices

To get numerical elasticity measures for the various effects on the demand for commodity i following a "marginal" *expected* change in the price of commodity j, all we need is to appropriately normalize the $\mathbf{r}_{ij}$'s and $\mathbf{t}_{ij}$'s defined above. The natural choice for this normalization is to divide them respectively by $\|\mathbf{x}_i\| \times \|\mathbf{p}_j\| = \sqrt{E\mathbf{x}_i^2} \times \sqrt{E\mathbf{p}_j^2}$ the product of the norms of $\mathbf{x}_i$ and $\mathbf{p}_j$ in $L_2(\Omega, \mathbb{R})$. Hence, for the *precautionary* risk substitution effects the elasticities are:[71]

$$\epsilon_{ij}^{rs} \equiv \frac{1}{\sqrt{E\mathbf{x}_i^2} \times \sqrt{E\mathbf{p}_j^2}} \times cov\left[\left[\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j}\mathbf{p}_j + \frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial m} \circ \mathbf{x}_j(\mathbf{p},m)\mathbf{p}_j\right],\ \mathbf{p}_j\right] \qquad \textbf{(2.6.1)}$$

For the *precautionary* risk aversion effects the elasticities are:

$$\epsilon_{ij}^{rm} \equiv -\frac{1}{\sqrt{E\mathbf{x}_i^2} \times \sqrt{E\mathbf{p}_j^2}} \times cov\left[\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial m} \circ \mathbf{x}_j(\mathbf{p},m)\mathbf{p}_j\ ,\ \mathbf{p}_j\right] \qquad \textbf{(2.6.2)}$$

The uncompensated *precautionary* risk elasticities are hence:

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[70] We recall again the behavioral and mathematical differences between an *expected* change in a random variable and a change in the *expected* value of a random variable.

[71] We recall that by the self-dual property of $L_2(\Omega, \mathbb{R})$ we can identify linear continuous forms defined on $L_2(\Omega, \mathbb{R})$ with its elements.

$$\epsilon_{ij}^{r} \equiv \frac{1}{\sqrt{E\mathbf{x}_i^2} \times \sqrt{E\mathbf{p}_j^2}} \times cov\left[\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j}\cdot\mathbf{p}_j\ ,\ \mathbf{p}_j\right] = \epsilon_{ij}^{rs} + \epsilon_{ij}^{rm} \qquad \textbf{(2.6.3)}$$

Similarly, the taste substitution elasticities and the elasticity measures corresponding to the pure income effect which corresponds to the nonrisk effects are respectively:

$$\epsilon_{ij}^{ts} \equiv \frac{1}{\sqrt{E\mathbf{x}_i^2} \times \sqrt{E\mathbf{p}_j^2}} \times E\left[\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j}\cdot\mathbf{p}_j + \frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial m} \circ \mathbf{x}_j(\mathbf{p},m)\cdot\mathbf{p}_j\right] \times E\mathbf{p}_j \qquad \textbf{(2.6.4)}$$

$$\epsilon_{ij}^{tm} \equiv -\frac{1}{\sqrt{E\mathbf{x}_i^2} \times \sqrt{E\mathbf{p}_j^2}} \times E\left[\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial m} \circ \mathbf{x}_j(\mathbf{p},m)\cdot\mathbf{p}_j\right] \times E\mathbf{p}_j \qquad \textbf{(2.6.5)}$$

and

$$\epsilon_{ij}^{t} \equiv \frac{1}{\sqrt{E\mathbf{x}_i^2} \times \sqrt{E\mathbf{p}_j^2}} \times E\left[\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j}\cdot\mathbf{p}_j\right] \times E\mathbf{p}_j = \epsilon_{ij}^{ts} + \epsilon_{ij}^{tm} \qquad \textbf{(2.6.6)}$$

The overall price elasticities of demand which includes all effects are hence:

$$\epsilon_{ij} = \epsilon_{ij}^{r} + \epsilon_{ij}^{t} \qquad i,j = 1,\ldots,n. \qquad \textbf{(2.6.7)}$$

The expenditure elasticities can be computed as:

$$\eta_i \equiv \frac{1}{\|\mathbf{x}_i\|} \times \frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial m}\cdot m = \frac{m}{\sqrt{E\mathbf{x}_i^2}} \times \frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial m} \quad i = 1,\ldots,n. \quad \textbf{(2.6.8)}$$

The second equality follows from our identification of linear continuous functionals from **R** to $L_2(\Omega, \mathbf{R})$ with elements of $L_2(\Omega, \mathbf{R})$. Thus, the expenditure elasticities are random variables with means:

$$E\eta_i \equiv \frac{m}{\sqrt{E\mathbf{x}_i^2}} \times E\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial m} \qquad i=1,\dots,n. \tag{2.6.9}$$

and variances:

$$var[\eta_i] \equiv \frac{m^2}{E\mathbf{x}_i^2} \times var\left[\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial m}\right] \qquad i=1,\dots,n. \tag{2.6.10}$$

We note here that the risk effects are irrelevant because the predetermined total expenditure m is not random. The signs and relative magnitudes of the means $E\eta_i$ $i=1,\dots,n.$ (with respect to one), can be used to classify the commodities as usual into normal, inferior, luxury and necessary goods.[72]

Naturally, all the elasticities computed above reduce to their deterministic analogues when there is no uncertainty. They are easily calculated when one is given a differentiable demand function $\mathbf{x}(\mathbf{p}, m)$. Fréchet-differentiation in general Banach spaces follows the same rules as in $\mathbb{R}^n$ (additivity, chain rule etc...). However, in this context it is important to keep track of the different spaces, and keep in mind that in general derivative are linear continuous functions. The use of isomorphic identifications usually simplifies the expressions and/or notations. In chapter 3 we work out in detail examples of demand functionals, and derive the

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[72] Note that, as explained in section 2.2.7, if instead in term of *expected* total expenditure m, we want the expenditure elasticities in terms of total expenditure $\mathbf{m}$ which is a random variable, we can use the fact that m is linked to $\mathbf{m}$ by $m = \langle 1, \mathbf{m}\rangle = E\mathbf{m}$, and the fact that $\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{m}} = \frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial m} \circ E$, E being the expectation operator, to get the risk and nonrisk components of the expenditure elasticities in terms of $\mathbf{m}$:

$$\tilde{\eta}_i^r \equiv \frac{E\mathbf{m}}{\sqrt{E\mathbf{x}_i^2} \times \sqrt{E\mathbf{m}^2}} \times cov\left[\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial m}\ ,\ \mathbf{m}\right] \qquad i=1,\dots,n.$$

$$\tilde{\eta}_i^t \equiv \frac{(E\mathbf{m})^2}{\sqrt{E\mathbf{x}_i^2} \times \sqrt{E\mathbf{m}^2}} \times E\left[\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial m}\right] \qquad i=1,\dots,n.$$

corresponding analytical expressions of all the elasticities. In chapter 5, we show how all the elasticities can be estimated with existing econometric methods.

2.6.2 Elasticity measures for non-marginal change in random prices

As in deterministic demand theory, the elasticities computed above are valid only for "small" marginal changes in prices and expenditure. The issue of, to what extent a "change" in prices that are random can be considered as marginal, becomes even more important in this uncertainty context.

For simplicity, we will consider only four types of "change" in random prices which are of interest for the evaluation of the impacts of policies:

(i) **Linear transformation of $\mathbf{p}_j$.** That is, the "change" in the price of the j^{th} commodity from $\mathbf{p}_j$ to $\mathbf{q}_j$ can be written as

$$\mathbf{q}_j = \alpha \mathbf{p}_j + \beta \qquad \text{with } \alpha, \beta \in \mathbf{R} \tag{2.6.11}$$

This type of "change" is very common and characterizes price changes like those induced by sale tax, tariff, etc....

(ii) **Piece-wise linear transformation of $\mathbf{p}_j$.** That is, the "change" in the price of the j^{th} commodity from $\mathbf{p}_j$ to $\mathbf{q}_j$ can be written as

$$\mathbf{q}_j = \begin{cases} \alpha & \text{if } \mathbf{p}_j(\omega) > a \\ \gamma \mathbf{p}_j + \delta & \text{if } b \leq \mathbf{p}_j(\omega) \leq a \\ \beta & \text{if } \mathbf{p}_j(\omega) < b \\ & \text{with } \alpha, \beta, \gamma, \delta, a, \text{ and } b \in \mathbf{R} \end{cases} \tag{2.6.12}$$

This type of "change" is also common and corresponds to price stabilization schemes, insurance, call and put options etc...

(iii) Addition of "noise" to $\mathbf{p}_j$. That is:

$$\mathbf{q}_j = \mathbf{p}_j + \mathbf{z}_j \qquad \text{with } \mathbf{z}_j \in L_2(\Omega, \mathbb{R}) \tag{2.6.13}$$

This type of price "change" includes in particular the move from a regime of price control where $\mathbf{p}_j(\omega) = \alpha \in \mathbb{R}$ for all $\omega \in \Omega$, to a liberalized price regime $\mathbf{q}_j \in L_2(\Omega, \mathbb{R})$.

In the classification of Meyer and Ormiston (1988), the first two "changes" are in the class of *deterministic transformations* while the latter is in the class of *stochastic transformations*.

The first type of price "change" can be considered marginal if α and β are such that $\| \mathbf{q}_j - \mathbf{p}_j \| = \sqrt{E(\mathbf{q}_j - \mathbf{p}_j)^2}$ is "small". In this case, the local elasticities computed above can be used. The two other types of price "changes" cannot in general be considered as marginal.

For nonmarginal "changes" in prices, elasticity measures can be computed by appropriately modifying and normalizing the $\tilde{\mathbf{r}}_{ij}^{s}$'s and $\tilde{\mathbf{t}}_{ij}^{s}$'s derived in the global comparative risk section. It is easily seen that for a "change" in the price of the j^{th} commodity from $\mathbf{p}_j$ to $\mathbf{q}_j$, elasticity measures of the "changes" in the demand for the i^{th} commodity is obtained by replacing $\mathbf{q}_j$ - $\mathbf{p}_j$ with $\mathbf{p}_j$ in the $\tilde{\mathbf{r}}_{ij}^{s}$'s and $\tilde{\mathbf{t}}_{ij}^{s}$'s, and by dividing by $\|\mathbf{x}_i\| \times \| \mathbf{q}_j - \mathbf{p}_j \| = \sqrt{E\mathbf{x}_i^2} \times \sqrt{E(\mathbf{q}_j - \mathbf{p}_j)^2}$. Hence, in a way similar to the local measures, the elasticities for the various effects are:

$$\tilde{\epsilon}_{ij}^{rs} \equiv \frac{1}{\sqrt{E\mathbf{x}_i^2} \times \sqrt{E(\mathbf{q}_j - \mathbf{p}_j)^2}} \times cov[\mathbf{x}_i(\mathbf{q}, m) - \mathbf{x}_i(\mathbf{p}, \mu(\mathbf{p}, \mathbf{q}, m)) \ , \ \mathbf{p}_j] \tag{2.6.14}$$

$$\tilde{\epsilon}_{ij}^{rm} \equiv \frac{1}{\sqrt{E\mathbf{x}_i^2} \times \sqrt{E(\mathbf{q}_j - \mathbf{p}_j)^2}} \times cov[\mathbf{x}_i(\mathbf{p}, \mu(\mathbf{p}, \mathbf{q}, m)) - \mathbf{x}_i(\mathbf{p}, m) \ , \ \mathbf{p}_j] \tag{2.6.15}$$

$$\bar{\epsilon}^{r}_{ij} \equiv \frac{1}{\sqrt{E\mathbf{x}_i^2} \times \sqrt{E(\mathbf{q}_j - \mathbf{p}_j)^2}} \times cov[\mathbf{x}_i(\mathbf{q},m) - \mathbf{x}_i(\mathbf{p},m)\ ,\ \mathbf{p}_j] = \bar{\epsilon}^{rs}_{ij} + \bar{\epsilon}^{rm}_{ij} \quad \textbf{(2.6.16)}$$

$$\bar{\epsilon}^{tm}_{ij} \equiv \frac{1}{\sqrt{E\mathbf{x}_i^2} \times \sqrt{E(\mathbf{q}_j - \mathbf{p}_j)^2}} \times E\big(\mathbf{x}_i(\mathbf{p}, \mu(\mathbf{p},\mathbf{q},m)) - \mathbf{x}_i(\mathbf{p},m)\big) \times E\mathbf{p}_j \quad \textbf{(2.6.17)}$$

$$\bar{\epsilon}^{t}_{ij} \equiv \frac{1}{\sqrt{E\mathbf{x}_i^2} \times \sqrt{E(\mathbf{q}_j - \mathbf{p}_j)^2}} \times E\big(\mathbf{x}_i(\mathbf{q},m) - \mathbf{x}_i(\mathbf{p},m)\big) \times E\mathbf{p}_j = \bar{\epsilon}^{ts}_{ij} + \bar{\epsilon}^{tm}_{ij} \quad \textbf{(2.6.18)}$$

and

$$\bar{\epsilon}_{ij} = \bar{\epsilon}^{r}_{ij} + \bar{\epsilon}^{t}_{ij} \qquad i,j = 1,\ldots,n. \quad \textbf{(2.6.19)}$$

These elasticities are global in the sense that they are valid for all types of price "changes", including the marginal and nonmarginal ones. Their disadvantages compared to the local elasticity measures derived in the previous section, is that they require knowledge of both the before-change and after-change prices.

2.7 Measures of welfare changes under uncertainty

For a pure theoretical analysis, the indirect utility functional is all that is needed to assess welfare changes resulting from policy changes that affect prices and/or expenditure. In particular, the monotonicity in prices and expenditure property of the indirect utility functional gives the direction of welfare change following a price/expenditure change from (**p**, *m*) to (**q**, *m'*). However, this information is only of qualitative value because the difference V(**q**, *m'*) - V(**p**, *m*) is an ordinal measure which depends on the choice of the utility representation of the consumer's preference and is not even observable. Thus for quantitative numerical assessments of welfare changes, it is necessary to come up with alternative measures which are independent of the choice of the utility representation of consumer's preference, and which, if not observable, can at least be calculated from observable demand data.

McKenzie (1983, p. 2) lists the following five criteria that, according to him, any operational measure of welfare changes should satisfy:[73]

(1) For an individual or homogenous group of individuals (i.e., a group possessing identical tastes and expenditure levels) the measure must be capable of ranking all relevant price/quantity situations according to the preferences of the individual or homogeneous group.
(2) The measure must take the form of a single metric or scale.
(3) The measure or scale must be expressed in monetary units.
(4) The welfare indicator must be amenable to calculation in terms of the parameters of ordinary, observable demand functions.
(5) Once an indicator meeting the preceding four criteria is constructed, it must be such that it can be aggregated across individuals or homogeneous groups so as to obtain an overall measure of the social desirability of the project or policy under consideration.

In deterministic demand theory, it is known that the equivalent variation (EV), derived from the *compensation function* or *money-metric utility function*, meets all five criteria; while the compensating variation (CV), also derived from the same compensation function, essentially fail the first criterion (see, for example, McKenzie, 1983; or Varian, 1984, pp 122-125 and pp 263-276). In contrast, the (Marshallian) *consumer's surplus* obtained by integrating the uncompensated demand function can be a valid measure of welfare change only if preferences are homothetic. But, homothetic preference is an unnecessarily very restrictive assumption.

The only disadvantage of EV, CV is that they are hard to get because they require the econometric estimation of a full demand system. And, even if a full demand system is estimated, their theoretical properties which make them economically appealing cannot be guaranteed unless all the restrictions implied by demand theory (homogeneity, adding up, and symmetry and negativity of the Slutsky matrix) are imposed during the econometric estimation. While it is relatively easy to impose the first three restrictions, the negativity of

[73] It should be emphasized that, while the first four criteria involve only purely technical problems, the fifth one necessarily involves value judgments (i.e., interpersonal "utility" comparisons).

the Slutsky matrix is very difficult to impose globally except for very simple and restrictive functional forms.[74] Detailed derivations and theoretical properties of **EV**, **CV**, and the associated living standard and cost-of-living indices in the absence of uncertainty can be found in Deaton and Muellbauer (1980a), Diewert (1981), Mckenzie (1984), and Varian (1984). In the next sections we will extend these measures of welfare change to the case of uncertainty and show that they preserve their theoretical economic interpretations and properties. We will also provide measures of risk aversion and monetary measures of risk premium and cost of risk. Furthermore, as for the comparative statics and elasticities, we will decompose the total welfare change resulting from a "change" in random prices into two components. One being the risk premium measuring in monetary units the sensitivity of welfare to risk aversion (i.e., "dislike" of uncertainty), and the other being the change in welfare that would obtain if the consumer were welfare-indifferent to uncertainty. The living standard and cost-of-living indices will also be decomposed similarly.

2.7.1 The money-metric or compensation functionals[75]

As in the deterministic case, one can use the expenditure functional and defines two compensation or money-metric functionals:

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[74] One is usually content to verify after estimation that the eigenvalues of the Slutsky matrix are negative. However, negativity of the Slutsky matrix at the point estimates does not guarantee negativity within the range of prices considered.

[75] The following developments are extensions to uncertainty of Varian's (1984) treatment of the deterministic case. The choice of the welfare indicators follows McKenzie (1983).

(i) **The *direct compensation functional*** $\mathbf{M}$: $L_2(\Omega, \mathbb{R}^n)_{++} \times L_2(\Omega, \mathbb{R}^n)_{++} \rightarrow \mathbb{R}_{++}$ with:

$$\mathbf{M}(\mathbf{p}, \mathbf{x}) \equiv \mathbf{E}(\mathbf{p}, \mathbf{U}(\mathbf{x})) \tag{2.7.1}$$

which measures the minimum the consumer facing prices $\mathbf{p}$ would need to spend in order to be as well off as consuming the commodity bundle $\mathbf{x}$.

It is straightforward to see from its definition and from the properties of the expenditure functional, that for given $\mathbf{x}$, $\mathbf{M}(., \mathbf{x})$ behaves like an expenditure functional, and for given $\mathbf{p}$, $\mathbf{M}(\mathbf{p}, .)$ is just another utility representation of preference.

(ii) **The *indirect compensation functional*** μ: $L_2(\Omega, \mathbb{R}^n)_{++} \times L_2(\Omega, \mathbb{R}^n)_{++} \times \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ with:

$$\mu(\mathbf{p}, \mathbf{q}, m) \equiv \mathbf{E}(\mathbf{p}, \mathbf{V}(\mathbf{q}, m)) \tag{2.7.2}$$

which measures the minimum the consumer facing prices $\mathbf{p}$, would need to spend in order to be as well off as when facing prices $\mathbf{q}$ and spending m (i.e., when consuming the demanded bundle $\mathbf{x}(\mathbf{q}, m)$).

Again, it is straightforward to see from its definition and from the properties of both the expenditure and indirect utility functional, that for given $\mathbf{q}$ and m, $\mu(., \mathbf{q}, m)$ behaves like an expenditure functional, and for given $\mathbf{p}$, $\mu(\mathbf{p}, ., .)$ behaves like an indirect utility functional with:

$$\mu(\mathbf{p}, \mathbf{p}, m) \equiv \mathbf{E}(\mathbf{p}, \mathbf{V}(\mathbf{p}, m)) = m \tag{2.7.3}$$

Also, from Shephard's lemma (equation (2.3.8)) and the second equation in (2.3.5) we have:

$$\frac{\partial \mathbf{E}(\mathbf{p}, \mathbf{V}(\mathbf{q}, m))}{\partial \mathbf{p}_i} = \mathbf{h}_i(\mathbf{p}, \mathbf{V}(\mathbf{q}, m)) = \mathbf{x}_i(\mathbf{p}, \mathbf{E}(\mathbf{p}, \mathbf{V}(\mathbf{q}, m)) \quad i = 1, \ldots, n \tag{2.7.4}$$

Or, stated in terms of μ:

$$\frac{\partial \mu(\mathbf{p}, \mathbf{q}, m)}{\partial \mathbf{p}_i} = \mathbf{x}_i(\mathbf{p}, \ \mu(\mathbf{p}, \mathbf{q}, m)) \quad i = 1, \ldots, n \tag{2.7.5}$$

where **h** and **x** are the Hicksian and Marshallian demand functionals respectively.

The equations in (2.7.5) are known in standard consumer theory as the "integrability equations". It represents a system of partial differential equations in μ and **p** with initial condition given by the identity $\mu(\mathbf{q}, \mathbf{q}, m) = m$. This leads to the practical question of, when and how is it possible to solve the system of partial deferential equations in μ, with the only knowledge of the Marshallian demand functionals? This is the "integrability problem". For the infinite dimensional case, the problem was solved by Berger and Meyers (1966), and Berger (1971) within the framework of reflexive Banach spaces which include our choice space.[76]

Hence, under some regularity conditions a solution will exist so that knowledge of the Marshallian demand functional allows to recover μ, which will be independent of the choice of the utility representation of $\succeq$. In general, an analytical solution for μ is difficult to obtain. However, numerical analysis methods can be used to numerically solve for μ.

Since the indirect compensation functional μ involves only observable variables and is independent of any choice of utility representation of preference, it is not a surprise that it is the basis of *operational* measures of welfare changes.

[76] A complete treatment for the finite dimensional case is provided by Hurwicz and Uzawa (1971), and Berger and Meyers (1971).

2.7.2 The equivalent and compensating variations under uncertainty

By proposition **2.3.6**, for given $\mathbf{p}_0$ $\mathbf{E}(\mathbf{p}_0, .)$ is a strictly increasing function of utility u. Thus, we have:

$$\mathbf{V}(\mathbf{q},m') \geq \mathbf{V}(\mathbf{p},m) \quad \text{if and only if} \quad \mathbf{E}(\mathbf{p}_0,\mathbf{V}(\mathbf{q},m')) \geq \mathbf{E}(\mathbf{p}_0,\mathbf{V}(\mathbf{p},m)) \qquad \textbf{(2.7.6)}$$

or stated in terms of the indirect compensation functional:

$$\mathbf{V}(\mathbf{q},m') \geq \mathbf{V}(\mathbf{p},m) \quad \text{if and only if} \quad \mu(\mathbf{p}_0,\mathbf{q},m') \geq \mu(\mathbf{p}_0,\mathbf{p},m) \qquad \textbf{(2.7.7)}$$

Hence, there is a one to one correspondence between changes in **V** and changes in μ with the signs of $\mu(\mathbf{p}_0, \mathbf{q}, m') - \mu(\mathbf{p}_0, \mathbf{p}, m)$ and $\mathbf{V}(\mathbf{q}, m') - \mathbf{V}(\mathbf{p}, m)$ being the same for any given base price $\mathbf{p}_0$. In other words, measures of welfare changes based on changes in the indirect utility functional are consistent with measures of welfare changes based on changes in the money-metric or indirect compensation functional.[77]

There are two known measures of welfare changes that are based on the difference $\mu(\mathbf{p}_0, \mathbf{q}, m') - \mu(\mathbf{p}_0, \mathbf{p}, m)$:

(i) **The *equivalent variation* (EV).** It is defined by taking as base price $\mathbf{p}_0 \equiv \mathbf{p}$. That is:

$$\mathbf{EV} \equiv \mu(\mathbf{p},\mathbf{q},m') - \mu(\mathbf{p},\mathbf{p},m) = \mu(\mathbf{p},\mathbf{q},m') - m \qquad \textbf{(2.7.8)}$$

EV measures the change in expenditures, measured at initial price **p**, required in order to leave a consumer having expenditure m′ and facing the new price vector **q**, as well off as when consuming the demanded bundle $\mathbf{x}(\mathbf{p}, m)$. It is usually considered as a measure of "willingness to pay".

[77] Note that since the indirect utility and compensation functionals are real valued, the meaning of the word "change" in their values is the same as usual.

(ii) **The *compensating variation* (CV).** It is defined by taking as base price $\mathbf{p}_0 \equiv \mathbf{q}$. That is:

$$\mathbf{CV} \equiv \mu(\mathbf{q},\mathbf{q},m') - \mu(\mathbf{q},\mathbf{p},m) = m' - \mu(\mathbf{q},\mathbf{p},m) \quad \textbf{(2.7.9)}$$

CV measures the change in expenditures, measured at the new price **q**, needed for a consumer with expenditure m' and facing the new price **q**, to maintain the same level of satisfaction as when consuming the previously demanded bundle **x(p**, m).

Although the **EV** and **CV** can take different values because of the difference in the base price, we see from above that they always have the same sign: the sign of **V(q**, m') - **V(p**, m).

2.7.3 Living standards and Cost-of-living indices under uncertainty

The changes in welfare as measured by the **CV** or **EV** can also be expressed in percentage terms by calculating **EV**/m and **CV**/m' which are normalizations by the respective base expenditures. These normalizations give unit-free measures of welfare changes, and, like price and expenditure elasticity measures of demand, facilitate comparisons across consumers of different expenditure levels (in terms of how they are relatively affected by the policy change). Such information is often more useful than the total welfare change obtained by aggregating the different individual welfare changes.

The expression in percentage terms also shows directly how the **EV** and the **CV** are related to the numerous price and quantity indices that have been proposed as measures of *living standards* or *cost-of-living*. Indeed from the definition of **EV** we have:

$$\frac{\mathbf{EV}}{m} = \frac{\mu(\mathbf{p},\mathbf{q},m')}{m} - 1 = Q_A - 1 = \frac{m'}{m} \times \frac{1}{P_M} - 1 \quad \textbf{(2.7.10)}$$

where $Q_A \equiv \frac{\mu(\mathbf{p},\mathbf{q},m')}{m} = \frac{\mathbf{EV}+m}{m}$ is by definition the Allen quantity index, which is a preference-based measure of *living standard*; and $P_M \equiv \frac{m'}{\mu(\mathbf{p},\mathbf{q},m')} = \frac{m'}{\mathbf{EV}+m}$ is the McKenzie-Pearce price index which is a preference-based measure of *cost of living*. The difference between the Allen's quantity index and the McKenzie-Pearce price index is that the former is obtained through a normalization of **EV** by *m*. It takes into account changes in both prices and expenditure (making it a *true* indicator of "real income"). The latter on the other hand is obtained through a normalization of **EV** by m'. It adjusts the expenditure change to take into account of only changes in prices (making it a *true* indicator of cost-of-living).[78]

Mckenzie and Pearce (1976) proposed their cost-of-living index arguing that it is the one that the Allen quantity index should be paired with, instead of the commonly used Konüs cost-of-living index. The Konüs cost-of-living index can be derived from the **CV** by using the fact that:

$$\frac{\mathbf{CV}}{m'} = 1 - \frac{\mu(\mathbf{q},\mathbf{p},m)}{m'} = 1 - \frac{m}{m'} \times P_K \qquad \textbf{(2.7.11)}$$

where $P_K \equiv \frac{\mu(\mathbf{q},\mathbf{p},m)}{m'} = \frac{m' - \mathbf{CV}}{m}$ is by definition the Konüs cost-of-living index.

The pairing (P_M, Q_A) satisfies two essential properties thought to be necessary for any good and economically meaningful pair of quantity/price index numbers (MacKenzie; 1983, chapter 7):

[78] In contrast, the easily constructed and commonly used living standard and cost-of-living indicators such as "real income", statistically constructed quantity and price indices (based on the Laspeyres or Paasche formula, or simply on commodities shares weighted price averages) are not good economic measures of living standards and cost-of-living. This is because these easily constructed indices by not incorporating substitution and income effects do not take into account individual preferences. It is only when the substitution effect is zero that the Laspeyres and Paasche quantity and price indices are equal to the *true* living standard and cost-of-living indices respectively. Zero substitution effect occurs only with a linear cost function which is generated by a Leontief utility functional form (right-angle indifference curves).

(i) **The transitivity criterion**: For given m', Q_A is a monotonic, increasing function of V(**q**, m') and P_M is a monotonic, decreasing function of V(**q**, m'). That is, the quantity index measuring the "real income" should be directly related to the level of welfare and the price index measuring the cost-of-living should be inversely related to the level of welfare.

(ii) **The weak factor reversal criterion**:

$$Q_A \cdot P_M = \frac{m'}{m} \tag{2.7.12}$$

The criterion requires that the product of the quantity and price indices be equal to the ratio of the expenditures. This is for consistency.

On the other hand, McKenzie (1983) shows that the pairing (P_K, Q_A) would satisfy either criterion if and only if preferences are homothetic; that is, if and only if

$$\mu(\mathbf{p}, \mathbf{q}, m') \times \mu(\mathbf{q}, \mathbf{p}, m) = m \times m' \tag{2.7.13}$$

But such a condition is unnecessarily restrictive.

Moreover, the fact that the Konüs Price index is based on the compensating variation which uses the after-change prices as a basis for valuing the welfare change, makes it more difficult to compare more than two alternatives. McKenzie also reviewed several other quantity and price indices that have been proposed. None of them had the desired properties.

Finally, to conclude this section on index numbers, we note that they are often misleadingly discussed in the applied benefit/cost analysis literature as if they constitute an approach to measuring economic welfare different from the one based on the money-metric or compensation functional. But it should be clear from the above relationships, that the EV, CV and index numbers are just different scaling methods or ways of presenting the same quantifiable information embodied in the compensation functional.

2.7.4 Measures of the sensitivity of the consumer's welfare to price-risk

In section **2.5** we have followed Kimball (1990) and distinguished the implications of the two related concepts of *risk aversion* ("dislike" of uncertainty) and *prudence* or *precaution* ("reaction" to uncertainty), because the first concept relate to the sensitivity of *ex-ante* welfare to uncertainty and the second one to the sensitivity of *ex-ante* optimal choices to uncertainty. We have already analyzed the sensitivity of *ex-ante* optimal choices to uncertainty in that section and identified and separated the consumer's *precautionary* response to uncertainty from usual taste substitution and income effects. In this section we complete our analysis of consumer's behavior under uncertainty by analyzing and finding measures of the sensitivity of the consumer's welfare to price uncertainty.

Defining the risk premium in the many-commodity case and finding economically meaningful measures of risk aversion having the desired properties analogous to the one-commodity Arrow-Pratt measure of risk aversion: $\mathbf{r}(x) \equiv -\frac{\mathbf{u}''(x)}{\mathbf{u}'(x)}$, where $\mathbf{u}(x)$ is the utility of money income or wealth x, is not an obvious task. This is especially true in a non-expected utility framework. Within the expected utility framework, many-commodities generalizations of the Arrow-Pratt measure of risk aversion have been proposed by Khilstrom and Mirman (1974), Duncan (1977), and Kerni (1979). The Khilstrom and Mirman measure is a scalar based on the direct utility while Duncan and Kerni's measures are matrix measures based respectively on the direct utility and the indirect utility. They all justify their proposed measures of risk aversion by relating them to some measure of risk premium. A vector of risk premium, one for each commodity, for Khilstrom and Mirman, and Duncan; and a scalar measure based on the expenditure function for Kerni.

Since risk aversion to price uncertainty is about the sensitivity of the consumer's welfare to price-risk, measures of it and of the implied risk premium measuring the "equivalent" change in welfare can be based on a scalar measure of how changes in the indirect utility functional following changes in random prices are related to the variability of prices. But, such measures although perfectly adequate for theoretical purposes, will not be *operational* because their units will be in utility terms. Hence, given the one to one correspondence between changes in the indirect utility functional and changes in the indirect compensation functional as shown by (2.7.7), it is preferable to base the measures of risk aversion and risk premium on a scalar measure of how changes in the indirect compensation functional, following changes in random prices, are related to the variability of prices. Such a scalar measure allows the measurement of the risk premium in monetary units. We will derive first the scalar measure of the risk premium, and derive from it a measure of risk aversion.

2.7.4.1 Measures of risk premia

As in the case of the measurement of total welfare change, two measures of risk premium can be obtained depending on the choice of the base price used to evaluate the change in the indirect compensation functional. If the before-change price vector is used as base price we get an uncertainty analogue of the equivalent variation which corresponds to what Kimball (1990) and Pope and Chavas (1985) call the *equivalent risk premium* and to what Pratt (1964) called the *asking price of risk*. On the other hand, if the after-change price vector is used as base price we get an uncertainty analogue of the compensating variation which corresponds to what Kimball (1990) and Pope and Chavas (1985) call the *compensating risk premium* and to what Pratt (1964) called the *bid price of risk*. Since the derivations in the two cases are identical, we will focus on the *equivalent risk premium* that we simply call the risk premium.[79]

Following the same reasoning as in section **2.5**, under the metric $\| \; \|_2$ a scalar measure of how changes in the indirect compensation functional, when the random price vector changes from **p** to **q**, is related to the variability of prices is provided by the scalar product of the deviation of the marginal change in the indirect compensation functional from its *expected* value and the deviation of the vector of prices from its *expected* value. That is by:

$$\begin{aligned}\tilde{R} &\equiv \langle D_p\mu(\mathbf{p},\mathbf{q},m') - ED_p\mu(\mathbf{p},\mathbf{q},m'), \mathbf{p} - E\mathbf{p}\rangle \\ &= \sum_{i=1}^{n} \langle \frac{\partial\mu(\mathbf{p},\mathbf{q},m')}{\partial \mathbf{p}_i} - E\frac{\partial\mu(\mathbf{p},\mathbf{q},m')}{\partial \mathbf{p}_i}, \mathbf{p}_i - E\mathbf{p}_i\rangle \\ &= \sum_{i=1}^{n} cov\left[\frac{\partial\mu(\mathbf{p},\mathbf{q},m')}{\partial \mathbf{p}_i}, \mathbf{p}_i\right]\end{aligned} \qquad (2.7.14)$$

But, given the quantity involved in the product scalar, this scalar is in terms of the same monetary unit as the indirect compensation functional. Hence, it can be used as the measure of the risk premium. However, because of Corollary **2.7.1** below, we will define the total risk premium, corresponding to a price/expenditure change from (**p**, m) to (**q**, m') as $R(\mathbf{p},\mathbf{q},m,m') \equiv -\tilde{R}(\mathbf{p},\mathbf{q},m,m')$. The negative sign is to make its interpretation conform to the convention that interprets it as the positive amount that the consumer is willing to pay to avoid facing uncertainty, instead of what is lost under uncertainty. Moreover, by using the integrability equations in (2.7.5), we can expressed the total risk total premium in term of the observable Marshallian demands. That is,

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[79] Moreover, given the fact that the risk premium is intuitively perceived as a monetary measure of "willingness to pay" for not having to deal with uncertainty, a natural choice for its *operational* measure is to base it on the equivalent variation which, itself, is an *operational* monetary measure of "willingness to pay", equivalent to the change in welfare resulting from a price/income change.

$$R \equiv R(\mathbf{p},\mathbf{q},m,m') = -\sum_{i=1}^{n} cov[\mathbf{x}_i(\mathbf{p},\, \mu(\mathbf{p},\mathbf{q},m')),\, \mathbf{p}_i] = -\mathrm{tr}\, cov[\mathbf{x}(\mathbf{p},\, \mu(\mathbf{p},\mathbf{q},m')),\, \mathbf{p}] \quad \textbf{(2.7.15)}$$

The individual or *directional* risk premiums attached respectively to each commodity can also be defined as:

$$R_i \equiv \mathrm{R}_i(\mathbf{p},\mathbf{q},m,m') \equiv -\, cov[\mathbf{x}_i(\mathbf{p},\, \mu(\mathbf{p},\mathbf{q},m')),\ \mathbf{p}_i]$$

Thus, $R = \sum_{i=1}^{i=n} R_i$.

The next result, which follows from the monotonicity of the Hicksian demand, tells us that the total risk premium is always non-negative. That is, a consumer whose preference satisfies axioms **A1-A6** is always willing to pay a positive risk premium to avoid having to face the uncertainty involved in any price change (regardless of the direction of the price change). In other words, uncertainty can only reduce welfare.

Corollary 2.7.1

$$R \equiv -\,\mathrm{tr}\, cov[\mathbf{x}(\mathbf{p},\, \mu(\mathbf{p},\mathbf{q},m')),\ \mathbf{p}] \geq 0.$$

2.7.4.2 Disentanglement of the risk premium from the total welfare change

We can use the same procedure as the one used in the disentanglement of the *precautionary* risk effects in the comparative statics equations to decompose the equivalent variation corresponding to a price/expenditure change from (**p**, *m*) to (**q**, *m'*) into its two components: one being the total risk premium measuring the sensitivity of welfare to price-risk, and the other being the nonrisk part of the welfare change (noted *T*) which obtains when the consumer's welfare is not affected by uncertainty.

From the definitions of the indirect compensation functional μ and of the expenditure functional **E** we have:

$$\begin{aligned}\mu(\mathbf{p},\mathbf{q},m') \equiv \mathbf{E}(\mathbf{p}, \mathbf{V}(\mathbf{q},m')) &= \langle \mathbf{h}(\mathbf{p}, \mathbf{V}(\mathbf{q},m'))\ ,\ \mathbf{p}\rangle \\ &= \langle \mathbf{x}(\mathbf{p}, \mu(\mathbf{p},\mathbf{q},m'))\ ,\ \mathbf{p}\rangle \\ &= \operatorname{tr} cov[\mathbf{x}(\mathbf{p}, \mu(\mathbf{p},\mathbf{q},m')),\ \mathbf{p}] + E\mathbf{x}(\mathbf{p}, \mu(\mathbf{p},\mathbf{q},m'))\cdot E\mathbf{p}\end{aligned} \tag{2.7.16}$$

Thus,

$$\begin{aligned}\mathbf{EV} &= \operatorname{tr} cov[\mathbf{x}(\mathbf{p}, \mu(\mathbf{p},\mathbf{q},m')), \mathbf{p}] + E\mathbf{x}(\mathbf{p}, \mu(\mathbf{p},\mathbf{q},m'))\cdot E\mathbf{p} - m \\ &= -R + T\end{aligned} \tag{2.7.17}$$

2.7.4.3 Risk premium and cost-of-risk indices

The total risk premium R can be normalized and made unit free by dividing it by the base income level m so that it is interpreted as percentage of income. In fact, the normalized value of the total risk premium is nothing else than the risk component in the Allen living standard index Q_A. Hence, the normalized total risk premium can be defined to be the *risk premium index* Q_A^r. That is:

$$Q_A^r \equiv \frac{R}{m} = -\frac{1}{m}\operatorname{tr} cov[\mathbf{x}(\mathbf{p}, \mu(\mathbf{p},\mathbf{q},m')),\ \mathbf{p}] \tag{2.7.18}$$

Similarly, the risk component in the Mackenzie-Pearce cost-of-living index, P_M can be defined as the *cost-of-risk index* P_M^r. That is:

$$P_M^r \equiv -\frac{m'}{\mu(\mathbf{p},\mathbf{q},m')} \times \frac{\operatorname{tr} cov[\mathbf{x}(\mathbf{p},\ \mu(\mathbf{p},\mathbf{q},m')),\ \mathbf{p}]}{E\mathbf{x}(\mathbf{p}, \mu(\mathbf{p},\mathbf{q},m'))\cdot E\mathbf{p}} = \frac{R}{(T+m)} \times P_M \tag{2.7.19}$$

The *risk premium index* measures in relative terms how much in the change in living standard is due to risk aversion, while the *cost-of-risk index* measures in relative terms how much in the change in the cost of living is due to risk aversion. It is straightforward to check that Q_A^r and P_M^r.are non-negative and bounded above respectively by Q_A and P_M. [80]

For completeness, we have the nonrisk components in the Allen living standard index and Mackenzie-Pearce cost-of-living index respectively:

$$Q_A^l \equiv \frac{T}{m} + 1 = \frac{1}{m} E\mathbf{x}(\mathbf{p}, \mu(\mathbf{p},\mathbf{q},m')) \cdot E\mathbf{p} \tag{2.7.20}$$

$$P_M^l \equiv \frac{m'}{T+m} = \frac{m'}{E\mathbf{x}(\mathbf{p}, \mu(\mathbf{p},\mathbf{q},m')) \cdot E\mathbf{p}} \tag{2.7.21}$$

We note that $Q_A = Q_A^l - Q_A^r$, $P_M = P_M^l + P_M^r$, and that both pairs (P_M^r, Q_A^r) and (P_M^l, Q_A^l) satisfy the transitivity criterion, while only the pair (P_M^l, Q_A^l) satisfy the weak factor reversal criterion. The fact that the pair (P_M^r, Q_A^r) does not satisfy (2.7.12) is just natural since we want the risk indices to be zero if there is no uncertainty.

2.7.4.3 A matrix measure of risk aversion

Finally, the matrix

$$\begin{aligned} R^A \equiv \mathrm{R}^A(\mathbf{p},\mathbf{q},m,m') &\equiv -\frac{1}{m} cov[\mathbf{x}(\mathbf{p}, \mu(\mathbf{p},\mathbf{q},m')), \mathbf{p}] \\ &= -\frac{1}{m}\left(cov[\mathbf{x}_i(\mathbf{p}, \mu(\mathbf{p},\mathbf{q},m')), \mathbf{p}_j]\right)_{\substack{1 \le i \le n \\ 1 \le j \le n}} \end{aligned} \tag{2.7.22}$$

can be defined as a matrix measure of risk aversion. Then, by definition, a consumer will be said to be locally risk averse at a point (**p**, **q**, m, m') if and only if $\mathbf{R}^A$(**p**, **q**, m, m') is a

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[80] Since the *risk premium* and the *risk premium index* entail a loss in welfare but are reported as positive value (so as to conform to the conventional interpretation), they should always be subtracted from the respective nonrisk components in order to get the EV which measures the (total) welfare change in monetary units, and the (total) living standard index respectively. In contrast, since it is an index of cost, the *cost-of-risk index* must always be added to the non-risk component in order to get the (total) cost-of-living index.

positive semi-definite matrix. He or she is globally risk averse if and only $\mathbf{R}^A(\mathbf{p}, \mathbf{q}, m, m')$ is positive semi-definite at all points $(\mathbf{p}, \mathbf{q}, m, m')$.

With this definition of risk aversion, it is straightforward to see that risk aversion implies a positive total risk premium. But, the implication does not run the other way. Also, because the off-diagonal terms of R^A may be different from zero, zero risk premium does not mean that the consumer is indifferent to uncertainty. Indifference to uncertainty should be measured as when all the elements of R^A are identically zeros.

Because of Roy's identity, the matrix R^A is closely related to Kerni's matrix measure of risk aversion, which is based on the first and second derivatives of the von Neumann-Morgenstern indirect utility function (defined in the outcome space).

Along the same lines one can compare the relative risk averness between two consumers by using the differences of their matrix measures of risk aversion. We would define a consumer 1 with matrix of risk aversion R_1^A to be more risk averse than a consumer 2 with matrix of risk aversion R_2^A if and only if the matrix $(R_2^A - R_1^A)$ is positive semi-definite. It is straightforward then to see that the more risk averse consumer is willing to pay a higher total risk premium for the same change in price.

2.8 Conclusion

In conclusion, we will very briefly summarize the results derived in this chapter, and indicate natural extensions of the analysis in this static framework.

We have started with a general model of decision making under uncertainty which is based on Savage's (1954) conceptual framework, and which justified our choice of the consumer's choice space. Functional analysis methods then enabled us to extend all the results of static deterministic demand theory to the case of uncertainty. In particular, we derived a general comparative statics results that generalizes the familiar decomposition of the effects of a price change into a substitution and an income effects. Moreover, the *precautionary* risk substitution and risk aversion effects were disentangled from the nonrisk taste substitution and pure income effects which obtain when the consumer's behavior is not affected by uncertainty. All the comparative statics results derived above make reference only to observable demand data. No reference to either the original primitive concept of *preference* or to the derived but unobservable concept of utility is made. Thus the empirical validity of the comparative statics statements can be tested. Moreover, numerical elasticity measures corresponding to the total, *precautionary* risk, and nonrisk effects resulting from an *expected* change in prices were derived. We have also derived monetary measures of welfare changes under uncertainty as well as monetary measures of risk premium and cost of risk. All these elasticities and measures of welfare change, risk premium, and cost of risk can be calculated from observable demand data. In chapter 5 we show how they can be estimated using existing econometric methods. We also outline in chapter 4 how all the results derived in this chapter are extended to intertemporal consumption choices.

Extension of the analysis in this chapter to the case of random income stream, profit maximization, and joint determination of consumption and production is not difficult and is already apparent from the functional analysis setting. Another extension of the analysis in this chapter is made by noting that by a simple change of domains of definition for the different functionals we have derived, we can move from an analysis using random prices as arguments of the functionals, to an analysis using their probability distribution as arguments of the functionals. Machina's (1983) preference functional can thus be derived. One key link between the two approaches is the budget constraint:

$\langle \mathbf{p}, \mathbf{x} \rangle \equiv \int_{\Omega} \mathbf{p}(\omega)\mathbf{x}(\omega)d\mathcal{P}(\omega) = E(\mathbf{p}\mathbf{x}) = m$ which depends on the probability distribution $\mathcal{P}$

which in turn induces **F(p)**, the cumulative distribution function for the vector of prices **p**. However, within our framework of *subjective* probability, the behavioral and psychological interpretations of stochastic dominance concepts will be different. In addition, as we argue in Appendix **4**, working with cumulative distribution functions in multivariate decision problems is mathematically cumbersome and at best problematic from a behavioral perspective. In contrast, recognizing that the demand, indirect utility, expenditure, and compensation functionals are all dependent on the expectation operator E (which was identified with the consumer's "expectation" process) and the *subjective* probability distribution $\mathcal{P}$ which induces it, we can easily use functional analysis methods to quantify (both analytically and empirically) the impact on behavior and welfare of "changes" in "expectations" and "beliefs".[81] These "expectations" and "beliefs" issues will be dealt in our future work.

In the next chapter and in chapter **5**, we continue the static analysis by working in detail explicit examples of expected and "non-expected" utility preferences to illustrate the abstract theory in this chapter and show how it is implemented in practice. In the examples in chapter **3** we follow the primal approach by starting with direct utility functionals that are maximized to derive the associated demand, indirect utility, and expenditure functionals, and the elasticity and measures of welfare change formula. In contrast, in chapter **5** we follow the duality approach by starting with the indirect utility or expenditure functional to derive the associated demand functionals and elasticity formula. It is in this chapter where we deal with some of the econometric issues involved in the estimation of demand systems that are derived under uncertainty.

[81] The dependence of the derived functionals on E and $\mathcal{P}$ were not made explicit because we were not interested in that dependence in this chapter.

CHAPTER 3

EXAMPLES OF UTILITY FUNCTIONALS AND THE IMPLIED RESTRICTIONS ON BELIEFS, EXPECTATIONS, BEHAVIOR, AND WELFARE

3.1 Introduction

The specific aim of this chapter is to work in detail through examples of utility and expenditure functionals and derive their associated Marshallian and Hicksian demand functionals along with the preference parameters (demand elasticities, measures of attitude toward risk and welfare changes). In fact, one example could have been sufficient to illustrate the abstract theory of chapter 2. But, we thought that working in detail, step by step, through several examples and deriving the numerical elasticities necessary for applied work, might be helpful for someone not used to calculus in general normed linear spaces.

In chapter 2 we derived general comparative statics results (equivalent to the symmetry and negative semi-definiteness of the Slutsky matrix when demands are differentiable) which we said exhausted all the implications of the assumptions of monotonic and convex preferences. The additional restrictions on preference embodied in the following examples will lead to less general comparative statics conclusions.

Moreover, the examples are not mere examples because they give insights into some aspects of choice under uncertainty, and such insights are not perceptible from the general theory of chapter 2. The examples show clearly the qualitative differences between choice under uncertainty and under certainty. In particular, three results stand out from the examples: (i) For a consumer maximizing an expected utility with a multivariate Von

Neumann-Morgenstern Cobb-Douglas utility that is homogenous of degree one, under price uncertainty and a budget constraint, both the indirect utility and demands are independent of the probability distribution of prices. This also implies that some nonlinear function of the random prices must be independent of the state of nature (i.e, must have a degenerate probability distribution). In the univariate case, this restriction on the probability distribution of prices leads to the striking result that (Arrow-Pratt) risk neutrality is inconsistent with uncertainty, in the sense that for a maximizing consumer with a linear Von Neumann-Morgenstern univariate utility, price must be nonrandom. These results seem at first surprising, but if we recall that in the deterministic case homothetic preference implies severe restrictions on behavior (unitary income elasticity, zero cross-price elasticities, etc...), then we can understand the logic of these restrictions on "beliefs"and expectations when they are introduced into the analysis. **(ii)** The apparent instability of behavior and "beliefs" for preferences generated by expected utility, as illustrated for example by the CRRA functional form, for which a "small change" in the preference parameter can lead to some degeneracy in beliefs and to a qualitatively different type of behavior. **(iii)** In contrast to the deterministic case where the optimal solution of maximizing a univariate utility subject to a budget constraint is determined solely by the budget constraint (with the utility function playing no role), in the uncertainty case the utility functional and "beliefs" (as represented by the *subjective* probability distribution of prices) play important roles in determining the optimal solution.

The restrictions implied by some of the expected utility examples seem to be implied by the independence axiom which leads to its linear form with respect to the probabilty distribution of outcomes, in conjunction with the assumption that the Von Neumann-Morgerstern utility is homogenous of degree one. Indeed, both the Cobb-Douglas and constant elasticity of substitution (CES) expected utility functional forms lead to the same type of restrictions on "beliefs" and choices if they are homogenous of degree one, while there is no restriction on beliefs and choices if their degree of homogeneity is less than one. The log-linear expected utility does not lead to degenerate beliefs, but the corresponding Marshallian demands are independent of "beliefs". "Non-expected" utility versions of Cobb-Douglas and CES functional forms that are homogenous of degree one do not lead to neither degenerate "beliefs" nor independence of demands with respect to beliefs.

Of course, all these restrictions on beliefs are inconsistent with the concept of an "objective" probability distribution of uncertain events facing all consumers. That is why we place these mathematical results within the Bayesian or "subjectivistic" framework of choice under uncertainty as laid down by Ramsey (1931), De Fenetti (1937) and Savage (1954), among others. The restrictions may represent very strong restrictions on individual "beliefs". Nevertheless, it makes sense within this *subjectivistic* view where decision makers are assumed to be free to choose whatever "beliefs" they want to hold. What is surprising here is the fact that these restrictions are obtained without using the *optimality of belief assessments* postulated in Appendix 4, which says that for a choice to be *ex-ante optimal*, the decision maker should have no incentive to change his or her beliefs or expectations. Indeed like the behavioral restrictions implied by the *preference maximization* postulate in deterministic models, this postulate will necessarily imply some restrictions on beliefs and expectations which can be analytically derived for most expected and "non-expected" utility functionals. However, in this chapter we will not deal with the analytical implications of this postulate, because it requires dealing first with few mathematical technicalities that have not been discussed so far.

The chapter is organized as follows. In section **3.2** we briefly review some preliminary "technicalities" briefly discussed in chapter **3** and give the necessary first order condition for optimality for the consumer's maximization problem. In section **3.3** we derive the optimality condition for a consumer maximizing a multivariate expected utility under a budget constraint. We then proceed to derive the demand, indirect utility, and expenditure functionals for the popular univariate expected utility models (CARA, CRRA, and Risk neutral). We use these univariate examples to compare our definition of the risk premium given in chapter **2**, with the standard definition based on expected values. In these examples, the two risk premia are always positive, but with different magnitudes. The multivariate generalizations of these univariate functional forms are then treated with all the elasticities and welfare indicators derived. The multivariate generalization of the CARA utility will coincide with the Cobb-Douglas expected utility functional form. In Section **3.4** we show how to generate and solve for "non-expected" utility functional forms. The Cobb-Douglas "non expected" utility is then treated with all the functional, elasticities, and welfare indicators derived. Finally, we briefly conclude the chapter with some remarks in section **3.5**. Appendix **3** contains the proofs of two lemmas used in the chapter.

3.2 First order necessary conditions for optimality for a utility defined in $L_2(\Omega, \mathbf{R}^n)_+$

We recall from chapter **2** that when the utility functional **U** is defined on a weakly compact and convex subset **X** of $L_2(\Omega, \mathbf{R}^n)_+$ then the consumer's problem:

$$\mathbf{V}(\mathbf{p}, m) = \max_{s.t.\ \mathbf{p}\cdot\mathbf{x} = m} \mathbf{U}(\mathbf{x}) \qquad (3.2.1)$$

has a solution in **X** that is unique if **U** is strictly quasi-concave. Here, **p** is the vector of prices, the dot sign "$\cdot$" is the inner or scalar product of the Hilbert space $L_2(\Omega, \mathbf{R}^n)$, m is the nonrandom predetermined total expenditure, and **V** is the indirect utility functional defined on

$L_2(\Omega, \mathbb{R}^n)_{++} \times \mathbb{R}_{++}$.[82] Furthermore, Proposition **2.3.1** in chapter **2** guaranteed the existence of a Lagrange multiplier $\lambda \in \mathbb{R}_+$, such that (3.2.1) is equivalent to the maximization problem

$$\begin{gathered} \mathbf{V}(\mathbf{p}, m) = \max_{\mathbf{x} \in \mathbf{X}} \{\mathbf{U}(\mathbf{x}) - \lambda(\mathbf{p} \cdot \mathbf{x} - m)\} \\ \text{with } \mathbf{p} \cdot \mathbf{x}^* - m = 0 \text{ for any optimal solution } \mathbf{x}^* \end{gathered} \tag{3.2.2}$$

However, since $\mathbf{X} \subset L_2(\Omega, \mathbb{R}^n)_+$ has an empty interior, we cannot speak of the Fréchet or Gâteaux differentiability of **U** in the classical sense. Hence, we cannot directly use Corollary **A2.1** in Appendix **2** which generalizes to the infinite dimensional case, the usual necessary "first order conditions" for an interior solution of a constrained optimization problem. Furthermore, since we are interested in explicit solutions, we will replace the abstract requirement $\mathbf{x} \in \mathbf{X}$ by simply imposing the non negativity constraints $\mathbf{x} \geq \mathbf{0}$ (i.e., $\mathbf{x} \in L_2(\Omega, \mathbb{R}^n)_+$).

To get around the differentiability problem, in chapter **2** we extended the definitions of Fréchet and Gâteaux differentiability to functions whose domains of definition have empty interiors. Briefly, a functional or operator **F** is said to be Fréchet differentiable at a point $\mathbf{x} \in \mathbf{D} \subset L_2(\Omega, \mathbb{R}^n)_+$ if and only if there exists an open set containing **D** and a functional $\mathbf{F}^*$ that extends **F** to the open set and that is Fréchet differentiable at **x**. The Fréchet derivative of **F** at **x** is then defined as $\mathbf{F}'(\mathbf{x}) \equiv \mathbf{F}^{*\prime}(\mathbf{x})$. For all the functional and operators in the examples below (direct or indirect utility, expenditure, demands, etc....), the open set will be the whole space $L_2(\Omega, \mathbb{R}^n)$ and the extending function will be defined as $\mathbf{F}^*(\mathbf{x}) \equiv \mathbf{F}(|\mathbf{x}|)$. Where, $|\mathbf{x}| \equiv (|\mathbf{x}_1|,\ldots, |\mathbf{x}_n|)$ and $|\mathbf{x}_i|$ is defined by $|\mathbf{x}_i|(\omega) \equiv \sup(\mathbf{x}_i(\omega), -\mathbf{x}_i(\omega))$ for all $\omega \in \Omega$. Thus defined, $\mathbf{F}^*$ will always be Fréchet differentiable at all $\mathbf{x} = (\mathbf{x}_1,\ldots,\mathbf{x}_n) \in L_2(\Omega, \mathbb{R}^n)$ such that $\mathbf{x}_i \neq \mathbf{0}$ $i = 1,\ldots, n$. Hence, given our definition, simple differentiation with the chain

[82] See chapter **2** for how to handle the case of random total expenditure.

rule will always give us $\mathbf{F}'(\mathbf{x}) = \mathbf{F}^{*\prime}(\mathbf{x})$ for all $\mathbf{x}$ such that $\mathbf{x}_i > \mathbf{0}$ (i.e $\mathbf{x}_i \in L_2(\Omega, \mathbb{R})_{++}$) $i = 1, \ldots, n$.[83] The only exception to this extension procedure is the CARA multivariate direct expected utility which is simply defined as the restriction to $L_2(\Omega, \mathbb{R}^n)_+$ of a functional that is defined and Fréchet differentiable on the entire space (including the origin). This takes care of corner solutions which arise in the multivariate CARA case. So, for all practical purposes we can carry out our differential calculus with $\mathbf{F}$ as if $L_2(\Omega, \mathbb{R}^n)_{++}$ had an non empty interior. However, all the differentiations carried out below must always be understood in this extended sense, even if the extending $\mathbf{F}^*$ will rarely be mentioned for shortness and for the sake of simplifying the notation.

When the direct utility functional $\mathbf{U}$ is Fréchet differentiable (in the extended sense) at $\mathbf{x}^* = (\mathbf{x}_1^*, \ldots, \mathbf{x}_n^*) \in L_2(\Omega, \mathbb{R}^n)_+$, a direct application of Corollary **A2.1** in Appendix **2** to (3.2.2) yields the following necessary first order conditions for $\mathbf{x}^*$ to be a solution of (3.2.2):

$$\begin{gathered} \mathbf{U}_i'(\mathbf{x}^*) - \lambda \mathbf{p}_i + \mathbf{q}_i^* = \mathbf{0}, \\ \mathbf{q}_i^* \geq \mathbf{0} \quad \text{and } \mathbf{q}_i^* = \mathbf{0} \quad \text{if } \mathbf{x}_i^* > \mathbf{0} \qquad i=1,\ldots,n \end{gathered} \tag{3.2.3}$$

$$\mathbf{p} \cdot \mathbf{x}^* - m = 0 \tag{3.2.4}$$

where $\mathbf{q}^* = (\mathbf{q}_1^*, \ldots, \mathbf{q}_n^*) \in L_2(\Omega, \mathbb{R}^n)_+$ is the vector of Lagrange "multipliers" corresponding to the non negativity constraint $\mathbf{x} \geq \mathbf{0}$ (see Appendix **3**). By an "interior" solution of (3.2.2) we always means a solution $\mathbf{x}^* = (\mathbf{x}_1^*, \ldots, \mathbf{x}_n^*) \in L_2(\Omega, \mathbb{R}^n)_+$ such that $\mathbf{x}_i^* > \mathbf{0}$, $i = 1, \ldots, n.$, while by a "corner" solution we refer to the case where $\mathbf{x}_i^* = \mathbf{0}$ for at least one i.

[83] This is based on Lemma 3.3.1 below, and on the fact that the real valued function on the real line g: t → |t| is differentiable everywhere except at 0, with g′(t) = sign(t).

In all the examples below, the utility functional **U** is Fréchet differentiable (in the extended sense) at all points **x** such that $\mathbf{x}_i > \mathbf{0}$ for all i. Hence, (3.2.3) will always apply for an "interior" solution in which case it reduces simply to

$$\mathbf{U}_i'(\mathbf{x}^*) - \lambda \mathbf{p}_i = \mathbf{0} \quad i=1,\ldots,n \qquad \textbf{(3.2.5)}$$

Moreover, except for the multivariate CARA functional form, the unique "interior" solution that solves the operator equations (3.2.5) and (3.2.4) will be the optimal solution of the consumer's maximization problem.[84] For the multivariate CARA expected utility functional, the possibility of a "corner" solution will arise. But since the CARA expected utility functional will be Fréchet differentiable at the origin **0**, (3.2.3) will still be a necessary condition for the optimal solution.

3.3 The special case of an expected utility maximizing consumer

The expected utility of a random consumption bundle $\mathbf{x} \in L_2(\Omega, \mathbf{R}^n)_+$ is given by $E\mathbf{u}(\mathbf{x}) = \int_\Omega \mathbf{u}(\mathbf{x}(\omega))d\mathbf{P}(\omega)$. Where E is the expectation operator and **u** is a real valued strictly concave and continuously differentiable von Neumann-Morgenstern utility defined on the *outcome* space $\mathbf{R}^n$, with its partial derivatives satisfying: $\mathbf{u}_i'(\mathbf{x}) > 0 \quad i=1,\ldots,n$. $E\mathbf{u}(\mathbf{x})$ is usually expressed in the equivalent form that uses **F(x)**, the *subjective* cumulative distribution

[84] This will be because if for some i $\mathbf{x}_i = \mathbf{0}$, either $\mathbf{U}(\mathbf{x}_1,\ldots,\mathbf{x}_n) = 0$ or $\mathbf{U}(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ is not defined. For the univariate CARA, **0** cannot be optimal.

function of $\mathbf{x}$ instead of the *subjective* probability distribution $\mathcal{P}$; i.e, as

$$Eu(\mathbf{x}) = \int_{\mathbb{R}^n} u(x)\, d\mathbf{F}(\mathbf{x})(x)$$ [85]

We can define on $L_2(\Omega, \mathbb{R}^n)_+$ the utility functional $\mathbf{U}$ associated with expected utility as $\mathbf{U}(\mathbf{x}) \equiv E\mathbf{u}(\mathbf{x})$. $\mathbf{U}(\mathbf{x})$ can be expressed in a way suitable for functional analysis as:

$$\mathbf{U}(\mathbf{x}) = E(\mathbf{u}\circ\mathbf{x}) \equiv (E\circ\mathbf{T}_{\mathbf{u}})(\mathbf{x}) \qquad \textbf{(3.3.1)}$$

where $\circ$ is the composition operator, and $\mathbf{T}_{\mathbf{u}}: L_2(\Omega, \mathbb{R}^n) \to L_2(\Omega, \mathbb{R})$ is the functional defined by: $\mathbf{T}_{\mathbf{u}}(\mathbf{x}) = \mathbf{u}\circ\mathbf{x}$.

It is easy to see that the concavity of $\mathbf{u}$ implies the concavity of $\mathbf{U}$. Also, since the expectation operator E is linear and continuous, it is Fréchet differentiable everywhere with $E'(\mathbf{t}) = E$ for all $\mathbf{t} \in L_2(\Omega, \mathbb{R})$. The following Lemma shows that the Fréchet differentiability of $\mathbf{T}_{\mathbf{u}}$ follows from the differentiability of $\mathbf{u}$.

Lemma 3.3.1 *The operator* $\mathbf{T}_{\mathbf{u}}: L_2(\Omega, \mathbb{R}^n) \to L_2(\Omega, \mathbb{R})$ *defined by* $\mathbf{T}_{\mathbf{u}}(\mathbf{x}) = \mathbf{u}\circ\mathbf{x}$ *is everywhere Fréchet differentiable with* $\mathbf{T}_{\mathbf{u}}'(\mathbf{x}) = \mathbf{T}_{\mathbf{u}'}(\mathbf{x}) = \mathbf{u}'\circ\mathbf{x} \in L_2(\Omega, \mathbb{R}^n)$. *Furthermore, if* $\mathbf{u}$ *is* n *times differentiable then so is* $\mathbf{T}_{\mathbf{u}}$ *with* $\mathbf{T}_{\mathbf{u}}^{(n)}(\mathbf{x}) = \mathbf{T}_{\mathbf{u}^{(n)}}(\mathbf{x}) = \mathbf{u}^{(n)}\circ\mathbf{x} \in L_2(\Omega, \mathbb{R}^n)^n$.

Hence, by the chain rule, the differentiability of the utility functional $\mathbf{U}$ follows from the differentiability of the utility function $\mathbf{u}$ with:

[85] It is important not to confuse the random variable $\mathbf{x}$ with the dummy variable x. In standard notation, the dependence of the *subjective* cumulative distribution $\mathbf{F}$ on $\mathbf{x}$ is not made explicit as we do here, so that instead of writing $\mathbf{F}(\mathbf{x})(x)$, one would simply write $\mathbf{F}(x)$. Formally, $\mathbf{F}$ is a functional defined on $L_2(\Omega, \mathbb{R}^n)_+$ and which associates to each vector valued random variable $\mathbf{x}$ its *subjective* cumulative distribution function $\mathbf{F}(\mathbf{x})$.

$$\mathbf{U}'(\mathbf{x})\cdot\mathbf{h} = (E\circ\mathbf{T}_{\mathbf{u}}'(\mathbf{x}))\cdot\mathbf{h} = \int_{\Omega}\mathbf{u}'(\mathbf{x}(\omega))\cdot\mathbf{h}(\omega)d\mathcal{P}(\omega) = \langle\mathbf{u}'\circ\mathbf{x},\mathbf{h}\rangle \quad \text{for all } \mathbf{h} \in L_2(\Omega,\mathbb{R}^n) \qquad (3.3.2)$$

Thus, by identification we have: $\mathbf{U}'(\mathbf{x}) = \mathbf{u}'\circ\mathbf{x}$, which can be written in the usual shorthand notation as $\mathbf{U}'(\mathbf{x}) = \mathbf{u}'(\mathbf{x})$.[86]

This presentation of expected utility shows in particular that it is just a special functional form of the general utility functional **U** of the previous section; in the same way as linear or Cobb-Douglas utility in deterministic consumer theory are special utility functions that lead to some restrictions in behavior. In particular, the linear dependency of expected utility on the cumulative distribution **F(x)** is generally seen as showing the restrictive nature of expected utility. On the other hand, both expressions of expected utility shows its nonlinear dependence on the random variable **x** (except when **u** is linear).[87]

At this point, it is perhaps worth recalling that the von Neumann-Morgenstern utility **u** that gives rise to **U** is unique only up to a positive affine transformation. In this sense, **u** is a cardinal utility, because a monotonic non decreasing transformation of **u** that leaves the ordinal ranking in the outcome space unchanged, will in general change the ordinal ranking of the random variables in the choice set. In other words, although $\mathbf{f} \circ \mathbf{U}$ and **U** represents the

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[86] This shorthand notation can be misleading because, unlike $\mathbf{U}'$ whose domain of definition is $L_2(\Omega, \mathbb{R}^n)_+$, the domain of definition of $\mathbf{u}'$ is rather $\mathbb{R}^n_+$. In other words, $\mathbf{U}'(\mathbf{x}) = \mathbf{u}'(\mathbf{x})$ means $\mathbf{U}'(\mathbf{x})(\omega) = \mathbf{u}'(\mathbf{x}(\omega))$ for all $\omega \in \Omega$. The same remark applies when one writes $\mathbf{u}(\mathbf{x})$.

[87] See Machina (1982) for a comprehensive review of the behavioral restrictions implied by the independence axiom leading to the linearity, and the experimental evidences against it. Another illuminating way to see the restrictiveness of the expected utility functional form is to note that in the expression $\mathbf{U}(\mathbf{x}) \equiv E\mathbf{u}(\mathbf{x})$, the utility functional **U** is a linear functional of the decision maker's "expectations" as represented by the expectation operator E. And, since from a functional analysis perspective E is just an element of the dual of $L_2(\Omega, \mathbb{R})$, hence an element of $L_2(\Omega, \mathbb{R})$ itself given the fact that this space is an Hilbert space, with our extension of the concept of Fréchet differentiability one can see that the Fréchet derivative of **U** with respect to E is just $\mathbf{u}(\mathbf{x})$ which is a random variable independent of E and $\mathcal{P}$. That is, the "marginal utility" from changing "expectations" is independent of "expectations" and "beliefs". Hence, with respect to "expectations" and "beliefs" expected utility is like linear utility with respect to the choice variable in deterministic models.

same preference in the choice set if **f** is a monotonic non decreasing function, **f** o **U** and E o **f** o **u** do not represent the same preference unless **f** is a positive affine function.[88] Many of the experimental evidences against expected utility can be traced to this non invariance to nonlinear monotonic transformations of outcomes. The examples below will show analytically some of the behavioral consequences of this non invariance to monotonic transformations.

3.3.1 The optimality condition for expected utility

The first order conditions for an "interior" solution $\mathbf{x}^*$ (3.2.5) and (3.2.4) becomes here:

$$\mathbf{u}'\circ\mathbf{x}^* - \lambda\mathbf{p} = \mathbf{0} \tag{3.3.3}$$

$$\mathbf{p}\cdot\mathbf{x}^* - m = 0 \tag{3.3.4}$$

(3.3.3), (3.3.4) is an equation in $L_2(\Omega, \mathbf{R}^n)$ which can be expressed in terms of "marginal rates of substitution" as:

$$\frac{\mathbf{u}_i'\circ\mathbf{x}^*}{\mathbf{u}_j'\circ\mathbf{x}^*} = \frac{\mathbf{p}_i}{\mathbf{p}_j} \qquad i,j = 1,\ldots,n \tag{3.3.5}$$

Equations (3.3.3), (3.3.4) and (3.3.3), (3.3.4) also imply:

$$\langle \mathbf{u}'\circ\mathbf{x}^*, \mathbf{x}^*\rangle = \lambda\langle \mathbf{p}, \mathbf{x}^*\rangle = \lambda m \tag{3.3.6}$$

Hence, the optimality condition for an expected utility maximizing consumer is:

[88] In fact, it is well known from expected utility analysis that if **f** is strictly concave and monotonic increasing, then, in the sense of the Arrow-Pratt measure of risk aversion, the preference represented by E o **f** o **u** is more risk averse than the one represented by E o **u**. The Arrow-Pratt risk premium under E o **f** o **u** is also greater.

$$\mathbf{u}'\circ\mathbf{x}^* = \frac{1}{m}\langle \mathbf{u}'\circ\mathbf{x}^*, \mathbf{x}^*\rangle \mathbf{p} = \frac{\mathbf{p}}{m}E\big(\mathbf{x}^*\cdot\mathbf{u}'(\mathbf{x}^*)\big) \tag{3.3.7}$$

or, in terms of the expected budget shares,

$$E\omega_i \equiv \frac{\langle \mathbf{p}_i, \mathbf{x}_i\rangle}{m} = \frac{\langle \mathbf{u}_i'\circ\mathbf{x}^*, \mathbf{x}_i\rangle}{\langle \mathbf{u}'\circ\mathbf{x}^*, \mathbf{x}^*\rangle} \qquad i=1,\ldots,n \tag{3.3.8}$$

If $\mathbf{u}$ is homogeneous of degree α, then $\langle \mathbf{u}'\circ\mathbf{x}, \mathbf{x}\rangle = \alpha E\mathbf{u}(\mathbf{x})$. Hence, the optimality condition for a consumer with an homogeneous expected utility of degree α reduces to:

$$\mathbf{u}'\circ\mathbf{x}(\mathbf{p}, m) = \frac{\alpha}{m}\mathbf{V}(\mathbf{p}, m)\mathbf{p} \tag{3.3.9}$$

which can be solved for the indirect utility functional $\mathbf{V}(\mathbf{p}, m) \equiv E\big(\mathbf{u}\circ\mathbf{x}(\mathbf{p}, m)\big)$ to get:

$$\mathbf{V}(\mathbf{p}, m) = m\frac{\mathbf{u}_i'\circ\mathbf{x}(\mathbf{p}, m)}{\alpha\mathbf{p}_i} \qquad i=1,\ldots,n \tag{3.3.10}$$

From (3.3.10) we can see that for an homogeneous expected uility the independence of demand with respect to beliefs and "expectations", as represented by the subjective probability distribution $\mathcal{P}$ and the corresponding expectation operator E, will have two implications: **1)** Welfare will also be independent of beliefs and "expectations", and **2)** There will be some restrictions on beliefs and "expectations".

3.3.2 Examples of univariate expected utility functionals

The following univariate results can be obtained as special cases of their multivariate counterparts in the next section. We derive them here in order to compare our definitions of the risk premium and risk aversion given in chapter 2, and which are based on the equivalent variation, with the traditional (univariate) definitions based on the Von Neumann-Morgenstern utility $\mathbf{u}$. We recall that the Arrow-Pratt measures of absolute and relative risk aversion are defined respectively as $r_a \equiv -\frac{u''(x)}{u'(x)}$ and $r_r \equiv -x\frac{u''(x)}{u'(x)}$; while the risk premium is defined as the real number π that satisfies $\mathbf{u}(E\mathbf{x} - \pi) = E\mathbf{u}(\mathbf{x})$.

The derivation of the univariate results also puts in contrast the qualitative differences compared to the deterministic case. In static deterministic consumer theory, the demand function corresponding to a univariate utility is given by $x(p,m) = \frac{m}{p}$ and is determined solely by the budget constraint (3.3.3), (3.3.4) regardless of preference. In this uncertainty context, the shape of the univariate demand function will depend in a important way on preference, beliefs and expectations.

3.3.2.1 Demand function for CRRA expected utility

The Arrow-Pratt constant relative risk aversion (CRRA) expected utility class is defined by taking the class of real functions defined on the set of positive real numbers by:

$$\mathbf{u}(x) = \begin{cases} x^{\alpha} & \text{for } 0 < \alpha \leq 1 \\ Log\,x & \text{for } \alpha = 0 \end{cases} \qquad \textbf{(3.3.11)}$$

For $\alpha = 0$, $\mathbf{u}$ is defined only for strictly positive numbers. The Arrow-Pratt coefficient of relative risk aversion is $1 - \alpha$. For $\alpha = 1$ $\mathbf{u}$ is linear and we have a risk neutral preference, otherwise $\mathbf{u}$ is strictly concave and preference is said to exhibit risk aversion. As a function of the random variable $\mathbf{x}$, the corresponding risk premiums are

$$\pi(\mathbf{x}) = \begin{cases} E\mathbf{x} - \left(E\mathbf{x}^{\alpha}\right)^{\frac{1}{\alpha}} & \text{for } 0 < \alpha \leq 1 \\ E\mathbf{x} - e^{ELog\mathbf{x}} & \text{for } \alpha = 0 \end{cases} \qquad \textbf{(3.3.12)}$$

Case 1: $0 < \alpha < 1$. In this case the expected utility optimality condition (3.3.7) becomes

$$\alpha \mathbf{x}^{\alpha-1} = \alpha \frac{\mathbf{p}}{m} E\mathbf{x}^{\alpha} \tag{3.3.13}$$

which can be solved for **x** to get

$$\mathbf{x} = \left(\frac{\mathbf{p}}{m}\right)^{\frac{1}{\alpha-1}} \left(E\mathbf{x}^{\alpha}\right)^{\frac{1}{\alpha-1}} \tag{3.3.14}$$

Raising both sides of (3.3.14) to the power of α, then taking the expectation of both sides, and then solving for $E\mathbf{x}^{\alpha}$ yields

$$E\mathbf{x}^{\alpha} = m^{\alpha}\left(E\mathbf{p}^{\frac{\alpha}{\alpha-1}}\right)^{1-\alpha} \tag{3.3.15}$$

Substituting (3.3.15) in (3.3.14) gives the demand functional

$$\mathbf{x}(\mathbf{p},m) = \frac{m\mathbf{p}^{\frac{1}{\alpha-1}}}{E\left(\mathbf{p}^{\frac{\alpha}{\alpha-1}}\right)} \tag{3.3.16}$$

Hence, we see that the demand functional depends not only on the preference parameter α, but also it depends on the consumer's *beliefs* and "expectations" as represented by the subjective probability distribution $\mathcal{P}$ and the expectation operator E since $E\mathbf{p}^{\frac{\alpha}{\alpha-1}} = \int_{\Omega} \mathbf{p}(\omega)^{\frac{\alpha}{\alpha-1}} d\mathcal{P}(\omega)$.

The right hand side of (3.3.15) gives directly the expression for the indirect utility functional $\mathbf{V}(\mathbf{p},m) = E\mathbf{x}(\mathbf{p},m)^{\alpha}$. The expression for the expenditure functional **E** is obtained by inverting **V**:

$$\mathbf{E}(\mathbf{p},u) = u^{\frac{1}{\alpha}}\left(E\mathbf{p}^{\frac{\alpha}{\alpha-1}}\right)^{\frac{\alpha-1}{\alpha}} \tag{3.3.17}$$

The compensation functional μ and the Hicksian demand are given respectively by

$$\mu(\mathbf{p},\mathbf{q},m) \equiv \mathbf{E}(\mathbf{p}, \mathbf{V}(\mathbf{q},m)) = m\left(\frac{E\mathbf{p}^{\frac{\alpha}{\alpha-1}}}{E\mathbf{q}^{\frac{\alpha}{\alpha-1}}}\right)^{\frac{\alpha-1}{\alpha}} \tag{3.3.18}$$

and

$$\mathbf{h}(\mathbf{p},u) = \mathbf{x}(\mathbf{p}, \mathbf{E}(\mathbf{p},u)) = \frac{u^{\frac{1}{\alpha}}\mathbf{p}^{\frac{1}{\alpha-1}}}{\left(E\mathbf{p}^{\frac{\alpha}{\alpha-1}}\right)^{\frac{1}{\alpha}}} \tag{3.3.19}$$

The expression for the risk premium corresponding to a price/expenditure change from ($\mathbf{p}$, m) to ($\mathbf{q}$, m'), as we have defined it in chapter 2, is

$$R \equiv -cov\left(\mathbf{x}(\mathbf{p}, \mu(\mathbf{p},\mathbf{q},m')), \mathbf{p}\right) = -m'\left(\frac{E\mathbf{p}^{\frac{\alpha}{\alpha-1}}}{E\mathbf{q}^{\frac{\alpha}{\alpha-1}}}\right)^{\frac{\alpha-1}{\alpha}} \times \frac{cov\left(\mathbf{p}^{\frac{1}{\alpha-1}}, \mathbf{p}\right)}{E\left(\mathbf{p}^{\frac{\alpha}{\alpha-1}}\right)} \tag{3.3.20}$$

Similarly, the corresponding risk aversion matrix which reduces here to a scalar equal to the risk premium index is

$$R^A \equiv \frac{R}{m} \equiv Q_A^r = -\frac{m'}{m}\left(\frac{E\mathbf{p}^{\frac{\alpha}{\alpha-1}}}{E\mathbf{q}^{\frac{\alpha}{\alpha-1}}}\right)^{\frac{\alpha-1}{\alpha}} \times \frac{cov\left(\mathbf{p}^{\frac{1}{\alpha-1}}, \mathbf{p}\right)}{E\left(\mathbf{p}^{\frac{\alpha}{\alpha-1}}\right)} \tag{3.3.21}$$

Since the preference associated with the expected utility functional **U** satisfies all the assumptions on preference in chapter 2, we already know from the general theory that the risk premium R is positive.[89] Hence, by our definition of risk aversion (which reduces here to the positivity of R^A), this preference exhibits risk aversion.

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[89] One can show directly that $cov\left(\mathbf{p}^{\frac{1}{\alpha-1}}, \mathbf{p}\right) < 0$ by using the fact that $\langle \mathbf{p}^{\frac{1}{\alpha-1}} - (E\mathbf{p})^{\frac{1}{\alpha-1}}, \mathbf{p} - E\mathbf{p}\rangle < 0$ which follows from the monotonicity of the operator $\mathbf{p} \to \mathbf{p}^{\frac{1}{\alpha-1}}$.

To facilitate the comparison with the standard measures of risk premium, let us assume that expenditure remains unchanged, and that the price change from **p** to **q** is a deterministic homothetic transformation so that it can be expressed in terms of a γ percentage change, with $\gamma \in \mathbb{R}$.[90] In this case, $\mathbf{q} = (1+\gamma)\mathbf{p}$ and the risk premium R simplifies to

$$R = -\frac{m'}{1+\gamma}\frac{cov\left(\mathbf{p}^{\frac{1}{\alpha-1}}, \mathbf{p}\right)}{E\left(\mathbf{p}^{\frac{\alpha}{\alpha-1}}\right)} = -\frac{m'}{(1+\gamma)E\left(\mathbf{p}^{\frac{\alpha}{\alpha-1}}\right)} \times \left(E\mathbf{p}^{\frac{\alpha}{\alpha-1}} - E\mathbf{p}^{\frac{1}{\alpha-1}}E\mathbf{p}\right) > 0 \qquad \textbf{(3.3.22)}$$

On the other hand, substituting the expression for the compensated demand functional evaluated at utility $\mathbf{V}(\mathbf{q}, m')$ (3.3.16) in (3.3.12) gives the Arrow-Pratt risk premium in terms of **p** and m':

$$\pi = -\frac{m'}{(1+\gamma)E\left(\mathbf{p}^{\frac{\alpha}{\alpha-1}}\right)} \times \left[\left(E\mathbf{p}^{\frac{\alpha}{\alpha-1}}\right)^{\frac{1}{\alpha}} - E\mathbf{p}^{\frac{1}{\alpha-1}}\right] > 0 \qquad \textbf{(3.3.23)}$$

Clearly, the magnitudes of the two risk premia are different although they are both strictly positive.[91] However, while R depends only on the underlying ordinal preference and is thus invariant to monotonic increasing transformations of **U**, π is dependent on the cardinal utility **u** which is unique only up to affine transformations of **u**. The same comments apply to the measures of risk aversion. In addition, while the Arrow-Pratt measure is constant, R^A depends on **p**. For completeness, the risk premium index and the cost-of-risk index as defined in chapter 2 are respectively

[90] This type of change includes sales tax, tariff, and devaluation of a currency.

[91] The positivity of π follows from the concavity of the real valued function $t \to t^\alpha$ in conjunction with Jensen's inequality.

$$R^A \equiv \frac{R}{m} = -\frac{m'}{(1+\gamma)m}\frac{cov\left(\mathbf{p}^{\frac{1}{\alpha-1}}, \mathbf{p}\right)}{E\left(\mathbf{p}^{\frac{\alpha}{\alpha-1}}\right)} \tag{3.3.24}$$

and

$$P_M^r \equiv \frac{m'R}{\mu(\mathbf{p},\mathbf{q},m')\left(E\mathbf{x}(\mathbf{p},\mu(\mathbf{p},\mathbf{q},m'))E\mathbf{p}\right)} = -(1+\gamma)\frac{cov\left(\mathbf{p}^{\frac{1}{\alpha-1}}, \mathbf{p}\right)}{E\left(\mathbf{p}^{\frac{1}{\alpha-1}}\right)E\mathbf{p}} \tag{3.3.25}$$

Case 2: $\alpha = 0$. In this case the expected utility optimality condition (3.3.7) becomes

$$\frac{1}{\mathbf{x}} = \frac{\mathbf{p}}{m}E\left[\mathbf{x}\frac{1}{\mathbf{x}}\right] = \frac{\mathbf{p}}{m} \tag{3.3.26}$$

Hence the demand function is

$$\mathbf{x}(\mathbf{p},m) = \frac{m}{\mathbf{p}} \tag{3.3.27}$$

and the indirect utility functional is

$$\mathrm{V}(\mathbf{p},m) = ELog\frac{m}{\mathbf{p}} \tag{3.3.28}$$

The other functionals are easily derived from (3.3.27) and (3.3.28). One thing to note here is that the demand function is similar to the deterministic case; the only difference is that here **p** is a random variable. Also, the demand function is independent of beliefs and expectations while, in contrast, the indirect utility functional depend on beliefs and expectations. Finally, one can check that again the two risk premia are positive but different in magnitude.

Case 3: α = 1. In this case the expected utility optimality condition (3.3.7) becomes

$$1 = \frac{\mathbf{p}}{m} E\mathbf{x} \tag{3.3.29}$$

Equation (3.3.29) gives the indirect utility functional

$$\mathbf{V}(\mathbf{p}, m) = \frac{m}{\mathbf{p}} \tag{3.3.30}$$

from which, by Roy's identity, follows the demand function

$$\mathbf{x}(\mathbf{p}, m) = \frac{m}{\mathbf{p}} \tag{3.3.31}$$

However, since **V** takes its values in **R**, the right hand side of (3.3.30) must be nonrandom. That is **p** is non random, and the consumer's beliefs are degenerate. Hence, in this case the demand function is the same as in the deterministic case. In other words, Arrow-Pratt risk neutrality is inconsistent with uncertainty in the sense that the consumer makes his or her choice while believing that price is independent of the state of nature. This of course implies that the risk premium is zero.

3.3.2.2 Demand functional for the CARA expected utility

The Arrow-Pratt constant absolute risk aversion (CARA) expected utility class is defined by taking the class of real functions defined on the set of positive real numbers by:

$$\mathbf{u}(\mathbf{x}) = 1 - e^{-\alpha \mathbf{x}} \quad \text{for } \alpha > 0 \tag{3.3.32}$$

The Arrow-Pratt coefficient of absolute risk aversion is α and the corresponding risk premium is

$$\pi(\mathbf{x}) = E\mathbf{x} + \frac{1}{\alpha} Log E e^{-\alpha \mathbf{x}} \qquad \textbf{(3.3.33)}$$

In this case, the expected utility optimality condition (3.3.7) becomes

$$e^{-\alpha \mathbf{x}} = \frac{\mathbf{p}}{m} E\left(\mathbf{x} e^{-\alpha \mathbf{x}}\right) \qquad \textbf{(3.3.34)}$$

which can be solved for **x** to get

$$\mathbf{x} = -\frac{1}{\alpha} Log \left[\frac{\mathbf{p}}{m} E\left(\mathbf{x} e^{-\alpha \mathbf{x}}\right) \right] \qquad \textbf{(3.3.35)}$$

Taking the scalar product of both sides of (3.3.35) with **p** gives

$$\begin{aligned} m = \langle \mathbf{p},\ \mathbf{x} \rangle &= -\frac{1}{\alpha} \langle \mathbf{p}, Log \left[\frac{\mathbf{p}}{m} E\left(\mathbf{x} e^{-\alpha \mathbf{x}}\right) \right] \rangle \\ &= -\frac{1}{\alpha} \left\{ E \left[\mathbf{p} Log \frac{\mathbf{p}}{m} \right] - E\mathbf{p} Log E\left(\mathbf{x} e^{-\alpha \mathbf{x}}\right) \right\} \end{aligned} \qquad \textbf{(3.3.36)}$$

Solving for $E\mathbf{x}^{\alpha}$ and substituting back into (3.3.35) gives the demand function for the CARA expected utility

$$\mathbf{x}(\mathbf{p}, m) = \frac{m}{E\mathbf{p}} + \frac{1}{\alpha E\mathbf{p}} \left\{ E \left[\mathbf{p} Log \frac{\mathbf{p}}{m} \right] - (E\mathbf{p}) Log \frac{\mathbf{p}}{m} \right\} \qquad \textbf{(3.3.37)}$$

Again, we see that the demand function depends on both the preference parameter α and on the consumer's beliefs and "expectations" represented by $\mathcal{P}$ and E respectively.

The indirect utility, compensation, Hicksian demand, risk premium, and risk aversion functionals are easily derived from (3.3.37). They are respectively[92]:

[92] The risk premium and risk aversion are evaluated asuuming that $\mathbf{q} = (1 + \gamma)\mathbf{p}$.

$$\mathbf{V}(\mathbf{p}, m) = 1 - e^{-\frac{1}{E\mathbf{p}}\left\{\alpha m + E(\mathbf{p} Log \mathbf{p})\right\}} E\mathbf{p} \tag{3.3.38}$$

$$\mathbf{E}(\mathbf{p}, u) = \frac{1}{\alpha}\left\{E\mathbf{p} Log E\mathbf{p} - E\mathbf{p} Log(1-u) - E(\mathbf{p} Log \mathbf{p})\right\} \tag{3.3.39}$$

$$\mu(\mathbf{p}, \mathbf{q}, m) = \frac{m}{\alpha}\left\{\frac{E\mathbf{p}}{E\mathbf{q}} + E\mathbf{p} Log \frac{E\mathbf{p}}{E\mathbf{q}} + \frac{E\mathbf{p}}{E\mathbf{q}} E(\mathbf{q} Log \mathbf{q}) - E(\mathbf{p} Log \mathbf{p})\right\} \tag{3.3.40}$$

$$\mathbf{h}(\mathbf{p}, u) = \frac{1}{\alpha} Log\left[\frac{E\mathbf{p}}{(1-u)\mathbf{p}}\right] \tag{3.3.41}$$

$$R = \frac{1}{\alpha} cov(Log \mathbf{p}, \mathbf{p}) \tag{3.3.42}$$

$$R^{A} = \frac{1}{\alpha m} cov(Log \mathbf{p}, \mathbf{p}) \tag{3.3.43}$$

One can verify that the risk premium is indeed positive by using the concavity of the Log function, the convexity of the function $t \to t\, Log\, t$, and Jensen's inequality. The Arrow-Pratt risk premium (evaluated at the compensated demand) is

$$\pi = -\frac{1}{\alpha}\left(E Log \mathbf{p} - Log E\mathbf{p}\right) \tag{3.3.44}$$

which is also strictly positive by Jensen's inequality. Both risk premiums are independent of expenditure, hence of income. All the comments made in the CRRA case apply equally for the CARA. Again for completeness, the risk premium index and the cost-of-risk index as defined in chapter 2 are respectively

$$Q_A^r \equiv \frac{\mathbf{R}}{m} = \frac{1}{\alpha m} cov(Log\mathbf{p}\,,\,\mathbf{p}) \tag{3.3.45}$$

and

$$P_M^r \equiv \frac{m'\mathbf{R}}{\mu(\mathbf{p},\mathbf{q},m')(E\mathbf{x}(\mathbf{p},\mu(\mathbf{p},\mathbf{q},m'))E\mathbf{p})} = \frac{\alpha(1+\gamma)\,cov(Log\mathbf{p}\,,\,\mathbf{p})}{\frac{m'}{\gamma+1} + E\mathbf{p}Log\mathbf{p} - E(\mathbf{p})Log\mathbf{p}} \tag{3.3.46}$$

Finally, one can verify that all the expressions above reduce to their deterministic counterparts when no uncertainty is assumed. The elasticity measures for the CRRA will be calculated in the more general and interesting multivariate context.

3.3.3 Examples of multivariate expected utility functionals

3.3.3.1 Multivariate generalizations of the (CRRA) expected utility

In a multivariate context, Cobb-Douglas and log-linear functional forms for **u** are the natural generalization of the univariate utility with a constant relative Arrow-Pratt risk aversion coefficient (CRRA).

3.3.3.1.1 Demand functionals for the Cobb-Douglas expected utility

Let **u** be defined by

$$\mathbf{u}(\mathbf{x}) = \prod_{j=1}^{n} \mathbf{x}_j^{\alpha_j} \quad \text{with } \alpha_j \geq 0 \text{ and } \alpha \equiv \sum_{j=1}^{n} \alpha_j \leq 1 \tag{3.3.47}$$

Then, $\mathbf{u}_i' \circ \mathbf{x} = \alpha_i \mathbf{x}_i^{\alpha_i - 1} \prod_{j \neq i}^{n} \mathbf{x}_j^{\alpha_j}$ and the optimality conditions (3.3.7) and (3.3.9) reduce respectively to:

$$\frac{\alpha_i \mathbf{x}_j}{\alpha_j \mathbf{x}_i} = \frac{\mathbf{p}_i}{\mathbf{p}_j} \qquad i,j = 1,\ldots,n \tag{3.3.48}$$

$$\alpha_i x_i^{\alpha_i - 1} \prod_{j \neq i} x_j^{\alpha_j} = \frac{\alpha}{m} V(\mathbf{p}, m) \mathbf{p}_i \tag{3.3.49}$$

which by substituting (3.3.48) into (3.3.49) reduce to

$$x_i^{\alpha - 1} = \frac{\alpha}{m} \left[\frac{\alpha_i}{\mathbf{p}_i} \right]^{\alpha - 1} \left(\Upsilon(\mathbf{p}) \right)^{-1} V(\mathbf{p}, m) \quad i = 1, \ldots, n. \tag{3.3.50}$$

where $\alpha \equiv \sum_{j=1}^{n} \alpha_j$ and $\Upsilon(\mathbf{p}) \equiv \prod_{j=1}^{n} \left[\frac{\alpha_j}{\mathbf{p}_j} \right]^{\alpha_j}$.

At this point there are two cases to consider: the homothetic case $\alpha = 1$, and the non homothetic case $\alpha < 1$. Unlike in deterministic consumer theory, here the two cases lead to qualitatively different results.

Case 1: $\alpha = 1$.

In this case, (3.3.50) can be solved for the indirect utility functional to get:

$$V(\mathbf{p}, m) = m \Upsilon(\mathbf{p}) \tag{3.3.51}$$

We note that, since V takes its values in **R**, (3.3.51) implies that the nonlinear function of price $\Upsilon(\mathbf{p})$ is a non random scalar. That is, it has a degenerate probability distribution. In other words, the expected utility consumer believes that this non linear function of prices is independent of the state of nature. This result can be seen as a generalization of the univariate risk neutrality case.

The partial derivatives of the indirect utility functional are given by:

$$\frac{\partial V(\mathbf{p}, m)}{\partial \mathbf{p}_i} = -\frac{\alpha_i m}{\mathbf{p}_i} \prod_{j=1}^{n} \left[\frac{\alpha_j}{\mathbf{p}_j} \right]^{\alpha_j} = -\frac{\alpha_i m \Upsilon(\mathbf{p})}{\mathbf{p}_i} \quad i = 1, \ldots, n \tag{3.3.52}$$

and

$$\frac{\partial \mathbf{V}(\mathbf{p},m)}{\partial m} = \prod_{j=1}^{n} \left[\frac{\alpha_j}{\mathbf{p}_j}\right]^{\alpha_j} = \Upsilon(\mathbf{p}) \tag{3.3.53}$$

Hence, since $\frac{\partial \Upsilon(\mathbf{p})}{\partial \mathbf{p}_i} = -\frac{\alpha_i}{\mathbf{p}_i}\Upsilon(\mathbf{p})$, by Roy's identity we get the n Marshallian demand functionals:

$$\mathbf{x}_i(\mathbf{p},m) = \frac{\alpha_i m}{\mathbf{p}_i} \qquad i = 1,\dots,n \tag{3.3.54}$$

The above formula are similar to the ones in the deterministic case except the fact that here, $\mathbf{x}_i$ and $\mathbf{p}_i$ are random variables. In addition, the demand functions in this case are independent of the consumer's "beliefs" and "expectations" represented respectively by $\mathcal{P}$ and E.

Using the fact that $\mathbf{V}(\mathbf{p}, \mathbf{E}(\mathbf{p},u)) = u$, we deduce the expenditure and compensation functionals:

$$\mathbf{E}(\mathbf{p},u) = \frac{u}{\Upsilon(\mathbf{p})} \tag{3.3.55}$$

$$\mu(\mathbf{p},\mathbf{q},m) = \mathbf{E}(\mathbf{p}, \mathbf{V}(\mathbf{q},m)) = m\frac{\Upsilon(\mathbf{q})}{\Upsilon(\mathbf{p})} \tag{3.3.56}$$

Finally, the Hicksian or compensating demand functionals are:

$$\mathbf{h}_i(\mathbf{p},u) = \frac{\partial \mathbf{E}(\mathbf{p},u)}{\partial \mathbf{p}_i} = \frac{\alpha_i u}{\Upsilon(\mathbf{p})\,\mathbf{p}_i} \qquad i = 1,\dots,n \tag{3.3.57}$$

We note that all the above functionals are independent of the subjective probability distribution $\mathcal{P}$. In particular, the consumer's welfare is independent of his or her "beliefs" and "expectations".

The total, *precautionary* risk, and nonrisk elasticities

From the expression of the Marshallian demand in (3.3.53), we get:

$$\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_i} = -\frac{\alpha_i m}{\mathbf{p}_i^2}, \quad \frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j} = 0 \quad \text{for } i,j=1,\ldots,n \text{ and } i \neq j \tag{3.3.58}$$

and

$$\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial m} = \frac{\alpha_i}{\mathbf{p}_i} \quad i=1,\ldots,n \tag{3.3.59}$$

Hence, the uncompensated *precautionary* risk elasticities are:

$$\epsilon_{ij}^r \equiv \frac{1}{\sqrt{E\mathbf{x}_i^2} \times \sqrt{E\mathbf{p}_j^2}} \times cov\left[\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j}\cdot \mathbf{p}_j\ ,\ \mathbf{p}_j\right] = 0 \quad \text{for } i \neq j \tag{3.3.60}$$

and

$$\epsilon_{ii}^r \equiv \frac{1}{\sqrt{E\mathbf{x}_i^2} \times \sqrt{E\mathbf{p}_i^2}} \times cov\left[\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_i}\cdot \mathbf{p}_i\ ,\ \mathbf{p}_i\right] = \frac{1}{\sqrt{E\mathbf{x}_i^2} \times \sqrt{E\mathbf{p}_i^2}} \times cov\left[\frac{-\alpha_i m}{\mathbf{p}_i^2}\mathbf{p}_i\ ,\ \mathbf{p}_i\right]$$
$$= \frac{-\alpha_i m}{\alpha_i m\sqrt{E\left[\frac{1}{\mathbf{p}_i^2}\right]} \times \sqrt{E\mathbf{p}_i^2}} \times cov\left[\frac{1}{\mathbf{p}_i}\ ,\ \mathbf{p}_i\right] \tag{3.3.61}$$

Hence, after simplification we have:

$$\epsilon_{ii}^r = -\left(1 - E\left[\frac{1}{\mathbf{p}_i}\right]E\mathbf{p}_i\right)\left(E\left[\frac{1}{\mathbf{p}_i^2}\right]E\mathbf{p}_i^2\right)^{-\frac{1}{2}} \quad i=1,\ldots,n \tag{3.3.62}$$

The nonrisk uncompensated demand elasticities are:

$$\epsilon^{t}_{ij} \equiv \frac{1}{\sqrt{E\mathbf{x}_i^2} \times \sqrt{E\mathbf{p}_j^2}} \times E\left[\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j}\cdot \mathbf{p}_j\right] E\mathbf{p}_j = 0 \qquad \text{for } i \neq j \tag{3.3.63}$$

$$\epsilon^{t}_{ii} \equiv \frac{1}{\sqrt{E\mathbf{x}_i^2} \times \sqrt{E\mathbf{p}_i^2}} \times E\left[\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_i}\cdot \mathbf{p}_i\right] E\mathbf{p}_i = \frac{1}{\sqrt{E\mathbf{x}_i^2} \times \sqrt{E\mathbf{p}_i^2}} \times E\left[\frac{-\alpha_i m}{\mathbf{p}_i^2}\mathbf{p}_i\right] E\mathbf{p}_i \tag{3.3.64}$$

which after substitution and simplification gives:

$$\epsilon^{t}_{ii} = -\frac{E\left[\frac{1}{\mathbf{p}_i}\right] E\mathbf{p}_i}{\sqrt{E\left[\frac{1}{\mathbf{p}_i^2}\right] E\mathbf{p}_i^2}} \qquad i = 1, \ldots, n \tag{3.3.65}$$

The total uncompensated price elasticities are hence:

$$\epsilon_{ij} = \epsilon^{r}_{ij} + \epsilon^{t}_{ij} = 0 \qquad \text{for } i \neq j \tag{3.3.66}$$

and

$$\epsilon_{ii} = \epsilon^{r}_{ii} + \epsilon^{t}_{ii} = -\frac{1}{\sqrt{E\left[\frac{1}{\mathbf{p}_i^2}\right] E\mathbf{p}_i^2}} \qquad i = 1, \ldots, n \tag{3.3.67}$$

One can verify that with no uncertainty $\epsilon^{r}_{ii} = 0$ and $\epsilon^{t}_{ii} = \epsilon_{ii} = -1$, which is indeed the value found with the Cobb-Douglas utility in deterministic consumer theory. But, in general $\epsilon_{ii} \neq -1$, except in very special cases where $E\left[\frac{1}{\mathbf{p}_i^2}\right] = \frac{1}{E\mathbf{p}_i^2}$.

For the income and *precautionary* risk aversion effects, we have:

$$\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial m} \circ \mathbf{x}_j(\mathbf{p},m)\cdot \mathbf{p}_j = \frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial m}\cdot\left(\mathbf{x}_j(\mathbf{p},m)\cdot \mathbf{p}_j\right)$$

$$= \frac{\alpha_i}{\mathbf{p}_i} \times \frac{\alpha_j m}{\mathbf{p}_j}\mathbf{p}_j \tag{3.3.68}$$

$$= \frac{\alpha_i \alpha_j m}{\mathbf{p}_i} \qquad \text{for } i,j=1,\ldots,n$$

Hence, in terms of elasticities the *precautionary* risk aversion and pure income effects are respectively:

$$\epsilon_{ij}^{rm} \equiv -\frac{1}{\sqrt{E\mathbf{x}_i^2} \times \sqrt{E\mathbf{p}_j^2}} \times cov\left[\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j} \circ \mathbf{x}_j(\mathbf{p},m)\cdot \mathbf{p}_j \ , \ \mathbf{p}_j\right]$$

$$= -\alpha_j \left[E\left(\frac{\mathbf{p}_j}{\mathbf{p}_i}\right) - E\left(\frac{1}{\mathbf{p}_i}\right)E\mathbf{p}_j\right]\left[E\left(\frac{1}{\mathbf{p}_i^2}\right)E\mathbf{p}_j^2\right]^{-\frac{1}{2}} \qquad i,j=1,\ldots,n \tag{3.3.69}$$

and

$$\epsilon_{ij}^{tm} \equiv -\frac{1}{\sqrt{E\mathbf{x}_i^2} \times \sqrt{E\mathbf{p}_j^2}} \times E\left[\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j} \circ \mathbf{x}_j(\mathbf{p},m)\cdot \mathbf{p}_j\right]E\mathbf{p}_j = -\alpha_j \frac{E\left(\frac{1}{\mathbf{p}_i}\right)E\mathbf{p}_j}{\sqrt{E\left(\frac{1}{\mathbf{p}_i^2}\right)E\mathbf{p}_j^2}} \qquad i,j=1,\ldots,n \tag{3.3.70}$$

The total income/risk aversion effects are then:

$$\epsilon_{ij}^{m} = \epsilon_{ij}^{rm} + \epsilon_{ij}^{tm} = -\alpha_j E\left(\frac{\mathbf{p}_j}{\mathbf{p}_i}\right)\left[E\left(\frac{1}{\mathbf{p}_i^2}\right)E\mathbf{p}_j^2\right]^{-\frac{1}{2}} \qquad i,j=1,\ldots,n \tag{3.3.71}$$

For the substitution elasticities, one can either subtract the income effects from the uncompensated elasticities or directly take the partial derivatives of the Hicksian demands at (**p**, **V**(**p**, m)). In either case, the *precautionary* risk and taste substitution elasticities are respectively:

$$\epsilon_{ij}^{rs} = \epsilon_{ij}^{r} - \epsilon_{ij}^{rm} = \begin{cases} \alpha_j \left[E\left[\frac{\mathbf{p}_j}{\mathbf{p}_i}\right] - E\left[\frac{1}{\mathbf{p}_i}\right] E\mathbf{p}_j \right] \left[E\left[\frac{1}{\mathbf{p}_i^2}\right] E\mathbf{p}_j^2 \right]^{-\frac{1}{2}} & i \neq j \\ (\alpha_i - 1)\left[1 - E\left[\frac{1}{\mathbf{p}_i}\right] E\mathbf{p}_i \right] \left[E\left[\frac{1}{\mathbf{p}_i^2}\right] E\mathbf{p}_i^2 \right]^{-\frac{1}{2}} & i = 1,\ldots,n \end{cases} \tag{3.3.72}$$

and

$$\epsilon_{ij}^{ts} = \epsilon_{ij}^{t} - \epsilon_{ij}^{tm} = \begin{cases} \alpha_j E\left[\frac{1}{\mathbf{p}_i}\right] E\mathbf{p}_j \left[E\left[\frac{1}{\mathbf{p}_i^2}\right] E\mathbf{p}_j^2 \right]^{-\frac{1}{2}} & i,j = 1,\ldots,n \text{ and } i \neq j \\ (\alpha_i - 1) E\left[\frac{1}{\mathbf{p}_i}\right] E\mathbf{p}_i \left[E\left[\frac{1}{\mathbf{p}_i^2}\right] E\mathbf{p}_i^2 \right]^{-\frac{1}{2}} & i = 1,\ldots,n \end{cases} \tag{3.3.73}$$

Hence, the total substitution elasticities are:

$$\epsilon_{ij}^{s} = \epsilon_{ij} - \epsilon_{ij}^{m} = \epsilon_{ij}^{rs} + \epsilon_{ij}^{ts} = \begin{cases} \alpha_j E\left[\frac{\mathbf{p}_j}{\mathbf{p}_i}\right] \left[E\left[\frac{1}{\mathbf{p}_i^2}\right] E\mathbf{p}_j^2 \right]^{-\frac{1}{2}} & i,j = 1,\ldots,n \text{ and } i \neq j \\ (\alpha_i - 1) \left[E\left[\frac{1}{\mathbf{p}_i^2}\right] E\mathbf{p}_i^2 \right]^{-\frac{1}{2}} & i = 1,\ldots,n \end{cases} \tag{3.3.74}$$

Hence, as predicted by the general theory, the total own-price substitution effects, ϵ_{ii}^{s} $i = 1,\ldots,n$, are negative. But, there are several features implied by the Cobb-Douglas homothetic expected utility form that can be noted. First, as in the deterministic case, the own-price taste substitution elasticities are negative; however, because of Jensen's inequality, the own-price *precautionary* risk substitution elasticities are positive. This means that even when the consumer is compensated for the possible loss of welfare due to risk aversion and to an *expected* increase in the price of a commodity, his or her *precaution* or *prudence* induces him or her not to decrease his or her *ex-ante* demand of that commodity by the full amount implied by the own-price taste substitution elasticity. In other words, the *precautionary* *motive* dampens the substitution effect.[93] The same positive *precautionary* risk substitution effect is found by Drèze and Modigliani in the two-period consumption choice case with one aggregated commodity and second period income uncertainty, under the assumption of decreasing Arrow-Pratt risk aversion coefficient (see also Kimball,1990).

