This is to certify that the

dissertation entitled

CREATING AND STUDYING A PRACTICE
OF TEACHING
ELEMENTARY MATHEMATICS FOR UNDERSTANDING

presented by

Ruth M. Heaton

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Teacher Education

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Date March 23, 1994

CREATING AND STUDYING A PRACTICE OF TEACHING ELEMENTARY MATHEMATICS FOR UNDERSTANDING

Volume I

By

Ruth M. Heaton

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Teacher Education

1994

ABSTRACT

CREATING AND STUDYING A PRACTICE OF TEACHING ELEMENTARY MATHEMATICS FOR UNDERSTANDING

By

Ruth M. Heaton

What would it take to teach elementary mathematics in ways envisioned by the current reforms in mathematics education? What struggles would be experienced by teachers as they transform their teaching? Wanting to understand these questions from both the inside and the outside, from the perspective of a researcher and a teacher, the author embarked on a year of teaching fourth-grade mathematics. In her efforts to enact a vision of the reforms, she used an innovative textbook, taught new topics, experimented with new methods, encountered new mathematics. During the year, she documented her teaching. Audio tapes and videotapes of her math lessons were made, observations by two colleagues who visited her class regularly were recorded, interviews with her were audio taped, and a daily teaching journal was kept by her. Three years later she revisited these data in an effort to understand the learning she had to do to make changes in her teaching practice.

The results of this study shed light on several important questions. First, what does it take--intellectually and emotionally--for teachers to embrace new visions of mathematics teaching? What might they need to learn about mathematics? What changes in relationships with students and curriculum do they need to undergo in order to teach in these ways? The author found *learning*

to teach mathematics for understanding entails, simultaneously, learning to *continuously invent a practice* of teaching mathematics for understanding while doing it. Her expectations for what was entailed in making changes were revised throughout her experiences. She learned that she needed to develop a repertoire of pedagogical options, representing a mix of old and new teaching practices, to use as a resource when making decisions in practice about what to do next. She also learned that pedagogy is deeply connected to understanding mathematics, both what it is and how one knows it. Her study implies the need to make curriculum materials educative for teachers and to focus teacher education on learning a balance of particulars about mathematics, students, and pedagogy as well as what it means to know in any one of those domains.

To Steve and my parents

ACKNOWLEDGEMENTS

This dissertation has taken a strength of mind, courage of heart, and trust of self which I have been unable to generate or sustain on my own. I have relied heavily on a variety of people from the moment I decided to teach fourth grade mathematics through the final printing of this document. It is to these people, with my deepest gratitude, I now turn.

I begin with the people who made the opportunity to create a practice possible: the classroom teacher in whose classroom I worked and the fourth grade students with whom I worked. I am thankful to the other classroom teacher for graciously sharing her classroom, students, time, and insights with me. I am deeply appreciative to my students, who became instrumental in helping me understand what it means to teach mathematics for understanding.

I am thankful to David Cohen for his vision of what it would mean to take my experiences trying to teach mathematics differently and turn them into a study of learning to teach. Not only did David envision what this dissertation could be, he has worked closely with me on developing the central ideas of each of the parts. I am deeply appreciative to him for asking the hardest questions of me and my work and teaching me the benefits of writing to figure out what I mean. David and the other members of my committee, Bill McDiarmid, Jay Featherstone, Deborah Ball and Magdalene Lampert, are an extraordinary collection of individuals who have encouraged and helped me to take intellectual risks as a teacher and scholar that I would not have ventured to do on my own.

As my advisor and the chair of my committee, Bill McDiarmid has helped me think through many of the decisions surrounding my work over the past five years. I am thankful for Bill's patience and insightfulness. His unending support has given me the courage to act when what lay before me seemed just too hard. I am thankful to Jay Featherstone for helping me to see how I could delve deeply into mathematics and still hold fast to my belief that elementary teachers ought to be generalists. He is also responsible for reintroducing me to *Life on the Mississippi* and helping me understand the relevance of Twain's work to mine. Deborah Ball has been invaluable to me as a teacher and friend in every stage of this work. She has invested countless hours in every aspect of my work, from data analysis through drafts of every chapter, too numerous to count, to a close edit of the final text. For her insight and support through the highs and lows of doing this work, when I know she had a million other things she could have been doing, I am deeply thankful. Maggie Lampert, the director of my dissertation, my mentor and friend, knows many of the voices in this dissertation better than I do. Over the past five years, she's spent many hours talking to each of them, taken the time to learn of their strengths and weaknesses, and found ways to nurture and develop each of them in me as well as this text. For the trust and respect that has grown between us and for the time and energy Maggie has committed to my learning as a teacher and a scholar, I am deeply and everlastingly grateful.

There are several other faculty members, not on my committee, who have contributed in important ways to this work as well. Suzanne Wilson did not have much of a choice when she lived with me in California during the formative stages of this work. I am very grateful for her willingness to listen, ask questions, offer insight, and give general moral support at a moment's notice or with no notice at all. I spent a year in a study group with Susan Florio-Ruane. I am thankful for what she and the graduate students in that group taught me about critically watching video tape and analyzing classroom interactions. I am thankful to Helen Featherstone, who took a number theory course in the math department with me, for helping me to think about what it means to be a learner of mathematics. I also appreciate her general knack at reading me and saying just what I needed to hear. I am also thankful to Chris Clark, who walked slowly enough through the halls of the College of Education so that I could talk to him about the experience of doing this work. For his understanding of the risks and support of me in the process, I am thankful. I am also grateful to Glenda Lappan and Dan Chazan, faculty members who understand more mathematics than I do. They helped me understand some of the complex mathematical ideas my students and I confronted without making me feel like these were simple ideas I should have understood. Claire Heidema, while not a faculty member, was also helpful in explaining the meaning of the mathematics as it was represented in the textbook I used.

I am also thankful to the members of the research projects on which I have worked while doing this study. The people on these projects formed intellectual communities from which many of the ideas in this thesis have grown. My work on the Mathematics and Teaching through Hypermedia Project and Communication in and about School Mathematics, funded by the National Science Foundation, co-directed by Deborah Ball and Magdalene Lampert, particularly the Lampert Team (Maggie Lampert, Nan Jackson, Ginny Keen, and Thom Dye), has contributed in significant ways to my understanding of what it means to teach mathematics for understanding. My understandings of the relationship between policy and practice in the context of the reforms in mathematics education has stemmed from my involvement with faculty and graduate students on the Educational Policy and Practice Study, co-directed by

Deborah Ball, David Cohen, Penelope Peterson, and Suzanne Wilson, funded by the National Science Foundation, PEW Charitable Trusts, and the Consortium for Policy Research in Education. The Center for the Learning and Teaching of Elementary Subjects, funded primarily by the Office of Educational Research and Improvement, U. S. Department of Education, co-directed by Penelope Peterson and Jere Brophy, the site of my first assistantship, is where I met Jim Reineke, the researcher who videotaped, audio taped and interviewed me during part of the year of teaching under study in this project. For Jim's involvement in this project and the many conversations we had during my first few months of teaching, I am deeply grateful.

My friends have offered me invaluable support. I am thankful to SG Grant who has read countless drafts and with whom I have shared bottomless cups of coffee over the ideas of his dissertation and mine. I am also deeply grateful and appreciative of Kara Suzuka who started out doing the graphics for my work and became interested in the ideas. For the intellectual support and laughter she's brought into the office we've shared in the last six months of working on this, I am thankful. I consider myself most fortunate that Kara has treated the format and printing of my dissertation with the care and attention to detail as if it were her own. I am also thankful to Dirck Roosevelt who has pushed my thinking on many of the ideas presented here and has never not asked me how my work was going and listened to the answer. Nancy Jennings and Jimmy Spillane entered this doctoral program with me and have seen me through to the finish. For their friendships, I am thankful. I am also appreciative of Lauren Pfeiffer, who has graciously listened to me talk about the struggles of doing this work. I am grateful for Sam Larson's friendship which began the day I moved to East Lansing. She has helped me to maintain some perspective on my life by continually reminding me that there is more to life than graduate school and dissertations. I am also thankful for Marilyn May, a teaching colleague and friend in Vermont, who has managed to support me, long distance, every step of the way.

I am deeply thankful for my family's love and support. I am grateful to my parents, who have always wanted and made it possible for me to have opportunities they never had. I am also appreciative of my siblings, Michael, Jane, and Julie, who were my first teachers.

Steve Swidler, who married me in the course of writing this thesis, has provided the day to day balance in my life, which has so often felt out of balance, while working on this dissertation. He has spent long hours listening to and helping me work through the intellectual and emotional challenges presented by this thesis. I am thankful for his keen and deep insight into me and this work. He has also given me all the time and space I ever wanted. For his endurance and endless patience, support, love, humor, and good cooking, I am also deeply grateful. While it is a joy to be at the end of my dissertation, it will be an even greater joy when Steve is finished with his, too. In the meantime, I have mighty large shoes to fill.

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CHAPTER 1

REFORMING MATHEMATICS EDUCATION: A NATIONAL MOVEMENT AND A PERSONAL DECISION

This is both a story and a study of the process of learning to teach from a learning teacher's perspective, a perspective on experience rarely found in educational scholarship. This is a story about a good teacher who took on the task of trying to teach mathematics in a way that respects students' thinking and mathematical ideas. What you are about to read comes from my own experiences. I am the teacher in this story. This is, however, more than just a telling of my teaching experiences. It is, in addition, a study of those experiences for I am also a researcher interested in teaching, learning, and learning to teach. When I lived the story you are about to read, I did so as a teacher trying to make changes in the way I taught mathematics. It was only in retrospect that I decided, as a researcher, that a study of my own experiences as I tried to make changes in my mathematics teaching could be an interesting site for inquiry into what is entailed in learning to teach mathematics differently. This study of my experiences learning to teach has the potential to inform broader questions of teacher knowledge and teacher learning as well as adding to understanding of the relationship between the reform ideas associated with mathematics education and the implementation of those ideas in practice.

No Longer Feeling Like a Good Teacher

I did not dream up a new way of teaching elementary mathematics. My efforts to change my mathematics teaching connect to a whole host of new ideas about mathematics teaching sweeping the country at local, state, and national levels. For example, the California State Department of Education (1985, 1992) developed the *Mathematics Framework* while the national organization of mathematics educators compiled curriculum and teaching standards (National Council of Teachers of Mathematics, 1989, 1991). These new ideas about teaching mathematics have fueled revisions of old texts and tests and encouraged the development of new curriculum materials and alternative forms of assessment. People interested in bettering schools for children have tinkered with these traditional tools of educational change for many years. The current reforms in mathematics education have provided yet another opportunity.

I first learned of the widespread reform movement in mathematics education when I entered my doctoral program in teacher education at Michigan State University five years ago. I worked as a research assistant with a group of other graduate students and faculty on a project designed to study the relationship between instructional policy and teaching practice. In the early years of our study, we concentrated our efforts on the reform of elementary mathematics education in the state of California (California State Department of Education, 1985).[1] I began my work on this project, new to graduate school, with very little knowledge about the making of instructional policy and no real interest in mathematics. I did, however, have nine years of elementary teaching experience and a deep appreciation for the realities of teaching.

The purpose of that research project was to understand the relationship between instructional policy and mathematics instruction. What does teaching

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[1] In the last five years, this research has expanded from looking at policy and practice in the context of elementary mathematics in California to include reading as well. Michigan and South Carolina have been added as research sites. For additional studies related to this project, see Jennings (1992) for a study of teacher learning in the context of reading policy and practice, Spillane (1993) for a study of reading policy across school districts, Grant (in preparation) for a study of the ways in which teachers manage multiple reforms, Remillard (in preparation) for a study of teachers' use of innovative mathematics textbooks, and Geist (in preparation) for a study of how the reform ideas "travel" through one teacher's practice.

practice look like as teachers try to implement the current reform ideas in mathematics education? How are the ideas about mathematics teaching interpreted and implemented in practice? We observed and interviewed elementary teachers and talked with various state, district, and school personnel. We found wide variation in people's interpretations of the policy and witnessed a range of teaching practice. We found many of our teachers trying to teach mathematics in meaningful ways and struggling--sometimes from their perspectives, other times from ours (Ball, 1990; Cohen, 1990; Cohen and Ball, 1990; Peterson, 1990; Wiemers, 1990; Wilson, 1990; Prawat, Remillard, Putnam, & Heaton, 1992; Putnam, Heaton, Prawat, & Remillard, 1992; Heaton, 1992; Prawat, 1992; Putnam, 1992; Remillard, 1992).

As a backdrop to our efforts as researchers to understand mathematics teaching practice, we tried to understand what it meant to teach mathematics for understanding. What was teaching mathematics for understanding? What was the underlying conception of mathematical knowledge? What were students supposed to learn? What was meant by understanding? What did this imply for teaching? Like most of the teachers with whom we spoke, we were confronting as researchers a way of teaching and learning mathematics that was foreign to our past experiences as learners. To help ourselves, we drew on the work of two mathematics educators, colleagues within our own college: Magdalene Lampert and Deborah Ball. We read articles they had written about the teaching and learning of mathematics (Lampert, 1987, 1989; Ball, 1988a, 1988b) and analyzed and discussed video tapes of their elementary mathematics teaching. While we never reached consensus on what it might mean to teach mathematics for understanding, we did gain an appreciation for the complexity of interpreting these new ideas about mathematics teaching and learning.

The more involved I became as a researcher on that study of instructional policy and others' teaching practice, the less content I felt with myself as an elementary teacher. This worried me. I entered graduate school thinking I was prepared to learn to be a teacher educator because I was a good teacher. I no longer felt that way. I had taught mathematics in traditional ways for nine years unaware of an alternative. As I studied the practice of other teachers for this project, a jumble of questions raced through my mind. What was so difficult about trying to put the ideas about teaching and learning mathematics into practice that seemed so appealing in theory within the reform documents? Why were the teachers we were studying struggling? Why was teaching mathematics in a meaningful way so difficult to do? Was I any different than any of these teachers we were studying? It was true that I had taken mathematics classes through calculus, but what mathematics did I really understand? Would I have the same troubles I saw other teachers having if I tried to enact these reforms? What would it take for me to try to teach mathematics in the ways these teachers were aiming to do? What challenges would I face? How would my own mathematical understanding or lack thereof influence what I could do with students? Could I proceed with becoming a teacher educator and ignore these uneasy feelings about myself as a teacher? For a short while, I tried.

Questioning My Future as a Teacher of Teachers

Questions about my past experiences as a teacher of elementary mathematics nagged at me throughout the first year of my doctoral program in teacher education. Issues surrounding my own credibility concerned me. How could I be studying to be a teacher educator and feel so dissatisfied with an aspect of my own teaching practice? Would I become a teacher educator who could not and did not practice what she preached? Would I soon find myself, a teacher educator, pushing new ideas about mathematics teaching and learning without a sense of what these ideas meant in practice? Wasn't this typical of the practices of many of the teacher educators I had encountered in my past experiences learning to teach that I had grown to despise? If I did not have some sense of what it meant to implement these ideas about teaching in practice--my own practice--could I propose them as ideas for others to implement? This was certainly an alternative and one pursued by many people learning to be teacher educators--first, through graduate course work, to read research and learn of ideas related to teaching, and later to teach these same research findings and ideas in teacher education courses. This route worked for some but would not for me. The strong tug of my identity as a teacher pulled me away from following that path to reach my goal of becoming a teacher educator. I needed to find a way to build consistency between my own teaching practice and what I would be teaching others about teaching.

Voicing this concern about my credibility as a teacher educator feels risky among friends and colleagues who are also teacher educators. Hearing my personal concerns about credibility could raise questions about the practices of other teacher educators, many of whom do not have relevant or recent classroom teaching experience. The alternative would have been for me to remain silent and try to ignore the problem I saw emerging in my evolving role as a teacher educator with an increasing responsibility to teach others about teaching. I chose, however, to speak up and, in doing so, broke a conspiracy of silence that has tended to loom around the gap between theory and practice in teacher education. Few university academics confront, head on, the fact that they are teaching others to teach and doing research and writing about teaching and learning, while they remain distanced from experiences with teachers, students, and classrooms. In breaking this silence and raising these issues, I do not mean to suggest that all teacher educators should become classroom teachers. Rather, I wish to insert a bit of humility into the role of a university professor. In addition to our responsibilities to teach students theory about teaching, we have a responsibility to learn what we can about teaching from practice, whether it be through our own work in schools or through the written accounts or close observations of other teachers' practice.

Cuban (1990) and Eisner (1992) are examples of two university professors who have attempted to bring their university work closer to the realities of teaching after long absences from the classroom. They illustrate what I am suggesting. Both men have written about the humbling nature of their experiences, the reasons they had for entering a classroom, and the influence of what they learned from their encounter with teachers and students on their work at the university. Cuban was drawn back into the classroom by issues of credibility, similar to ones that concerned me. He wrote,

> First, I wanted to maintain my credibility both as a teacher and as an academic who writes about teaching and public schools. I believe deeply in the idea of a scholar-practitioner--that is, someone who can bridge the two very different worlds of the university and the public school. Such switch hitters are uncommon, and I wanted to be one of that breed. (p. 480)

I share Cuban's desire to be someone who can move between the worlds of practice and scholarship. Eisner (1992) notes the nature of the gap between theory and practice, "there is a profound difference between knowing something in the abstract and knowing it through direct experience" (p. 263). He uses his experiences in classrooms with teachers and students to describe his new appreciation for the complexity of teaching and learning and to argue against the usefulness of prescriptive products from educational research. His work is an example of using what can be learned in one world to inform work in another.

My decision to go back into the classroom in order to make changes in my own mathematics teaching was not made in haste. During my nine years of teaching, I did not readily jump on bandwagons of new teaching ideas. In this case, my feelings about the need to change my practice were being stirred not only by reformers who thought math teaching should change and by my conscience as a teacher educator. When I considered my past practice as a teacher, I could see that while my view and teaching of other subjects had changed in nine years, nothing had really changed in the way I taught mathematics. I had never perceived a need for change or run across any real options. I had taught mathematics for nine years basically the way I had learned it. My encounter with the math reforms was my first hint that mathematics could be taught differently.

By the end of the first year in my doctoral program, I found myself making plans to go back into the classroom as a teacher. I arranged to spend my second year of doctoral study in teacher education, my tenth year as a teacher, in a fourth grade math class. The teacher in me wanted to try to enact the vision of mathematics teaching and learning I had constructed through my work as a researcher. I hoped my move back into the classroom would help to resolve the gap I felt growing between theory and my practice.

The Math Reforms

What were these new ideas about mathematics teaching and learning that I was going to try to enact in the classroom? I constructed a vision of what I wanted my practice to be like from reading reform documents and observing teachers trying to implement the ideas. These experiences offered me new perspectives on what it means to know mathematics, what is important to know, and how students learn mathematics. I wanted to learn what these new ideas about knowledge, content, and learning implied for my teaching practice. I wanted to experience for myself what it was like to try to teach mathematics for understanding. To understand the story and study of my efforts, it is important to provide some explanation of what the reform documents looked like to me as a scholar and an experienced teacher before I taught.

New Ideas about Mathematical Knowledge

Before starting my doctoral program, I never considered that there were multiple ways to think about the answers to the following questions. What is mathematics? What does it mean to know mathematics? In fact, I did not even recognize these as questions. In my past experiences as a student and a teacher, mathematics was found in the textbook. Knowing mathematics was memorizing a set of rules and procedures and applying them to computation to come up with right answers. The view of mathematical knowledge I have had most of my life is one where knowledge is fixed and hierarchical. For years, I thought mathematics resided in the textbook. The image I had of doing mathematics, as a teacher or student, was one of following the textbook, page by page, year after year. I thought knowing mathematics was being able to recall and apply rules and procedures to get right answers.

Mathematical knowledge as conceptualized by the current reforms in mathematics education is of a much different nature. In the context of these reforms, mathematical knowledge is considered dynamic. It is something which is constructed and re-constructed through an on going process of sense-making by the learner and emphasizes both content and process. Knowing mathematics is understanding particular domains of knowledge (Hiebert & Behr, 1988), topics and concepts, and the connections among them. Knowing mathematics is also having an understanding of mathematical processes or how it is that knowledge is acquired and used. This means having an understanding of mathematical argument or what counts as evidence and justification for a particular

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mathematical point of view. Knowing mathematics is using and creating mathematical tools, language, and other representations to construct and communicate understandings of particular domains of mathematical knowledge.

This view of mathematical knowledge does not entirely dismiss mathematical rules and procedures but treats them as dynamic tools rather than static bits of knowledge. It places their use in the context of some purposeful activity (National Council of Teachers of Mathematics, 1989) where the short term goal is for students to solve problems in ways that make sense to them and the long term goal is for students to become mathematically literate (National Research Council, 1989). "Mathematically powerful students think and communicate, drawing on mathematical ideas and using mathematical tools and techniques" (California State Department of Education, 1992, p. 3). This means that at times students draw on their understandings of formal mathematical rules and procedures. At other times, it means they make use of their intuitive understandings of mathematics. However a mathematical problem is solved, the aim is for students to construct their own understandings of mathematics--to know why a particular solution and method for solving a problem makes sense in a particular context. "As an ongoing product of human activity, mathematics is a dynamic and expanding system of connected principles and ideas constructed through exploration and investigation" (National Council of Teachers of Mathematics, 1991, p. 133).

New Ideas about Mathematical Content

Changes in what is important for students to know would accompany changes in what it means to know mathematics. The mathematics curriculum commonly taught in schools now includes "more and different content" (California State Department of Education, 1992, p. 78). Consistent with views of knowledge, the reform documents give curricular attention to particular content as well as mathematical processes. The California State Department of Education (1992) and the National Council of Teachers of Mathematics (1989, 1991) provide lists of mathematical topics which include: estimation, number sense and numeration, number and number relationships, whole numbers, geometry and spatial sense, measurement, statistics and probability, fractions and decimals, patterns and relationships, and functions. Statistics and patterns represent topics new to elementary curricula. The traditional topic of long division might be subsumed under whole numbers, decimals, fractions, or number and number relationships. The push of the new conception of content is to consider mathematics more broadly and move beyond a conception of mathematics defined entirely by the rules and procedures necessary to perform algorithms. In traditional math classes, doing mathematics meant memorizing, recalling, and applying rules and procedures to endless algorithms. In the context of the current reforms, memorizing is not doing mathematics. Rather, mathematical processes that more closely resemble the work of mathematicians within a mathematical community define the doing of mathematics. The National Council of Teachers of Mathematics (1991) outlines the mathematical curriculum to include,

> Examining patterns, abstracting, generalizing, and making convincing mathematical arguments. . . definitions, examples, and counterexamples and the use of assumptions, evidence, and proof. Framing mathematical questions and conjectures, constructing and evaluating arguments, making connections, and communicating mathematical ideas are all important aspects of mathematical discourse. (p. 133)

Mathematical processes and content are intended to be taught in tandem, one providing the context for the other. "Mathematics must be approached as a whole. Concepts, procedures, and intellectual processes are interrelated" (National Council of Teachers of Mathematics, 1989, p. 11) and are to be taught with the expectation that part of what is to be learned are the ways in which content and process are related. "The curriculum should include deliberate attempts, through specific instructional activities, to connect ideas and procedures both among different mathematical topics and with other content areas" (National Council of Teachers of Mathematics, 1989, p. 11).

Apart from the newness of what was considered content and process in the context of these reforms, I anticipated that this new way of conceiving of the integration of content and process was going to push on the limits of my own mathematical experiences. I had no experiences as a teacher or learner of mathematics on which to draw that could give me a sense of what it would mean to weave together content and process in practice. I had no experiences with knowing mathematics in these ways.

New Views about Learning Mathematics

In the past, learning mathematics meant passively memorizing the rules and procedures that were necessary to get right answers. Now that what is to be learned is changing, what does it mean to be actively engaged in learning mathematics in some meaningful way that pushes beyond or beneath rules and procedures? The view of learning mathematics implied by the reform documents represents a "shift from learning mathematics as accumulating facts and procedures to learning mathematics as an integrated set of intellectual tools for making sense of mathematical situations" (National Council of Teachers of Mathematics, 1991, p. 2). The work of Putnam, Lampert, and Peterson (1989) offers me a way of sorting out these various interpretations of what it could mean to understand mathematics in fundamental ways.

One interpretation of learning mathematics for understanding is learning how to represent and communicate a mathematical idea or interpret the representations of others, through the use of language, diagrams, pictures,

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manipulatives, and other tools to aid communication (Kaput, 1987). A second interpretation of what it means to understand mathematics is learning particular models or structures of knowledge necessary to perform mathematical tasks. The analysis of specific tasks and students' performance on tasks around addition and subtraction, the foundation of Cognitively Guided Instruction (Carpenter and Fennema, 1988) is an example of this way of thinking about the learning of mathematics. A third way of considering what it means to understand mathematics centers on building connections among types of knowledge or attending to the links between procedural and conceptual knowledge. For many years, educators believed that to learn school mathematics, students had to learn basic skills and procedures before progressing to activities that required higher level thinking or some conceptual understanding (Resnick, 1987). Learning basic skills and how to reason through activities and problems with concrete materials and language as tools are integrated goals for all students in the current reforms (National Council of Teachers of Mathematics, 1989). Learning mathematics is learning how to link "formal symbol systems with representations of quantities and actions" (Hiebert & Behr, 1988). A fourth interpretation of learning is to consider mathematical understanding as the active construction of knowledge (Cobb, 1988). In mathematics, "successful learners understand the task to be one of constructing meaning, of doing interpretive work rather than routine manipulations" (Resnick, 1987, p. 12). The sense learners make of ideas plays a primary role. The math reforms call for teachers to draw on children's intuitions or informal mathematical sense-making and to have these ways of knowing be part of what is valued in school mathematics. This is based on a belief about learning that the place to begin constructing meaning is with what the child knows, whatever form that knowledge takes. Learning mathematics is done as students actively construct meaning for themselves (National Research Council, 1989) by connecting new ideas with previous understandings. The construction of knowledge, as advocated by these reforms, is not done in isolation. Making meaning out of mathematics is a social activity (Greeno, 1989) and the reforms are based on the belief that "students' learning of mathematics is enhanced in a learning environment that is built as a community of people collaborating to make sense of mathematical ideas" (National Council of Teachers of Mathematics, 1989, p. 58).

In theory, I could see how a shift in an emphasis on right and wrong answers or what counts as mathematical knowledge was necessary, given perspectives on learning mathematics which emphasized interpretation, construction of meaning, and representation and communication of ideas within a community of learners. I had begun to value the sense students made of ideas in social studies and language arts but was that possible in mathematics--a subject that had always been a straightforward and clearly-defined subject to teach and learn? Was there room for students' interpretation of mathematical ideas? I had to stretch my mind to consider the possibility. I tried to envision what implementing such ideas about knowledge in a math class would actually entail.

Implications for Teaching

As I read through these reform documents in the university setting, I could grasp some meaning of these new ideas about mathematics teaching and learning. Reading through the literature in the fields of cognitive psychology, policy studies, curriculum, theory, research on teaching, and the like, I was able to understand the reforms from several scholarly perspectives. But, as an experienced teacher who contemplated doing them, I was much less certain what all of this rhetoric implied for teaching. The reform documents offered some theoretical ideas about what teaching aligned with these reforms would entail. "Teaching for understanding emphasizes the relationships among mathematical skills and concepts and leads students to approach mathematics with a commonsense attitude, understanding not only how but also why skills are applied" (California State Department of Education, 1985, p. 12). Reasoning, communication, problem solving, and connections (National Council of Teachers of Mathematics, 1989) play a primary role.

To my teacher self, this language seemed abstract. How do these new views of knowledge, learning, and content translate into the actual work of teaching? I looked for further information on the implications of these ideas in practice. "Teaching is a complex practice and hence not reducible to recipes or prescriptions. . . teaching mathematics draws on knowledge from several domains: knowledge of mathematics, of diverse learners, of how students learn mathematics, of the context of classroom, school and society" (National Council of Teachers of Mathematics, 1991, p. 22). The more I read, the more I felt a conflict with what I had understood to be good practice. Mathematics teaching had not seemed too complex in the past for it was based on straightforward rules, procedures, and algorithms that my students either knew or had to learn. I was unsure what was meant by these various domains of knowledge. How were these ideas about the knowledge needed to teach different from what had guided my teaching in the past?

A change in practice. The *Professional Standards for Mathematics Teaching* (National Council of Teachers of Mathematics, 1991) offers a comparison of traditional practice and teaching mathematics for understanding. To move in the direction of the kind of math teaching envisioned by this reform, they state the following as the shifts in teaching and learning mathematics that would need to occur:

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> -toward classrooms as mathematical communities--away from classrooms as simply a collection of individuals;
> -toward logic and mathematical evidence as verification--away from the teacher as the sole authority for the right answers;
> -toward mathematical reasoning--away from merely memorizing procedures;
> -toward conjecturing, inventing, and problem solving--away from an emphasis on mechanistic answer-finding;
> -toward connecting mathematics, its ideas, and its applications--away from treating mathematics as a body of isolated concepts and procedures. (p. 3)

I knew all too well from my own teaching experiences what this document was advocating. The difficulty was understanding in a practical way what teachers were supposed to move toward and how the domains of knowledge they were supposed to draw on would help them do the work.

The *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989) offers another cut on expectations of the sorts of changes that are going to need occur to implement these reforms. The following are lists of instructional practices that need increased or decreased attention:

Grade level	Increased Attention	Decreased Attention
K-4	use of manipulative materials cooperative work discussion of mathematics questioning justification of thinking writing about mathematics problem-solving approach to instruction content integration use of calculators and computers	rote practice rote memorization of rules one answer and one method use of worksheets written practice teaching by telling
5-8	actively involving students individually and in groups in exploring, conjecturing, analyzing, and applying mathematics in both a mathematical and a real-world context using appropriate technology for computation and exploration using concrete materials being a facilitator of learning assessing learning as an integral part of instruction	teaching computations out of context drilling on paper-and-pencil algorithms teaching topics in isolation stressing memorization being the dispenser of knowledge testing for the sole purpose of assigning grades

(pp. 20, 21, 72, 73)

Figure 1.1

Teaching practices to increase and decrease in implementing the math reforms

When I looked at these columns, I understood the areas in which teachers were to decrease attention. They had been focal in *my* practice for years. What I was less certain about was what it would mean to "decrease attention" in these areas and "increase attention" in others. Without even trying this, I could see several problems from my point of view as an experienced classroom teacher. First, some of the areas where it was suggested teachers increase attention, like justification of thinking and writing about mathematics, or exploring, conjecturing, or analyzing, were aspects of practice to which I had never attended in math class. How could I increase attention on these particular aspects of practice when I was uncertain what it would mean to attend to them in the first place? And if I could not imagine what this would look like in the classrooms where I had taught, how could I help others learn to do it?

Another thing that was a bit troublesome as I looked at what was to be decreased and what was to be increased was that there was not a one-to-one correspondence between them. In many instances, to teach in these new ways might be a matter of decreasing my attention on some aspect of my old practice while increasing my attention in the direction of something I had never done before. My role seemed full as a teacher in my old practice, plenty to do and to manage. It seemed that not only were my responsibilities as a teacher changing, but they were expanding, and expanding into quite unfamiliar territory.

A change in role. As I tried to get a conceptual and practical understanding of the reforms, I made use of the idea of "teachers' role" to try to get a more vivid image of what was expected. It was difficult for me to imagine just what sort of role I would construct for myself, given all of the changes in practice that were being suggested. Scattered throughout various reform documents are brief sketches or images of the role of the teacher. For example, *Reshaping School Mathematics* (National Research Council, 1990) describes the teacher as an "intellectual coach" (p. 40) and outlines the various dimensions of the envisioned role. The National Research Council (1990) proposes that at various times the teacher will be required to be a role model, a consultant, a moderator, an interlocutor, and a questioner. Teachers are to be "facilitators of learning rather than imparters of information" (California State Department of Education, 1992, p. 41). The National Council of Teachers of Mathematics (1991) offers the image of the teacher as musical conductor, someone who plays a central role in "orchestrating the oral and written discourse in ways that contribute to students' understanding of mathematics" (p. 35). In the context of these various images, teachers are responsible for leading discussions of mathematical ideas and constructing learning environments where students can be engaged in meaningful tasks alone or in collaboration with others. These metaphors were useful in imaging my role but I was still uncertain what it would take or actually be like to learn to be in it.

It felt to me that implementing these new ideas about mathematics teaching and learning placed high demands on teachers. My own sense was that trying to teach in these new ways was going to be hard. I saw hints throughout the documents that others thought so, too. "The type of mathematics instruction that involves students actively and intellectually requires much from the teacher" (California State Department of Education, 1985, p. 13). Teachers who helped to develop the *Mathematics Framework* (California State Department of Education, 1992) state what they think are the most difficult tasks confronting teachers in these reforms. The following list is extracted from their statement:

> --change in how we practice our profession
> --abandon traditional roles, become chief questioners and facilitators for students
> --reflect, experiment, and accept uncertainty
> --share doubts and confusions with colleagues
> --through professional explorations and examination gain the experience and confidence necessary to implement the new programs and take the curriculum into our own hands (p.12)

What these teachers did not mention was what would it mean for any teacher to try to do these things in everyday practice. What would it mean to "accept uncertainty" in a class of fourth graders, for example? Or, what would it mean to "take the curriculum into your own hands"? The National Council of Teachers of Mathematics (1991) state that, "The kind of instruction needed to implement the National Council of Teachers of Mathematics standards requires a high degree of individual responsibility, authority, and autonomy--in short professionalism on the part of each teacher" (p. 4). Again, practical questions surfaced. What would it be like for me to learn to shoulder the responsibilities it appears would be demanded of me? It was beginning to feel like an enormous weight and I wasn't even in the classroom yet.

It seemed to me that the successful implementation of the reforms rested with me, the teacher. In describing what is entailed in making these changes, the National Research Council (1989) notes the teacher as "the central person who can bring about meaningful and lasting change" (p. viii). The *Mathematics Framework* (California State Department of Education, 1992) states, "teachers will inevitably carry the greatest burden" (p. viii). The idea of what it will take to make these changes is pushed a bit further and some ideas are offered about what ought to be done by administrators in schools and districts to help teachers make these changes. For example, the National Research Council (1990) suggests that school districts pay more attention to inservice training while the *Mathematics Framework* (California State Department of Education, 1985) proposes that teachers learn more mathematics and provides descriptions of possible content courses for high school and elementary teachers. I was also concerned with the issue of not knowing enough mathematics but I was not entirely convinced that what I needed to know could be found in a math course, at least not in the kind of math courses I had taken earlier as a student. The only thing I felt certain of was that there was much I did not know and even more I had to learn.

Creating a Practice

Through my efforts to create a practice of teaching mathematics for understanding for myself, I hoped to better understand the difference between my scholarly understandings and my practical sense of these reforms. Without trying to actively teach mathematics for understanding, myself, the best I could do in helping others learn to teach mathematics differently was compare and

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contrast my sense of the scholarly ideas behind these reforms to my past, more traditional ways of teaching and learning mathematics. In theory, I could note the differences. Creating a practice allowed me to understand the differences and see the problems in trying to teach mathematics differently from a teacher's point of view.

As I imagined all that was going to be demanded of me, I was thankful to have arranged to teach only mathematics. Such concentration in a single subject area was something I never had the luxury to do in my years as a full time elementary teacher. The part of me that always felt rushed as a teacher was looking forward to slowing down, thinking about what I was doing, and maybe even learning something; it seemed like a rare opportunity. Another part of me was worried. Maybe I was not smart enough to teach mathematics in these new ways. I knew I needed to learn more mathematics but I was uncertain just what it was that I should learn because I was still unclear about what it was that I did not know.

After nine years of teaching, to find myself in a position of learning to teach left me feeling uneasy. As the school year approached, my feelings about teaching varied. There were some summer days when I was anxious to be back in the classroom, eager for a new challenge. There were other days when my uncertainty about the subject matter got the better of me and I wondered whatever had possessed me to make arrangements to try to teach mathematics differently. When I began my doctoral program to learn to be a teacher educator, I never expected that I would find myself, within a year, headed back into the classroom to learn to teach.

Problems of Practice

When I entered teaching, I felt like I had some understanding of what was entailed in teaching mathematics differently from the reform documents I had read and the practices I had observed. I had some sense of the major problems one might face in trying to implement these reforms. By the end of the year, the problems had not disappeared but my understanding of them had deepened.

Knowing more mathematics. As I have already mentioned, I worried a lot about not knowing enough mathematics to do this kind of teaching but I do not think I really understood the nature of the mathematical knowledge I needed to learn. I had the impression, prior to teaching, that the mathematics I needed to learn was "out there." I just had to figure out what mathematics I was lacking and learn it. I imagined that it was quantity that I needed. Anyone who could teach mathematics for understanding would have to possess more mathematical knowledge than I had. My efforts to teach mathematics differently make visible, through the absence of mathematical knowledge in my practice, the importance of it in teaching mathematics for understanding. What one needs to know, however, is not necessarily more mathematics. What one needs is to be able to understand the nature of mathematical knowledge differently.

Constructing understandings. Before I tried to make changes in my practice, I was also under the impression that a lot of the process of learning would be left up to students since they were supposed to be responsible for constructing their own understandings of the mathematics. I wondered if this was something all students could do. Were the times I observed in other classrooms and saw students making sense of mathematical ideas rare moments or were the students unusual? What I quickly learned when I tried to teach mathematics differently was that my students, when given the opportunity, could readily come up with their own understandings of mathematical ideas. That was not the challenge. The problem was making sense of their understandings and finding ways to link their sense with what I wanted them to learn and my instruction. I was surprised at the demands placed on my mathematical knowledge in the context of a kind of teaching that centered on students' construction of knowledge.

Learning to teach. I approached the year I spent teaching with the expectation that what I was going to do was learn to teach differently. I also entered teaching, as I just indicated, unaware that I did not know the subject matter I was going to be teaching well enough. What I did not appreciate, prior to teaching, was the problem of trying to learn a new pedagogy while also trying to learn the subject matter in deeper and more connected ways than I had ever experienced as a teacher or learner. As researchers and teacher educators have gotten wiser about the need for subject matter knowledge to teach, teacher education programs have tried to address the need to know prior to students efforts to learn to teach. As an experienced teacher, however, I found myself faced with the challenge of simultaneously trying to acquire subject matter knowledge while learning to teach in new ways.

Studying a Practice

These problems, which I have come to have a deep appreciation for as a practitioner, are many of the same problems I have tried to understand through a study of my own practice of learning to teach mathematics differently. As a researcher, I have tried to understand what it is that I, an experienced teacher, could have learned or might have known and how it is that could have made use of this knowledge to ease the difficulties posed to me by the problems of creating a new practice of teaching elementary mathematics.

Gaps in Research

The problems I faced in practice and the ones I address in my research are related to ones that other researchers, through their research on teaching or learning to teach, have addressed in various ways. A review of this research illustrates the ways in which my study complements and fills the gaps of existing research.

Learning to acquire and use knowledge about teaching. My work builds on the explorations of teacher knowledge done by other researchers. It furthers insight into what knowledge about mathematics teaching is and how one goes about acquiring and learning to use it from a teacher's perspective. A closer look at specific examples of research done over the last ten years illustrates the ways I see my work building on and helping to fill in the gaps in what we currently understand about teacher knowledge.

Early research on teaching worked to define the observable behaviors of teachers, the technical features of teaching (Brophy & Good, 1986). Clark and Peterson (1986) dug beneath the surface of teacher actions and talked with teachers about the thought processes that go into doing teaching, the intentions and motivation that guide and drive teachers. What Clark and Peterson (1986) revealed was an elaborate process of thinking and decision making guiding the observable actions of teachers. Their research has helped us to see teaching as a complex cognitive activity requiring constant deliberation and judgment, before, during, and after the act of teaching.

Elbaz's (1983) detailed account of an English teacher and Lampert's (1986) introduction of the idea of dilemma management in teaching are two examples of research on teaching that began to reveal the complexities of the relationship between teachers' thoughts and actions and the dynamic nature of teacher knowledge. Elbaz's study helps us to see the personal, practical, and constructive nature of teacher knowledge while Lampert's work provides insight into the nature of the moment to moment judgments demanded by teaching. Schön (1983) has looked, specifically, at the reflective nature of teachers' work and how it is that practitioners make decisions and learn from them. Shulman

(1986) introduced "content knowledge, pedagogical content knowledge, and curricular knowledge" (p. 9) as constructs for the knowledge teachers acquire and use to make these ongoing decisions in practice. Subject specific case research on teacher knowledge (i.e., Ball 1993a, 1993b; Grossman 1990; Gudmundsdottir, 1990; Wilson & Wineberg, 1988) has served to ground and explicate Shulman's theoretical ideas about teacher knowledge, what it is and how it is used.

An understanding which has grown out of this work is that what teachers need to know is not a fixed body of knowledge learned once and used repeatedly. Rather, the knowledge needed for teaching exists in a context and its usefulness lies in its value for making judgments and guiding actions (Shulman, 1986). Therefore, much recent research has tried to represent teacher knowledge through narratives of practice where knowledge is represented as it exists, "contextual, interactive, and speculative" (Lampert & Clark, 1990, p. 21).

Up to now, research on teacher knowledge has been invaluable in helping us to see what is meant by knowledge about teaching and how it is used in practice. Shulman (1987) describes the knowledge growth in novices learning to teach as a "commute from the status of a learner to that of a teacher" (p. 12). The case studies in different subject matters paint vivid pictures of that commute for novices. But we still have little understanding of what this journey of growth is like for experienced teachers as they attempt to change their practice. The study of novices learning to teach centers on the transformation of subject matter knowledge into pedagogical content knowledge. One is learned before the other. The learning of experienced teachers might be somewhat different and more complicated because they are trying to learn subject matter knowledge in the midst of efforts to acquire pedagogical content knowledge. What does knowledge growth look like for experienced teachers who once thought they had learned to teach but find themselves trying to make changes in their practice? My work begins to address this question through its detailed account of the process of acquiring and learning to use knowledge about teaching and mathematics in practice from an experienced teacher's perspective.

Learning to teach mathematics for understanding. In the current research on teaching mathematics for understanding, the seams are smooth between the intersection of mathematical knowledge and the various tasks of teaching. My study of learning to teach mathematics for understanding begins to reveal the rough edges. How is mathematical knowledge used in the work of teaching? The bumps and bruises of my process of learning to teach allows a particular view of teaching mathematics for understanding and the need for mathematical knowledge which may go unnoticed in the practice of teachers more experienced in teaching mathematics for understanding. Shulman (1987) notes the advantages of learning about teaching in the context of the practice of beginners, "The neophyte's stumble becomes the scholar's window" (p. 4). My practice is a place to understand the importance of mathematical knowledge in teaching mathematics for understanding by studying the struggles that occur, in part, because of its absence.

The studies of teaching mathematics for understanding do much for helping us to understand mathematics teaching in the context of the current reforms. Cobb, Wood, Yackel, and McNeal (1992) and Yackel, Cobb, and Wood (1991), for example, have attempted to clarify what it means to teach mathematics for understanding from cognitive and sociological perspectives through an analysis of interaction. My work illustrates the challenge of guiding classroom interactions without a command of the subject matter and helps to show their interdependence. Lampert (1987, 1988, 1989, 1990, 1992a, 1992b), Ball (1993a, 1993b) and Schoenfeld (1988, in press) all write about their own practices teaching mathematics. Schoenfeld (1988) focuses on the influence of social context and what it means to do mathematics (Schoenfeld, in press). How the social aspects of doing mathematics intersect with understanding the subject matter are explored from a beginner's perspective in my work as I struggle to integrate them. Lampert's work includes attention to choosing and posing mathematical problems, developing mathematical tools for facilitating communication between the teacher and students and understanding mathematical knowledge in the context of classroom discourse. What my work illustrates is the ways in which my understanding or lack of understanding of mathematics impacts my choice of problems, my ability to make tools accessible to my students for solving mathematical problems, and my skill at guiding class discussions. Ball (1993b) has looked closely at instructional representations, representational contexts, and pedagogical content knowledge in the context of mathematics teaching. What my work does is highlight the need for the interaction of content with pedagogy. This is especially vivid in places where I am at a loss for my next pedagogical move because of my lack of understanding of what mathematics I am trying to teach.

Another way in which researchers have tried to understand teaching mathematics for understanding is through studies of learning mathematics for understanding. Some who do research on learning and draw connections to teaching suggest the feasibility of applying knowledge of learning to instruction. Hiebert and Wearne (1988; 1992), for example, have done extensive work on students' understanding of rational numbers. From their research on students' understandings, implications for teaching are drawn. Cognitively Guided Instruction (Carpenter & Fennema, 1988) is another example of researchers on learning applying their understandings of students' understanding to instruction, in particular to the topics of addition and subtraction. What these

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researchers are doing as they consider the ways knowledge of students' understanding can inform teaching is one of the major tasks I faced on a daily basis as a classroom teacher. We need more understanding of how teachers are to learn to make use of knowledge of students' understandings in teaching.

Learning to teach as an experienced teacher. Teacher development is a field of educational research that has been examined from a variety of perspectives (Borko & Putnam, in press; Carter, 1990; Feiman-Nemser, 1983; Richardson, 1990). For example, teacher development includes understandings of how it is that teachers are socialized into the profession (Lortie, 1975), the stages of teachers' careers (Fuller, 1969; Huberman, 1988), staff development (Lieberman & Miller, 1979), professional development (Little, 1985; Little, Galagran, & O'Neal, 1984), and teacher change from a social and organizational perspective (McLaughlin, 1976). Across this literature, teacher development has been seen primarily as a generic process (not subject specific) and rarely is the topic of how teachers develop over time addressed directly (Feiman-Nemser, 1983) or in any extended detail that reveals its complexity (Borko & Putnam, in press). There a few exceptions. Featherstone (1993) is an example of a closer, more detailed look at the learning of beginning teachers. Current research projects by researchers at the National Center for Research on Teacher Learning (National Center for Research on Teacher Education, 1990) have been designed to begin to address some of the holes in current research on teacher learning.

There is some research in the area of learning to teach in the context of mathematics education that spans both the practice of pre-service and beginning teachers (Borko & Livingston, 1989; Eisenhart, Borko, Underhill, Brown, Jones, & Agard, 1993; Schram 1992) as well as teachers with years of experience attempting to make changes in practice (i.e., Cohen, Peterson, Wilson, Ball, Putnam, Prawat, Heaton, Remillard, & Wiemers, 1990; Featherstone, Pfeiffer, &

Smith, 1993; Hart, 1991; Schifter & Fosnot, 1993; Simon & Schifter, 1991; Stein, Grover, and Silver, 1991; Thompson, 1985; Wood, Cobb, & Yackel, 1991). A general theme across all of these studies is the difficulty of learning to teach mathematics in meaningful ways whether you are a brand new teacher or an experienced teacher trying to make changes in practice. But we have much more to learn about what makes it so hard. I expected that trying to teach mathematics differently to be difficult when I started. What I had little appreciation for was what makes it so challenging. My study shines a bright beam on the difficulties of an experienced teacher faced with the challenge of learning new pedagogy and subject matter.

Studying and Creating a Practice of Teaching Mathematics

My creation and study of a practice complements existing research and is an attempt on my part to travel between the worlds of university scholarship and teaching practice. My study also connects to autobiographical traditions, in particular, autobiographies of teachers. Teachers have been writing about their own teaching in autobiographical accounts of teaching for a long time (Ashton-Warner, 1963; Calkins, 1983; Dennison, 1969; Eggleston, 1899; Kohl, 1967, 1984; Paley, 1979, 1981, 1990; Sprague-Mitchell, 1963; Stewart, 1949; Wiggington, 1986) and this continues to be a growing category of literature. What these accounts offer is a highly personal and contextual account of teaching, stories illustrative of the interaction between thought and action. They provide us with stories of teaching. Through their stories, one can learn about these authors' lives, their work with students, what they value as teachers, how it is they choose to live their lives. Teaching is portrayed from an insider's perspective. These autobiographical accounts represent the cognitive work of teaching as well as the emotional demands of the work. While much can be learned about teaching and the knowledge needed to teach from them, these autobiographies do not--nor did their authors intend them to--fall into the realm of academic research.

The writing by teachers that is finding its way into academia is the work being done by teacher researchers. Teacher research is an effort by teachers to construct their own knowledge about teaching from their day-to -day experiences. It is gradually becoming a legitimate way of knowing about teaching (Lytle & Cochran-Smith, 1992). Teacher's questions "often emerge from discrepancies between what is intended and what occurs" (Cochran-Smith & Lytle, 1990, p. 5) and often focus on issues central to the learning of their students. The questions posed and the solutions generated tend to help the teacher who is doing the research cope with immediate problems in the day-to-day work of teaching. For the most part, knowledge about teaching generated by teacher research is not considered "high status information," like the knowledge generated by university researchers. Cochran-Smith and Lytle (1990) are leaders among people in this country promoting the value of teacher research and the knowledge about teaching produced by it because of its potential to inform the personnel of schools and universities.

While most autobiographies by teachers to date are not considered by their authors or the academy as attempts at educational scholarship, I intend my study as both an autobiographical account of learning to teach and a piece of scholarship. What I have written is not just a story of my experience. It is also a close and careful study of my learning experiences over two different points in time based on an analysis of a variety of sources of documentation of my teaching.

The chapters which follow represent my journey of learning to teach differently and what I have learned along the way. My dissertation is divided into two parts. Part I, A New Vision, an Experienced Practitioner, and a Method

of Inquiry, includes the first three chapters. Chapter 1 has outlined the latest ideas for reforming mathematics education at a national level as well as the new goals I set for myself as a teacher. I took on the task of trying to learn to do a new kind of mathematics teaching and arranged to teach mathematics in a fourth-grade classroom. Chapter 2 is my pedagogical autobiography. In this chapter, I describe my past teaching practice which includes an account of my old ways of teaching mathematics. These reflections on my past work as a teacher, the changes I experienced in teaching practice in nine years, and how I learned to teach, set a historical context for the changes in practice I tried to make in my mathematics teaching. Chapter 3 describes my method of inquiry. I outline the data that was collected and how it was collected. I explain the process I went through to make the decisions about which stories to tell and how I analyzed them. As an example, I briefly discuss the analytic work that went into Chapter 5, the second of four chapters on my teaching. I also describe the purposes of my study and the risks associated with doing it.

Part II, The Creation and Study of a Practice, begins with a Prologue. In this introduction, I describe the tools and context of my mathematics teaching. I describe the context in which I worked and provide a description of the mathematics curriculum with which I began the school year. The next four chapters are narratives and analyses of my practice. Together, the chapters span the school year and provide insight into the challenges I confronted and the learning I experienced during that year of teaching and in the three years between the time I did the teaching and began work on this dissertation. Chapter 8 concludes with a look back at the previous chapters. I discuss what it is that I have learned about learning to teach and the nature of the practice of teaching mathematics for understanding. I examine the expectations I had for making changes in my teaching prior to teaching and the ways in which those expectations were revised through my efforts to create and study a practice. The chapter concludes with suggestions for educating teachers to teach mathematics for understanding.

While this is a story and study of what it means to learn to teach, it is also a story and study of what it means for me to be a learner. And, it is around this last point that I speak from my heart and mind when I say this study of my own experiences has been difficult to do. One of the hardest parts has been stepping back and seeing all that I am revealing in these chapters about who I am, what I understand, and what I have left to learn. While a certain vulnerability is inherent in authoring any piece of scholarship, in this case, a story and study of my experiences as a learner, the stakes feel especially high. Intellectually, I understand this is what makes mine a powerful story, a unique and valuable study. Personally, I just need to say one more thing before you read any further. This work feels like an incredible risk. One, however, worth taking.

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CHAPTER 2

MY EXPERIENCES AS A TEACHER AND LEARNER ABOUT TEACHING

I want to provide some background on who I was and the experiences I brought with me as a teacher and learner when I began teaching fourth grade mathematics in the year of teaching under study in this dissertation. I do this to shed light on my own story as well as point out the similarities and differences in my experiences as I see them compared with other experienced teachers who may attempt to make comparable changes in their mathematics teaching. I could discuss only my formal experiences learning to teach in an elementary teacher education program along with what I have learned about teaching in my nine years of teaching experience in varied places with different ages of students. This, however, would omit all that I learned about teaching as a student, the years I spent in classrooms as a student observing my teachers. These experiences as a student learning about teaching left lasting impressions on me with regard to the roles of the teacher and students and what it means to teach and learn in schools. So, it is with these experiences that I shall begin, followed by a discussion of my formal teacher education experiences and the nine years of teaching experience I brought with me to my doctoral studies.

Learning about Teaching

I am not alone in thinking that my experiences as a student influenced my view of teaching. Dan Lortie (1975), a sociologist who did extensive research on teaching, coined this way of learning about teaching as "the apprenticeship of observation" (p. 61). From an observer's position, the learner about teaching has no idea about the demands and difficulties of teaching from the teacher's perspective. Behaviors are what the learner observes as a student and later imitates as a teacher. What is missing at the time for the learner, and unfortunately later as the teacher, is an analytic perspective on teaching. It is often not only that students lack such perspective, but that they may not even realize that teaching has an intellectual side.

Experiences in Elementary School

My three older siblings and I spent grades one through eight in our neighborhood Catholic grade school in St. Paul, Minnesota. For most of those years, I was one of forty students in a crowded classroom. Our wall-to-wall desks were arranged in rows that faced the front of the room. My teachers tended to conduct class from a podium at the front of the room where they kept us under strict control and followed one of the numerous teacher's guides that defined the curriculum. We spent most of our school days quietly working out of these textbooks, answering questions posed by the teacher or the text. In either case, the questions always had one right answer.

Not all of my childhood learning about teaching was done in school. Many of my out-of-school hours were spent playing with my two sisters, who are two and three years older than me. We spent a lot of our time playing school with discarded textbooks and recycled worksheets. As the youngest, there was never any question about what role I would play. I was the eternal student. I watched as my two older sisters squabbled over who would be the teacher and imitated the words and actions of our real teachers. Between my practices here and my efforts at school, I perfected school learning. I knew what was expected and how to perform.

It was not until I thought carefully about my school experiences while writing this thesis that I realized that much of what happened in my school experiences as a student was similar to what happened when my sisters and I played school. For many years as a student, I did school. This meant r

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memorizing rules and procedures, getting right answers, and turning in neat papers. Much later in my life, I experienced a different kind of learning--I felt what it was like to curious, to be engaged in ideas, and to think and reason for myself.

My early school experiences shaped the way I thought about teaching and learning and provided me with an image of the role of a teacher. I viewed teaching as a didactic process aimed at reproducing a fixed body of knowledge (Jackson, 1986). Learning was primarily about memorization and recall. Teaching was telling, and as a student, my job was to listen. There was little that was demanded of me in terms of real intellectual engagement as I later experienced it. Schools were not about teaching and learning things that mattered to me. Schools were about learning a defined set of curricula represented primarily by textbooks in a directed fashion.

Much of my elementary school learning was done around textbooks and worksheets. Learning entailed searching for right answers, recalling them accurately, and recording them neatly. There was no space for my creativity alongside textbook knowledge and worksheets with right answers. It was a long time before I realized that textbooks had authors and the text that appeared was constructed by people. Where knowledge came from was not a mystery for me as a child. It was not even a question. If I wanted to know something, I looked it up in a textbook. The subject did not matter. We had textbooks for everything including art, music, religion, reading, history, science, and mathematics. Since this is a study of learning to teach mathematics, I will not discuss my experiences in each subject. Instead, I will just focus on my experiences as a learner of mathematics.

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Experiences in Mathematics Classes

The last formal math class I took was a full year of calculus as a college freshman at the small liberal arts women's college my oldest sister had chosen to attend. A friend from my high school was also in the course. We studied together on a regular basis outside of math class. I had done well in high school math yet found calculus somewhat of a challenge. Unlike my math classes in the past, doing well was not merely a matter of memorization and recall. I had to apply rules and formulas to understand mathematical change and relationships. Calculus confronted me with a new school experience. I was expected to think. Even with this new requirement, all that I needed to learn came pre-packaged in the textbook. We were even given problem sheets, similar to the worksheets I did in elementary school. They, too, had single right answers. The difference was that there were just many more steps to solving these problems.

Before my sophomore year, I switched colleges. The size and diversity of the large public university where I transferred was much more to my liking. At the time I made the transfer, I had already decided to be an elementary teacher and learned that the credits earned by taking one more math class would qualify me for a concentration in a particular subject area, a requirement for elementary education majors. I enrolled in a course called Differential Equations, for which Calculus was a prerequisite. In a room full of men majoring in science or mathematics, I was one of a few women enrolled in the course. I could not follow the lectures and felt completely alone and lost in the content. I grew increasingly unable to even formulate questions for myself about what it was that I did not understand. This was my first experience feeling like a complete failure in school. After the mid-term, I dropped the class in shame. I concluded from this experience that I had finally reached the highest level of mathematics I was capable of understanding.

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I continued to feel badly about myself and this math experience until the following year when I was admitted into the elementary teacher education program. There, I found that I had taken more mathematics than most of my peers. When I heard their horror stories about learning mathematics, I realized that many people carried bad mathematical memories that went as far back as elementary school. Hearing their stories, I no longer thought of myself as Someone-Who-Dropped-Differential-Equations. Rather, I came to view myself as Someone-Who-Made-it-Through-Calculus. I completed my concentration in mathematics by taking a BASIC computer science course. I had had my fill of mathematics.

I initially took math courses because I felt like I knew how to work with numbers and enjoyed it. I think I was lulled into believing in high school that fields of study were not gender-specific and my attendance at the women's college delayed any understanding of inclusion or exclusion of women in mathematics (McDade, 1988) for a year. It was not until I was enrolled in the public co-ed university that exclusion by gender in mathematics was apparent to me. My experiences in college mathematics are, unfortunately, not unusual. While math was not my major, I did experience similar feelings to the women who McDade (1988) studied who were math majors and dropped out of math. McDade (1988) writes, "most women who left these majors described their futile attempts to improve ailing GPAs as debilitating indicators of their inferior intelligence, poor study habits, poor high school preparation, or general inability to remain confident and competent at their tasks as students" (p. 99). I had always thought of myself as a good student. My performance in this math class shook my confidence. In terms of the general population, my math experiences were typical. Women were not expected to succeed. However, relative to the population of people who become elementary teachers, I was seen as successful and had more math experiences than most of my peers in elementary education (Ball & McDiarmid, 1990).

My Decision to Teach

As a young white woman from a working middle class background with educational opportunities my parents never had, I share much in common with others who decide to become teachers (Lanier & Little, 1986; Goodlad, 1984). My decision to be a teacher was one encouraged by my parents. I was going to college to learn to be something that, once I had the degree and a teaching job, would place me firmly in the middle class (Lortie, 1975), perhaps in a slightly better socio-economic place than my parents were in at the time. My orientation to schooling and the purposes for getting a college degree, were based on the economic promise (Goodlad, 1984) that education could give me.

My older siblings and I all went on to college and acquired degrees, an opportunity neither of my parents had themselves. Like many first generation college students, I had limited economic resources (Lortie, 1975) and my siblings and I all chose to attend colleges near home. I started out at a college my oldest sister attended and then switched to the university my brother attended. My oldest sister became a teacher. My other sister became a nurse. Like many others in the profession, I became a teacher more by default than by deliberate choice (Johnson, 1990). I tended to follow my sisters' leads and since I had no desire to be a nurse, I became a teacher. While I do not recall the decision as a choice, it made sense. In spite of the fact that school was not very engaging, I liked the school scene, engaged the way I was expected to, and imagined myself in the role of the teacher. Lortie (1975) refers to this as "the continuation theme:" I liked school as a child, why not continue to be part of school life and become a teacher?

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My sister and I became part of the second generation of teachers in my family. My grandmother had been a schoolteacher and principal in the early 1900's. Although neither my sister nor I know much about my grandmother's teaching practice, by choosing to be teachers we were following in her footsteps. Many who become teachers have family members and relatives who are also teachers (Lortie, 1975). My grandmother lived at a time, however, when women could be wives or teachers, not both. A sentence from the announcement of her wedding in her local newspaper which highlighted her decision has lingered with me since the day I read it.

> Those who have come to esteem and regard highly the young lady who served as principal of Ferryville's schools the past year feel pleased that she will not depart from this vicinity but has been won by a progressive young farmer as his life partner.

How had she felt about this? What would she think of the choices I have had, the life I have made for myself as a teacher? My only window into my grandmother's practice as a teacher comes from one of her book's, *A Brief Course in the Teaching Process*, (Strayer, 1919), with pages well-worn by her thumb and margins marked with her pencil. I have searched the passages she noted that I would have noticed, too. I found this one.

> Pupils at work forming habits of thought, feeling, and action; acquiring knowledge of nature and of society; forming ideals which make for social well being; and learning in all of this work to act independently, to function in the society of which they are a part: this is education, and these are the goals which we should strive to achieve every day and every hour that we teach. (p. 10).

It is interesting to me that something my grandmother read and marked with her pencil more than 70 years ago makes sense to me today. Would my grandmother be surprised that decades later, I share similar beliefs? But, interestingly, these are not the values I held about teaching when I entered my teacher education

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program. The transformation of my beliefs and values about teaching can be followed through my experiences in the life I made for myself as a teacher in the nine years of teaching experiences explained in the remaining part of this chapter and the year of math teaching described and analyzed in following chapters.

Experiences in a Teacher Education Program

In the late 1970's, I went through typical undergraduate elementary teacher education courses whose content and instructors I can barely remember. I student taught in a first grade classroom with a competent, well-organized, supportive cooperating teacher. I gradually took on the role of the teacher, taking more of the cooperating teacher's responsibilities as the term progressed. By the middle of the ten week term, I was teaching full-time. It was a successful experience. I left thinking that I would be a good teacher. I played the role well and had accumulated a collection of materials and activities for teaching, and strategies for managing a class. Like so many other teachers, I left college feeling prepared to teach an elementary class of my own and my principle teacher had been experience in the classroom.

There was nothing in my teacher education program to challenge my childhood views of teaching and learning. I was part of a program in which the primary objective of practice was acquiring technical skills. I left with the same views of teaching and learning with which I entered. Lortie (1975) notes, "Teachers seem to emerge from their induction experiences with a strongly biographical orientation to pedagogical decision-making" (p. 81). Teachers' own life histories as students shape the sort of teachers they become. What students learn about teaching, Lortie reported, tends to be "intuitive and imitative rather than explicit and analytical; it is based on individual personalities rather than pedagogical principles" (p. 62). I was no different than most.

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I graduated from college in the middle of a school year and spent my time substituting. I bounced from one grade level to another, jumping in and out of teacher's guides. Curriculum was the least of my concerns. Daily survival was my objective. This was a rough way to start teaching but it made me especially appreciative when I was hired the following year to teach a class of first graders in a classroom that was all my own.

My Teaching Adventures

In the section that follows, I relay experiences from my nine years of elementary teaching. In those nine years, I taught in three different locations: Minnesota, Germany, and rural Vermont. The level of detail at which I will consider my teaching experiences in these three different places varies. I discuss aspects of teaching specific subjects that I think, in retrospect, influenced my conception of knowledge, and the learning and teaching of particular subjects.

Minnesota: Teaching the Way I was Taught

I began teaching within 30 miles of where I grew up and went to college. Like most students who finish a teacher education program, I did not view learning to teach as "a long term enterprise" (Lanier & Little, 1986, p. 561). Rather, my perspective at the time was that my preservice program was about learning to teach. I finished that. My first job was about teaching.

I have fond recollections of my first two years of teaching in a newly developing suburban area in Minnesota. It was both exciting and overwhelming to realize that I now had full responsibility for a class, the same responsibility as the veteran teacher of 20 years across the hall. For these two years, I worked behind the closed door of my classroom and experienced success by teaching the way I had been taught. I closely followed the textbooks for each subject, taught in a didactic manner, and rewarded my students' right answers with stickers and smiling faces drawn with sweet smelling markers. There was nothing in these

experiences to challenge the view of teaching I had acquired as a student and had been perpetuated by my teacher education program. I learned that year by trial and error. Lortie (1975) refers to this as the "sink or swim" approach (p.60). Fortunately, I swam. I was playing school in my own classroom and I finally got to be the teacher.

Germany: Broadening My Views of Teaching and Learning

After my first year of teaching, I married a teacher. The two of us taught in Minnesota for another year. The following year, we moved to Berlin, Germany. It was quite a big step considering the fact that I had never lived more than 30 miles from my parents. During the three years I lived in Berlin, my perspective on many things changed, including some of the views about teaching and learning I had developed as a student and teacher in Minnesota. I came to have a new appreciation for what school learning could be. I also came to appreciate what it would mean for me to admit as a teacher that I was also a learner.

At the time of our move, my spouse spoke German and was hired to teach in a bilingual class in a Berlin K-12 public school, half of whose faculty and student population were American, half were German. The American students came primarily from the families of diplomats and businessmen while the German students were locals who were admitted to the school by lottery. Instruction, especially in the primary grades, occurred in both German and English and a mix of German and American curriculum materials were used. I did not speak German and had no job when we left the United States.