Second, also as in the deterministic case, all commodities are taste substitutes, while the cross-price *precautionary* risk substitution effects can be either positive or negative. However, they are all substitutes in total (i.e., when one adds the *precautionary* risk and taste substitution effects). Third, still as in the deterministic case, the pure income effects are negative, while the *precautionary* risk aversion effects can be either positive or negative except for $i=j$, in which case it is positive. Furthermore, Schwarz's inequality implies that the total income/risk aversion effects are between -1 and 0. Again, the positivity of the *precautionary* risk aversion effect was also found by Drèze and Modigliani. This means that when the price of a commodity is *expected* to increase, risk aversion alone induces the consumer to take some *precaution* by decreasing his or her *ex-ante* demand of that commodity by less than the amount implied by the decrease in "real income". Moreover, although there is no <u>taste</u> Giffen good (i.e., $\epsilon_{ii} < 0$), all the uncompensated *precautionary* risk effects are positive. This implies that when the consumer *expects* the price of a commodity to increase, the *precautionary* risk substitution across possible states of nature and risk aversion effects work in the same direction to dampen the overall decrease in his or her *ex-ante* demand for that commodity which would have occurred if he or she were indifferent to uncertainty. Finally, one can verify that with no uncertainty, all the risk elasticities are zero and the taste

[93] Note that because the total own price substitution effect is always negative, the *precautionary* risk effect must be less than the absolute value of the own-price taste substitution effect.

elasticities reduce to their respective deterministic analogues; but, in general the magnitudes are respectively different.

Finally, the income (or more precisely budget) elasticities for the Cobb-Douglas expected utility are:

$$\eta_i \equiv \frac{m}{\sqrt{E\mathbf{x}_i^2}} \times \frac{\partial \mathbf{x}_i(\mathbf{p}, m)}{\partial m} = \frac{1}{\mathbf{p}_i} \times \left[E\left(\frac{1}{\mathbf{p}_i^2} \right) \right]^{-\frac{1}{2}} \qquad i = 1, \ldots, n \tag{3.3.75}$$

they are real valued random variable which reduce to 1, the value found in the deterministic case, if there is no uncertainty. Their means tell us that there are no inferior goods with a Cobb-Douglas homothetic expected utility.

Welfare indicators and risk premium

From the compensation functional given by (3.3.56) we get the equivalent variation (EV), the Allen quantity (living standard) index (Q_A), and the Mackenzie-Pearce (cost of living) index (P_M) respectively:

$$\mathbf{EV} \equiv \mu(\mathbf{p}, \mathbf{q}, m') - m = m' \frac{\Upsilon(\mathbf{q})}{\Upsilon(\mathbf{p})} - m \tag{3.3.76}$$

$$Q_A \equiv \frac{\mu(\mathbf{p}, \mathbf{q}, m')}{m} = \frac{m'}{m} \frac{\Upsilon(\mathbf{q})}{\Upsilon(\mathbf{p})} \tag{3.3.77}$$

$$P_M \equiv \frac{m'}{\mu(\mathbf{p}, \mathbf{q}, m')} = \frac{\Upsilon(\mathbf{p})}{\Upsilon(\mathbf{q})} \tag{3.3.78}$$

The i[th] directional risk premium is given by:

$$R_i \equiv -cov\left[\mathbf{x}_i(\mathbf{p}, \mu(\mathbf{p}, \mathbf{q}, m'), \mathbf{p}_i\right] = -\alpha_i m' \frac{\Upsilon(\mathbf{q})}{\Upsilon(\mathbf{p})} cov\left[\frac{1}{\mathbf{p}_i}, \mathbf{p}_i\right]$$
$$= -\alpha_i m' \frac{\Upsilon(\mathbf{q})}{\Upsilon(\mathbf{p})}\left(1 - E\left(\frac{1}{\mathbf{p}_i}\right) E\mathbf{p}_i\right) > 0 \tag{3.3.79}$$

Hence, the total risk premium is:

$$R = \sum_{i=1}^{n} R_i = -m' \frac{\Upsilon(\mathbf{q})}{\Upsilon(\mathbf{p})} \sum_{i=1}^{n} \alpha_i \left(1 - E\left(\frac{1}{\mathbf{p}_i}\right) E\mathbf{p}_i\right) > 0 \tag{3.3.80}$$

The positivity of the risk premiums, which is predicted by the general theory, can be shown directly by using Jensen's inequality. Normalizing by the base expenditure *m*, gives the risk premium index which is also the risk component in the in the Allen quantity index Q_A:

$$Q_A^r \equiv \frac{R}{m} = -\frac{m'}{m} \frac{\Upsilon(\mathbf{q})}{\Upsilon(\mathbf{p})} \sum_{i=1}^{n} \alpha_i \left(1 - E\left(\frac{1}{\mathbf{p}_i}\right) E\mathbf{p}_i\right) \tag{3.3.81}$$

Similarly, the cost-of-risk index, which is the risk component in the Mackenzie-Pearce price index is given by :

$$P_M^r \equiv \frac{m'R}{\mu(\mathbf{p}, \mathbf{q}, m')\left(E\mathbf{x}(\mathbf{p}, \mu(\mathbf{p}, \mathbf{q}, m'))\cdot E\mathbf{p}\right)} = -\frac{\Upsilon(\mathbf{p})}{\Upsilon(\mathbf{q})} \left\{\sum_{i=1}^{n} \alpha_i E\left(\frac{1}{\mathbf{p}_i}\right) E\mathbf{p}_i\right\}^{-1} \sum_{i=1}^{n} \alpha_i cov\left[\frac{1}{\mathbf{p}_i}, \mathbf{p}_i\right] \tag{3.3.82}$$

The $(i,j)^{th}$ element of the matrix of risk aversion, R^A, is given by:

$$R_{ij}^A \equiv -\frac{1}{m} cov\left[\mathbf{x}_i(\mathbf{p}, \mu(\mathbf{p}, \mathbf{q}, m'), \mathbf{p}_j\right] = -\frac{m'}{m} \frac{\Upsilon(\mathbf{q})}{\Upsilon(\mathbf{p})} \alpha_i cov\left[\frac{1}{\mathbf{p}_i}, \mathbf{p}_j\right]$$
$$= -\frac{m'}{m} \frac{\Upsilon(\mathbf{q})}{\Upsilon(\mathbf{p})} \alpha_i \left(E\left(\frac{\mathbf{p}_j}{\mathbf{p}_i}\right) - E\left(\frac{1}{\mathbf{p}_i}\right) E\mathbf{p}_j\right) \tag{3.3.83}$$

We recall that tr $R^A = Q_A^r$. But we could not determine whether or not R^A is a positive definite matrix.

The nonrisk components in the equivalent variation, Allen quantity index, and the Mackenzie-Pearce price index are respectively:

$$T = E\mathbf{x}(\mathbf{p}, \mu(\mathbf{p},\mathbf{q},m')) \cdot E\mathbf{p} - m = m' \frac{\Upsilon(\mathbf{q})}{\Upsilon(\mathbf{p})} \sum_{i=1}^{n} E\left[\frac{1}{\mathbf{p}_i}\right] E\mathbf{p}_i - m \tag{3.3.84}$$

$$Q_A' \equiv \frac{T}{m} + 1 = \frac{m'}{m} \frac{\Upsilon(\mathbf{q})}{\Upsilon(\mathbf{p})} \sum_{i=1}^{n} \alpha_i E\left[\frac{1}{\mathbf{p}_i}\right] E\mathbf{p}_i \tag{3.3.85}$$

$$P_M' \equiv \frac{m'}{T+m} = \frac{\dfrac{\Upsilon(\mathbf{p})}{\Upsilon(\mathbf{q})}}{\sum_{i=1}^{n} \alpha_i E\left[\dfrac{1}{\mathbf{p}_i}\right] E\mathbf{p}_i} \tag{3.3.86}$$

Case 2: $\alpha < 1$.

In this case, (3.3.50) can be solved for $\mathbf{x}_i$ to get

$$\mathbf{x}_i = \frac{\alpha_i}{\mathbf{p}_i} \left[\frac{\alpha}{m} (\Upsilon(\mathbf{p}))^{-1} \mathbf{V}(\mathbf{p}, m)\right]^{\frac{1}{\alpha - 1}} \quad i = 1, \ldots, n. \tag{3.3.87}$$

Taking the scalar product of both sides of (3.3.87) with $\mathbf{p}_i$, and summing across i gives

$$m = \sum_{i=1}^{n} \langle \mathbf{p}_i, \mathbf{x}_i \rangle = \sum_{i=1}^{n} E\left\{\frac{\mathbf{p}_i \alpha_i}{\mathbf{p}_i} \left[\frac{\alpha}{m} (\Upsilon(\mathbf{p}))^{-1} \mathbf{V}(\mathbf{p}, m)\right]^{\frac{1}{\alpha - 1}}\right\} \tag{3.3.88}$$

solving (3.3.88) for the indirect utility functional $\mathbf{V}$ yields

$$\mathbf{V}(\mathbf{p},m) = \left[\frac{m}{\alpha}\right]^{\alpha}\left[E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)\right]^{1-\alpha} \tag{3.3.89}$$

Substituting back into (3.3.87) gives the Marshallian demands

$$\mathbf{x}_i(\mathbf{p},m) = \frac{\alpha_i m \Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}}{\alpha \mathbf{p}_i E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)} \qquad i = 1,\ldots,n \tag{3.3.90}$$

In this non-homothetic case, $\Upsilon(\mathbf{p})$ is not degenerate. Hence, the Marshallian demands depend on the consumer's "beliefs" and "expectations". Moreover, each individual demand depends on the entire price vector $\mathbf{p}$ through $\Upsilon(\mathbf{p})$. This is in sharp contrast with both the homothetic and the deterministic cases for which the (uncompensated) cross-price elasticities are always zero.

From (3.3.89) and (3.3.90) follows easily the expenditure, compensation, and Hicksian demand functional respectively:

$$\mathbf{E}(\mathbf{p},u) = \alpha u^{\frac{1}{\alpha}}\left[E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)\right]^{\frac{\alpha-1}{\alpha}} \tag{3.3.91}$$

$$\mu(\mathbf{p},\mathbf{q},m) = m\left[\frac{E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)}{E\left(\Upsilon(\mathbf{q})^{\frac{1}{1-\alpha}}\right)}\right]^{\frac{\alpha-1}{\alpha}} \tag{3.3.92}$$

$$\mathbf{h}_i(\mathbf{p},u) = \frac{\alpha_i u^{\frac{1}{\alpha}} \Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}}{\alpha \mathbf{p}_i \left[E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)\right]^{\frac{1}{\alpha}}} \qquad i = 1,\ldots,n \tag{3.3.93}$$

They all depend on the consumer's "beliefs" and "expectations" as represented respectively by the subjective probability distribution $\mathcal{P}$ and the expectation operator E.

Total, nonrisk, and *precautionary* risk elasticities

From the expression of the Marshallian demand in (3.3.90), and the fact that $\frac{\partial \Upsilon(\mathbf{p})}{\partial \mathbf{p}_i} = -\frac{\alpha_i}{\mathbf{p}_i}\Upsilon(\mathbf{p})$ we get:

$$\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j} = \begin{cases} \dfrac{\alpha_i m \Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}}{\alpha \mathbf{p}_i^2 E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)} \left\{ \dfrac{\alpha_i \Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}}{(1-\alpha) E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)} - \dfrac{\alpha_i}{1-\alpha} - 1 \right\} & \text{if } i=j \\[2ex] \dfrac{\alpha_i \alpha_j m \Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}}{\alpha(1-\alpha)\mathbf{p}_i \mathbf{p}_j E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)} \left\{ \dfrac{\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}}{E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)} - 1 \right\} & \text{if } i \neq j \end{cases} \tag{3.3.94}$$

and

$$\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial m} \circ \mathbf{x}_j \cdot \mathbf{p}_j = \frac{\alpha_i \alpha_j m}{\alpha^2 \mathbf{p}_i} \Upsilon(\mathbf{p})^{\frac{2}{1-\alpha}} \left[E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)\right]^{-2} \tag{3.3.95}$$

Hence, the *precautionary* risk aversion and pure income effects are:

$$\epsilon_{ij}^{mr} = -\frac{\alpha_j}{\alpha} E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right) cov\left[\frac{\Upsilon(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i}, \mathbf{p}_j\right] \left[E\mathbf{p}_j^2 E\left[\frac{\Upsilon(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i^2}\right]\right]^{-\frac{1}{2}} \tag{3.3.96}$$

$$\epsilon_{ij}^{mt} = -\frac{\alpha_j}{\alpha} E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right) \left[E\mathbf{p}_j^2 E\left[\frac{\Upsilon(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i^2}\right]\right]^{-\frac{1}{2}} E\left[\frac{\Upsilon(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i}\right] E\mathbf{p}_j \tag{3.3.97}$$

Hence the total income/risk aversion effects are:

$$\epsilon_{ij}^{m} = \epsilon_{ij}^{mr} + \epsilon_{ij}^{mt} = -\frac{\alpha_j}{\alpha} E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right) \left[E\mathbf{p}_j^2 E\left[\frac{\Upsilon(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i^2}\right]\right]^{-\frac{1}{2}} E\left[\frac{\Upsilon(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i}\mathbf{p}_j\right] \tag{3.3.98}$$

Similarly, the uncompensated *precautionary* risk, nonrisk and total elasticities are:

$$\epsilon^{r}_{ij} = \begin{cases} \left[E\mathbf{p}_i^2 E\left[\dfrac{\Upsilon(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i^2} \right] \right]^{-\frac{1}{2}} \left\{ \dfrac{\alpha_i cov\left[\dfrac{\Upsilon(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i}, \mathbf{p}_i \right]}{(1-\alpha)E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)} - \left[1 + \dfrac{\alpha_i}{1-\alpha} \right] cov\left[\dfrac{\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}}{\mathbf{p}_i}, \mathbf{p}_i \right] \right\} & \text{if } i=j \\ \dfrac{\alpha_j}{(1-\alpha)} \left[E\mathbf{p}_j^2 E\left[\dfrac{\Upsilon(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i^2} \right] \right]^{-\frac{1}{2}} \left\{ \dfrac{cov\left[\dfrac{\Upsilon(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i}, \mathbf{p}_j \right]}{E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)} - cov\left[\dfrac{\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}}{\mathbf{p}_i}, \mathbf{p}_j \right] \right\} & \text{if } i \neq j \end{cases} \qquad \textbf{(3.3.99)}$$

$$\epsilon^{t}_{ij} = \begin{cases} \left[E\mathbf{p}_i^2 E\left[\dfrac{\Upsilon(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i^2} \right] \right]^{-\frac{1}{2}} \left\{ \dfrac{\alpha_i E\left[\dfrac{\Upsilon(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i} \right] E\mathbf{p}_i}{(1-\alpha)E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)} - \left[1 + \dfrac{\alpha_i}{1-\alpha} \right] E\left[\dfrac{\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}}{\mathbf{p}_i} \right] E\mathbf{p}_i \right\} & \text{if } i=j \\ \dfrac{\alpha_j}{(1-\alpha)} \left[E\mathbf{p}_j^2 E\left[\dfrac{\Upsilon(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i^2} \right] \right]^{-\frac{1}{2}} \left\{ \dfrac{E\left[\dfrac{\Upsilon(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i} \right] E\mathbf{p}_j}{E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)} - E\left[\dfrac{\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}}{\mathbf{p}_i} \right] E\mathbf{p}_j \right\} & \text{if } i \neq j \end{cases} \qquad \textbf{(3.3.100)}$$

$$\epsilon_{ij} = \epsilon^{r}_{ij} + \epsilon^{mr}_{ij} = \begin{cases} \left[E\mathbf{p}_i^2 E\left[\dfrac{\Upsilon(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i^2} \right] \right]^{-\frac{1}{2}} \left\{ \dfrac{\alpha_i E\left(\Upsilon(\mathbf{p})^{\frac{2}{1-\alpha}}\right)}{(1-\alpha)E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)} - \left[1 + \dfrac{\alpha_i}{1-\alpha} \right] E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right) \right\} & \text{if } i=j \\ \dfrac{\alpha_j}{(1-\alpha)} \left[E\mathbf{p}_j^2 E\left[\dfrac{\Upsilon(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i^2} \right] \right]^{-\frac{1}{2}} \left\{ \dfrac{E\left[\dfrac{\Upsilon(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i} \mathbf{p}_j \right]}{E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)} - E\left[\dfrac{\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}}{\mathbf{p}_i} \mathbf{p}_j \right] \right\} & \text{if } i \neq j \end{cases} \qquad \textbf{(3.3.101)}$$

The substitution elasticities are then:

$$
\epsilon_{ij}^{sr} = \epsilon_{ij}^{r} - \epsilon_{ij}^{mr} = \begin{cases} \left[E\mathbf{p}_i^2 E\left[\dfrac{T(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i^2}\right]\right]^{-\frac{1}{2}} \left\{ \dfrac{\alpha_i cov\left[\dfrac{T(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i}, \mathbf{p}_i\right]}{\alpha(1-\alpha)E\left(T(\mathbf{p})^{\frac{1}{1-\alpha}}\right)} - \left[1+\dfrac{\alpha_i}{1-\alpha}\right] cov\left[\dfrac{T(\mathbf{p})^{\frac{1}{1-\alpha}}}{\mathbf{p}_i}, \mathbf{p}_i\right] \right\} & \\ \dfrac{\alpha_j}{(1-\alpha)} \left[E\mathbf{p}_j^2 E\left[\dfrac{T(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i^2}\right]\right]^{-\frac{1}{2}} \left\{ \dfrac{cov\left[\dfrac{T(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i}, \mathbf{p}_j\right]}{\alpha E\left(T(\mathbf{p})^{\frac{1}{1-\alpha}}\right)} - cov\left[\dfrac{T(\mathbf{p})^{\frac{1}{1-\alpha}}}{\mathbf{p}_i}, \mathbf{p}_j\right] \right\} & \text{if } i \neq j \end{cases} \quad \text{(3.3.102)}
$$

$$
\epsilon_{ij}^{st} = \epsilon_{ij}^{t} - \epsilon_{ij}^{mt} = \begin{cases} \left[E\mathbf{p}_i^2 E\left[\dfrac{T(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i^2}\right]\right]^{-\frac{1}{2}} \left\{ \dfrac{\alpha_i E\left[\dfrac{T(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i}\right] E\mathbf{p}_i}{\alpha(1-\alpha)E\left(T(\mathbf{p})^{\frac{1}{1-\alpha}}\right)} - \left[1+\dfrac{\alpha_i}{1-\alpha}\right] E\left[\dfrac{T(\mathbf{p})^{\frac{1}{1-\alpha}}}{\mathbf{p}_i}\right] E\mathbf{p}_i \right\} & \\ \dfrac{\alpha_j}{(1-\alpha)} \left[E\mathbf{p}_j^2 E\left[\dfrac{T(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i^2}\right]\right]^{-\frac{1}{2}} \left\{ \dfrac{E\left[\dfrac{T(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i}\right] E\mathbf{p}_j}{\alpha E\left(T(\mathbf{p})^{\frac{1}{1-\alpha}}\right)} - E\left[\dfrac{T(\mathbf{p})^{\frac{1}{1-\alpha}}}{\mathbf{p}_i}\right] E\mathbf{p}_j \right\} & \text{if } i \neq j \end{cases} \quad \text{(3.3.103)}
$$

and

$$
\epsilon_{ij}^{s} = \epsilon_{ij}^{rs} + \epsilon_{ij}^{ts} = \begin{cases} \left[E\mathbf{p}_i^2 E\left[\dfrac{T(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i^2}\right]\right]^{-\frac{1}{2}} \left\{ \dfrac{\alpha_i E\left(T(\mathbf{p})^{\frac{2}{1-\alpha}}\right)}{\alpha(1-\alpha)E\left(T(\mathbf{p})^{\frac{1}{1-\alpha}}\right)} - \left[1+\dfrac{\alpha_i}{1-\alpha}\right] E\left(T(\mathbf{p})^{\frac{1}{1-\alpha}}\right) \right\} & \\ \dfrac{\alpha_j}{(1-\alpha)} \left[E\mathbf{p}_j^2 E\left[\dfrac{T(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i^2}\right]\right]^{-\frac{1}{2}} \left\{ \dfrac{E\left[\dfrac{T(\mathbf{p})^{\frac{2}{1-\alpha}}}{\mathbf{p}_i}\mathbf{p}_j\right]}{\alpha E\left(T(\mathbf{p})^{\frac{1}{1-\alpha}}\right)} - E\left[\dfrac{T(\mathbf{p})^{\frac{1}{1-\alpha}}}{\mathbf{p}_i}\mathbf{p}_j\right] \right\} & \text{if } i \neq j \end{cases} \quad \text{(3.3.104)}
$$

It is difficult to determine directly the sign of ϵ_{ii}^{s} although we know from the general theory that it is negative.

Finally, the expenditure elasticities for the nonhomothetic Cobb-Douglas expected utility are:

$$\eta_i \equiv \frac{m}{\sqrt{E\mathbf{x}_i^2}} \times \frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial m} = \frac{\dfrac{\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}}{\mathbf{p}_i}}{\sqrt{E\left[\dfrac{\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}}{\mathbf{p}_i}\right]^2}} \qquad i=1,\dots,n \tag{3.3.105}$$

they are real valued random variable which reduce to 1 if there is no uncertainty. Like in the previous case, their means tell us that there are no inferior goods.

Welfare indicators and risk premium

From the compensating functional given by (3.3.92) we get the equivalent variation (EV), the Allen quantity (living standard) index (Q_A), and the Mackenzie-Pearce (cost of living) index (P_M) respectively:

$$\mathbf{EV} \equiv \mu(\mathbf{p},\mathbf{q},m') - m = m'\left[\frac{E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)}{E\left(\Upsilon(\mathbf{q})^{\frac{1}{1-\alpha}}\right)}\right]^{\frac{\alpha-1}{\alpha}} - m \tag{3.3.106}$$

$$Q_A \equiv \frac{\mu(\mathbf{p},\mathbf{q},m')}{m} = \frac{m'}{m}\left[\frac{E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)}{E\left(\Upsilon(\mathbf{q})^{\frac{1}{1-\alpha}}\right)}\right]^{\frac{\alpha-1}{\alpha}} \tag{3.3.107}$$

$$P_M \equiv \frac{m'}{\mu(\mathbf{p},\mathbf{q},m')} = \left[\frac{E\left(\Upsilon(\mathbf{q})^{\frac{1}{1-\alpha}}\right)}{E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)}\right]^{\frac{\alpha-1}{\alpha}} \tag{3.3.108}$$

The total risk premium is:

$$R = \sum_{i=1}^{n} R_i = -\frac{m'}{\alpha}\left[\frac{E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)}{E\left(\Upsilon(\mathbf{q})^{\frac{1}{1-\alpha}}\right)}\right]^{\frac{\alpha-1}{\alpha}} \frac{\sum_{i=1}^{n}\alpha_i cov\left[\frac{\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}}{\mathbf{p}_i}, \mathbf{p}_i\right]}{E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)} > 0 \qquad (3.3.109)$$

The risk premium index is

$$Q_A^r \equiv \frac{R}{m} = -\frac{m'}{\alpha m}\left[\frac{E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)}{E\left(\Upsilon(\mathbf{q})^{\frac{1}{1-\alpha}}\right)}\right]^{\frac{\alpha-1}{\alpha}} \frac{\sum_{i=1}^{n}\alpha_i cov\left[\frac{\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}}{\mathbf{p}_i}, \mathbf{p}_i\right]}{E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)} \qquad (3.3.110)$$

Similarly, the cost-of-risk index is given by :

$$P_M^r \equiv \frac{R}{T+m}P_M = -E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)\left[\frac{E\left(\Upsilon(\mathbf{q})^{\frac{1}{1-\alpha}}\right)}{E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)}\right]^{\frac{\alpha-1}{\alpha}} \frac{\sum_{i=1}^{n}\alpha_i cov\left[\frac{\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}}{\mathbf{p}_i}, \mathbf{p}_i\right]}{\sum_{i=1}^{n}\alpha_i E\left[\frac{\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}}{\mathbf{p}_i}\right] E\mathbf{p}_i} \qquad (3.3.111)$$

The $(i,j)^{th}$ element of the matrix of risk aversion, R^A, is given by:

$$R_{ij}^A = -\frac{\alpha_i m'}{\alpha m}\left[\frac{E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)}{E\left(\Upsilon(\mathbf{q})^{\frac{1}{1-\alpha}}\right)}\right]^{\frac{\alpha-1}{\alpha}} \frac{cov\left[\frac{\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}}{\mathbf{p}_i}, \mathbf{p}_j\right]}{E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)} \qquad (3.3.112)$$

Risk aversion could not be determined.

The nonrisk components in the equivalent variation, Allen quantity index, and the Mackenzie-Pearce price index are respectively:

$$T = \frac{m'}{\alpha}\left[\frac{E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)}{E\left(\Upsilon(\mathbf{q})^{\frac{1}{1-\alpha}}\right)}\right]^{\frac{\alpha-1}{\alpha}}\frac{\sum_{i=1}^{n}\alpha_i E\left[\frac{\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}}{\mathbf{p}_i}\right]E\mathbf{p}_i}{E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)} - m \tag{3.3.113}$$

$$Q_A' \equiv \frac{T}{m} + 1 = \frac{m'}{\alpha m}\left[\frac{E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)}{E\left(\Upsilon(\mathbf{q})^{\frac{1}{1-\alpha}}\right)}\right]^{\frac{\alpha-1}{\alpha}}\frac{\sum_{i=1}^{n}\alpha_i E\left[\frac{\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}}{\mathbf{p}_i}\right]E\mathbf{p}_i}{E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)} \tag{3.3.114}$$

$$P_M' \equiv \frac{m'}{T+m} = \alpha\left[\frac{E\left(\Upsilon(\mathbf{q})^{\frac{1}{1-\alpha}}\right)}{E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)}\right]^{\frac{\alpha-1}{\alpha}}\frac{E\left(\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}\right)}{\sum_{i=1}^{n}\alpha_i E\left[\frac{\Upsilon(\mathbf{p})^{\frac{1}{1-\alpha}}}{\mathbf{p}_i}\right]E\mathbf{p}_i} \tag{3.3.115}$$

3.3.3.1.2 Demand functionals for the Log-linear expected utility

The multivariate log-linear generalization of the CRRA expected utility is obtained by taking the logarithm of the Cobb-Douglas cardinal utility. Hence, it is given by $\mathbf{U} \equiv E \circ \mathbf{u}$ where

$$\mathbf{u}(\mathbf{x}) = \sum_{j=1}^{n}\alpha_j Log\,\mathbf{x}_j \quad \text{with } \alpha_j \geq 0 \text{ and } \alpha \equiv \sum_{j=1}^{n}\alpha_j \leq 1 \tag{3.3.116}$$

We recall that since this is a non affine transformation of the Von Neumann-Morgenstern utility, the preference generated by this expected utility functional is different from the one generated by the Cobb-Douglas expected utility. What follows will tell us how different the two preferences are.

Since $\mathbf{u}_i^{\prime}\circ\mathbf{x} = \frac{\alpha_i}{\mathbf{x}_i}$, the optimality conditions (3.3.7) and (3.3.9) reduces respectively to:

$$\frac{\alpha_i \mathbf{x}_j}{\alpha_j \mathbf{x}_i} = \frac{\mathbf{p}_i}{\mathbf{p}_j} \qquad i,j = 1,\ldots,n \tag{3.3.117}$$

$$\frac{\alpha_i}{\mathbf{x}_i} = \frac{\alpha}{m}\mathbf{p}_i \tag{3.3.118}$$

Hence, the Marshallian demand functionals are:

$$\mathbf{x}_i(\mathbf{p},m) = \frac{\alpha_i m}{\alpha \mathbf{p}_i} \qquad i = 1,\ldots,n \tag{3.3.119}$$

The above formulas are similar to the ones in the deterministic and homothetic Cobb-Douglas expected utility cases. In particular, the cross price elasticities are zero and the demand functions are independent of the consumer's "beliefs" and "expectations" as represented by $\mathcal{P}$ and E respectively.

The indirect utility functional is given by:

$$\mathbf{V}(\mathbf{p},m) = \alpha Log\frac{m}{\alpha} + ELog\Upsilon(\mathbf{p}) \tag{3.3.120}$$

where $\Upsilon(\mathbf{p}) \equiv \prod_{j=1}^{n}\left[\frac{\alpha_j}{\mathbf{p}_j}\right]^{\alpha_j}$

We note that here, although the Marshallian demands are independent of "beliefs" and "expectations", the indirect utility, thus welfare, is not. That is, as in the non-homothetic Cobb-Douglas case, $\Upsilon(\mathbf{p})$ needs not be degenerate even if $\alpha = 1$.

The expenditure, compensation, and Hicksian demand functionals are respectively:

$$\mathbf{E}(\mathbf{p},u) = \alpha e^{\frac{1}{\alpha}(u - ELog\Upsilon(\mathbf{p}))} \tag{3.3.121}$$

$$\mu(\mathbf{p},\mathbf{q},m) = me^{-\frac{1}{\alpha}ELog\Upsilon(\mathbf{p})} \tag{3.3.122}$$

They all depend on the consumer's beliefs and expectations.

$$\mathbf{h}_i(\mathbf{p},u) = \frac{\alpha_i}{\mathbf{p}_i} e^{\frac{1}{\alpha}(u - ELog\Upsilon(\mathbf{p}))} \qquad i=1,\ldots,n \tag{3.3.123}$$

The expressions for the elasticities are exactly the same as in the case of the homothetic Cobb-Douglas expected utility, except that α_i is replaced by $\frac{\alpha_i}{\alpha}$. Similarly, the expressions for the monetary welfare indicators are the same as in the homothetic Cobb-Douglas expected utility case except α_i is replaced by $\frac{\alpha_i}{\alpha}$ and $\frac{\Upsilon(\mathbf{q})}{\Upsilon(\mathbf{p})}$ is replaced by $e^{-\frac{1}{\alpha}ELog\frac{\Upsilon(\mathbf{q})}{\Upsilon(\mathbf{p})}}$.

3.3.3.2 Multivariate generalization of the CARA expected utility

In a multivariate context, a natural generalization of the univariate utility with a constant relative Arrow-Pratt risk aversion coefficient (CARA) is obtained by defining on $L_2(\Omega, \mathbf{R}^n)$ the functional $\mathbf{U}^* \equiv E \circ \mathbf{u}$ where $\mathbf{u}$ is defined as:

$$\mathbf{u}(\mathbf{x}) = \exp\{-\sum_{j=1}^{n} \alpha_j \mathbf{x}_j\} \quad \text{with } \alpha_j > 0 \tag{3.3.124}$$

The restriction $\mathbf{U}$ of $\mathbf{U}^*$ to the positive cone $L_2(\Omega, \mathbf{R}^n)_+$ is then defined as the multivariate generalization of the univariate CARA expected utility. By Lemma **3.3.1** $\mathbf{U}$ is Fréchet differentiable (in the extended sense) at all points of $L_2(\Omega, \mathbf{R}^n)_+$ including the origin, with $\mathbf{U}_i'(\mathbf{x}) = \mathbf{u}_i' \circ \mathbf{x} = \alpha_i \exp\{-\sum_{j=1}^{n} \alpha_j \mathbf{x}_j\}$. Hence, the optimality conditions (3.2.3) and (3.2.4) reduces respectively to:

$$\alpha_i \exp\{-\sum_{j=1}^{n} \alpha_j \mathbf{x}_j\} - \lambda \mathbf{p}_i + \mathbf{q}_i^* = 0 \tag{3.3.125}$$

$$\mathbf{p}\mathbf{x} - m = 0 \tag{3.3.126}$$

with $\mathbf{q}_i^* \geq 0$, and $\mathbf{q}_i^* = 0$ if $\mathbf{x}_i > 0 \quad i=1,\ldots,n$. We recall that the Lagrange "multipliers" $\mathbf{q}_i^*$ $i=1,\ldots,n$ are real valued random variables.

Let $I^* \equiv \{i:\ \mathbf{x}_i > \mathbf{0},\ i=1,\ldots,n\}$, $I^0 \equiv \{i:\ \mathbf{x}_i = \mathbf{0},\ i=1,\ldots,n\}$, and $\mathbf{z} \equiv \sum_{j=1}^{n} \alpha_j \mathbf{x}_j$. Then for all $i, j \in \mathbf{I}^*$ $\mathbf{q}_i^* = \mathbf{0}$, $\mathbf{q}_j^* = \mathbf{0}$, and (3.3.125) implies:

$$\alpha_i e^{-\mathbf{z}} = \lambda \mathbf{p}_i \tag{3.3.127}$$

$$\frac{\mathbf{p}_j}{\alpha_j} = \frac{\mathbf{p}_i}{\alpha_i} \tag{3.3.128}$$

Taking the scalar product of both sides of (3.3.127) with $\mathbf{x}_i$, summing across i, and solving for λ gives

$$\lambda = \frac{E\left(\mathbf{z}e^{-\mathbf{z}}\right)}{m} \tag{3.3.129}$$

Substituting back into (3.3.127) gives after taking log

$$\sum_{j \in I^*} \alpha_j \mathbf{x}_j = -Log\left[\frac{\mathbf{p}_i}{\alpha_i m}\right] - Log\,E\left(\mathbf{z}e^{-\mathbf{z}}\right) \tag{3.3.130}$$

Taking the scalar product of both sides of (3.3.130) with $\mathbf{p}_i$, using (3.3.128), and solving for $Log\,E\left(\mathbf{z}e^{-\mathbf{z}}\right)$ gives

$$Log\,E(\mathbf{z}e^{-\mathbf{z}}) = -\frac{1}{E\mathbf{p}_i}\left\{E\left[\mathbf{p}_i Log\frac{\mathbf{p}_i}{\alpha_i m}\right] + \alpha_i m\right\} \tag{3.3.131}$$

which substituted back to (3.3.130) yields

$$\sum_{j\in I^*}\alpha_j\mathbf{x}_j = -Log\left[\frac{\mathbf{p}_i}{\alpha_i m}\right] + \frac{1}{E\mathbf{p}_i}\left\{E\left[\mathbf{p}_i Log\frac{\mathbf{p}_i}{\alpha_i m}\right] + \alpha_i m\right\} \tag{3.3.132}$$

Now, if $k \in \mathbf{I}^0$ and $\mathbf{q}_k \neq \mathbf{0}$, then (3.3.125) implies

$$\frac{\alpha_k}{\mathbf{p}_k} < \frac{e^{\mathbf{z}}E(\mathbf{z}e^{-\mathbf{z}})}{m} = \frac{\alpha_i}{\mathbf{p}_i} \quad \text{for all } i \in \mathrm{I}^* \tag{3.3.133}$$

That is

$$\frac{\mathbf{p}_k}{\alpha_k} > \frac{\mathbf{p}_i}{\alpha_i} \quad \text{for all } i \in \mathrm{I}^* \tag{3.3.134}$$

Hence, the Marshallian demands for the multivariate expected utility generalization of CARA are given by

$$\begin{aligned} &\mathbf{x}_i(\mathbf{p},m) = \mathbf{0} \qquad \text{if } \frac{\mathbf{p}_i}{\alpha_i} > \tilde{\mathbf{p}} \\ &\mathbf{z}(\mathbf{p},m) = -Log\left[\frac{\tilde{\mathbf{p}}}{m}\right] + \frac{1}{E\tilde{\mathbf{p}}}\left\{m + E\left[\tilde{\mathbf{p}}Log\frac{\tilde{\mathbf{p}}}{m}\right]\right\} \end{aligned} \tag{3.3.135}$$

where $\tilde{\mathbf{p}} \equiv \min_{j=1,\dots,n}\{\frac{\mathbf{p}_j}{\alpha_j}\} = \frac{\mathbf{p}_i}{\alpha_i}$ for all $i \in \mathrm{I}^*$, and $\mathbf{z}(\mathbf{p},m) \equiv \sum_{j\in \mathrm{I}^*}\alpha_j\mathbf{x}_j(\mathbf{p},m)$

There are two things to notice here. First, (3.3.128) implies some restriction on the consumer's beliefs, which is that the consumer believes that the prices for all goods with non zero demands are positively perfectly correlated (i.e., have pairwise coefficients of correlation of 1); and second, the quantities for the individual goods with non zero demand are indeterminate. This expresses some indifference among these goods. The indirect utility and other functionals follow easily from (3.3.135). But, it would be meaningless to compute elasticities in this context.

In summary, the expected utility examples treated here reveal the potentially restrictive nature of preferences generated by expected utility. Three striking features emerge from these examples: **(1)** the restriction on consumer's beliefs implied by some preferences. **(2)** The possible non dependence of consumer's behavior and welfare on his or her beliefs and expectations. **(3)** The apparent instability of behavior and beliefs that is implied by the fact that a "small change" in preference can lead to some degeneracy in beliefs and to a qualitatively different type of behavior. These three features combined can explain many of the expected utility paradoxes found through constructed examples and experimentation.

3.4 Examples of non-expected utility, expenditure and demand functionals

Given the sometimes severe restrictions on both behavior and beliefs that can emerge so easily from preferences generated by expected utility functionals (when only the *optimizing* behavior is used), it is of interest to see whether "non-expected" utility functionals will generate the same kinds of restrictions when the *optimizing* behavior assumption is used alone with no additional assumption on beliefs formation.[94] By "non-expected" utility we mean the impossibility to write the utility functional **U** as $\mathbf{U}(\mathbf{x}) = \mathbf{E}\mathbf{u}(\mathbf{x})$ for some cardinal utility function **u**, so that it depends linearly on the *subjective* probability distribution, the

[94] As already discussed, all functional forms including "non-expected" ones will generate restrictions on beliefs and expectations when the assumption of *optimality* of belief assessments (i.e., at optimal choices the decision maker should not have *ex-ante* any incentive to change his or her "beliefs" or "expectations") is used. When this assumption is mathematically formalized, one can use calculus to analytically derive the implied restrictions on "beliefs" and "expectations" for each functional form. The issue here is whether we can get restrictions on beliefs and expectations without this assumption.

expectation operator E, or the *subjective* cumulative distribution function of $\mathbf{x}$. In terms of the axioms of expected utility theory, it means dropping the controversial, so-called "independence axiom" that leads to this representation.

The method we use to generate "non-expected" utility functionals is based on the norm defining the topology of the choice space $L_2(\Omega, \mathbf{R}^n)$. In Appendix **4** we explain the rationale for making the utility functional depends explicitly on the topology of the choice space. In essence, because the ability to numerically represent the underlying ordinal preference $\succeq$ by a utility functional **U** depends on the topological structure of the choice space, directly and through the assumed continuity of $\succeq$, the functional form for **U** should exhibit this dependence. Hence, one might expect utility functionals of the form

$$\mathbf{U}(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \mathbf{F}(\|g_1(\mathbf{x}_1)\|,\ldots,\|g_n(\mathbf{x}_n)\|) \qquad \textbf{(3.4.1)}$$

to be less restrictive than expected utility functionals. Here $\| \; \|$ is the norm of $L_2(\Omega, \mathbf{R})$ so that $\|\mathbf{x}\|^2 = \sum_{i=1}^{n} \|\mathbf{x}_i\|^2$ for all $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in L_2(\Omega, \mathbf{R}^n)$, **F** is a real-valued function defined on the Euclidean space $\mathbf{R}^n$, and the $\mathbf{g}_i$'s are functions appropriately chosen so that **U** is at least quasi-concave.[95]

In the following derivations, we use the differentiability property of the Hilbertian norm of $L_2(\Omega, \mathbf{R}^n) : \mathbf{x} \rightarrow \|\mathbf{x}\|$. We recall that $\|\mathbf{x}\|^2 = \int_\Omega \|\mathbf{x}(\omega)\|^2 dP(\omega) = E(\|\mathbf{x}\|^2) = \sum_{i=1}^{n} E\mathbf{x}_i^2$, where the $\| \; \|$ inside the integral or the expectation operator is the usual Euclidean norm. But, it is analytically convenient to keep the notation $\|\mathbf{x}_i\|$, and substitute for $E\mathbf{x}_i^2$ only for computational purposes. The following result is proved in appendix **3**.

[95] We have dropped the respective subscripts from $\| \; \|$ to simplify the notation, since the respective spaces the norms refer to are clear from the context.

Lemma 3.4.1 *Let* $f: \mathbf{x} \to \|g \circ \mathbf{x}\|$ *be a functional on* $L_2(\Omega, \mathbb{R})$ *where* **g** *is a differentiable real valued function on the real line* $\mathbb{R}$. *Then f is Fréchet differentiable everywhere except the origin* **0** *with*

$$f'(\mathbf{x}) = \frac{(\mathbf{g} \circ \mathbf{x})(\mathbf{g}' \circ \mathbf{x})}{\|\mathbf{g} \circ \mathbf{x}\|} \quad \text{for all } \mathbf{x} \in L_2(\Omega, \mathbf{R}) \backslash \{\mathbf{0}\} \tag{3.4.2}$$

In the "non-expected" utility examples below, the utility functional **U** will be defined as the restriction to the positive cone, $L_2(\Omega, \mathbf{R}^n)_+$, of a Fréchet differentiable functional in the form of (3.4.1) that is defined on the entire space. Thus, by Lemma **3.4.1** and by the chain rule, **U** will be Fréchet differentiable (in the extended sense) except at the origin. Likewise, all the derived functional (demands, expenditure, etc...) will be Fréchet differentiable (in the extended sense) except at the origin.

Hence, if the utility functional **U** is given by (3.4.1), the first order conditions for an "interior" solution (3.2.5) and (3.2.4) reduce respectively to

$$\mathbf{F}_i'(\|g_1(\mathbf{x}_1)\|, \ldots, \|g_n(\mathbf{x}_n)\|) \frac{(\mathbf{g}_i \circ \mathbf{x}_i)(\mathbf{g}_i' \circ \mathbf{x}_i)}{\|\mathbf{g}_i \circ \mathbf{x}_i\|} - \lambda \mathbf{p}_i = \mathbf{0} \quad i = 1, \ldots, n \tag{3.4.3}$$

$$\mathbf{p}\mathbf{x} - m = 0 \tag{3.4.4}$$

We recall that $\mathbf{F}_i'(\|g_1(\mathbf{x}_1)\|, \ldots, \|g_n(\mathbf{x}_n)\|)$ is a real number, while $\frac{(\mathbf{g}_i \circ \mathbf{x}_i)(\mathbf{g}_i' \circ \mathbf{x}_i)}{\|\mathbf{g}_i \circ \mathbf{x}_i\|}$ is a random variable.

3.4.1 Demand functionals for the Cobb-Douglas non-expected utility functional

We define the Cobb-Douglas non-expected utility functional as:[96]

$$U(\mathbf{x}) = \prod_{j=1}^{n} \|\mathbf{x}_j^{\alpha_j}\| \quad \text{with } \alpha_j \geq 0, \text{ and } \alpha \equiv \sum_{j=1}^{n} \alpha_j \leq \frac{1}{2} \tag{3.4.5}$$

Since U can be written as the product of strictly positive concave functionals, it is quasi-concave.

The partial derivatives of U are

$$U_i'(\mathbf{x}) = \frac{\mathbf{x}_i^{\alpha_i} \times \alpha_i \mathbf{x}_i^{\alpha_i - 1}}{\|\mathbf{x}_i^{\alpha_i}\|} \times \prod_{j \neq i}^{n} \|\mathbf{x}_j\|^{\alpha_j} = \frac{\alpha_i \mathbf{x}_i^{2\alpha_i - 1}}{\|\mathbf{x}_i^{\alpha_i}\|^2} U(\mathbf{x}) \quad i = 1,\ldots,n \tag{3.4.6}$$

Hence, the necessary conditions (3.4.3) becomes here

$$\frac{\alpha_i \mathbf{x}_i^{2\alpha_i - 1}}{\|\mathbf{x}_i^{\alpha_i}\|^2} V(\mathbf{p}, m) = \lambda \mathbf{p}_i \quad i = 1,\ldots,n \tag{3.4.7}$$

Taking the scalar product of both sides of (3.4.7) with $\mathbf{x}_i$, summing across i and using (3.4.4) gives:

$$\sum_{i=1}^{n} \alpha_i \frac{\langle \mathbf{x}_i^{2\alpha_i - 1}, \mathbf{x}_i \rangle}{\|\mathbf{x}_i\|^2} V(\mathbf{p}, m) = \sum_{i=1}^{n} \alpha_i V(\mathbf{p}, m) = \lambda \sum_{i=1}^{n} \langle \mathbf{p}_i, \mathbf{x}_i \rangle = \lambda m \tag{3.4.8}$$

Hence

$$\lambda = \frac{\alpha}{m} V(\mathbf{p}, m) \tag{3.4.9}$$

and

[96] Here $F(t_1,\ldots,t_n) = \prod_{j=1}^{n} t_j$ and $g_i(t_i) = |t_i|^{\alpha_i}$.

$$\mathbf{x}_i = \left(\frac{\alpha \mathbf{p}_i}{\alpha_i m}\right)^{\frac{1}{2\alpha_i - 1}} \| \mathbf{x}_i^{\alpha_i} \|^{\frac{2}{2\alpha_i - 1}} \qquad i = 1,\ldots,n \tag{3.4.10}$$

Raising both sides of (3.4.10) to the power α_i, taking the norm of both sides of the resulting equation, and then solving for $\| \mathbf{x}_i^{\alpha_i} \|$ gives

$$\| \mathbf{x}_i^{\alpha_i} \| = \left(\frac{\alpha_i m}{\alpha}\right)^{\alpha_i} \| \mathbf{p}_i^{\frac{\alpha_i}{2\alpha_i - 1}} \|^{(1 - 2\alpha_i)} \tag{3.4.11}$$

which substituted back into (3.4.10) gives the Marshallian demands

$$\mathbf{x}_i(\mathbf{p}, m) = \frac{\alpha_i m}{\alpha} \frac{\mathbf{p}_i^{\frac{1}{2\alpha_i - 1}}}{\| \mathbf{p}_i^{\frac{\alpha_i}{2\alpha_i - 1}} \|^2} \qquad i = 1,\ldots,n \tag{3.4.12}$$

The corresponding indirect utility functional is

$$\mathbf{V}(\mathbf{p}, m) = \left(\frac{m}{\alpha}\right)^{\alpha} \phi(\mathbf{p}) \tag{3.4.13}$$

where $\phi(\mathbf{p}) = \prod_{i=1}^{n} \| \left(\frac{\alpha_i}{\mathbf{p}_i}\right)^{\frac{\alpha_i}{1 - 2\alpha_i}} \|^{1 - 2\alpha_i}$.

The expenditure, compensation and Hicksian demand functionals are respectively:

$$\mathbf{E}(\mathbf{p},u) = \alpha \left[\frac{u}{\phi(\mathbf{p})} \right]^{\frac{1}{\alpha}} \tag{3.4.14}$$

$$\mu(\mathbf{p},\mathbf{q},m) = m \left[\frac{\phi(\mathbf{q})}{\phi(\mathbf{p})} \right]^{\frac{1}{\alpha}} \tag{3.4.15}$$

$$\mathbf{h}_i(\mathbf{p},\mathrm{u}) = \alpha_i \left[\frac{u}{\phi(\mathbf{p})} \right]^{\frac{1}{\alpha}} \frac{\mathbf{p}_i^{\frac{1}{2\alpha_i - 1}}}{\| \mathbf{p}_i^{\frac{\alpha_i}{2\alpha_i - 1}} \|^2} \quad i = 1,\ldots,n \tag{3.4.16}$$

We can note that here we no longer have the restriction about the consumer's beliefs regarding prices, and the indirect utility, expenditure and demand functionals all depend on consumer's "beliefs" and expectations as represented by $\mathcal{P}$ and E.

3.1.1 The *precautionary* risk, nonrisk, and total elasticities

From the expression of the Marshallian demands in (3.4.12), we have:

$$\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial m} = \frac{\alpha_i}{\alpha} \| \mathbf{p}_i^{\frac{\alpha_i}{2\alpha_i - 1}} \|^{-2} \mathbf{p}_i^{\frac{1}{2\alpha_i - 1}} \quad i = 1,\ldots,n \tag{3.4.17}$$

and

$$\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j} = \begin{cases} \dfrac{\alpha_i m}{\alpha(2\alpha_i - 1)} \left[\| \mathbf{p}_i^{\frac{\alpha_i}{2\alpha_i - 1}} \|^{-2} \mathbf{p}_i^{\frac{2(1-\alpha_i)}{2\alpha_i - 1}} - 2\alpha_i \| \mathbf{p}_i^{\frac{\alpha_i}{2\alpha_i - 1}} \|^{-4} \mathbf{p}_i^{\frac{2}{2\alpha_i - 1}} \right] & \text{for } i = j \\ = 0 \quad \text{for } i \neq j \end{cases} \tag{3.4.18}$$

Hence, the uncompensated *precautionary* risk and nonrisk elasticities are

$$\epsilon_{ij}^{t} = \begin{cases} 0 \quad \text{for } i \neq j \\ \frac{1}{(2\alpha_i - 1)} \| \mathbf{p}_i \|^{-1} \| \mathbf{p}_i^{\frac{1}{2\alpha_i - 1}} \|^{-1} \left\{ cov\left[\mathbf{p}_i^{\frac{1}{2\alpha_i-1}}, \mathbf{p}_i\right] - 2\alpha_i \| \mathbf{p}_i^{\frac{\alpha_i}{2\alpha_i-1}} \|^{-2} cov\left[\mathbf{p}_i^{\frac{1+2\alpha_i}{2\alpha_i-1}}, \mathbf{p}_i\right] \right\} \quad \text{for } i=j \end{cases} \quad \textbf{(3.4.19)}$$

and

$$\epsilon_{ij}^{t} = \begin{cases} 0 \quad \text{for } i \neq j \\ \frac{1}{(2\alpha_i - 1)} \| \mathbf{p}_i \|^{-1} \| \mathbf{p}_i^{\frac{1}{2\alpha_i - 1}} \|^{-1} \left\{ E\left[\mathbf{p}_i^{\frac{1}{2\alpha_i-1}}\right] E\mathbf{p}_i - 2\alpha_i \| \mathbf{p}_i^{\frac{\alpha_i}{2\alpha_i-1}} \|^{-2} E\left[\mathbf{p}_i^{\frac{1+2\alpha_i}{2\alpha_i-1}}\right] E\mathbf{p}_i \right\} \quad \text{for } i=j \end{cases} \quad \textbf{(3.4.20)}$$

The total uncompensated elasticities are then:

$$\epsilon_{ij} = \epsilon_{ij}^{r} + \epsilon_{ij}^{t} = \begin{cases} 0 \quad \text{for } i \neq j \\ \frac{1}{(2\alpha_i - 1)} \| \mathbf{p}_i \|^{-1} \| \mathbf{p}_i^{\frac{1}{2\alpha_i - 1}} \|^{-1} \left\{ E\left[\mathbf{p}_i^{\frac{2\alpha_i}{2\alpha_i-1}}\right] - 2\alpha_i \| \mathbf{p}_i^{\frac{\alpha_i}{2\alpha_i-1}} \|^{-2} E\left[\mathbf{p}_i^{\frac{4\alpha_i}{2\alpha_i-1}}\right] \right\} \quad \text{for } i=j \end{cases} \quad \textbf{(3.4.21)}$$

Similarly, the *precautionary* risk aversion and pure income effects are:

$$\epsilon_{ij}^{rm} = -\frac{\alpha_j}{\alpha} \| \mathbf{p}_j \|^{-1} \| \mathbf{p}_i^{\frac{1}{2\alpha_i-1}} \|^{-1} \| \mathbf{p}_j^{\frac{\alpha_j}{2\alpha_j-1}} \|^{-2} cov\left[\mathbf{p}_i^{\frac{1}{2\alpha_i-1}} \mathbf{p}_j^{\frac{2\alpha_j}{2\alpha_j-1}}, \mathbf{p}_j\right] \qquad i,j = 1,\ldots,n \quad \textbf{(3.4.22)}$$

and

$$\epsilon_{ij}^{tm} = -\frac{\alpha_j}{\alpha} \| \mathbf{p}_j \|^{-1} \| \mathbf{p}_i^{\frac{1}{2\alpha_i-1}} \|^{-1} \| \mathbf{p}_j^{\frac{\alpha_j}{2\alpha_j-1}} \|^{-2} E\left[\mathbf{p}_i^{\frac{1}{2\alpha_i-1}} \mathbf{p}_j^{\frac{2\alpha_j}{2\alpha_j-1}}\right] E\mathbf{p}_j \qquad i,j = 1,\ldots,n \quad \textbf{(3.4.23)}$$

The total income effects are thus:

$$\epsilon_{ij}^{m} = \epsilon_{ij}^{rm} + \epsilon_{ij}^{tm} = -\frac{\alpha_j}{\alpha} \| \mathbf{p}_j \|^{-1} \| \mathbf{p}_i^{\frac{1}{2\alpha_i-1}} \|^{-1} \| \mathbf{p}_j^{\frac{\alpha_j}{2\alpha_j-1}} \|^{-2} E\left[\mathbf{p}_i^{\frac{1}{2\alpha_i-1}} \mathbf{p}_j^{\frac{4\alpha_j-1}{2\alpha_j-1}}\right] \qquad i,j = 1,\ldots,n \quad \textbf{(3.4.24)}$$

Finally, the *precautionary* risk substitution and taste substitution elasticities are respectively:

$$\epsilon_{ij}^{rs} = \begin{cases} \dfrac{\alpha_j}{\alpha} \| \mathbf{p}_j \|^{-1} \| \mathbf{p}_i^{\frac{1}{2\alpha_i - 1}} \|^{-1} \| \mathbf{p}_j^{\frac{\alpha_j}{2\alpha_j - 1}} \|^{-2} cov\left[\mathbf{p}_i^{\frac{1}{2\alpha_i - 1}} \mathbf{p}_j^{\frac{2\alpha_j}{2\alpha_j - 1}}, \mathbf{p}_j \right] & \text{for } i \neq j \\ \dfrac{1}{(2\alpha_i - 1)} \| \mathbf{p}_i \|^{-1} \| \mathbf{p}_i^{\frac{1}{2\alpha_i - 1}} \|^{-1} \left\{ cov\left[\mathbf{p}_i^{\frac{1}{2\alpha_i - 1}}, \mathbf{p}_i \right] \right. & \\ \qquad \left. - \dfrac{\alpha_i}{\alpha}(1 + 2(\alpha - \alpha_i)) \| \mathbf{p}_i^{\frac{\alpha_i}{2\alpha_i - 1}} \|^{-2} cov\left[\mathbf{p}_i^{\frac{1 + 2\alpha_i}{2\alpha_i - 1}}, \mathbf{p}_i \right] \right\} & \text{for } i = j \end{cases} \tag{3.4.25}$$

and

$$\epsilon_{ij}^{ts} = \begin{cases} \dfrac{\alpha_j}{\alpha} \| \mathbf{p}_j \|^{-1} \| \mathbf{p}_i^{\frac{1}{2\alpha_i - 1}} \|^{-1} \| \mathbf{p}_j^{\frac{\alpha_j}{2\alpha_j - 1}} \|^{-2} E\left[\mathbf{p}_i^{\frac{1}{2\alpha_i - 1}} \mathbf{p}_j^{\frac{2\alpha_j}{2\alpha_j - 1}} \right] E\mathbf{p}_j & \text{for } i \neq j \\ \dfrac{1}{(2\alpha_i - 1)} \| \mathbf{p}_i \|^{-1} \| \mathbf{p}_i^{\frac{1}{2\alpha_i - 1}} \|^{-1} \left\{ E\left[\mathbf{p}_i^{\frac{1}{2\alpha_i - 1}} \right] E\mathbf{p}_i \right. & \\ \qquad \left. - \dfrac{\alpha_i}{\alpha}(1 + 2(\alpha - \alpha_i)) \| \mathbf{p}_i^{\frac{\alpha_i}{2\alpha_i - 1}} \|^{-2} E\left[\mathbf{p}_i^{\frac{1 + 2\alpha_i}{2\alpha_i - 1}} \right] E\mathbf{p}_i \right\} & \text{for } i = j \end{cases} \tag{3.4.26}$$

The total substitution elasticities are:

$$\epsilon_{ij}^{s} = \begin{cases} \dfrac{\alpha_j}{\alpha} \| \mathbf{p}_j \|^{-1} \| \mathbf{p}_i^{\frac{1}{2\alpha_i - 1}} \|^{-1} \| \mathbf{p}_j^{\frac{\alpha_j}{2\alpha_j - 1}} \|^{-2} E\left[\mathbf{p}_i^{\frac{1}{2\alpha_i - 1}} \mathbf{p}_j^{\frac{4\alpha_j - 1}{2\alpha_j - 1}} \right] & \text{for } i \neq j \\ \dfrac{1}{(2\alpha_i - 1)} \| \mathbf{p}_i \|^{-1} \| \mathbf{p}_i^{\frac{1}{2\alpha_i - 1}} \|^{-1} \left\{ E\left[\mathbf{p}_i^{\frac{2\alpha_i}{2\alpha_i - 1}} \right] - \dfrac{\alpha_i}{\alpha}(1 + 2(\alpha - \alpha_i)) \| \mathbf{p}_i^{\frac{\alpha_i}{2\alpha_i - 1}} \|^{-2} E\left[\mathbf{p}_i^{\frac{4\alpha_i}{2\alpha_i - 1}} \right] \right\} & \end{cases} \tag{3.4.27}$$

Interpretations similar to the expected utility case can be made here. Also, one can verify that with no uncertainty, all the *precautionary* risk elasticities are zero and the nonrisk elasticities reduce to their respective deterministic analogues.

The income or budget elasticities for the Cobb-Douglas "non-expected" utility are:

$$\eta_i \equiv \frac{m}{\|\mathbf{x}_i\|} \times \frac{\partial \mathbf{x}_i(\mathbf{p}, m)}{\partial m} = \| \mathbf{p}_i^{\frac{1}{2\alpha_i - 1}} \|^{-1} \mathbf{p}_i^{\frac{1}{2\alpha_i - 1}} \qquad i = 1, \ldots, n \tag{3.4.28}$$

they are real valued random variable which reduce to 1, the value found in the deterministic case, if there is no uncertainty.

3.1.2 Welfare indicators and risk premium

From the compensating functional given by (3.4.15) we get the equivalent variation (**EV**), the Allen quantity (living standard) index (Q_A), and the Mackenzie-Pearce (cost of living) index (P_M) respectively:

$$\mathbf{EV} \equiv \mu(\mathbf{p}, \mathbf{q}, m') - m = m' \left[\frac{\phi(\mathbf{q})}{\phi(\mathbf{p})} \right]^{\frac{1}{\alpha}} - m \tag{3.4.29}$$

$$Q_A \equiv \frac{\mu(\mathbf{p}, \mathbf{q}, m')}{m} = \frac{m'}{m} \left[\frac{\phi(\mathbf{q})}{\phi(\mathbf{p})} \right]^{\frac{1}{\alpha}} \tag{3.4.30}$$

$$P_M \equiv \frac{m'}{\mu(\mathbf{p}, \mathbf{q}, m')} = \left[\frac{\phi(\mathbf{p})}{\phi(\mathbf{q})} \right]^{\frac{1}{\alpha}} \tag{3.4.31}$$

The i[th] directional risk premium is given by:

$$R_i \equiv -cov\left(\mathbf{x}_i(\mathbf{p}, \mu(\mathbf{p}, \mathbf{q}, m')), \mathbf{p}_i\right) = -\frac{\alpha_i m'}{\alpha} \left[\frac{\phi(\mathbf{q})}{\phi(\mathbf{p})} \right]^{\frac{1}{\alpha}} \| \mathbf{p}_i^{\frac{\alpha_i}{2\alpha_i - 1}} \|^{-2} cov\left[\mathbf{p}_i^{\frac{1}{2\alpha_i - 1}}, \mathbf{p}_i \right] \geq 0 \tag{3.4.32}$$

and, the total risk premium is $R = \sum_{i=1}^{n} R_i \geq 0$. The positivity of the risk premiums follows since the preference generated by **U** satisfies the axioms in chapter **2**. Alternatively, this can be shown directly by using the monotonicity of the operator $\mathbf{p} \mapsto \mathbf{p}^{\frac{1}{2\alpha - 1}}$. The risk premium index which is the risk component in the Allen quantity index Q_A is:

$$Q_A^r = \frac{R}{m} = -\frac{m'}{m}\left[\frac{\phi(\mathbf{q})}{\phi(\mathbf{p})}\right]^{\frac{1}{\alpha}} \sum_{i=1}^{n} \frac{\alpha_i}{\alpha} \left\| \mathbf{p}_i^{\frac{\alpha_i}{2\alpha_i - 1}} \right\|^{-2} cov\left[\mathbf{p}_i^{\frac{1}{2\alpha_i - 1}}, \mathbf{p}_i\right] \qquad \textbf{(3.4.33)}$$

Similarly, the cost-of-risk index which is the risk component in the Mckenzie-Pearce cost-of-living index is given by :

$$P_M^r \equiv \frac{R}{T+m} P_M = -\left[\frac{\phi(\mathbf{p})}{\phi(\mathbf{q})}\right]^{\frac{1}{\alpha}} \frac{\sum_{i=1}^{n} \alpha_i \left\| \mathbf{p}_i^{\frac{\alpha_i}{2\alpha_i - 1}} \right\|^{-2} cov\left[\mathbf{p}_i^{\frac{1}{2\alpha_i - 1}}, \mathbf{p}_i\right]}{\sum_{i=1}^{n} \alpha_i \left\| \mathbf{p}_i^{\frac{\alpha_i}{2\alpha_i - 1}} \right\|^{-2} E\left[\mathbf{p}_i^{\frac{1}{2\alpha_i - 1}}\right] E\mathbf{p}_i} \qquad \textbf{(3.4.34)}$$

The (i,j)[th] element of the matrix of risk aversion is given by:

$$R_{ij}^A \equiv -\frac{1}{m} cov\left[\mathbf{x}_i(\mathbf{p}, \mu(\mathbf{p}, \mathbf{q}, m')), \mathbf{p}_j\right] = -\frac{\alpha_i m'}{\alpha m}\left[\frac{\phi(\mathbf{p})}{\phi(\mathbf{q})}\right]^{\frac{1}{\alpha}} \left\| \mathbf{p}_i^{\frac{\alpha_i}{2\alpha_i - 1}} \right\|^{-2} cov\left[\mathbf{p}_i^{\frac{1}{2\alpha_i - 1}}, \mathbf{p}_j\right] \qquad \textbf{(3.4.35)}$$

The nonrisk components in the equivalent variation, Allen quantity index, and the Mckenzie-Pearce price index are respectively

$$T = \mathbf{x}(\mathbf{p}, \mu(\mathbf{p}, \mathbf{q}, m')) \cdot E\mathbf{p} - m = m'\left[\frac{\phi(\mathbf{q})}{\phi(\mathbf{p})}\right]^{\frac{1}{\alpha}} \sum_{i=1}^{n} \frac{\alpha_i}{\alpha} \left\| \mathbf{p}_i^{\frac{\alpha_i}{2\alpha_i - 1}} \right\|^{-2} E\left[\mathbf{p}_i^{\frac{1}{2\alpha_i - 1}}\right] E\mathbf{p}_i - m \qquad \textbf{(3.4.36)}$$

$$Q_A^t = \frac{\mathbf{T}}{m} + 1 = \frac{m'}{m}\left[\frac{\phi(\mathbf{q})}{\phi(\mathbf{p})}\right]^{\frac{1}{\alpha}} \sum_{i=1}^{n} \frac{\alpha_i}{\alpha} \left\| \mathbf{p}_i^{\frac{\alpha_i}{2\alpha_i - 1}} \right\|^{-2} E\left[\mathbf{p}_i^{\frac{1}{2\alpha_i - 1}}\right] E\mathbf{p}_i \qquad \textbf{(3.4.37)}$$

$$P_M^t \equiv \frac{m'}{T+m} = \left[\frac{\phi(\mathbf{p})}{\phi(\mathbf{q})}\right]^{\frac{1}{\alpha}} \left[\sum_{i=1}^{n} \frac{\alpha_i}{\alpha} \left\| \mathbf{p}_i^{\frac{\alpha_i}{2\alpha_i - 1}} \right\|^{-2} E\left[\mathbf{p}_i^{\frac{1}{2\alpha_i - 1}}\right] E\mathbf{p}_i\right]^{-1} \qquad \textbf{(3.4.38)}$$

3.4.2 Demand functionals for the CES non-expected utility

We define the constant elasticity of substitution (CES) non-expected utility functional as:

$$U(x) = \left[\sum_{j=1}^{n} \alpha_j \| x_j^{-\rho} \| \right]^{-\frac{1}{\rho}} \quad \text{with } \|x_j\| \neq 0, \alpha_j \geq 0 \ j=1,\ldots,n \ \text{ and } \rho > 0 \tag{3.4.39}$$

The partial derivatives of U are:

$$U_i'(x) = \alpha_i \frac{x_i^{-2\rho-1}}{\|x_i^{-\rho}\|} \left[\sum_{j=1}^{n} \alpha_j \| x_j^{-\rho} \| \right]^{-\frac{1}{\rho}-1} \qquad i=1,\ldots,n \tag{3.4.40}$$

Following the same procedures as in the Cobb-Douglas case, one can solve the first order conditions (3.4.3) and (3.4.4) to get the Marshallian demands and the indirect utility functional:

$$x_i(p,m) = \frac{m}{\Psi(p)} \left[\left\| \left[\frac{p_i}{\alpha_i}\right]^{\frac{\rho}{1+2\rho}} \right\| \right]^{-\frac{1}{1+\rho}} \left[\frac{p_i}{\alpha_i}\right]^{-\frac{1}{1+2\rho}} \qquad i=1,\ldots,n \tag{3.4.41}$$

$$V(p,m) = m(\Psi(p))^{-\frac{(1+\rho)}{\rho}} \tag{3.4.42}$$

where $\psi(p) = \sum_{j=1}^{n} \alpha_j \left[\left\| \left[\frac{p_j}{\alpha_j}\right]^{\frac{\rho}{1+2\rho}} \right\| \right]^{\frac{1+2\rho}{1+\rho}}$.

The other functionals, elasticities, and welfare indicators are easily obtained from (3.4.41) and (3.4.42) by following the same procedures as in the Cobb-Douglas example.

3.5 Conclusion

We briefly conclude by noting that the procedures used in the above examples can be used for any deterministic utility or expenditure function to get its corresponding "expected" or "non-expected" functional. However, generation of "non-expected" functionals is not limited to the method used here. These examples show clearly that the econometric estimation of the Marshallian demand functionals proceeds essentially in the same way as in the deterministic case. Moreover, here one need not introduce ex-post a so-called "error term" into the demand model in order to justify the statistical procedures. Nevertheless, one still may want to add an "error term", because of the inevitable presence of "noise" or measurement errors in collected data. In chapter 5 we use the duality approach to start from an expenditure or indirect utility functional to derive the corresponding demand functionals, elasticities and welfare indicators. We will also discuss in that chapter some of the econometric issues involved in the estimation of the demand functionals. However, before discussing the econometric estimation of demand functionals under uncertainty, we outline in the next chapter how to extend the the static results thus far derived to intertemporal choice under uncertainty.

CHAPTER 4

EXTENSION TO INTERTEMPORAL CONSUMPTION CHOICES

4.1 Introduction

In this chapter we give a detailed outline of the mathematical framework that will enable the extension of the static results derived in chapters **2** and **3** to intertemporal consumption choices. In Appendix **4** we present the conceptual framework and methodology for analyzing intertemporal choices under uncertainty. In particular, we have argued that in general the consumer's *preference* can always be defined in a choice space which has the general form $L_0(\Omega, \mathbf{F})$, the space of random variables (or functions) defined on Ω (the set of possible states of natures) and taking values in **F** (the outcome space). In the static analysis thus far conducted, **F** was taken to be the Euclidean space $\mathbf{R}^n$ (the usual commodity space), and $L_0(\Omega, \mathbf{F})$ was endowed with an appropriate mathematical structure which made possible the use of functional analysis and differential calculus tools to derive the theoretical results. Hence, what we need to do for the intertemporal case is to identify the relevant outcome space **F**, and endow $L_0(\Omega, \mathbf{F})$ with an appropriate mathematical structure that will allow the derivation of similar results using the same tools.

We have already mentioned in chapter **2** that, although in their present form our theoretical results does not make explicit references to intertemporal choices, the mathematical approach is valid for intertemporal choice models as long as the choice space is a *uniformly convex Banach* space, so that the two basic theorems **A2.1** and **A2.2** in Appendix **A2** can be applied. Hence, the basic strategy is to select **F** and endow $L_0(\Omega, \mathbf{F})$ with a mathematical structure that makes it a uniformly convex Banach space. However, any mathematical structure on $L_0(\Omega, \mathbf{F})$ will depend to a large extend on the one on **F** (this is especially true for the uniform convexity property), which in the (continuous time) intertemporal case, is an infinite dimensional space (see Appendix 4). Hence, unlike the static case where **F** was the finite dimensional Euclidean space $\mathbf{R}^n$, the choice of a topology or metric on **F** is a crucial step for determining the overall mathematical structure of the choice space.[97] Furthermore, with the Hindy, Huang and Kreps (1992) (HHK hereafter) criticism of the standard intertemporal consumption choice models, defining the economic structure and significance of **F** is a crucial first step for determining the economic structure of the choice space $L_0(\Omega, \mathbf{F})$ and interpreting its elements. The chapter is divided in two main parts. The first part focusses on the economic and mathematical structures of the outcome space **F**, and the second part on the ones of the choice space $L_0(\Omega, \mathbf{F})$. The second part concludes with a brief discussion of the issues related to the utility representation of preference, intertemporal constraints, and set-up of the consumer's utility maximization problem.

4.2 The economic and mathematical structure of the outcome space

In chapter 2 we briefly alluded to HHK's arguments on why standard continuous time models are inappropriate for describing continuous time intertemporal consumption choices. Without going into detail in their economic and mathematical arguments, we will summarize briefly their economic arguments and adapt their proposal to view consumers as choosing among alternative *cumulative* *consumption paths or patterns* to our conceptual and mathematical framework.

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[97] Because all metrics defined on $\mathbf{R}^n$ are equivalent, the choice of a topology on $\mathbf{R}^n$ is not an issue (see the related discussion in the introduction of chapter 2).

4.2.1 The outcome of intertemporal consumption choices

HHK's arguments are twofold. We discuss their first argument in this section and the second argument in the next section. HHK argue that to be realistic, in modelling consumption over time one should allow for continuous consumption at "rates" as well as consumption in lumps or "gulps" at discrete points in time. However, the standard model, by representing consumption over the time interval **T** by a function $\mathbf{c}: \mathbf{T} \subset \mathbb{R}_+ \to \mathbb{R}^n_+$, where $\mathbf{c}(t)$ is the rate of consumption at time t (a vector of commodity bundles), does not allow for consumption in lumps at discrete points in time, which, in fact is a more realistic description of consumption overtime.[98] Hence they propose representing consumption over the time interval **T** by a right-continuous function $\mathbf{x}: \mathbf{T} \subset \mathbb{R}_+ \to \mathbb{R}^n_+$, where $\mathbf{x}(t)$ is the *cumulative* consumption from time zero to time t under the *consumption pattern* $\mathbf{x}$.[99] The time derivative of a *cumulative consumption* function (if it exists in the classical sense of derivative) gives the associated consumption rate function **c**. In this way, continuous consumption in *rate* as well as discrete consumption in *"gulp"* or *lump sum* is allowed. However, because the time derivative of the *cumulative consumption* function need not exist in the classical sense of a derivative (as when *cumulative consumption* is made of *lump sum* consumption at discrete points in time), the relevant concept of derivative to use here is the one of *generalized or*

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[98] Hindy, Huang, and Kreps (1992) and Hindy and Huang (1992) argue that discrete consumption in "gulp" or lump sum (such as taking meals at discrete time for example) is the most realistic picture of the way people consume commodities in the real world. However, this is ruled out by the standard practice of modelling continuous time intertemporal consumption choice as a choice among alternative patterns of consumption rates.

[99] In fact, HHK treated the one (aggregated) commodity case; but their arguments carry over without change to the many commodities case.

distributional derivative.[100] Hence, in this general and more realistic framework **c**, the usual consumption "rate" function, becomes what is known in functional analysis as a *distribution or generalized function.*[101] Moreover, as it is recognized that *consumption smoothing* issues (including issues of seasonality in consumption) are very important in the analysis of intertemporal consumption choices, the consumption "rate" function is not the only relevant function related to the *cumulative consumption pattern* **x**. Indeed, the concept of *consumption smoothing* relates to the "curvature" of the consumption "rate" function; that is, to various degrees of "the change in the change in the change ... in the change in the consumption "rate" function". In short, this means that the first and higher time distributional derivatives up to an arbitrary order k of the *cumulative consumption pattern* **x** are all relevant for the analysis of intertemporal consumption choices. These economic arguments indicate that for intertemporal consumption choices, the relevant consumption outcomes are *cumulative consumption paths or patterns*, and that mathematically they lie in a space known in functional analysis as a *Sobolev* space. In other words, the outcome space **F** containing all the possible *cumulative consumption patterns* is a *Sobolev* space of order k defined by

$$W_p^k = W_p^k(\mathbf{T}, \mathbf{R}^n) \equiv \{\mathbf{x};\ D^\alpha \mathbf{x} \in L_p(\mathbf{T}, \mathbf{R}^n),\ \alpha = 0,1,\ldots,k;\}$$

where **T** is an open (time) interval of the positive orthant of the real line, $D^\alpha\mathbf{x}$ is the α^{th} time *distributional* derivative of the *cumulative consumption pattern* **x**, with $D^0\mathbf{x} \equiv \mathbf{x}$; and $L_p(\mathbf{T}, \mathbf{R}^n)$

[100] The *distributional derivative* concept generalizes the classical concept of derivative to functions that are merely (locally) integrable. For example, continuous functions as well as step functions have *distributional derivatives* of infinite order. The *distributional derivative* of a function is called *distribution* or *generalized function* which also has *distributional derivatives* of infinite order (see, for example, Rudin, 1991).

[101] HHK and Hindy and Huang (1992) did not use *distributional* derivatives in their analysis. But, from a functional analysis perspective it is more convenient to work with the distributional derivative. Working with *distributional* derivatives allows one also to resolve some mathematical technicalities that could prevent the application of theorems **A2.1** and **A2.2**.

is the space of Lebesgue p-integrable functions defined on $\mathbf{T}$ and taking value in $\mathbb{R}^n$.[102] The metric of the *Sobolev* space W^k is given by the *Sobolev* norm defined by:

$$\|\mathbf{x}\|_{p,k} \equiv \|\mathbf{x}\|_{W_t^k} \equiv \left[\sum_{\alpha=0}^{k} \int_{\mathbf{T}} \|D^\alpha \mathbf{x}(t)\|^p dt \right]^{\frac{1}{p}} \quad \text{for all } \mathbf{x} \in W_p^k \tag{4.2.1}$$

where the norm $\| \; \|$ appearing inside the integral is the usual Euclidean norm. In the terminology of the conceptual framework of Appendix **4**, the norm $\| \; \|_{k,p}$ formalizes the consumer's *perception* of "similarities/dissimilarities" among the various possible *cumulative consumption patterns*. The time (*distributional*) derivatives involved in this *perception* of "similarities/dissimilarities" means that the degrees of "*smoothness*" of the various possible *cumulative consumption patterns* are economically relevant *intrinsic* characteristics that are *perceived* to distinguish the *cumulative consumption patterns* one from another. One notes that since by definition a possible *cumulative consumption pattern* is a positive monotonically increasing function of time, the set $\mathbf{X}_o$ of all possible *cumulative consumption patterns* is a subset of the (natural) positive cone of the Sobolev space W_p^k endowed with its natural order (i.e., the order $\geq$ defined by $\mathbf{x} \geq \mathbf{y}$ *iff* $\mathbf{x}(t) \geq \mathbf{y}(t)$ for all $t \in \mathbf{T}$). In other words, $\mathbf{X}_o \subset W_{p+}^k$. One also notes that $\mathbf{X}_o$ itself is a cone, and thus can be used to define another ordering $\geq_o$ on W_p^k by the relation $\mathbf{x} \geq_o \mathbf{y}$ *iff* $\mathbf{x} - \mathbf{y} \in \mathbf{X}_o$.[103]

It is a standard result of functional analysis that for $1 \leq p < \infty$ (resp $1 < p < \infty$) the Sobolev space W_p^k is a separable Banach space (resp separable uniformly convex Banach

[102] Note in particular that by definition $W_p^k \subset L_p(\Omega, \mathbb{R}^n)$.

[103] HHK take the choice space to be the linear span of $\mathbf{X}_o$ and work with the ordering defined by $\geq_o$. One should note that the meaning of the usual monotonicity of preference assumption is different depending which ordering one uses to define it.

space).[104] Moreover, for $p=2$, W_2^k is a Hilbert space with an inner or scalar product given by:

$$\langle \mathbf{x},\mathbf{y}\rangle_{k,2} \equiv \sum_{\alpha=0}^{k} \int_T \langle D^\alpha \mathbf{x}(t), D^\alpha \mathbf{y}(t)\rangle dt \quad \text{for all } \mathbf{x},\mathbf{y} \in W_2^k \tag{4.2.2}$$

where the scalar product $\langle\ \rangle$ appearing inside the integral is the usual Euclidean scalar product.[105] Hence, if we take prices to come from the dual space, then for $\mathbf{F} = W_2^k$ a possible price $\mathbf{p}$ that values the possible *cumulative consumption patterns* is an element of the same space W_2^k, and under $\mathbf{p}$ the *shadow* price or cost of a possible *cumulative consumption pattern* $\mathbf{x} \in W_2^k$ is given by:

$$\langle \mathbf{p},\mathbf{x}\rangle_{k,2} = \sum_{\alpha=0}^{k} \int_T \langle D^\alpha \mathbf{p}(t), D^\alpha \mathbf{x}(t)\rangle dt \tag{4.2.3}$$

One notes in particular that the *shadow* cost of a possible *cumulative consumption pattern* includes not only the observable costs of the consumption rates, but also the unobservable *shadow* cost of the "degree of *smoothness*" of the *consumption pattern*.

4.2.2 The nature of intertemporal substitution in consumption

The second argument of HHK in their criticism of standard intertemporal consumption choice models is related to what they imply about substitution of consumption across time. HHK argue that commonly observed real world consumption decisions indicate that consumption at nearby dates are close substitute. However, the standard models imply just

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[104] For these properties of Sobolev spaces, see, for example, Zeidler (1990) and the literature cited there (W_∞^k is a Banach space but is not separable). The uniform convexity of W_p^k which obtains when $1<p<\infty$ means, roughly speaking, that the "aggregation" across time of the Euclidean distances is "smooth" (uniform convex spaces are also called *smooth* spaces).

[105] Note that uniform convexity implies reflexivity (see Appendix A2).

the opposite: consumptions at nearby dates are perfect nonsubstitutes! This strange implication is due to the fact that the standard models view intertemporal consumption as choice among alternative consumption rate functions and use a very strong metric on the choice space. Furthermore, HHK argue that a realistic intertemporal consumption choice model should not only allow consumption at nearby dates to be close substitutes, but also should be such that substitutability of consumption across time involves some (smooth) trade-off between the amount substituted and the timing of consumption. That is, consumers should be able to tolerate shifting a given amount of consumption across short periods of time, as well as to tolerate shifting an increasingly larger amount of consumption over a decreasingly shorter period of time, provided the rate of growth of the amount shifted is sufficiently low with respect to the shift in the timing of consumption. These types of "flexible" substitution across time, which are commonly observed in real life consumption behavior, are also ruled out by the standard intertemporal consumption choice models.

Hence, to accommodate these more reasonable types of intertemporal substitution in consumption not allowed by the standard models, HHK propose to view consumers as choosing among alternative *cumulative consumption patterns* as described in the previous section, and use an appropriate metric in this choice space. They introduce a class of metrics that generate choice spaces that are known in functional analysis as *Orlicz* spaces. Moreover, they show that the choice space must be an Orlicz space if one is to accommodate the type of intertemporal substitution in consumption related to the trade-off between the amount of consumption substituted across time and the timing of consumption. The Orlicz spaces generalize the L_p spaces, in the sense that instead of being based on the particular family of convex functions defined from $\mathbf{R}_+$ to $\mathbf{R}_+$ by $x \to x^p$; $1 \leq p < \infty$, they are based on the much larger family of convex functions, $\varphi: \mathbf{R}_+ \to \mathbf{R}_+$, which satisfy the conditions $\varphi(0) = 0$ and

$\lim_{x\to\infty}\varphi(x) = \infty$. A convex function φ having such properties is called a Young function. If in addition of being a Young function φ satisfies the two properties: $\lim_{x\to 0}\frac{\varphi(x)}{x} = 0$ and $\lim_{x\to\infty}\frac{\varphi(x)}{x} = \infty$, then it is called an *N-function*; and can be shown to be a strictly increasing function. The (*gauge*) norm of the Orlicz space associated with the Young function φ is defined by:

$$\|\mathbf{x}\|_{\varphi} \equiv \inf\left\{\epsilon > 0:\ \int_{\mathbf{T}}\varphi\left[\frac{\|\mathbf{x}(t)\|}{\epsilon}\right]dt \leq 1\right\} \tag{4.2.4}$$

The corresponding Orlicz space noted L_{φ} is defined as

$$L_{\varphi} = L_{\varphi}(\mathbf{T},\mathbf{R}^n) \equiv \left\{\mathbf{x}: \mathbf{T}\to\mathbf{R}^n;\ \int_{\mathbf{T}}\varphi(\epsilon\,\|\mathbf{x}(t)\|)dt < \infty \ \text{ for some } \epsilon > 0\right\} \tag{4.2.5}$$

When φ is the convex function $x \to x^p$; $1 \leq p < \infty$, L_{φ} and $\|\ \|_{\varphi}$ reduce to the familiar L_p space and norm respectively.[106] Hence, since the L_p spaces, $1 \leq p \leq \infty$, are commonly used choice spaces in the standard intertemporal consumption choice models, their failure to capture real life intertemporal substitution in consumption reflects more the practice of viewing consumers as choosing among alternative consumption rate functions than the metric defining the topological structure of the choice space. That is to say that the fundamental departure of HHK from the standard models is to view consumers as choosing among alternative *cumulative consumption paths or patterns*, and to exclude other non-Orlicz spaces as inappropriate "pathological" choice spaces for modelling intertemporal consumption choices. The exclusion of the "pathological" non-Orlicz spaces (which are sometimes used in some economic problems) is important because it underscores the economic importance of the choice of an appropriate topology when working with infinite dimensional choice spaces. For

[106] We note that $L_{\infty} = L_{\varphi_{\infty}}$ where φ_{∞} is the Young function defined by $\varphi_{\infty} = 0$ if $0 \leq x < 1$ and $= \infty$ if $x > 1$.

example, L_∞ is considered as the natural choice space for infinite dimensional economic problems for the fact that its positive cone has a nonempty interior. In contrast, the L_p spaces, $1 \leq p < \infty$ are generally considered unattractive for economic problems because of the fact that the interiors of their positive cone are empty (see, for example, Stokey and Lucas (1989)). We have already discussed (in the introduction of chapter **2**) the problems caused by the emptiness of the positive cone of the choice space and showed how one can get around them. We have also given some mathematical reasons why L_∞ and L_1 are inappropriate infinite dimensional choice spaces if one wants to generalize the methods and results of static deterministic consumer theory. HHK's arguments provide us the economic reasons for dismissing L_∞ and L_1 as inappropriate choice spaces because they are at the border line for the satisfaction of HHK's requirements related to intertemporal substitution in consumption (i.e., at the border line of being Orlicz spaces).[107] Moreover, in order for the Orlicz space L_φ to be uniformly convex (thus allowing the extension of the theoretical results of chapters **2** and **4** to L_φ), we need to restrict further the φ function to the class of *N-functions* satisfying the additional so-called (global) Δ_2 condition given by:

[107] Note that L_1 correspond to a linear φ function (see Rao and Ren (1991) for a comprehensive account of the properties of Young functions and Orlicz spaces).

$$\varphi(2x) \leq a\varphi(x) \ \forall x \geq 0, \text{ for some } a > 0 \qquad (\Delta_2)$$

The Δ_2 condition is a restriction on the rate of growth of φ.[108] Roughly speaking, it excludes from consideration *N-functions* which grow faster than an exponential function.[109] For the remainder of this chapter, the *N-function* φ is assumed to satisfy the Δ_2 condition.[110]

With φ satisfying (Δ_2), L_φ is a separable and uniformly convex Banach space (hence is reflexive). Its dual (which is the price space) is the Orlicz space L_{φ^*}, where $\varphi^*: \mathbf{R}_+ \to \mathbf{R}_+$ is the *complementary* or *conjugate* function of φ defined by:

$$\varphi^*(y) \equiv \sup_x \{xy - \varphi(x)\,;\ x > 0\}$$

φ^* is a *N-function* that satisfies another growth condition called the ∇_2 condition and which is given by:[111]

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[108] HHK did also restrict the φ function to the class of *N-functions* in order to make the consumption space a separable subspace of L_φ (so that *preference* can be represented by a utility functional via Debreu's theorem), and characterize its dual (the price space). Hindy and Huang (1992) also used a similar restriction for the same reasons. However, the restriction of φ to a *N-function* is not sufficient to make L_φ a uniformly convex Banach space. In fact, the Δ_2 condition is necessary for uniform convexity (see, for examples, Rao and Ren (1991), Turett (1980), and Hudzik (1983)).

[109] Besides the L_p spaces, examples of *N-functions* that satisfy the Δ_2 condition are: $\varphi_1(x) = x^p \ln(1 + x^p)$, $1 < p < \infty$ and $\varphi_2(x) = (1 + x)\ln(1 + x) - x$. Examples of *N-functions* that do not satisfy the Δ_2 condition are: $\varphi_3(x) = e^x - x - 1$, $\varphi_4(x) = e^{x^2} - 1$, and $\varphi_4(x) = x^{\ln x}$ for $x > e$ and $= \frac{x}{e}$ for $0 \leq x \leq e$ (see Rao and Ren (1991) for more examples and other growth related conditions).

[110] When φ satisfies (Δ_2), L_φ reduces to $L_\varphi = \left\{\mathbf{x} : \mathbf{T} \to \mathbf{R}^n;\ \int_{\mathbf{T}} \varphi(\|\mathbf{x}(t)\|)\,dt < \infty\right\}$

[111] For example, the *complementary* function of the *N-function* $x \to |x|^p$, $1 < p < \infty$ corresponding to the L_p space is the function $y \to |y|^q$ with $p^{-1} + q^{-1} = 1$. Likewise, the *complementary* function of the *N-function* $\varphi_2(x) = (1 + |x|)\ln(1 + |x|) - |x|$ is $\varphi_3(y) = e^{|y|} - |y| - 1$. Note that neither φ nor φ^* need to satisfy both growth conditions (Δ_2) and (∇_2) at the same time. The L_p cases are examples where they both do. Another *N-function* that satisfies both (Δ_2) and (∇_2) is the function $\varphi_5(x) = |x|^p(1 + |\ln|x||)$, $p > 1$.

$$\varphi^*(x) \leq \frac{1}{2b}\varphi^*(bx) \;\; \forall x \geq 0, \text{ for some } b > 1 \qquad (\nabla_2)$$

With the Orlicz space L_φ generalizing L_p, $1<p<\infty$, the Sobolev space W_p^k, introduced in the previous section in order to incorporate consumption smoothing issues, is generalized in a obvious way as:

$$W_\varphi^k \equiv \{\mathbf{x};\; D^\alpha \mathbf{x} \in L_\varphi(\mathbf{T}, \mathbf{R}^n),\; \alpha = 0,1,\ldots,k;\}$$

W_φ^k is called an *Orlicz-Sobolev* space, and its norm is defined by:

$$\|\mathbf{x}\|_{k,\varphi} \equiv \inf\left\{ \epsilon > 0 : \sum_{\alpha=0}^{k} \int_{\mathbf{T}} \varphi\left[\frac{\|D^\alpha \mathbf{x}(t)\|}{\epsilon}\right] dt \leq 1 \right\} \quad \text{for all } \mathbf{x} \in W_\varphi^k \qquad \textbf{(4.2.6)}$$

When φ satisfies (Δ_2), W_φ^k is a separable and uniformly convex Banach space.[112]

To introduce the Orlicz space $L_\varphi(\mathbf{T}, \mathbf{R}^n)$, we have followed HHK and taken the associated Young function φ as being a real-valued function defined on $\mathbf{R}_+$. But, a more general and convenient way to introduce $L_\varphi(\mathbf{T}, \mathbf{R}^n)$ is to take the Young function φ as being an *even* real-valued function defined on the Euclidean space $\mathbf{R}^n$. That is, $\varphi: \mathbf{R}^n \to \mathbf{R}_+$ is a continuous convex function which satisfies $\varphi(-\mathbf{x}) = \varphi(\mathbf{x})$, $\varphi(\mathbf{0}) = 0$, and $\lim_{s\to\infty}\varphi(s\mathbf{x}) = \infty$ for $\mathbf{x} \neq \mathbf{0}$. The definition of an *N-function* and the growth conditions (Δ_2) and (∇_2) are generalized accordingly (see Kozek (1977) and Turett (1980)). With this general definition of φ, the expressions in (4.2.4) to (4.2.6) are still correct with $\varphi: \mathbf{R}^n \to \mathbf{R}_+$ being defined as $\varphi(\mathbf{x}) \equiv \tilde{\varphi}(\|\mathbf{x}\|)$; where $\tilde{\varphi}$ is a Young function defined on the real line. Moreover, this general definition of the Young function is still valid when instead of $\mathbf{R}^n$ we have an infinite dimensional separable Banach space $\mathbf{X}$ like in the next section.

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[112] See, for examples, Kozek (1977), and Hudzik (1979, and 1983) for these results and for the characterization of its dual.

This more general definition of the Orlicz space $L_\varphi(\mathbf{T}, \mathbf{R}^n)$ allows us to use the conceptual framework of Appendix 4 and interpret the Young function φ as representing the consumer's *perception* of the *intrinsic* characteristics distinguishing the various *cumulative consumption paths or patterns* one from another (i.e., the consumer's *perception* of "similarities/dissimilarities" among the alternative *cumulative consumption patterns*). In other words, as explained in Appendix 4, the Young function φ represents the consumer's *perception* of "closeness" (a primitive concept) in the same way that the utility function and the *subjective* probability represent the primitive concepts of *preference* and *beliefs* respectively.[113] Furthermore, the general definition of the Orlicz space $L_\varphi(\mathbf{T}, \mathbf{R}^n)$ allows us to conveniently incorporate "time-dependent *perception*" issues[114] by letting the associated Young function φ depend on the time parameter t. That is, by letting φ be defined as $\varphi: \mathbf{T}\times\mathbf{R}^n\rightarrow\mathbf{R}_+$ with $\varphi(t, .)$ being a Young function for all $t \in \mathbf{T}$, and $\varphi(., \mathbf{x})$ being a Lebesgue measurable function for all $\mathbf{x} \in \mathbf{R}^n$. When φ is defined in this way, it is known as a *generalized* Young

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[113] HHK assumed that φ is the same for all consumers in a given economy, but reported (in footnote 10) that some have found the role played by φ not very natural and have asked the question of how would one discover the appropriate φ for a given population of consumers? Here, the conceptual framework of chapter 2 has provided us with an intuitive interpretation of φ. Moreover, since we take the concept of *perception* to exist independently of *preference* and *beliefs* (see the arguments on the separate existence of three concepts made in chapter 2), HHK's assumption that φ is given uniformly for all consumer's is not too unrealistic; considering the fact that the *perception* of the *intrinsic* properties of the objects of choice, and the relations among them, should not be very different from one consumer to another (in most cases these *intrinsic* properties are physical characteristics). Regarding the question of how to discover the appropriate φ, our answer would be to treat φ in the same way we treat individual *preferences* and *beliefs*, and discover its effects by observing overt choices.

[114] See Appendix 4 for the distinction between "time/state-dependent *perception*" and "time/state-dependent *preference*".

function, and L_φ and W^k_φ are called *generalized Orlicz* and *generalized Orlicz-Sobolev* spaces respectively.[115]

Thus, economic and mathematical arguments have led us to the generalized Orlicz-Sobolev space as the appropriate choice space for intertemporal consumption choices under certainty, with consumers being viewed as choosing among alternative *cumulative consumption paths or patterns*. Since under uncertainty the outcomes of intertemporal consumption choices are the *cumulative consumption paths*, we conclude that $L_0(\Omega, \mathbf{F}) = L_0(\Omega,\ W^k_\varphi)$. That is the consumer's objects of choice are positive right-continuous monotonically increasing vector valued random functions defined on Ω, the set of possible states of nature, and taking values in W^k_φ (endowed with its Borel σ-algebra), a separable and uniformly convex generalized Orlicz-Sobolev space.

Before moving to the characterization of the topological structure of $L_0(\Omega,\ W^k_\varphi)$ we will make few comments on the different approach we have taken to arrive at the linear space W^k_φ containing the set of possible *cumulative consumption patterns* $\mathbf{X}_o$, compared to the HHK's approach. HHK defined the choice space $\mathbf{X}_l$ for intertemporal consumption choices under certainty as the linear span of $\mathbf{X}_o$ and endow it with an Orlicz norm.[116] As they have noted, thus constructed $\mathbf{X}_l$ is not a Banach space (i.e., is not complete), which makes it much harder to apply directly the well known mathematical theorems for solving optimization

[115] Hindy and Huang (1992) used generalized Young functions for their analysis in order to take care of state-dependent "trade-off of consumption across time" (what we have called state-dependent *perception* in Appendix 4). Examples of generalized Young functions are the functions defined by $\varphi_1(t,\mathbf{x}) = \tilde{\varphi}(p(t)\mathbf{x})$ and $\varphi_2(t,\mathbf{x}) = (\tilde{\varphi}(\mathbf{x}))^{p(t)}$ where $\tilde{\varphi}: \mathbf{R}^n \to \mathbf{R}_+$ is a Young function and $p: \mathbf{T} \to \mathbf{R}_+$ is a function that satisfies $1 < p(t) < \infty$ for all $t \in \mathbf{T}$ (see, for examples, Kozek (1977), Hudzik (1979 and 1983), and Musielak (1983) for more on the generalized Orlicz and generalized Orlicz-Sobolev spaces).

[116] Hindy and Hung (1992) who extended HHK's analysis to uncertainty also used a similar construction of the choice space.

problems and characterizing their solutions in $\mathbf{X}_l$.[117] In particular, Theorems **A2.1** and **A2.2** in the appendix, which we have used to derive the results in the chapters **2** and **3**, cannot be applied in $\mathbf{X}_l$. Instead of taking linear span, we have followed a different approach by simply considering $\mathbf{X}_o$ as a subset of an appropriately chosen Banach space which we have taken to be a Sobolev space. This approach is simpler and avoid the mathematical difficulties that can arise otherwise in many applications. We use Sobolev spaces because not only it allows to conveniently incorporate consumption smoothing issues, but also they are the most efficient tools for working with functions defined in the real line and which are not differentiable in the classical sense.[118] Moreover, even if one is restricted to smooth functions, the Sobolev spaces are required for the general existence of solutions of optimization problems.[119]

4.3 The structures of the choice space and consumption set

4.3.1 The mathematical structure of the choice space

With a general set of possible states of nature, Ω (with no particular order or topological structure), little can be said concerning substitution across possible states of nature. That is, it would be difficult to rely on economic arguments similar to those used for the outcome space **F** to determine the appropriate topological structure of the choice space

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[117] It is also harder to prove the existence of Walrasian equilibria in $\mathbf{X}_l$ (see Mas Collel and Richard 1991).

[118] HHK and Hindy and Huang (1992) did not consider the Sobolev spaces.

[119] The Sobolev spaces complete the class of functions that are continuously differentiable in the classical sense by including functions that are not differentiable in the classical sense but admit *distributional or generalized derivatives*, in a way similar to how the irrational numbers complete the set of rational numbers to form the set of real numbers. In particular, it is a known fact that one cannot have a general existence theorem for classical calculus of variation problems unless one works within a Sobolev space. Outside the framework of Sobolev spaces, one only gets case by case existence theorems (see Zeidler, 1990, pp. 1-5, and 1985, p. 200).

$L_0(\Omega,\ W_\varphi^k)$.[120] Hence, we will proceed by simply endowing $L_0(\Omega,\ W_\varphi^k)$ with a topology which makes it satisfy the mathematical requirements for extending the results of chapters **2** and **3** to intertemporal consumption choices under uncertainty.[121] For that, we choose a *N-function* $\psi\colon W_\varphi^k \rightarrow \mathbf{R}_+$ that satisfies the Δ_2 condition, so that $L_\psi(\Omega,\ W_\varphi^k)$ is a separable uniformly convex Orlicz space.[122] That is, we take the choice space to be the Orlicz space $L_\psi(\Omega,\ W_\varphi^k)$ defined by

$$L_\psi(\Omega, W_\varphi^k) \equiv \left\{ \mathbf{x}: \Omega \rightarrow W_\varphi^k;\ \int_\Omega \psi\big(\mathbf{x}(.,\omega)\big) d\mathcal{P}(\omega) < \infty \right\} \tag{4.3.1}$$

and endowed with the (gauge) norm defined by:

$$\|\mathbf{x}\|_{\psi,\varphi,k} \equiv \inf\left\{ \epsilon > 0:\ \int_\Omega \psi\left[\frac{\mathbf{x}(.,\omega)}{\epsilon}\right] d\mathcal{P}(\omega)\ \leq 1 \right\} \tag{4.3.2}$$

with the *N-function* ψ given by

$$\psi(\mathbf{x}(.,\omega)) = \int_{\mathbf{T}} \varphi(\mathbf{x}(t,\omega))dt \equiv \sum_{\alpha=0}^{k} \int_{\mathbf{T}} \tilde{\varphi}(\|D^\alpha \mathbf{x}(t,\omega)\|)dt \quad \forall \omega \in \Omega \tag{4.3.3}$$

where $\tilde{\varphi}$ is a *N-function* defined on the real line and which satisfies the Δ_2 condition.

As in the previous section, we interpret the *N-function* ψ (which depends on the *N-function* φ an k) as representing the consumer's *perception* of the intrinsic characteristics distinguishing the alternative random *cumulative consumption paths* in $L_\psi(\Omega,\ W_\varphi^k)$ (i.e., the *ex-*

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[120] Some plausible arguments about substitutability of consumption across events can be used (see Hindy and Huang, 1992).

[121] We still assume as in chapters **2** and **3** that the consumer's uncertain environment is described by the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where Ω is the set of possible states of nature, $\mathcal{F}$ is the set of events, and $\mathcal{P}$ is a *subjective* probability distribution representing the consumer's *belief*.

[122] There are some regularity conditions on the *subjective* probability distribution $\mathcal{P}$.

ante consequences of his or her consumption decisions) one from another. We emphasize here that the *perception* represented by ψ is independent of the consumer's *beliefs* concerning which state of nature is likely going to be realized (see Appendix **4**). This independence is illustrated in (4.3.3) by the fact that ψ does not depend on the consumer's *beliefs* as represented by the *subjective* probability distribution $\mathcal{P}$. However, as explained in Appendix **4**, before any consumption decision is made the consumer uses his or her *beliefs* about the relative likelihoods of the various events, to "weight" his or her *perception* of the *intrinsic* distinguishing characteristics of the alternative random *cumulative consumption paths* so as to arrive at a "beliefs-adjusted" *perception* of the "similarities/dissimilarities" among them. The "belief-adjustment" of *perception* insures that the consumer's *preference* ordering of the alternative random *cumulative consumption paths* is consistent with his or her *belief* about which state of nature is likely going to be realized. As in chapter **2**, this "belief-adjusted" *perception* is formalized by the norm $\| \ \|_{\psi,\varphi,k}$ given by (4.3.2).

One allows for state-dependent *perception*, by taking ψ to be a generalized *N-function* $\psi : \Omega \times W^k_\varphi \to \mathbf{R}_+$ (with $\psi(.,\mathbf{x})$ being an integrable function for each $\mathbf{x} \in W^k_\varphi$). In this case, ψ will be given by:

$$\psi(\omega, \mathbf{x}(.,\omega)) = \int_{\mathbf{T}} \varphi(\omega, \mathbf{x}(t,\omega))dt \equiv \sum_{\alpha=0}^{k} \int_{\mathbf{T}} \tilde{\varphi}(\omega, \|D^\alpha \mathbf{x}(t,\omega)\|)dt \tag{4.3.4}$$

We note that if in (4.3.4) we let $n = 1$ and $k = 0$ (the one commodity case with consumption smoothing issues not built into the topology), then we will get the norm used by Hindy and Huang (1992). If we want to allow for both state-dependent and time-dependent perception, then we have to take both ψ and φ to be generalized *N-functions* defined on the product spaces $\Omega \times W^k_\varphi$ and $\mathbf{T} \times \mathbf{R}^n$ respectively. In this case, ψ will be given by

$$\psi(\omega, \mathbf{x}(., \omega)) = \int_{\mathbf{T}} \varphi(\omega, t, \mathbf{x}(t, \omega))dt \equiv \sum_{\alpha=0}^{k} \int_{\mathbf{T}} \tilde{\varphi}(\omega, t, \|D^{\alpha}\mathbf{x}(t, \omega)\|)dt \qquad \textbf{(4.3.5)}$$

with $\tilde{\varphi}(\omega,.,.): \mathbf{T} \times \mathbf{R}_{+} \to \mathbf{R}_{+}$ being a generalized *N-function* defined on the real line for each $\omega \in \Omega$.[123] To simplify the notation, and without loss of generality, we will continue with simple *N-functions* instead of generalized *N-functions*.

Because $L_{\psi}(\Omega, W_{\varphi}^{k})$ is uniformly convex, it is reflexive and its dual (the space of *shadow* prices of random *cumulative consumption paths*) is given by $L_{\psi}(\Omega, W_{\varphi}^{k})^{*} = L_{\psi^{*}}(\Omega, W_{\varphi}^{k*})$, where W_{φ}^{k*} is the dual of the Orlicz-Sobolev space W_{φ}^{k} (i.e., the space of possible *shadow* prices of *cumulative consumption paths*), and where $\psi^{*}: W_{\varphi}^{k*} \to \mathbf{R}_{+}$ is the *complementary* or *conjugate* function of ψ defined by:

$$\psi^{*}(\mathbf{p}) \equiv \sup_{\mathbf{x}} \left\{ \langle \mathbf{p}, \mathbf{x} \rangle_{k\varphi} - \psi(\mathbf{x}) \; ; \; \mathbf{x} \in W_{\varphi}^{k} \right\}$$

where $\langle \mathbf{p}, \mathbf{x} \rangle_{k,\varphi}$ is the value of the continuous linear functional $\mathbf{p} \in W_{\varphi}^{k*}$ at the point $\mathbf{x} \in W_{\varphi}^{k}$ (see. Kozek (1977) and Turett (1980))[124] Moreover, the value of an element $\mathbf{p} \in L_{\psi^{*}}(\Omega, W_{\varphi}^{k*})$ at the point $\mathbf{x} \in L_{\psi}(\Omega, W_{\varphi}^{k})$, noted $\langle \mathbf{p}, \mathbf{x} \rangle_{\psi,\varphi,k}$, is given by (see Kozek (1977):[125]

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[123] Examples of state/time-dependent *perception* are obtained by taking the generalized Young functions defined by $\varphi_1(\omega, t, \mathbf{x}) \equiv \varphi_0(p(\omega, t)\mathbf{x})$ and $\varphi_2(\omega, t, \mathbf{x}) \equiv (\varphi_0(\mathbf{x}))^{p(\omega,t)}$ where $\varphi_0: \mathbf{R}^n \to \mathbf{R}_{+}$ is a Young function and $p: \Omega \times \mathbf{T} \to \mathbf{R}_{+}$ is a measurable function that satisfies $1 < p(\omega, t) < \infty$ for all $(\omega, t) \in \Omega \times \mathbf{T}$

[124] We recall that ψ^{*} is a *N-function* that satisfies the ∇_2 condition.