Learning to speak German. At the time of our move, I had little idea what a turning point and transformation this experience in Germany would become with regard to my views of teaching and learning. The process I went through to learn German to get a job, my insight in to what it takes to become a fluent

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speaker of a second language in a culture that demands it of you, and what I learned from working closely with teaching colleagues all contributed to challenging my previous school experiences as a teacher and a learner. My experiences in Germany reshaped my visions of what teaching and learning could be.

It turned out that my only hope for a job that year was to be hired as a substitute for a first grade teacher at the bilingual school who had plans to take an extended maternity leave beginning in November. Speaking German was a prerequisite for the job. When I learned of this potential opportunity to teach, I enrolled in a German language school. I spent four hours a day, five days a week, immersed in German classes. I spent time outside of class studying and living in a culture where speaking German was essential to my survival. By the end of three months, I had learned enough German to be hired. The intensity of these experiences had given me a good start with learning the language even though I acknowledged and others knew I still had plenty more to learn. There was no way I could learn all the German I would need to before I started teaching. The learning of a language never stops. What I had learned in those three months was a foundation to help me to continue to learn on my own.

This was notably different from any sort of learning I had done in the past where the expectations that others held for me and that I held for myself were that I would "get" whatever it was that was being taught to me. This was the first time it occurred to me that learning was not about "getting" a certain body of knowledge. In the case of learning a language, it was about acquiring a certain foundation that would enable me to keep on learning vocabulary and make use of the language I understood. I would never completely master the language. As with English, I would always have the need for a dictionary. There would always be more I could learn.

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Much of the German I continued to learn was acquired through interactions with my students. Of the 20 children in my class, half spoke German as a first language, half spoke English. Several of the children were products of bilingual homes. These six and seven-year-old bilingual children were marvelous teachers. In their presence, I realized for the first time how much I could learn from my students. I was forced to for I found myself in situations, repeatedly, where I had to depend on their help with the language. A comfortable sort of interdependence grew among us. We each had things to teach and learn from one another. I openly acknowledged that I was learning and the students readily offered their assistance. Within a short time, I found myself respected as a learner of German as well as a first grade teacher by my students and their parents.

Teaching as a member of a team. Teaching in this school, in a new setting, in a new country, with new colleagues, speaking a new language, I felt in some ways as if I was a brand new teacher being inducted into teaching all over again. This time, unlike in my first two years of teaching, I did not feel like I was doing it alone. The five teachers at my particular grade level worked as a team and this made finding my way through the curriculum somewhat different than it had been in the past. Instead of having to pretend I understood everything that I was supposed to teach, I could be a learner and ask questions in the context of my work with these other teachers. I could admit to not knowing with these teachers. It seemed similar to being able to ask questions about the German I did not understand. Pretending to understand something I did not in German, often times, got me into immediate trouble and I quickly learned it was better to be honest about what I did not understand. It was easier to hide my ignorance with colleagues. But, quite surprising, I discovered there was no need to. My questions about content were encouraged. In fact, I learned much from this team of teachers and it was in the context of working with them, learning from them, that I had my first inkling that learning for my students could be different than what I had experienced. My work in Berlin was a turning point in that my experiences working with this team of teachers to develop and teach social studies, science, art, and music changed my images of curriculum and learning and broadened my perspective on what could be done with students and counted as school learning. I also got a taste of how I could be viewed as a teacher and not have to know or pretend to know everything.

I taught reading to all English speakers in my class using the same basal reading series I had used in Minnesota. The teacher's guide for mathematics helped to make mathematics the easiest subject to teach relative to the other subjects I was expected to teach in two languages, like science and social studies which had no ready-made guides. I could look over the guide ahead of time and prepare myself for the German vocabulary that might come up in the context of the lesson the next day. I made my way through a consumable math workbook of computation with my students by demonstrating how to do problems, assigning pages, and correcting the previous day's work, all the while sticking closely to the teacher's guide.

The ways in which a new way of thinking about school learning played out for me can be seen in the ways in which my colleagues and I worked together with our students to teach and learn social studies and science. I depended heavily on the creativity of my colleagues to construct the curriculum. They arranged for us to take numerous field trips to interesting places throughout the city of Berlin. It was new for me to think that adventures, similar to the ones I had exploring the city on the weekend, could be arranged during the week for students and counted as "school learning." When we went on field trips, I found myself learning right along with my students.

My colleagues cared that students learn something from our experiences but there was not the expectation that all students learn the same thing. There was the expectation that students use some medium to represent and share what they learned from our experiences. The medium and the representations varied. Students wrote stories in their own languages about their experiences and did accompanying art work--pictures, murals, and objects created from water colors, crayons, charcoal, pastels, and clay intended to capture students' experiences and perspectives. The diversity in what they had learned was celebrated. I enjoyed not having the curriculum tied to a textbook and at the same time feeling like my students were learning things of interest to all of us. I saw that school experiences did not have to be driven by a textbook. In fact, these learning experiences, not derived from a textbook, were more exciting for me and many of my students than anything else we did in school. I found myself engaged with my students in a kind of learning I had never before experienced in school.

Vermont: Change and Growth as a Teacher and Learner

After our third year in Berlin, we picked a beautiful place in the United States and moved to Vermont, where I taught for the next four years. As I consider those experiences now, I can see that Vermont became a site to continue the changes in my teaching I had begun in Berlin. My colleagues in Berlin had offered me a different view of what a social studies and science curriculum could be. In Vermont, I encountered Man: A Course of Study, an ambitious social studies curriculum, as well as "whole language," a new way of conceiving of reading. These curricular ideas continued to influence my changing views of social studies and offered me a new way of thinking about language arts. In the sections that follow, I describe my first year in Vermont. Several days before the school year began I was hired as a long term substitute. During that year, the least of my concerns was curricula. This is followed by a discussion of the next three years of my teaching in Vermont by subject area--social studies, language arts, and mathematics--and a comparison and contrast of my views of teaching, learning, curriculum, and knowledge in each of these areas.

A year of substitute teaching. Through a chain of fast moving events, I was hired as a long term substitute for a class of sixth-graders two days before the school year started. I was thrilled to have a job, even temporarily, but frightened. Up until now, teaching first and second graders was the extent of my experiences. I knew nothing about sixth graders--what they were like or what they needed to learn. But, given my overseas adventure, I did not think this endeavor could be any more challenging.

The sixth-grade teachers in the school shared students and responsibility for teaching different subjects. The teacher for whom I was substituting was responsible for teaching U.S. history and English. In addition, we exchanged students for reading. I was charged with managing a room full of high-ability readers. For the next four months, I taught the highlights of U.S. history drawing on history books and the notes of the teacher for whom I was substituting. I used his dittos and followed his ideas for activities and projects. In English, I followed the teacher's guide, and in reading, students read and did worksheets from the upper levels of the school's basal reading series.

During those four months, I attended to my relationships with these sixth grade students and found safety in the teachers' guides and the regular teacher's notes for what to teach. I knew this was only a temporary position and I did not invest myself in the content. I worked, instead, on managing my interactions with the students. I saw content and social interaction separate at the time. They were two different issues with which to deal. Keeping myself detached from the curriculum lessened my feelings of responsibility. Following someone else's plans answered the question of what to do next and helped me deal with the question of whether or not we were making progress. Knowing that I was "covering" what I was supposed to lessened the anxiety on days when things did not go well and I questioned what my students had learned. I was managing survival. For the third time in five years, I felt like a new teacher.

For the rest of that school year, I continued to be employed as a full time substitute teacher within the school. My time was focused on building relationships with teachers and students with minimal investment in the curriculum of whatever class of students I happened to be teaching. I was never in one place long enough to feel like I could call anything my own. I spent the entire school year teaching someone else's students using someone else's plans. This was unsettling. I felt unsettled for personal reasons, too. During this year, my marriage ended. Fortunately, the next year, I retired as a substitute and was hired into a permanent position teaching fifth grade in the same school. In the next three years, I dealt with curricular issues around social studies and language arts. In addition to classroom teaching, I started and completed a master's degree. I had a student teacher and received an award for my teaching. Each of these experiences are described in the remaining sections of this chapter.

<u>Man: A Course of Study.</u> When I started teaching fifth grade in Vermont, I encountered Man: A Course of Study[1] (MACOS), a social studies curriculum developed by Jerome Bruner and his colleagues in the 1960's. The course is organized around the fundamental question, 'What makes man human?' The question is pursued through a year long comparative study of salmon, herring gulls, baboons, the Kalahari Bushmen, and the Netsilik Eskimos. As for the intended goals of MACOS, Dow (1991) writes, "asking questions was to be as

[1]See Dow's book, *Schoolhouse Politics* (1991) for a discussion of the development and implementation of MACOS.

important as finding answers, and new knowledge about human nature was to be combined with issues of value, namely, how human beings can improve the quality of life" (p. 79). This was my first encounter with a written curriculum that did not emphasize right answers.

In the set of MACOS curriculum materials there was nothing that resembled any textbook or teacher's guide I had used to teach any subject in the past. The guides offered open-ended discussion questions and the student texts were, in many cases, condensed versions of primary source documents for students to read and interpret. I found the number and variety of materials overwhelming but the experiences they offered quite interesting. The set of curriculum materials consisted of "nine teacher's guides, 36 children's booklets, 16 films, four records, 5 filmstrips, 3 games, 54 artifact cards, 2 wall-sized maps, a caribou hunting strategy chart, a kinship chart, a sea ice camp chart, 11 enlarged photographs taken from the Netsilik films, several poster-sized photo murals, and a take-apart seal" (Dow, 1991, p. 134). My students learned of other cultures and ways of living without ever having to leave the classroom through reading, films, and simulations .

Over the three years during which I taught MACOS, I felt like I learned more about the curriculum as a whole as well as the parts. I constructed connections between topics and relationships among themes. I also learned how to use the curriculum materials to teach ideas. Through my attempts to teach the curriculum, I gained a sense of its structure as well as the culture of inquiry that drove it. By the third year, I found myself and my students delving deeper and deeper into the ideas. Individually, in small groups, and as a whole class, we explored questions without right or wrong answers and students learned to generate and investigate questions as a means of constructing their own connections and relationships.

I loved teaching MACOS. It was intellectually exciting for me as well as my students. My students and I were engaged in ideas together, real ideas, genuine questions, without simple answers. The last time I had felt similarly engaged with something I was teaching was in the context of teaching social studies and science in Berlin. Having experiences with students around ideas that were thought-provoking to me was a treat. As I look back now, I think my work with MACOS definitely influenced the changes I went on to make the following year in language arts. Our discussions gave me a feel for what it could mean to be engaged with students around ideas and text.

Reading real books. In my second year in fifth grade, with MACOS underway, I turned my attention toward reading or language arts. A teacher noted for doing interesting things in language arts with elementary students in a nearby school was hired to teach fifth grade with me. Under her guidance and my principal's support, we did some experimenting with teaching reading. My goal was to make reading as interesting as social studies had become.

We received some grant money from the school district to purchase trade books. We replaced the reading series with real books and did away with reading groups. To prepare to teach these new books, we read them ourselves and spent hours talking about what we had read, what interested us, and questions we could ask to get at the sense our students made of the story. I found myself thinking back to my experiences in Berlin for ways to integrate writing and art into our discussions of text and experiences. Students wrote stories, created works of art, and did assorted projects to represent what they understood about what we read. I was amazed at what my students could produce and their interpretations of the text when I gave them the opportunity to express themselves.

Reading and MACOS fast became my two favorite subjects to teach. As I look back on this now, I think that I was seeing that teaching in ways that were of intellectual interest to me were also of interest to my students. There was nothing pretend about what any of us were learning. These changes in my teaching were changing my views on teaching and learning. MACOS and language arts had become occasions for me to "acquire new standards, reverse earlier impressions, ideas, and orientations" (Lortie, 1975, p. 81). In social studies and reading, my ideas about teaching and learning had changed. In each of these subjects, I had tasted, what Cohen (1988) refers to as, "adventurous" teaching.

Mathematics stands alone. While these changes were happening in other curricular areas, my mathematics instruction remained untouched. I had been teaching math pretty much the same for nine years. And, the way I was teaching it resembled the way I had learned it. Mathematical knowledge had always been a fixed body of rules and procedures to me. Something to be memorized and recalled when needed. Questions had only right answers. That I could have such different views of what and how to teach math and social studies surprises me when I look back on my practice now. At the time, I did not question it. In fact, many other teachers' practice follows a similar pattern. Stodolsky (1988) found in her study of fifth grade social studies and math teachers that their instruction across subject areas varied. "When individual teachers shifted from one content area to the other, they varied cognitive goals, instructional formats, student behaviors, materials, and the extent to which pupils worked together" (p. 74).

Stodolsky (1988) hypothesizes why in her study it seemed to be the case that other subjects were taught in more ambitious ways while mathematics remained skill and procedure driven. "Perhaps traditional forms of teaching are more regularly found in sequential subjects or subjects where knowledge is perceived as such" (p. 116). That was definitely my view of mathematical knowledge at the time. In fact, I did not know there was an alternative view. My math instruction was similar to much of what she observed. Some of the fifth grade teachers she studied also taught MACOS and their mathematics teaching, like mine, was much different than the teaching they did in social studies. Here she describes their mathematics teaching,

> Most of the classes we observed had competent teachers, but most of them seemed to hold a very limited view of what should be learned in school and appeared very skeptical of children's motivation and ability to learn. More often than not, students were treated as receptacles for knowledge that teachers transmit, not active participants in learning. And the nature of the knowledge to be transmitted, often defined by textbooks, and other curriculum materials, was chunked so small that is significance was not at all clear. (p. 135)

At the foundation of my math teaching in Vermont was a mathematics textbook and a school-wide set of math performance objectives. I scheduled math for an hour each day. During that hour, I spent the first 20 to 30 minutes correcting the previous day's computation homework and working through sample problems on the board. It was a time when I taught rules and outlined the steps, through examples, of particular procedures for working algorithms. During the remainder of the class, students worked quietly on computation problems assigned from the textbook or toward the completion of one of their undone math objectives.

I juggled time on these objectives with doing problems from the textbook. The work in the textbook always came first. I made my way through the textbook, about two pages a day, four days per week. We spent Wednesdays on "problem solving." This was the day the students did the word problems from the textbook that on other days I told them they did not have to do. Word problems caused such a ruckus in my class that I tried to confine them to one day per week. Students put up a fight. Every week they asked me the same questions. Did they have to show their work? Did they have to label their answers? Would they get partial credit? Every week my answers were the same--yes, yes, yes. There were lots of questions about what each problem meant and what sort of computation was needed. I was always exhausted after math on Wednesdays.

The school's math objectives covered the kinds of computation the district administrators had decided fifth graders were supposed to learn and coincided with topics in the math textbook like addition, subtraction, multiplication, division, fractions, and decimals. I always felt like there was a lot to manage with this objective system. What objective were individual students working on? Which ones had they passed? What was their score? If a child did not pass an objective, what plans were there for that child to retake it? If skills needed reteaching, when was that going to happen? Who was going to do it? The management only increased as the school year progressed. By spring, some students had passed all of their objectives while others still had countless unmastered ones awaiting them. It was a continual struggle to keep the finished students busy and motivate the others. By the end of the school year, I was tired of the objectives, as were my students.

The work we did out of the textbook the other four days was fairly straightforward. Class often began with the correction of the previous days homework which amounted to me reading the correct answers out of the teacher's guide. At least twice a week, this was followed by a timed test--one hundred multiplication or division facts--one of the many math objectives at this grade level. Some days I did the mental arithmetic "warm-up" activities found within the teacher's guide and not in the students' texts. This was usually followed by sample problems done by me on the board for the purpose of introducing or reviewing rules and procedures.

I came up with a creative on a way for working math problems together. Somewhere I got the idea of making individual chalkboards for students out of squares of cardboard and chalkboard paint. I made one for each of my students and equipped them with chalk and a felt square as an eraser. The individual chalkboards enabled everyone to try out the problems I worked on the board. Using chalk on the little boards offered an alternative to the usual paper and pencil routine and appeared to add a little life to what seemed like for many, including myself, a dull subject. If the content was not exciting, maybe the process could be fun.

For a similar reason, I dabbled a bit in manipulatives. One summer, I took a workshop sponsored by the developers of *Mathematics: A Way of Thinking* (Baratta-Lorton, 1977). I was excited about the ideas but wondered how I was going to add these ideas and materials into an already tight schedule. As it was, math spilled over into other times of the day toward the end of the year in an effort to complete all of those objectives. I did assemble a collection of beans and cups and used them a few times. I either lost or never found the point of using them and soon they were just collecting dust on my bookshelf. I purchased a couple of buckets of plastic cubes that became favorite toys during free time but never found their way into math class.

Long division was a topic in fifth grade that we seemed to work on all year. The long division objectives were the hardest for students to pass and we went over and over the steps of the process--divide, multiply, subtract, bring down, divide, multiply, subtract, bring down. There were so many numbers to attend to, places to keep track of, and operations to perform. Working problem after problem on the chalkboards together seemed to help a bit. At least I could see who had not yet mastered the procedure. Long division was a hurdle that any fifth grade teacher or student had to jump. My emphasis on computation paid off each year. Each spring my students took standardized tests and did just fine, even on the long division problems. I had mastered techniques for efficiently teaching my students rules and procedures. They had learned how to recall them when they needed them. In the traditional sense of mathematics instruction, my students were successful and so was I.

My mathematics teaching was driven by the need to teach my students what others had defined that they needed to know. Knowing was getting right answers on what seemed like an infinite number of mathematics objectives and achieving high scores on standardized tests. These were the same goals I was asked to meet as a student. I knew of no other way of knowing mathematics. I believed there was no way around it. Teaching and learning mathematics was just tedious work . And, my students and I had to keep at because of all the content we were expected to cover.

<u>My master's degree.</u> In my first three years in Vermont, in addition to being a teacher, I was also a student in a master's program in curriculum and instruction at the University of Vermont. I took one class each semester with many of my colleagues as I continued to teach. We all chose to take these classes, yet the atmosphere carried with it a tone of resentment. Who did these university people think they were to tell us the way things ought to be done? Most of them had not taught in schools for years, if ever. The group was not so much openly hostile as highly skeptical. Every theory was questioned and collectively confirmed or dismissed by someone's real classroom experiences. Lortie (1975) found such complaints to be commonplace for teachers taking university course work. Teachers often felt as if their professors held "impractical expectations and a utopian conception of classroom reality" (p. 69)

and they were "too remote from classroom exigencies" (p. 69). My classmates and I looked for and found ways to make most projects and papers directly applicable to our own teaching.

My master's project focused on MACOS. I conducted a qualitative survey of people who had taken MACOS sometime during the thirteen years it had been used in my school district. The curriculum, it seemed, had made a lasting impression on most of the people who had experienced it. Years later it stood out to them as different from other subjects they took. Students seemed to appreciate the absence of single right answers. Most students viewed MACOS as one of the rare school experiences in which they were asked to think for themselves and reason about problems.

Supervising a student teacher. In my fourth year in Vermont, I found myself at a point in my teaching where I felt like I was ready to help someone else to learn to teach. I took on the responsibility of my first student teacher. I went into the experience thinking I would teach someone to teach. I came out of the experience surprised to have learned so much about my own practice. The student teacher with whom I worked was older than most undergraduates and seemed more mature than I remembered myself being when I student taught. She was not afraid to ask me questions about anything having to do with my practice. At first, I felt threatened by her questions. No one had ever asked such pointed questions about my decisions. What I came to realize was that her questions were genuine and she was interested in understanding my practice because she wanted to develop her own. Her questions about my practice helped to surface much of what I took for granted, assumptions I held, implicit beliefs that guided my actions. I groped for language to talk about my teaching and realized what a rare opportunity it was for teachers to have someone with w

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whom they could talk in detail about their work. In eight years, no one had ever paid such close attention to my instruction, not even me.

An award for teaching. In my fourth year in Vermont, the same year I took on a student teacher, I was honored as an "outstanding teacher." It was in the context of an annual statewide recognition program sponsored by the University of Vermont for teachers in schools and school districts across the state. While it was nice to have my work noticed and appreciated, it was somewhat embarrassing to be recognized publicly. Such extrinsic rewards are rare in teaching and Lortie (1975) notes that the publicity around such awards tend to be awkward for all involved. Even to write about it here feels a bit pretentious.

My purpose of including this experience, however, is to provide further support for the point I wish to make. I entered a doctoral program and this year of math teaching as a competent teacher, viewed as successful in many people's eyes. When you read of the difficulties and struggles I faced in trying to learn to enact the math reforms in the remaining chapters, I do not want you to explain them away by assuming that, in general, I was an incompetent elementary teacher. That was just not the case.

Reflections on My Experiences as a Teacher and Learner about Teaching

As I look back on my experiences as a teacher and learner about teaching, I recognize that the kind of adventurous teaching as described in the current reform documents in mathematics education is of a spirit similar to the teaching I did in other subjects, first in Berlin, and then in Vermont. The adventures in science and social studies in Berlin, the talk of ideas in language arts, and the focus on discussions of questions without right answers in MACOS seems similar to what I was aiming for in mathematics. It is interesting to me that while I had made changes in the way I taught other subjects, mathematics had remained virtually untouched. I had never considered that there could be questions as fundamental and thought-provoking in mathematics as I had found in social studies, like, what makes man human?

Accompanying changes in the nature of my teaching came a new sense of what it means to learn, new possibilities for what school learning could be and a new view of myself as a learner. The relationship between learning and teaching in the role of a teacher have come to interest me. In the beginning of my teaching career, my images of teaching were largely influenced by my views of teaching and learning--my role as a teacher and what I could expect of my students--as shaped by my early experiences as a learner. Throughout my nine years of teaching, my views of teaching and what it means to learn certain subjects have changed and the lines between my role as a teacher and learner of those subjects have blurred as I found myself teaching students about ideas I was curious about, myself, as a learner. My story and study in the chapters to follow continues to push at the boundaries and my understandings of the roles of teachers and learners. I will return explicitly to the relationship between teachers and learners in Chapter 8, the conclusion to this dissertation. In the meantime, I invite you to keep reading.

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CHAPTER 3

LEARNING ABOUT LEARNING TO TEACH

My study of what it is that I--an experienced and successful teacher--have had to learn to make fundamental changes in my mathematics teaching raises interesting issues about the relationship between teaching and research and the fragility of studying one's own practice. It also illustrates the intellectual challenges and personal risks posed by doing research on one's own teaching practice. The kind of teaching I was trying to do was novel to me and relatively unusual in school practice. But, research on learning to teach mathematics differently from a learning teacher's perspective is an even more novel practice with few images or guidelines for going about the process. I used no single method of inquiry. Instead, I drew from a variety of qualitative research traditions, made links to a broad range of literature in areas outside educational research, and relied heavily on my own insight and intuition. This dissertation, in both its content and its method, represents a process of self-education.

A Temporal Context

I taught fourth grade mathematics during the 1989-90 school year. I made the decision to make my teaching experiences a study of learning to teach almost two years after I did the teaching. This relationship between my teaching and research is shown in Figure 3.1.

Given this is a historical study of learning, it was especially important that I "use a temporal context of actions to make sense of them" (Hammersley & Atkinson, 1989, p. 192) and be explicit about this context in my analysis. I needed to find a way to acknowledge my learning as dynamic while simultaneously establishing a methodological way to hold my learning still long enough to analyze and write about it. To do that, I identified two points of time and used them as reference points for an analysis of learning.

The analysis is divided into two sections: what I was learning at the time of my teaching based on the sense I made of my practice then and what I am learning now, three years later, as I revisit my teaching in the course of studying it. When I taught, I learned. I have documentation that supports the conjectures I make in this dissertation about the sense I was making of my practice at the time. Three years later, as I looked back on that year of teaching, I learned more--both about my own teaching and about the process of learning to teach differently. The time line in Figure 3.2 shows that "three years later" is conceptually a broader time frame than the single point in time which it might appear. The actual time of analysis falls on either side of September 1992, which is approximately three years after the start of my teaching and when I began writing this text.

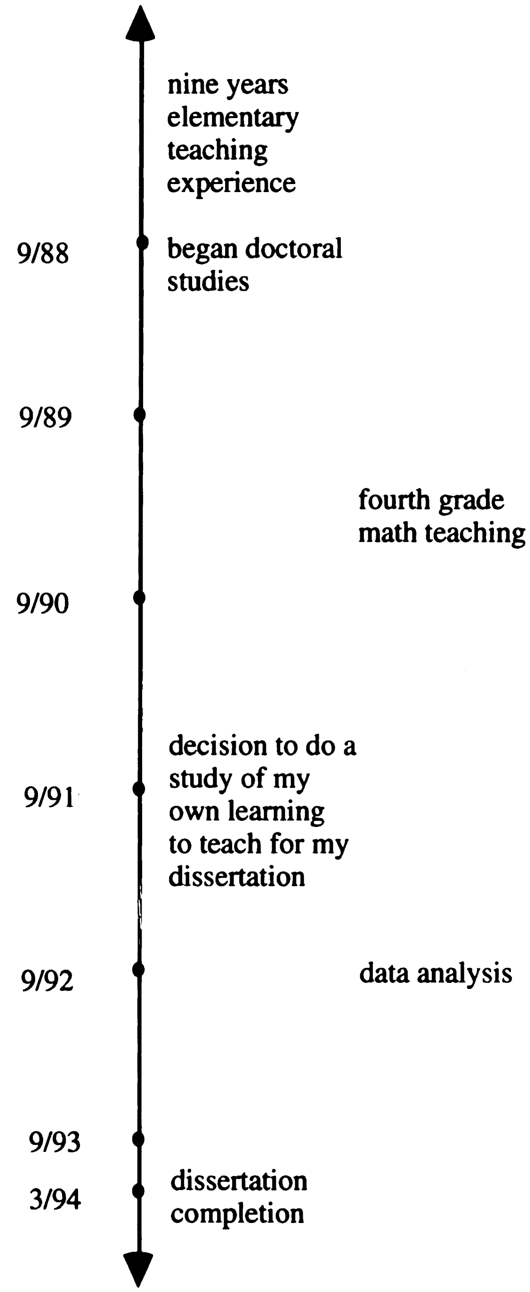

Figure 3.1

Timeline of my teaching and research

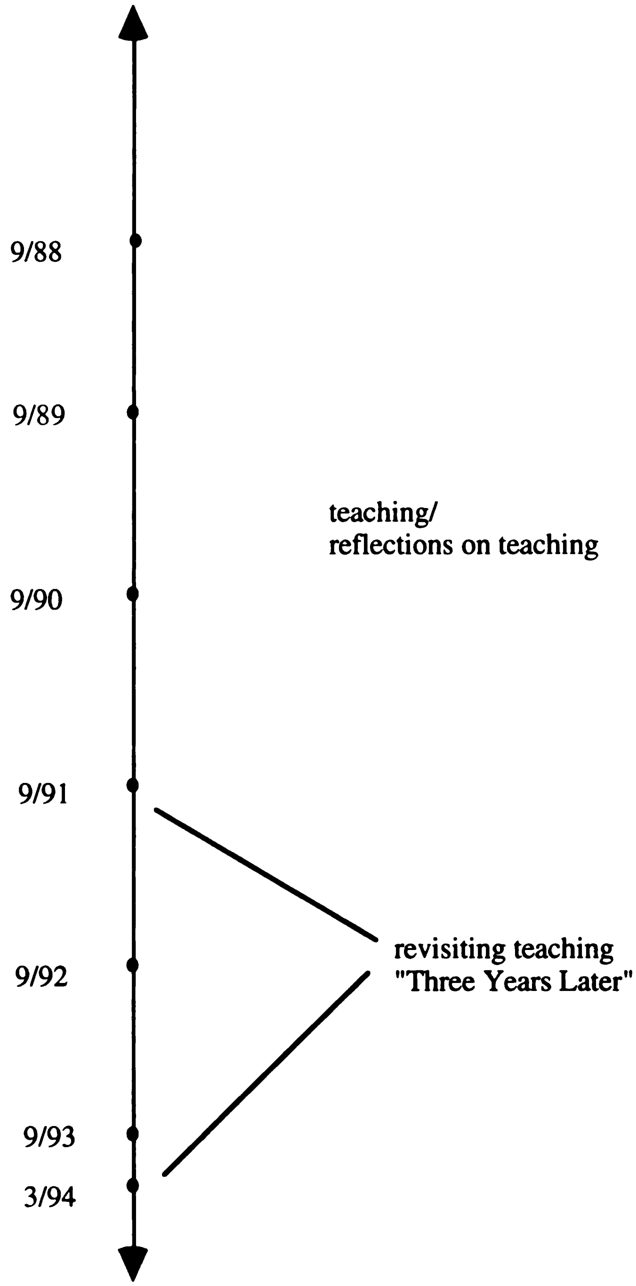

Figure 3.2

Timeline of my teaching, research, and learning

Implications of this temporal context for my analysis and writing can be seen represented in the structure of the teaching chapters, Chapters 4 through 7. Chapters 4 and 5 both contain accounts and analyses of teaching that are organized around four main sections: planning, teaching, reflections on teaching, and looking back on that teaching, from my perspective three years later. Chapters 6 and 7 contain accounts of teaching organized around two main sections: teaching and looking back on that teaching, from my perspective three years later. The difference in the design of these chapters, the absence of separate sections for planning and reflections on teaching, can be explained by looking at the teaching featured in each chapter. Chapters 4 and 5 are both about teaching events that occurred on single days. Chapters 6 and 7 are about multiple days of teaching where reflections on teaching blur readily into plans for the next lesson. In the text of Chapters 6 and 7, the days become dividers of time. What is common across all four chapters is the analysis from my perspective three years later.

This temporal structure helps to sort out when I learned what--what I understood or did not understand at a particular point in time. This is important given that this study aims to understand what teachers need to learn to make changes in their practice. Learning can be defined as moving from a point in time of not understanding to understanding. In other words, learning has occurred when I now understand something that at an earlier point in time I did not understand. It could also be said that learning needs to occur when I recognize there is something that I do not understand. In these situations, sometimes I know what I could learn to move myself from a point of not understanding to understanding. Other times, I do not. Given that learning is a dynamic process, at least two points of comparison are always necessary.

Working within a temporal context and representing what has been learned or needs to be learned in text requires attention to voice. This meant that who I thought I was, to whom I thought I was speaking, at what particular point in time, and how I made this identifiable to the reader became important. Connelly and Clandinin (1990) note, "Narrative writers frequently move forward and back several times in a single document as various threads are narrated" (p. 7). The next section is an attempt to sort out the issues of identity, audience, and tense which were raised by my efforts to understand and write about my own learning.

Issues of Voice

Most academic work is written in a voice that distances the work from the author through the use of rhetorical devices such as passive voice, past tense, and the third person. Eisner (1988) writes about the distant and impersonal nature of most academic writing.

> We write and talk in a voice void of any hint that there is a personal self behind the words we utter: the author, the subject, the researcher. . . we somehow multiply our individuality and write about what "we" found. All of these linguistic conventions are, paradoxically, rhetorical devices designed to persuade the reader that we, as individuals, have no signature to assign to our work. (p. 18)

What results, Eisner describes, is "the creation of a language of research that only researchers can understand and that, even for them, is difficult to use to reconstruct images of classroom life having semantic content" (p. 18). Those researchers who adhere to the rhetorical devices that have become norms in academic research tend to have their work accepted and judged as legitimate by the academic community. Those who choose to do other than the norm risk rejection and alienation. Traditional research methods carry power, status, and control within academia without consideration for the value of what can be learned from them.

In contrast to traditional academic work which tries to hide or disguise the presence of the author, my work embraces my signature and in doing so provides a vivid, personal portrait of teaching, learning, and learning to teach mathematics for understanding. Because I have chosen not to follow the norms of traditional academic research, it does not mean that my research was done without a method or that my writing has been done haphazardly. Creating consistent ways to analyze my learning and represent who I am in the text has been an important part of the work, critical to telling my story and studying my experiences. To accomplish this, one of the things I did was devise my own set of rhetorical devices to identify the multiple voices present in my work.

Multiple Selves

My dissertation is written in the first person. "I" is found throughout the text, in each of the chapters. But the "I" that is speaking in some places is not the same "I" as speaking in others. Connelly and Clandinin (1990) note the necessity and complexity of trying to "sort out which voice is the dominant one when we write " (p. 9). We all have multiple I's that represent "all of the ways each of us have of knowing" (p. 10). What makes the work of this study somewhat more complicated is that not only do the multiple selves at any given point in time matter, a particular self changes over time. Since the point of this dissertation is to understand my learning, the changes in self are important to analyze and represent. What this implies is that it is important to know which voice is talking, to whom, and when. The temporal context, as described above, provides a context for recognizing a particular self, of which there are many.

Ruth the researcher
Ruth the teacher educator
Ruth the scholar
Ruth the teacher

Ruth 1--the Ruth who did the teaching in 1989/90
Ruth 2--the Ruth who was learning about teaching in 1989/90
Ruth 3--the Ruth who is learning about teaching, 3 years later

Figure 3.3

Multiple selves

Locating myself in time and relating the immediate past and distance past, as well as the present, and future relative to given points in time is something I have struggled with throughout this text. A challenge has been to understand and try to communicate who I am in relationship to myself and others, over time.

I planted implicit and explicit clues within the text to help the reader understand which self is dominant at a particular point in time. For example, I alter how Deborah Ball and Magdalene Lampert are referred to throughout the text. At times, they are addressed in academic formality, by last names, in a more scholarly voice, as I would address any scholar who has influenced my thinking. Other times they are referred to more informally, by first names, as a teaching colleague or a mentor. Another variation in the text, related to the representation of voices within the text, is found within the various excerpts of transcripts from classroom interactions and conversations with Jim Reineke, a researcher who studied my teaching. In transcripts, I reference myself by the name that the people with whom I am interacting refer to me by. So, in transcripts representing classroom interactions, I am known as Ms. Heaton. In transcripts of conversations with Jim, I refer to myself as Ruth. Different voices dominate different sections and chapters of the text. In general this is true, however, I make no guarantees that you will not find multiple voices in the same paragraph or the same sentence. In some places I am explicit about which self is speaking by referring to myself "as a researcher" or "as a teacher." In other places, knowing the temporal context and the situation provides sufficient context for a reader to figure out which of my selves is speaking.

These issues around voice are not something I was able to figure out before writing text. The times I tried to define the multiple I's for a particular section in advance of writing did not work. I had to write text and then examine it in an effort to sort out the voices that were speaking. Over time, some patterns have emerged. For example, after many months of writing, I was working on Chapter 7, the fourth and last chapter on my teaching. In the midst of my struggle to create the text, I realized there was a pattern of voices in the teaching chapters and a relationship among voices across the teaching chapters and the conclusion.

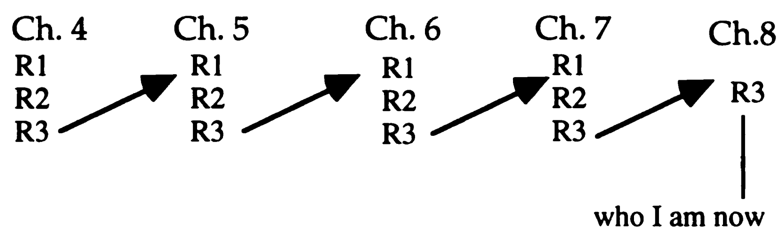

Figure 3.4

A pattern of voices on teaching

Figure 3.4 represents the pattern on teaching I noticed. Each chapter on my teaching contains at least three voices: Ruth 1, Ruth 2, and Ruth 3. As I explained in Figure 3.3, Ruth 1 is the teacher who taught fourth grade mathematics. Ruth 2 is the self who was making sense of teaching, then. Ruth 3 is the self who is making sense out of teaching, now. In each of the four chapters on my teaching, Ruth 1 narrates the story of teaching. Ruth 2 is the voice of the self reflecting or analyzing the teaching right after it happened, while Ruth 3 looks at the teaching from a perspective of three years later.

As the stories from my teaching move chronologically across the year, from Chapter 4 to Chapter 8, the voice of Ruth 3, while supposedly speaking in the here and now, switches back to Ruth 1, as the reader moves from one story to the next. Ruth 3 is the voice that carries with it the most risk because it is what defines what it is that I now understand as well as what I have left to learn. Ruth 1 and Ruth 2 are older voices. In a sense, I have outgrown them. Ruth 3 is closer to who I am right now, especially the Ruth 3 at the end of Chapter 7. Bruner (1990) noticed a similar pattern in writing about ones own experiences and explains how he sees time interacting with identity.

> There is something curious about autobiography. It is an account given by a narrator in the here and now about a protagonist bearing his name who existed in the there and then, the story terminating in the present when the protagonist fuses with the narrator. (p. 121)

Tense

Another rhetorical device I used throughout the text was the tense I chose to use at any given point in time, in any single section, paragraph, or part of a chapter. The stories from my teaching (the first half of each of the four chapters on my teaching) are primarily written in past tense. If in telling the story, I refer to a time prior to the moment, the lesson, or that year, I tended to use past perfect or a tense that represented one step further into the past from the present. I used present tense in the section written from my perspective, three years later.

While this is in general how the text appears, now, it is not necessarily the way I initially generated it. One of the rhetorical tricks I played on myself to just plunge into writing the story of a particular teaching episode was to write in present tense about a past event, what is formally called present historical tense. By the time I wrote about the teaching in Chapter 6, the patterns in the development of my use of different tenses was being used as a method of analysis. For example, I struggled to get the story started in Chapter 6. I did not want to write pages to lead into the story. I just wanted to get started because I had so much to tell, this being the first chapter that I had a series of lessons rather than a single day around which to construct a story. Initially, I wrote the whole first half of Chapter 6, a series of lessons about making labels for cans, in present tense. I made the decision to switch it all to past tense, once I was satisfied that I had communicated what I was trying to say, before doing the analysis from a perspective three years later. I think the images of my interactions with students are the most vivid in this story or at least some of the students, as characters, come alive.

Others' Voices

My understanding of reasons to use the voices of others' in my work has sharpened in the process of writing this dissertation. I have played around with metaphors, quotes, and references throughout the course of creating text. What I have come to see is that I have used other people's voices in the writing of my dissertation in at least two ways. One way is as tools for helping me to make sense of my own experiences and the other way is to communicate my ideas to others. This corresponds to the way I am beginning to define the writing process for myself. I think I have probably always viewed writing as a means of communicating my ideas to someone else. In the course of writing and rewriting the text of these chapters countless times, I have begun to see writing also as a tool for helping me to figure out what it is that I think. There is a writing I do for myself and a writing I do for others. The numerous drafts of the text of each of the chapters moved me from one to the other. Other's voices seem strongest in my work in earlier drafts. As my voice gets stronger and clearer on what it is that I want to say, the voices of others tend to fall into the background. This makes my work quite different from conventional academic writing which tends to value references and quotes of others at the expense of the author's own voice.

Data Collection

When I decided to make the year of teaching a study of my own learning to teach, I gathered together all of the documentation that had been done around my teaching that year. Jim Reineke, a graduate student in educational psychology, had been in my classroom to do a practicum of his own design. He studied students' and a teacher's changing conceptions of mathematical knowledge (Reineke, 1993). He videotaped my math class twice a week, audio taped all of the lessons, took field notes, and did informal tape recorded interviews from September through December.[1] Magdalene Lampert observed in my classroom several times a week. She took observation notes and responded throughout the year to a daily journal I kept on my teaching. Her reason for observing my math class was to help me learn to teach mathematics differently. By the end of the school year, I had collected the following data: the teaching journal I kept all year, Maggie's annotations on my teaching journal all year, Maggie's observation notes from observing my teaching twice a week from September through December, and a week in May, video tapes and audio tapes of my teaching between September and December, three weeks of video tapes of my teaching in April, an audio tape of a lesson in June, and audio tapes of

[1]See (Heaton, Reineke, and Frese, 1991) for a description and analysis of what these conversations were like for both of us.

interviews with Jim. In addition, I had talked into a tape recorder before and after my teaching, doing "think alouds," for three weeks of my teaching in the spring. The following is a list of the data collected and the quantity of each data type.

Table 3.1

Data Collected

DATA TYPE	QUANTITY
video tapes of lessons	22
audio tapes of lessons	43
audio tapes of interviews with Jim Reineke	19
my teaching journal entries	97 days represented
Maggie Lampert's annotations of my journal	97 days represented
Maggie Lampert's observation notes	22 days observed
think alouds	12

Glaser and Strauss (1967) refer to these as "slices of data" (p. 65). Different kinds of data provided me with different views of the same event. The perspectives offered by Jim and Maggie on my teaching allowed me to triangulate (Hammersley & Atkinson, 1989) my analysis. I could draw on the interpretations of others to challenge or affirm my own.

Choosing What Stories to Tell

Any teacher would have quite a few stories to tell from a year's worth of teaching experiences. Russell Baker (1987) noted the problem of having too much information when trying to write about your own experiences. "The biographer's problem is that he never knows enough. The autobiographer's problem is that he knows too much"(p. 49). My challenge was to figure out which stories to tell and how best to tell them. My concern was choosing stories to tell that were relevant to my question of what it was that I had to learn to make changes in my teaching. I had to tell the story in a way that would allow me to analyze it. To do this, I moved back and forth between constructing the story of my teaching, then, and analyzing it from a perspective three years later. How I thought about my organization of one influenced my work on the other. I went through a similar process on a larger scale as I moved back and forth between working on parts of this dissertation and conceiving of it as a whole. When choosing what stories to tell and how to analyze them, I considered their power individually and their richness collectively.

> Being autobiographical always forces us to select from the multitude of our minute-by-minute, day by day experiences, to give shape and order to our selections, and to convey to our audience some sense of why we have chosen to emphasize this particular aspects of our lives (Lyons, 1984, p. 5)

So, how did I choose which stories I would tell? I started by trying to get a picture of the whole year. I made multiple representations of the year that showed when I taught, what I taught, and noted anything that stood out to me for any reason for each day I taught from September to December and March to June. I taught an undergraduate teacher education course at the university between January and mid-March.

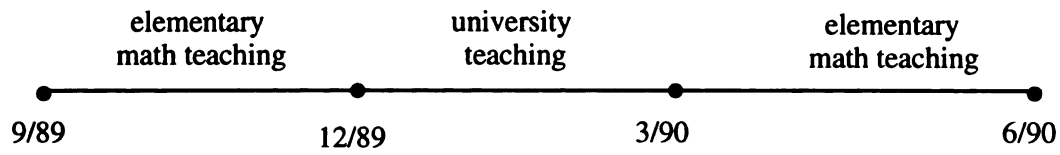

Figure 3.5

Overview of the school year

In addition to making overviews of the year, I read through three different data types--my journal, Maggie's observation notes, and Jim's interviews--to identify themes. A search for themes or categories is a common activity within the qualitative research tradition. I read through the data looking for particular words, phrases, patterns of behavior, ways I thought about things and events that repeated themselves or stood out for some reason (Bogdan & Bilken, 1982). I sorted these findings into like kinds as a way to look for regularities and patterns (Bogdan & Bilken, 1982).

One category I came up with was the textbook and ideas related to it throughout the data. For example, this included things I noticed in the data directly related to particular aspects of the textbook, like my understanding of particular representations. It also included my ideas at the time about what the textbook was or was not addressing as a mathematics curriculum. It contained my thoughts about making use of the teacher's guide and points at which I made conscious decisions not to make use of the teacher's guide, either to do things myself or draw on other resources. I also included my ideas related to knowledge, authority, and certainty at the time, and my understanding of these in relationship to the textbook. A second category encompassed what I seemed to be noticing about students. One student was featured quite a bit in my journal and throughout my conversations with Jim. This was Sipho, a student introduced in Chapter 6, in the context of making labels with cans. I included things I noticed about other students in this category as well. A third category had to do with the mathematics. This included my troubles with understanding it myself, my lack of clarity on mathematical purposes for what I was doing with students, and my students' understanding of the mathematics. A fourth category had to do with me or my role in teaching. What was it that I was supposed to be doing? This category linked to the others as I questioned my relationship to the textbook, to students, and to the mathematics. The textbook was the largest of categories and seemed to be a thread throughout the others as well. This led me to consider it as an overarching theme.

With the textbook as a prominent theme, I read back through the data and wrote an analytic memo about what I thought was going on with the textbook. I pulled from across data sources and the year . I noted particular days or events of interest and why they seemed relevant. From this memo, I tried to write several pages about what I thought a story of the textbook, based on my experiences, would be. Schatzman and Strauss (1973) write about the importance of pushing ones ideas into story form during data analysis even before you are entirely certain of the point of the story.

> We urge the novice in analysis to convert relatively inert abstractions into stories--even with plots--in order to induce themes and models that link datum to datum . . . This way, the analyst escapes the formal stereotype inherent in the concepts; he deals with very human and live phenomena that are amenable to story-making and probably productive to new constructs. The story line can always, later, be reconverted to formal terminology, should the analyst find it necessary. In the meantime, he deals comfortably and naturally with what appears only as description and illustration, but which is but a short distance, conceptually, from generalized, social process. (p. 121)

Trying to formulate the story of the textbook helped me to see my changing relationship with the textbook--what I thought I could expect from it, what I thought it could or would do for me. While my relationship with the textbook was changing, so were my relationships with students, the mathematics, and myself. Considering the story of the textbook in terms of a changing relationship helped me to see the other categories in terms of relationships, too.

From this I narrowed the data to particular dates of days and events of interest across the year which amounted to still more data than I could include. From these I chose the ones you see written about in Chapter 4 through Chapter 8. I decided the stories and my analysis of my learning ought to span the school year based on the assumption that there is some chronology to learning. As I contemplated choices, I decided to have the different teaching events feature different lengths of time: a day, a moment, a series of days, a series of days followed by a six week break and then the second to last day of math class.

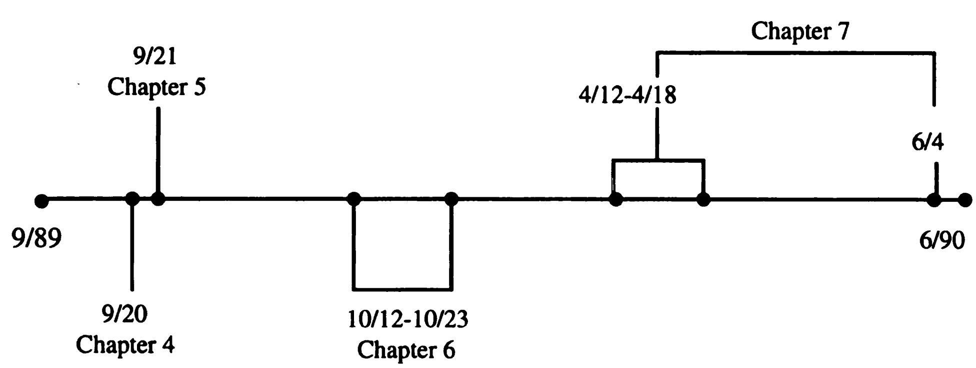

Figure 3.6

The events I chose to write about and where they fall in the school year

For example, Chapter 4 is about my frustrations in the early weeks of school and the examination of what I am learning is based one lesson. What I am learning in Chapter 5 comes from examining a single moment in my teaching, the first moment that I felt like I was teaching that year. Chapter 6 spans a series of five math lessons or eleven calendar days, a series of lessons improvised from a one day lesson in the math textbook. Chapter 7 is about four days in April followed by a six week interlude and concludes with the second to the last day of my teaching in June, a series of lessons in which I struggled with the question of what I should do next.

Together, these chapters offer a rich picture of teaching and learning to teach from a fine grained look at what I am learning in a single moment to a detailed look at a series of lessons that span almost a two month period. In spite of the difference in scale or size of teaching being featured, each unit of analysis has been treated equally. Each chapter is approximately 50 pages long with the first half devoted to the story of teaching and the second half to my analysis from a perspective three years later. In each chapter what gets featured changes in much the same way that one can buy different maps, all the same size in terms of the paper on which they have been printed, but each one enables a view of different things. Consider for example what can be learned from a world map, a state map, a county map, or a detailed city map. In each of four chapters on my teaching, different features are made prominent and each offers a different perspective. Together, they offer a rich view of the terrain of what an experienced teacher needs to learn to make fundamental changes in mathematics teaching.

date--day of week	my journal (with Maggie's comments)	video tape	audio tape	interview with Jim	Maggie's observation notes
Chapter 4					
9/20/89--W	X	X	X	X	X
Chapter 5					
9/21/89--Th	X		X		X
Chapter 6					
10/12/89--Th	X		X		
10/16/89--M	X	X	X	X	X
10/17/89--T	X		X		
10/18/89--W	X	X	X	X	
10/23/89--M	X		X	X	
Chapter 7				think alouds	
4/12/90--Th	X	X		X	
4/16/90-M	X	X		X	
4/17/90-T	X	X		X	
4/18/90-W	X	X		X	
6/4/90-M	X		X		

Figure 3.7

Available data for the events I chose to write about

I transcribed tapes and annotated transcripts of all lessons, myself, and Jim and I shared the task of transcribing the tapes of our conversations. Listening to myself on tape while doing the transcriptions was enormously helpful in becoming reconnected with my teaching after a long time away from it. What was particularly helpful was to hear the tone and inflection of my voice. It helped me to recollect the events as well as some of the emotions associated with them. Maggie had commented on my daily reflections on teaching in a copy of my journal over the course of the school year. For purposes of analysis, it was useful to have two copies of my journal, the original with only my reflections and a copy with Maggie's comments. I could choose whether I wanted to work with Maggie's comments as well as mine. I could easily just focus on one or the other. I was glad to have the option. Maggie's comments on my journal were separate from her observation notes. She gave me the originals of those when the school year was over.

The Use of Metaphors

While I was doing this narrowing down of data, I was also reading. During this time is when I came across pieces of literature that I thought contained powerful metaphors for what it was that I was learning in my efforts to make fundamental changes in the way I taught elementary mathematics. The thing that excited me about particular metaphors was that they were drawn from texts far afield from educational research yet seemed so connected and relevant. This seemed to me to be useful in creating new categories, images, and language for thinking about what is demanded of teachers who are trying to make fundamental changes in their mathematics teaching practice. Metaphors are useful tools in narrative inquiry (Connelly & Clandinin, 1990). Abrahams (1972), a folklorist, defines metaphor as,

> A product of man's desire to make a whole out of parts, especially when those parts have the appearance of discontinuity and therefore potential chaos. Metaphor by definition does exactly this, bringing together things from different and apparently irreconcilable realms, in such a way that we are forced to recognize their essential sameness. Metaphor is a special kind of stylization, for it artificially accentuates certain congruent features of things being compared. (p. 79)

Glaser and Strauss (1967) note that, "the search for comparisons, involving the discovery of useful comparison groups, was essential to the generation of theory" (p. 170). The metaphors I have used have been invaluable tools in helping me understand and communicate my ideas to others. Lakoff and Johnson (1980) write,

> A large part of self-understanding is the search for appropriate personal metaphors that make sense of our lives. Self-understanding requires

> unending negotiation and renegotiation of the meaning of your experiences to yourself. (p. 233)

Choosing Metaphors

The metaphors I eventually used as tools of my analysis were chosen from a wide variety of reading I did in the course of doing this study. Glaser and Strauss (1967), in writing about qualitative methods, recommend looking for and reading anything you think bears on the areas you are studying. Bogdan and Bilken (1982) write about the importance of reading widely to see "how authors present data, how they construct their arguments, how they arrange their sentences, how they organize their format" (p. 182). I read widely in fields that seemed relevant, fields both inside and outside of traditional academic research.

I was curious about other's ideas on methods of representing experience and others' written representations of experience. For considering ways of representing experience, I read the work of people who have thought extensively about narrative and narrative ways of knowing within educational research (Bruner, 1985, 1990; Carter, 1993; Connelly & Clandinin, 1990). I also read about the tradition of autobiography (Buckley, 1984; Rosen; 1988) and the challenges of writing about one's own experiences (Lyons, 1984; Zinsser, 1987).

I also read many representations of experience. In addition to the autobiographies of teachers, some of which I listed in Chapter 1, I read autobiographical classics like, *The Education of Henry Adams* (Adams, 1971/1918), *The Narrative of the Life of Frederick Douglass*, (Douglass, 1968/1845), and *The Souls of Black Folks* (Du Bois, 1990/1903), as well as two different collections of autobiographies of American women (Conway, 1992; Culley, 1992). These were all different and complementary examples of people trying to make sense of their own lives.

I also read, specifically, to examine ways people have represented experiences: autobiographical accounts of people learning specific practices or writing about the practices of others. The nature of the experiences varied greatly and I pushed myself, when it seemed reasonable, to draw comparisons and contrasts to teaching. For example, I read about learning to do the work of artists (Halpern, 1988), writers (Plimpton, 1984; Steinbeck, 1990; Sternberg, 1991), tennis players (Gallwey, 1974), skiers (Gallwey, 1977), automobile mechanics (Harper, 1987), the mother of a handicapped child (Featherstone, 1980), riverboat pilots (Merrick, 1987; Twain, 1990/1883), jazz musicians (Dean, 1989; Mack 1970; Mehegan, 1959; Stormen, 1955; Sudnow, 1978), and improvisational dancers (Blom & Chaplin, 1988; Novack, 1990). Many of these works offered me comparative and contrasting images of a practice or learning a practice that were useful in thinking about teaching and what it was that I was learning to do. From this array of representations of experience, I chose three metaphors which captivated my interest and around which I focused the analysis of my learning.

One was Twain's (1990/1883) account of learning to be a riverboat pilot, another was Sudnow's (1978) work on learning to play improvisational jazz, and a third was Novack's (1990) and Blom's and Chaplin's (1988) work on learning to do improvisational dance. Learning to pilot a riverboat, improvise jazz, or do improvisational dance with others are all on the surface very different activities. What I have found is that the knowledge and learning demanded by the participants in each situation strongly parallels and highlights different aspects of the knowledge and learning demanded of me in learning to teach mathematics for understanding.

Finding examples of other people writing about their own learning helped me to think about how it was that I wanted to write about my own learning--both what I wanted to say and how I wanted to say it. The strength of these pieces of text is that they are autobiographical accounts of learning and the nature of the work these authors are trying to learn to do has notable connections to the kind of teaching I was aiming to learn to do. "We need a language that will permit us to talk about ourselves in situations and that will also let us tell stories of our experiences" (Lakoff and Johnson, 1980, p. 59). These metaphors have given me images and a language to understand what I was learning and communicate my understandings to others. No single metaphor captures all that I have learned about what is entailed in learning to teach mathematics differently. No single representation is sufficient. Taken together, these three metaphors offer a rich picture of the complexity of the work and demands of learning to do the work of teaching mathematics for understanding.

Learning to Navigate the River

The first metaphor I found comes from Mark Twain's (1883/1990) experiences learning to be a riverboat on the Mississippi River. Twain's experiences learning to navigate parallel my beginning adventures making changes in my practice. Naive preconceptions of the work, continual struggles with learning to do the work, and insight into what one needs to know to be able to navigate are themes found throughout Twain's story as well as my own. Twain writes,

> I supposed that all a pilot had to do was to keep his boat in the river, and I did not consider that that could be much of a trick, since it was so wide. (p. 38)

It did not take Mark Twain long to realize that learning to be a riverboat pilot was not merely a matter of learning to keep the boat afloat on what looked from the shore to be a predictably wide body of water. On the Mississippi River, from behind the wheel of the boat, Twain came to have a different understanding of the work. He saw complexities in navigating the river and learning to be a riverboat pilot that he never envisioned from the shore. The ever-changing river and the sort of understanding needed to navigate it required his continual attention. I was intrigued by Twain's conception of what it meant to pilot a riverboat prior to trying it himself. His simple view of the work from the shore seemed comparable to the simplicity and ease with which the idea of teaching and learning to teach mathematics for understanding is written and talked about through the eyes of many advocates of the current reforms in mathematics education. Twain's learning experiences from behind the wheel of the riverboat quickly altered his perceptions of the work and his role.

What does learning to pilot a riverboat have to do with my learning to teach mathematics for understanding? How does the work of navigating a riverboat parallel the work of teaching mathematics for understanding on a daily basis with fourth graders? What connections can be drawn between what is demanded of Twain in learning to navigate the river and what it is that I am learning is demanded of me in my efforts to teach mathematics for understanding? In mathematics teaching, what is the river? What is the boat? What is the shore? What is it that I am learning to navigate? What does it mean for Twain to learn the river? What is the nature of the knowledge he is trying to learn to navigate? These are the many of the questions I consider as I draw connections between Twain's experiences and my own in two chapters on my teaching, Chapter 4 and Chapter 7.

Learning to Improvise

In Chapter 5, the second chapter on my teaching, I move away from Twain and make use of a metaphor generated from Sudnow's (1978) detailed account of learning to improvise jazz. What I seemed to be learning in this episode of teaching had something to do with understanding what it was like to be connected to the work of teaching. To create an image of what it feels like to

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teach mathematics for understanding and what it might mean to learn to be at a point in teaching where you have the "feel" of it, I draw on David Sudnow's account of his own learning to play improvisational jazz. The parallel I find in his story and mine has to do with what we are both learning about what is demanded in improvisation--an acute sense of hearing, ability to act in the moment, and a repertoire of possible pathways to take depending on the music that came before and one's sense for where one is headed.

Learning to Share the Dance

In Chapter 6, the third chapter on my teaching, the emphasis moves from a focus on the discovery of a moment of improvisation in teaching to trying to understand the interactions that occur within such moments and what must be learned to construct and maintain or sustain constructive interactions. Several books on improvisational dance Novack (1990) and Blom and Chaplin (1988) have been helpful in pushing my thinking about what teacher-student and student-student interactions entail and what is required of the participants. In the context of the image of improvisational dance, the sensory nature of teaching is vivid, the constant need to get a "feel" for the situation is prominent, and the need of the participants to be responsive is ever present. And, as in jazz, decisions about what to do next are made in the moment of movement.

Shifting the Metaphors

My use of these metaphors throughout the text was much more prominent in earlier drafts than what you see in the following chapters. As I contemplated lightening their use and making their connection to my work become explicitly part of what I learned, I began to see a distinction between using metaphors as tools for my understanding and how that might be different than using them as tools to communicate my understandings to someone else. Shifting the metaphors from their use throughout the entire text of a chapter to the second half, to become part of what I had learned from a perspective three years later, strengthened my own voice throughout the entire chapter. Connelly and Clandinin (1988) offer some insight into what happened as I recognized metaphors as a tool in my own learning. Shifting the metaphors from dominating my work to a much quieter place in the background of my voice makes sense. Using metaphors, Connelly and Clandinin predict, "will probably lead you back to your images, principles, and rules and may give you new insight into your personal philosophy" (p. 75). The metaphors played a major role in helping me figure out what I thought and a lesser role, over time, in communicating my ideas to others. The role of metaphors throughout this text is another example of using and altering the role of other people's voices as a means of strengthening my own.

A Study of a Moment of Teaching

This section is about the difficulties, challenges, and choices I confronted in doing analysis--writing the chronologies, constructing the narratives, and figuring out what I was learning by using metaphors--in each of the four chapters on my teaching. Rather than describe the writing of each of the chapters, I tell of my experiences with one of those chapters, Chapter 5, what I identified as the first moment of teaching. What this story also shows is how a major source of knowledge about my method came from my own intuitions about when to do what and what to do next. Hammersley and Atkinson (1989) call this "common sense knowledge" (p. 17), while Glaser and Strauss (1967) refer to insight as a source of theory, "The root sources of all significant theorizing is the sensitive insights of the observer himself" (p. 251). Over time, mainly through checking out the reasonableness of my ideas with others, I learned to trust my own insights and intuitions.

<u>What is the Point of the Story?</u>

Nearly three weeks into school and there had been really only one felt moment of teaching. It seemed worth trying to understand why this moment on September 21 was different from the rest. In my memory, I had recollected the moment as an "aha" in teaching. Maggie's observation notes from that day and a tape recorded conversation with Jim a few days later confirmed my recollections. There was something notable about this moment in teaching. It seemed particularly noteworthy after the frustrations of the day before, September 20, which I had written about in the previous chapter, Chapter 4. I thought focusing on the next day, which had been as good as the day before had been bad, would provide an interesting contrast to the day before and a useful site for examining my learning. I proceeded to transcribe the tape. I made use of Maggie's observation notes to guide me to the interaction where this moment began. I listened to it over and over again while making notes on the transcript.

It struck me that the feeling in this moment of being connected to teaching in a way that I had not previously experienced was similar to what I had read about in Sudnow's (1978) book about his experiences learning to improvise jazz. I made the decision to try to use Sudnow's experiences to frame a story of my own around this moment in teaching. In one of my earliest drafts, I began with a quote from Sudnow in which he talks about what it felt like to improvise. My plan was to draw parallels to Sudnow's experience by writing about what that moment felt like to me when things came together in teaching. I went to my journal as this was the best source of data for how I was feeling at the time. It was often what I wrote about, particularly in those first few weeks of school.

In looking back over my teaching journal from this day of teaching, it became clear to me that the way I was holding this moment in my memory did not quite fit the way it had occurred. The moment was mine, all right, but the initial recognition of it was not. In my journal I saw my musings about Maggie's question from her observation notes about whether or not at a particular moment I had gotten some "bright idea." I also saw my own notes from listening to the audio tape of the lesson. Given the placement of my notes in my journal, I assume this was something I did before the next day's math class. What all of this meant to me was that this moment was something Maggie noticed at the time, not me. I was deeply disappointed. How could I write about this as a moment of improvisation? It was not anything I recognized at the time. With this new discovery, the story I thought was here had just slipped away. The events were still there but the point was going. Carter (1993) notes the phenomenon of constructing stories from events,

> Stories are not merely raw data from which to construct interpretations but products of a fundamentally interpretive process that is shaped by the moralistic impulses of the author and many narrative forces and requirements. (p. 9)

If I were going to try to make use of this day of teaching, I needed to create a new plot. After many conversations with Maggie, I decided that I could still write about this moment. It could still be my moment, an "aha" in my teaching, even if she had been the one to help me see it. The point was that in this moment, regardless of who noticed it, I was connected to teaching in ways I had not been before.

A part of me was still disappointed because Sudnow's experiences, when I thought this was a moment I had noticed, seemed so perfect for explaining what was happening to me. With this new discovery of the events, I worried that Sudnow might not be so useful in helping me understand what was happening.

What is My Relationship to the Metaphor?

Before giving up on drawing connections to Sudnow's experiences, I decided to go back and reread the text. Maybe the "aha" was that I was now able to see teaching as improvisation, thus making an understanding of the metaphor's relationship to teaching part of what I was learning. Improvisation was not a way that I would have described teaching back then. It was a way I could see to make sense of it, now.

In rereading Sudnow's book, I tried to leave myself open to understanding his experiences differently in relation to mine now that I had a new sense of my experiences. Originally, I had tried to see myself as Sudnow, the learner. This time, I made a connection between myself and Sudnow's teacher. I came across a passage where Sudnow recollected experiences watching his instructor play jazz. Sudnow wrote,

> I would ask, 'what was that?' He would say 'what was what?' I said 'that little thing you just did over the G minor chord there.' Now a characteristic 'trouble' occurred, whose significance I did not appreciate at the time and for several years. I would say, 'that little thing you just did on the G minor chord,' and he would have a hard time finding what he had 'just done.' He would at times frankly say, 'I'm not following rules so I don't really know what I just did' (and on other occasions admit, 'I just improvise, I really cannot tell you how, you have to have a feel for it'). (p. 25)

The fact that I had not noticed this moment when it occurred may actually be evidence to strengthen my claim that what I experienced was an improvised moment in teaching. What the discovery did was reshape how I thought about using the metaphor. In this moment, it made more sense to connect myself with Sudnow's teacher, someone who knew how to improvise, rather than Sudnow, the learner. For a moment, like Sudnow's teacher, I had mastered the art.

Slowly, I was constructing the story I would tell and the analysis I would do on my learning. The point was to tell the story in a way that made this moment look different than the others but not identify it as a moment of improvisation until the second half of the chapter. The analysis would explain the moment by trying to understand what it was that Maggie might have noticed to see this as a moment of improvisation. In doing so, I would draw on images and language associated with learning to improvise jazz.

What am I Learning?

If what was happening was that I was learning what improvisation meant in teaching, what was it that I was learning? I did several things to help myself with the analysis. One thing I did was to learn more about what was going on in this lesson mathematically. How did the activity I was doing with students relate to some bigger mathematical ideas? In the previous chapter, I had figured out that having a purpose in mind was important. What was the purpose here? Another thing I did was read other books about learning to play improvisational jazz. Doing this did two things. One is that it helped me to understand Sudnow's experiences better. Another thing it did was help me understand my teaching. Once I saw that the connection between improvisational jazz and teaching was reasonable, reading books about jazz was really like reading books about teaching. Through this, I gained some insight into what it was that I was learning when I was able to see improvisation as an important part of teaching mathematics for understanding.

I used the making of a table of contents for each chapter as an analytic tool. For example, for this chapter, I tried to phrase what I was learning using language from images of improvisational jazz.

Figure 3.8

Table of contents from Chapter 5

This turned out to be an incredibly helpful device for focusing my analysis and providing a vivid image for what it was that I was learning. As with the other tools I have used in my method, this way of using a table contents is not something I had defined prior to writing text. It developed out of many drafts of text in which I tried to express my ideas. In turn, the tool then became a device for helping me to express my ideas to others.

Does My Analysis Make Sense?

My test for whether or not I was successful in communicating my ideas to others came through the multiple readings that I had various people do of the text. As far as the story of my teaching, it mattered to me that people who read it could understand what had occurred and that my description of what had occurred made sense to them. Connelly and Clandinin (1990) note that, "A plausible account is one that tends to ring true" (p. 8). As far as a check for the analysis, Schatzman and Strauss (1973) discuss something called "phenomenon recognition" (p.135). Is the reader of the text able to understand with new clarity what the researcher has analyzed? "Does the researcher's analysis, which was probably based upon a different perspective or framework from theirs, actually help the audience explain--albeit in a new way--their own experiences?" (p. 135). For anyone who has read excerpts of my work, the images offered by the metaphors and the connections drawn to my experience offer new and rich images and understandings of what is entailed in learning to teach mathematics for understanding differently.

Continued Learning

There are at least two ways I can see to extend this study. One would be to revisit this data in another few years. I have no doubts that I would be able to learn things from my teaching that I have been unable to see up to now. Connelly and Clandinin note (1990), "the narrative insights of today are the chronological events of tomorrow" (p. 9).

Another way to extend this study is to step back and look at the intersection of my method of inquiry and what I am learning. By this I mean, what have I been learning about learning to teach, as a teacher, through the method of studying my own teaching? And, how has my work as a researcher been transformed through this study of teaching? In a collection of essays by writers on artists (Halpern, 1988), Aldous Huxley made the following observations about the layers of insight one can glean from an artist's work as he wrote about the artwork of El Greco,

> A picture always expresses more than is implicit in its subject. Every painter who tells a story, tells it in his own manner, and that manner tells another story superimposed, as it were, upon the first. (p. 62)

My dissertation is about unpacking layers of learning from my own experience. While I have unearthed more than I ever thought possible, there is still much left to be learned.

Reflections on Learning about Learning to Teach

Admittedly, there is a certain fragility to a method of studying one's own practice. One danger with such self reports is that they are not always valid. This is where it is helpful that there were others observing in my classroom and I have documentation of what they observed.

Choosing to analyze my own learning as a teacher is about as close to or inside the phenomenon of learning to teach as one can get--a study of learning to teach from a learning teacher's perspective. It offers a different perspective than what someone, other than me, might see who tried to examine my learning and what I could see or interpret in another teacher's experiences. My study of learning to teach differently offers a perspective on experience rarely found in educational scholarship--learning from the learner's perspective. This work represents a historical study of learning from the learner's point of view.

The Purpose

I considered the appropriateness of this study given my question: what is it that I--an experienced and successful elementary school teacher--have had to learn to make fundamental changes in my mathematics teaching? More broadly, I was asking: What do teachers need to know, care about and be disposed to do to teach mathematics for understanding? In the context of the mathematics reforms, we have little understanding of how it is that teachers acquire the knowledge needed to teach and use it in practice, especially experienced teachers trying to make changes in practice. Since this is a relatively new field of inquiry, exploratory research is needed. The aim of this study is try to describe and understand my process of learning to teach with primary attention to the nature of what I was learning. The usefulness of such a study stems from the concepts, categories, hypotheses, and language it generates to think about teacher learning (Glaser & Strauss, 1967; Schatzman & Strauss, 1973). It is not a study intended to verify or test hypotheses and make broad generalizations.

Reconsidering the role of the different voices in the text is useful. What the community of scholars and reformers can learn from Ruth 1 and Ruth 2 is about the experience of changing teaching from the inside. What they can learn from Ruth 3 is how to think about that experience in relation to teacher education and reform efforts. In other words, Ruth 3 constructs the "theory" and "questions" that studies such as mine, as noted by Glaser and Strauss (1967) and Schatzman and Strauss (1973), are supposed to generate. What these multiple selves allow for in this study is theory and questions about teaching generated in a rigorous way by an insider.

As in medical research, when charting new territory, the value of the work does not come only from the questions answered but from the new questions that get asked because of it.

> When generation of theory is the aim, however, one is constantly alert to emergent perspectives that will change and help develop his theory. These perspectives can easily occur even on the final day of study or when the manuscript is reviewed...so the published word is not the final one, but only a pause in the never-ending process of generating theory. When verification is the main aim, publication of the study tends to give readers the impression that this is the last word. (Glaser & Strauss, 1967, p. 40)

My work is not the last word on anything. In fact, I think just the opposite. My hope is that my work will initiate conversations among educators from a variety of perspectives at various levels who care about reforming mathematics education.

The Risks

There are personal risks to the task of analyzing my own learning. What it involves is recognizing and admitting what it is that I don't know. To talk about my own learning, I have had to admit that there were times when there were things which I did not understand. After seeing myself and having others view me as a good teacher, this has been extremely difficult to bring myself to do. It leaves me in a vulnerable position. For in analyzing what I have learned or need to learn, I am forced to expose and examine my own weaknesses and then make them public. This is a practice foreign to teachers and academicians or most people, for that matter.

Why have I subjected myself to this vulnerability? One risk studying my own practice avoids is the danger of being accused, as a researcher, of being a "teacher basher." This descriptor is directed at researchers who write descriptions of teaching for the purpose of understanding practice, but whose accounts of experience come across to some as evaluative and prescriptive, insensitive to the complex set of demands placed on teachers. Some complaints seem justified. At times in an effort to explain what is understood about classrooms, researchers do write about the practice of particular teachers in ways that seem highly critical, somewhat offensive, and bordering on disrespect. Carter (1993), an educational researcher disturbed by some researchers' accounts of teaching, wrote,

> In most conventional stories told about teachers, the narrator, however invisible, assumes a superior, more knowing attitude toward the characters. It is the narrator who has access to the relevant literatures, who frames the study, who provides the interpretations, and who modulates the teacher's voice. (p. 9)

Having written about another teacher's practice (Heaton, 1992), I know the dilemma faced by researchers trying to construct a narrative of a teacher's ex

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experience. It is a challenge to know how to manage the examination of teaching in ways that respects the practitioner and the ideas being studied while also pushing at the status quo of theory and practice.

While its true that in this study I can escape the accusation of teacher basher, what I risk doing is being extremely hard on myself. This is a danger and something I have had to deal with throughout this entire study. The strength of this study is letting people see who I am as a learner, my process of learning, what it is I know and what it is that I have yet to learn. In moments of doubt, I have had to remind myself this is worth doing.

Eisner (1988, 1993) and Barone (1992) are two educational researchers who have recently encouraged people to push at the boundaries and question the traditions of educational research. Krieger (1991) is someone who has struggled with the relationship between social science and the self. Cochran-Smith and Lytle (1990) are encouraging teachers to generate their own knowledge, to speak for themselves. (Berger, 1990) put together a collection of essays by established sociologists for the purpose of discussing the connections between their personal and professional lives. Berger wrote the following in the introduction to the collection of essays,

> It was obvious to me that asking sociologists to write about their lives constituted a substantial departure from the standard practices of academic writing, which constrain sociologists to keep themselves as decently or discreetly invisible as possible. Outside the entertainment pages, narcissism does not have good press, particularly in those fields whose "discipline" recognizes the logical irrelevance of the personal qualities of the author to the objective qualities of the work, and a radical separation between them. One of the aims of this book is to narrow that gap. (p. xv)

In education, I would argue, and in my work, as an example, the personal qualities of a teacher or a researcher are deeply intertwined with the objective qualities of the work of learning to teach or doing research on teaching. What my dissertation does, in both its content and its method, is make visible these personal and intellectual connections. Eisner (1993) warns, "Working at the edge of incompetence takes courage" (p. 10). I would add that writing a dissertation about being on the edge of incompetence takes a special kind of courage.

PART II

THE CREATION AND STUDY OF A PRACTICE

PROLOGUE

CONTEXT AND TOOLS FOR CREATING A PRACTICE

Before continuing with the story and study of my practice, it is important that I explain a bit about the context in which I attempted changes in my practice and tools I used to facilitate that process. The background necessary to understanding my practice includes information on the kind of school in which I worked, the sorts of students with whom I worked, and the observers in my math class that year. The tools I used to create my practice include the mathematics curriculum with which I began the year and the teaching journal I kept all year. An explanation of the curriculum, the Comprehensive School Mathematics Program (CSMP) (McREL, 1986), and the process I went through to keep a journal are included here.

The Context

My teaching, like all teaching, was situated in an environment with certain features. To understand my story and study of learning to teach, it is important that I explain the sort of school I worked in, the kind of students with whom I worked, and the interests and involvement of the people who observed my math class on a regular basis.

A Professional Development School

I attempted to make changes in my practice within a supportive environment. The public elementary school in which I taught was one of many Professional Development Schools (Holmes Group, 1990) working closely with Michigan State University faculty and graduate and undergraduate students. The school and classroom provided a setting where norms and routines were established for what it might mean for someone from Michigan State University to teach mathematics in someone's classroom. Magdalene Lampert and Deborah Ball[1] had been teaching math in other teachers' classrooms for a number of years. This meant that classroom teachers in this school had past experiences sharing their classrooms and students with university people and no one looked at me too oddly when I inquired whether I could teach mathematics to fourth graders for the purpose of trying to make changes in my teaching. The teachers, as a whole, in this school were committed and thoughtful practitioners. Many of them had spent a number of years considering what it might mean to teach mathematics in more conceptual ways. Many had new visions of what mathematics teaching could be and some were even trying to enact these visions in their classrooms.

I was pleased to be entering such a receptive environment. Trying to teach mathematics in new ways within this school meant that I did not have to try to convince people that the way I wanted to try to teach mathematics was a good idea. What this setting allowed me to do was start right away with trying to teach in new ways. I was not faced with the work of having to argue people that the ideas I had about mathematics teaching were worth trying. Maggie and Deborah had already paved the way for me in this context. Given my decision to work in a context where people were already somewhat familiar with the current reforms in mathematics education, mine is not a study of learning to convince people of the value of thinking about mathematics teaching differently. There is much to be learned about what that work entails but given the context in which I was working, that is not what I am doing here. My choice to work in this school

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[1] My choice of what to call Magdalene Lampert and Deborah Ball throughout my dissertation varies. At times, in referring to their scholarly work, I refer to them by their surnames, as I do any other scholars. In this portion of the text, however, they are my teaching colleagues and I refer to them, initially, by first and last names. I later refer to them by just their first names. The switch is intentional and conveys the complexity of our relationship.

allowed me to jump right into teaching, to focus my study of learning to teach on the day to day work of teaching fourth graders mathematics in meaningful ways within a supportive environment.

The fourth grade teacher in whose classroom I taught treated me as a peer and colleague. We were both experienced teachers, she with a few more years of experience than me.[2] She knew from the onset that I was going to be attempting to make changes in my practice. I negotiated my teaching schedule--four days a week, Monday through Thursday, a 60 minute class period right before lunch. I made the choice to attempt changes in my practice while teaching only mathematics. The alternative would have been to teach all or multiple subjects. Given my other responsibilities as a doctoral student and research assistant, this was all of the time I felt like I could responsibly commit to elementary teaching. When I made this decision, I realized that my experiences teaching only math would not resemble the curricular demands placed upon most elementary teachers. Given these conditions, what my study ends up being is a best case scenario--a case of an experienced, successful elementary teacher attempting changes in mathematics teaching within a supportive school environment without other teaching responsibilities. Even under conditions that I think most elementary teachers would call ideal, trying to make changes in my practice proved to be hard.

While meeting my university obligations of teaching an undergraduate course during winter term, I was able to "back away" from my fall term teaching experiences and take stock of them. This too, I realize, is a luxury that does not come with the usual responsibilities of an elementary teacher.

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[2]See *Collective Reflection: An Account of Collaborative Research* (Heaton, Reineke, and Frese, 1991) for how we defined our goals, jointly and separately, in our work together.

Throughout the year, I had weekly meetings with the classroom teacher. During those meetings, we focused on the progress of individual students, the pace of the group as a whole, as well as my plans for the coming week. These were important conversations for me in that this teacher brought a wealth of information about the children as learners in contexts outside of math class. I also benefited from her years of experience with fourth graders and the Comprehensive School Mathematics Program (CSMP). In addition to our planned meetings, we had informal conversations at the end of each class period in which each of us shared our own observations about what had just occurred. Even though we agreed she did not have to stay in the room while I taught math, she opted to observe every day. She also video taped my teaching, at my request, for three weeks during the spring term.

My Students

The students in the school in which I taught come from families where one or both parents are university students and have moved to Michigan from places all over the United States and the world to attend the university. Many different nationalities and first languages are represented by the students in the school. It is common for students to leave or new students to arrive in the course of the school year as their parents complete or begin their studies at the university. For most of the year I taught, I had 23 students . For a short while, there were 25. Of these 25 students, 12 were English speaking, United States citizens. The other 13 were from various countries around the world. Five of these 13 foreign students were new to the school the year I taught and learned to speak English in the course of the school year.

In the figure below, the students (by pseudonym) are listed by characteristics of their diversity: gender, the language they spoke at the start of the school year, and whether they were United States citizens or international students.

	International students who started in my fourth grade class as non-English speakers	International students who started in my fourth grade class as English speakers	United States Citizens
females	Fenice Katrina Olivia	Ana Jamilia Pili Wu Lee	Jennifer Valerie Lucy
males	Adim Jang Gee	Asim Arif Faruq Naruj Sipho	Bob David John Luke Mike Ron Richard Scott

Figure P.1

My fourth grade students by gender, birth place, and language capabilities

Working with this group of students allowed me to see what it is like to try to make changes with a diverse group of students with a range of ability, similar to what would be found in many classrooms. The fact that five students started the school year not speaking English provided me with experiences in helping students for whom English is not their first language communicate about mathematical ideas.

A Variety of Observers

There were several regular observers in my math class. My purpose in describing their participation is not to imply causal links between their presence and my learning to teach. Rather, I explain their involvement in my teaching as a means of further understanding features of the context in which I attempted changes in my practice. Having other perspectives on my teaching benefited me intellectually as I tried to understand what it meant to make changes in my practice, figure out what was happening in class, and make decisions about what to do next.

My teacher. As my feelings of dissatisfaction over my own teaching practice grew in my first year as a doctoral student, I introduced myself to Maggie and told her of my desire to try to teach mathematics differently. From January to June, in the year before I took on the task of teaching fourth-grade mathematics, I met weekly with Maggie to view her math teaching on video tape or observe in her elementary math class. We used this as an occasion to talk about her teaching and how it compared and contrasted to my past practice. In these conversations, I tried to make sense of what I saw Maggie doing and construct an image of the mathematics teaching I wanted to try to do.

Maggie put me in touch with a teacher who she thought would allow me teach her students mathematics. Kindly, the teacher agreed. I arranged for Maggie to observe in my classroom twice a week. I wanted her to observe so we could then talk about my teaching from a shared experience. She took observation notes that she shared with me each time she observed. In these notes, she raised questions about my actions and intentions and pushed me to think hard about the mathematics I was trying to teach and what my students seemed to be understanding. We discussed her notes weekly.[3] Maggie's notes were filled with different voices. Her notes represented her understandings as a scholar as she directed me toward related readings. They represented her work as a teacher educator as she offered her perspective on the difficulties she could see I was having with pedagogy and the mathematics. They represented her understanding of mathematics as she asked me specific questions to try to

[3]See Heaton and Lampert (1993) for a discussion of our relationship around my teaching.

understand my understanding of the mathematical purpose of what I was doing and my interpretation of the sense students made of ideas. And, finally, her notes represented her perspective as a colleague and teacher, empathetic to the difficulties inherent in trying to teach mathematics for understanding. In the spring term, rather than coming twice a week, Maggie observed consecutive days for a full week. I wanted Maggie's help in answering a question that arose from a project I did on my planning that I did for a course I took earlier in the year.[4] I was curious to understand how to go about constructing the curriculum from one day to the next.

During this school year, I also observed in Maggie's fifth grade classroom two or three days a week. I did this out of my interest in her teaching as well as my responsibilities as a research assistant on the Mathematics and Teaching through Hypermedia Project (Lampert, Heaton, & Ball, in press; Ball, Lampert, & Rosenberg, 1991). During that year, my responsibilities for the project included taking field notes during Maggie's math class, writing observation summaries afterward, video taping math classes, interviewing students, and talking with Maggie about her teaching in formal and informal conversations several times a week.

A researcher. Jim Reineke, a doctoral student and researcher in educational psychology, observed in my classroom twice a week and talked with me on a weekly basis about my teaching during fall term. He designed his research project to study students' and my own changing conceptions of mathematical knowledge.[5] He videotaped my math class twice a week, audio taped all of the lessons, took field notes, and did informal interviews of students.

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[4] See *Continuity and Connectedness in Teaching and Research: A Self-Study of Learning to Teach Mathematics for Understanding* (Heaton, 1991) for my research report on this project.
[5] See Reineke (1993) for an account of my teaching from Jim's perspective.

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I audio taped the lessons on the days when he was not there. He also interviewed me and recorded our conversations[6] after each lesson he observed.

The decision to allow Jim to do his project in my classroom was advantageous for me. At the time of my teaching, I was glad to have him to talk with about my teaching. I made copies of the audio tapes and video tapes made of my teaching fall term as well as the interviews he did with me about my teaching. There were days when his observations overlapped with Maggie's. What this means for my study is that on certain days, I have multiple sources of data for the same teaching event. From a research perspective (Cochran-Smith & Lytle, 1990; Hammersley & Atkinson, 1989), this enriches the data set and strengthens the evidence for the arguments I have made throughout my study.

Tools of My Teaching Practice

Many teachers make use of a mathematics textbook to teach mathematics. I did for the nine years I taught mathematics in a fairly traditional manner, prior to this year of teaching. I also made use of a textbook during the year I attempted to change my practice. In the chapters which follow, my relationship with the teacher's guide is a theme. To better understand this relationship, I describe the nature of the curriculum--its underlying philosophy, how the curriculum is organized, and what the teacher's guide is like. During that year of teaching, I also kept a journal on my teaching. This, unlike making use of a teacher's guide, is not something a lot of teachers do. Writing about my practice on a daily basis is something I had to learn to do. In the chapters which follow, I provide evidence for my story and study by drawing from my journal. In this chapter, I will describe a bit of the process and struggle of learning to collect my

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[6]See *Collective Reflection: An Account of Collaborative Research* (Heaton, Reineke, and Frese, 1991) for what these conversations were like for both of us.

daily thoughts about my teaching on paper. I begin with background on the mathematics curriculum.

A Mathematics Curriculum

Before the school year began, I made the decision that I was going to use a math textbook when I started my efforts to teach mathematics for understanding. The alternative was to do as Maggie and Deborah did, and not use a textbook. I did not feel qualified to do that. I feared I did not know enough mathematics to teach on my own without using a textbook. I decided to use the math textbook that was adopted by the school district, the Comprehensive School Mathematics Program (CSMP) (McREL, 1986). Like most teachers who make use of textbooks, I did so with minimal support outside of what appeared in the text of the teacher's guide.

This curriculum was developed prior to the current math curriculum and teaching standards (National Council of Teachers of Mathematics, 1989, 1991) yet, in many ways, the pedagogy implied within it seemed consistent with "new" ways of thinking about math teaching being advocated in the reform documents. For example, problem solving is a central focus. Communication or mathematical discourse plays an important role. A major aim is to help all children learn to reason about mathematical ideas and understand how mathematical ideas relate to one another. And, it is not only from my perspective that CSMP seemed aligned with the current ideas about mathematics education. In the publisher's current catalog of CSMP curriculum materials the following appears, "CSMP already addresses, and has for years, every one of the NCTM objectives or goals" (McREL, 1992b, p. 1).

CSMP is an innovative mathematics curriculum that looks different from the math textbooks I used in the past. It is not filled with pages of computation. There are no actual student textbooks. There is only a detailed teacher's guide

and student worksheets to accompany individual lessons. Using a curriculum that looked different from ones I had used in the past and one that was advertised as aligned with the latest mathematics curriculum and teaching standards (National Council of Teachers of Mathematics, 1989, 1991) seemed like a good way to go about supporting the changes I wanted to make in my teaching. I thought I knew how to teach mathematics with a textbook. I had done so for nine years. Using a curriculum that represented the new ideas for teaching mathematics advocated by the reforms seemed like a reasonable way to launch myself into teaching mathematics differently.

In this study of the learning I have had too to teach differently, many questions surfaced for me around this curriculum and my use of the teacher's guide. What was its role in my learning to teach? How did my interpretation of particular lessons match the authors' intentions? How did my relationship with the teacher's guide change over time? What did I expect it would do for me? How did I think it would help? To understand my study of these questions and others that arose around my use of the textbook in subsequent chapters, I provide background on the curriculum. When and why was it developed? What are the conceptions of mathematics and pedagogy implied within it? How is the content organized? To answer these questions, I draw primarily on documents published by the developers of the curriculum.

The development of CSMP. I wondered how it could be that CSMP was developed prior to the current NCTM standards yet was consistent with them. What was the state of mathematics education at the time of CSMP's initial development in 1966? The launching of Sputnik in 1957 prompted people to do something to upgrade mathematics and science instruction in the United States. Shortly after Sputnik, Jerome Bruner (1977/1960), a powerful spokesperson for thinking in new ways about learning, teaching, and subject matter, and his c

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colleagues became leaders in a new wave of curriculum development. Bruner and his colleagues thought that students were capable of thinking about more sophisticated ideas than they were given credit or opportunity to do in schools. They encouraged the development of curricula that was responsive to offering students new opportunities to learn. What Bruner thought students ought to and were capable of understanding was the structure of the discipline. That is, students could learn the fundamental ideas of the discipline and understand the connections among the ideas. Bruner (1977/1960) noted, "To learn structure, in short, is to learn how things are related" (p. 7). The initial wave of math reform, in response to Sputnik and these ideas of Bruner's, came in the form of "new math." This reform movement was an attempt to enrich the content of the mathematics students were learning by developing curricula that represented the structure of the discipline (Bruner, 1977/1960). The implementation of these new mathematics curricula[7] did not meet with huge success (Sarason, 1982; Stake & Easley, 1978). New math was taught in old ways (Sarason, 1982).

One response to this failure of "new math" was to move from content with a conceptual focus back to the basic skills. This resulted in curricula designed around sequenced behavioral objectives intended to be taught, assessed, and mastered (Kaufmann & Sterling, 1981). Another response was to preserve the mathematical content of new math but give more serious thought to issues of pedagogy. This is what the developers of CSMP intended to do. CSMP was developed under the commitment to "dual emphasis on mathematical content and pedagogy designed to support mathematical reasoning" (McREL, 1992a, p. 4).

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[7] These first wave of "new math" curriculum projects did not include CSMP. It did include projects and programs like SMSG, UICSM, The Arithmetic Project, the Madison Project, and Minnemath (Kaufmann & Sterling, 1981).

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The underlying assumptions. CSMP is built on the assumption that being mathematically literate is knowing something about calculus or understanding real numbers, the ideas of functions, and logic (CEMREL, 1981b). Working backward from that goal, the purpose of the curriculum is to "give students an appreciation and feeling for algebraic structure." The concepts of set, relation, and function, without the formal mathematical terminology, hold a prominent place in this curriculum. In addition to the specific content, CSMP wants students to have an appreciation for the "power and elegance" (CEMREL, 1981b) of mathematics.

CSMP is built upon a conception of mathematics as a unified body of knowledge (McREL, 1992a). Mathematics, from CSMP's perspective, is about ideas, not notation (CEMREL, 1981b). Computation is still important but the developers of CSMP (CEMREL, 1981b) did not believe that students need endless practice of algorithms. Instead, algorithms are introduced in the context of what they believe to be interesting problem situations for children.

What are the assumptions about how children learn that were popular around the time of the development of CSMP? Bruner and his colleagues thought that students ought to learn mathematics through the discovery approach. That is, given rich mathematical experiences, students would be able to intuitively grasp or discover the structure of the discipline. The experiences need to be carefully sequenced and revisited repeatedly (Bruner, 1977/1960).

Curricular implications. CSMP uses a problem solving approach designed around "a pedagogy of situations." That is, CSMP has aimed to situate the learning of mathematics in contexts they believe are accessible to children. They make use of story situations and concrete representations (i.e. the Minicomputer, arrow roads, and string pictures). These representations are described in detail a bit later. There are no student textbooks, only a teacher's

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guide and student worksheets to accompany individual lessons. The curriculum is designed to "help children use their curiosity, their intuition, their powers of reasoning to formulate and think about interesting problems and situations" (McREL, 1986, p. 30). The problem situations are intended to guide the sequencing of content and provide a means for developing computation skills (McREL, 1986). The belief is based on the theory that students learn by intuition in the context of problem situations that they find intriguing. "The content is presented not as an artificial structure external to the experience of children but rather as an extension of experiences children have encountered in their development" (Herbert, 1984, p. 3). Doing these problems "could easily lead in several ways to important mathematical ideas or ways of thinking" (Herbert, 1984, p. 6).

The mathematical content is organized into four strands: The World of Numbers, Geometry and Measurement, Probability and Statistics, and The Language of Strings and Arrows. These strands are organized in a spiral. This means that in a week's time, the developers intend students to encounter situations from a different strand on each day. The next week, students return to the strand and encounter similar ideas in new lessons over again. Herbert (1984) explains the rationale behind this approach,

> CSMP believes that different children learn at different times and at different rates and since learning is not necessarily a linear process, this spiral organization gives each student a new chance to work with an idea at each turn of the spiral. Thus, according to the developers, when students return to a topic a week or two later, some who did not understand the concepts the first time around may now be better prepared to work on the ideas. (p. 22)

The intention of the spiral organization is for students to have experiences which "combine brief exposures to a topic (separated by several days before the topic appears again) with a thorough integration of topics from day to day" (CEMREL, 1982, p. 3). Introducing, leaving, and returning to concepts and ideas is intentional and related to beliefs about learning. "The gap between the segments provides time for the material to "sink in"; later segments proved a natural review of earlier segments" (CEMREL, 1982, p. 3). The assumption underlying CSMP is that learning occurs in a spiral, rather than a linear, process.

A description of mathematical content within each of the strands includes a description of the three non-verbal languages found within the strands: string pictures, arrow diagrams and the Minicomputer. The non-verbal languages are intended to help students consider more complicated mathematical ideas without encountering writing, reading, or formal terminology as obstacles. A description of each of the strands and the non-verbal tools within them follow.

The World of Numbers. The World of Numbers is a site for children to have a variety of numerical experiences, including addition, subtraction, multiplication, division, negative numbers, decimals, and fractions. The Papy Minicomputer is a tool for helping students to construct and understand ideas about place value when doing basic computations. The Minicomputer is not like a computer or a calculator. It is a large square board divided into four squares of different colors and different values. Each of the smaller squares within the large square has a different value--1, 2, 4, or 8. From right to left, the values of the boards increase as their place in the base ten system increases.

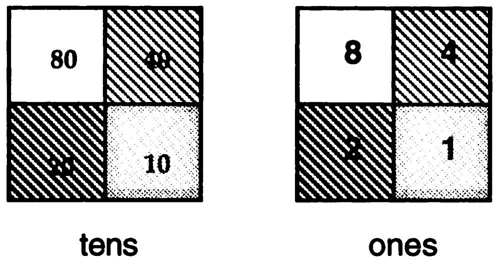

Figure P.2

The values of the Minicomputer

Checkers placed on the squares have the value of the square. To find the number on the Minicomputer you add up the values of each of the checkers. For example, the number on the Minicomputer below is 16 or 2 + 4 + 10.

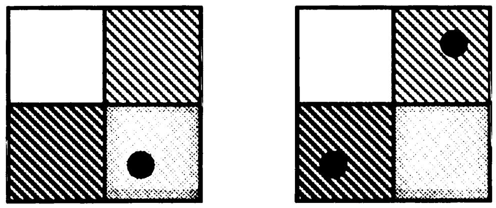

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Figure P. 3

2 + 4 + 10 or 16 as represented by the Minicomputer

Students use the Minicomputer to represent numbers, perform calculations, and solve problems. The Minicomputer is intended (CEMREL, 1981a) to be a way for young students to be able to work with relatively large numbers early in their study of mathematics.

<u>The Language of Strings and Arrows.</u> The Languages of Strings and Arrows is a way to represent and study sets and relations and use classification

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to understand concepts and solve problems. This is important in the study of mathematics because, "The definition of math is the study of sets and relations principally involving numbers and geometrical objects. This is in children's experiences and mathematics, therefore key in elementary mathematics" (McREL, 1986, p. vi). There are two non-verbal languages within this strand, the language of arrows and the language of strings.

The Language of Arrows is used to "represent relations among numbers or objects" (Herbert, 1984, p. 9). Children can "read and draw the relationship before they can present the same information in words" (McREL, 1986, p. vii). Arrow diagrams are used to represent the mathematical concepts of relations and functions, numerical and non-numerical relations like adding to, subtracting from, multiplying by, and sharing equally among. They are also used for logical thinking activities.

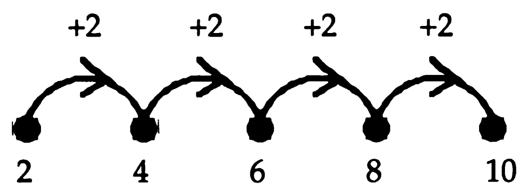

Figure P. 4

An arrow road

The Language of Strings is used to represent the mathematical notion of set. Colored strings provide non-verbal language for classification and logical thinking and deductive reasoning exercises. This is important because, "The ability to classify, to reason about classification, to extract information about classification are important skills for every day life" (McREL, 1986, p. vi).

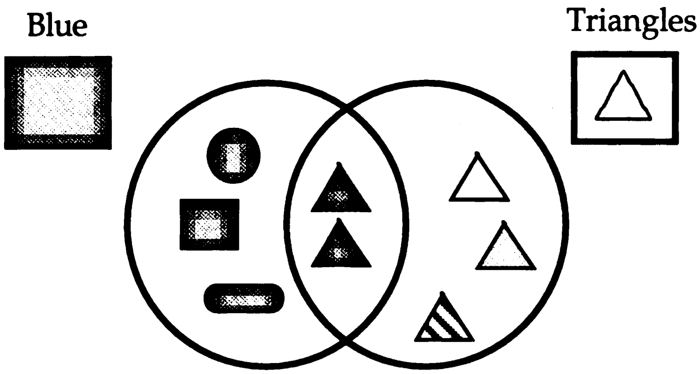

Figure P. 5

A string picture

Geometry and Measurement. The study of mathematics is not limited to the study of numerical situations. In the Geometry and Measurement strand, measurement becomes a means for investigating problems and developing concepts and provides an opportunity to "explore the rich connections between arithmetic concepts and physical concepts" (McREL, 1986, p. v).

Probability and Statistics. The strand of Probability and Statistics provides challenging problem situations and the opportunity for students to understand the world around them by dealing with ideas like chance and prediction. In this strand, students investigate the notion of probability and learn to think logically as a means of predicting the likelihood of future events.

How the lessons are organized. The individual lessons within a strand begin with a short summary of the lesson. "Indirectly, these summaries suggest goals but, of course, the goals of the curriculum spread over whole series of lessons and cannot be broken into specific goals for individual lessons" (McREL, 1986, p. 32). These are not goals in the form of objectives to be mastered. Mastery would run counter to the beliefs about learning supported by the spiral o

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organization of the curriculum. CSMP makes some assumptions about how it is teachers will figure out the goals of a particular lesson, "If you read the introduction to the strand in which a lesson appears as well as the Capsule Lesson Summary for that lesson, we expect you will easily see the direction and intent of the lesson. We decline to predict for any one lesson what will happen in terms of student learning" (McREL, 1986 p. 32). The spiraling that occurs across strands, as described earlier, is found within strands as well. The mathematical ideas found within a single lesson reoccur in lessons found later in a particular strand.

The teacher's role. How does the teacher guide "students to a desired mathematical experience or conclusion" (McREL, 1992a, p. 5)? What is the teacher's role in guided discovery? The way the teacher helps the students to learn or to discover the structure of what it is they are learning intuitively is through the asking of questions that help students to think about the problem. Bruner (1977/1960), however, saw this as no easy task for teachers.

> Given a particular subject matter or a particular concept, it is easy to ask trivial questions or to lead the child to ask trivial questions. It is also easy to ask impossibly difficult questions. The trick is to find the medium questions that can be answered and that take you somewhere. This is the big job of teachers and textbooks. (p. 40)

In order to address this problem, CSMP provides teachers with a scripted teacher's guide--teacher questions followed by several examples of students' responses you might hear students say. These teacher's guides place a heavy emphasis on the discovery approach and question-asking techniques.

> CSMP is very teacher directed. Teachers are encouraged to follow the teacher's guide fairly carefully until they become comfortable with the kinds of questions and procedures intended. Because of CSMP'S highly structured schedules and lessons, the teachers' guide is **the** crucial program vehicle. It provides support and instruction to the teacher from

> training, through practice of the lessons, and on to eventual mastery of the content and pedagogy. (Herbert, 1984, p. 25)

What this implies is that a teacher might be able to learn the content and pedagogy of CSMP through repeated use of the teacher's guide. Herbert (1984) makes it sound like CSMP is not only a guide for teaching but also a tool for learning to teach. CSMP suggests, "You may wish to follow it at the beginning of the school year and then deviate from it as the needs of your students may dictate" (McREL, 1986, p. 39). The detailed lessons, according to the developers, are only given for teachers who feel they need them.

> You should never feel obligated to follow any lesson word for word as it is written in the guide. By the same token, you should never insist that your students give the same response as those given in the hypothetical dialogues. Use your own creativity in drawing out students' responses as the situations suggest. Read the lesson plans very carefully beforehand to get an idea how the lessons are expected to proceed. (McREL, 1986, p. 39)

One of the purposes of the teacher's guide is to model good questioning techniques which is, from CSMP's perspective, a critical skill in problem solving and one that teachers often find difficult to do.

The significance of CSMP to my study. My study is not a study of the implementation of CSMP. I am studying what is entailed in learning to teach mathematics in ways advocated by the current reform. I began my efforts by using a math textbook, the CSMP teacher's guide, as a tool in that process. The way I made use of CSMP is the way most teachers, I conjecture, will try to make use of any textbook. I tried to glean what I could from the text in the teacher's guide. When I did not understand something within the teacher's guide, I made an educated guess about its meaning or dismissed what I did not understand as unimportant. I used the textbook under the best of intentions with minimal support for making sense of the content and the pedagogy as it appeared explicitly and implicitly within the teacher's guide.

My Teaching Journal

I kept a journal on my teaching all year. Keeping a teaching journal is something I observed both Maggie and Deborah doing. Neither of them used a teacher's guide or lesson plan book. But they kept journals that contained detailed and extensive notes on their teaching written mainly before and after each math class. I tried to keep such a journal and gave a copy of it to Maggie to read and respond to on a weekly basis throughout the year I taught.

The decision to keep this journal, at the time, was a difficult one to follow through on. I had kept personal diaries as a child and even during periods of time in my life as an adult. Trying to write about my teaching, on a daily basis, however, was a far greater challenge. It was not as easy for me as it looked to be for Maggie and Deborah. The difficulty was not so much in finding the time as knowing what there was to write about. As I was learning that it was possible to discuss mathematical ideas with students, I was also learning that there were issues related to mathematics teaching and learning worth thinking and even writing about.

When I started the school year, I only wrote in my journal after teaching. I used a separate notebook more like a plan book to record what it was that I was going to do. I saved my journal for reflections on what happened. As my use of the textbook changed throughout the school year, so did the way I kept track of what I planned to teach and what happened when I taught. Over time, the separate notebook for plans went by the wayside. I started keeping track of everything in one journal and wrote in it before and after each class period as I tried to think through what had happened during a class as part of the process of figuring out what to do next. Many days, it was hard to face pen and paper after class when what I wanted to do was as quickly as possible put the challenging experiences of the class behind me. The last thing I wanted to do was reflect on them. Often, I had to let a few hours pass after teaching before I could write at any length about the lesson and prepare for the next class period. Keeping a journal was not just about reflecting on practice. Journal writing became a tool of my practice.

By the end of the school year, I had some sort of journal entry for every day that I taught. Since deciding to study my learning for this year of teaching, I have been thankful I struggled through keeping a journal. It is a record of my teaching that is unlike what I find in any of my other sources of data. On many days, it is a record of my feelings. Another advantage to my journal and Maggie's written response to it is that it is the only record of any of the exchanges we had over my teaching apart from what I may have noted about those conversations in my journal. At the time of my teaching, the act of audio recording our conversations was not an activity relevant to doing the teaching. At the time, my purpose was teaching, not research. My conversations with Maggie and the immediate needs they served were what mattered to me.[8] In retrospect, I regret not having documentation of those conversations with Maggie.

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[8]See Heaton and Lampert (1993) for a discussion of the nature of those conversations.

CHAPTER 4

WHAT IS A PATTERN?:

FRUSTRATION IN THE EARLY WEEKS OF SCHOOL

Introduction

The first few weeks of teaching were wrought with difficulties and frustration as I tried to enact the vision I held for the kind of mathematics teaching I wanted to be doing. I continually felt as if I was falling short of my goal. At the start of that year, the time of this lesson, I found myself tethered to the textbook. I acted as if it were my lifeline to teaching. Ironically, a bit later I found myself needing to sever the connection to make progress in learning to teach differently. The textbook did not, however, drift completely out of my sight. Rather, throughout the year, I reconsidered my relationship to it and worked at recasting its place in my teaching.

The difficulties and confusion I confront in the teaching described and analyzed in this chapter, based on a lesson (09/20/89) in the third week of the 1989-90 school year, are representative of the challenges I confronted on a daily basis in the early weeks of school. The frustrations I felt throughout this lesson, however, were more intense than what I had experienced in previous lessons. My curiosity over the source of these frustrations drew me to study this particular lesson. Painful as it has been to look back at this period in my teaching, doing so has offered a rich site for exploring the demands of learning to teach mathematics for understanding.

I have chosen to organize this chapter into three main parts. The first part is my telling of the story of the teaching on this particular day. The story, itself, has three sections based on the way I organized my teaching in those early weeks of school: planning , teaching, and reflections on teaching. Through my descriptions of events, I intend to portray the confusion and frustrations I felt as well as the sense, or lack of sense, I made of what was happening at the time. Any frustration you may feel as a reader as you try to follow my teaching experience on the day focused on in this chapter is intentional. The lack of smoothness in the flow of the narrative is characteristic of what I was experiencing in the teaching. Bear with me. The tenor of my teaching smoothes out somewhat in later chapters.

In the second part of this chapter, I look back on this lesson and what I am learning from my present perspective, three years after teaching the lesson. I re-examine my frustrations and try to understand what it is that I could learn or have learned since then, that might have eased my frustrations at the time I experienced them. I am able to see and hear much more now in my teaching than I was able to then. A primary focus is on what I have learned about the mathematical ideas contained with the lesson, the text of the teacher's guide, and the territory in which the students and I found ourselves: the mathematical meaning and significance of functions and their composition as well as the relevance of patterns.

Finally, in the third part, I draw parallels between what I am learning from this particular math lesson about learning to teach and Mark Twain's adventures learning to navigate a riverboat as portrayed in *Life on the Mississippi* (Twain, 1990/1883). I do this because Twain faces frustrations in his early efforts to learn the river that seem similar to mine. He struggles with how to make use of his notebook, his written guide for navigation. I do battle with the textbook. Through his struggles, he begins to recognize the depth of what he needs to know and the fluid and flexible way he needs to make use of what he knows as he attempts to navigate the river. He also begins to see the ever-changing quality of that which he is trying to learn and the never ending journey of his own learning. I have to come to have a similar view of the content and process of my own learning as a teacher.

Planning to Teach

Following the Teacher's Guide

The lesson I did with my students on this particular day came from "The World of Numbers," one of the content strands that make up the spiral curriculum of CSMP. These strands were described in the Prologue to Part II. At the time of this lesson, I had been following the spiral organization of the CSMP teacher's guide from one strand to another, from one lesson to the next. As I looked over this particular lesson and noticed the title, "Composition of Functions," I felt the need to get some sense of what this lesson was intended to be about as well as what my students were supposed to do.

There had always been a clear match between what the math textbook said I was supposed to teach and what my students were to learn and do. In my past teaching, what my students were supposed to learn was what I taught them to do. For example, if students were supposed to learn to do long division, I taught them the rules and procedures for computing the algorithm, and they practiced finding the answers to many long division problems. Knowing when and what they understood was also more straightforward. Answers were either right and I moved on or wrong and students did the work over. If many students had difficulty, I retaught the lesson. In my past teaching, the mere title of a math lesson in the teacher's guide gave me sufficient information to plan, teach, and assess the successfulness of any lesson (i.e. two-digit division, three-digit multiplication, addition of fractions). There was never a question about what I was supposed to do, what my students were to learn, and whether or not we had accomplished anything.

In this lesson on the composition of functions from CSMP, the relationship between what to teach, what was to be learned, my role, and what my students were supposed to do did not seem so clear to me. I saw a need to construct connections between the activity in the teacher's guide and the mathematical ideas the activity was intended to teach. The title of this lesson, the "Composition of Functions," gave me little information about either. The language was foreign to my past experiences as an elementary teacher. In my nine years of teaching, I could not recall any topic, chapter, or lesson in an elementary math textbook that used this same language.[1] While I had learned something about functions somewhere in my own learning of mathematics, what my fourth graders were supposed to learn about them was a mystery to me. Seeing the need for connections and making them are two different things. I looked for clues throughout the teacher's guide to help me build the connections I knew were important to make.

I turned to the lesson summary, a description found at the start of each new lesson in the teacher's guide.

> Using arrow diagrams and the Minicomputer, investigate the composition of certain numerical functions, for example, + 10 followed by +2 and 3 x followed by 2 x. (p. 11)

Reading this summary generated more questions for me than it answered. I knew what it meant to find an answer to a math problem but what did it mean to investigate a mathematical idea? What did it mean to investigate

[1]Leinhardt, Zaslavsky, and Stein (1990) note the marginality of functions as a topic in elementary math textbooks. Open Court's *Real Math* (Willoughby, Bereiter, Hilton, & Rubinstein, 1981) and CSMP(McREL, 1986) are among the exceptions.

the composition of certain numerical functions? What was there to investigate? What were we supposed to be looking for? How was I supposed to assess what was learned in the investigation? Were there right and wrong ways to investigate? Finding the right answer to a problem had signaled being done in the past. How was I to decide when we were finished? How would I know when it was time to move on?

In an attempt to further understand what was beneath the phrase "composition of functions", I read through the content overview of The World of Numbers, in the section entitled, Composition of Functions.

> Several lessons in this strand deal with what happens when you compose a sequence of functions, that is, apply the functions in order one at a time. These compositions lead to many general, powerful insights into the properties of numbers and operations. (p. xix)

After reading this, I questioned what sort of powerful insights we were searching for and why. What constituted a "powerful insight"? I read a bit further and found that my fourth graders, in previous years, should have had experiences with the composition of functions.

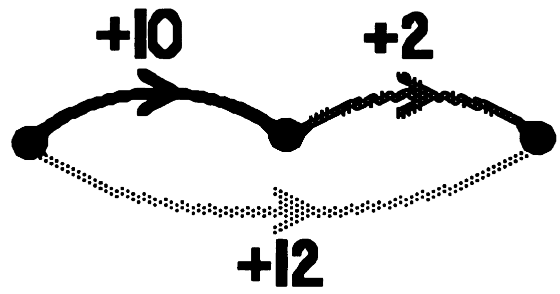
> Your students' extensive experiences with the composition of functions in the CSMP Upper Primary Grades curriculum led them to many insights that involved the development of algorithms, the discovery of number patterns, and efficient mental arithmetic techniques. A goal in this strand is to review these discoveries and to apply composition to new situations and problems. (p. xx)

The fact that my students might be somewhat familiar with the content of the lesson I was about to teach was of some comfort. But, if this was true, would they remember what they had learned? And, what about the handful of students in my class who were new to the school? What could I count on in terms of any of my students' past experiences?

Most of the CSMP lessons were designed to be completed in a single class period. In the first few weeks of my teaching that year, I found that lessons took much longer than CSMP or I planned. CSMP typically allowed one day for each lesson while I spent at least two days, sometimes longer, on each one. I had begun to feel like I was falling behind. I looked for logical ways to divide the single day CSMP lessons over a couple of math classes. One way to lessen the feeling of falling behind was to become more realistic about moving ahead.

This particular lesson on functions involved a sequence of problems around addition, subtraction, and multiplication. I decided to do the three problems from the teacher's guide that dealt with addition and subtraction (defined as Exercise 1) one day, and multiplication (defined as Exercise 2) in CSMP. Even with splitting up this lesson over two days, I remained skeptical as to whether or not I would actually get through what I had planned. These days my judgments about pace felt off. This was unsettling. As I found myself falling further behind in terms of numbers of problems finished and pages covered in the teacher's guide, I began to question if these were reasonable ways to measure progress in my current practice. Why were things taking so much longer? What were realistic expectations? Was I moving anywhere? In those first few weeks, I experienced a strange sense of feeling behind in a context where I felt unsure of where I was supposed to be going.

The problems in the teacher's guide under the heading of "composition of functions" looked like the following example. The labels of the arrows varied from one problem to the next.



starting number	ending number

At the time I planned the lesson, the table seemed easy enough for my students to complete. I was certain they could follow the arrows, one red and one blue in the teacher's guide, and come up with numbers to fill in the table. Once the table was filled with numbers, the students were to notice patterns in the relationship between the input (starting) numbers and the output (ending) numbers. I decided to do the first problem together with the whole group and assign the remaining two problems in the teacher's guide for the students to work on their own.

starting number	ending number

starting number	ending number

I planned to follow the problems as they appeared in the teacher's guide. Students would find starting and ending numbers to complete the tables, identify the composition arrow, an arrow which would encompass the other two, for each table of numbers, and notice patterns.

Plans to Enact a Vision of the Math Reforms

In my early weeks of teaching, CSMP was a guide for my practice as were ideas that I got elsewhere for the kind of math teaching that I wanted to learn to do. When I came across the question of patterns in the script of the CSMP lesson, my familiarity with the math reforms at the time helped me to see this as an important question even though a study of patterns was foreign to my past experiences as a teacher or student. Based on my observations in Maggie's class and what I had read in the *Mathematics Framework* (California State Department of Education, 1985), I thought that a discussion of patterns was a way to consider mathematical relationships. Patterns and functions appeared as one of seven mathematical content strands used to organize the mathematics curriculum in the *Mathematics Framework* (California State

Department of Education, 1985). The importance of the study of the relationship between patterns and functions was described this way:

> The study of mathematical patterns and functions enables students to organize and understand most observations of the world around them. It involves discovery of patterns and relations, identification and use of functions, and representation of relations and functions in graphs, mathematical sentences or formulas, diagrams, and tables. (p. 10)

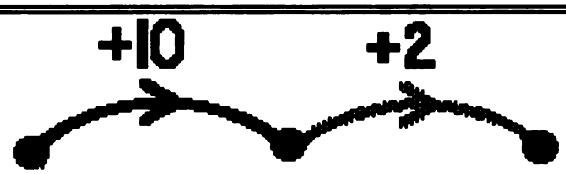
I had observed Maggie engage her fifth graders on a regular basis in discussions of patterns. I was eager to have the opportunity to attempt a discussion of patterns with my own students. I had become intrigued with the idea of mathematical discussions and wanted to get my students talking. I thought patterns would give us something to talk about.

Before I became acquainted with the reforms, I never knew there was anything that would lend itself to a discussion in math class. I had learned through observing Maggie's class and reading the *Mathematics Framework* (California State Department of Education, 1985), that it was possible to discuss mathematical ideas. I could ask students to explain how they solved a problem, why they solved a problem in a particular way, and to share alternative means of finding solutions. I had led discussions in other subjects and could not imagine that mathematical discussions would be that much different to conduct, now that I knew there were ideas to discuss.

At the onset of this school year, I expected that moving away from a primary focus on rules and procedures would deepen my students' understandings of mathematics and push at the limits of my own mathematical experiences and understandings. I was worried about the mathematics I did not understand. I was reminded of the limits of my understanding as I searched for the meaning of "composition of functions" within the text of CSMP. These concerns about my lack of mathematical understanding did not, however, keep me from going ahead with this lesson. With the teacher's guide in hand, I felt more or less prepared. Even though I was unclear exactly what I was supposed to teach, I thought having something to do would get us through the hour. The teacher's guide provided me with problems I thought my students could do, questions for me to ask, and even responses I might hear from my students. I hoped that working on three different tables would fill the time and that asking the question of patterns would encourage a different way of thinking about number and spark a discussion--two aspects of the reforms I was aiming to make prominent in my practice that day.

<u>Teaching</u>

At the start of class, I wrote the following problem on the chalkboard:



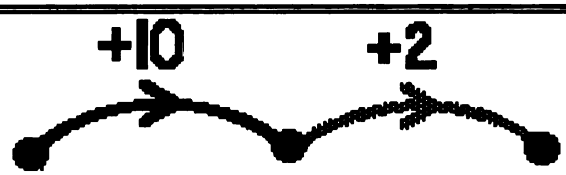
starting number	ending number

I said, "I'd like you to take a look at the chart. Think of a number. Everyone just think of a number." Bob inquired about the parameters for choosing a starting number. "Between what?" he asked. We barely began and already there was a question. I had not considered putting conditions on the problem nor was there any indication in the teacher' guide that I should. Why was

Bob asking the question? Given my own uncertainty about why we were doing what we were doing, the last thing I wanted was for a student to question me about the task. I said to him, "just think of a number and add 10 to it and then add 2 more to it." After a few moments of silence while students thought of numbers, I asked for someone to give the beginning number they chose. I started with Richard, whose hand was in the air.

Richard: 99.
Ms. Heaton: And what did you end with?
Richard: 111.
Ms. Heaton: And can you tell us how you got that?
Richard: Well, I did 99 plus 10 equals 109 and then plus 2 is like 9 plus 2 equals 11.[2]

We followed a similar routine for adding the first four pairs of numbers to the table.



starting number	ending number
99	111
8000	8012
250	262
1	13

Asking my students the question, "How did you get that?" was my idea. It did not appear in CSMP but it was a question I had heard Maggie ask in order to

[2]Please note that I refer to myself as "Ms. Heaton" in the excerpts of transcripts from lessons found throughout this chapter and the chapters on my teaching which follow. I do this because this is how I was referred to by my students. In excerpts of transcripts of conversations with Jim Reineke, I refer to myself as "Ruth." Jim and I are peers and refer to one another by first names. This switch in how I refer to myself is a complicated issue and related to issues of voice and identity which I discuss in Chapter 3.

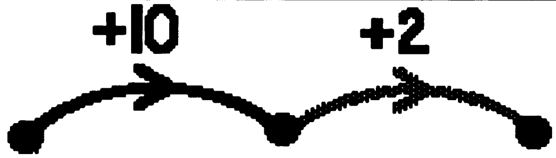
prompt her fifth graders to talk. Rather than just have students give me beginning and ending numbers, I added this question thinking that asking it would get students talking about how they solved the problem. What happened was that individuals, like Richard, responded with procedural explanations that did not seem to hold anyone's interest. Why was this happening? When Maggie asked the question of her students, they always offered interesting explanations.

As the teacher's guide suggested, I varied the process of filling the table with numbers by asking some students to give ending numbers first, followed by an explanation of how they generated them. The teacher's guide also suggested that I call on someone different to offer these explanations. These variations in the task and ways of calling on people appealed to me. I thought that starting with the ending number might make the explanations about how people got from one number to the other more interesting and calling on a variety of people would get more people talking. Maybe this would help the talk seem more like a discussion in which many people were engaged. Here is a sample of the interactions that occurred.

Ms. Heaton:	Can somebody give us their ending number? The number they ended with, David?
David:	19.
Ms. Heaton:	And what was your starting number?
David:	7.
Ms. Heaton:	Can you tell us how you got that?
David:	Because 7 plus 10 is 17 and 17 plus 2 is 19.
Ms. Heaton:	O.K., can someone else give an ending number? Jennifer?
Jennifer:	215
Ms. Heaton:	Can someone else tell us what she might have started with? Bob?
Bob:	203.
Ms. Heaton:	O.K., how did you get that?

Bob: Because 203 plus 10 is 213, 213 plus 2 is 215 and I have another number.

I continued calling on people and filled in starting and ending numbers until I ran out of space on the chalkboard.

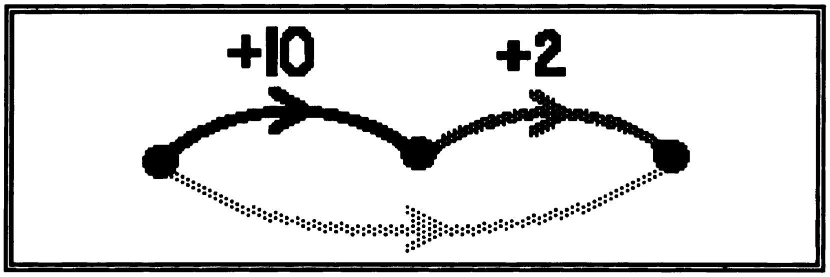


starting number	ending number
99	111
8000	8012
250	262
1	13
7	19
203	215
~~4088~~ 4988	5000

We ended up with a table of numbers but my efforts to generate a discussion led us no where.

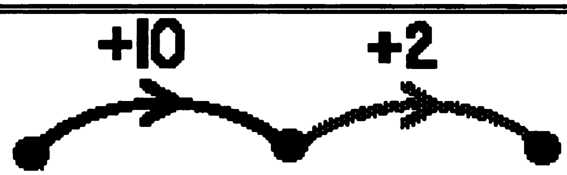
<u>Do You See Any Patterns?</u>

Disappointed with the lack of interesting talk that had occurred thus far, I held high hopes for the next part of the activity in which I planned to ask students to notice patterns. I followed the script in the teacher's guide and said to the students, "I want you to look at these numbers. Do you see any patterns?" For the next 15 minutes, students proceeded to give their ideas about patterns. I called on Valerie first. She said, "Each of them have a beginning number and then they have an ending number that is 12 more." I asked her how she knew the ending number was 12 more. She said, "Because you have to add 10, you find a number, you add 10 to it and then 2." I added the green arrow, the composition arrow, and said, "If I were to put in another arrow here, what would I put, plus what?"



Valerie responded, "Plus 12." I added the label to the arrow and, given my goal to initiate a discussion, proceeded to find out what others were thinking.

Pili said, "I see, I think I see a pattern. It is 80, 80." She pointed to the "80" in 8000 and the "80" in 8012 as she said, "80 right here and 80 right here."

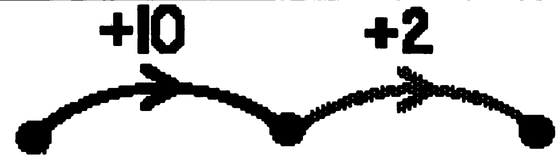


starting number	ending number
99	111
8000	8012
250	262
1	13
7	19
203	215
~~4088~~ 4988	5000

Hearing Pili's idea, I thought to myself, what sort of patterns were we looking for? Definitely not ones like this. She was right. There was an 8 and a zero in each of those numbers but taken together, they did not mean "80." The 8 was actually 8000 and the zero was in the hundreds place. In another attempt to promote discussion, I asked people to comment on Pili's idea. This call for comments from other students was a pedagogical move I had observed Maggie make with her students. In addition to being able to explain their own ideas, I wanted to teach my students to listen and build on the ideas of others. I thought that what Pili had said seemed like a meaningless pattern and I hoped that one of her classmates might say something to challenge her idea.

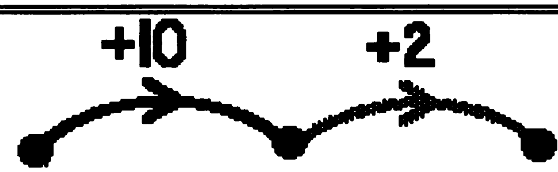
I repeated words not found in CSMP but ones I had often heard Maggie ask, "Okay, what do other people think about that?" I called on Jennifer, who said, "I agree with her." This, unfortunately, was not the response I wanted. I hoped Jennifer would disagree with Pili. I could have asked Jennifer why she agreed with Pili but I was not in a mood to hear explanations for what I thought was a meaningless pattern. Wasn't it clear that Pili was wrong? Should I just come out and tell her so? But, wasn't doing that counter to my goal for students to construct their own meanings? How did the categories of "right" and "wrong" fit into my desire to give students space to make their own sense of ideas? Should I accept all of my students' ideas, even the ones I thought were wrong?

I moved on without pursuing Jennifer's comment or dealing with this bigger issue of right and wrong answers. I addressed the class, "Okay, are there any other patterns that you see?" I called on Lucy, who said, "There is 8 and then 8 going across and then 2 and then 2 going across." She went to the chalkboard and pointed out what she was talking about. She pointed to the 8's at the beginning of 8000 and 8012 and the 2's at the beginning of the pairs of numbers, 250, 262 and 203, 215.



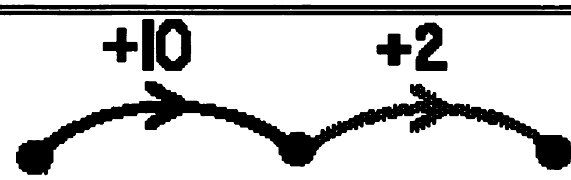
starting number	ending number
99	111
8000	8012
250	262
1	13
7	19
203	215
~~4088~~ 4988	5000

She was right. These numerals matched one another but what did this observation have to do with the question of patterns in the context of functions? Reluctantly, I returned to Pili, who was waving her hand frantically, ready to give another "pattern." She said, "I see three zeroes here and here," and she underlined the three zeroes in 8000 and 5000.



starting number	ending number
99	111
8000	8012
250	262
1	13
7	19
203	215
~~4088~~ 4988	5000

I was worried. This "pattern" seemed even more irrelevant than her last one. Maybe I should have come out and said what I thought of her ideas. She attempted to draw connections between a pair of numbers not even in the same row. Since the ordered pairs were generated independent of one another, I knew there was no mathematical reason why there would be meaningful connections between numbers not found in the same row. Maybe Pili would see for herself the unreasonableness of what she was doing, if I could get her to talk about her idea. I asked her to explain. She said, "In 8000 there are three zeroes and in 5000 there are three zeroes." But what did this have to do with patterns? What was a pattern? I knew that what I was hearing were not patterns. Jennifer followed Pili's comment and said, "I have something similar to hers but it is not exactly the same. She went to the board and underlined the zeroes in the hundred's place in 8000, 8012, 4088, and 5000.



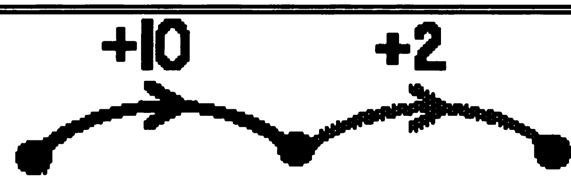
starting number	ending number
99	111
8000	8012
250	262
1	13
7	19
203	215
~~4088~~	5000
4988	

I felt a mix of emotions. On the one hand, I was excited that a student had listened to another student and was making an attempt to build on a classmate's idea. On the other hand, I was growing frustrated as I watched one irrelevant pattern lead to another. In fact, Jennifer just included a number that did not even belong on the chart. It was there with a line through it because it had been revised. The idea of taking all ideas seriously, revised or not, was something else I had picked up from Maggie. Seeing Jennifer making use of the error, however, made me question my decision not to just erase it like I would have done in the past. Was that revised number just adding to the confusion and the fact that it was revised insignificant in this context? Were there some contexts where revised thinking and evidence of it was more important than others?

These questions about revisions seemed like the least of my worries. How could I stop this generation of meaningless patterns without just reverting back to my past ways of telling students they were wrong? I was trying to search for mathematical sense in what students were saying but was this an instance where there was none to be found? I was curious how Jennifer would describe the pattern she noticed, so I asked, "What would you say that pattern is. How would you describe it?"

Jennifer:	One here and one here, and one here and one here.
Ms. Heaton:	I don't understand what you mean by one here, what are you showing me?
Jennifer:	There is a zero in the hundreds place and one zero in the hundreds place, and there is one zero in hundreds place and one zero in the hundreds place.

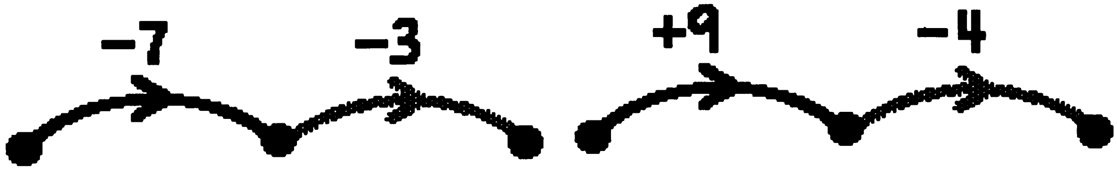
Hearing Jennifer talk of place value gave me a momentary sense of relief from the meaninglessness of all of this. I continued to ask myself, however, what any of what she or the other students were saying had to do with patterns or functions. What was a meaningful pattern? What was a function? I called on Richard. He explained, "I have another pattern. Can I go up there? Right here is three zeroes in a row and three ones in a row, and then three zeroes in a row." I asked what the pattern was. He said, "000, 111, 000."



starting number	ending number
99	111
8000	8012
250	262
1	13
7	19
203	215
~~4088~~ 4988	5000

I felt troubled as I listened to Richard. I knew these were not patterns in a mathematical sense but what were they? And, more importantly, what were mathematically meaningful patterns in this table of numbers? These students noticed "patterns" in pairs of numbers in the same row, pairs of numbers that spanned rows, and now senseless patterns in numbers in each of the columns. I knew that "meaningful" patterns could not be found in columns of numbers in which pairs of numbers had been generated by beginning or ending numbers chosen at random.

After more than 15 minutes of these patterns, when I could bear this "discussion" no longer, I announced, "I want to move on to another example." I wrote the following problems on the chalkboard for students to work on.



starting number	ending number

starting number	ending number

As the students worked on these problems, I had a few moments of my own to think and asked myself once more, what is a pattern? I knew that I had already made plans to teach the second half of this lesson the next day. Could I bring myself to come back the next day and face this teaching and, quite possibly, the question of patterns again? I left school feeling perturbed with myself for having trusted the teacher's guide and annoyed with myself for having thought I entered this lesson prepared to teach.

<u>Reflecting on Teaching</u>

That evening, I was distraught and troubled by the day's events. My mind was filled with questions as I tried to sort out the difficulties I experienced that day. Was I teaching? What were my students learning? What were they supposed to be learning? What would teaching and learning in this situation look like? Why was I unable to get an interesting discussion started? Why was I asking my students the question of patterns anyway? What was a pattern? Did the patterns my students noticed have any relevance? What is a function? Why didn't my students come up with responses that matched the teacher's guide? How was I supposed to be making use of this teacher's guide?

In my interview with Jim that afternoon, I said,

> I felt like I was floundering today. . . I was asking myself the question, what do I mean by a pattern? What does the word pattern mean? I am not sure I have an answer to these questions but that's what I was thinking as I stood up there, listening to these kids. And the sort of student responses the book says will come up--that CSMP puts in, what kids would respond when you say, "What's a pattern?"--none of that came up. I felt really like I didn't know what to do to move a discussion about patterns. (Reineke interview, 09/20/89)

I had trusted that the teacher's guide was going to help me with this work. I felt like it was letting me down. It was, at once, too much of a guide and not enough of a guide. It had given me enough guidance to lead me to believe we could do this activity without providing some broader sense of its purpose. The only reason I had for doing the lesson was because it was the next thing in the teacher's guide. In the midst of teaching, I found myself feeling lost and helpless with no way to respond to students.

As soon as I heard my students' responses, I realized I did not know what I meant by 'pattern' or really why I was asking the question. I initially asked the question because it was in the teacher's guide and, given what I knew about the reforms, it had seemed like a good idea to me. But my students' responses did not match what the teacher's guide had predicted. It frustrated me to have an intuitive sense about what was not a pattern but no real sense of what would be sensible patterns or how to get students to notice and discuss them.

The predicted responses in the teacher's guide were not useful to me in the situation. Here is what appeared in the script of the teacher's guide:

T: Look closely at this chart. What patterns do you notice?

S: An ending number is always larger than the starting number.

S: If you start with an even number, you end with an even number. If you start with an odd number you end with an odd number.

S: An ending number is always 12 larger than the starting number.