[125] Note that to simplify the notation we are using the same notation for a generic element of W_{φ}^{k} (resp W_{φ}^{k*}) and a generic element of $L_{\psi}(\Omega, W_{\varphi}^{k})$ (resp $L_{\psi^{*}}(\Omega, W_{\varphi}^{k*})$). But, there is no confusion in doing that because the context is always made clear.

$$\langle \mathbf{p}, \mathbf{x} \rangle_{\psi,\varphi,k} = \int_{\Omega} \langle \mathbf{p}(\omega,.), \mathbf{x}(\omega,.) \rangle_{k\varphi} d\mathbb{P}(\omega) \tag{4.3.6}$$

Hence, when $\varphi: \mathbb{R}^n \to \mathbb{R}_+$ is the *N-function* $\mathrm{x} \to \|\mathrm{x}\|^2$, so that W_φ^k is the Hilbert space W_2^k, then (4.3.6) becomes:

$$\begin{aligned} \langle \mathbf{p}, \mathbf{x} \rangle_{\psi,\varphi,2} &= \int_{\Omega} \langle \mathbf{p}(\omega,.), \mathbf{x}(\omega,.) \rangle_{k,2} d\mathbb{P}(\omega) \\ &= \sum_{\alpha=0}^{k} \int_{\Omega} \int_{T} \langle D^\alpha \mathbf{p}(\omega,t), D^\alpha \mathbf{x}(\omega,t) \rangle \, dt \, d\mathbb{P}(\omega) \end{aligned} \tag{4.3.7}$$

Furthermore, when $\psi: W_2^k \to \mathbb{R}_+$ is the *N-function* $\mathbf{x} \to \|\mathbf{x}\|^2$, then $L_2(\Omega,\ W_2^k)$ is an Hilbert space with the symmetric bilinear functional $\langle\ ,\ \rangle_{\psi,\varphi,k} = \langle\ ,\ \rangle_{2,\varphi,k}$ defined in (4.3.7) being its scalar product (see, for example, Zeidler, 1990, p. 1072). In this case, the norm of $L_2(\Omega,\ W_2^k)$ defined in (4.3.2) simplifies to[126]

$$\|\mathbf{x}\|_{2,2,k} = \left(\int_{\Omega} \|\mathbf{x}(\omega,.)\|_{2,k}^2 d\mathbb{P}(\omega)\right)^{\frac{1}{2}} = \left[\sum_{\alpha=0}^{k} \int_{\Omega} \int_{T} \|D^\alpha \mathbf{x}(\omega,t)\|^2 dt \, d\mathbb{P}(\omega)\right]^{\frac{1}{2}} \tag{4.3.8}$$

This ends our discussion of the structure of the choice space for intertemporal consumption choices under uncertainty. In the next section we conclude the outline by briefly discussing the structure of the consumption set (i.e., the subset of $L_\psi(\Omega,\ W_\varphi^k)$ containing the random *cumulative consumption patterns*), the representation of the consumer's preference by a utility functional, the constraints facing the consumer, and other relevant modelling issues.

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[126] A similar simplification of the norm is obtained for $L_p(\Omega,\ W_p^k)$, $1 \le p < \infty$.

4.3.2 The consumption set and the existence of a utility functional

Following the arguments and notation in section **4.2**, we take the consumer's consumption set, noted **X**, as the subset of the positive cone of $L_\psi(\Omega,\ W_\varphi^k)$ defined by:

$$\mathbf{X} \equiv \left\{\mathbf{x} \in L_\psi(\Omega, W_\varphi^k):\ \mathbf{x}(\omega,.) \in \mathbf{X}_o \subset W_\varphi^k\ \forall\omega \in \Omega\right\} \qquad \textbf{(4.3.9)}$$

where $\mathbf{X}_0$ is the set of positive, monotonically increasing, and right continuous functions of time defined on the open interval $\mathbf{T} \subset \mathbf{R}_+$ and taking values in $\mathbf{R}_+^n$; with *distributional* derivative up to order k. In short, **X** is the subset of $L_\psi(\Omega,\ W_\varphi^k)$ made of positive and monotonically increasing stochastic processes with right-continuous sample paths. It is clear that **X** is a convex cone since $\mathbf{X}_0$ is. Moreover, we can assume that **X** is a weakly compact subset of $L_\psi(\Omega,\ W_\varphi^k)$ (by taking the closure of the right hand side of (4.3.9) if necessary, and restricting ourself to a bounded subset of $\mathbf{X}_0$).

With the consumer's *preference*, $\succeq$, being defined on **X** and satisfying the standard assumptions **A1-A6** in chapter **2**, and $L_\psi(\Omega,\ W_\varphi^k)$ being a separable space, Debreu's theorem guarantees the existence of a continuous, bounded, and strictly quasi-concave utility functional $\mathbf{U}: \mathbf{X} \subset L_\psi(\Omega, W_\varphi^k) \rightarrow \mathbf{R}_+$ such that for all **x**, **y** $\in$ **X** $\mathbf{x} \succeq \mathbf{y}$ *iff* $\mathbf{U}(\mathbf{x}) \geq \mathbf{U}(\mathbf{y})$. Furthermore, we allow for "time-dependent" *preference* by letting $\succeq$ to be a function of time defined on **T**. In this case, Debreu's representation theorem can be applied for each $\succeq(t)$, $t \in \mathbf{T}$ to get a time dependent utility functional $\mathbf{U}(t,.) \equiv \mathbf{U}(\succeq(t),.): \mathbf{X} \subset L_\psi(\Omega, W_\varphi^k) \rightarrow \mathbf{R}_+$ such that for all **x**, **y** $\in$ **X** $\mathbf{x} \succeq(t)\, \mathbf{y}$ *iff* $\mathbf{U}(t,\mathbf{x}) \geq \mathbf{U}(t,\mathbf{y})$ (see Appendix **4**).

With this utility representation of *preference*, one can solve the consumer's utility maximization problem for an optimal random *cumulative consumption paths* satisfying the *feasibility* constraints (discussed below), and derive the corresponding demand, indirect utility, and expenditure functionals. One can then use Theorems **A2.1** and **A2.2** to establish the analogues of the various propositions derived in chapter 2 (including the generalized versions of Roy's identity and Shephard's lemma). From these propositions will follow the comparative statics results relating "changes" in the random *cumulative consumption path* of a commodity to "changes" in its *shadow* price and the various measures of total welfare change. However, unlike the static case where the only constraint facing the consumer is the budget constraint, in the intertemporal case there are additional constraints and boundary conditions that complicate the formulation and solution of the consumer's utility maximization problem. The (*lifetime*) budget constraint will still be of the same form:

$$\begin{aligned} \langle D\mathbf{p}, D\mathbf{x}\rangle_{*} &\equiv \int_{\Omega}\int_{T} \langle D\mathbf{p}(\omega,t), D\mathbf{x}(\omega,t)\rangle \, dt\, d\mathbb{P}(\omega) \\ &\leq \int_{\Omega}\int_{T} \mathbf{w}(\omega,t)\, dt\, d\mathbb{P}(\omega) \equiv \langle \mathbf{1}, \mathbf{w}\rangle^{*} \end{aligned} \tag{4.3.10}$$

where $\langle D\mathbf{p}, D\mathbf{x}\rangle_{*}$ is the value of the continuous linear functional $D\mathbf{p}$, the usual observable stochastic price vector (being the *distributional* derivative of the *shadow* price of *cumulative consumption patterns*) at the point $D\mathbf{x}$, the usual observable vector-valued stochastic rates of consumption of the commodities; $\mathbf{w}$ is the stochastic process of the *cumulative net wealth* measured in term of the numeraire; and $\mathbf{1}$ is the stochastic process that takes the constant value of 1 at all time and all possible states of nature. As the static case, this budget constraint lies in the real line.[127] On the other hand, the additional "liquidity" constraint which links the change in $\mathbf{w}$ to the "flow variables" (labor income, expenditure, borrowing/saving, etc...) and "borrowing" constraint are stochastic constraints that must be satisfied in each possible state of nature and at each point in time. These two constraints

[127] Note however that unlike the static case, $\langle \mathbf{1}, \mathbf{w}\rangle$ is not an exogenously predetermined scalar. Instead, it is endogenously determined by the labor supply, saving, production and investment decisions.

together with the boundary conditions govern the dynamic of intertemporal consumption choices. They constitute an example of a case of an infinite number of constraints which can be written in the form $N(\mathbf{u}) \leq \mathbf{0}$, with $\mathbf{u}$ being a vector of stochastic process defining the constraints and **N** being a (possibly nonlinear) functional taking values in a reflexive Banach space **Y** and satisfying some regularity conditions. Hence, they can be handled using the generalized versions of the Kuhn-Tucker conditions guaranteed by Theorem **A2.1** in Appendix **2**. One notes that in this case the "Lagrange multipliers" corresponding to the constraints are stochastic processes.

However, the appropriate formulation of the stochastic constraints facing the consumer requires an explicit account of how the partial resolution of uncertainty as time evolves (i.e., information revelation) affects consumption decisions. To see the importance of information revelation in the formulation of these stochastic constraints, we recall from Appendix **4** that a given random *cumulative consumption path* is a *consequence* of a consumer's *contingent plan* made of a sequence of *contingent acts* or *actions* affecting consumption (saving, borrowing, and labor supply decisions for example), and which are implemented over time conditional on the occurrence of uncertain events in the future. Hence, since the taking of any *contingent act* depends on the resolution of the uncertain event upon which it is conditioned, the structure of the flow of information, which determines the resolution of future events, must be explicit in the formulation of any stochastic constraint facing the consumer for a model to realistically represent intertemporal consumption choices under uncertainty. We have briefly discussed in chapter **2** some of the probabilistic concepts involved in the mathematical modelling of these information issues (see also Appendix **4**).

However, the full discussion of all the relevant concepts will not be considered in this outline.[128]

From the above discussion, it should be clear that the knowledge that the consumption set **X** is a weakly compact subset of the positive cone of $L_\psi(\Omega,\ W_\varphi^k)$ is sufficient to derive the theoretical results extending the static analysis of chapter **2**. For the analytical expressions extending the results in chapter **3**, one can proceed by finding analytical constraints that can describe the elements of **X**. For that, one can represent the positive and monotonically increasing conditions by:

$$\mathbf{x} \geq \mathbf{0} \quad \text{and} \quad D\mathbf{x} \geq \mathbf{0} \tag{4.3.11}$$

where D is the (time) *distributional* derivative operator (a continuous linear operator) and $\geq$ represents the natural ordering of the two spaces involved. (4.3.11) can be written in the form $\mathbf{M(u)} \leq \mathbf{0}$, where $\mathbf{M} \equiv -(\mathbf{I}\ ,\ \mathbf{D})$ is a continuous linear operator taking values in the reflexive Banach space $\mathbf{Y} \equiv L_\psi(\Omega,\ W_\varphi^k)\times L_\psi(\Omega,\ W_\varphi^k)$, and **I** is the identity operator defined on $L_\psi(\Omega,\ W_\varphi^k)$. Hence, like in chapter **3**, Corollary **A2.1** in Appendix **2** can always be applied to obtain the generalized version of the Euler equation characterizing the solution of the consumer's utility maximization problem. For many explicitly given utility functionals, the Euler equation can be solved to get an analytical expression of the optimal solution. Moreover, like in the static case, one need not be restricted to time-additive and/or expected utility functional to get tractable analytical solutions. However, for "non time-additive\non-expected" utility functionals generated as function of the norm $\|\ \|_{\psi,\varphi,k}$ like in the static case, one notes that the Fréchet derivative of $\|\ \|_{\psi,\varphi,k}$ (which exists since $L_\psi(\Omega,\ W_\varphi^k)$ is a uniformly

[128] On the mathematical modelling of information issues relevant to economic problems, see, for example, Harrison and Kreps (1979), Huang (1985a and 1985b), Duffie and Huang (1986), and Hindy and Huang (1992).

convex Banach space) is a function of the duality mapping of $L_\psi(\Omega,\ W^k_\varphi)$.[129] The duality mapping is closely related to the Fréchet derivative of the *N-function* ψ. Turett (1980) gives the explicit analytical expression of the Fréchet derivative of the norm of $L_\psi(\Omega,\ \mathbf{Y})$, with **Y** being a uniformly convex Banach space, as a function of the Fréchet derivative of ψ (see also Rao and Ren, 1991, for the case of $L_\psi(\Omega,\ \mathbb{R})$. However, since ψ depends on the *N-function* φ and k which together define the norm of the Orlicz-Sobolev space W^k_φ, one still needs an analytical expression of the Fréchet derivative of the norm of W^k_φ. This can be derived by using the chain rule and noting that the *distributional* derivative operator $\mathbf{L} = \sum_{\alpha=0}^{k} D^\alpha$ is a continuous linear operator.[130] Needless to say, even for time-additive expected utility functionals, the optimal solutions will in general depend on the time distributional derivative operator D, since most of the constraints are expressed in term of D.

In the same way as in the static case treated in the next chapter, one can develop a duality approach to derive demand systems involving the random *cumulative consumption paths* as functions of the stochastic vector of *shadow* prices, and other exogenously determined stochastic processes determining the constraints facing the consumer (eg., interest rate, exogenous income, etc...). Thus, versions of the QUAIDS demand system, derived in the next chapter, can be derived for intertemporal consumption choices under uncertainty. The intertemporal QUAIDS versions can be empirically implemented by noting that the time *distributional* derivatives of a *cumulative consumption path* and of its (per unit) *shadow* price give the usual observable consumption rates for the various commodities and their per unit prices respectively. However, to properly derive the consumption rates, one needs to

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[129] In the case of the Hilbert space $L_2(\Omega,\ W^k_2)$ the duality mapping reduces to the identity operator (see Cioranescu, 1990).

[130] For the case of W^k_p, $1<p<\infty$), see Lions (1969) and Cioranescu (1990).

explicitly take into account the informational issues related to the progressive resolution of uncertainty as time evolves. Again, as in the case of the stochastic constraints, the need to take into consideration these informational issues arises because the observable consumption rates for the various commodities are the result of the implementation of the optimal *contingent plan* leading to the optimal random *cumulative consumption path*.

Finally we conclude this outline by noting three sets of issues that are relevant to intertemporal consumption choices under uncertainty but have not been discussed. The first set of issues is about the disentanglement in both the comparative statics and measures of welfare change of the effects due respectively to taste substitution, income, attitudes toward risk, and time preference (i.e., intertemporal substitution). This separation can be done following the same procedure as in the static case. The second set of issues relate to the analytical assessment of how the progressive revelation of information as time evolves affects the consumer's *beliefs* and *expectations*. The third and final set of issues concerns the pricing of information, the characterization and measurement of its value to the consumer, and the characterization and measurement of the effects of the preference for early versus late resolution of uncertainty. These information-related issues will be addressed in our future work using the concepts and methods of martingale theory. In the next chapter we continue the static analysis of chapter 2 and 3 by using the duality theory of chapter 2 to derive flexible demand systems under uncertainty.

CHAPTER 5

GENERATION AND ECONOMETRIC ESTIMATION OF INTEGRABLE AND "FLEXIBLE" DEMAND SYSTEMS UNDER UNCERTAINTY

In all the examples in chapter **3**, we started with a direct utility functional and derived the corresponding demand, indirect utility and expenditure functionals. Alternatively, as in deterministic consumer theory, one can use the duality theory of chapter **2** and start with either an expenditure or an indirect utility functional and recover the corresponding demand and direct utility functionals. This duality approach has been found to be more convenient for applied micro-economics. In what follows we will be working with the quadratic AIDS (QUAIDS) functional form of Banks, Blundell and Lewbel (1994) (B.B.L hereafter). However, the derivations can be applied to any one of the other flexible functional forms that have been proposed such as the translog or the generalized Leontief. The chapter is organized as follows. In section **5.1** the general functional forms for the integrable QUAIDS shares equations, and elasticity and welfare indicator formula are derived in a general framework which encompasses both the static deterministic and uncertainty cases. In section **5.2** we specialize the general QUAIDS to derive the shares, elasticities, and welfare indicators of what we call the "expected" version of QUAIDS under uncertainty. Here, the term "expected" is not related to expected utility theory, but is used only to describe the way the "expected" version of QUAIDS is obtained. In section **5.3** we derive the shares, elasticities, and welfare indicators for what we call the "non-expected" version of QUADS under uncertainty. In section **5.4** we discuss the econometric estimation issues related to the empirical implementation of general QUAIDS model. In particular, we provide in this section a new method for estimating the "nuisance" parameter which appears in the QUAIDS share equations and seemed to be difficult to estimate in previous empirical implementation of the QUAIDS model.

5.1 The QUAIDS functional form under Uncertainty

The QUAIDS functional form is the rank three generalization of the popular AIDS model of Deaton and Muellbauer (1980). The rank of a demand system has been defined by Gorman (1981) as the dimension of the space spanned by its Engel curves. In essence, the QUAIDS model adds to the AIDS share equation a quadratic term in income with price dependent coefficients. Building on Gorman (1981), and Lewbel (1991), B.B.L show that the rank three QUAIDS is the most general integrable demand system for which the shares are polynomial functions of the logarithm of deflated total expenditure. Although B.B.L's results are derived for the deterministic consumer model within the context of the finite dimensional Euclidean space, we expect their results to hold in any separable and uniformly convex Banach space. Hence, the uncertainty versions of the QUAIDS model presented below will all have the same type of generality and integrability as their deterministic counterpart.

5.1.1 The general rank three integrable QUAIDS model

The indirect utility function of the general rank three QUAIDS model is given by

$$\left(\ln V(\mathbf{p},m)\right)^{-1} = \left(\ln V^{*}(\mathbf{p},m)\right)^{-1} - \lambda(\mathbf{p}) \tag{5.1.1}$$

with

$$\ln V^{*}(\mathbf{p},m) = \frac{1}{b(\mathbf{p})} \times \ln\left[\frac{m}{a(\mathbf{p})}\right] \tag{5.1.2}$$

where $a(\mathbf{p})$, $b(\mathbf{p})$, and $\lambda(\mathbf{p})$ are Fréchet differentiable functions defined from the positive cone of the price space to the real line, m is predetermined total expenditure, and $\mathbf{p}$ is the vector of prices. For the deterministic QUAIDS, the price space is the finite dimensional Euclidean space, $\mathbb{R}^n$, while for the uncertainty QUAIDS models, the price space is the space of vector valued random variables with finite variances (i.e., $L_2(\Omega, \mathbb{R}^n)$).[131]

The associated expenditure and compensation functions are respectively given by:

$$\ln E(\mathbf{p},u) = \ln a(\mathbf{p}) + b(\mathbf{p}) \times \left\{\lambda(\mathbf{p}) + \left(\ln u\right)^{-1}\right\}^{-1} \tag{5.1.3}$$

and

$$\ln \mu(\mathbf{p},\mathbf{q},m) = \ln a(\mathbf{p}) + \frac{b(\mathbf{p})}{b(\mathbf{q})} \ln\left[\frac{m}{a(\mathbf{q})}\right] \times \left\{1 + \frac{\lambda(\mathbf{p}) - \lambda(\mathbf{q})}{b(\mathbf{q})} \ln\left[\frac{m}{a(\mathbf{q})}\right]\right\}^{-1} \tag{5.1.4}$$

The budget shares and compensated demands (evaluated at expenditure $\mu(\mathbf{p}, \mathbf{q}, m)$) for the general QUAIDS model are

[131] For differentiability matters in the uncertainty case, $a(.)$, $b(.)$ and $\lambda(.)$ are considered as the respective restrictions to the positive cone of the functionals $\mathbf{p} \mapsto a(|\mathbf{p}|)$, $\mathbf{p} \mapsto b(|\mathbf{p}|)$, and $\mathbf{p} \mapsto \lambda(|\mathbf{p}|)$ defined on the whole space $L_2(\Omega, \mathbb{R}^n)$; where $|\mathbf{p}| \equiv (|\mathbf{p}_1|, \ldots, |\mathbf{p}|_n)$, and $|\mathbf{p}_i|(\omega) = \sup(\mathbf{p}_i(\omega), 0)$ $i = 1,\ldots n$, for all $\omega \in \Omega$.

$$\omega_i = \mathbf{p}_i \left\{ \frac{\partial \ln a(\mathbf{p})}{\partial \mathbf{p}_i} + \frac{b_i'(\mathbf{p})}{b(\mathbf{p})} \ln \frac{m}{a(\mathbf{p})} - \frac{\lambda_i'(\mathbf{p})}{b(\mathbf{p})} \left[\ln \frac{m}{a(\mathbf{p})} \right]^2 \right\} \quad i=1,\ldots,n \tag{5.1.5}$$

and

$$cd_i = \mu \left\{ \frac{\omega_i}{\mathbf{p}_i} + \frac{b_i'(\mathbf{p})}{b(\mathbf{p})} \ln \frac{\mu}{m} - \frac{\lambda_i'(\mathbf{p})}{b(\mathbf{p})} \left[\ln \frac{m\mu}{a(\mathbf{p})^2} \right] \ln \frac{\mu}{m} \right\} \quad i=1,\ldots,n \tag{5.1.6}$$

The AIDS model corresponds to the case where $\lambda(\mathbf{p}) = 0$ for all $\mathbf{p}$. Also, one can check that the symmetry, adding up, and homogeneity restrictions are the same as the ones in the AIDS model, except that there are additional restrictions on $\lambda(\mathbf{p})$ (in particular, $\lambda(\mathbf{p})$ must be homogenous of degree zero).

5.1.2 Elasticities and welfare indicators for the general QUAIDS model

From (5.1.4), (5.1.5), and (5.1.6) we calculate the various elasticities, welfare indicators, and risk aversion measures as derived in chapter **2**. For notational convenience, and for the purpose of GAUSS programming, we let

$$\varphi_i^m \equiv \frac{\partial \mathrm{x}_i(\mathbf{p},m)}{\partial m} = \frac{\omega_i}{\mathbf{p}_i} + \frac{b_i'(\mathbf{p})}{b(\mathbf{p})} - 2\frac{\lambda_i'(\mathbf{p})}{b(\mathbf{p})} \ln \frac{m}{a(\mathbf{p})} \quad i=1,\ldots,n \tag{5.1.7}$$

$$\psi_i \equiv \frac{\partial \ln a(\mathbf{p})}{\partial \mathbf{p}_i} = \frac{\omega_i}{\mathbf{p}_i} - \frac{b_i'(\mathbf{p})}{b(\mathbf{p})} \ln \frac{m}{a(\mathbf{p})} + \frac{\lambda_i'(\mathbf{p})}{b(\mathbf{p})} \left[\ln \frac{m}{a(\mathbf{p})} \right]^2 \quad i=1,\ldots,n \tag{5.1.8}$$

and

$$\phi_{ij} \equiv \frac{1}{m}\frac{\partial \mathbf{x}_i(\mathbf{p},m)}{\partial \mathbf{p}_j} = \frac{\partial^2 \ln a(\mathbf{p})}{\partial \mathbf{p}_i \partial \mathbf{p}_j} + \left[\frac{\omega_i}{\mathbf{p}_i} - \varphi_i^m\right]\psi_j + \left[\frac{b_{ij}''(\mathbf{p})}{b(\mathbf{p})} - \frac{b_i'(\mathbf{p})b_j'(\mathbf{p})}{b(\mathbf{p})^2}\right]\ln\frac{m}{a(\mathbf{p})}$$
$$- \left[\frac{\lambda_{ij}''(\mathbf{p})}{b(\mathbf{p})} - \frac{\lambda_i'(\mathbf{p})b_j'(\mathbf{p})}{b(\mathbf{p})^2}\right]\left[\ln\frac{m}{a(\mathbf{p})}\right]^2 \qquad i,j=1,\ldots,n \tag{5.1.9}$$

With these notations, the various elasticities and welfare indicators for the general QUAIDS model are

Uncompensated (Marshallian) elasticities

Precautionary risk elasticities:

$$\epsilon_{ij}^r \equiv \left[E\left[\frac{\omega_i}{\mathbf{p}_i}\right]^2 E\mathbf{p}_j^2\right]^{-\frac{1}{2}} \times cov\left[\phi_{ij}\mathbf{p}_j \text{ , } \mathbf{p}_j\right] \qquad i,j=1,\ldots,n \tag{5.1.10}$$

Nonrisk elasticities:

$$\epsilon_{ij}^t \equiv \left[E\left[\frac{\omega_i}{\mathbf{p}_i}\right]^2 E\mathbf{p}_j^2\right]^{-\frac{1}{2}} \times E(\phi_{ij}\mathbf{p}_j)E\mathbf{p}_j \qquad i,j=1,\ldots,n \tag{5.1.11}$$

Total uncompensated elasticities:

$$\epsilon_{ij} = \epsilon_{ij}^r + \epsilon_{ij}^t = \left[E\left[\frac{\omega_i}{\mathbf{p}_i}\right]^2 E\mathbf{p}_j^2\right]^{-\frac{1}{2}} \times E\left(\phi_{ij}\mathbf{p}_j^2\right) \qquad i,j=1,\ldots,n \tag{5.1.12}$$

Income/risk aversion effects elasticities

Precautionary risk aversion effects:

$$\epsilon_{ij}^{rm} \equiv -\left[E\left[\frac{\omega_i}{\mathbf{p}_i}\right]^2 E\mathbf{p}_j^2\right]^{-\frac{1}{2}} \times cov\left[\varphi_i^m\omega_j \text{ , } \mathbf{p}_j\right] \qquad i,j=1,\ldots,n \tag{5.1.13}$$

Nonrisk pure income effects:

$$\epsilon_{ij}^{tm} \equiv -\left[E\left(\frac{\omega_i}{\mathbf{p}_i}\right)^2 E\mathbf{p}_j^2\right]^{-\frac{1}{2}} \times E\left(\varphi_i^m \omega_j\right) E\mathbf{p}_j \qquad i,j=1,\ldots,n \tag{5.1.14}$$

Total income/risk aversion effects:

$$\epsilon_{ij}^{m} = \epsilon_{ij}^{rm} + \epsilon_{ij}^{tm} = -\left[E\left(\frac{\omega_i}{\mathbf{p}_i}\right)^2 E\mathbf{p}_j^2\right]^{-\frac{1}{2}} \times E\left(\varphi_i^m \omega_j \mathbf{p}_j\right) \qquad i,j=1,\ldots,n \tag{5.1.15}$$

Compensated (Hicksian) elasticities

The compensated elasticities are calculated by using the Slustky decomposition (see chapter 2) as the differences between the uncompenasted elasticities and the income/risk aversion effects respectively.

Precautionary risk substitution elasticities: $\epsilon_{ij}^{rs} = \epsilon_{ij}^{r} - \epsilon_{ij}^{rm} \qquad i,j=1,\ldots,n$

Taste substitution elasticities: $\epsilon_{ij}^{ts} = \epsilon_{ij}^{t} - \epsilon_{ij}^{tm} \qquad i,j=1,\ldots,n$

Total substitution elasticities: $\epsilon_{ij}^{s} = \epsilon_{ij}^{rs} + \epsilon_{ij}^{ts} = \epsilon_{ij} - \epsilon_{ij}^{m} \qquad i,j=1,\ldots,n$

Budget elasticities

We recall from chapter 2 that in a uncertainty context the budget elasticities are random variables. So their means and variances are what will be numerically computed.

Budget elasticities:

$$\eta_i \equiv \left[E\left(\frac{\omega_i}{\mathbf{p}_i}\right)^2\right]^{-\frac{1}{2}} \varphi_i^m \qquad i=1,\ldots,n \tag{5.1.16}$$

$$\text{means: } E\eta_i \equiv \left[E\left(\frac{\omega_i}{\mathbf{p}_i}\right)^2\right]^{-\frac{1}{2}} E\varphi_i^m \qquad i=1,\ldots,n$$

and

$$\text{Variances: } var(\eta_i) \equiv \left[E\left[\frac{\omega_i}{\mathbf{p}_i}\right]^2 \right]^{-1} var(\varphi_i^m) \qquad i=1,\ldots,n$$

5.1.3 Indicators of welfare change and risk attitude

The welfare indicators and measures of risk premium, cost of risk, and risk aversion are calculated for a price/budget change from $(\mathbf{p}, m)$ to $(\mathbf{q}, m')$. Their expressions are obtained by using the general formulas of chapter 2, along with the expressions of the compensation function and compensated demands as given by (5.1.4) and (5.1.6) respectively. We give them here for completeness.

Indicators of total welfare change

Equivalent variation: $\mathbf{EV} = \mu(\mathbf{p}, \mathbf{q}, m') - m$

Allen Living-standard index: $Q_A = \dfrac{\mu(\mathbf{p}, \mathbf{q}, m')}{m}$

Mckenzie-Pearce cost-of-Living index: $P_M = \dfrac{m'}{\mu(\mathbf{p}, \mathbf{q}, m')}$

Decomposition of the total welfare indicators

Total risk premium (part of EV due to risk aversion): $R = -\sum_{i=1}^{n} cov(cd_i, \mathbf{p}_i)$

Total nonrisk component of EV: $T = \sum_{i=1}^{n} Ecd_i E\mathbf{p}_i$

Risk premium index (part of the Allen index due to risk aversion): $Q_A^r = \dfrac{R}{m}$

Nonrisk component of the Allen index: $Q_A^t = \dfrac{T}{m} + 1$

Cost-of-risk index (part of the Mckenzie-Pearce index due to risk aversion):

$$P_M^r = \frac{R}{T+m} P_M$$

Nonrisk component of the Mckenzie-Pearce index: $P_M^t = \dfrac{m'}{T+m}$

Measures of risk aversion

Risk aversion matrix: $R^A = -\dfrac{1}{m}\left(cov(cd_i, \mathbf{p}_j)\right)_{\substack{1\le i\le n \\ 1\le j\le n}}$

The expressions above are general and they apply to both the deterministic and the uncertainty QUAIDS models. Naturally, in the deterministic case the terms referring to risk are all zero.

The deterministic QUAIDS model of B.B.L is obtained by defining $a(\mathbf{p})$, $b(\mathbf{p})$, and $\lambda(\mathbf{p})$ as respectively:

$$\ln a(\mathbf{p}) = \alpha_0 + \sum_{k=1}^{n} \alpha_k \ln \mathbf{p}_k + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n} \gamma_{kl} \ln \mathbf{p}_k \ln \mathbf{p}_l \tag{5.1.17}$$

$$b(\mathbf{p}) = \prod_{k=1}^{n} \mathbf{p}_k^{\beta_k} \tag{5.1.18}$$

and

$$\lambda(\mathbf{p}) = \sum_{k=1}^{n} \lambda_k \ln \mathbf{p}_k \tag{5.1.19}$$

where the α's, β's,γ's, and λ's are parameters to be estimated.

5.2 The "expected" version of the QUAIDS model under uncertainty

Under price uncertainty, where the right hand sides of (5.1.17), (5.1.18), and (5.1.19) become random, we can define an "expected" version of the QUAIDS model under uncertainty (EQUAIDS) by taking the expected values of $a(\mathbf{p})$, $b(\mathbf{p})$ and $\lambda(\mathbf{p})$. Since the general QUAIDS model is integrable, there always exists a direct utility functional defined in the choice space $L_2(\Omega, \mathbb{R}^n)$, which corresponds to the "expected" version of the QUAIDS model. However, this utility functional need not be an expected utility.[132]

Hence, we define the EQUAIDS model by taking $\boldsymbol{a}(\mathbf{p})$, $\boldsymbol{b}(\mathbf{p})$, and $\lambda(\mathbf{p})$ as respectively:

$$\ln \boldsymbol{a}(\mathbf{p}) = \alpha_0 + \sum_{k=1}^{n} \alpha_k E \ln \mathbf{p}_k + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n} \gamma_{kl} E(\ln \mathbf{p}_k \ln \mathbf{p}_l) \qquad \textbf{(5.2.1)}$$

$$\boldsymbol{b}(\mathbf{p}) = E\left[\prod_{k=1}^{n} \mathbf{p}_k^{\beta_k}\right] \qquad \textbf{(5.2.2)}$$

and

$$\lambda(\mathbf{p}) = \sum_{k=1}^{n} \lambda_k E \ln \mathbf{p}_k \qquad \textbf{(5.2.3)}$$

where the α's, β's, γ's, and λ's are parameters to be estimated. Then, using lemma **3.3.1**, it is easy to see that $\boldsymbol{a}(\mathbf{p})$, $\boldsymbol{b}(\mathbf{p})$, and $\lambda(\mathbf{p})$ are Fréchet differentiable (in the extended sense) with

$$\frac{\partial \ln \boldsymbol{a}_i(\mathbf{p})}{\partial \mathbf{p}_i} = \alpha_i + \sum_{k=1}^{n} \gamma_{ik} \ln \mathbf{p}_k \quad i = 1,\ldots,n \qquad \textbf{(5.2.4)}$$

$$\frac{\partial^2 \ln \boldsymbol{a}(\mathbf{p})}{\partial \mathbf{p}_i \partial \mathbf{p}_j} = \frac{\gamma_{ij}}{\mathbf{p}_i \mathbf{p}_j} - \frac{\delta_{ij}}{\mathbf{p}_i^2}\left[\alpha_i + \sum_{k=1}^{n} \gamma_{ik} \ln \mathbf{p}_k\right] \qquad \textbf{(5.2.5)}$$

$$\boldsymbol{b}_i'(\mathbf{p}) = \frac{\beta_i}{\mathbf{p}_i} \boldsymbol{b}(\mathbf{p}) \quad i=1,\ldots,n \qquad \textbf{(5.2.6)}$$

and

[132] Whether or not the EQUAIDS model is generated by an expected utility preference is an open question. At this point, all we can say is that it is integrable and the corresponding direct utility functional may be a "non-expected" utility which cannot be reduced to an expected utility form.

$$b''_{ij}(\mathbf{p}) = \frac{\beta_i}{\mathbf{p}_i}\left[\frac{\beta_j}{\mathbf{p}_j} - \frac{\delta_{ij}}{\mathbf{p}_i}\right] b(\mathbf{p}) \quad i,j=1,\ldots,n \tag{5.2.7}$$

$$\lambda_i'(\mathbf{p}) = \frac{\lambda_i}{\mathbf{p}_i} \quad i=1,\ldots,n \tag{5.2.8}$$

$$\lambda''_{ij}(\mathbf{p}) = -\delta_{ij}\frac{\lambda_i}{\mathbf{p}_i^2} \quad i,j=1,\ldots,n \tag{5.2.9}$$

where δ_{ij} is the Kroneker delta which is equal to 1 if $i=j$ and 0 otherwise.

Hence, allowing for the household characteristics to affect some of the preference parameters, the budget shares to be estimated are:

$$\omega_i = \alpha_i(\mathbf{z}) + \sum_{k=1}^{n} \gamma_{ik} \ln \mathbf{p}_k + \beta_i \ln \frac{m}{a(\mathbf{p})} - \frac{\lambda_i}{b(\mathbf{p})}\left[\ln \frac{m}{a(\mathbf{p})}\right]^2 \quad i=1,\ldots,n \tag{5.2.10}$$

where $\mathbf{z}$ is the r-dimensional vector of the time variables, the geographical variables, and the other household characteristics, while $\alpha_i(\mathbf{z})$ represents the total effect of $\mathbf{z}$ on the i^{th} share, and is given by

$$\alpha_i(\mathbf{z}) = \alpha_{i0} + \sum_{k=1}^{r} \alpha_{ik} \mathbf{z}_k \tag{5.2.11}$$

The α's, γ's, β's and λ's are the parameters to be estimated.

The above expressions are identical to the ones in the deterministic QUAIDS model, except here ω_i and $\mathbf{p}$ are random variables, while they are theoretically nonrandom in the deterministic case. However, in terms of econometric estimation it is now customary to consider both the dependent and independent variables of deterministic models as stochastic (econometric estimation will be discussed below). Note that the shares depend on consumers "expectations" and "beliefs" only through the "price aggregators" $a(\mathbf{p})$, $b(\mathbf{p})$, and $\lambda(\mathbf{p})$.

For the computation of the elasticities and welfare indicators, the relevant expressions are:

$$\varphi_i^m = \frac{\omega_i}{\mathbf{p}_i} + \frac{\beta_i}{\mathbf{p}_i} - \frac{2\lambda_i}{\mathbf{p}_i b(\mathbf{p})}\ln\frac{m}{a(\mathbf{p})} \qquad i=1,\ldots,n \tag{5.2.12}$$

$$\psi_i = \frac{\omega_i}{\mathbf{p}_i} - \frac{\beta_i}{\mathbf{p}_i}\ln\frac{m}{a(\mathbf{p})} + \frac{\lambda_i}{\mathbf{p}_i b(\mathbf{p})}\left[\ln\frac{m}{a(\mathbf{p})}\right]^2 \qquad i=1,\ldots,n \tag{5.2.13}$$

$$\phi_{ij} = \frac{\gamma_{ij}}{\mathbf{p}_i\mathbf{p}_j} + \left[\frac{\omega_i}{\mathbf{p}_i} - \varphi_i^m\right]\psi_j + \frac{\lambda_i\beta_j}{\mathbf{p}_i\mathbf{p}_j b(\mathbf{p})}\left[\ln\frac{m}{a(\mathbf{p})}\right]^2 - \delta_{ij}\frac{\omega_i}{\mathbf{p}_i^2} \qquad i,j=1,\ldots,n \tag{5.2.14}$$

and

$$cd_i = \mu\left\{\frac{\omega_i}{\mathbf{p}_i} + \frac{\beta_i}{\mathbf{p}_i}\ln\frac{\mu}{m} - \frac{\lambda_i}{\mathbf{p}_i b(\mathbf{p})}\left[\ln\frac{m\mu}{a(\mathbf{p})^2}\right]\ln\frac{\mu}{m}\right\} \qquad i=1,\ldots,n \tag{5.2.15}$$

5.3 The "non-expected" versions of the QUAIDS model under uncertainty

Under uncertainty, there are at least two possible "non-expected" versions of the QUAIDS model. One is obtained by replacing in the right hand sides of (5.1.17), (5.1.18), and (5.1.19) the $\mathbf{p}_k$'s with their respective L_2 norms; the other by replacing the logarithm of the $\mathbf{p}_k$'s with their respective L_2 norms. However, with respect to uncertainty, the latter version is more "flexible" in the sense that the derived shares, elasticities, and welfare indicators are functions of moments of prices, with the order of the moments being determined empirically. On the other hand, with the former version the shares, elasticities and welfare indicators are constrained to be functions of the second moments of prices only.[133] Nevertheless, for completeness the results pertaining to the former "non-expected" version are given in Appendix 5.

Thus, we define the "non-expected" version of the QUAIDS model under uncertainty (NEQUAIDS) by taking $\boldsymbol{a}(\mathbf{p})$, $\boldsymbol{b}(\mathbf{p})$, and $\lambda(\mathbf{p})$ as respectively:

$$\ln \boldsymbol{a}(\mathbf{p}) = \alpha_0 + \sum_{k=1}^{n} \alpha_k \|\ln \mathbf{p}_k\| + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n} \gamma_{kl} \|\ln \mathbf{p}_k\| \, \|\ln \mathbf{p}_l\| \qquad (5.3.1)$$

$$\boldsymbol{b}(\mathbf{p}) = \beta_0 \prod_{k=1}^{n} \|\mathbf{p}_k^{\beta_k}\| \qquad (5.3.2)$$

and

$$\lambda(\mathbf{p}) = \sum_{k=1}^{n} \lambda_k \|\ln \mathbf{p}_k\| \qquad (5.3.3)$$

where $\|\mathbf{x}_i\|^2 = E\mathbf{x}_i^2$ is the second moment of the random variable $\mathbf{x}_i$, and the α's, β's, γ's, and λ's are parameters to be estimated.[134] By Lemma **4.4.1** $\boldsymbol{a}(\mathbf{p})$, $\boldsymbol{b}(\mathbf{p})$, and $\lambda(\mathbf{p})$ are Fréchet differentiable (in the extended sense) with

$$\frac{\partial \ln \boldsymbol{a}_i(\mathbf{p})}{\partial \mathbf{p}_i} = \left[\alpha_i + \sum_{k=1}^{n} \gamma_{ik} \|\ln \mathbf{p}_k\|\right] \frac{\ln \mathbf{p}_i}{\mathbf{p}_i \|\ln \mathbf{p}_i\|} \qquad i = 1,\dots,n \qquad (5.3.4)$$

[133] Another theoretical reason for preferring the latter version is the fact that in some of the examples originally used in chapter 3, which were derived in the same way as with the former version, the total risk premium ended up being globally negative. This indicated that the preferences generated by those examples were not convex.

[134] For convenience, we will keep the notation $\|\mathbf{x}_i\|$, and substitute for $E\mathbf{x}_i^2$ only for computational purposes.

$$\frac{\partial^2 \ln a(\mathbf{p})}{\partial \mathbf{p}_i \partial \mathbf{p}_j} = \frac{1}{\mathbf{p}_i \mathbf{p}_j} \left\{ \gamma_{ij} \frac{\ln \mathbf{p}_i \ln \mathbf{p}_j}{\|\ln \mathbf{p}_i\| \, \|\ln \mathbf{p}_j\|} + \delta_{ij} \left[\alpha_i + \sum_{k=1}^{n} \gamma_{ik} \|\ln \mathbf{p}_k\| \right] \left[\frac{1}{\|\ln \mathbf{p}_i\|} - \frac{\ln \mathbf{p}_i}{\|\ln \mathbf{p}_i\|} - \frac{(\ln \mathbf{p}_i)^2}{\|\ln \mathbf{p}_i\|^3} \right] \right\} \tag{5.3.5}$$

$$b_i'(\mathbf{p}) = \beta_i b(\mathbf{p}) \frac{\mathbf{p}_i^{2\beta_i - 1}}{\|\mathbf{p}_i^{\beta_i}\|^2} \qquad i=1,\ldots,n \tag{5.3.6}$$

$$b_{ij}''(\mathbf{p}) = \beta_i \beta_j b(\mathbf{p}) \frac{\mathbf{p}_i^{2\beta_i - 1} \mathbf{p}_j^{2\beta_j - 1}}{\|\mathbf{p}_i^{\beta_i}\|^2 \|\mathbf{p}_j^{\beta_j}\|^2} + \delta_{ij} \beta_i b(\mathbf{p}) \left\{ (2\beta_i - 1) \frac{\mathbf{p}_i^{2\beta_i - 2}}{\|\mathbf{p}_i^{\beta_i}\|^2} - 2\beta_i \frac{\mathbf{p}_i^{4\beta_i - 2}}{\|\mathbf{p}_i^{\beta_i}\|^4} \right\} \tag{5.3.7}$$

$$\lambda_i'(\mathbf{p}) = \lambda_i \frac{\ln \mathbf{p}_i}{\mathbf{p}_i \|\ln \mathbf{p}_i\|} \qquad i=1,\ldots,n \tag{5.3.8}$$

and

$$\lambda_{ij}''(\mathbf{p}) = \delta_{ij} \frac{\lambda_i}{\mathbf{p}_i \mathbf{p}_j} \left[\frac{1}{\|\ln \mathbf{p}_i\|} - \frac{\ln \mathbf{p}_i}{\|\ln \mathbf{p}_i\|} - \frac{(\ln \mathbf{p}_i)^2}{\|\ln \mathbf{p}_i\|^3} \right] \qquad i,j=1,\ldots,n \tag{5.3.9}$$

where δ_{ij} is the Kroneker delta which is equal to 1 if $i=j$ and 0 otherwise.

Hence, allowing for the household characteristics to affect some of the preference parameters, the budget shares for the NEQUAIDS to be estimated are:

$$\omega_i = \alpha_i(\mathbf{z}) \frac{\ln \mathbf{p}_i}{\|\ln \mathbf{p}_i\|} + \sum_{k=1}^{n} \gamma_{ik} \frac{\ln \mathbf{p}_i}{\|\ln \mathbf{p}_i\|} \ln \|\mathbf{p}_k\| + \beta_i \frac{\mathbf{p}_i^{2\beta_i}}{\|\mathbf{p}_i^{\beta_i}\|^2} \ln \frac{m}{a(\mathbf{p})} - \frac{\lambda_i \ln \mathbf{p}_i}{\|\ln \mathbf{p}_i\| \, b(\mathbf{p})} \left[\ln \frac{m}{a(\mathbf{p})} \right]^2 \tag{5.3.10}$$

where **z** is the *r*-dimensional vector of the time variables, the geographical variables, and the other household characteristics, while $\alpha_i(\mathbf{z})$ represents the total effect of **z** on the i^{th} share, and is given by

$$\alpha_i(\mathbf{z}) = \alpha_{i0} + \sum_{k=1}^{r} \alpha_{ik}\mathbf{z}_k \tag{5.3.11}$$

The α's, γ's, β's and λ's are the parameters to be estimated.

For the computation of the elasticities and welfare indicators, the relevant expressions are:

$$\varphi_i^m = \frac{\omega_i}{\mathbf{p}_i} + \beta_i \frac{\mathbf{p}_i^{2\beta_i}}{\mathbf{p}_i \| \mathbf{p}_i^{\beta_i} \|^2} - \frac{2\lambda_i \ln \mathbf{p}_i}{\mathbf{p}_i \| \ln \mathbf{p}_i \| b(\mathbf{p})} \ln \frac{m}{a(\mathbf{p})} \qquad i=1,\ldots,n \tag{5.3.12}$$

$$\psi_i = \frac{\omega_i}{\mathbf{p}_i} - \beta_i \frac{\mathbf{p}_i^{2\beta_i}}{\mathbf{p}_i \| \mathbf{p}_i^{\beta_i} \|^2} \ln \frac{m}{a(\mathbf{p})} + \frac{\lambda_i \ln \mathbf{p}_i}{\mathbf{p}_i \| \ln \mathbf{p}_i \| b(\mathbf{p})} \left[\ln \frac{m}{a(\mathbf{p})} \right]^2 \qquad i=1,\ldots,n \tag{5.3.13}$$

$$\begin{aligned}
\phi_{ij} = {} & \gamma_{ij} \frac{\ln \mathbf{p}_i \ln \mathbf{p}_j}{\mathbf{p}_i \mathbf{p}_j \| \ln \mathbf{p}_i \| \| \ln \mathbf{p}_j \|} + \left[\frac{\omega_i}{\mathbf{p}_i} - \varphi_i^m \right] \psi_j \\
& + \frac{\lambda_i \beta_j \ln \mathbf{p}_i}{\mathbf{p}_i \mathbf{p}_j \| \ln \mathbf{p}_i \| b(\mathbf{p})} \frac{\mathbf{p}_j^{2\beta_j}}{\| \mathbf{p}_j^{\beta_j} \|^2} \left[\ln \frac{m}{a(\mathbf{p})} \right]^2 \\
& + \frac{\delta_{ij}}{\mathbf{p}_i^2} \left\{ \omega_i \left[\frac{1}{\ln \mathbf{p}_i} - 1 - \frac{\ln \mathbf{p}_i}{\| \ln \mathbf{p}_i \|^2} \right] \right. \\
& \left. + \beta_i \frac{\mathbf{p}_i^{2\beta_i}}{\mathbf{p}_i^2 \| \mathbf{p}_i^{\beta_i} \|^2} \left[2\beta_i \left[1 - \frac{\mathbf{p}_i^{2\beta_i}}{\| \mathbf{p}_i^{\beta_i} \|^2} \right] - \frac{1}{\ln \mathbf{p}_i} + \frac{\ln \mathbf{p}_i}{\| \ln \mathbf{p}_i \|^2} \right] \ln \frac{m}{a(\mathbf{p})} \right. \\
& i,j=1,\ldots,n
\end{aligned} \tag{5.3.14}$$

and

$$cd_i = \mu \left\{ \frac{\omega_i}{\mathbf{p}_i} + \beta_i \frac{\mathbf{p}_i^{2\beta_i}}{\mathbf{p}_i \| \mathbf{p}_i^{\beta_i} \|^2} \ln\frac{\mu}{m} - \frac{\lambda_i \ln\mathbf{p}_i}{\mathbf{p}_i \| \ln\mathbf{p}_i \| b(\mathbf{p})} \left[\ln\frac{m\mu}{a(\mathbf{p})^2} \right] \ln\frac{\mu}{m} \right\} \quad i=1,\ldots,n \qquad (5.3.15)$$

Before moving to the econometric estimation issues, we note that all the expressions pertaining to the uncertainty versions of QUAIDS reduce to their deterministic counterparts when one assumes no uncertainty.

5.4 Econometric estimation

5.4.1 The Econometric estimation issues

For the "expected" and "non-expected" versions of the QUAIDS model under uncertainty the econometric estimation of the share equations requires prior estimates of the unobservable consumer's *expected* values for the various random variables which appear in the right hand side of the share equations. In this static framework where we have abstracted from information issues and identified the consumer's "expectation" process with the *unconditional* expectation operator E, a natural way to proceed is to estimate the unobservable *unconditional* moments by their sample analogues. That is, for any random variable $\mathbf{x}$ the *unconditional* moments $E\mathbf{x}$, and $\|\mathbf{x}\|^2 = E\mathbf{x}^2$ are estimated respectively by $\frac{1}{T}\sum_{t=1}^{T} \mathrm{x}_t$, and $\frac{1}{T}\sum_{t=1}^{T} \mathrm{x}_t^2$, where the x_t's are the T realizations of the random variable $\mathbf{x}$. Then, standard asymptotic theory tells us that the resulting share equations are asymptotically equivalent to the original ones. However, to the extent that the geographical variability in prices is irrelevant for the purpose of analyzing the impact of price uncertainty on consumption choices, in a static framework only one realization of the random vector of prices $\mathbf{p}$ can be observed for a given consumer (or group of consumers living in the same geographical areas). This makes it impossible to consistently estimate the *unconditional* moments unless one interpret the different values of the price vector across time as realizations of the same random vector of price p; and take sample averages across the time dimension. This is a reasonable procedure if one wants to remain within the static framework and avoid the econometric issues related to the estimation of *conditional* moments. The misspecifications resulting from this procedure can then be taken to be part of the measurement errors and/or omitted variables.

The second estimation issue relates to the endogeneity of total expenditure m which is computed by summing the expenditures of the individual commodities.[135] In the deterministic case the endogeneity of total expenditure is usually taken care of by instrumenting with variables that, while correlated with total expenditure, are orthogonal to the measurement error. In this uncertainty context, a natural extension of this procedure is to include in the list of instruments some of the moments of prices.

The deterministic QUAIDS functional form generates a nonlinear demand system, and the one generated by the "non-expected" QUAIDS is even "more" nonlinear - if one can say so - because of the price dependent coefficients in the first income term of the share equations. However, existing Nonlinear Least Squares (NLS) methods can be used to estimate all the parameters if the system is not too large and the sample size not too big (see, for example, Amemiya (1985), chapter 8). Alternatively, one can note that conditional on the terms $a(\mathbf{p})$, $b(\mathbf{p})$, and $\frac{\mathbf{p}_i^{2\beta_i}}{\|\mathbf{p}_i^{\beta_i}\|^2}$, the shares equations are linear in the parameters. Furthermore, conditional on α_0, these parameters uniquely determine the values of the above terms. Hence, given α_0, one can use an iterative estimation method that uses initial starting values for $a(\mathbf{p})$, $b(\mathbf{p})$, and $\frac{\mathbf{p}_i^{2\beta_i}}{\|\mathbf{p}_i^{\beta_i}\|^2}$, to get least squares estimates of the parameters, which are in turn used to get new values for the above terms. The procedure is to be repeated until convergence. The identification of α_0 is what usually causes problems in this iterative

[135] The other sources of endogeneity of total expenditure will be discussed in the next chapter.

procedure. But, in the following section we discuss in detail an iterative procedure in which α_0 can be consistently estimated within each iteration.

5.4.2 The Iterated Moment Estimator and the Minimum Chi-square method

For demand systems with large sample size, Full Information Maximum likelihood or NLS methods for full systems such as those described in chapter 5 of Amemiya (1985), are in general computationally expensive. To impose the symmetry restriction implied by the theory, one usually needs a system-wide estimation method that enables the imposition of cross-equation restrictions. Fortunately, we can circumvent both the non-linearity of the system and the need for a system-wide estimation by using a two-stage estimation procedure.[136] In the first stage we use the Iterated Moment Estimator of Blundell and Robin (1993) which exploits the conditional linearity of the system as discussed above to allow an equation-by-equation OLS estimation with little efficiency loss compared to an NLS estimation. And in the second stage we use the Minimum Chi-square method to impose the symmetry restriction without loss of efficiency compared to a system-wide estimation (Ferguson (1958) and Rothenberg (1973)).[137] The other two restrictions implied by the theory which we are concerned with - homogeneity and adding-up - can be imposed with the equation-by-equation estimation method. Indeed, the homogeneity restriction can be imposed equation by equation by dividing one of the prices through the remaining vector of prices, while the adding-up restriction can be imposed by dropping one of the equations from the

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[136] In fact, it is a three-stage estimation if one takes care of the endogeneity of total expenditure by instrumenting.

[137] We note that in contrast to the deterministic and "expected" cases, the vector of regressors in the "non-expected" case varies from one equation to another, so equation-by-equation estimation does not yield the same estimator as the unrestricted GLS estimator produced by a system-wide estimation.

estimation.[138] This two-stage procedure is also used by B.B.L (1994), Browning and Chiappori (1994), Browning and Meghir (1991), and Blundell et al., (1993). In following sections we will give a summary of the two stages.

5.4.2.1 The Iterated Moment Estimator for conditionally linear systems

To simplify the discussion and notation, we will confine ourself to the deterministic AIDS case since there is no loss of generality in doing so.

Recall that if we allow for the household characteristics to affect some of the preference parameters, the n AIDS budget shares to be estimated are:

$$\omega_i = \alpha_i(\mathbf{z}) + \sum_{k=1}^{n} \gamma_{ik} \ln \mathbf{p}_k + \beta_i \ln \frac{m}{a(\mathbf{p})} + u_i \quad i = 1,\ldots,n \tag{5.4.1}$$

where $\mathbf{z}$ is the q-dimensional vector of the time variables, the geographical variables, and the other household characteristics, while $\alpha_i(\mathbf{z})$ represents the total effect of $\mathbf{z}$ on the i^{th} share, and is given by

$$\alpha_i(\mathbf{z}) = \alpha_{i0} + \sum_{k=1}^{q} \alpha_{ik} \mathbf{z}_k \tag{5.4.2}$$

The budget deflator $a(\mathbf{p})$ is given by:

$$\ln a(\mathbf{p}) = \alpha_0 + \sum_{i=1}^{n} \alpha_i(\mathbf{z}) \ln \mathbf{p}_i + \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \gamma_{kl} \ln \mathbf{p}_k \ln \mathbf{p}_l \tag{5.4.3}$$

α_0, and the α's, γ's, and β's are the parameters to be estimated, and u_i is an i.i.d.error term with mean $Eu_i = 0$, and variance $\text{var}(u_i) = \sigma_i^2$, $i = 1,\ldots n$. Let $\text{cov}(u_i, u_j) = \sigma_{ij}$ be the covariances of the errors for different equations and $\Sigma \equiv cov(u) = (\sigma_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le n}}$.

[138] Note, however, that when updating the price "aggregators" $a(\mathbf{p})$, and $b(\mathbf{p})$ in each iteration of the IME procedure, one needs to recover all the parameters of the system (including those that were not estimated).

Conditional on the budget deflator $a(\mathbf{p})$, the share equations (5.4.1) are linear in the parameters; so that except for α_0 they can all be consistently estimated with OLS equation by equation. The Iterated Moment Estimator (IME) of Blundell and Robin (1993) exploits this conditional linearity. This is well known for the AIDS model and has been exploited by using the Stone price index to approximate $\mathbf{a}(\mathbf{p})$. The IME allows one to estimate $\mathbf{a}(\mathbf{p})$ instead of estimating it. To present the IME we fixe α_0 at some constant value and write the regression equations in (5.4.1) as:[139]

$$w_{it} = g(\mathbf{x}_t; \alpha_0, \theta)'\theta_i \ - u_{it} \quad i=1,\dots,n; \quad t=1,\dots,T \qquad \textbf{(5.4.4)}$$

where $\theta=(\theta_1,\dots, \theta_n)$, $\theta_i=(\alpha_{i.}, \gamma_{i.}, \beta_i)$, $\mathbf{x}_t=(\mathbf{z}_t, \ln\mathbf{p}_{1t},\dots, \ln\mathbf{p}_{nt}, \ln m_t)$, g is a nonlinear function of its arguments taking values in $\mathbf{R}^{q+1+n+1}$, and T is the number of observations. For the endogeneity of total expenditure m, let $\mathbf{h}_t$ be the full K-dimensional vector of instruments such that $K > q+1+n+1$ and $\mathbf{h}_t$ satisfies the identification conditions:

$$E(u_t \,|\, \mathbf{h}_t) = 0 \quad t=1,\dots,T \qquad \textbf{(5.4.5)}$$

$$var(u_t \,|\, \mathbf{h}_t) = \Sigma \quad t=1,\dots,T \qquad \textbf{(5.4.6)}$$

where $\mathrm{u}_t = (\mathrm{u}_{1t},\dots, \mathrm{u}_{nt})'$. Furthermore, let

$$w_i = \begin{bmatrix} w_{i1} \\ \cdot \\ \cdot \\ \cdot \\ w_{iT} \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_T \end{bmatrix}, \quad G_T(\alpha_0, \theta) = \begin{bmatrix} g(\mathbf{x}_1, \alpha_0, \theta)' \\ \cdot \\ \cdot \\ \cdot \\ g(\mathbf{x}_T, \alpha_0, \theta)' \end{bmatrix}, \quad H_T = \begin{bmatrix} h_1 \\ \cdot \\ \cdot \\ \cdot \\ h_T \end{bmatrix}, \quad \text{and } P_{H_T} = H_T\left(H_T'H_T\right)^{-1} H_T$$

With these notations, the IME for (5.4.4) is defined as the solution $\hat{\theta}_T$ of the following fixed point problem:

[139] To simplify the notation we use the same notation for the true unknown parameters and generic elements of the corresponding parameter space.

$$\Psi_T(\theta) = \theta! \tag{5.4.7}$$

where $\Psi_T(\theta) \equiv \left(\Psi_{1T}(\theta), \ldots, \Psi_{nT}(\theta)\right)$, and

$$\Psi_{iT}(\theta) \equiv \left(G_T(\theta)' P_{H_T} G_T(\theta)\right)^{-1} G_T(\theta)' P_{H_T} w_{iT} \quad i=1,\ldots,n \tag{5.4.8}$$

When (5.4.7) has a unique solution, the IME estimate $\hat{\theta}_{iT}$ for the i^{th} equation can be obtained as the converging value of the following iterative procedure:

$$\hat{\theta}_{iT}^{(s+1)} = \Psi_{iT}(\hat{\theta}_T^{(s)}) \quad i=1,\ldots,n; \quad s=1,2,3,\ldots. \tag{5.4.9}$$

Hence each iteration consists of an equation-by-equation 2SLS linear regression of (5.4.1).

Under some standard regularity conditions, Blundell and Robin (1993) prove that $\hat{\theta}_T$ is consistent and asymptotically normal with an asymptotic variance-covariance matrix given, in the homoscedastic case, by:

$$Asv(\hat{\theta}_T) \equiv \left[A_T(\theta)' \left[\Sigma^{-1} \otimes \left(P_{H_T} G_T(\theta) \left(G_T(\theta)' P_{H_T} G_T(\theta)\right)^{-1} G_T(\theta)' P_{H_T}\right)\right] A_T(\theta)\right]^{-1} \tag{5.4.10}$$

with

$$A_T(\theta) = I_n \otimes G_T(\theta) + F_T(\theta) \tag{5.4.11}$$

where I_n is the identity matrix of size n and $F_T(\theta)$ is the $n \times n$ block matrix whose i, j^{th} block is the $T \times (q+1+n+1)$ matrix the t^{th} row of which is $\theta_i' \frac{\partial g(\mathbf{x}_t, \alpha_0, \theta)}{\partial \theta_j}$, and $\otimes$ is the Kronecker product. To derive an alternative heteroscedastic-robust version of (5.4.10) we follow White (1982) to get :

$$Asv_h(\hat{\theta}_T) \equiv \left[A_T(\theta)' \left(I_n \otimes P_{H_T} G_T(\theta)\right) B_T(\theta)^{-1} \left(I_n \otimes P_{H_T} G_T(\theta)\right)' A_T(\theta)\right]^{-1} \tag{5.4.12}$$

with

$$B_T(\theta) = E\left[\left(I_n \otimes P_{H_T} G_T(\theta)\right) uu' \left(I_n \otimes P_{H_T} G_T(\theta)\right)\right] \tag{5.4.13}$$

Feasible consistent estimates of $Asv(\hat{\theta}_T)$ and $Asv_h(\hat{\theta}_T)$ are obtained by replacing the unknown parameters θ and Σ by $\hat{\theta}_T$ and $\hat{\Sigma} = \frac{1}{nT}\sum_{t=1}^{T} \hat{u}_t' \hat{u}_t$ respectively; where $\hat{u}_t = (\hat{u}_{1t}, \ldots, \hat{u}_{nt})'$ is the n-dimensional vector of residuals corresponding to the t^{th} observation. The matrix $B_T(\theta)$ can be estimated by:

$$\hat{B}_T(\hat{\theta}_T) = \left(I_n \otimes P_{H_T} G_T(\hat{\theta}_T)\right) \hat{D} \left(I_n \otimes P_{H_T} G_T(\hat{\theta}_T)\right) \tag{5.4.14}$$

where $\hat{D}$ is the $n \times n$ block matrix whose i, j^{th} block is the $T \times T$ diagonal matrix the t^{th} diagonal element of which is $\hat{u}_{it}\hat{u}_{jt}$.

Note, however, that the computation of $Asv(\hat{\theta}_T)$ and $Asv_h(\hat{\theta}_T)$ require a system-wide computation which can be prohibitive if the sample size is large. In the empirical analysis reported in the next chapter we compute the heteroscedastic-robust $Asv_h(\hat{\theta}_T)$. However, we were able to save significant computational time by reorganizing the system by observation instead of the standard equation-by-equation grouping of the observations required by (5.4.10) and (5.4.12). The reorganization of the system by observation significantly reduces the dimensions of the matrices to be multiplied and inverted to less than $n \times (q+1+n+1)$).[140] Indeed, with an organization by observations, the feasible estimate of $Asv_h(\hat{\theta}_T)$ can be computed using the following algebraically equivalent formula:

[140] Also, the asymptotic properties of the vector of parameters in a system of equations are easier to derive when the system is organized by observation rather than by equation (see Wooldridge, 1991).

$$A\hat{s}v_h(\hat{\theta}_T) \equiv \left[\sum_{t=1}^{T} A_t(\hat{\theta}_T)'\hat{g}_t(\hat{\theta}_T)\right]^{-1} \sum_{t=1}^{T} \hat{g}_t(\hat{\theta}_T)'\,\hat{u}_t'\,\hat{u}_t\hat{g}_t(\hat{\theta}_T) \left[\sum_{t=1}^{T} \hat{g}_t(\hat{\theta}_T)'A_t(\hat{\theta}_T)\right]^{-1} \quad \textbf{(5.4.15)}$$

where $\hat{g}_t(\hat{\theta}_T) = I_n \otimes \hat{v}_t(\hat{\theta}_T)$ and $\hat{v}_t(\hat{\theta}_T)$ is the t^{th} row of the matrix $P_{H_T}G_T(\hat{\theta}_T)$; $A_t(\hat{\theta}_T)$ is the $n \times n(q+1+n+1)$ matrix given by

$$A_t(\hat{\theta}_T) = \begin{bmatrix} A_{1t}(\hat{\theta}_T) \\ \cdot \\ \cdot \\ \cdot \\ A_{nt}(\hat{\theta}_T) \end{bmatrix}$$

with $A_{it}(\hat{\theta}_T)$ being the t^{th} row of the i^{th} $T \times n(q+1+n+1)$ rectangular block submatrix of $A(\hat{\theta}_T)$.

The converging parameter estimates of the iterative procedure just outlined depends on the fixed value of the "nuisance" parameter α_0. Hence, we need to find an estimator of it which allows us to update the deflator $a(\mathbf{p})$ in each iteration in a non arbitrary way. In the remainder of this section we detail a method that allows a consistent estimation of α_0 within each iteration.

Following an idea originally suggested by Deaton and Muellbauer (1980), B.B.L (1994) estimate the deterministic QUAIDS model by choosing α_0 as a number just below the lowest value of ln(m) in their sample. An earlier version of their paper used a less arbitrary choice of α_0 which consisted of a grid search in which the conditional log likelihood function for α_0 (given estimates of the α's, γ's, and β's parameters) is maximized. A similar choice of α_0 is used by Browning and Chiappori (1994), with the difference that instead of maximizing the conditional loglikelihood, they used the GMM estimator, while holding constant the GMM weighting matrix.

To find a less arbitrary and simpler estimator of α_0, we rewrite the share equations (5.4.1) as:

$$\omega_i = \alpha_i(\mathbf{z}) + \sum_{k=1}^{n} \gamma_{ik} \ln \mathbf{p}_k + \beta_i \ln \frac{m}{r(\mathbf{p})} - \beta_i \alpha_0 + u_i \quad i = 1, \ldots, n \tag{5.4.16}$$

where

$$\ln r(\mathbf{p}) \equiv \sum_{i=1}^{n} \alpha_i(\mathbf{z}) \ln \mathbf{p}_i + \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \gamma_{kl} \ln \mathbf{p}_k \ln \mathbf{p}_l \tag{5.4.17}$$

Equation (5.4.16) can be rewritten as:

$$y_i = \beta_i \alpha_0 - u_i \quad i = 1, \ldots, n \tag{5.4.18}$$

where y_i is defined as

$$y_i \equiv \alpha_i(\mathbf{z}) + \sum_{k=1}^{n} \gamma_{ik} \ln \mathbf{p}_k + \beta_i \ln \frac{m}{r(\mathbf{p})} - \omega_i \quad i = 1, \ldots, n \tag{5.4.19}$$

In terms of the observations, the regression equations are:

$$y_{it} = \beta_i \alpha_0 - u_{it} \quad i = 1, \ldots, n; \quad t = 1, \ldots, T \tag{5.4.20}$$

where T is the number of observations.

Hence, if y_i and β_i are known, then α_0 can be estimated from (5.4.20), which, conditional on the α's, γ's, and β's parameters, is a set of seemingly unrelated linear regressions (SUR) with the same parameter α_0 appearing in all equations. Hence, using the IME procedure just outlined, SUR methods can be applied to (5.4.20) to get a consistent estimate of α_0 (OLS or GLS) within each iteration. Another straightforward estimator of α_0 is given by the average:

$$\hat{\alpha}_o^A = \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{y_{it}}{\beta_i} = \frac{1}{n}\sum_{i=1}^{n}\frac{\bar{y}_i}{\beta_i} \tag{5.4.21}$$

where $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_n)'$, and $\bar{y}_i = \frac{1}{T}\sum_{t=1}^{T} y_{it}$. Under the standard assumptions for the error term u_i, this estimator of α_0 is consistent if the estimator of the α's, γ's, and β's parameters are consistent.

Furthermore, it is easy to check that the OLS and GLS for α_0 in the SUR regression in (5.4.20) simplify respectively to

$$\hat{\alpha}_o^{OLS} = (\beta'\beta)^{-1}\beta'\bar{y} \tag{5.4.22}$$

and

$$\hat{\alpha}_o^{GLS} = (\beta'\Sigma^{-1}\beta)^{-1}\beta'\Sigma^{-1}\bar{y} \tag{5.4.23}$$

where $\beta = (\beta_1, \ldots, \beta_n)$.

Expressions (5.4.22) and (5.4.23) show clearly that OLS and GLS applied to the SUR in (5.4.20) are merely OLS and GLS applied respectively to the following n-observations regression with heteroscedastic and autocorellated errors:

$$\bar{y}_i = \beta_i\alpha_0 + v_i \quad i = 1, \ldots, n \tag{5.4.24}$$

where $v_i = -\frac{1}{T}\sum_{t=1}^{T} u_{it}$, The v_i's have zero means and variance-covariance matrix equal to $\frac{1}{T}\Sigma$.

A straightforward application of the central limit theorem shows that α_0^A, α_0^{OLS}, and α_0^{GLS} are asymptotically normal with asymptotic variances given respectively by:

$$var[\hat{\alpha}_o^A] = \frac{1}{n^2T}\beta'\Sigma\underline{\beta} \tag{5.4.25}$$

$$var[\hat{\alpha}_o^{OLS}] = (\beta'\beta)^{-1}\beta'\Sigma\beta(\beta'\beta)^{-1} \tag{5.4.26}$$

and

$$var[\hat{\alpha}_o^{GLS}] = (\beta'\Sigma^{-1}\beta)^{-1} \qquad \textbf{(5.4.27)}$$

where $\underline{\beta} = (\frac{1}{\beta_1}, \ldots, \frac{1}{\beta_n})$.

Feasible estimates of α_0^A, α_0^{OLS}, and α_0^{GLS} and their respective asymptotic variances are obtained by replacing Σ and the α's, γ's, and β's parameters by their consistent estimates. These consistent estimates are obtained from the OLS equation-by-equation of the IME procedure applied to the conditionally linear share equations in (5.4.20). By the Slutsky theorem, the resulting estimators have respectively the same asymptotic distribution as the original ones under standard assumptions.

Interestingly, for our system of equations, it is the case that $\bar{y} = c\hat{\beta}$; that is $\frac{\bar{y}_i}{\hat{\beta}_i}$ is the same for all i. Therefore, all three feasible estimators of α_0 are algebraically identical and equal the common ratio $\frac{\bar{y}_i}{\hat{\beta}_i}$. This in turn implies that:

$$\hat{\alpha}_o^A = \frac{1}{n}\sum_{i=1}^{n}\frac{\bar{y}_i}{\hat{\beta}_i} = c \qquad \textbf{(5.4.28)}$$

$$\hat{\alpha}_o^{OLS} = (\hat{\beta}'\hat{\beta})^{-1}\hat{\beta}'\bar{y} = c(\hat{\beta}'\hat{\beta})^{-1}\hat{\beta}'\hat{\beta} = c \qquad \textbf{(5.4.29)}$$

and

$$\hat{\alpha}_o^{GLS} = (\hat{\beta}'\Sigma^{-1}\hat{\beta})^{-1}\hat{\beta}'\Sigma^{-1}\bar{y} = c(\hat{\beta}'\Sigma^{-1}\hat{\beta})^{-1}\hat{\beta}'\Sigma^{-1}\hat{\beta} = c \qquad \textbf{(5.4.30)}$$

Hence, we have derived a very simple estimator of α_0 allowing the full implementation of the IME procedure to the conditionally linear QUAIDS share equations.[141] It thus remains to impose the homogeneity, symmetry, and adding-up implied by the theory. Because the homogeneity condition is a within-equation, it can be imposed within each iteration of the IME procedure. Similarly, although the adding-up restrictions involve parameters from different equations, they can still be imposed within each iteration of the IME procedure by dropping one of the equations in the system. In fact, it is necessary to drop one of the equations in order to have a nonsingular variance-covariance estimate for error terms in the estimated system. However, in each iteration of the IME procedure one need to recover the non estimated parameters in order to update $r(\mathbf{p})$ which requires the full α and γ matrices. But, the homogeneity and the adding-up restrictions must still be maintained (if they are imposed during estimation) when computing any of the estimates of α_0 (i.e., when computing $\frac{\bar{y}_i}{\hat{\beta}_i}$ and taking average across equations). Because otherwise α_0 cannot be identified.

On the other hand, the IME procedure does not allow a direct imposition of the symmetry restrictions. Indeed, to get the symmetry-restricted parameter estimates, one needs to impose cross equation restrictions using simultaneously the relevant information contained

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141 For the general QUAIDS with or without uncertainty, the analogue of the equation (5.4.18) has the form

$$y_i = \left[\frac{\mathbf{p}_i b_i'(\mathbf{p})}{b(\mathbf{p})} - \frac{\mathbf{p}_i \lambda_i'(\mathbf{p})}{b(\mathbf{p})} \ln \frac{m}{r(\mathbf{p})} \right] \alpha_0 - u_i \qquad i=1,\ldots,n$$

where

$$y_i \equiv \mathbf{p}_i \left\{ \frac{\partial \ln a(\mathbf{p})}{\partial \mathbf{p}_i} + \frac{b_i'(\mathbf{p})}{b(\mathbf{p})} \ln \frac{m}{r(\mathbf{p})} - \frac{\lambda_i'(\mathbf{p})}{b(\mathbf{p})} \left[\ln \frac{m}{r(\mathbf{p})} \right]^2 - \alpha_0^2 \frac{\lambda_i'(\mathbf{p})}{b(\mathbf{p})} \right\} - \omega_i \qquad i=1,\ldots,n$$

In this case where the dependent variable y_i depends itself on α_0^2, one can still use a one-step linear procedure within each iteration by replacing α_0^2 by the square of $\hat{\alpha}_0$ estimated from the previous iteration (or by using the starting value of α_0 in the case of the first iteration). However, the simplifying formula for α_0^{OLS}, and α_0^{GLS} are not valid anymore for the "non-expected" version of the QUAIDS model under uncertainty. In fact, the equivalent of (5.4.22) and (5.4.23) give two other alternative estimators for α_0.

in each equation. But, this is not possible with the equation-by-equation estimation involved in the IME procedure. However, the consistent symmetry-unrestricted parameter estimates and their estimated (nonsingular) variance-covariance matrix obtained from the IME procedure can be used with the Minimum Chi-square method to obtain the symmetry-restricted parameter estimates without a joint estimation of the system, and yet with no loss of efficiency.

5.4.2.2 The Minimum Chi-square Estimator for imposing cross-equation restrictions

The Minimum Chi-square (MCS) method is a general but computationally simple method for generating efficient constrained estimates and test statistics from unconstrained consistent estimates.[142] The method requires only the availability of a consistent estimate of the unknown parameter and a nonsingular estimate of its variance-covariance matrix. To present the MCS method, let θ^0 be a l-dimensional vector of unknown parameters in a parameter space $\Theta \subset \mathbf{R}^l$, and which is known to satisfy the constraint $\mathbf{R}(\theta^0) = \mathbf{0}$, where $\mathbf{R}$ is a (possibly nonlinear) function taking values in $\mathbf{R}^k$ with $k \leq l$. Suppose that $\hat{\theta}$ is a consistent and asymptotically normal estimator of θ^0 with $\hat{\Omega}$ being its estimated asymptotic variance-covariance matrix. Then the minimum distance estimator defined by:

$$\hat{\theta}_R \equiv \underset{\theta}{\operatorname{argmin}}\left\{(\hat{\theta} - \theta)'\hat{\Omega}^{-1}(\hat{\theta} - \theta);\ \mathbf{R}(\theta) = 0\right\} \qquad \textbf{(5.4.31)}$$

is a consistent and asymptotically normal estimator of θ^0 with an estimated asymptotic variance-covariance given by

$$\hat{\Omega}_R = \hat{\Omega} - \hat{\Omega}\frac{\partial\mathbf{R}(\hat{\theta})'}{\partial\theta}\left[\frac{\partial\mathbf{R}(\hat{\theta})}{\partial\theta}\hat{\Omega}\frac{\partial\mathbf{R}(\hat{\theta})'}{\partial\theta}\right]^{-1}\frac{\partial\mathbf{R}(\hat{\theta})}{\partial\theta}\hat{\Omega} \qquad \textbf{(5.4.32)}$$

[142] It is also sometimes referred to as the Minimum Distance method.

Moreover, $\hat{\theta}_R$ is relatively more efficiency compared to the unconstrained $\hat{\theta}$ with:

$$T(\hat{\theta} - \hat{\theta}_R)'\hat{\Omega}^{-1}(\hat{\theta} - \hat{\theta}_R) \overset{D}{\rightarrow} \chi^2(d) \qquad \textbf{(5.4.33)}$$

where T is the sample size and d is the number of independent restrictions generated by the constraint. Thus, (5.4.33) provides a mean for testing the validity of the restriction $\mathbf{R}(\theta^0) = \mathbf{0}$ (see Ferguson (1958), Rothenberg (1973), and Lee (1992) for the proof of these results). Furthermore, when $\mathbf{R}$ is linear $\hat{\theta}_R$ is given by the simple and well known formula:

$$\hat{\theta}_R = \hat{\theta} - \hat{\Omega}\mathbf{R}'\left(\mathbf{R}\hat{\Omega}\mathbf{R}'\right)^{-1}\mathbf{R}\hat{\theta} \qquad \textbf{(5.4.34)}$$

The main attraction of the Minimum Chi-square method is that the dimensions of the matrices involved in (5.4.31) to (5.4.34) are all less or equal to l^2 which, in most practical situations, is significantly less than the sample size.

To conclude this chapter we will give a summary of the I.M.E/MCS procedure which apply to both the deterministic and "expected" AIDS models empirically implemented in the next chapter using the food consumption data from the ISRA/IFPRI survey.

Summary of the Iterated Moment Estimation procedure.

STEP 1: *Choice of the starting values of the iterative procedure.*

a(p) = Stone price index (alternatively one can directly choose appropriate starting values for the parameters α_0, α, γ, and β and calculate a(p)).

STEP 2: *Conditional TLS equation by equation estimation.*

Using the starting values a(p), compute the TLS-equation-by-equation instrumental variables (for the endogeneity of ln(m)) estimates of $\alpha \equiv (\alpha_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le r+1}}$, $\gamma \equiv (\gamma_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le n}}$, and $\beta \equiv (\beta_1, \ldots, \beta_n)$. At this stage, impose the homogeneity restriction by dividing the

r^{th} price through the remaining *n-1* prices. Also, impose the adding up restriction at this stage by dropping one of the equations (say the d^{th} equation) from the estimation (this avoid having an estimated singular covariance matrix for the error term). Note that *d* needs not equal to *r*.

STEP 3: *Recovering of the non estimated parameters from the homogeneity and adding up restrictions.*

Recover the *n-1* parameters $(\gamma_{1r},\ldots,\gamma_{nr})$ related to the r^{th} price by using the homogeneity restrictions $\gamma_{jr} = -\sum_{k \neq r}^{n} \gamma_{jk} \quad j=1,\ldots,n$ and the $(n\text{-}1)\times(n\text{-}1)$ estimated parameters of the γ matrix. Also, use the adding up restrictions $\alpha_{d0} = 1 - \sum_{i \neq d}^{n} \alpha_{i0}$, $\alpha_{dk} = -\sum_{i \neq d}^{n} \alpha_{ik} \quad k=1,\ldots,q$, $\gamma_{dj} = -\sum_{i \neq d}^{n} \gamma_{ij} \quad j=1,\ldots,n$, and $\beta_d = -\sum_{i \neq d}^{n} \beta_i$ to estimate the $q+1+n+1$ parameters of the dropped equation *d* from the $(n\text{-}1)(q+1+n\text{-}1+1)$ estimated parameters.

STEP 4: *Updating of the "partial" budget deflator* $r(\mathbf{p})$.

Use the full matrices of TLS estimates of α, γ, and β obtained from STEP 3 to compute the predicted values $\hat{r}(\mathbf{p})$ of $r(\mathbf{p})$ as given by (5.4.17) (for the "expected" AIDS, the relevant sample analogues of $E\ln\mathbf{p}_k$ and $E(\ln\mathbf{p}_k\ln\mathbf{p}_l)$ are used in place of $\ln\mathbf{p}_k$ and $\ln\mathbf{p}_k\ln\mathbf{p}_l$ respectively).

STEP 5: *Estimation of* α_0.

Use the TLS estimates of α, γ, and β obtained from STEP 2 (i.e., not including the parameters recovered in STEP 3) along with the predicted values $\hat{r}(\mathbf{p})$ of $r(\mathbf{p})$ obtained in STEP 4 to compute one of the equivalent feasible estimate $\hat{\alpha}_0 = \frac{\bar{y}_i}{\hat{\beta}_i}$ of α_0.

STEP 6: *Updating of the starting values and iteration until convergence.*

Use the estimate of α_0 obtained from STEP 5 along with the predicted values $\hat{r}(\mathbf{p})$ of $r(\mathbf{p})$ obtained in STEP 4 to compute the predicted values

$\hat{a}(\mathbf{p}) = \exp(\alpha_0 \hat{r}(\mathbf{p}))$ of $a(\mathbf{p})$. Then, use the predicted values $\hat{a}(\mathbf{p})$ as new starting values and repeat steps **2-5** until convergence of all parameters of the model.

STEP 7: *Computation of the estimate of the variance-covariance matrix of the IME*

Once convergence is obtained, use the converging estimates of α_0, α, β, and γ, to compute the *T structural* 2SLS residuals $\hat{u}_t = (\hat{u}_{1t}, \ldots, \hat{u}_{nt})$ and the first derivative of each estimated equation with respect to the *(n-1)(q+1+n-1+1)* vector of parameters using (5.4.11). For the deterministic AIDS $F_T(\theta)$ is given by

$$F_T(\theta) = \beta \otimes \left(\ln\mathbf{p} *\sim [\mathbf{z}, \ln\mathbf{p}, \mathbf{0}_T]\right)$$

while for the "expected" AIDS $F_T(\theta)$ is given by

$$F_T(\theta) = \beta \otimes \left[(\mathbf{1}_T \otimes E\ln\mathbf{p}) *\sim \mathbf{z}, \mathbf{1}_T \otimes E(\ln\mathbf{p} *\sim \ln\mathbf{p}), \mathbf{0}'_{T(n-1)}\right]$$

where $\ln\mathbf{p}$ is the $T\times(n\text{-}1)$ price matrix (not including the r^{th} price)[143], $\mathbf{z}$ is the $T\times(q+1)$ household characteristics matrix (including the ones for the constant term), $\mathbf{1}_T$ is the T dimensional vector made of ones, $\mathbf{0}_{T\,(n-1)}$ is the $T\times(n\text{-}1)$ matrix made of zeros, "$*\sim$" is the GAUSS horizontal product operator and $\otimes$ is the Kronecker product.[144] Then compute the heteroscedastic-robust asymptotic variance-covariance matrix estimate of the iterated moment estimates using the formula in (5.4.15).

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[143] Note that $\ln\mathbf{p}$ is not the $T\times(n\text{-}1)$ matrix of price ratio used in the O.L.S regressions.

[144] If A is a $r\times s$ matrix and B is a $r\times k$ matrix, A $*\sim$B is the $r\times sk$ matrix obtained by multiplying element-by-element every column of A with all columns of B.

STEP 8: *Recovering of the non estimated elements of the full variance-covariance matrix from the homogeneity and adding up restrictions.*

Use the homogeneity restrictions $\gamma_{jr} = -\sum_{k \neq r}^{n} \gamma_{jk} \quad j = 1,\ldots,n$, the adding up restrictions $\alpha_{d0} = 1 - \sum_{i \neq d}^{n} \alpha_{i0}$, $\alpha_{dk} = -\sum_{i \neq d}^{n} \alpha_{ik} \quad k = 1,\ldots,q$, $\gamma_{dj} = -\sum_{i \neq d}^{n} \gamma_{ij} \quad j = 1,\ldots,n$, and $\beta_d = -\sum_{i \neq d}^{n} \beta_i$, along with the $(n\text{-}1)(q+1+n\text{-}1+1) \times (n\text{-}1)(q+1+n\text{-}1+1)$ estimated IME variance-covariance matrix, to recover the full $n(q+1+n+1) \times n(q+1+n+1)$ variance-covariance matrix corresponding to the full $n \times (q+1+n+1)$ matrix of parameters recovered in STEP 3.

STEP 9: *Rearranging of the full γ matrix of parameters so as to have a coherent n-1 by n-1 system.*

Rearrange if necessary (i.e., in the case where $d \neq r$) the full $n \times n$ γ matrix (i.e., the estimated parameters augmented with the parameters $\gamma_{jr} = -\sum_{k \neq r}^{n} \gamma_{jk} \quad j = 1,\ldots,n$ and $\gamma_{dj} = -\sum_{i \neq d}^{n} \gamma_{ij} \quad j = 1,\ldots,n$ recovered in STEP 3) and drop either the d^{th} column and d^{th} row or the r^{th} column and r^{th} row so as to arrive at a coherent $(n\text{-}1) \times (n\text{-}1)$ share/price γ matrix. Also, extract the corresponding $(n\text{-}1)^2 \times (n\text{-}1)^2$ variance-covariance matrix from the full variance-covariance matrix recovered in STEP 8 (note that the $n^2 \times n^2$ variance-covariance matrix corresponding to the full γ matrix is singular).

STEP 10: *Imposing the Symmetry restriction on the gamma parameters*

Finally, obtain the $(n\text{-}1) \times (n\text{-}1)$ symmetry restricted γ matrix by using the minimum Chi-square formula:

$$\gamma_R = \gamma_U - cov(\gamma_U)R'\left(Rcov(\gamma_U)^{-1}R'\right)^{-1}R\gamma_U$$

where γ_R is the vec of the $(n\text{-}1) \times (n\text{-}1)$ symmetry restricted matrix of the γ parameters, γ_U is the vec of the $(n\text{-}1) \times (n\text{-}1)$ matrix of unrestricted γ parameters obtained from STEP 9, and R

is the $((n\text{-}1)n/2) \times (n\text{-}1)^2$ matrix of symmetry restrictions made of 0, 1, and -1, such that $R'\text{vec}(\gamma_R) = 0$. The variance-covariance matrix of γ_R is given by:

$$cov(\gamma_R) = cov(\gamma_U) - cov(\gamma_U)R'\left(Rcov(\gamma_U)^{-1}R'\right)^{-1}Rcov(\gamma_U)$$

This concludes our theoretical economic and econometric analysis of static consumption choices under price uncertainty. The next chapter starts our empirical analysis.

CONSUMER BEHAVIOR AND WELFARE MEASUREMENT UNDER UNCERTAINTY: THEORY AND EMPIRICAL EVIDENCE FROM SENEGAL

Volume II

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CHAPTER 6

THE CONSUMPTION BEHAVIOR AND RISK ATTITUDES OF SENEGALESE HOUSEHOLDS: EMPIRICAL EVIDENCE

6.1 Introduction

This chapter uses the food consumption data from the 1988-90 ISRA/IFPRI household survey in Senegal to empirically implement the econometric estimation procedure described in chapter 5. The chapter also gives and interpretes the elasticity estimates, and comments on the policy implications of their relative magnitudes. However, before discussing the econometric analysis, we discuss briefly the macroeconomic and institutional environment within which the Senegalese agricultural household operates and describe the ISRA/IFPRI household survey and the resulting data set. Discussing first the policy environment in the Senegalese agricultural sector will give the reader a general understanding of the sources and natures of the uncertainties in food prices, and, more generally how the Senegalsee government marketing and pricing policies influence household consumption patterns in Senegal. The brief discussion will also help put the policy implications derived from the econometric analysis in a global agricultural and food policy framework.

The focus of the econometric estimation is the "expected" version of the AIDS model under uncertainty defined in chapter 5. However, since the "expected" AIDS is a generalization to uncertainty of the frequently estimated deterministic AIDS, we also estimate the latter to compare the parameter and elasticity estimates from the two models. The comparison of the estimates from uncertainty and deterministic model is of interest because, to the best of our knowledge, a demand system derived under price uncertainty has never been estimated. We were not successful in estimating the more flexible "non-expected" QUAIDS" model because of the computational difficulties we had in our early attempt to estimate it.[145]

The organization of the chapter is as follows. Section **6.2** gives a brief overview of the instutitional arrangements and policies in the Senegalese agricultural sector. Section **6.3** describes the ISRA/IFPRI survey, sampling method, and data set. The section also contains a brief description of the food consumption data used in the econometric analysis. Section **6.4** discusses the specification of the econometric model and presents the results of the econometric estimation. Section **6.5** presents the uncompensated and compensated total, *precautionary* risk and nonrisk elasticity estimates, and discusses the policy implications of their relative magnitudes. Finally, section **6.6** is a brief conclusion summarizing the empirical findings of this chapter.

6.2 The Institutional and macroeconomic environment

In describing the macroeconomic and institutional environment within which Senegalese farmers operate, our emphasis is not on the details of the institutional arrangements and macroeconomic policies, but rather our intention is to give a general sense

[145] We suspect that the computational difficulties we encountered early on with the "non-expected" QUAIDS were largely due to over-parametrization. Indeed, in addition to the α parameters, we also allowed the β and λ parameters to depend on the full vector of household characteristics **z**, which tended to cause a lot of multicollinearity given the functional form of the "non-expected" QUAIDS (see chapter 5). For the other QUAIDS models, we had a convergence problem for the IME estimator. However, later on when we had already reverted to the AIDS functional form, we discovered a bug in our program which was probably causing the non-convergence in the QUAIDS case. Furthermore, even for the AIDS models we get singularity whenever we use the seasonally aggregated data and allow the β parameters to depend on **z**. Hence, with the bug corrected and a minimal parametrization for the vector of household characteristics, we expect to be able to estimate the "non-expected" QUAIDS in the future.

of the sources of price uncertainty in Senegal, and show how the behavior of the agricultural household is shaped by the global institutional arrangement in the agricultural sector.

6.2.1 The institutional organization of the agricultural sector

At the macro and institutional level, the Senegalese agricultural sector can be characterized by the existence of three entities that interact strategically. The first entity is made of a large pool of subsistence and semi-commercial farm households who constitute 65% of the Senegalese work force according to the recent national surveys, "Enquêtes sur les priorités (EP)" (République du Sénégal, 1992). They produce mainly food crops (millet, sorghum, maize, and cowpeas) and cash crops (peanut and cotton) with poor technology and low productivity. The second entity is a central government that influences to some degree the marketing system through a system of public or semi-public enterprises, developmental agencies, and/or state-sponsored cooperatives. These state-controlled organizations or agencies buy the cash crops produced by farmers at prices officially set by the central government, and until 1984 supplied credit and most of the inputs used by farmers at subsidized prices. The third entity is the national central bank, which although distinct to some extent from the central government, plays a relatively active role in the agricultural sector. Indeed, the monetary authority is not directly under the control of the government because the national central bank is just a branch of BCEAO, the central bank of the monetary union that links France with its seven West African former colonies. The central bank intervenes in the agricultural sector through the channel of what is called "the agricultural campaign" by which it lends money to the government so that it can buy the cash crops from the farmers.

The food produced by the Senegalese agricultural sector is not enough both in terms of quantity and variety to feed the whole country. On average, total cereal production covers only 50 to 60% of consumption per year (Kite, 1993), despite the 80% food self-sufficiency goal set by the Government in 1984.[146] Thus, Senegal must import food (mostly cereals) to make up the deficit and satisfy the consumer preferences. Cereal imports, 90% of which are rice, are centralized and carried out by a government agency known by its French acronym CPSP[147]. The situation in the agricultural sector is further complicated by the uncertainties resulting from price fluctuations in world market and more importantly, from the year to year variability in the weather. Figure 6.1 and Figure 6.2 give simplified descriptions of the institutional arrangements in the agricultural sector that prevailed since 1986. More detailed descriptions can be found in BCEAO (1984-1990), Gersovitz and Waterbury (1987), Berg (1990), Delgado and Jammeh (1991), and Kite (1993).

6.2.2 The Senegalese Government's Marketing and Pricing Policies

The government intervenes in the agricultural sector through a number of public enterprises and agencies, and through government-initiated farmers' organizations (see Figure 6.1 and Figure 6.2). The range of interventions covers production, extension, marketing, transport, and processing. However, the government's role has been diminishing in many areas since the adoption of the New Agricultural Policy (NPA) in 1984. Export crops is one area where the government continues to be very active. The two major cash

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[146] Aggregate statistics usually vary from one source to another. For consistency, most of the aggregate statistics cited in the thesis are based on Kite's (1993) recent review of the cereal subsector in Senegal.

[147] Caisse de Péréquation et de Stabilisation des Prix. The CPSP is also in charge of importing rice and/or distributing licenses to selected private importers.

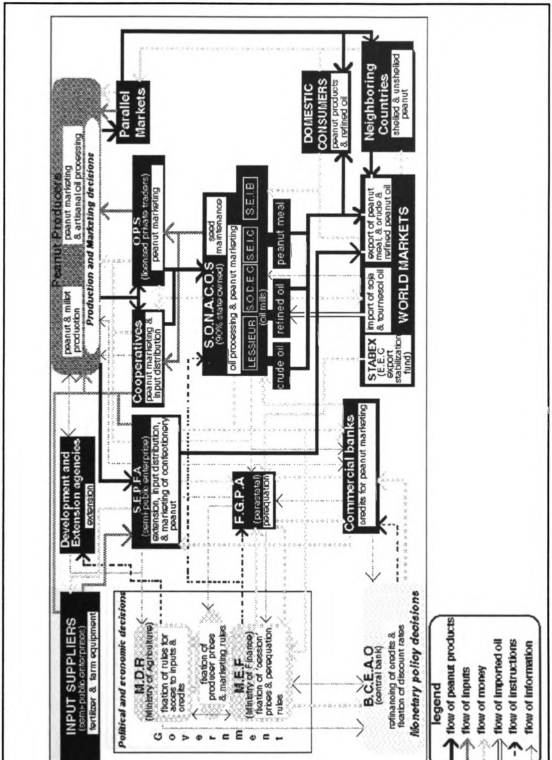

Figure 6.1 Senegal: Institutional organization of the peanut sector (1986-92).

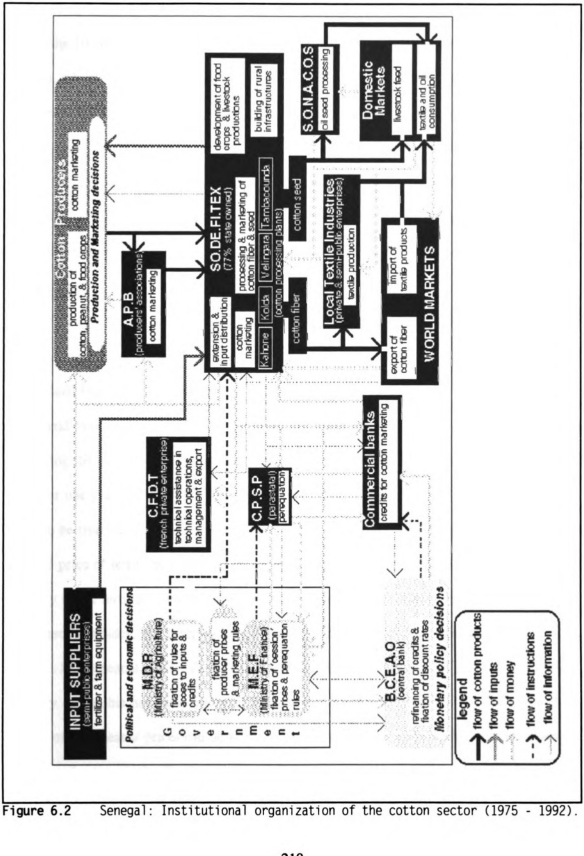

Figure 6.2 Senegal: Institutional organization of the cotton sector (1975 - 1992).

crops are groundnut and cotton. Groundnut is grown in the Peanut Basin which covers five out of the 10 administrative regions of the country (Diourbel, Louga, Thies, Kaolack, and Fatick), while cotton is grown mainly in the Southeastern part of the country. The sale of cash crops in international markets is an important source of foreign exchange for the country, and, in the past, of revenues for the government's budget. Moreover, because of the industries and services revolving around it, the peanut sector covers a relatively large share of Senegal's GDP. The critical importance of the export crops to the Senegalese economy may explain the heavy involvement of the Senegalese government in their development and marketing.

Each year, before the growing season, the Senegalese government announces the prices at which it guarantees to buy the different cash crops at harvest. It also sets by decree the official beginning and ending dates for their marketing (usually in December and between April and May respectively). Two state-controlled monopolies have the exclusive legal rights of buying the cash crops from farmers. In general, the official prices are meant to be valid only for one year, although they tend to last, on average, more than two years. The same used to be true for the prices of some of the inputs used by farmers (fertilizer and seed). The official price of fertilizer used to be maintained by a subsidy (phased out in 1989) paid directly to the fertilizer processing plant. Until the mid 1980's, inputs including seed, fertilizer, pesticides, and farm equipments were distributed on credit and paid back at harvest.

In the peanut sector, an unofficial parallel market which extends to the neighboring country of Gambia has developed over the years. This parallel market limits the extent to which official peanut prices can be kept low. For example, in 1984 when the government introduced a tax which effectively lowered the price paid to farmers from 60 CFA/kg to 50 CFA/kg, the parallel market's share of total peanut marketed jumped from 3.5 % in the 1982-

83 season to 18% in the 1983-84 season. The parallel market's share reached its highest level (41%) in the 1984-85 season despite the government's promise to return 10 CFA/kg to farmers who market through official channels. This prompted the government to raise the peanut official price in the middle of the marketing season in order to increase the supply of peanuts to the state controlled oil processing plants whose capacity is underutilized (BCEAO, 1984).

The prices of internationally non-traded commodities (millet, sorghum, maize, and cowpeas) are in general determined by the law of supply and demand in domestic markets, although the government has attempted several times in the past - without success - to enforce officially set prices (Ndoye, Boughton, and Crawford, 1991). In its effort to boost domestic cereal production in support of its goal of 80% food self-sufficiency, the government adopted in 1986 a policy of setting only floor and ceiling producer prices for the food crops. In theory, a government agency (CSA)[148] was supposed to intervene in the food crop markets whenever market prices were outside the range delimited by their respective official ceiling and floor prices. In practice, this price stabilization scheme rarely worked because of lack of sufficient funds, low storage capability, and inappropriate timing of the intervention. In fact, farmers complained that instead of stabilizing prices, CSA brought more instability because of its erratic intervention (Ndoye and Ouedraogo, 1987 and 1988). The price stabilization scheme was abandoned in 1988. Similarly, most of the laws that prohibited the transaction of more than 200 kg of cereals without authorization were repealed in 1988.

On the consumption side, price setting is not year by year. Instead, the prices for imported food are changed infrequently, usually to respond to significant price changes in international markets, pressures in the balance of payment and/or government budget, or

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[148] Commissariat à la Sécurité Alimentaire.

political and social pressures. Imported food prices can remain stable for a relatively long period of time. However, they are not set independently of cash crop prices because there are cross-subsidizations among commodities. The cross-subsidization schemes (which include other commodities such as sugar, vegetable oil, gas, etc...) are relatively complex and are managed by CPSP. For example, until the early 1980s most of the benefits realized from peanut exports were used to subsidize rice imports. With the collapse of the world market prices of both rice and peanut in the mid 1980's combined with the overvaluation of the CFA francs, the rice sector has been generating a surplus that is partly used (until the January 1994 devaluation of the CFA franc) to subsidize peanut growers, who have been receiving a producer price well above the world market price.[149]

Recently, important policy and institutional changes have been taking place since 1987, so that it is expected that by 1996 the importation and marketing of rice will be completely liberalized with the role of CPSP reduced to monitoring the evolution of the privatized sector (Kite, 1993; République du Sénégal, 1994). Also, although the government intends to continue to be involved in the setting of peanut producer prices because of their importance for the economy, in the future they will be set at levels consistent with world market prices, and the direct link of the peanut sector to the government budget will be severed with the creation of a separate self-financing fund (FGPA) that will finance or absorb any deficit or surplus in the peanut sector (République du Sénégal, 1994). In fact the fund has been in place since 1989, but it has not been able to operate as intended because of the

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[149] France, the major trading partner of Senegal (and the other 6 UMOA countries) has guaranteed the convertibility of the CFA with the French franc at a fixed rate since 1948. Until the 50% devaluation of the CFA francs in January 1994, The rate has remained constant at 1 Franc for 50 CFA. To counterbalance the loss of competitiveness of Senegalese exports crops and domestically produced cereals resulting from the surevaluation of the CFA, the government resorted to a policy that pays producer prices for peanut and cotton above world market prices while imposing high tariffs on rice imports.

chronic deficit of the peanut sector. Similar reforms are planned for the cotton sector. However, there is no sign that the government intends to liberalize peanut marketing which continues to be dominated by representatives of the oil processing industry and a limited number of authorized private wholesalers called "collectors". Marketing peanut in the parallel market is still an illegal activity in Senegal. Farmers caught selling outside of the official channels run the risk of having their peanuts confiscated.

Another important policy change has taken place in 1989 in the monetary side. Indeed, the deterioration of the balance of payments during the 1980s led the Central Bank to change its monetary policy in 1989 by requiring the government to prefinance up front before the release of the "agricultural campaign" credits any deficit that might results from the projected values of the export crops and the total amount paid to farmers (UMOA, 1991). In addition, along with eliminating the preferential interest rate, BCEAO policy now requires that the "agricultural campaign" credits be within the bounds of the ceiling on the country's yearly "volume of credits" (UMOA, 1991). This change in BCEAO's monetary policy is a very important development in the agricultural sector. Indeed, one can identify at least three consequences.

First, the ex-ante financing of the deficit - which used to be written off by BCEAO - and the inclusion of the "agricultural campaign" credits on the ceiling on total credits to the economy, effectively puts a yearly ceiling on the official producer prices that are announced before planting.

Second, the stringent requirement that the "gap" be ex-ante financed before the "agricultural campaign" credits are released is likely to create delays in the payments of farmers for their crops, as the governments must struggle to find very scarce funds in these days of tight budgetary constraints.[150] These delays are likely to push cash-constraint farmers into the parallel market, and/or force them to sell part of their food stocks at very low prices.

Third, and perhaps more importantly, official producer prices announced before planting can no longer be guaranteed to hold at harvest. That is, the government is likely to renege on its pre-announced guaranteed producer price in years of good harvest and/or when the "gap" is expected to be large.[151] Farmers have long been accustomed to high yield uncertainty for their peanuts; now they have to face policy-induced price uncertainty that is resolved only once they have harvested their crops. This type of price uncertainty is more difficult to cope with compared to market-induced price uncertainty which is usually caused by variations in supply induced by changes in the weather. Indeed, with the latter type of price uncertainty, as time goes and nature progressively reveals itself, farmers can smoothly adjust their behavior by altering their labor supply, inputs use, consumption and savings, in anticipation of prices that are likely to prevail at harvest given expected market supplies.

Hence, the new monetary policy regime while bringing a much needed discipline in the government's financing of the "agricultural campaign", introduces more uncertainty in farmers' crop mix decisions at planting time, and in their labor supply, consumption and saving decisions during the growing and harvest season.

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[150] This seems to be what happened this year according to the December 14, 1993 edition of the independent daily newspaper Walfadjri.

[151] This is exactly what happened this year when the government, faced with one of the best harvest in a decade, decided to reduce the pre-announced official price of peanuts from 80 to 70 CFA/kg just when farmers were starting to market their peanuts, with the express promise to return to farmers the 10 CFA/kg differential if after commercialisation the sector turns out be in surplus. According to the December 6, 1993 edition of the independent daily newspaper Walfadjri, the announcement of the price decrease has taken farmers by surprise because just one month before, they have obtained reassurances from the Minister of Agriculture that the government will maintain its before-planting official prices.

With this background on the institutional setting of the Senegalese agricultural sector, in the next section we describe and interpret some of the empirical evidences contained in the data set resulting from the 1988-1991 survey conducted in Senegal by the Institut Sénégalais de Recherches Agricoles (ISRA) and the International Food Policy Research Institute (IFPRI).

6.3 Description of the ISRA/IFPRI survey and data set

6.3.1 General description of the sample and data.

The ISRA/IFPRI survey was a three-year project financed by the United States Agency for International Development (USAID). The objective of the project was to study "The Consumption and Supply Impacts of Agricultural Price Policies" in Senegal. The project was mostly focussed on rural households. However, an urban component focussing on the consumption and income generation activities of urban households was added in the second year of the project. The data collection for the rural households started in October 1988 and ended in December 1990, while for the urban households the data collection started in June 1990 and ended in July 1991.

The survey collected household-level income, consumption, market transactions, and credit data as well as plot-level crop input/output crops data. The sample consisted of 360 households in 30 villages and 6 agro-climatic regions plus the two urban cities of Kaolack and Tambacounda (see map in Figure 6.3)[152]. The 6 agro-climatic regions cover the Peanut Basin and the Eastern region of Senegal. According to Kelly et al., (1993), they cover 85% of

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[152] The map is taken from Kelly et al., (1993).