(p. 14)

I had these responses in hand when I went into class. In the past, I would have been intent on listening for these responses. If I had not heard them, I would have encouraged students to produce these "right" answers. Like many good "traditional" teachers, I have had years of experience asking convergent questions. I know how to lead students to the "right" answer. But, in this situation I wanted to be open to the idea that my students' answers might vary from the teacher's guide. How was I to know which of my students' responses were reasonable variations of the responses found in the script and which were not? How would I figure out how to be responsive to these variations? Was there a way that I could be respectful of all of my students' ideas and at the same time let them know that many of their observations were irrelevant to the question of patterns? How could I respond to my students' ideas in ways that would push their thinking? The script in the CSMP teacher's guide was designed with questions for me to initiate but I felt frustrated by what little help it offered me in figuring out what to do next in the situation, especially when it seemed that what my students were saying did not match what the developers of CSMP had anticipated they might.

My frustrations went beyond my students' responses not matching the ones in the teacher's guide. I was bothered that the talk that was happening felt out of my control. Moreover, I did not have any sense of what it would mean for it to be *in* my control. I wanted to do something to guide a discussion but, not knowing what to do, I remained silent. Inside, I was struggling. I wanted us to have a meaningful discussion. I knew that we were not having one yet I did not know what we could have talked about or how I could have shaped what my students were saying into a meaningful discussion. While troubled by my students' responses to the question of patterns, I was also wrestling with my own response. I asked myself over and over again, what is a pattern? I had not seen this question asked or answered in the teacher's guide but it was a question that concerned me.

The script in CSMP offered me no assistance with responding to the students or to the questions I asked myself. If I had been more certain of the point of the lesson, I might have been more willing and able to move away from the script and ask the students a different question--*my* question--or I might have had a way of responding to their responses. As it was, the script in my hands, intended to help me in my role as the teacher, felt disconnected from my students and myself and the sense any of us were making of the question of patterns.

Questioning the Question

When I asked my students if they noticed any patterns, my response was to ask myself, 'What is a pattern?'[3] I asked my students one question--what patterns do you notice--and myself another--what is a pattern--repeatedly throughout this lesson, yet I never considered asking my students

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[3]Recognizing the question of, 'What is a pattern?' in the context of asking the question of noticing patterns as well as struggling with the fundamental nature of that question seems similar to the question of, 'What is a whole number?' which comes up in a lesson one week later. Maggie and I wrote about that lesson (09/27/89) in, "Learning to Hear Voices: Inventing a New Pedagogy of Teacher Education," (Cohen, McLaughlin, & Talbert, 1993).

the question I asked myself. I was surprised when Jim brought up the idea in our interview.

Ruth:	The question for me became what is a pattern? And, are these things that the kids are giving me patterns?
Jim:	So, why didn't you ask that?
Ruth:	Why didn't I ask- -?
Jim:	The class.
Ruth:	What?
Jim:	What is a pattern?
Ruth:	That would have been a good question. That is the question that I had on my mind.

As I said the last sentence, I heard myself laugh on the audio tape. I think I was realizing how slow I was to see that what Jim was suggesting was that I ask my students the question that had been on my mind. It did not occur to me, prior to this moment in my conversation with Jim, that asking it would have been a reasonable thing to do. I did not even consider it an option. Why? One reason might have had to do with my concern for time and my own need to get through the lesson I had planned. How could I have justified spending time on what patterns were when, according to CSMP, students were supposed to be searching for them? The teacher's guide defined the task as noticing patterns, not defining them. Would the students have gotten through the task of noticing patterns if we had stopped to define them? I began, however, to question how worthwhile the task of noticing patterns had been without some shared understanding of what we were looking for and why.

I was relieved to find out that I was not the only one who asked what a pattern was. Jim said that he had asked himself the same question. Maggie had also observed this lesson and as I read her observation notes for the day, I saw that she had also been playing around with the definition of a pattern. What follows is an excerpt from her notes:

> Asking "do you see any patterns?" What do you want to get out of that? Was the idea to get someone to say that the ending number is 12 more than the beginning number? There are relevant and irrelevant patterns (i.e. like 8-2-2 and 8-2-2). Hard to exactly explain what I mean by "irrelevant patterns." There are things you could get out of almost every pattern the kids come up with. . .what the kids are saying are more like observations than patterns. I think you were trying to get at this a bit when you asked, "Does that pattern apply to any other set of numbers? (Lampert observation notes, 09/20/89)

During the lesson, I assumed that I was the only one who didn't know what a pattern was. It was comforting to know that I was not the only one raising the question and that from Maggie's perspective, there was not an easy answer. The definition of a pattern is related to questions of relevance. Questions of relevance are related to questions of purpose and questions of purpose are related to the mathematics (i.e. *both* mathematical ideas and ways of knowing) to be learned. Where might this discussion have gone if I had taken my own question seriously? How might things have turned out if I had asked the question that made sense to me rather than the one with little meaning for me found in the teacher's guide?

I went into this teaching thinking that I did not know enough mathematics and that I was going to be learning a lot. That was in the abstract, before I started this teaching. Reality hit when I was faced with a question for which I really did not know the answer. Could I ask my students a question that was genuine for me?[4] Admitting to myself and others that I

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[4]I had had the experience of asking genuine questions as a teacher and learner in the context of Man: A Course of Study (MACOS). My experiences with this curriculum are described in Chapter 2. For example, "What makes man human?" the question around which the entire curriculum is organized, is a question for which I did not have an answer when I began teaching MACOS. *Schoolhouse Politics* (Dow, 1991), especially chapter 4 on teacher workshops,

had something to learn clashed with my view of role and responsibilities as a mathematics teacher. I was the teacher. I was supposed to ask questions for which I had the answers. Or, was I? Could I legitimately ask a question as a learner while also being the teacher? Could I take the risk? I needed to learn to set aside the teacher's guide, value what I did not know, and trust myself to ask a different question--my own question--even if it meant raising questions about questions in the teacher's guide.

The Limitations of Following a Teacher's Guide

My frustrations with the teacher's guide in this lesson prompted me to reconsider why it was I thought following this teacher's guide was a good idea in the first place. The content and pedagogy represented by the CSMP teacher's guide looked different from any curriculum materials I had used in the past. Since I wanted to move away from "traditional" teaching, "non-traditional" curriculum materials seemed like a useful tool for making fundamental changes in my view of content as well as how to teach it. I was coming to see that making the changes in my teaching to achieve what I was aiming for was not going to be as easy as merely following the CSMP teacher's guide as I had followed teacher's guides in the past.

How was I making use of the CSMP teacher's guide in my teaching? As in the past, I used the CSMP teacher's guide to make my decisions about what and how to teach. In this lesson, what to teach was defined as the "composition of functions" and how to teach was to "investigate." I had a good understanding of neither of these at the time. In my past teaching, what I did was the same as what I wanted students to learn and there was less room

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supports the idea that other teachers found themselves, often accompanied by much anxiety, in a similar position.

for interpretation of either. Now, everything seemed wide open to interpretation and much more complicated.

With little success, I had tried to figure out the point of the lesson by using the teacher's guide when I planned this lesson. At the time, I ended up putting aside my questions about the mathematical idea of composition of functions and what my students were supposed to learn and proceeded with giving my students the problems from the text. The problems looked doable without an understanding of some larger purpose and I trusted that doing them would lead to something meaningful. Once I asked the question of patterns, I questioned my decision to continue discussing them without understanding why they were important. How could the teacher's guide have helped me to be better prepared for this lesson? Were there ways that these problems could have been situated in terms of broader mathematical goals that I could have understood? What would I have needed to know outside of the teacher's guide to have felt prepared and gotten the most out of these problems and the question of patterns? Were there aspects of this teaching that CSMP or any textbook could not prepare me for?

Another set of frustrations accompanied my use of the teacher's guide when I tried to generate a discussion and interact with my students. The script of the teacher's guide offered me one possible route through the material. It offered enough detail of the activity to get my students and I started even though I did not understand exactly where I was going or why. This had worked fine in my past math teaching and it would have been fine this time if what had happened in my class had matched exactly what was anticipated by CSMP. But this was not the case. At the time, I thought there was a complete mismatch between the anticipated student responses in the teacher's guide and what my students said. In my journal I wrote,

> What my kids came up with did not resemble these responses. . . I am wondering what the role of the teacher's manual is in this type of teaching. A lot of the space in CSMP is taken up with examples of student/teacher interaction. I'm not finding this particularly useful and at times troublesome. When I read it, I think I have an idea of what is going to happen in class and today was an example, last Thursday was too, of things not going as I anticipated. I base my plans on the book but that doesn't seem to be working. (Heaton journal, 09/20/89)

This frustrated me to no end. Were there alternative routes to the one in the teacher's guide? If so, what were they and how could I connect them to the varied sense my students were making of the question of patterns?

Before this year, I would have steered us right back on the course defined by the textbook, regardless of how my students responded to my questions. Now, however, I wanted to be doing things differently. I wanted to be responsive to students' ideas. Was there a way I could do that and still get to where we were headed by choosing alternate routes prompted by my students' sense-making? What sort of guide could give me enough of a feel for where I was headed so as to guide my decisions about how to get there yet do so in ways that were responsive to my students' ideas?

<u>What is There to Talk About?</u>

While I was teaching, I was frustrated with my attempts to generate a discussion. Even though my students were talking, both as they filled the table with numbers and as they searched for patterns, it felt to me like there was nothing to talk about or what they proposed was meaningless. In the first part of the lesson, the discussion felt flat and in the second part of the lesson, the substance being discussed seemed irrelevant. Teaching as telling dominated my past practice as a teacher. At the time of this lesson, only a few weeks into the school year, I was determined to change my ways of interacting with students and avoid doing anything that resembled telling. I thought asking questions was a way to make this happen. I even inserted questions of my own into the script of the teacher's guide. There was still no discussion.

Given my lack of success at generating a discussion through questions, I began to think that there had to be more to my role than just asking questions. There was something to be learned about how to comment on students' ideas in the situation. How do I decide what to say, about what, to whom, and when? I discovered that I could get students talking with the questions I read about and heard Maggie use in her class. These questions--Do you see any patterns? How did you get it? What do others think?--along with the questions in the CSMP guide were pedagogical tools for getting people to talk. Asking them was a way for me to try on the role of a different sort of teacher. The questions got students talking but then what? Asking the questions was one thing. Knowing what to do with the responses was quite another.

I began to think that the decisions I could see I needed to make were related to where I wanted to go with a lesson which was dependent on an understanding of the purpose for attempting a discussion in the first place. If I had known why we were looking for patterns, I might have been able to consider the relevance of the students' patterns. When I asked a question about patterns in my journal, Maggie commented in the margin, "Isn't a larger question, why are we looking for patterns in math class in the first place???" (Lampert annotation of Heaton journal, 09/20/89).

A sense of purpose might have helped me to figure out what there was to talk about or possible ways to help my students think about their ideas. I opened up the discourse with questions I had heard Maggie ask but I did so without a sense of where I wanted to go. I found myself with lots of students' ideas before me, plenty of my own thoughts in my head, but no way to respond and the CSMP teacher's guide that carried me this far was of little use. I did not know what to do to make a discussion out of what my students were saying. I did not know how to respond in ways that would push their thinking. I also lacked a sense of where I wanted to push their thinking and why. What was it that I was trying to teach them about the composition of functions? What was important for students to learn? I did not have answers to these questions. I was trying to get a discussion of patterns going without a sense of what there was to talk about.

Three Years Later: Revisiting My Frustrations

At least part of my frustration at the time was directed at the teacher's guide and what I thought was its inability to help me respond to my students. In going back to this lesson, three years later, I wanted to understand more about my relationship with the textbook--what I expected of it or what I thought it would provide me with or do for me while I was teaching, the trust I had in it and the people who wrote it, and the way I made use of it in class. To do so, I have re-examined my frustrations and tried to understand what it is that I could have known or understood that would have eased or avoided my frustrations. What mathematics was I supposed to be teaching? What was mathematically significant about the patterns students gave me? What is a pattern? What is a function? What is the connection between patterns and functions? What is mathematically significant about the predicted student responses in the teacher's guide? What was the intent of this lesson from CSMP's perspective? How do the mathematical ideas in this lesson connect to broader mathematical ideas? How does doing the activity address what it is that students are supposed to learn? What is important for students to learn?

In the section that follows, I draw on resources outside of the CSMP teacher's guide to answer these questions. It is unclear to me whether or not the authors of CSMP expected me to know, prior to teaching this lesson, what I now understand. From my conversations with people who know about CSMP and the written commentary on the curriculum, I have no doubt that the people who designed the individual lessons, the content strands, and the curriculum had a sense for the mathematical connections and relationships at many different levels that I am only now coming to understand. This mathematical structure, however, was not presented in ways that were accessible to me nor am I certain that it was intended to be. It seems that the developers may have overestimated what teachers know and understand and underestimated the sheer difficulty of the mathematical ideas represented by the curriculum.

In the course of pursuing an understanding of the mathematics for myself, I have discovered a set of connections and relationships among mathematical ideas in CSMP that were invisible to me at the time I taught the lesson. If the authors of CSMP thought I understood what follows, I am uncertain how they thought I would have learned it. If they assumed the information was in CSMP, it was not in a form that I found accessible. The areas of mathematics I pursue are organized under three broad questions. Why are patterns and functions an important topic to teach? Why teach about patterns and functions in a particular way? What sense might students make out of a study of patterns and functions?

Learning the Importance of Patterns and Functions

The presence of a lesson on the "composition of functions" within CSMP implies that it was something worth teaching. But, why is this topic worth teaching? What is the importance of learning about functions? What is a pattern? Why study patterns? What is meant by the composition of functions? How do patterns and functions connect to broader mathematical ideas? Why does CSMP link patterns and functions? What is the relationship between patterns and functions? I had begun to ask myself various forms of these questions at the time of the lesson. Three years later, I have begun to construct some answers.

Why spend time on patterns? I have recently begun to appreciate that mathematics is a science of patterns and a search for patterns drives the work of mathematicians. Steen (1990) writes, "Seeing and revealing hidden patterns are what mathematicians do best." (p. 1). In an effort to broaden and deepen students' understandings of mathematics, reformers advise a similar search for patterns in mathematics by all students at every grade level (National Research Council, 1989; National Council of Teachers of Mathematics, 1989, 1991; California State Department of Education, 1985, 1992). For example, the *Professional Standards for Teaching Mathematics* note that as "teachers shift toward the vision of teaching presented by these standards, one would expect to see teachers asking, and stimulating students to ask, questions like "Do you see a pattern?" (p. 4). Patterns, according to the *Mathematics Framework* (California State Department of Education, 1992), "help children to see order and make sense of underlying structures of things, situations, and experiences. Patterns help children predict what will happen" (p. 108). A hunt for patterns expands the concept of doing mathematics beyond the search for a single right answer toward an understanding of mathematical relationships, an aim of the current reforms in mathematics education.

Recognizing, describing and creating a wide variety of patterns provides a foundation for exploring mathematical relationships in numbers in later grades. The *Curriculum and Evaluation Standards for School Mathematics* put it this way:

> From the earliest grades, the curriculum should give students opportunities to focus on regularities of events, shapes, designs, and sets of numbers. Children should begin to see that regularity is the essence of mathematics. The idea of a functional relationship can be intuitively developed through observations of regularity and work with generalizable patterns. (p. 60)

In the context of a study of patterns, the doing of school math bears strong resemblance to the work of real mathematicians. The study of patterns now permeates the latest version of the *Mathematics Framework* (California State Department of Education, 1992) as a "unifying idea" across content strands. There are good reasons why patterns have come to take a prominent role in elementary mathematics curricula.

> What matters in the study of mathematics is not so much which particular strands one explores, but the presence in these strands of significant examples of sufficient variety and depth to reveal patterns. By encouraging students to explore patterns that have proven their power and significance, we offer them broad shoulders from which they will see farther than we can. (Steen, 1990, p. 8)

<u>What is a pattern?</u> There was no indication in the teacher's guide that my question about what a pattern is was reasonable. To have asked it would have required me to value my own question above what I found in the teacher's guide. This would have required a confidence that I did not have at the time. In the throes of changing my practice, my self-confidence was at low point. I thought looking to the CSMP teacher's guide as an authority for what and how to teach was a safe way to proceed with making changes in my math teaching. Doing the teaching was hard enough without taking on all of the responsibility for decisions about what to do next. I assumed that the

teacher's guide knew better than I what questions were reasonable to ask. At the time, I also assumed that my questioning the definition of a pattern was related to my being new to investigations of patterns. I realize now that noticing patterns and figuring out what a pattern is are related tasks. Contemplating one informs the other. Constructing the definition of a pattern, as you notice patterns, is part of the work of figuring out whether a pattern is relevant or irrelevant.

Regularity and predictability are the two fundamental characteristics of a pattern. Something is a pattern if you can observe regularity in it. The regularity allows you to be predictive about the pattern's behavior. A numerical pattern with regularity and predictability enables you to describe a relationship between two variables. Identifying a pattern allows you to manipulate one variable and predict what will happen with the other. A relationship between two variables with this kind of regularity and predictability is a function.

I have learned several ways to think about this relationship called a function. One way is to describe this relationship of variables as a relationship between sets as shown in Figure 4.1. If for any member of one set (Set A), you can describe the predictability that enables you to tell with certainty the corresponding member of a second set (Set B), then the relationship between the two sets is a function. In other words, given $f(x) = y$. For any x ($x_1, x_2, x_3, \ldots$), you can predict any y ($y_A, y_B, y_C, \ldots$).

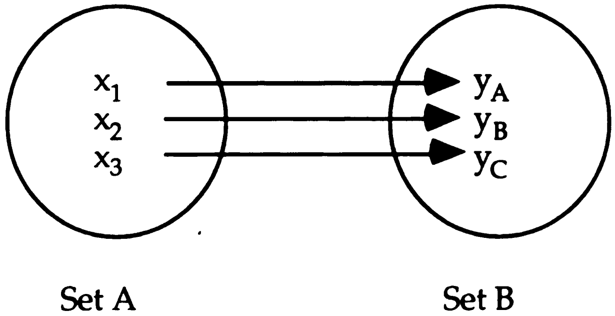


Figure 4.1

A functional relationship between two sets

Leinhardt, Zaslavsky, and Stein (1990) offer a similar explanation of this relationship using slightly different language. A function is a "special type of relation or correspondence, a relation with a rule that assigns to each member of Set A exactly one member of Set B" (p.27). A well defined function will allow one to say what y value will go with a particular x. For each x there is only one value of y. Each element in the domain (Set A) corresponds to only one element in the range (Set B). When starting with a particular value, you can predict what the output will be with certainty. Functions carry with them the notion of predictability and regularity, the essence of pattern.

Why spend time on functions? Understanding the composition of functions encourages students to build flexibility into the way they look at numbers. Studying functions in elementary school is a way to prepare for algebra or "the study of operations and relations among numbers through the use of variables" (Karush, 1989, p. 4). It is also a way to prepare for operating "with concepts at an abstract level and then applying them, a process that often foster generalizations and insights" (National Council of Teachers of

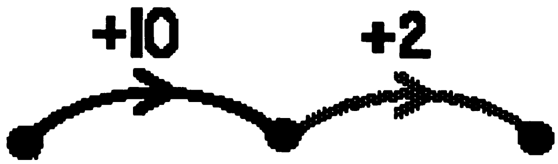
Mathematics, 1989, p. 150). CSMP is a curriculum oriented around functions. It attempts to build some conceptual understandings of algebra without using terminology through work with arrow roads. From this foundation, the expectation is that students, in later grades, will be able to do the kind of abstract generalizing required in the explicit study of algebra. A major goal is to build the kind of thinking and flexibility that will make a transition to the more advanced world of mathematics easier, especially aspects of mathematics having to do with functions.

An overall goal of CSMP and this lesson was to learn to look at calculation in different ways. Composing functions offers an alternative way of thinking about computation through the practice of mental strategies for the purpose of combining different numbers. For example, adding 12 can be thought of as adding 10 plus adding 2. A developer of CSMP admitted that the problems given in the CSMP teacher's guide, + 10 followed by + 2, + 7 followed by - 3, and + 9 followed by - 4 are not particularly strong examples of good practice in mental arithmetic. The numbers are easy to calculate and you do not have to have very complex mental strategies to figure them out. A more difficult example might be to add 38 and think of it, for example, as the composition of two functions, + 40 followed by - 2.

Looking back on these problems now, I can see how mental arithmetic could be emphasized but this was not apparent to me as a goal when I read through the teacher's guide. When I asked students to explain how they figured out starting and ending numbers, it could have been understood as an attempt to make public students' mental strategies for figuring out these problems. The calculations they did, however, were not too difficult. Therefore, the mental strategies that students revealed were not too interesting for a student to report or the rest of us to consider. Perhaps larger and more difficult numbers to compose would have made completion of the table more interesting and the goal of building flexibility in mental strategies more obvious to me.

Learning the Purpose of the Task

What is the meaning of the task of searching for patterns in a table? Why are tables useful in learning about functions? What were students supposed to notice when they looked for patterns in the table? These were not questions I asked myself three years ago. At that point in time, I assumed that by doing the lesson, students would learn something. Rather, I hoped they would. Someone else had decided it was a worthwhile task. Three years later, I decided, once again, to find out for myself what was important about it. I approached my investigation of the meaning of this task with two questions. First, what sort of representation of a function is a table? Second, what does it mean to look for patterns within such a table? The problems under the heading of "composition of functions" in CSMP looked like the problems in Figure 4.2. The labels of the arrows varied. Once the table was filled with numbers, the students were to notice patterns.



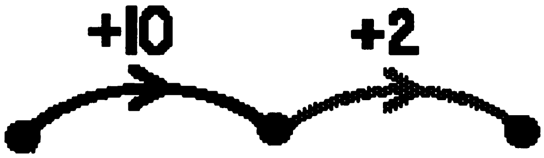
starting number	ending number

Figure 4.2

The first problem I gave students

My current understandings of the purpose of the search for patterns within this problem comes from my own investigation into its meaning by using resources outside of the CSMP teacher's guide.

What were students learning about functions when they completed the table with the numbers as shown in Figure 4.3? To complete the table, my students chose inputs and predicted outputs or chose outputs and predicted inputs. Picking any starting number, they could be certain of the ending number by adding 12 to it. They could also predict the starting number by subtracting 12 from any ending number. The only uncertainty came if a student had difficulty performing the addition or subtraction. When students started with the ending number, they performed the inverse function, arrived at by subtracting first 2 and then 10. Doing enough of these, students could have learned about inverse relationships among functions.



starting number	ending number
99	111
8000	8012
250	262
1	13
7	19
203	215
~~4088~~ 4988	5000

Figure 4.3

The beginning and ending numbers generated by students

Commutativity is also a possible focus. The students could have asked the question, if the two functions are applied to a starting number in the opposite order, do you get the same ending number? In other words, if I do + 2 followed by + 10 do I end up in the same place as + 10 followed by + 2? The table is normally thought of as the simplest representation of a function. In the example of + 10 followed by + 2, the table represents a composition of functions. The first operation, add 10, gives an output that becomes the input for the operation, add 2. The first function is $f(x) = x + 10$. The output of this function is symbolized by $f(x)$. The input is x. The second function is $g(x) = f(x) + 2$ where $f(x)$ is the input for $g(x)$. It can be written algebraically as shown in Figure 4.4.

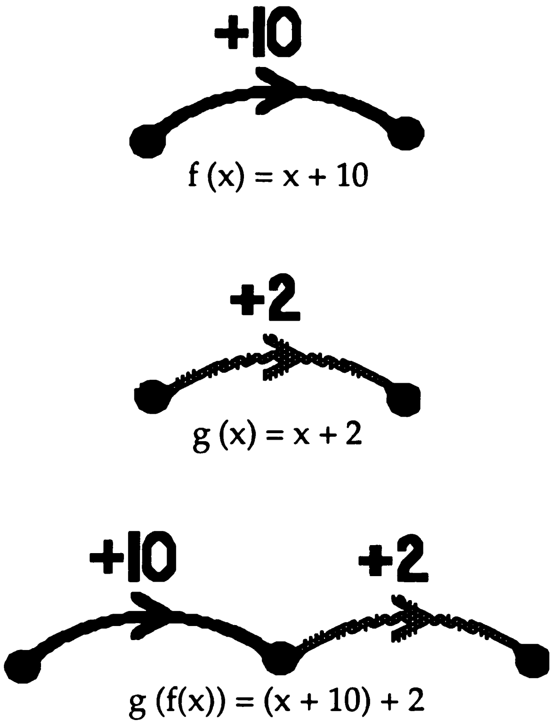


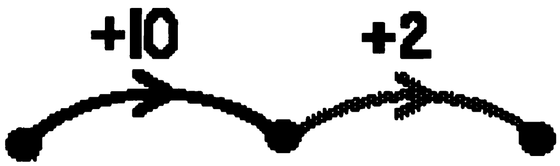
Figure 4.4

The composition of functions represented algebraically

In the case of the composition of functions, the two functions are composed such that the outputs for the first function become the inputs for the second function.

Looking for the kinds of patterns that lead to an understanding of functional relationships involves multiple steps as well. The two columns of a table come in rows which have a beginning number and an ending number. The beginning number determines the ending number because the labels of the arrows allow me to predict an ending number from a beginning number or vice versa. To notice patterns, as they relate to functions, I begin by making observations about a pair of numbers in one row. If these observations hold true for more than one row, they are patterns.

There is another layer of sophistication to the analysis of patterns in tables. Looking, for example, at a set of starting and ending numbers not chosen randomly--in the example below, the starting numbers are all multiples of 3--I can look for patterns down the columns as well as across the rows.



starting number	ending number
3	15
6	18
9	21
12	24

Figure 4.5

Starting and ending numbers as multiples of 3

By putting the beginning numbers in order, and choosing them with some regularity, I can make use of patterns in the column of starting numbers to predict the column of ending numbers. Since the outputs depend on the inputs, I can look to see if there are patterns in the ways that the outputs change when there is a certain orderly change in the inputs. I could ask, for example, is there a constant difference between the outputs when there is a constant difference between the inputs? Patterns in a table where the numbers are ordered and not randomly chosen have the potential to be more complex because they represent a coordination between changes in inputs and outputs. Patterns and functions are also a way to consider classification. Functions are objects and as objects they can be classified and sorted. Different functions will have different kinds of patterns. For example, the function + 12 has certain patterns. The functions + 10 or + 2 will have other patterns.

Learning to Make Sense of the Search for Patterns

What sense can be made out of the question of looking for patterns in the context of a table of numbers? To answer this question, I look at both the responses that are a part of this lesson in the teacher's guide and my students' responses made during the lesson. How did I understand these responses in the teacher's guide then and how do I understand them now? What sense did I make out of the students' responses then and how do I interpret them now? What do the inconsistencies between my perceptions and understandings then and now say about what I am learning?

Responses in the teacher's guide. If the CSMP dialogue is intended to be a guide and a way to prepare for what might happen along the way, an understanding of the text seems important. What I found is that just reading the text did not guarantee that I understood it the way CSMP may have intended. In my explorations of the intentions of CSMP, I discovered a wealth of mathematical ideas and connections within the teacher's guide that were not apparent to me at the time and only recently became visible to me in a conversation with a math educator involved in curriculum development and familiar with the development and design of CSMP.

I focused my conversation with a developer of CSMP on the three student responses listed in CSMP in response to the question of noticing patterns.

```
T:  Look closely at this chart.  What patterns do you notice?

S:  An ending number is always larger than the starting number.

S:  If you start with an even number, you end with an even number.
If you start with an odd number you end with an odd number.

S:  An ending number is always 12 larger than the starting number.

(p. 14)
```

Figure 4.6

Student responses to the question of patterns from CSMP

When I taught the lesson, I was under the impression that these responses were ones I should expect from my students because these represented how other students have responded to the question. What I have since discovered is that the student responses that appear throughout CSMP come either from the developers' experiences in classrooms with students or they represent ideas that are mathematically interesting from the perspective of the person or people responsible for writing the text of the lesson. I learned that the responses in this particular lesson were mathematically interesting ones and not ones that I could necessarily expect from students. I wish that the teacher's guide had included information on where these responses had come from and why they were considered mathematically interesting.

Here is how I now understand the responses in CSMP. The third response in the text, that the ending number is always 12 larger than the starting number, is a response at the level of a generalization and the most related to the function notion. This response represents an important mathematical idea through its description of the relationship between the starting and ending number. The first response, that the ending number is always larger than the starting number, is an observation that is a more general variation on the third response. The second response, if I start with an even number, I end up with an even number or the parity of the odd/even relationship of the starting number and ending number, is an interesting observation. If I add 11 rather than 12, the parity would be different. The significance of each response and how they are related to one another is something I have learned but not from the teacher's guide. If I had known any of this going into the lesson, would it have helped me along the way? At this point, I can only hypothesize. I think the answer to this question would be yes.

The questions of how much, what kind, and in what ways to provide information for the teachers is an on going dilemma in curriculum development and a problem that the developers of CSMP have considered. When I voiced my concern of not having had access to certain information and ideas through the teacher's guide, the CSMP curriculum developer explained concerns about giving too much background on the responses might make the use of the guide too tedious for teachers. What is the balance between enough and too much information in a teacher's guide? In this particular example, how useful are the suggested students' responses without the additional information I learned on my own? If I had had some understanding of these responses, I think I would have had a better sense of what made for sensible answers. I expected the students' responses to vary but without an understanding of the significance of the mathematics, I had no idea what reasonable variations might be.

<u>What were my students seeing?</u> What am I able to see and hear in my students' responses now that I was unable to see or hear at the time I was teaching? I was curious to see if it really was the case that what my students

had said in response to the question, "Do you see any patterns?' was unlike any of the students' predicted responses in the teacher's guide. In examining this lesson three years later, I have gone back to the video tape of this lesson and replayed it with the scripted lesson from CSMP in one hand and a transcript in the other.

Valerie's response. In Figure 4.7 is an excerpt from the teacher's guide side-by-side the excerpt from the lesson's transcript where I asked the question of patterns.

The CSMP Script	Transcript 09/20/89	
T: Look closely at this chart. What patterns do you notice?	Ms. Heaton:	OK, I want you to look at these numbers, do you see any patterns? Valerie?
S: An ending number is always larger than the starting number	Valerie:	Each of them have a beginning number and then they have an ending number that is 12 more.
S: If you start with an even number, you end with an even number. If you start with an odd number, you end with an odd number.	Ms. Heaton:	O.K., and how do you know that it is 12 more?
S: An ending number is always 12 larger than the starting number. (McREL, p. 14)	Valerie:	Because you have to add ten, you find a number, you add ten to it and then 2.
	Ms. Heaton:	If were to put in another arrow here, what would I put, plus what?
	Valerie:	plus 12.

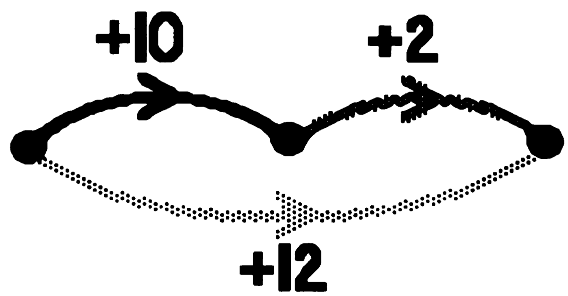


Figure 4.7

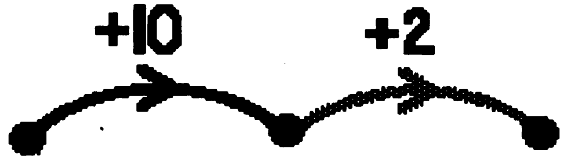
CSMP student responses compared to Valerie's response

As I look back at the teacher's guide and the transcript, now, I see that Valerie, the first person to respond to my question of patterns, gave one of the answers that is in the teacher's guide. I even responded to her, added the composition arrow, and labeled it with plus 12. Yet, I continued with what resulted in a pointless discussion of patterns and I ended that class feeling frustrated that my students had not responded in any of the ways predicted by the teacher's guide. What does this discovery of the similarity of Valerie's comment to what appears in the teacher's guide say about what I understood to be my frustrations at the time and what I have learned since then that might help me to understand my feelings? I wanted to try to understand rather than discount the sense I made out of the situation at the time. How could it be that I thought all of my students' responses to the question of patterns were meaningless and bore no resemblance to what was in the teacher's guide when Valerie's response was really such a close match?

One explanation that comes immediately to my mind was my focus on having a discussion. If this was the case, then I was not really focused on the right answer and it might make sense that I did not "hear" Valerie or want to "hear" her. If I had, she was the first person to respond and her "right answer" may have ended the discussion. What more was there to talk about? It also seems quite possible that I might have "heard" Valerie but did not really understand what she was telling me. At the time, I was uncertain about the point of the lesson. If the question of noticing patterns relates to the reasons why one would look for them, it might follow that I was listening for the responses in the teacher's guide but I did not really understand their significance to the question of patterns in this context. I did not really know what I was listening for. My ability to hear Valerie now may have something to do with what I have learned about patterns and functions since that year of teaching.

Valerie's response, that each of the beginning numbers has an ending number that is 12 more, seems on the surface to be rather mundane. But, I appreciate now ways in which it is mathematically significant. Her response is a generalization that defines the composition of the two functions in this problem. If I had asked her to write down what she was saying in the shortest form possible, she would probably have written something like, "number + 12". This could be thought of as the equation, "X + 12." I now see that looking for patterns in a table is like looking for the equation that defines the function. This is a move in the direction of algebra. Valerie's generalization meets the criteria of regularity and predictability, the two key characteristics of patterns. It is a statement that holds true for the pairs of numbers in all rows. You could choose any starting number and predict with certainty, using Valerie's pattern, the ending number.

Other students' responses. What about the responses of the other students? Were those patterns? If not, what were they? What the other students were observing were not mathematically relevant patterns. They were observing regularities in the numbers. Take the following "pattern" a student noticed as shown in Figure 4.8 for example--there are 3 zeroes in 8000 as well as 5000.



starting number	ending number
99	111
8000	8012
250	262
1	13
7	19
203	215
~~4088~~ 4988	5000

Figure 4.8

A student's example of a pattern

There is nothing predictable about this observation. It does not tell me anything about what else is happening with numbers in the table. If I make use of the criterion that to find patterns I must start out with observations about a pair of numbers in one row, the responses where my students made observations about pairs of numbers not in the same row are eliminated immediately as patterns. The "pattern" identified in the table above is an example of this. Even when students were looking at pairs of numbers in the same rows, the sorts of regularities they were noticing were not predictive.

When my students explained how they found beginning and ending numbers to fill in the table, they gave procedural explanations of how they got from one to the other. Their explanations, which I first interpreted as dull and pointless, reveal the meaning of the composition of functions and the relationship between outputs and inputs. The following examples illustrate this point.

Richard:	Well, I did 99 plus 10 equals 109 and then plus 2 is like 9 plus equals 11.
Lucy:	1 plus 10 is 11 plus 2 is 13
David:	Because 7 plus 10 is 17 and 17 plus 2 is 19.
Bob:	Because 203 plus 10 is 213, 213 plus 2 is 215.

If their procedures were represented in a chart of inputs and outputs they would look as they appear in Figure 4.9 :

input	output	input	output
99	109	109	111
1	11	11	13
7	17	17	19
203	213	213	215

Figure 4.9

The relationship between inputs and outputs

Perhaps there was something interesting to notice in their responses to the question of how they got from their beginning number to their ending number. Mathematically, the composition of functions is a layer of complexity that is quite significant even though the mental arithmetic necessary to fill in the chart was rather simple for my fourth graders.

Bob's question about what kinds of numbers they should choose as beginning numbers pointed out to me that there were no constraints on the numbers to use in the chart as the task was presented in the teacher's guide. When either the starting or ending numbers are chosen at random, as was the case in the problem I gave my students, the chance of noticing interesting patterns is reduced. Constraining the choice of numbers and ordering the numbers chosen could help to make obvious some interesting patterns between rows and columns which might otherwise go unnoticed.

Learning the Shape of the River

Mark Twain's writing about what he had to learn to navigate the river in *Life on the Mississippi* (1990/1883) offers me images of what it is that I needed to learn as I attempted to make changes in my math teaching. In this section, I construct parallels between his early learning experiences and mine mainly in the sense of his growing understanding of the nature of what he needed to learn and the usefulness of a text in doing the work. I draw further connections between Twain's experiences and mine in Chapter 7, a description and analysis of a series of lessons later in the year, and in Chapter 8, the conclusion to this dissertation.

In this particular lesson, I found myself dependent on the CSMP teacher's guide. I trusted that it was going to help me do the kind of mathematics teaching I envisioned. In a similar way, Twain began his adventures on the river with a notebook in his hands on which he depended. Twain, like me, thought this text was going to be the key to navigating the river. He wrote about what was in his notebook and his sense of its contents,

> I had a notebook that fairly bristled with the names of towns, 'points,' bars, islands, bends, reaches, etc.; but the information was to be found only in the notebook--none of it was in my head. (p. 43)

For both Twain and I, our work while trying to follow these guides did not go as planned. We both became frustrated. Within a short time on the river, Twain found that the information in his notebook was insufficient for the navigation he needed to do. I found the same to be true in my efforts to make use of the teacher's guide. Following its directions was not helping me teach in the way I imagined it would when I found myself trying to interact with students around mathematical ideas. Twain wrote of the frustration his reliance on his notebook caused him.

> The boat came to shore and was tied up for the night . . . I took my supper and went immediately to bed, discouraged by my day's observations and experiences. My late voyage's notebook was but a confusion of meaningless names. It had tangled me all up in a knot every time I had looked at it in the daytime. I now hoped for respite in sleep; but no, it reveled all through my head till sunrise again, a frantic and tireless nightmare. (p. 48)

I can understand how he felt. The teacher's guide and the sense I made of it seemed to hinder my teaching more than help it. I was naive to think that the teacher's guide could carry aspects of this teaching that I was learning were my responsibility. For example, I needed to have an understanding of the mathematical purposes for what I was doing, a mathematical sense of why I was asking the questions I was asking. Simply following the script in the teacher's guide was not working. Without a sense of purpose or an understanding of what mathematics was important for my students to learn, I was lost as far as being able to figure out what to do next.

Mr. Bixby intended for Twain's notebook to be useful. But, how did he expect him to make use of it? What role did he think it should play in Twain's navigation of the river? What did the writers of CSMP know that made them think that what was in the teacher's guide would be helpful to me? What was it that either Twain or I needed to know to make use of our texts? Was it the text that needed to change or the way we made use of it? Or, both?

I think that Mr. Bixby knew much more about the river than Twain realized and I doubt that Mr. Bixby, an experienced riverboat pilot, ever imagined that Twain would rely so heavily on his notebook. In my case, I did not want to be so dependent on the textbook but, in the heat of the moment, in the face of uncertainly, I fell back on old and familiar ways of using a math textbook. I trusted it. I followed it without an understanding that was meaningful to me of what was important for students to learn and why. Like Mr. Bixby, I do not think the developers of CSMP ever expected me to cling so tightly to the text. That was part of my problem. But, to stray from the text in purposeful ways, without wandering too far from important mathematics, I needed a much stronger and clearer sense of purpose for what I was doing. I started out searching for that when I planned the lesson. When I failed to turn up much that seemed useful, my past experiences pushed me to go ahead with the lesson anyway. That is when I found myself in trouble, unprepared and ill-equipped.

Twain wrote about what he learned about the way he needed to know what was in his notebook.

> I have not only to get the names of all of the towns and islands and bends, and so on, by heart, but I must even get up a warm personal acquaintanceship with every old snag and one-limbed cotton-wood and obscure wood pile that ornaments the banks of this river for twelve hundred miles; and more than that, I must actually know where these things are in the dark. (p. 47)

The idea that he learned he needed a "personal acquaintanceship" with the river seems related to what I learned about the sense of purpose I needed to teach mathematics for understanding. The authors of CSMP had a much greater sense of what was important for students to learn and why particular concepts were important than I could see in the teacher's guide. The information in Twain's notebook was a representation of the river. My teacher's guide was a representation of the mathematical terrain. Mr. Bixby knew the river. The authors of CSMP knew the terrain. What would Twain's notebook look like if it represented what Mr. Bixby knew about the river in a way that was accessible to Twain? Could it be represented in a text? What would it look like for the teacher's guide to represent what the developers of CSMP knew about the mathematics in a way that was accessible to me?

After Twain had been learning particulars on the river for a while, Mr. Bixby spoke in more general terms about what he thought Twain was learning and why it was important,

> You learn the shape of the river; and you learn it with such absolute certainly that you can always steer by the shape that's in your head, and never mind the one that's before your eyes. (p. 54)

From this particular lesson, I learned that I needed to develop a sense of purpose for what I was teaching that was my own. I learned, specifically, about what it would mean to have this sense of purpose for teaching patterns and functions by looking closely at why patterns and functions are important to teach, why this particular lesson was important, and what students could learn from it. But what I have learned goes beyond the specifics of patterns and functions. What I take from this lesson is the need for a new sense of purpose, the questions to ask myself to go about constructing a sense of purpose for whatever mathematical idea I am trying to teach, and an understanding of why this sense of purpose is important in the kind of math teaching I am learning to do. What I have learned is that the river or the mathematical terrain has a shape that I must learn to enable myself to navigate my way around in it. What is important is my understanding of the mathematics I am trying to teach for that will help me "hear" my students and build connections from their ideas to the mathematics I want them to learn.

CHAPTER 5

I WANT TO SHOW YOU SOMETHING: THE FIRST MOMENT OF TEACHING

Introduction

Not all math lessons were as frustrating as what transpired in the last chapter. In fact, the day following the lesson on patterns was not such a bad day. Because of their contrast, I chose to study these two lessons back-to-back: 09/20/89 in Chapter 4 and 09/21/89 in this chapter. My intense frustrations of the day before are followed here by a fulfilling moment, the first time since the school year began that "I felt like I was teaching" (Heaton journal, 09/21/89). From my perspective as the teacher, this chapter is about that moment, a moment when I had a sense a purpose for what to do, I had something to do, and I knew why I was doing it. What I am learning here is how a sense of purpose can inform teaching and the ways in which teaching is moment-to-moment work.

At the time of this lesson, as was the case in the previous chapter, I basically followed math lessons as they appeared in CSMP without a sense of purpose that was my own for what lessons I did with students. I made use of the teacher questions provided in the script together with the problems and activities as they appeared in the teacher's guide. Following the lessons in the teacher's guide gave me a starting point, offered me problems that I could give my students, and questions that I could ask them. As I began to understand the improvisational nature of teaching, I could see that making use of a textbook, as I had in the past, limited rather than strengthened my ability to improvise. In addition to questioning the way I made use of a textbook, I began to question what sort of resources would be most helpful to me as I tried to be responsible to the mathematics I wanted to teach as well as responsive to the sense my students made of ideas.

The chapter begins with a description of my planning for the lesson and the interactions with students that occurred during the lesson. This is followed by reflections on the lesson at the time. This is a lesson Maggie observed and her observations prove instrumental to me in identifying what happened in a moment that seemed significant to Maggie immediately after teaching the lesson. Three years later, I look back on this moment and examine how I view its significance now. The nature of the moment, a moment of improvisation, serves as another image of the teaching I tried to learn to do. I conclude by drawing connections between learning to do this math teaching and learning to improvise jazz (Sudnow, 1978; Mehegan, 1959; Mack, 1970). Learning to play improvisational jazz is a useful metaphor because of the parallels I see with teaching in terms of the spontaneity of the work, the demands on the person learning to do the improvisation, and what it means to be able to respond in immediate and meaningful ways.

Planning to Teach

I was in the midst of the third week of school and had yet to feel like I was teaching. I had my doubts that anything I was doing in math class came close to teaching mathematics for understanding. As I prepared for the next day's lesson, the questions and difficulties that arose for me with the question of patterns earlier that day weighed heavily on my mind. I thought back over what my teaching had been like in these first few weeks with the intense frustrations of the day before reminding me of everything I thought was wrong with my practice.

I had been using the CSMP curriculum to make my plans, to provide my directions, and then I tried to follow them in the classroom. I could feel, however, that this was not working. Even though I had lesson plans, I found that little of what happened went as planned. Everything seemed to be "out of synch." I had been following a teacher's guide yet felt lost. I had the urge to move but lacked a clear sense of direction. I kept asking questions of my students that I thought would move us to a new place but we remained where we were or moved in an unexpected direction. I wanted to engage my students in a mathematical discussion but found myself not knowing what there was to talk about. I trusted the teacher's guide but had little confidence in myself. I had been going through what I thought were the motions of planning and teaching mathematics for understanding but I was uncertain what my students were learning or supposed to be learning. I had expected that learning to teach mathematics for understanding would be difficult but I never thought it would be as challenging as I was finding it to be. Why was this so hard?

In the last math class, as described in the previous chapter, I had done the first part of Exercise 1 from the CSMP lesson on the composition of functions. The activity involved finding and describing patterns. I planned to do the second exercise from the same CSMP lesson the next day. It continued with the composition of functions but this time the context was multiplication rather than addition and subtraction.

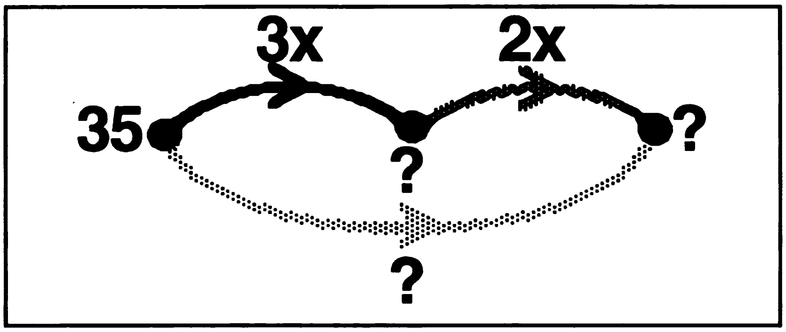
I paged through the teacher's guide and searched the script for any signs of a question about patterns, similar to the one I had asked the day before. After my frustration with the question in the first part of the lesson when I asked students to notice patterns the day before, I had decided to pay closer attention to the script as I planned the next half of the lesson. There was no further mention of patterns in this second half of the lesson. I did not have to attempt to initiate another discussion around the idea of patterns.

I reread the lesson summary in the teacher's guide that I had tried to understand several days earlier.

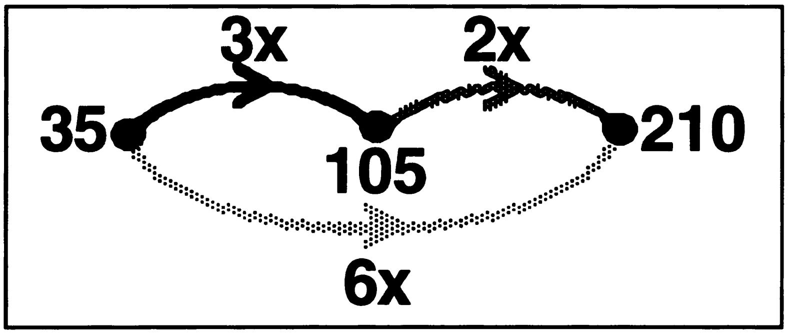
> Using arrow diagrams and the Minicomputer, investigate the composition of certain numerical functions, for example, +10 followed by + 2 and 3 x followed by 2 x. (p. 11)

I found myself no clearer about the point of this lesson as I looked it over a second time and considered what we had covered and what remained for us to do. What had my students learned through the identification of meaningless patterns we had done in the first half of this lesson? What were students supposed to investigate about the composition of functions in the context of multiplication? Why was the Minicomputer used in this part of the lesson and not the previous one?

As I glanced through the teacher's guide, the kinds of arrow problems I saw did not look too different from the ones we had just done and, like the others, these looked like something my students could do even though I was still unclear about the point of the lesson. Here is an example of one of the problems:



Again, one arrow was red and the other was blue. I turned the page in the teacher's guide and saw the green or composition arrow labeled 6x for this particular problem. When I first saw the green arrow, my initial thought had been to label it 5x rather than 6x, adding 3x and 2x. This was consistent with finding the label for the composition arrows in the addition and subtraction problems in the first half of this CSMP lesson. I wondered if my students would do the same thing.

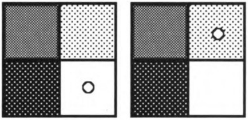


The CSMP worksheet I planned to assign, intended by CSMP to accompany the entire CSMP lesson on the composition of functions, gave me a purpose for getting through this lesson: to be able to do the worksheet. The worksheet contained problems dealing with the composition of functions in the context of addition and subtraction, and multiplication. Each problem consisted of a set of arrows with a table to be completed by composing two functions. I know my purpose was quite narrow and shallow but it was a purpose, nevertheless, and one quite familiar from my past more traditional math teaching experiences. Is some purpose better than none? What this worksheet did for me at the time was give me a measuring stick against which I could evaluate my accomplishments. Were my students learning anything? Giving students the worksheet was one way to find out. If my students did well on it, it might help to ease my mounting uncertainty about what it was my students were learning.

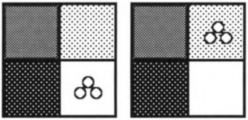
The Minicomputer

The Minicomputer, a representational tool described in the prologue to Part II, The Creation and Study of a Practice, was intended, according to what appeared in the teacher's guide, to accompany this portion of the lesson. There was no explicit discussion in the teacher's guide as to why this was a useful tool for the multiplication problems in this part of the lesson. Its presence implied its importance and I came up with my own reasons why. I thought that arranging checkers on the Minicomputer, as pictured in the teacher's guide, illustrated the number of groups involved in the process of multiplication and that being able to keep track of that might help students see the number of groups involved when composing two functions in the context of multiplication. Knowing the number of groups involved was essential to labeling the arrows, something students had to do to work these problems and complete the worksheet.

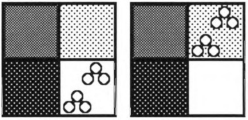
For example, multiplication of 14 by 3 can be thought of as 3 groups of 14 objects or 42. Fourteen can be represented on the Minicomputer with 1 checker on the 10 square and 1 on the 4 square or 10 + 4. Three times 14 or 3 groups of 14 can be represented by repeating the pattern of checkers for 1 group of 14, 3 times. If this whole thing is multiplied by 2 or doubled, this means the same as 2 groups of 3 groups of 14 or 2 x (3 x 14) or 6 x 14. This can be represented on the Minicomputer by repeating the pattern of checkers used to represent 3 x 14, twice.



14
or
10+4



3x14



2 x (3x14)

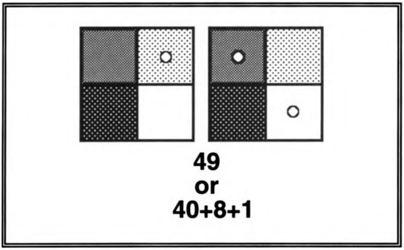
The pattern of the arrangement of checkers on the Minicomputer was a way to keep track of the number of groups when two functions in the context of multiplication were composed and that was why the Minicomputer seemed to me to be a useful tool here. Seeing the checkers in a physical pattern showed the number of groups. This way of considering 'pattern' was not something discussed in the CSMP teacher's guide. It was something that occurred to me as I planned for the lesson.

As in earlier lessons, I remained committed to following the script closely. I followed the teacher's guide and insisted my students use the Minicomputer for every problem just as outlined in the text. There was nothing in the teacher's guide that indicated I should question the relevance of the Minicomputer as it related to my students' work on particular problems or take into account other ways students might come up with solutions. I went into the lesson having drawn my own conclusions as to why the Minicomputer was a useful tool for working these multiplication problems.

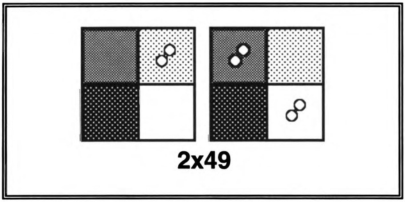
The Minicomputer was a new tool for me and the handful of students in my class who were new to the school. We had used the Minicomputer once before this lesson. When we did, I had found it a challenge to consider the placement of the checker, the value of the square, and do the mental arithmetic necessary to keep track of the total value of the checkers on the Minicomputer. As I glanced at the pages of pictures in CSMP of the checkers arranged on the Minicomputer, I could see that the number of checkers increased as functions were composed. Reluctant to take on the mental concentration I thought was required to keep track of the checkers, I decided to stick closely to the script and use the numbers and checkers exactly as they appeared in the teacher's guide. If I did this, I could just copy the placement of the checkers as they appeared in the guide and not worry about keeping track of the value of the checkers myself.

Teaching

Before math class began, I stood two Minicomputer boards along the chalkboard tray at the front of the room. One board represented tens, the other represented ones. I began with the first example in the teacher's guide and placed three checkers on the Minicomputer to show 49.



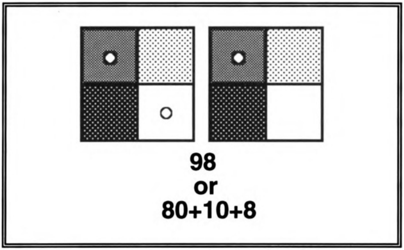
Following the script, I asked students what number was represented on the Minicomputer. John responded, "49," and at my request explained how he knew, "Because the checker on the purple space on the second Minicomputer is 40 and the ones Minicomputer, that has a checker on the 8 and 1 on the ones and that is 9." I moved to the next problem in the teacher's guide and asked students to show "double 49." Bob volunteered to arrange checkers on the Minicomputer.



His arrangement of checkers matched what was shown in the teacher's guide. The teacher's guide, however, contained this additional note.

NOTE: There are other correct configurations for 2 x 49, such as the standard configuration for 98, but the one above emphasizes that there are 2 49's on the Minicomputer. (p. 17)

There was no explanation in the teacher's guide as to why the configuration emphasizing two 49's was preferred over other arrangements, like 98 made with the fewest checkers possible.



Given this preference, I assumed the difference in the two representations was in using the Minicomputer as a tool to illustrate the process of doubling versus using it as a representation of the answer to the doubling. I could see a connection between showing the process and labeling the arrows on the worksheet. Students had no way of knowing that CSMP preferred one way of arranging the checkers over another. For example, Mike announced, "I have an easier way of doubling 49 than just putting another checker on the 40's and another checker on both the 8 and the 1," and proceeded to arrange 98 with the fewest possible checkers as shown above. His way was easier if the purpose of using the Minicomputer was, as he might have interpreted it, to show the solution once it had been computed by some other means. It was not easier if the intention of its use was, as I interpreted it, to understand the meaning of multiplication by representing the number of equal groups.

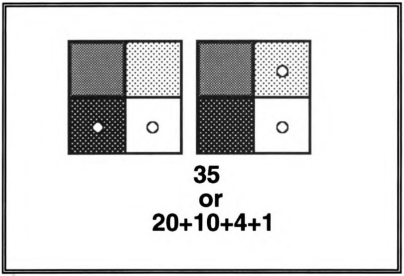
As in the lesson of the day before, I inserted my own questions in the script of the teacher's guide. I asked if people had other ways of thinking about the problem and Ron explained his mental strategy, "50 and 50 is 100 and then take away 1 and it is just 9. And, then another 1 and it is 98." There was no mention in the teacher's guide that students might figure out the totals to the multiplication problems using alternative methods of composing and decomposing numbers and not attend to or need the Minicomputer. I was a little annoyed that students, like Ron, were not making use of the Minicomputer and that students, like Mike, were not using it the way I thought the teacher's guide intended. It was going to become harder to make it through my plans if students kept offering these alternatives.

On the outside, I accepted all students' responses whether or not they matched the teacher's guide. On the inside, my own frustration was mounting. I was confronted with a conflict. Should I get the students to do as the script said and trust that this would cause learning? Or, should I encourage students to be analytic, creative, and innovative? What was I trying to do here? I could see that the students were able to do the math in their heads, that they had ways of figuring out the answer that did not require using the Minicomputer. So, in one sense they could do the math. But in another sense, I wanted them to follow the directions of using the Minicomputer because of where I thought the lesson was headed. It seemed to me that using the Minicomputer to solve one-step multiplication problems was building up to using the Minicomputer for the composition of functions. I continued to use it as an accompaniment throughout the entire lesson even though I was one of the few people in the class engaged in its use. Most students used other ways to find the solutions to the multiplication problems. What was the right thing to do?

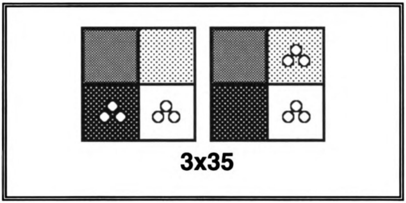
I persisted with the Minicomputer and worked through the next two examples in the teacher's guide with the numbers 65 and 27. I arranged checkers to represent these numbers and then students, with some prompting on my part, doubled or tripled the pattern of checkers to represent 2 or 3 times the number. This was a painstakingly slow process because I insisted that students arrange the checkers in patterns as illustrated in the CSMP teacher's guide. I thought that if the checkers were randomly placed in the squares on the Minicomputer, the idea that multiplication was about repeated numbers of equal groups would be lost. Arranging the checkers on the Minicomputer seemed like a tedious chore when students already had figured out the answers in their heads but I could foresee the benefit of being able to see the number of groups once we got into composing functions. Besides, it was the teacher's guide's idea to make use of the Minicomputer for each of these problems, not mine. Since I had no reason beyond the worksheet to be doing these problems, it made sense for me to follow the textbook closely and do things that would help students successfully complete it.

Most students wanted to share how they computed the solutions to the multiplication problems in their heads or show the ways they had of arranging checkers to represent the total with the fewest checkers possible. There was no indication in the teacher's guide that this was an acceptable way to work the problems. I could not see how their mental arithmetic strategies connected with either my interpretation of the problems or the role of the Minicomputer in figuring them out. An interest in repeating patterns of checkers on the Minicomputer continued to be more mine than theirs. This was understandable, given the worksheet I knew awaited them but they knew nothing about.

The next problem began like the others. I arranged a number on the Minicomputer.



Sipho explained how he knew this was 35. "Because there are 2 checkers on the tens board. One of them is on the 2 and the other is on the 1. One plus 2 equals 3 and on the ones board you have a checker on the 4 and there is another checker on the 1. Four plus 1 equals 5." I assumed that Sipho meant 30 when he said "3" on the tens board. I asked Jamila to show 3 times this number on the Minicomputer. Under my direction, she carefully repeated the pattern of checkers for 35, 3 times.



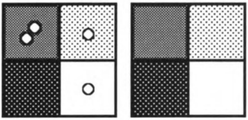
Jennifer, who had been working the problems in her head and paying little attention to the checkers and the Minicomputer, announced that she knew the answer was 105. She explained how she knew, "Because I did it in my head again. I did 35 plus 35 plus 35 and then I went 5 plus 5 plus 5 is 15, put down 5 carry the 10. 30 plus 30, plus 30 equals 90 and I would add 1 more 10 and that's 100." Jennifer, like many of the other students, had no reason to think the Minicomputer and checkers were important when she had mental strategies for figuring out the answers.

We went through the same tedious work of arranging checkers for more than 30 minutes and I seemed to be the only one who could see the relevance in what we were doing. This worried me for I knew we were approaching an important part of the lesson. We had been doing relatively simple, one-step multiplication problems that were about to get to more complex. If students did not understand the number of groups involved when multiplying they might face difficulty in labeling the composition arrows on the worksheet.

The Composition of Functions

I asked, "How would you double this (3 x 35)?" In response, Ron quickly replied, "210." I wished these students would stop giving the total. It revealed nothing about the process of multiplication. Knowing the totals or having a way to figure them out mentally was not necessarily helpful in figuring out what to label the composition arrow. If they knew what was on the worksheet, maybe they would stick with me and attend to the process. I wanted students to see why 2 x (3 x 35) was equal to 6 x 35 not 5 x 35. I had no direct evidence that they thought it would be 5 x 35. I was projecting what I thought they might do based on my own experience. I asked for someone to show how to double the checkers on the Minicomputer. If students understood that 2 groups of 3 groups of 35 was the same as 6 groups of 35 they might understand how to label the composition arrow and understand the meaning of 3 x followed by 2 x.

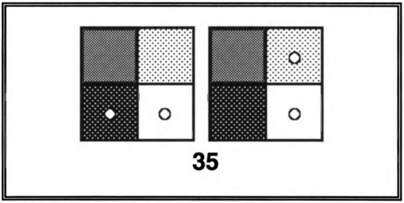
In response to my request for someone to double 3 x 35 on the Minicomputer, Ron volunteered to show 210.



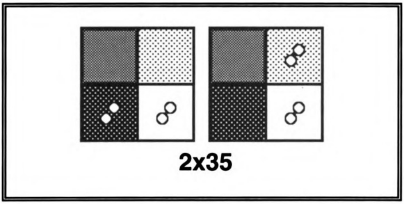
That was fine even though from my perspective I thought it had nothing do with learning about the composition of functions. I just wanted to get the totals out of the way, so we could concentrate on the numbers of groups. Arif said that he had another way to show it. I welcomed it. I assumed since Ron had already shown the total, Arif had in mind to show the number of groups of 35. We all watched as Arif stared at the Minicomputer for a bit, moved some checkers around, added others. After an awkward silence, he said, "I am confused." He appeared to be uncertain how to add and organize checkers on the Minicomputer to represent the doubling of 3 times 35. I asked him why he was confused. He responded, "Because over here we added 3 of them and we got 105 and I thought over here we were supposed to add 2 more of them." I assumed "them" meant 35's. Arif was doing what I had done--addition rather than multiplication. Adding 2 35's to the product of 3 x 35 or 105 was not the same as multiplying 3 x 35 or 105, by 2. How were other people thinking about this problem?

I asked, "Does anyone have thoughts for Arif about this?" I called on Bob who said, "I know how you can make one a little easier instead of putting all of those checkers on. You could make it easier." I assumed that Bob wanted to show the total with the fewest number of checkers and I did not want to go in that direction. I cut him off. Fewer checkers might be "easier" in the sense of trying to keep track of the total value of the number on the Minicomputer but Bob's sense of "easier" did not mean that the groups making up a multiplication problem would be visible, which from my perspective, is what would make the problem "easier." Students needed to know the numbers of groups to complete the worksheet.

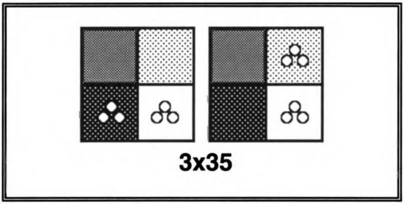
I finally reached a point in the lesson where I felt like I had something to teach. Someone was thinking about the problem the way I thought they would and so I could make use of the representation in the way it had made sense to me. I would show Arif and the others the number of groups I wanted them to see by using the Minicomputer. I put aside the teacher's guide and moved beside Arif. As I started to talk, he returned to his desk. I said, "I would like to show you something here." I removed all the checkers Arif had put on the board and put back only enough checkers to show 35.



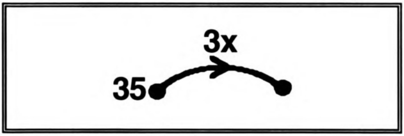
Then I asked, "What number is this on the Minicomputer? Jennifer?" Jennifer said, "35." Next, I repeated the pattern of 35 on the Minicomputer.



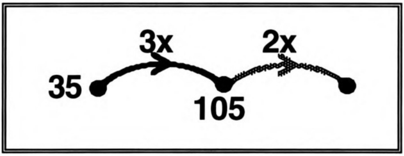
I said, "Okay. If I want to make this a multiplication problem, what would I say? Using a multiplication sign, how could I make a problem out of this, what could I say? Faruq?" Faruq said, "2 times 35." I repeated what he said and wrote 2 x 35 on the chalkboard. Then, I arranged the checkers to show 3 times 35. I repeated the pattern of checkers for 35 a third time.



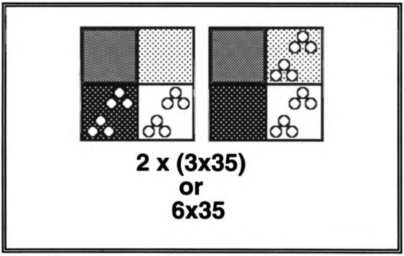
I asked, "What would I say now with multiplication? Using the multiplication sign, what could I say now? Arif?" He said, "3 times 35." I drew an arrow on the chalkboard and labeled it 3 x.



I said, "Now, and this is what some people were having problems with, I want to double this." After seeing Arif at a loss for how to arrange checkers to represent the composition of functions on the Minicomputer, I assumed that I understood the source of Arif's confusion and what others might be thinking based on my own experiences with this problem. I added and labeled a second arrow with 2x.

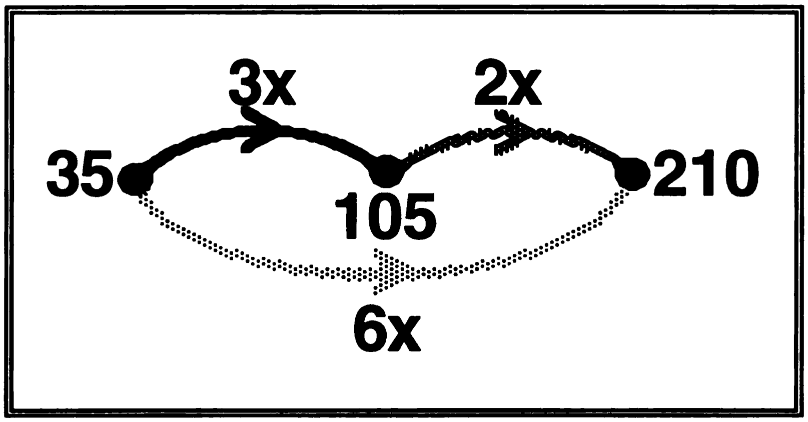


As I did so, the students chorused, "Ohhhhhhhh," as if they had just seen something in new way. I said, "Two times what's here," as I pointed to the Minicomputer arranged with checkers illustrating 3 x 35 and the dot for 105 on the arrow road, "How could I do that without moving these checkers and adding something else to the Minicomputer? Arif?" Arif said, "I remembered now." He stepped forward and repeated the pattern of 3 x 35 on the Minicomputer.



When he was finished, he said, "I have doubled the whole thing." When I asked, "Why did you do it that way? Why does that make sense to you?" He said, "Because you have to double the 105."

When Arif said this, I did not follow up with any further questions. I was content to leave things this way. I wanted to believe that Arif understood. I labeled the arrows like this:



The other students were restless and I needed to get them engaged in something. If I did not believe Arif or others understood, what would we do in the remaining time if I could not give them the worksheet? I had no other way of representing the meaning of multiplication and its relationship to addition. I passed out the worksheet and students spent the remaining 15 minutes of the class working on it independently.

<u>Reflecting on Teaching: A Noticed Moment</u>

Maggie observed class that day. As usual, she had to hurry off after my math class to prepare for her own math teaching in the fifth grade class next door. She handed me her observation notes on her way out the door. We had a scheduled time to discuss them later.[1] I was always anxious to read what she

[1] See Heaton and Lampert (1993) for more information about the interactions between us around my teaching.

had written during her observations. On that day, I read her notes in my car, in the school parking lot, before heading back to the university. In the midst of what she wrote, I found this,

> Did you get some kind of "bright idea" about how to pull all this together when you said "I want to show you something"? Or were you following the script? It seemed to me as if you were more engaged here, more thinking about the kids and the subject matter and the representation rather than following the manual. It seemed like you saw what they needed in order to get the point. (Lampert observation notes, 09/21/89)

Did I get some bright idea? What moment was she referring to? At the time, as I thought back on the lesson, I was uncertain. Her observation peaked my curiosity and came at a time when I was searching for anything I might have been doing that resembled the teaching I was aiming to do.

I had audio taped the lesson and had the tape with me. I decided that I would replay the tape and try to hear the moment Maggie had noticed before I taught the next day. I thought if I replayed the moment I might understand what it was that she had noticed. At home that evening I replayed the audio tape. I located my statement, "I want to show you something," and played and replayed the part of the lesson that surrounded it. The moment that Maggie noticed was when I interacted with Arif around the problem of doubling 3 x 35.

As I listened and considered what had gone on, I decided that, indeed, something new was happening here. I made notes in my journal as I listened to the tape.

> I did see what they needed. The point was to see the connection between addition and multiplication. By the time I stepped in, the students had already wandered off the track. They were missing the point and I could see that. I think part of what was happening was that I was clear on the purpose--connection between addition and multiplication--(make use of patterns on the Minicomputer). I also trusted myself. I had the confidence to make a decision about what to do independent of the teacher's guide. (Heaton journal, 09/21/89)

I felt the need to do something and, unlike moments when I had felt like this in the past, I had something in mind that I could do that connected to the efforts of one of my students to make sense of the problem as well as what I understood about the mathematics they were supposed to be learning. I felt the need to respond to Arif's confusion over representing 2 x (3 x 35) on the Minicomputer. The teacher's guide--specifically, its use of the Minicomputer to show the number of groups--provided me with a means of communicating what it was I thought Arif and the other students needed to see.

After listening to the tape, I wrote in more general ways about what it was that I thought was happening in that moment.

> Things came together in that moment because I was thinking about the subject, listening to the students and trying to make sense of what they were saying and then I acted. I think this last part is what has been missing up until now--the teaching (as a verb, an action) part of teaching for understanding. How do I keep it up? (Heaton journal, 09/21/89)

I struggled to make a distinction between the teaching I did prior to this moment and the teaching that occurred in the moment. "Teaching" was coming to have a new meaning for me, a meaning that depended on my action or participation in the moment as well as some sense of purpose. My teaching was responsive in a way that was "sensitive to the moment and place" (Yinger, 1988). I was learning that I needed to be responsive to the sense my students made of the mathematics as well as what made sense to me as I considered the question of what to do next. I redefined teaching to include such moments.

After examining this moment Maggie had noticed, I saw myself connected to the work of teaching in ways that I had not experienced before. I had a new feel for teaching and decided at the time that what I had experienced, what Maggie had noticed, was what I was aiming to do. In my journal I wrote, "Maggie spotted the moment. It's been over two weeks of school and one moment of teaching mathematics for understanding!" (Heaton journal, 09/21/89). The significance of the moment for me at the time was grander than the moment, itself. In fact, it probably would have gone unnoticed if not for Maggie's keen eye. It was a moment in which I made the decision to do something I thought needed to be done and had a means for following through on my decision even though my resources for responding to the question of what to do next were limited at the time. In my journal, I wrote that it felt like for a moment, "I took control. I knew what I was doing" (Heaton journal 09/21/89).

Three Years Later: Revisiting the Moment

The moment of interaction with Arif and the other students around the problem of doubling 3 x 35, however short-lived, was powerful for me in that it gave me a new feel or image for what it was I was aiming to do. In this section, three years later, I go back to that moment with the aid of an audiotape and transcript to try to understand the significance of the moment or my relationship to teaching, from a perspective three years later. What led up to the moment? What evidence is there, beyond Maggie's observation, that this was a significant moment for me? In what ways was the teaching in this moment different from the teaching I had done before? What was changing? What was happening with my interactions with students? What was happening in my interactions with the mathematics? What was my understanding of the mathematical ideas at issue in that moment at the time? What do I understand about the mathematics now? What would be required of me to sustain such moments in my teaching for longer periods of time?

In an effort to try to understand my interactions with students prior to the moment, I begin with an examination of the discourse. What were my interactions with students like? How did our interactions compare with the suggested teacher-student dialogue in the script of the teacher's guide? How did our interactions change throughout the lesson? This is followed by an analysis of what I am learning about the mathematical territory being explored in the composition of functions in the context of multiplication. How does my understanding of the mathematics at the time influence my role? In what ways does my understanding of the mathematics help or hinder me from hearing the significance of the moment, at the time of its occurrence and now, some three years later? What would it mean to have multiple ways to respond in the moment, to have choices of pathways to take? As it was, I had only one way to respond. While it was a way that made sense to me, I stopped before I found out the sense it made to others. Because I lacked alternatives, I did not know what I would have done, other than repeat what I had done, if someone had said they did not understand. And finally, I look at the tensions that existed over my interpretation of the point of the lesson and the students' interpretation of the task prior to that moment. What happened to our intentions or sense of purpose in that moment? How were intentions negotiated prior to and during that moment?

Learning What to Do in the Moment

In this section, I try to understand what I was learning about what was demanded of me in that moment, by analyzing my interaction with students and the mathematics being discussed in several different situations throughout the class. What was I learning about what I needed to do in order to be, in Yinger's words, "sensitive to the moment and place"? How did my role vary in different moments? How did it change in the moment Maggie noticed? How were these interactions different from earlier ones? What sort of routines existed for the ways in which my students and I interacted? How did these routines vary from one another and from the teacher's guide throughout the lesson?

In the first section, I give some further background on particular aspects of my vision of math teaching that were in the foreground on that particular day. In the second section, I look at the social interaction on that day of teaching in two different exchanges prior to the moment Maggie noticed. In the third section, I look at interactions within the moment Maggie noticed.

What I thought I needed to do differently. When I began this school year, based on what I knew of the math reforms and my observations of and conversations about Maggie's teaching, I thought one of the things I needed was to change my role in the classroom. In an interview with Jim (09/89), before the school year began, I talked about my vision of that role.

Jim:	The next question I was going to ask is if you plan to change your instruction. I think you just answered that to some extent. Is there anything else that you'd like to add to that?
Ruth:	One thing would be to have them explain their thinking and how they got their answers. I think another change would be for me to take a different role in the classroom, one where the students would do the talking. The students would talk with each other.

One way to change my role was to change the kinds of things I said, what I meant by them and when I said them. The way I managed this in the beginning of the year, as evidenced in this lesson and the lesson in the previous chapter is that I stopped all telling and tried to do nothing but ask questions.

I hoped the questions I asked would serve at least two specific purposes. First, I wanted the questions I asked to influence the responses of my students. I thought that asking students to respond to my questions, which were intended to open up the discourse, would be a way for them to be pushed into taking on a different role. I wanted the students to begin to take some responsibility for interpreting how to do the problems I gave them and deciding for themselves when something made sense. Secondly, asking questions and listening to students' responses would give me a way to begin to understand the sense my students were making of what we were doing. My questions got students talking but with this change came a whole new set of difficulties for me.

It was true that asking questions put my students in a different role and led to some insight into how my students were thinking. But, I was learning that asking questions was one thing, responding to students' answers to the questions was quite another. Once I had their ideas on the table, what was I to do with them? Which ideas should I pursue? Is one answer as meaningful as another? What should I do with ideas that seemed wrong to me? Questions like these had surfaced for me the day before as I heard my students' responses to the question of patterns. On the outside, I tried to remain neutral to all students responses, accepting of all answers, in an attempt to let students construct their own meanings. On the inside, I was churning with questions that troubled me.

In this lesson, for example, while I accepted all of my students' responses to the problems I posed with the Minicomputer, I was growing quite frustrated with the students who were able to answer my questions without attending to the Minicomputer. Did it really matter how they found the answers? Was accepting all of their answers and their means for coming up with them the right thing to do? Was not responding being irresponsible? What was the responsible thing to do? Was it alright if what they came up with did not match what was in the teacher's guide? When that happened, what was I supposed to do? How was I supposed to figure out what to do next?

I was beginning to see that it was necessary for me to respond to the question of what to do next in the moment. What I needed to learn was how to be listen to my students and be responsive to the mathematics in the moment.

Sudnow (1978) writes of learning a similar kind of moment to moment responsiveness in the playing of improvisational jazz.

> For a long while, I guided my hands through the terrain of the keyboard by moving my fingers along the various routes and scales I had conceived. My looking, even when directed at the keyboard after looking was not especially necessary to follow the paths, was so involved in the style of activity under way that I didn't see my fingers' doings as I now see them. The doings were different, and the lookings, 'inside or outside,' were different, for together they were part of a way of proceeding other than what I do now. I am not using pathways to make up melodies. Now I find places too in the course of going to them, each particular next place at a time, doing improvisation. (p. xii)

If improvisation[2] was to characterize my teaching, then my planning also needed to change. I found that the interactions were such that could not follow the script in the teacher's guide closely during class and therefore could not plan everything in advance. Some of my decisions for what to do next had to wait until I was in the situation because they were tied to the sense my students were making of the mathematics. What could I do to prepare myself to respond in the situation in ways that may or may not coincide with what is in the teacher's guide?

At the time, my way of responding was to accept every response. In class, I did not want to do what I had done in the past which was to simply evaluate students' answers and declare them as either right or wrong. I was searching for other ways to think about and respond to students' comments. But I was beginning to feel that math needed to be more than just a time to share ideas. I was a *teacher*. I had things that I had planned to teach and I struggled to figure out a way to implement my plans. I tried to think of my students as sense-

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[2]Marvin Hoffman, an educator who has also drawn an analogy between teaching and playing jazz (1992), describes the key element in teaching "that final, irreducible element in good teaching--soul" (p. 6).

makers, both in terms of what they said to me and how I responded to them. How could I show respect for them and their ideas and at the same time fulfill my responsibilities as the teacher who knows things I wanted them to learn? I found myself worrying much of the time about my responsibilities and questioning what my role ought to be on a daily basis.

Varying the script. My initial ideas about what I thought I needed to do to change my teaching influenced the way I made use of the teacher's guide. The teacher questions in the CSMP teacher's guide served as starting points for me. But if I wanted to be responsive to what my students were saying, beyond an acknowledgment of their answers as right or wrong, it looked like I was on my own to figure out what to do. I had already begun to feel the limits of a predetermined scripted dialogue and started doing things to "fill in" between the questions the teacher's guide indicated that I should ask in hopes of initiating a discussion. I found the teacher's guide to be of little help beyond the first question in my attempts to generate a conversation about mathematical ideas.

In the two tables that follow, an excerpt from the teacher's guide is in the first column and compared in the second column to excerpts from the lesson transcript representing the interaction that occurred between myself and my students. I use these tables to analyze my role in the turns involved in "doing one." "Doing one" included establishing a particular number on the Minicomputer and then multiplying it by 2 or 3. The two exchanges that are examined here, occurred prior to the moment Maggie noticed. My words are in italics in the transcript that follows.

The script in the teacher's guide represents the *typical* way the class could go while the transcript represents what actually happened. They are two different perspectives on role. Goffman (1961) addresses these differences.

> Role may now be defined . . . as the *typical* response of individuals in a particular position. Typical role must of course be distinguished from the actual role performance of a concrete individual in a given position. Between typical response and actual response we can usually expect some difference, if only because the position of an individual, in the terms now used, will depend somewhat on the varying fact of how he perceives and defines his situation" (p. 92).

What do the differences between the teacher's guide and the actual discourse show? What sort of "fill-in" was I doing? What was my interaction with the students like during this lesson prior to the moment Maggie noticed? What sorts of questions did I ask? How did students respond? What did I do with their responses? How did this vary from what was in the teacher's guide? What does this say about the usefulness of the teacher's guide in my teaching of this particular lesson and this kind of teaching in general?

<u>The first exchange.</u>

CSMP script	**Lesson transcript (09/21/89)**	
Clear the Minicomputer and put on this configuration. (35)	*Ms. Heaton:*	*Um, let's do another one here. What number is that? Sipho?*
	Sipho:	37.
T: What number is on the minicomputer?	*Ms. Heaton:*	*How do you know that?*
	Sipho:	No wait, 35.
S: 35	*Ms. Heaton:*	*How do you know that?*
	Sipho:	Because there are 2 checkers on the tens board. One of them on the 2 and the other is on the 1. One plus 2 equals 3 and on the ones board you have 4 and there is another checker on the 1. Four plus 1 equals 5.

Figure 5.1

The first exchange: The CSMP script compared to the lesson transcript

As directed by the teacher's guide, I arranged the checkers on the Minicomputer. The script in the teacher's guide assumed that students would correctly identify the number represented by checkers on the Minicomputer. There were no alternatives presented, like, for example, what to do if someone gave a wrong answer. No matter how a student in my class responded to the initial question, right or wrong, I asked the follow up question, 'How do you know that?' This was a question not found in the teacher's guide and gave me some insight into how my students made sense of what they were doing.

In the first instance above, the alternative would have been to tell Sipho he had the wrong answer. Instead, in this particular example, it is possible that my question gave him a moment to think and the space to revise his answer. Asking him a second time how he knew was a way to see what he understood about the answer he gave. He had a way of explaining why the answer was 35. In general, during the initiation of a new number or problem on the Minicomputer, I was fairly certain about my role. I asked students how they knew in response to whatever it was that they said. This was true in most previous lessons with regard to asking any of the questions the CSMP guide offered me. This is, however, where the turn ended because I was not sure how to follow up on their responses. I could ask the questions, listen to their responses and push them to elaborate, but figuring out what to do with their responses beyond that was a continual challenge. Was it enough to hear how they understood their own answers? Were there things I should be doing to follow up on their responses to push their thinking?

The second exchange. Once the number was established, I asked students to double it or multiply it by 3. In the exchange that follows, I asked students to multiply 35 by 3.

CSMP Script	**Lesson transcript (09/21/89)**	
T: Who can add some checkers so that 3 x 35 is on the Minicomputer?	*Ms. Heaton:*	*O.K., what if I want to show now, three times ? Can someone show me that on the minicomputer? Jamilia?*
	Jamilia:	[Jamilia arranges checkers on the Minicomputer.]
	Ms. Heaton:	*O.K., can you tell me,--us why that is three times 35? Can you explain what you did? Can someone help her out? Do you agree with that? Do you disagree? What do you think? Bob?*
	Bob:	I agree because see 35 times 3 is just like 3 35's. It is sort of like, it is tripled.
	Ms. Heaton:	*Okay, what do people think about that? Jennifer?*
	Jennifer:	I know how much it is.
	Ms. Heaton:	*And what is it?*
	Jennifer:	105
	Ms. Heaton:	*How do you know that?*
	Jennifer:	Because I did it in my head again. I did 35 plus 35 plus 35 and then I went 5 plus 5 plus 5 is 15, put down 5 carry the 10. 30 plus 30, plus 30 equals 90 and I would add more 1 more 10 and that's 100.
	Ms. Heaton:	*O.K. and what if I want to show that now on the minicomputer, 105 times two? Ron?*
	Ron:	210
	Ms. Heaton:	*How do you know that?*
	Ron:	Two times 105 equals, can I show it?
	Ms. Heaton:	*Sure. You can do both. You can show us and tell us.*
	Ron:	100 plus 100 is 200 and then plus 5 and 5 is ten.
	Ms. Heaton:	*Do you want to show us with this on the Minicomputer a way you can multiply by 2?*

Figure 5.2

The second exchange: The CSMP script compared to the lesson transcript

The next thing to appear in the teacher's guide after what you see in the first column was a Minicomputer with 3 x 35 neatly arranged on it followed by a new problem. There were assumptions embedded in the teacher's guide that students would know how to arrange the checkers. There was a note about alternative ways of arranging checkers. But, there was no explanation as to why one way was preferred over others or what to do if the preferred way was not what a student offered. The script in the CSMP guide assumes that what is being asked of students is a mathematical task. What the transcript recognizes is the cognitive complexity of the task.

My students knew by the time of this lesson that I accepted all ways they thought about the answers to the questions I asked. Looking back, I see that my students responded to my request to show 3 x 35 or one-step multiplication problems in one of 3 ways. They knew that I accepted more than one way of thinking about the problem and they volunteered, regardless of how the first person responded, with "I have another way." This valuing of "other ways" is something I had been trying to encourage from the start of the school year. It is also something that many of my students experienced in their math class the year before.

What a Student Did	What I Did
1) A student gave me the total or the answer to the multiplication problem.	1) I asked, "Can you tell us how you figured it out?"
2) A student showed the total or the answer to the multiplication problem on the Minicomputer.	2) I asked, as the checkers were placed on the Minicomputer, "Could you tell us what you are doing as you do it?"
3) A student repeated the pattern of checkers of the number we started with either twice or three times on the Minicomputer, depending on whether a number was multiplied by 2 or 3.	3) I hovered over their shoulder and helped them to arrange the checkers to create and maintain a pattern. This turn took much longer than the others.

Figure 5.3

A predictable set of my responses to students

The list of ways I could respond grew as the variance in my students' responses increased. Once we exhausted the students' alternatives for thinking about a particular problem, I said, "Let's do another one" and the pattern of exchange would repeat itself. My inner restlessness with students' responses continued. I tended to remain silent as students offered their individual ways for thinking about the problem. I did not know what else to do. When I started the school year, asking questions and listening to students' responses had been fine. I was thrilled to have students doing much of the talking. But as time went on, I had begun to feel like I ought to be doing something with their responses. I needed to expand my role and it was becoming clearer to me that the CSMP teacher's guide was going to be of limited assistance.

The various ways I created to respond to students' responses in Figure 5.3 above could be thought of as a form of improvisation. "The use of formulae-like chord sequences may often simply be a way of facilitating maximally relaxed improvising" (Dean, 1989, p. 89). While I did not think of these as formulas for improvising my actions at the time, I can now see how these routines, not found in the teacher's guide, could be interpreted that way.

> I must emphasize occasion and the situation in which an improvisation occurs, because the social setting of an improvisation will usually tell us the kind of music that needs to be improvised. (Mack, 1970, preface)

For example, in the first exchange, no matter what someone said, I asked, "How do you know that?" In the second exchange, I constructed an if-then system, if a student does this, then I do this. Even with these formulas, I found myself getting less certain of my role. If I follow up on whatever someone says with the question, "How do you know that?" then what do I do? I realized that I could not anticipate all of the possible classroom scenarios in advance and even if I could, trying to hold all of the possible ways to respond in my head would be unrealistic.

I was learning that there was a need for me to respond and decisions about how to respond had to happen once I was in the situation. The deeper I got into the teaching, the less certain I became of my role. I found myself less dependent on the CSMP guide and more dependent on myself to figure out what to do next. The teacher's guide was a starting point and between the guide and my experiences I could get through the first few moves and be fairly certain of them ahead of time. Further than that, it was hard to predict what situations might arise and even harder to decide ahead of time what to do next. The situations and decisions about my next moves were getting more complex. At the time, I wrote in my journal (09/21/89), "Somehow I thought following CSMP was going to help me in all of this. Now, I am beginning to question what it means to follow a book in this kind of teaching."

Hearing and seeing what I needed to do next. What happened in my interactions with Arif? In what ways were they like what had been happening in these two exchanges? In what ways were they different? This was a new situation. It was the first time I asked a question about the composition of functions in the context of multiplication. When I asked how to double 3 x 35, the first student to respond gave the total, 210. This was like what students had done on earlier problems. The solution was found by doing an easy mental addition problem of 105 plus 105. Ron offered to show the solution on the Minicomputer. He cleared the checkers that had been showing 3 x 35 from the board and started again. He put up the total of 210 with the fewest checkers possible. This, too, was like what students had done on earlier problems and as had been the case, students wanted to offer other ways of thinking about the problem.

Arif said he had another way to show the total on the Minicomputer. Since, Ron had already put up the total, I assumed that Arif was going to show the "other way" of representing it on the Minicomputer which entailed repeating the pattern of checkers for 35, 6 times. Arif went to the Minicomputer and started rearranging checkers. He had difficulty trying to show what he wanted. With the teacher's guide open before me, I could see that arranging 6 x 35 required lots of checkers and patience. I had prompted students to arrange the checkers in patterns on previous problems. Given the number of checkers and the pattern needed to show 2 x (3 x 35), it would not be too easy to prompt him without some sense of his understanding of what it was he was to try to represent.

Arif became confused. At the time, I felt like I could understand his confusion on several levels. First of all, I could see why he might find it difficult to keep track of all of the checkers on the Minicomputer. If he were to get all of the checkers in the right place for 6 x 35, what he was aiming for can be seen in Figure 5.4--which was at least twice as many checkers as we had dealt with in the past.

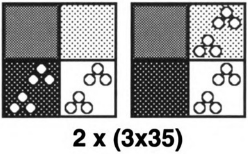

Figure 5.4

A Minicomputer representing 2 x (3 x 35)

And, because Ron had taken off all of the checkers and put up the total, Arif had to start with an empty Minicomputer without a teacher's guide to show him the way. I could also understand his confusion with the problem of doing 3 x 35 and then doubling that. I had been caught off guard by the task, myself, when I planned this lesson. Without much thought, I had assumed the composition of these two functions resulted in 5 groups of 35, not 6.

With empathy for his confusion, I asked him if he could talk about why he was feeling confused. He said, "Because over here we added 3 of them and we got 105 and I thought over here we were supposed to add 2 more of them." I took this to mean that Arif understood "doubling 3 x 35" to be adding on 2 more 35's. I knew this was not the solution. Doing what Arif suggested would result in 5 35's not 6 35's. How could I help him see the correct solution?

I tried to push his thinking with a question--a question specific to this situation--that if Arif tried to answer might lead him to see what I thought he needed to learn. I asked, "Does that sign tell you to add, what are you doing?" Looking at my question from my perspective now, it is as if I thought the difficulty was in his literal understanding of the multiplication sign. In a sense, I was asking, why are you adding when it says to multiply? I do not think this was the difficulty. I was fumbling for a question to ask to push his thinking. I think the difficulty he faced had to do with his understanding of the relationship between addition and multiplication. Perhaps the question to be asking here was what was being doubled? The answer to that question is what Arif needed to see.

I continued by asking if anyone had thoughts for Arif on the difficulty he was having. This was a way to buy myself some time and see if other students could help push his thinking in ways I felt I had failed to do. Bob said he had a way he could make it easier for Arif, a way for him to not put all of those checkers on the Minicomputer. I could see that Bob, like Ron, was probably aiming for the fewest number of checkers on the Minicomputer. I knew this move was not going to take us where I wanted to go. It would not help Arif and others see what I wanted them to see--the number of groups. I could see no way around setting up the Minicomputer with all of those checkers. I thought seeing the groups on the Minicomputer would clear up Arif's confusion. I decided that I was not going to leave this up to another student. Instead, I interrupted Bob and said, "I want to show you something." I proceeded to put back all of the checkers, myself, showing students, step by step, how what I was doing with the checkers related to the multiplication problem I posed.

This declarative move by me was in stark contrast to what I had been doing earlier this day and during the three weeks prior to this lesson. I had been doing nothing but asking questions and remaining silent when students responded. For the first time, I felt like I had a way to respond to a response a student made to my question for which I had not planned. How had my brush with the problem when I was planning the lesson helped me understand Arif's difficulties? In the situation, I had some understanding of the mathematical issue on the table, the sense students were making of it, and I had a way to show students what I wanted them to learn. I was joining in the doing of mathematics with my students in ways that I had not done before. I was now in the situation as a more knowledgeable other in a way unlike I had been in the past. I had something to contribute to the discussion. Rather than continuing as a silent bystander, I was now engaged with my students, responding to questions as well as asking them.

Teaching as telling. As I look back on this moment now, I find it ironic that the first moment I felt like I was teaching, I was telling. Telling was a teacher move I consciously deleted from my repertoire when I started changing my practice. It felt like an integral part of my past teaching that needed to be omitted. I replaced my statements with countless questions and listened to students' ideas but, then what? I felt frustrated that we were going nowhere with the ideas. In this moment, I learned that showing or telling had a place in this teaching, I could choose to do it. Erickson (1982) refers to a break in the routine of teacher-student interaction such as happened here when I added telling to my options as an "adaptation to the exigencies of the moment--actions that make sense within an adequately specified context" (p. 169).

The telling I was doing felt different from the telling I had done in the past. Here, I was showing and telling my students something I knew and thought would help them in response to a student who had found the courage to admit he was confused. This way of being in the situation as a teller felt different to me than my role as a teller in the past.[3] In this situation, I was taking into account my students first and, on that basis, making a decision to tell, followed by a choice of what to tell. In this instance, I had different intentions for my telling (Olson and Astington, 1993). The telling I did in this situation was done for reasons other than standing in front of a group students and telling them what to do or what to learn without any sense of how what I was telling connected to what the students did or did not understand or what it was I wanted them to learn. In the teaching standards (National Council of Teachers of Mathematics, 1991) there is a discussion of the teacher's role in classroom discourse and the place of telling is given attention.

> Beyond asking clarifying or provocative questions, teachers should also, at times, provide information and lead students. Decisions about when to let students struggle to make sense of an idea or a problem without direct teacher input, when to ask leading questions, and when to tell students something directly are crucial to orchestrating productive mathematical discourse in the classroom. (National Council of Teachers of Mathematics, 1991, p. 36)

How do you decide when the appropriate response is to tell? Are there other ways to push students' thinking? If so, what are they? And, how do you decide the appropriateness of one response over another in a given moment?

Another way to interpret my telling in that moment of teaching is that it marked my entrance into the "conversation of practice" (Yinger, 1988, p. 74). Interaction in the context of improvisation, Yinger (1988) writes, "is reactive and responsive; thought and action are adapted to the dynamics of social interaction and conversation" (p. 87), therefore, "instructional conversations and the strategy

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[3]There are other math educators considering the role of telling in teaching. There was a 1993 NCTM research presession, March 30, 1993 entitled, *Telling in Teaching? Thoughts on Teacher Telling in Constructivist-Influenced Teaching*. Daniel Chazan, Deborah Ball, Paul Cobb, Erna Yackel, Jere Confrey, John Smith, Marcella Perlwitz and Robert Davis were presenters .

of patterns of teaching can be described only generally" (p.87). If this is so, how do you plan for these conversations? *Can* you plan?

Several days after this lesson (09/25/89), I noted in my journal that I was "writing my plans differently--making decisions." When Maggie read through my journal, next to this comment she wrote, "Noting the difference in how you plan seems crucial." Why? Yinger (1988) argues that if we draw an analogy between teaching and improvisation, then we need to consider planning as preparation.

> Preparation acknowledges our limited ability to predict and the constructive nature of life. Preparation expects diversity, surprise, the random, and the wild. To prepare is to get ready, to become equipped, and to become receptive. The focus of preparation is on oneself, not on a framework to constrain possibility. In a sense, preparation enlarges the future. (p. 88)

It seems that Maggie was already anticipating changes in the nature of my planning given the nature of the changes happening in my teaching.[4]

Learning to Hear the Significance of the Moment

Preparation for improvising jazz music involves learning to "hear" or listen. Mehegan (1959) wrote,

> The problem of developing the ear for what is called prehearing is of major importance in performing jazz. It goes without saying that the hearing demands in jazz are extremely high and no effort should be spared in the development of the ear. (p. 200)
>
> Prehearing means a memorized hearing--heard in anticipation of the moment of playing. This is the ideal we all seek and is the reason why fine jazz playing is a challenge both to play and appreciate. (p. 201)

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[4]Later during that year, I did a research project for a course that I was taking. I studied my own practice and looked at the relationship between my planning and teaching (Heaton, 1991).

Is there a way that learning to hear can help you improvise in the context of teaching? From my experience with patterns in the last chapter, I would say that learning to hear is critical and related to how it is that you decide how to respond. During this lesson, I thought I heard Arif and that I had a sense of his difficulty with the mathematics. Based on what I thought I heard from him, what I thought I understood about the mathematics, and what I wanted him to learn, I responded. From the chapter on patterns, it seems that not only does what you know about the mathematics help you to respond, but it helps you hear what it is that you need to respond to.

In my analysis of patterns and functions, three years later, I learned things that helped me to hear Valerie differently. In this section I explore similar questions. Are there things that I now know about the composition of functions in the context of multiplication that would have helped me to hear Arif and his classmates differently at the time? What is the sort of hearing that you need to do moment to moment in teaching mathematics for understanding? What is it that you can learn to help you hear in anticipation of the moment? How would such hearing help you participate in the conversation? What is the relationship between a sense of purpose and the ability to hear and respond? Since doing the teaching, I have learned much about what my students were doing and the complexities of the mathematics. As with patterns and functions in the last chapter, exploring the mathematics of the composition of functions in the context of multiplication has given me a greater appreciation for what I was hearing and broadened my understanding of ways in which I could have responded.

The difficulty of the task. I did not give much thought to my own confusion when I first encountered the task of labeling the composition arrow while planning for this lesson. At the time, my reaction was to add. I attributed my mistake to the similarity of this problem to the ones we did the day before which had required adding. I have learned since then that the tendency to add can also be explained mathematically. I explored the significance of the mathematics within these problems to try to understand from a mathematical perspective why it was important to go through the tedious work of using the Minicomputer when students had perfectly good mental computation strategies. What is the Minicomputer about anyway? What follows is what I learned about the meaning of the task, the significance of the mathematics being learned, and alternative representations of what is being learned. As with the case of my study of patterns and functions discussed in Chapter 4, if what I have since learned about the composition of functions in the context of multiplication was to be found within the CSMP teacher's guide then it was in a form inaccessible to me. What I have learned has come from my own investigation in to the mathematics using resources outside of the CSMP teacher's guide.

The composition of functions in the context of multiplication pushes at a piece of fundamental mathematics, namely the move from additive to multiplicative structures. Leinhardt, Zaslavsky, and Stein (1990) describe this transition as "one of the critical moments in early mathematics" (p.2). [5] The critical moments are opportunities for "powerful learning" that have two interesting features: "They are often unmarked in the 'normal' course of teaching; on the other hand, they are fundamental to other more sophisticated parts of mathematics"(p.2). So, what 3 x (2 x 35) represents is a move from additive to multiplicative structures and this shift marks moments that hold the potential for powerful learning.

As far as I can see, the significance of the mathematical territory this problem gets the learner into and the significance of what could be learned goes

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[5]The other critical moments they mention include functions and graphs, regrouping, and expanding the number system from counting numbers to rationals.

unmarked in the CSMP teacher's guide and follows the pattern of what Leinhardt, Zaslavsky, and Stein (1990) found in most other math curricula. It is unclear to me that any of the critical moments, as defined by Leinhardt, Zaslavsky, and Stein (1990) are marked in the curriculum. Perhaps I could identify these critical moments within the CSMP curriculum if I had a picture of the whole curriculum, across a year as well as relative to other grade levels. This, however, is not an understanding I had at the time I was teaching. It seems that any critical moments, shifts or markers within the mathematical territory are diffused by the nature of moving through a spiral curriculum where goals and objectives are intentionally vague and less well defined and mastery is not expected in the context of any particular lesson.

What sort of critical moment is represented by the problems in this half of the CSMP lesson? Moving from the additive world to multiplicative structures is an enormous conceptual leap for students to make (Vergnaud, 1988).[6] One of the fundamental problems is that students bring many experiences with addition when they make their way into the multiplicative world. An understanding of multiplication is often built on an understanding of addition. It is common for students to make connections between addition and multiplication or to think of multiplication as repeated addition. For example, in a problem like 3 x 35, rather than think of it as multiplying by three, students tend to think of it in terms of addition--adding 35 three times. This gets you an answer but what does it help you understand about the meaning of multiplication? What is the underlying meaning of multiplication? What is the difference between additive and

[6]Multiplicative structures is an area of mathematics education research that has been difficult for researchers to conceptualize and represent exactly how it is that children understand this domain. See Vergnaud (1988) for a discussion of multiplicative structures. He writes that the study of children's understanding of multiplicative structures includes the study of the concepts of "linear and non-linear functions, vector spaces, dimensional analysis, fraction, ratio, rate, rational number, and multiplication and division" (p. 141).

multiplicative change? And, does using the Minicomputer help students to understand this shift? If so, how?

Multiplicative change. What is multiplicative change? How does the idea of multiplicative change relate to what my students were doing? I now see that composing the functions, 3 x followed by 2 x, put my students in the multiplicative world. They immediately reverted back to the additive world, a place where, understandably, given what they have learned about mathematics in school up to this point, they feel quite comfortable because of the focus on additive structures in earlier grades. Most of the time dealing with multiplication as repeated addition works. For example, 3 x 35 is the same as 35 + 35 + 35. But, the relationship between addition and multiplication becomes more complicated when you try to multiply 3 x 35 by 2. To Arif, from an additive perspective, it looked like he needed to add two 35's on to three 35's. In response to a question of why he was adding two 35's he said, "Because over here we added three of them and we got 105 and I thought over here we were supposed to add two more of them." There are multiple incorrect but predictable ways someone with a limited view of multiplication might try to symbolically represent this multiplication problem.

(35 + 35 + 35) + 35 + 35

or

(5 x 35)

or

(3 x 35) + (2 x 35)

or

(105) + 35 + 35

Figure 5.5

Predictable but incorrect ways to represent 2 x (3 x 35)

What Arif and I did not understand, at first, is that the composition of functions brings out the complexity and difficulty of multiplication. Consider the problem 2 x (3 x 35) in terms of inputs and outputs. If I begin with 35 as the input, apply times three to this, and consider multiplication as repeated addition, I get one group of 35 plus another group of 35 plus another group of 35. These 3 groups of 35 (3 x 35) or (35 + 35 + 35) become the new input to which I apply times two. I end up with two sets of three 35's or six 35's, not two more 35's added to a set of 3 groups of 35. The fact that the input changes from 35 to (35 + 35 + 35) is the difficult part.

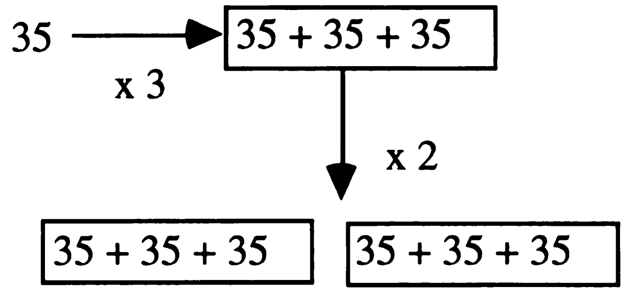

Figure 5.6

The input changes from 35 to 35 + 35 + 35

Adding two more to the set is very different from duplicating the set. Hiebert and Behr (1988) describe the significance of multiplicative change. "It is not a trivial shift, because it represents a change in what counts as a number. A group or composite of, for instance, three, now can count as a unit, as one. A change in the nature of the unit is a change in the most basic entity of arithmetic"(p.2).

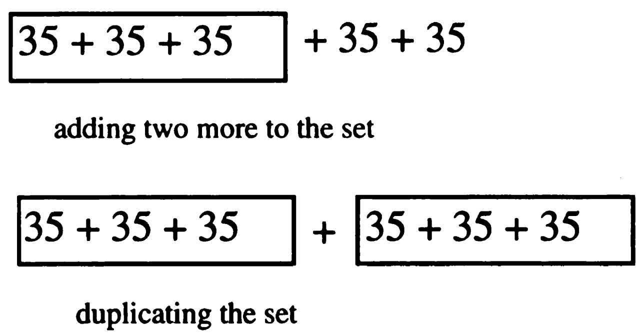

Figure 5.7

The difference between adding two more to the set and duplicating the set

What did I do in the moment? Using the Minicomputer gave me a way to show students the number of groups involved in the multiplication process. The first stage involves repeating the set of checkers representing 35, three times or 3 x 35. This is then taken as the input (3 x 35) and doubled. At the time, I did not have a clear understanding of the symbolic relationship between addition and multiplication represented by the possible interpretations for the problem. I was able to set up the Minicomputer and illustrate the numbers of groups but I did not have multiple ways to consider the symbolic representation of what was happening. What I realize now is that there are other ways, in addition to the Minicomputer, to represent what was happening there. There was only one place in the script of the teacher' s guide where an alternative symbolic representation for multiplication was offered. It came in the context of the following teacher question.

T: What is 3 x 35? What is 35 + 35 + 35?

Figure 5.8

Multiple symbolic representations of multiplication in the textbook

There was no discussion about how these two symbolic representations, 3 x 35 and 35 + 35 + 35, were related or how the composition of functions effects the symbolic representation. If I had worked through the mathematics beforehand, I might have rejected the Minicomputer as a representation or used it more strategically, or not felt so completely dependent on it.

Learning to Construct New Pathways

At the time, I was only marginally certain of what was happening here, mathematically. I had one way of teaching what I thought I understood and that is what I showed my students. How might this lesson have turned out if I had

had multiple ways of representing the mathematical issues embedded in the composition of functions? I realize that I was making a huge assumption about what it was that Arif understood. The alternative, at the time, would have been to question what I thought he understood. But, where would that have left me? I might have found out that he did not understand and then I would have had to think of a way to deal with that. What would I have done then? And, what about the other students? What did they understand? Did my demonstration on the Minicomputer make sense to students? Pursuing these questions at the time would not have helped. I did not know what I would do if I learned that students did not understand. This was the only way of illustrating the composition of functions with multiplication in the teacher's guide and I had not considered alternatives. Opening up the discourse to find out in more detail what students did or did not understand only complicated my job as a teacher.

If I had had multiple ways of approaching the difficulty, as I saw it at the time, I might have been more willing to spend time on it, pursued a bit more of what it is that Arif or others understood. If I had had more ways to think about the problem, to represent the mathematical issues to students, I might have been more willing to be less certain about what it was that anyone understood. As it was, I was just happy to have a way to respond. I wanted to believe Arif understood because I did not know what I would do if he or others had not understood. I only had one way of teaching what it was I thought the students needed to learn. What I thought Arif needed to learn was based on the my understanding of the sense he made out of the problem. Because he did what I did, I thought he needed to learn what I could see, in the way I had come to see it. Were there other ways I could have shown him? At the time, I did not consider that question. I was just excited to have had one way to show him what I thought he needed to learn. I wanted to believe that his comment meant that he understood that 105 was the same as three groups of 35 and doubling that was the same as two times three groups of 35 or "six times" or six times 35. Questions about what I did or the sense another made of it would have to be saved for the next step, which I was not prepared to take.

What I was lacking at the time was a knowledge of representations, "analogies, illustrations, examples, explanations, demonstrations, the ways of representing and formulating the subject to make it comprehensible to others" (Shulman, 1986, p.9). A repertoire of representations would prepare me to be responsive to the situation--to the students and the mathematics--to construct new pathways to understanding. How can I use what I understand about the mathematics together with what I hear in my students' responses and interactions to draw on appropriate representations to help students learn what I think they need to? In other words, representations become a resource for improvisation.

What are these resources and how do I collect them for myself? How can I make them be useful to me as the teacher? How can I learn to make use of them? The Minicomputer was a representation I had at my disposal in this context but my understanding of its usefulness and limitations was limited by the information or lack of it offered with the teacher's guide. The symbolic representation of the relationship between addition and multiplication came out of my conversations with someone who tried to explain the mathematics to me. The representation came out of my own learning experiences. Representations, therefore, are something I expect to learn over time, through my teaching practice and the opportunities I create for myself to learn more mathematics.

> The important overall considerations are to permit oneself the opportunity of all the possibilities you can envisage; to realise that freedom is possible within nearly all the improvising contexts we have discussed; and to be

> continually on the lookout (by listening to yourself and your co-improvisers, by hearing other improvisers perform) for new inputs. (Dean, 1989, p. 111).

Learning to Form New Intentions

My uncertainty about what I was teaching, how to make use of the Minicomputer, and why I was using it may have contributed to my clinging to the teacher's guide and using the Minicomputer as it appeared it in the teacher's guide even if it did not connect in meaningful ways to the sense my students were making of the problems. I saw the Minicomputer as a helpful tool in working the composition of functions problems but what about for the easier multiplication problems? Was it useful for solving these? This is a question I ask now but at the time I never questioned whether or not it was useful, I just used it. Was it necessary to use the Minicomputer, especially given the mental arithmetic strategies my students had for working the problems? The usefulness of the Minicomputer was not apparent to the students until much later in the lesson. Was there a way to discriminate on the use of the representation? Were there reasons to use or not to use it other than because it was or was not in the teacher's guide?

I am not certain that students saw any point to doing the multiplication problems beyond getting the right answer. And for the most part, they had their own strategies to help themselves to come up with the right answers. They had no need to mess around with all of those checkers if the only purpose was to get the right answer. And, as I look back on the lesson now, they had no reason to believe otherwise. Was it necessary for them to make use of the Minicomputer for problems they could already do in their heads?

From CSMP's perspective, what were students supposed to get out of doing simple multiplication problems prior to the more complex problems around the composition of functions? It was not explained within the teacher's guide. I constructed a connection. The simple multiplication problems I did with students in the beginning seemed too easy for them if the point was just to do the multiplication. Was I missing the point of CSMP? Was I missing the point of what my students were showing me they were capable of doing? Was my own sense making of what I was doing with students and why missing?

For much of the lesson, the students and I proceeded along parallel tracks, with different interpretations of the task, playing different tunes. I was intent on using the Minicomputer for each problem we did. The process of arranging these checkers was tedious and I expected students to follow along without any understanding as to why it was important to do so. I was blindly following, why couldn't they? They were intent on getting answers and had complex mental strategies for finding them. I was not really interested. I tolerated their alternative ways of thinking in much the same way they tolerated my pursuit of the placement of countless checkers. For much of the lesson, none of us were really engaged in what the other was doing. The mental computation strategies they used to get the right answers were not necessarily going to help them complete the worksheet where I knew they would be asked to identify the number of groups being multiplied.

Shared intentions. When the problem being worked on changed from simple multiplication to the composition of two functions, we all took notice.

> I began to join up in a new partnership with the hands and the sounding terrain, as where we were going together began to slowly integrate into an altogether new way of doing singing at the piano: a new way for intentions to be formed, a new sort of synchrony and directionality of linkage between my head's aiming for singable sounds, becoming progressively shaped and refined. (Sudnow, 1978, p. 95)

I listened and responded when Arif said he was confused and stopped Bob when he wanted to talk about totals. The total, I knew, masked what was happening in the process. The students were about to see why the process mattered from my perspective and, with luck, why I had been pushing them to make use of the Minicomputer and all of those checkers. Arif's comment and my response gave the students a reason to focus on the Minicomputer. It is interesting to me that when the students had a reason for looking, they saw or at least indicated through their "Ohhhhhhhh," that they might have understood something in a new way.

If I had had a better sense of the intent of the lesson from CSMP's perspective and ways to make use of the Minicomputer, I might have been able to find ways to connect to students sense making much earlier in the lesson. Since teaching this lesson, I found out from one of the developers of CSMP that improving students' ability to do mental arithmetic was a goal here. For example, if I could see 6 x as the composition of 3 x and 2 x I might be able to think of the computation as finding the double of 3 x. Composition of functions is about building flexibility in the way one looks at numbers and mental arithmetic requires being flexible with computation. It makes sense to me now that this lesson could be about learning to look at calculation in different ways and in doing so include attention to mental computation strategies. But that seems like a goal of a much broader mathematical nature than what appeared in the lesson summary:

> Using arrow diagrams and the Minicomputer, investigate the composition of certain numerical functions, for example, +10 followed by + 2 and 3 x followed by 2 x. (p. 11)

If I had attended to the mental strategies students were using and been more thoughtful about when and why to use the Minicomputer, in what ways would this lesson have turned out differently? These both seem like things that would have been good to have done yet there was nothing in the teacher's guide to indicate that these were even things to be considered. What would it be like for students and the teacher to jointly construct the agenda of a math class in ways that connected to what we all understood? In the moment of my exchange with Arif and what immediately followed, we were all focused on working the problem with the Minicomputer. Together, we examined a mathematical problem identified by a student, of interest to his peers, which I thought I could explain through a representation that had helped me to understand.

Learning to Improvise

I found myself *in* the teaching in ways I had not experienced before. For a moment I felt what it was like to improvise in teaching, to be responsive to students' understanding and to the mathematics. What is that state of being I am learning to be in as a part of teaching? I felt it. How do I describe it? This is where Sudnow's (1978) description of learning to play improvisational jazz is useful. Sudnow describes the way his hands have to be, the state his mind and his hands have to be in order to improvise. He and his hands must learn to be "singingly present" (p. 90) with his fingers in the terrain--seeing, hearing, feeling, prepared to choose a path.

One of the first steps in preparing myself to choose a path is learning what a pathway is. I think this is something that I was learning. I now see the teacher's guide is a path through the terrain. I think up until this point, I saw the teacher's guide in mathematics as the terrain. Sudnow (1978) describes how he sees the relationship between the path and the terrain. "I saw the path as a figure against the background of the terrain" (p. 21). Viewing the teacher's guide this way, as one path through the terrain, opens up the possibility that there may be other paths to follow. It makes following the teacher's guide a choice, one that I may decide is or is not appropriate given the exigencies of the moment. The same idea of choice can be extended to tools, like the Minicomputer, that can help you move about in the terrain, along different paths. Again, the Minicomputer is a tool or a path, a figure in the terrain, not the terrain. The purpose is not to teach the Minicomputer, it is a means to an end, and an end. Rather, what ways can this tool be used to help you see something that might otherwise go unnoticed in the terrain? What do I want my students to notice? What are they noticing? What tools can I use to help them see what I see? I always run the risk that what enables me to see might not be what enables someone else. That is why I need a repertoire of tools or pathways for helping others see. Learning what pathways are possible and learning how to decide which path to take is part of the work entailed in learning to improvise.

What can I do to prepare myself to improvise? Mack (1970) offers the following advice under a section in the beginning of the book called, "general suggestions."

> Don't try to "learn" improvisations. Once you've decided on an approach, try to vary the key, the tempo, the mood, the meter, the register. Try to use your wits and don't rely on rote or repetition. On the other hand, you may find it useful to keep a notebook of good ideas--material that serves well for certain purposes.

What would such a "notebook" look like for teaching? The notebook should not just be a collection of pathways. How to construct pathways and make choices relative to specific purposes would also need to be part of such a notebook. It would be a mistake to think that I could ever compile all of the possible pathways and anticipate all of the possible choices and their advantages and disadvantages. There could always be something new added to the list and contexts are always changing. Leonard Bernstein (Mehegan, 1959), a famous musician, recognizes the wish and impossibility of ever being able to write down all the knowledge that is needed to improvise music in a textbook. Bernstein writes in the preface to Mehegan's book,

> There has long been a need for a sharp, clear, wise textbook which would once and for all codify and delineate that elusive procedure known as jazz improvisation. Of course, no improvisation can ever be explained down to its roots; therein lies the mystery and joy of spontaneous creation. And any improvisation will vary greatly in proportion to talent, mood, colleagues, and endless personal factors.

A solution in mathematics is to construct a textbook that can serve as a guide, rather than a script, for improvisation. The guide should not be the response. The exact response is constructed in the situation. Rather than being a how-to guide for action, curriculum materials should help the teacher prepare to act.

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CREATING AND STUDYING A PRACTICE OF TEACHING ELEMENTARY MATHEMATICS FOR UNDERSTANDING

Volume II

By

Ruth M. Heaton

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Teacher Education

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CHAPTER 6

NOT READY TO PART WITH THE CANS: A SERIES OF IMPROVISATIONS IN TEACHING

Introduction

I had begun to get the feel for what it meant to improvise in teaching. Unlike the previous two chapters that were confined to single lessons, this chapter focuses on a series of five lessons in the middle of October, a series of improvised lessons. What was intended by CSMP to be a one day lesson to make labels for soup cans evolved into a series of five days of lessons for my students. Improvisation played a role in the creation of these lessons and how they unfolded. I decided to focus on these lessons, because, during them, I felt like I was really teaching.

In this chapter, I deepen my understanding of the interaction contained within moments of improvisation in teaching. Drawing on what I learned about improvisation as a general characteristic of this kind of teaching, I begin to see that I was not learning to do solo improvisation. Rather, my interactions were connected in important ways to my students' interactions. We were learning to improvise together.

> Talk among teachers and students in lessons--talk that is not only intelligible but situationally appropriate and effective--can be seen as the collective improvisation of meaning and social organization from moment to moment. (Erickson, 1982, p. 153)

The chapter is organized into two main parts. The first part is an account of five moments of improvisation and what I was learning from them at the time. The second part revisits these moments to reinterpret them from my perspective now, three years later. And finally, I consider what it is that I am learning by drawing connections between learning to do improvisational dance (Blom & Chaplin, 1988; Novack, 1990) and learning to teach mathematics. The metaphor of improvisational dance enriches my image of improvisation in teaching as it focuses on the interdependent relationship among the participants.

Moments of Improvisation

What follows are five episodes from my teaching of a lesson on measurement from the "Geometry and Measurement" strand of CSMP. This lesson summary appears as follows in the CSMP teacher's guide:

> CAPSULE LESSON SUMMARY
> Make labels for cans. Discuss how size problems might be reduced by measuring to the nearest millimeter instead of to the nearest centimeter. (p. 6)

I indicate each episode's relationship to what was found in the script of the teacher's guide.

Rectangles

By the time I taught this lesson (10/12/89), I had already learned that talk alone was not enough to make a mathematical discussion. The lesson around patterns in Chapter 4 was an example of where I saw that it was necessary for the talk to be about something mathematically interesting. I found out the hard way that it was impossible to initiate and sustain a mathematical discussion with my students when I, as the teacher, had no sense of what there was to talk about. The questions in the teacher's guide were a starting point, in the sense that they provided me with an initial question to pose to my students. I also had a pool of general follow-up questions from which to draw. But without a vision of the purpose, it was hard for me to know, specifically, how to respond to students' responses and push their thinking on particular mathematical ideas.

Without knowing what steps to take to move forward, I began feeling like I was falling behind. In my past teaching, moving ahead meant keeping up with the pace of two math book pages a day. That year, I had been judging my progress based on CSMP's master schedule of lessons. I was in the midst of the fifth week of school and doing a lesson that was to have come at the end of the second week of school according to CSMP's schedule. By CSMP's standards I was far behind. But was I? If I could have abandoned the CSMP calendar as a benchmark for progress, what would I have put in its place? How could I have decided for myself whether or not we were making progress?

On our first day of working with the cans, I stood at the front of the room about to ask a question from the CSMP teacher's guide with one of the soup cans the students brought to school in my hands. "What shape do you think the label will be when I pull it off the can?" I asked. Ron's arm immediately shot into the air. His was the only hand raised. I nodded in his direction and he proclaimed with great certainty, "A rectangle." I asked him how he knew. His personal experience with labels surprised me. "My mom collects soup can labels. She saves them down flat." Ron explained that when left to their own devices, the labels "automatically curl up because they have been like that too long." But, "if you stretch them out, the shape of the label is a rectangle."

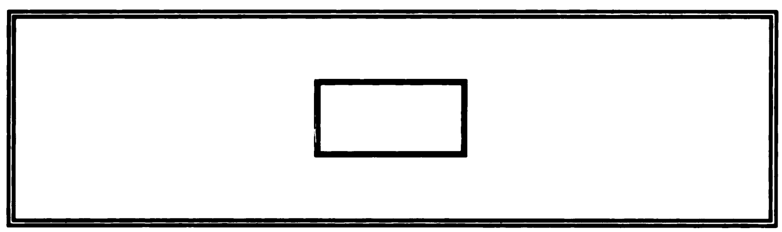
I followed up Ron's response with one of the questions that had become part of my repertoire of moves whenever someone responded to one of my initial questions. I asked, "What do other people think?" In the face of Ron's evidence, how could anyone disagree? How often does anyone see labels that have been peeled off of cans? In a way, we were lucky that Ron's mom was a den mother for Boy Scouts and hit upon saving soup can labels as a fund raiser. Everyone agreed with Ron. I was left to wonder how the other students would have answered the question without their classmate's experience to draw on. I went into this class thinking that students would use the shape of the can to figure out the shape of the label and that we might have had an interesting discussion. I never expected any of my students to answer the question of the shape of the label so quickly. I was interested in finding something mathematical to talk about so, I was not prepared to drop the idea of discussing rectangles altogether. I followed up on Ron's comment by asking, "What makes something a rectangle?" This was not a question in the teacher's guide. It was a question that I came up with in the situation. By fourth grade, most students were able to identify shapes as rectangles but what was a rectangle? Identifying a shape as a rectangle was one thing. Being able to articulate the properties of a rectangle was quite another.

When I asked my students, "What makes something a rectangle?" several students, including John, raised their hands. He was a rather shy and timid about offering his ideas but when he did, we always learned a lot from him. I had privately told him how much I valued his contributions. I was not sure that he believed me. Whenever I saw John's hand raised, I called on him. He said, "If two ends are the same size, like if two ends are little and two ends are littler than the other two ends." What John said sounded like the beginning of a definition from which one could begin to infer properties of rectangles (i.e. opposite sides are equal). He struggled for language. I hoped other people could offer language that would help John articulate what he meant.

Ms. Heaton:	O.K., what do people think about John's idea? So, two ends are littler than the other two ends. Are there other things that make something a rectangle or do you have other comments about John's idea? Luke?
Luke:	I challenge. I think like if two ends are littler than the other two ends then it would be going down like a triangle. It would be, it just wouldn't be a rectangle.

With a somewhat frustrated look on his face, John asked, "Can I show him on the board?" I wondered when and how often I should encourage students to use pictures to explain their ideas. A picture might help John explain his idea. It also might help someone else come with language to help clarify their own or John's idea. Sometimes it seemed worthwhile groping for language. Other times it seemed important to draw pictures to help communicate your ideas to someone else. When to do what and for what purposes were questions I continued to ask myself. There were no rules to follow about what to do when.

At that moment, John's cheeks were flushed, he seemed at a loss for words, and I did not want him to give up. It was something I had seen him do at other times when someone disagreed with him. I handed him a piece of chalk and he drew a rectangle. "This is what I mean. These two ends are little and these two ends are long," he explained.



Rising in his chair to see the board, Luke said, "Oh. That's not what you said." With a slightly pained look on his face, John softly replied, "Well, I tried to say it."

I wanted to support John's efforts. What he was trying to articulate was difficult. In other situations, I had seen Luke try to understand something from another's perspective so I assumed that he had really tried to picture in his mind what John had described. And apparently what he saw in his mind did not resemble a rectangle. I did not want John to drop out of this exchange because of what I knew about him as well as what I knew about Luke. Often John ret

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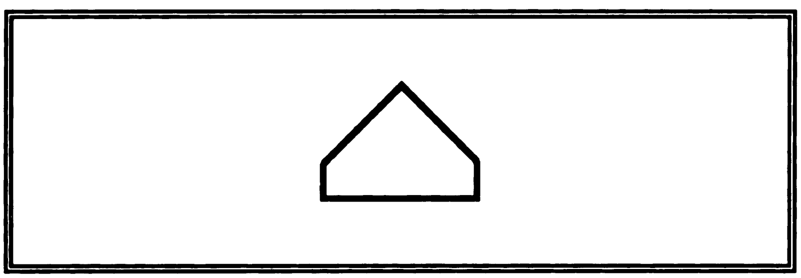
retreated into silence when someone questioned him. I wanted him to keep talking and try to respond to Luke. Luke appeared to be genuinely puzzled by John's definition. He also had patience for listening to other's ideas.

I said, "John, how could you make it clearer to Luke what you meant?" John chose a couple of new words and tried again, "Two of the ends are shorter and two of the ends are longer." It was not clear to me whether this made any more sense to Luke. I thought it might have been a good time to have others join this discussion. I noticed David on his knees in his chair with his hand waving in the air. It was clear he had something to say. I invited him to share what he was thinking. David said, "Can I show something up on the board? I disagree with John when he said that because, like, see, like this, if you have five ends and these two ends are longer, it is not in the shape of a rectangle."



He had a good point. This did fit John's definition but it was definitely not a rectangle. I grew increasingly concerned, however, about the tone of David's voice. To me, it hinted at competition rather than cooperation. I wondered what John was thinking. How was he feeling? In his quiet way, John was a leader in this class. While more vocal, David was also a leader. The interactions between the two of them and several other pairs of students in the class this year had prompted me to begin noticing the fine distinction between disagreeing with someone for the sake of disagreeing with the *person* and disagreeing with someone because you disagreed with their *ideas*. I tended to step in and take the lea

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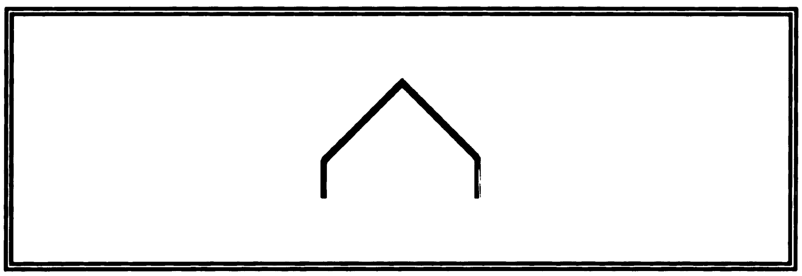
lead when I thought it was the former and remain quiet during the later. It was never easy for me to decide which was which. I often recognized what seemed like pickiness after it had gone too far. I held back here to see if I could tell which this was.

John looked at David's drawing and said, "Well, you'd need four." David ran his finger around the sides of the figure he drew and counted, " One, two, three, four, five." On his own, John revised his definition. "There is two short ends and two long ends and there is not five ends. There is four ends." David took an eraser, removed a fifth side and said, "There, that has four ends."



Was David really helping John to revise and construct a definition of a rectangle or was he picking at John and his ideas, trying to make John look foolish? As a teacher, it was a hard call. David's questions were forcing John to be more explicit about the assumptions embedded in his definition but at what cost? How was John going to feel when this was over? How would I know? In response to David's drawing, John asked, "Can you connect it?" To this David responded, "It is connected," and ran his finger around the sides.

My sense was that David did understand what John said and tried to find holes in his statement. This could be interpreted as a reasonable way to help John revise his ideas and for the group to construct some shared definition of a rectangle. But I felt uneasy about the exchange and noticed that others did, too. There was a sharpness in David's voice and a redness in John's face that I was

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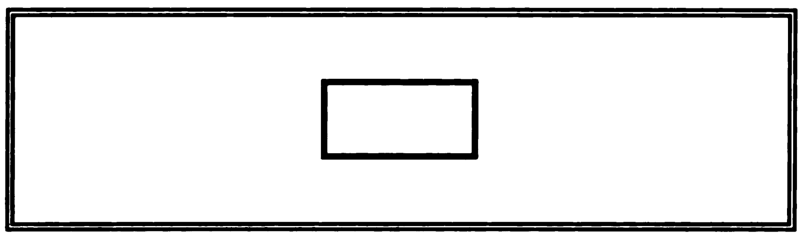
certain was apparent to others. Some students squirmed in their seats and glanced from the two boys at the chalkboard, to me at the side of the room, and then back to the boys at the front. I read their movements as a request for me to do something.

I moved to the front of the room and asked, "Can we think of a way to talk about a rectangle that describes a shape that looks like this?"



"John has given us a good start. Two ends are short and two ends are long and there are four ends. How else could we, what could we do to add to that description to make it clearer that we are talking about something like this?" I pointed to the rectangle John drew. David and John returned to their seats. The group relaxed. The tension in the air eased. It was now a group task to use John's ideas to construct a definition of a rectangle.

After class. In my journal (10/12/89), I reported on a conversation I had with the other classroom teacher immediately after class. She said to me, "After this class, I don't care if you spend two weeks on the can." I responded to her that in the past I would have just told students how to make the labels and we would have gotten it done in one day. She said she thought this was an example of what can happen when you give students time to talk about their ideas.

When I started this lesson, to make labels for cans was the driving goal for what I was doing but I was beginning to think that actually making them might not be the point. This was an example of a rich activity with the potential for leading us in lots of different, mathematically interesting directions. I decided I

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would spend a few days playing around with whatever came up. This was a whole new way for me to think about a math curriculum. On the one hand, the structured freedom to move as I pleased made me a bit nervous. On the other hand, the discussion of rectangles was nothing I planned yet it created the longest sustained feel for this teaching that I had experienced since I started this teaching back in September.

The following is what had appeared in the teacher's guide:

> Hold up a can with an easily removable label. Trace straight down the side of the can; the seam is a natural guide.
>
> T: If I cut the label straight down the side and then flatten it, what shape do you think the label will have?
>
> S: A circle.
> S: A rectangle.
> S: An oval.
> (p. 7)

I had taken my student's response of a rectangle and turned it into a discussion by asking people to talk about their assumptions about rectangles. After one day, we had not moved beyond discussing the rectangular shape of the label. I had already made a decision not to rush into making the label for the sake of getting it done. There seemed to be a lot we could explore in the process. What else could I make into a discussion?

Strings

Unbeknownst to the students, the next day we had math (10/16/89), I had a long piece of string in my pocket and plans in my head for using it to measure the circumference of a can. Here were the directions as they appeared in the CSMP teacher's guide:

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Measure to the nearest centimeter the circumference of your
can and its height.  Use a metric tape measure, or cut
pieces of string in the appropriate lengths and then measure
the string.  Label the measurements of your can. (p. 8)
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I decided to keep the string in my pocket until I saw how someone measured the circumference of the can. I wanted the students to see the need for an alternative method before I gave them one. This was much different than what I would have done in the past. In the past, I would have been focused on getting the label made. I was now trying to be responsive to my students' sense-making and open to alternative ways of thinking about the mathematics.

Arif volunteered to show us a method of measuring a can that he thought would work. All eyes were glued on him as he stood before the group rolling the can along his ruler. The ruler was a familiar measurement tool for my students. Each of them had one stored in their desks. The measuring of a round object with a straight edge was an awkward process to watch and it appeared an even more awkward process to do. The can slipped, the ruler slid and shook, fingers got in the way, eyes could not read numbers. Arif started, stopped, and began to measure the can several times. He decided to kneel on the floor, eye level with the can which rested atop an empty desk at the front of the room. One hand held the ruler straight, parallel to the top of the desk, perpendicular to and in front of the can. The other hand reached behind the ruler and rotated the can along the edge of the ruler. He slowly rotated the can one time as he marked the start of the rotation on the side of the can and tried to watch as the number of centimeters increased. When the can rotated back to where it started, Arif announced, "about 22 centimeters." As he returned to his seat, I recorded Arif's finding under the word 'circumference' on the chalkboard and noted his name nex

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next to it. His noble but cumbersome attempt set a perfect stage for where I wanted to move next.