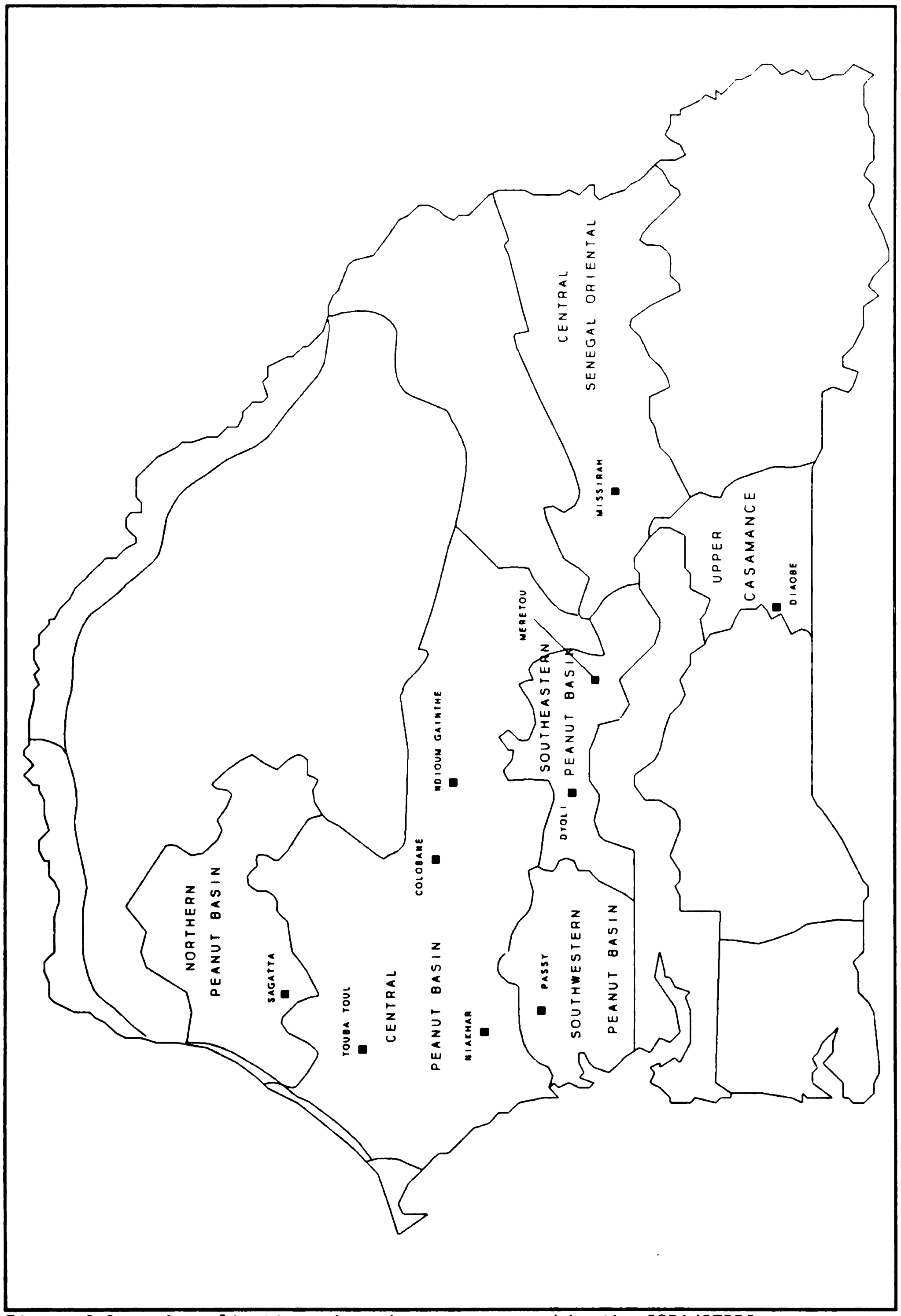

Figure 6.3 Agroclimatic and study zones covered by the ISRA/IFPRI survey.

YEAR	1988												1989											
MONTH	JAN	FEB	MAR	APR	MAY	JUN	JUL	AUG	SEP	OCT	NOV	DEC	JAN	FEB	MAR	APR	MAY	JUN	JUL	AUG	SEP	OCT	NOV	DEC
Agricultural production activities			land prep.		planting		weeding		harvesting						land prep.		planting		weeding		harvesting			
Marketing Activities	←marketing→									peak marketing		←marketing→										peak marketing		
Climate	--cool winds→		←hot winds→			←rains→					←cool winds→				←hot winds→			←rains→						←cool winds-
local reference terms for sub-seasons	Noor		Thioron			Nawet							Noor		Thioron			Nawet				Lolli bu toy		Looli bu wow

DATA COLLECTION ACTIVITIES

DATA COLLECTION ↓ MONTH →	JAN	FEB	MAR	APR	MAY	JUN	JUL	AUG	SEP	OCT	NOV	DEC	JAN	FEB	MAR	APR	MAY	JUN	JUL	AUG	SEP	OCT	NOV	DEC
Zones selection																								
Sub-zones selection																								
Villages selection																								
Households selection																								
Q1: Village reconnaissance surveys								▬	▬															
Q2: Compound Census of Village								▬	▬															
Q3: Household Census											■						□			■	semestrial			
Q4: Stocks of Agric. Products									■			■				■				■	every 4 months			
Q5: Food consumption										■■	■■	■■	■■	■■	■■	■■	■■	■■	■■	■■	fortnightly / 24 hours recall			
Q6: Agric. products transactions											■■	■■	■■	■■	■■	■■	■■	■■	■■	■■	fortnightly/ 2 weeks recall			
Q7: Livestock transactions											←recalled	----	----	■	■	■	■	■	■	■	monthly			
Q8: Infrequent non-ag. purchases											←recalled	----	----	■	■	■	■	■	■	■	monthly			
Q9: Livestock input costs														■	■	■	■	■	■	■	monthly			
Q10: Wage income														■	■	■	■	■	■	■				
Q11: Cash and credit flows											←recalled	----	----	■	■	■	■	■	■	■	monthly/ 2 months recall			
Q12: Non-agricultural income																	■■	■■	■■	■■	fortnightly/ 2 weeks recall			
Q13: Frequent non-ag. purchases																	■		■		■	bimonthly		
Q14: Migration income																■	←endogenous frequency→							
Q15: Agricultural capital census																	■	←once a year→						
Q16: Plot (field) identification																		■	←once a year before rainy season→					
Q17: Agricultural production																				■■	■■	fortnightly during cropping season		
Q18: Plot area and characteristics																		←each cropping season→						
Q19: Livestock census																		■						

Note: The figure layout is adapted from Goetz (1990), and the information is taken from Fall et al., (1989).

Figure 6.4 The questionaires and timing of the ISRA\IFPRI survey.

Senegal's rural population, 60% of its total population, and 90% of both its peanut and coarse grains (millet, sorghum, maize) production.

The sample selection proceeded by purposely selecting 10 rural study zones within the 6 agro-climatic regions based on physical, socio-economic, and representativeness criteria. The selected rural study zones were: Sagatta in the Northern Peanut Basin, Niakhar, Colobane, Ndioum Gainthe, and Touba Toul in the Central Peanut Basin, Passy in the Southwestern Peanut Basin, Dioly and Meretou in the Southeastern Peanut Basin, Missirah in Eastern Senegal, and Diaobe in upper Casamance. Then in each rural study zone, 3 representative villages were selected purposely, with one of the villages being always a market village. The resulting oversampling of market villages was intended for assessing the influence of market infrastructures on household behavior. The last stage of the sample selection consisted of randomly selecting in each village 12 households from the list of all households in the village. The two urban cities were purposely selected based on the fact that they were the two main focal cities for respectively the Peanut Basin and the Eastern region of Senegal. In each city, 40 households were randomly selected from a list of households based on documents prepared for the 1988 national census. However, several of the study zones were eliminated from the final data set for various reasons including poor performance of the enumerators (Touba Toul) lack of transport during the second year of the survey (Sagatta), and lack of time for data entry and cleaning (Ndioum gainthe, Meretou and Diaobe). But, the first year of data from Sagatta was kept in the final analysis.

The households were interviewed every 2 weeks for 2 years. Household members were also interviewed individually whenever such a level of disaggregation was warranted. Figure 6.4 describes the questionnaires and the timing of the data collection activities in the ISRA/IFPRI survey. A full description of the objectives, characteristics of the study zones, sampling methods, and questionnaires of the ISRA/IFPRI survey can be found in Diagana et al., (1990), Kelly and Reardon (1989), and Fall et al., (1989). A complete documentation of the original data base resulting from the survey can be found in Dakono et al., (1993).

The data series used in our descriptive and econometric analysis were constructed from the original data base which is completely documented by Dakono et al., (1993). The files in the original data base are organized by topic (corresponding to the titles of the questionnaires illustrated in Figure 6.4) and by study zone. The data in the files in the original data base are still in coded form. The original SPSS programs written to construct the data series used for the analysis in the original project reports were completely rewritten. The only exceptions are the programs for the income series. Furthermore, the new SPSS programs include codes that correct for some minor data entry errors discovered after the analysis in the original project reports was completed. Instead of detailing the methods we used to construct our data series - which will be too long to do here - we refer the interested reader to the SPSS program files which are well documented.[153] The data series used in our econometric analysis is decribed in the next section.

6.3.2 Some characteristics of the households in the sample

A detailed descriptive analysis of the ISRA/IFPRI data set is contained in the project reports by Kelly et al. (1993). Hence, our descriptive presentation will be brief and is only intended to introduce the reader to the data set and give some feeling about the characteristics

[153] Because of our experience in working with the previously written programs, we took the pain to document line by line the codes in the new programs, so that potential future users of the data base who want to use our series or want to reconstruct theirs can have a detailed knowledge of how they were constructed. This documentation seems reasonably adequate, since some of the new programs have been used and/or revised at IFPRI in their ongoing analysis of the data..

of the households in the ISRA/IFPRI study zones.[154] Furthermore, we restrict ourself to a strict presentation of the "facts" with no substantive discussion. This is because the only reasons for this short description are to show the central tendencies of household income, cereal transactions, and food consumption pattern, and to provide a context for rationalizing and interpreting the estimated elasticies and welfare changes following the January devaluation of the CFA franc. Moreover, the tables show only averages because; we chose not to display standard errors, minimums, maximums, etc... to avoid overloading the tables.[155] Thus, there is no pretense that the apparent differences or similarities among classified groups shown by the tables, and sometimes mentioned in our discussion, are statistically significant. We preferred to concentrate our efforts in addressing the statistical issues in the econometric analysis, which allows a much more complete and satisfactory assessment of the respective effects of the various exogenous variables on household consumption choices.

6.3.2.1 Average levels of household income

Table 6.1 gives the average levels of household incomes and food, nonfood, and total expenditures. In harvest year 88-89 which was a drought year except for the Dioly and Missirah zones, the average net income per adult equivalent for the rural zones is

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[154] A more detailed presentation of the data used in our present analysis will be available in a forthcoming document (Diagne et Kelly, 1994) which contains an extensive set of descriptive tables complementing the set of tables published in the original reports.

[155] In our exploration of the data we did a detailed study of the distribution of all the variables used in our analysis. In fact, we have histograms, stem and leaf plots, boxplots, robust estimates of location, standard errors, etc.. for all the important variables. But, the printed document (over 400 pages) is too big to be included here.

Table 6.1 Food, non-food, total expenditures, and total income per adult equivalent: Variation across years and zones

in CFA francs ($1 =280 CFA)	Rural zones						Urban zones		Average
	Sagatta	Niakhar	Colobane	Passy	Dioly	Missirah	Kaolack	Tamba	
Harvest year 88-89									
Food................	24.244	24.311	23.472	28.071	26.055	19.760	.	.	24.542
Nonfood.............	3.316	2.049	4.074	3.789	7.175	3.663	.	.	3.367
Total expenditure...	27.559	26.360	27.546	31.861	33.230	23.408	.	.	27.911
Net income..........	38.420	27.113	26.034	42.555	45.863	30.024	.	.	30.956
Ratio of net income to per capita GDP	21%	15%	14%	23%	25%	16%	.	.	17%
Rainfall (in mm)....	449	644	625	669	810	950	.	.	629
Harvest year 89-90									
Food................	.	28.207	36.932	30.601	38.673	27.853	.	.	32.292
Nonfood.............	.	4.411	9.641	8.266	17.324	6.013	.	.	7.498
Total expenditure...	.	32.617	46.573	38.867	55.997	33.866	.	.	39.790
Net income..........	.	31.621	49.212	63.607	57.237	28.088	.	.	43.581
Ratio of net income to per capita GDP	.	17%	27%	35%	31%	15%	.	.	24%
Rainfall (in mm)....	.	802	556	717	736	756	.	.	694
Harvest year 90-91									
Food................	.	.	.	.	.	.	73.594	71.166	73.109
Nonfood.............	.	.	.	.	.	.	27.395	23.025	26.522
Total expenditure...	.	.	.	.	.	.	100.989	94.191	99.631
Net income..........	.	.	.	.	.	.	134.819	100.371	127.936
Ratio of net income to per capita GDP	.	.	.	.	.	.	78%	58%	74%
Rainfall (in mm)....	.	.	.	.	.	.	462	676	505
Average across years									
Food................	24.244	26.444	30.417	29.302	32.463	23.544	73.594	71.166	34.838
Nonfood.............	3.316	3.342	6.946	5.967	12.329	4.779	27.395	23.025	8.461
Total expenditure...	27.559	29.786	37.363	35.269	44.793	28.374	100.989	94.191	43.301
Net income..........	38.420	29.581	37.993	52.797	51.640	29.118	134.819	100.371	50.312
Ratio of net income to per capita GDP	21%	16%	21%	29%	28%	16%	78%	58%	28%
Rainfall (in mm)....	449	731	589	692	772	859	462	676	639

Source: ISRA/IFPRI survey.
Notes: Each cell contains the weighted mean of the annual levels across all households. A harvest year runs from October to September; However, 88-89 started in January 89. For Kaolack and Tamba, the period covered is July 90 - June91. Per capita GDP for Senegal (from World Bank Tables) were respectively: $650, $650 and $615.

Table 6.2 Cereals and Pulses transactions per adult equivalent: Variation across zones

	Rural zones							Urban zones		
	Sagatta	Niakhar	Colobane	Passy	Dioly	Missirah	Average	Kaolack	Tamba	Average
Quantity sold										
Cereals										
Millet	0	2	8	10	21	1	6	0	0	0
Sorghum	0	0	0	0	12	2	1	0	0	0
Maize	0	0	0	0	1	3	0	0	0	0
Rice	0	0	0	0	0	0	0	0	0	0
Pulses										
Peanut	20	14	67	59	167	18	42	0	0	0
Niebe	1	0	0	0	0	0	0	0	0	0
Quantity bought										
Cereals										
Millet	14	7	15	7	29	8	11	11	5	10
Sorghum	0	0	0	0	7	7	0	0	4	1
Maize	3	1	1	0	0	26	1	1	5	2
Rice	29	10	22	15	15	11	16	39	45	40
Pulses										
Peanut	1	0	1	0	7	35	1	4	4	4
Niebe	2	0	0	0	0	1	0	1	0	1
Net quantity Sold										
Cereals										
Millet	-13	-5	-6	4	-9	-7	-5	-11	-5	-10
Sorghum	0	0	0	0	5	-5	0	0	-4	-1
Maize	-3	-1	-1	0	1	-23	-1	-1	-5	-2
Rice	-29	-10	-22	-15	-15	-11	-16	-39	-45	-40
Pulses										
Peanut	18	13	66	58	160	-17	41	-4	-4	-4
Niebe	-1	0	0	0	0	0	0	-1	0	-1
Ratio of net income to per capita GDP	18%	14%	20%	27%	28%	15%	18%	71%	77%	72%
Rainfall (in mm)	449	724	592	693	773	854	666	462	676	506

Source: ISRA/IFPRI survey.
Notes: Each cell contains the weighted mean across all households of the season averages of the respective quantity transacted.

approximately 31,000 CFA ($110).[156] This average increases by 42% to 44,000 CFA ($157) in the following 89-90 harvest year which was a better year for all zones except Dioly and Missirah. This is an indication of the yearly fluctuations in income that rural households are subject to. The two-year average is approximately 37,000 CFA ($133) per year. This represents only 20% of Senegal's $650 national GDP per capita. For the urban zones, the 1990-1991 average net income per adult equivalent is 128,000 CFA ($454) which represents 74% of Senegal's per capita GDP ($615). This illustrates the large income gap between rural and urban households in Senegal.[157] Table 6.1 also shows the income disparities among agro-ecological zones, with the Passy zone having the highest two-year average net income per adult equivalent - 53,000 CFA ($189 or 29% of per capita GDP) - and Missirah zone having the lowest - 29,000 CFA ($104 or 16% of per capita GDP).[158]

6.3.2.2 Household cereal and pulses transactions

Table 6.2 gives the average kilograms per adult equivalent of cereals and pulses transacted in the different zones. On average, during the two-year period, households have sold 6 kg of millet and 42 kg of peanuts per adult equivalent. They also bought on average 11 kg of millet per adult equivalent. Thus, as the table shows on average households in the

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[156] The weights used in the calculation of adult equivalents were adapted by Kelly et al., (1993) from FAO 1975 standards. They were calculated as follows: 1 for adult male more than 15 years old, .81 for boy 5-14 years old, .73 for female more than 5 years old, and .44 for boy or girl less than 5 years old.

[157] This big income disparity between rural and urban households in Senegal is confirmed by the 1991 E.P. nationwide survey (République du Sénégal, 1992). However, the E.P. numbers are not comparable with ours because they estimated only monetary income.

[158] This is very puzzling because Missirah has the highest rainfall level among all zones (it also has -with Dioly- the best land in Senegal (USAID/SENEGAL, 1991)). Moreover, although the 1988-89 rainfall level was almost twice as high in Missirah (950 mm) than in Sagatta (449 mm), the 1988-89 average net income per adult equivalent was 28% lower in Missirah than in Sagatta, which experienced a drought.

study zones are net buyers of food. Only the households in Passy who are also the richest households are on average net sellers of food. On average, the households in Missirah who are also the poorest households have the largest food deficits, both in terms of coarse grains and in terms of total cereals (i.e., including imported rice). However, they buy mostly maize, while in the other zones households buy mostly rice and millet.

6.3.2.3 Household food consumption patterns

Table 6.3 gives the average household shares of food in total expenditures for each zone. What strikes one first in this table are the very high average food shares. They significantly depart from the conventional wisdom's general estimates (for African rural households) of about 70%.[159] Except for the rich zone of Dioly which has a two-year average food share of 77%, all other rural zones have food shares that average more than 80% for the two years. One also would note that the average food shares for the 1988 drought year are noticeably higher (by an average of 6%) than the ones for 1989, which had better rainfall. Altough the average food shares for the two cities of Kaolack (73%) and Tambacounda (78%) are less than 80%, they are still relatively high.

Table 6.4 gives the average household shares of the main food product out of total food expenditure. In all zones except for the city of Kaolack, the two single most important food items are the coarse grains and imported rice. On average for the rural households in the sample 39% of total food expenditure is spent on the coarse grains, 16% on imported rice, 11% on pulses (peanut and niebe), 7% on fresh and dried fish, 5% on vegetable oil, and 1% on meat. The remaining 20% of the food budget is spent on other food items (vegetables,

[159] Note, however, that the food expenditures included the imputed values of food consumption out of home production.

Table 6.3 Food and non-food shares of total expenditure: Variation across years and zones

	Rural zones						Urban zones		Average
	Sagatta	Niakhar	Colobane	Passy	Dioly	Missirah	Kaolack	Tamba	
Harvest year 88-89									
Food................	88%	93%	86%	89%	81%	86%	.	.	89%
Nonfood............	12%	7%	14%	11%	19%	14%	.	.	11%
Ratio of net income to per capita GDP	21%	15%	14%	23%	25%	16%	.	.	17%
Rainfall (in mm)....	449	644	625	669	810	950	.	.	629
Harvest year 89-90									
Food................	.	87%	80%	79%	74%	83%	.	.	83%
Nonfood............	.	13%	20%	21%	26%	17%	.	.	17%
Ratio of net income to per capita GDP	.	17%	27%	35%	31%	15%	.	.	24%
Rainfall (in mm)....	.	802	556	717	736	756	.	.	694
Harvest year 90-91									
Food................	.	.	.	.	.	.	73%	78%	74%
Nonfood............	.	.	.	.	.	.	27%	22%	26%
Ratio of net income to per capita GDP	.	.	.	.	.	.	78%	58%	74%
Rainfall (in mm)....	.	.	.	.	.	.	462	676	505
Average across years									
Food................	88%	90%	83%	84%	77%	85%	73%	78%	84%
Nonfood............	12%	10%	17%	16%	23%	15%	27%	22%	16%
Ratio of net income to per capita GDP	21%	16%	21%	29%	28%	16%	78%	58%	28%
Rainfall (in mm)....	449	731	589	692	772	859	462	676	639

Source: ISRA/IFPRI survey

Notes: Each cell contains the weighted mean of the annual shares across all households. A harvest year runs from October to September; However, 88-89 started in January 89. For Kaolack and Tamba, the period covered is July 90 - June91. Per capita GDP for Senegal (from World Bank Tables) were respectively: $650,$650 and $615.

Table 6.4 Food products shares of total food budget: Variation across zones

	Rural zones							Urban zones		
	Sagatta	Niakhar	Colobane	Passy	Dioly	Missirah	Average	Kaolack	Tamba	Average
Cereals										
Millet............	17%	51%	27%	36%	40%	11%	39%	11%	2%	9%
Sorghum...........	0%	0%	0%	0%	5%	13%	0%	0%	2%	1%
Maize.............	2%	0%	0%	0%	1%	24%	0%	1%	4%	1%
Rice..............	33%	10%	21%	18%	9%	6%	16%	21%	23%	21%
Pulses										
Peanut............	5%	11%	6%	14%	10%	16%	9%	4%	7%	5%
Niebe.............	6%	2%	2%	0%	2%	1%	2%	1%	0%	1%
Animal proteins										
Meat/Chiken.......	0%	1%	1%	2%	1%	2%	1%	7%	16%	9%
Fresh fish........	2%	3%	5%	7%	0%	0%	4%	9%	6%	9%
Dried fish........	5%	3%	3%	5%	5%	5%	3%	2%	1%	2%
Others										
Vegetables........	3%	1%	2%	3%	4%	0%	1%	11%	6%	10%
Condiments........	3%	0%	5%	3%	8%	0%	3%	3%	5%	3%
Vegetable oil.....	11%	3%	9%	3%	1%	0%	5%	13%	7%	12%
Other food........	17%	13%	24%	12%	19%	23%	17%	19%	20%	19%
Ratio of net income to per capita GDP.....	18%	14%	20%	27%	28%	15%	18%	71%	77%	72%
Rainfall (in mm)......	449	724	592	693	773	854	666	462	676	506

Source: ISRA/IFPRI survey
Notes: Each cell contains the weighted mean across all households of the season averages of the respective daily shares (the food consumption survey collected 24 hours-recall data every 15 days for two years).

Table 6.5 Food products shares of total food budget: Variation across income groups

	Rural zones				Urban zones (90-91)			
Income groups	low 25%	25 - 50%	50 - 75%	top 25%	low 25%	25 - 50%	50 - 75%	top 25%
Cereals								
Millet	27%	35%	30%	32%	7%	6%	8%	5%
Sorghum	4%	4%	2%	3%	1%	2%	1%	1%
Maize	10%	3%	3%	2%	4%	2%	2%	1%
Rice	15%	13%	16%	15%	22%	24%	21%	19%
Pulses								
Peanut	11%	11%	10%	10%	7%	5%	6%	4%
Niebe	2%	2%	2%	1%	1%	1%	0%	0%
Animal proteins								
Meat/Chiken	2%	1%	1%	1%	10%	10%	11%	15%
Fresh fish	3%	3%	3%	4%	7%	8%	8%	8%
Dried fish	4%	4%	4%	5%	2%	2%	2%	1%
Others								
Vegetables	1%	1%	2%	3%	8%	8%	9%	9%
Condiments	1%	3%	3%	5%	4%	3%	4%	4%
Vegetable oil	5%	3%	4%	3%	10%	12%	11%	8%
Other food	15%	19%	21%	19%	16%	18%	18%	25%
Ratio of net income to per capita GDP	7%	14%	20%	39%	13%	39%	58%	189%

Source: ISRA\IFPRI survey
Notes: Each cell contains the mean across all households of the season averages of the respective daily shares (the food consumption survey collected 24 hours-recall data every 15 days for two years).

Table 6.6 Characteristics of households in the sample: Consumption and production patterns by income groups

	Rural zones				Urban zones			
	Income quartile				Income quartile			
	low 25%	25 - 50%	50 - 75%	top 25%	low 25%	25 - 50%	50 - 75%	top 25%
Food share in total exp.								
< 70%........	0 (0%)	2 (2%)	2 (2%)	11 (11%)	2 (22%)	3 (18%)	6 (35%)	4 (29%)
70 - 80%......	3 (5%)	14 (17%)	13 (14%)	40 (42%)	4 (44%)	7 (41%)	4 (24%)	8 (57%)
80 - 90%......	33 (59%)	38 (45%)	53 (58%)	41 (43%)	2 (22%)	7 (41%)	6 (35%)	2 (14%)
> 90%........	20 (36%)	30 (36%)	23 (25%)	4 (4%)	1 (11%)	0 (0%)	1 (6%)	0 (0%)
Rice share in food exp.								
< 10%........	31 (55%)	41 (49%)	19 (21%)	40 (42%)	0 (0%)	0 (0%)	0 (0%)	1 (7%)
10 - 20%......	7 (13%)	24 (29%)	42 (46%)	22 (23%)	4 (44%)	5 (29%)	11 (65%)	8 (57%)
20 - 30%......	15 (27%)	15 (18%)	24 (26%)	29 (30%)	5 (56%)	9 (53%)	5 (29%)	4 (29%)
> 30%........	3 (5%)	4 (5%)	6 (7%)	5 (5%)	0 (0%)	3 (18%)	1 (6%)	1 (7%)
Coarse grain share in food exp.								
< 10%........	0 (0%)	2 (2%)	7 (8%)	4 (4%)	3 (33%)	10 (59%)	7 (41%)	12 (86%)
10 - 20%......	7 (13%)	7 (8%)	8 (9%)	8 (8%)	4 (44%)	6 (35%)	8 (47%)	2 (14%)
20 - 30%......	6 (11%)	9 (11%)	13 (14%)	14 (15%)	2 (22%)	1 (6%)	2 (12%)	0 (0%)
> 30%........	43 (77%)	66 (79%)	63 (69%)	70 (73%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)
Coarse grain share in crop income								
< 50%........	33 (59%)	45 (54%)	51 (56%)	83 (86%)	9 100%	17 100%	17 100%	14 100%
50 - 60%......	8 (14%)	14 (17%)	22 (24%)	12 (13%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)
60 - 70%......	11 (20%)	17 (20%)	8 (9%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)
> 70%........	4 (7%)	8 (10%)	10 (11%)	1 (1%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)
Peanut share in crop income								
< 50%........	34 (61%)	49 (58%)	44 (48%)	19 (20%)	9 100%	17 100%	17 100%	14 100%
50 - 60%......	9 (16%)	14 (17%)	20 (22%)	28 (29%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)
60 - 70%......	6 (11%)	10 (12%)	7 (8%)	29 (30%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)
> 70%........	7 (13%)	11 (13%)	20 (22%)	20 (21%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)

Source: ISRA/IFPRI survey.
Notes: Each cell contains the number of households in the group and the column percentage in parentheses (the rural and urban households were separately classified into income quartiles).

condiments, etc...). The corresponding average shares for the two cities are 11% for the coarse grains, 21% for imported rice, 6% for the pulses, 11% for fish, 12% for vegetable oil, 9% for meat, and 32% on the other food items. However, the pattern of food consumption is not uniform across all rural zones. For example, the pattern of food consumption in Sagatta is closer to the one in urban zones than to the ones in the other rural zones. Finally, Table 6.5 and Table 6.6 give the break down of the household consumption patterns by income quartiles (based on average household annual income per adult equivalent).

6.4 Data, econometric estimation, and empirical results

6.4.1 The data used in the econometric analysis

The data used in the empirical estimation consists of the 24-hour recall food consumption data from the ISRA/IFPRI survey. The food consumption data was complemented with the income, livestock assets, production and other socio-economic variables from the other components of the survey. Because the household level price data collected by the survey was very thin, we used weekly market price data from two other sources to construct monthly price series for the cereals (millet, sorghum, maize, and rice) and pulses (peanut and cowpeas) for each zone in the stratification of the survey. The first source of price data is the Market Information System of CSA which has been collecting weekly consumer and producer price data for cereals and pulses in Dakar and other 40 local markets around Senegal since 1986. The second source of price data is the ISRA/MSU cereal marketing research project which collected producer and consumer price data for cereals and pulses every two weeks in 18 rural markets around Senegal and Gambia from 1984 to 1990. For the other food commodities, we used the price data in the food consumption survey to construct monthly weighted prices for each group of commodities and each zone, the weights

Table 6.7 Food (consumer) prices: Zone mean prices(coefficient of variation in parenthesis)

in CFA francs ($1 =280 CFA)	Rural zones						Urban zones	
	Sagatta	Niakhar	Colobane	Passy	Dioly	Missirah	Kaolack	Tamba
Cereals								
Millet........	91 (8%)	82 (16%)	72 (12%)	73 (19%)	69 (14%)	85 (18%)	119 (23%)	100 (12%)
Sorghum.......	. .	. .	. .	. .	66 (16%)	70 (15%)	. .	94 (21%)
Maize.........	107 (9%)	94 (27%)	102 (12%)	112 (12%)	73 (15%)	73 (28%)	114 (7%)	96 (16%)
Rice..........	135 (0%)	135 (0%)	135 (0%)	135 (0%)	138 (2%)	137 (2%)	138 (8%)	138 (2%)
Pulses								
Peanut........	178 (18%)	203 (17%)	145 (13%)	223 (22%)	143 (26%)	170 (23%)	214 (27%)	185 (33%)
Niebe.........	102 (14%)	95 (52%)	123 (23%)	115 (28%)	131 (18%)	147 (23%)	231 (22%)	403 (31%)
Animal proteins								
Meat/Chiken...	771 (4%)	728 (23%)	642 (13%)	654 (4%)	644 (2%)	526 (6%)	851 (9%)	772 (10%)
Fresh fish....	162 (18%)	104 (11%)	100 (0%)	50 (0%)	274 (54%)	312 (36%)	760 (56%)	323 (16%)
Dried fish....	143 (21%)	129 (37%)	100 (0%)	181 (13%)	152 (25%)	190 (20%)	226 (13%)	455 (43%)
Others								
Vegetables....	116 (15%)	172 (16%)	284 (30%)	136 (12%)	266 (13%)	207 (19%)	371 (25%)	467 (27%)
Condiments....	113 (7%)	122 (19%)	370 (16%)	188 (11%)	787 (29%)	27 (15%)	474 (45%)	429 (28%)
Vegetable oil.	367 (4%)	358 (2%)	340 (8%)	423 (3%)	367 (5%)	391 (10%)	370 (2%)	371 (4%)
Other food....	398 (2%)	426 (8%)	642 (12%)	386 (21%)	905 (11%)	457 (16%)	593 (10%)	372 (10%)

Sources: ISRA/IFPRI, ISRA/MSU, and CSA surveys.

Notes: The prices for vegetables, condiments and Other food are weighted average prices of the products making up the group. All units are in kilo except for vegetable oil (liter), condiments and other food which have mixed units of kilo, liter, and pieces.

being the zone median expenditure shares of the commodities making up the group. For months with missing weighted price for the other food commodities, we used the variations in the Dakar monthly price indices to reconstruct the corresponding monthly weighted prices. Table 6.7 gives the two-year average food prices and coefficients of variation for each zone.

Despite a painstaking and time-consuming effort, we have not been successful in constructing acceptable monthly weighted prices for the nonfood commodities except for fuel. This is mostly due to the fact that the units of measurement for the nonfood commodities are so numerous and diverse (with no information for possible standardization), and the price differentials resulting from quality differences so pronounced, that the monthly and geographical variations in the so-far constructed series (sometimes very large with no clear pattern) can hardly be considered as "genuine". Hence, because of this problem we were led to concentrate our demand analysis on the food commodities only.

In deterministic consumer theory, the separability conditions which theoretically justify considering the food commodities as a separate system are standard (see, for examples, Deaton and Muellbauer, 1980a, Varian, 1984, and Blundell, 1988). In our theoretical analysis of static consumption choices under uncertainty, we did not address the separability issues, but given the data constraints, we have no choice but to assume that the same separability conditions hold. Alternatively, we can also justify the restriction of our analysis to the food commodities by the weak separability assumption of food and nonfood commodities via the conditional cost function approach of Browning and Meghir (1991). In this case, the nonfood commodities are conditioning variables that affect the demand for the food commodities only through the income effects, and their demands need not be explicitly modelled (see Browning and Meghir, 1991). Additional information on the data and methods used for constructing the price series and other critical variables are in the data appendix in Appendix 6.

The sample used in the analysis is constituted by 269 households who were interviewed every other week for about two years. The pooled data has about 8000 observations, with the number of time periods for a household ranging from a minimum of 9 to a maximum of 45. The average number of time periods for a household is 32. If we aggregate at the monthly level by taking the monthly averages of the daily food consumption data, the number of observations of the pooled data reduces to about 4000, with the number of time periods per household ranging from 6 to 23 with 58% of the households having some gaps between months. If we aggregate at the seasonal level (by taking the season averages of the daily data), the sample size of the pooled data drops to about 1700 observations, with the number of time periods per household dropping to between 2 to 7, with 4 households only having a 1 season gap. Before the estimation, the sample was trimmed of outliers constituted by: 1) households who reported very low income compared to total expenditures (less than 70%); 2) households who have incomplete data for any of the variables used in the regressions, and 3) for each household, observations for which the total food expenditure per adult equivalent was twice as high or twice as low as the household season median total food expenditure per adult equivalent. At the end, we had 247 households, and 3889 pooled cross section-time series observations for the monthly aggregated data, and 1558 pooled cross section-time series observations for the seasonally aggregated data. Because of computational constraints, the estimated model is based on the seasonally aggregated data.

6.4.2 Estimation and emprical results[160]

As explained in the introduction of this chapter, the focus of the econometric estimation is the "expected" version of the AIDS model under uncertainty derived in chapter 5. But, we also estimate the deterministic AIDS model to compare the parameter and elasticity estimates from the two models. The estimation proceeds by following the steps of the Iterated Moment Estimator (IME) and Minimum Chi-square method (MCS) outlined at the end of chapter 5. Estimating a static model with panel data is a standard practice although there are problems brought about by ignoring intertemporal substitution issues.[161] Furthermore, we have already concluded in chapter 5 that in calculating the sample estimates of the moment of prices that appear in the uncertainty model, we have to rely on the price variation across the time dimension since the geographical price variation is irrelevant for measuring the impact of uncertainty on the individual consumer. To shorten the discussion of these intertemporal issues, we simply consider their implications in terms of mispecification and measurement errors to be incorporated in the error terms of the regressions.

Closely related to these theoretical economic considerations are the econometric implications and difficulties underlying the analysis of panel data, in which the correlation between unobserved household and/or time specific effects and the observed variables can yield inconsistent estimates of the parameters of interest when OLS is applied to the pooled data. Ideally, one would like to use an efficient instrumental variable estimator in the context of panel data such as those discussed in Amemiya and MaCurdy (1986), Wooldridge (1991),

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[160] All the computations were carried out in GAUSS.

[161] These intertemporal substitution issues are those related to the known conditions for intertemporal separability of consumption choices. However, in light of the Hindy, Huang, and Kreps (1992) criticism of the standard models of intertemporal consumption choice with respect to what they imply about intertemporal substutitability of consumption, the standard conditions for intertemporal separability have to reassessed.

and Cornwell, Schmidt and Wyhowski (1992), in order to minimize the effects of the correlation between measurement errors and unobserved household and time specific effects. However, given the computational difficulties in implementing these procedures with our relatively large sample size, we simply use pooled Least Squared estimators by invoking the assumption of strict exogeneity conditional on the unobservable effects (Wooldridge, 1991). We do use the heteroscedastic-robust covariance estimator of the IME estimator given in chapter 5 to guard against possible heteroscedasticity.

6.4.2.1 Estimation of the food demand system

The system of five food commodities estimated using the IME and MCS methods consists of the coarse grains (millet, sorghum, and maize), imported rice, the pulses (peanut and cowpeas), fish/meat (fresh fish, dried fish, meat, and chicken), and other food (vegetable oil, vegetables, condiments, and all other food commodities not listed here).[162] We normalized the daily total household food budget by dividing by the number of household members present the day of interview to obtain the daily total food expenditure per capita. This grouping of the food commodities was largely dictated by the important policy issue in Senegal of the substitutability beteween imported rice and the coarse grains, the quality of the data in some of the series, the frequency of zeros in some of the commodity series when isolated, and the need to keep the size of the system moderate for computational reasons.[163]

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[162] The complete list of food commodities on which data are collected by the ISRA/IFPRI survey can be found in Dakono et al., (1993).

[163] Because of its nature and its share in food expenditure, one would have preferred to keep vegetable oil separate from the other food category. However, the quality of the vegetable oil series is not particularly good because of some data collection problems during the survey.

Because the price of rice is fixed by the government at the same level for all zones and during the two-year period of the survey (see Table 6.7), we took relative prices by dividing by the price of rice and use the homogeneity conditions to recover the gamma coefficients related to the price of rice. Hence, with the homogeneity condition imposed, the lack of variation in the price of rice entails only a loss in efficiency. In addition, since we want to calculate the various welfare indicators, we imposed the adding up and symmetry restrictions during the estimation so as to get as close as possible to the integrability condition. The adding up restriction was imposed by dropping the equation for the other food category, while the symmetry restriction was imposed using the Minimum Chi-squared method.

Finally, we need at least one identifying restriction to take care of the endogeneity of total food expenditure per capita, resulting both from the weak separability assumption underlying the two-stage budgeting or conditional demand approach, and the fact that the total food expenditure was obtained by summing the individual food expenditures (thus implying some common measurement errors between total food expenditure and the shares). For that purpose, the annual household (gross) income seems to be a natural instrument for the daily total food expenditure per capita. We also used another set of instruments that initially included various types of income, livestock assets, and household characteristic variables. However, many of them were rejected after a series of overidentification tests. Likewise, many of the household socio-demographic variables were not included in our final specification, mostly because they either tended to introduce multicollinearity or they had very insignificant effects (on both food expenditures and the food shares). Table 6.8 gives the coefficient estimates of the first stage with the logarithm of total food expenditure per capita as the dependent variable and with the additional instruments being the logarithm of total

Table 6.8 Per capita food budget (first stage) and food share regressions

	Per capita food expenditure		Food share	
Constant	1.324	(1.454)	-.876	(1.081)
DPEUL	-.221	(.045)	-.008	(.007)
DSERERE	.058	(.035)	-.037	(.006)
DSARAKH	-.518	(.060)	-.010	(.010)
DOTHER	-.079	(.049)	-.029	(.009)
HHsize	-.028	(.002)	-.004	(.000)
RHadult	.212	(.104)	.035	(.017)
RFadult	.375	(.142)	-.042	(.026)
RGirls	.123	(.090)	-.056	(.015)
DMARKET	.143	(.028)	-.014	(.005)
DKaolack	.014	(.072)	.005	(.017)
DTamba	.011	(.069)	.017	(.018)
DNorth	-.524	(.039)	.025	(.007)
DSagatta	.174	(.071)	.000	(.013)
Total	-.365	(.074)	.009	(.013)
RPPulses	.238	(.061)	-.002	(.012)
RPFish\Meat	-.698	(.052)	.073	(.008)
RPOther	-.674	(.066)	-.049	(.012)
CATLSTK	.080	(.021)	.	.
HRDKSTK	-.059	(.071)	.	.
SHGTSTK	.025	(.014)	.	.
PLTRSTK	-.022	(.008)	.	.
OTHASTK	-.141	(.062)	.	.
ROFFINC	-.142	(.049)	.	.
RMIGINC	-.466	(.123)	.	.
LTotinc	-.283	(.268)	.	.
LTotinc2	.023	(.013)	.	.
LTotexp	.	.	.422	(.207)
Ltotexp2	.	.	-.025	(.010)
Diagnostic tests				
Unadjusted R^2	.4796		.3587	
Amemiya's adjusted R^2	.4613		.3421	
F-statistics	$F_{26,\ 1532} = 56$		$F_{19,\ 1539} = 48$	

Notes: Heteroscedastic-robust standard errors in parentheses annual income (LTotinc), the square of the logarithm of total annual income (LTotinc2), the ratio of off-farm income to total income (ROFFINC), the ratio of migration income to total income (RMIGINC), and the stocks of cattle (CATLSTK), horses/donkeys (HRDKSTK), sheeps/goats (SHGTSK), poultry, other animals (OTHASTK) (the list and labels for all the variables used in the econometric analysis is given in the data appendix).

6.4.2.2 The expected and deterministic AIDS coefficient estimates

We used the Stone price index (calculated using the zone median expenditure shares) as the starting value for the logarithm of the budget deflator a(p). With this choice of initial value for a(p), the first iteration of the IME method gives effectively the usual linear AIDS parameter estimates based on the Stone price index.[164] In general, the IME estimator, with α_0 estimated in each iteration, converges quickly depending on the sample size and tolerance level. Furthermore, when we fix α_0 to some constant value that is used in each iteration to update the budget deflator, the estimate of α_0 obtained from the procedure outlined in chapter 5 converges eventually to the same fixed α_0. However, in all our specifications, the converging values of α_0 were always different from the value suggested by Bank, Blundell and Lewbel (1994) which in our data would have been 3.2. Moreover, in our data some of the estimated parameters were very sensitive to the arbitrary choice of a fixed α_0. Finally, for any given tolerance level and choice of instruments and household demographics, the "expected" AIDS always converged faster than the deterministic AIDS. Table 6.10 presents the IME parameter estimates of α_0 of 1.49 for the "expected" AIDS and -2.57 for the deterministic model (with the OLS standard errors).

[164] In all our specifications, these initial parameter estimates were in general different from the converging parameter values (see Browning and Meghir, 1991, for a similar finding).

6.4.2.2.1 Model specification tests

At the bottom of Table 6.9 we present the specification tests for the "expected" and deterministic AIDS models. Both the symmetry and the over-identifying restrictions are rejected with a high confidence level. This is a typical finding in demand analysis using household level data (Blundell, 1988; Blundell et al., 1993; and Browning and Chiappori, 1994).[165] The rejection of the over-identifying restrictions is perhaps an indication of the rejection of the separability assumption and/or the invalidity of the remaining instruments as being not orthogonal to the measurement errors.[166] For the rejection of symmetry, Browning and Chiappori (1994) give some theoretical and empirical evidence suggesting that its systematic rejection in demand systems estimated with household level data may be due to the application of an individual choice model to data aggregated at the household level. They use the Pareto efficiency criterion to develop a model of household collective choices. In this "collective" model the symmetry restriction of the individual choice model becomes a symmetry and rank condition.

In the next sections we compare the price, income, and household characteristics coefficient estimates from the "expected" and deterministic AIDS models, and discuss Vuong's (1989) model selection test.

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[165] With demand systems using aggregated time series data, the rejection of homogeneity and symmetry is prevalent (Deaton and Muellbauer, 1980a). There is some evidence that the rejection of homogeneity may be due to aggregation problems, and that it may not be rejected with household level data (Blundell et al., 1993). Of course we cannot test for homogeneity here because of the lack of variation in the price of rice.

[166] We later discovered (private communication from Dr. Valerie Kelly) that because the data collection started after the 1988 harvest, part of the food consumption data originating from households' food stocks were used to estimate the household 1988-1989 income and production figures. This may explain the relatively high Chi-squared statistics for the coarse grains and pulses equations.

Table 6.9 Price and income coefficients for the uncertainty and deterministic AIDS models.

	"Expected" AIDS					Deterministic AIDS				
	Shares					Shares				
	Coarse grains	Pulses	Fish/ Meat	Other food	Rice	Coarse grains	Pulses	Fish/ Meat	Other food	Rice
	Gamma matrix					Gamma matrix				
Coarse grains...	.070	-.063	.022	-.108	.079	-.102	-.157	.032	.270	-.043
	(.035)	(.048)	(.068)	(.206)	(.066)	(.129)	(.152)	(.195)	(.659)	(.196)
Pulses..........	.076	.026	.017	-.112	-.007	.100	.038	.016	-.165	.010
	(.028)	(.037)	(.053)	(.161)	(.054)	(.069)	(.093)	(.127)	(.413)	(.129)
Fish/Meat.......	.109	-.020	-.075	.099	-.114	.130	-.008	-.076	.053	-.098
	(.024)	(.031)	(.043)	(.135)	(.045)	(.018)	(.023)	(.033)	(.099)	(.035)
Other...........	-.082	-.085	-.008	.146	.029	.044	-.015	-.016	-.135	.121
	(.077)	(.107)	(.149)	(.473)	(.147)	(.104)	(.141)	(.192)	(.623)	(.192)
Rice............	-.173	.142	.043	-.025	.012	-.172	.141	.044	-.023	.011
	(.150)	(.219)	(.307)	(.971)	(.306)	(.072)	(.112)	(.165)	(.496)	(.166)
	Symmetry restricted Gamma matrix					Symmetry restricted Gamma matrix				
Coarse grains...	.091	-.033	.060	-.168	.051	.055	.008	.078	-.077	-.064
	(.017)	(.008)	(.009)	(.012)	(.019)	(.039)	(.013)	(.010)	(.019)	(.028)
Pulses..........	-.033	.020	.000	-.049	.062	.008	.003	.005	-.074	.058
	(.008)	(.009)	(.006)	(.010)	(.014)	(.013)	(.010)	(.006)	(.009)	(.013)
Fish/Meat.......	.060	.000	-.046	.057	-.070	.078	.005	-.046	.027	-.064
	(.009)	(.006)	(.009)	(.009)	(.012)	(.010)	(.006)	(.009)	(.008)	(.011)
Other...........	-.168	-.049	.057	.051	.109	-.077	-.074	.027	-.030	.154
	(.012)	(.010)	(.009)	(.018)	(.027)	(.019)	(.009)	(.008)	(.016)	(.021)
Rice............	.051	.062	-.070	.109	-.151	-.064	.058	-.064	.154	-.085
	(.019)	(.014)	(.012)	(.027)	(.050)	(.028)	(.013)	(.011)	(.021)	(.038)
	Beta coefficients					Beta coefficients				
	-.103	-.059	.008	.230	-.077	-.109	-.060	.006	.239	-.078
	(.063)	(.107)	(.043)	(.258)	(.054)	(.083)	(.141)	(.033)	(.222)	(.129)
Symmetry test:		$\chi^2_{(6)} = 95$						$\chi^2_{(6)} = 98$		
Over-identification test: $\chi^2_{(9)} =$	201	167	94	.	53	196	169	94	.	52
Vuong's variance test					$\omega = .00$					

Notes: Heteroscedastic-robust standard errors in parentheses

6.4.2.2.2 Price and income coefficients

Price and income (more precisely expenditure) effects are best and commonly evaluated by using the price and inome elasticities, which are discussed in the last section of this chapter. However, we use this section to briefly compare the price and expenditure coefficient estimates from the "expected" and deterministic AIDS model.

In Table 6.9 we present the parameter estimates for the gamma matrix from both the "expected" and the deterministic AIDS model. The first thing to note is that estimates of the gamma parameters from the two models seem to be different, although only a few of the unrestricted parameters are statistically significant at the 5% significance level.[167] But, when we apply the symmetry restriction all but one of the "expected" AIDS restricted gamma parameter estimates become statistically significant at the 5% significance level, while for the deterministic AIDS estimates one-third of the restricted gamma parameters remain statistically insignificant at the 5% level. Also, by and large the two models' restricted gamma parameter estimates seem to be different. In contrast to the restricted gamma parameter estimates, none of the expenditure (beta) coefficients are statistically significant at the 5% level, although the estimates from the "expected" AIDS and deterministic AIDS models seem to be very close.

6.4.2.2.3 The coefficients and impacts of household characteristics

Table 6.10 reports the "expected" and deterministic AIDS estimates for the alpha matrix which measures the impacts of the household characteristics variables on the food shares. We note that for most of the variables, the estimated parameters from the two models are very close. When the estimated coefficients from the two models seem to diverge, they

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[167] Here, for convenience, we refer to the adding-up and homogeneity restricted parameters as unrestricted.

Table 6.10 The household characteristics parameter estimates for the uncertainty and deterministic AIDS models.

	"Expected AIDS"					Deterministic AIDS				
	Shares					Shares				
	Coarse grains	Pulses	Fish/ Meat	Other food	Rice	Coarse grains	Pulses	Fish/ Meat	Other food	Rice
	alpha0 = 1.489 (.289)					alpha0 = -2.568 (.277)				
	Alpha matrix					Alpha matrix				
Constant....	.467	.216	.151	.046	.120	.778	.387	.132	-.638	.342
	(.072)	(.063)	(.107)	(.153)	(.043)	(.163)	(.083)	(.141)	(.242)	(.033)
DPEUL.......	-.005	-.007	-.022	.008	.025	-.011	-.010	-.021	.023	.020
	(.011)	(.092)	(.079)	(.147)	(.149)	(.010)	(.180)	(.104)	(.267)	(.192)
DSERERE.....	.046	.017	-.002	-.053	-.008	.036	.012	-.001	-.032	-.015
	(.009)	(.009)	(.130)	(.049)	(.109)	(.014)	(.009)	(.219)	(.091)	(.134)
DSARAKH.....	-.012	.035	.014	-.049	.012	-.024	.029	.015	-.024	.004
	(.019)	(.006)	(.014)	(.110)	(.133)	(.013)	(.014)	(.014)	(.215)	(.215)
DOTHER......	-.012	.007	.028	-.025	.002	-.020	.002	.029	-.007	-.004
	(.012)	(.023)	(.009)	(.044)	(.015)	(.010)	(.010)	(.019)	(.019)	(.015)
HHsize......	-.002	-.004	.000	.010	-.004	-.002	-.004	.000	.010	-.004
	(.003)	(.011)	(.034)	(.048)	(.009)	(.003)	(.007)	(.017)	(.021)	(.019)
RMadult.....	.078	.035	-.028	-.084	.001	.087	.038	-.028	-.102	.005
	(.041)	(.003)	(.018)	(.036)	(.033)	(.057)	(.004)	(.013)	(.067)	(.015)
RFadult.....	-.008	.009	.068	-.150	.081	.010	.017	.068	-.187	.092
	(.060)	(.044)	(.004)	(.080)	(.018)	(.071)	(.065)	(.005)	(.130)	(.012)
RGirls......	-.074	-.029	.015	.050	.038	-.069	-.026	.015	.039	.041
	(.023)	(.067)	(.061)	(.126)	(.004)	(.025)	(.078)	(.087)	(.169)	(.005)
DMARKET.....	-.145	-.014	.041	.052	.066	-.147	-.015	.041	.055	.065
	(.008)	(.014)	(.095)	(.157)	(.062)	(.009)	(.017)	(.105)	(.193)	(.085)
DKaolack....	-.418	-.060	.115	.260	.104	-.387	-.043	.113	.192	.126
	(.023)	(.005)	(.023)	(.121)	(.094)	(.017)	(.009)	(.027)	(.115)	(.103)
DTamba......	-.385	-.036	.173	.064	.183	-.395	-.043	.175	.089	.174
	(.044)	(.032)	(.008)	(.083)	(.025)	(.018)	(.015)	(.012)	(.030)	(.026)
DNWest......	-.155	-.047	.031	.042	.129	-.158	-.049	.031	.049	.127
	(.012)	(.062)	(.046)	(.098)	(.008)	(.019)	(.012)	(.026)	(.028)	(.013)
DSagatta....	-.364	-.064	.012	.192	.225	-.331	-.045	.009	.117	.250
	(.052)	(.009)	(.086)	(.079)	(.047)	(.030)	(.022)	(.019)	(.037)	(.026)

Notes: Heteroscedastic-robust standard errors in parentheses. The household characteristic variables are: Dummies for ethnicity (Wolof is the omitted group): Peul, Serere, Sarakhole (DSARAKH), and Others. Household composition: total household population (HHsize), ratio of male adults (age > 15) population to total household population, ratio of female adult (age >15) population to total household population, and ratio of girl (age <15) population to total household population. Geographic location dummies: Market village, City of Kaolack, City of Tambacounda, Western zones(Colobane, Niakhar, and Passy) tend to be not statistically significantly different from zero at the usual 5% significance level. This should not be a surprise since in terms of econometric analysis the share equations of the two models are identical except for the budget deflator $a(\mathbf{p})$. This statistical finding will be confirmed more formally by Vuong's (1989) model selection test discussed below. But, before discussing Vuong's test we will briefly discuss the impacts of the household characteristic variables on the food commodity shares.

Almost all the statistically significant estimated coefficients have the expected sign. In particular, the "expected" AIDS model tells us that everything else being equal, the share of coarse grains (millet, sorghum, and maize) in food expenditure is .046 higher for a Serere than for a Wolof. Surprisingly, the corresponding coefficient in the rice share equation, although negative as expected, is not statistically different from zero even at the 10% significance level. Furthermore, ethnicity does not seem to have a statistically significant impact on food consumption patterns when household composition, geographic location, relative prices, and deflated total food expenditure per capita are accounted for.[168]

Household composition also does not seem to have a statistically significant impact on the daily shares when the other variables are controlled for. In the "expected" AIDS model the coefficient for the total household population is never statistically significant for any of the shares. The only household composition coefficients that are statistically significant are the household population ratio of girls (age 15 or younger) in the coarse grains and rice equations (-.074 and +.038) respectively, the household population ratio of female adults (age 15 or older) in the fish\meat and rice equations (+.068 and +.081) respectively, and the household

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[168] The exceptions are the share of pulses (peanut and cowpeas) - which is .035 higher for Sarakhole compared to Wolof when everything else is controlled for - , and the share of animal protein (fish, meat, and chicken) - which is .028 higher for households not in the Wolof, Serere, and Sarakhole ethnic groups than for Wolof households, when everything else is controlled for.

population ratio of male adults (age 15 or older) in the pulses equations (+.035). This means for example, when everything else is being controlled for, the increase in the household population ratio of girls age 15 or younger by one unit has the effect of decreasing the share of coarse grains in the household's total food expenditure by .074, and increasing the share of rice by .038. The other household composition coefficients are interpreted similarly.

In contrast, the geographic location dummy variables seem to have a significant impact on the household daily food commodity shares. This is especially true for the shares of coarse grains and rice. Moreover, they all have the expected signs. From the "expected" AIDS model, everything else being equal the share of the coarse grains is .145 lower for a household in a market village than for one in a nonmarket village, although there does not seem to be any statistical difference when it comes to the share of rice. Similarly, when the other variables are held constant, households in the cities of Kaolack and Tambacounda, and those in the Northern and Western rural zones of Senegal (Sagatta, Colobane, Niakhar, and Passy) tend to have significantly lower shares of coarse grains and higher shares of rice compared to those in the Eastern rural zones (Dioly and Missirah).

6.4.2.2.4 Vuong's model selection test

In this section we use Vuong's (1989) series of model selection tests for strictly non-nested, overlapping, and nested models to compare the "expected" AIDS and deterministic AIDS models. Vuong's tests are loglikelihood ratio based tests which are valid even for misspecified models.

As we have already said, although different from an economic theoretical view, in terms of econometric estimation the share equations of the two models are identical except for the budget deflator in the income terms. Hence, if we assume that the distribution of the measurement error is the same in the two models, the relevant test to apply here is the one for overlapping models. Vuong's test for overlapping models is a sequential test which tests first the null hypothesis that the variance of the loglikelihood ratio (LR) is equal to zero, against the alternative that it is different from zero. Vuong (1989) shows that the null hypothesis is equivalent to the condition that the conditional density of the two models being compared are equal, and that under it the LR statistic is asymptotically distributed as a Weighted Sums of Chi-Squared Distributions. On the other hand, under the alternative the LR statistic is asymptotically normal. If the null hypothesis is not rejected, then the testing is over and one concludes that the two models are statistically indistinguishable given the data. But, if the null hypothesis is rejected, one uses the distribution of the LR statistic under the alternative to test the null hypotheses that the expected value of the loglikelihood ratio is equal to zero, meaning that the two models are equivalent, against either one of the two alternatives that the expected value of the LR is positive or negative, meaning that one of the two models is "better" than the other.[169] To perform the variance test in the first stage, Vuong gives an estimator of the LR variance, which he shows to be asymptotically distributed as a Weighted Sums of Chi-Square Distributions under the null hypotheses. Unfortunately, the distribution of the Weighted Sums of Chi-Square Distributions is non-standard in the sense that its critical values are not available in published statistical tables. They need to be computed through programming.[170] Hence, in the absence of critical values to compare with, we simply give in

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[169] Note that the null hypotheses in this LR test contains the null hypotheses in the variance test in the first stage of the sequential testing procedure.

[170] Vuong (1989) refers to the availability of two Fortran subroutines (one in Johnston and Kotz, 1969; and the other in Dubin and Rivers (1986)) that can be used to compute the relevant critical values for the Weighted Sums of Chi-Square Distributions. Vuong also shows that under a condition called Information Matrix Equivalence, the distribution of the Weighted Sums of Chi-Square Distributions reduce to a standard Chi-Square distribution. White (1982) outlines a procedure for testing the Information Matrix Equivalence condition. However, this procedure is difficult to implement.

Table 6.9 the computed variance statistics, $\hat{\omega}$, and informally conclude given its practically zero value that the "expected" and deterministic AIDS models are statistically indistinguishable in our data.[171] Despite the lack of critical values to formally compare, this conclusion is not surprising given the facts that the two models would be econometrically identical if the beta coefficients in both models were equal to zero, and that the beta estimates in both models are not statistically different from zero at the 5% significance level.

Besides this argument based on the beta coefficients, there are at least three more reasons why the "expected" and deterministic AIDS models might be statistically indistinguishable. 1) Since the uncertainty model reduces to the deterministic model, there may not be enough price variation in our data to allow the statistical distinction of the two models. 2) Even if there were sufficient price variation in the data, the econometric models resulting from the two economic theoretical formulations are so close to prevent the statistical distinction of one from another. 3) The behavior implied by the two apparently different theoretical functional forms is the same. Indeed, we saw in the examples in chapter 3 that "expected" functional forms sometimes lead to restrictive behavior that is very close to behavior in the absence of uncertainty. This is especially relevant given the fact that the "expected" AIDS shares are only functions of the first moments of prices, which are

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Nevertheless, by direct computation and without formal statistical testing we found the Information Matrix Equivalence condition not likely to be satisfied in our data. In any case, even if the condition was satisfied, the implied degrees of freedom was found to be zero, making the application of the Chi-Square test impossible.

[171] Note that zero is the lower bound of the variance statistics, and we are testing the null hypothesis of equality to zero.

moreover only in the budget deflator $a(\mathbf{p})$. The estimation of the more uncertainty-flexible "non-expected" AIDS will perhaps shed more light on this issue.[172]

6.5 The consumption behavior and risk attitudes of Senegalese households

6.5.1 Numerical computation of the pre-devaluation elasticities

We used the models' (symmetry restricted) parameter estimates in Table 6.9 and Table 6.10 to calculate the elasticities for both the "expected" and the deterministic AIDS model using the pre-devaluation price series and the elasticity formulas derived in chapter 5. The pre-devaluation elasticities were computed for each household. The calculation by household eliminates the aggregation bias that would result if we were to use average values across households for the variables (Blundell et al., 1993). This is particularly important given the nonlinearity of the AIDS model. But, perhaps more importantly, the calculation by household enables us to analyze the distribution of each elasticity across households through the use of histograms, boxplots, and cross tabulations.[173]

For the deterministic model the calculation was done for each observation first before aggregating at the household level (by taking average). For the uncertainty model the elasticities and welfare indicators are functions of the moments of the variables. Hence, for each household we need a series of observations for each variable in order to calculate the required sample analogues. In principle, the number of observations for the computation of

[172] Again we emphasize that even if the two models are statistically indistinguishable, from the economic theoretical view they may still be different if theoretical behavioral restrictions do not reduce the uncertainty model to the deterministic model. Moreover, as the formula shows, we can still expect significant differences in the respective computed elasticities and welfare indicators unless the price variation in the data is insignificant.

[173] All the elasticity and welfare indicator calculations were done in GAUSS. The household level figures were then transferred to SPSS where they were matched with the other household characteristic variables before analysis. All the tables, histograms and boxplot diagrams are direct SPSS outputs.

the elasticities need not be the same as for the regression analysis. That is, we can use monthly or yearly observations even if the regression analysis uses seasonal data. However, the number of observations must be large enough so as permit the computation of reliable sample estimates. For the pre-devaluation elasticities, we used the same monthly price series used in the econometric estimation, the per capita household total food expenditure series, and the vector of household characteristics series. To avoid incorporating the irrelevant (cross sectional) price variations across zones, the sample moments for the prices were calculated for each zone along the time dimension, with one value for each zone and for each sample moment.[174] Whenever relevant, we have weighted by the respective village sampling weights when calculating the median across households in order to take into account the oversampling of market villages and the relative population weights of the study zones. The weights were calculated using the 1988 national census data. Four types of weights were used depending on the level of aggregation used in the calculation of the medians (within zones, across zones, within markets and nonmarket villages respectively, and within rural and urban zones respectively). Hence, the weighting schemes make the elasticity figures representative for the population of the study zones. The only figures that were not weighted are those related to the income groups, due to the fact that the income classification is sample based.

The elasticity formulas in chapter 5 are expressed in terms of the food commodity shares. This raises the question of whether to use the actual shares or the predicted shares. The elasticities reported here are based on the predicted shares because they were found to be more stable and reasonable, and less prone to outliers. In calculating the predicted shares we

[174] These constant price sample moments suppose that the price series are stationary so that their moments are independent of time. Alternatively, we could use a Generalized Autoregressive Conditional Heteroscedastic (GARCH) model for the price series to estimate their time-varying conditional moments.

included the reduced-form residuals (Amemiya, 1985, chapter 9). With the endogeneity of per capita total food expenditure, including the reduced-form residuals in predicting the shares is equivalent to forecasting with the over-identifying restrictions imposed (Blundell et. al., 1993). Finally, since the elasticity calculations involve dividing by the commodity shares, whenever a predicted value share was negative, we set the corresponding value to 0.01. This is a reasonable practice given the level of commodity aggregation and the commodity share statistics in Table 6.4, Table 6.5, and Table 6.6. In any case, only a few shares were predicted to be negative (less than 3% of all cases), and a quick sensitivity analysis revealed that the elasticity estimates were not sensitive to this choice.

It is perhaps worth noting that although it is frequently done in the literature, the statistical significance of the price and expenditure effects, as given by the gamma and beta parameters in Table 6.9 and Table 6.10, alone cannot be used to judge the statistical significance of the respective price and expenditure (or budget) elasticities. In fact, one gamma parameter can be statistically not different from zero, and yet the corresponding price elasticity be statistically significantly different from zero. This is because the standard error of any individual elasticity depends on the variances and covariances of all the gamma and beta parameter estimates in the model, as is clear from the elasticity formulas. Despite the nonlinearities in the formula, asymptotic standard errors for the elasticities can be analytically calculated using the "delta method" (i.e., by a second order Taylor expansion of the nonlinear functions of the parameters around the true parameter values). But, because of time constraints, we did not calculate these standard errors.

Finally, before presenting the elasticity estimates, we briefly recall, for the convenience of the reader, their interpretations in the context of uncertainty. The interpretations of price and budget elasticities in deterministic demand models is standard: the percentage change in quantity demanded corresponding to 1% change in one price (or in total expenditure) while holding all the other prices, total expenditure, and the household characteristic variables constant.[175] In chapter **2** we showed that the elasticities derived under uncertainty have essentially the same interpretations. There are only two differences. First, the meaning of the word "percentage change" is governed by the metrics of the respective spaces. In deterministic models, the distance between two points is given by the usual Euclidean distance. In the uncertainty context, we defined the distance between two random variables to be, roughly speaking, the average across all states of nature of the state-by-state Euclidean distances of their respective possible values. Or, in other words and more precisely, the "expected" value of their Euclidean distance. Moreover, as discussed in section **2.2.5**, under this metric, we can always interpret a "percentage change in a random variable" as a percentage *expected* change in the random variable. Hence, in this framework elasticities are interpreted as: the percentage *expected* change in demands resulting from one percent *expected* change in a price.[176] Second, with uncertainty the budget elasticities are random variables. Hence, only their means and variances are reported (see chapters **2** and **5** for more details). Moreover, since the demand system includes the food commodities only, the budget elasticities reported here are for the food budget, not the total household budget.

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[175] Here, we are using the term "budget or expenditure elasticity" instead of the more commonly used term of "income elasticity". See section **2.2.7** in chapter **2** (or, for example, Deaton and Muelbauer, 1980a, chapter 1) for the theoretical issues involved in the difference between total income and total expenditure). Although "income elasticity" is still the prevalent terminology in the literature and the reporting of elasticity estimates, the term "budget elasticity" is being increasingly used to recognize the difference between total income and total expenditure. Although in the thesis we do use the terms "income elasticity" and "income effects" for convenience, they always refer to total expenditure.

[176] We recall again the behavioral and mathematical differences between an *expected* change in a random variable and a change in the *expected* value of a random variable (see chapter **2** for more details).

Furthermore, in section **2.5** we have identified the degree of risk in a random price by the consumer's *perception* of its degree of variability which we have measured by its distance to the consumer's *expected* value for it under the metric defined above. The elasticity measures of how changes in the consumer's *ex ante* optimal consumption choices, resulting from an *expected* change in a random price, is related to the variability of the random price in question is then defined as the elasticity measure of the consumer's overall *precautionary* response to the risk in the random price. We have also decomposed each price elasticity into its *precautionary* risk component (which is equal to zero in the absence of uncertainty or when the consumer's behavior is not affected by uncertainty) and its nonrisk component (which reduces to its deterministic counterpart in the absence of uncertainty). The term *precautionary* that is attached to these price-risk effects describes the *ex-ante* precautionary measures that the consumer takes to deal with the price uncertainty (i.e., the precautionary consumption decisions made before the resolution of the price uncertainty). The reader is referred to chapter **2** for more details.

6.5.2 Distribution of the households' matrices of elasticity estimates

6.5.2.1 Sample median estimates of the matrices of elasticities

6.5.2.1.1 The uncompensated and budget elasticities

Table 6.11 gives the pre-devaluation sample median uncompensated elasticity matrices and budget elasticities for both the "expected" and the deterministic AIDS models. The elasticities from the deterministic AIDS model seem to be respectively different from their (total) analogues from the "expected" AIDS model, but they are not too far apart. From the total elasticity matrix of the "expected" we can see that only the fish/meat food group has a

Table 6.11 Pre-devaluation median uncompensated and budget elasticities for the uncertainty and deterministic AIDS models: All zones

	% change in quantities				
	Coarse grains	Pulses	Fish/ Meat	Other food	Rice
	"Expected" AIDS model				
Total					
Coarse grains...	-.776	.160	.094	-.214	.169
Pulse...........	.107	-.819	.053	-.160	.101
Fish/Meat.......	-.198	-.203	-1.172	-.536	-.205
Other food......	.732	.706	.560	-.941	.713
Rice............	.074	.066	.019	-.228	-.904
Precautionary risk					
Coarse grains...	.013	-.003	-.002	-.004	-.002
Pulse...........	-.002	.021	-.001	-.001	-.001
Fish/Meat.......	.003	.003	.017	.003	.003
Other food......	-.004	-.004	-.004	.005	-.004
Rice............	.000	.000	.000	.000	.000
Nonrisk component					
Coarse grains...	-.792	.163	.096	-.210	.173
Pulse...........	.109	-.844	.055	-.159	.105
Fish/Meat.......	-.202	-.207	-1.145	-.537	-.202
Other food......	.730	.722	.574	-.970	.722
Rice............	.073	.066	.019	-.228	-.904
Budget elasticity					
Mean	.809	.796	.958	1.746	.807
Variance	.051	.073	.126	.311	.047
	Deterministic AIDS model				
Coarse grains...	-.651	.273	.242	-.851	-.018
Pulse...........	.140	-.873	.009	-.597	.236
Fish/Meat.......	.300	.107	-1.159	-.143	-.091
Other food......	-.272	-.471	.095	-.821	.342
Rice............	.026	.557	-.226	-.059	-1.114
Budget elasticity ..	.711	.685	1.021	1.777	.788

Notes: The elasticities were calculated for each household. Each cell contains the weighted median of the respective elasticities across all households. The two models were estimated with the pooled season averages of the daily food consumption data, using the Iterated Moment Estimator of Blundell and Robin (1993). (the food consumption survey collected 24 hours-recall data every 15 days for two years).

median (total) uncompensated own price elasticity greater than 1 in absolute value (-1.172), while from the deterministic AIDS model, both fish\meat and rice have median uncompensated own price elasticities greater than 1 in absolute value (-1.159 and -1.114 respectively). Out of the 25 elasticities from the deterministic AIDS elasticity matrix, 6 do not have the same sign as their respective analogues from the "expected" AIDS total elasticity matrix. Similarly, from the sample median means of the budget elasticities from the "expected" AIDS model, only the other food group is a luxury with a sample median budget elasticity of 1.746. All other food groups (coarse grains, pulses, fish/meat, and rice) are necessities, with respective budget elasticities of .809, 0.796, 0.958, and 0.807. In the deterministic AIDS model fish/meat becomes a luxury commodity along with the other food group with respective budget elasticities of 1.021 and 1.777. The "deterministic" figures for the other food groups are 0.711 for coarse grains, 0.685 for pulses, and 0.788 for rice.

The discrepancies of the elasticities from the two models despite the fact that the two models are theoretically close and statistically indistinguishable, indicate biases in elasticities calculated from estimated deterministic models, when price uncertainty is relevant. Moreover, from Table 6.11 it is clear that the direction of the bias is uniform.[177] The potential biases were already clear from the theoretical formulas in chapters 2 and 5, and from the examples in chapter 3. Furthermore, on the empirical side, for all the different models we estimated (with differences in specification among models being only on the grouping of the food commodities and the included sets of household characteristics and instrumental variables), the elasticities and welfare indicators from the "expected" AIDS were always found to be more "reasonable" and stable (i.e., less sensitive to minor changes in

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[177] Given the facts that the standards errors are not provided and that we cannot formally test the statistical significance of differences, the extent to which the apparent differences between the two models estimated elasticities can be considered significant is mostly a matter of personal judgement.

specification) than the ones from the identically specified deterministic AIDS. Hence, to shorten the discussion, for the remainder of this chapter we will focus only on the elasticity and welfare indicator estimates from the "expected" AIDS model. However, in most of the tables below, we have provided estimates from both models so as to let the reader appreciate the discrepancies between the estimates for the two models.

Table 6.11 indicates that all the total uncompensated own price elasticities are negative, and that the total uncompensated cross price elasticities between coarse grain and rice are positive. This means in particular that for a median household a 1% *ceteris paribus* expected increase in the price of coarse grains will decrease the average daily consumption of coarse grains and other food by 0.78%, and 0.21% respectively, and will increase the average daily consumption of pulses, fish/meat, and rice by 0.16%, 0.09%, and 0.17% respectively. Similarly, for a median household a 1% *ceteris paribus* expected increase in the price of rice will decrease the average daily rice and other food consumption by 0.9% and 0.23% respectively, and increase the average daily coarse grains, pulses and fish/meat consumption by 0.074%, 0.066%, and 0.02% respectively.

Table 6.11 also gives the uncompensated precautionary risk and nonrisk price elasticities. The interpretation of the nonrisk-price elasticities is similar to the one for the total elasticities. For the precautionary risk price elasticities, with respect to the coarse grains for example, a 1% *ceteris paribus* expected increase in its price will lead a median household to respond to the uncertainty in its price (measured by its variability) by increasing the average daily consumption of coarse grains by 0.013% and decreasing the average daily consumption of pulses, fish/meat, other food, and rice by 0.003%, 0.002%, 0.004%, and 0.002% respectively. The explanation is that, as the household "expects" the coarse grains prices to rise, the fact that the prices are uncertain, leads it to take some precautionary measures by, for example, producing more coarse grains or building up enough stocks so as to end up consuming over time more of the available coarse grains on stocks than if coarse grain prices were not random. Hence, with respect to the price of the coarse grains, the median household's precautionary risk response to its uncertainty has, in general, the effect of dampening its overall price responsiveness.[178] The same pattern of responses to price uncertainty appears for pulses and the other food group. The only exceptions are rice, for which the median price risk elasticity is zero because of the the constancy of its price in most of the zones, and fish/meat for which the median household responds to the variability in its price by increasing the daily average consumption of all food commodities.[179] However, the magnitudes of the precautionary elasticities are very small compared to their nonrisk respective counterparts, indicating that uncertainty in food prices is not an important issue in the Senegalese household food consumption choices. This perhaps is not a surprising empirical finding given the low variation in food prices in Senegal during the two-year period of the ISRA/IFPRI survey (see Table 6.7). But still, from a policy perspective it is an important empirical confirmation given the Senegalese government's past and failed attempts to stabilize coarse grain prices.[180]

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[178] Note that except for the other food group, the precautionary risk and nonrisk elasticities have opposite signs.

[179] We note that the theoretical results in chapter 2 did not put any sign restrictions on either the uncompensated or compensated precautionary risk elasticities.

[180] To avoid misundertanding, our conclusion does not say that uncertainty is not important in the Senegalese household food consumption choices. Our conclusion concerns only <u>price</u> uncertainty. The effect of income uncertanty (which may be important) is not covered by this conclusion.

6.5.2.1.2 The compensated elasticities

Table 6.12 gives the median compensated elasticity matrices. All the total own price compensated elasticities are negative as predicted by the theory in chapter 2, but none of them seems to be elastic (although some are "sizeable" by some standards). Also, the eigenvalues of the Slutsky matrix were found to be all negative except the one related to rice. We note also that all the food commodities (including rice and the coarse grains) are substitute excepts for fish\meat and the other food group for which the sign reversal of the cross price elasticities seems to be in contradiction to what the theory predicts.[181] However, the -0.037 might indicate that it is not significantly different from zero.

An important fact to note here are the relatively large differences between the compensated and the uncompensated elasticities. This is an indication that there is a relatively strong income or budget effect in the median household's response to expected change in prices.[182] For example, in the case of the coarse grains for a 1% *ceteris paribus* expected change in the prices of the coarse grains, there is a 0.52% decrease in the average daily consumption of coarse grains due to the expected loss of purchasing power only, while only 0.25% decrease in the average daily consumption of the coarse grains is due to the pure price effect (i.e., to the fact that the coarse grains are expected to become relatively more expensive). In contrast, for the same 1% *ceteris paribus* expected increase in coarse grains prices, there is a relatively modest substitution toward rice (0.67% increase) due only to the pure price effect, which ends up being reduced to a mere 0.17% increase by the equally

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[181] Note that unlike the Slutsky matrix, the (total) compensated elasticity matrix needs not be symmetric. However, the cross price elasticities in each respective pair should always have the same sign.

[182] We recall that in terms of elasticity the income or budget effect is the difference between the uncompensated elasticity and the compensated elasticity (see chapter 2 and 5 for more details).

Table 6.12 Pre-devaluation median compensated elasticities for the uncertainty and deterministic AIDS models: All zones

	% change in quantities				
	Coarse grains	Pulses	Fish/ Meat	Other food	Rice
	"Expected" AIDS model				
Total					
Coarse grains...	-.252	.652	.675	.837	.667
Pulse...........	.403	-.466	.419	.536	.397
Fish/Meat.......	.022	.010	-.858	-.037	.009
Other food......	.882	.869	.780	-.482	.869
Rice............	.440	.440	.463	.636	-.499
Risk component					
Coarse grains...	.005	-.006	-.002	-.006	.001
Pulse...........	-.005	.010	-.003	-.004	-.003
Fish/Meat.......	.002	.001	.015	.003	.001
Other food......	-.004	-.003	-.005	.003	-.004
Rice............	.000	.000	.000	.000	.000
Nonrisk component					
Coarse grains...	-.257	.655	.677	.836	.658
Pulse...........	.408	-.478	.417	.555	.393
Fish/Meat.......	.020	.006	-.836	-.039	.008
Other food......	.895	.881	.786	-.477	.882
Rice............	.440	.440	.463	.636	-.499
	Deterministic AIDS model				
Coarse grains...	-.274	.618	.760	-.092	.323
Pulse...........	.400	-.601	.275	-.116	.501
Fish/Meat.......	.564	.371	-.811	.503	.196
Other food......	.000	-.216	.507	-.184	.618
Rice............	.378	.932	.276	.897	-.710

Notes: The elasticities were calculated for each household. Each cell contains the weighted median of the respective elasticities across all households. The two models were estimated with the pooled season averages of the daily food consumption data, using the Iterated Moment Estimator of Blundell and Robin (1993). (the food consumption survey collected 24 hours-recall data every 15 days for two years).

relatively strong income or budget effect. This contrasting impact that the income effect has on the own price and cross price responses of the median household is a general pattern applying to almost all the commodities, and is clearly discernable from the total compensated and uncompensated elasticity matrices. That is, when it comes to the own-price response of the median household, most of the response is due to a relatively strong income or budget effect, but when it comes to its cross-price response, most of the potential price substitution effect is wiped out by the same relatively strong income or budget effect. This is perhaps not surprising given the fact that the households in the sample are on average very poor with mean net income of 50,000 CFA ($180) per year or just 28% of average GDP per capita; and that 21% of the households in the sample spend on average more than 90% of their total budget on food, 24% of them spend between 80 to 90% on food, and only 8% of them spend less than 70% on food (see Table 6.3 and Table 6.5).

This empirical finding has important policy implications regarding the cereal pricing policy debate in Senegal. The government has so far maintained a policy of high official prices for imported rice in the hope that higher relative price for imported rice compared to the locally grown coarse grains will push consumers to switch to the latter. So far the policy has not worked either in curtailing consumer demand for imported rice, or in boosting the demand for the coarse grains. But, some policy observers and researchers maintain that is because the relative price of imported rice has not been high enough. Consequently, they advocate further increase in the official price of rice. But, the figures in these tables clearly indicate that one of the reasons for the failure of the higher rice price policy lies rather in the resulting loss in purchasing power. The numbers also indicate that for these poor households in the ISRA/IFPRI study zones, further *ceteris paribus* increases in the price of imported rice would come at such a great cost that it would not leave them much for buying more millet.

For example, a 1% *ceteris paribus* expected increase in the price of rice could have increased the median household's average daily coarse grains consumption by as much as 0.44% if there were no reduction in purchasing power. But, because of the resulting loss in purchasing power the increase in the average daily coarse grains consumption will only be a mere 0.07%! Meanwhile, 44% of the 0.9% resulting decrease in the average daily rice consumption will be due to a loss in purchasing power, not to substitution toward other food commodities.

Finally, the compensated precautionary risk price elasticity matrix in Table 6.12 shows that the median own-price precautionary risk subtitution elasticity, altough very small, is positive for all commodities except rice for which it is zero as expected. Moreover, for almost all commodities the cross-price precautionary risk substitution and nonrisk elasticities have opposite signs. Hence, in general, the *ex ante* precautionary price-risk substitution effects dampen the household's overall ability to substitute accross commodities. In other words, even when the household is compensated for the possible loss of welfare due to risk aversion and to an *expected* increase in the price of a commodity, the *ex ante* precautionary risk substitution consumption decisons, taken to deal with the price uncertainty, reduce the substitution effects that would have obtained in the absence of uncertainty. The only exception to this is rice with respect to the price of the coarse grains and the price of fish/meat.

For each commodity, the difference between the uncompensated precautionary effects and the *precautionary* risk substitution effects give the *precautionary* risk aversion effects which measures part of the changes in consumption of that commodity resulting from the loss in *ex ante* welfare associated with risk aversion (i.e., "dislike" of uncertainty). Hence, conceptually the precautionary risk aversion effect is similar to the pure income effects However, the overall magnitudes of the precautionary price-risk elasticities are so small compared to their nonrisk counterparts, that they can be considered as negligible. Their importance lies in knowing the direction in which they might affect substitution across commodities. Hence, in the remainder of the chapter we will focus our discussion on the total elasticities.

In this section we have focussed on the price and budget responses of the sample median household. In the next sections we will study how these responses vary across geographic locations and income groups.

6.5.2.2 Variation of the median elasticity estimates across geographic locations

Table 6.13 and Table 6.14 give the median uncompensated and compensated elasticity matrices and budget elasticities for both the rural and urban households. A comparison of the figures from the two tables with those in Table 6.11 and Table 6.12 indicates that the rural median elasticities are all respectively close to their sample median counterparts. In contrast, some of the urban median elasticities seem to be different from their sample median counterparts. This is especially true for the other food column in the total uncompensated matrix where most of the elasticities have changed signs. A notable difference between the rural and urban households - which is not perhaps surprising - is in the compensated own-price elasticity for the coarse grains (-0.25 for rural and -0.46 for urban). Indeed, one can expect that the rural households who grow most of their coarse grains consumption to be less responsive to change in its price than urban households who have to buy the coarse grains. However, a comparison of the corresponding uncompensated elasticities (-0.78 and -0.72 respectively) indicates that the difference in responses results mainly from the income or budget effect, with the rural households having stronger income or budget effects. The stronger income or budget effects for the rural households apply in general for all the other

Table 6.13 Pre-devaluation median uncompensated and budget elasticities for the uncertainty and deterministic AIDS models: Rural versus urban zones

	Rural zones					Urban zones				
	% change in quantities					% change in quantities				
	Coarse grains	Pulses	Fish/ Meat	Other food	Rice	Coarse grains	Pulses	Fish/ Meat	Other food	Rice
	"Expected" AIDS model									
Total										
Coarse grains...	-.777	.155	.092	-.237	.167	-.718	.201	.179	.109	.198
Pulse...........	.106	-.822	.049	-.176	.101	.119	-.748	.092	.029	.108
Fish/Meat.......	-.209	-.214	-1.172	-.538	-.208	-.160	-.181	-1.182	-.517	-.190
Other food......	.720	.706	.560	-.933	.713	.808	.750	.508	-1.081	.703
Rice............	.073	.066	.018	-.229	-.903	.092	.082	.023	-.148	-.912
Precautionary risk										
Coarse grains...	.013	-.003	-.002	-.004	-.002	.015	-.005	-.006	-.009	-.006
Pulse...........	-.002	.019	-.001	-.001	-.001	-.007	.047	-.006	-.005	-.006
Fish/Meat.......	.003	.003	.017	.003	.003	.006	.005	.034	.011	.005
Other food......	-.004	-.004	-.004	.005	-.004	-.007	-.009	-.009	-.001	-.009
Rice............	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
Nonrisk component										
Coarse grains...	-.793	.160	.096	-.230	.171	-.734	.207	.185	.122	.206
Pulse...........	.108	-.845	.050	-.173	.104	.125	-.796	.099	.034	.114
Fish/Meat.......	-.203	-.208	-1.140	-.537	-.206	-.167	-.188	-1.212	-.533	-.193
Other food......	.728	.711	.581	-.952	.722	.812	.766	.519	-1.081	.709
Rice............	.073	.066	.018	-.229	-.904	.092	.082	.023	-.148	-.912
Budget elasticity										
Mean	.811	.800	.956	1.746	.807	.688	.690	.990	1.761	.794
Variance	.049	.073	.133	.318	.048	.055	.084	.073	.130	.035
	Deterministic AIDS model									
Coarse grains...	-.663	.272	.241	-.852	-.017	.264	.495	.339	-.704	-.082
Pulse...........	.139	-.876	.009	-.597	.244	.496	-.736	.018	-.543	.200
Fish/Meat.......	.293	.107	-1.158	-.143	-.091	1.388	.279	-1.216	-.160	-.128
Other food......	-.263	-.465	.094	-.823	.344	-1.547	-1.520	.130	-.743	.317
Rice............	.026	.531	-.226	-.063	-1.114	.032	1.481	-.303	.054	-1.104
Budget elasticity ..	.724	.698	1.021	1.776	.788	-.448	.047	1.029	1.982	.795

Notes: The elasticities were calculated for each household. Each cell contains the weighted median of the respective elasticities across all households. The two models were estimated with the pooled season averages of the daily food consumption data, using the Iterated Moment Estimator of Blundell and Robin (1993). (the food consumption survey collected 24 hours-recall data every 15 days for two years).

Table 6.14 Pre-devaluation median compensated elasticities for the uncertainty and deterministic AIDS models: Rural versus urban zones

	Rural zones					Urban zones				
	% change in quantities					% change in quantities				
	Coarse grains	Pulses	Fish/ Meat	Other food	Rice	Coarse grains	Pulses	Fish/ Meat	Other food	Rice
"Expected" AIDS model										
Total										
Coarse grains...	-.249	.655	.681	.845	.673	-.456	.451	.535	.722	.492
Pulse...........	.411	-.449	.443	.542	.421	.278	-.569	.317	.425	.288
Fish/Meat.......	.016	.010	-.838	-.042	.009	.033	.028	-.901	-.013	.036
Other food......	.876	.860	.780	-.479	.869	.959	.924	.762	-.593	.903
Rice............	.447	.448	.469	.637	-.490	.376	.368	.437	.567	-.565
Precautionary risk										
Coarse grains...	.005	-.006	-.002	-.006	.001	.012	-.006	.000	-.008	-.002
Pulse...........	-.005	.009	-.003	-.003	-.003	-.006	.035	-.012	-.017	-.009
Fish/Meat.......	.002	.001	.012	.003	.001	.006	-.001	.022	.004	.001
Other food......	-.003	-.003	-.005	.003	-.004	-.005	-.009	-.007	.002	-.008
Rice............	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
Nonrisk component										
Coarse grains...	-.252	.661	.687	.839	.662	-.468	.459	.532	.736	.491
Pulse...........	.422	-.476	.446	.563	.421	.286	-.612	.327	.440	.297
Fish/Meat.......	.016	.006	-.824	-.042	.008	.028	.026	-.917	-.014	.033
Other food......	.889	.879	.788	-.474	.882	.959	.943	.768	-.593	.909
Rice............	.447	.447	.469	.637	-.490	.376	.367	.436	.567	-.565
Deterministic AIDS model										
Coarse grains...	-.289	.618	.795	-.084	.333	.259	.536	.436	-.495	.004
Pulse...........	.397	-.596	.315	-.106	.535	.476	-.669	.141	-.363	.300
Fish/Meat.......	.554	.371	-.808	.503	.196	1.279	.405	-.972	.469	.054
Other food......	.012	-.185	.508	-.184	.639	-1.696	-1.473	.465	-.162	.583
Rice............	.389	.894	.276	.885	-.695	-.139	1.506	.083	.929	-.786

Notes: The elasticities were calculated for each household. Each cell contains the weighted median of the respective elasticities across all households. The two models were estimated with the pooled season averages of the daily food consumption data, using the Iterated Moment Estimator of Blundell and Robin (1993). (the food consumption survey collected 24 hours-recall data every 15 days for two years).

food commodities. It is a reflection of their relative poverty and higher share of food in total expenditure, making them lose more in purchasing power whenever there is a *ceteris paribus* expected change in prices.

6.5.2.3 Sample and zone empirical distributions of the households' elasticities

In the previous sections we have focussed on the median elasticities because unlike the mean, the median is robust to outliers. However, the median does not use all the sample information. Hence, it will be interesting to see the entire empirical distribution of the elasticities through histograms and boxplots. However, an histogram and a boxplot for every element of the matrices would be too detailed and too long. Hence, we will focus only on the own price elasticities of the coarse grains and rice, and on their cross price elasticities.

Figure 6.5 shows the histograms and the sample statistics of the rice and coarse grains (total) compensated elasticities for all the households in the sample grouped together respectively. All the values for the compensated own price elasticities are negative as predicted by the theory, with the one for the coarse grains ranging from -0.57 to -0.38 and the one for rice ranging from -0.73 to -0.14. They both have a standard error of about 0.008. Meanwhile, the values for the compensated rice-coarse grains cross price elasticity are all positive confirming that rice and the coarse grains are indeed substitutes, although the degree of substitution is limited. The rice-coarse grains compensated cross-price elasticities range from a minimum of 0.21 to a maximum of 0.75, with standard error 0.007. But, none of the elasticities seems to have an empirical distribution close to the normal distribution. The empirical distribution for the compensated own price elasticity of the coarse grains is skewed to the left (i.e., toward less responsiveness) with an index of skewness of -0.19, while the ones for the compensated own price elasticity of rice and for the rice-coarse grains compensated cross-price elasticity are skewed to the right (i.e., toward more responsiveness

COARSE GRAINS COMPENSATED OWN-PRICE ELASTICITY

```
Frequency   Bin Center                                    Valid cases:       247.0
                                                          Missing cases:      .0
   1.00    -.57500                                        Percent missing:     .0
  11.00    -.52500   *****
  20.00    -.47500   **********                           Mean         -.2841  Std Err      .0083
  27.00    -.42500   *************                        Median       -.2692  Variance     .0171
  24.00    -.37500   ************                         5% Trim      -.2827  Std Dev      .1308
  26.00    -.32500   *************                        Min          -.5685  Skewness    -.1912
  27.00    -.27500   *************                        Max          -.0381  S E Skew     .1549
  26.00    -.22500   *************                        Range         .5304  Kurtosis   -1.0877
  44.00    -.17500   **********************               IQR           .2254  S E Kurt     .3086
  26.00    -.12500   *************
  13.00    -.07500   ******                                        M-Estimators
   2.00    -.02500   *                                    Huber  (1.339)                   -.2790
    .00     .02500                                        Tukey  (4.685)                   -.2797
                                                          Hampel (1.700,3.400,8.500)       -.2815
Bin width :     .05000                                    Andrew (1.340 * pi)              -.2797
Each star:        2 case(s)
```

RICE COMPENSATED OWN-PRICE ELASTICITY

```
Frequency   Bin Center                                    Valid cases:       247.0
                                                          Missing cases:     0
   3.00    -.72500   *                                    Percent missing:   0.0
  11.00    -.67500   *****
  38.00    -.62500   *******************                  Mean         -.4809  Std Err      .0084
  39.00    -.57500   *******************                  Median       -.5065  Variance     .0176
  40.00    -.52500   ********************                 5% Trim      -.4857  Std Dev      .1327
  30.00    -.47500   ***************                      Min          -.7251  Skewness     .5688
  20.00    -.42500   **********                           Max          -.1355  S E Skew     .1549
  16.00    -.37500   ********                             Range         .5896  Kurtosis    -.5475
  18.00    -.32500   *********                            IQR           .1983  S E Kurt     .3086
  16.00    -.27500   ********
  11.00    -.22500   *****                                           M-Estimators
   3.00    -.17500   *                                    Huber  (1.339)                   -.4987
   2.00    -.12500   *                                    Tukey  (4.685)                   -.5005
    .00    -.07500                                        Hampel (1.700,3.400,8.500)       -.4930
                                                          Andrew (1.340 * pi)              -.5006
Bin width :     .05000
Each star:        2 case(s)
```

RICE-COARSE GRAINS COMPENSATED CROSS-PRICE ELASTICITY

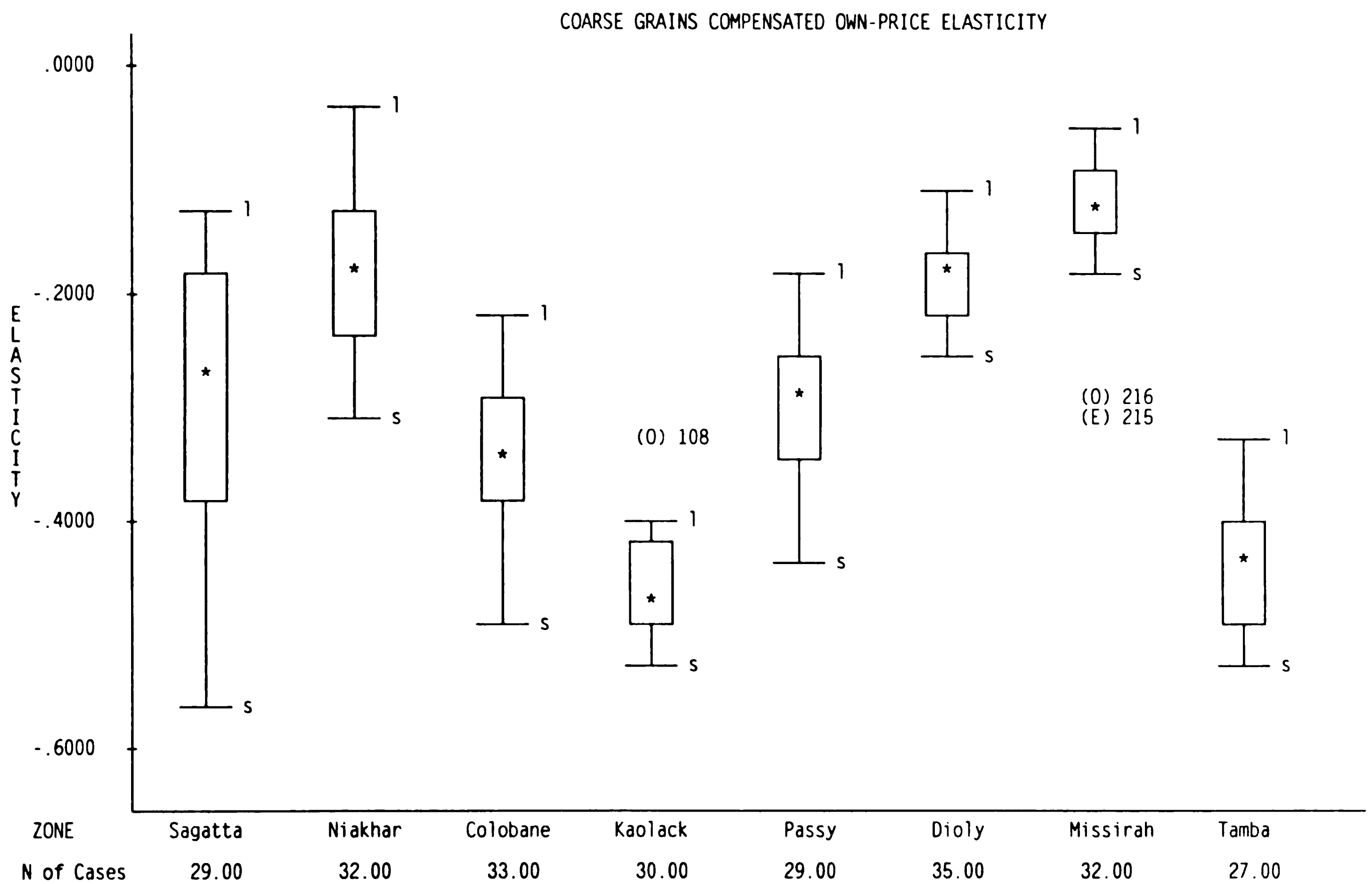
```
Frequency   Bin Center                                    Valid cases:       247.0
                                                          Missing cases:      .0
   4.00     .22500   **                                   Percent missing:    .0
   5.00     .27500   **                                   Mean          .4425  Std Err      .0071
  45.00     .32500   **********************               Median        .4166  Variance     .0125
  53.00     .37500   **************************           5% Trim       .4376  Std Dev      .1117
  45.00     .42500   **********************               Min           .2122  Skewness     .6629
  25.00     .47500   ************                         Max           .7461  S E Skew     .1549
  22.00     .52500   ***********                          Range         .5339  Kurtosis    -.2455
  24.00     .57500   ************                         IQR           .1682  S E Kurt     .3086
   9.00     .62500   ****
   9.00     .67500   ****                                            M-Estimators
   6.00     .72500   ***
                                                          Huber  (1.339)                    .4247
Bin width :    .05000                                     Tukey  (4.685)                    .4186
Each star:        2 case(s)                               Hampel (1.700,3.400,8.500)        .4287
                                                          Andrew (1.340 * pi)               .4184
```

Figure 6.5 Sample empirical distributions of the compensated coarse grains and rice price elasticities: histograms and sample statistics.

and less responsiveness respectively) with an index of skewness of 0.57 and 0.66 respectively. Figure 6.5 also shows that the means and other robust estimates of location for the elasticities are very close to the medians which we have adopted.

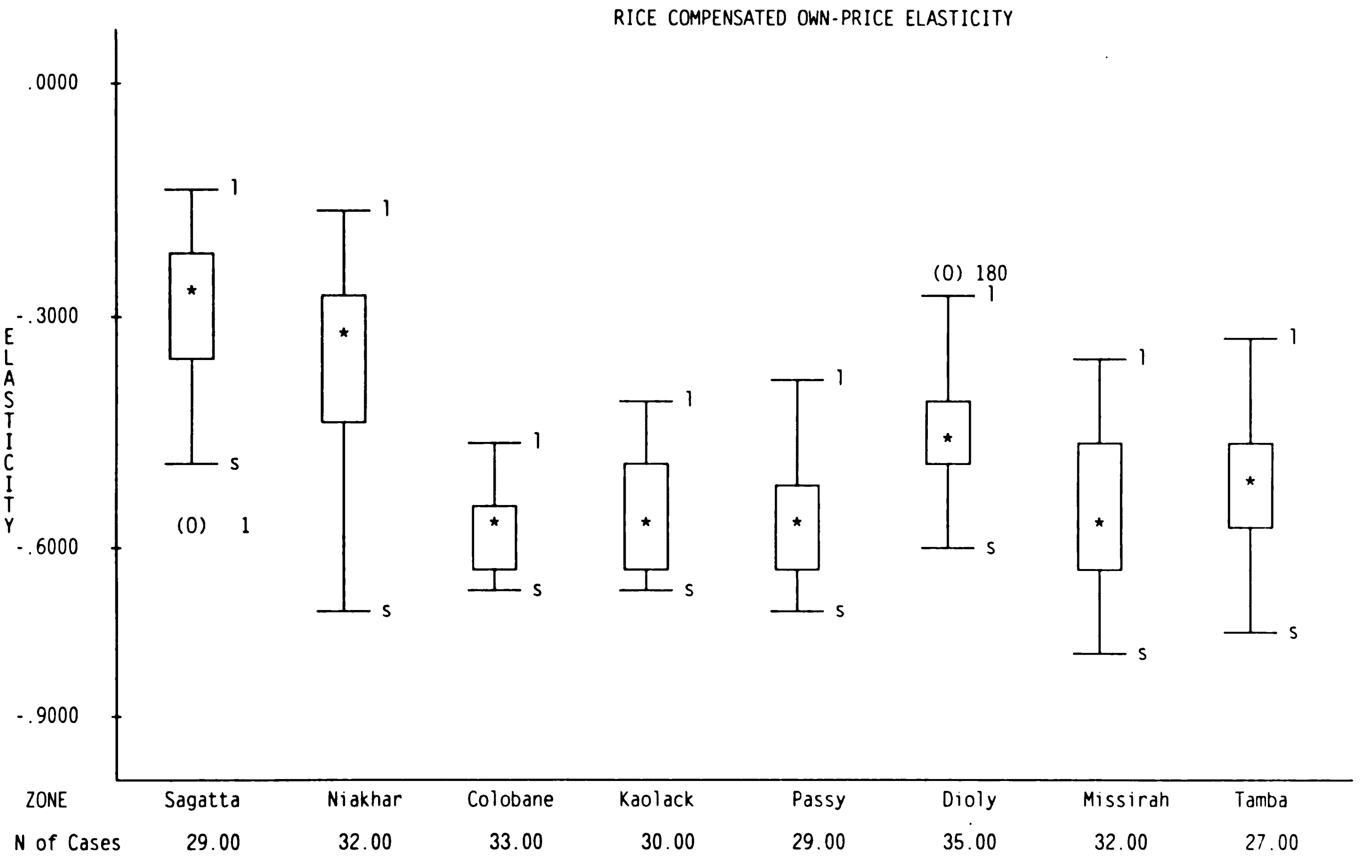
Figure 6.6, Figure 6.7, and Figure 6.8 use boxplot diagrams to compare the empirical distributions of the compensated elasticities across zones. From Figure 6.6 we can see that households in the urban zones have the highest own-price compensated response for the coarse grains and households in Niakhar, Dioly and Missirah have the lowest. In fact, the respective empirical distributions of the coarse grains compensated own price elasticity for these two extreme groups do not intersect. By and large, households in Sagatta, Colobane, and Passy have similar own-price compensated response for the coarse grains, with Sagatta having the lowest median response among the three zones, but also with the greatest variability among all the zones. In fact, the extreme responses of some households in Sagatta are even higher than any other response in the urban zones. This is perhaps a reflection of the fact that among the rural zones Sagatta has on average the lowest share of coarse grains in the food budget (see Table 6.4). The variability in the own-price compensated response for the coarse grains is the lowest in Missirah, Dioly, and Kaolack, with Colobane, Passy, Tamba, and Niakhar having intermediate similar variability.

In contrast, Figure 6.7 shows that the empirical distribution of the rice own-price compensated responses of urban households is not by and large much different from the ones of rural households, except the ones for households in Sagatta and Niakhar who in general have lower rice own-price compensated responses. However, the variability of the rice own-price compensated responses is noticeably lower in Colobane, and noticeably higher in Niakhar and Missirah. Again, the higher variability in the rice own-price compensated



Symbol Key: * - Median. For each zone, 50% of cases have values within the box, the length of the box is the interquartile range, and the lower boundary (resp upper boundary) of the box is the 25th (resp 75th) percentile.
s - smallest observed value that is not outlier.
l - largest observed value that is not outlier.
(O) - Outlier: value less (resp more)than 1.5 box-lengths from 25th (resp 75th) percentile.
(E) - Extreme. value less (resp more)than 3 box-lengths from 25th (resp 75th) percentile

Figure 6.6 Empirical distributions of the compensated own-price elasticity for coarse grains: Boxplot diagrams by zone.



Symbol Key: * - Median. For each zone, 50% of cases have values within the box, the length of the box is the interquartile range, and the lower boundary (resp upper boundary) of the box is the 25th (resp 75th) percentile.

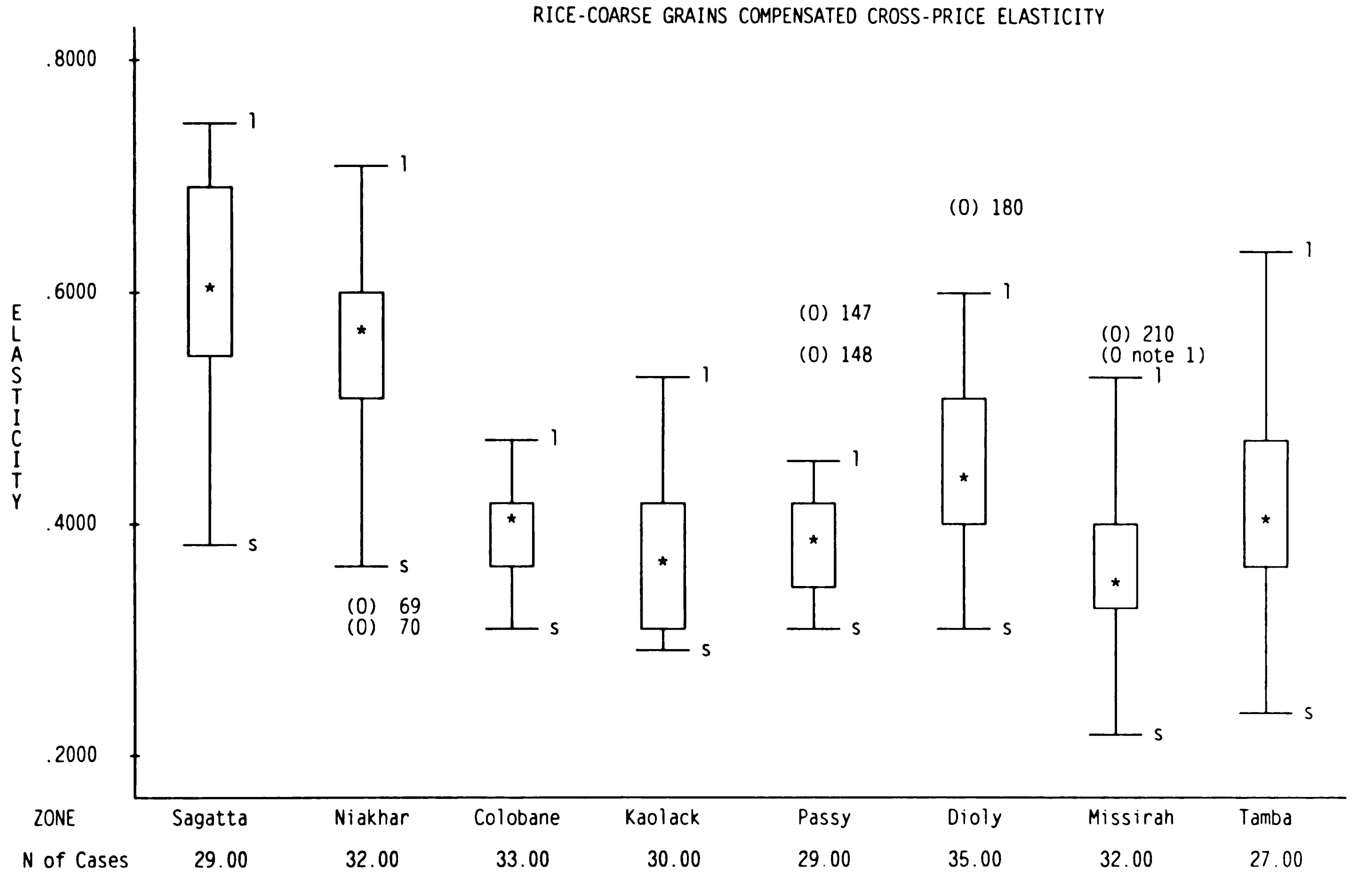
s - smallest observed value that is not outlier.

l - largest observed value that is not outlier.