The students decided, after witnessing the problems Arif faced with the ruler, that an alternative means, preferable something that would bend around the can, would be useful. Arif concurred. I pulled the string from my pocket and offered it as a means for measuring the circumference. Wu Lee volunteered to come forward and demonstrated a way to use the string. All eyes were fixed on her as she held the ends of the string and wrapped it around the middle of the can. She found that the string was longer than the circumference of the can. With intense concentration, she used the fingers of one hand to mark the length of string she needed. As she pulled the string from the can, she tried to keep her fingers positioned in the same place. I noticed her fingers slip. She glanced at me over the top of her glasses and replaced her fingers at about the same spot. I suspected this slip of her fingers was troublesome to her. I watched her strive for precision and perfection in everything she did. I decided not to say anything. She laid the length of string between her fingers along the edge of her ruler and announced, "26 centimeters." As I noted this on the chalkboard along with her name, many hands rose in the air. Others wanted to try measuring the can with the string.

Rather than spend any time critiquing Wu Lee's method, I invited Luke forward. After watching Wu Lee, he disclosed his concern for the tension of the string. "I think it might work to place the string at the bottom of the can, wrap it around, and then make sure it is nice and tight." I asked, "Why does it need to be tight?" Luke reasoned, "Because if it is loose, it could add a few more centimeters than what it really is." Luke tried out another method. Placing the string at the bottom of the can let him use the desk to steady the string. His fingers marked the spot and he laid the string on the ruler and pulled it tight. "21

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centimeters," he announced. As I put his name and measurement on the board, there were many hands in the air again.

This time I saw Faruq's hand. I could hardly believe it. Faruq never volunteered and I was often uncertain about how to draw him in. He represented a hard issue for me in choosing to focus on a conceptual understanding of mathematics. I thought that doing mathematics in way that emphasized language and reasoning was very difficult for him. He was someone that could probably "do" math if I would just teach rules and let him practice them. Morally, this seemed irresponsible for I knew that he would not have known how to work a problem if he forgot the rule. And I had already seen that remembering rules and following procedures was hard for him. What would he do when faced with a problem for which there were no rules? I knew that the thinking and reasoning I asked him to do was demanding but how could I expect less of him than the others? Faruq stepped forward. The room fell silent. Everyone recognized this was a rare event.

Faruq carefully measured the circumference of the can. He bent to his knees as Arif did, pulled the string not quite as tight as Luke, and watched so his fingers did not slip nearly as much as Wu Lee's. Faruq laid the string along the ruler and whispered, "23 centimeters." I added his name and measurement to the growing list on the board. As Faruq returned to his seat, I noticed that he glanced several times over his shoulder at the chalkboard. When I saw him do this I realized this was probably the first time his name had been attached to an idea in math class. I had never seen him look more confident. I felt awful. I had been thinking that learning math for understanding was just too hard for him. It had taken until mid-October for me to see that he had something to contribute.

I quickly called on Jennifer who, it turned out, had been thinking about the finger slippage problem. She had noticed that centimeters appeared to get

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lost or found somewhere in the process between wrapping the string around the can and placing it along the edge of the ruler. She inquired, "Could I cut the string?" This was a new idea for all of us. I asked her if she would like to demonstrate. She fetched a pair of scissors. Carefully, like the students before her, she wrapped the string around the can. This time, however, the string was snipped at the point where it made its way around the can. The leftover piece of string fell to the floor. Jennifer measured the cut piece of string along the edge of a ruler and declared, "22 and a half centimeters."

By now, more than 20 minutes into the lesson, some students were getting restless while others appeared listless. I had been having such difficulty with pacing. How long should I stay with something? How should I decide when to move on? I had planned to stop after Luke but then I saw Faruq's hand. How could I have not called on him? I had been so preoccupied with thoughts of Faruq, I just called on someone else without much deliberate thought about what I was doing or why. The measuring of a single can had gone on way too long. It had held the attention of the five students who had had a chance to measure but what about the others? I decided that I needed to get a can and string into the hands of everyone as quickly as possible. Out of the corner of my eye I saw Sipho's hand raised. I wondered what I should do. I began the year with the problem of getting a discussion going. Now, I found myself with the problem of how to end one. How long should I let this discussion go on? How should I bring it to a close? What should I do about the students who were not contributing? What were they doing? How could I tell if those who were silent were engaged?

At the time, in addition to my concerns about how to end discussions in general, I was reluctant to call on Sipho in particular. His way of contributing had been bothering me. Perhaps it was more accurate to say that what was bothering me was the way I had been feeling when I saw that Sipho had something he wanted to contribute. I hated to admit that at times I felt like I did not want to call on him. But I did. From my perspective, he almost never gave a concise answer. It was always a long, detailed, rambling explanation, hard for me to follow. If I was looking for a way to end a discussion, calling on Sipho would not bring closure. Often times, because his ideas did not necessarily follow on those that had come before, or it was not apparent to me that they did, I questioned whether he had been listening. I knew I was going to need to listen carefully for a long period of time if I did decide to call on him. Did I have the energy? Could I spare the time? Did I have the patience? I had already let this part of the lesson go on too long. This was reason enough to move on. It was also reason to take a few more minutes and hear what Sipho had to say. "O.K., Sipho, this is the last comment before we all get cans. What are you thinking?"

Sipho began, "I wanted to say that they each got different answers. I notice that they each got different answers and they used the same string. I am wondering why they got different answers if they measured the same can." He had made a keen observation. I scolded myself for having thought of not letting him participate. The accuracy of measurement was an important idea and his observation of the inaccuracy of measurement was insightful. I suspected that many people tuned out of this activity long ago. I repeated what Sipho said and asked what others thought about his observation. Ron interpreted Sipho's observation as a question about why it was that the circumference of each of the cans would have a different measurement and attributed it to differences among manufacturing companies. Mike spoke up and disagreed. This was not how he understood Sipho's question. He pointed to the chalkboard and paraphrased what he saw, "Jennifer got 22 1/2, Faruq got 23, Luke got 21, Wu Lee got 26 and Arif got 22. They got all different answers for just this one little can." Students dis

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discussed Sipho's comment. They offered their own reasons why the measurements were inconsistent and someone made the suggestion that they measure their own cans more than once just to be certain of the size. I looked around the room and saw the new life that Sipho's comment had brought to his classmates and the activity of the previous 20 minutes.

After class. In my interview with Jim (10/16/89) later that day, I recounted what had been going on just prior to calling on Sipho. I explained my hesitation to do so.

> There were kids that were up out of their desks. I had other kids, like Jennifer, who were practically asleep. I wanted to move it and get cans into all of their hands. And this is why . . . I hesitated and almost did not call on Sipho for that reason . . . you know he brought up an interesting question and I almost did not call on him . . . and the thing is I was ready to bulldoze ahead. If I had, I would have missed that question . . . I think it is a good question. . .why we got all different answers when different people have been measuring the same can with the same string. . . You know what else I found interesting about Sipho's question? Did you notice that they talked to each other and did not go through me during that part of today's discussion?

Not only had Sipho asked a good question. It appeared to give others something interesting to talk about.

A Bendable Ruler

Not until Sipho stopped me on the way into class the next day (10/17/89), did I realize that he took home with him his worries about the discrepancies in measurement he noticed the day before. With his face beaming, he lunged out of his seat when he saw me, dashed across the room, and said, "Ms. Heaton, I bet you can't guess what I brought with me today." He was right, I could not guess nor was I sure that I wanted to. There was another lesson going on with the other classroom teacher and I thought he should pay closer attention to that instead of talking to me. For the second time in two days I was confronted with my feelings about Sipho. I could have reprimanded him for springing out of his seat and talking to me. Math class had not started and according to the rules he was supposed to be sitting down and participating in whatever the group was doing with the other teacher. I recalled his contribution of yesterday, saw the excitement on his face, and suppressed the urge to send him back to his seat. I whispered, "Let's move over here," and shuffled us over as far to the side of the room as we could get. I turned to face him, crouched so I was at his eye level, and noticed that he had one hand behind his back.

The hand behind Sipho's back brought to my mind the assortment of trinkets that he had at school almost every day. He defended them as falling into the school supply category whenever I objected to their presence on the top of his desk. For example, he owned erasers that looked like circus animals, pens that sparkled like magic wands, pencil sharpeners that resembled miniature automobiles, and pencils that balanced tiny cartoon characters on the end where normally sat an eraser. Anytime I suggested that he put his toys away because they appeared to me as distractions, he reminded me of their role in school work and immediately put them to their intended use. While my history with Sipho made me a bit skeptical about what occupied the hand behind his back that day, the situation seemed different. In his interactions with me, Sipho usually made an effort to conceal rather than reveal his treasures. The fact that he had made a move to show me what he brought led me to think that maybe what he had was not just one of his trinkets which, by the stretch of the imagination, was a useful tool.

Given the time, I was also a bit hesitant to prolong my conversation with him. In just a few minutes I was supposed to start teaching math. Sipho looked as if he would burst if he could not tell me what he brought. It was clear to me that whatever he had was very important to him and he thought it was going to be to me, too. "No, Sipho," I said, "I can't guess what you brought, why don't you just tell me." Sipho could hardly contain his enthusiasm. He was no longer whispering. "Well, remember how we were measuring cans yesterday and we were doing it with strings and rulers and everybody was getting different measurements because it was hard trying to use the string. Well, I was talking to my dad and he has this special kind of ruler. I asked and he said I could bring it today. I think it's going to help us." Taking his hand from behind his back, he held out a shiny gray green metal ruler for me to admire. "See what it can do," he said as he bent it into a semi-circle. Sipho was right. This was no ordinary straight edge. It was a bendable ruler.

I immediately altered my plans for math class. I decided that Sipho would begin the class by showing everyone this new tool and demonstrate how he thought it would be useful in measuring a can. Then, we would discuss the measurements students made of their cans the day before. I realized that I rarely gave Sipho and his ideas this sort of attention. He was never the first to be called on and he was lucky to be the last. Somehow, he and I had started the year off on the wrong foot.

<u>Dimensions</u>

By the end of the class period (10/17/89), all students had figured out how to measure their cans with strings and had done so several times. We had discussed their measurements. The time had come to transfer the measurements of each can into dimensions for making labels out of construction paper. The CSMP teacher's guide directed,

> Ask the students to draw a rectangle on a slip of paper and to record the measurements of their cans near the sides of the rectangles. (p. 8)

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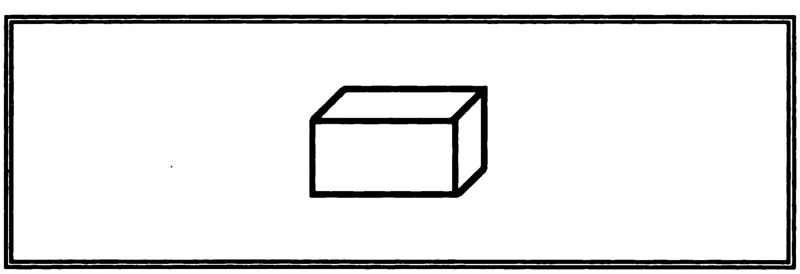
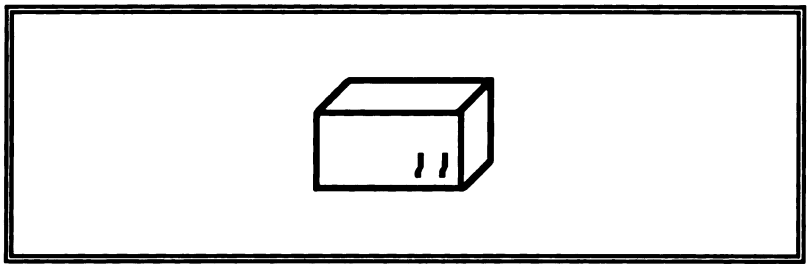
These directions seemed simple. There was no indication in the teacher's guide that following this part of the directions might cause some confusion. How to represent the height of the can--a three dimensional object--on the drawing of a rectangle--a two dimensional representation of the label--turned out to be a difficult task for many of my students and more conceptually complex than either the teacher's guide or I expected.

Rather than just have my students go off on their own and record their measurements as directed by the teacher's guide, I had a hunch there might be something we could discuss about these directions. I started the next phase of the label making process by drawing a rectangle on the board. I told students that the rectangle represented a label and they needed to take the dimensions of their cans and apply them to the label. Recording the measurement of these dimensions on this rectangle was an intermediate step. I asked if someone would record the height of their can on the rectangle I had drawn on the board. Bob volunteered and went to the chalkboard. We all watched as he stood at the chalkboard and looked puzzled. He could not figure out where to record the height of the can on the rectangle. He took a piece of chalk and added a few lines to the rectangle to make something that looked like this.



He said, "I made a 3-D picture of a rectangle so you can see where the height is . . . because it looks more realistic than just the rectangle," as he pointed to and labeled the height of what he had just drawn. "11 centimeters," he announced.

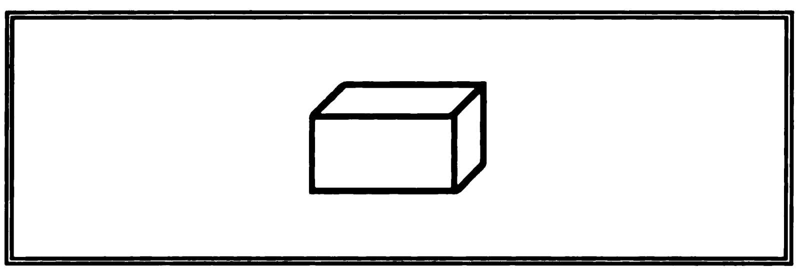


I heard someone say, "Oh, 3-D, that is neat." I was not sure that I thought it was so neat. What were we getting ourselves into? Thinking about the label as something that looked like what Bob drew was not going to be too helpful in making one but the students seemed interested in the idea of three dimensions. What should I do? I could understand Bob's confusion although the possibility of this interpretation was nothing I had considered before class. A can has a height, the rectangle I drew has a length and width. I had asked him to record his can's height.

I was curious what these students understood about dimensions and decided that I would lead us away from making labels for a moment. I asked, "What do you mean by 3-D?" Ron responded, "Three dimensions." I pushed, "What do you mean by three dimensions?" As I asked, I wondered how I would have responded to the question. Ron did not hesitate, "The first dimension is just a line in space." While Ron talked, the bell rang that marked the end of math and the start of lunch. I signaled for the class to remain quiet. I had not realized it was so close to lunch when I started us in this new direction. Ron continued, "The second is depth. It is how we can be able to pick up a pencil. Without it, let's say our pencil is right there, your hand would go right through it. We wouldn't be able to pick it up. And the third dimension is. . ." Inside the classroom the students were restless and hungry. Outside the classroom other students were talking on their way to the lunchroom. All of this made it difficult to hear Ron. He continued to talk through the noise, "They are also working on a fourth dimension, which is time, I think."

Before I dismissed the students for lunch, I announced, perhaps a bit too hastily, that we would continue this talk of dimensions during the next math class. Class often ended the way this one did, at the sound of the bell, in the midst of a conversation. Lots of times I said we would continue what we were doing but we did not, at least not immediately. This time I meant it. The talk of dimensions seemed to have sparked quite an interest among the students and I was impressed by Ron's competent definition of the dimensions. On these grounds, alone, I decided it was worth spending more time on the next day.

The following day (10/18/89), I asked Bob to draw the "3-D" picture he had made during the last math class.



I returned to Ron and asked him to tell us, once again, how he understood dimensions. I thought starting with his definition would be a good place to begin a discussion.

Ron:	3-D means three dimensional, that's just a short name for it. And a couple of the other dimensions are [*pause*]. Can I go up to the board? Well, the second dimension Bob already has up there. But the first, he doesn't.
Ms. Heaton:	Where is the second dimension?
Ron:	The one that Bob just drew.
Bob:	No, that is third dimensional.
Other students join Bob:	That's third, Ron.
Ms. Heaton:	What are you thinking about, do you want to tell us more?
Ron:	The first dimension is just a line through space. The second dimension is depth. If you are just flying through

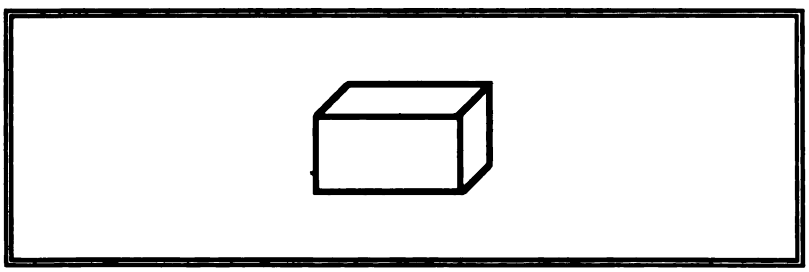
> space and you just keep on trying to grab a pencil it would, your hand would just go right through it so it has to have depth.

Bob interrupted Ron, again, and started to say what he thought. I told Bob to let Ron talk and when Ron was done he was more than welcome to comment on what had been said. Bob was annoyed. Ron continued,

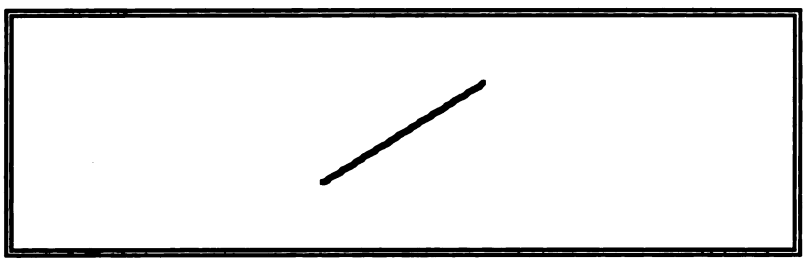
> See, the second dimension is depth. If this was just a line through space it wouldn't have depth. Depth means how deep it is. But if we add depth here, even paper has a depth of about a oneth of a centimeter, less than a millimeter. Everything has a depth if we just look . . . without it we could walk through walls. We couldn't build houses.

The more Ron talked, the harder it became for me to follow what he was saying. What was going on here? Maybe his understanding of dimensions was not as sophisticated as what he first said had led me to believe. Maybe it would help if we focused on one dimension at a time.

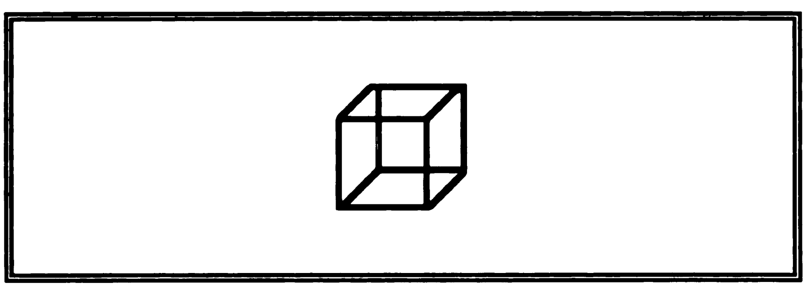
Ms. Heaton:	Could you tell me again, what you are thinking when you think of the first dimension? What are you thinking about?
Ron:	The first dimension is just a line through space.
Ms. Heaton:	And the second dimension?
Ron:	Depth.
Ms. Heaton:	And the third dimension?
Ron:	I am not sure of that.
Ms. Heaton:	Are there other people that want to comment on these dimensions? David, what are you thinking?
David:	Can I go up there? I want to challenge Ron on what he said. He said that this is from the second dimension. This is the third dimension.



Ms. Heaton:	Can you explain?
David:	I don't really know what the third dimension is I just, I think there, the second dimension looks almost like this and Ron might have got them mixed up or I don't know. But I do know this is three dimensional because I used to watch a T.V. show and the guy, he always talked while he drew and he would draw a 3-D square and he would do something like this.
Ms. Heaton:	What do other people think? Jamila?
Jamila:	I don't know what Ron said.
Ms. Heaton:	Do you want to ask him a question?
Jamila:	I just don't understand what you were saying.
Ron:	Can I go to the board? The first dimension is this line through space. It looks like that.



The second dimension is depth. [*He drew the following on the board*.]



Ms. Heaton:	And how does that show depth?
Ron:	You know how like the depth of, the depth of air, air doesn't have depth. Well it does but not the depth that I mean. What I am trying to say is that the third dimension is kind of like the power to pick things up. This drawing is the second dimension because you can't reach in there and grab it and pull it out...It may look like the third dimension but it is not.

The discussion continued. A handful of other people joined in. There was a struggle with the question of dimensions and a debate about the second and third dimensions.

At times like these I wondered what the students were thinking who were not raising their hands. Were they feeling lost? Some of them might have been daydreaming but others could have been really trying to make sense of the ideas of those who were talking. I was never quite sure. I thought one part of the difficulty with what we were doing was that everything that was drawn on the chalkboard was two-dimensional. Some of the things drawn on the chalkboard were two dimensional drawings of three-dimensional objects. The drawings were made in two-dimensions but looked three-dimensional. I thought this added to the confusion. At least it made things more confusing for me.

I saw Jennifer, who had been unusually quiet today, raise her hand. I made a space for her in the conversation, "Jennifer, do you have something you want to say?" "Yeah," she replied, "I don't understand what Ron and the others are talking about. What is a dimension?" Jennifer sounded sincere, as though she really wanted to know. I admired her courage for asking the question. I asked her to repeat it. "I don't understand what Ron and the others are talking about. They said a dimension. I want to know what a dimension is." Jennifer asked a fundamental question[1], a type that rarely gets asked or explored in schools.

After class. By the time I talked with Jim that afternoon, I wondered what it was that I understood about dimensions. When I met up with him I said, "I don't know about you, but right now I feel so confused about what dimensions

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[1] This is a question of the type that I had asked myself in the teaching described in Chapter 4, what is a pattern? It is also similar to the question, what is a whole number, which Wu Lee asked in the lesson from my teaching described in "Learning to Hear Voices: Inventing a New Pedagogy of Teacher Education," (Heaton & Lampert, 1993).

are" (Reineke interview, 10/18/89). It also seemed to me as if I had lost the reason why we were pursuing a discussion of dimensions. I just knew that as eager as I had been to pursue dimensions, I was ready to drop them. Why had I followed Ron in the first place? How do you decide which ideas to pursue? I had not figured out that part of this teaching yet. I talked to Jim about how I thought Maggie might help me.

> One of things that I did with Maggie was to look at her tapes of teaching and I would stop them and ask her why she did this or what she was thinking when she asked that question. I think it would be really interesting for her to watch a tape like this one of today's class and for her to like, maybe she has clues about the mathematics that she could sort of help me with this discussion and tell me like what I could have followed and what wouldn't have been a good idea to follow.

I wondered what role my own knowledge of dimensions played in this discussion. Was that the problem? I said to Jim,

> I am wondering if the reason that I am floundering with this and I don't know what to do, is it because my own knowledge of dimensions is not helping me, that I am missing that and is that why I don't know where to steer this?

Dimensions had seemed like a good idea to pursue the day before. Having done it, I was not so certain anymore.

Labels

The next day we had math class (10/23/89), I decided it was time to make the labels. I began class by placing four sheets of brightly colored construction paper in the center of each group of desks as students took cans, scissors, strings, and rulers from their desks. There was a lot of noise in the room as students assembled these supplies on their desks. Over the clanging of cans, I heard students choosing the color of construction paper they wanted. Deals were struck when two people wanted the same color. I did not involve myself in the exchange.

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exchange. I just watched. I saw that Sipho, as usual, was equipped with more than the necessary supplies. He noticed me watching. He smiled and held up his bendable ruler. I smiled back and silently wondered if his father would ever see it again. The pitch in the room seemed high. The students were visibly excited. They seemed satisfied talking and listening the last four days but now seemed anxious to actually make the labels. I thought we had already gotten a lot of mileage out of the cans. I begged students to please keep the cans quiet for just a few more moments. I wanted to discuss their plans for making the labels.

I wanted students to draw on their experiences over the last four days to make labels. I assumed that they would either measure the height and circumference of the can again or use the measurements they made and recorded the day before. I wanted them to make a record of the measurements on a rectangle they drew in their notebooks and then use the measurements to cut the size of the label needed. The process, however, did not seem so obvious to Wu Lee.

Wu Lee:	Do you make them [labels] very small or very big?
Ms. Heaton:	What do you think?
Wu Lee:	I think make them big because, do we have to cut them out?
Ms. Heaton:	What do you think?
Wu Lee:	I don't know.

I called on Bob, who looked eager to respond to Wu Lee. Bob said, "Well, like you could, like when you take it like, you put it to the very bottom of your can and to the very top of your can and then you take it around and then when it is here you get it and then you cut all of it that's around it and then you fasten it to the can." He was right. Wrapping the paper around the can and cutting the extra off was a way to make a label. But doing it this way did not draw on any of the ideas we had explored over the last four days.

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Here was an example of what I meant by the point of the lesson being something more than just making the label. The students had done a lot of measuring over the past few days. It was important to me that they try to use what they had learned about measuring a can to make their label. Wrapping the paper around the can would get the label made but would not draw on any of their experiences of the past few days. If I thought the only point to what we were doing was to make a label for the can, I could have just told them to wrap paper around the can days ago.

Jennifer spoke up. "I disagree. You can just look on your paper, see what you wrote, measure it with this [the ruler] and cut what is leftover and then it will be much easier than having to wrap it [the paper] around and try to cut the thing." I assumed her way seemed easier to her as she imagined the struggle to wrap the paper around the can, hold it in place with one hand, and cut around the can with a pair of scissors in her other hand. I told the students that Jennifer and Bob had both come up with viable ways to make a label. Bob's way was something they could have done without knowing anything about height and circumference. Jennifer's way made use of what we had learned about measuring. Since measuring height and circumference were things we had worked on in the past few class periods, I said, "I want you to try to come up with a way of using the measurements of height and circumference to figure out how to make the label for the can." I was learning that I could never assume that the connections I saw across days and within activities were obvious to anyone else. Part of my role was to make them explicit. Sometimes this meant identifying students like Jennifer to help me.

Having promised to measure, students began making their labels. I walked around from one cluster of desks to the next. I watched and listened as students worked. Something I saw a student do reminded me that I had forgotten part of t

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forgotten part of the directions. I raised my voice above the students' talk to one another and said, "I want to add one more thing. When you make your label, can I borrow someone's paper for a minute? I want you to try to make it so that when it wraps around the can the edges just meet. I do not want them to overlap. Make them so they just meet, so that they don't overlap." I held up the paper and said, "See the difference between them just meeting and overlapping?" Luke asked, "But why don't you want us to overlap?" Before this year, I would have interpreted Luke's question as a smart remark. But knowing Luke and what I was trying to teach him, I knew he was sincere. I hoped that these students were learning not to accept anything without reasons, even from me.

I realized that what I was instructing students to do was not the way real labels were attached to cans. They overlapped. But I continued and in an authoritative voice said, "Because I would like you to try to get them to meet." Sipho pushed on me. "Well, if they do not overlap, how are we going to fasten them to the can?" He had a point. "You will be able to put a piece of tape across the back," I said in a tone that implied there would be no further discussion. The students went back to work. I wondered if I should have told them the real reason why I wanted them to make labels that did not overlap. It was not my idea, it was in the teacher's guide. This is what it said in the teacher's guide:

> T: Today we're going to make labels for cans you've brought. We'll try to make the labels so that they just fit the cans--so that they don't overlap themselves. What measurements of the cans will we need to make? (p. 7)

It did not say why students should make labels that did not overlap. I constructed my own reason. If they made labels that did not overlap it increased the likelihood that the labels would not fit. The point of the lesson was to measure the labels in centimeters and when they did not fit, see that measuring in millimeter

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in millimeters, a smaller unit, would lead to a more accurate, better fitting label. Here is what was noted a bit later in the teacher's guide.

> After everyone has cut out a label, discuss possible problems with size.
>
> T: Some of you found that your labels did not fit quite right. The label might have been too wide, too short, or too long. You may or may not have tried to make an adjustment. What might have been some reasons for the size differences?
>
> Accept reasonable comments. If no one suggests or hints at size problems caused by measuring to the nearest centimeter, mention it yourself. (p. 10)

I wanted my students to make labels and be able to judge the accuracy of their own measurements, so from that perspective, making labels that did not overlap seemed important. I was uncertain whether or not we would get to a discussion of millimeters. I would be satisfied if everyone got labels made and we took time to assess their fit.

I had already decided, though, that I would move on to something new the next day no matter what happened that day. Five days seemed like plenty of time to have spent on this. I wanted to stop before we got to the point where I was thinking that this had gone on for too long. I wanted to stop while it still felt like we had things to talk about. I wanted to move on before it felt like the students had lost interest. I also wanted us to be at a point where it felt like we had accomplished something. I wanted to end with the feel that there was still more to learn.

It took students most of the class period to make the labels. After Sipho brought it to our attention a few days earlier that five people got five different measurements of the same can, it looked as if most people did not trust their first measurement. I saw many of the students measure their cans multiple times, before measu

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before measuring and cutting the construction paper. The teacher's guide noted that students might want to decorate their labels if they had the time. Most students used markers or colored pencils to decorate the labels before attaching them to the can. Some people, like Wu Lee, strove to replicate the lettering and illustrations of the label as it appeared on the can when they brought it in. Others put aside that idea and made pictures of their own choosing. Army tanks, outdoor scenes, and animals were popular. With 15 minutes until the bell was to ring, I pushed students to finish or at least get their label attached to their can. I wanted the last 10 minutes of class to discuss what they had done.

The activity turned out just as the teacher's guide had predicted, most labels did not fit. Some were off by a few centimeters while others were off by millimeters. As I walked around the room, I encouraged students to finish their work. I had to do some fast talking with several students who looked disappointed and visibly upset that their labels did not quite fit. They thought they had done bad work. I needed to convince them otherwise. I asked them to tell me why they thought their label was the wrong size and how they would make it differently if they were to do it again. Each person had ideas about the source of their errors and had already come up with remedies if they were to try it again. I told them their answers to those questions mattered more to me than whether or not their label fit.

I decided to repeat this conversation of what mattered to me with the whole group and dismissed the idea of talking about millimeters. There were only 10 minutes remaining and I did not want anyone leaving today feeling like they were not successful. From my perspective, the last five days of making labels for cans had been the best math classes thus far this year. But I only saw it that way when I redefined the meaning of success. Given the richness of the discussions of the last few days, I did not mind if we never discussed millimeters even though

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even though it was the point of the lesson from the perspective of the teacher's guide. In a similar way, I wanted my students to feel successful because of what they had learned in spite of the fact that the labels they just made did not quite fit.

I began the whole group discussion by asking the students with whom I had already spoken to share how they were now thinking about their labels. They talked about the process of making their labels, showed what resulted, and described what they were thinking they would do differently if they were to make labels again. It was a time to share and listen as students evaluated their own work. In the midst of the talk, the bell rang. Before I could say anything, someone asked if they could take the cans home now that the labels had been made. "No," I said, "Please keep them in your desk. We might use them again." After class, I admitted to the other classroom teacher that I had no immediate plans for making use of the cans again. I just did not feel right about letting them go, at least not that day. At that moment, these cans and labels represented the longest sustained feeling of teaching mathematics I had experienced that year. I was not ready to part with the cans just yet. I wanted to hold on to them.

After class. Jim did not observe on the day we made labels. When I talked to him the next day (10/24/89), I tried to explain how I was feeling.

Ruth:	Yesterday, that was the end of the cans.
Jim:	Yeah?
Ruth:	But, not really the end of the cans, I don't think.
Jim:	Why?
Ruth:	I don't feel like it has ended.
Jim:	You'll come back to the cans?
Ruth:	I think so. I don't know how or what I'll do with them but I don't feel, I don't want to send those cans home.
Jim:	Why?
Ruth:	I mean I don't want to send them home now because that somehow that signals an end.

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It felt more like a beginning in terms of the feel I was acquiring for this kind of teaching.

Three Years Later: Revisiting these Moments of Improvisation

The events highlighted in these five episodes were all interactions that happened independent of the script in the teacher's guide. They were improvised movements created in moments of interactions with my students. Perret-Clermont (1992) describes each classroom encounter as a new event.

> Each teacher-student interaction in a classroom remains a new event, in which each partner has an active role and specific goals and attributes meaning according to his or her specific past experience and present understanding of the situation (including understanding of his or her partner's actions and reactions). (p. 335)

The following section is about what I am learning about my role, the students' roles, and our interactions as I revisit these movements from a perspective three years later. Through a review of the moments and a reinterpretation of the movements, I reveal what I am learning about what is required of me. This is not independent of what is required of my students as together we were learning what it might mean to teach and learn mathematics for understanding.

Learning the Meaning of Mathematical Argument

My efforts to understand the exchange around rectangles has become an occasion for me, three years later, to more deeply understand what it means to do mathematics and engage with intellectual ideas associated with teaching and learning mathematics for understanding. John's struggle with language as he attempted to define a rectangle and his interactions with others around his conjecture is an occasion for me to further my understanding of proof and what it means do mathematics. His interactions with Luke and David in the context of trying to define a rectangle gives me an opportunity to explore the meaning of mathematical argument, particularly the role of disagreement. How does the

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doing of mathematics in ways that attend to mathematical argument and disagreement interact with the social organization of a classroom in which the aim is to teach and learn mathematics for understanding? What is required of teachers and students? What am I learning about the demands and meaning of the social interactions within the math class and the relationship between the social factors and the intellectual ideas?

John offered the following as a definition of a rectangle in response to my question, What makes something a rectangle? "If two ends are the same size, like if two ends are little and two ends are littler than the other two ends." Luke challenged his idea with the response, "I challenge. I think like if two ends are littler than the other two ends then it would be going down like a triangle. It would be, it just wouldn't be a rectangle." At the time, I wanted John to work on his definition of a rectangle. I wanted him to try to improve on what was the start of a good conjecture. I thought this would be a good class activity. Generally, John had many good ideas that I thought others could benefit from if he participated more. I also thought that in this particular situation his classmates could help him revise his definition. This would not occur, though, without my consideration of the complex social interactions surrounding the situation. What I knew about John indicated to me that the only way to keep him from dropping out of the discussion was if the pushing on his ideas could be done in a way that allowed him to feel secure yet open to criticism. John needed to know that the disagreements were with his ideas for the purpose of improving his initial conjecture, not with him. I thought Luke was a gentle soul and just the right person to help John articulate his ideas more clearly.

In response to Luke, John asked to draw a picture on the chalkboard and use it to explain what he meant to say. "This is what I mean," he said, "These two ends are little and these two ends are long."

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Figure 6.1

John's drawing of a rectangle to match his definition

Figure 6.1 seemed to help Luke see a little more clearly what John meant. It, however, did not lead John to further develop his language for expressing his ideas. His encounter with David pushed him a bit more in that direction.

David was a much different person than either Luke or John. He was someone who did not hesitate to speak his mind, especially to John. I noticed signs of a rivalry between these two boys from the start of that school year. They were both academically strong and I think they continually checked to see who was stronger. They had been together in classes for several years. My sense was that the competition between them had a history that predated fourth grade.

David offered a counterexample as a way to refute John's definition of a rectangle. He said, "Can I show something up on the board? I disagree with John when he said that because, like, see, like this, if you have five ends and these two ends are longer, it is not in the shape of a rectangle."

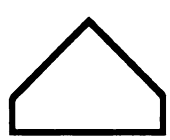

Figure 6.2

David's drawing of what he thought John had defined

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This is a perfectly defensible move, mathematically. Proof or ways of verifying, justifying, and accepting something as reasonable knowledge (Lakatos, 1976) are important aspects of doing mathematics. What David came up with fits John's original conjecture but reveals some hidden assumptions. After hearing what David had to say, John revised his initial conjecture by adding the condition that there were four, not five, "ends." Lakatos refers to what John did as "the method of monster-barring" (p. 23). "Using this method," Lakatos writes, "one can eliminate any counterexample to the original conjecture by a sometimes deft and always ad hoc redefinition" (p. 23). David posed an interpretation of John's definition that led to a different figure (what Lakatos would call a "monster") than John intended. As a way of dealing with this "monster," John revised his definition of a rectangle thereby eliminating the "monster." David, however, was not through. He revealed yet another of John's hidden assumptions. David took an eraser, removed the fifth side and said, "There, that has four ends."

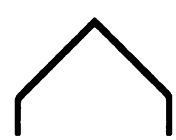

Figure 6.3

David's interpretation of John's revised definition

In response to David's drawing, John asked, "Can you connect it?" To this David responded, "It is connected," and ran his finger around the sides.

At the time, I was less certain of what to make of John's exchange with David than I was about John's exchange with Luke. I felt like the exchange with

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Luke was a good thing. I was less sure about David. One interpretation of what David did is that he played the "refutationist" (Lakatos, 1976), someone who calls attention to possible "unintended interpretations" (p. 84). He interpreted John's conjecture in ways that stretched the meaning of the conjecture to include shapes beyond what John intended. The outcome of this was that David's method of surfacing unintended meanings moved John to be more explicit about what it was that he did intend. Mathematically, this was a good thing to have happen. A second interpretation of what David did was that he picked on John as a person, looked for ways to point out his errors for the purpose of embarrassing him. In one case, the aim was to further the mathematical thinking of the group. In the other case, the aim was to humiliate John. Given what I know about the boys' relationship as well as their mathematical ability, both could be going on and both were reasonable interpretations of the situation. Lakatos describes such differences in refutations as "refutations which only reveal a silly mistake and refutations which are major events in the growth of knowledge" (p. 86). How do I know which is which? What was going on here? It mattered because I wanted the criticism in my math class to be mathematically constructive rather than personally destructive.

There are no rules to follow which would guarantee that a set of interactions would lead to one or the other. Such calls were left up to my judgment based on my ongoing assessment of the situation. What do I know about the individuals involved? Do I let the interaction continue? Do I invite other students to join in? Do I interfere? Do I call a halt to the whole thing? In the situation, I need to sense the feel of the interaction and based on a complex set of variables decide what to do next. Any decision and subsequent action is based on a host of reasons having to do with what it means to do mathematics, what I know about individuals, what I sense from their interactions and what my concerns

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my concerns are for the mathematical progress of the group as a whole at a given moment. Acquiring a clearer sense of the role of disagreement in doing mathematics and the potential benefits and risks for the participants helps me to make the moment to moment, student by student decisions demanded in the situation.

There are two things that I am trying to manage here--the content and the discourse. The first thing I needed to learn was what it meant to take control of these in my relationship with the teacher's guide. I needed to have control of both of these myself. Now I am learning what it means to give up control of these in my interactions with students. What is my role when the substance and the discourse is in the control of my students? Does it mean that it is completely out of my control? I need to figure out when to intervene and when to let students take over the lead.

Why is it important to learn how to argue? In the context of mathematics, it is a way for mathematical ideas to develop. But I am learning the reasons are broader than that. The similarities in the way I want my students to learn to do mathematics connects to the ways in which I want them to engage with intellectual ideas in general. Fawcett (1938), writing about the nature of proof, addresses some broader purposes.

> What these teachers really want is not only that these young people understand the nature of proof but that their way of life should show that they understand it. Of what values is it for a pupil to understand thoroughly what a proof means if it does not clarify his thinking and make him more "critical of new ideas presented"? The real value of this sort of training to any pupil is determined by its effect on his behavior, and for purposes of this study we shall assume that if he clearly understands these aspects of the nature of proof his behavior will be marked by the following characteristics:
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> 1. He will select the significant words and phrases in any statement that is important to him and ask that they be carefully defined.

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2. He will require evidence in support of any conclusion he is pressed to accept.
3. He will analyze that evidence and distinguish fact from assumption.
4. He will recognize stated and unstated assumptions essential to the conclusion.
5. He will evaluate these assumptions, accepting some and rejecting others.
6. He will evaluate the argument, accepting or rejecting the conclusion.
7. He will constantly re-examine the assumptions which are behind his beliefs and which guide his actions. (pp. 11-12).

So, on one level, I am learning how to work with students to define rectangles. On another level, I am learning how to work with students to have mathematical arguments. On yet another level, I am learning how to work with students for the purpose of engaging in intellectual ideas. What is learned here is not preparation for something else; it is learning a way of being, of engaging with other people and ideas, of living as a member in a democracy (Dewey, 1966/1916).

Learning to Appreciate Individuals

In my past teaching, I always gave the directions and students were supposed to follow them. The point was to get through the directions as quickly as possible and move onto the task. I considered time spent on directions a waste. My attitudes about what is worth spending time on have been changing. In my continual search for interesting mathematical ideas to talk about, I have been learning to have a new appreciation of the directions of a task as a site for mathematical exploration. I now see that directions of a task are usually based on an implicit set of assumptions about an appropriate way to perform a task as well as an appropriate outcome or product. What happens if you start questioning the directions? If the directions are opened to interpretation, these hidden assumptions get explored. The process becomes open to interpretation and part of the task. The idea of questioning the directions surfaces assumptions in ways simi

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in ways similar to what I learned about the power of questioning a question about patterns in Chapter 4. Questioning questions or directions in the teacher's guide becomes a way for me to begin to take some control of the content. That is, it is a way to not accept questions or directions in the teacher's guide as givens. I was learning that I have the power and ability to use the teacher's guide in ways that make sense to me.

The students can do the same, if they are given the chance. The exchange with Sipho over making the labels such that they do not overlap is an example of where Sipho is questioning my directions. But because I am the teacher, I have more authority in the situation when I want to make use of it than he does. When exercised, that authority can stifle students' attempts to make meaning for themselves--something I am learning to do as I learn to question the directions given to me around my work. The ways in which I am trying to renegotiate my position with the textbook and its authority is similar to what the students are trying to learn to do with me. In a sense, their work ought not be any more about blindly following directions than mine is. I am negotiating a new relationship with teacher's guide. They are negotiating a new relationship with me.

We spent much of the class period from which the episode "Strings" is drawn dealing with the directions for making the label. The question of how to measure a can with a string was the focus. Here is what the directions in the CSMP teacher's guide said:

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> Measure to the nearest centimeter the circumference for your can and its height. Use a metric tape measure, or cut pieces of string in the appropriate lengths and then measure the string. (p. 8)

Rather than just give students a string and send them off to measure their cans, we had a discussion which included a series of demonstrations by several different students on how you could use a string to measure a can. This occurred after someone tried measuring a can with a ruler, a more familiar tool That a ruler would present difficulties and therefore other tools might be needed is implicit in CSMP's directions. I wanted my students to see this for themselves so that they could appreciate, first hand, the need for another kind of tool.

There is a risk in opening up the directions to interpretation. It means moving away from the teacher's guide and hoping that something useful will come from the venture. I could feel that as my dependence on the teacher's guide was decreasing, my dependence on the students was increasing. For example, I had a hunch that the directions and the string could lead to something interesting but I could not be certain how the lesson would turn out until I tried it with my students. A lot was riding on how my students would make sense of measuring the can. Would they be intrigued by it? Would they be able to make interesting observations about the process? Would they appreciate an attention to the directions as much as me? I trusted that my students would make something of the activity and they did. Was this because of what I asked them to do? Was this because of the way I phrased the task? Was this because of who I called on and what they did?

I am learning to see my students as integral players in all that I attempt to do as a teacher. Not only do I need something interesting for my students to talk about but I am dependent on their involvement. Cohen (1988) writes about how such adventur

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such adventurous teaching makes distinctive demands on teachers. It requires that teachers "depend on students to produce an unusually large share of instruction" (p. 58). Learning to depend on my students means learning to trust that they each have something to offer and that together we will construct mathematical meaning out of a task. I want students to appreciate the process at least as much as the product. "Each partner participates mostly unconsciously and implicitly in the definition of the task and of what is at stake in the interaction" (Perret-Clermont, 1992, p. 335). As the teacher, I have the responsibility to look for tasks that hold potential for engaging my students in meaningful mathematics.

I am also learning that I have to recognize potential in my students. Learning what it is that my students have to offer is an ongoing task. I learned repeatedly that year that my students have more to offer than I ever knew and what they have to offer is in continual flux and development. By the middle of October, when these lessons occurred, I was getting to know my students fairly well. I had a sense of who they were as individuals, the nature of their personalities, and the shape of their relationships with one another. I also had a sense for what they understood about mathematics. There were certain things that I came to expect of each of them.

Fixed expectations did something to ease my uncertainty but at times my experiences blinded me from my students' growing potential. I needed to see my expectations as being as dynamic and evolving as the growth and development of my students because I used what I knew about individual students as one means of guiding my actions during discussions. Decisions about who to call were based on reasons that included the feel of the moment, what the other students were doing, what mathematical ideas I wanted to teach, what ideas I had for teaching them, how much time had passed and remained in the math clas

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the math class, where we had been, where we were at the moment, and where we were headed. The question of who to call on was continually in flux as were the variables that influenced my decision.

From what I can tell the dynamics of the teacher student interaction is not something explicitly addressed in CSMP. In places where typical students answers are given in response to a teacher question, there is no indication that the relationship between the teacher and student and among the students plays a role in the learning of mathematics. Here is how the interaction among the participants is represented in the CSMP teacher's guide:

T: Question
S: Response
S: Response
S: Response

This means a teacher question is often followed by three possible student responses to the question. The responses come from real classroom experiences or were considered mathematically interesting by the curriculum designers. While I am certain CSMP intended for these relationships to be more complex than they appear here, I am not certain they would argue that they play as central a role in the learning of mathematics as I propose. What does CSMP assume about these relationships? Does it matter who these students are as people and what their relationships are to one another? If so, how could the complexity of the relationships and ways for dealing with them be represented in a teacher's guide? What could be done to help the teacher learn to understand and respond to these interactions? I was not dealing with generic fourth graders. These were real students with real growing and changing personalities and abilities that played a central role in my teaching.

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The fact that I am dealing with growing, changing human beings and I am one myself contributes to the inherent uncertainty in my job as a teacher (Cohen, 1988). How are my students going to respond and how am I going to respond back to them? Sometimes students respond as I predict, other times they do not. Sometimes the unpredictability leads to exciting new places, other times to dead ends. Sometimes it takes me longer to see where we are headed than others. Are there ways a teacher's guide can help me deal with the uncertainty? I found myself moving away from the teacher's guide I was using. Its certainty was not helping me deal with the uncertainty of this kind of teaching. I found myself, oddly enough, looking to my students, the source of the uncertainty. I was learning to see my students not only as the source of the difficulties I confronted in this teaching but as resources for dealing with them.

My first exchange with Sipho over the discrepancies in measurements of the same can is a good example of my coming to see a student as a resource. It was an unpredictable exchange that led in a totally unexpected, surprisingly productive direction that almost did not happen because of my fixed expectations of Sipho and my momentary drive to make progress on the task of making the labels. I happened to make the space for Sipho's idea and was amazed by his insightfulness. I worry about how many times this was not the case. How many times did I not give Sipho the opportunity to be heard? How many of his insights went unnoticed? Were there other students who were not given the opportunity to be heard? I worry about the reasons why this might have happened. Did it have to do with their color, their gender, what I perceived as their mathematical ability? I am not sure that I know the answers to these questions but I am learning that they are important questions to ask as I learn to consider the potential of all students as resources.

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Sipho noticed something about the measurements of his classmates that went unnoticed until he brought it to our attention through his question. His curiosity and willingness to raise the question moved the whole group forward by bridging our past experiences of measuring and the inaccuracy of the process with the measuring all of the students were about to do. For a moment, Sipho stepped into the role of the teacher. For a moment, he was the teacher in ways similar to the moment I experienced with Arif in Chapter 5. He heard what the students said, connected what he heard to the mathematical idea of measurement, and had a way to respond. His way of responding was with a thought-provoking question. I was learning that students could be teachers. They, too, could be improvisers. They only needed to be given the opportunity.

To give students opportunities to be teachers means, as the teacher, I need to be prepared to be responsive, spontaneous, and willing to take risks, and put myself in the role of a learner. Just when I think I might have been able to predict a student's response, I was taken by surprise. But it is the surprises, I am learning, that can be opportunities for teaching and learning. And those surprises often come from the students I least expect when I structure my work to allow for that to happen. In a sense, what I have to do is learn to structure my work in ways that allows for uncertainty such that my students and I can move along in this teaching together with the direction being jointly constructed as we go.

Learning to Share the Lead

Dealing with the surprises that I am learning are an important part of this teaching and learning evoke a range of emotion. At times the surprises feel threatening, like when a student sees something I overlooked. This was the case with Sipho and what he made of his observations of the students' measurements. A first impulse could have been to stifle him and hide the fact that I was learning from him. H

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from him. He saw something I did not. I was coming to see that celebrating insights, even when they were not my own, could benefit the whole group. If I could say that a student had helped me to see something that otherwise would have gone unnoticed for me, I would be offering a powerful model for the kind of learning from one another that I wanted my students to be doing.

The surprises of teaching are also not always timed the way I might want them to be, as was the case with Sipho's bendable ruler. From Sipho's perspective it was perfectly timed. It came the day after he noticed the inaccuracy of the measurements with string and understood the difficulties presented by an ordinary ruler. A bendable ruler is ingenious. But I know few teachers who would appreciate any surprise, ingenious or not, when revealed moments before they are about to teach, a time when their minds are filled with their own plans for what they are about to do.

In this case, my interactions with Sipho the day before are what saved the moment of Sipho's surprise for him and me. After hearing Sipho's thoughtful observation that day, I had new respect for Sipho the next time we interacted. Sipho was a student whom I had earlier pegged as difficult to understand and a burden on the discourse. He was someone who stretched the limits of my patience. Letting Sipho open the class with a demonstration of his bendable ruler was a signal to the group that I was willing to share the lead with him, of all people, someone with whom I had always felt like I was competing for attention. Through this exchange, I came to see Sipho in a new light. I had always perceived him as a follower, doing anything to be accepted, wanting desperately to be heard. Giving up the lead to him seemed like a way to begin to meet his needs and allowed him to construct a different role for himself. Is it just a coincidence that Sipho and I had a relationship filled with friction and he also had troubles with his peers? How much influence can I, as the teacher, have over how student

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how students are perceived by their peers? What did my show of respect for Sipho and my move to give him the lead do for our relationship and his relationship with his peers?

Learning to share the leadership with students was not so easy to do. It required that I feel confident enough in what I was doing to give up control. I was learning that giving up control was actually a sign of being in control, something it seems that both teachers and students need to learn. I am learning to take responsibility for the content and discourse and not rely solely or blindly on CSMP while also learning to give up control for the content and discourse to students.

Learning which Leads are Useful to Follow

I was very excited when I first heard Ron's elaborate explanation of dimensions and thought that his ideas might hold the potential to take us to interesting mathematical places.[2] At the time, the talk of dimensions came up at the very end of a class period. I decided that to do any justice to the idea, to honor Ron's explanation, we needed to begin the next class period where this one had left off. This is what we did. I started the next class period with Ron reviewing his definition of the various dimensions. Ron seemed much less clear about his understanding of dimensions than I had first assessed. And it had been based upon that initial assessment that I had decided to begin the class with a discussion of dimensions. I assumed it would be grounded in Ron's explanation. The talk turned out to be confusing and I came to see that what I thought was a deep understanding of dimensions was actually quite fragile and formalistic. Rather than being an opportunity for Ron to shine, I started to worry that he was

[2] Recently, I came across a chapter by Thomas Banchoff entitled, "Dimensions," in the book, *On the Shoulder of Giants*, (Steen, 1990) which explores a range of mathematical ideas contained within the concept of dimension.

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losing face. In the situation, I needed to think about ways to help Ron say what he understood and try to understand for myself what I thought he understood. At the time, I felt as if we were moving in circles with the mathematics and backing Ron into a corner.

As I look back on this now, I can more clearly see a layer of social issues that sits on top of the talk of dimensions. I was reminded of the interplay between the mathematical content and social interaction in looking back over the conversation I had with Jim around this lesson (10/18/89) and Ron, who led off the discussion of dimensions.

Ruth:	Did you notice his posture? I guess that's maybe what I would want to look at the tape--how it changes throughout the class. At another point he is sort of sitting back in his chair with his arm on the counter and my thought when I saw him like that was he feels, it was a posture like he was saying, "I will just let these people talk but I know that what I am saying is right." There was something about his posture there.
Jim:	O.K., what does that say about the construction of knowledge?
Ruth:	That maybe it had ended at that point for him. . . I am just not sure how much other people's ideas were effecting his ideas.
Jim:	Is that important?
Ruth:	I think that is important. Because it seems like there could be a two-way thing also to the degree that his, other people's ideas are affecting his and how are his ideas influencing someone else--it seemed like that sort of stopped or something. I just found the whole thing really interesting both body movements and ideas. The way they were sharing their ideas and the implications underlying these things.
Jim:	What does Ron's pompousness, sitting back in his chair putting his elbow up and things like that and kind of gazing over the masses. . .what does that do or how does that effect conjecture and refutation as part of mathematical dialog?

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Ruth:	I think then that people might be addressing that feeling that he is giving off rather than really thinking about his ideas.
Jim:	Does it say anything about Ron's authority of knowledge?
Ruth:	Ron thinks he has the authority. . . rather than, 'we are going to figure this out together.' It is like he has the idea and he is just waiting for people to pick it up.

I was not sure what to do. I did not feel like I knew enough about dimensions at the time, myself, to take back the lead I had given Ron. I could not figure out how to make a graceful link back to the cans. Jennifer saved us. She said what I think many of us were thinking at the time. She said she did not understand what was being talked about. She was lost in a discussion of multiple dimensions when she was not sure what a dimension was. She had the courage to say she did not understand something and in the process, she cut to the heart of the difficulty of the discussion and put a halt to our spinning. She asked a fundamental question that sits at the core of the confusion as well as understanding. Her question allowed me to put the question of dimensions gracefully on hold. Maybe we would return to it as a group, maybe students would meet ideas about dimensions on their own. And, if we or they did, maybe they would recall this discussion and be able to make new sense of it. In any case, we did not have to spend any more time with the idea then and could return to making labels for cans. Some improvised moments, like this one, lead nowhere in particular, or at least at the time they appear this way. Some may be later recalled and lead somewhere in the future. Others will never be touched again. I was learning that deciding when and how to do what was an ongoing challenge.

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Learning to Assess My Own Progress

CSMP intended this one day lesson on making labels for cans to be one in which students would make labels that did not fit and see that measuring in millimeters would lead to a more accurate measurement than centimeters. This point was actually only made in the midst of our fifth day of working with these cans in a private conversation with one of the students. We took five days to do a one day lesson in CSMP and as a group did not even get to the point of the lesson from CSMP's perspective. Does that mean this lesson was a failure? Could I say I was making progress if I never got to the point of the lesson as defined by the teacher's guide even when interesting things happened along the way?

I view these five days as a success. I base my assessment on what was accomplished in these five days with my students. I have a sense where we started as a group and as individuals and where we ended up. We made our own meaning out of the activity of making labels for cans. The fact that our meaning did not match CSMP's does not take away from its success. This was a new way for me to be thinking about success and progress. Improvisational teaching and the process of mathematical thinking cannot be assessed with answer books. But are there ways assessments can be constructed and given that are a bit less ambiguous than just my sense that something meaningful happened and my students made progress?

Self-assessment was something my students were learning to deal with as well. Many of my students thought they had done poorly because their labels did not fit. I needed to convince these students that whether or not their labels fit was not the point. A lot had been learned to get to this point and they all had ideas of what they would do differently, or what they had learned from doing

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this that would inform their work if they were to make these labels over again. I found myself trying to convince my students of ways of assessing their own learning that I was only beginning to understand in terms of my own teaching.

Are there expectations of what students are to learn in an area like geometry that are different from other areas they are to learn about, like multiplication and division, for example? Something I found in CSMP leads me to think that CSMP sees a difference. A benefit of the "Geometry and Measurement" strand, from CSMP's perspective (McREL, 1986), is that it provides alternative opportunities for mathematical reasoning. Geometric reasoning in the context of spatial and measurement activities provides an alternative area for the success of students for whom numerical reasoning is a challenge. "The focus of this strand is on experience . . . no goals for mastery can be listed here. The effects of such an informal approach must be judged by the long term effects on the students' knowledge, confidence, intuition, and interest in the world of geometry and measurement" (McREL, 1986, p. vi). Why does CSMP imply that geometry is an area that could be taken less seriously? Could such goals be extended to all areas of mathematics? Could the interactions that I experienced here happen in the context of a different mathematical topic?

Learning to Dance

What I am learning here is that my interactions with the students and the mathematics are about being in control and taking or giving up control at a moment's notice. What does this mean? What does it imply for my responsibilities? What does it imply for the responsibilities of my students? The dance metaphor is helpful in explaining what I mean. In the language of improvisational dance, being in control means "being centered" (Blom & Chaplin, 1988, p. 31) and this means being "responsive and ready" (Blom & Chaplin, 1988, p. 31), prepared to "give or take weight, to support, to resist, or to yield, as cal

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yield, as called for by the interaction" (Novack, 1990, p. 128). As in Chapter 5, I am learning about a way of being with students and subject matter that is ever-changing and uncertain. One of the things I need to do is learn to deal with that state of being I need to learn how to deal with the fact that I cannot decide until I am in the situation what my next move will be.

Traditional dance is such that you learn the moves or steps and as the music plays you do the right steps, repeating them over and over, in a rote fashion. Jackson (1986) gives the example of dance lessons, such as these, as the closest example he can think of to "pure mimetic teaching." This is much closer to the way I learned mathematics and taught it when I followed the textbook.

Improvisational dance offers a completely different image of what mathematics teaching and learning can be. Improvisational dancers prepare themselves to be in the situation with other dancers and decide in the moment, based on their feel for their interactions with other dancers, what to do next.

> In contact improvisation, each person is conceived of as an individual yet cannot do the dance unless it is shared with another. Contact improvisation defines the self as the responsive body and also as the responsive body listening to another responsive body, the two together spontaneously creating a third force that directs the dance. The boundaries of the individual are crossed by "seeing through the body" and "listening through the skin," allowing the dance to unfold. (Novack, 1990, p. 189)

This kind of reading of and responding to the situation is something that I am learning to do. As Novack's (1990) title of her book, *Sharing the Dance*, implies, my students and I are not just learning to dance as individuals. Collectively, we are learning to share the dance.

Understanding the nature of this partnership is not easy, though. For while it is true that we are learning to share the dance, as the leader, I have responsibilities which differ from my those of my students. The image of the role of the leader c

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of the leader of a group of improvisational dancers seems useful in understanding certain aspects of my role and its responsibilities.

> Your job is to insure that the conditions conducive to such a response are present. It is for you to generate an atmosphere where explorations, experiments, and risks can take place safely. You must see that each person is accepted, respected and dealt with equally; that no one is judged or compared; and that each mover is helped to shape his experience so as to make it meaningful for him. (Blom & Chaplin, 1988, p. 54)

CHAPTER 7

WHAT SHOULD I DO NEXT?:

EFFORTS TO CONSTRUCT THE CURRICULUM

Introduction

Every teacher is faced with the question of what to do next. This chapter looks at how I deal with the question during four lessons in April (04/12/90, 04/16/90 - 04/18/90) and the second to the last day of math class during the first week of June (06/04/90). In dealing with the question, I find myself learning to integrate what I have learned in the previous chapters. That is, I am learning how to listen and respond to students in the situation. I am doing this while I learn to take the lead of what and how to teach away from the CSMP teacher's guide and give up some control of the content and conversation to my students. I do this as I struggle to understand what there is besides the rule for my students to learn about adding fractions. I know there is something more on the mathematical terrain for students to learn than just rules and procedures. I am just uncertain what. This chapter is my search. What are the underlying concepts that help one to understand why the rules governing operations on fractions work? How does my struggle to understand the mathematics and to see what is important for students to learn interact with my search for an answer to the question of what to do next? And how does what transpires as a result of my decisions about what to do next contribute to my increasing understanding of the mathematics, my growing sense of the terrain?

Nearly six months elapsed between the series of lessons studied in this chapter and those lessons described and analyzed in the previous chapter. Those lessons occurred in the middle of October. I continued to teach fourth grade math thro

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math through December. As I explained in Chapter 3, from January through the middle of March I taught an introductory teacher education course at the university. During the time I taught at the university, the other classroom teacher taught mathematics on a daily basis. I observed once a week as a way to stay somewhat connected to the math content my students were learning and the progress of my students. At the time of the lesson focused on in this chapter, I had been back teaching the math class for two weeks.

When I returned, I no longer used CSMP exclusively. I had begun to take much more responsibility for answering the question of what to do next. CSMP became one of several resources for math problems to give my students. I stopped following the CSMP script for particular lessons. I was frustrated with the spiral organization of the CSMP curriculum--switching from one content strand to another from one day to the next. I began to consider ways I could reorganize the curriculum such that I could stay with a single topic over a series of days as one way to deal with the question of what to teach in the remaining several months of school. At the time, I wrote,

> I am very frustrated with CSMP at this point. This jumping around from topic to topic doesn't fit with what I am trying to do. I am thinking about how to reorganize it. (Heaton journal, 04/10/90)

I chose this series of lessons in this chapter because of my interest, as a teacher and a learner, in the mathematical problem the students were working on in this series of lesson. I knew there was more for my students to learn than the rule for adding fractions. This series of lesson is about trying to teach mathematics for understanding while trying to figure out what mathematics is important to learn. This series of lessons has become an opportunity to examine the relationship between my struggles to figure out what is important for students to learn and how to decide what to do next.

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The question of what to do next propels the story of my teaching. My analysis focuses on the math problem being worked on in these lessons: 2/4 + 3/4. I use the problem as a way to look back at my past teaching and understanding of mathematics. I consider the changes I experienced in this series of lesson and my new conceptions of mathematics and teaching. I also reflect on my study of fractions and my new appreciation of the nature of mathematical knowledge. I return to Twain's adventures on the Mississippi(1990/1883) and Merrick's (1987) writing about the craft of navigating a riverboat to consider what it means to know and the nature of what is to be known in the context of teaching mathematics. What it means for a riverboat pilot to come to know the river and make use of what he knows offers images of the knowledge I have had to learn and make use of in my teaching.

The Rain Problem

As I planned for the next day's math lesson (04/12/90), I wanted to continue with the topic of fractions we had begun working on several days earlier. I found myself searching for a math problem that would require my students to add fractions with like denominators and provide a challenge for those who I thought could already add fractions. I found a problem in the Addison-Wesley textbook (Eicholz, O'Daffer, Feelnor, Charles, Young, & Barnett, 1986) which seemed appropriate.

> The Rain Problem
>
> On Saturday it rained 3/8 of an inch. On Sunday it rained 7/8 of an inch. How much did it rain in the two days?
>
> (p. 251)

The fractions were of a reasonable size and the numbers were manageable. I thought an interpretation of how to solve the problem would be straightforward.

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I wanted my students to focus on the meaning of the addition of fractions, not on an interpretation of the problem, even though that was something I was learning could have value. My reason for choosing a particular problem had become part of what I wrote in my journal as I planned for the next lesson.[1]

> I chose the problem because I want to stick with fractions. In the lessons I've been doing on finding fractions of a whole number, addition and subtraction of fractions has come up. So, I think they need a bit more of a challenge. This problem allows them to add fractions but also gives those who are ready the opportunity to think about something more--the numerator larger than the denominator, leading us into mixed numbers. It also seems like the opportunity to talk about where fractions fall on the number line. (Heaton journal, 04/11/90)

How Would I Solve the Problem?

I was asking students to make sense of rules and procedures about fractions that I could recall and apply but did not understand. Consequently, I found myself rethinking my own understanding of the mathematics I was asking my students to learn. How would I make sense of the meaning of the addition of fractions? I thought a picture would help students find a solution to this problem and communicate their ideas to others. I imagined that my students would draw a rain gauge. I imagined that they would divide it into eighths, start at 3/8, add on 7/8 more, and end up with 10/8 as the solution.

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[1]This was another change--that my plans were now part of my journal. At the beginning of the year, I had one notebook in which I wrote my plans and another notebook where I kept my reflections on the lesson. By the time of this lesson, my plans and journal were integrated into a single notebook and I was in the midst of collecting and analyzing data on my planning for a research course I was taking. See Heaton (1991) for what I learned from that research.

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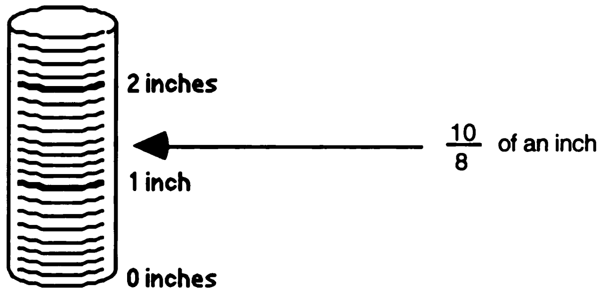

RAIN GAUGE

I thought the challenge for my students would be to see 10/8 as an amount greater than an inch or equivalent to one whole inch of rain plus 2/8 of an inch.

Students sat at desks arranged in groups of four, and talked quietly as they worked on the problem. On that particular day, their talk sounded like it was about the math problem. This was not always the case. Some days I heard the buzz of their voices and was fooled into thinking they were working together until I listened more closely and heard them discussing anything and everything but mathematics. In the past it had been much easier to make judgments about my students' engagement. They were either talking and off task or working and silent. Silence, however, was no longer a goal. Talking in meaningful ways about mathematics had taken its place.