(O) - Outlier: value less (resp more)than 1.5 box-lengths from 25th (resp 75th) percentile.

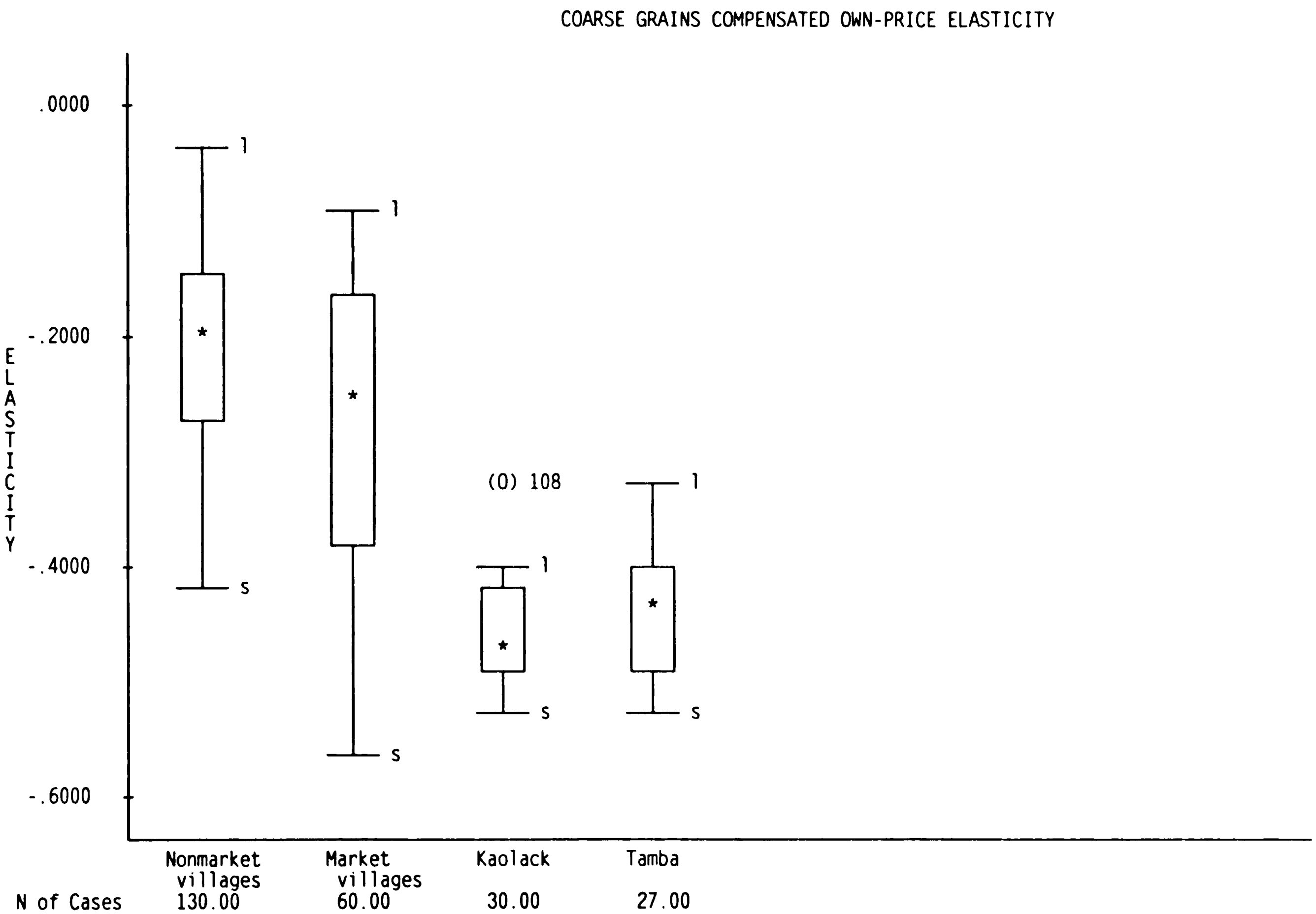
(E) - Extreme. value less (resp more)than 3 box-lengths from 25th (resp 75th) percentile

Figure 6.7 Empirical distributions of the compensated own-price elasticity for rice: Boxplot diagrams by zone.



Symbol Key: * - Median. For each zone, 50% of cases have values within the box, the length of the box is the interquartile range, and the lower boundary (resp upper boundary) of the box is the 25th (resp 75th) percentile.
s - smallest observed value that is not outlier.
l - largest observed value that is not outlier.
(O) - Outlier: value less (resp more)than 1.5 box-lengths from 25th (resp 75th) percentile.
(E) - Extreme. value less (resp more)than 3 box-lengths from 25th (resp 75th) percentile

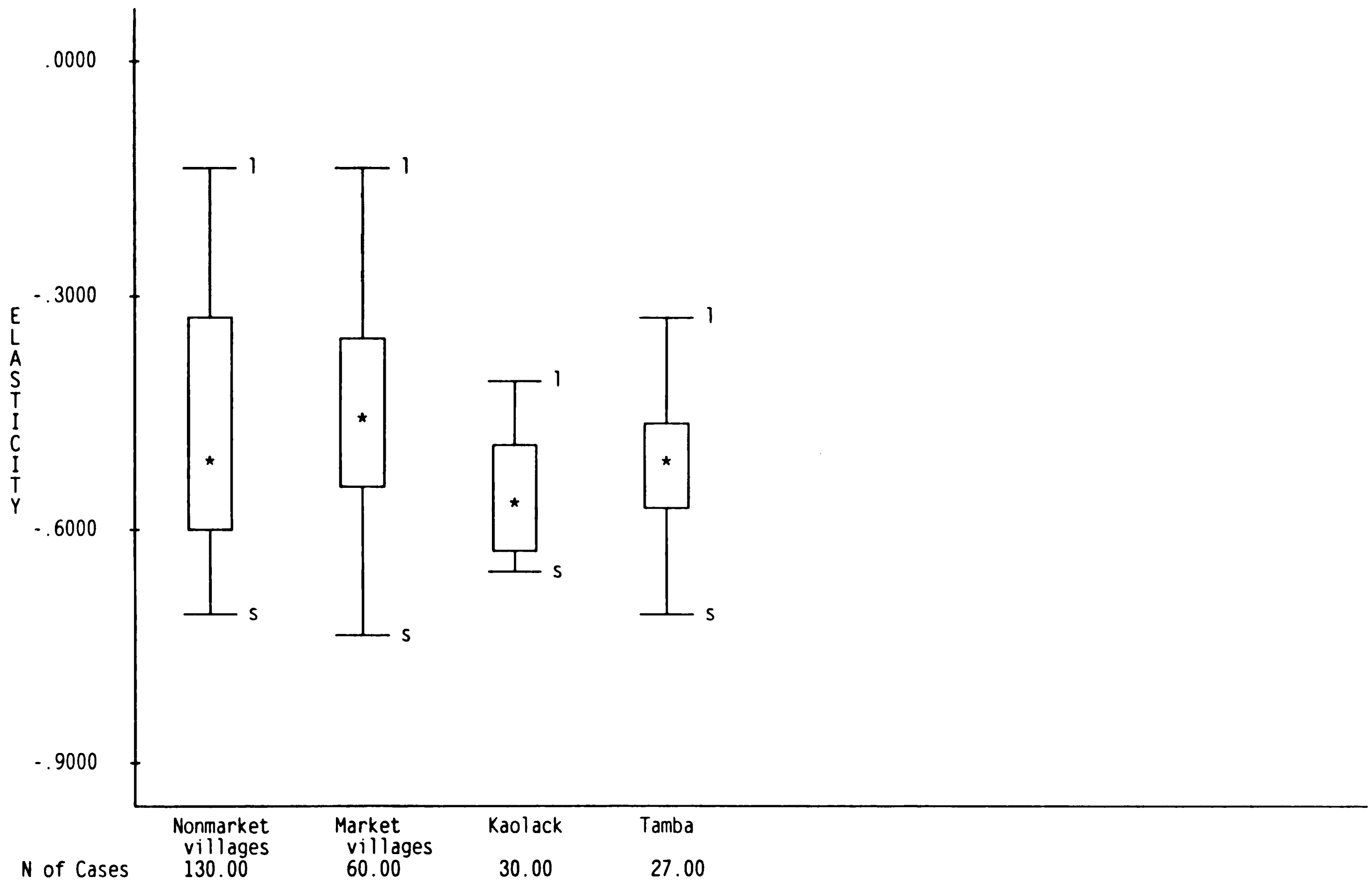
Figure 6.8 Empirical distributions of the compensated cross-price elasticity for rice and the coarse grains: Boxplot diagrams by zone.



Symbol Key: * - Median. For each village type 50% of cases have values within the box, the length of the box is the interquartile range, and the lower boundary (resp upper boundary) of the box is the 25th (resp 75th) percentile.
s - smallest observed value that is not outlier.
l - largest observed value that is not outlier.

Figure 6.9 Empirical distributions of the compensated own-price elasticity for coarse grains: Boxplot diagrams by market/nonmarket villages

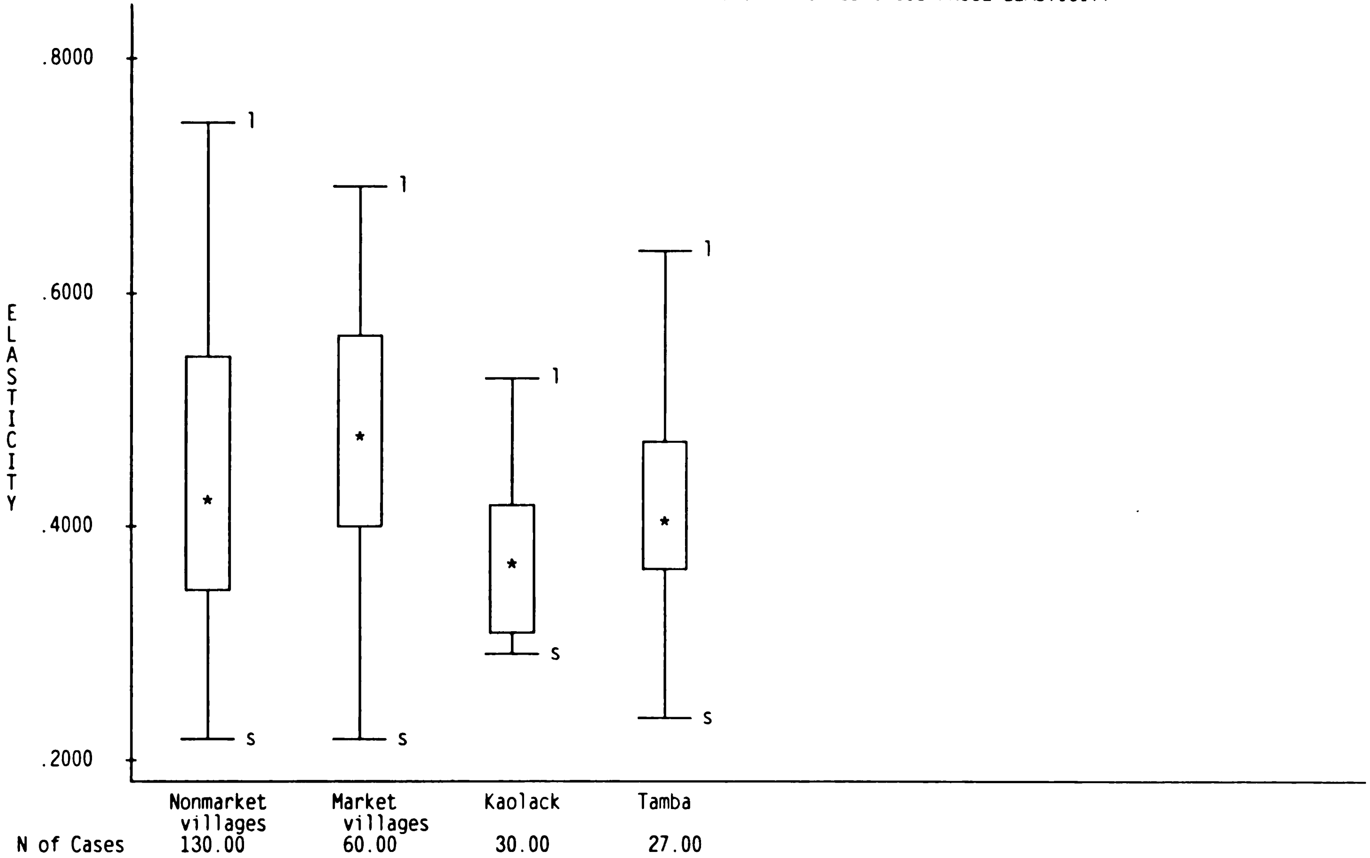
RICE COMPENSATED OWN-PRICE ELASTICITY



Symbol Key: * - Median. For each village type 50% of cases have values within the box, the length of the box is the interquartile range, and the lower boundary (resp upper boundary) of the box is the 25th (resp 75th) percentile.
s - smallest observed value that is not outlier.
l - largest observed value that is not outlier.

Figure 6.10 Empirical distributions of the compensated own-price elasticity for rice: Boxplot diagrams by Market/nonmarket villages

RICE-COARSE GRAINS COMPENSATED CROSS-PRICE ELASTICITY



Symbol Key: * - Median. For each village type 50% of cases have values within the box, the length of the box is the interquartile range, and the lower boundary (resp upper boundary) of the box is the 25th (resp 75th) percentile.
s - smallest observed value that is not outlier.
l - largest observed value that is not outlier.
(O) - Outlier: value less (resp more)than 1.5 box-lengths from 25th (resp 75th) percentile.
(E) - Extreme. value less (resp more)than 3 box-lengths from 25th (resp 75th) percentile

Figure 6.11 Empirical distributions of the compensated cross-price elasticity for rice and the coarse grains: Boxplot diagrams by market/nonmarket villages responses in Niakhar and Missirah reflects the fact that they have the lowest share of rice in expenditures on food among all the zones (see Table 6.4). Another thing to note here is the fact that the empirical distribution of the rice own-price compensated response in Colobane is noticeably skewed to the left (i.e., toward less responsiveness).

The patterns of rice-coarse grains cross-price compensated responses shown in Figure 6.8 are very similar to the ones for the rice own-price compensated responses, with the notable difference that instead of having the lowest responses, Sagatta and Niakhar have here the highest compensated responses. Indeed, here again by and large in terms of substitution between rice and the coarse grains urban households behave much like rural households except those in Sagatta and Niakhar. Another difference compared to the rice own-price compensated response case is the noticeable increase in the variability of compensated responses in Sagatta, Dioly, and Tamba, and its equally noticeable decrease in Missirah and Niakhar.

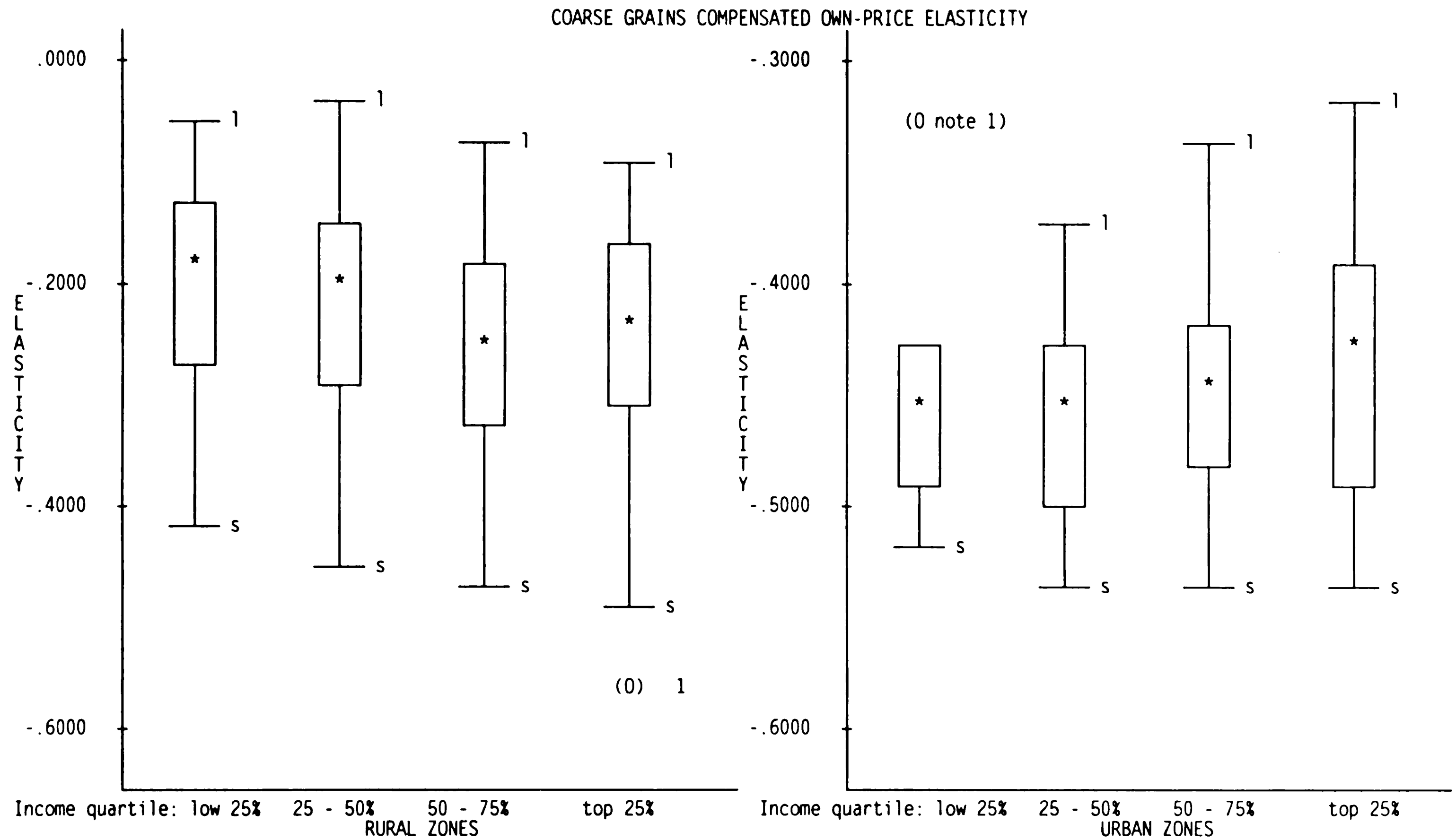
What is intriguing in the above figures is the similarity of responses of households in Sagatta and those in Niakhar with respect to both the rice own-price and the rice-coarse grains cross-price compensated elasticities. This is puzzling because in terms of the respective shares of rice and coarse grains in total food expenditures, Sagatta and Niakhar are at the opposite extremes (see Table 6.4). Indeed, Sagatta has the highest average rice share among all zones including the urban zones (33%) and the lowest average coarse grains share among the rural zones (19%), while Niakhar has the highest coarse grains share among all zones (51%) and the second lowest rice share (10%) after Dioly (9%). Frankly, given these contrasting figures we cannot explain why households in Sagatta and Niakhar have the lowest rice own-price compensated responses, and at the same time have the highest rice-coarse grains cross-price compensated responses.

Finally, to conclude this section, Figure 6.9, Figure 6.10, and Figure 6.11 compare the empirical distribution of the coarse grains and rice compensated elasticities across market/nonmarket villages. For the coarse grains own-price compensated response, nonmarket villages tend to have both lower median response and lower variability in the responses compared to market villages. In contrast, for both the rice own-price and rice-coarse grains cross-price compensated responses, market villages have lower median responses and lower variability. Figure 6.9, Figure 6.10, and Figure 6.11 also confirm the patterns of price responses that emerged from the zone-by-zone analysis. That is, the empirical distribution of the coarse grains own-price compensated elasticities of urban households is quite far apart from the ones of both market and nonmarket villages, while the respective empirical distributions for the rice own-price and rice-coarse grains cross-price compensated elasticities are by and large closer. However, in all cases the variability in responses is always lower for urban households.

6.5.2.4 Variation of the elasticity estimates across income groups

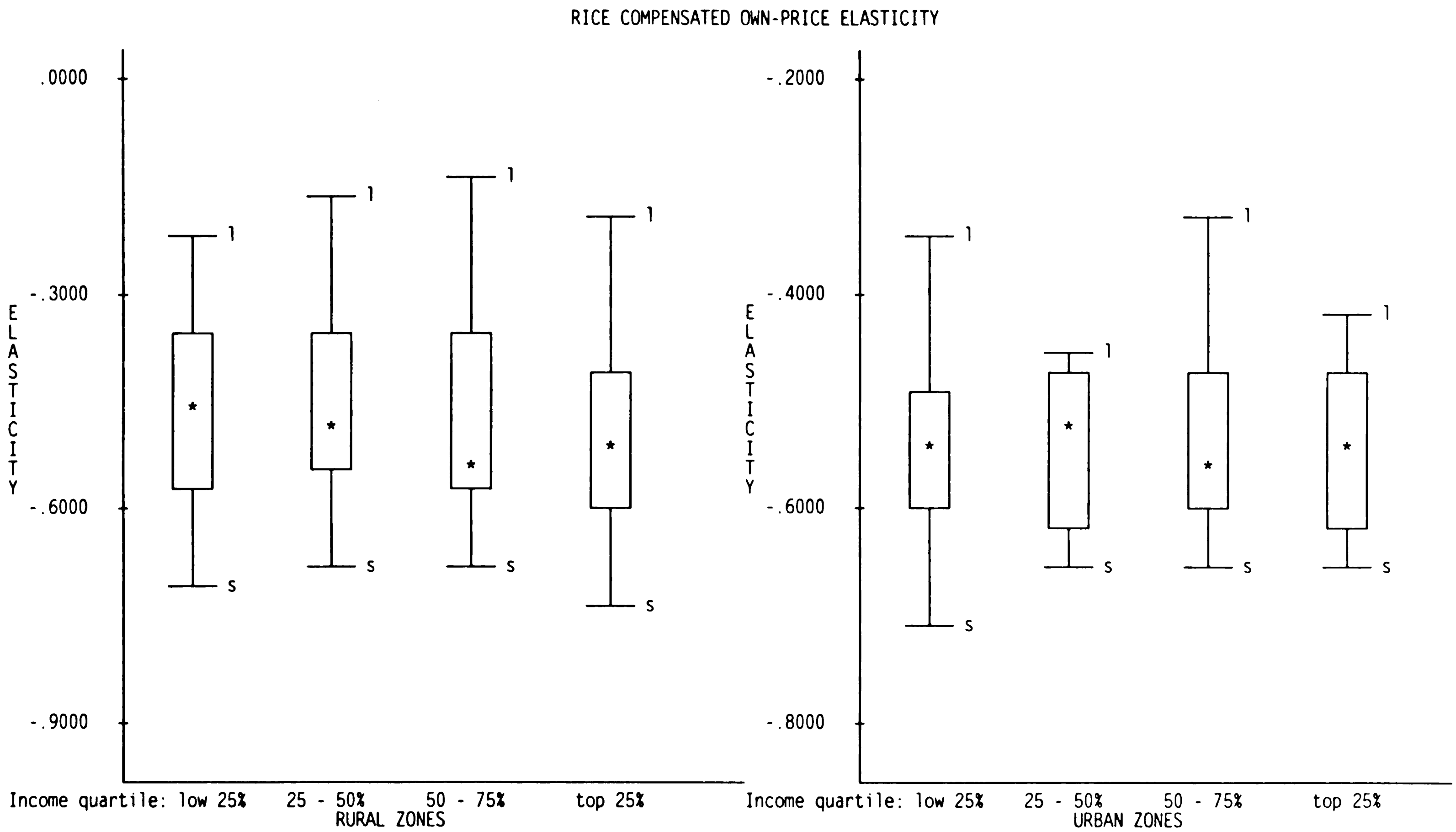
As in the previous section, in this section we focus only on the (total) compensated elasticities of the coarse grains and rice because an analysis for each element of the elasticity matrices is too long to be carried out here. Hence, we will focus on the two main commodities: imported rice and the coarse grains.

First, for the coarse grains own-price compensated elasticities Figure 6.12 shows the empirical distributions within each income quartile for the rural and urban households



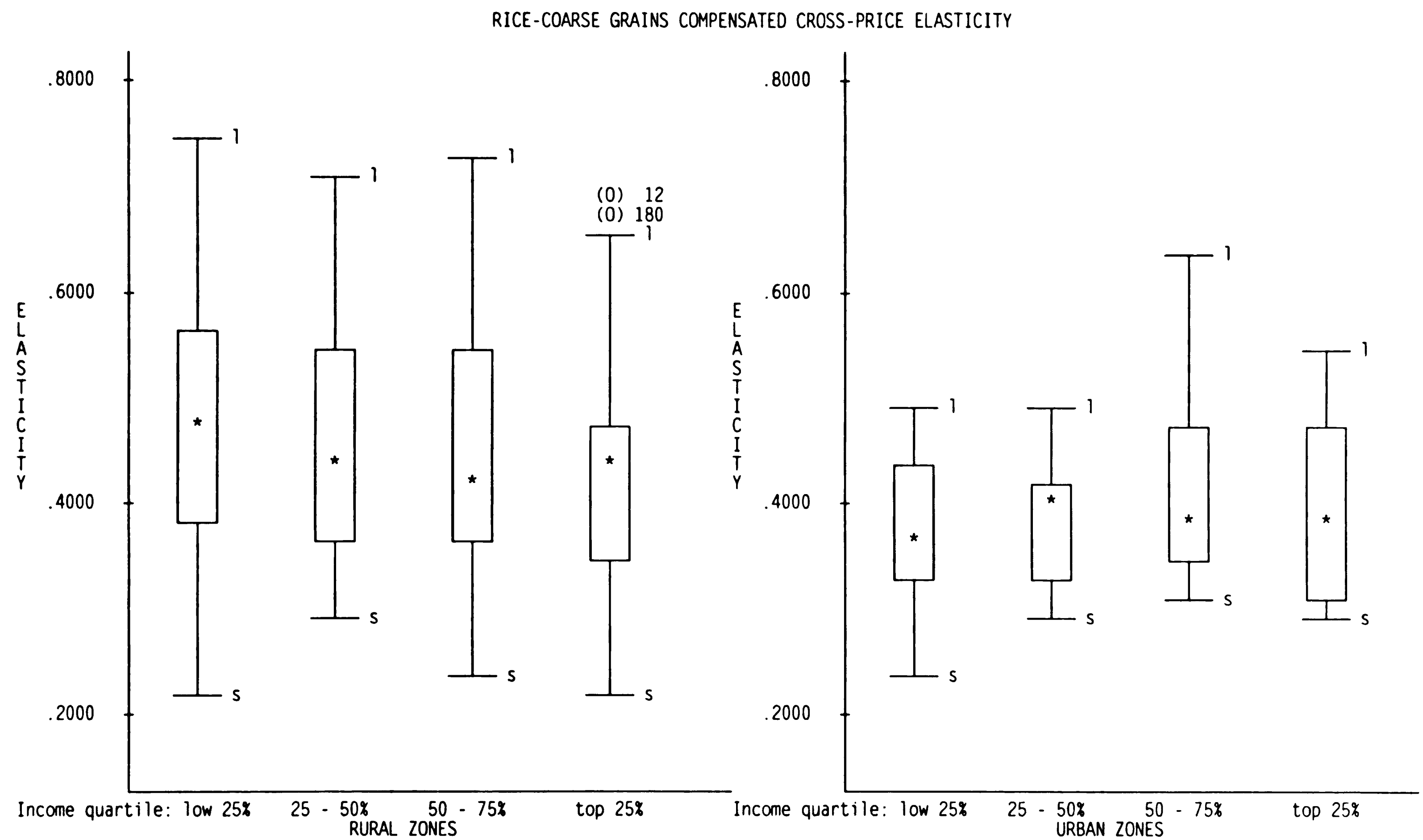
Symbol Key: * - Median. For each income quartile 50% of cases have values within the box, the length of the box is the interquartile range, and the lower boundary (resp upper boundary) of the box is the 25th (resp 75th) percentile.
s - smallest observed value that is not outlier.
l - largest observed value that is not outlier.
(O) - Outlier: value less (resp more)than 1.5 box-lengths from 25th (resp 75th) percentile.
(E) - Extreme. value less (resp more)than 3 box-lengths from 25th (resp 75th) percentile

Figure 6.12 Empirical distributions of the compensated own-price elasticity for coarse grains: Boxplot diagrams by income groups.



Symbol Key: * - Median. For each income quartile 50% of cases have values within the box, the length of the box is the interquartile range, and the lower boundary (resp upper boundary) of the box is the 25th (resp 75th) percentile.
s - smallest observed value that is not outlier.
l - largest observed value that is not outlier.
(O) - Outlier: value less (resp more)than 1.5 box-lengths from 25th (resp 75th) percentile.
(E) - Extreme. value less (resp more)than 3 box-lengths from 25th (resp 75th) percentile

Figure 6.13 Empirical distributions of the compensated own-price elasticity for rice: Boxplot diagrams by income groups.



Symbol Key: * - Median. For each income quartile 50% of cases have values within the box, the length of the box is the interquartile range, and the lower boundary (resp upper boundary) of the box is the 25th (resp 75th) percentile.
s - smallest observed value that is not outlier.
l - largest observed value that is not outlier.
(O) - Outlier: value less (resp more)than 1.5 box-lengths from 25th (resp 75th) percentile.
(E) - Extreme. value less (resp more)than 3 box-lengths from 25th (resp 75th) percentile

Figure 6.14 Empirical distributions of the compensated cross-price elasticity for rice and the coarse grains: Boxplot diagrams by income groups.

Table 6.15 "Expected" AIDS pre-devaluation total compensated and budget elasticities for coarse grains and rice: Variation across income groups.

	Rural zones												Urban zones			
	Sagatta		Niakhar		Colobane		Passy		Dioly		Missirah		Kaolack		Tamba	
	Coarse grains	Rice	Coarse grains	Rice	Coarse grains	Rice	Coarse grains	Rice	Coarse grains	Rice	Coarse grains	Rice	Coarse grains	Rice	Coarse grains	Rice
	% change in quantity															
Income quartile																
low 25%																
Coarse grains.	-.222	.594	-.188	.705	-.269	.654	-.345	.693	-.242	.598	-.110	.722	-.456	.518	-.412	.466
Rice..........	.581	-.270	.573	-.333	.437	-.534	.435	-.519	.542	-.356	.340	-.571	.387	-.511	.342	-.556
Mean budget elasticity.	.772	.862	.799	.853	.822	.811	.811	.813	.732	.863	.833	.739	.688	.803	.634	.796
25 - 50%																
Coarse grains.	-.246	.598	-.164	.697	-.373	.600	-.281	.656	-.195	.693	-.115	.786	-.482	.487	-.445	.501
Rice..........	.578	-.293	.562	-.333	.399	-.582	.400	-.537	.418	-.451	.363	-.560	.321	-.622	.412	-.513
Mean budget elasticity.	.769	.842	.813	.819	.791	.804	.819	.801	.830	.765	.818	.741	.690	.781	.672	.800
50 - 75%																
Coarse grains.	-.257	.632	-.205	.676	-.326	.654	-.312	.630	-.242	.649	-.126	.735	-.470	.493	-.418	.535
Rice..........	.674	-.235	.527	-.404	.401	-.568	.363	-.568	.395	-.490	.412	-.422	.378	-.568	.495	-.474
Mean budget elasticity.	.767	.846	.814	.826	.815	.789	.798	.773	.800	.757	.804	.785	.707	.794	.723	.835
top 25%																
Coarse grains.	-.377	.510	-.179	.749	-.325	.607	-.276	.657	-.176	.723	-.091	.813	-.469	.469	-.411	.498
Rice..........	.576	-.324	.649	-.287	.374	-.567	.350	-.616	.448	-.464	.292	-.655	.345	-.588	.411	-.510
Mean budget elasticity.	.688	.839	.795	.878	.795	.781	.808	.770	.826	.800	.849	.670	.626	.780	.698	.806

Notes: The elasticities were calculated for each household. Each cell contains the weighted median of the respective elasticities across all households. The two models were estimated with the pooled season averages of the daily food consumption data, using the Iterated Moment Estimator of Blundell and Robin (1993). (the food consumption survey collected 24 hours-recall data every 15 days for two years.

respectively.[183] We can see from Figure 6.12 that for the rural households the empirical distribution of the coarse grains own-price compensated elasticity is very similar across income quartiles. There is practically the same variability in price responses across income quartiles, with the two lower income quartiles having slightly lower median responses than the two upper income quartiles. In contrast, for the urban households the empirical distribution of the coarse grains own-price compensated elasticity varies across income quartiles, with households in the upper income quartile having the lowest median response and the highest variability in responses.

Second, for the rice own-price compensated elasticities Figure 6.13 indicates that for the rural households the patterns of the empirical distribution within income quartiles are similar to the ones for the coarse grains own-price compensated elasticity. But, for urban households the pattern across income quartiles shown by the empirical distributions is noticeably different compared to the one for the coarse grains own-price compensated elasticity, with the most notable difference being for the lowest income quartile.

Third, for the rice-coarse grains cross-price compensated elasticities, Figure 6.14 shows that for the rural households the pattern of the within income quartile empirical distributions is like the two previous cases. The only noticeable difference is the reversal in the order of price responsiveness of the income quartiles. Here, coarse grains and rice are substitutes for all rural households, but those in the lowest income quartile have a slightly higher cross-price compensated responses. For urban households, the pattern of the within income quartile empirical distributions is again different from any of the two previous cases. Moreover, the empirical distribution of the rice-coarse grains cross-price compensated

[183] We recall that the urban and rural households were classified into income quartiles separately.

elasticity is noticeably skewed to the left (i.e., toward more responsiveness) for the lower-medium income quartile.

In conclusion, when it comes to difference in price responsiveness among income quartiles, rural households show more similarities than urban households. Table 6.15 gives the median figures for each income quartile in each zone.

6.6 Conclusion

We conclude this chapter by a brief summary of the empirical findings regarding the consumption behavior and risk attitudes of Senegalese households.

6.6.1 Impacts of household characteristics on the food commodity shares

In general, household ethnicity and demographic composition have negligible impacts on the relative daily shares of the food commodities. In contrast, the geographic location variables were found to have a significant impact on the food commodity shares. In particular, when the other variables are controlled for, households in the urban areas and those in the Northern and Western rural zones of Senegal tend to have significantly lower shares of coarse grains and higher shares of rice compared to those in the Eastern rural zones.

6.6.2 The total and nonrisk elasticities estimates

We found that all the food commodities (including rice and the coarse grains) are substitute except for fish\meat and the other food group for which the cross-price elasticity is practically zero. However, we found relatively important income effects that significantly limit the ability of the median household to substitute across commodities. Indeed, when it comes to the own-price response of the median household, most of the response is due to a relatively strong income or budget effect, but when it comes to its cross-price response, most of the potential price substitution effect is wiped out by the same relatively strong income or budget effect. For example, a 1% *ceteris paribus* expected increase in the price of rice could have increased the median household's average daily coarse grains consumption by as much as 0.44% if there were no reduction in purchasing power. But, because of the resulting loss in purchasing power the increase in the average daily coarse grains consumption will only be a mere 0.07%! Meanwhile, 44% of the 0.9% resulting decrease in the average daily rice consumption will be due to a loss in purchasing power, not to substitution toward other food commodities.

This empirical finding is pertinent to the Senegalese government's cereal pricing policy which is presently designed to promote the consumption of coarse grains by maintaining high official prices for imported rice. The strong income effect clearly indicates that one of the reasons that the policy has not worked either in curtailing consumer demand for imported rice, or in boosting the demand for the coarse grains, is the relatively big loss in purchasing power resulting from higher prices for imported rice.

6.6.3 The *precautionary* risk elasticities

We found that in general the price-risk effects dampen the median household's overall ability to substitute across commodities. However, overall the magnitudes of the precautionary price-risk elasticities are so small compared to their nonrisk counterparts that they can be considered as negligible. Since so much is said about the effects of uncertainty on the food security of the poor, finding empirically that the effects of price uncertainty are negligible (at least for Senegal) and the direction in which they might affect substitution across commodities, has itself some value.

6.6.4 Differences in consumption patterns between rural and urban households

We found that the empirical distribution of the coarse grains own-price compensated elasticities of urban households is quite far apart from the ones of rural households in both the market and nonmarket villages. On the other hand, the respective empirical distributions for the rice own-price and rice-coarse grains cross-price compensated elasticities were by and large closer. However, in all cases the variability in responses is always lower for urban households.

6.6.5 Differences in consumption patterns across income groups

Finally, when it comes to difference in price responsiveness among income quartiles, rural households show more similarities than urban households. However, although coarse grains and rice are substitutes for all rural households, those in the lowest income quartile have a slightly higher cross-price compensated responses. For urban households, the empirical distribution of the rice-coarse grains cross-price compensated elasticity is noticeably skewed to the left (i.e., toward more responsiveness) for the lower-medium income quartile.

CHAPTER 7

THE CONSUMPTION AND WELFARE IMPACTS OF THE CFA DEVALUATION ON SENEGALESE HOUSEHOLDS

7.1 Introduction

In this final chapter we use the estimated models to evaluate the impacts of the price changes following the January 1994 devaluation of the CFA currency on households' consumption behavior and welfare. To evaluate the impact on the households' consumption behavior, we use the post-devaluation prices to calculate the elasticity estimates corresponding to these new prices. However, since elasticities are valid only for marginal changes in prices, neither the pre-devaluation nor the post-devaluation elasticities can be used to evaluate the impact of the non-marginal change in prices which followed the CFA devaluation. Hence, to evaluate the consumption impact of the one-time non-marginal change in prices resulting from the CFA devaluation, we use the estimated model parameters to predict the new food commodity shares corresponding to the post-devaluation prices, and use the predicted new shares to calculate the estimated change in the shares and quantity demanded. Nonetheless, the post-devaluation elasticities can be used to evaluate the impacts of the eventual marginal changes in prices following the one-time change brought about by the devaluation.

To evaluate the welfare impact of the devaluation, we used the indicators of welfare change derived in chapters 2 and 5. Unlike the elasticity measures, these measures of welfare changes are valid for nonmarginal as well as marginal changes in prices and income. The main difficulty here (as well as in the computation of the post devaluation elasticities) is in getting estimates of the post-devaluation vector of prices and expenditure level (**q** and m' respectively in the notation of chapters **2** and **5**). Normally, if one does not have the actual new price vector **q** and expenditure level m', one would have to use models of aggregate supply and demand to predict the post-devaluation vector of prices **q** and the change in the income of producers. Alternatively by default, one would make plausible assumptions about income changes and the likely values for the different prices depending on whether the respective commodities are internationally traded or not, and how the relative prices of internationally nontraded commodities may adjust in response to domestic supply and demand adjustments. We initially chose to proceed by the alternative method of making plausible assumptions complemented by a sensitivity analysis on the possible ranges of prices. This choice was mostly dictated by the micro-economic focus of our study and by the lack of reliable aggregate time series data needed for the aggregate supply and demand method. The devaluation of the CFA took place in January 1994 when we just started our econometric analysis, allowing us then to use the actual resulting price changes and estimate directly the welfare changes, instead of simulating them based on hypothetical scenarios of price changes.

The key assumption in the calculations of the post-devaluation elasticities and indicators of welfare changes is that the underlying consumer preference and "beliefs" parameters did not change as a result of the CFA devaluation. This is a very reasonable assumption in the short run, because households need time to adjust their consumption patterns. It is perhaps worth recalling here that both in the deterministic and uncertainty case, demand functions give the schedules of quantity demanded for prices varying from zero to (practically) infinity. Besides prices, these functions depend (in a parametric way) on consumer preference and "expectations" (see the examples in chapter **3** and the share equations for the "expected" AIDS in chapter 5). Hence, once the demand functions are estimated, elasticities and indicators of welfare changes (which are also functions of prices) can be calculated for any price level, assuming that the preference parameters remain unchanged. There is no reason to assume that preference parameters such as taste and risk attitude would necessarily change when prices change. As seen in chapters 2 and 3, even change in prices need not change "expectations".[184]

The chapter is organized as follows. Section 7.2 describes the methods we used to compute the post-devaluation elasticities and welfare indicators. Section 7.3 evaluates the consumption impact of the price changes following the CFA devaluation by comparing the post and pre-devaluation elasticities, and presenting the estimated changes in the quantities demanded and shares for the food commodities. Section 7.4 evaluate the welfare and risk impacts of the price changes following the CFA devaluation. The section gives the median households' estimated changes in total welfare and costs of food, and risk premia and cost of risk for each agro-climatic zone in the ISRA/IFPRI survey. The section also compares the empirical distributions of the estimated welfare changes and risk measures across income groups. Finally, section 7.5 concludes the chapter by summarizing the empirical finding of the consumption and welfare impacts of the change in prices following the CFA devaluation.

[184] To the extent that our theoretical and econometric models capture household consumption behavior reasonably as embodied in the ISRA/IFPRI data set, there still remains the problem of how reliable out-of-sample predictions can be. But this is a problem inherent in any statistical analysis. All we can have are best estimates given the available data and information. In any case, one should keep in mind that from a policy perspective this type of exercise is the primary justification for spending funds to collect household consumption data and estimate demand systems. Testing the implications of economy theory alone can hardly be justified in the policy arena.

7.2 Computation of the post-devaluation elasticities and welfare indicators

For the calculation of the post-devaluation elasticities and welfare indicators in the uncertainty model, the changes in prices from **p** to **q** can be either random or deterministic. In the case where the change in prices is assumed to be a deterministic (homothetic) transformation of random prices then the empirical price series are just scaled up (or down) by some (known) nonrandom scalars (the moments of prices are also just scaled up or down). The scaling factors which can be expressed in terms of percentages may not be the same for all commodities (i.e, in general we have $\mathbf{q}_i = (1 + t_i)\mathbf{p}_i$). This can be the case for price changes resulting from a sales tax, a tariff, or a devaluation of a currency. When the change in prices is random (i.e, caused by some random shocks), the new price series **q** must be given either by random generation or through a time series model that uses the random shock to predict the values for the new price series **q**.

In our computations of the post-devaluation elasticities and welfare indicators we assumed that the change in prices following the 50% devaluation of the CFA franc in early January 1994 is deterministic. More precisely, for the government-controlled prices we calculated the percentage changes in prices based on the official list of new prices published in January 1994 (33% for rice, 23% for vegetable oil, 43% for peanut, and 30% for cotton). For the internationally nontraded food commodities such as the coarse grains (millet, sorghum, and maize) and cowpea, we assumed no price change. We also assumed no price change for fish and meat since they are largely domestically traded. But, for the other food group, we used the 30% average percentage change in prices in the official list (which included other food and nonfood commodities). A review of the monthly price series published by CSA from January to June 1994, indicates that the price change assumptions for the cereals and pulses are reasonable ones. For the other food commodities we have no means for checking the plausibility of our assumptions. But we feel that the only assumption that might be problematic is the one for the other food category.

For the total food budget m' corresponding to the new price vector $\mathbf{q}$, the new predicted values are obtained by estimating for each household the percentage change in the per capita food budget as a result of the increase in total income due to the increases in the producer prices of peanuts and cotton. The percentage change in the per capita food budget is calculated as the percentage change in per capita total income times the elasticity of per capita food expenditure with respect to per capita total expenditure (i.e. per capita total expenditure in food and nonfood commodities). The percentage change in total income is calculated as the percentage change in peanut prices times the share of peanut in total income plus the percentage change in cotton prices times the share of cotton in total income. The elasticity of per capita food expenditure was calculated by estimating a quadratic Engel curve with food share as dependent variable and per capita total expenditure and squared value as regressors. The regressors also included the same relative prices and household socio-demographic variables used in the demand system. We also used the same instruments as in the main model to take care of the endogeneity of total expenditure. The coefficient estimates for this regression are reported in Table 6.8.

As in the computation of the pre-devaluation elasticities, the post-devaluation elasticities and the welfare indicators were calculated for each household for both the deterministic and uncertainty AIDS model (for the deterministic model the calculation was done for each observation first before aggregating at the household level). As for the elasticities, the calculation of the welfare indicators by household not only avoids the aggregation bias that occurs when we calculate using averages, but also it enables us to analyze the distribution of each welfare indicator across households through the use of histograms and cross tabulations.

7.3 Evaluation of the consumption impact of the CFA devaluation

7.3.1 Comparison of the post and pre-devaluation elasticities

Table 7.1 and Table 7.2 give the sample median post-devaluation uncompensated and budget elasticities, and the median post-devaluation compensated elasticities respectively. In general, the post-devaluation elasticities are not much different from the pre-devaluation ones.[185] There is only a slight trend of increase in the uncompensated price and budget elasticities toward more responsiveness. In particular, the median uncompensated own price elasticities for the coarse grains and rice change from -0.776 and -0.904 to respectively -0.791 and -0.911, and the medians of the means of their budget elasticities increase from 0.809 and 0.807 to respectively 0.829 and 0.823. In contrast, for the compensated elasticities there is a slight trend of reduction in the own-price responsiveness, contrasting with a slight trend of increase in the cross-price responsiveness. For example, the median compensated own price elasticities for the coarse grains and rice change from -0.252 and -0.499 to respectively -0.214 and -0.481, while their median compensated cross-price elasticities increase from 0.44 and 0.667 to respectively 0.461 and 0.709. These trends combined with the one of the uncompensated elasticities indicate the increased impact of the already identified relatively strong income or budget effect which dominates the median household's response to relative price changes. In other words, the higher the levels of prices, the more the median household

[185] Note however that the signs of the compensated cross price elasticities for fish\meat and the other food group which were theoretically inconsistent are now theoretically consistent. Hence, our conjecture that the -0.037 figure was probably not statistically significantly different from zero appears largely correct.

Table 7.1 Post-devaluation median uncompensated and budget elasticities for the uncertainty and deterministic AIDS models: All zones

	% change in quantities				
	Coarse grains	Pulses	Fish/ Meat	Other food	Rice
	"Expected" AIDS model				
Total					
Coarse grains...	-.791	.154	.088	-.389	.170
Pulse...........	.099	-.835	.047	-.272	.101
Fish/Meat.......	-.177	-.184	-1.179	-.588	-.184
Other food......	.871	.829	.703	-.981	.838
Rice............	.072	.065	.017	-.332	-.911
Precautionary risk					
Coarse grains...	.012	-.002	-.002	-.005	-.002
Pulse...........	-.002	.020	-.001	-.002	-.001
Fish/Meat.......	.003	.003	.016	.003	.003
Other food......	-.005	-.005	-.005	.003	-.006
Rice............	.000	.000	.000	.000	.000
Nonrisk component					
Coarse grains...	-.808	.160	.091	-.382	.172
Pulse...........	.101	-.857	.049	-.274	.101
Fish/Meat.......	-.181	-.188	-1.151	-.613	-.187
Other food......	.872	.840	.731	-.990	.865
Rice............	.072	.065	.017	-.332	-.911
Budget elasticity					
Mean	.829	.822	.971	2.025	.823
Variance	.026	.054	.109	.268	.032
	Deterministic AIDS model				
Coarse grains...	-.667	.212	.201	-.651	-.008
Pulse...........	.142	-.888	.007	-.481	.146
Fish/Meat.......	.291	.099	-1.134	-.143	-.054
Other food......	-.245	-.348	.079	-.924	.208
Rice............	.064	.438	-.192	-.133	-1.025
Budget elasticity ..	.732	.767	1.018	1.576	.881

Notes: The elasticities were calculated for each household. Each cell contains the weighted median of the respective elasticities across all households. The two models were estimated with the pooled season averages of the daily food consumption data, using the Iterated Moment Estimator of Blundell and Robin (1993). (the food consumption survey collected 24 hours-recall data every 15 days for two years).

Table 7.2 Post-devaluation median compensated elasticities for the uncertainty and deterministic AIDS models: All zones

	% change in quantities				
	Coarse grains	Pulses	Fish/ Meat	Other food	Rice
	"Expected" AIDS model				
Total					
Coarse grains...	-.214	.708	.724	1.032	.709
Pulse...........	.438	-.452	.456	.637	.431
Fish/Meat.......	.060	.063	-.842	.007	.069
Other food......	1.003	.979	.875	-.506	.989
Rice............	.461	.465	.486	.755	-.481
Precautionary risk					
Coarse grains...	.003	-.006	-.002	-.005	.001
Pulse...........	-.005	.009	-.003	-.004	-.003
Fish/Meat.......	.002	.001	.013	.003	.001
Other food......	-.004	-.005	-.006	.001	-.005
Rice............	.000	.000	.000	.000	.000
Nonrisk component					
Coarse grains...	-.223	.701	.726	1.042	.717
Pulse...........	.438	-.471	.454	.650	.432
Fish/Meat.......	.059	.062	-.822	.003	.067
Other food......	1.014	1.003	.893	-.512	.993
Rice............	.461	.465	.486	.755	-.481
Estimated % change in quant.demand .	17.270	-2.700	25.740	-15.41	-4.780
Estimated % change in the shares ...	-1.030	8.960	6.620	-8.470	4.310
	Deterministic AIDS model				
Coarse grains...	-.220	.624	.827	.073	.365
Pulse...........	.443	-.562	.290	.003	.482
Fish/Meat.......	.631	.429	-.745	.528	.279
Other food......	.094	.014	.570	-.193	.629
Rice............	.568	.958	.484	.993	-.432

Notes: The elasticities were calculated for each household. Each cell contains the weighted median of the respective elasticities across all households. The two models were estimated with the pooled season averages of the daily food consumption data, using the Iterated Moment Estimator of Blundell and Robin (1993). (the food consumption survey collected 24 hours-recall data every 15 days for two years).

is sensitive to loss in purchasing power, and the more its price substitution ability is negatively affected by the loss in purchasing power.[186] Hence, the post-devaluation elasticities just confirm the conclusion we have already reached after the analysis of the median pre-devaluation elasticities in the previous section. That is, a cereal policy based on a *ceteris paribus* higher relative prices for rice compared to the ones for the coarse grains is not going to lead to more substitution toward the coarse grains, unless one finds a way to compensate the median household for the resulting loss in purchasing power.

7.3.2 Estimated changes in food commodities quantities demanded and shares

Table 7.2 gives the estimated changes in the quantities demanded and shares of the food commodities after the change in prices following the 50% devaluation of the CFA. For the sample median household, the average daily consumption of the coarse grains and fish\meat for which prices have remained unchanged, are estimated to have increased by 19% and 26% respectively. Meanwhile, the average daily consumptions of pulses, other food, and rice for which prices have increased by 43%, 30%, and 33% respectively, are estimated to have decreased by 3%, 15%, and 5% respectively. The low decrease in rice consumption indicates that most of the 17% increase in the consumption of coarse grains is not due to substitution away from rice, but rather to an increase in purchasing power for rural households as a result of the increases in their peanut income. Indeed, this is confirmed by the figures in Table 7.3 which gives the separate median estimated changes for rural and

[186] Note that despite the 33% increase in the relative price of rice compared to the coarse grains, the median of their uncompensated cross-price elasticities with respect to the price of rice has slightly decreased from 0.074 to 0.072, while the one with respect to the price of the coarse grains has increased only from 0.169 to to 0.170. In contrast, the corresponding compensated elasticities have increased from 0.440 to 0.461 and from 0.667 to 0.709 respectively. This indicates that most of the median household's increased pure ability to substitute rice for coarse grains has been wiped out by the relatively strong loss in purchasing power.

urban households who did not experience an increase in income. In fact, for the median urban household the average daily coarse grains consumption are estimated to decrease by 3% despite the 16% estimated decrease in the average daily rice consumption! Meanwhile, for the median rural household, with a 19% estimated increase in the average daily coarse grains consumption, there is a mere 4% decrease in rice consumption. These figures merely confirm our earlier findings based on the elasticities: increases in the relative price of rice with respect to the coarse grains do not lead to substitution toward the coarse grains unless consumers are compensated for the resulting loss in purchasing power.

Table 7.3 also shows that the median urban household is substituting away from the other food group (a luxury), the pulses, and rice (-32%, -4%, and -16%, respectively) toward fish\meat (+10%) whose price has remained unchanged. On the other hand, the 27% increase in the median rural household's average daily consumption of fish\meat is more of a reflection of the increase in its purchasing power, than a reflection of a substitution away from pulses (-3%), the other food (-15%), and rice (-4%).

Finally, we note from Table 7.2 that despite the 17% estimated increase in the sample median household's average daily consumption of coarse grains, its share in total expenditures on food is estimated to decrease by 1%. In contrast, the share of rice is estimated to increase by 4% despite its 5% decrease in consumption. Even in rural areas where the coarse grains consumption is estimated to increase by 19%, its share in total food expenditures rises only by a negligible 0.03%, compared to rice whose share is estimated to rise by 4% despite the estimated 4% decrease in its consumption. Of course, there is no contradiction in these figures given the fact that the price of rice has risen by 33% and the one for the coarse grains remains unchanged.

Table 7.3 Post-devaluation median compensated elasticities for the uncertainty and deterministic AIDS models: Rural versus urban zones

	Rural zones					Urban zones				
	% change in quantities					% change in quantities				
	Coarse grains	Pulses	Fish/ Meat	Other food	Rice	Coarse grains	Pulses	Fish/ Meat	Other food	Rice
	"Expected" AIDS model									
Total										
Coarse grains...	-.200	.742	.738	1.032	.738	-.473	.471	.542	.832	.496
Pulse...........	.460	-.428	.467	.642	.455	.311	-.557	.353	.526	.328
Fish/Meat.......	.048	.041	-.842	.004	.043	.096	.078	-.842	.041	.093
Other food......	.994	.969	.875	-.505	.989	1.103	1.028	.904	-.712	1.003
Rice............	.468	.465	.488	.761	-.476	.402	.429	.475	.710	-.530
Risk component										
Coarse grains...	.003	-.006	-.002	-.005	.002	.012	-.006	-.001	-.006	-.002
Pulse...........	-.005	.008	-.003	-.004	-.002	-.004	.035	-.013	-.019	-.009
Fish/Meat.......	.002	.001	.013	.003	.001	.005	-.002	.020	.004	.001
Other food......	-.004	-.004	-.006	.001	-.005	-.007	-.009	-.008	.003	-.008
Rice............	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
Nonrisk component										
Coarse grains...	-.209	.733	.737	1.043	.743	-.487	.480	.537	.848	.497
Pulse...........	.474	-.458	.483	.651	.459	.310	-.593	.366	.557	.335
Fish/Meat.......	.046	.039	-.817	.001	.039	.082	.080	-.860	.040	.092
Other food......	1.012	.999	.893	-.510	.993	1.103	1.038	.903	-.704	1.013
Rice............	.468	.465	.488	.761	-.476	.402	.429	.475	.710	-.529
Estimated % change in quant.demand .	18.900	-2.650	26.700	-14.70	-4.080	-2.950	-4.000	10.200	-32.31	-16.13
Estimated % change in the shares ...	.030	8.120	6.600	-7.020	3.930	-2.150	21.040	10.010	-17.31	9.820
	Deterministic AIDS model									
Coarse grains...	-.240	.631	.847	.103	.394	-.020	.433	.414	-.484	.012
Pulse...........	.454	-.537	.348	.015	.534	.364	-.677	.130	-.379	.282
Fish/Meat.......	.614	.429	-.735	.533	.288	1.054	.427	-.903	.504	.138
Other food......	.117	.016	.578	-.193	.643	-1.241	-1.308	.399	-.193	.515
Rice............	.579	.951	.491	.993	-.430	.029	1.414	.241	1.033	-.638

Notes: The elasticities were calculated for each household. Each cell contains the weighted median of the respective elasticities across all households. The two models were estimated with the pooled season averages of the daily food consumption data, using the Iterated Moment Estimator of Blundell and Robin (1993). (the food consumption survey collected 24 hours-recall data every 15 days for two years).

7.4 Evaluation of the welfare and risk impacts of the CFA devaluation

Before presenting the estimates of the welfare and risk impacts of the price changes following the CFA devaluation, we briefly recall, for the convenience of the reader, the measures and interpretations of welfare changes and risk attitudes derived in chapter **2**.

With a combination of price and income changes leading to a welfare change, the equivalent variation (EV) measures the maximum amount in monetary units that the consumer is willing to pay (or accept if he or she will gain from the change) to prevent the change from taking place. On the other hand, the compensated variation (CV) measures the minimum amount in monetary units that the consumer must be compensated (or taken away from if he or she has gained from the change) so that he or she is at least as well off as before the change. Similarly, the Allen living standard index and the Mckenzie-Pearce cost-of-living index measure respectively the after-change consumer's welfare and cost of achieving that welfare relative to the before-change levels. These welfare indicators are preference-based true indicators of welfare changes, when the changes are through the market.

In chapter **2** we have extended these *true* indicators of welfare change to the case of uncertainty and have shown that they preserve their theoretical economic interpretations and properties.[187] Furthermore, as for the elasticities, we have decomposed the equivalent variation measuring the total welfare change resulting from a "change" in random prices into two components. One being the risk premium measuring in monetary units the sensitivity of welfare to risk aversion (i.e., "dislike" of uncertainty), and the other being the change in welfare that would obtain if the consumer were welfare-indifferent to uncertainty. The risk

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[187] Again, as for the elasticities, the meaning of the word "change" in the uncertain prices is governed by the metrics of the price space. The distance between two random prices is defined to be, roughly speaking, the average across all states of nature of the state-by-state Euclidean distances of their respective possible values. Note that since total expenditure and all the welfare indicators are real nonrandom scalars, the meaning of the word "change" for these figures is the same as usual.

premium, which was shown to be always positive regardless of the direction of the price change, gives the maximum amount in monetary units the consumer is willing to pay in order not to face uncertainty. The living standard and cost-of-living indices are also decomposed similarly. The risk component of the living standard index measuring in relative terms how much in the change in the living standard is due to uncertainty is the *risk premium* index, while the risk component of the cost-of-living index measuring in relative terms how much in the change in the cost of living is due to uncertainty is the *cost-of-risk index*.[188]

Because we have estimated a demand system with only the food commodities included, all the indicators of welfare change numbers presented in the following tables refer to the satisfaction derived from food consumption and to the cost of food only. With the maintained assumption of weak separability between the food group and the nonfood commodities, the indices are *subindices* in standard terminology (Deaton, 1980a). That is, they do not incorporate the loss of welfare resulting from the increase in the prices of nonfood commodities.[189] Hence, to be more precise in the labelling in the tables, we should have had "index of satisfaction from food consumption" instead of "living standard index", and "cost-of-food index" instead of "cost-of-living index", etc... For the sake of brevity, we have maintained the standard terminology, but the reader should always keep this in mind when interpreting the numbers. However, given the high food shares emerging from the

[188] Since the *risk premium* and the *risk premium index* entail a loss in welfare but are reported as positive values (so as to conform to the conventional interpretation), they should always be subtracted from the respective nonrisk components in order to get the EV which measures the (total) welfare change in monetary units, and the (total) living standard index respectively. In contrast, since it is an index of cost, the *cost-of-risk index* must always be added to the nonrisk component in order to get the (total) cost-of-living index.

[189] Also, since the estimated model is a static model with uncertainty in prices only, the risk measures refer to price uncertainty only. They do not incorporate the loss of welfare due to income uncertainty.

ISRA/IFPRI survey (see Table 6.3 and Table 6.5), the indicators of welfare change in the following tables should be very close to the total figures which would have been obtained if our demand system included the nonfood commodities.

Because the food consumption data was aggregated by taking the average daily food commodity expenditures and the daily total food expenditure per capita in CFA francs, the unit for the equivalent variation, the compensated variation, and the risk premium is CFA per capita per day, while the percentage changes are average daily percentages per capita.

Finally, we emphasize that the indicators of welfare change in the following tables are intended to measure only the short term welfare impact of the change in prices following the January 1994 50% CFA devaluation. Indeed, in the long run, preferences, production patterns, relative prices, and income may change in such a way that households that are shown as losers in the tables may end up gaining. The appropriate time frame for which the welfare impacts evaluation can be considered as reasonably valid is the period from just after the devaluation to June 1994. Beyond the starting date of the 1994 harvest, the validity of the welfare change estimates is questionable.[190]

7.4.1 Estimates of the median welfare and risk impacts

Table 7.4 gives the various indicators of welfare change due to the price changes following the 50% CFA devaluation for the median household in each zone, and from both the "expected" and the deterministic AIDS models. What we can notice first, are the big

[190] Another issue here is whether or not the actual changes in the various prices are caused by the devaluation only. We have no way of distinguishing the price changes due to the devaluation from the ones due to other possible factors (rainfall for example). Hence, strictly speaking it is not the welfare impacts of the devaluation we are estimating, but rather the welfare impacts of the change in prices following the devaluation.

Table 7.4 Indicators of welfare changes after the 50% CFA devaluation for the uncertainty and deterministic AIDS models: All zones

	Rural zones							Urban zones		
	Sagatta	Niakhar	Colobane	Passy	Dioly	Missirah	average	Kaolack	Tamba	average
	"Expected" AIDS model									
Equivalent variation (CFA/day/capita)										
Total.............	-.925	8.207	10.447	4.353	17.488	9.668	8.603	-20.00	-15.82	-19.63
Risk premium......	.239	1.430	2.933	1.336	1.508	2.612	1.579	1.561	.866	1.333
Nonrisk component	-.797	10.236	13.099	6.275	19.560	13.054	10.605	-17.75	-15.57	-17.09
Equivalent variation in %	-1.510	9.280	11.500	5.890	26.130	14.860	9.440	-14.28	-11.02	-13.79
Compensated variation (CFA/day/capita)										
Total.............	-7.351	.920	8.407	-.652	12.031	.905	3.372	-32.85	-27.03	-29.71
Compensated variation in %	-8.790	1.020	6.590	-.740	15.420	.990	2.970	-20.59	-17.06	-20.27
Living-standard index										
Total.............	.985	1.093	1.115	1.059	1.261	1.149	1.094	.857	.890	.862
Risk premium index	.003	.017	.029	.017	.021	.039	.019	.011	.008	.010
Nonrisk component	.987	1.131	1.148	1.067	1.282	1.192	1.125	.870	.892	.876
Cost-of-living index										
Total.............	1.148	1.032	1.131	1.118	1.043	.995	1.095	1.167	1.124	1.160
Cost-of-risk index	.004	.015	.029	.019	.020	.035	.018	.015	.008	.014
Nonrisk component	1.147	1.020	1.095	1.103	1.026	.947	1.066	1.149	1.121	1.142
	Deterministic AIDS model									
Equivalent variation (CFA/day/capita)...	-9.724	-2.986	-.204	-3.671	3.682	-3.062	-2.234	-30.49	-29.79	-29.79
Equivalent variation in %	-11.99	-3.610	-.150	-4.960	4.600	-4.060	-2.550	-18.94	-18.19	-18.76
Compensated variation (CFA/day/capita).	-12.42	-3.533	-.262	-4.471	4.357	-3.669	-2.673	-38.60	-37.29	-37.67
Compensated variation in %	-14.01	-3.820	-.170	-5.310	4.470	-4.330	-2.690	-23.94	-23.05	-23.81
Living-standard index	.880	.964	.999	.950	1.046	.959	.975	.811	.818	.812
Cost-of-living index	1.233	1.170	1.209	1.192	1.184	1.167	1.194	1.234	1.223	1.231
	Other Statistics									
Share of Peanut in total income	24.100	33.800	50.300	30.900	65.100	9.400	40.900	.000	.000	.000
% change in nominal income	10.360	14.530	21.630	13.290	29.150	13.650	18.020	.000	.000	.000
Estimated % change in food exp........	9.570	13.050	20.780	12.260	25.300	12.660	17.350	.000	.000	.000
Laspeyres Price index	1.282	1.258	1.246	1.230	1.237	1.312	1.249	1.262	1.227	1.242
Paasche Price index .	.424	.398	.613	1.458	2.365	1.412	.528	.778	1.153	1.070

Notes: The welfare indicators were calculated for each household. Each cell contains the weighted median of the respective welfare indicators across all households.

Table 7.5 Indicators of welfare changes after the 50% CFA devaluation for the uncertainty and deterministic AIDS models: Market versus nonmarket villages

	Nonmarket villages	Market villages
	"Expected" AIDS model	
Equivalent variation (CFA/day/capita)		
Total	9.199	5.044
Risk premium	1.508	1.885
Nonrisk component	10.925	6.824
Equivalent variation in %	11.380	6.450
Compensated variation (CFA/day/capita)		
Total	3.986	-.958
Compensated variation in %	4.750	-1.150
Living-standard index		
Total	1.114	1.065
Risk premium index	.020	.017
Nonrisk component	1.131	1.084
Cost-of-living index		
Total	1.081	1.124
Cost-of-risk index	.018	.018
Nonrisk component	1.061	1.108
	Deterministic AIDS model	
Equivalent variation (CFA/day/capita)	-1.146	-5.255
Equivalent variation in %	-1.330	-5.540
Compensated variation (CFA/day/capita)	-1.426	-6.691
Compensated variation in %	-1.380	-5.980
Living-standard index	.987	.945
Cost-of-living index	1.185	1.215
	Other Statistics	
Share of Peanut in total income	43.400	32.600
% change in nominal income	18.920	14.020
Estimated % change in food exp.	17.770	12.810
Laspeyres Price index	1.249	1.243
Paasche Price index	.530	.475

Notes: The welfare indicators were calculated for each household. Each cell contains the weighted median of the respective welfare indicators across all households.

differences between the respective estimates from the two models. Indeed the deterministic AIDS model shows a net welfare loss for the median household in all zones except Dioly, while for the "expected" AIDS model only the median households in the urban zones and in Sagatta experience a net welfare loss. Given the fact that the "expected" AIDS model reduces to the deterministic AIDS one in the absence of uncertainty, we did not expect such big differences between estimates from the two models because of the fact that the fluctuations in food prices are not very great. In any case, this is another illustration of the potential biases in estimates from deterministic models when uncertainty is relevant to the choice situation. We can note here that although their estimates are not theoretically comparable with ours, the estimates of changes in "real income" after the CFA devaluation given by Kelly et al., (1994) are closer to the ones from the "expected" AIDS than the ones from the deterministic AIDS.[191]

7.4.1.1 Estimates of the median total welfare changes

As we can expect given the consumption and production patterns in the different zones and the direction and magnitude of the changes in relative prices, households in Niakhar, Colobane, Passy, Dioly and Missirah who have a relatively high share of peanut income, and a relatively low share of rice and/or a relatively high share of coarse grains in total food expenditures are the ones who have experienced a net gain in welfare. On the other hand, households in the urban zones and in Sagatta, who have no or relatively low peanut income

[191] Kelly et al., (1994) did not estimate a demand model. Instead they used the average food commodity shares in each zones and assumed they were constant. Hence, their estimates did not incorporate any substitution or income effects. Also, their analysis covered only peanut, rice, and the coarse grains in the consumption side, and peanut in the cropping income side. Their estimates of "changes in real income" were -8% for Sagatta, 5% for Niakhar, 14% for Colobane, 0% for Passy 16% for Dioly, 2% for Missirah, -8% for Kaolack, and -10% for Tamba.

but have high rice shares and low coarse grains shares in total food expenditure experienced a net welfare loss. As a percentage of the pre-devaluation average daily per capita food budget, households in Dioly have experienced the most gains in welfare, with a median net gain of +26% (or 17 CFA per capita per day), while households in the city of Kaolack have the highest lost in welfare with a median net lost of -14% (or 20 CFA per capita per day). The median net losses or gains for the other zones are respectively -1.5% (or -0.9 CFA per capita per day) for Sagatta, +9% (or 8 CFA per capita per day) for Niakhar, +11.5 % (or 10 CFA per capita per day) for Colobane, +6% (or 4 CFA per capita per day) for Passy, 15% (or 10 CFA per capita per day) for Missirah, and -11% (or 16 CFA per capita per day) for Tamba.[192]

In terms of market and nonmarket villages, Table 7.5 shows that nonmarket villages have a higher median net welfare gain (11 CFA per capita per day or 11% of the pre-devaluation average daily food budget per capita) than market villages (7 CFA per capita per day or 6% of the pre-devaluation average daily food budget per capita).

7.4.1.2 Estimates of the median total cost-of-food indices

Similarly to the total welfare impacts, as we can expect, it is in the zones with the highest shares of rice and the lowest shares of coarse grains in total food expenditures where the cost of living has risen the most. In contrast, in the low rice and high coarse grains consuming zones of Niakhar, Dioly, and Missirah, the cost of living (more precisely the cost of food) has practically remained unchanged. The cost of food has risen the most in Kaolack

[192] Note that the living-standard index is obtained by adding one to the ratio of the equivalent variation to the pre-devaluation average daily food budget per capita (that is the equivalent variation in % row divided by 100). For more details on the relationships among the various welfare indicators, see chapter 3.

(+17%), Sagatta (+15%), Colobane (+13%), and Passy (+12%). The median rise in the cost of food for Niakhar and Dioly are respectively +3% and +4%. In Missirah, there is practically 0% rise in the cost of food.[193] We note also that despite the fact that Sagatta has experienced a median net welfare loss of 0.9% while Colobane has experienced a median net welfare gain of 13%, the median increase in the cost of living in the two zones are very close (15% and 13% respectively). This is a reflection of the fact that the consumption patterns in the two zones are close, while, as Table 7.4 shows, on average the share of peanut income in total income in Colobane (50%) is more than twice higher than the one in Sagatta (25%).[194]

Finally, Table 7.5 shows that the median increase in the total cost of food is lower in nonmarket villages (8%) than in market villages (12%). This is, again, what one would expect given the fact that nonmarket villages have a higher average share of coarse grains (44%) and a lower average share of rice (14%) in total food expenditures compared to market villages (28% and 20% respectively).

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[193] From the theory in chapter 2, the 0.005% decrease in the cost of living of the median household in Missirah is theoretically impossible because for a consumer whose preference satisfies the standard axioms the true cost-of-living index is monotonically increasing in prices. Hence, we take this small decrease as an statistical artifact. That is, we consider it as statistically significantly not different from zero. In fact, if we do not weight (i.e., if we do not correct for the oversampling of market villages) then the estimated median total cost-of-food index in Missirah is 1.012. Moreover, since in the absence of uncertainty (i.e., in the deterministic case) the true cost-of-living index is still monotonically increasing in prices, it is really the the median increase in the nonrisk component of the cost-of-living index that must be taken as practically zero, so that the median net increase in the total cost-of-living index in Missirah is reduced to the cost-of-risk index, which is estimated to be 0.035 (see the theoretical formula in chapter 3). In summary, because of theoretical and statistical considerations for Missirah we take the estimated median cost-of-food index to be 1.035, and the risk and nonrisk component of it to be 0.035 and 1 respectively.

[194] The average shares of coarse grains and rice in total expenditures on food are respectively 19% and 33% in Sagatta, and 27% and 21% in Colobane. The respective figures for the other zones are 51% and 10% for Niakhar, 36% and 18% for Passy, 40% and 9% for Dioly, 48% and 6% in Missirah, 12% and 21% in Kaolack, and 8% and 23% in Tamba.

7.4.1.3 Estimates of the median risk premiums and cost-of-risk indices

Table 7.4 also gives the median risk premium and risk premium index for each zone. They are all positive as predicted by the theory in chapter 2. The risk premium measures in CFA per capita per day the total welfare loss associated with the uncertainty in the food commodity prices. The risk premium index measures the same welfare loss, but as a ratio to the pre-devaluation average daily food budget per capita. As a percentage of the pre-devaluation average daily food budget per capita, the median risk premium is highest in Missirah (3.9% or 2.6 CFA per capita per day), and lowest in Sagatta (0.3% or 0.24 CFA per capita per day). This perhaps reflects the fact that in terms of the respective shares of rice and coarse grains in total food expenditure Sagatta and Missirah are at opposite extremes among the zones (see Table 6.4), along with the fact that the price of rice is fixed and controlled by the government, while the coefficients of variation of the coarse grains prices are almost twice as high in Missirah than in Sagatta. (see Table 6.7). Moreover, as Table 6.7 shows, for almost all the food products the price variability is noticeably higher in Missirah than in Sagatta. However, the relative importance of rice and coarse grains in total food expenditure alone cannot explain the differences in the median risk premiums among zones. For example, Niakhar which has the highest average coarse grains share (51%) and the third lowest average rice share (10%), has a noticeably lower median risk premium (1.7% of the pre-devaluation average daily food budget per capita or 1.4 CFA per capita per day) than Colobane which has respective average coarse grains and rice shares of 27% and 21% respectively, and a median risk premium (in percentage terms) of 2.9% (or 3 CFA per capita per day). Indeed, in general the relative shares and variability of individual prices alone do not allow one to predict the relative magnitude of the risk premia. In fact, as the complicated expressions for the "expected" AIDS compensated demands in chapter 5 - which are used to calculate the total risk premiums - show, not only the individual variances of prices enter the computation of each respective directional risk premium, but also covariances with nonlinear function of other prices as well as nonlinear functions of moments for all prices enter in the computation of each of them. The respective risk premium figures for the other zones are 1.7% or 1.3 CFA per capita per day for Passy, 2.1% or 1.5 CFA per capita per day for Dioly, 1% or 1.6 CFA per capita per day for Kaolack, and 0.8% or 0.9 CFA per capita per day in Tamba.

The nonrisk components of the welfare gains or losses in Table 7.4 indicate that if there were no price uncertainty, the median net welfare gains - as percentage of the pre-devaluation average daily food budgets per capita - would have been 13% instead of 9% in Niakhar, 15% instead of 11.5% in Colobane, 7% instead of 6% in Passy, 28% instead of 26% in Dioly, and 19% instead of 15% in Missirah. Meanwhile, in Sagatta, Kaolack, and Tamba, price uncertainty has further increased the median net welfare losses (as percentage of the pre-devaluation average daily food budget per capita) from 1.3% to 1.5%, 13% to 14%, and 10.8% to 11.2% respectively.

The pattern shown in Table 7.4 by the zone median cost-of-risk indices is similar to the one shown by the risk premium figures. That is, the cost of price-risk is highest in Missirah and lowest in Sagatta. In other terms, price uncertainty increased the median cost-of-food indices in Missirah and Sagatta by respectively 0.035 points (or 4%) and 0.004 points (or 0.3%).[195] The respective increases for the other zones are 0.015 points or 1.5% for Niakhar, 0.029 points or 2.6% for Colobane, 0.019 points or 1.7% for Passy, 0.02 points 1.9% for Dioly, 0.015 points or 1.3% for Kaolack, and 0.014 points or 0.7% for Tamba.

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[195] For Missirah, see footnote 193.

With regard to market and nonmarket villages, Table 7.5 indicates that while the median risk premium index is higher in nonmarket villages (0.02) than in market villages (0.017), the median cost-of-risk index is the same in the two groups of villages (0.018).

7.4.1.4 Estimates of the median total compensated variations

Table 7.4 gives also the zone median compensated variations both in levels and as percentage of the post-devaluation average daily food budget per capita. We recall that the compensated variation measures the same welfare change as the equivalent variation, but with the difference that it uses the post-devaluation prices instead of the pre-devaluation ones to value, in monetary terms, the welfare changes. Hence, since it takes the post-devaluation situation as the status-quo, the compensated variation measures how much the median household in each zone must be compensated (if there is welfare loss) or taken away from it (if there is welfare gain) so as its level of welfare remains the same as before the devaluation. What we first note is the case of Passy where the estimated median compensated variation is negative (-0.65 CFA per capita per day or -0.7% of the post-devaluation average daily food budget per capita). However, we know from the theory in chapter 2, that although they may be of different magnitudes because of the fact that they use a different set of prices to value the change in welfare, the equivalent variation and the compensated variation must always have the same signs. Hence, we take the small negative value for the median compensated variation in Passy to be just an statistical artifact, and take it to be probably not statistically significantly different from zero. In other words, considering the prevailing post-devaluation level of prices as the status-quo, by and large households in Passy need no compensation or taxation if the median level of welfare is to remain the same as before the devaluation. The cases of Niakhar and Missirah where the estimated median compensated variations are respectively 0.92 and 0.90 CFA per capita per day (or 1.02% and 0.99% of their respective post-devaluation average daily food budgets per capita) are very close and also similar to the one of Passy.[196] On the other hand, if their level of welfare is to be the same as before the devaluation, households in Sagatta, Kaolack and Tamba must be compensated by respectively 7, 33, and 27 CFA per capita per day (or 9%, 21%, and 17% of their respective post-devaluation average daily food budgets per capita). In contrast, by and large households in Colobane and Dioly have to be taxed by respectively 8 and 12 CFA per capita per day (or 6.5% and 15% of their respective post-devaluation average daily food budgets per capita), if their respective median levels of welfare is to remain the same as before the devaluation.

Finally, in terms of market and nonmarket villages, Table 7.5 tell us that by and large households in nonmarket villages have to be taxed and those in market villages be compensated by respectively 4 and 1 CFA per capita per day (or 5% and 1% of their respective post-devaluation average daily food budgets per capita), if their respective median levels of welfare is to remain the same as before the devaluation.

For completeness, at the end of Table 7.4 we give for comparison the estimated zone median percentage changes in nominal income and the estimated resulting median changes in the average daily food budget per capita. The estimated changes in nominal incomes are based on the share of peanut income in total income (also given in the same table) and the one for cotton income, along with the percentage changes in peanut and cotton prices. As already mentioned, the estimated percentage changes in the average daily food budget per capita were obtained by estimating an Engel curve of food share on total expenditure (that is food plus nonfood expenditure). There are two things to note here. First, all rural zones experience an

[196] For Missirah, given the estimated low median cost-of-food index, the median compensated variation estimate does not seem to be plausible.

increase in nominal income, with Dioly having the highest median increase (29%) and Sagatta the lowest (10%). The increases in nominal income for the other zones are respectively 15% for Niakhar, 22% for Colobane, 13% for Passy, 14% for Missirah, and 0% for Kaolack and Tamba. But, the noticeable thing here is the similarity of the median increases in nominal income with the estimated median increases in the average daily food budgets per capita which are respectively 9.5% for Sagatta, 13% for Niakhar, 21% for Colobane, 12% for Passy, 25% for Dioly, 13% for Missirah, and 0% for Kaolack and Tamba. Second, for some of the zones like Dioly, Missirah, and Niakhar, the estimated median increase in the average daily food budgets per capita are very close to their estimated median net gain in welfare (which were 26%, 15%, and 9% respectively). On the other hand, for other zones like Colobane and Passy, their estimated median net welfare gains (which were 11.5% and 6% respectively) are about half their estimated median increases in their average daily food budget per capita. Moreover, for Sagatta, despite a 9.5% median increase in its average daily food budget per capita as a result of the net increase in total income, there is still a median net welfare loss of 1.5% in terms of food consumption. Of course, these differences among zones are explained by the differences in their cost-of-food indices.

Still for completeness, we have given in Table 7.4 the median estimates of the Laspeyres and Paasche price indices. As the figures show, they both tend to grossly over or underestimate the true cost-of-living indices.

7.4.2 Empirical distributions of the welfare and risk impacts

In the analysis in the previous section, we have focussed exclusively on the median welfare impacts on each zone. However, within each zone the change in prices following the devaluation has affected households differently depending on their income status, consumption patterns, production patterns and other socio-economic variables. One way to see how households with different characteristics have been affected by the change in price, is to study the empirical distribution of the indicators of welfare changes in each zone. However, to avoid lengthy discussions, we will focus only on the empirical distributions of the equivalent variation as percentage of the pre-devaluation average daily food budget per capita and the total cost-of-food index. In the next section where we study the welfare impacts of the change in prices following the devaluation across income groups, we will come back to the other indicators of welfare changes.

Figure 7.1 gives the histograms and sample statistics respectively for both the equivalent variation as percentage of the pre-devaluation average daily food budget per capita and the total cost-of-food index. In the overall sample, the welfare losses or gains range from - 17% to +38% with an standard error of 0.91, while the total cost-of-food index range from 0.92 to 1.17 with an standard error of 0.004. With respect to the cost-of-food index, as already discussed in footnote 193 the 16 cases (5% of the sample) for which the cost-of-food index is estimated to be below one are ruled out by the theory. For both the welfare change and the cost-of-food index, the sample median is higher than either the simple mean or the other robust estimates of locations. The sample empirical distribution of the welfare changes is close to being bimodal, with one mode being in the welfare loss range between -12.5% to -7.5%, and the other mode being in the welfare gain range between +7.5% to +12.5%. On the other hand, the sample empirical distribution of the total cost-of-food index is noticeably skewed to the right (i.e., toward higher values).

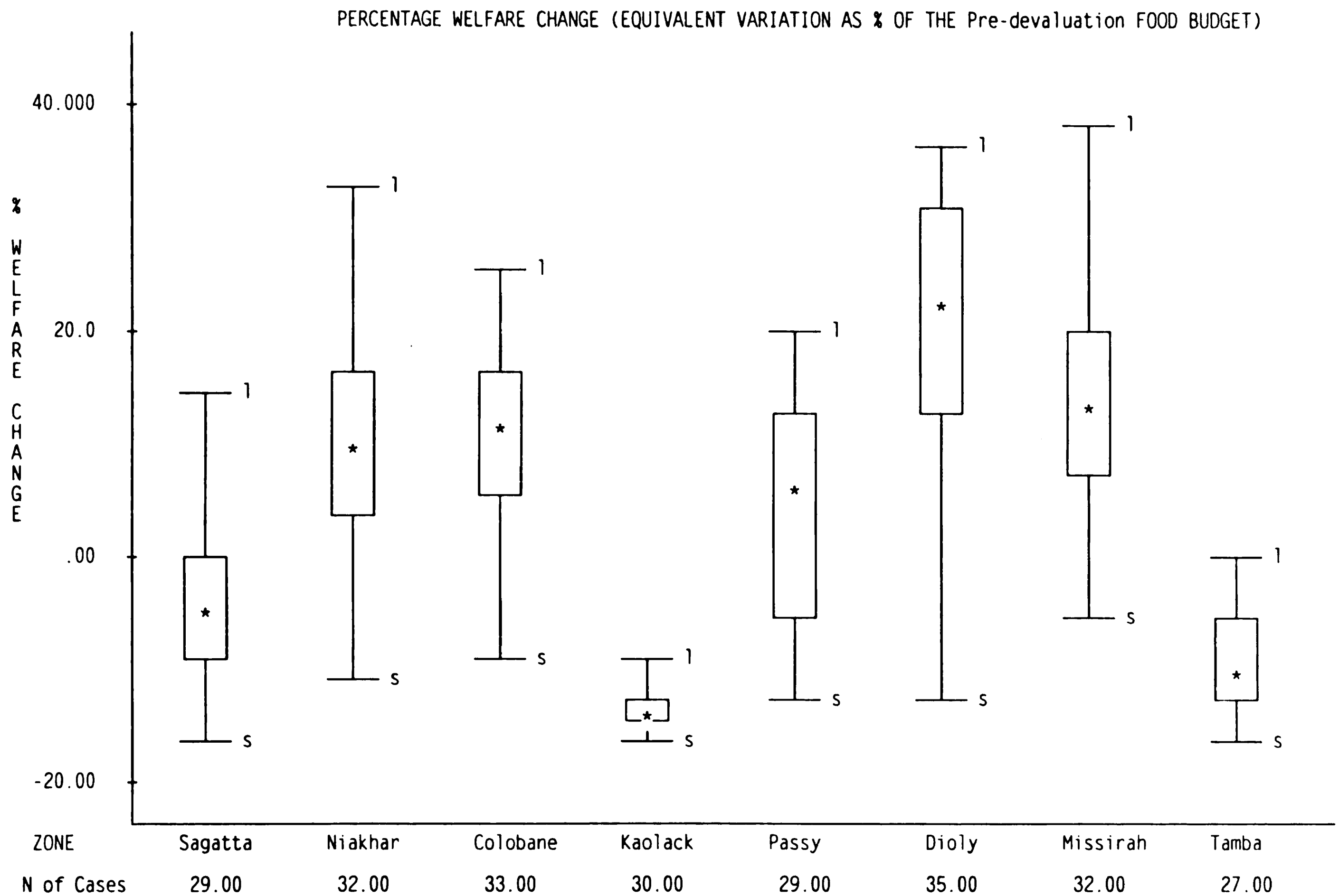
```
WELFARE CHANGE AS % OF THE PRE-DEVALUATION FOOD BUDGET

Frequency   Bin Center                                  Valid cases:       247.0
                                                        Missing cases:        .0
  11.00  -17.5000   *****                               Percent missing:      .0
  41.00  -12.5000   *********************
  33.00  -7.50000   *****************                   Mean         4.6134  Std Err        .9076
  19.00  -2.50000   **********                          Median       5.2900  Variance    203.4480
  16.00   2.50000   *********                           5% Trim      4.1675  Std Dev      14.2635
  34.00   7.50000   ******************                  Min        -16.7100  Skewness       .2693
  31.00  12.50000   ****************                    Max         38.0400  S E Skew       .1549
  19.00  17.50000   **********                          Range       54.7500  Kurtosis      -.9771
  21.00  22.50000   ***********                         IQR         24.0700  S E Kurt       .3086
   9.00  27.50000   ****
  10.00  32.50000   *****                                           M-Estimators
   3.00  37.50000   *                                   Huber  (1.339)                     3.7551
                                                        Tukey  (4.685)                     4.0560
Bin width :    5.00000                                  Hampel (1.700,3.400,8.500)         4.1177
Each star:        2 case(s)                             Andrew (1.340 * pi)                4.0612


COST-OF-FOOD INDEX

Frequency   Bin Center                                  Valid cases:       247.0
                                                        Missing cases:        .0
   1.00    .91000                                       Percent missing:      .0
   1.00    .93000
   6.00    .95000   ***                                 Mean         1.0967  Std Err        .0040
   4.00    .97000   **                                  Median       1.1114  Variance       .0039
   4.00    .99000   **                                  5% Trim      1.0995  Std Dev        .0623
  17.00   1.01000   *********                           Min           .9170  Skewness     -.5952
  23.00   1.03000   ************                        Max          1.2097  S E Skew       .1549
  13.00   1.05000   ******                              Range         .2927  Kurtosis     -.3855
  18.00   1.07000   **********                          IQR           .0968  S E Kurt       .3086
  23.00   1.09000   ************
  28.00   1.11000   ***************                                  M-Estimators
  38.00   1.13000   *******************                 Huber  (1.339)                     1.1066
  32.00   1.15000   *****************                   Tukey  (4.685)                     1.1082
  27.00   1.17000   **************                      Hampel (1.700,3.400,8.500)         1.1028
  10.00   1.19000   *****                               Andrew (1.340 * pi)                1.1082
   2.00   1.21000   *

Bin width :    .02000
Each star:        2 case(s)
```

Figure 7.1 Empirical distributions of the percentage welfare change (equivalent variation) and cost-of-living index: histogram and sample statistics.

Figure 7.2 and Figure 7.3 show the boxplot diagrams for the estimated welfare changes and total cost-of-food indices in each zone. First, as Figure 7.2 shows there is no zone where all households experienced a net welfare gain. However, except for the zone of Passy, in all the zones that showed a median net welfare gain, more than 75% of the households have experienced a net welfare gain. For the zone of Passy about 50% of the households have experienced a net welfare loss. Naturally, virtually all the households in the urban zones have experienced a net welfare loss. But, in Sagatta where the median net welfare change is negative, about 25 % of the households have experienced a net welfare gain. Second, the variability in the net welfare changes across households is higher in zones where the median net welfare change is positive than in zones where the median net welfare change is negative. The variability in the net welfare changes across households is particularly low in Kaolack where the empirical distribution is particularly skewed toward the values of greater net welfare losses.

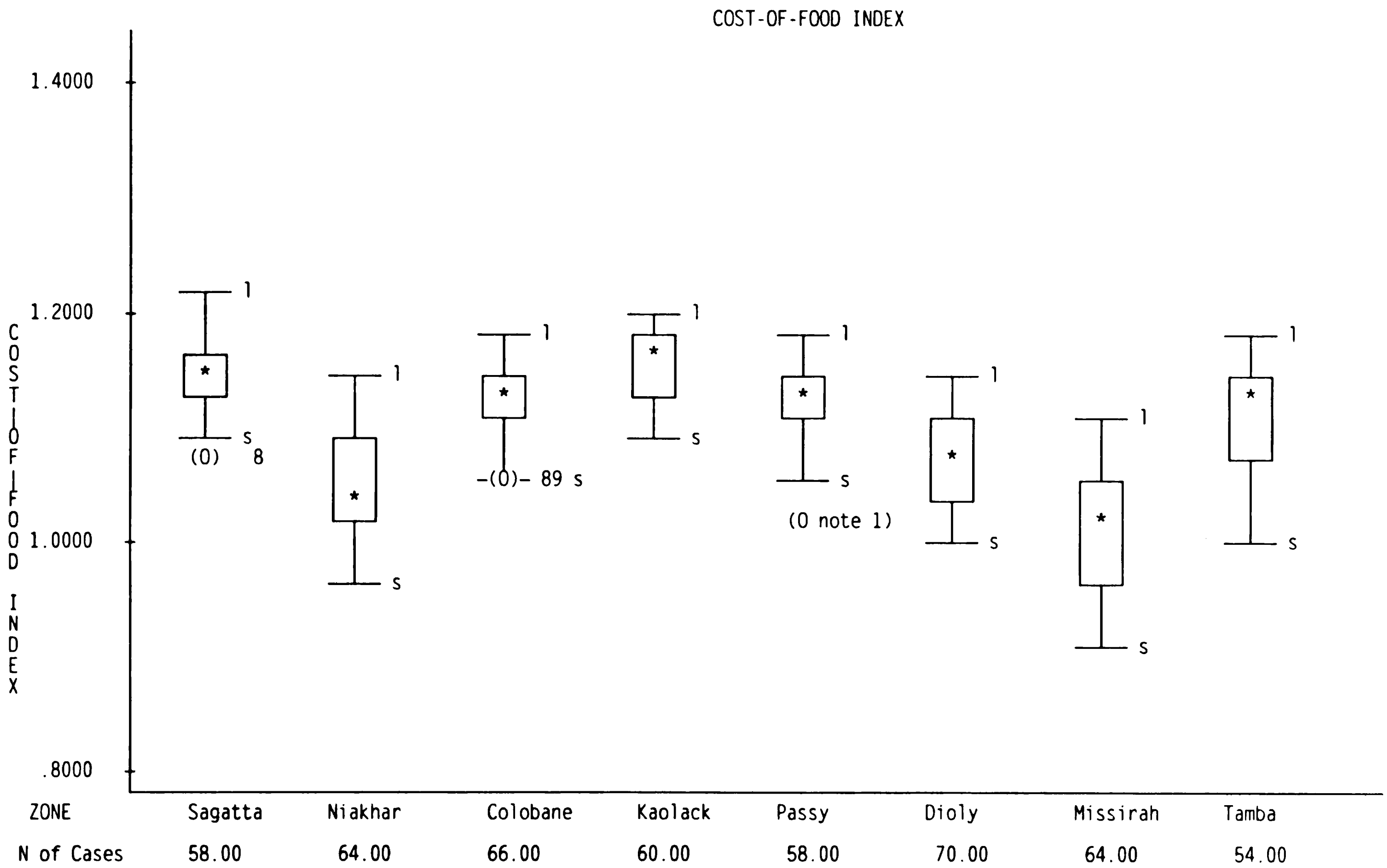
In contrast, Figure 7.3 shows that by and large the empirical distributions of the total cost-of-food indices are similar across zones, especially in terms of variability. It is only in Missirah where close to 50% of the estimated cost-of-food indices have theoretically wrong magnitudes (i.e., less than 1). Few cases in Niakhar also have theoretically wrong estimated cost-of-food indices. Figure 7.3 also confirms the trend in the increases of the cost of food as shown by the median estimates. That is, by and large the cost of food has increased the least in Niakhar, Dioly, and Missirah, and increased the most in Kaolack.

Finally, Figure 7.4 and Figure 7.5 compare the empirical distributions in market and nonmarket villages of the welfare changes and total cost-of-food indices respectively. Although in both cases the empirical distributions in market and nonmarket villages do intersect, the boxplot diagrams merely confirm the trends shown by the median estimate.



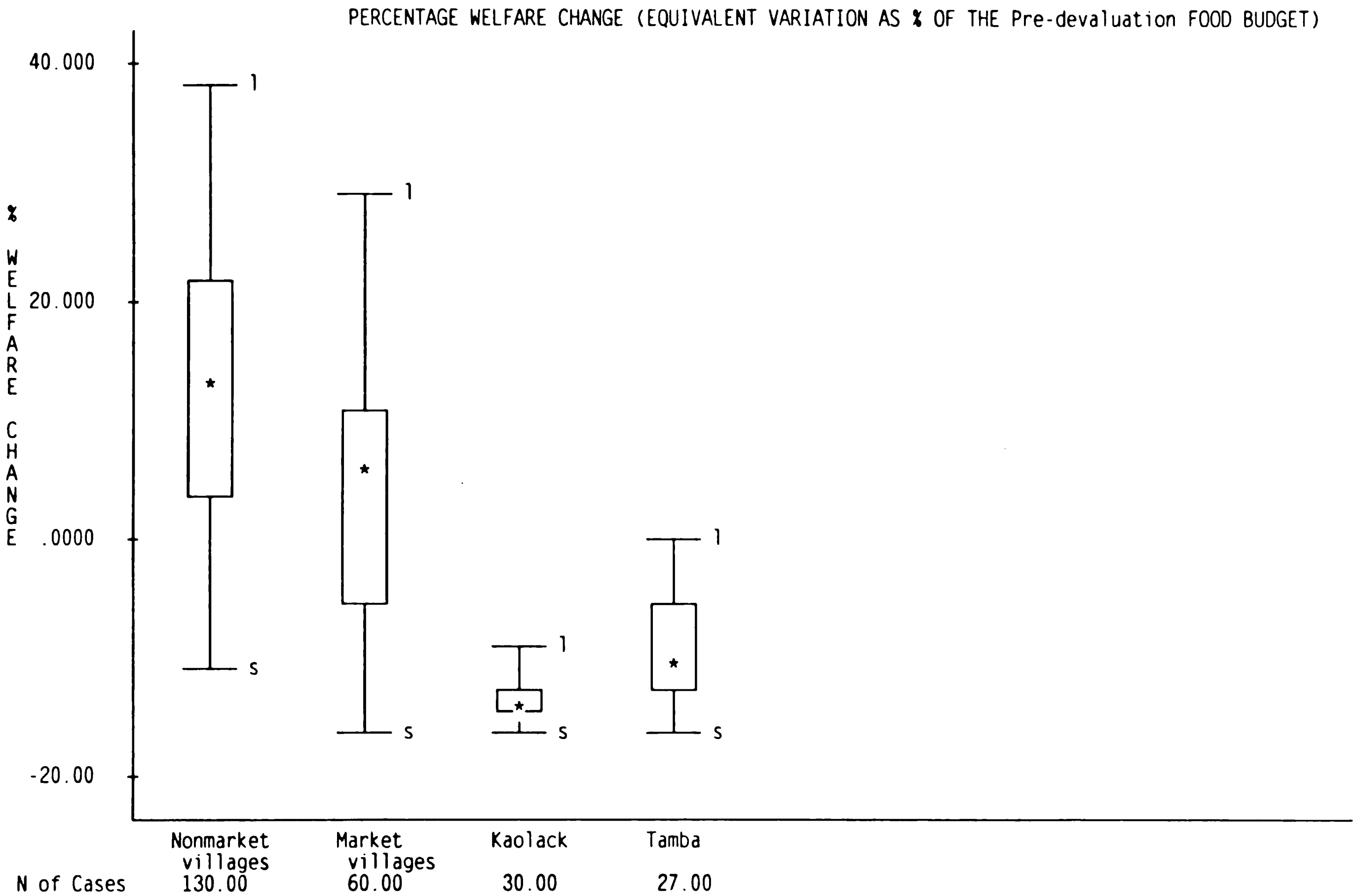
Symbol Key: * - Median. For each zone, 50% of cases have values within the box, the length of the box is the interquartile range, and the lower boundary (resp upper boundary) of the box is the 25th (resp 75th) percentile.
s - smallest observed value that is not outlier.
l - largest observed value that is not outlier.
(O) - Outlier: value less (resp more)than 1.5 box-lengths from 25th (resp 75th) percentile.

Figure 7.2 Empirical distributions of the percentage change in welfare: Boxplot diagrams by zone.



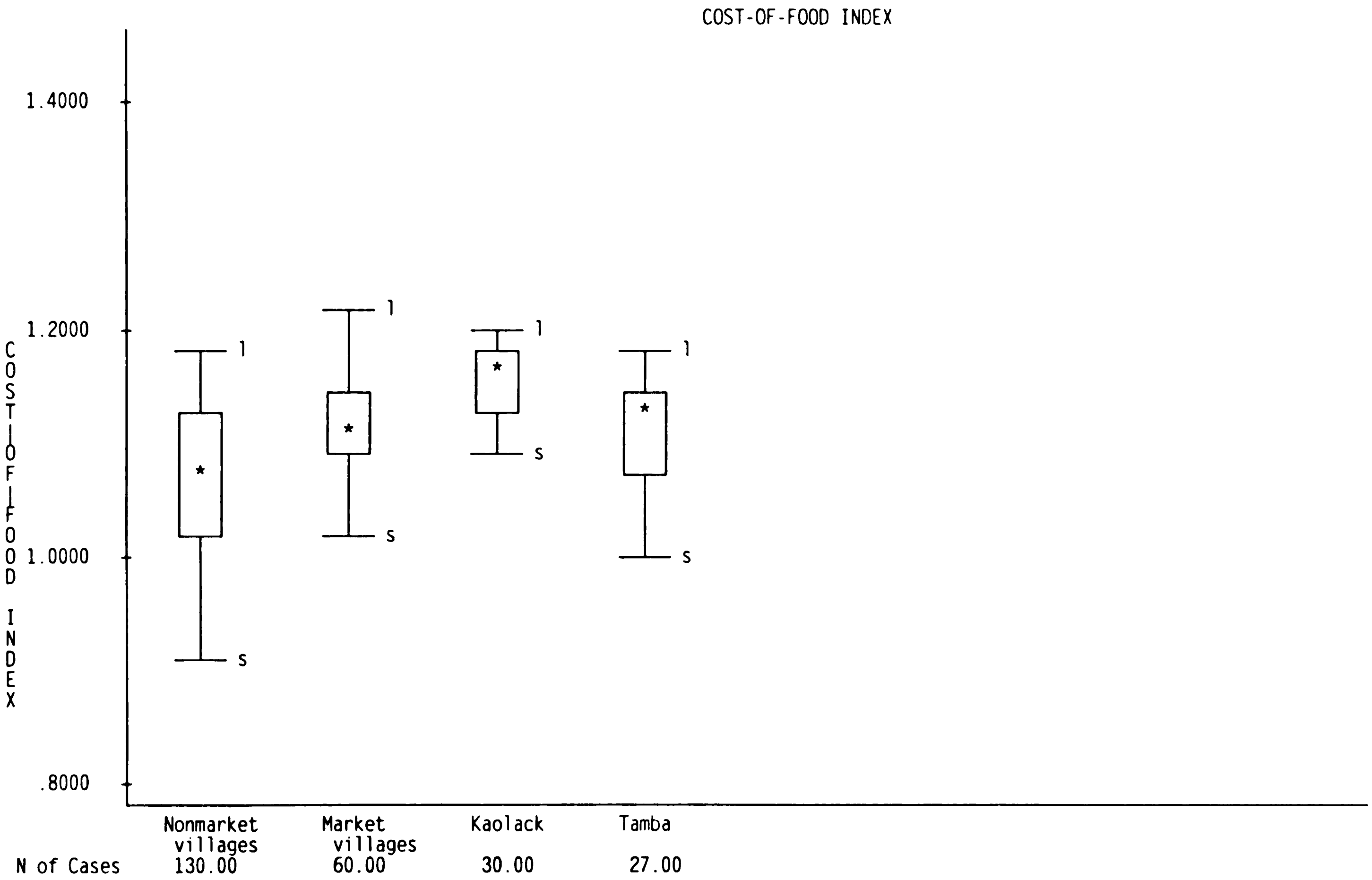
Symbol Key: * - Median. For each zone, 50% of cases have values within the box, the length of the box is the interquartile range, and the lower boundary (resp upper boundary) of the box is the 25th (resp 75th) percentile.
s - smallest observed value that is not outlier.
l - largest observed value that is not outlier.
(O) - Outlier: value less (resp more)than 1.5 box-lengths from 25th (resp 75th) percentile.
(E) - Extreme. value less (resp more)than 3 box-lengths from 25th (resp 75th) percentile

Figure 7.3 Empirical distributions of the cost-of-food index: Boxplot diagrams by zone.



Symbol Key: * - Median. For each village type 50% of cases have values within the box, the length of the box is the interquartile range, and the lower boundary (resp upper boundary) of the box is the 25th (resp 75th) percentile.
s - smallest observed value that is not outlier.
l - largest observed value that is not outlier.
(O) - Outlier: value less (resp more)than 1.5 box-lengths from 25th (resp 75th) percentile.
(E) - Extreme. value less (resp more)than 3 box-lengths from 25th (resp 75th) percentile

Figure 7.4 Empirical distributions of the percentage change in welfare: Boxplot diagrams by Market/nonmarket villages.



Symbol Key: * - Median. For each zone, 50% of cases have values within the box, the length of the box is the interquartile range, and the lower boundary (resp upper boundary) of the box is the 25th (resp 75th) percentile.
s - smallest observed value that is not outlier.
l - largest observed value that is not outlier.
(O) - Outlier: value less (resp more)than 1.5 box-lengths from 25th (resp 75th) percentile.
(E) - Extreme. value less (resp more)than 3 box-lengths from 25th (resp 75th) percentile

Figure 7.5 Empirical distributions of the cost-of-food index: Boxplot diagrams by Market/nonmarket villages

That is, by and large households in nonmarket villages experienced higher net welfare gains and lower increase in their total cost of food than households in market villages. Perhaps one last thing to notice here is the low variability across households in the total cost-of-food index in market villages compared to nonmarket villages.

7.4.3 Variation of the welfare and risk impacts across income groups

Table 7.6 gives the median welfare impacts for each income quartile in the rural and urban zones separately, while Table 7.7 gives the same information for market and nonmarket villages separately. However, as in the previous section, to shorten the discussion we will focus on the total equivalent variation and cost-of-food index only.[197]

7.4.3.1 Welfare impacts across income groups in rural and urban zones

Table 7.6 shows that in the urban zones, the median net welfare loss as a percentage of the pre-devaluation average daily food budget per capita and the increase in the total cost of food are greatest for the lowest income quartile (15% or 16 CFA per capita per day and 18% respectively). They are lowest for the lower-medium income quartile (12.5% or 14 CFA per capita per day and 14% respectively). For the two higher income quartiles, the corresponding figures are respectively 13% (or 15 CFA per capita per day) and 15% for the upper-medium quartile, and 13% (or 25 CFA per capita per day) and 15.5 % for the upper income quartile.

In the rural zones, the median net welfare change is positive for all income quartiles. In percentage of the pre-devaluation average daily food budget per capita, the median net

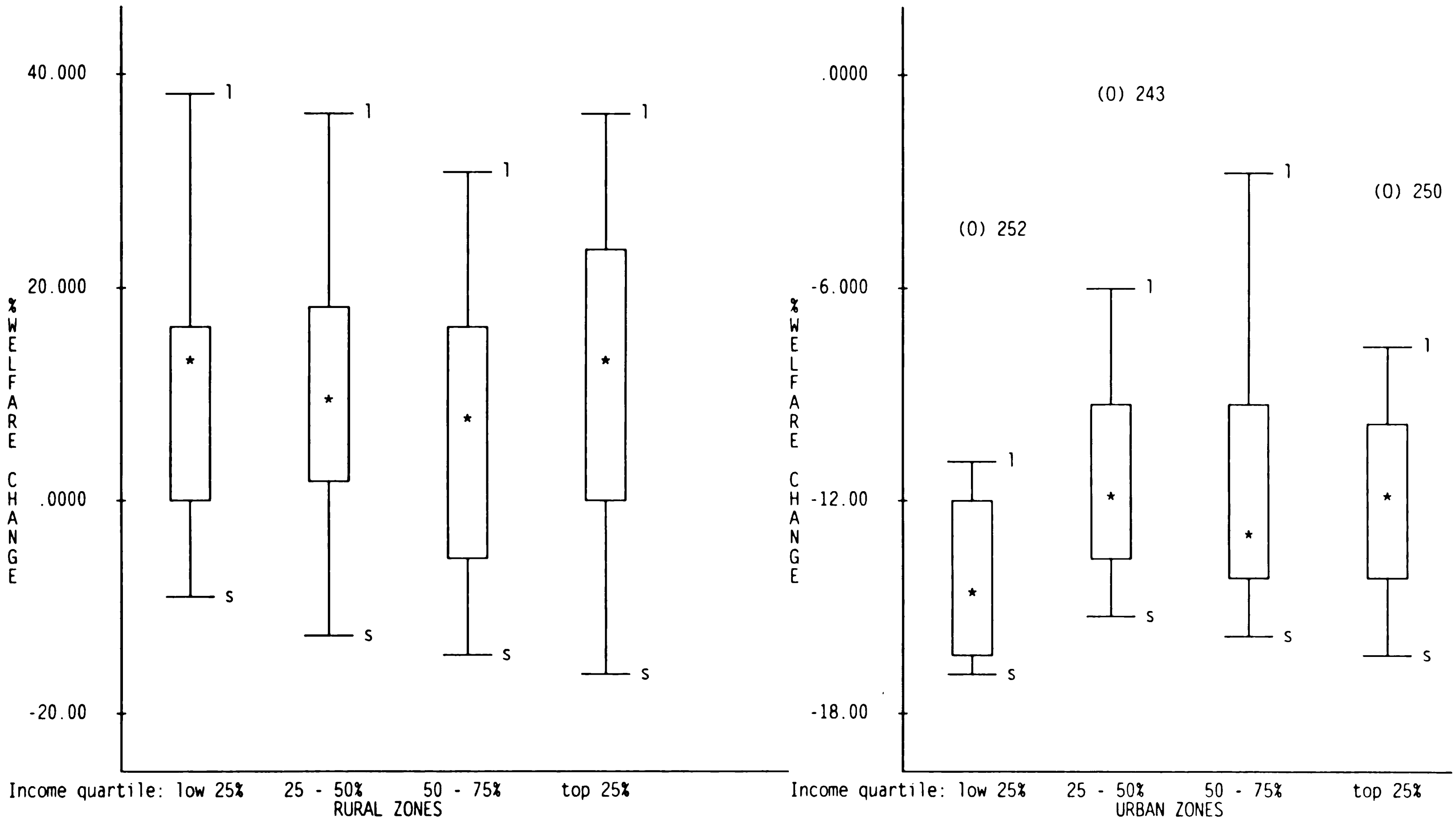
[197] One noticeable fact here is the theoretically wrong sign for the risk figures for the lowest income quartile in the market villages.

Table 7.6 Indicators of welfare changes after the 50% CFA devaluation for the uncertainty and deterministic AIDS models: Variation across income groups

	Rural zones				Urban zones			
	Income quartile				Income quartile			
	low 25%	25 - 50%	50 - 75%	top 25%	low 25%	25 - 50%	50 - 75%	top 25%
	"Expected" AIDS model							
Equivalent variation (CFA/day/capita)								
Total.............	7.998	7.090	10.447	10.256	-15.82	-13.78	-15.31	-24.79
Risk premium......	1.430	1.228	1.764	1.863	1.200	2.034	.662	2.034
Nonrisk component	8.265	8.706	11.854	14.607	-15.07	-12.26	-15.57	-23.08
Equivalent variation in %	12.410	8.340	10.130	8.960	-15.06	-12.51	-12.98	-13.43
Compensated variation (CFA/day/capita)								
Total.............	1.650	1.293	5.670	7.805	-27.69	-24.05	-30.86	-37.71
Compensated variation in %	2.970	1.310	5.830	5.690	-19.23	-20.40	-19.90	-20.72
Living-standard index								
Total.............	1.124	1.083	1.101	1.090	.849	.875	.870	.866
Risk premium index	.022	.016	.020	.022	.010	.015	.007	.010
Nonrisk component	1.168	1.103	1.119	1.153	.868	.887	.873	.876
Cost-of-living index								
Total.............	1.089	1.072	1.095	1.126	1.177	1.143	1.149	1.155
Cost-of-risk index	.017	.015	.020	.020	.015	.019	.009	.013
Nonrisk component	1.066	1.058	1.076	1.098	1.152	1.127	1.146	1.142
	Deterministic AIDS model							
Equivalent variation (CFA/day/capita)..	-2.495	-3.705	-.268	-1.146	-29.78	-22.28	-31.21	-39.89
Equivalent variation in %	-2.230	-4.350	-.320	-.960	-18.70	-18.61	-18.88	-19.27
Compensated variation (CFA/day/capita)...	-3.154	-4.594	-.335	-1.426	-37.67	-28.15	-39.53	-50.69
Compensated variation in %	-2.330	-4.650	-.330	-1.000	-23.81	-23.42	-23.85	-24.74
Living-standard index	.978	.957	.997	.990	.813	.814	.811	.807
Cost-of-living index	1.181	1.184	1.199	1.199	1.230	1.229	1.233	1.239
	Other Statistics							
Share of Peanut in total income	37.500	34.700	48.200	49.900	.000	.000	.000	.000
% change in nominal income	17.330	14.920	20.730	21.460	.000	.000	.000	.000
Estimated % change in food exp.	16.050	13.590	18.980	19.840	.000	.000	.000	.000
Laspeyres Price index	1.246	1.257	1.249	1.236	1.204	1.272	1.264	1.265
Paasche Price index .	.445	.446	.541	.751	.999	.966	1.142	1.111

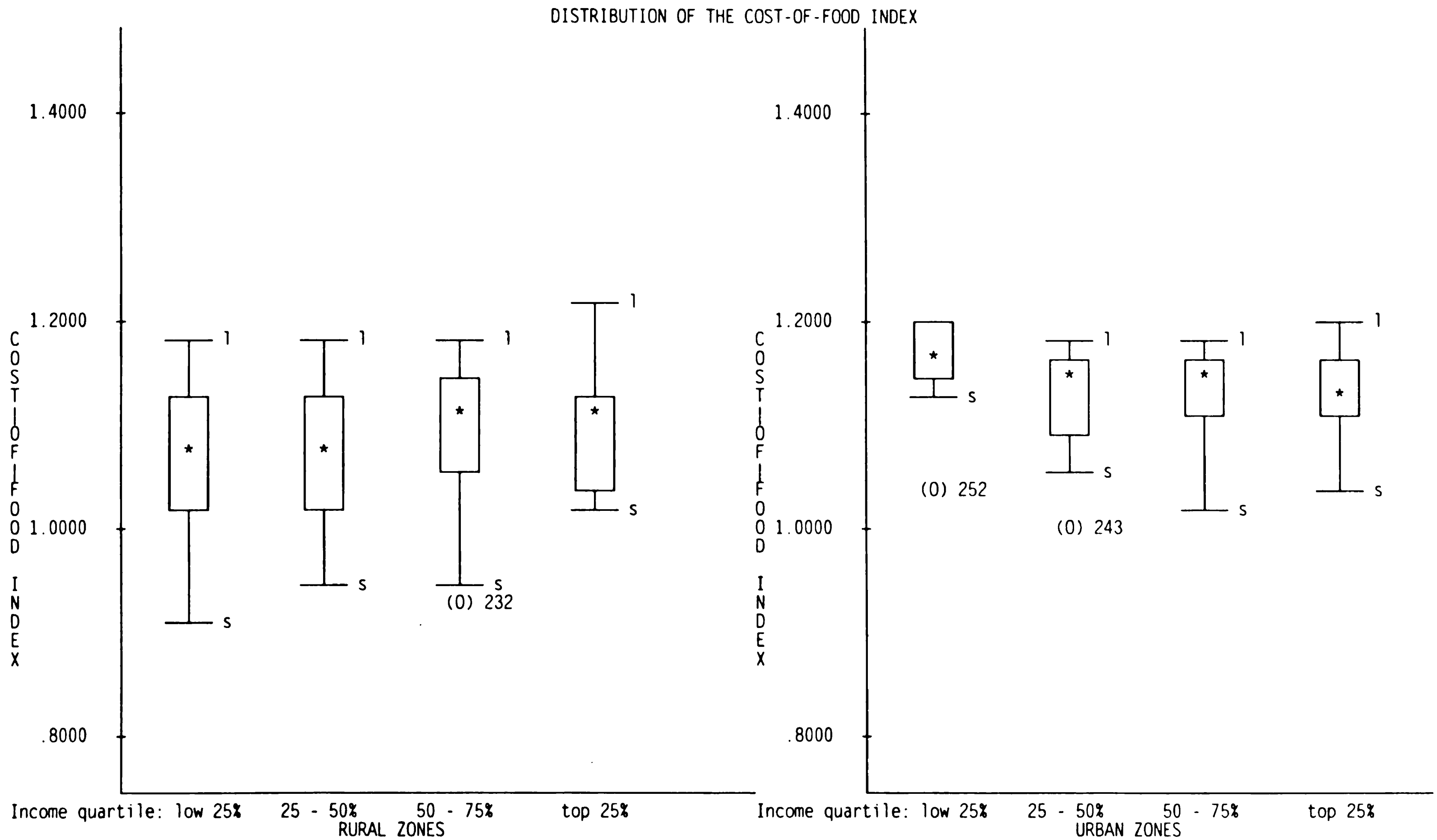
Notes: The welfare indicators were calculated for each household. Each cell contains the median of the respective welfare indicators across all households.

DISTRIBUTION OF THE PERCENTAGE CHANGE IN WELFARE (EQUIVALENT VARIATION AS % OF THE INITIAL FOOD BUDGET)



Symbol Key: * - Median. For each income quartile 50% of cases have values within the box, the length of the box is the interquartile range, and the lower boundary (resp upper boundary) of the box is the 25th (resp 75th) percentile.
s - smallest observed value that is not outlier.
l - largest observed value that is not outlier.

Figure 7.6 Empirical distributions of the percentage change in welfare: Boxplot diagrams by income groups.



Symbol Key: * - Median. For each income quartile 50% of cases have values within the box, the length of the box is the interquartile range, and the lower boundary (resp upper boundary) of the box is the 25th (resp 75th) percentile.
s - smallest observed value that is not outlier.
l - largest observed value that is not outlier.

Figure 7.7 Empirical distributions of the cost-of-food index: Boxplot diagrams by income groups.

welfare gain is greatest for the lowest income quartile (12% or 8 CFA per capita per day) and lowest for the lower-medium income quartile (8% or 7 CFA per capita per day). The figures for the two higher income quartiles are respectively 10% (or 10 CFA per capita per day) for the upper-medium income quartile, and 9% (or 10 CFA per capita per day) for the upper income quartile. In contrast, the median increase in the total cost of food is the lowest for the lower medium income quartile (7%), and the highest for upper income quartile (12.6%). The median increase in the total cost of food for the lowest and upper-medium income quartile are 9% and 9.5 % respectively. The fact that households in the lower-medium income quartile have the lowest median net welfare gain despite having the lowest median increase in the total cost of food, is mostly a reflection of their lower median share of peanut income in total income (35%) compared to the ones for the other income quartiles (37.5% for the lowest quartile, 48% for the upper-medium quartile, and 50% for the upper quartile).

Finally, Figure 7.6 and Figure 7.7 show the boxplot diagrams of the empirical distributions of the percentage net welfare changes and cost-of-food indices within each income quartile for the rural and urban households respectively.[198] First, for the rural zones, in each income quartile 75% of all households have experienced a positive net welfare gain with more than 25% having a positive net welfare gain of 15% of their pre-devaluation average daily food budget per capita or greater. The only exception to this pattern is in the upper-medium income quartile where close 40% of the households have experienced a net welfare loss. Overall, in the rural zones households in the upper income quartile have the highest net welfare gain with a positive net welfare gain of more than 20% of the pre-devaluation average daily food budget per capita for more than 25% of them. For the cost-

[198] We recall that the urban and rural households were classified into income quartiles separately.

of-food index, the within-income quartile empirical distributions seem to be similar, except for the upper income quartile where it is noticeably skewed upward. Overall, for all income quartiles the cost of food has risen less than 12% for 75% of the households; except for the upper-medium income quartile where the corresponding percentage of households is about 60%.

Second, in contrast to the rural zones, in the urban zones where all households have a net welfare loss, in all income quartiles for 75% of the households the net welfare loss is greater than 9% of their respective pre-devaluation average daily food budget per capita. Furthermore, households in the lower income quartile are hurt the most with all households having a net welfare loss of more than 11% of their respective pre-devaluation average daily food budget per capita. The cost of food has also risen the most for these households (more than 12% for all the households). However, in all income quartiles, for 75% of the households the cost of food has risen by more than 10%. Finally, Figure 7.6 and Figure 7.7 show that overall the within-income quartile empirical distributions of the welfare impacts show relatively less variability in the urban zones than in the rural zones. Hence, in contrast to the within-income quartile empirical distributions of the pre-devaluation elasticities in the previous chapter, when it comes to the differences among income quartiles of the welfare impacts of price changes following the CFA devaluation, urban households show more similarities than rural households.

7.4.3.2 Welfare impacts across income groups in market and nonmarket villages

In the previous section, the assessment of the differences across income quartiles of the welfare impacts on the rural households was done by pooling households in market and nonmarket villages together. Table 7.7 gives the welfare impact figures for the income

Table 7.7 Welfare changes after the devaluation for the uncertainty and deterministic AIDS models: Market versus nonmarket villages by income group

	Nonmarket villages				Market villages			
	Income quartile				Income quartile			
	low 25%	25 - 50%	50 - 75%	top 25%	low 25%	25 - 50%	50 - 75%	top 25%
				"Expected" AIDS model				
Equivalent variation (CFA/day/capita)								
Total.............	7.998	7.090	10.447	11.196	5.673	1.496	6.316	-3.457
Risk premium......	1.701	1.228	1.636	1.863	-1.227	1.072	2.493	4.081
Nonrisk component	10.673	10.236	11.854	14.607	7.599	4.724	10.231	.503
Equivalent variation in %	13.560	9.280	12.210	13.630	6.600	2.460	5.180	-2.760
Compensated variation (CFA/day/capita)								
Total.............	1.650	1.566	7.533	8.407	2.644	-4.239	-4.044	-5.613
Compensated variation in %	2.970	1.690	7.840	6.100	2.490	-6.360	-2.670	-4.090
Living-standard index								
Total.............	1.136	1.093	1.122	1.136	1.066	1.025	1.052	.972
Risk premium index	.022	.016	.020	.021	-.013	.014	.019	.034
Nonrisk component	1.173	1.128	1.148	1.156	1.099	1.078	1.084	1.004
Cost-of-living index								
Total.............	1.077	1.069	1.095	1.123	1.122	1.094	1.145	1.136
Cost-of-risk index	.018	.015	.020	.020	-.017	.014	.020	.038
Nonrisk component	1.056	1.034	1.076	1.098	1.139	1.081	1.126	1.126
				Deterministic AIDS model				
Equivalent variation (CFA/day/capita)	-.579	-3.400	1.034	-.916	-2.838	-5.717	-6.389	-8.996
Equivalent variation in %	-1.140	-4.030	1.090	-.960	-3.550	-9.310	-7.170	-10.21
Compensated variation (CFA/day/capita)	-.688	-4.174	1.266	-1.108	-3.492	-6.965	-8.283	-11.56
Compensated variation in %	-1.190	-4.300	1.100	-1.000	-3.750	-10.46	-7.880	-11.64
Living-standard index	.989	.960	1.011	.990	.965	.907	.928	.898
Cost-of-living index	1.179	1.176	1.197	1.192	1.203	1.198	1.226	1.242
				Other Statistics				
Share of Peanut in total income	35.300	34.700	54.100	50.300	45.000	20.600	31.900	29.900
% change in nominal income	15.180	14.920	23.260	21.630	19.350	8.860	13.720	12.860
Estimated % change in food exp......	14.480	13.590	21.740	19.840	18.130	8.270	11.940	11.490
Laspeyres Price index	1.249	1.257	1.247	1.242	1.216	1.279	1.258	1.231
Paasche Price index .	.439	.446	.541	1.264	.445	.421	.558	.632

Notes: The welfare indicators were calculated for each household. Each cell contains the median of the respective welfare indicators across all households.

Table 7.8 Equivalent variation in CFA per capita per day and as % of the per capita food budget: by income groups, consumption, and production patterns

Risk premium index in parentheses	Rural zones						Urban zones	
	Sagatta	Niakhar	Colobane	Passy	Dioly	Missirah	Kaolack	Tamba
Income quartile								
low 25%	1.005 (.005)	1.097 (.027)	1.185 (.028)	1.129 (-.01)	1.344 (.009)	1.135 (.045)	.836 (.011)	.884 (.002)
25 - 50%	1.007 (.001)	1.093 (.013)	1.045 (.029)	1.057 (.018)	1.213 (.021)	1.106 (.036)	.874 (.018)	.905 (.010)
50 - 75%	.937 (.003)	1.100 (.020)	1.161 (.027)	1.044 (.015)	1.078 (.019)	1.081 (.031)	.864 (.007)	.881 (.003)
top 25%	.896 (.004)	1.075 (.073)	1.172 (.033)	1.044 (.016)	1.246 (.021)	1.073 (.037)	.857 (.009)	.891 (.010)
Food share in total exp.								
< 70%	.902 (.006)	. .	. .	.914 (.012)	1.320 (.027)	1.013 (.036)	.859 (.010)	.899 (.003)
70 - 80%	1.041 (.004)	1.083 (.014)	1.071 (.028)	1.044 (.013)	1.212 (.023)	1.073 (.039)	.855 (.013)	.892 (.012)
80 - 90%	.937 (.004)	1.090 (.033)	1.141 (.031)	1.093 (.020)	1.255 (.016)	1.135 (.036)	.859 (.009)	.877 (.010)
> 90%	.999 (.002)	1.107 (.015)	1.085 (.010)	1.097 (.015)	1.102 (.018)	1.109 (.033)	. .	.866 (.00)
Rice Share in Food exp.								
< 10%	. .	1.128 (.016)	1.183 (.019)	1.080 (.037)	1.264 (.021)	1.156 (.043)	. .	.892 (.012)
10 - 20%	1.069 (.00)	1.060 (.014)	1.090 (.032)	.950 (.016)	1.030 (.020)	1.021 (.022)	.857 (.011)	.878 (.008)
20 - 30%	1.004 (.003)	1.100 (.020)	1.110 (.030)	1.113 (.015)	1.036 (.031)	1.020 (.035)	.856 (.009)	.891 (.007)
> 30%	.918 (.003)	. .	.987 (.012)	. .	. .	1.101 (.060)	.900 (.020)	.930 (.008)
Coarse grains share in food exp.								
< 10%	.890 (.003)	. .	1.079 (.029)	. .	. .	. .	.870 (.011)	.922 (.009)
10 - 20%	.927 (.004)	1.124 (-.02)	1.065 (.031)	.915 (.009)	. .	.947 (.037)	.854 (.010)	.866 (.003)
20 - 30%	.985 (.005)	. .	1.168 (.033)	1.147 (.013)	.983 (.019)	1.091 (.041)	.852 (.013)	.878 (.002)
> 30%	1.073 (.002)	1.092 (.017)	1.108 (.027)	1.047 (.016)	1.239 (.021)	1.129 (.036)	. .	. .
Coarse grains share in cropping income								
< 50%	.951 (.003)	1.230 (.011)	1.138 (.029)	1.097 (.015)	1.261 (.022)	1.124 (.036)	.857 (.011)	.890 (.008)
50 - 60%	.999 (.005)	1.148 (.017)	1.143 (.030)	.935 (.026)	1.215 (.011)	1.119 (.031)	. .	. .
60 - 70%	. .	1.083 (.023)	.953 (.032)	.908 (.016)	1.225 (.027)	1.135 (.052)	. .	. .
> 70%	.881 (.003)	.993 (.014)	1.094 (-.01)	1.059 (.039)	.900 (.012)	1.071 (.066)	. .	. .
Peanut share in cropping income								
< 50%	1.003 (.004)	1.090 (.017)	1.069 (.032)	.946 (.020)	1.215 (.016)	1.152 (.035)	.857 (.011)	.890 (.008)
50 - 60%	.904 (.003)	1.272 (.011)	1.124 (.029)	.972 (.013)	1.222 (.019)	1.080 (.035)	. .	. .
60 - 70%	1.069 (.003)	. .	1.118 (.021)	1.066 (.016)	1.284 (.023)	1.087 (.050)	. .	. .
> 70%	.937 (.002)	. .	1.142 (.030)	1.139 (.006)	1.261 (.027)	1.077 (.037)	. .	. .

Notes: The values were calculated for each household using the "expected" AIDS model. Each cell contains the median of the respective values across all households.

Table 7.9 The living standard and risk premium indices: by income groups, consumption, and production patterns

Risk premium index in parentheses	Rural zones												Urban zones			
	Sagatta		Niakhar		Colobane		Passy		Dioly		Missirah		Kaolack		Tamba	
Income quartile																
low 25%	1.005	(.005)	1.097	(.027)	1.185	(.028)	1.129	(-.01)	1.344	(.009)	1.135	(.045)	.836	(.011)	.884	(.002)
25 - 50%	1.007	(.001)	1.093	(.013)	1.045	(.029)	1.057	(.018)	1.213	(.021)	1.106	(.036)	.874	(.018)	.905	(.010)
50 - 75%	.937	(.003)	1.100	(.020)	1.161	(.027)	1.044	(.015)	1.078	(.019)	1.081	(.031)	.864	(.007)	.881	(.003)
top 25%	.896	(.004)	1.075	(.073)	1.172	(.033)	1.044	(.016)	1.246	(.021)	1.073	(.037)	.857	(.009)	.891	(.010)
Food share in total exp.																
< 70%	.902	(.006)	.	.	.	.	.914	(.012)	1.320	(.027)	1.013	(.036)	.859	(.010)	.899	(.003)
70 - 80%	1.041	(.004)	1.083	(.014)	1.071	(.028)	1.044	(.013)	1.212	(.023)	1.073	(.039)	.855	(.013)	.892	(.012)
80 - 90%	.937	(.004)	1.090	(.033)	1.141	(.031)	1.093	(.020)	1.255	(.016)	1.135	(.036)	.859	(.009)	.877	(.010)
> 90%	.999	(.002)	1.107	(.015)	1.085	(.010)	1.097	(.015)	1.102	(.018)	1.109	(.033)	.	.	.866	(.00)
Rice Share in Food exp.																
< 10%	.	.	1.128	(.016)	1.183	(.019)	1.080	(.037)	1.264	(.021)	1.156	(.043)	.	.	.892	(.012)
10 - 20%	1.069	(.00)	1.060	(.014)	1.090	(.032)	.950	(.016)	1.030	(.020)	1.021	(.022)	.857	(.011)	.878	(.008)
20 - 30%	1.004	(.003)	1.100	(.020)	1.110	(.030)	1.113	(.015)	1.036	(.031)	1.020	(.035)	.856	(.009)	.891	(.007)
> 30%	.918	(.003)	.	.	.987	(.012)	.	.	.	.	1.101	(.060)	.900	(.020)	.930	(.008)
Coarse grains share in food exp.																
< 10%	.890	(.003)	.	.	1.079	(.029)	.	.	.	.	.	.	.870	(.011)	.922	(.009)
10 - 20%	.927	(.004)	1.124	(-.02)	1.065	(.031)	.915	(.009)	.	.	.947	(.037)	.854	(.010)	.866	(.003)
20 - 30%	.985	(.005)	.	.	1.168	(.033)	1.147	(.013)	.983	(.019)	1.091	(.041)	.852	(.013)	.878	(.002)
> 30%	1.073	(.002)	1.092	(.017)	1.108	(.027)	1.047	(.016)	1.239	(.021)	1.129	(.036)	.	.	.	.
Coarse grains share in cropping income																
< 50%	.951	(.003)	1.230	(.011)	1.138	(.029)	1.097	(.015)	1.261	(.022)	1.124	(.036)	.857	(.011)	890	(.008)
50 - 60%	.999	(.005)	1.148	(.017)	1.143	(.030)	.935	(.026)	1.215	(.011)	1.119	(.031)	.	.	.	.
60 - 70%	.	.	1.083	(.023)	.953	(.032)	.908	(.016)	1.225	(.027)	1.135	(.052)	.	.	.	.
> 70%	.881	(.003)	.993	(.014)	1.094	(-.01)	1.059	(.039)	.900	(.012)	1.071	(.066)	.	.	.	.
Peanut share in cropping income																
< 50%	1.003	(.004)	1.090	(.017)	1.069	(.032)	.946	(.020)	1.215	(.016)	1.152	(.035)	.857	(.011)	.890	(.008)
50 - 60%	.904	(.003)	1.272	(.011)	1.124	(.029)	.972	(.013)	1.222	(.019)	1.080	(.035)	.	.	.	.
60 - 70%	1.069	(.003)	.	.	1.118	(.021)	1.066	(.016)	1.284	(.023)	1.087	(.050)	.	.	.	.
> 70%	.937	(.002)	.	.	1.142	(.030)	1.139	(.006)	1.261	(.027)	1.077	(.037)	.	.	.	.

Notes: The values were calculated for each household using the "expected" AIDS model. Each cell contains the median of the respective values across all households.

Table 7.10 The cost-of-living and cost-of-risk indices: by income groups, consumption, and production patterns

Cost-of-risk index in parentheses	Rural zones												Urban zones			
	Sagatta		Niakhar		Colobane		Passy		Dioly		Missirah		Kaolack		Tamba	
Income quartile																
low 25%	1.150	(.005)	1.062	(.024)	1.124	(.025)	1.117	(-.01)	1.077	(.007)	1.009	(.042)	1.196	(.015)	1.132	(.003)
25 - 50%	1.175	(.002)	1.032	(.011)	1.122	(.031)	1.119	(.019)	1.032	(.019)	1.004	(.032)	1.144	(.023)	1.106	(.012)
50 - 75%	1.137	(.003)	1.055	(.019)	1.113	(.025)	1.149	(.018)	1.093	(.020)	1.037	(.023)	1.157	(.010)	1.136	(.004)
top 25%	1.140	(.005)	1.073	(.067)	1.150	(.031)	1.124	(.016)	1.067	(.019)	1.021	(.036)	1.167	(.012)	1.122	(.013)
Food share in total exp.																
< 70%	1.113	(.008)	.	.	.	.	1.123	(.015)	1.057	(.021)	1.071	(.037)	1.164	(.014)	1.113	(.004)
70 - 80%	1.145	(.005)	1.080	(.014)	1.146	(.026)	1.126	(.017)	1.082	(.022)	1.025	(.036)	1.170	(.018)	1.122	(.014)
80 - 90%	1.156	(.005)	1.069	(.031)	1.127	(.031)	1.121	(.020)	1.044	(.014)	1.009	(.032)	1.165	(.012)	1.140	(.012)
> 90%	1.135	(.002)	1.026	(.014)	1.108	(.009)	1.149	(.018)	1.043	(.017)	.998	(.033)	.	.	1.155	(.00)
Rice share in food exp.																
< 10%	.	.	1.026	(.015)	1.065	(.017)	1.063	(.034)	1.041	(.016)	.995	(.036)	.	.	1.122	(.015)
10 - 20%	1.148	(.00)	1.072	(.016)	1.113	(.035)	1.153	(.016)	1.121	(.020)	1.077	(.022)	1.167	(.015)	1.140	(.010)
20 - 30%	1.162	(.004)	1.122	(.017)	1.146	(.029)	1.124	(.016)	1.114	(.032)	1.109	(.037)	1.168	(.013)	1.122	(.008)
> 30%	1.133	(.004)	.	.	1.094	(.013)	.	.	.	.	1.107	(.057)	1.111	(.024)	1.076	(.009)
Coarse grains share in food exp.																
< 10%	1.135	(.004)	.	.	1.141	(.030)	.	.	.	.	.	.	1.149	(.015)	1.085	(.012)
10 - 20%	1.133	(.004)	1.146	(-.02)	1.146	(.032)	1.160	(.011)	.	.	1.104	(.041)	1.170	(.015)	1.155	(.004)
20 - 30%	1.165	(.005)	.	.	1.133	(.031)	1.128	(.017)	1.115	(.021)	1.098	(.039)	1.174	(.018)	1.140	(.002)
> 30%	1.166	(.002)	1.041	(.016)	1.113	(.025)	1.120	(.018)	1.055	(.020)	1.009	(.033)	.	.	.	.
Coarse grains share in cropping income																
< 50%	1.148	(.004)	1.032	(.010)	1.134	(.028)	1.126	(.016)	1.067	(.020)	1.021	(.033)	1.167	(.015)	1.124	(.008)
50 - 60%	1.089	(.005)	1.033	(.016)	1.113	(.029)	1.135	(.031)	1.035	(.010)	.996	(.024)	.	.	.	.
60 - 70%	.	.	1.069	(.023)	1.121	(.037)	1.109	(.019)	1.013	(.022)	.934	(.042)	.	.	.	.
> 70%	1.135	(.004)	1.090	(.012)	1.052	(-.01)	1.061	(.038)	1.133	(.015)	.998	(.059)	.	.	.	.
Peanut share in cropping income																
< 50%	1.156	(.005)	1.048	(.016)	1.113	(.034)	1.113	(.023)	1.035	(.012)	.987	(.033)	1.167	(.015)	1.124	(.008)
50 - 60%	1.136	(.004)	1.032	(.009)	1.108	(.027)	1.136	(.017)	1.055	(.017)	1.045	(.033)	.	.	.	.
60 - 70%	1.148	(.003)	.	.	1.114	(.022)	1.127	(.016)	1.081	(.021)	1.064	(.047)	.	.	.	.
> 70%	1.145	(.003)	.	.	1.161	(.029)	1.116	(.006)	1.093	(.023)	1.052	(.033)	.	.	.	.

Notes: The values were calculated for each household using the "expected" AIDS model. Each cell contains the median of the respective values across all households.

quartiles separately for market and nonmarket villages. In nonmarket villages, it is the upper and lowest income quartile that have the highest median net welfare gains (13.6% of the pre-devaluation average daily food budget per capita or 11 CFA and 8 CFA per capita per day for the upper and lowest quartile respectively). For the upper income quartile, this is despite having the highest median increase in the total cost of food (12%). For the lowest income quartile, the median increase in the total cost of food is only 8%. The lower-medium income quartile has still the lowest median net welfare gain (9% or 7 CFA per capita per day) despite having the lowest median increase in the total cost of food (7%). Again this is largely explained by its lower share of peanut income in total income (34.7%) compared to 35.3%, 54%, and 50.3% for the lowest, upper-medium, and upper income quartiles respectively. The median net welfare gain and increase in the total cost of food for the upper-medium income quartile are 12% of the pre-devaluation average daily food budget (or 10 CFA per capita per day) and 9.5% respectively.

In contrast to the case of the nonmarket villages, in market villages the median net welfare change for the upper income quartile is negative (-3% of the pre-devaluation average daily food budget per capita or -3.5 CFA per capita per day). The upper income quartile also has the second highest median increase in the total cost of food (13.6%) after the upper-medium quartile (14.5%). However, despite having the highest increase in the total cost of food, the upper-medium quartile has a positive median net welfare change (+5% of the pre-devaluation average daily food budget per capita or +6 CFA per capita per day), which is even higher than the one for the lower lower-medium quartile (+2.5% of the pre-devaluation average daily food budget per capita or +1.5 CFA per capita per day) which still has the lowest median increase in the total cost of food (9%). This fact is explained once again by the lower median share of peanut income in total income for the lower-medium income quartile (20.5%) compared to 45% for the lowest quartile, 32% for the upper-medium quartile, and 30% for the upper quartile. The lowest income quartile has the highest median net welfare gain (+6.6% of the pre-devaluation average daily food budget per capita or +6 CFA per capita per day) and the second lowest median increase in the total cost of food (12%).

Finally, Table 7.8, Table 7.9, and Table 7.10 give the income breakdown of the welfare impacts by zone. A noticeable fact in Table 7.8 is the fact that in Sagatta where the median net welfare change is negative, the lowest and lower-medium income quartiles do in fact have a median net increase in welfare of 1% of their respective pre-devaluation average daily food budget per capita. The tables also give cross tabulations of the welfare impacts according to the household's shares of coarse grains and rice in total food expenditures, shares of food in total expenditures, as well as shares of coarse grains and peanut incomes in the total cropping incomes.

7.5 Conclusion

To conclude the chapter we summarize our empirical findings regarding the consumption and welfare impacts of the changes in prices following the CFA devaluation.

7.5.1 Consumption impact of the CFA devaluation

In general, the post-devaluation elasticities were found to be not much different from the pre-devaluation ones in the previous chapter However, we found a slight trend of increase in the uncompensated price and budget elasticities toward more responsiveness. For example, the median uncompensated own price elasticities for the coarse grains and rice changed from -0.776 and -0.904 to respectively -0.791 and -0.911, and the medians of the means of their budget elasticities increased from 0.809 and 0.807 to respectively 0.829 and

0.823. In contrast, for the compensated elasticities we found a slight trend of reduction in the own-price responsiveness, contrasting with a slight trend of increase in the cross-price responsiveness. These trends combined with the one of the uncompensated elasticities indicate an increased impact of the relatively strong income or budget effect, which was found in the previous chapter to dominate the median household's response to relative price changes. In other words, the higher the levels of prices, the more the median household is sensitive to loss in purchasing power, and the more its price substitution ability is negatively affected by the loss in purchasing power.

Using the post-devaluation food commodity price levels, we have estimated the changes in the quantities demanded and shares of the food commodities after the change in prices following the 50% devaluation of the CFA. For the sample median household, the average daily consumption of the coarse grains and fish\meat for which prices have remained unchanged, are estimated to have increased by 19% and 26% respectively. Meanwhile, the average daily consumptions of pulses, other food, and rice for which prices have increased by 43%, 30%, and 33% respectively, are estimated to have decreased by 3%, 15%, and 5% respectively. The low decrease in rice consumption indicates that most of the 17% increase in the consumption of coarse grains is not due to substitution away from rice, but rather to an increase in purchasing power for rural households as a result of the increases in their peanut income. Indeed, for the median urban household who did not experience an increase in income the average daily coarse grains consumption are estimated to decrease by 3% despite the 16% estimated decrease in the average daily rice consumption! Meanwhile, for the median rural household, with a 19% estimated increase in the average daily coarse grains consumption, there is a mere 4% decrease in rice consumption.

Hence, the post-devaluation elasticities and the estimated changes in the quantities demanded of the food commodities after the CFA devaluation confirm the conclusion we reached after the analysis of the median pre-devaluation elasticities in the previous chapter. That is, a cereal policy based on *ceteris paribus* higher relative prices for rice compared to the ones for the coarse grains will not lead to more substitution toward the coarse grains, unless one finds a way to compensate the median household for the resulting loss in purchasing power.

7.5.2 Welfare and risk impacts of the CFA devaluation

As we can expect given the the direction and magnitude of the changes in relative prices, households in Niakhar, Colobane, Passy, Dioly and Missirah who have a relatively high share of peanut income, and a relatively low share of rice and/or a relatively high share of coarse grains in total food expenditures are the ones who have experienced a net gain in welfare. On the other hand, households in the urban zones and in Sagatta, who have no or relatively low peanut income but have a high rice shares and low coarse grains shares in total food expenditure, experienced a net welfare loss. As a percentage of the pre-devaluation average daily per capita food budget, households in Dioly have experienced the most gains in welfare, with a median net gain of +26% (or 17 CFA per capita per day), while households in the city of Kaolack have the highest lost in welfare with a median net lost of -14% (or 20 CFA per capita per day). Similarly to the total welfare impacts, as we can expect, it is in the zones with the highest shares of rice and the lowest shares of coarse grains in total food expenditures where the cost of living has risen the most. In contrast, in the low rice and high coarse grains consuming zones of Niakhar, Dioly, and Missirah, the cost of food has practically remained unchanged. The cost of food has risen the most in Kaolack (+17%), Sagatta (+15%), Colobane (+13%), and Passy (+12%). The median rise in the cost of food for Niakhar and Dioly are respectively +3% and +4%. In Missirah, there is practically 0% rise in the cost of food.

As percentage of the pre-devaluation average daily food budget per capita, the median risk premium is highest in Missirah (3.9% or 2.6 CFA per capita per day), and lowest in Sagatta (0.3% or 0.24 CFA per capita per day). Similarly, the cost of price-risk, is highest in Missirah and lowest in Sagatta. In other terms, price uncertainty increased the median cost-of-food indices in Missirah and Sagatta by respectively 0.035 points (or 4%) and 0.004 points (or 0.3%). This reflects the fact that both in terms of the respective shares of rice and coarse grains in total food expenditure and variability of coarse grains prices, Sagatta and Missirah are at the opposite extremes among the zones.

- Geographic distribution of the welfare impact

The sample empirical distribution of the total welfare changes was found to be bimodal, with one mode being in the net welfare loss range between -12.5% to -7.5% (of the pre-devaluation average daily food budget per capita), and the other mode being in the welfare gain range between +7.5% to +12.5%. On the other hand, the sample empirical distribution of the total cost-of-food index is noticeably skewed to the right (i.e., toward higher values). There is no zone where all households experienced a net welfare gain. However, except for the zone of Passy, in all the zones that showed a median net welfare gain, more than 75% of the households have experienced a net welfare gain. For the zone of Passy about 50% of the households have experienced a net welfare loss. Naturally, virtually all the households in the urban zones have experienced a net welfare loss. But, in Sagatta where the median net welfare change is negative, about 25 % of the households have experienced a net welfare gain. Second, the variability in the net welfare changes across households is higher in zones where the median net welfare change is positive than in zones where the median net welfare change is negative. By and large the empirical distributions of the total cost-of-food indices are similar across zones, especially in terms of variability.

- Distribution of the welfare impact across income groups

In the urban zones, the median net welfare loss as percentage of the pre-devaluation average daily food budget per capita and the increase in the total cost of food are greatest for the lowest income quartile (15% or 16 CFA per capita per day and 18% respectively). They are lowest for the lower-medium income quartile (12.5% or 14 CFA per capita per day and 14% respectively). For the two higher income quartiles, the corresponding figures are respectively 13% (or 15 CFA per capita per day) and 15% for the upper-medium quartile, and 13% (or 25 CFA per capita per day) and 15.5 % for the upper income quartile.

In the rural zones, the median net welfare change is positive for all income quartiles. In percentage of the pre-devaluation average daily food budget per capita, the median net welfare gain is greatest for the lowest income quartile (12% or 8 CFA per capita per day) and lowest for the lower-medium income quartile (8% or 7 CFA per capita per day). The figures for the two higher income quartiles are respectively 10% (or 10 CFA per capita per day) for the upper-medium income quartile, and 9% (or 10 CFA per capita per day) for the upper income quartile. In contrast, the median increase in the total cost of food is the lowest for the lower medium income quartile (7%), and the highest for upper income quartile (12.6%). The median increase in the total cost of food for the lowest and upper-medium income quartile are 9% and 9.5 % respectively. A noticeable fact is the fact that in Sagatta where the median net welfare change is negative, the lowest and lower-medium income quartiles do in fact have a median net increase in welfare of 1% of their respective pre-devaluation average daily food budget per capita.

CHAPTER 8

SUMMARY, POLICY IMPLICATIONS, AND CONCLUSIONS

The research conducted in this dissertation was motivated by our desire to understand and assess both analytically and empirically to what extent the Senegalese Agricultural household's food security strategy is dictated by the fact that the Senegalese government guarantees fixed producer prices for peanuts and cotton, which are known before the start of the growing season, while at the same time it maintains a fixed and stable price for imported rice, which happens to compete, on the demand side, with locally grown coarse grains, whose prices are random. At the micro level, this is an important policy issue in Senegal that is related to the debate about whether or not the coarse grains are substitutes for imported rice in consumption. It is also a key policy issue at the macroeconomic level because of the importance of imported rice in the country's balance of payments, and the government's stated goal of achieving 80% cereal self-sufficiency by the year 2000.

In short, the question we have asked is how uncertainty affects substitution among commodities. Or more precisely, to what extent the agricultural household's levels of consumption of the various food commodities are driven by differences in the levels of uncertainty in the prices of the commodities produced and consumed? In this thesis, we have provided analytical and empirical answers to this question, working within a general multivariate framework of choice under uncertainty which can handle multiple sources of uncertainty as well as "non-expected" utility preferences. The conceptual framework underlying the mathematical analysis was based on Savage's 1954 conceptual framework of choice under uncertainty which, is based on the concepts of *beliefs*, *acts*, *consequences*, and *outcomes*, and which we have conveniently modified and extended to intertemporal choices. Our framework contrasts with the prevalent univariate expected utility framework which could not be used to answer our basic question. In this research, we have derived several new theoretical results related to consumer behavior under uncertainty which we summarize first before discussing the policy implications of our empirical findings.

8.1 Summary of the Theoretical Results and Econometric Analysis

8.1.1 The theoretical results

In the process of answering our basic question, we have extended all the main results of deterministic demand theory to the case of uncertainty, including Roy's identity, Shephard's lemma, the Slutsky equation, and a comparative statics result that generalizes the well-known symmetry and negativity condition of the Slutsky matrix. We have also derived elasticity formulas which generalize the elasticity formulas of deterministic models and have essentially the same interpretations. Furthermore, in the derived comparative statics results as well as in the elasticity formulas, we have disentangled the effects due to attitudes toward risk or uncertainty from the usual ones that are due to taste substitution and income and which obtain in the absence of uncertainty. This separation was a requirement if we were going to answer our basic question. Following Kimball (1990), we have called the consumption effects due to attitudes toward risk "*precautionary* risk effects", because they measure the effects on consumption of the consumer's *ex ante* (i.e., before the resolution of uncertainty) precautionary consumption decisions taken in response to uncertainty alone. We have also extended the standard measures of welfare changes (the equivalent variation, the compensated variation, the living standard index, and the cost-of-living index) to the case of uncertainty, and show that in the uncertainty context, they still maintain their theoretical properties as well as their practical appeal as ideal measures of welfare changes. Moreover, as in the elasticity case, we have decomposed each welfare indicator into a part that is purely due to attitude toward risk, and which reduces to zero in the absence of uncertainty or when the consumer is indifferent to uncertainty, and a part that reduces to the deterministic measure in the absence of uncertainty. The risk part of the equivalent variation, which was proved to be always positive, is the risk premium measuring the sensitivity of the consumer's welfare to risk aversion (i.e., "dislike" of uncertainty). With our derivation, the risk premium gives directly in monetary units how much the consumer is willing to pay in order not to face price uncertainty. Similarly, the risk parts in the living standard and cost-of-living indices were naturally defined as the risk premium index and the cost-of-risk index respectively.

To illustrate the theoretical framework, we have worked out in detail several examples of univariate and multivariate expected utility functionals as well as multivariate "non-expected" utility ones. For each example, we have derived the corresponding demand, indirect utility, and expenditure functionals, and have calculated the various corresponding elasticities and measures of welfare change and risk attitude.[199] We have also shown through these examples, on one hand, the qualitative differences between consumer behavior with and without uncertainty, and on the other hand, the numerous possible restrictions on consumers' "beliefs", expectations, behavior, and welfare that are implied by the expected utility hypothesis. In particular, we have shown that in contrast to the deterministic case where the demand function of a consumer maximizing a univariate utility is independent of the shape of

[199] A functional is simply a function whose argument is another function. Here demand is a function of price which in turn is a function of the state of nature.

the utility function and is determined solely by the budget constraint, in the uncertainty case the demand functional corresponding to the same univariate utility not only depends on the shape of the utility functional, but also on the consumer's beliefs and expectations. The other notable fact emerging from the examples is that homogeneous of degree one expected utility functions imply that behavior and welfare are independent of beliefs and expectations, and that some nonlinear functions of random prices must have a degenerate probability distribution (i.e., must be nonrandom). The restrictiveness of the expected utility hypothesis is by now well documented by the experimental evidence. The analytical results provided through some of these examples - and which are new to the best of our knowledge - shed new light on the restrictive implications of the expected utility hypothesis.

We have also used the duality theory under uncertainty developed in the thesis to show how to generate "flexible" and integrable demand systems in the context of price uncertainty. In particular, we have extended the recently proposed QUAIDS model to uncertainty. The QUAIDS model is the rank three integrable quadratic extension of the popular AIDS model of Deaton and Muellbauer (1980a), proposed recently by Banks, Blundell, and Lewbel (1994). We have derived three uncertainty versions of the QUAIDS model which have the same type of generality and integrability as the deterministic QUAIDS one. One of the versions was defined as the "expected" QUAIDS, and the two other versions as the "non-expected" QUAIDS versions 1 and 2 respectively. For the three uncertainty versions we have derived the respective formulas for computing the precautionary risk and nonrisk elasticities, the indicators of total welfare changes, the risk premiums, and cost-of-risk indices. We have also discussed the econometric estimation of the QUAIDS model under uncertainty. In the process, we have derived a simple method for estimating the "nuisance" parameter α_0 associated with the QUAIDS model which seemed to be difficult to identify in previous work.

8.1.2 The econometric analysis

The "expected" AIDS model was estimated using biweekly 24-hour recall food consumption data collected by the Institut Sénégalais de Recherches Agricoles (ISRA) and the International Food Policy Research Institute (IFPRI) two-year survey in Senegal. For each household in the sample we estimated the risk and nonrisk elasticity matrices, as well as estimates of the consumption, welfare, and risk impacts of the recent devaluation of the CFA currency. Our empirical findings with regards to household consumption behavior and risk attitudes in Senegal as well as their policy implications will be discussed in the next section. Meanwhile, we have also estimated the deterministic AIDS model so as to compare the parameter, elasticities, and welfare indicators estimates from the two models.

For the parameter estimates, our finding was that by and large the two models' estimated coefficients for the household characteristics variables and budget were very close, while the ones for prices tended to be different. However, using Vuong's (1989) model selection test for overlapping models we concluded that given the data, the two models were not statistically distinguishable. We also noted that this conclusion was not a surprise, because, although they are different from a economic theoretic view, in terms of econometric estimation the two models differ only in the AIDS budget deflator. For the elasticity estimates, we found that the set of elasticities from the deterministic AIDS model tended to be different from the comparable "expected" AIDS ones, although they were not too far apart. For example, out of the 25 sample median elasticities from the deterministic AIDS uncompensated matrix, 6 did not have the same sign as their comparable analogues from the "expected" AIDS model. The biggest differences between the two models were in the estimates of the indicators of welfare change. The comparable estimates from the two models were significantly different in magnitude, and most of them did not have the same signs. These empirical findings confirm the conclusion that was already apparent in the theoretical expressions of the elasticities and welfare indicators. That is, when price uncertainty is relevant, there are potential biases in elasticities and indicators of welfare changes estimated from deterministic models.

In next section, we discuss our empirical findings in terms of consumer behavior in Senegal and the consumption and welfare impacts of the change in prices following the CFA devaluation, and their policy implications.

8.2 Summary of the empirical findings and their policy implications

The empirical findings in the dissertation have some policy implications with respect to the Senegalese government food policy. The main policy contribution of the dissertation, however, is to provide demand elasticity estimates for Senegal that are disaggregated by geographic location and income group. One of the main difficulties in doing policy analysis in Senegal has always been the lack of reliable demand and supply elasticity estimates (Kramer, 1984; Braverman and Hammer,1986; and Kite, 1993). The fact that the lack of reliable elasticity estimates has been a constraining factor for policy analysis is understandable because a large part of the debated policy reforms in Senegal during the 1980s turned around the government's cereal marketing and pricing policies (see Berg, 1990). In fact, one of the main goals of the ISRA/IFPRI project was to provide demand and supply elasticities for Senegal (Kelly et al., 1993). It is hoped that the household-level elasticity estimates provided in the dissertation will fill a gap and be useful for policy analysts dealing with food policy issues in Senegal. We particularly hope that the level of disaggregation will be useful in assessing and monitoring the impacts of the pending liberalization of the rice market for the different regions of Senegal. Indeed, one important policy issue of concern to the Senegalese government is how the liberalization will impact the different regions of Senegal (Kite, 1993). In what follows, we summarize the conclusions derived from our interpretations of the signs and relative magnitudes of the estimated elasticities and from our estimates of the consumption and welfare impacts of the change in prices following the CFA devaluation.

8.2.1 Empirical evidence on the consumption related issues

8.2.1.1 The total and nonrisk elasticities estimates

Based on the pre-devaluation price levels, we found that all the food commodities (including rice and the coarse grains) are substitutes except for fish\meat and the other food group for which the cross-price elasticity is practically zero. Perhaps the most important finding, however, was the relatively strong income effect that significantly limits the ability of the median household to substitute across commodities. Indeed, we found that most of the own-price response of the median household is due to a relatively strong income or budget effect. At same time, most of the median household potential price substitution effect is wiped out by the same relatively strong income or budget effect. This relatively strong income effect is found for all commodities. For example, a 1% *ceteris paribus* expected increase in the price of rice could have increased the median household's average daily coarse grains consumption by as much as 0.44% if there were no reduction in purchasing power. But, because of the resulting loss in purchasing power the increase in the average daily coarse grains consumption will only be a mere 0.07%! Meanwhile, 44% of the 0.9% resulting decrease in the average daily rice consumption will be due to a loss in purchasing power, not to substitution toward other food commodities.

This pattern of response to price changes of the median household was confirmed by the post-devaluation elasticities and the estimated changes in the quantities demanded of the food commodities calculated using the prices prevailing after the 50% devaluation of the CFA francs. Indeed, we found the post-devaluation elasticities to be not much different from the pre-devaluation ones. There was only a slight trend of increase in the uncompensated price and budget elasticities toward more responsiveness. For example, the median uncompensated own price elasticities for the coarse grains and rice changed from -0.776 and -0.904 to respectively -0.791 and -0.911, and the medians of the means of their food budget elasticities increased from 0.809 and 0.807 to respectively 0.829 and 0.823. In contrast, for the compensated elasticities we found a slight trend of reduction in the own-price responsiveness, contrasting with a slight trend of increase in the cross-price responsiveness. These trends, combined with the one of the uncompensated elasticities, indicate an increased impact of the relatively strong pre-devaluation income effect, which was found to dominate the median household's response to relative price changes. Thus, the higher the levels of prices, the more the median household is sensitive to loss in purchasing power, and the more its price substitution ability is negatively affected by the loss in purchasing power.

Moreover, using the post-devaluation food commodity price levels, we estimated the changes in the quantities demanded and shares of the food commodities after the change in prices following the 50% devaluation of the CFA. For the sample median household, the average daily consumption of the coarse grains and fish\meat for which prices have remained unchanged, are estimated to have increased by 19% and 26% respectively. Meanwhile, the average daily consumption of pulses, other food, and rice for which prices have increased by

43%, 30%, and 33% respectively, is estimated to have decreased by 3%, 15%, and 5% respectively. The low decrease in rice consumption indicates that most of the 17% increase in the consumption of coarse grains is not due to substitution away from rice, but rather to an increase in purchasing power for rural households as a result of the increases in their peanut income. Indeed, for the median urban household which did not experience an increase in income, the average daily coarse grains consumption is estimated to decrease by 3% despite the 16% estimated decrease in the average daily rice consumption! Meanwhile, for the median rural household, with a 19% estimated increase in the average daily coarse grains consumption, there is a mere 4% decrease in rice consumption. Thus, the impacts of the nonmarginal change in prices following the CFA devaluation confirm again the importance of the income effects found in the elasticity estimates.

Differences in consumption patterns between rural and urban households

We found that the empirical distribution of the coarse grains own-price compensated elasticities of urban households is quite far apart from the ones of rural households in both the market and nonmarket villages. On the other hand, the respective empirical distributions for the rice own-price and rice-coarse grains cross-price compensated elasticities were by and large closer. However, in all cases the variability in responses is always lower for urban households (i.e., urban households tend to behave similarly to rural households).

Differences in consumption patterns across income groups

When it comes to difference in price responsiveness among income quartiles, rural households show more similarities than urban households. However, although coarse grains and rice are substitutes for all rural households, those in the lowest income quartile have slightly higher cross-price compensated responses. For urban households, the empirical distribution of the rice-coarse grains cross-price compensated elasticity is noticeably skewed to the left (i.e., toward more responsiveness) for the lower-medium income quartile.

Policy implications

The relatively strong income effects found in the pre- and post-devaluation elasticity estimates, and in the estimated changes in the quantities demanded of the food commodities after the CFA devaluation lead to an important policy implication with respect to the Senegalese government's cereal pricing policy, which is presently designed to encourage the domestic production of rice and to induce Senegalese consumers to switch to locally produced coarse grains by progressively increasing the relative price of imported rice. A debate regarding the economic rationale and effectiveness of the cereal pricing policy has been going on since the early 1980s among policy makers and researchers working on Senegalese issues (see, Berg, 1990, and Kite, 1993). Opponents of the policy have questioned its economic rationale given its high cost to the Senegalese consumers and economy. They have also questioned its effectiveness since it has so far given no sign of curbing rice demand or boosting the consumption of the coarse grains. Its only achievement, they point out, is the increased local production of irrigated rice in the Fleuve region at a very high cost in terms of subsidy and resource allocation.[200] Hence, they argue, instead of trying to produce its own rice at a very high cost to the Senegalese consumer and economy, Senegal should take advantage of the cheap imported rice and concentrate on the production of crops like peanut, where it has known comparative advantage. Advocates of the cereal policy justify it by pointing to the need for Senegal to reduce its chronic balance of payments deficit (for which the importation of rice is a main component), and develop the domestic market for the coarse

[200] The Domestic Resource Cost (DRC) of producing rice in Senegal has been estimated to be in the range of 2.3 to 6.8 (Kite, 1993). Kite (1993) also estimated the annual resource misallocation cost involved in the development of irrigated rice to be in the range between $25 and $70 million.

grains as part of its agricultural development strategy and goal of 80% food self-sufficiency by the year 2000.

Advocates of the cereal policy have always argued that the reason why it has not worked yet is that the relative price of imported rice has not been high enough. Opponents of the policy have argued that the main reason for its failure lies in the so-called "nonprice" factors (such as urbanization, cultural practices, and convenience and cost of preparation).[201] The empirical evidence provided by the relative magnitudes of the elasticity estimates indicates another reason why the cereal policy has not succeeded in decreasing the consumption of imported rice and increasing the one for the coarse grains. That is, the relatively important loss of purchasing power resulting from higher prices for rice, significantly reduces the ability of the poor Senegalese household to substitute toward the other food products. In other words, a cereal policy based on a *ceteris paribus* higher relative prices for rice compared to the ones for the coarse grains is not going to lead to more substitution toward the coarse grains, unless one finds a way to compensate the median household for the resulting loss in purchasing power. Hence, since income compensation of consumers is out of the question, the Senegalese government should reassess its cereal policy and focus on other nonprice means for encouraging consumers to decrease their rice consumption and switch to the local cereals. Then comes the question of what type of nonprice food policy can be suggested

An important step toward the direction of a nonprice-based cereal policy is the growing interest, both from the Senegalese government and the donor community, in the development of industrial and semi-industrial processing technologies that would make the

[201] One nonprice factor often cited as the main reason for the limited substitutability of rice and the coarse grains, is the urban Senegalese cultural practice of almost always eating rice at noon, and the coarse grains only in evening meals.

coarse grains more attractive to consumers in terms of convenience in preparation (USAID/SENEGAL, 1991). However, except for a limited consumer acceptance test program, much of this effort is concentrated on the technological aspect of the problem and in reducing the coarse grains' processing costs compared to imported rice. The program should be more consumer oriented by trying to influence consumer preferences through educational programs on alternative culinary practices that rely on new and creative use of coarse grains in the Senegalese diet.[202] Moreover, the focus of the Senegalese food policy on the cereals is too narrow both from a policy and nutritional perspective. The development of other local food products (vegetables, livestock products, and fruits, etc.) should be vigorously pursued to promote a much more diversified diet for the Senegalese consumer.[203] The use of the coarse grains in livestock feeding should also be promoted.

8.2.1.2 The *precautionary* risk elasticities

We found that in general the price-risk effects dampen the median household's overall ability to substitute across commodities. However, the overall magnitudes of the precautionary price-risk elasticities are so small, compared to their nonrisk counterparts, that they can be considered as negligible. The empirical finding that the effects of price

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[202] ITA, the Senegalese Food Technology Institute, has been developing alternatives uses of the coarse grains. But, so far no vigorous program to educate Senegalese consumers on these alternative culinary practices has been developed. The impact of changes in preference on consumption is not usually dealt with in quantitative economic analysis. But, as the examples in chapter 3 show, a small change in preference can lead to nonmarginal change in consumption. Hence, policy designed to influence consumers preferences can be more effective in changing demand than policies based on change in relative prices.

[203] The ISRA/IFPRI survey revealed that the share of cereals in caloric intake ranged from 65% for households in the urban zones and Northern Peanut Basin to more than 80% in the Southwest and Southeast Peanut Basin (Kelly et al., 1993). Goetz (1990) reports some empirical evidence which shows that rural households would diversify their diet if they were given the opportunity to do so (see also, Abt Associates, 1985, for a similar finding in Dakar).

uncertainty on food consumption in Senegal are negligible has nonetheless some policy value in the sense that it gives a justification for the Senegalese government to deemphasize trying to stabilize coarse grain prices as it did in the past. Moreover, given the recent trend of promoting village cereal banks in Senegal, from a policy perspective, the empirical evidence suggests that they can be justified only if they are intended not to deal with price uncertainty, but rather to solve the liquidity constraints facing many farmers, which force them to sell their food crops at low prices at harvest and buy back later when prices are high. However, given the subsidies and hidden opportunity costs of setting and operating these cereals banks, there may be a better and more cost efficient way for solving the liquidity constraints facing farmers (by providing credit that is not tied to production activities for example).[204] Hence, the economic rationale behind the promotion of these cereal banks should be questioned; especially considering the fact that they impede on the long-term development of a well working and efficient cereal marketing system which would allow farmers to specialize in production.

8.2.2 Median estimates of the welfare and risk impacts of the CFA devaluation

As we can expect given the direction and magnitude of the changes in relative prices, households in the rural zones with a relatively high share of peanut income, and a relatively low share of rice and/or a relatively high share of coarse grains in total food expenditures (Niakhar, Colobane, Passy, Dioly and Missirah) are the ones who have experienced a net gain in welfare. On the other hand, households in the urban zones and in Sagatta, who have no or

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[204] As of 1990, a total of 571 cereal banks have been created through the help of non-governmental organizations and international donor organizations which provide grants and subsidize some of the operating costs. However, 64% of them are operating at an average of 10% capacity (USAID/SENEGAL, 1991). This low utilization rate may be an indication that they are trying to solve a problem that, practically speaking, is not really very important for farmers.

relatively low peanut income but have a high rice share and low coarse grains share in total food expenditure experienced a net welfare loss. As a percentage of the pre-devaluation average daily per capita food budget, households in Dioly have experienced the most gains in welfare, with a median net gain of +26% (or 17 CFA per capita per day), while households in the city of Kaolack have the highest loss in welfare with a median net loss of -14% (or 20 CFA per capita per day). Similarly, it is in the zones with the highest shares of rice and the lowest shares of coarse grains in total food expenditures where the cost of food has risen the most. In contrast, in the low rice and high coarse grains consuming zones of Niakhar, Dioly, and Missirah, the cost of food has remained practically unchanged. The cost of food has risen the most in Kaolack (+17%), Sagatta (+15%), Colobane (+13%), and Passy (+12%). The median rise in the cost of food for Niakhar and Dioly is respectively+3% and +4%. In Missirah, there is practically 0% rise in the cost of food.

As a percentage of the pre-devaluation average daily food budget per capita, the median risk premium was highest in Missirah (3.9% or 2.6 CFA per capita per day), and lowest in Sagatta (0.3% or 0.24 CFA per capita per day). If there were no price uncertainty, the median net welfare gains - as percentage of the pre-devaluation average daily food budgets per capita - would have been 13% instead of 9% in Niakhar, 15% instead of 11.5% in Colobane, 7% instead of 6% in Passy, 28% instead of 26% in Dioly, and 19% instead of 15% in Missirah. Meanwhile, in Sagatta, Kaolack, and Tamba, price uncertainty has further increased the median net welfare losses (as a percentage of the pre-devaluation average daily food budget per capita) from 1.3% to 1.5%, 13% to 14%, and 10.8% to 11.2% respectively. Hence, on average, price uncertainty added 2 percentage points to the total welfare change.

Similarly, the cost of price risk was highest in Missirah and lowest in Sagatta. In other terms, price uncertainty increased the median cost-of-food indices in Missirah and

Sagatta by respectively 0.035 points (or 4%) and 0.004 points (or 0.3%). This reflects the fact that in terms of **a)** the respective shares of rice and coarse grains in total food expenditure and **b)** variability of coarse grains prices, Sagatta and Missirah are at the opposite extremes among the zones.

Empirical distributions of the estimated welfare impacts

The sample empirical distribution of the total welfare changes was found to be bimodal, with one mode being in the net welfare loss range between -12.5% to -7.5% (of the pre-devaluation average daily food budget per capita), and the other mode being in the welfare gain range between +7.5% to +12.5%. On the other hand, the sample empirical distribution of the total cost-of-food index is noticeably skewed to the right (i.e., toward higher values). There is no zone where all households experienced a net welfare gain. However, except for the zone of Passy, in all the zones that showed a median net welfare gain, more than 75% of the households have experienced a net welfare gain. For the zone of Passy about 50% of the households have experienced a net welfare loss. Naturally, virtually all the households in the urban zones have experienced a net welfare loss. But, in Sagatta where the median net welfare change is negative, about 25 % of the households have experienced a net welfare gain. Second, the variability in the net welfare changes across households is higher in zones where the median net welfare change is positive than in zones where the median net welfare change is negative. By and large the empirical distributions of the total cost-of-food indices are similar across zones, especially in terms of variability.

Distribution of the welfare impact across income groups

In the urban zones, the median net welfare loss as a percentage of the pre-devaluation average daily food budget per capita and the increase in the total cost of food were greatest for the lowest income quartile (15% or 16 CFA per capita per day and 18% respectively).

They were lowest for the lower-medium income quartile (12.5% or 14 CFA per capita per day and 14% respectively). For the two higher income quartiles, the corresponding figures are respectively 13% (or 15 CFA per capita per day) and 15% for the upper-medium quartile, and 13% (or 25 CFA per capita per day) and 15.5 % for the upper income quartile.

In the rural zones, the median net welfare change was found to be positive for all income quartiles. As a percentage of the pre-devaluation average daily food budget per capita, the median net welfare gain was greatest for the lowest income quartile (12% or 8 CFA per capita per day) and lowest for the lower-medium income quartile (8% or 7 CFA per capita per day). The figures for the two higher income quartiles are respectively 10% (or 10 CFA per capita per day) for the upper-medium income quartile, and 9% (or 10 CFA per capita per day) for the upper income quartile. In contrast, the median increase in the total cost of food is the lowest for the lower medium income quartile (7%), and the highest for upper income quartile (12.6%). The median increases in the total cost of food for the lowest and upper-medium income quartile were 9% and 9.5 % respectively. It is noteworthy that in Sagatta where the median net welfare change was negative, the lowest and lower-medium income quartiles did in fact have a median net increase in welfare of 1% of their respective pre-devaluation average daily food budget per capita.

Policy implications

The debate on the economic benefits and cost of a devaluation of the CFA franc has been going on for almost a decade now. Now that the devaluation has taken place, the debate has shifted to questions about what policies should be promoted in order to take advantage of the CFA devaluation and alleviate its short term impacts on the poor. The estimates of the size and distribution of the welfare impacts given in this dissertation enable one to draw a profile of particular households that need to be targeted for help in mitigating the short term negative welfare impacts of the CFA devaluation. Whether or not the negative welfare impacts are large enough to warrant designing targeting programs to help the vulnerable households is a policy judgment that has to be made taking into account the effectiveness of targeting programs in reaching the needy, and their administrative costs.

However, focussing too much on the short-term negative welfare impacts of the devaluation may, at the end, distract from designing and implementing the economic policy reforms that should go with the devaluation in order for it to have a long-term positive impact on the economic well-being of Senegalese households. Even the lower income quartile households in the urban zones who, from our estimates, seem to be the ones who have been the most hurt in the short run by the devaluation, may be better helped in the long run by macroeconomic policy reforms that encourage job creation rather than targeted food distribution. Indeed, finding jobs for unemployed household members is likely to have more impact for the well being of the poor household than food handouts. The ISRA/IFPRI sample shows that urban households have an average adult population of 5, of which, according to the EP survey (République du Sénégal, 1992), two at least are likely to be unemployed. Moreover, the Senegalese government is increasingly under pressure from the World Bank to significantly reduce its reliance on the high taxation of rice and gasoline as the principal means for financing its budget. However, given the bad state of its finances it is presently unrealistic to expect the removal of the taxation on both products at the same time. Although the price of imported rice is directly related to the food security of the poor, however, if there is a choice between the two products, the government should choose to significantly reduce if not eliminate taxation on gasoline because of economic growth linkages and potential for job creation. Indeed, gasoline is a necessary input for many economic activities that benefit the poor, including the transport of food from the rural areas. On the other hand, for imported rice, there is always the availability of cheap coarse grains and other local food products, if policies are designed to educate urban consumers on alternative culinary practices that substitute local food products for rice. Along the same lines, another policy reform that would help the urban poor benefit from the devaluation in the long run, is to stimulate the housing construction sector by removing the present overpricing of cement. Indeed the housing construction and related sectors constitute a potential source of jobs for the large pool of unskilled labors making up most of the poor urban households. Moreover, unlike upper income skilled laborers, unskilled laborers are more likely to spend their earnings on locally produced commodities, thus generating more employment through intersectoral linkages.

For rural households that are negatively affected by the short-term effects of the CFA devaluation, our welfare impact estimates show that they are mostly located in the Northern Peanut Basin. Moreover, the estimates show that the net welfare impact is positively correlated with peanut production. Hence, the best way to help them is to give them access to peanut seed and other related inputs. This is not a trivial issue because the ISRA survey shows that poor households in the Peanut Basin did not benefit from the better 1989 rainy season following the 1988 drought, compared to the higher income households (see Kelly et al., and the tables in the data appendix). It is likely that these poor households do not have good access to inputs in general, and to peanut seed in particular, the lack of which has been a major source of complaints from farmers since the adoption of the New Agricultural Policy. Hence, the better peanut producer prices following the CFA devaluation may not benefit the poor household in the Peanut Basin unless a conscious and targeted effort is made to ensure sure it has access to the necessary inputs. At the end, what would help rural households and the agricultural sector take the most advantage of the change in relative prices brought about by the CFA devaluation, is for the Senegalese government to establish an efficient rural credit system that would enable farmers to have timely access to agricultural inputs and contribute to solving their consumption-related liquidity constraints.

8.2.3 Other policy-relevant research questions not addressed

Finally, before moving to the discussion of the theoretical limitations of our empirical analysis, and of directions for future research toward improving it, we will mention two policy-relevant research questions that we originally proposed to shed light on, but which we have not yet dealt with.

As stated in the introduction of the dissertation, one of the policy questions that we were interested in was how the liberalization of the rice market (by allowing its price to fluctuate randomly as do the ones for the coarse grains) would affect the Senegalese agricultural household's choice of crop mix and level of consumption of the various food commodities. This is a timely policy-relevant research question given the pending decision of the Senegalese government to liberalize the rice market (Kite, 1993, and République du Sénégal, 1994).

The comparative statics and elasticity formulas derived in chapter 2 give a theoretical answer to the part of the question related to consumption. For a simulated empirical answer, more data related to the price of imported rice are needed (since in our price series the price of rice is constant in most zones during the period of the survey, most of the estimated precautionary risk elasticities that depended on its variation were zero). To simulate the consumption impacts of the liberalization of the rice market on the households, one can use the variations in the world market prices (or Dakar C.I.F prices). That is, for each zone the price of rice can be assumed to be just $t\%$ greater than the Dakar C.I.F price, where t is the per unit tariff plus marketing and transport costs from Dakar to the zone. Monthly C.I.F price series for imported rice can be obtained from C.P.S.P. For the transport and marketing costs, Kite (1993) provides estimates for the different regions of Senegal. Once the random price series for rice are generated for each zone, the calculations can proceed essentially in the same way as for the evaluation of the consumption and welfare impact of the change in prices following the devaluation, with the difference that here we have a random change in prices, instead of a deterministic change in prices. It is perhaps worth noting that from the (theoretical) comparative statics results in chapter 2, the sign of the precautionary price-risk elasticities was ambiguous. Hence, to even know the direction of the impact of the liberalization of the rice market on rice consumption, an empirical analysis is needed. But, if rice is like the other commodities in our empirical analysis, which all had small but positive precautionary price-risk elasticities, then one should expect the liberalization of the rice market to increase the consumption of rice by a negligible amount.[205]

On the other hand, the production-related part in our basic question as well as in the rice liberalization question, which refers to the agricultural household's choice of crops mix, has not been theoretically answered because we have not yet treated producer behavior within our static framework of choice under uncertainty. However, except for minor conceptual issues that need to be settled, the extension of our theoretical results on consumer behavior under uncertainty to their production equivalents (Hotelling's lemma, and the comparative statics related to output and input uses) is similar to the way results of deterministic consumer theory translate to the ones in deterministic producer theory. Hence, once the comparative statics results related to static producer behavior under uncertainty are derived, we would be

[205] One can see this by noting that when the price of rice is uncertain consumers are likely to take precautionary measures by buying in bulk and stocking more often than presently.

able to answer (at least theoretically) the part of our basic question related to how differential price uncertainty affects choice of crop mix. In fact, it is in the production side where the payoffs of our mathematical framework in terms of uncertainty-related policy implications will be the most apparent. Indeed, the framework enables one to study crop diversification strategies in response to price and weather uncertainties, and to derive empirically estimable measures of crop-specific risk premiums. That is, in the same way that we have given analytical and empirical answers to the question of how much a consumer is willing to pay to eliminate price uncertainty in the consumption side, with this framework it is possible to give an answer to the question of how much a farmer is willing to pay for crop insurance. However, an empirical analysis will be impossible based on the ISRA/IFPRI data, since it contains only two years of production data. Time series aggregate production and price data may be the only thing available here if one is to discard the uncertainty-irrelevant cross-sectional variations contained in household level production data. Still, another area where our mathematical framework can be usefully applied is in evaluation of the behavioral and welfare impacts of price stabilization schemes.

8.3 Limitations of the Empirical Analysis and Direction of Future Research

Because of time constraints and the computational difficulties we encountered, we did not estimate our preferred "non-expected" QUAIDS version. Instead, we have estimated only the "expected" and deterministic AIDS models which are rank two demand systems. There is ample empirical evidence that the AIDS functional form is restrictive, especially when it comes to the budget or income responses of households (Blundell et al., 1993, and Banks et. al., 1994). This is very clear from the formula for the AIDS budget elasticities, which depend on the level of total expenditure only through the respective commodity shares.

Moreover, Banks et al., (1994) found significant differences between the welfare impacts of a sales tax estimated using the AIDS model and ones estimated using its rank three quadratic extension. This brings to mind the question of whether and how our empirical findings would change if we had estimated the QUAIDS functional forms. We expect the QUAIDS estimates to be possibly different in magnitude. However, the differences should not change the general conclusions drawn from our empirical results because the "expected" and deterministic AIDS functional forms are special cases of their respective QUAIDS extensions. On the other hand, results from either the "non-expected" AIDS or the "non-expected" QUAIDS are not only expected to be different in magnitudes from both the deterministic and the "expected" AIDS ones, but also may change some of our general conclusions. This is especially true for the conclusions related to the statistical indistinguishability between the estimated uncertainty and deterministic models given the data, and the small magnitude of the precautionary price-risk elasticities. Indeed, as it is clear from the theoretical expressions, the "non-expected" QUAIDS is very different from both the deterministic and "expected" QUAIDS, both from theoretical economic and econometric estimation considerations. Hence, the estimation of the "non-expected" QUAIDS is our next step in this research.

The other main limitation of our empirical analysis is the fact that we have estimated a static model - limited to the food group only - despite the fact that the biweekly food consumption data was generated by households making intertemporal and interrelated food and nonfood consumption as well as production choices. The estimation of a static demand system limited to the food group only was dictated by two factors. The first factor is the poor quality of the nonfood prices series. The second is the fact that the production, saving, and transaction data which need to be included in a full intertemporal household model require further cleaning before they are usable in a econometric analysis. The estimation of our static model can only be theoretically justified by the assumptions of separability between consumption and production decisions, intertemporal separability, and weak separability between the food group and the nonfood commodities.

However, even if the production, saving, and transaction data were ready for an econometric analysis, the existing models of intertemporal choices under uncertainty which can be empirically implemented are all univariate (i.e., one commodity) models which are inappropriate for the purpose of studying substitutions across both time and commodities. Moreover, except the ones based on Epstein and Zin (1989), most of the existing empirically implementable intertemporal choice models are based on expected utility and time additivity. They are also all subject to the Hindy, Huang, and Kreps (1992) criticism related to how they unrealistically restrict intertemporal substitution in consumption. In particular, they all imply that consumptions at nearby dates are perfect nonsubstitutes.

Although in their present form our theoretical results do not make explicit references to intertemporal choices, they can be extended to intertemporal choice as outlined in chapter 4. In particular, the general form of the uncertainty versions of the QUAIDS derived here for a static model of choice under uncertainty is still valid in a intertemporal choice context, as long as we follow Hindy Huang and Kreps (1992) and view consumers as choosing among alternative *cumulative consumption patterns* rather than among alternative consumption rates as it is in standard practice. Hence, extending our results to intertemporal choice along the lines of chapter 4 is another area of our future research.

APPENDICES

APPENDIX 1 SUMMARY OF THE THEORETICAL RESULTS

ITEM	Without Uncertainty	With Uncertainty
1- SET UP		
Consumption (or choice) space	the Euclidean space $\mathbf{R}^n$	the space of vector valued random variables with finite variances $L_2(\Omega, \mathbf{R}^n)$ (Ω is the set of possible states of nature)
Price space	the Euclidean space $\mathbf{R}^n$	the space of vector valued random variables with finite variances $L_2(\Omega, \mathbf{R}^n)$
Scalar (or dot) product	$x \cdot y = \sum_{i=1}^{n} x_i y_i$	$\langle x, y \rangle = \int_\Omega x(\omega) \cdot y(\omega) d\mathcal{P}(\omega) = E(x \cdot y) = \sum_{i=1}^{n} E(x_i y_i)$ (where $\mathcal{P}$ is the consumer's *subjective* probability measure defined on the set of events associated with Ω, and E is the expectation operator)
Metric	the Euclidean norm: $\lVert x \rVert^2 = \sum_{i=1}^{n} x_i^2$	the L_2 norm: $\lVert x \rVert_{L_2}^2 = \int_\Omega \lVert x(\omega) \rVert^2 d\mathcal{P}(\omega) = E(\lVert x \rVert^2) = \sum_{i=1}^{n} E x_i^2$
Assumptions on Preference and Behavior	Complete, reflexive, transitive, continuous, monotone, and convex preference; and preference maximization	Complete, reflexive, transitive, continuous, monotone, and convex preference; and preference maximization
Budget constraint	$p \cdot x = \sum_{i=1}^{n} p_i x_i = m$ (where m is total expenditure)	$\langle p, x \rangle = \int_\Omega p(\omega) \cdot x(\omega) d\mathcal{P}(\omega) = m$ or equivalently, $E(p \cdot x) = \sum_{i=1}^{n} E(p_i x_i) = m$ (where m is the non-random expected total expenditure. Random total expenditure can incorporated into the analysis)

ITEM	Without Uncertainty	With Uncertainty
2- Existence Results		
Utility function	$u: \mathbb{R}^n_+ \to \mathbb{R}_+$	$U: L_2(\Omega, \mathbb{R}^n)_+ \to \mathbb{R}_+$
Marshallian Demand	$x: \mathbb{R}^n_{++} \times \mathbb{R}_{++} \to \mathbb{R}^n_{++}$ $(p,m) \to x(p,m) = \operatorname{argmax}\{u(x): p \cdot x = m\}$	$x: L_2(\Omega, \mathbb{R}^n)_{++} \times \mathbb{R}_{++} \to L_2(\Omega, \mathbb{R}^n)_{++}$ $(p,m) \to x(p,m) = \operatorname{argmax}\{U(x): \langle p, x \rangle = m\}$
Hicksian Demand	$h: \mathbb{R}^n_{++} \times \mathbb{R}_{++} \to \mathbb{R}^n_{++}$ $(p,u) \to h(p,u) = \operatorname{argmin}\{p \cdot x: u(x) \geq u\}$	$h: L_2(\Omega, \mathbb{R}^n)_{++} \times \mathbb{R}_{++} \to L_2(\Omega, \mathbb{R}^n)_{++}$ $(p,u) \to h(p,u) = \operatorname{argmin}\{\langle p, x \rangle: U(x) \geq u\}$
Indirect Utility	$v: \mathbb{R}^n_{++} \times \mathbb{R}_{++} \to \mathbb{R}$ $(p,m) \to v(p,m) = \max_{s.t.\ p \cdot x = m} u(x)$	$V: L_2(\Omega, \mathbb{R}^n)_{++} \times \mathbb{R}_{++} \to \mathbb{R}$ $(p,m) \to V(p,m) = \max_{s.t.\ \langle p, x \rangle = m} U(x)$
Expenditure function	$e: \mathbb{R}^n_{++} \times \mathbb{R}_{++} \to \mathbb{R}$ $(p,u) \to e(p,u) = \min_{s.t.\ u(x) \geq u} p \cdot x$	$E: L_2(\Omega, \mathbb{R}^n)_{++} \times \mathbb{R}_{++} \to \mathbb{R}$ $(p,u) \to E(p,u) = \min_{s.t.\ U(x) \geq u} \langle p, x \rangle$
Compensation function	$\mu: \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \times \mathbb{R}_{++} \to \mathbb{R}$ $(p,q,m) \to \mu(p,q,m) = e(p, v(q,m))$	$\mu: L_2(\Omega, \mathbb{R}^n)_{++} \times L_2(\Omega, \mathbb{R}^n)_{++} \times \mathbb{R}_{++} \to \mathbb{R}$ $(p,q,m) \to \mu(p,q,m) = E(p, V(q,m))$

ITEM	Without Uncertainty	With Uncertainty
3 -Global Properties	*wherever they are defined*	*wherever they are defined*
Demand	x (.,.) and h(.,.) are continuous functions	x (.,.) and h(.,.) are continuous functional
Indirect Utility	v(.,.) is continuous and homogeneous of degree 0; nonincreasing and quasi-convex in prices; and nondecreasing in income.	V(.,.) is continuous and homogeneous of degree 0; monotone nonincreasing and quasi-convex in prices; and nondecreasing in income.
Expenditure	e(.,.) is continuous; homogeneous of degree 1, nondecreasing and concave in prices, and nondecreasing in utility.	E(.,.) is continuous; homogeneous of degree 1, monotone nondecreasing and concave in prices, and nondecreasing in utility.
4 - Generic Properties	*"almost everywhere" they are defined* *(formally, except possibly in a set of zero Lebesgue measure).*	*"almost everywhere" they are defined* *(formally, except possibly in a countable union of closed sets with empty interiors).*
Differentiability	v(.,.) and e(.,.) are differentiable functions	V(.,.) and E(.,.) are differentiable functional
Roy's identity	$x_i(p,m) = -\dfrac{\frac{\partial v(p,m)}{\partial p_i}}{\frac{\partial v(p,m)}{\partial m}} \quad i=1,\ldots,n$	$x_i(p,m) = -\dfrac{\frac{\partial V(p,m)}{\partial p_i}}{\frac{\partial V(p,m)}{\partial m}} \quad i=1,\ldots,n$
Shephard's lemma	$h_i(p,u) = \dfrac{\partial e(p,u)}{\partial p_i} \quad i=1,\ldots,n$	$h_i(p,u) = \dfrac{\partial E(p,u)}{\partial p_i} \quad i=1,\ldots,n$

ITEM	Without Uncertainty	With Uncertainty
5 - Comparative Statics Results		
5.1- Global comparative statics	*differentiability of the demand function is not needed*	*differentiability of the demand functional is not needed*
(i)- Monotonicity property	The Hicksian demand is a monotone operator: $(h(q,u) - h(p,u)) \cdot (q - p) \leq 0$	The Hicksian demand is a monotone operator: $\langle h(q,u) - h(p,u) , q - p \rangle \leq 0$
(ii)-Potential property	The Hicksian demand is a potential operator: $F(p) = \int_0^1 p \cdot h(tp, u) dt$ for some differentiable function **F**	The Hicksian demand is a potential operator: $F(p) = \int_0^1 \langle p , h(tp, u) \rangle dt$ for some differentiable functional **F**

ITEM	Without Uncertainty	With Uncertainty
5.2- Local comparative statics	*need the demand function to be differentiable*	*need the demand functional to be differentiable*
(i)- Slutsky equation	$\frac{\partial h_i(p,\ v(p,m))}{\partial p_j} = \frac{\partial x_i(p,m)}{\partial p_j} + \frac{\partial x_i(p,m)}{\partial m} x_j(p,m) \quad i,j = 1,\ldots,n.$	$\frac{\partial h_i(p,\ V(p,m))}{\partial p_j} = \frac{\partial x_i(p,m)}{\partial p_j} + \frac{\partial x_i(p,m)}{\partial m} \circ x_j(p,m) \quad i,j = 1,\ldots,n.$ where $\circ$ is the composition operator between functions
(ii)- Negative semi-definitness (equivalent to 5.1 (i))	the Slutsky matrix $\left[\frac{\partial x_i(p,m)}{\partial p_j} + \frac{\partial x_i(p,m)}{\partial m} x_j(p,m)\right]_{\substack{1\le i\le n\\ 1\le j\le n}}$ is a negative definite matrix. In particular, the own price effects are negative.	the linear functional represented by the Slutsky matrix of linear functional $\left[\frac{\partial x_i(p,m)}{\partial p_j} + \frac{\partial x_i(p,m)}{\partial m} \circ x_j(p,m)\right]_{\substack{1\le i\le n\\ 1\le j\le n}}$ is a negative linear operator. In particular, the own price effects are negative.
(iii)- Symmetry (equivalent to 5.1 (ii))	the Slutsky matrix $\left[\frac{\partial x_i(p,m)}{\partial p_j} + \frac{\partial x_i(p,m)}{\partial m} x_j(p,m)\right]_{\substack{1\le i\le n\\ 1\le j\le n}}$ is a symmetric matrix.	the linear functional represented by the Slutsky matrix of linear functional $\left[\frac{\partial x_i(p,m)}{\partial p_j} + \frac{\partial x_i(p,m)}{\partial m} \circ x_j(p,m)\right]_{\substack{1\le i\le n\\ 1\le j\le n}}$

ITEM	Without Uncertainty	With Uncertainty
6 - Calculation of Elasticities		
Uncompensated price elasticities	$\epsilon_{ij} = \frac{\partial x_i(p,m)}{\partial p_j} \times \frac{p_j}{x_i} \quad i,j = 1,\ldots,n.$	$\epsilon_{ij} = \frac{1}{\sqrt{Ex_i^2} \times \sqrt{Ep_j^2}} \times E\left[\left(\frac{\partial x_i(p,m)}{\partial p_j} \cdot p_j\right) \times p_j\right] = \epsilon_{ij}' + \epsilon_{ij}'$
-Part due to response to uncertainty (uncompensated *precautionary* risk elasticity)	0	$\epsilon_{ij}' = \frac{1}{\sqrt{Ex_i^2} \times \sqrt{Ep_j^2}} \times cov\left[\frac{\partial x_i(p,m)}{\partial p_j} \cdot p_j \, , \, p_j\right]$
- Part excluding uncertainty effects (uncompensated nonrisk elasticity)	$\frac{\partial x_i(p,m)}{\partial p_j} \times \frac{p_j}{x_i} \quad i,j = 1,\ldots,n.$	$\epsilon_{ij}' = \frac{1}{\sqrt{Ex_i^2} \times \sqrt{Ep_j^2}} \times E\left(\frac{\partial x_i(p,m)}{\partial p_j} \cdot p_j\right) \times Ep_j$

ITEM	Without Uncertainty	With Uncertainty
Substitution effects (elasticities)	$\epsilon'_{ij} = \frac{\partial x_i(p,m)}{\partial p_j} \times \frac{p_j}{x_i} + \frac{\partial x_i(p,m)}{\partial m} x_j(p,m) \times \frac{p_j}{x_i} \quad i,j = 1,\ldots,n.$	$\epsilon'_{ij} = \frac{E\left[\left[\frac{\partial x_i(p,m)}{\partial p_j} p_j + \frac{\partial x_i(p,m)}{\partial m} \circ x_j(p,m) \cdot p_j\right] \times p_i\right]}{\sqrt{Ex_i^2} \times \sqrt{Ep_i^2}} = \epsilon''_{ij} + \epsilon''_{ij}$
-Part due to response to uncertainty (*precautionary* risk substitution elasticity)	0	$\epsilon''_{ij} = \frac{cov\left[\left[\frac{\partial x_i(p,m)}{\partial p_j} \cdot p_j + \frac{\partial x_i(p,m)}{\partial m} \circ x_j(p,m) \cdot p_j\right], p_i\right]}{\sqrt{Ex_i^2} \times \sqrt{Ep_i^2}}$
- Part excluding uncertainty effects (taste substitution elasticity)	$\frac{\partial x_i(p,m)}{\partial p_j} \times \frac{p_j}{x_i} + \frac{\partial x_i(p,m)}{\partial m} x_j(p,m) \times \frac{p_j}{x_i}$	$\epsilon''_{ij} = \frac{E\left[\frac{\partial x_i(p,m)}{\partial p_j} \cdot p_j + \frac{\partial x_i(p,m)}{\partial m} \circ x_j(p,m) \cdot p_j\right] \times Ep_i}{\sqrt{Ex_i^2} \times \sqrt{Ep_i^2}}$

ITEM	Without Uncertainty	With Uncertainty
Income effects (elasticities)	$\epsilon_{ij}^{m} = -\frac{\partial x_i(p,m)}{\partial m} x_j(p,m) \times \frac{p_j}{x_i} \quad i,j = 1,\ldots,n.$	$\epsilon_{ij}^{m} = -\frac{1}{\sqrt{Ex_i^2} \times \sqrt{Ep_j^2}} \times E\left[\left(\frac{\partial x_i(p,m)}{\partial m} \circ x_j(p,m) \cdot p_j\right) \times p_j\right]$
-Part due to response to uncertainty (*precautionary* risk aversion effects)	0	$\epsilon_{ij}' = -cov\left[\frac{\partial x_i(p,m)}{\partial m} \circ x_j(p,m) \cdot p_j\ ,\ p_j\right] \times \frac{1}{\sqrt{Ex_i^2} \times \sqrt{Ep_j^2}}$
- Part excluding uncertainty effects (nonrisk pure income effects)	$-\frac{\partial x_i(p,m)}{\partial m} x_j(p,m) \times \frac{p_j}{x_i}$	$\epsilon_{ij}'^{m} = -\frac{1}{\sqrt{Ex_i^2} \times \sqrt{Ep_j^2}} \times E\left(\frac{\partial x_i(p,m)}{\partial m} \circ x_j(p,m) \cdot p_j\right) \times Ep_j$
Expenditure or budget elasticities	$\eta_i = \frac{\partial x_i(p,m)}{\partial m} \times \frac{m}{x_i} \quad i = 1,\ldots,n$	$\eta_i = \frac{1}{\lVert x_i \rVert} \times \frac{\partial x_i(p,m)}{\partial m} \cdot m = \frac{m}{\sqrt{Ex_i^2}} \times \frac{\partial x_i(p,m)}{\partial m} \quad i = 1,\ldots,n.$

ITEM	Without Uncertainty	With Uncertainty
7- Welfare Indicators		
Equivalent variation (EV) (a measure of "willingness to pay")	$EV = \mu(p,q,m') - \mu(p,p,m) = \mu(p,q,m') - m$	$EV = \mu(p,q,m') - \mu(p,p,m) = \mu(p,q,m') - m = R + T$
- Part of EV due to uncertainty (the equivalent risk premium)	0	$R = -tr\,cov[x(p,\mu(p,q,m')),\ p] = -\sum_{i=1}^{n} cov[x_i(p,\ \mu(p,q,m')),\ p_i]$
- Part of EV excluding uncertainty	$\mu(p,q,m') - m$	$T = Ex(p,\ \mu(p,q,m'))\cdot Ep - m = \sum_{i=1}^{n} Ex_i(p,\ \mu(p,q,m'))\cdot Ep_i - m$
Allen quantity index (Q_A) (a living-standard index)	$Q_A = \frac{\mu(p,q,m')}{m} = \frac{EV+m}{m}$	$Q_A = \frac{\mu(p,q,m')}{m} = \frac{EV+m}{m} = -Q_A' + Q_A'$
- Part of Q_A due to uncertainty (the risk premium index)	0	$Q_A' = -\frac{1}{m} tr\,cov[x(p,\ \mu(p,q,m')),\ p] = \frac{1}{m}R$
- Part of Q_A excluding uncertainty	$\frac{\mu(p,q,m')}{m}$	$Q_A' = \frac{1}{m} Ex(p,\ \mu(p,q,m'))\cdot Ep = \frac{T}{m} + 1$
Mackenzie-Pearce price index (P_M) (a cost-of-living index)	$P_M = \frac{m'}{\mu(p,q,m')} = \frac{m'}{EV+m}$	$P_M = \frac{m'}{\mu(p,q,m')} = \frac{m'}{EV+m} = P_M' + P_M'$
- Part of P_M due to uncertainty (the cost-of-risk index)	0	$P_M' = -\frac{m'\,tr\,cov[x(p,\mu(p,q,m')),\ p]}{\mu(p,q,m')Ex(p,\mu(p,q,m'))\cdot Ep} = \frac{R}{T+m}P_M$

ITEM	Without Uncertainty	With Uncertainty
- Part of P_M excluding uncertainty	$\frac{m'}{\mu(p,q,m')}$	$P'_M = \frac{m'}{Ex(p, \mu(p,q,m'))\cdot Ep} = \frac{m'}{T + m}$
Matrix of risk aversion coefficients	0	$R^A = -\frac{1}{m} cov[x(p, \mu(p,q,m')), p]$

Note: One can verify that with no uncertainty, the results in column (3) collapse to their deterministic analogues in column (2). In particular, this is the case for all the elasticities and welfare indicators calculated. For the definition of deferentiability of functions whose domains of definitiion has an empty interior, see Chapter 2 and Appendix 2

APPENDIX 2

PROOFS OF THEOREMS AND PROPOSITIONS

Proofs of theorems and propositions

The proofs of the theorems in this appendix require some standard concepts and facts of functional analysis. For the reader who is not familiar with these concepts and facts, they are summarized in the second part of the appendix. Also, for convenience, we have put there three additional theorems referred to in chapter 2 and in the first part of the appendix.

PART I: Theorems and Proofs.

The proofs of the propositions stated in chapter 2 are based on the following theorems (all the facts referred to are in the second part of this appendix).

Theorem A2.1: *Let* **B** *and* **Y** *be real reflexive Banach spaces,* $\mathbf{B}^*$ *and* $\mathbf{Y}^*$ *their respective duals,* **M** *a non empty subset of* **B**, *and* **K** *a cone of* **Y** *with* $\mathbf{K}^+$ *its dual cone. Let* $\mathbf{F}: \mathbf{M} \to \mathbf{R}$ and $\mathbf{N}: \mathbf{M} \to \mathbf{Y}$ *be functionals on* **M**. *Consider the following minimization problem:*[206]

$$\inf_{x \in M} \mathbf{F}(\mathbf{x}) = \alpha, \quad \mathbf{Nx} \leq \mathbf{0} \qquad \textbf{(P1)}$$

and Define the functional $\mathbf{L}: \mathbf{M} \times \mathbf{K}^+ \to \mathbf{R}$ *by*: $\mathbf{L}(\mathbf{x},\mathbf{q}) \equiv \mathbf{F}(\mathbf{x}) + \langle \mathbf{q}, \mathbf{Nx} \rangle$ *for all* $(\mathbf{x}, \mathbf{q}) \in \mathbf{M} \times \mathbf{K}^+$. *Assume:*

(H1) **M** *is closed, convex, and bounded.*

(H2) **K** *is closed and convex.*

(H3) **F** *is weakly lower semicontinuous.*

(H4) **N** *is weakly lower semicontinuous in the sense that the functionals*: $\mathbf{x} \to \langle \mathbf{q}, \mathbf{Nx} \rangle$ *are weakly lower semicontinuous for all* $\mathbf{q} \in \mathbf{K}^+$.

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[206] We recall that, **K** being a cone, $N(x) \leq \mathbf{0}$ means $\mathbf{0}- N(x) = -N(x) \in K$; where **0** is the origin of **Y** . Also we recall the standard convention of noting N(x) as simply Nx (given N is an operator from a Banach space to another), By the same convention, when $y \in Y$ and $y^* \in Y^*$ the value $y^*(y)$ is noted as $\langle y^*, y \rangle$. When Y is a Hilbert space so that Y^* is identified with Y, $\langle y^*, y \rangle$ is simply the inner product of y and y^*.

(H5) *There exists* $\mathbf{x}_0 \in \mathbf{M}$ *such that* $\mathbf{L}(\mathbf{x}_0\,,\,.)$ *is weakly coercive in* $\mathbf{K}^+$ in the sense that

$$\lim_{|q|\to\infty} \mathbf{F}(\mathbf{x}_0) + \langle \mathbf{q}\,,\mathbf{N}\mathbf{x}_0\rangle = -\infty\,.$$

Then:

(1) Existence of solution and Langrangian for **(P1)**:

(i) *There exists a solution* $\mathbf{x}^*$ *of the minimization problem* **(P1)** *such that*

$$\mathbf{F}(\mathbf{x}^*) = \alpha, \quad \mathbf{x}^* \in \mathbf{M}, \text{ and } \mathbf{N}\mathbf{x}^* \leq \mathbf{0} \tag{A2.1}$$

(ii) **L**, *as defined above, is a Lagrangian for* **(P1)**.

(2) Existence of Lagrange "multipliers" and generalized Kuhn-Tucker conditions: *The following two conditions are equivalents*:

(i) $\mathbf{x}^*$ *is a solution of the minimization problem* **(P1)**.

(ii) *There exists a Lagrange "multiplier"* $\mathbf{q}^* \in \mathbf{K}^+$ *and a* $\mathbf{x}^* \in \mathbf{M}$ *such that*:

$$\mathbf{L}(\mathbf{x}^*,\mathbf{q}^*) = \min_{x\in M} \mathbf{L}(\mathbf{x},\mathbf{q}^*), \quad \mathbf{N}\mathbf{x}^* \leq \mathbf{0}, \quad \text{and } \langle \mathbf{q}^*, \mathbf{N}\mathbf{x}^*\rangle = 0. \tag{A2.2}$$

Before proving the theorem, we give a condition that implies the assumption **(H5)**, which, in some instances is easier to obtain than **(H5)**. The condition is the so called Slater condition:

(SC) There exist a $\mathbf{x}_0 \in \mathbf{M}$ such that $-\mathbf{N}(\mathbf{x}_0)$ belongs to the interior of **K**.

However, this condition is very restrictive and is impossible to meet when **K** has an empty interior like in the case of $L_2(\Omega, \mathbf{R}^n)_+$. When **Y** is the Euclidean space $\mathbf{R}^n$ **(SC)** is generally met in most practical problems. Aubin (1979, p. 452) gives another weaker condition (the so called "constraint qualification") which in some instances can replace **(H5)**.[207]

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[207] For a proof that **(SC)** implies **(H5)**, see Zeidler (1986, p. 481). In Propositions **2.3.3** and **2.3.5** below where we apply the theorem, SC will be easily satisfied. However, in chapter **3** where we work out specific examples in which the non negativity constraint $\mathbf{x} \geq \mathbf{0}$ needs to be explicitly imposed, it is impossible to satisfy SC.

Finally, we note that when $\mathbf{Y} = \mathbb{R}^n$, the Euclidean space, and **K** is its positive cone, the Lagrange "multiplier" $\mathbf{q}^*$ is just the familliar vector of non-negative real numbers. Whereas, when $\mathbf{Y} = L_2(\Omega, \mathbb{R}^n)$ and **K** its positive cone, the Lagrange "multiplier" $\mathbf{q}^*$ is a n-dimensional vector of non-negative random variables. Moreover in these two cases, the conditions $\mathbf{Nx}^* \leq \mathbf{0}$ and $\langle \mathbf{q}^*, \mathbf{Nx}^* \rangle = 0$ lead to the usual principle of *complementary slackness:*[208]

$$\mathbf{Nx}^* \leq \mathbf{0},\ \mathbf{q}^* \geq 0,\ \text{ and } \mathbf{q}^* = \mathbf{0} \text{ if } \mathbf{Nx}^* < 0. \qquad \textbf{(A2.3)}$$

Theorem **A2.1** is essentially Theorem **50.A** and Corollary **50.1** of Zeidler (1986, p. 481). we have reformulated them to weaken the convexity of **F** and **N**, and drop the Slater Condition which they originally required. In doing that, we have added the requirement that **M** be bounded.[209]

Proof of theorem A2.1:

Proof of **(1.ii)**. Since **K** is a closed convex cone, $\mathbf{K}^{++} = \mathbf{K}$. Hence using **Fact 5** we have:

$$\sup_{\mathbf{q}\in\mathbf{K}^*} \{\mathbf{F}(\mathbf{x}) + \langle \mathbf{q}, \mathbf{Nx}\rangle\} = \mathbf{F}(\mathbf{x}) + \sup_{\mathbf{q}\in\mathbf{K}^*} -\langle \mathbf{q}, -\mathbf{Nx}\rangle$$
$$= \mathbf{F}(\mathbf{x}) - \inf_{\mathbf{q}\in\mathbf{K}^*} \langle \mathbf{q}, -\mathbf{Nx}\rangle = \begin{cases} \mathbf{F}(\mathbf{x}) & \text{if } -\mathbf{Nx} \in \mathbf{K} \\ \infty & \text{if } -\mathbf{Nx} \notin \mathbf{K} \end{cases} \qquad \textbf{(A2.4)}$$

Hence, $\inf_{\mathbf{x}\in\mathbf{M}} \sup_{\mathbf{q}\in\mathbf{K}^*} \mathbf{L}(\mathbf{x},\mathbf{q}) = \inf_{\mathbf{x}\in\mathbf{M}} \mathbf{F}(\mathbf{x}) = \alpha$ which shows that **L** is a Lagrangian of (P1).

Moreover, using (A2.4), one can see that $\mathbf{x}^*$ is a solution of (P1) if and only if $\sup_{\mathbf{q}\in\mathbf{K}^*} \mathbf{L}(\mathbf{x}^*, \mathbf{q}) = \alpha$. Hence, **(P1)** is equivalent to the mini-max problem:

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208 This is always true when **Y** is an ordered Banach space, and **K** is its positive cone. We recall that in this case, $\mathbf{y} < \mathbf{0}$ means that $\mathbf{y} \leq \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$, and that if $\mathbf{q}^* \in \mathbf{K}^+\backslash\{\mathbf{0}\}$ and $\mathbf{y} > \mathbf{0}$ then $\langle \mathbf{q}^*, \mathbf{y}\rangle > 0$ (by definition of the stricly positive elements of **Y***).

209 This was done in oder to accomodate quasi-concave utility functionals or constraints, and/or instances where the slater condition is impossible to be met.

$$\alpha = \inf_{\mathbf{x}\in \mathbf{M}} \sup_{\mathbf{q}\in \mathbf{K}^{\cdot}} \mathbf{L}(\mathbf{x},\mathbf{q}) \qquad (P)$$

For future references, let the dual of (P) be:

$$\beta = \sup_{\mathbf{q}\in \mathbf{K}^{\cdot}} \inf_{\mathbf{x}\in \mathbf{M}} \mathbf{L}(\mathbf{x},\mathbf{q}) \qquad (P^{\cdot})$$

The proof of the remaining assertions is based on the fact that the assumptions of the theorem guarantee that **L** has a saddle point with respect to $\mathbf{M}\times\mathbf{K}^{+}$. This, follows from the following lemma (for a proof see Zeidler (1986, p. 459, Theorem **49.A**).

Lemma A2.1. *Let* **X** *and* **Y** *be real reflexive Banach spaces, and* **L**: $\mathbf{M}\times\mathbf{Q} \subseteq \mathbf{X}\times\mathbf{Y} \to \mathbb{R}$ *a functional on the subset* $\mathbf{M}\times\mathbf{Q}$ of $\mathbf{X}\times\mathbf{Y}$. *Assume*:

(1) **M** *and* **Q** *are closed and non empty.*

(2) *The functional* $\mathbf{x} \to \mathbf{L}(\mathbf{x}, \mathbf{y})$ *is weakly sequentially lower semicontinuous on* **M** *for all* $\mathbf{y} \in \mathbf{Q}$.

(3) *The functional* $\mathbf{y} \to \mathbf{L}(\mathbf{x}, \mathbf{y})$ *is weakly sequentially upper semicontinuous on* **Q** *for all* $\mathbf{x} \in \mathbf{M}$.

(4) *Either* **M** *is bounded or there exists* $\mathbf{y}_0 \in \mathbf{Q}$ *such that the functional* $\mathbf{x} \to \mathbf{L}(\mathbf{x}, \mathbf{y}_0)$ *is weakly coercive on* **M**.

(5) *Either* **Q** *is bounded or there exists* $\mathbf{x}_0 \in \mathbf{M}$ *such that the functional* $\mathbf{y} \to -\mathbf{L}(\mathbf{x}_0, \mathbf{y})$ *is weakly coercive on* **Q**.

Then **L** *has a saddle point with respect to* $\mathbf{M}\times\mathbf{Q}$.

Proof of **(1.i)**. Since (P1) is equivalent to (P), **(1.i)** follows from lemma **A2.1** and the following so called *strongest duality assertion* (the proof is easy and uses only the definitions of inf, sup, and saddle point; i.e., no particular property of **L**, **M**, and $\mathbf{K}^{+}$ are used; see Zeidler (1986, p. 460)).

Lemma A2.2. *If* **L**: $\mathbf{M}\times\mathbf{K}^{+} \subseteq \mathbf{X}\times\mathbf{Y} \to \mathbb{R}$ *is a functional on the non empty subset* $\mathbf{M}\times\mathbf{K}^{+}$ of $\mathbf{X}\times\mathbf{Y}$, *Then the following two assertions are equivalents*:

(i) (x^*, q^*) *is a saddle point of* L *with respect to* $M \times K^+$.

(ii) x^* *is a solution of* (P), q^* *is a solution of* (P^*), *and* $\alpha = \beta$.

Proof of **(2)**. First we will prove that **(2.ii)** is equivalent to the assertion that **L** has a saddle point at (x^*, q^*) with respect to $M \times K^+$. If (x^*, q^*) is a saddle point of **L**, then $L(x^*, q^*) = \min_{x \in M} L(x, q^*)$ and, by lemma **A2.2**, x^* is a solution of (P1) with

$$Nx^* \leq 0, \quad \text{and } \alpha = F(x^*) = L(x^*, q^*) = F(x^*) + \langle q^*, Nx^* \rangle \tag{A2.5}$$

from which it follows that $\langle q^*, Nx^* \rangle = 0$ and **(2.ii)** is satisfied.

Conversely, if **(2.ii)** is satisfied, then from $Nx^* \leq 0$ and $\langle q^*, Nx^* \rangle = 0$ we deduce $\langle q, Nx^* \rangle \leq 0$ for all $q \in K^+$, and $L(x^*, q^*) = F(x^*)$ respectively. Hence,

$$L(x^*, q) = F(x^*) + \langle q, Nx^* \rangle \leq F(x^*) = L(x^*, q^*) \quad \text{for all } q \in K^+ \tag{A2.6}$$

Thus, $L(x^*, q^*) = \max_{q \in K^*} L(x^*, q)$, and since $L(x^*, q^*) = \min_{x \in M} L(x, q^*)$, (x^*, q^*) is a saddle point of **L**.