There were reasons for listening closely to what my students were doing that went beyond judging on-task and off-task behavior. I used what I heard as they worked the problem to help me to decide what to do next. I borrowed a structure that I had observed in Maggie's class that supported the listening and decision making in action that I was learning to do. Maggie started each class by giving her students time to work on a math problem in small groups. They could work on it alone or talk with the people in their group. I was beginning to see how this was more than just an alternative way to organize a math class. I could not do as I had in the past and plan everything I did ahead of time. My work depended

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depended on my students' participation. Roaming the room and eavesdropping while students worked in small groups was a way to get some sense of students' understanding of the problem.

Multiple Representations

As I wandered around the room and looked over my students' shoulders, I was amazed by the variety of pictures I saw. The pictures were mainly a collection of shaded rectangles and circles that looked different from the rain gauge I had imagined. Maybe fourth graders were not quite as familiar with rain gauges as I was. What did they understand about the process of measuring rainfall? It was a context that made sense to the writers of the word problem and it made sense to me but what sense did it make to these fourth graders? Imagining a picture I might draw forced me to consider my own solution to the problem but did not prepare me for these other points of view.

As a way to build on the variety of students' work I saw as I circled the room, I asked five students to draw on the chalkboard what was in their notebooks. We would discuss their solutions for 7/8 + 3/8 in a large group discussion in the remaining 20 minutes of class.

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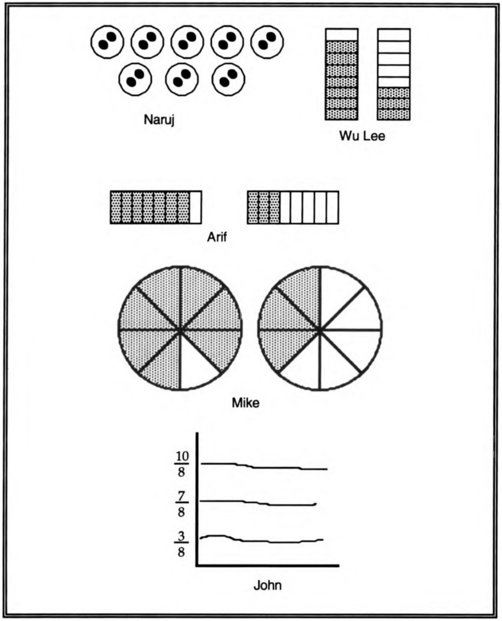

We discussed four different ways of thinking about the problem in the time that remained in the class period. Three of the four explanations discussed had been among the five initially drawn on the chalkboard. What follows are the four we discussed.

<u>Naruj's perspective.</u> A stranger might wonder what Naruj's picture had to do with figuring out the rain problem but I recognized where it came from. It

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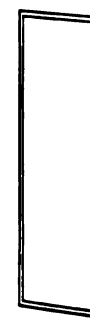

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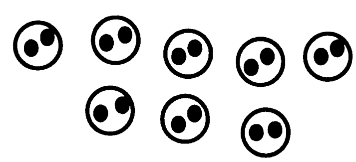
was similar to the drawings that students had done over the past several days to figure out fractions of whole numbers. Naruj said the answer was 10/8 of 16 which was the same as 20. He worked this problem in a similar way to the ones we had worked on in the past several days. I understood that Naruj had done something reasonable given what I knew about what we had been working on in earlier math classes. He got a right answer but his was not an answer that made sense with the rain problem.

To find 10/8 of 16, divide 16 into eight equal groups of two each, although I am uncertain where he got the 16 from. One eighth of 16 is 2. Naruj's picture shows 8/8 of 16, 8 groups of 2, or 16.



Each circle represents one eighth of 16.

9/8 of 16 is found by adding another one eighth of 16 or 2 to 16 which equals 18. 10/8 of 16 is found by adding another one eighth of 16 or 2 to 18 which equals 20. Naruj's statement that 10/8 of 16 equals 20 was correct but it was unclear to me how he saw this related to the rain problem.

Bob's perspective. Bob thought the solution was 10/16 and explained why.

> Well, I knew it says three eighths of an inch and seven eighths of an inch and I knew it could not be higher than two inches because it says three eighths of *an inch*. I mean there are 16 millimeters in an inch and I mean there are only three millimeters that we are using in one eighth so I knew it could not be higher than one inch. Actually, I thought it couldn't be higher than an inch because we

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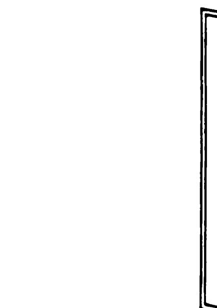

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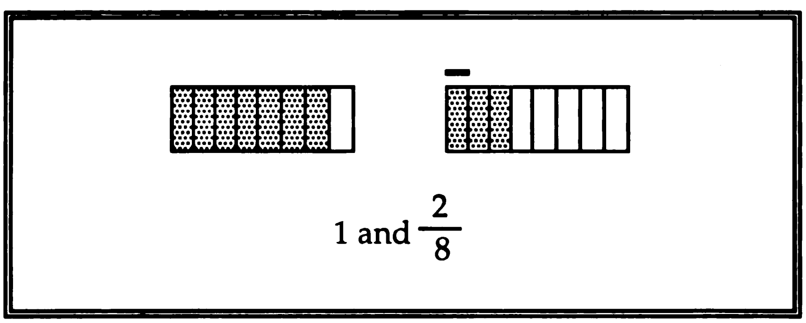
are only using three out of 16 and seven out of 16 and that is only 10.

As I heard Bob's mix of inches, millimeters, and fractions of inches, I felt frustrated. Why didn't I consider the use of a ruler for this problem when I planned the lesson? If I had, I would have warned students to stay clear of the metric measurements on their rulers. Students were always confused by the metric and standard units. The different units embedded in each of these (i.e., centimeters, millimeters, inches, half inches, quarter inches, and eighths of an inch) tended to confuse them even further. Where did Bob get the 16 from? Sixteen might have seemed reasonable to him because it matched the denominator if the 8's were added together. He might also have counted the shortest lines between the 0 and 1 on the inch side of his ruler and hit upon 16 sixteenths and called them millimeters.

<u>Arif's perspective.</u> Arif agreed with Bob that the answer was 10/16. He drew the following picture on the chalkboard and explained how looking at his picture one way represented 1 $\frac{2}{8}$, while looking at it another way represented 10/16.



He said, "Well, they add up to be the same. Like I had seven eighths right here. Well, I had to get three more eighths so I colored this space in and that is one whole and then I colored two here and that is one whole and two eighths." I asked, "What about ten sixteenths?" Arif said, "Well, I did the same thing except

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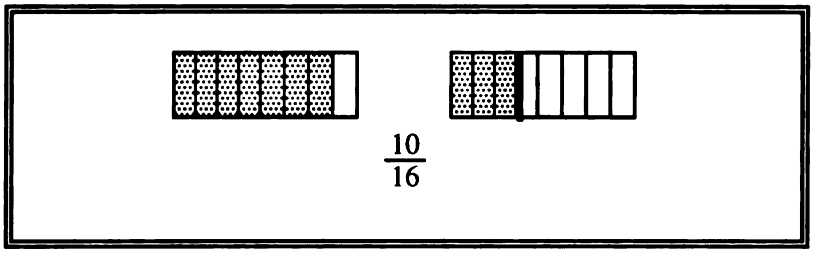
I did not put this line here. I put it over here and just added them like I said, like it is 16 spaces all over here and ten are colored in."



I was amazed that Arif was able to view his picture in terms of eighths as well as sixteenths. If I viewed the two rectangles together as the whole, as he did in the his second explanation, I could see 10/16. If I saw each rectangle as one whole, I could see $1\frac{2}{8}$ of two rectangles. From my own mastery of rules, I knew it was not right to add the denominators but how would I explain why the answer was 10/8 or $1\frac{2}{8}$ rather than 10/16? On what grounds would I make my argument? Was there a way to use Arif's picture to explain why you did not add the denominators? What should I do with Arif's sound explanation of a wrong solution? Should I proceed as usual and accept all answers? Could I let students think the solution was 10/16?

<u>Mike's perspective.</u> Mike agreed with Arif and disagreed with Naruj. This was fine until I realized that he thought Naruj's answer was 10/8. He said, "Naruj, you can't get ten eighths because if you get ten eighths you are going to have one whole and two eighths just like me and Arif got. Because if you have seven eighths all you need is one more eighth, one eighth to fill that in and that would be one whole. You can't go over eight eighths." He seemed to think that 10/8 was equivalent to $1\frac{2}{8}$ and I was glad of that but where had he gotten the idea that the answer could not "go over eight eighths"? I had chosen this problem deliberately because the answer was more than 8/8 or one whole. I was curious to hear more of what Mike thought so, I asked him to elaborate. He said,

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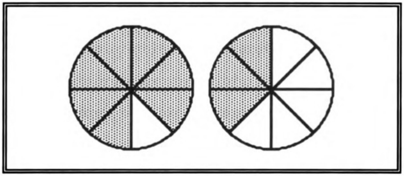

He continued, "I am talking about my picture. It is about pies and eight eighths is your whole." He did not seem to notice that he had slipped from the language of inches of rain to pieces of pie. "Well you have eight pieces in each pie. Seven pieces are colored in for one of the pies. So you take one of them pieces and you eat that other piece but you don't. You add that in there." I tried to follow what he was saying by moving my fingers from circle to circle and piece of pie to piece of pie. I asked, "Put this one in here?" as I pretended to take one piece from the circle with 3/8 shaded in and move it to the circle with 7/8 shaded in. He nodded and said, "And that's your whole. That's eighth eighths of the whole pie. Then you have two eighths left over here of that pie so, it is one and two eighths." What did he mean? How could he say that the answer was $1\frac{2}{8}$ but use the argument that it was not 10/8 because it could not go over 1? What did he think was the relationship between 10/8 and $1\frac{2}{8}$? Why was he unable to see it was possible to have a fraction greater than one? What did 10/8 mean to him?

The class ended when the other classroom teacher motioned to me that it was already past the time to go to lunch. None of us heard the bell. Later in my journal, I wrote,

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> I think it is interesting that I didn't hear the bell ring for lunch and I was not able to pick up from the kids that they were interested in ending class. I think this says something about their engagement as well as my own. (Heaton journal, 04/12/90)

What Should I Do Next?

Giving the students a problem with an answer greater than one whole was intended to present a challenge for anyone who thought they knew how to add fractions. Instead, it seemed to be viewed as an impossibility. I began to question what it was that anyone understood who I assumed was able to add fractions. If they understood something more than the rule, what was it that they understood? I considered my options for what to do next. I could continue with the rain problem but could I really bring myself to do that? I was so far off in what I planned students would do with that problem compared to what they actually did. The rulers seemed to only add to the confusion. A second option was to move to an entirely new topic, leave the addition of fractions behind. This was not really what I wanted to do. I wanted to try to stick with one topic over consecutive days. A third option was to stay with the topic of adding fractions but find a new problem to work on. What should I do next?

I felt this way before--not knowing what to do. But it seemed that this time I had gotten myself in to this spot differently. Earlier in the year when I tried to follow the script of the teacher's guide closely and my students came up with ideas not in the teacher's guide, I found myself not knowing what to do. It had not worked when I tried to follow the teacher's guide closely and control the content and conversation with the teacher's guide. This was different. Here was an example of where I had opened up the content and conversation wider around a problem than I had done before. And I was doing it on my own, independent of any teacher's guide. But was it that I opened things up too far? I found myself overwhelmed by the deep differences in what and how students worked the p

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worked the problem. What was important for my students to learn? What was the purpose of what we were doing?

I had assumed 10/8 was a given, an obvious answer to the addition of 3/8 and 7/8. I was wrong. If you know the rule about adding numerators and keeping the denominators the same when you have like denominators, the answer is obvious. But, what if you were like many of my students and had not learned the rule yet? What were ways to reason about the addition of fractions? I assumed the challenge for students was to see how 10/8 could be equivalent to $1\frac{2}{8}$. Instead, some students reasoned that the answer was $1\frac{2}{8}$ but saw it as equivalent to 10/16. Conceptualizing 10/8 seemed troublesome to my students. Why? What did the numerators and denominators mean to my students? What were they not seeing that would have helped them to understand that $1\frac{2}{8}$ was not equivalent to 10/16--that one answer was more and the other was less than one? I decided I wanted to continue with adding fractions but move as far away from rulers and linear measurement as possible.

On a page neighboring the rain problem in Addison-Wesley, I found a another problem. It situated the addition of fractions in cups of flour rather than inches of rain. The problem, as it appeared in the textbook, used eighths. I decided to replace the eighths with fourths rather than risk the chance of anything other than the idea of addition carrying over from the rain problem. I wanted to keep a bit more control on what students did with this problem. I did not want to open things up as much with this problem as I had done the day before. I seriously hoped that cups of flour were going to be less messy than inches of rain. I wrote in my journal (04/15/90),

> *I* hope that the unit "cup of flour" will not complicate the problem like "inch" did last time, bringing in millimeters and rulers. This got messy.

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Here is the problem I modified to use the next day as it appeared in my journal.

> The Muffin Problem
>
> One batch of muffins needed 3/4 cup of flour. The second batch needed 2/4 cup of flour.
>
> 1) How much flour was used in both batches?
> 2) How much more flour was used in the first batch than the second batch?
>
> For each problem:
>
> How do you know?
> Draw a picture.

The Muffin Problem: Day 1

I watched and listened as students worked on the problem in the next class period (04/16/90). What sort of pictures were they drawing? What sort of computation were they doing? What were they doing with the denominators? What words were they using to explain their reasoning? Students had spent part of each math class working in small groups. Some days they talked more than others. I noticed that people within the same small group on this day were thinking quite differently. I thought if students talked with each other about the problem they might be able to push each other's thinking. I had watched students influence each other's thinking on a daily basis in Maggie's class.

To initiate this interaction, I interrupted students as they worked on the problem and told them to talk with the other people in their group. To help structure their talk, I suggested they see if their small group could agree upon a solution to the problem. I stressed the word "agree" and hoped that trying to reach consensus within a small group would help to limit the range of possible conjectures introduced once the whole group discussion began. I appreciated the variability of students' ideas but I did not always know what to do with them. Maybe the students could begin to eliminate some of one another's "wrong" ideas. The s

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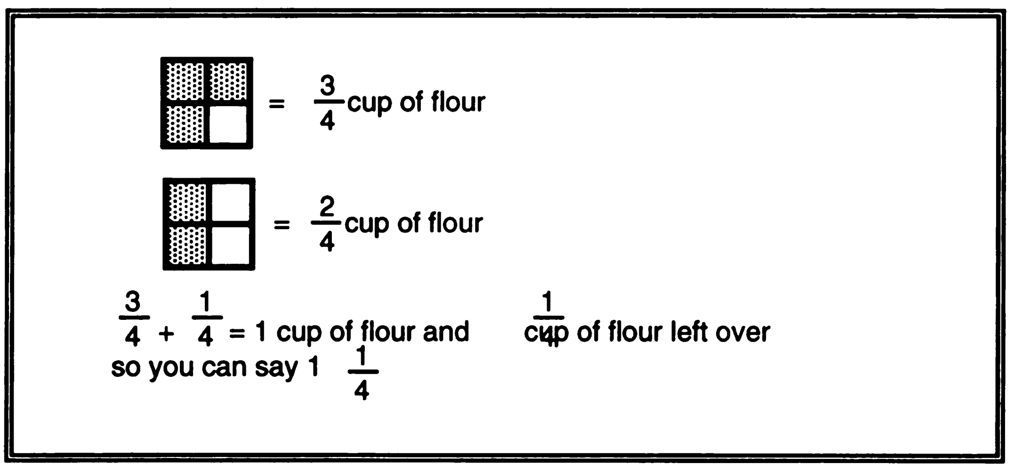
ideas. The small groups never reached consensus. When I introduced this task, I did not consider the possibility that the students with "wrong" answers might have arguments as persuasive as those with the "right" answers.

Multiple Conjectures

Given the range of ideas that surfaced in earlier class periods, I began this whole group discussion with a single drawing which I thought was simple, clear, and representative of the "right" answer. David's work caught my eye. I asked him to copy his work on the chalkboard, as it appeared in his notebook.



Afraid that we might end up in a mess similar to the one created by the rain problem, I decided to go through the features of David's picture in a deliberate manner with the students. I did this in an attempt to keep things from opening up too much and the discussion in my control. I verified that the squares represented cups of flour even though they did not look like cups of flour. We also discussed why it made sense, given the context of the word problem, to add the cups. When I thought we were clear on the squares and the reason for adding, I asked if anyone could explain how they knew that four fourths was the same as one. I asked this because if the students could understand that one was the same as four fourths, then they might be able to see that five fourths was one fourth more than one.

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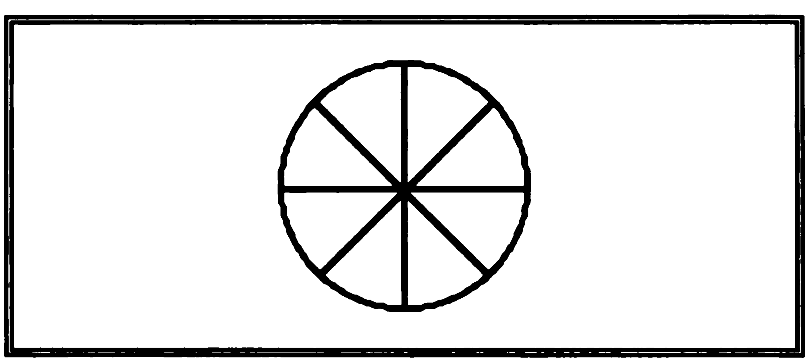
Ana had the "right" answer when I talked to her during the small group time. I called on her hoping she could explain her understanding of the relationship between four fourths and one. She said, "Because if you have four out of four you have a whole." Bob shot his hand in the air. I could see that my efforts to constrain the discussion were about to be thwarted. The moment I called on Bill, I knew the possibilities of where this might lead were no longer completely in my control.

Reluctantly, I nodded in Bob's direction and he said, "I am challenging Ana when she says four fourths equals a cup." Why was Bob doing this? I wanted to disagree with him. I knew, however, that telling him he was wrong would not necessarily mean that he would change the way he thought. And besides, this teaching was not about right and wrong answers, alone. It was about understanding why answers were right or wrong. I let Bob talk and hoped his classmates were listening. He continued, "I am thinking, well it can be a lot of different numbers because of, if you have a circle and you cut it into eight pieces, four fourths isn't the whole of that."

My patience was fading. What did he mean four fourths was not the whole? Why was he bringing up eighths? Why couldn't he see that it was still a whole circle whether the was cut into four or eight pieces? I drew the following picture on the board.



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I asked, " What would four fourths of this circle be?" Bob, in all sincerity responded, "That would be half of the whole." I asked calmly on the outside but frustrated on the inside, "Then, what is the whole?" He responded, "One whole is eight eighths." Did Bob really think four fourths was half of the whole circle when the whole circle was cut into eighths. What did he think was the relationship between four fourths and eight eighths? Could he imagine them both equivalent to one whole circle? What should I do next?

I wanted to push him on the relationship between fourths and eighths. In an attempt to have him describe their relationship to the whole and parts of this circle, I went back to the circle I drew and asked, "What would half of this circle be?" Bob said, "Four eighths." Did that mean he thought four fourths and four eighths were the same thing? Could another student help me raise questions about his idea? I saw Asim's hand. Asim did not speak often but when he did, he always made thoughtful contributions. I called on him. He said, "I agree because four eighths means like four things, like four pieces of cake or pie colored in out of eight." I wanted Asim to speak to the idea of four fourths not being half of the whole circle. Rather than follow up directly on Asim's ideas, I called on Mike who had his hand in the air and a puzzled look on his face. He said, "I have something for Bob. Bob, were you thinking about the rain?" Mike seemed to be really trying to understand Bob's point of view. Mike assumed Bob had a good reason for thinking what he did and he wanted to understand it.

Bob responded to Mike, "I know. I revised with Ana. Because I was just thinking that she might have said that all the time [it's fourths] but now she only said for this one so I think four fourths would be the whole when you have fourths but if you were using eighths the whole would be eight eighths." Hearing this, I wanted to think that, at least for the moment, this question of whether the whole circle was four fourths or four eighths was settled. We could get back to

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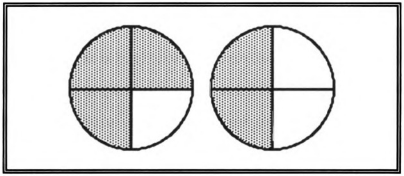
get back to the question of 3/4 plus 2/4. What sort of differences existed in students' answers to this question now? How close were we to reaching a consensus?

I asked, "Did anyone come up with something different than David who got one and one fourth?" Arif's arm was stretched in the air. "I think this is the same thing. I came up with five eighths," he announced. Did he really think $1\frac{1}{4}$ was the same as 5/8? How could he be thinking that? When I asked him to explain, he asked if he could draw something on the board. He drew two circles and divided each into four sections.



He continued with his explanation, "See together there's eight pieces in both. Eight pieces of pie in both and I colored in three from here and two from here so I just add the three plus two is five and if you like, you can use this from here and that would be four fourths and one fourth." He motioned with his hand as though he was taking one of the two fourths that had been shaded in the circle and moving it over to fill in the circle with 3/4 already shaded in. Arif thought the answer was 5/8 <u>or</u> $1\frac{1}{4}$! Did he really believe that was possible? This was similar to what he did the day before when he decided that the solution to the rain problem was 10/16 or $1\frac{2}{8}$.

His picture was convincing. Looking at these circles as two wholes, I saw 5/4. But, looking at it as one whole (two circles), I could see eight pieces with

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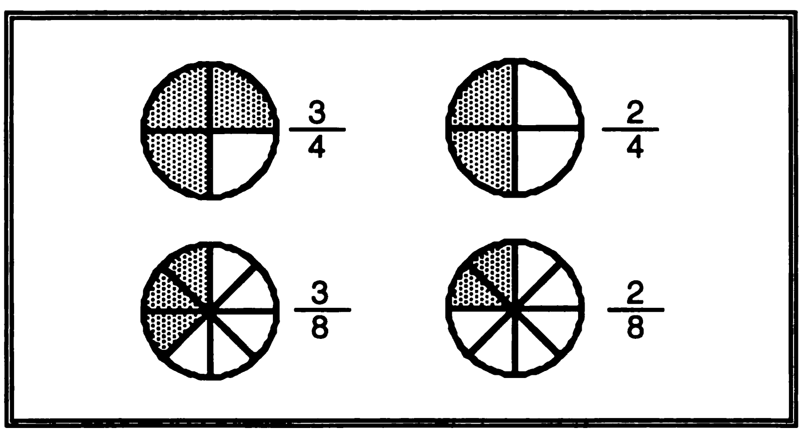
five shaded in. With that latest conjecture, there were now students who thought 5/4, or $1\frac{1}{4}$, or 5/8. There were also some students who thought that 5/4 was the same as $1\frac{1}{4}$ and others who thought 5/8 was the same as $1\frac{1}{4}$. What should I do next? Should I just tell them the "right" answer? How would I explain the "right" answer using Arif's picture? What was the reason that the answer was 5/4 and not 5/8? *I* wasn't sure.

Bob had more to say. He said, "Well, I sort of challenge Arif on this in that also I think I have a hypothesis. I challenge Arif because I agree with the five for the numerator but I challenge what he put for the denominator because we're talking in fourths right now, so when you're talking in fourths we're not using eighths. So I think it is five fourths and also my hypothesis is I think, I can't remember what the name is but those numbers that connect two of them. I think maybe five fourths and one and one fourth might be the same." He was referring to a conversation we had the week before about equivalent fractions. I said "equivalent fractions," he nodded and continued, "We are talking with fourths, not eighths. I mean then we'd have to switch the three fourths to three eighths and two fourths to two eighths so I think it would have to be five fourths." Did he really think 3/8 was equivalent to 3/4 and 2/4 equivalent to 2/8?



What had he learned in our discussions of equivalent fractions the week before?

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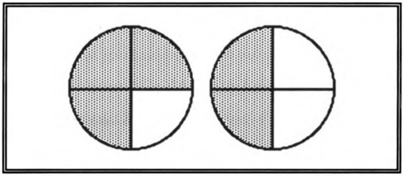
I returned to Arif, who looked at Bob and said, "Well, Bob, okay, we're talking about if, like the first batch, in the first batch there's three fourths, so we're including fourths and then the second batch that's like totally different, it's like two fourths." Bob responded, "Yes, but we're still talking fourths." To which Arif replied, "I know we're still talking about fourths except we're just adding them." Bob continued, "Yes, but when you add fourths, I mean you're getting eighths so then you're going to have to have three eighths and two eighths. So then that would get five eighths but we're talking about fourths, not eighths." Arif said, "Yes, but you're just adding the fourths. Like over there, what do you think the answer is to this?" He pointed to his picture.



"I think it would either be one and one fourth or five eighths." I interrupted and said in a questioning tone, "or five eighths," hoping that he meant 5/4. He said, "five fourths." I addressed the whole group, "What do other people think here? We've got two ideas--Arif's idea that this is five eighths and Bob's idea that this is five fourths." I intended to open up the conversation to others but I noticed Arif. His waving hand told me he had more to say and I gave him another chance to speak. Arif addressed Bob directly, again, "Well, Bob, first of all it can't be five fourths because there's four pieces and you have to color in five of them. How can you do that? You've got four pieces and you're coloring five of them?" How could I show Arif what was possible? His understanding of fractions as shaded regions over total number of regions convinced him the answer was 5/8 and kept him from seeing 5/4 as a possibility.

Bob and Arif just carried on a conversation without me. Each of them took a series of turns addressing one another directly. I was curious what others made of their exchange. I started to detect a bit of tension between the two boys. I could feel ourselves approaching one of those moments when I had to decide whether we were engaged in an argument of ideas or personalities.

Pili's hand was in the air. She often asked insightful questions. I called on her to ease the tension. Pili said, "I think it is five fourths." This was just what I hoped she would say. Maybe Bob's reasoning did make some sense to others besides Bob. We needed to develop a sound argument to counter Arif's. I had my hopes set on Pili's reasoning. "Pili, why do you think it is five fourths?" I asked. She said, "Because you said you aren't supposed to add the denominators so I don't think you add them." Could I really have said that? I responded, "I don't remember if I ever said that, but why does that make sense or why do you think somebody would say that? Could you explain why you wouldn't add those denominators?" Pili shrugged her shoulders and in a quiet voice said she did not know why you did not add the denominators.

My hopes quickly vanished. Pili made no connection to Bob's reasoning and where did she get the idea that I had given her the rule? I was certainly tempted. I had to remind myself why I had made the choice not to tell my students the rule. I wanted Pili and the others to appreciate what it meant to add fractions. This was a nice example of a problem that could be easily done by following a rule but not one easily understood. It was also an interesting problem because the circles and rectangles that people used to represent the problem could be read in two ways, either as 5/4 or 5/8. Why was that? Why was it that this rule made sense? What did it mean to understand how to add fractions?

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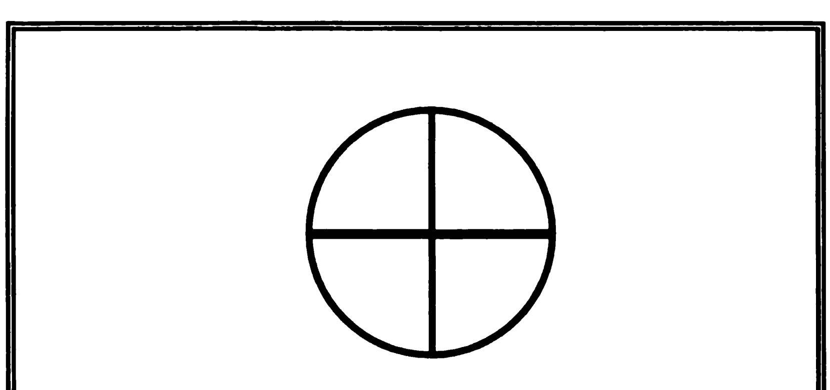
fractions? What were you understanding if you could do the muffin problem or the rain problem independent of following a rule? How would I explain why the denominators were not added? How could I help my students see the significance of keeping the denominators the same?

Changing Minds

As lunch time drew near, I surveyed the group to see who was thinking what. This information was helpful in thinking about what to do the next day. How many people still needed to be convinced that the answer was 5/4? As I watched whose hands raised for what solutions, I was astonished by the hand I saw raised for 5/8. Moments ago Bob had been arguing for 5/4 but his hand just went up for 5/8. What was going on? I was so thrown by Bob's change of heart that I did not pay close attention to who, exactly, was thinking what. I did notice that the class, as a group, was fairly evenly divided between 5/8 and 5/4 with a few people opting to remain unsure.

I wanted to hear what Bob was thinking. He said, "I think I revise to five eighths. Well, I think it's five eighths because when you have like, a thing like this, and your whole is fourths which is the denominator and five is the numerator which you have to shade in." He turned in his chair and faced the students rather than me and continued, "Will someone who thinks that it is five fourths come up here and shade in five out of these four?"



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Before I had a chance to call on anyone to respond to Bob's question, he started to explain why he thought what he had proposed could not be done. "And when you have five eighths, like, okay and when you have five eighths, you have eight and then you just have to color in five and you can't do that here because you can't shade in five out of this four." Arif's picture had convinced Bob that the answer was 5/8. I had not anticipated that revisions might go in this direction. As the bell rang, I decided that we had to continue with this same problem the next day.

I wrote in my journal after class (04/16/90) about the conversation the other classroom teacher and I had once the students went to lunch,

> After class, we stood staring at the board, shaking our heads in amazement. This discussion was filled with ideas. The class ended with the unresolved issue of whether $3/4 + 2/4 = 5/4$ or 5/8. . .the difficulty is in their understanding of what the whole is--it occurs to me that maybe Arif and company do not recognize that a solution can be more than one whole.

<u>What Should I Do Next?</u>

I got very nervous when Bob changed his mind. I started questioning whether we were getting anywhere with learning mathematics when people with right answers revised to wrong answers. I contemplated the question of what I should do next? What was the responsible thing to do? Should I allow students like Bob to be persuaded to revise to a "wrong" answer? But, what good is the right answer, in the context of teaching mathematics for understanding, if you have no idea why it is right? And, I was even beginning to wonder how "wrong" 5/8 was in terms of the representation. Reading the representation as regions shaded in over total number of regions, I could understand Arif's argument that the solution was 5/8. Mathematically, what was I after? Pedagogically, what was the right thing to do?

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I began to think there must be some underlying mathematical idea to be understood related to the addition of fractions that was independent of whether we were considering inches of rain or cups of flour. Were the representations of these contexts as shaded rectangles or circles confusing students? What would it mean for students to see that one whole was the same as 4/4 and 5/4 greater than one whole? What would they be seeing? What was important for students to understand about the meaning of 4/4 and its relationship to 5/4? What would be an appropriate representation to illustrate this?

Unfortunately, Maggie had not observed on these two days. I wanted her assistance in understanding both the difficulty we were having and what I could do about it. Why was it that my students could use the same representation to justify two very different solutions? I met with her and explained how my students had used the same representation to construct two different explanations for two different solutions. From her perspective, the difficulty was some mix of the students' part-whole interpretation of fractions and the limitations of shaded regions of circles and rectangles as representations of fractions.

Area models, circles and rectangles with parts shaded in are commonly used as representations when students are first introduced to fractions in the primary grades. My students used circles and rectangles in the rain and muffin problems and had found fractions of whole numbers using discrete objects (i.e. bananas and dollars) in earlier problems. In the rain and muffin problems, the students' part/whole interpretations together with their geometric representations served to reinforce their solutions. For example, 3/4 + 2/4 could be understood as 5/8 by considering the following representation as five regions shaded in out of the eight total number of regions.

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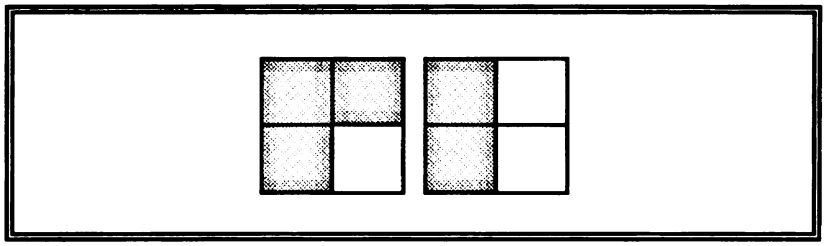


The only way to recognize the above representation as 5/4 is to recognize that the unit of the fractions being added is fourths. This picture represents five one-fourths or 1/4 + 1/4 + 1/4 + 1/4 + 1/4. Adding two fractions that are part of a whole divided into fourths should result in a solution in terms of fourths. Students who see the answer in eighths are changing the unit from fourths to eighths. The unit is what gives the fraction meaning but it is often implicit. There is nothing within this representation that unquestionably defines the unit. Students can easily use this representation to justify 5/8 or 5/4.

Once I figured out that the problem students were struggling with was bigger than inches of rain or cups of flour, I moved away from the idea of switching contexts as a way to solve the problem. I wrote in my journal (04/16/90),

> I guess I was surprised that we ended up in the same place with this problem as we did with the rain problem. I thought the confusion was with the unit--because it was an inch--but now I think it is with any unit--because they are confused about the whole--what that means. Whether we are talking about inches or cups of flour, it doesn't matter.

I needed to come up with a way to help the students who thought the solution was 5/8 understand this issue of the unit. I was not sure what to do. From previous classes, I remembered how persuasive peers could be. Maybe some of the strong advocates of 5/4 would be able to explain what they understood about the unit and convince those who thought the solution was 5/8 to change their minds.

The Muffin Problem: Day 2

The next day (4/17/90), I asked students to figure out the answer, formulate reasons through pictures and words, and then see if the members of their small group could come to some agreement. The students worked on this together in small groups for quite a while. I was excited to see students invested in ideas even if what they were invested in happened to be 5/8. I had worked for months to achieve something that resembled a mathematical argument with my students. Grappling with the question of whether the solution was 5/4 or 5/8 was the closest we had come. Students did not reach consensus within their small groups but the task did serve to get students talking with each other about the problem.

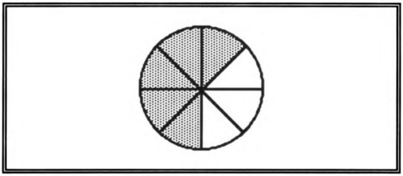
A Mathematical Argument

I started the whole class discussion with a survey similar to the one I did at the end of the last class. Were we any closer to reaching a consensus? As I asked students to raise their hands, I could not keep track of what everyone thought in the brief moment their hands were in the air. But I did notice that Bob was back to thinking 5/4, along with Wu Lee, John, Ana, Luke, Katrina, and Ron and those that thought 5/8 included Arif, Sipho, Faruq, and Mike. Naruj, alone, held firmly to $1\frac{1}{4}$ as the only possible solution.

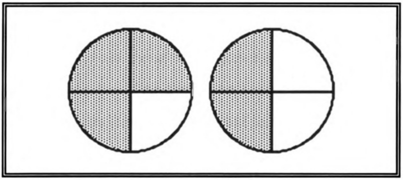
Bob's conjecture. The class discussion evolved into a series of people, including me, trying to communicate points of view we barely understood ourselves. Language, pictures, examples, and counterexamples all become part of the effort. As I heard each response, I had to think about what to do next. For example Bob proposed,

> Well, I was thinking that I challenge everybody that thinks one and one fourth and five fourths because you have one cup and only one cup of flour, not two cups or anything, and so like, say this is one cup, I think it

is five eighths, and if this is the only cup that you use because you only have one cup then I think it is five eighths because when you have this, you can do this with one cup, you can shade in five. You can do that.



[He divided a circle into 8 and shaded in 5.] But you can't do one and one fourth or five fourths with only one *[He divided another circle into four.]* because you would have to have two [circles] to get five fourths.



How could I explain to Bob that you could use two wholes or two circles, each divided into fourths and that was different than looking at two circles together as one whole, each divided into four to make eighths? How could I explain the importance of the unit and at the same time explain that there could be more than one whole of a particular unit--two circles, each representing fourths?

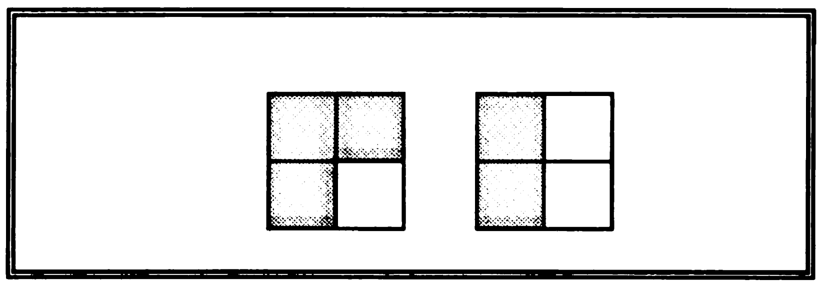
Ana's conjecture. I asked Ana, who had her hand up, to comment on what Bob had just said. She immediately disagreed, "I think you can't change the denominator. You are not [*pause*], can I go up there?" Ana was at a loss for words and wanted to use the chalkboard. As Ana went to the board, Bob responded directly to her, "You are not changing the denominator when you have five eighths. You are not changing it." Several of the other students jumped in, "Yes, you are." I did not want Bob to feel like people were ganging up on him and I wanted to hear what Ana had to say. I knew from watching what she did in her small group that she understood the answer as 5/4. But could she explain it? "Let's see what Ana has to say," I said.

Ana responded to Bob, "You started out with three fourths plus two fourths and now you have got five eighths." Bob admitted, "O.K., Well, yeah, you did change the denominator but let's see you with one, only one cup, make five fourths." I wanted to hear what Ana had to say about 5/4 and did not want her to get side-tracked by Bob's question. I tried once again to give space for Ana to talk. I said, "Let's see what Ana's explanation is. Could you tell us what you are thinking and why?" Ana said, "Well, um, I think, like you can't add two denominators you can't make these two together and have eight for a denominator." I gently asked, "And why can't you? Why do you think you can't?" I wanted her to find the language that I felt I lacked at that moment.

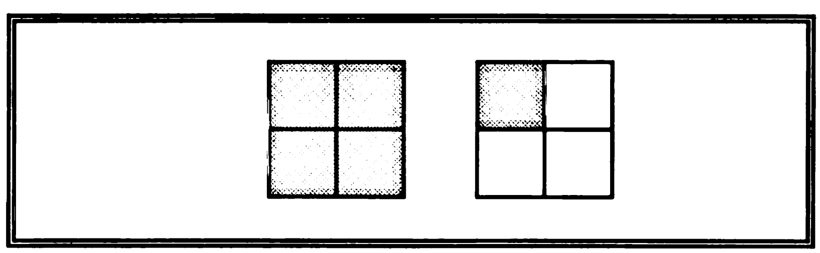
There was a long pause while Ana stood at the chalkboard and made a few attempts to start talking. Her lips moved but no words come out. She started to draw a picture and then erased what she drew. She let out an enormous sigh. I understood exactly how she felt. I did not know how to explain why you did not add the denominators beyond the rule at this point either. Pedagogically, I did not know how to explain or show students what I wanted them to see. What was it I wanted them to see? The only thing I felt I could do was reassure her that this was, indeed, a hard question for everyone. I said, "It's tough. It really is hard and you are doing a really good job of trying to think this through."

<u>Arif's conjecture.</u> I was curious what Arif was thinking. We had not heard from him yet that day. Was this discussion having any influence on his

thinking? I called on him and he said, "I still think it is 5/8 and I have a picture to show." He drew two squares and divided them each into fourths and said, "These are the cups of flour and I will divide it into fourths. Well, now first color in here. Some people color this in and call this three fourths and this is two fourths. Then you are supposed to move it like this." He shaded in three of the four parts in one square and two of the four parts in the other square.

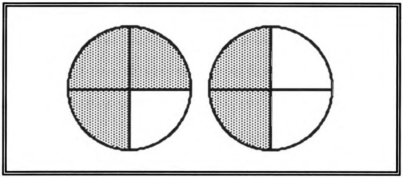


He said, "If you get this one fourth and put it right here, you have four pieces, and then I said you got it like cut it in half and that is one whole but you have to add one more fourth to get five fourths.

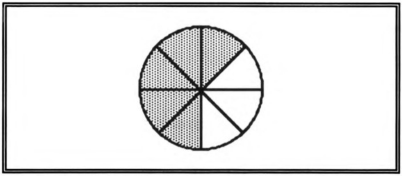


Ana responded to him, "If you are adding, if you are taking, if you are taking this one from here, you are not putting this whole thing together you are just taking this part to here. And that makes one and one fourth." Arif said, "Some people are doing that. But I am not talking about one fourth. Some people are just going, put this one fourth over here and then they have one fourth left right there. But it is one whole right there." Arif saw adding 1/4 to 4/4 as adding another whole which is right as long as you keep the whole as fourths no matter how many wholes you have.

I tried another approach. At that moment, it occurred to me that comparing the size of 5/8 and 3/4 as they were drawn and labeled here might be a way to see that there was less area shaded in something that was 5/8 of a whole circle than 3/4 of the same whole circle. How could 5/8 be the answer to adding two fractions together when one was 3/4? 3/4 was larger than 5/8. I said, "O.K., I have a question. I have seen people draw three fourths and two fourths and come up with this picture.

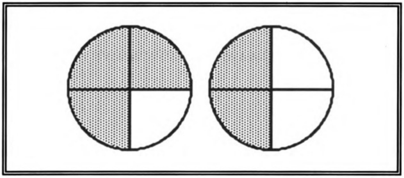


I really wonder, I am questioning when you draw, and here is a circle with five eighths in it,



could you show me how you use two fourths plus three fourths in these pictures to come up with five eighths? Because in addition, this is like the total when I add these two pieces together, so could you show me how you could add three fourths and two fourths and get a picture that looks like this?" I struggled with language and groped for words.

Mike volunteered to draw a picture to match his conjecture of five eighths. He drew this.



"He said, "There is three fourths right there." I asked, "Why isn't that eighths?" He said, "Because this is one, this is one cup of flour and this is another cup of flour and this is two fourths and if you add them together it is just like on this picture, you have these two cups of flour and there is eight one fourth spaces of flour and if you, so it is eight and three plus two is five so it would be five eighths." What Mike said was further evidence to me that the unit was implicit and slippery. He said that there were eight *one fourth spaces*, yet, concluded that the answer was five *eighths*. How could this be?

Scott entertained the possibility of making use of more than one cup. He said, "I could challenge because of what Bob said. Bob said if you, you can't have two cups of flour because it says cup of flour." Bob joined in, "Yeah, cup of flour. That is only one cup." Bob interrupted. I wanted to hear Scott. I said, "And what are you thinking, Scott?" Scott said, "Well, Mike used two cups to get the answer." Bob said, "I know but I only used, on that picture, I only used one because I cut it up into eighths instead of fourths and besides the way I got it with eighths is because with one cup of flour, whoever thinks it is five fourths, with one cup of flour, because it only says cup of flour, five fourths of a cup of flour, then let's see someone draw five out of four." We were back to where the class began--hindered by the part/whole interpretation of fractions. The discussion had gone no where. We were no further along in reaching a consensus by the end of the class than we had been at the beginning. We seemed stalled in the argument of whether the solution was 5/4 or 5/8. Looking at fourths and eighths in the context of circles and squares was taking us no where.

Homework

When I heard the bell ring, all I could think was that maybe the homework, which I always gave on Tuesdays, would make a difference. I assigned students a page of computation on the addition of fractions (Addison-Wesley, 1987, p. 247). They were to find the answer to each problem and I asked them to draw a picture to show how they thought about it. The fractions all had like denominators and added up to less than one. I considered giving them problems in which the answers would be something greater than one but since I was no longer sure what anyone understood about adding fractions, I went back to easier problems.

Samples from the Homework

1/3 + 1/3

3/6 + 2/6

4/8 + 3/8

2/4 + 1/4

2/6 + 2/6

What Should I Do Next?

I could afford to spend one more day on the muffin problem. I was beginning to feel the pressure of time and the need to move on to a new problem and topic. I wrote in my journal (04/17/90),

> How long is the other teacher (or am I for that matter) going to have patience for this? I think she wonders why I don't just tell them. Sometimes, so do I. Would parents say the same thing? What will the students have found out from their parents about fractions in the process of working on the homework?

I had run out of ideas for how to help students see what I wanted them to learn. What did I want them to learn? How could I help students see that the solution involves two wholes and that some them were changing the unit? Counting the number of parts shaded in and the total number of parts did nothing to define the whole or ensure that it would remain consistent.

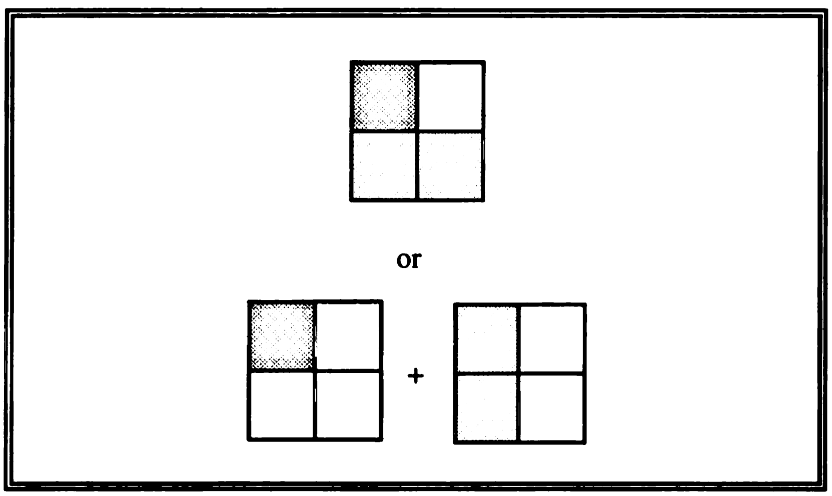
I knew the answer to the muffin problem was 5/4 and was not content to leave the problem in a state where we agreed to disagree. Yet, I was not content with just telling students the rule and moving on. Applying rules to get right answers was not the same thing as having reasons for why an answer seemed right. I wrote in my journal before class the next day,

> I want to try to help the students reason why you do not add the denominators without telling them that you don't add them. If I tell them and they don't understand what good is it? It would help them do the problems in the Addison-Wesley textbook and I think that would make both of us feel better. But, would it?

I decided to see how students did on the homework and build on that in the class. No matter what happened the next day, it would be our last one on this problem for a while. I planned to use the time we worked on different problems and new topics to learn more about fractions, myself, and then try to return to this problem before the school year was over. I was not going to lay this problem to rest until I had reasonable ways to explain to myself and others why the answer was 5/4.

The Muffin Problem: Day 3

I arrived at school a bit early the next day (04/18/90) to look at their homework assignments before math class. As I looked through their papers, I noticed several things. No one had added the denominators. Everyone had "right" answers. Why was it that they could get these problems right but they saw it as reasonable to add the denominators for 3/4 + 3/4 and get 5/8? As I examined their papers more carefully, I realized that all of their representations were in the form of circles or rectangles. The only variation was in how the shapes were used. For example, 1/4 + 2/4 was represented in one of two ways with either squares or circles.



I decided to start with 1/4 + 2/4 and try to move us from there to 3/4 + 2/4.

Whose Understanding is It?

To start the class, I asked if someone would share how they thought about 1/4 + 2/4. Olivia raised her hand. This was a first! What gave her the confidence to finally speak? Olivia, like several other students in my class, entered as a non-English speaker. For the last couple of months, I noticed her talking in English to the people who sat near her. This was the first time she volunteered to speak in math class in front of the whole group. I called on her and she told the group that the answer was 3/4 because "you can't change the denominators." I found myself in a precarious situation. Just about anyone else in the class would be faced with my usual question of why, but in this case, I did not want to take the risk of silencing her. Instead, I complemented her on her response, something I rarely did these days, then said, "Can anyone add to what Olivia said about not adding the denominators? Sipho, what do you think about this?"

Sipho, whose hand was raised while Olivia spoke, relayed a conversation he had with his father the night before.

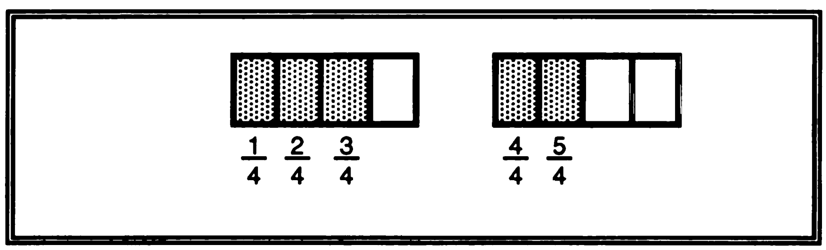
> Well, last night when I was doing my homework, I was doing a problem that I got stuck on so I asked my dad about the denominators, just like what everybody says about the denominators. As we were talking I just realized that you can never *[He shakes his head back and forth.]* add the denominators. Even if it is the same, you still never add them.

Parent involvement and student interest in mathematics outside of math class was something I wanted to encourage. I could not help but be a bit worried, however, that Sipho's father might not have appreciated the time we were spending on 3/4 + 2/4. Sipho explained his understanding as to why you do not add the denominators.

> Well, if you use two big numbers, real big numbers, and you add the denominators it is going to seem to come out to some big huge number and it wouldn't seem reasonable for a small number at the top and a huge, huge number at the bottom and if you add then it gets too large of a number. What reason would you have to do it like that?

Was Sipho trying to reason about the numerator and denominator as a single quantity or was he talking just about the denominator? If he meant the quantity, could it be that he understood that the bigger the denominator the smaller the quantity? For example, the quantity 1/8 is less than the quantity 1/4 even though 8 is larger than 4. Did he understand that when you add fractions the sum should be bigger not smaller? Adding two fractions and getting a bigger as the denominator (or a smaller fraction) does not make sense. Or, was he thinking of adding just the denominators? Sipho was certain that you did not add the denominators. I was uncertain whether he had a mathematical understanding of why. I wasn't sure why myself.

David reported on a conversation he had with his father. He drew this on the board.



He asked, "Okay, Mike, what is this?" David directed his comment to Mike, who just announced that when you draw two squares and divide them each into fourths, together you have eight. David moved his finger from one fourth to the next. As he did this, Mike counted fourths, "One fourth, two fourths, three fourths, four fourths." He paused briefly as David moved his finger to the final fourth shaded in and announced, "Five eighths." Many of the students, including Mike and David, laughed. It seemed that to many students, and perhaps even Mike, it made sense that the next fraction would be 5/4. I was uncertain why Mike clung, in spite of this evidence, to 5/8. David presented a convincing representation but Mike did not change his mind. I asked David to say something about how he would count these fractions. David said, "One fourth, two fourths, three fourths, four fourths, five fourths. Either that or one and one fourth." Through this exchange, David had convinced me that he knew that the answer to the muffin problem was 5/4 and could articulate a reason why. His father taught him to count and add on fractions as a way to reason about addition. Was Mike being stubborn or did David's representation really not make sense to him?

Arif announced that he also talked with his father. He shared what his father told him. "It would equal one and one fourth but, then I asked if it was five eighths or five fourths. First, he thought five eighths and then he said it was five fourths. It is just like David said [*Arif counted the fourths on his fingers*] one fourth, two fourths, three fourths, four fourths, five fourths." Was Arif really convinced that the solution was 5/4? Were the explanations my students gave me their understandings of the problem or their parents'?

I faced new issues. Could I conclude that a student understood something because his father told him? Was that reason enough? How was a parent's authority any different than the authority of the teacher or the textbook when it came to the "telling" of mathematical knowledge? I was teaching the students to think of themselves as authorities for what made sense. Did that mean they should question the authority of their parents? How could I show respect for their parents' ideas yet at the same time push my students to understand mathematics in ways their parents never experienced?

The bell rang. I told the students of my plans to move onto something else the next day. It seemed that many of them had the idea that the denominators were not to be added together. I wanted them to push harder on reasons why that was the case. I suggested that they might even want to think this through with their parents again. David, who I thought learned a reason from his father for thinking 5/4, made a striking remark. He said, "I think I know what the answer is but I just don't know whether or not to change the denominator. I think it is five fourths." Even though David thought the answer was five fourths, and his father had a reason, he apparently still did not feel like *he* understood or could articulate his own reasoning. Were my students learning to value their own reasons?

What Should I Do Next?

After class I wrote in my journal,

> I was surprised that students were still engaged in the discussion even after some of them were told by their parents not to add the denominators. Figuring out why was truly the focus. (Heaton journal, 04/18/90)

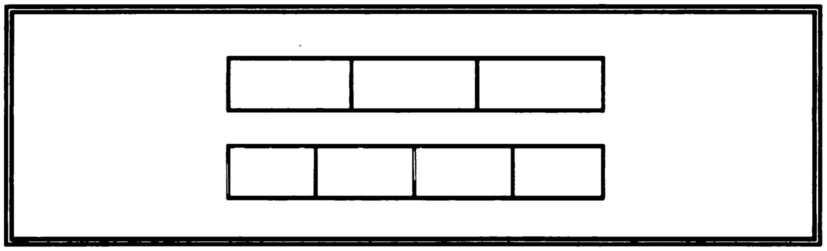
I needed to get a better grip on this math problem myself before we continued. I felt all year like I needed to learn more mathematics. I finally felt like I had a real mathematical question that came from my students and intrigued me. I had a specific area of mathematics I wanted to learn more about. In my journal (04/18/90), I wrote about my decision to stick with one topic over time and what I thought the benefits were to this way of thinking about curriculum.

> This is truly an intriguing problem to me. I think we got as far as we did because of the intensity of the discussion over the last few days. I think if I had dropped it and tried to pick it up at a later date the outcome may have been different. I think there is something about what has been building over the last few days. Two problems in four days--this is a first for me. It was clear that some students had discussed the problem at home. Would this have happened if they had not been given homework dealing with fractions?

During the next six weeks, I moved on to other mathematical topics with my students, pursuing a single topic over a series of days. I made a promise to myself to return to fractions, specifically this problem, before the school year ended. In the meantime, I wanted to think about why we were stuck. It seemed to me that I needed alternative ways to think about the problem, alternative representations, or I needed a way to construct an argument for the correct solution with the representations the students had been using.

The Muffin Problem: Six Weeks Later

It was the first week of June and there were only two math classes left in the school year. During the last few days of math class, we returned to the topic of fractions and I was committed to finding a way to return to 3/4 + 2/4. Most recently we folded paper strips to represent different fractions and compared their sizes. For example, we folded a paper strip into a fourths and another into thirds and then asked, which is larger, one fourth or one third? When I cut the paper strips for this activity, I did not make them a particular length. I did make sure, however, for the purposes of comparing units that the strips were all the same length.



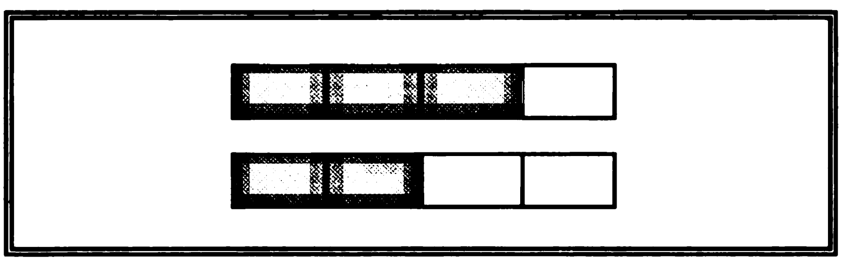
When students folded the strips, some just moved the folds in the strip around until the parts were even. Others used a ruler first, divided the length of the strip into equal parts, marked where the folds ought to go, and then folded the strip. Some students were challenged when I asked them to divide a paper strip that was a bit longer than nine inches into thirds. Some students using rulers folded back the portion of the strip that was greater than nine inches so that each thirds would come out an even three inches long. I wrote in my journal (05/31/90) after the class in which this occurred.

> Ron took a ruler and measured the strip--9 $\frac{1}{2}$ inches. He turned back the excess and divided nine by three. I thought he gave a great explanation why he did it. It was too messy. He wanted to get the work done. He wanted it to turn out even. . . I asked people to respond to Ron. Ana said, "I don't think he can do that. You can't change the whole." This is the

point. This is one of those statements that people might think you paid or planted a child to say. It was wonderful.

I asked students how they would have done it if they could not change the length of the strip or the whole. I told them that folding back part of the strip and not using it was, as Ana said, changing the size of the whole. To follow up on this idea, I gave each student an eight inch strip of yellow paper and asked them, for homework, to try to divide it into tenths and write about how they did it. I wanted to begin the next class with a discussion of what they had done. I had been waiting and watching for an opportune time to connect back to the problem of 3/4 + 2/4. Talk around whether or not you could change the whole seemed like it might be the opportunity. Relating 3/4 + 2/4 to the idea of not changing the whole seemed like a reasonable connection to make.

The school year was quickly drawing to a close and I wanted to return once more to the problem. I decided that in the second part of the next class, I would give students paper strips and ask them to show their solutions to 3/4 + 2/4. I thought they would make two strips that looked like this.



The strip could be used to consider what it would mean to add fractions together and not change the whole with the whole being one paper strip or four fourths. I wrote the following in my notes as the plan for the next math class:

1) What did you do with the yellow strips?
   Were you able to make tenths out of an 8 inch strip?
   If so, how?
   If not, why not?

2) Is there a way to use paper strips to show 3/4 + 2/4?
   If so, how?
   If not, why not?
   How many fourths are in one whole?

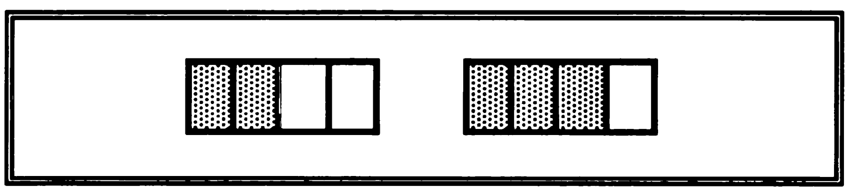
When the class began, students took out their yellow strips of paper and talked about ways they found to divide the strip into 10 equal pieces. Some had folded the paper, others had measured. I opted not to spend too much time on this problem. I was eager to move on to 3/4 + 2/4.

Students put away the yellow strips and took out two of the white paper strips we used in class the day before. I reintroduced the problem of 3/4 + 2/4. I was prepared to hear moans and groans of fourth grade students being asked to reconsider something they thought they had finished and put to rest long ago. This was not the case at all. There were no signs of resistance. It appeared that I was not the only one who still cared about this question. I told students to try to use paper strips to explain their solutions.

Students immediately started talking with one another about the problem. I stood back and watched. I felt no need to direct their talk to be about mathematics or to provide any additional structure to the task. They were doing fine themselves. Students seemed intrigued by the question and invested in figuring out how to use paper strips to represent their thinking. There was a life within these small groups that I witnessed on only a few other occasions. With about fifteen minutes remaining in class, I decided to begin a whole group discussion even though it looked as if students could carry on talking about this among themselves for much longer.

<u>A New Way of Seeing</u>

Luke was on his knees, almost on top of his desk with his hand in the air, when I asked if there was someone who wanted to begin. "I think it is five fourths," he said. I asked, "And do you have a way to talk about it?" Luke started with, "My dad. . . " I abruptly interrupted. We had heard fathers' thoughts on this problem the last time we worked together on this. I wanted to discourage an appeal to outside authorities. It mattered to me that the students learn to construct their own reasons. I reminded Luke of the paper strips. "Do you have a way to talk about it with paper strips?" Luke was willing to put aside his father's ideas and talk about his own. "I took two strips. One with two fourths colored. I made four pieces out of one and four pieces out of the other. I colored in two for two fourths. I colored in three to make three fourths on the other strip.



I added the colored in ones, two plus three is five. And when you have two denominators that are the same, adding one fraction to another fraction, you do not add the denominators." He had slipped into the rule. I really wanted students to try to reason about the denominators with the paper strips and without using the rule. I asked, again, "O.K., how can you make sense of that with paper strips? What does the four in the denominator have to do with those paper strips?" Luke said, "They show you how many pieces you are supposed to divide it into." I replaced his "it" with the word, "whole" and repeated what he had said, "They show you how many pieces you are supposed to divide the whole into." I was trying to emphasize the connection between the fourths that the strip was divided into and the fourths as a unit or the whole.

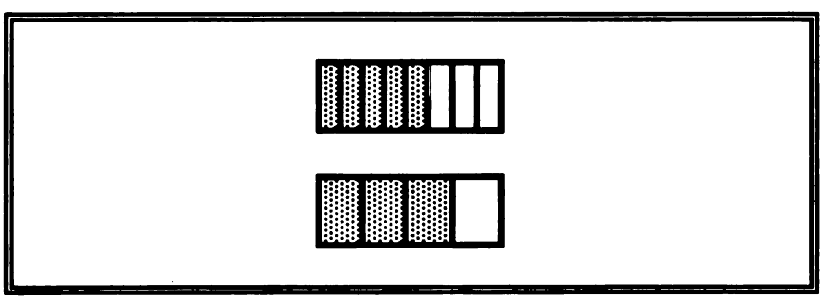
I asked, "O.K., does someone want to comment or add to Luke's idea?" I was curious what Arif, who had earlier been the leading proponent for 5/8, was thinking. He said, "Well, I am not sure but I think I agree with Luke, one and one fourth. But, I think five fourths and one and one fourth are the same. I am not sure about five eighths." A few students shot their hands into the air and started gasping in ways that interrupted Arif. I asked students to put their hands down and let him finish what he was saying. Arif continued, "Well, I like, I had four fourths and that was one whole and then like I added another strip just like Luke did. I got like one and one fourth but there were eight parts in all of the strips that we used, there were eight parts." I tried to say back to Arif what it was that I thought he said he understood, "So you think five fourths is the same as one and one fourth but then you are wondering about this five eighths." Arif added, "They might all be the same." Arif had reasons for thinking 5/4 and $1\frac{1}{4}$ and yet he did not seem convinced to abandon the possibility of 5/8.

I had watched Maggie teach fifth graders all year and been paying attention to how it was she responded to students' ideas. One of the things that I noticed she did was repeat or paraphrase what she heard students say and in doing so added or replaced words in subtle ways to alter their meaning. Her changes were subtle but substantive. They pushed the mathematical ideas by building on the ideas of students in the direction of the mathematics she intended to teach. It seemed like a way to work with the students' ideas that was responsive to students as well as the mathematics she wanted to teach.

I called on Ana, "What do you think?" She said, "I think it is five fourths or one and one fourth." Rather than have her explain why she thought these were the solutions, I hoped she might be able to help Arif and so I asked, "And why not five eighths?" She responded, "Because you are adding two fourths and three fourths. And when you do, this would be one whole and this would a little bit bigger." I tried to say what I thought she meant to say, "It sounds like when you are adding two fourths plus three fourths you are getting something bigger than the whole. Are you getting something longer than one paper strip?" Ana nodded in agreement. If we could talk about the solution as being something more than one paper strip, I thought it might help students to see that the answer was greater than one whole.

It also seemed like one way to address Bob's issue of long ago that the answer could only be in terms of one whole, *an* inch or *a* cup of flour. I wondered what he thought now. I called on him. He said, "I think it is five fourths. Four fourths equals a whole. Since it is four fourths and that is your whole and then you have one more fourth. Five fourths is one fourth more than your whole." Two paper strips, each divided into fourths, gave a way to visualize the answer as more than one whole, as some number of fourths more than four fourths or the length of one paper strip, one whole.

Naruj caught my eye. He said, "I don't think five eighths should be up there because when I look on my paper, one whole is much bigger than five eighths and I don't think that five eighths should be there." I saw that he had two paper strips in his hands and said, "You made up two strips here, can I hold these up." I held up his paper strips.



I continued to talk, "You have one divided into fourths and one divided into eighths. I am going to fold it back to where he has up to here is five eighths and here is the whole. What do you think about that? Naruj, would you tell them again why you don't think it could be five eighths?" Naruj responded, "Because five eighths is smaller than three fourths or a whole. Because it says three fourths plus two fourths so I don't think it can be one of the numbers." I repeated what I thought Naruj was saying, "So, you are saying one of the numbers is three fourths and you are adding something to it so it is going to be bigger than three fourths but here when you look at these two strips, three fourths, all by itself, without adding anything to it, is bigger than five eighths. What do other people think?" The bell rang. I wished we had more time. It seemed as if we were finally getting somewhere. Students were beginning to compare the size of fractions. They also seemed to be trying to wrap their minds around the idea that the unit was fourths not eighths and that the solution was more than one whole.

Three Years Later: Revisiting the Question of What to Do Next

During the past three years, when someone asked me a question about what my math teaching experiences in fourth grade had been like, I often relayed this series of lessons and my struggles with the question of what to do next as an example of what I was learning about the nature of mathematical knowledge and teaching and learning mathematics for understanding. The math problem, 2/4 + 3/4, is a powerful example for me because I can use my understanding of it over time to trace and explain how my conceptions about mathematics and teaching have changed. In this part of the chapter, I examine this math problem and what it represents in terms of what I am learning about a conception of mathematics and the relationship between my changing conception of mathematics and my understanding of teaching. I can trace my answers to the question of what to do next from prior to this year of teaching, during the teaching of the series of lessons described in this chapter, and from my perspective now, three years later.