Hence, to complete the proof of **(2)**, we need only to show that a $x^* \in M$ is a solution of (P1) if and only if there exists a $q^* \in K^+$ such that (x^*, q^*) is a saddle point of **L**. But, Let (z^*, q^*) be a saddle point of **L** given by lemma **A2.1**. Then, by lemma **A2.2** q^* is a solution of the dual problem (P^*) and $\alpha = \beta$. Hence, by the same lemma **A2.2**, a $x^* \in M$ is a solution of (P1) if and only if (x^*, q^*) is a saddle point of **L**. ■

Corollary A2.1. *Along with assumptions* **(H1)-(H5)** *assume that the interiors of the domaines of definition of* **F** *and* **N** *are both non empty and contain* **M**. *Then, the following holds*:

(1) First order (necessary condition). *If* $x^* \in M$ *is a local solution of* (P1), *then there exists a Lagrange "multiplier"* $q^* \in K^+$ *such that*

$$L_x'(x^*, q^*) = F'(x^*) + q^* \circ N'(x^*) = 0, \quad Nx^* \leq 0, \quad \text{and } \langle q^*, Nx^* \rangle = 0. \tag{A2.7}$$

when both $F'(x^*)$ *and* $N'(x^*)$ *exist as a Gâteaux derivative or as a Fréchet derivative; where* $\circ$ *is the composition operator.*

(2) Sufficient condition. *Let n be a even number,* $n \geq 2$. *Then* x^* *is a strict local solution of* (P1) *when the following two conditions are fulfilled:*

(i) *There exists a Lagrange "multiplier"* $q^* \in K^+$ *such that for all* $h \in B$ *and fixed* $c > 0$

$$\begin{aligned} L_x^{(k)}(x^*, q^*) &= F^{(k)}(x^*) + q^* \circ N^{(k)}(x^*) = 0 \quad k = 1, \ldots, n-1, \\ L_x^{(n)}(x^*, q^*)h^n &= \left(F^{(n)}(x^*) + q^* \circ N^{(n)}(x^*)\right)h^n \geq c\,\|h\|^n, \\ Nx^* &\leq 0, \text{ and } \langle q^*, Nx^* \rangle = 0. \end{aligned} \tag{A2.8}$$

(ii) F *and* N *are n-times Fréchet differentiable in a neighborhood of* x^*, *and* $F^{(n)}$ *and* $N^{(n)}$ *are continuous at* x^*.

(3) Necessary and sufficient condition. *If in addition it is assumed that* F *is convex, and* N *is convex in the sense that* $N(\alpha x + (1-\alpha)z) \leq \alpha Nx + (1-\alpha)Nz$ *for all* $x, z \in B$ *and* $0 \leq \alpha \leq 1$; *then the necessary condtion of* **(1)** *is also sufficient for* $x^* \in M$ *to be a global solution of* (P1). *Moreover, if either* F *or* N *is strictly convex, then* x^* *is unique.*

Proof. From Theorem **A2.1 (2)** it follows that if x^* is a solution of (P1) then for some $q^* \in K^+$ $Nx^* \leq 0$, $\langle q^*, Nx^* \rangle = 0$, and

$$L(x^*, q^*) = \min_{x \in M} L(x, q^*) \tag{$\bar{P}$}$$

Thus, assertions **(1)** and **(2)** follows by applying Theorem **A2.3** of Part **II** below to the minimization problem $(\bar{P})$, and from the fact that

$$L_x'(x^*, q^*)h = F'(x^*)h + \langle q^*, N'(x^*)h\rangle = F'(x^*)h + (q^* \circ (N'(x^*))h \quad \text{for all } h \in B \qquad \textbf{(A2.9)}$$

so that

$$L_x'(x^*, q^*) = F'(x^*) + q^* \circ N'(x^*) \qquad \textbf{(A2.10)}$$

which, by the chain rule and the fact that the mapping $y \mapsto q^* \circ y$ is linear and continuous, gives the expression for the higher derivatives. Finally, since $L(., q^*)$ is (strictly) convex when **F** (or) and **N** (is) are (strictly) convex, assertion **(3)** follows by applying Theorem **A2.4** of Part **II** below. ■

Theorem A2.2. (Ekeland and Lebourg): *Let* **X** *be a topological space,* **B** *be a real locally uniformly smooth Banach space,* B^* *its dual. Let* $g: X \to R$ *be a lower semicontinuous function which is bounded below, and*

$A: X \to B^*$ *be a continuous and proper function such that* $A(X)$ *is norm bounded in* B^*.

Finally, *define the function* $f: X \times B \times R \to R$ *by* $f(x, u, b) \equiv g(x) + \langle Ax, u\rangle + b$ for all $(x, u, b) \in X \times B \times R$, *and consider the following minimization problem*:

$$F(u, b) = \inf_{x \in X} f(x, u, b) \qquad (P_{u,b})$$

Then **F** *is locally lipschitzian on* $B \times R$ *and there exists a dense* G_δ *subset* $T \subset B \times R$, *at every point* (u, b) *of which*:

(1) **F** *is Fréchet differentiable,*

(2) *the set* $S_{u,b}$ *of optimal solutions for* $(P_{u,b})$ *is compact nonempty,*

(3) $f_u'(x, u, b) = F_u'(u, b)$ and $f_b'(x, u, b) = F_b'(u, b)$ *for all* $x \in S_{u,b}$,

(4) *the restriction to* **T** *of the mapping* $F': B \times R \to B^* \times R$ *is continuous,*

(5) *the set-valued mapping* $(u, b) \mapsto S_{u,b}$ *from* **T** to 2^X *is upper semicontinuous.*

Proof: This theorem is a minor modification of Proposition **3.7** of Ekeland and Lebourg (1976, p. 208). In the original propositipon $b = 0$. But, an examination of the proof of their more general Theorem **3.4** (p. 205) on which the proposition is based shows that this minor modification does not alter the conclusions of the proposition. ■

Proofs of Lemma and Propositions.

As a rule, the proof of any property that does not involve the topological structure of the spaces is identical to the finite dimensional case. Only properties that involve topological arguments need a careful examination, although for many properties the topology does not make difference. Consequently, properties that do not call upon topological arguments and have counterparts in the finite dimensional case will not be reproved as most of them are proved in Varian (1984) chapter **1** and **3**. The continuity of the indirect utility and expenditure functional follows from Theorem **A2.2 (5)** above (see also Aubin, 1979, p. 70, Theorem 3.). The fact that **V** is strictly increasing in *m* follows directly from the strict monotonicity assumption.

Proof of proposition 2.2.1: Since $L_2(\Omega, \mathbf{R}^n)$ is perfectly separable, we can apply Debreu (1954) general theorem which gives the numerical representation of a complete, reflexive, and transitive ordering on a perfectly separable topological space. The strict quasi-concavity of **U** follows directly from the assumption of strict convexity of preference, while the boundness of **U** follows from its continuity and from the fact that **X** is weakly compact.[210]
■

Proof of lemma 2.3.1: The proof is straightforward and uses the same arguments as in the finite dimensional case. To prove unicity, suppose there are two distinct optimal solutions $\mathbf{x}^*$ and $\mathbf{y}^*$ belonging to $\mathbf{S}^*$, then $\frac{1}{2}\mathbf{x}^* + \frac{1}{2}\mathbf{y}^* \in \mathbf{B}(\mathbf{p},\mathbf{m})$ and by strict convexity is strictly preferred to either $\mathbf{x}^*$ or $\mathbf{y}^*$ contradicting the fact that they are optimal. To prove the

[210] Mas-Colell (1986) and Richard and Zame (1986) give other proofs that do not use the separability assumption. Instead, they exploit the order structure of the space. However, Mas-Colell did assume that **X** is a order interval, while Richard and Zame assumed that the preference is proper.

binding of the budget constraint at the optimum, suppose there is an $\mathbf{x}^* \in \mathbf{S}^*$ with $\mathbf{p}\cdot\mathbf{x}^* < m$. Then since $\mathbf{p}$ is continuous, the set $\mathbf{BO} \equiv \{\mathbf{x} \in L_2(\Omega, \mathbf{R}^n): \mathbf{p}\cdot\mathbf{x} < m\}$ is open and contains $\mathbf{x}^*$. Thus it contains an open ball $\mathbf{B}(\mathbf{x}^*, \epsilon)$ for some $\epsilon > 0$. Take $y \equiv x^* + \frac{\epsilon}{2\|x_0\|}x_0$ for some $\mathbf{x}_0 \in \mathbf{B}(\mathbf{x}^*, \epsilon) \cap L_2(\Omega, \mathbf{R}^n)_{++}$ so that $\|y - x^*\| = \frac{\epsilon}{2} < \epsilon$, that is $y \in \mathbf{B}(\mathbf{x}^*, \epsilon)$, and hence $\mathbf{p}\cdot y < m$. But $y > \mathbf{x}^*$, which by strict monotonicity implies that $y \succ \mathbf{x}^*$, contradicting the fact that $\mathbf{x}^*$ is optimal. ■

Proof of proposition 2.3.1: We apply Theorem **A2.1** where we take $\mathbf{B} = \mathbf{B}^* = L_2(\Omega, \mathbf{R}^n)$, $\mathbf{Y} = \mathbf{Y}^* = \mathbf{R}$, $\mathbf{M} = \mathbf{X}$, $\mathbf{K} = \mathbf{K}^+ = \mathbf{R}_+$, $\mathbf{F} = -\mathbf{U}$, and $\mathbf{N}$ as the linear (affine) mapping $\mathbf{N}$: $\mathbf{M} \rightarrow \mathbf{R}$ defined by $\mathbf{N}\mathbf{x} = \langle \mathbf{p}, \mathbf{x} \rangle - m$. To check that all the assumptions of the theorem are satisfied we note that $L_2(\Omega, \mathbf{R}^n)$ and $\mathbf{R}$ are reflexive, $\mathbf{X}$ is convex, closed bounded by assumption, $-\mathbf{U}$ is quasi-convex and continuous, thus is weakly lower semicontinuous (see Facts **1-4**). Furthermore since by assumption the budget set $\mathbf{B}(\mathbf{p}, m)$ is nonempty for every $(\mathbf{p}, m)$, there exists $\mathbf{x}_0 \in \mathbf{X}$ such that $\langle \mathbf{p}, \mathbf{x}_0 \rangle = \frac{m}{2} < m$. Hence, $\mathbf{N}\mathbf{x}_0 = \langle \mathbf{p}, \mathbf{x}_0 \rangle - m < 0$ so that $-\mathbf{N}\mathbf{x}_0 \in \text{int } \mathbf{K} = \mathbf{R}_{++}$. This shows that the Slater condition **(SC)** is satisfied, and hence, all the conditions required by the theorem are fulfilled.[211] ■

Before proving Proposition **2.3.3**, we recall the definition of Fréchet and Gâteaux differentiabilty of a function, not necessarily defined on a open set, that we gave in chapter **2**.

Definition A2.1. Let **X** and **Y** be two real Banach spaces, **M** a subset of **X** which is not necessarily open, and **f** a function $\mathbf{f}: \mathbf{M} \subseteq \mathbf{X} \rightarrow \mathbf{Y}$. **f** is said to be n-times *Fréchet (resp Gâteaux) differentiable* at $\mathbf{x}_0 \in \mathbf{M}$ if and only if there exists a neighborhood $\mathbf{O}(\mathbf{x}_0)$ of $\mathbf{x}_0$ that contains **M**, and a function $\mathbf{f}^*: \mathbf{O}(\mathbf{x}_0) \subseteq \mathbf{X} \rightarrow \mathbf{Y}$ which is n-times Fréchet (resp Gâteaux) differentiable at $\mathbf{x}_0$, such that $\mathbf{f}^*(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{M}$. The k^{th} Fréchet (resp Gâteaux) derivative of **f** at $\mathbf{x}_0$ is then defined as: $\mathbf{f}^{(k)}(\mathbf{x}_0) \equiv \mathbf{f}^{*(k)}(\mathbf{x}_0)$, $k = 1,...,n$. **f** is said to be *n-times*

[211] We note that here we did not need to impose the non negativity constraint, $\mathbf{x} \geq \mathbf{0}$. It would be redundant to do so since we already have $\mathbf{X} \subseteq L_2(\Omega, \mathbf{R}^n)_+$.

continuously Fréchet (resp Gâteaux) differentiable if and only if the function $\mathbf{x} \mapsto \mathbf{f}^{(n)}(\mathbf{x})$ is continuous on **M**.[212]

In general, the open neighborhood and the differentiable function extending **f** is the same for all elements of **M**. In the propositions that follow, and in all the examples worked out in chapter **3** where **M** has an empty interior, the open neighborhood is the whole space and **f** is naturally extended to the whole space. Hence, in what follows, whenever we speak of the Fréchet or Gâteaux differentiabilty of a function $\mathbf{f}: \mathbf{M} \subseteq \mathbf{X} \rightarrow \mathbf{Y}$, and **M** has an empty interior, it must be understood in the sense of the definition given above, even if the open neigborhood and the extended function are not explicitly given.

Proof of Proposition 2.3.3: this is an easy application of Theorem **A2.2** of Ekeland and Lebourg.

We take $\mathbf{g} = -\mathbf{U}$, $\mathbf{B} = L_2(\Omega, \mathbf{R}^n)$, $(\mathbf{u}, \mathbf{b}) = \lambda(\mathbf{p}, m)$, and $\mathbf{A} = \lambda\mathbf{I}$ (with **I** being the identity from $L_2(\Omega, \mathbf{R}^n)$ to itself). We then define $\mathbf{f}: \mathbf{X} \times L_2(\Omega, \mathbf{R}^n) \times \mathbf{R} \rightarrow \mathbf{R}$ as:

$$\mathbf{f}(\mathbf{x}, \mathbf{p}, m) \equiv \mathbf{g}(\mathbf{x}) + \langle \mathbf{A}\mathbf{x}, \mathbf{u} \rangle + \mathbf{b} = -\mathbf{U}(\mathbf{x}) + \lambda\langle \mathbf{x}, \mathbf{p} \rangle - \lambda m \qquad \textbf{(A2.11)}$$

Since $L_2(\Omega, \mathbf{R}^n) \times \mathbf{R}$ is an Hilbert space it is uniformly smooth. $\lambda\mathbf{I}$ is trivially continuous and proper, and since **X** is weakly compact by assumption, $\lambda\mathbf{X}$ is norm bounded in $L_2(\Omega, \mathbf{R}^n) \times \mathbf{R}$ (see Fact **2**). Finally, -**U** being quasi-convex and continuous, and **X** weakly compact, it is lower semicontinuous and bounded below (see Fact **4**). Hence, all the assumptions of the theorem are satisfied, and we can apply the theorem to the function **F** defined by

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[212] It is clear that for a given $\mathbf{x}_0$ $\mathbf{f}^{(k)}(\mathbf{x}_0)$ is well defined since if there are two pairs of neighborhood of $\mathbf{x}_0$ and function $(\mathbf{O}_1(\mathbf{x}_0), \mathbf{f}_1^*)$ and $(\mathbf{O}_2(\mathbf{x}_0), \mathbf{f}_2^*)$ satisfying the definition then it immediately follows that $\mathbf{f}_1^{*(k)}(\mathbf{x}_0) = \mathbf{f}_2^{*(k)}(\mathbf{x}_0)$, and $\mathbf{f}^*_1 = \mathbf{f}_2^*$ on **M**. It is also clear that when **M** is an open set the definition reduces to the usual defintion of Fréchet and Gâteaux differentiabilty (since it suffices to take $\mathbf{O}(\mathbf{x}) = \mathbf{M}$, and $\mathbf{f}^* = \mathbf{f}$ for all $\mathbf{x} \in \mathbf{M}$).

$$F(\mathbf{p}, m) = \inf_{x \in X} f(\mathbf{x}, \mathbf{p}, m) \tag{A2.12}$$

to get

$$F_p'(\mathbf{p}, m) = f_p'(\mathbf{x}^*, \mathbf{p}, m) = \lambda \mathbf{x}^* \quad \text{and} \quad F_m'(\mathbf{p}, m) = f_m'(\mathbf{x}^*, \mathbf{p}, m) = -\lambda \tag{A2.13}$$

for all (**p**, m) in a G_δ subset $\mathbf{Q}\times\mathbf{D}$ of $L_2(\Omega, \mathbf{R}^n)\times\mathbf{R}$. Where $\mathbf{x}^*$ is the unique optimal solution of (A2.12). Now, it is clear that the indirect utility functional **V** is merely the restriction of -**F** to $L_2(\Omega, \mathbf{R}^n)_{++}\times\mathbf{R}_{++}$. Hence, taking into consideration Definition **A2.1**, if we show that $(\mathbf{Q}\times\mathbf{D})\cap(L_2(\Omega, \mathbf{R}^n)_{++}\times\mathbf{R}_{++})$ is a G_δ subset of $L_2(\Omega, \mathbf{R}^n)_{++}\times\mathbf{R}_{++}$ (endowed with its induced topology), then the assertions of the proposition will follow, with

$$\frac{\partial V(\mathbf{p}, m)}{\partial \mathbf{p}} = -\lambda \mathbf{x}(\mathbf{p}, m), \quad \text{and} \quad \frac{\partial V(\mathbf{p}, m)}{\partial m} = \lambda \tag{A2.14}$$

for all (**p**, m) $\in S_1\times S_2 \equiv (\mathbf{Q}\cap L_2(\Omega, \mathbf{R}^n)_{++})\times(\mathbf{D}\cap\mathbf{R}_{++})$. But since $\mathbf{Q}\times\mathbf{D}$ is a G_δ, clearly we have $S_1\times S_2 = \bigcap_{i\in I}(\mathbf{Q}_i\cap L_2(\Omega, \mathbf{R}^n))\times(\mathbf{D}_i\cap\mathbf{R})$ where $\{\mathbf{Q}_i \,;\, i \in I\}$ and $\{\mathbf{D}_i \,;\, i \in I\}$ are countable sets of open dense subsets of $L_2(\Omega, \mathbf{R}^n)$ and **R** respectively. Furthermore, if (**p**, m) $\in L_2(\Omega, \mathbf{R}^n)_{++}\times\mathbf{R}_{++}$ then for each $i \in I$ there exists a sequence $\{(\mathbf{p}_n, m_n)\}_{n\in N} \subset \mathbf{Q}_i\times\mathbf{D}_i$ which converges to (**p**, m). For each $n \in N$ let $\mathbf{p}_n^+ \equiv \sup\{\mathbf{p}_n, \mathbf{0}\} \in \mathbf{Q}_i\cap L_2(\Omega, \mathbf{R}^n)_{++}$ and $m_n^+ \equiv \sup\{m_n, 0\} \in \mathbf{D}_i \cap\mathbf{R}_{++}$, then since both the norms of $L_2(\Omega, \mathbf{R}^n)$ and **R** have the lattice property (i.e., $\sup(z, -z) \leq \sup(y, -y)$ implies $\|z\| \leq \|y\|$) so that the sup function is continuous, the sequence $\{(\mathbf{p}_n^+, m_n^+)\}_{n\in N}$ converges to $(\mathbf{p}^+, m^+) = (\mathbf{p}, m)$. This shows that each $(\mathbf{Q}_i\cap L_2(\Omega, \mathbf{R}^n))\times(\mathbf{D}_i\cap\mathbf{R})$ is dense in $L_2(\Omega, \mathbf{R}^n)_{++}\times\mathbf{R}_{++}$ endowed with its induced topology, and hence, $S_1\times S_2$ is a G_δ of $L_2(\Omega, \mathbf{R}^n)_{++}\times\mathbf{R}_{++}$. ■

Proof of proposition 2.3.6: We again apply Theorem **A2.1** by taking $\mathbf{B} = L_2(\Omega, \mathbf{R}^n)$, $\mathbf{Y} = \mathbf{R}$, $\mathbf{M} = \mathbf{X}$, $\mathbf{K} = \mathbf{K}^+ = \mathbf{R}_+$, $\mathbf{F} = \langle \mathbf{p}, . \rangle$, and $\mathbf{N} = -(\mathbf{U} - u)$. Because of the strict monotonicity of $\succeq$, the facts that $\mathbf{U}(\mathbf{X})$ is bounded and $u \in \text{int}\, \mathbf{U}(\mathbf{X})$ guarantee that there always exists a $\mathbf{x}_0 \in \mathbf{X}\backslash\{\mathbf{0}\}$ such that $u < \mathbf{U}(\mathbf{x}_0) < b$ for some $b > 0$. Hence, $-\mathbf{N}(\mathbf{x}_0) \in \mathbf{R}_{++}$ and the Slater condition (**SC**) is satisfied. By following the same steps as in the proof of Proposition **2.3.3**, it is easy to see that all the other assumptions required by the theorem are satisfied. ■

Proof of proposition 2.3.7: Define f: $\mathbf{X} \times L_2(\Omega, \mathbb{R}^n) \times \mathbb{R} \rightarrow \mathbb{R}$ by :$\mathbf{f}(\mathbf{x}, \mathbf{p}, u) = -\lambda U(\mathbf{x}) + \langle \mathbf{p}, \mathbf{x} \rangle + \lambda u$ and the rest of the proof is the same as in Proposition **2.3.3**. ■

Proof of proposition 2.4.1: This is an immediate consequence of Propositions **2.3.5** and **2.3.7** and of Theorems **A2.3-A2.5** below (note the sign difference in the definition of monotonicity; we should have said monotonic decreasing in the proposition to be more precise).

Proof of Corollary 2.6.1: Since the Hicksian demand is strictly monotonic with respect to prices, we have

$$\begin{gathered}\langle \mathbf{h}(\mathbf{q}, u) - \mathbf{h}(\mathbf{p}, u), \mathbf{q} - \mathbf{p} \rangle < 0 \\ \textit{for all } \mathbf{p}, \mathbf{q} \in \mathbf{Q}_1 \textit{ and } u \in \mathbf{Q}_2 \textit{ with equality if } \mathbf{p} = \mathbf{q}.\end{gathered} \tag{A2.15}$$

Hence, for $\mathbf{p} \in \mathbf{Q}_1$ and $\mathbf{q} = E\mathbf{p}$. we have

$$\langle \mathbf{h}(E\mathbf{p}, u) - \mathbf{h}(\mathbf{p}, u), E\mathbf{p} - \mathbf{p} \rangle \leq 0 \tag{A2.16}$$

But, since the range of the restriction of $\mathbf{h}(., u)$ to $\mathbb{R}^n_{++}$ is $\mathbb{R}^n_{++}$, we have $\langle \mathbf{h}(E\mathbf{p}, u), \mathbf{p} \rangle = \mathbf{h}(E\mathbf{p}, u) \times E\mathbf{p}$. Hence, after expension and simplification, (A2.16) is equivalent to:

$$E(\mathbf{h}(\mathbf{p}, u) \times \mathbf{p}) - E\mathbf{h}(\mathbf{p}, u) \times E\mathbf{p} \leq 0 \tag{A2.17}$$

That is,

$$\operatorname{tr} cov(\mathbf{h}(\mathbf{p}, u), \mathbf{p}) \leq 0 \tag{A2.18}$$

Now, taking $u = \mathbf{V}(\mathbf{q}, m')$ and using the fact that $\mathbf{h}(\mathbf{p}, \mathbf{V}(\mathbf{q}, m')) = \mathbf{x}(\mathbf{p}, \mu(\mathbf{p}, \mathbf{q}, m'))$, yield the desired result. ■

PART II

A. The L_p Spaces.

For $1 \leq p < \infty$, $L_p(\Omega, \mathbb{R}^n)$ (resp $L_\infty(\Omega, \mathbb{R}^n)$) is defined as the space of vector valued random variables **x** defined on Ω, taking values in $\mathbb{R}^n$ and satisfying $\int_\Omega \|\mathbf{x}(\omega)\|^p dP(\omega) < \infty$ (resp $\|\mathbf{x}\| \leq \mathbf{M}$ for some real $\mathbf{M} < \infty$) with $\| \ \|$ being the Euclidean norm of $\mathbb{R}^n$. Endowed with the norms $\| \ \|_p$ defined respectively by $\|\mathbf{x}\|_p = \{\int_\Omega \|\mathbf{x}(\omega)\|^p dP(\omega)\}^{\frac{1}{p}}$ for $\mathbf{x} \in L_p(\Omega, \mathbb{R}^n)$, $1 \leq p < \infty$ and $\|\mathbf{x}\|_\infty = \inf\{\mathbf{M}: \|\mathbf{x}\| \leq \mathbf{M}$ a.e$\}$ for $\mathbf{x} \in L_\infty(\Omega, \mathbb{R}^n)$, the L_p spaces are Banach spaces (i.e complete normed vectorial spaces). Also, for $1 \leq p \leq p' < \infty$ we have $L_{p'}(\Omega, \mathbb{R}^n) \subseteq L_p(\Omega, \mathbb{R}^n)$. Another basic result is that for $1 < p < \infty$, $L_p(\Omega, \mathbb{R}^n)^*$, the topological dual of $L_p(\Omega, \mathbb{R}^n)$, which is defined as the space of continuous linear forms defined on $L_p(\Omega, \mathbb{R}^n)$ endowed with the norm $\| \ \|_p$, is equal to $L_q(\Omega, \mathbb{R}^n)$ with q satisfying $\frac{1}{p} + \frac{1}{q} = 1$ (i.e, $q = \frac{p}{p-1}$).

B. Concepts and Facts from Functional and Convex analysis.

For all the following concepts, see Zeidler (1985, 1986 and 1990), Aubin (1979), and Rudin (1991) for more details. In the definitions below, the symbols $\rightarrow$ and $\rightharpoonup$ stand respectively for the strong or norm convergence and the weak convergence. For more details on these concepts and on the concept of weak topology, see, for example, Rudin (1991, chapter 3). For more on the concepts of uniformly convex and smooth Banach spaces, see, for example, Cioranescu (1990).

To ensure that our extremum problems are always well defined, real valued functionals are occasionally allowed to take the two values $+\infty$ and $-\infty$. So, let $\bar{\mathbf{R}} = \mathbf{R} \cup \{-\infty, +\infty\}$.

Definition A2.2: Let **B** be a real Banach space, and $\mathbf{f}: \mathbf{M} \subseteq \mathbf{B} \rightarrow \bar{\mathbf{R}}$ a functional on the subset **M** of **B**. Also, for $\mathbf{r} \in \mathbf{R}$ let $\mathbf{M}_r = \{ \mathbf{x} \in \mathbf{M}: \mathbf{f}(\mathbf{x}) \leq \mathbf{r} \}$. Then:

(1) f is said to be *weak sequentially continuous* if and only if

$p_n \rightharpoonup p$ implies $f(p_n) \rightharpoonup f(p)$ for all sequences (p_n) of M.

(2) f is said to be *sequentially lower semicontinuous* (resp *weak sequentially lower semicontinuous*) if and only if

$p_n \to p$ implies $\underline{\lim}_{n\to\infty} f(p_n) \geq f(p)$ (resp $p_n \rightharpoonup p$ implies $\underline{\lim}_{n\to\infty} f(p_n) \geq f(p)$ for all sequences (p_n) of M).

(3) f is said to be *lower semicontinuous* (resp *weakly lower semicontinuous*) if and only if M_r is closed (resp weakly closed) relative to M for all $r \in \mathbb{R}$.

(4) f is said to be *lower semicompact* (resp *weakly lower semicompact*) if and only if M_r is compact (resp weakly compact) for all $r \in \mathbb{R}$.[213]

(5) f is said to be *(weakly) upper semicontinuous* (resp *(weakly) upper semicompact*) if and only if -f is *(weakly) lower semicontinuous* (resp *(weakly) lower semicompact*).

(6) f is said to be *weakly coercive* if and only if $\lim_{\|p\|\to\infty} f(p) = \infty$

We recall that f is *quasi-convex* if and only if M_r is convex for all $r \in \mathbb{R}$; and that f is *quasi-concave* if and only if -f is *quasi-convex*. Also, it is easy to see that if f is (weakly) lower semicontinuous, and M is (weakly) compact, then f is (weakly) lower semicompact.

Definition A2.3: Let X be a topological space, and $f: M \subseteq B \to \bar{\mathbb{R}}$ a functional on the subset M of X.

(1) $x_0 \in M$ is said to be a *global minimum* of f if and only if $f(x_0) \leq f(x)$ for all $x \in M$, i.e., $f(x_0) = \min_{x\in M} f(x)$.

(2) Provided the interior of the domain of definition of f is non empty and contains M, $x_0 \in M$ is said to be a *local (resp strict local) minimum* of f if and only if there exists a

[213] In some books (e.g Aubin 1979) lower semicompactness is defined by the **relative** compactness of M_r (by definition, a set is relatively compact if its closure is compact). Here we are using the definition given in Zeidler (1985, page 150). The two definitions coincide when f is lower semicontinuous (resp weakly lower semicontinuous).

neighborhood of x_0, $N(x_0)$, such that $f(x_0) = \min_{x \in N(x_0)} f(x)$ (resp $f(x_0) = \min_{x \in N(x_0)} f(x)$ and $f(x_0) \neq f(x)$) for all $x \in N(x_0)$).

Definition A2.4: Let **B** and **Y** be real Banach spaces, and $g: M \times Q \subseteq B \times Y \to R$ a functional on the subset $M \times Q$ of $B \times Y$. Then **g** is said to have a *saddle point* with respect to $M \times Q$ if and only if there exists $(a, b) \in M \times Q$ such that

$$\min_{x \in M} g(x,b) = g(a,b) = \max_{y \in Q} g(a,y)$$

Definition A2.5: Let **B** be a real Banach space, B^* its dual, and $f: M \subseteq B \to \bar{R}$ a functional on the subset **M** of **B**. Let the minimization problem:

$$\inf_{x \in M} f(x) = \alpha \qquad (P_0)$$

(1) $x^* \in M$ is said to be a *global solution* of (P_0) if and only if x^* is a global minimum of **f**.

(2) Provided the interior of the domain of definition of **f** is non empty and contains **M**, $x^* \in M$ is said to be a *local (resp strict local) solution* of (P_0) if and only if x^* is a local (resp strict local) minimum of **f**.

(3) A functional $L: M \times Q \subseteq B \times B^* \to \bar{R}$ is said to be a *Lagrangian of the minimization problem* (P_0) if and only if

$$\alpha = \inf_{x \in M} \sup_{q \in Q} L(x,q) \qquad (P)$$

(4) If **L** is a Lagrangian of the minimization problem (P_0) then an element $q^* \in Q$ is called a *Lagrange "multiplier"* if and only if

$$\alpha = \inf_{x \in M} \sup_{q \in Q} L(x,q) = \inf_{x \in M} L(x,q^*)$$

(5) The *dual problem* of the mini-max problem defined by (P) is defined as:

$$\beta = \sup_{q \in Q} \inf_{x \in M} L(x,q) \qquad (P^*)$$

(6) $x^* \in M$ (resp $q^* \in Q$) is said to be a *solution* of **(P)** (resp (P^*) if and only

$\sup_{q \in K^*} L(x^*, q) = \alpha$ (resp $\inf_{x \in M} L(x, q^*) = \beta$).

Definition A2.6: Let **X** be a toplogical space, and **M** a subset of **X**. Then:

(1) **M** is called a G_δ subset of **X** if and only if **M** is the countable intersection of dense open subsets of **X** (which is dense by the Baire category theorem).

(2) A property (P) defined on **X** is said to be *generic* if and only if there is a G_δ subset **M** of **X** at every point of which (P) is true.

Definition A2.7: Let **X** and **Y** be two topological spaces.

(1) A continuous map **g**: $X \to Y$ is said to be *proper* if and only if it is closed (ie. maps closed subsets onto closed subsets) and $g^{-1}(y)$ is compact for all $y \in Y$.

(2) A set-valued mapping S: $X \to 2^Y$ is said to be *upper semicontinuous* at $x \in X$ if and only if for any neighborhood N(S(x)) of S(x) there exists neighborhood N(x) of **x** such that $S(N(x)) \subset N(S(x))$.

Definition A2.8: Let **X** and **Y** be two real Banach spaces with respective norms $\| \ \|_X$ and $\| \ \|_Y$, **O** an open subset of **X**, and **f** a function $f: O \subseteq X \to Y$.

(1) **f** is said to be *Fréchet differentiable* at $x \in O$ if and only if there exists a continuous linear function from **X** to **Y**, noted **f′(x)**, such that $\lim_{\|h\| \to 0} \frac{\| f(x+h) - f(x) - f'(x) \cdot h \|_Y}{\|h\|_X} = 0$. **f′(x)** is then called the *Fréchet derivative* of **f** at **x**.

(1) **f** is said to be *Gâteaux differentiable* at $x \in O$ if and only if there exists a continuous linear function from **X** to **Y**, noted **f′(x)**, such that

$\lim_{\theta \to 0} \frac{\| f(x+\theta h) - f(x) - f'(x) \cdot h \|_Y}{\theta} = 0$ for all $h \in X$. **f′(x)** is then called the *Gâteaux derivative* of **f** at **x**.

We recall that if **f** is Fréchet differentiable at **x** then it is Gâteaux differentiable at **x** with the derivative being the same.[214]

Definition A2.9: Let **B** be a Banach space with $\| \ \|$ its norm, $\mathbf{B}^*$ its dual, and **f**, **g**: $\mathbf{B} \rightarrow \mathbf{R}$ the functionals defined by:

$$\mathbf{f}(\mathbf{x}) = \|\mathbf{x}\| \quad \text{and} \quad \mathbf{g}(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|^2$$

Then:[215]

(1) **B** is said to be *smooth* if and only **f** is Gâteaux differentiable at all $\mathbf{x} \in \mathbf{B}\backslash\{\mathbf{0}\}$.

(2) **B** is said to be *locally uniformly smooth* if and only **f** is Fréchet differentiable at all $\mathbf{x} \in \mathbf{B}\backslash\{\mathbf{0}\}$.

(3) **B** is said to be *uniformly smooth* if and only **B** is locally uniformly smooth and

$$\lim_{\theta \to 0} \sup_{|\mathbf{x}| = |\mathbf{y}| = 1} \left| \frac{\|\mathbf{x} + \theta\mathbf{y}\| - \|\mathbf{x}\|}{\theta} - \mathbf{f}'(\mathbf{x})\mathbf{y} \right| = 0$$

(2) **B** is said to be *uniformly convex* if and only if **g** is strictly convex.

Definition A2.10: Let **B** be a real Banach space, $\mathbf{B}^*$ its dual, and **K** a subset of **B**.

(1) **K** is said to be a *cone* if and only if $\alpha\mathbf{x} \in \mathbf{K}$ for all real $\alpha > 0$, and $\mathbf{x} \in \mathbf{K}$.

(2) The set $\mathbf{K}^+$ defined by $\mathbf{K}^+ \equiv \{\mathbf{A} \in \mathbf{B}^*: \mathbf{A}(\mathbf{x}) \geq 0 \text{ for all } \mathbf{x} \in \mathbf{K}\}$ is called the *dual cone* of **K**. $\mathbf{K}^+$ is itself a convex cone in $\mathbf{B}^*$ that contains **0**.

It is a fact that if **K** is a closed convex cone then $\mathbf{K}^{++} \equiv (\mathbf{K}^+)^+ = \mathbf{K}$. Also, with **K** being a cone, by $\mathbf{x} \leq \mathbf{y}$ one always means $\mathbf{y} - \mathbf{x} \in \mathbf{K}$.

Definition A2.11: Let **B**, and **Y** be real Banach spaces, and **A** the operator **A**: $\mathbf{B} \rightarrow \mathbf{Y}$ an operator on **B**, Then:

(1) **A** is said to be *hemicontinuous* if and only if

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[214] Since **O** is an open set, we can always find a $\delta > 0$ such that the quantities $\mathbf{f}(\mathbf{x}+\mathbf{h})$ and $\mathbf{f}(\mathbf{x}+\theta\mathbf{h})$ are well defined whenever $\|\mathbf{h}\| < \delta$ (resp $\|\theta\mathbf{h}\| < \delta$). This justifies the requirement that **O** be open in the definition of Fréchet and Gâteaux differentiability.

[215] There are numerous equivalent statements for each one of the definitions given here. These were chosen for their simplicity and shortness. See Cioranescu (1990) for more details and other related concepts.

the function $t \to \langle A(p + tq), s \rangle$ is continuous for all $\mathbf{p}, \mathbf{q} \in \mathbf{B}$, and $s \in \mathbf{Y}$.

(2) **A** is said to be *demicontinuous* if and only if

$\mathbf{p}_n \to \mathbf{p}$ implies $\mathbf{A}(\mathbf{p}_n) \rightharpoonup \mathbf{A}(\mathbf{p})$ for all sequences $(\mathbf{p}_n)$ of **B**.

(3) **A** is said to be *bounded* (resp *locally bounded*) if and only if

A maps bounded sets into bounded sets (resp each $\mathbf{p} \in \mathbf{B}$ has a neighborhood $\mathbf{N}(\mathbf{p})$ such that $\mathbf{A}(\mathbf{N}(\mathbf{p}))$ is bounded).

(4) **A** is said to be *Lipschitz* on $\mathbf{M} \subset \mathbf{B}$ if and only if there exists $k \geq 0$ such that

$\|\mathbf{Ap} - \mathbf{Aq}\| \leq k\|\mathbf{p} - \mathbf{q}\|$ for all $\mathbf{p}, \mathbf{q} \in \mathbf{M}$.

(5) **A** is said to be *locally Lipschitz* if and only if each $\mathbf{x} \in \mathbf{B}$ has a neighborhood $\mathbf{V}(\mathbf{x})$ on which **A** is Lipschitz.

If $\mathbf{Y} = \mathbf{B}^*$, the dual of **B**, Then

(6) **A** is said to be *monotone* (resp *strictly monotone*) if and only if

$\langle \mathbf{Ap} - \mathbf{Aq}, \mathbf{p} - \mathbf{q} \rangle \geq 0$ (resp > 0) for all $\mathbf{p}, \mathbf{q} \in \mathbf{B}$; with equality if $\mathbf{p} = \mathbf{q}$.

(7) **A** is a *potential* operator if and only if

there exists a Gâteaux-differentiable functional $\mathbf{F}: \mathbf{B} \to \mathbb{R}$ such that $\mathbf{A}=\mathbf{F}'$. **F** is then called a potential of **A** and is obtained from **A** by:

$$\mathbf{F}(\mathbf{p}) = \int_0^1 \langle \mathbf{A}(t\mathbf{p}), \mathbf{p} \rangle dt \qquad \text{for all } \mathbf{p} \in B$$

(8) if **A** is a linear and continuous operator, then **A** is *positive* (resp *strictly positive*) if and only if

$$\langle \mathbf{A}(\mathbf{p}), \mathbf{p} \rangle \geq 0 \ \ (\textit{resp}\ \langle \mathbf{A}(\mathbf{p}), \mathbf{p} \rangle > 0)\ \textit{for all}\ \mathbf{p} \in \mathbf{B}.$$

A is *negative* (resp *strictly negative*) if and only if -**A** is positive (resp strictly positive).

(9) if **A** is a linear and continuous operator, then **A** is *symmetric* if and only if

$$\langle \mathbf{A(p)}, \mathbf{p} \rangle = \langle \mathbf{A(q)}, \mathbf{p} \rangle \textit{ for all } \mathbf{p}, \mathbf{q} \in \mathbf{B}.$$ [216]

We recall the following facts the proofs of which are standard and can be found, for example, in Rudin (1991), Aubin (1979), Zeidler (1986) and Cioranescu (1990).

Fact 1: A closed and convex subset of a Banach space is weakly closed.

Fact 2: A subset of a Banach space is norm bounded if and only if it is weakly bounded.

Fact 3: A weakly closed and bounded subset of a reflexive Banach space is weakly compact.

Fact 4: let **B** be a real reflexive Banach space, and $\mathbf{F}: \mathbf{M} \subseteq \mathbf{B} \to \bar{\mathbf{R}}$ a quasi-convex functional on the closed convex subset **M** of **B**. Then :

(1) The following three assertions are equivalents:

- **(i)** **F** is lower semicontinuous.
- **(ii)** **F** is weakly lower semicontinuous.
- **(iii)** **F** is weakly sequentially lower semicontinuous.

(2) If **M** is bounded, then **F** is bounded below.

(For the equivalence between **(ii)** and **(iii)** the reflexivity of **B**, and the quasi-convexity of **F** are not needed).

Fact 5 (Generalized Weierstrass theorem): let **X** be a topological space, and $\mathbf{F}: \mathbf{M} \subseteq \mathbf{X} \to \bar{\mathbf{R}}$ a functional on subset **M** of **X**. Then, the minimization problem:

$$\min_{\mathbf{x} \in \mathbf{M}} \mathbf{F}(\mathbf{x}) = \alpha$$

has a solution when **F** is lower semicompact.

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[216] The definitions of positivity and strict positivity correspond in finite dimension linear spaces to the concepts of positive semi-definite and positive definite of matrices respectively. Likewise for the definition of symmetry (recall that linear functions in finite dimension linear spaces are continuous and can be represented by matrices of finite number of rows and columns). Also, for a linear operator positivity is equivalent to monotonicity.

Fact 6: let **B** be a real Banach space, $\mathbf{B}^*$ its dual and **K** a cone of **B** Then,

$$\inf_{\mathbf{x}\in\mathbf{K}} \langle \mathbf{x}, \mathbf{q} \rangle = \begin{cases} 0 & \text{if } \mathbf{q} \in \mathbf{K}^+ \\ -\infty & \text{if } \mathbf{q} \notin \mathbf{K}^+ \end{cases}$$

Fact 7: let **B** be a real Banach space, and $\mathbf{B}^*$ its dual. Then:

(1) **B** is uniformly convex (resp uniformly smooth) if and only if $\mathbf{B}^*$ is uniformly smooth (resp uniformly convex).

(2) **B** is reflexive if it is uniformly convex.

(3) $L_p(\Omega, \mathbf{R}^n)$ is uniformly convex if and only if $1 < p < \infty$.

Finally, the following theorems are proved in Zeidler (1986, p. 194, Proposition **40.B**; p. 247, Proposition **42.6**; p. 250, Proposition **42.11**; and 1990, p. 555, Propositions **26.3** and **26.4**).

Theorem A2.3: *Let* $\mathbf{F}: O \subset \mathbf{B} \to \mathbf{R}$ *be a functional on the open subset of the real Banach space* B. *Then the following hold*:

(1) Necessary condition. *If* **F** *has a free local minimum at* $\mathbf{x}_0$, then

$\mathbf{F}'(\mathbf{x}_0) = \mathbf{0}$ (generalized Euler equation)

when $\mathbf{F}'(\mathbf{x}_0)$ *exists as a Gâteaux derivative or as a Fréchet derivative.*

(2) Sufficient condition. *Let n be a even number,* $n \geq 2$. *Then,* **F** *has a free strict local minimum at* $\mathbf{x}_0$ *when the following two conditions are fulfilled*:

(i) $\mathbf{F}^{(k)}(\mathbf{x}_0) = \mathbf{0}$, $k = 1,\ldots,n-1$, *and* $\mathbf{F}^{(n)}(\mathbf{x}_0) \geq c\|\mathbf{h}\|^n$ *for all* $\mathbf{h} \in \mathbf{B}$ *and fixed* $c > 0$.

(ii) **F** *is n-times Fréchet differentiable in a neighborhood of* $\mathbf{x}_0$ *and* $\mathbf{F}^{(n)}$ *is continuous at* $\mathbf{x}_0$.

Theorem A2.4: *Let* **F**: **B** → **R** *be a functional on the real Banach space* B. *Suppose the G-derivative* **F**′: **B** → **B*** *exists on* **B**. *Then the following hold*:

(1) *The following three assertions are equivalent:*

(i) **F** *is convex on* **B**.

(ii) **F**′ *is monotone on* **B**.

(iii) $\mathbf{F(q) - F(p)} \geq \langle \mathbf{F'(p)}, \mathbf{q - p} \rangle$ *for all* $\mathbf{p, q} \in \mathbf{B}$.

(2) *if* **F** *is convex on* **B** *and* **B** *is reflexive, then* **F**′ *is monotone and demicontinuous on* **B**.

(3) *The following three assertions are equivalent:*

(i) **F** *is strictly convex on* **B**.

(ii) **F**′ *is strictly monotone on* **B**.

(iii) $\mathbf{F(q) - F(p)} > \langle \mathbf{F'(p)}, \mathbf{q - p} \rangle$ *for all* $\mathbf{p, q} \in \mathbf{B}$; *with equality only if* $\mathbf{p} = \mathbf{q}$.

Theorem A2.5: *Let* **A**: **B** → **B*** *be an operator on the real Banach space* B.

(1) *The following three assertions are equivalent:*

(i) **A** *is a monotone potential operator.*

(ii) **A** *is monotone hemicontinuous and satisfies the following integrability condition*:

$$\int_0^1 \langle \mathbf{A}(t\mathbf{p}), \mathbf{p} \rangle dt - \int_0^1 \langle \mathbf{A}(t\mathbf{q}), \mathbf{q} \rangle dt = \int_0^1 \langle \mathbf{A}(\mathbf{q} + t(\mathbf{p} - \mathbf{q})), \mathbf{p} - \mathbf{q} \rangle dt$$

$$\text{for all } \mathbf{p}, q \in B$$

(iii) **A** *is a potential operator, i.e.,* **A** = **F**′, *and* **F** *is convex and weakly sequentially lower semicontinuous on* **B**.

(2) *if the demicontinuous G-derivative* **A**′ *exists on* **B**, *then the following two assertions are equivalent.*

(i) **A** *is a monotone potential operator.*

(ii) **A**′ *satisfies the following two properties:*

(a) *Symmetry:* $\langle \mathbf{A}'(\mathbf{p})\mathbf{q}, \mathbf{s}\rangle = \langle \mathbf{A}'(\mathbf{p})\mathbf{s}, \mathbf{q}\rangle$ for all $\mathbf{p}, q, s \in B$,

(b) *Positivity:* $\langle \mathbf{A}'(\mathbf{p})\mathbf{q}, \mathbf{q}\rangle \geq 0$ for all $\mathbf{p}, q \in B$.

(3) *if* **A** *is a monotone potential operator, then* **A** *is demicontinuous.*

Theorem A2.6: *Let* $\mathbf{A}$: $\mathbf{B} \to \mathbf{B}^*$ *be an operator on the real Banach space* B. *Then:*

(i) *If* **A** *is monotone, then* **A** *is locally bounded.*

(ii) *If* **A** *is linear and monotone, then* **A** *is continuous.*

(iii) *If* **A** *is monotone and hemicontinuous on the real reflexive Banach space* **B**, *then* **A** *is demicontinuous.*

(iv) *If* **A** *is demicontinuous on the real reflexive Banach space* **B**, *then* **A** *is locally bounded.*

APPENDIX 3

NECESSARY FIRST ORDER CONDITION FOR OPTIMALITY AND DIFFERENTIABILITY

Necessary First Order Condition for Optimality and Proofs of Differentiabity Lemma

A3.1 The Necessary First Order Condition for Optimality

To apply Corollary **A2.1** of Appendix **2** to the maximization problem (3.2.2), we set, in the notation of the corollary, $\mathbf{F}(\mathbf{x}) = -\mathbf{U}(\mathbf{x}) + \lambda\langle \mathbf{p}, \mathbf{x}\rangle$ and $\mathbf{N} = -\mathbf{I}$, where $\mathbf{I}$ is the identity of $L_2(\Omega, \mathbf{R}^n)$.[217] Thus the condition $\mathbf{Nx} \leq \mathbf{0}$ reduces to the non negativity constraint $\mathbf{x} \geq \mathbf{0}$. Since $\langle \mathbf{q}, \mathbf{x}\rangle = \sum_{i=1}^{n} \langle \mathbf{q}_i, \mathbf{x}_i\rangle$, we have for all $\mathbf{q}, \mathbf{x} \in L_2(\Omega, \mathbf{R}^n)_+$

$\langle \mathbf{q}, \mathbf{x}\rangle = 0$ *iff* $\langle \mathbf{q}_i, \mathbf{x}_i\rangle = 0$ $i=1,\ldots,n$ and $\mathbf{q}_i = 0$ if $\langle \mathbf{q}_i, \mathbf{x}_i\rangle = 0$ and $\mathbf{x}_i > \mathbf{0}$. In addition, the weak coerciveness is trivially satisfied (since $\lim_{|\mathbf{q}|\to\infty} \langle \mathbf{q}, \mathbf{x}_0\rangle = +\infty$ for any $\mathbf{x}_0 \in L_2(\Omega, \mathbf{R}^n)_{++}$). Hence, **(3.2.3)** follows from the necessary condition **(A2.7)** of the corollary. ■

A3.2 Proofs of the Differentiabity Lemma

Proof of Lemma 3.3.1

To show that the functional $\mathbf{T_u}: L_2(\Omega, \mathbf{R}^n) \to L_2(\Omega, \mathbf{R})$ defined by: $\mathbf{T_u}(\mathbf{x}) = \mathbf{u}\circ\mathbf{x}$ is Fréchet differentiable in $L_2(\Omega, \mathbf{R}^n)$ with $\mathbf{T_u}' = \mathbf{u}'\circ\mathbf{x}$, we need to show that

$\lim_{\theta\to 0} \dfrac{\mathbf{u}\circ(\mathbf{x} + \theta\mathbf{h}) - \mathbf{u}\circ\mathbf{x} - (\mathbf{u}'\circ\mathbf{x})\cdot\mathbf{h}}{\theta} = \mathbf{0}$ uniformly with respect to $\mathbf{h}$ on each bounded set of $L_2(\Omega, \mathbf{R}^n)$.[218]

Let $\epsilon > 0$, and let $\mathbf{B}$ be a bounded subset of $L_2(\Omega, \mathbf{R}^n)$. Then the set $\mathbf{R}(\mathbf{B}) \equiv \{\mathbf{h}(\omega)\text{: } \mathbf{h} \in \mathbf{B}$ and $\omega \in \Omega\}$ is a bounded subset of $\mathbf{R}^n$ (otherwise $\mathbf{B}$ would not be bounded). Now, since $\mathbf{u}$ is differentiable at each $\mathbf{x}(\omega)$, $\omega \in \Omega$, we have

$\lim_{\theta\to 0} \dfrac{\mathbf{u}(\mathbf{x}(\omega) + \theta\mathbf{h}(\omega)) - \mathbf{u}(\mathbf{x}(\omega)) - (\mathbf{u}'(\mathbf{x}(\omega))\cdot\mathbf{h}(\omega)}{\theta} = 0$ uniformly with respect to $\mathbf{h}(\omega)$ on $\mathbf{R}(\mathbf{B})$.

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[217] We could have applied the corollary directly to the maximization problem (3.2.1). The end result would be the same.

[218] This general definition of Fréchet differentiability (which valid even for non normed space) is equivalent to the usual definition given for normed spaces (see Yamamuro, 1974).

Thus it exists $\eta > 0$ (which depends only on ϵ and **B**) such that $|\theta| < \eta$ implies $\left\| \frac{\mathbf{u}(\mathbf{x}(\omega) + \theta\mathbf{h}(\omega)) - \mathbf{u}(\mathbf{x}(\omega)) - (\mathbf{u}'(\mathbf{x}(\omega))\cdot\mathbf{h}(\omega)}{\theta} \right\| < \epsilon$ for all $\omega \in \Omega$, and $\mathbf{h} \in \mathbf{B}$. The positivity of the integral operator then implies $\int_\Omega \left\| \frac{\mathbf{u}(\mathbf{x}(\omega) + \theta\mathbf{h}(\omega)) - \mathbf{u}(\mathbf{x}(\omega)) - (\mathbf{u}'(\mathbf{x}(\omega))\cdot\mathbf{h}(\omega)}{\theta} \right\|^2 d\mathbb{P}(\omega) < \epsilon^2$. That is, $\left\| \frac{\mathbf{u}\circ(\mathbf{x} + \theta\mathbf{h}) - \mathbf{u}\circ\mathbf{x} - (\mathbf{u}'\circ\mathbf{x})\cdot\mathbf{h}}{\theta} \right\|_{L_2} < \epsilon$ for all $\mathbf{h} \in \mathbf{B}$, whenever $|\theta| < \eta$. Hence showing the uniform convergence with respect to **h** on **B**. ■

Proof of Lemma 3.4.1

This lemma follows directly from Lemma 3.3.1, and the fact that the Hilbertian norm $h: \mathbf{x} \mapsto \|\mathbf{x}\|$ is Fréchet differentiable everywhere except the origin **0** with

$$h'(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|} \quad \text{for all } \mathbf{x} \in L_2(\Omega, \mathbf{R})\backslash\{\mathbf{0}\} \tag{A3.1}$$

Hence, by the chain rule we have

$$f'(\mathbf{x}) = h'(\mathbf{T_g}(\mathbf{x})) \circ \mathbf{T_g}'(\mathbf{x}) \quad \text{for all } \mathbf{x} \in L_2(\Omega, \mathbf{R})\backslash\{\mathbf{0}\} \tag{A3.2}$$

But,

$$\begin{aligned}
\left(h'(\mathbf{T_g}(\mathbf{x})) \circ \mathbf{T_g}'(\mathbf{x})\right)\cdot\mathbf{h} &= h'(\mathbf{T_g}(\mathbf{x}))\cdot(\mathbf{T_g}'(\mathbf{x})\cdot\mathbf{h}) \\
&= \langle h'(\mathbf{T_g}(\mathbf{x})), \mathbf{T_g}'(\mathbf{x})\cdot\mathbf{h}\rangle \\
&= E\left\{h'(\mathbf{T_g}(\mathbf{x}))(\mathbf{T_g}'(\mathbf{x})\cdot\mathbf{h})\right\} \\
&= E\left\{\left(h'(\mathbf{T_g}(\mathbf{x}))\mathbf{T_g}'(\mathbf{x})\right)\mathbf{h}\right\} \\
&= \langle h'(\mathbf{T_g}(\mathbf{x}))\mathbf{T_g}'(\mathbf{x}), \mathbf{h}\rangle
\end{aligned} \tag{A3.3}$$

for all $\mathbf{h} \in L_2(\Omega, \mathbf{R}^n)$.

Hence, by identification $f'(\mathbf{x}) = h'(\mathbf{T_g}(\mathbf{x}))\mathbf{T_g}'(\mathbf{x}) = \frac{(\mathbf{g} \circ \mathbf{x})(\mathbf{g}' \circ \mathbf{x})}{\|\mathbf{g} \circ \mathbf{x}\|}$ for all $\mathbf{x} \in L_2(\Omega, \mathbf{R})\backslash\{\mathbf{0}\}$. ■

APPENDIX 4

CONCEPTUAL FRAMEWORK OF INTERTEMPORAL CHOICE UNDER UNCERTAINTY

APPENDIX A4.1

A RECONSIDERATION OF SAVAGE'S CONCEPTUAL FRAMEWORK OF CHOICE UNDER UNCERTAINTY AND ITS EXTENSION TO INTERTEMPORAL CHOICE

A Reconsideration of Savage's Conceptual Framework of Choice under Uncertainty and its Extension to Intertemporal Choice[219]

A4.1.1 Introduction

In this appendix we a develops conceptual framework based on Savage's (1954) framework of choice under uncertainty. The motivation for this conceptual framework was a need to justify the mathematical approach to choice under uncertainty developed in chapter 2. Besides justifying the mathematical framework, the interpretation of some of the mathematical and economic results derived in chapter 3 makes sense only within the Bayesian framework of Ramsey (1926), De Finetti (1937), Savage (1954), and Anscombe and Aumann (1963) among others, which adopts the view of *subjective* or *personalistic* probability. For example, it is natural to think that if a decision maker's welfare is affected by the outcome of a random phenomenon outside of its control, then both welfare and behavior should depend on the information related to the random phenomenon along with its probability distribution. This dependency should hold regardless of the *subjective* or *objective* nature of this probability distribution. But, in some of the examples worked out in chapter 3, this is not the case. In chapter 3 it was shown that for a consumer maximizing an expected utility with a multivariate Von Neumann-Morgenstern Cobb-Douglas utility that is homogenous of degree one, under price uncertainty and budget constraint, both the indirect utility and demands are independent of the probability distribution of prices. More than that, the model implies that some nonlinear function of the random prices must be independent of the state of nature (i.e, must have a degenerate probability distribution). Clearly this last result cannot be interpreted within the *objective probability* view within which prices and their (*objective*) joint probability

[219] The discussion in this appendix is strictly at the conceptual level, and a conscious effort was made to keep it informal.

distribution are assumed to be exogenously given to the consumer, so that there is no reason why the (*objective*) probability distribution of a transformation of random prices must be degenerate for a particular consumer. If there are many consumers with different preferences, we end up with several probability distributions for the same random vector of prices, which is self-contradictory for a concept of a unique *objective* probability distribution of prices facing all consumers.

Within the *subjective probability* view, one can interpret the above result by saying that this particular consumer, who has a homogenous of degree one Cobb-Douglas expected utility, *believes* that the value taken by this nonlinear function of price is independent of the state of nature. In other words, he or she has degenerate "beliefs" for this nonlinear transformation of prices regardless of any pertinent information. The implications of the examples may represent very strong restrictions on individual "belief". Nevertheless, it makes sense within this *subjectivistic* view where decision makers are assumed to be free to choose whatever "beliefs" they want to hold.

Although the *subjectivistic* view provides a much more appropriate context for interpreting the results of the mathematical analysis, further intuitive, conceptual, and mathematical considerations led us to reconsider Savage's original framework by modifying some of his terminology and adding some new concepts. From the mathematical side, these conceptual changes allow us to extend more easily Savage's framework to cover intertemporal choice, thus having a unified treatment of the four main decision models in economics: static deterministic choice, dynamic deterministic choice, static choice under uncertainty, and dynamic choice under uncertainty. The mathematical tools which allow this unified treatment are indicated.

Another departure from Savage's framework is our treatment and formalization of the primitive concept of *beliefs*. Savage's *subjective probability* representing *beliefs* is based on the underlying *preference*. Here, we argue that the two primitive concepts of *beliefs* and *preference* are two distinct psychological concepts that exist independently from one another. Consequently, we postulate at the outset the independent existence of a *likelihood ordering* among events, and hence adopt the approach of De Finetti (1937) and Chateauneuf (1985) among others, who represent it by a *subjective* probability distribution independently of preference.

Finally, we identify a third psychological concept distinct from the ones of *beliefs* and *preference*, and which is called *perception*. We argue that many aspects of behavior that are usually attributed to either *beliefs* or *preference*, are in fact the results of the decision maker's *perception* of the *intrinsic* properties of the objects of choices, as well as the *intrinsic* relationships existing among them. Moreover, the disentanglement of the concept of *perception* from *preference* and *beliefs* clarifies the true meaning of the so-called phenomenon of "state-dependent preference". The separation of the three concepts also simplifies the mathematical formalization and analysis of intertemporal choice under uncertainty.

The appendix is organized as follows: Section **A4.1.2** presents briefly the main concepts used to describe decision processes. Section **A4.1.3** discusses Savage's conceptual framework of choice under uncertainty along with changes introduced here. Section **A4.1.4** uses the concept of *perception* to clarify the true meaning of so called "state-dependent" and "time-dependent" *preferences*. Section **A4.1.5** extends Savage's conceptual framework to cover intertemporal choice. Section **A4.1.6** discusses the methodological implications of the conceptual framework. Finally, Section **A4.1.7** is a short conclusion. Appendix **A4.2** contains part of an axiomatic presentation of the conceptual framework.

A4.1.2 Aspects of Decision Making: *Preference, Beliefs, Perception, and Act*

In any choice or decision making situation, we conceptually recognize the interaction of two types of objects: *physical objects* which are elements of the physical environment, and *psychological objects* which can be attributed to the decision maker. The link between the physical and psychological objects is *behavior*, which consists of the *acts* or *actions* taken by the decision maker.

The relevant objects describing the physical environment of the decision maker are the *time*, the (uncertain) *states of nature*, and the things which are usually referred in economics as *commodities or goods* and which are the objects of choice. Philosophical issues aside, there is unanimous agreement on the existence and nature of these physical objects, because they tend to have an "objective" character.

There are three psychological objects in possession of the decision maker: *preference, beliefs,* and *perception*. In contrast to the physical ones, the nature and properties of these psychological concepts are sources of disagreement among behavioral scientists.

Preference embodies three aspects that guide behavior: the aspect of *taste,* which describes relative strengths of desire for commodities of different types, the aspect of *attitude toward uncertainty* or *risk*, and the aspect of *patience/impatience* or *time preference*. These three aspects combined dictate substitutions across commodities, states of nature, and time. An important analytical and empirical issue that is partly addressed in this thesis, is how one can isolate their respective effects by observing overt choices. In general, the disagreements about preference concern its properties (ordering, completeness, transitivity, etc....), not its assumed existence.

Beliefs describe the decision maker's appraisal of the relative likelihoods of the different events which contain the various possible states of nature. The issues regarding the concept of *beliefs* that are debated in the economic literature can be classified into two broad questions: (1) Can the concept of *beliefs* have any objective meaning, or should it be considered exclusively as a personal or subjective thing? In other words, do individuals always agree on the same set of *beliefs* when evaluating the relative likelihoods of various events? **(2)** Given that one adopts the subjectivistic view, **(i)** to what extent are *beliefs* independent from preference? **(ii)** To what extent are individuals free to choose the *beliefs* they want to hold? **(iii)** And, if indeed they are free to choose their *beliefs*, which criteria do they use to choose them? In this thesis, we adopt the subjectivistic view of Ramsey (1926), De Finetti (1937) and Savage (1954). That is, we consider *beliefs* as a personal matter which exist independently of preference, and that decision makers choose freely their *beliefs* about the relative likelihoods of the different events.[220]

Perception is another psychological concept that is not usually distinguished (at least explicitly) from the two other concepts of *preference* and *beliefs*. We argue that a decision maker's *perception* of the intrinsic physical characteristics of objects, along with the intrinsic relationships among them, exists independently of his or her *preference* and *beliefs*, and must be analyzed and formalized separately.[221] We identify four aspects pertaining to *perception*: a *"set"* aspect embodying the rules for which some objects are perceived to belong to a given set and others not;. an *"algebraic"* aspect embodying the rules for combining objects

[220] To avoid misunderstanding regarding what we mean by this sentence, we want to clarify two points. First, the level of independence between *preference* and *belief* argued here is at the existence level as two different concepts. We are not ruling out the fact that they jointly determine behavior, in the sense that someone's optimal choices must be consistent with his or her *beliefs* (see the examples in chapter 3). We are also not ruling out the fact that one may have preference among the different possible states of nature. But still, having a preference for one possible state of nature over another is different from your opinion about which one is likely to occur. Second, by assuming free choice of *beliefs*, we are not ruling out the fact that the chosen *beliefs* may be influenced by one's own past *beliefs* and other individuals' *beliefs*. The argument here is similar to the fact that while economists recognize that an individual's *preference* may be influenced by others' preferences (e.g., envy or altruism) they still take *preference* as an individual matter.

[221] Again, the independence of the three concepts here is at the existence level, and we are not ruling out the fact that they jointly determined optimal choices. For instance, one has to *perceive* first differences among the possible states of nature (i.e., differences between rainy, sunny, and cloudy days for example) before one can formulate *preference* among them and/or has an *opinion* about their relative likelihood of occurrence.

belonging to the same set; an *"order"* aspect ranking objects according to some intrinsic characteristics (weight for example); and, a *"similarity"* or *"closeness"* aspect defining the "similarities/dissimilarities" among objects, including all the relevant intrinsic characteristics that distinguish objects of a same set from one another. All these aspects of the decision maker's *perception* are independent of his or her *preference* and *beliefs*. For example, with respect to the *"order"* aspect, the decision maker can perceive a relationship between one kilogram of rice and two kilograms of rice, regardless of his or her *preference* and his or her *beliefs* about the various possible states of nature. With respect to the *"similarity"* or *"closeness"* aspect, the intrinsic similarities and dissimilarities existing among chicken, turkey, and beef, can lead one to perceive that one kilogram of turkey is "closer" to one kilogram of chicken than one kilogram of beef is. This perception is regardless of any *preference* one may hold for any one of them, and regardless of any *beliefs* on the relative likelihoods of events.

An *act* is a behavioral concept which characterizes the choice or decision making process. It is a mechanism by which the psychological objects are related to the physical ones. The question of what guides *acts* (that is, what rule decision makers follow when making their choices) is a controversial question. In economics, the most commonly used assumption is that of *preference* or *satisfaction maximization*. That is, a decision maker ranks all the elements in his or her choice set according to his or her *preference*, and then *acts* so as to reveal the element with the highest rank as his or her most preferred choice. If one admits the existence of a *preference* that can rank all the elements in the choice set, and if this choice set is finite, then it is not difficult to imagine a finite-step mechanism by which the element with the highest rank can be selected. However, when the choice set is infinite and/or uncountable, the existence of a finite-step mechanism that effectively makes the selection of the element with the highest rank feasible can be at best an assumption.[222] *Satisfaction maximization* is the assumption we have adopted in this thesis, both with respect to *acts*, and with respect to the choice of *beliefs*.[223]

A4.1.3 Reconsideration of Savage's Framework

A4.1.3.1 Acts, Consequences, and Outcomes

Besides the concepts of *states of nature*, *events*, *beliefs*, and *preference*, the main primitive concepts which distinguish Savage's conceptual framework of choice under uncertainty from others are the concepts of *acts* and *consequences* (which he used interchangeably with *outcomes*). Savage defines an *act* as a mapping from the set of *states of nature* to the set of *consequences* (or *outcomes*). The interpretation is that: in a choice situation involving uncertainty, you act, and nature determines the consequence (outcome) of your act. *Preference* is then among alternative feasible *acts*. We found it intuitively and mathematically more convenient to make the distinction between *ex-ante consequence* that we simply call *consequence*, and *ex-post consequence* that we call *outcome*. We also introduce the additional primitive concept of *options* which have no counterpart in Savage's framework and which allow us to reinterpret Savage's *acts* in a way that we found more intuitive. We take an *act* to be a mapping from the set of *options* to the set of *consequences*, while a *consequence* is a mapping from the set of *states of nature* to the set of *outcomes* (see

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[222] See Dudley (1989, p.14) for a discussion of a similar problem related to the axiom of choice in mathematics.

[223] Note that here there is only one objective, *satisfaction*, which is maximized by both the chosen *acts* and *belief*. By this we mean that at the moment where decisions are being made, there no other feasible *actions* you can take and no other *belief* you can hold and which can *ex-ante* increase your *satisfaction*. However, this assumption of *optimality* of *beliefs* is not formally used in any part of the thesis. It is adopted and stated here for the sole purpose of rationalizing the restrictions on consumer's *beliefs* implied by some of the expected utility examples in chapter 3. That is, those type of *beliefs* on prices are the only ones consistent with the maximization of homogenous-of-degree one expected utility functionals. Note that even if the *optimality* of *beliefs* assumption is imposed, they may be a multitude of beliefs that are consistent with the implied restrictions.

Appendix **A4.2** for more precise definitions). Hence, what we now call *consequence* corresponds to Savage's *act* while our *outcome* is Savage's *consequence*. These changes lead to the following alternative interpretation: in a choice situation you have a set of options, and you act by taking an option; the consequence of your act is ex-ante random; nature will ex-post determine the outcome of your act. *Preference* is among alternative *consequences* of your *acts*".

Savage's choice of the domain of an act as being the set of all possible states of nature, does not sound intuitive to us because it suggests that there is something random in decision making per se.[224] But, we think of an act (at least from the decision maker's perspective) as a *deterministic* action that a decision maker takes or is planning to take. In a intertemporal setting, an act may be contingent on the occurrence of an event in the future. But even in this case, when comes time to *act* or not to *act* (doing nothing being considered as an act), there will be nothing random in the process of deciding. Hence, we suggest that both *ex-ante* and *ex-post*, there is nothing random in the process of making a decision or acting (i.e, the decision to do something). For us, what is *ex-ante* random is really the *consequence* of the *act*, which can yield several possible *outcomes* depending on which state of nature gets realized. There is no *ex-ante* uncertainty in Savage's definition of the *consequence* of an *act* since mathematically it is the realized value of a random variable. While we consider this to be true *ex-post* when the true state of nature is revealed, and the many possible *ex-ante* outcomes reduce to a single *ex-post* outcome, we feel that intuitively, in the face of uncertainty, the consequence of an act or action is *ex-ante* inherently random. This is the reason why we choose the term *outcome*, which is synonym of consequence in Savage's framework, to designate the *ex-post* realized value of the *consequence* of an *act*.

[224] In mathematical terms, Savage's *act* is basically a random variable with one possible realized value being what he calls a *consequence*.

The intuitive justifications for these changes can be illustrated by the following examples. The first example is the case of a person who is making a choice of what to have for dinner. She has two possibilities to choose from: either she can go to a restaurant specialized in chicken or she can go to a restaurant specialized in beef. We suppose that the uncertainty is about the quality of the beef q_b and that of chicken q_c. q_b and q_c can take either the value 1 (for good) or the value 1/2 (for bad). Table A4.1.3.1 contrasts Savage's terminology for describing the choice situation with the changes we have proposed. Table A4.1.3.2 makes the same comparison with an example borrowed from Karni (1993). It is about a person who is deciding whether or not to go to the stadium to watch a football game, or to stay at home and watch the game on television. The uncertainty in this example is about the weather, ω, which can be of two states: sunny or rainy. This example will be used later to discuss the so called phenomenon of "state-dependent preference".

We note that in the proposed changes illustrated in the two examples, each act leads to a well defined consequence so that by choosing one act, the decision maker is revealing his or her preference for the consequence it leads to. We also note that the decision maker cannot choose among the four outcomes, because it cannot "control" any one of them. All he or she can determine by his *acts* are the two consequences. In general, this will always be the case.[225] By looking at the two tables it may seem to some readers that nothing is gained from the changes introduced here, since formally in both descriptions *preference* is still defined on the same objects which are just given different names. However, there are some gains in clarity both from conceptual and formal perspectives. First, one has to admit that there is a conceptual difference between the concepts of an *act* and its associated (*ex-ante*) *consequence*.

[225] We note that so far the concept of *beliefs* has not been introduced in the discussion. This contrasts with the standard practice in most of the theoretical risk literature that uses Savage's framework, where often an Savage's act is reinterpreted as a probability distribution over the sets of *outcomes* (i.e., as the probability distribution of Savage's act) (see, for examples, Aumann, 1963, Machina and Schmeidler, 1992).

Table A4.1.3.1: Options, acts, consequences, and outcomes: example of a person making her choice for dinner with uncertain quality.

Concepts	Savage's description	The proposed description
Options:	not applicable	two options: - option 1: chicken for dinner - option 2: beef for dinner
Acts:	two acts: - act 1: have a chicken for dinner with quality q_c - act 2: have beef for dinner with quality q_b	two acts: - act 1: the process of having chicken for dinner (i.e going to the chicken restaurant) - act 2: the process of having beef for dinner (i.e going to the beef restaurant)
Consequences:	four consequences (same as outcomes): - consequence 1: chicken for dinner with good quality ($q_c=1$) - consequence 2: chicken for dinner with bad quality ($q_c=1/2$) - consequence 3: beef for dinner with good quality ($q_b=1$) - consequence 4: beef for dinner with bad quality ($q_b=1/2$)	two consequences: - consequence 1: chicken for dinner with quality q_c - consequence 2: beef for dinner with quality q_b
Outcomes:	same as the consequences above	four possible outcomes (two for each consequence): - outcome 1: chicken for dinner with good quality ($q_c=1$) - outcome 2: chicken for dinner with bad quality ($q_c=1/2$) - outcome 3: beef for dinner with good quality ($q_b=1$) - outcome 4: beef for dinner with bad quality ($q_b=1/2$)
Preference:	between the two acts	between the two consequences

Table A4.1.3.2: Options, acts, consequences, and outcomes: example of a person deciding whether to watch a football game on TV, or go to the stadium to watch it there. The uncertainty is about the weather.

Concepts	Savage's description	The proposed description
Options:	not applicable	two options: - option 1: watch the football game at the stadium - option 2: watch the football game on TV
Acts:	two acts: - act 1: watching the football game at the stadium under weather ω - act 2: watching the football game on TV under weather ω	two acts: - act 1: the process of going to the stadium and buying a ticket - act 2: staying at home and making arrangements for watching the game on TV
Consequences:	four consequences (same as outcomes): - consequence 1: watching the football game at the stadium in a rainy day - consequence 2: watching the football game at the stadium in a sunny day - consequence 3: watching the football game on TV in a rainy day - consequence 4: watching the football game on TV in a sunny day	two consequences: - consequence 1: watching the football game at the stadium under weather ω - consequence 2: watching the football game on TV under weather ω
Outcomes:	same as the consequences above	four possible outcomes (two for each consequence): - outcome 1: watching the football game at the stadium in a rainy day - outcome 2: watching the football game at the stadium in a sunny day - outcome 3: watching the football game on TV in a rainy day - outcome 4: watching the football game on TV in a sunny day
Preference:	between the two acts	between the two consequences

The conceptual difference is even more perceptible in a intertemporal context where the (*ex-ante*) *consequence* of one single *act* taken at one point in time can span an entire period of time. Second, the concept of *option* is a very relevant concept which must be part of any realistic description of a decision process. It is different from the concept of *act* because it exists independently of the decision maker, while an *act* is always associated with a decision maker. Third, although in common parlance *consequence* is synonym of *outcome*, in a uncertainty context there exists a conceptual difference between *ex-ante consequence* (i.e., the perceived consequence of an act before the uncertainty is resolved) and *ex-post consequence* (i.e., what is perceived to be the consequence of an act once the uncertainty is resolved). Indeed, in a uncertainty context when one takes an act, he or she will not know the exact consequence until the uncertainty is resolved. The effect of uncertainty is to associate a list of several possible consequences to the same act with each consequence made possible by one possible state of nature. However, once the uncertainty is resolved there can only be one consequence associated with the realized state of nature.[226] For lack of better terms to differentiate between the two concepts of *ex-ante consequence* and *ex-post consequence*, we have called the first concept simply *consequence*, and the second one *outcome*.

Hence, the changes add some clarity in the conceptualization of decision making processes. For example, in the context of agricultural production decisions, the *options* are the inputs such as seed, fertilizer, labor, etc... The *acts* consist of the planting and weeding activities, and fertilizer applications. The (*ex-ante*) *consequences* are the levels of output respectively associated to the different levels of input use. The *outcomes* are the various levels of output at harvest respectively associated to the different levels of input use and possible states of nature (determined by the climatic conditions). The changes introduced here has also some empirical relevance, because they enable to better link directly the objects of

[226] If each possible consequence is a vector, then we have one vector of consequences associated to the realized state of nature.

the conceptual framework with the observable variables used in econometric analysis. For example, *consequence* is not observable but *outcome* is both observable and measurable. Similarly, *act* is observable but it is not measurable. On the other hand *option* is both observable and measurable.

Mathematically, these conceptual changes make it possible to analyze choice situations by defining a preference ordering in the consequence space which will invariably have the form $L^0(\Omega, \mathbf{F})$, the space of vector (or function) valued random variables with domain Ω, the set of states of nature, and taking value in $\mathbf{F}$, the outcome space.[227] In particular, the mathematical differences among the four general decision models used in economics (static choice under certainty, dynamic choice under certainty, static choice under uncertainty and dynamic choice under uncertainty) result only from a change in the consequence space and/or outcome space. Indeed, static decision making with perfect and complete information corresponds to the case where the consequence space and outcome space are both equal to the commodity space $\mathbf{R}^n$.[228] Intertemporal decision making with perfect and complete information corresponds to the case where the consequence and outcome spaces are both equal to $L^0([t, T], \mathbf{R}^n)$, the set of Borel measurable functions with domain an interval $[t, T]$ of the real line and taking values in $\mathbf{R}^n$ (that is, the space of time paths of cumulative consumptions, with t and T

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[227] The term "space" is merely a synonym for a set with some mathematical structure that goes beyond what makes it mathematically a set (i.e., a set in the set theoretic sense). The underlying structure of $L^0(\Omega, \mathbf{F})$ in question involves the partitioning of both the set of events Ω and the set of outcomes $\mathbf{F}$ into subsets (events in the case of Ω) that satisfy some "stability" properties. Roughly speaking, the elements of $L^0(\Omega, \mathbf{F})$ are those functions from Ω to $\mathbf{F}$ that "preserve" the structure of the two partitioning. $L^0(\Omega, \mathbf{F})$ is formally called a measurable space and its elements measurable functions or random variables (vectors or functions in general).

[228] We recall that the commodity space in the Arrow-Debreu framework is mathematically identified with the finite dimensional Euclidean space $\mathbf{R}^n$. Moreover, to accommodate uncertainty with a finite number of possible states, Arrow (1964) and Debreu (1959 and 1983) extended the definition of a commodity as follows: "...The definition of an economic commodity now specifies in addition to its physical characteristics, its date and its location, the state of the world in which it will be available."

being respectively the initial (decision) time, and the time horizon).[229] In static decision making under uncertainty the outcome space will be equal to the commodity space $\mathbf{R}^n$, while the consequence space becomes $L^0(\Omega, \mathbf{R}^n)$, the space of vector valued random variables defined on Ω and taking values in $\mathbf{R}^n$ (that is, the space of random commodity bundles). Finally, in intertemporal decision making with uncertainty the outcome space will be $L^0([t, T], \mathbf{R}^n)$ and the consequence space will be $L^0(\Omega, L^0([t, T], \mathbf{R}^n))$, the space of vector valued stochastic processes (or random functions) defined on Ω and taking values in $L^0([t, T], \mathbf{R}^n)$ (that is, the space of random time paths of cumulative consumptions).[230]

Hence, by considering a general choice space of the form $L^0(\Omega, \mathbf{F})$, and using functional analysis methods, one gets a unified mathematical treatment of the four principal decision models. This is because with the standard assumptions on preference (completeness, transitivity, continuity, etc...), a general numerical utility representation of preference on $L^0(\Omega, \mathbf{F})$ can be obtained, and general comparative statics results which depend only on the mathematical structure of the consequence space can then be derived using the usual calculus and convex analysis methods, with no particular attention to time or uncertainty issues. Moreover, a general and unified treatment allows one to avoid restrictive assumptions such as time additivity, recursivity, and/or "expected utility" and yet get very tractable models, both analytically and empirically. The analysis can be pursued such that time and uncertainty issues will only appear at the end of the mathematical analysis when one wants to assess and disentangle the respective effects due to attitude toward uncertainty, time preference, and

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229 Here we are following Hindy, Huang and Kreps (1992), and Hindy and Huang (1992) who argue that for both mathematical reasons and intuitive economic reasons, intertemporal consumption choice should be modelled as choice among alternative cumulative consumptions instead of the standard practice which take the choice to be among consumption rates.

230 With models of commodity differentiation (Mas-Collel, 1975; and John, 1984), $\mathbf{R}^n$ is replaced in all the spaces above by $M(\mathbf{R}^n)$ the space of finite signed measures defined on the Euclidean space $\mathbf{R}^n$, which is the commodity space in these models.

nature of the information flow (that is, when one wants to do comparative risk and/or comparative dynamic analysis).

A4.1.3.2 Preference and Beliefs

The second departure from Savage's conceptual framework stems from our view on *preference* and *"beliefs"*. Savage derives his *subjective probability* simultaneously with an expected utility representation of preference.[231] This is achieved by defining a *likelihood ordering* among events through the intermediary of the preference ordering among *acts*. The behavioral postulates leading to the derivation are such that likelihood ordering among events is consistent with preference ordering among acts, in the sense that a decision maker's preference for an act agrees with the event that he or she believes is most likely to occur. Hence, Savage's primitive concept of *preference* contains an expression of the decision maker's *beliefs*, and the separation of *beliefs* from *preference* as represented by the derived subjective probability and Von Neumann-Morgerstern utility respectively, is a result following from the axioms.[232]

In our setting, we keep strictly separate the *preference* ordering among consequences of acts and the *likelihood* ordering among events because we feel that they represent different psychological concepts. Thus, their separation is postulated at the outset. We view the *preference* among consequences as strictly expressing *desires* or *wants* and the relative *likelihood* of events as expressing *opinion* or *judgment* about uncertain events. It is important

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[231] Machina and Schmeidler (1992) have extended Savage's derivation to the case of "non-expected" utility.

[232] See also, Debreu (1959, p. 101), who, in his treatment of consumer theory under uncertainty includes in the usual definition of a commodity the state of nature under which it is available and postulates consumer preference to apply to this extended commodity space. He then states: "This *preference preordering* reflects the tastes of the consumer for goods and services (including, in particular, their spatial and temporal specifications), his personal appraisal of the likelihoods of the various events, and attitude toward risk."

at this conceptual level to distinguish between the primitive concept of *preference* defined on *ex-ante consequence*, and its representation by a numerical utility functional. Indeed, as we will argued later, continuity assumptions will require the utility functional representing *preference* among *ex-ante consequences* to depend on the decision maker's *beliefs*. Moreover, the von Neumann-Morgenstern utility in Savage's representation needs not depend on *beliefs* because it represents a "preference" ordering defined on *outcomes* and which is different from the one defined on *ex-ante consequences*.[233]

In many choice situations that involve uncertainty, the decision maker is only concerned with the consequences or possible set of outcomes of his or her decision regardless of which state of nature finally prevails (example: desire to earn some money, be it in a rainy or sunny day). In other situations, the decision maker can have a preference for a particular outcome being realized in a given state of nature (example: desire to earn some money while being in good state of health rather than being sick with the money). In this situation, the domain of the "preference" ordering over *outcomes* can be thought as extending to include the set of possible states of nature. But, even in this case, we can argue that the restriction of the extended "preference" ordering over *outcomes* to the set of possible states of nature is conceptually distinct from the likelihood ordering that is defined on the same set of possible states of nature. Indeed, the latter ordering is an assessment of which event or state of nature is likely to occur, while the former expresses a wish or desire for having a particular event to occur. For example, an individual may have the perception or belief that it is more likely to

[233] To be more precise, in chapter 3 we wrote the expected utility of an *ex-ante consequence* **x** as $U(\mathbf{x}) \equiv E\mathbf{u}(\mathbf{x}) \equiv \int_{\Omega} \mathbf{u}(\mathbf{x}(\omega))\, d\mathcal{P}(\omega)$. Where U is the utility functional representing preference among *ex-ante consequences*, E is the expectation operator, **u** is the von Neumann-Morgenstern utility function defined on *outcomes*, and $\mathcal{P}$ is the *subjective* probability distribution representing the decision maker's *beliefs*. As it is clear in this expected utility case, the Utility functional U depends on the *subjective* probability distribution $\mathcal{P}$, but the von Neumann-Morgenstern utility **u** does not. Savage's representation theorem is about the separation of **u** and $\mathcal{P}$. Throughout the thesis, when we speak of preference without qualification we always mean the one that ranks *ex-ante consequences*, which is the relevant one in choice under uncertainty (since one cannot *ex-ante* choose among (*ex-post) outcomes*).

rain tomorrow yet, would prefer tomorrow to be a sunny rather than a rainy day. Nevertheless, his or her preference over possible states of nature has presumably no effect on nature's choice.[234] These are the reasons why we maintain a conceptual and mathematical distinction between preference ordering and likelihood ordering. For us, they describe two conceptually distinct aspects of the decision making process which we want to formalize and analyze separately. The mathematical formalization of *preference* will be along the traditional lines which identify it with a binary ordering defined on the choice space (the set of *consequences*), and which ranks *consequences* according to their orders of preference. *Beliefs* will be mathematically identified with a separate binary ordering defined on the set of *events*, which ranks *events* according to their relative likelihoods of occurrence.

A4.1.3.3 The criterion for choosing "beliefs"

Savage's framework takes as given the fact that *beliefs* are personal matters, and that individuals are free to choose whatever *beliefs* about the possible states of nature they want to hold. This of course supposes that a decision maker has access, not just to one, but to a set of *likelihood* or *belief assessments*, from which he or she can choose one to rank all the events. How realistic is this assumption?

One can argue that in any decision making situation several alternative *belief assessments* are available to the decision maker (since different people do hold different *beliefs*), and that *beliefs* in themselves are matter of individual choice (i.e., not forced on individuals). In other words, individual decision makers choose to have a particular opinion or appraisal of the likelihood of possible events, which guides their actions given their

[234] Note that we do not rule out the fact that nature's choice may affect *preference* afterward. This phenomenon called *state-dependent preference* is discussed in section A4.1.4 below. What we do rule out is the possibility that an individual's *preference* itself may affect nature's choice. In other words, we take the source of uncertainty to be exogenously given (i.e., outside of the control of the decision maker).

preferences among alternative consequences of their acts. Indeed, it is natural to think that, while individual actions can be controlled or limited by outside forces (like in controlled economies or under dictatorship), so as to limit their freedom of choice (i.e., free to act in a manner that will yield the most preferred consequence); one can argue that individual beliefs are very difficult if not impossible to be totally controlled by outside forces. To be sure, *beliefs* can be influenced or manipulated. But, *ultimately, it is a matter of the individual decision maker freely choosing the set of belief assessments under which actions taken are likely to yield the most preferred outcome.*[235] For example, the emergence of "black" markets for currencies (or any other commodity for that matter) in some economies under a fixed and controlled exchange rate regime, is a clear indication of economic agents who, being concerned with the value of a currency, refuse to *believe* that the "official" exchange rate that is being forced on them reflects the true exchange value of the currency.

Outside of economics, there are several real life examples that indicate that *beliefs* are indeed matter of individual choices. One example is provided by the commonly observed organization of citizens in democratic societies along political party lines. Individuals in each political party can be considered as sharing the same set of *beliefs* concerning the likely outcomes resulting from political and economic decisions. In this context, an individual's free choice of a political party to belong can be considered as choice of a set of *beliefs*. Another example is provided by the environmental battles opposing groups of individuals sharing different set of *beliefs* about how human actions are affecting the ecological balance of the planet. In this case, someone who is taking side, is freely choosing a set of *beliefs* to hold regarding the likely impact of human actions on the environment.

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[235] One can draw here a parallel between the way an individual's *beliefs* is influenced by outside forces, and the way an individual's *preference* is influence by outside forces or other individuals' *preferences* (as in the case of *envy* or altruism), In both cases, one recognizes the interdependence of individuals' *preferences* and *beliefs* respectively. However, except in special problems where one is particularly interested on this interdependence, in most of economic analysis, one always takes *preference* as a personal matter.

It is important to see that the question of choosing among alternative *beliefs* is different from the more usual concept of Bayesian updating of given *beliefs* in light of new information. The distinction between the two concepts are perhaps best illustrated by quoting the following sentences from De Finetti (1937, p. 146).

> Whatever be the influence of observation on predictions of the future, it never implies and never signifies that we correct the primitive evaluation of the probability $\mathbf{P}(E_{n+1})$ after it has been *disproved* by experience and substitute for it another $\mathbf{P}^*(E_{n+1})$ which conforms to that experience and is therefore probably *closer to the real probability*; on the contrary, it manifests itself solely in the sense that when experience teaches us the result A on the first *n* trials, our judgment will be expressed by the probability $\mathbf{P}(E_{n+1})$ no longer, but by the probability $\mathbf{P}(E_{n+1} \mid A)$, i.e. that which our initial opinion would already attribute to the event E_{n+1} considered as conditioned on the outcome A. Nothing of this initial opinion is repudiated or corrected; It is not the function **P** which has been modified (replaced by another $\mathbf{P}^*$), but rather the argument E_{n+1} which has been replaced by $E_{n+1} \mid A$, and this is just to remain faithful to our original opinion (as manifested in the choice of the function **P**) and coherent in our judgment that our predictions vary when a change takes place in the known circumstances.

In the above sentences, De Finetti is essentially explaining that updating one's set of *beliefs* in light of new information does not mean changing your *beliefs*. Nonetheless, one still has to choose which set of *beliefs* to use and update in the first place (as it is implicit in De Finetti's statement: "as manifested in the choice of the function **P**)[236].

If indeed decision makers are free to choose their *beliefs,* which criterion do they use to select a *belief* from all the possible ones? There is a natural answer to this question if we follow the tradition in economics: The *optimizing behavior* assumption. That is, if individual beliefs are objects of choice, it must be the case that they are chosen *optimally* (with respect to the same objective of maximizing the *satisfaction* derived from the objects of choice).[237] In other words, among all the belief assessments available to the decision maker, the one actually chosen to assess the relative likelihood of events is the one that ***ex-ante*** maximizes his

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[236] Again, we do not rule out the possibility that the chosen *beliefs* are influenced by others' *beliefs* (see footnote 235).

[237] see footnote 223

or her welfare for all possible actions that he or she can take.[238] In a more general context, the *optimality of beliefs* postulate gives an hypothetical answer to the following question: why do people have a particular appraisal or opinion of the world which guides their actions in decision making situations? Or restated in other terms, why do they choose to hold some particular beliefs instead of others? The answer we give here is that, it is because the chosen beliefs, in a decision making situation, are the ones that *ex-ante* give them the most satisfaction for all possible actions they can take.[239]

In principle, with this *optimality of belief* assumption it is mathematically possible to derive the restrictions that characterize the *optimal beliefs* of individual decision makers by observing their overt choices or reactions to observable exogenous random variables.[240]

A4.1.3.4 Perception of the outside world

The concept of *perception* is not present in Savage's framework. But, one can argue that many aspects of behavior that are usually attributed to *preference* and/or *beliefs* can be directly link to the concept of *perception* as discussed in section **A.4.1.2** above. The formal analysis of decision problems usually requires the assumption that the decision maker's objects

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[238] The word "*ex-ante*" is very important here. Indeed, in light of new information one can have *regrets* about past decisions. But these *ex-post regrets* are *ex-ante* irrelevant to the choice situation. That is, even if I might *regret* my present decision in the future, I still have to make a choice.

[239] The general idea that the postulate conveys is more or less implicit in the literature dealing with the refinement of perfect Bayesian Nash equilibriums in game theory, where it is often referred as "rationality of belief assessments" (see Fundemberg and Tirole (1991) on equilibrium refinements). Moreover, in this individual choice context, one can still reinterpret the optimality of actions postulate in terms of equilibrium concepts as follows: A decision maker is in a *state of equilibrium* in a decision making situation if and only if he or she has *ex-ante* no incentive to change neither his or her belief assessments nor his or her planned actions. In other words, being in a *state of equilibrium* means that he or she cannot *ex-ante* improve his or her welfare by choosing alternative belief assessments and/or course of actions.

[240] Some may think that the characterization of *optimal beliefs* in terms of observable choice variables is not possible. But the principles are the same as the ones guiding demand analysis where we characterize preference through observable variables such as quantity, prices, and expenditures.

of choice satisfy some mathematical rules and properties. The statements of these rules and properties invariably involve the use of mathematical objects and concepts such as "set", "$\in$" (member), "$\subset$" (inclusion), "$\cup$" (union), "$\cap$" (intersection), "$+$" (addition), "-" (subtraction), "$\times$" (multiplication), and "$\geq$" (greater than). These mathematical objects give the decision maker's choice set (and the set of states of nature) an "algebraic" and "order" structure. Another mathematical concept that is often used (though sometimes not explicitly) is the concept of "topology" or "metric" which formalizes some intuitive notion of "*closeness*" or "degrees of *similarity* and *dissimilarity*" among the objects of choice (and possibly among states of nature).

Implicit in this modelling practice, is the assumption that the decision maker *perceives* these rules and properties (or what they are supposed to represent in his or her consciousness) as applying to his or her objects of choice and states of nature. For example, in choice under uncertainty, it is assumed that the decision maker fully *perceives* the set of all possible states of nature and its partitioning into events. These *perceived* properties can be called the *intrinsic* properties of the choice set in the sense that they are independent of the decision maker's *preference* and *beliefs*, and depend only on his or her *perception* of how objects are related. The plausibility of any *intrinsic* property or rule involved can always be decided with no reference to his or her *preference* and set of *beliefs*.

The statements of some other properties such as *continuity*, *monotonicity*, *convexity*, *optimism, pessimism, etc...*, which are used in formal analysis, involve *perception* in conjunction with *preference* and/or *beliefs*. These properties can be called the "*connecting*" properties in the sense that they establish connections among the three concepts of *perception*, *preference*, and *beliefs*. Many behavioral conclusions of economic nature that are usually attributed to the optimizing behavior assumption are in fact the result of one or several of these connecting properties. For example, Hindy, Huang, and Kreps (1992), and Hindy and

Huang (1992) show how the continuity assumption alone can be used to infer behavioral conclusions regarding the substitutability of consumption across time and/or states of nature.

Taking *perception* as a psychological concept separate from *preference* and *beliefs* reveals many implicit assumptions that are usually embodied in formal mathematical model. This is particularly true for the assumed linear structure of choice sets. The separation also simplifies the mathematical formalization of each of the three concepts and facilitates the intuitive justifications of the assumptions that are made during the mathematical analysis. In particular, this separation helps clarify many of the issues related to the so called problems of state or time-dependent *preference*, by relating them directly to *perception* instead of *preference*.

A4.1.4 State/time-dependent *Preference* and *Perception*

In most choice situations, what is really meant by "state-dependent" *preference*, is not that the decision maker's *preference* changes when the *state of nature* changes. But rather that the decision maker's *perception* of the *intrinsic* characteristics which distinguish the *consequences* (objects of choice) from one another, and define the *degree of similarities* among the corresponding *outcomes*, is "state-dependent". We can illustrate this point by using Karni's (1993) football game example given above. Here, the decision maker's objects of choice are the two *consequences*: watching a football game at the stadium under the uncertain weather ω, and watching the same football game on TV under the uncertain weather ω. In this situation, it can be said that watching a football game at the stadium when it is sunny is *perceived* to be different from watching it at the stadium when it rains. This *perceived* difference is regardless of *preference* and *beliefs*. Similarly for the *consequence* consisting of watching a football game on TV under the uncertain weather ω. Hence, in the *perception* of the decision maker, the *intrinsic* characteristics distinguishing the two consequences include the possible states of nature where they can be realized. In other terms, the two *consequences* are *perceived* to be different not only because watching a football game in a stadium is different from watching it on TV, but also because the state of nature under which they may possibly be watched introduces further "similarities" or "dissimilarities" between the two *consequences*. The foregoing point can be made clearer if instead of assuming that the uncertain weather ω can only be of two states, we assume that it varies continuously. If we suppose for example that ω stands for the temperature, and can takes values ranging from minus ten to one hundred and ten degrees. Then, watching a football game in a stadium at sixty degrees can reasonably be *perceived* as "closer" to watching the same game in the stadium at fifty five degrees than it is to watching it in the stadium at one hundred degrees or at minus five degrees. There may also be a temperature (say, forty degrees for example) at which watching the football game in the stadium is *perceived* to be very "similar" to watching it on TV. Still possible is the case where watching the game on TV at seventy degrees is *perceived* to be "similar" to watching it at the stadium at twenty five degrees.

The differences between the two *consequences* discussed above reflect their *intrinsic* features, and their discussion does not necessitate any reference to *preference* or *beliefs*. In other words, it is not because the decision maker prefers one over the other that makes the two *consequences* different, but rather, they are *perceived* different first before the decision maker formulates any preference between them. However, prior to any ranking of the *consequences*, the *perceived intrinsic* "similarities" or "dissimilarities" among them must be "adjusted" with the decision maker's *beliefs* to take into account the *relative likelihood* of the different *states of nature* they may possibly be realized. That is, prior to any ranking of the consequences, the decision maker uses his or her *beliefs* about the *relative likelihood* of the possible states of nature to "adjust" his or her *perception* of the *intrinsic* characteristics distinguishing the possible *outcomes* to arrive at a *beliefs-weighted perception* of the differences among the corresponding consequences (which are the objects of choice). In the football game example above, the decision maker ranks the two alternatives (watching the game at the stadium and watching it on TV), its *perception* of the differences among the four possible outcomes (watching the game at the stadium under the sun, watching it at the stadium under the rain, watching it on TV in a sunny day, and watching it on TV in a rainy day) is to be "weighted" by its *belief* about the relative likelihoods of rainy and sunny days, to arrive at a *belief-adjusted perception* of the differences between watching the game at the stadium and watching it on TV. This prior "weighting" of the *perception* of *intrinsic* distinguishing characteristics of the possible *outcomes* by *beliefs*, insures that the decision maker's preferred choices are always consistent with his or her *opinion* or *judgement* about which state of nature is likely going to be realized. For example, in the example above, if the decision maker prefers always to watch the game on TV when it rains, then we cannot observe him or he going to the stadium to watch it when he or he believes that there is a one percent chance that the game will be played under the rain.

Hence, in most cases when one speaks of "state-dependent *preference*", the intended meaning is really "state-dependent" *perception*. That is, *consequences* are *perceived* to be different because, among other things, the state of nature in which they may be realized counts. True "state-dependent" *preference* occurs when *preference* changes because of the realization or non realization of a given state of nature that can affect *taste*, *attitude toward uncertainty*, or *time preference*. But, this true "state-dependent" *preference* is not really relevant in the context of a "one-shot" static choice. This is because even if *preference* changes afterward, by the time the true state of nature is revealed the decision has already been made. In other words, the "new" *preference* that results in the realization or not of a given state of nature is not relevant to the old choice problem, which was resolved by using the ranking given by the "old" *preference*. True "state-dependent" *preference* is relevant only in a intertemporal decision making context, where a sequence of interrelated choices has to be made over time. In the football game example above, it may happen that the decision maker chooses to go to the stadium to watch the game, but, unfortunately, it rained during the game. It can be that because of this "bad" experience, the decision maker's *preference* between watching a football game on TV and watching it in a stadium changes completely, so as to make him or her now prefer always watching it on TV, even if there is a one hundred percent chance that there will be no rain. This "new" *preference* is relevant only for subsequent football game decisions. Moreover, even in the intertemporal context where true "state-dependent" *preference* is relevant, the term "state-dependent" does not correctly describe the phenomenon. Indeed, in a intertemporal setting, where outcomes span an entire period of time, a given realized state of nature is supposed to describe all sources of uncertainty for the entire length of time considered (up to infinity for infinite time horizon decision problems). Hence, the true state of nature is not fully revealed until the end of the period considered. Only partial information (subsets of events) about the true state of nature is revealed as time evolves. It is this partial resolution of uncertainty, through the occurrence of events that can possibly affect the mental and/or physical state of the decision maker as time goes, that eventually changes his or her *preference* by changing its *taste*, *attitude toward uncertainty*, and/or *time preference* aspects. Hence, the term "event-dependent" *preference* describes this phenomenon more correctly.

Essentially, the same distinction applies when one speaks of "time-dependent" *preference* in a intertemporal context. That is, there exists a distinction between "time-dependent" *perception* and true "time-dependent" *preference*. "Time-dependent" *perception* reflects the fact that the factor "*time*" is among the features that are *perceived* to distinguish the decision maker's objects of choice from one another. True "time-dependent" *preference* represents the case when a decision maker's *preference* is changing as time goes, in reflection of changes in his or her *taste*, *attitude toward uncertainty*, and/or *time preference (or patience)*. This latter case is best illustrated by the phenomenon of the impact of aging on *preference*.

The distinction between the phenomena of "time-dependent" and/or "state-dependent" *perception* on one hand, and the ones of "time-dependent" and/or "state-dependent" *preference* on the other hand, leads to the recognition of two distinct problems involved in the analysis of decision making processes. The first problem reads as follows: For a given fixed triplet *preference*, *beliefs*, and *perception*, what is the optimal choice resulting from an optimizing behavior? The second problem poses the following question: given that the optimal choice that solves the first problem depends on the given *preference* (and on *beliefs* and *perception*), how does it change when *preference* changes? In short, how does optimal choice depends on *preference*. Clearly, these two problems, though related, are very different, and perhaps are best analyzed separably. In fact, one can solve the first without worrying about the second. In the first problem, *preference* is merely a parameter, among others, on which the solution will depend. The second problem may be interesting for its own sake. However, answers to the question that go beyond trivial statements may require a theory of how *preference* changes over time, as the sequential occurrences of events partially reveal the true state of nature, and/or as factors affecting the environment of the decision making process change.[241]

It is very important to note that by this recognition of the two distinct problems that are involved in the analysis of the decision making process through time, we allow decision makers to optimize as many times as they wish. In other words, decision makers are allowed to "reoptimize" as time goes, when there are significant changes in their physical and/or mental states, and or in the environment of the decision making situation that warrant such a "reoptimization". All we are interested in, as an analyst, is what are the optimal choices?

[241] In the absence of a theory of how and why preference change, one may posit (reasonably) that the law of motion of *preference* is given by a stochastic differential equation (see, for example, Duffie and Epstein, 1992).

(i.e., the first problem). And how these, eventually numerous, optimal choices relate among themselves? (i.e., the second problem).[242]

A4.1.5 The Nature of Optimal Choices in a Intertemporal Context

In an intertemporal choice context the decision maker's *preference* is ranking *consequences* (sets of possible *outcomes*) which are stochastic processes. Each *consequence* is uniquely determined by a *contingent plan*, which, by definition, is a sequence of *contingent or conditional acts* that consist of the "taking" of *options* over time contingent or conditional on the occurrence of events (see Appendix **A4.2** for precise definitions). For example, a *contingent act* may read as follows: "at time *t*, take this given option if that particular event occurs". The exact timing of a *contingent act* may or may not be random. In this example, the timing of the *contingent act* is given by a fixed time *t*. But, if the example was formulated in the following way: "take this given option when (or if) that particular event occurs", then the timing of the *contingent act* will be random, because it is not known when that particular event will occur. Moreover, as is apparent from the examples, the timing of *contingent acts* is endogenous, that is, they are conditionally chosen by the decision maker. In some instances, a *contingent act* may reduce to a *deterministic act*, that is, an *act* that is unconditional to the occurrence of events. For example, a *deterministic act* may read as follows: "at time *t*, take that option regardless of the prevailing state of nature". Furthermore, a *contingent act* may simply consist of the *"do nothing" act*, that is, the *act* that consists of not taking any *option* at all. For example, "do not take any option if that particular event occurs".

Hence, in the context of intertemporal choice under uncertainty, an optimal choice consists of the choice of a *contingent plan* that leads to a unique optimal *consequence*. In this

[242] It is assumed that at each point in time the decision maker solves at most one optimization problem.

sense, one can speak of an "optimal" *contingent plan*, which would mean that the unique *consequence* the *contingent plan* leads to is optimal (i.e., has the highest rank in the *preference* ranking). These optimal *contingent plans* and the respective *consequences* they lead to are functions of the variable *time* and are, in addition, both indexed by the time at which they were found to be optimal.[243]

In this intertemporal choice under uncertainty context, it is important to see the different roles the occurrences of uncertain events have on one hand, in the "updating" of *beliefs* and in the implementation of *contingent plans* over time, and on the other hand, in changing the decision maker's preference as time goes by. Indeed, when the taking of an *act* is contingent upon the occurrence of an event, at each time the decision maker's information set is used to determine whether or not that event has occurred. Hence, an *act* that is planned, but contingent upon the occurrence of an event may not be taken, even if the decision maker's *preference* (i.e., the ranking of *contingent plans*) does not change at the time when the *act* is supposed to be taken. The crucial point here is in defining what the objects of choice are, i.e., what *preference* is supposed to rank. If *preference* is ranking *consequences* that are uniquely determined by sequences of *acts* that are conditional on the state of nature, then using the information available at each time *t* to implement a *contingent plan*, does not preclude a given *contingent plan* that was found to be optimal at the initial time, to remain optimal throughout the entire period of the decision making process.[244] However, there is also no reason to assume that optimal *contingent plans*, once chosen, are going to remain optimal for the entire remaining period, regardless of changes in *preference*

[243] Since *preference* ranks objects that are functions of both time and state of nature, optimal *contingent plans* and their *consequences* are functions defined on the set $\Omega \times [t_d, T]$. Where, Ω is the set of states of nature, T is the time horizon, and t_d stands for the "decision" or "optimization" time. Hence, *optimal contingent plans* and their corresponding *consequences* depend on t_d and T.

[244] Again, by the term "optimal *contingent plan*" we implicitly mean that the uniquely determined *consequence* that the *contingent plan* leads to is optimal.

and/or changes in the environment of the decision making process. In other words, there is nothing that indicates "inconsistency" or "naivety" in the behavior of a decision maker who formulates an optimal *contingent plan* while knowing that it may not be implemented in the future, given that *preference* can change at any time. In fact, not knowing whether or not and when *preference* will change in the future, the most logical thing to do for a decision maker, is to have an optimal *contingent plan* on hand at all time. This type of behavior, is at least consistent with the optimizing-behavior assumption, that is made in most of economic analysis, and which essentially postulates that decision makers behave optimally at all times.

The decision maker also uses new information to continuously update his or her *beliefs* regarding the relative likelihoods of future events. It is the "updated" *beliefs* that are used in subsequent determinations of optimal *contingent plans* and their implementations, following changes in *preference* and/or in time varying "parameters" describing the environment of the decision making process.

The roles of events and information in the decision making process discussed above are different from the possible roles they have in changing *preference*, through changes in the mental and/or physical states of the decision maker.[245] When the occurrence of an event induces *preference* to change to a "new" one, it is this "new" *preference* that is used in another subsequent optimization problem that solves for a new optimal *contingent plan* which would uniquely lead to a new set of possible *outcomes*, that is to say, to a "new" *consequence*. The question addressed in the second problem involved in the analysis of decision making processes, is how this "new" *contingent plan*, which is the one that is implemented over time starting the period at which preference did change, is different from the "old" *contingent plan*, which is abandoned now. It may be the case that two different *preferences* yield the same ranking of *consequences*, so that the *optimal contingent plan* is the

[245] For example, one may learn unexpectedly that he or she has a terminal illness.

same for the two different preferences. Also, although unlikely, it is conceivable that *preference* remains invariant throughout, so that there is only one optimal *contingent plan* that is implemented during the entire period. There is also the possibility of a *stationary preference* which, beyond a given time, becomes invariant with respect to time (Koopmans, 1960). But, the more realistic case is the one of a *recursive preference*. That is, a *preference*, the structure of which (taste, attitude toward uncertainty, and impatience) exhibits some persistence over time, so as to make *preference* at any time depend on *preference* at previous times (Koopmans, 1960; Epstein and Zin, 1989 and 1991; Duffie and Epstein 1992). These characterizations of different types of *preferences* represent restrictions on *preference* that may or may not be plausible, depending on the nature of the choice problem on hand.

The issues involved in the foregoing discussion can be best seen by taking *preference*, or the utility function representing it, as one additional argument (among the other usual arguments) of the function defining a given optimal *contingent plan*. In this way, one can see that there are two types of driving forces that determine the respective (sample) trajectories or *sample paths* of *outcomes* and actually realized optimal *acts* through time. The first driving force operates through the "usual" variables that are arguments of the function defining optimal *contingent plans*. In consumer theory, these "usual" variables are the vector of prices and income (or more precisely cumulative net income or wealth). They are themselves functions of time and state of nature, and thus their trajectories or *sample paths* directly determine the *sample paths* of actually realized *acts* and *outcomes*. The second driving force operates through *preference* and other time varying "parameters" of the decision making process. To see that, we can consider a sequence of optimal *contingent plans* and their respective *consequences* which are indexed by a time varying *preference* (or more generally by a set of time varying "parameters") simply as functions of a function of time that indicates for which *preference* each optimal *contingent plan* and its respective *consequence* refer to. Then, as time evolves one can see how the effect of change in *preference* can determine, from which optimal *contingent plan* and *consequence*, *ex-post* actually realized *acts* and *outcomes* come from respectively. This is just to say that when *preference* changes through time, *ex-post* actually realized *acts* and *outcomes* may respectively reflect the partial realization of several optimal *contingent plans* and their respective *consequences*.

In the above discussion, we have been very vague on what do we mean by the term *information*. That is because a precise and satisfactory definition necessarily requires the formal use of concepts from probability theory. However, in Appendix **A4.2** where we give the formal definition of *information set*, we make the careful distinction among the three concepts of *information*, *knowledge*, and *learning*. *Information* is one component of *knowledge*; with the other component of *knowledge* being what is usually referred as *expertise*, "*know how*", or simply as *human capital*. *Learning* is then taken to be the "rate" of acquisition of *knowledge*. In essence, the rationale behind these distinctions lies in the fact that, intuitively, the notion of *information* is about (historical) *facts* or passed *events* that have to be taken (passively) as they are; but which helps the decision maker *assess* the relative likelihoods of future *events* and *predict outcomes*. Whereas, the notions of *expertise "know how"* and *human capital* are about knowing various ways of "doing things", taking *acts* or *actions*, and of interpreting *facts* (i.e., processing *information*); including those "extraneous" *facts* that are not contained in *information sets*. On the other hand, the notion of *learning* incorporates some dynamism that reflects both the passage of time and the "activism" involved in the acquisition of *knowledge*. At this point, we recall the implicit, but important, assumption that is made in mathematical models of choice under uncertainty. That is, the fact that decision makers are assumed to *fully perceive* the set of *all possible states of nature* (Ω in our notation), and its partitioning into events. The source of uncertainty is supposed to result only from the fact that he or she does not know which *state of nature*, among all the possible ones, is going to be realized. Moreover, as already emphasized, in the intertemporal context a given state of nature is supposed to describe one possible realization of all the *perceived* exogenous sources of uncertainty during the entire period of the decision making process (including the actions of other decision makers). This assumption is a potentially limiting factor for present mathematical models of intertemporal choice under uncertainty to realistically describe decision making processes. Indeed, it is conceivable that as time goes the decision maker *perceives* the possibility of other *states of nature* that were not part of the initial set of possible states of nature Ω.[246]

The role of the decision maker's *physical endowments* in the choice process has not been discussed explicitly. In Appendix A4.2 where the term is formally defined, *physical endowments* are taken to be part of the *opportunity set*; that is, the set of *options* available to the decision maker. Hence, they are important determinant of the set of *feasible acts* that can be taken at any given point in time. In other words, they determine what *contingent plans* are implementable.

Finally, before we turn to the methodological implications of this conceptual framework in terms of mathematical modeling of intertemporal choice under uncertainty, we present, as a way of summarizing the previous discussion, a brief description of the decision making process that captures the essential features involved in it (see Appendix 2 for a more detailed description).

We have postulated that whenever an individual decision maker is in a choice situation, the environment of the choice situation is described by three types of *physical objects* consisting of the set of all possible *states of nature*, the *time horizon*, and the *commodities or goods* which are the objects of his or her choices. He or she is also supposed to possess three *psychological objects* consisting of his or her *perception* which enables him or her to perceive the *intrinsic* properties of the physical objects, along with the relationships

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[246] Note that this phenomenon is different from the phenomenon of *information "surprises"* which has been mathematically formalized by Huang (1985a and 1985b), and Hindy and Huang (1992); through the use of concepts from martingale theory. The formal definition of what constitutes an *information "surprise"* is based on given partitions of the (initial) set of all possible states of nature Ω.

among them; a *preference* which allows him or her to rank alternative *consequences* of his or her set of possible *acts* taken during the time period; and a set of *beliefs* which allows him or her to subjectively assess the relative likelihoods of the different possible states of nature.

We have also postulated that whenever a decision maker is in a *choice situation*, the following sequence of phenomenon occurs.

(i) *Awareness of consequences of acts and the constraints on them*: he or she is fully aware that any of his or her possible *act* or *actions* will lead to a well defined *consequence*, a possible realization of which is an *outcome* which not only depends on the *act* taken, but also on the future realized *state of nature*. Here, an *act* consists of taking an *option*. He or she is also fully aware that at any time, the set of possible *acts* he or she can take is limited to a subset of *feasible acts* that is determined by his or her set of *endowments*.

(ii) *Subjective assessment of the relative likelihood of events*: He or she is able to freely select one *belief* from his or her set of *beliefs* to completely assess the relative likelihood of the *events* that he or she *perceives* as forming a "partition" of the set of all possible *states of nature*.

(iii) *Preference ordering of the consequences of sets of possible acts*: he or she uses his or her *preference* and his or her "beliefs-adjusted" *perception* to completely and consistently rank all the consequences that result from the respective *contingent plans*. A *contingent plan* consists of a sequence of *acts* that are taken over time conditional on the occurrence of *events*.

(iv) *Optimality of actions or decisions*

(iv.a) *Optimality of belief*: The choice of a *belief* from the set of *beliefs* to assess the relative likelihood of the *events* is *optimal* in the sense that given his *knowledge* at the time of decision, the upheld *belief* is the one, among all the possible *beliefs*, that yields the *maximum satisfaction* for any *contingent plan* the decision maker can *choose*. In other words, for any *contingent plan* that might be selected, he or she cannot improve his or her *welfare* by changing his or her *belief*.

(iv.b) *Preference maximization*: finally, given the optimal *subjective assessment of the relative likelihoods* of the various *events*, and the *ranking* of the *consequences* resulting from the respective *contingent plans*, he or she *chooses* a *feasible contingent plan* which uniquely determines the most *preferred consequence* among all the *consequences* that respectively result from all the *feasible contingent plans*. In other words, given the decision maker's set of *endowments*, the chosen *feasible contingent plan* leads to the *consequence* with the *highest rank* in the ranking given by his or her *preference*.

The decision making process just described, is supposed to be repeated whenever there is a change in *preference*. The *optimal contingent plan* and *consequence* corresponding to each repetition are indexed (or dated) by the time the repetition occurs. It is also assumed that an *optimal contingent plan* is implemented starting the time when it is found *optimal* until the time the next repetition occurs, at which time the newly found *optimal contingent plan* starts to be implemented, and so on until the end of the *time horizon*.

A4.1.6 Methodological Implications of the Conceptual Framework

The conceptual framework presented in this appendix has major methodological implications in terms of the mathematical analysis of intertemporal choice under uncertainty. Given that the discussion in this appendix is intended to be informal, and that any detailed discussion of all the methodological implications would require some formalism and the use of more mathematical concepts and definitions, we will only give a brief and informal account of some of the implications. A full account along the lines of chapter **4** will be provided in future work.

We have already mentioned that choice under uncertainty is best described by defining a *preference* ordering on a choice having the form $L^0(\Omega, \mathbf{F})$, the space of vector (or function) valued random variables with domain Ω, the set of states of nature, and taking value in $\mathbf{F}$, the outcome space. The advantage of this over the more standard practice which consists of defining *preference* on a set of probability measures is twofold.[247] The first advantage stems from difference in the structure of the two choice spaces. An appropriate choice of a topology on $L^0(\Omega, \mathbf{F})$ makes it a Banach space (i.e. a complete normed linear space). This allows easy applications of standard functional and convex analysis, and differential calculus methods to obtain general existence results for optimal solutions of optimization problems and their characterization by some form of generalized "first order" or Kuhn-Tucker conditions (see the introduction and appendix to chapter 2). In most cases, these generalized conditions are easily solved to obtain closed form analytical solutions (see chapter 3). Even in the intertemporal case, Dynamic Programming arguments and/or restrictions are not needed to get analytically and empirically tractable solutions. On the other hand, the space of probability measure is not a linear space, so that one often needs to embed it in a linear space in order to use the standard methods of functional analysis or differential calculus (e.g., Machina, 1982). Moreover, even if it is embedded in a linear space, the formulation of the constraints of optimization problems in terms of probability distributions is difficult and cumbersome. The second advantage results from the frequent need to empirically estimate or test the dependence of optimal choice variables on the "exogenous" variables of the model. With a choice space having the form $L^0(\Omega, \mathbf{F})$, the choice variables and the "exogenous" ones are in terms random variables whose realizations are directly observable. Hence, traditional econometric procedures can be used to directly estimate the relationships derived from the theoretical model (see Chapter 5). On the other hand, when the choice variables are in terms of

[247] There have been two approaches of modelling choice uncertainty. The general equilibrium and mathematical finance literatures used a choice set of the form $L^0(\Omega, \mathbf{F})$, while in the more traditional literature of risk, choice sets in terms of sets of probability distributions, mixture sets, or sets of cumulative distribution functions are most commonly used.

unobservable probability distributions, a direct econometric estimation of the relationships derived from the theoretical model is at best very difficult.

Once a choice space of the form $L^0(\Omega, \mathbf{F})$ is selected, the next step is to represent separately *belief* and *preference* respectively by a (*subjective*) probability distribution so as to make $L^0(\Omega, \mathbf{F})$ a measure space, and by a continuous utility functional defined on a appropriately topologized $L^0(\Omega, \mathbf{F})$. These representations can be done by applying the results of Chateauneuf (1985), and Debreu (1954) or Mas-Collel (1986) respectively.[248]

Next, in solving the problem of maximizing utility subject to constraints, as argued in the previous section, one needs to distinguish the two separate problems involved in the analysis of intertemporal choice problems. Indeed, in finding the optimal choice for a given *preference*, one need only to worry about the problems of "time-dependent" and/or "state-dependent" *perception*, which can be taken care of by the choice of an appropriate "topology" that incorporates "time-dependent" and/or "state-dependent" phenomena (see, for example, Hindy and Huang, 1992). The chosen "topology" will mathematically formalize the "similarity" and "dissimilarity" aspects of *perception*. Moreover, the choice of a topology concerns only the *"beliefs-adjusted"* structure of the choice set ("algebraic", "order", and "topological"), as *perceived* by the decision maker. In particular, there is no need to make reference to *preference* at this level unless one wants to check the appropriateness of the chosen topology by testing the implications of continuity of *preference* in it.

However, although *preference* itself exists independently of the decision maker's *perception* of the "topological" structure of the choice set, the possibility to represent it by a (continuous) numerical utility function depends, directly and through the *continuity* assumption, on this topological structure. Hence, the functional form of a utility function that pretends to represent *preference* should exhibit this dependence, so that analysis done

[248] For the representation of *belief* see also Fishburn (1972) and the literature cited there.

using it reveals all the behavior-relevant features built into the "topology". The resulting mathematical model need not rely on restrictive assumptions such as "expected utility", time separability, and/or recursivity, in order to yield optimal solutions that are both analytically and empirically tractable[249]. If despite the dependence of the utility function on the topological structure of the choice set, the analysis yields nonsensical or unjustified restrictive behavior, then one should question the appropriateness of the chosen "topology", and/or functional form for the utility.[250]

In order to carry out properly the analysis in the (optional) second problem involved in decision making process, it is important in solving the optimization problem, to make explicit the dependence of the problem on the initial decision time t. This dependence is allowed by using a time interval of the form $[t, T]$ or $[t, \infty[$ instead of the more common practice of using a time interval of the form $[0, T]$, $[0, 1]$, or $[0, \infty[$. In solving the optimization problem, the decision time t is just a parameter on which the topology of the choice space (and the one of its dual) and some other variables (which are random functions of time) and parameters depend explicitly. Hence, in general, the optimal solution will be a function of the initial decision time t both directly and through the "exogenous" variables and parameters.

Once the optimal choice is derived, the problem becomes how that optimal choice depends on time and *preference* (and possibly on other time varying "parameters"). When *preference* is represented by a parametrized utility function, this would mean studying the

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[249] Other restrictive assumptions on the subjective probability distributions of the random variable such as normality, independence, etc..., are also not needed.

[250] In light of the foregoing discussion, one can suspect that most awkward or restrictive behavior implied by models based on "expected utility" and/or time separability, result from the fact that the functional form for the utility function does not use all the features implicit in the topology of the chosen choice space, or the chosen topology does not take into account all the relevant information pertaining to the choice situation.

dependence of the optimal choice on the parameters defining the utility function.[251], In an intertemporal choice context, one can formulate a "law of motion" that describes how *preference* changes as time goes and new information partially reveals the true state of nature. In general, this "law of motion" will be in the form of stochastic differential equations in terms of utility functionals. But, if the utility functionals are parametrized, with the Euclidean space $\mathbf{R}^n$ being the parameter space, then the stochastic differential equation in terms of utility functionals can be reduced to a system of stochastic differential equations in terms of real numbers. In principle, the "law of motion" describing how *preference* changes over time can be solved and/or used to infer how optimal choices depend on initial *preference* and on time. Alternatively, one may not need to worry about how *preference* changes over time, but instead just study the dependence of the optimal choice on the initial time t of decision.[252]

In the intertemporal context, a proper and correct analysis of this second stage necessarily requires the use of the methods and concepts of martingale theory, which enables to take into account the partial resolution of uncertainty over time through the revelation of new information at each time t.

Finally, one methodological issue regards the choice between discrete time and continuous time models of intertemporal choice under uncertainty. Discrete time models are

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[251] For example, for a consumer maximizing a bivariate Cobb-Douglas utility function $\mathbf{u}(x_1, x_2) = x^{\alpha} x^{\beta}$ under the budget constraint $p_1 x_1 + p_2 x_2 = m$ with no uncertainty, this means studying the dependence of the demand functions $x_i(p_1, p_2, m, \alpha, \beta) = \dfrac{\alpha m}{(\alpha + \beta) p_i}$ on the parameters α and β.

[252] In general the parametrized intertemporal utility functional **U** is of the form $U(t, \mathbf{x}) = f(t, \mathbf{x}, \alpha(t))$. where t is the initial decision time, f is a real valued functional, **x**, the choice variable, is a vector valued stochastic process defined on $[t, T] \times \Omega$, and $\alpha(t)$ is a vector of real parameters which depends on the initial time t. In maximizing utility subject to constraints, t and $\alpha(t)$ are fixed parameters on which the optimal solution will depend in the form $\mathbf{x}^*(t) = g(t, \mathbf{z}(t), \alpha(t))$, where g is an operator that takes values in a function space, and $\mathbf{z}(t, .)$ is a "exogenous" stochastic process defined on $[t, T] \times \Omega$ and that is given in the constraints. The main issue involved in the second stage, is how the optimal solution $\mathbf{x}^*$depends on t and $\alpha(t)$ when t varies over time. The dependence of the *preference* parameter α on the time variable can (optionally) be given by a system of stochastic differential equations.

usually viewed as more realistic because of the facts that real life decision making processes are discrete. In the conceptual framework of this appendix, one can see that although time is continuous, due to biological and institutional constraints in most realistic intertemporal decision situations, *contingent acts* that are not *"do nothing" acts* are taken only at *discrete times*. However, this fact does not imply that mathematical discrete time models are more appropriate than continuous time ones for describing real life intertemporal decision making processes. On the contrary, since the *timing* of the decisions or acts are themselves endogenous, that is, (conditionally) chosen by the decision maker, continuous time models are more appropriate. In fact, using discrete time models under uncertainty, with *decision or acting times* that are exogenously determined at given discrete times, may be very restrictive. This is because they rule out the possibility for decision makers to delay the taking of an *option*, "just a little bit", when new information indicate a certain "gain" in doing so. Hence, for mathematical modeling purposes, continuous time models are more appropriate for describing intertemporal choice under uncertainty. Moreover, they are much easier to work with. However, in order to reflect real decision making situations continuous time models should embody enough restrictions that lead to optimal *contingent plans* that are discrete (i.e., discontinuous) functions of time, with discontinuity (or jump) points endogenously determined by the model.

A4.1.7 Conclusion

This appendix we have presented a conceptual framework of choice under uncertainty that is based on Savage's (1954) conceptual framework. The meanings of some of Savage's concepts were changed and other concepts were introduced in order to better describe the decision making process in the presence of uncertainty. These changes were justified both intuitively and mathematically. The conceptual framework leads to a choice space that allows the application of functional and convex analysis, and differential calculus methods to extend all the results of deterministic consumer and producer theory to the case of uncertainty. It is only within this Bayesian conceptual framework where decision makers choose freely their *beliefs*, that the mathematical results of chapter **3** can be interpreted.[253] The conceptual changes made to Savage's original static framework have allowed us to extend it easily to intertemporal decision making situations.

Although the discussion in this appendix was informal, the extension shows clearly that the way uncertainty is partially resolved over time plays an important role in describing intertemporal decision making processes. The theory started in Appendix **A4.2** and chapter **4** will provide a formal treatment of intertemporal choice under uncertainty which will based on functional analysis and martingale theory. The approach in Appendix **A4.2** and chapter **4** is axiomatic and more mathematically oriented. Nevertheless, despite the abstract setting, the theory started in this appendix will lead to empirically testable comparative statics results that parallel the ones derived in chapters **2**, **3**, and **5**.[254] The abstract theory started in Appendix **A4.2** and chapter **4** will also provide elasticity, welfare indicators, and other measures related to intertemporal choice, and that will be directly estimable using standard econometric techniques. Indeed, this is already apparent in chapter **4** where the conceptual framework is used to outline an extension to the intertemporal case of the static theoretical results derived in the thesis. Starting with the simpler case of a static choice under uncertainty allows one to get most of the new results allowed by the use of functional analysis and differential calculus in general linear space methods, without the additional complications that would result from the incorporation of information revelation and intertemporal substitution aspects in the analysis.

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[253] We emphasize again that the free choice of *beliefs* does not preclude the chosen *beliefs* to be influenced past by *beliefs* and/or other individuals' *beliefs*. That is, interdependence of individual beliefs is not ruled out.

[254] An intertemporal agricultural household model is outlined at the end of Appendix **2** to illustrate the concepts of the axiomatic setting.

APPENDIX A4.2

AN AXIOMATIC FOUNDATION OF INTERTEMPORAL CHOICE UNDER UNCERTAINTY

An Axiomatic Foundation of Intertemporal Choice under Uncertainty:

The Primitive Concepts and Behavioral/Psychological Postulates

"…. First, the primitive concepts of the economic analysis are selected, and then, each one of these primitive concepts is represented by a mathematical object…. Second, assumptions on the mathematical representations of the primitive concepts are made explicit and are fully specified. Mathematical analysis then establishes the consequences of these assumptions in the form of theorems… An axiomatized theory substitutes for an ambiguous economic concept a mathematical object that is subject to entirely definite rules of reasoning. No doubt the economic interpretation of the primitive mathematical objects of the theory is free, and this is indeed one of the sources of the power of the axiomatic method… But while a primitive concept of an axiomatic theory admits different interpretations a theorist who has chosen one of them succeeds in communicating his intended meaning with little ambiguity because of the completely specified formal context in which he operates… The axiomatization of a certain part of economic theory also requires a full specification of the assumptions under which any one of its conclusions is asserted… More positively, the complete specification of assumptions, the exact statement of conclusions, and the rigor of the deductions of an axiomatized study provide a secure foundation on which the construction of economic theory can proceed… The reasons I have listed alone would explain why the proponents of a particular theory welcome its axiomatization. Several of these reasons also explain why the opponents of the same theory may welcome its axiomatic form since this brings out more sharply the aspects of that theory to which they object… Thus axiomatization facilitates the detection of logical errors within the model, and perhaps more importantly it facilitates the detection of conceptual errors in the formulation of the theory and in its interpretations.[255]

A4.2.1 The Primitive Concepts

A4.2.1.1 Physical primitive concepts

Time*, *state of nature*, and *commodities

We postulate that the physical environment of the decision maker can be described by the following three conceptual objects: **T**, **Ω**, and **K**. Where, **T** is the *time set or period*. **Ω** is the set of *all possible states of the world or nature* he or she is aware of; with an element ω

[255] Gerard Debreu on "The Axiomatization of Economic Theory" ; cited by W. Hildenbrand in the introduction of Debreu (1983, p. 4).

of Ω being a *possible state of nature* that could occur *during* the *time interval* **T**.[256] **K** is the set of *all commodities* or *goods available* for the entire *time set* **T**. To fix ideas, one can take **T** to be an interval of the real line ℝ, and **K** to be the Euclidean space $\mathbb{R}^n$.

A4.2.1.2 Conceptual primitive concepts

We postulate that in any choice situation, the decision maker is conscious of the following conceptual objects:

Events

$\mathcal{F}$ is the set of all *possible events* as perceived by the decision maker. It is a set of subsets of Ω, an element of which is a subset of *possible states of nature*.[257] It is postulated that both Ω and the empty set ∅ belong to $\mathcal{F}$. Ω is the *certain event* and ∅ is the *event that never occurs*.

Information profile*, *information structure*, and *information set

$\Im$ is a set-valued mapping $\Im$: $\mathbf{T} \rightarrow 2^{\mathcal{F}}$ called a possible *information profile* (or *information path*) for the decision maker. It associates to each time t the unique subset $\Im(t) \subset \mathcal{F}$, called the *information set* of the decision maker at time t.[258] The range of $\Im$, which is by definition $\Im(\mathbf{T}) \equiv \{\Im(t): t \in \mathbf{T}\}$, is called the decision maker's *information structure* associated with $\Im$. Hence, an *information structure* is merely a family of subsets of the set of

[256] The state of nature is interpreted here in a broad sense and includes the actions or decisions taken by all relevant economic agents. *Uncertainty* arises only because of the fact that among all the *possible states* the consumer does not *know exactly* which one is the *true state of nature*.

[257] An event is the *possibility* that the *true state of nature* belongs to a given subset A of Ω. A is then mathematically identified with that *event*.

[258] An *information set* at a time t is the *fact known with certainty* by time t that the *true state of nature* belongs to one of the subsets of Ω that are in a given subset $\Im_t$ of $\mathcal{F}$ (i.e., $\Im_t$ is a set of events contained in $\mathcal{F}$). $\Im_t$ is then mathematically identified with that *information set*.

events $\mathcal{F}$. Let Ψ be the set of *all possible information profiles* that can be *chosen* by the decision maker.

Human capital stock or expertise level* and *expertise profile

$\mathcal{H}$ is the set of *all possible "human capital stocks"* or set of *all possible levels of "expertise"*, which consists of all possible level of *expertise* that the decision maker can possess during the entire time interval **T**. A generic element of $\mathcal{H}$ is noted h and is called an *expertise level* or *education level* or simply *skills*.

h is a mapping $\mathbf{h}$: $\mathbf{T} \times \Omega \rightarrow \mathcal{H}$ called the *expertise profile* of the decision maker. It associates to each *time t* and *state of nature* ω the unique element $\mathbf{h}(t, \omega) \in \mathcal{H}$ which consists of *the expertise or education level achieved by the decision maker at time t under the state of nature* ω. For a given *state of nature* ω, the mapping $\mathbf{h}(., \omega)$: $\mathbf{T} \rightarrow \mathcal{H}$ that associates to each time t the *expertise level* $\mathbf{h}(t, \omega)$, is called an *expertise sample path*. Let $\ddot{\mathrm{H}}$ designate the set of all *expertise profiles*.

Knowledge*, *knowledge profile*, and *learning curve

For given $\mathbf{h} \in \ddot{\mathrm{H}}$, and $\Im \in \Psi$, the pair $(\mathbf{h}(t, \omega), \Im(t))$ consisting of the *expertise level at time t under state of nature* ω and of the *information set at time t*,is called the decision maker's *knowledge at time t under state of nature* ω. The pair $(\mathbf{h}, \Im)$ is then called a *knowledge profile* for the decision maker, and for given state of nature ω, the pair $(\mathbf{h}(.,\omega), \Im)$ is what can be called a *sample learning curve* or *sample learning path*. Hence, a *knowledge profile* uniquely determines at each *time* and each *state of nature*, the *expertise level* and *information set* possessed by the decision maker. It also determines the "path" at which the decision maker *acquires* "new" *information* and *improves* his or her *expertise level* corresponding to any *state of nature*.

Option* and *Option process

Θ is a set-valued mapping $\Theta: \mathbf{T}\times\Omega \to 2^{\mathbf{K}}$ called the *set of options process* for the decision maker. It associates to each *time t* and *state of nature* ω the unique subset (possibly empty) $\Theta_t(\omega)$ of the set of commodities **K**. $\Theta_t(\omega)$ is the decision maker's set consisting of *all options available at time t under the state of nature* ω. When $\Theta_t(\omega)$ is empty, then it is called the *empty option set*, consisting of "*the case when the decision maker has no option available (or has run out of options) to take*". For simplicity, we assume that $\Theta_t(\omega)$ is non empty for all $t \in \mathbf{T}$ and $\omega \in \Omega$.[54] Let the set $\mathbf{\Theta} = \bigcup_{t\in\mathbf{T}} \Theta_t \subseteq \mathbf{K}$ be the decision maker's set of *all available options for the entire time interval* **T**.[55] A generic element of $\mathbf{\Theta}$ is noted $\boldsymbol{\theta}$. It is assumed that there exists an *option* $\mathbf{0} \in \bigcap_{t\in\mathbf{T}} \Theta_t$ which is called the "*zero option*".[56]

$\boldsymbol{\theta}$ is a mapping $\boldsymbol{\theta}: \mathbf{T}\times\Omega \to \mathbf{\Theta}$ called an *option process*. It associates to each *time t* and *state of nature* ω one *possible available option* $\boldsymbol{\theta}(t, \omega) \in \Theta_t(\omega)$. For a given state of nature $\omega \in \Omega$, the mapping $\boldsymbol{\theta}(., \omega): \mathbf{T} \to \mathbf{\Theta}$ is called a *sample option path*. Let $\tilde{\mathbf{O}}$ be the set of *all options processes*.

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[54] Ruling out cases where $\Theta_t(\omega)$ is empty is not restrictive for all practical purposes. The assumption can be relaxed, but at the cost of additional complexities in notations and concepts.

[55] It can be noted that the non empty *option sets* may (and are likely to) be overlapping. This is because an *option* $\theta(t, \omega) \in \Theta_t(\omega)$ may still be available at future time if it is not taken at time t (either by the decision maker or by exogenous factors). Likewise, *options* that were not available in the past can become available in the future. Hence, in general the non empty *option sets* do not form a monotonic family of non empty sets. An example where they do form a non-increasing monotonic family, is the case of extraction of non-renewable resources. In this case, the *option set* $\Theta_t(\omega)$ can be thought as being the set of resources remaining at time t, so that $\Theta_t(\omega)$ can only shrink over time.

[56] As it will be clear below, here we are making the difference between the case when $\Theta_t(\omega)$ is empty so that the decision maker has no available *options* to take, and the case when $\Theta_t(\omega)$ is non empty so that *options* are available, but the decision maker decides to take no *option*. In this later case we would say that the decision maker is taking the *zero option*.

Act, *action*, *act process*, and *line of action*

A is the set of all *possible acts* that can be taken by the decision maker. A generic element of **A** is called an *act* and is noted *a*.

a is a mapping **a**: $\mathcal{H} \times \boldsymbol{\Theta} \times \mathbf{T} \rightarrow \mathbf{A}$ called an *action*. It associates to each *expertise level* $h \in \mathcal{H}$, *option* $\theta \in \boldsymbol{\Theta}$, and *time* $t \in \mathbf{T}$, the unique *act* $\mathbf{a}(h, \theta, t) \in \mathbf{A}$ consisting of *"using the expertise level h to take the option θ at time t"*. For all *expertise level h*, the *act* $\mathbf{a}(h, \mathbf{0}, t)$ is the *"do nothing" act* consisting of *"taking the 'zero option' at time t"*.[262], [263]

For given *action* **a**, *expertise profile* **h**, and *option process* $\boldsymbol{\theta}$, the mapping η: $\mathbf{T} \times \boldsymbol{\Omega} \rightarrow \mathbf{A}$ is called the *act process* of the decision maker. It associates to each *time t* and *state of nature* ω the unique element $\eta(t, \omega) \equiv \mathbf{a}(\mathbf{h}(t, \omega), \boldsymbol{\theta}(t, \omega), t) \in \mathbf{A}$ consisting of *"the act taken at time t under the state of nature* ω.

For a given state of nature $\omega \in \boldsymbol{\Omega}$, the mapping $\eta(., \omega)$: $\mathbf{T} \rightarrow \mathbf{A}$ is called a *sample action path* or *sample line of action*. It associates to each *time t* the *act* $\eta(t, \omega) \equiv \mathbf{a}(\mathbf{h}(t, \omega), \boldsymbol{\theta}(t, \omega), t)$ *which is taken at time t under the state of nature* ω.

Acting time and *contingent acting time*

The time $t \in \mathbf{T}$ in the *act* $\mathbf{a}(h, \theta, t)$ is called the *acting* (or *implementation) time* for the *option* θ.[264] For a fixed option $\theta \in \boldsymbol{\Theta}$, its *acting or implementation time* $t(\theta)$, may

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[262] When $\Theta_t(\omega)$ is the *empty option set*, then one would say that we have a *"'no option' case at time t"*. But, outsiders who cannot observe $\Theta_t(\omega)$ may not be able to distinguish between a *"do nothing" act* and a *"no option" case*.

[263] For all practical purposes, we can always identify the *act* $\mathbf{a}(h, \theta, t)$ with the triplet (h, θ, t). This is because **a** is not observable (or at least not measurable) whereas (h, θ, t) can be observed and measured in most practical situations (one can use proxies like education level to get a measure of h)

[264] Although time is continuous, because of biological and institutional constraints in most realistic intertemporal decision situations *acts* that are not *"do nothing" acts* are taken only at *discrete times*. That is, the set $\{ t \in \mathbf{T}: \mathbf{a}(h, \theta, t) \neq \mathbf{a}(h, 0, t) ; h \in \mathcal{H}, \theta \in \boldsymbol{\Theta}\}$ is countable. However, this does not mean that discrete time sets (i.e $\mathbf{T} \subseteq \mathbf{N}$) are more appropriate for modelling intertemporal choice. On the contrary, since the *timing* of the decisions or acts are themselves endogenous (i.e chosen by the decision maker), continuous time sets (i.e $\mathbf{T} \subseteq \mathbf{R}$) are more appropriate for intertemporal choice

possibly depend on the state of nature ω. In this case, the mapping $t(\theta)$: $\Omega \rightarrow \mathbf{T}$ which associates to each state of nature ω, $t(\theta)(\omega)$, the actual *implementation time* of θ, is called the *contingent acting time* for θ. For example, the decision maker may condition the *taking* of the *option* θ on the occurrence of an *event* B, element of $\mathscr{F}$, by saying that: "*option* θ *will be taken if event B occurs*". The time t(B) $\in$ **T** when event B occurs is then, by definition, the actual *implementation time* of θ, and is equal to $t(\theta)(\omega)$.

Contingent act

For a fixed *time t* the mapping $\eta(t, .)$: $\Omega \rightarrow \mathbf{A}$ is called a *contingent act*. It associates to each *state of nature* ω the *act* $\eta(t, \omega) \equiv \mathbf{a}(\mathbf{h}(t, \omega), \theta(t, \omega), t)$ *which is taken at time t if the state of nature* ω *prevails*. The *time t* is then called the *constant acting time* or *timing* of the contingent act $\eta(t, .)$. If $\eta(t,)$ takes a constant value independent of the state of nature ω, then $\eta(t, .)$ is called a *constant or non-contingent act*. When t: $\Omega \rightarrow \mathbf{T}$ is a *contingent acting time* for some option θ, then the mapping $\eta(t(.), .)$: $\Omega \rightarrow \mathbf{A}$ is still called a *contingent act*. Basically, a contingent *act* $\eta(t, .) \equiv \mathbf{a}(\mathbf{h}(t, .), \theta(t, .), t)$ is a "projected" act whose realization is conditioned on the occurrence of an event B element of $\mathscr{F}$: "*option* θ*(B) will be taken if event B occurs*"; the time $t(\theta$(B)) when event B occurs is then, by definition, a *contingent acting time* for the *option* θ(B).[265] Moreover, if one considers a non *contingent act* as an special *contingent act* whose realization is conditioned on the occurrence of the event Ω, then all acts are *contingent acts*.[266]

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situations.

[265] A *contingent act* may consist of the *"do nothing" act*: "do nothing if event B occurs". Furthermore, if we allow for the possibility of the set of *available options* to be empty, then a *contingent act* need not lead to a "real" observable *act* .

[266] Mathematically, a *contingent act* is a random variable. Since any random variable with a constant value independent of the state of nature can still be considered as a random variable, a *constant or non-contingent act* is also a *contingent act*. Similarly, we can consider all *acting time* as

Contingent plan

For given action **a**, *expertise profile* **h**, and *option process* $\boldsymbol{\theta}$, the mapping $\mathbf{a}(\mathbf{h}, \boldsymbol{\theta})\colon \Omega \to \mathbf{A}^{\mathrm{T}}$ defined by $\mathbf{a}(\mathbf{h}, \boldsymbol{\theta})(\omega) \equiv \eta(., \omega) \equiv \mathbf{a}(\mathbf{h}(., \omega), \boldsymbol{\theta}(., \omega), .)$ is called the *contingent plan* of the decision maker corresponding to **h** and $\boldsymbol{\theta}$ for the *entire time period* **T**. It associates to each *state of nature* ω the unique *sample action path or line of action* $\eta(., \omega)$. Let $\boldsymbol{\Pi}$ be the set of *all possible contingent plans*.[267]

Informed act, informed act process, informed contingent act,* and *informed contingent plan

For given *action* **a**, *expertise profile* **h**, *option process* $\boldsymbol{\theta}$, and *information profile* $\Im$, the pair $(\eta, \Im)$ is called an *informed act process*. For each *time t* and *state of nature* ω the pair $(\eta(t, \omega), \Im(t)) \equiv (\mathbf{a}(\mathbf{h}(t, \omega), \boldsymbol{\theta}(t, \omega), t), \Im(t)) \in \mathbf{A}\times 2^{\mathscr{F}}$ is the *informed act* consisting of "*the act taken at time t under the state of nature* ω *and using the information set* $\Im(t)$". Or more precisely, of "*the option* $\boldsymbol{\theta}(t,\omega)$ *taken at time t based on the expertise level* $\boldsymbol{h}(t, \omega)$ *and information set* $\Im(t)$". Similarly, the pair $(\eta(t,.)\ \Im(t))$ is called an *informed contingent act* and the pair $(\mathbf{a}(\mathbf{h}, \boldsymbol{\theta}), \Im)$ is called an *informed contingent plan* or *knowledgeable contingent plan*.[268]

Strategy* or *game plan

For a given *action* **a**, the mapping $\zeta\colon \mathrm{H}\times\tilde{\mathbf{O}}\times\boldsymbol{\Psi} \to \boldsymbol{\Pi}\times\boldsymbol{\Psi}$ defined by $\zeta(\mathbf{h}, \boldsymbol{\theta}, \Im) \equiv (\mathbf{a}(\mathbf{h}, \boldsymbol{\theta}), \Im)$ is called the decision maker's *strategy* or *game plan*. It associates to each triplet

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contingent acting time . Alternatively, one can consider *contingent acts* and their *timing* to be conditioned on the occurrence of the *event* Ω: "option $\theta(t)$ will be taken at time t regardless of the prevailing state of nature".

[267] Again, since the action **a** is not observable, for all practical purposes we can identify the *contingent plan* $\mathbf{a}(\mathbf{h}, \boldsymbol{\theta})$ with the pair *expertise profile-option process* $(\mathbf{h}, \boldsymbol{\theta})$ whose values are observable.

[268] Since we can identify the *contingent plan* $\mathbf{a}(\mathbf{h}, \boldsymbol{\theta})$ with the pair $(\mathbf{h}, \boldsymbol{\theta})$, and the pair $(\mathbf{h}, \Im)$ has already been called a *knowledge profile*, there is no problem for calling the pair $(\mathbf{a}(\mathbf{h}, \boldsymbol{\theta}), \Im)$ both an *informed contingent plan* and a *knowledgeable contingent plan*.

expertise profile, option process, and *information profile* the corresponding *informed contingent plan*. Let Σ be the set of *all possible strategies*.[269]

Hence, given the decision maker's *action* **a**, his or her *choice* of a *strategy* $\varsigma \in \Sigma$ consists essentially of: (1) the *selection* of an *expertise or skill profile* **h** from $\mathbb{H}$, (2) the *selection* of an *option process* $\boldsymbol{\theta}$ from $\tilde{\mathbf{O}}$, and (3) the *selection* of an *information profile* $\mathfrak{I}$ from $\boldsymbol{\Psi}$. Then **a** gives the *contingent plan* $\mathbf{a}(\mathbf{h}, \boldsymbol{\theta})$ which *tells* the decision maker *how to implement the option process* $\boldsymbol{\theta}$ *over the entire time set* **T** *using the expertise profile* **h**. The *contingent plan* $\mathbf{a}(\mathbf{h}, \boldsymbol{\theta})$ is then *implemented over time*, based on the *information structure* $\mathfrak{I}(\mathbf{T}) \equiv \{\mathfrak{I}(t) \colon t \in \mathbf{T}\}$ provided by the *selected information profile* $\mathfrak{I}$.

Act-consequence mapping, consequence, outcome,* and *outcome process

x is a mapping $\mathbf{x} \colon \mathbf{A} \to \mathbf{K}^{\Omega}$ called the *act-consequence mapping*. It associates to each *action* $a \in \mathbf{A}$, the unique element $\mathbf{x}(a, .) \in \mathbf{K}^{\Omega}$ called *the consequence of the act a* which itself, is a mapping from Ω to the set of *commodities* **K** that associates to each *state of nature* ω the *commodity* $\mathbf{x}(a, \omega)$ called the *outcome of action a when the state of nature* ω *prevails*.

For a given *(informed) contingent plan* $(\mathbf{a}(\mathbf{h}, \boldsymbol{\theta}), \mathfrak{I})$, the mapping $\mathbf{o} \colon \mathbf{T} \times \Omega \to \mathbf{K}$ defined by $\mathbf{o}(t, \omega) \equiv \mathbf{x}((\mathbf{a}(\mathbf{h}(t, \omega), \boldsymbol{\theta}(t, \omega), t), \mathfrak{I}(t)), \omega)$ is called the *outcome process* corresponding to the *(informed) contingent plan*. It associates to each *time t* and *state of nature* ω the unique *commodity* $\mathbf{o}(t, \omega) \equiv \mathbf{x}((\mathbf{a}(\mathbf{h}(t, \omega), \boldsymbol{\theta}(t, \omega), t), \mathfrak{I}(t)), \omega)$ consisting of "*the outcome at time t of the (informed) contingent plan* $(\mathbf{a}(\mathbf{h}, \boldsymbol{\theta}), \mathfrak{I})$ *when the state of nature* ω *prevails*". For a given state of nature $\omega \in \Omega$, the mapping $\mathbf{o}(., \omega) \colon \mathbf{T} \to \mathbf{A}$ is called a *sample outcome path*. It associates to each *time t* the *outcome* $\mathbf{o}(t, \omega) \equiv \mathbf{x}((\mathbf{a}(\mathbf{h}(t, \omega), \boldsymbol{\theta}(t, \omega), t), \mathfrak{I}(t)), \omega)$.

[269] If for the same reasons as in footnote 267 we identify the *informed contingent plan* $(\mathbf{a}(\mathbf{h}, \boldsymbol{\theta}), \mathfrak{I})$ with the triplet $(\mathbf{h}, \boldsymbol{\theta}, \mathfrak{I})$, then a *strategy* always reduces to the identity mapping.

For a given *(informed) contingent plan* $(\mathbf{a}(\mathbf{h}, \theta), \Im)$, the mapping $\mathbf{x}(\mathbf{a}(\mathbf{h}, \theta), \Im): \Omega \rightarrow \mathbf{K}^T$ uniquely defined by $\mathbf{x}(\mathbf{a}(\mathbf{h}, \theta), \Im)(\omega) \equiv \mathbf{o}(., \omega) \equiv \mathbf{x}((\mathbf{a}(\mathbf{h}(., \omega), \theta(., \omega), .), \Im), \omega)$ is called the *consequence* associated with the *(informed) contingent plan* $(\mathbf{a}(\mathbf{h}, \theta), \Im)$. It associates to each *state of nature* ω the unique *sample outcome path* $\mathbf{o}(., \omega)$. Let **C** be the set of *all possible consequences*.

Endowment profile* and *endowment set

$\mathscr{E}$ is a set-valued mapping $\mathscr{E}$: $\mathbf{T} \times \Omega \rightarrow 2^{\mathbf{K}}$ called the decision maker's *endowment profile*. It associates to each *time t* and *state of nature* ω the unique subset $\mathscr{E}_t(\omega) \subseteq \mathbf{K}$ called the decision maker's *endowment set*, and which consists of "*all the commodities the decision maker is endowed with at time t when the state of nature* ω *prevails*.[270] We can assume that $\mathscr{E}_t(\omega) \subseteq \Theta_t(\omega)$ for all $t \in \mathbf{T}$ and $\omega \in \Omega$. That is, possible *endowments* are part of the corresponding *option sets*, and hence are *available options* that can be *taken* by the decision maker.

A4.2.1.3 Psychological primitive concepts

We postulate that whenever an individual decision maker is in a *choice situation*, he or she possesses the following three conceptual objects: $\mathbf{P}$, $\succeq_p$, and $\succeq_\ell$. $\mathbf{P}$ is the decision maker's perception of the intrinsic properties of the physical and conceptual objects along with the relationships among them. $\succeq_p$ is the decision maker's *preference* over alternative *consequences*. $\mathscr{L}$ is his or her set of *beliefs* or *subjective assessments of the relative likelihoods* of the different *events*;

[270] Since the decision maker is always endowed with his or her *labor*, the *endowment set* is always non empty.

A4.2.2 Behavioral/psychological Postulates Linking the Primitive Concepts

We postulate that whenever the decision maker is in a *choice situation*, the following happens:

A4.2.2.1 Subjective assessment of relative likelihood of events

He or she is able to *freely select* an element $\succeq_\ell \in \mathscr{L}$ to completely *assess the relative likelihood* of the events, elements of $\mathscr{F}$ with: for **E**, **F** $\in \mathscr{F}$, **E** $\succeq_\ell$ **F** meaning the *event* **E** is *more likely to occur than* the *event* **F**.

A4.2.2.2 Awareness of consequences of acts and the constraints on them

He or she is *fully aware* that given his or her *knowledge profile* $(\mathbf{h}, \Im) \in \mathrm{H} \times \Psi$ and set of *available options* Θ, any of his or her possible *informed contingent plan* $(\mathbf{a}(\mathbf{h}, \theta), \Im) \in \Pi \times \Psi$ will lead to a well defined *consequence* $\mathbf{x}(\mathbf{a}(\mathbf{h}, \theta), \Im)$ the realization of which is an *outcome* $\mathbf{o}(., \omega) \equiv \mathbf{x}((\mathbf{a}(\mathbf{h}(., \omega), \theta(., \omega), .), \Im), \omega) \in \mathbf{O}$, which not only depends on the chosen *informed contingent plan* $(\mathbf{a}(\mathbf{h}, \theta), \Im)$, but also on the future realized *state of nature* ω. Moreover, he or she is also *fully aware* that the *implementation* of any *informed contingent plan* must be within the *constraints* $\mathbf{A}(\mathscr{E})$ determined by his or her *endowment profile* $\mathscr{E}$.

A4.2.2.3 "Beliefs-adjusted" perception of the consequences

He or she uses the selected *likelihood ordering* $\succeq_\ell$ to "adjust" his or her *perception* **Þ** of the consequences in **C** and the *intrinsic relationships* among them, according to the *relative likelihood* of the possible outcomes of each of the consequences in **C**.

A4.2.2.4 Preference ordering of the consequences

He or she uses $\succeq_p$ to *completely rank in a consistent manner* all the elements of **C**, taking into account his or her "beliefs-adjusted" *perception* of them and the *intrinsic relationship* existing among them with: for **x**, **y** $\in$ **C**, **x** $\succeq_p$ **y** meaning **x** is *preferred* to **y**.

A4.2.2.5 Optimality of actions or decisions

A4.2.2.5a Optimality of belief assessments

The selection of an element $\succeq_\ell$ from $\mathscr{L}$ to *assess the relative likelihood* of the elements of $\mathscr{F}$ is *optimal* in the sense that given his *knowledge* (h_0, $\Im_0$) at the *initial time* of decision, the chosen *likelihood assessment* is the one, among all likelihood assessments in $\mathscr{L}$ that yields the *maximum satisfaction* for any *strategy (informed contingent plan)* the decision maker can *choose*. In other words, for any *informed contingent plan* that might be selected from **A**($\mathscr{E}$), he or she cannot improve his or her *welfare* by selecting, instead of $\succeq_\ell$, an alternative *likelihood assessment* $\succeq_\ell'$ from $\mathscr{L}$

A4.2.2.5b Preference maximization

Finally, given the optimal *subjective assessment of the relative likelihoods* of the elements of $\mathscr{F}$, and the above *ranking* of the elements of **C**, he or she *chooses* an element from the set **A**($\mathscr{E}$) of *feasible informed contingent plans*, in a manner consistent with the fact that among all the *consequences* in a given subset **C**($\mathscr{E}$) of **C** corresponding to **A**($\mathscr{E}$), a particular one is his or her *most preferred consequence*. In other words, the particular *consequence* that the *act* will lead to has the *highest rank* in this *choice situation* when $\succeq_p$ is restricted to **C**($\mathscr{E}$).

A4.2.3 Application: An agricultural household model

The agricultural household model outlined below is an application of the conceptual framework of decision making under uncertainty just outlined. It is general in nature and does not attempt to represent all types of behavior or cover every aspect of the agricultural household environment. Nonetheless, its empirical relevance is not diminished because it can be easily adapted to any particular situation once the empirical data is on hand.

We assume that the objective of the household is to maximize satisfaction from its lifetime accumulated level of consumption of goods which include both food and non-food commodities (more on this shortly). As explained in Appendix **A4.1** and in the previous section, this objective is achieved through the choice and implementation of optimal *informed contingent plans*. Also, as explained in Appendix **A.4.1** we assume that at any point in time the agricultural household is operating along one optimal informed contingent plan. The agricultural household can choose or abandon a given optimal informed contingent plan at any point in time. However, a crucial stage in the decision process is at the beginning of the growing season where production decisions or plans must be made or revised. Another crucial stage in the decision process is at harvest where the consumption, saving, marketing, and off-farm employment decisions or plans must be made or revised in light of the harvested crops. The following paragraphs describe more in detail the agricultural household's decision process.

The uncertain environment of the agricultural household is described by a set Ω containing all relevant possible states of nature. A possible state of nature $\omega \equiv (s, g, b)$, includes the (random) vector of government policy variables g, the (random) vector of the central bank's policy variables b, and all the remaining sources of uncertainty s which includes the climatic variables (level and distribution of rainfall, winds, insect invasions, etc...), and the actions taken by other farmers and market participants (which affect the diversity and aggregate level of output and relative prices at harvest), etc... At any point t of its lifetime, the agricultural household information set $\Im(t)$ is made up of the historical evolution of the climatic variables, the condition of its fields (soil quality, erosion, etc...), its crop production, domestic market prices (both in the official and parallel markets), personal income, personal assets, and the realization of other relevant random variables up to time t. Also included in each farmer's information sets are the consumption outcomes of its past decisions, the history of the government agricultural policy, and the occurrence or not of any event that might affect the outcomes of its decisions.

At any point t of its lifetime, and for any possible state of nature ω, the agricultural household's expertise or human capital stock $\mathbf{h}(t, \omega)$ is made of its literacy and education levels, its farming skills and other skills in off-farm activities, trading, etc.... Also, at any point t of its lifetime, and for any possible state of nature ω, the set $\mathbf{\Theta}_t(\omega)$ of options available to the agricultural household is constituted by all the inputs of different kinds it can use in the production of the different crops, in the off-farm employment activities, and consumption related activities. Similarly, the agricultural household endowment set at time t under the state of nature $\omega, \mathscr{E}_t(\omega)$, - which determines the set of feasible acts - include its labor endowment, physical and biological assets (land, equipments, livestock, food stocks, cash, etc...), and entitlement or rights (access to credit, land, and other inputs), etc...

The set of acts $\mathbf{A}$ includes the selection of combination of crops to grow, the application of different inputs in each of them (seeding, fertilizer application, tilling, weeding, harvesting, etc...), as well as the performance of off-farm activities, and of consumption, trading, and storage-related activities, etc... The agricultural household's set of contingent acts are all the feasible acts the performance of which is conditioned on the occurrence or not of events determined by the set of possible states of nature $\mathbf{\Omega}$. The following are examples of possible contingent acts for the agricultural household.

(1) I will start planting my crops after the second rainfall. Or, I will start planting my crops when the soil is "very wet" (or, to be precise, when the rainfall reaches 20 mm)

(2) If by the end of June there is not "enough" water in the soil I will not apply fertilizer.

(3) If the drought persists until July 30th, I will abandon my fields. Or, if the drought persists until July 30th, and if I find someone who can lend me money, then I will borrow to buy seed and replant my fields.

(4) If the harvest is good this year, I will pay you back your money when I sell my crops 2 months after harvest. Or, I will pay you back your money when I sell my crops after harvest.

(5) If the harvest is not good this year I will migrate to the city.

(6) If this year's harvested food crop is good, and if the government does not decrease peanut prices next year, then I will reduce my food crop acreage next year.

(7) Starting next year I will rotate peanut and millet in my fields to limit the loss of soil nutrients.

(8) I will buy a sheep for fattening when I harvest my crops.

(9) I will sell one of my sheep to buy rice whenever I run out of food.

(10) I will sell half of my stocks of millet if its price reaches 100 CFA/kg.

(11) If the drought persists until July 30th, then I will start decreasing my level of food consumption so that I can save for next year.

(12) If there is another drought within the 3 years, then I will leave farming for good.

(13) If the government does not decrease peanut prices within the next 5 years, then I will start investing in more farm equipment.

(14) I will stop farming only when I am 60 years old.

Before continuing the description of the agricultural household's decision process, there are several behavioral concepts illustrated in the above examples that we want to comment on. First, most of the examples above are pure contingent acts. The only exceptions are the one in (7), (8), and (14), and the second example in (4) which will be implemented regardless of the prevailing state of nature. Second, we note that for most of them, their acting or implementation times are pure contingent acting times (i.e, the times when they are respectively implemented are random). The only exceptions are the examples which refer to July 30th, and the examples in (12) and (13) which will be implemented at the end of a 3 and 5 year periods, respectively. Also, we note that the timing of the contingent acts may vary from the immediate future to a relatively distant future. In general, they span the agricultural household's lifetime. Moreover, some of the contingent acts like the one in (7) are cyclical. Third, each of the acts in the second example in (3), and in (6) is conditional on the conjunction of two elementary events respectively. The others are conditional on one single event. In general, contingent acts are more complex than these. That is, there is usually a multitude of conditional "if" and/or "when" that are attached to any real world contingent act, and some of the "if" and/or "when" may not be consciously articulated. Fourth, some of the contingent acts are incompatible, and thus cannot belong to the same contingent plan. Contingent plans are made of a collection of compatible contingents acts which are not necessarily chronologically ordered. In fact, at each time t there is possibly a vector of contingent acts making up the part of the contingent plan corresponding to t, depending on whether the acts are conditioned on the same event or whether the events to which the acts are respectively conditioned have the same resolution time. The resolution time of an event is, by definition, the first time the agricultural household knows exactly (i.e, with probability one) that the event is going to occur or not.[271] Fifth, at each time t the agricultural household uses its information set $\Im(t)$ to check the occurrence or not of the various events that condition its contingent acts, and then decides whether or not to implement planned acts. We note also that in some cases like the first example in (4), the resolution time of the event conditioning the act is different from the implementation time of the act. Sixth, we note that not only does the formulation of a contingent act call upon the agricultural household's level of expertise **h**, but also the implementation of some of the acts calls upon more of this expertise than others.

Finally, to conclude these comments on the contingent acts examples, we note that from the above examples it seems natural to assume that at any point in time t, the information set of the agricultural household $\Im(t)$ allows it to tell if a contingent acting or implementation time has arrived, and whether or not a given contingent act has been

[271] See Duffie and Huang, 1986, for a formal definition of *resolution time* of an event that is based on martingale theory.

implemented. Roughly speaking, these two assumptions mean, in the formal language of martingale theory, that contingent acting times are *stopping or optional time*, and contingent plans are *adapted processes* with respect to the *information structure* defined by the agricultural household's collection of information sets $\{\Im(t);\ t \in \mathbf{T}\}$; where **T** is the agricultural household's lifetime. The two assumptions also rule out implicitly the possibility of *loss of memory* and *imperfect recall* of past acts in the part of the agricultural household.[272]

Continuing the description of the decision process, the set **C** of consequences of the agricultural household's possible feasible contingent plans encompasses the annual various possible levels of output at harvest for each crop grown, and the various possible lifetime accumulated feasible levels of consumption of the different goods. Each possible lifetime accumulated feasible level of consumption is called a *feasible consumption pattern* (following Hindy, Huang, and Kreps, 1992), and is determined, in particular, by one of its various possible lifetime accumulated levels of wealth, the stochastic vector of prices for all the commodities in the outcome space **O**, and the prevailing state of nature.

The agricultural household's preference ordering $\succeq$ which ranks the possible consequences corresponding to the various respective possible contingent plans, is restricted to the set of possible lifetime *feasible consumption patterns*, $\mathbf{X}(\mathcal{E})$ which is determined by its endowment profile $\mathcal{E}$. Here, the assumption of *optimality of actions or decisions* means that, given the agricultural household's perception of the outside world, its expertise level **h**, and its information structure $\{\Im(t);\ t \in \mathbf{T}\}$; its subjective assessment of the relative likelihood of

[272] Formally, it is also implicitly assumed that the *information structure* $\{\Im(t);\ t \in \mathbf{T}\}$ is a *filtering*. That is, the agricultural household's information set is monotonically increasing as time goes. In other word, there is no loss of memory, and information can only accumulate over time. However, for the government and other agricultural households since they may not know about this particular household's contingent plans, and may not observe the taking of some of its acts, its acting times and its contingent plans may not be *optional time* and *adapted processes* respectively with respect to their respective *information structures*

the different possible states of nature, made by selecting an element ℓ from its set of beliefs $\mathcal{L}$ at the time when optimal contingent plans were being chosen and during their implementation, was always the most satisfactory under all possible contingent plans the household may have considered; and that implemented contingent acts will be consistent with showing that a particular consequence in $\mathbf{X}(\mathscr{E})$ is its most preferred *consumption pattern*.

The set **O** of outcomes of the agricultural household's implemented optimal contingent plans include the actual annually realized levels of its harvested crops and bundles of consumption goods consumed during its lifetime.

The agricultural household model just outlined has been the inspirational force behind all the conceptual and theoretical developments in the thesis. However, time and space does not allow the full development of the model here, which cannot be properly done until the mathematical treatment of intertemporal choice under uncertainty is completed along the lines of chapter **4**, and with the introduction of additional concepts from martingale theory which allows an explicit and rigorous account of the impacts that the progressive revelation of information over time have on the agricultural household's expectations, behavior, and welfare.

APPENDIX 5

THE "NON-EXPECTED" QUAIDS VERSION 2

A5 Expressions for the "non-expected" QUAIDS version 2 model (NEQAIDS2)

The NQUEAIDS2 model is defined by taking $a(\mathbf{p})$, $b(\mathbf{p})$, and $\lambda(\mathbf{p})$ as respectively:

$$\ln a(\mathbf{p}) = \alpha_0 + \sum_{k=1}^{n} \alpha_k \ln\|\mathbf{p}_k\| + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n} \gamma_{kl} \ln\|\mathbf{p}_k\| \ln\|\mathbf{p}_l\| \tag{A5.1}$$

$$b(\mathbf{p}) = \prod_{k=1}^{n} \|\mathbf{p}_k\|^{\beta_k} \tag{A5.2}$$

and

$$\lambda(\mathbf{p}) = \sum_{k=1}^{n} \lambda_k \ln\|\mathbf{p}_k\| \tag{A5.3}$$

where $\|\mathbf{p}_i\|^2 = E\mathbf{p}_i^2$ is the second moment of the random variable $\mathbf{p}_i$, and the α's, β's, γ's, and λ's are parameters to be estimated. By Lemma **4.4.1** we have:

$$\frac{\partial \ln a_i(\mathbf{p})}{\partial \mathbf{p}_i} = \left[\alpha_i + \sum_{k=1}^{n} \gamma_{ik} \ln\|\mathbf{p}_k\|\right] \frac{\mathbf{p}_i}{\|\mathbf{p}_i\|^2} \qquad i=1,\ldots,n \tag{A5.4}$$

$$\frac{\partial^2 \ln a(\mathbf{p})}{\partial \mathbf{p}_i \partial \mathbf{p}_j} = \gamma_{ij} \frac{\mathbf{p}_i \mathbf{p}_j}{\|\mathbf{p}_i\|^2 \|\mathbf{p}_j\|^2} + \delta_{ij}\left[\alpha_i + \sum_{k=1}^{n} \gamma_{ik} \ln\|\mathbf{p}_k\|\right]\left[\frac{1}{\|\mathbf{p}_i\|^2} - 2\frac{\mathbf{p}_i^2}{\|\mathbf{p}_i\|^4}\right] \tag{A5.5}$$

$$b_i'(\mathbf{p}) = \beta_i b(\mathbf{p}) \frac{\mathbf{p}_i}{\|\mathbf{p}_i\|^2} \qquad i=1,\ldots,n \tag{A5.6}$$

$$b_{ij}''(\mathbf{p}) = \beta_i \beta_j b(\mathbf{p}) \frac{\mathbf{p}_i \mathbf{p}_j}{\|\mathbf{p}_i\|^2 \|\mathbf{p}_j\|^2} + \delta_{ij} \beta_i b(\mathbf{p}) \left\{\frac{1}{\|\mathbf{p}_i\|^2} - 2\frac{\mathbf{p}_i^2}{\|\mathbf{p}_i\|^4}\right\} \qquad i,j=1,\ldots,n \tag{A5.7}$$

$$\lambda_i'(\mathbf{p}) = \lambda_i \frac{\mathbf{p}_i}{\|\mathbf{p}_i\|^2} \qquad i=1,\ldots,n \tag{A5.8}$$

where δ_{ij} is the Kroneker delta which is equal to 1 if $i=j$ and 0 otherwise.

$$\lambda_{ij}''(\mathbf{p}) = \delta_{ij}\lambda_i \left\{ \frac{1}{\|\mathbf{p}_i\|^2} - 2\frac{\mathbf{p}_i^2}{\|\mathbf{p}_i\|^4} \right\} \quad i,j=1,\ldots,n \tag{A5.9}$$

Hence, allowing for the households characteristics to affect some of the preference parameters, the budget shares for the NEQAIDS2 to be estimated are:

$$\omega_i = \alpha_i(\mathbf{z})\frac{\mathbf{p}_i^2}{\|\mathbf{p}_i\|^2} + \sum_{k=1}^{n}\gamma_{ik}\frac{\mathbf{p}_i^2}{\|\mathbf{p}_i\|^2}\ln\|\mathbf{p}_k\| + \beta_i\frac{\mathbf{p}_i^2}{\|\mathbf{p}_i\|^2}\ln\frac{m}{a(\mathbf{p})} - \lambda_i\frac{\mathbf{p}_i^2}{\|\mathbf{p}_i\|^2 b(\mathbf{p})}\left[\ln\frac{m}{a(\mathbf{p})}\right]^2 \quad i=1,\ldots,n \tag{A5.10}$$

where $\mathbf{z}$ is the r-dimensional vector of the time variables, the geographical variables, and the other household characteristics, while $\alpha_i(\mathbf{z})$ represents the total effect of $\mathbf{z}$ on the i^{th} share, and is given by

$$\alpha_i(\mathbf{z}) = \alpha_{i0} + \sum_{k=1}^{r}\alpha_{ik}\mathbf{z}_k \tag{A5.11}$$

The α's, γ's, β's and λ's are the parameters to be estimated.

For the computation of the elasticities and welfare indicators, the relevant expressions are:

$$\varphi_i^m = \frac{\omega_i}{\mathbf{p}_i} + \beta_i \frac{\mathbf{p}_i}{\|\mathbf{p}_i\|^2} - \frac{2\lambda_i \mathbf{p}_i}{\|\mathbf{p}_i\|^2 b(\mathbf{p})} \ln\frac{m}{a(\mathbf{p})} \qquad i=1,\ldots,n \tag{A5.12}$$

$$\psi_i = \frac{\omega_i}{\mathbf{p}_i} - \beta_i \frac{\mathbf{p}_i}{\|\mathbf{p}_i\|^2} \ln\frac{m}{a(\mathbf{p})} + \frac{\lambda_i \mathbf{p}_i}{\|\mathbf{p}_i\|^2 b(\mathbf{p})} \left[\ln\frac{m}{a(\mathbf{p})}\right]^2 \qquad i=1,\ldots,n \tag{A5.13}$$

$$\begin{aligned} \phi_{ij} = {} & \gamma_{ij} \frac{\mathbf{p}_i \mathbf{p}_j}{\|\mathbf{p}_i\| \|\mathbf{p}_j\|} + \left[\frac{\omega_i}{\mathbf{p}_i} - \varphi_i^m\right] \psi_j \\ & + \frac{\lambda_i \beta_j \mathbf{p}_i \mathbf{p}_j}{\|\mathbf{p}_i\|^2 \|\mathbf{p}_j\|^2 b(\mathbf{p})} \left[\ln\frac{m}{a(\mathbf{p})}\right]^2 + \delta_{ij} \omega_i \frac{\|\mathbf{p}_i\|^2}{\mathbf{p}_i^2} \left[\frac{1}{\|\mathbf{p}_i\|^2} - 2\frac{\mathbf{p}_i^2}{\|\ln \mathbf{p}_i\|^4}\right] \\ & i,j=1,\ldots,n \end{aligned} \tag{A5.14}$$

and

$$cd_i = \mu \left\{ \frac{\omega_i}{\mathbf{p}_i} + \beta_i \frac{\mathbf{p}_i}{\|\mathbf{p}_i\|^2} \ln\frac{\mu}{m} - \frac{\lambda_i \mathbf{p}_i}{\|\mathbf{p}_i\|^2 b(\mathbf{p})} \left[\ln\frac{m\mu}{a(\mathbf{p})^2}\right] \ln\frac{\mu}{m} \right\} \quad i=1,\ldots,n \tag{A5.15}$$

APPENDIX 6

DATA APPENDIX

A6 Data Appendix

A6.1 Methods used to construct the price series

This section describes how we supplemented the price data in the original ISRA/IFPRI data set with market prices from other sources to construct monthly price series for the cereals (millet, sorghum, maize, and rice) and pulses (peanut and cowpeas) for each zone in the stratification of the survey. The first source of price data is the Market Information System of CSA which have been collecting weekly consumer and producer price data for cereals and pulses in Dakar and other 40 local markets around Senegal since 1986. The second source of price data is the ISRA\MSU cereal marketing research project which collected producer and consumer price data for cereals and pulses every two weeks in 18 rural markets around Senegal and Gambia from 1984 to 1990. For the other food commodities, the price data in the food consumption survey was used in conjunction with the variations in the Dakar monthly price indices for various food groups to construct monthly weighted prices for each group of commodities and each zone. The weights being the zone median expenditure shares of the commodities making up the group.

What follows describes how we proceeded to fill in the missing prices and calculate the zone median expenditure shares and price indices. However, we note that the zone median expenditure shares calculated in the process are approximate, and are used for the sole purpose of calculating the price indices. More accurate household level expenditures and expenditure shares series were constructed for each product category for the descriptive and econometric analyses.

A6.1.1. The cereal and pulse price series

The food products concerned here are Millet, Sorghum, Maize, Rice, Peanut, and Niebe. For peanut, only the price for shelled peanut were used in our analysis although price series for unshelled were also constructed.

For each CSA and MSU monthly market price series we calculated the median (across all months) of the differences between the market price and the estimated zone median transaction price from the ISRA/IFRI cereal transaction questionaire. These median differences are respectively added to each MSU or CSA market price we used to estimate the missing zone median month price. To fill in the missing prices we used the following procedure. First if a missing price is between 2 adjacent non-missing prices we fill in with the average of the two adjacent non-missing prices. If not, we use - whenever non-missing - in order of priority the CSA closest market's price, the MSU closest market's price, the MSU regional price, the CSA national price, and at last, the MSU national price. In the SPSS program that implements the procedure, the price with which the missing value was filled in, is documented by a flag (one for each product).

A6.1.2 The zone median expenditure shares series

First, we calculated the zone median expenditure for each product by calculating for each household the total yearly expenditure on that product. Then, for each product we took the median yearly expenditures across all years and across all households. Second, for each product 2 shares were calculated: the within category median expenditure shares (calculated by dividing the zone median expenditure on that product by the total expenditure on all products within the category), and the across products median expenditure shares (calculated by dividing the zone median expenditure on that product by the total expenditure on all products consumed by the household).

A6.1.3 The Price index series for the other food and nonfood commodities

There were several steps. First, for each product we chose only the units that are standard and can be found in almost all of the zones. Then we standardized the units for each product so that we will have only one unit for each product and five units for all products (kilo, liter, meter, months, and piece/box/unit). Second, we calculated the zone median price for each product as follows. For the food products, soap, cement, and fuel, we estimated the zone median price for each year, month, product, and unit. For the other products, we estimated for each product only the zone median price across all years, all months, and all units; we then later used the monthly variations in the Dakar price indices to estimate approximatively the monthly prices based on the single estimated median price for each product. We proceeded in this way because the units for these products are so diverse and the monthly price observations so thin so as to prevent any robust estimation of monthly prices for each product and unit.

Fourth, for each product and each month we filled in the missing prices by using the following procedure. For products other than food with no price at all (i.e prices for which all the months are missing), we took the Dakar price. Otherwise, if a missing price is between 2 adjacent non missing prices we filled in with the average of the two adjacent non missing prices. After that, if a price is still missing we estimated the price by using the monthly variations in the Dakar respective product prices.

Fifth, we eliminated from the calculation of the price indices those products for which no price could be estimated and we adjusted the within category expenditure shares accordingly.

Finally, we calculated the price index for each category as the weighted sum of prices of the products within the category, with the weights being their respective within category shares.[273]

The different product categories for which a price index was calculated are: Meat/Chicken, fresh fish, dried fish, Vegetables, Condiments, Vegetable oil, other food items (Milk, drink, fruits, stimulants etc..), Fuel, Cosmetics/Sanitary, Clothing/Shoes, Transport/telecommunication, Utensils/Kitchen materials, Furniture/Appliances and Housing related Expenses, Other nonfood items, and Agricultural inputs. From here, to have more aggregated group price indices, one only needs to take the weighted sum of the calculated respective price indices with the new weights being the respective shares of the product categories relative to the new larger group being constructed (i.e total expenditure on product categories divided by the total expenditure on the new larger group). For example, if one wants to combine in one new grouping the product categories 1, 2, and 3, with respective price indices p_1, p_2, and p_3, the price index of the new grouping is $(w_1/T)\ p_1 + (w_2/T)\ p_2 + (w_3/T)\ p_3$. Where wi is the total expenditure on product category i, and $T = w_1 + w_2 + w_3$.

[273] We note at this stage there is an option in the SPSS program that implements the procedure which allows, if one wishes, to select only a limited number of products within each category - for example, the 5 products with the highest shares - to be considered for the price index).

A6.2 Data series used in the econometric estimation

A6.2.1 Variable labels

A6.2.1.2 Choice (dependent) variables

Variable name **Label**

E1001 : Expenditure on Millet
E1002 : Expenditure on Sorghum
E1003 : Expenditure on Maize
E1004 : Expenditure on Rice
E1005 : Expenditure on Peanut
E1006 : Expenditure on Niebe
E2001 : Expenditure on Meat
E2002 : Expenditure on Fresh Fish
E2003 : Expenditure on Dried Fish
E2004 : Expenditure on Vegetables
E2005 : Expenditure on Condiments
E2006 : Expenditure on Vegetable Oil
E2007 : Expenditure on Other food items

Note: The seasonal averages of the biweekly 24-hours recall expenditure shares are used.

A6.2.1.2 Exogenous (independent) variables

Variable name **Label**

Prices

P1001 : Consumer price of Millet (CFA/KG) (monthly or seasonal data)
P1002 : Consumer price of Sorghum (CFA/KG) (monthly or seasonal data)
P1003 : Consumer price of Maize (CFA/KG) (monthly or seasonal data)
P1004 : Consumer price of Imported rice (CFA/KG) (monthly or seasonal data)
P1005 : Consumer price of sh.Peanut (CFA/KG) (monthly or seasonal data)
P1006 : Consumer price of Niebe (CFA/KG) (monthly or seasonal data)
PI2001 : Price index for Meat (CFA/KG) (monthly or seasonal data)
PI2002 : Price index for Fresh Fish (CFA/KG) (monthly or seasonal data)
PI2003 : Price index for Dried Fish (CFA/KG) (monthly or seasonal data)
PI2004 : Price index for Vegetables (CFA/KG) (monthly or seasonal data)
PI2005 : Price index for Condiments (CFA/KG) (monthly or seasonal data)
PI2006 : Price index for Vegetable oil (CFA/LITER) (monthly or seasonal data)
PI2007 : Price index for other food (CFA/UNIT) (monthly or seasonal data)

Total food expenditure (instrumented)

EFOODAE: Total food expenditure per adult equivalent (24-hours recall biweekly data)

Demographic variables

AGECM : Age of the head of household

POMH : Household male population (seasonal data)
POMF : Household female population (seasonal data)
POME : Household children population (less than 17 years old) (seasonal data)
POPNAV : Navetane (migrant workers) total population (seasonal data)
POPSRG : Sourghas (resident laborer) total population (seasonal data)

DPEUL : Dummy variable for Peul ethnic origine
DSERERE: Dummy variable for Serere ethnic origine
DSARAKH: Dummy variable for Sarakhole/Diakhanke ethnic origine
DOTHER : Dummy variable for Mand. Touc. and other ethnic origine
(The Wolof ethnicity is the control group)

Geographical variables

DMARKET: Dummy variable for market villages
DSAGATT: Dummy variable for the Sagatta zone
DNORTH: Dummy variable for the Nothwestern rural zones except Sagatta
(Colobane,Niakhar, and Passy)
DKAOLAC: Dummy variable for the Koalack zone
DTAMBA : Dummy variable for the Sagatta zone
(The Southeastern rural zones of Dioly and Missirah are the control variables).

A6.2.1.3 Other variables used as instruments for total food expenditures

Variable name **Label**

OFFINC : Harvest year off-farm income (annual data)
MIGINC : Harvest year migration income (annual data)
NTOTINC: Net harvest year total income (GTOTINC - total input costs) (annual data)

CATLSTK: number of cattles in stock (seasonal data)
HRDKSTK: number of horses, donkeys and cammels in stock (seasonal data)
SHGTSTK: number of sheeps and gots in stock (seasonal data)
PLTRSTK: number of poultry animals in stock (seasonal data)
OTHASTK: number of other animals in stock (seasonal data)

A6.2.2 Means, standard deviations, and ranges of variables

Number of Valid Observations (Listwise) = 1558.00

Variable	Mean	Std Dev	Minimum	Maximum	N	Label
Expenditures for the food commodities						
E1001	233.26	189.51	0	1108	1749	
E1002	21.02	67.52	0	648	1749	
E1003	36.96	107.32	0	1175	1749	
E1004	170.06	170.70	0	1621	1749	
E1005	82.09	68.78	0	925	1749	
E1006	14.16	23.78	0	298	1749	
E2001	50.89	171.18	0	2850	1749	
E2002	52.03	130.01	0	3913	1749	
E2003	33.29	35.46	0	588	1749	
E2004	36.61	88.94	0	1762	1749	
E2005	35.00	53.02	0	517	1749	
E2006	65.56	105.76	0	1050	1749	
E2007	214.78	229.05	0	1765	1749	
EFOODCAP	101.04	67.43	23.91	749.40	1711	Daily food total exp./cap.
ETOTCAP	30868.06	27435.19	4293	201523	1702	Annual total Exp./cap.
FOODSH	.82	.09	.254	.980	1749	Food share
Prices						
P1001	82.63	20.35	55	205	1749	
P1002	80.81	10.16	47	142	1749	
P1003	93.67	20.21	53	155	1749	
P1004	136.87	5.32	135	176	1749	
P1005	186.58	57.09	100	400	1749	
P1006	153.47	89.41	42	544	1749	
PI2001	691.71	128.56	491	1012	1749	
PI2002	227.69	200.09	50	1076	1749	
PI2003	178.87	102.60	33	700	1749	
PI2004	243.43	106.25	80	601	1749	
PI2005	306.90	260.50	21	1011	1749	
PI2006	374.40	31.08	300	452	1749	
PI2007	554.50	189.67	210	1041	1749	
STINDX	5.49	.27	5	6	1749	Stone price index
Household composition variables						
POMH	5.96	4.64	0	56	1711	

POMF	5.86	3.45	0	25	1711	
POMADL	5.19	2.86	1	21	1711	
POMENF	6.63	4.72	0	46	1711	
POMEH	3.52	3.60	0	43	1711	
POMEF	3.11	2.30	0	18	1711	
Livestock						
CATLSTK	.28	.68	0	5	1711	number of cattles in
HRDKSTK	.12	.13	0	1	1711	number of horses, do
SHGTSTK	.48	.76	0	10	1711	number of sheeps and
PLTRSTK	.59	1.04	0	13	1711	number of poultry an
OTHASTK	.01	.07	0	1	1711	number of other anim
Income Variable						
OFFINC	20438.89	55780.37	-4880	718496	1708	Annual off-farm income
MIGINC	1218.33	5419.12	-5313	61853	1708	Annual migration income
NTINCAP	41088.01	59786.94	-7713	812253	1708	Annual net income/cap.

A6.3 Additional descriptive tables on households in the ISRA/IFPRI Sample

Table A6.1 Characteristics of households in the sample: Ethnic origin and Sex of the head of household

	Rural zones						Urban zones		Sample total
	Sagatta	Niakhar	Colobane	Passy	Dioly	Missirah	Kaolack	Tamba	
Ethnic origin									
Wolof.........	20 (56%)	0 (0%)	27 (79%)	10 (32%)	32 (91%)	0 (0%)	16 (47%)	6 (19%)	111 (41%)
Mandingue.....	0 (0%)	0 (0%)	0 (0%)	1 (3%)	0 (0%)	4 (12%)	1 (3%)	2 (6%)	8 (3%)
Sarakhole.....	0 (0%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)	16 (48%)	0 (0%)	4 (13%)	20 (7%)
Toucouleur....	2 (6%)	0 (0%)	0 (0%)	1 (3%)	0 (0%)	0 (0%)	8 (24%)	1 (3%)	12 (4%)
Peul..........	14 (39%)	0 (0%)	0 (0%)	1 (3%)	3 (9%)	12 (36%)	4 (12%)	16 (50%)	50 (19%)
Serere........	0 (0%)	34 100%	7 (21%)	15 (48%)	0 (0%)	1 (3%)	5 (15%)	0 (0%)	62 (23%)
Other.........	0 (0%)	0 (0%)	0 (0%)	3 (10%)	0 (0%)	0 (0%)	0 (0%)	3 (9%)	6 (2%)
Total.........	36 100%	34 100%	34 100%	31 100%	35 100%	33 100%	34 100%	32 100%	269 100%
Sex of the head of household									
Male..........	36 100%	34 100%	33 (97%)	31 100%	35 100%	33 100%	26 (76%)	27 (84%)	255 (95%)
Female........	0 (0%)	0 (0%)	1 (3%)	0 (0%)	0 (0%)	0 (0%)	8 (24%)	5 (16%)	14 (5%)
Total.........	36 100%	34 100%	34 100%	31 100%	35 100%	33 100%	34 100%	32 100%	269 100%

Source: ISRA/IFPRI survey.
Notes: Ethnic distribution at the national level: Wolof (36%).Peul (17%). Serere (17%). Toucouleur (9%). Diola (9%). Mandingue (9%). Others (3%)

Table A6.2 Characteristics of households in the sample: Household population

	Rural zones						Urban zones		Sample average
	Sagatta	Niakhar	Colobane	Passy	Dioly	Missirah	Kaolack	Tamba	
Average age of the household head	47	52	49	50	41	48	51	46	48
Average household size	11	10	11	13	13	11	12	10	11
Average household children population	7	6	6	8	7	6	7	5	6
Average household adult population	5	4	5	5	6	5	6	4	5
Average household male population	5	5	5	6	7	6	6	5	6
Average household female population	6	5	6	6	6	6	6	5	6
Average household adults equivalent	8	7	8	9	10	9	10	7	9

Source: ISRA/IFPRI survey.

Table A6.3 Characteristics of households in the sample: Consumption and production patterns by income groups

	Rural zones							
	Nonmarket villages				Market villages			
	Income quartile				Income quartile			
	low 25%	25 - 50%	50 - 75%	top 25%	low 25%	25 - 50%	50 - 75%	top 25%
Food share in total exp.								
< 70%........	0 (0%)	0 (0%)	0 (0%)	8 (4%)	0 (0%)	2 (2%)	2 (2%)	3 (3%)
70 - 80%......	2 (1%)	9 (4%)	6 (3%)	22 (10%)	1 (1%)	5 (5%)	7 (7%)	18 (17%)
80 - 90%......	25 (11%)	26 (12%)	33 (15%)	33 (15%)	8 (8%)	12 (11%)	20 (19%)	8 (8%)
> 90%........	13 (6%)	26 (12%)	16 (7%)	3 (1%)	7 (7%)	4 (4%)	7 (7%)	1 (1%)
Rice share in food exp.								
< 10%........	27 (12%)	35 (16%)	15 (7%)	32 (14%)	4 (4%)	6 (6%)	4 (4%)	8 (8%)
10 - 20%......	3 (1%)	16 (7%)	22 (10%)	12 (5%)	4 (4%)	8 (8%)	20 (19%)	10 (10%)
20 - 30%......	7 (3%)	9 (4%)	14 (6%)	20 (9%)	8 (8%)	6 (6%)	10 (10%)	9 (9%)
> 30%........	3 (1%)	1 (0%)	4 (2%)	2 (1%)	0 (0%)	3 (3%)	2 (2%)	3 (3%)
Coarse grains share in food exp.								
< 10%........	0 (0%)	0 (0%)	1 (0%)	0 (0%)	0 (0%)	2 (2%)	6 (6%)	4 (4%)
10 - 20%......	4 (2%)	1 (0%)	2 (1%)	2 (1%)	3 (3%)	6 (6%)	6 (6%)	6 (6%)
20 - 30%......	4 (2%)	8 (4%)	9 (4%)	12 (5%)	2 (2%)	1 (1%)	4 (4%)	2 (2%)
> 30%........	32 (14%)	52 (23%)	43 (19%)	52 (23%)	11 (10%)	14 (13%)	20 (19%)	18 (17%)
Coarse grains share in crop income								
< 50%........	23 (10%)	34 (15%)	29 (13%)	58 (26%)	10 (10%)	11 (10%)	22 (21%)	25 (24%)
50 - 60%......	6 (3%)	10 (5%)	14 (6%)	8 (4%)	2 (2%)	4 (4%)	8 (8%)	4 (4%)
60 - 70%......	8 (4%)	13 (6%)	4 (2%)	0 (0%)	3 (3%)	4 (4%)	4 (4%)	0 (0%)
> 70%........	3 (1%)	4 (2%)	8 (4%)	0 (0%)	1 (1%)	4 (4%)	2 (2%)	1 (1%)
Peanut share in crop income								
< 50%........	26 (12%)	37 (17%)	29 (13%)	12 (5%)	8 (8%)	12 (11%)	15 (14%)	7 (7%)
50 - 60%......	7 (3%)	12 (5%)	14 (6%)	18 (8%)	2 (2%)	2 (2%)	6 (6%)	10 (10%)
60 - 70%......	4 (2%)	5 (2%)	5 (2%)	25 (11%)	2 (2%)	5 (5%)	2 (2%)	4 (4%)
> 70%........	3 (1%)	7 (3%)	7 (3%)	11 (5%)	4 (4%)	4 (4%)	13 (12%)	9 (9%)

Source: ISRA/IFPRI survey.
Notes: Each cell contains the number of households in the group and the correspoding percentage in parentheses. The rural and urban households were separately classified into income quartiles.

Table A6.4 Income per adult equivalent by sources: Variation across years and zones

in CFA francs ($1 =280 CFA)	Rural zones						Urban zones		Average
	Sagatta	Niakhar	Colobane	Passy	Dioly	Missirah	Kaolack	Tamba	
Harvest year 88-89									
Net on-farm income.	8.498	12.567	18.056	24.507	35.568	18.094	.	.	16.654
Migration income....	2.906	985	1.627	0	138	3.271	.	.	1.285
Off-farm income...	17.137	13.137	6.210	16.042	6.692	5.334	.	.	11.228
Other income........	6.046	424	141	1.625	2.668	3.195	.	.	1.262
Livestock income...	3.833	0	0	381	796	129	.	.	528
Net income..........	38.420	27.113	26.034	42.555	45.863	30.024	.	.	30.956
Rainfall (in mm)....	449	644	625	669	810	950	.	.	629
Harvest year 89-90									
Net on-farm income.	.	25.790	39.864	38.906	46.830	15.045	.	.	33.675
Migration income....	.	12	2.305	79	4	5.702	.	.	945
Off-farm income...	.	4.430	6.659	22.625	11.471	3.681	.	.	7.962
Other income........	.	1.389	384	470	-2.663	3.000	.	.	720
Livestock income...	.	0	0	1.527	1.596	659	.	.	279
Net income..........	.	31.621	49.212	63.607	57.237	28.088	.	.	43.581
Rainfall (in mm)....	.	802	556	717	736	756	.	.	694
Harvest year 90-91									
Net on-farm income.	.	.	.	.	.	.	622	978	693
Migration income....	.	.	.	.	.	.	5.569	-203	4.416
Off-farm income...	.	.	.	.	.	.	109.546	95.905	106.820
Other income........	.	.	.	.	.	.	17.289	3.691	14.572
Livestock income...	.	.	.	.	.	.	1.793	0	1.435
Net income..........	.	.	.	.	.	.	134.819	100.371	127.936
Rainfall (in mm)....	.	.	.	.	.	.	462	676	505
Average across years									
Net on-farm income.	8.498	19.806	29.308	31.512	41.288	16.668	622	978	21.530
Migration income....	2.906	452	1.977	38	70	4.408	5.569	-203	1.594
Off-farm income...	17.137	8.370	6.442	19.245	9.119	4.561	109.546	95.905	23.675
Other income........	6.046	952	266	1.063	-40	3.104	17.289	3.691	2.958
Livestock income...	3.833	0	0	939	1.203	377	1.793	0	554
Net income..........	38.420	29.581	37.993	52.797	51.640	29.118	134.819	100.371	50.312
Rainfall (in mm)....	449	731	589	692	772	859	462	676	639

Source: ISRA/IFPRI survey.

Notes: Each cell contains the weighted mean of the annual income across all households. A harvest year runs from October to September; However, 88-89 started in January 89. For Kaolack and Tamba, the period covered is July 90 - June 91.

Table A6.5 Shares of net income by sources: Rural versus urban zones

	Rural zones			Urban zones		
	Nonmarket villages	Market villages	Average	Kaolack	Tamba	Average
Harvest year 88-89						
Net on-farm income.	68%	43%	61%	.	.	.
Migration income....	5%	5%	5%	.	.	.
Off-farm income....	20%	48%	28%	.	.	.
Other income........	5%	3%	5%	.	.	.
Livestock income...	2%	1%	2%	.	.	.
Ratio of net income to per capita GDP	17%	18%	17%	.	.	.
Harvest year 89-90						
Net on-farm income.	82%	70%	79%	.	.	.
Migration income....	2%	3%	2%	.	.	.
Off-farm income....	12%	23%	16%	.	.	.
Other income........	3%	2%	3%	.	.	.
Livestock income...	0%	1%	1%	.	.	.
Ratio of net income to per capita GDP	25%	22%	24%	.	.	.
Harvest year 90-91						
Net on-farm income.	.	.	.	1%	2%	1%
Migration income....	.	.	.	5%	0%	4%
Off-farm income....	.	.	.	75%	93%	78%
Other income........	.	.	.	17%	6%	15%
Livestock income...	.	.	.	2%	0%	2%
Ratio of net income to per capita GDP	.	.	.	78%	58%	74%
Average across years						
Net on-farm income.	75%	58%	70%	1%	2%	1%
Migration income....	4%	4%	4%	5%	0%	4%
Off-farm income....	16%	35%	22%	75%	93%	78%
Other income........	4%	3%	4%	17%	6%	15%
Livestock income...	1%	1%	1%	2%	0%	2%
Ratio of net income to per capita GDP	20%	20%	20%	78%	58%	74%

Source: ISRA/IFPRI survey.
Notes: Each cell contains the weighted mean of the annual income shares across all households. A harvest year runs from October to September; However, 88-89 started in January 89. For Kaolack and Tamba, the period covered is July 90 - June 91. Per capita GDP for Senegal (from World Bank Tables) were respectively: $650, $650 and $615.

Table A6.6 Shares of net income by sources: Variation across years and income groups

	Rural zones				Urban zones				Average
	Income group				Income group				
	low 25%	25 - 50%	50 - 75%	top 25%	low 25%	25 - 50%	50 - 75%	top 25%	
Harvest year 88-89									
Net on-farm income.	65%	63%	59%	58%	.	.	.	.	61%
Migration income....	5%	3%	7%	3%	.	.	.	.	4%
Off-farm income....	9%	24%	24%	32%	.	.	.	.	23%
Other income........	20%	9%	5%	5%	.	.	.	.	9%
Livestock income...	2%	1%	5%	2%	.	.	.	.	3%
Ratio of net income to per capita GDP	9%	14%	17%	34%	.	.	.	.	20%
Harvest year 89-90									
Net on-farm income.	77%	77%	71%	74%	.	.	.	.	74%
Migration income....	4%	3%	6%	2%	.	.	.	.	4%
Off-farm income....	9%	13%	19%	24%	.	.	.	.	17%
Other income........	10%	4%	2%	-2%	.	.	.	.	3%
Livestock income...	0%	3%	3%	1%	.	.	.	.	2%
Ratio of net income to per capita GDP	8%	16%	24%	45%	.	.	.	.	26%
Harvest year 90-91									
Net on-farm income.	.	.	.	.	1%	2%	0%	0%	1%
Migration income....	.	.	.	.	1%	3%	5%	0%	3%
Off-farm income....	.	.	.	.	75%	74%	90%	91%	83%
Other income........	.	.	.	.	23%	20%	1%	8%	12%
Livestock income...	.	.	.	.	0%	0%	3%	0%	1%
Ratio of net income to per capita GDP	.	.	.	.	18%	39%	58%	150%	69%
Average across years									
Net on-farm income.	70%	70%	64%	66%	1%	2%	0%	0%	57%
Migration income....	4%	3%	6%	3%	1%	3%	5%	0%	4%
Off-farm income....	9%	19%	22%	28%	75%	74%	90%	91%	30%
Other income........	16%	7%	4%	2%	23%	20%	1%	8%	7%
Livestock income...	1%	2%	4%	2%	0%	0%	3%	0%	2%
Ratio of net income to per capita GDP	9%	15%	20%	39%	18%	39%	58%	150%	29%

Source: ISRA/IFPRI survey.
Notes: Each cell contains the weighted mean of the annual income shares across all households. A harvest year runs from October to September; However, 88-89 started in January 89. For Kaolack and Tamba, the period covered is July 90 - June 91. Per capita GDP for Senegal (from World Bank Tables) were respectively: $650, $650 and $615.

Table A6.7 Shares of net income by sources: Market versus nonmarket villages, versus urban zones by income groups

	Rural zones							
	Nonmarket villages				Market villages			
Income groups	low 25%	25 - 50%	50 - 75%	top 25%	low 25%	25 - 50%	50 - 75%	top 25%
Harvest year 88-89								
Net on-farm income.	69%	69%	74%	67%	52%	47%	37%	40%
Migration income....	6%	3%	8%	3%	0%	5%	5%	2%
Off-farm income....	10%	19%	9%	23%	7%	38%	45%	51%
Other income........	13%	10%	3%	6%	41%	7%	8%	5%
Livestock income...	3%	0%	6%	1%	0%	3%	5%	3%
Ratio of net income to per capita GDP	9%	14%	16%	33%	9%	13%	19%	35%
Harvest year 89-90								
Net on-farm income.	77%	84%	85%	78%	75%	60%	47%	65%
Migration income....	7%	3%	2%	3%	0%	5%	12%	0%
Off-farm income....	9%	8%	8%	21%	9%	25%	36%	32%
Other income........	7%	4%	2%	-2%	15%	5%	2%	-1%
Livestock income...	0%	1%	3%	1%	0%	6%	3%	4%
Ratio of net income to per capita GDP	8%	16%	24%	44%	9%	15%	23%	47%
Average across years								
Net on-farm income.	72%	76%	79%	72%	64%	54%	42%	51%
Migration income....	6%	3%	5%	3%	0%	5%	8%	1%
Off-farm income....	9%	14%	9%	22%	8%	31%	41%	42%
Other income........	11%	7%	2%	2%	28%	6%	5%	2%
Livestock income...	2%	1%	5%	1%	0%	4%	4%	4%
Ratio of net income to per capita GDP	9%	15%	20%	38%	9%	14%	21%	40%

Source: ISRA/IFPRI survey.
Notes: Each cell contains the weighted mean of the annual income shares across all households. A harvest year runs from October to September; However, 88-89 started in January 89. For Kaolack and Tamba, the period covered is July 90 - June 91. Per capita GDP for Senegal (from World Bank Tables) were respectively: $650, $650 and $615.

Table A6.8 Production and value of major crops per adult equivalent: Variation across years and zones

	Sagatta	Niakhar	Colobane	Passy	Dioly	Missirah	Average
Harvest year 88-89							
Millet (in Kg)......	34	116	109	139	165	41	109
Sorghum (in Kg).....	0	8	1	0	36	37	5
Maize (in Kg).......	0	0	0	1	8	75	1
Peanut (in Kg)......	90	86	196	217	367	76	154
Niebe (in Kg).......	20	0	2	0	0	3	3
Cotton (in Kg)......	0	0	0	0	27	93	2
Croping income (gross)..........	10.847	14.869	21.415	24.411	40.961	21.434	19.151
Rainfall (in mm)....	449	644	625	669	810	950	629
Harvest year 89-90							
Millet (in Kg)......	57	220	213	277	176	27	204
Sorghum (in Kg).....	0	27	7	7	81	50	18
Maize (in Kg).......	0	0	0	0	9	73	1
Peanut (in Kg)......	163	205	488	286	599	140	321
Niebe (in Kg).......	15	128	6	0	0	4	54
Cotton (in Kg)......	0	0	0	1	21	0	1
Croping income (gross)..........	15.815	36.490	48.793	35.434	57.601	18.520	39.044
Rainfall (in mm)....	474	802	556	717	736	756	671
Harvest year 90-91							
Millet (in Kg)......	.	134	202	203	114	16	165
Sorghum (in Kg).....	.	2	1	3	21	52	3
Maize (in Kg).......	.	0	0	2	0	69	1
Peanut (in Kg)......	.	41	224	426	60	173	160
Niebe (in Kg).......	.	2	2	0	0	0	2
Cotton (in Kg)......	.	0	0	3	4	133	2
Croping income (gross)..........	.	11.997	27.651	41.497	11.648	27.338	21.622
Rainfall (in mm)....	.	346	361	462	569	676	381
Average across years							
Millet (in Kg)......	46	159	175	205	152	29	160
Sorghum (in Kg).....	0	13	3	3	46	46	9
Maize (in Kg).......	0	0	0	1	6	73	1
Peanut (in Kg)......	126	112	308	308	341	126	216
Niebe (in Kg).......	18	46	3	0	0	3	21
Cotton (in Kg)......	0	0	0	1	17	75	2
Croping income (gross)..........	13.331	21.502	33.081	33.610	36.693	22.269	27.091
Rainfall (in mm)....	462	594	518	617	704	804	568

Source: ISRA/IFPRI survey.
Notes: Each cell contains the weighted mean of the annual production and value across all households.

Table A6.9 Major sources of cropping (gross) income: Variation across years and zones

	Sagatta	Niakhar	Colobane	Passy	Dioly	Missirah	Average
Harvest year 88-89							
Millet..............	21%	65%	40%	44%	28%	11%	46%
Sorghum.............	0%	0%	0%	0%	7%	13%	1%
Maize...............	0%	0%	0%	1%	2%	15%	0%
Peanut..............	65%	35%	58%	56%	59%	28%	50%
Niebe...............	13%	0%	1%	0%	0%	2%	2%
Cotton..............	0%	0%	0%	0%	4%	30%	1%
Cropping income.....	36%	55%	72%	65%	77%	63%	61%
Rainfall (in mm)....	449	644	625	669	810	950	629
Harvest year 89-90							
Millet..............	26%	52%	34%	47%	19%	7%	41%
Sorghum.............	0%	0%	1%	1%	7%	15%	1%
Maize...............	0%	0%	0%	0%	3%	23%	0%
Peanut..............	66%	35%	64%	51%	68%	53%	52%
Niebe...............	8%	13%	1%	0%	0%	2%	6%
Cotton..............	0%	0%	0%	0%	3%	0%	0%
Cropping income.....	.	96%	80%	62%	82%	72%	85%
Rainfall (in mm)....	474	802	556	717	736	756	671
Harvest year 90-91							
Millet..............	.	74%	41%	31%	57%	3%	55%
Sorghum.............	.	0%	0%	0%	7%	12%	1%
Maize...............	.	0%	0%	1%	0%	15%	0%
Peanut..............	.	25%	57%	67%	34%	40%	43%
Niebe...............	.	1%	1%	0%	0%	0%	1%
Cotton..............	.	0%	0%	0%	1%	30%	0%
Cropping income.....	.	.	.	.	.	.	.
Rainfall (in mm)....	.	346	361	462	569	676	381
Average across years							
Millet..............	23%	63%	38%	41%	35%	8%	47%
Sorghum.............	0%	0%	1%	1%	7%	13%	1%
Maize...............	0%	0%	0%	1%	2%	18%	0%
Peanut..............	66%	31%	60%	58%	54%	40%	48%
Niebe...............	11%	5%	1%	0%	0%	1%	3%
Cotton..............	0%	0%	0%	0%	3%	20%	0%
Cropping income.....	36%	77%	76%	64%	80%	67%	73%
Rainfall (in mm)....	462	594	518	617	704	804	568

Source: ISRA/IFPRI survey.
Notes: Each cell contains the weighted mean of the annual shares across all households (share of cropping income is gross cropping income divided by gross total income).

Table A6.10 Major sources of cropping (gross) income: Market versus nonmarket villages by income group

	Nonmarket villages				Market villages			
	Income group				Income group			
	low 25%	25 - 50%	50 - 75%	top 25%	low 25%	25 - 50%	50 - 75%	top 25%
Harvest year 88-89								
Millet	39%	43%	41%	28%	18%	28%	27%	20%
Sorghum	4%	3%	2%	3%	0%	9%	5%	6%
Maize	3%	3%	2%	3%	12%	3%	5%	1%
Peanut	32%	38%	48%	61%	60%	51%	62%	72%
Niebe	6%	4%	3%	1%	0%	1%	1%	0%
Cotton	17%	9%	4%	4%	9%	8%	1%	1%
Cropping income	68%	68%	73%	67%	71%	48%	39%	41%
Harvest year 89-90								
Millet	26%	35%	40%	22%	28%	43%	25%	24%
Sorghum	6%	4%	2%	5%	2%	8%	3%	3%
Maize	7%	3%	2%	1%	20%	10%	3%	4%
Peanut	56%	53%	52%	68%	48%	35%	60%	67%
Niebe	4%	5%	4%	1%	2%	5%	9%	2%
Cotton	0%	0%	1%	2%	0%	0%	0%	0%
Cropping income	94%	87%	86%	77%	87%	66%	54%	68%
Harvest year 90-91								
Millet	29%	45%	53%	41%	47%	47%	36%	45%
Sorghum	5%	2%	2%	6%	3%	6%	7%	2%
Maize	7%	2%	3%	2%	13%	0%	4%	0%
Peanut	30%	42%	37%	48%	36%	45%	53%	53%
Niebe	1%	1%	1%	0%	1%	1%	0%	0%
Cotton	29%	9%	5%	3%	0%	2%	0%	0%
Cropping income	.	.	.	.	.	.	.	.
Average across years								
Millet	32%	41%	44%	30%	30%	39%	29%	29%
Sorghum	5%	3%	2%	5%	1%	7%	5%	4%
Maize	5%	3%	2%	2%	15%	5%	4%	2%
Peanut	41%	45%	46%	59%	49%	43%	59%	65%
Niebe	4%	3%	3%	1%	1%	2%	4%	1%
Cotton	13%	6%	3%	3%	3%	3%	0%	0%
Cropping income	78%	77%	79%	72%	79%	57%	45%	53%

Source: ISRA/IFPRI survey.
Notes: Each cell contains the weighted mean of the annual shares across all households (share of cropping income is gross cropping income divided by gross total income).

Table A6.11 Cereals and Pulses transactions per adult equivalent: Variation across seasons and years

	Rural zones								Urban zones			
	88-89				89-90				90-91			
	Harvest season	Marketing season	Dry season	Rainy season	Harvest season	Marketing season	Dry season	Rainy season	Harvest season	Marketing season	Dry season	Rainy season
Quantity sold												
Millet	3	3	4	1	17	5	6	3	0	1	0	0
Sorghum	0	0	0	0	2	1	1	0	0	0	0	0
Maize	0	0	0	0	0	0	0	0	0	0	0	0
Rice	0	0	0	0	0	0	0	0	0	0	0	0
Peanut	56	17	0	8	152	54	0	0	0	0	0	0
Niebe	0	0	0	0	0	0	0	0	0	0	0	0
Quantity bought												
Millet	4	9	21	19	9	6	21	6	12	14	6	1
Sorghum	0	0	0	0	1	0	1	0	1	2	0	1
Maize	0	1	2	12	0	0	2	0	2	2	1	2
Rice	6	15	23	21	16	15	22	16	54	45	29	25
Peanut	1	1	1	1	3	1	1	4	6	4	3	1
Niebe	0	0	0	0	0	0	0	0	1	0	1	0
Net quantity Sold												
Millet	-2	-6	-18	-18	9	-1	-15	-3	-11	-13	-6	-1
Sorghum	0	0	0	0	1	0	1	0	-1	-1	0	-1
Maize	0	-1	-2	-12	0	0	-2	0	-2	-2	-1	-2
Rice	-6	-15	-23	-21	-16	-15	-22	-16	-54	-45	-29	-25
Peanut	55	17	0	6	149	53	-1	-4	-6	-4	-3	-1
Niebe	0	0	0	0	0	0	0	0	-1	0	-1	0

Source: ISRA/IFPRI survey.
Notes: Each cell contains the weighted mean across all households of the season averages of the respective quantity transacted.

Table A6.12 Livestock transactions and holdings: Variation across zones

	Sagatta	Niakhar	Colobane	Passy	Dioly	Missirah	Average
End of period holdings							
Cattle...........	3	8	1	3	2	8	5
Horse & Donkey....	1	1	2	1	2	2	2
Sheep & Goat......	9	6	6	6	8	7	6
Poultry...........	8	3	9	6	11	6	6
Other animals.....	0	0	0	1	0	0	0
Outflow							
Cattle...........	0	0	0	0	0	0	0
Horse & Donkey....	0	0	0	0	0	0	0
Sheep & Goat......	1	0	1	1	2	0	1
Poultry...........	0	0	3	0	0	0	1
Other animals.....	0	0	0	0	0	0	0
Inflow							
Cattle...........	0	0	0	0	0	0	0
Horse & Donkey....	0	0	0	0	0	0	0
Sheep & Goat......	1	0	2	1	2	1	1
Poultry...........	0	0	1	1	3	0	1
Other animals.....	0	0	0	0	0	0	0
Net Change in holdings							
Cattle...........	0	0	0	0	0	0	0
Horse & Donkey....	0	0	0	0	0	0	0
Sheep & Goat......	0	0	0	0	0	0	0
Poultry...........	0	0	-2	1	3	0	0
Other animals.....	0	0	0	0	0	0	0
Maintenance costs.....	1.051	1.590	1.306	883	958	2.060	1.359

Source: ISRA/IFPRI survey.
Notes: Each cell contains the weighted mean across all households of the season averages of the respective number of animals. Outflow include sale, consumption, death, and gift; while inflow include acquisition, gift, and birth. The maintenance costs include feed, water, medical expenses, housing, etc...

Table A6.13 Livestock transactions and holdings: Variation across years by income group in nonmarket villages

Income groups	Nonmarket villages							
	low 25%		25 - 50%		50 - 75%		top 25%	
	88-89	89-90	88-89	89-90	88-89	89-90	88-89	89-90
End of period holdings								
Cattle..........	4	5	3	4	7	8	4	4
Horse & Donkey..	1	2	1	2	1	2	2	2
Sheep & Goat....	7	7	5	7	6	8	9	13
Poultry.........	5	6	6	10	9	6	7	15
Other animals...	0	0	0	1	0	0	0	0
Outflow								
Cattle..........	0	0	0	0	0	0	0	0
Horse & Donkey..	0	0	0	0	0	0	0	0
Sheep & Goat....	1	1	1	1	1	1	1	2
Poultry.........	0	0	1	1	1	2	1	1
Other animals...	0	0	0	0	0	0	0	0
Inflow								
Cattle..........	0	0	0	0	0	0	0	0
Horse & Donkey..	0	0	0	0	0	0	0	0
Sheep & Goat....	1	1	1	1	1	1	1	3
Poultry.........	0	1	0	1	0	1	1	3
Other animals...	0	0	0	0	0	0	0	0
Net Change in holdings								
Cattle..........	0	0	0	0	0	0	0	0
Horse & Donkey..	0	0	0	0	0	0	0	0
Sheep & Goat....	0	0	0	0	0	0	0	1
Poultry.........	0	0	0	0	-1	-1	0	3
Other animals...	0	0	0	0	0	0	0	0
Maintenance costs...	1.132	2.122	645	931	817	800	1.496	1.697

Source: ISRA/IFPRI survey.
Notes: Each cell contains the mean across all households of the season averages of the respective number of animals. Outflow include sale, consumption, death, and gift; while inflow include acquisition, gift, and birth. The maintenance costs include feed, water, medical expenses, housing, etc...

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