Prior Conceptions of Mathematics and Teaching

In my past experiences, teaching students how to find the solution to a problem like 2/4 + 3/4 would not have been difficult. They probably would have mastered working similar problems within a couple of days. I would have told my students the rule for adding two fractions with like denominators--add the numerators, keep the denominators the same--and they would have applied it to numerous calculations. My hope would have been that when needed, they could recall the rule they had learned through practice. My challenge as the teacher was to find ways to help my students memorize not only this rule but the other rules that governed operations on fractions and whole numbers and develop strategies for storing and recalling the rules as needed. I was always on the lookout for the best manipulative, or trick to help my students learn. The point was to get the right answer. Students did not have to understand why rules made sense only that they worked.

What my students were learning was a procedural knowledge of mathematics, consistent with my conception of mathematics at the time. Hiebert and Lefevre (1986) define procedural knowledge as having two parts, "knowledge of symbols and syntax and knowledge of rules, algorithms, or procedures used to solve mathematical tasks" (p. 6). The procedures are hierarchically organized and stored in textbooks as a fixed body of knowledge. Jackson (1986) describes it as knowledge that "can be "passed" from one person to another or from one text to a person; thus we can see it as "detachable" from persons. . .preserved in books. . .it can be forgotten by those who once knew it" (p. 118). You either know something or you don't. He describes the teaching that accompanies this view of knowledge as teaching in the mimetic tradition. Mimetic teaching "gives a central place to the transmission of factual and procedural knowledge from one person to another, through an essentially imitative process" (p. 117). It is knowledge that is easily assessed and offers a certainty to questions of purpose and progress in teaching and learning.

> It can be judged right or wrong, accurate or inaccurate, correct or incorrect on the basis of a comparison with the teacher's own knowledge or with some other model as found in a textbook or other instructional material. Not only do judgments of this sort yield a measure of the success of teaching within this tradition, they also are the chief criterion by which learning is measured. (Jackson, 1986, p. 118)

This matches my views of mathematics and teaching and learning prior to my learning about the reforms in math education. I talked about my views with Jim in my initial interview with him before this school year began. At the time of this interview, I had been in graduate school for a full year, a year in which, as I explained in Chapter 1, I was introduced to the reforms in mathematics education.

Jim:	Has your thinking on what it means to know something mathematically changed since you were last in the classroom? (a year earlier)
Ruth:	Yes.
Jim:	Can you describe that change?
Ruth:	In the past, to know division would be for a child to be able to do the computation, get the right answer and not have to explain how they did it. I mean what's really happening with those numbers and not just the steps that they have do to get the answer. Another thing that has changed is probably the amount of reasoning that I would ask kids to do to explain how they come up with an answer. In the past, I would have been looking for the right answer and probably wouldn't have asked them very often how they got the answer. (Reineke interview, 08/89)

These are the views of mathematical knowledge, teaching, and learning that dominated my past experiences as a learner and teacher of mathematics and accompanied me in to this fourth grade math class at that start of the school year.

Changing Conceptions of Mathematics and Teaching

In the series of lessons described earlier in this chapter, I was not content to teach students to solve the problem of 2/4 + 3/4 the way I would have done in the past. I wanted to teach my students something more than the rules and procedures for operating with fractions but I was unsure just what was important for my students to learn. What was there besides rules for students to know? On what conceptual mathematical ideas was this rule based? Why and how did the rule work? There had to be some underlying meaning for these rules and that was what I was trying to figure out. Hiebert and Lefevre (1986) support the logic of what I was aiming to do, "If procedures are related to the underlying rationale on which they are based, the procedures begin to look reasonable. It is possible to understand how and why the procedures work" (p. 10). At the time, I did not understand why these rules worked.

In this series of lessons, I was struggling between two different conceptions of mathematical knowledge--loosening my hold on rules and procedures while searching for some deeper meaning. This state of being uncertain, knowing that I did not know something, was unsettling. I knew a way to put an end to mucking about with the addition of fractions and move on. I could teach my students the rule, have them practice it until they were proficient at getting right answers, and go on to the next topic. This was what I would have done prior to this year. In the midst of the mess I felt like I was in, the certainty of that view of mathematics and teaching practice appealed to me. If I thought about mathematical knowledge as something constructed, however, I could not so easily deal with the question of what to do next. Learning a rule for how to add fractions was not my ultimate goal. There had to be more to this. But what? How could I teach students to understand what I wanted them to learn? What, exactly, was it that I wanted them to learn? If reasoning was what I was after, the students who were thinking 5/8 had thoughtful explanations for why they thought so. Was I to value their reasons even if they supported a wrong solution? I did not know how to push students' reasoning to help them see that the solution was 5/4 and not 5/8. I could not quite put my finger on what made the difference, especially when I looked at the reasonableness of either solution as represented by these area models.

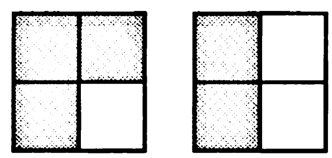


Figure 7.1

5/4 or 5/8?

My struggles with my conception of mathematics throughout that set of lessons are related to my questions of what to do next. I wanted what my students were learning about mathematics to be different but I did not quite know what it is that they should be learning. If I did not know what it was that I wanted them to learn, how would I know what I wanted to teach them? How do I decide what to do next? I had already figured out that the answer was not to be found in the teacher's guide. Throughout this series of lessons, I did make decisions about what to do next based on my best guess at the time as to what was going on with my students and their understandings of the mathematics as I struggled to figure out what it was that I wanted them to learn. What are the various ways I responded to the question of what to do next and what was the relationship between my response and what I understood about the mathematics throughout the series of lessons described in this chapter? In the section that follows, I examine this relationship in the context of several of my decisions about what to do next in this series of lessons and what I learned from looking back on those decisions from my perspective now, three years later.

A change in contexts. Between the day I gave the rain problem and planned to give the muffin problem, I thought that linear measurement and the use of rulers prompted by the context of measuring inches of rain were the main difficulties. I wanted to get as far away from linear measurement as possible. That is when I decided to move to the muffin problem and cups of flour. After switching contexts from inches of rain to cups of flour and seeing what students did with the muffin problem, I realized that the mathematics students were grappling with was not something that would be resolved by attending to flour rather than rain. The difficulty was with shifting units and it did not matter whether the context was rain or flour or the unit was an inch of rain or a cup of flour.

The irony to this sequence of events and my decision of what to do next that led me to move away from linear measurement was that it was probably counter to what people might have done who have a better understanding of the difficulty my students were facing. What I have since learned is that a context that pushed my students to consider the addition of fractions in the context of linear measurement might have been quite useful. "A number line can be used to model rational numbers greater than one (mixed numbers) provided the child already understands certain basic concepts about fractions before starting the number line" (Behr & Post, 1988, p. 221). Each fraction on the number line represents a distance. The interpretation of addition is finding the sum of two distances (Behr & Post, 1988). You can locate the distance 3/4 on the number line and add a distance of 2/4 to it.

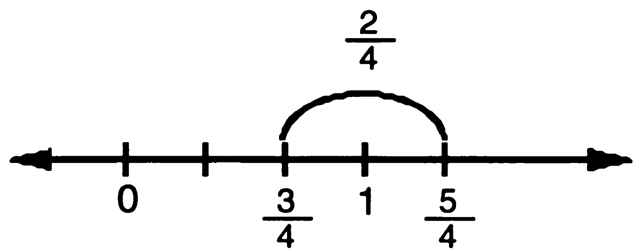


Figure 7.2

3/4 + 2/4 represented on a number line

The solution is 5/4 which can also be thought of as the iteration of the unit fraction, 1/4, five times or 1/4 + 1/4 + 1/4 + 1/4 + 1/4.

Helping each other see. Once I figured out that the my students' difficulty with adding fractions was something bigger than rain or flour, I gave up on the idea of switching contexts as the solution. I stuck with the same muffin problem over several days but was at a loss for how to push my students thinking on it. I hoped students would push one another. "Interacting with classmates helps children construct knowledge, learn other ways to think about ideas, and clarify their own thinking" (National Council of Teachers of Mathematics, 1989, p. 26). What I wasn't prepared for was that my students had convincing arguments for both 5/8 and 5/4. When I hoped that students would help one another see, I did not consider that they might persuade one another to see 5/8 as the solution.

I was at a loss for what to do with their ideas and I did not know what kind of problem to give them if I moved away from the muffin problem. I thought I should have some sense of what I wanted students to learn before I picked another problem. I did not know what mathematics I wanted to teach even though I had a sense there was something more to be learned besides the rule. Why was this such a hard thing to articulate? What Ana tried to communicate was that you need to hold the unit constant when you add together two fractions and that people who think that the answer is 5/8 are changing the unit. I was not as clear then as I am now how the area representations reinforced this idea and made it difficult to explain the differences in how people were thinking of the unit. I felt like the problem of trying to explain the issue of the unit with my students' representations pushed at the limits of understanding for anyone who thought the solution was 5/4, including myself. What do you do when you reach the limits of what you understand? In the situation, I hoped my students could help each other because I did not have an articulated sense of the mathematical terrain into which we had ventured.

Homework. My lack of understanding about the mathematical terrain and how tasks could move students into or out of certain parts of the terrain is something I learned through the homework assignment I gave. The homework, as a response to the question of what to do next, did not really work as I had planned. Students got plenty of practice with right answers by adding fractions that totaled less than one but the problems I gave them did not push them into the same mathematical territory as the fractions totaling more than one. I did not realize this until I had a stack of papers with all right answers in my hands. As I look back on this now, I am not certain just what difference I was hoping the homework would make. Perhaps a part of me hoped that just doing the computation, repeatedly, would somehow lead to understanding and drawing pictures would be some additional guarantee that something was learned.

As it was, students did not confront the same difficulties they had with problems where the fractions, added together, were greater than one. In the homework, students never had to deal with what to do when all of the regions of one whole had been shaded in. The part/whole interpretation and the circular and rectangular representations they knew so well worked without a hitch for all of these problems and repeatedly they came up with the correct solutions. The decontextualized computation problems I gave them did not encourage the use of any new sorts of representations. Maybe my choice of homework was not such a good idea. The students had used the same representations that were creating the difficulty with making distinctions between 5/8 and 5/4 and they were not being confronted with the same issues that they faced when they tried to add two fractions that totaled more than one.

A new representation. The area models of the circles and rectangles were problematic in that there was nothing about the representation that explicitly held the unit constant. The models represented a part-whole interpretation of fractions and there was nothing to clearly identify the whole. As I explained earlier, these representations can be read as 5/4 or 5/8 depending on whether you see the whole as divided into fourths and there are two wholes or if you see the whole as divided into eighths and there is just one whole (two squares) here with five shaded in.

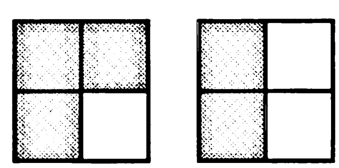


Figure 7.3

5/4 or 5/8?

Understanding the differences in the interpretation of the representation as 5/8 or 5/4 hinges on understanding what the unit is as well as understanding the interpretation of fractions supported by these shaded in circles and rectangles.[1]

I was excited when we returned to the problem of 2/4 + 3/4 near the end of the school year. The paper strips seemed useful. However, since then I have come to realize that paper strips really fall into the same family of representations as circles and rectangles. They are all area models and all involve partitioning a whole into equal parts. What I think was somewhat different with the paper strips was the difference between saying "shade 5 out of four" and "shade five one fourth parts" or recognizing that with the paper strips you are working with five one fourth parts. What the second phrasing does is make the whole--fourths--explicit.

When we were using circles and rectangles, we were all talking about fractions in terms of wholes and parts shaded in. For example, Bob's initial argument for why the solution could not be 5/4 was, "Will someone who thinks that it is 5/4 come up here and shade in five out of these four?" Whereas, six weeks later, Bob was thinking differently about fractions. "I think it is five fourths. Four fourths equals a whole. Since it is four fourths and that is your whole and then you have one more fourth. Five fourths is one fourth more than your whole." Students, like Bob, who could now see that the solution was 5/4 appeared to understand that five one fourth parts covered more area than one

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[1]What I figured out at the time is similar to what Silver (1986) discusses in an example of people coming up with similar solutions for 1/2 + 1/3. Silver writes, "we realized the resilience of this error was due in large measure to an underlying conceptual support system that reinforced not only the erroneous answer but also the strategy used to obtain the answer. In particular, the students were characterized by an over dependence on a single mental and physical model for the fraction concept. Almost universally, the students had one available model for a fraction--the part/whole model expressed as sectors of a circle...the dominance of this model combined with other elements of their knowledge base (e.g. a static interpretation of fraction addition, lack of distinction between fraction addition and ratio combination situations) to establish a conceptual knowledge base that was in direct support of the error" (p. 189).

whole unit and required the use of a second unit region (Behr & Post, 1988). "Five one-fourth parts" is a different mathematical conception of 5/4 than "Five shaded in out of four." Some students had a new understanding about why the solution was 5/4 and not 5/8. At the time, I thought it had to do with the representation. What I see now is that paper strips and the interpretations they support about fractions are not that much different than circles and rectangles. What seems more important is that students were learning some subtle and important variations and distinctions in the meaning of fractions that would allow them to see new things.

New Conceptions of Mathematics and Teaching

I have spent much time in the last three years exploring the mathematical terrain related to the problem of 2/4 + 3/4. What are the underlying conceptual ideas and issues that are embedded within this problem? What is the mathematical territory it leads me into? What I have learned falls into two main categories--multiple interpretations and representations of fractions and the history of rational numbers. What is it that I have learned about rational numbers, in particular, and how has that informed my reconception of mathematical knowledge?

Multiple interpretations and representations of fractions. One of the ways I pursued my interest in fractions was to delve into the research in math education on the topic of fractions. Fractions are an area in mathematics education that is known for the difficulty it causes students (Behr, Lesh, Post, & Silver, 1983) and teachers (Post, Behr, & Lesh, 1988). Some researchers argue that the difficulty stems from the multiple interpretations of rational numbers (Behr, Lesh, Post, & Silver, 1983). These authors note as many as six different interpretations of fractions including "a part-to-whole comparison, a decimal, a ratio, an indicated division (quotient), an operator, and a measure of continuous or discrete quantities" (p. 93).

Rational numbers are also part of a larger mathematical category called multiplicative structures (Vergnaud, 1988). Multiplicative structures include the more common mathematical topics of multiplication, division, fraction and ratio (Vergnaud, 1983) and are conceptually distinct from additive structures or the more common topics of addition and subtraction (Vergnaud, 1983). Children bring their whole number understanding embedded in additive structures to fractions. I draw on an example used by Behr and Post (1988). Take the fraction 3/5.

> There are at least two important relationships between 3 and 5; one is additive, the other is multiplicative. The additive relationship between 5 and 3 is what the child already knows; it is expressed by the difference between 5 and 3--2. The multiplicative relationship is something the child doesn't know, and this relationship is essential to understanding that 3/5 is a single number, that it has a size, and what the size actually is. (p. 196)

Why try to understand fractions as single numbers? When students add 3/4 + 2/4 and get 5/8, do they see each of 3/4 and 2/4 as one or two numbers (Behr & Post, 1988)? Do they have an understanding of "three fourths" or do they see this as the whole number three over the whole number four? Bob's initial interpretation of five parts shaded in out of four was more like seeing 5/4 as two whole numbers while the view of 5/4 as five one-fourth parts seems more like understanding 5/4 as a number, a quantity.

In my study of fractions, I came across several ways of interpreting the problem of 2/4 + 3/4 that would make the solution of 5/8 reasonable. For example, suppose you are playing softball and you get three hits in four times up to bat. This can be written as three hits out of four times at the bat or 3/4. The next time you get two hits in four times you are up to bat or 2/4. What you did altogether is like 3/4 + 2/4 or 5/8. These batting averages are ratios and combining these ratios is not the same as adding the quantities. In the case of adding batting averages, the total number of hits is placed over the number of times at bat. This is a case where the "denominators" are added.

A second way to justify the solution of 5/8 is by saying that two-fourths of one plus three-fourths of one is five-eighths of two. Consider the following scenario. Johnny has a cake and his cake is cut into four pieces. He eats two of them. Mary has the same size cake and also cuts it into four pieces. She eats three of them. How much cake have they eaten all together? They have eaten one and one fourth pieces of one cake. The difficulty with this solution is that you had to have two cakes to start in order for the second person to get three pieces. Even if the cakes happen to be the same size, they can easily be conceived of as different units or wholes.

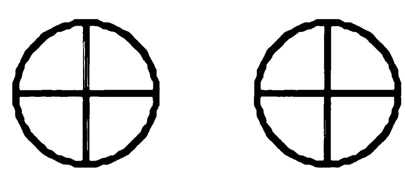

Figure 7.4

Two wholes divided into fourths or one whole divided into eighths?

<u>A historical context.</u> My study of fractions also led me into the history of rational numbers. I was led there through a mathematician's interpretation of the difficulty my students had with adding fractions. He drew connections between my students' struggles and the history of fractions.

> You know you're trying to teach them addition after all and not just ways of putting fractions together. You have to think, how did addition start

> historically? How did fractions first get added? They weren't always added. I mean people, Greeks, did not like to do arithmetic of any sort with fractions. They felt just like our kids. They hated them. The fractions were ratios. I mean you should read what they wrote about arithmetic with ratios, how they thought of it. They drew lines and said well its all measurement with numbers. I mean they started to think about measurements. There is no way to add fractions without thinking of them as measurement. . . So I guess with little kids you would have to think culturally, one has to think culturally about how things do progress. . . (Kutzko, personal communication, 05/17/90)

Culturally, how did the idea of fractions progress? I have selected several of the particulars of the history of fractions to include here. "Most of the ancient theory of fractions centered about the concept of ratio" (Smith, 1958, p. 210). The Egyptians and the Babylonians were the first two civilizations to contribute our present day understanding of mathematics (Kline, 1972). Both of these cultures used mathematics to answer questions that arose for them in their daily lives. The Egyptians, for example, wanted a means of being more precise in their measurement of quantities--length, weight, and time. Thus, arose the need and the development of the concept of a fraction in the context of measurement (Scott, 1969; Smith, 1958), for "seldom will a length appear to contain an exact number of linear units"(Eves, 1969, p. 58). If you think back to my class, what Ron did with his paper strip when he tried to divide a strip longer than nine inches into thirds is that he turned back the portion of the strip that was greater than nine. His reason for not dealing with the extra was that it was too messy. The Egyptians invented fractions to deal with that extra he folded back.

The Egyptians tried to avoid some of the computational problems of fractions by representing all fractions, except 2/3, as the sum of unit fractions, or fractions with unit numerators. For example, 2/7 is the same as 1/4 and 1/28 or 2/97 is the same as 1/56, 1/679, and 1/776. The Babylonians were the first to introduce fractions with numerators greater than one. They were also the ones to introduce fractions written around the scale of 60--the measures of time, angles, and arcs (Smith, 1958). For example, Smith points out that two hours, twenty minutes, and 45 seconds could be thought of as 2 + 20/60 + 45/3600 with hour as the unit, 60 minutes in an hour, and 3600 seconds in an hour.

While the historical "facts" associated with fractions or the particulars of mathematical history of fractions may be interesting, I think there is something more important that a study of a history of mathematics represents. For me, it has been a powerful means of seeing that mathematical knowledge is constructed and developed by humans, in a social and cultural context. Arcavi, Bruckheimer, and Bens-zvi (1987) discuss the benefits of looking historically at mathematics,

> The historical context may foster the creation of a reasonable image of mathematics and mathematical activity as a human, creative, and dynamic endeavor, as opposed to the more common view of mathematics dropping "ready-made-from-the-skies." (p. 19)

Mathematical knowledge is a human construction.

The possible interpretations and representations of fractions are numerous and vary in ways that lead to subtle and not so subtle differences in meaning. There is no such thing as the perfect representation with representations being "the most powerful analogies, illustrations, examples, explanations, and demonstrations--in a word, the ways of representing and formulating the subject that makes it comprehensible to others" (Shulman, 1986, p. 9). There are multiple representations for fractions.

> Since there are no single most powerful forms of representation, the teacher must have at hand a veritable armamentarium of alternative forms of representation, some of which derive from research whereas others originate in the wisdom of practice. (Shulman, 1986, p. 9)

Choosing the appropriate representation means weighing its strengths and limitations for a given concept, at a given moment, with a given group of students. And then, what one person may be able to see in the representation may not be visible to someone else.

What my study of rational numbers in the three years since I taught has done is push me to begin to see that there are multiple ways to construct, interpret, represent, and communicate the meaning of mathematical ideas. It happens in classrooms among teachers and students, in universities among mathematicians, and in the field of math education among researchers. Understanding an area of mathematics is ongoing work. Fractions, rational numbers, and multiplicative structures are areas of mathematics whose complexity of meaning continues to be explored by people who continue to learn new things. Fuson (1988) notes how as a community of math education researchers "we have not been able to agree on single way to structure these conceptual fields, either for multiplication or rational number situations" (p. 260). These are complex mathematical ideas, a terrain of deep uncertain, unsettled mathematical issues. Hiebert and Behr (1988), editors of *Number Concepts and Operations in the Middle Grades*, remind their readers of the tentativeness of knowledge and that even knowledge that seems certain is subject to change.

> But we must remember that theories always are tentative and sometimes wrong. They can never be cast in stone. They must be treated as working hypotheses, tested and refined and tested again. It is only though the cyclical process of building and testing theories that significant process can be made. (p. 16)

As I look back on my initial interview with Jim, before I started teaching that year, I am surprised by my view of mathematical knowledge and how I thought I was going to go about learning what I did not know. I started the year thinking that I did not know enough mathematics. Jim asked me how I saw myself getting over the hurdles before me, this hurdle of not knowing enough mathematics. This interview was done at the end of August, 1989, before I started teaching.

> I think the hurdle of my own knowledge of mathematics - - I don't feel like that's something I've been able to do ahead of time, that it's not possible for me to learn everything I need to know about mathematics before I teaching this. And that I have to deal with it as it comes up and figure out what it is I don't know and try to learn more about it. (interview, 08/89).

Three years later, I agree that my ongoing work as a teacher is to figure out what it is that I don't know and learn more about it. The difference now is that I have a new appreciation for what I do and do not understand. And, not knowing at this point in my work as a teacher, is not a shortcoming. Knowing what I don't know and having an interest in and means for investigating what I don't know is what keeps me developing as a teacher. Moreover, it is what keeps me growing as a person.

<u>Learning the River</u>

In the series of lessons in this chapter, I knew there was some mathematics that I did not know. I knew there was a mathematical territory with which I was unfamiliar. I knew there was a terrain or aspects of a terrain I could learn. I have learned some particulars of the terrain as well as how knowledge about the terrain is constructed. The way I learned about the terrain was to go exploring in it myself. How does having a sense of the terrain help you to navigate through it? George Byron Merrick (1987) in *Old Times on the Upper Mississippi* reflects on the knowledge of riverboat pilots during Twain's time on the river.

> The pilot of that day was absolutely dependent upon his knowledge of and familiarity with the natural landmarks on either bank of the river for guidance in working his way through and over the innumerable sand-bars and crossings. (p. 78)

Navigation demands learning the river in ways that helps you to move about easily on it, no matter what you may encounter. Merrick (1987) writes,

> To 'know the river' under those conditions meant to know absolutely the outline of every range of bluffs and hills, as well as every isolated knob or even tree-top. It meant that the man at the wheel must know these outlines absolutely, under the constantly changing point of view of the moving steamer; so that he might confidently point his steamer at a solid wall of blackness, and guided only by the shape of distant hills, and by the mental picture which he had of them, know the exact moment at which to put his wheel over and sheer his boat away from an impending bank. (p.79)

There is a paradox here about knowledge and knowing. How can a pilot ever know the river absolutely--if his point of view is constantly changing and the river is ever-changing? Doesn't this mean that we need to reconsider what it means to *know* something? What Twain tried to learn, the river, is something that can never, given its nature, be entirely known.

Many times on the river Twain thought he had learned the river or that he could know everything about the river necessary for navigation--once and for all he thought he had learned the river. At one point, he even tried to make a list of what he had learned under the assumption that he had finally completed his education. Twain writes,

> When I had learned the name and position of every visible feature of the river; when I had so mastered its shape that I could shut my eyes and trace it from St. Louis to New Orleans; when I had learned to read the face of the water as one would cull the news from the morning paper; and finally, when I had trained my dull memory to treasure up an endless array of sounding and cross-marks, and keep fast hold of them, I judged my education was complete: so I got to tilting my cap to the side of my head, and wearing a toothpick in my mouth at the wheel. (p. 67)

It was not too long before Mr. Bixby asked him a question about the river for which he did not have an answer. Twain addressed Mr. Bixby, "But what I want to know is, if I have got to keep up an everlasting measuring of the banks of this river, twelve hundred miles, month in and month out?" (p. 69). To which Mr. Bixby replied, "Of course."

The same infinity of the task that Twain found with learning to navigate the river applies to what I have learned about the nature of learning mathematics and learning to teach mathematics for understanding. Learning is inherent to the work. The account of my learning I offer in this dissertation, like Twain's account of his own learning in *Life on the Mississippi,* is unique. Like Twain, this was my first time on the river. Next time I will know from the start that "the whole river is bristling with exigencies in a moment" (Twain, 1990/1883, p. 92) and given what I have learned from this first journey, I will be prepared.

CHAPTER 8

LEARNING TO TEACH WHILE INVENTING A PRACTICE

Inventing a Practice

I now return to the question with which I began this study: What does it take for an experienced and successful elementary teacher to make changes in teaching mathematics? In this chapter, I draw on the analyses of this study to make conjectures about what it might take to change one's mathematics teaching in the direction of teaching mathematics for understanding. I argue that *learning to teach* mathematics for understanding entails, simultaneously, learning to *continuously invent a practice* of teaching mathematics for understanding while you do it. The intertwining of learning and invention is prominent throughout my efforts to embrace the reforms in mathematics education. Through the exploration of this theme of the inherent interconnection of learning to teach and continuously inventing the very thing one is trying to learn, I present and discuss a more theoretical understanding of what is involved in teacher change.

The air is filled these days with talk of "new" kinds of mathematics teaching and learning, of teaching "for understanding," of ambitious or adventurous pedagogy. That such a pedagogy requires continuous invention is true for novices as well as experts who may have years of experience teaching mathematics for understanding. I discuss evidence for this claim and explore how the metaphor of invention shapes understanding what is entailed in learning to teach mathematics for understanding. Looking back on my own experience, I realize now that I did not set out expecting that changing my practice would entail continuously inventing a practice. In this chapter, I analyze the expectations for the changes I anticipated making at the start of that year. I look at the discrepancies between my expectations and the realities of my efforts to learn to teach mathematics for understanding. Analyzing these discrepancies, I derive a new set of expectations for what it might take to teach mathematics for understanding and suggest implications for educating teachers.

Learning about Teaching while Learning to Teach

The question which drove this study was focused on what it takes to change teaching practice in the direction of teaching mathematics for understanding. While the analyses in Chapters 4 through 7 reveal new understandings of the process of change, they also imply new understandings of the practice itself. Through efforts to enact my vision of mathematics teaching, that vision itself was revised and reconstructed. Because of the changes I was trying to make, I found I needed to simultaneously change, teach, learn, and develop the very thing I was learning. For example, to begin to facilitate productive discussions, I had to develop concrete particulars about what might count as "productive" while the discussion was happening, ways to make such productivity possible on a moment-to-moment basis, and what it took for my students to engage in discussions. Initiating a mathematical discussion, I grew to see, is dependent on having interesting mathematical ideas to talk about. What counts as an "interesting mathematical idea" is, in part, situation-specific. I was also learning that there was more to sustaining a discussion than merely asking repeatedly what students thought, a simplistic image I had had going into this teaching. As the teacher, it was my responsibility to *do* something with students' ideas. On a moment-to-moment basis, I needed to make decisions about what I could do that would help students to construct their own understandings of mathematical ideas. My ability to do this was connected to what I understood about the mathematics I was aiming to teach as well as my capacity to hear and interpret my students' ideas. In the context of this kind of mathematics teaching, pedagogical skills and mathematical knowledge are connected to one another and continually constructed and reconstructed in the course of teaching.

Continuous invention of a practice is inherent to the work of teaching mathematics for understanding. Even teachers perceived by some as "experts" at teaching mathematics for understanding must invent a practice of teaching as they work. Teaching mathematics for understanding is not something that is ever completely learned. One can get better at it but it is not something that can be mastered for the "it," the teaching, is inherently under construction. Magdalene Lampert (e.g., 1986, 1990) and Deborah Ball (e.g., 1993a, 1993b), two experienced teachers of mathematics, offer in-depth analyses, based on their own practices, of what it means to invent a practice of teaching mathematics for understanding. Like Lampert and Ball, my work offers insight into what it means to do the work of teaching mathematics for understanding but does so from the perspective of a beginner, someone new to the teaching of mathematics for understanding and to the idea of teaching as invention, but not new to teaching.[1]

The Logic and Limits of Invention

The idea of teaching as continuous invention is an analogy that I would never have used to characterize my teaching of mathematics prior to this year of teaching. My past practice was prescribed, defined, and predictable. There was nothing creative and inventive about it. To say that the new way of teaching mathematics that I was trying to enact was one based on ideas of invention evokes helpful images of what it could mean to teach mathematics for understanding and the changes that might be necessary to move one's teaching in that direction. The metaphor is only useful, however, as long as the grounds

[1]The idea that inventing a practice of teaching is an inherent part of teaching is found in the work of those in the field of education who have made connections between teaching and improvisation--most notably, that of Robert Yinger (1988). Yinger recasts planning, implementation, and reflection into the language of preparation, improvisation, and contemplation. He writes in general terms about teaching as improvisation while my work conceptualizes the improvisational use of subject matter during and between lessons.

on which I make connections as well as the limits of any applications I intend are understood.

Why invention? Several factors underlie the argument that learning to teach mathematics for understanding must be understood as entailing invention as well as learning. First of all, the vision of mathematics teaching offered by the reform documents is underdetermined. This point is supported whether one looks at different policy documents that claim to represent the reforms in mathematics education or at the work of mathematics education researchers whose perspectives on teaching are based on their studies of students' learning.

Policy documents offer fairly clear visions of mathematical knowledge and theories of learning, but are less clear about the details of a practice of teaching rooted in these ideas about knowledge and learning. One such document, the *Mathematics Framework* (California Department of Education, 1992), does not prescribe a way to teach mathematics for understanding. The authors note the need for teacher exploration and experimentation as a means of implementing these reforms. Another such document, the *Professional Standards for Teaching Mathematics* (National Council of Teachers of Mathematics, 1991), also does not prescribe a way to teach. Instead, it offers a variety of vignettes that "do exemplify some specific worthwhile practices" (p. 11) but "they do not suggest one "correct" approach to teaching mathematics" (p. 11).

Some mathematics education researchers, through studying students' learning, have reached similar conclusions with regard to the lack of a single way to teach. For example, researchers who have done extensive study of children's understandings of addition and subtraction as part of the Cognitively Guided Instruction project (Carpenter & Fennema, 1988) have considered how to use knowledge of children's cognitions in teaching in ways that are responsive to students. "Although instructional practices have not been prescribed, there are broad principles of instruction that underlie our approach" (p.12). A similar assumption underlies the work of Yackel, Cobb, and Wood (1992), researchers in classrooms where teachers are trying to teach mathematics in conceptual ways. They note the nature of the interactions in these classrooms, "we speak not only of the negotiation of norms but their continual renegotiation" (p. 397). Thus teaching entails a continuous negotiation of social relationships that determine the nature of the teaching in the situation. Therefore, teaching is determined by the situation rather than defined and prescribed in advance. This way of understanding teaching is in sharp contrast to the prescriptive view of teaching underlying the development of "teacher proof" curriculum materials of the 1960's where little was intended to be left to the teacher's discretion. Even people who might be considered by some to be more experienced in enacting these reforms (e.g. Lampert and Ball) are having to invent their practice as they go along and their actual practices are not identical to one another. They are not implementing a fixed or predetermined model of practice nor do they consider themselves models of a practice. What these ideas about mathematics education and practices share is a commitment to a common set of ideas about mathematics, students, learning, and curriculum (Lampert, Ball, Rosenberg, & Suzuka, 1993).[2] At the level of rhetoric, there is general consensus about the aims of the reforms. At the practice level, there are multiple interpretations of what it means to enact them. Because of the kind of teaching this is, practice will always remain undetermined.

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[2]I have had the opportunity to note similarities and differences in Lampert's and Ball's mathematics teaching in the context of my five years of work on the Mathematics and Teaching through Hypermedia Project, a research project funded by the National Science Foundation. See (Lampert, Heaton, & Ball, 1991; Ball, Lampert, & Rosenberg, 1991) for additional information about this research project.

A second reason why it makes sense to view the process of teacher change as entailing invention is that this kind of teaching mathematics is so fundamentally dependent on being responsive to students. Of course one can learn how to anticipate what students might say. One can increase one's ability to predict what might happen. One can even learn to think through possible responses. But all this would still be insufficient to plan exactly what students will say, what one's responses as a teacher might be in any given moment in teaching. This makes the teaching of mathematics for understanding itself unpredictable, uncertain, subject to surprises, and requiring improvisation. What this means is that teaching that intends to be responsive to students must be invented locally, in the act of teaching. Invention allows the teacher to act with immediacy on and in ways specific to the ideas and interactions of the particular students with whom she is working.

A third reason why the idea of inventing a practice is reasonable centers on the nature of what it means to know and come to know the content being taught. It matters that what is being learned is mathematics and that the view of mathematical knowledge underlying these reforms is a constructivist one. Both what mathematics gets learned and how it gets learned is strongly determined by social interaction which can not be prescribed. An understanding of mathematics is constructed through a series of proofs and refutations (Lakatos, 1976) within a community of learners (Tymoczko, 1986). In other words, "a mathematician figures out what he or she is talking about in the process of trying to talk about it, not beforehand by some magical intuition" (Lampert, 1992a, p.306). One does not first learn the mathematics and then try to communicate it. One learns mathematics through efforts to simultaneously create and communicate a plausible argument, an argument invented in practice.

The limits of invention. Metaphors and analogies offer new ways to look at ideas. In the context of teaching mathematics for understanding, the idea of invention helps to reveal the moment-to-moment, compositional process of teaching and learning to teach that might otherwise go unnoticed. Metaphors and analogies can also, through their seductiveness, be obstacles as well as tools to understanding. In this section I raise two points of caution with respect to the "invention" metaphor.

On the one hand, inventing a practice of teaching mathematics while teaching mathematics implies that there is no rehearsal for the work of teaching. That is, one cannot rehearse precisely the exact teaching, the decision making, or the actions that will occur at a future point. Because one can not predict with certainty what will happen in interaction with students, this kind of mathematics teaching is not about executing a set of preplanned decisions and moves. But as stated above, there are certain things, like anticipating what students might say or do in the context of a particular math problem, for example, that *can* be learned outside the situation. Such learnings might help in anticipating how to act or teach in the situation. Learning this kind of teaching and doing this kind of teaching is learning to figure out what needs to be considered to make and act on moment-to-moment decisions about what to do next in the midst of teaching. One of the difficulties is that these decisions cannot be rehearsed out of the situation. Learning to make crucial decisions about what to do next--to improvise and invent--during the act of teaching is at the heart of this work.

On the other hand, one must not mistakenly assume that this means that there is nothing that teachers can learn. Quite the contrary. Later in the chapter, I will return to the question of what this implies for educating teachers. The image of inventing a practice of teaching could also suggest, erroneously, that all practice is equal, that there are not better or worse ways of working, no standards of quality or worth. As is common when developing criteria for assessing a new practice, it is often times easier to recognize and articulate what is not acceptable rather than what it is. While the standards for good practice remain ill-defined, judgment is both possible and necessary. Standards used to judge practice are not equivalent to recipes for how to do it.

I now return to what this all means for learning to teach. In my case, I did not embark on this year of teaching with a set of expectations that corresponded to the idea of invention nor did I anticipate that what I was aiming to do was to learn to invent a practice. In this next section, I examine the expectations I held for what I thought I would be learning to do that year and the related sets of expectations about what I thought I would need to change. I conclude with a new set of expectations, a set which reflects what I learned from my efforts to enact and study the enactment of my expectations in practice.

Expectations for Creating a Practice of Teaching Mathematics

My efforts to change how I taught mathematics was a continuous process of planning for teaching and not having things go as planned. In the beginning of the year, this was a source of great frustration. I was continually caught off guard when I started each class thinking that I was prepared but found that once I started interacting with students I was at a loss for what to do next. The lesson around the question of patterns is representative of those early frustrations. I did not enter this year of teaching expecting that what I needed to learn to do was invent a practice. Rather, I expected that I was learning to implement a way of teaching that was radically different from the teaching I had done in the past, but a way that was itself defined and articulated. My task, or so I imaged, was to learn to do it. In time and with experience I expected to learn it. What follows is a closer look at my fundamental expectation of implementation as it played out in three major areas--skills, knowledge, and dispositions--areas in which I perceived the need to make major changes.[3] I look at how my understanding of what was demanded of me in each of these areas was revised through my efforts to implement a practice of teaching mathematics for understanding.

A Series of Dichotomies

I expected the changes I needed to make in teaching to be dramatic ones. For example, I thought I would have to abandon many of my past classroom routines and ways of being with students that had always defined my role as a teacher. I based my perception of the degree of change necessary on observations of others' efforts to teach mathematics in ways aligned with the reforms, as well as my own interpretation of the reforms. Teaching mathematics for understanding seemed to bear little resemblance to my past teaching of mathematics. As a way of making concrete for myself the changes in practice that I imagined implementing these reforms would entail, I considered the sorts of behaviors and skills that characterized my past practice and tried to imagine what it was exactly that I would need to change. In many instances, I assumed that what I needed to do was the exact opposite of what I had been doing.

Images of the changes I thought necessary are arrayed below alongside patterns central to my former practice. The statements in column A are based on assumptions about the nature of the behaviors that characterized my old practice while the statements in column B are based on assumptions about the practice I imagined myself implementing.

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[3]These three categories, skills, knowledge, and dispositions, correspond to three alternative conceptual orientations to teacher preparation (Feiman-Nemser, 1990): the technological orientation, the academic orientation, and the personal orientation. Together they represent an eclectic view of what might be entailed in learning to teach.

A old practice of teaching mathematics	B goal--teaching mathematics for understanding
Teachers tell.	Teachers do not tell.
Teachers follow a prescribed curriculum.	Teachers do not follow a prescribed curriculum.
Math problems have a single right answer.	Math problems have no single right answer.
Students always work alone to do math.	Students do not work alone to do math.

Figure 8.1

Dichotomies of a changing practice of teaching mathematics

To envision change in terms of extreme behavioral dichotomies is not unusual. Dieters, seeking new svelteness, do it when they resolve to eliminate entirely certain foods--cookies, ice cream, butter--from their diets. Those who wish to make fast "crash" change in their weights may sometimes choose to cut out entire food groups--dairy products, for instance--even though these food groups can have nutritional benefits when consumed in moderation. Characterizing the needed change in such simple terms serves to make behavioral change both imaginable and manageable. It does nothing, however, to prepare one for discrepancies that tend to exist between the image of change and the realities of efforts needed to make change.

Two important points about teacher change have emerged from a study of my efforts to change my mathematics teaching. One major idea is that changing mathematics teaching as an experienced teacher does not come about by abandoning all past practices and trying to begin again as though one had never taught before. It is implausible to think that changing teaching is a simple matter of abandoning old practices or replacing old practices with new practices. Most current notions of learning suggest that people learn by building on prior experiences and bringing to bear what they already know as they make sense of new situations. Disregarding previous teaching experiences assumes that it is possible to make experienced teachers into blank slates who can begin creating a practice of teaching all over again with each new idea about teaching that comes along which they wish to embrace. This, however, is impossible. A second major idea is that changing practice does not come about by making announcements about what one should no longer do. Saying that one should do the opposite of what one has been doing does nothing to respect the complexity of what it is that one needs to learn. If a list of "don'ts" is offered it may imply that what is needed is simply a new and better list of "dos." It is important to understand that my struggle with change was not because I was operating from a wrong or inadequate list of "things to do." The difficulty was that conceptualizing change in terms of a fixed view of teaching--dichotomies and lists of dos and don'ts--seriously underestimated the process of change as well as the complex, constructive nature of the practice being learned. I turn now to a closer look at two specific efforts to abandon entirely past practices and to replace those practices with new ones.

Abandon past practices. There were some teaching behaviors I expected that I would need to abandon entirely: "Telling," for example. By this mean, I thought I needed to stop telling students how to work the math problems I gave them. I thought I needed to stop telling students whether their answers to the problems they worked were right or wrong. Trying not to tell, I focused on finding an alternative to telling. For example, rather than tell students exactly how to do a particular problem, I asked students questions to help them interpret the task. The lesson on making labels for cans is an example of this. Other times I asked students questions in hopes of helping them to decide whether their answers made sense. The series of lessons around adding fractions is a good example of students learning to express and consider the reasonableness of several different solutions to the same problem and decide for themselves which was the right answer and why. Efforts to completely eliminate a particular teaching behavior were extremely difficult and did not lead to the practice I imagined.

Failed efforts to eliminate telling completely from my practice and produce the image of teaching I was after forced me to reconsider the place of telling in my past teaching and the relationship between a particular teacher behavior and past and new views of mathematical knowledge, theories of learning, and images of teaching. I gained a new appreciation for the complexities of telling and asking questions. The boundaries of telling and not telling began to blur as I moved away from seeing them as opposite elements of a teacher's practice that could be easily switched on or off like eating dairy products or not would be for a dieter. Just as the intonation of a statement can denote a question, so can a question be constructed in a way so as to indicate telling. Many of the questions around the use of the Minicomputer for working multiplication problems could be said to be of a "telling nature," in that they pressed for convergence. Many had a single right answer and were not at all about exploring mathematical ideas. Another example of telling is found in Sipho's observation of the discrepancies of multiple measurements of a single can in the midst of a series of lessons around making labels for cans. In this example, Sipho's observation narrows the focus but does so in way that highlights a mathematical idea and moves the whole group forward in thinking about measurement. The telling I did in my interactions with Arif around the composition of functions in the context of multiplication and the use of the Minicomputer is another example of using telling to narrow the focus and, in doing so, illustrate something important about the mathematics. These latter two examples, with similar ones occurring in other lessons throughout the year, are what have helped me to reconsider the role of telling in teaching. Telling was more complex than I understood it to be in the context of my past practice. It was not simply an evil act to be shunned in teaching mathematics for understanding.

At the time, I did not appreciate the complexities of "telling" and "not telling." My view at the time was that if I did not tell, the alternative was to ask only questions. In trying to do this, I learned much about what it takes to ask questions in the context of teaching mathematics for understanding. I figured out that questions designed to promote a discussion of ideas are not generic. There is more, for example, to initiating a discussion than asking what people think. Ways to get people to talk about important mathematical ideas and what I was to do with students' ideas was closely tied to what I understood about what students understand about the mathematics and what I wanted students to learn. Asking better questions, ones that connected to the mathematics, was teaching; maybe even telling, because it is a way of telling people what to think about. Knowing what questions to ask, of whom, when, and for what purposes is intimately connected to my understanding of the mathematics. Examples of the relationship between my own understanding of the mathematics and my ability or inability to ask useful questions can be found throughout all four of the analysis chapters of this thesis.

Although I entered that school year expecting not to tell, I had not thought far enough ahead to be able to anticipate the consequences of asking students questions intending to probe their thinking rather than checking their ability to have found the right answer. I had to figure out how to respond to students' responses to my questions. It was not a simple matter of not telling students whether they were right or wrong. One might think that asking questions is a way for teachers to hide the mathematics they do not understand. I found, however, that asking good questions required me to understand more mathematics than did telling students the right answers. I could rely on the textbook's understanding of the mathematics for the answers, but my own understanding of the mathematics was essential for asking good questions.

My frustrations with only asking questions led me to reconsider telling in my practice including what it means to tell and when it might be appropriate to tell. As a teacher, I needed to become actively engaged in deciding from moment to moment whether or not it made sense to tell. If so, what could I tell that would make a difference? Deciding whether to tell, what to tell, to whom, and when was a challenge throughout my teaching. The series of lessons around the addition of fractions is a good example of what it means for a teacher to struggle to figure out what to tell that will make a difference. In that series of lessons, my difficulties with figuring out what to do next were connected in important ways to what it was that I understood about the mathematics and what it was that I understood about what my students understood. Decisions to tell depend on learning how to assess students' understanding as an ongoing part of teaching in ways that illustrate whether what was told made a difference in students' understanding. I was learning how to improvise as I made decisions about what to say or do next. Telling was one of the pathways I could choose to take in a moment when I had to act. As I learned to see telling as a choice, I was also learning that I needed to become thoughtful about the occasions in which I chose to tell. They were not necessarily moments I could predict prior to being in them.

The process of completely abandoning past practices as a way of changing teaching did not make sense in the context of the teaching I was aiming to do. Efforts to abandon this element of teaching did push me to reconsider the place of telling in my old practice as well as the new one I was trying to create. In contrast to my past practice where I never considered whether or not to tell, I just did it, I was learning that in teaching mathematics for understanding I could not predict out of the situation what I needed to do in the situation. I had to decide in the moment what I could do that would make a difference. Sometimes that might be telling and other times it might be something else. Whatever I decided to do, I learned that I needed to be clear on the reason given the surprises that might arise as a consequence of my actions.

Replace past practices. While there were some teaching practices, like telling, I thought I needed to completely abandon, there were other elements of my practice I thought I needed to replace. For example, I envisioned that the assumption from my old practice that a teacher follows a prescribed curriculum as represented by a mathematics textbook would be replaced by the expectation that a teacher construct the curriculum with her students. In a context where students were supposed to construct their own understandings of mathematical ideas as a means of learning, I expected that such a change in teaching would mean that authority for the curriculum would be transferred from the textbook to the students.

I was, however, frightened at the thought of giving students any control over the curriculum when I felt so uncertain about everything myself. At the time, a reasonable alternative was to make use of a new mathematics textbook, one in which the pedagogy and content was intended to be more like what the reformers had in mind than other more rule based and computation driven texts. Following a new textbook seemed like a reasonable way to begin to change my practice. It seemed useful to have a text that would begin to help me consider content and pedagogy in new ways. This line of thinking is not unlike the thoughts of many policy makers and school reformers who promote the adoption of new textbooks as a means of changing teaching. Changing practice, however, did not turn out to be as easy as changing textbooks.

I had two expectations for this new teacher's guide when I started using it. One was that I thought the textbook could cover for the mathematics I did not understand. Another was that I thought the textbook would help me to respond to students. In reality, I came to see that the textbook could not be ultimately responsible for either of these central tasks of teaching (Feiman-Nemser & Buchmann, 1986). Relying too heavily on the teacher's guide did not produce the teaching I imagined I was after. The need for my own understanding of the mathematics in this kind of mathematics teaching became evident through its absence. Without a deep understanding of the mathematics I wanted students to learn, I did not feel like I had the intellectual flexibility to interpret and build on a variety of students' ideas. Clinging to the text constrained what I felt I was able to do from moment to moment. Loosening my grip on the teacher's guide, left me free to respond to students in ways that made sense to me rather than in ways that made sense to the author of a textbook lesson who did not know my students.

Pedagogical tasks such as asking questions or guiding class discussions, that on the surface looked generic, were actually connected in deep ways to my understanding of the subject matter. To be able to carry out these tasks productively I needed to learn to be connected to what was happening with my students and their understandings of ideas in ways that allowed me to be constantly trying to learn what students were thinking or feeling at particular points in time. This kind of teaching placed intellectual and sensory demands on me in ways that no textbook could replace. Using an innovative textbook did not make much of a difference when I attempted to use it as a substitute for my own pedagogical and mathematical understandings. Rather than completely throwing out all textbooks, I started asking new questions. For example, in the context of teaching mathematics for understanding, what is the teacher's guide intended to *guide*? Could "follow" mean something new? What would it mean to try to use a textbook as a resource instead of a script? What would it mean to reconsider my expectations of a textbook? What new considerations could be entertained for the content and design of a teacher's guide? All important issues: I will return to consider what these might suggest for the development of curriculum materials at the end of this chapter.

As the school year progressed, I became less reliant on the textbook and more self-sufficient in terms of deciding for myself what to do next. Looking over and thinking through how I would implement the preplanned lessons in the textbook was inadequate preparation for the spontaneous sort of decision making and action demanded in this kind of mathematics teaching. I saw the need to learn to acquire and act on my own sense of the mathematics, my own sense of what my students understood, and my own sense of purpose for what we were doing. What I was beginning to see as a major part of my role as a teacher involved the sort of interactions and subsequent decision making and responsiveness that could not be wholly determined by a textbook. I, not the textbook, needed to be in control of what to do next.

I was learning how to take an active role in constructing the curriculum with students on a day to day and moment-to-moment basis. The making of labels for cans is an example of my students and I taking control over the curriculum on a day to day basis. What was intended as a one day lesson by the teacher's guide evolved into a series of lessons as I followed students' interests in the problem. The lesson about multiplication in the context of the composition of functions is an example of how I was learning to take control of the curriculum in the moment-to-moment work of teaching as I continually figured out how to interpret and build on students' ideas. Following the mathematics textbook two pages at a time, problem by problem, as I had done for many years as a teacher, and before that, as a student was no longer an appropriate thing to do. I was learning to put myself in a different relationship to the curriculum. The curriculum was not something disconnected from me, my interpretation of the immediate situation, my understandings of the mathematics, or my students' understandings.

I needed to learn to see myself as capable of constructing knowledge about content and pedagogy. I needed, however, to do more than just believe I was capable. I had to learn to think and reason about what made mathematical and pedagogical sense to do next. Making the change from following a prescribed curriculum to one in which the students' ideas and mine played a major role in curriculum construction was not a simple matter of transferring authority from the textbook to the students. I needed to see myself as and act as a key player who mediated the use of a text as well as students' ideas in the moment to moment and day to day decision making demanded by this kind of mathematics teaching.

Reconsider old practices, entertain new ones. In each case, the realities of practice changed my understanding of what was entailed in teaching mathematics for understanding and revised my expectations for what it might take to make changes in my practice. Framing the contrasts between old and new practice in terms of dichotomies of skills was not useful. Abandoning or replacing elements of teaching was a far too simple way to view teacher change. When I realized the inadequacy of the dichotomies, I found myself having to carve out and invent a practice. I needed to learn to figure out ways to decide in the moment what to do next. Sometimes the work was as specific as figuring out what to say or do next. Other times, the work entailed thinking in more general terms about purposes and possible next moves as I worked to construct the curriculum based on what my students understood and my sense of the purposes of what I was teaching.

I have a new appreciation for a chart I borrowed from the *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989) and used in the introductory chapter of this dissertation.

Grade level	**Increased Attention**	**Decreased Attention**
K-4	use of manipulative materials cooperative work discussion of mathematics questioning justification of thinking writing about mathematics problem-solving approach to instruction content integration use of calculators and computers	rote practice rote memorization of rules one answer and one method use of worksheets written practice teaching by telling
5-8	actively involving students individually and in groups in exploring, conjecturing, analyzing, and applying mathematics in both a mathematical and a real-world context using appropriate technology for computation and exploration using concrete materials being a facilitator of learning assessing learning as an integral part of instruction	teaching computations out of context drilling on paper-and-pencil algorithms teaching topics in isolation stressing memorization being the dispenser of knowledge testing for the sole purpose of assigning grades

(pp. 20, 21, 72, 73)

Figure 8.2

Practices to increase and decrease to teach mathematics for understanding

Without looking carefully, one could easily construe the left hand column to be representative of new practice and the right hand column to be representative of old practice. In short, this chart could be seen as a list of shoulds and should nots. What I now have an appreciation for are the headings--increased attention, decreased attention--and how they relate to each of the columns. These are not a list of dos and don'ts for teaching mathematics for understanding. They are not simply ways of acting as a teacher that one can either do or not do. They are new and old ideas about teaching mathematics that teachers learn to compose in a variety of ways to varying degrees at different points during teaching. Changing teaching is about learning to make judgments about what to do when in the act of teaching. Rather than abandoning the old or replacing the old with the new, changing teaching in the direction of teaching mathematics for understanding requires a skillful merging of old and new practices. This entails learning to enact new skills, routines, and behaviors as a teacher in ways that acknowledge, rather than ignore, previous practices.

A Deficit Model of Subject Matter Knowledge

The need for teachers to have more subject matter knowledge is a common refrain in teacher education (e.g., Ball, 1988a; Ball & McDiarmid, 1990; Grossman, 1990; Gudmundsdottir, 1990; Wilson & Wineberg, 1988). Few researchers, however, have confronted head on what mathematics it is that teachers need to know. Doing the analyses of teaching for my thesis, in part, was an opportunity for me to explore, in-depth, the mathematics it might have been useful to have known at the time I taught those lessons. Looking back on those chapters, I contemplate what I learned about the nature of the mathematics that I needed to learn. Shulman (1986) identifies theoretically the intersection between subject matter knowledge and teaching as "pedagogical content knowledge." He claims that it is one of several components of the broader construct of strategic knowledge (Shulman, 1986)--a theoretical construct for the knowledge needed to make moment-to-moment decisions in situations where principles of practice are not helpful. Many of the previous studies on subject matter knowledge in teaching looked at novices' development of these kinds of teacher knowledge, novices who had had opportunities to learn the subject matter followed by opportunities to learn how to use their subject matter knowledge in teaching.

My study tackles issues related to the simultaneous acquisition and use of mathematical knowledge in the course of teaching.

Not enough knowledge. I began this teaching expecting that what I would have to learn was more mathematics than what I knew. Years earlier, in college, I took mathematics classes through calculus though I could not recall much of what I had learned nor could I see explicit connections between higher level mathematics and the mathematics I would be teaching to fourth graders. I imagined that as I taught I would discover what I needed to learn. I expected the day to come, admittedly, perhaps not during that year, when I could say that I understood the mathematics I needed to know and the teaching of mathematics for understanding would, henceforth, become easier. In the meantime, I would take comfort in knowing that I was going to start off using a mathematics curriculum that attempted to teach and represent mathematics in ways less traditional and more conceptual than the previous mathematics textbooks I had encountered in the past. Even if I did not fully understand all of the mathematics I thought I needed to know, at least I could find some security in a textbook that did. However, I soon came to revise my expectations of the text as I reconsidered how reasonable it really was to attribute to the text all content and pedagogical authority, given the kind of mathematics teaching I was aiming to do.

As I made my way through the lessons in the mathematics textbook, mathematical questions arose for me as I tried to discuss math problems and solutions with my students. For example, I started being concerned that I did not understand the relevance of patterns as they related to functions. The connection and importance was implied within the text but it was not anything I could understand at the time. What was a function? What did it mean to compose two functions? Why look for patterns in the context of functions? What did one need to understand to make sense of the rules and procedures that governed work with fractions? These questions could not necessarily be answered by learning *more* math. Rather, the mathematics I needed to learn involved taking a new look at the mathematics I thought I understood. The difficulties of the changes I tried to make in my teaching were not clearly because I did not know *enough* mathematics. Rather, the difficulties arose because it seemed that the mathematics I needed to understand was qualitatively different from the mathematics I thought I understood.

A different kind of knowledge. When I expected that what I needed to know was more mathematics, I also expected that someday I would be able to say I had learned the mathematics I needed to know to teach mathematics for understanding. When my expectations shifted from the need to know more mathematics to the need to know a qualitatively different kind of mathematics, my expectation of ever knowing or mastering all the mathematics I needed to know vanished. Not, however, because the quantity seemed insurmountable. Rather, the nature of what I was learning I needed to understand was at a level of ideas fundamental to the study of mathematics. I was learning to look for and ask fundamental questions--questions without clear right or wrong answers which people with varying degrees of mathematical interest and expertise have struggled, argued about, and debated over through history.

An example of the complexity of the mathematical ideas to which I am referring can be found at the heart of the issues that arose in the context of the problem I gave students that dealt with the addition of two fractions with like denominators: 3/4 + 2/4. I went into this lesson knowing that the answer was 5/4 based on my past experiences with learning and teaching the rule needed to solve this problem: When adding fractions with like denominators, you add only the numerators and keep the denominators the same. I also anticipated that a common error students might make would be to add the denominators.

Therefore, 5/8 might be a common answer among my students but it obviously would be wrong. I thought that learning this math problem "for understanding" would mean being able to come up with a reason why it made sense to not add the denominators. I worked from the assumption that there was a single right answer and it was my responsibility in the context of this kind of mathematics teaching to help students come up with reasons to explain why that was the case.

Much to my surprise, the answer that I thought was undeniably wrong--5/8--was, under certain conditions, a reasonable answer. When adding two probabilities, such as batting averages, one adds the denominators. Five eighths also could be construed as reasonable when explained with certain representations where the unit is questionable. For example, 5/4 of one large square is equivalent to 5/8 of two large squares. See Figure 8.2.

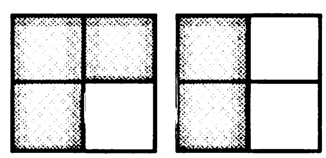

Figure 8.3

3/4 + 2/4 is 5/4 of one large square or 5/8 of two large squares

In this representation, similar to what my students argued, the whole or the unit is not obvious. Taking one large square as the whole, the unit is fourths. Taking both large squares as the whole, the unit is eighths. My understanding of what it could mean to understand the addition of fractions became more complex as I listened and observed the way my students interpreted this problem. Trying to come up with reasons why the answer is 5/4 is framing the problem differently than trying to understand under what conditions the solution could be 5/4 and under what conditions 5/8 could make sense. Understanding around what central mathematical ideas either argument hinged was an intriguing mathematical problem for me and my students. Under further investigation, I learned that what my students and I were trying to understand--issues about the unit--was a major theme in the history of the development of rational numbers. In the process of working on this problem with students, I had hit upon a key to understanding fractions--what is the whole? Through interactions with my students, I also expanded my expectations and sparked my own curiosities of what it could mean to understand the addition of fractions "for understanding" and more generally what would it mean to understand rational numbers.

New expectations for knowing subject matter. One of the major things I learned about subject matter through that year of teaching and the subsequent analysis of my experience is that acquiring it and using it is going to be an ongoing intellectual challenge in teaching. Knowing mathematics is never something I will master and be able to put aside and just teach (Kline, 1980; Lampert, 1992a; Putnam, Lampert, & Peterson, 1989). Learning mathematics is intimately connected to this kind of teaching. I give three reasons to support this claim. First, I learned that there is a mathematical territory defined by fundamental questions and ideas to be constructed and reconstructed through investigations by teachers and students in the context of teaching and learning mathematics for understanding that is qualitatively different from anything I learned in the past. Second, the problems, activities, and representations I used with my students were derivatives of or tools for learning the mathematical ideas that were important. They were not important in and of themselves. Therefore, part of the intellectual work of teaching related to subject matter becomes trying to see the fundamental mathematical ideas in textbook problems, activities, and representations where the connections to larger mathematical ideas are not necessarily made explicit. A third piece of intellectual work is learning to hear these fundamental ideas in students' efforts to communicate mathematical ideas they may be on the edge of understanding. It takes a keen ear and a flexible mind to interpret and build connections from what may be students' vague notions of mathematical sense to a vast territory of mathematical knowledge that is dynamic and fluid in nature.

As I look back across my analyses of teaching in the previous four chapters, I can see that I pushed my understanding of the mathematical ideas in each lesson to a place where I bumped up against a fundamental mathematical question or idea through close analysis of the activity I did with students and my students' responses to the problem or the activity. I confronted the fundamental mathematical ideas underlying patterns and functions, multiplication, and fractions, in the analyses of three different teaching episodes. In the analysis of a fourth series of lessons, I confronted an issue at the heart of doing mathematics within a mathematical community--the intermingling of the social and intellectual--and the tensions and issues that arise as the two interact as a means of learning mathematics. This view of mathematical knowledge (Davis & Hersh, 1981; Kitcher, 1984; Lakatos, 1976)--both what is to be known and how one comes to know-- revised my expectation that one could ever completely know mathematics. One's knowledge of the territory is tentative--continually constructed and reconstructed as one interacts with others around mathematical ideas.

Teaching for understanding necessitates the influence on what one knows mathematically to travel in multiple directions--from teachers to students, students to teachers, and students to students. All participants in a class aiming to teach mathematics for understanding need to expect to learn from one another. Teachers would still need to prepare for the situation by learning what they can about the mathematics they are about to teach prior to the situation. Questions still remain about where and how teachers are going to get the help learning what they need to know. I will return to this in the section on educating teachers at the end of this chapter. Prepared with some sense of the territory in which they might explore with students, teachers would enter the classroom curious about the sense students make of ideas expecting that what they learn about their students' understandings may reshape the way they see things, raise new questions about what it is that they think they understand, reshape the purposes of whatever it is they thought they were doing, and change anything that they may have had in mind to do. Teachers need to expect to revise and reconstruct their own mathematical understandings as they try to make use of what they and their students understand in the act of teaching. This is fundamentally different from thinking of mathematics as rules and procedures for doing problems with right and wrong answers.

An Intellectual Endeavor

I expected my efforts to make changes in my teaching of mathematics to be an intellectual challenge, given my sense of the mathematical knowledge I lacked. There was no doubt in my mind when I began teaching that year, that the changes I would face would be difficult to make. I would be trying to teach in a way that seemed radically different from what I had been doing. On the one hand, I had tried new ways of teaching in the past and so I was not too worried at the prospect of changing the way I did things. On the other hand, I was deeply concerned about how I was to going get by with the mathematical understandings I knew I lacked. What I did not expect, but soon came to learn, was that making change in teaching involved more than dealing with the intellectual challenges of my own knowing and not knowing of subject matter. I did not expect that making changes in practice would raise questions for me and others about who I thought I was and what I thought I knew about teaching.

The risks of being a learner. It felt like an enormous risk--to be headed into a fourth grade mathematics class when I knew I had much to learn about what I was supposed to be teaching. The risk felt magnified by the fact that I would not be trying to make these changes alone, with students, behind the closed door of my classroom. Instead, there would be people watching--the teacher who taught my students all of the other subjects, a researcher, and a teacher educator. At first it had felt threatening. Would I be able to take such risks in front of other people, people who represented broader categories of people who are traditionally considered knowers about teaching--experienced teachers, researchers, teacher educators? What would they think of me? What would they say about me to others? The only time I had had people watch me teach in the past, the purpose had been evaluation.

The people who had planned to observe me teach that year had all promised that evaluation was not their purpose. Could I trust them? Would they appreciate the problems I faced? The other classroom teacher had twice the number of years of teaching experience that I did. Would she view me as a rookie whose problems could easily be solved by experience? Jim, the researcher, had been interested in studying the students' and my changing conceptions of mathematics. At the time, I did not understand his question--math was math, wasn't it? Would he view me as less than qualified practitioner because I was unfamiliar with the theories that undergirded his work as a researcher? Maggie, the teacher educator, had offered to help me teach differently, to think through the pedagogical and mathematical problems I would face as I tried to make changes. She had assured me she would not be there to judge me. Could I believe her? She was a university professor. I was her student. Evaluation was always wrapped up in teacher-student relationships, wasn't it?

Much to my surprise, over time, I learned that I could trust each of these people. I learned to drop my defenses, relax, and confess to learning or needing to learn in front of each of them. For example, the other classroom teacher supported my efforts one hundred percent in front of those to whom my performance might have mattered--the students, the students' parents, other teachers, and the school's administrator. In our private conversations, she sometimes questioned what I did. She would wonder aloud about why I might have made one choice over another. But in the presence of others, she had voiced admiration for the changes I tried to make, sympathized with my struggles, and wanted others to understand what she and I were learning from my efforts. At any given moment, I felt as if we shared the responsibility for what our students were learning. Jim, the researcher, quickly abandoned his initial agenda of keeping a distanced position as he "researched" my changing conceptions of mathematics. My interviews with him became conversations in which we pondered together questions of mathematical content and pedagogy.[4] He entered graduate school when I did and was no further along in understanding this kind of mathematics teaching than me. Maggie was at once a wise teacher in the role of a teacher educator and an empathetic colleague who taught mathematics in the fifth grade classroom next-door to me. I found that she took my questions seriously and respected my past teaching experiences as well as all of my efforts to try something new. Oddly enough, she had seemed excited when I recognized things I did not know, when she and I could discuss questions I was able to articulate.[5]

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[4]See Heaton, Reineke, and Frese (1991) for additional analysis of our relationship.
[5]See Heaton and Lampert (1993) for additional analysis of our relationship.

In a strange way, over time within these relationships with people who I expected would intimidate me, people who I initially thought expected me to know, I found myself protected and sheltered, appreciated and celebrated for being a learner. Each grew sympathetic to my struggles and, together, we developed shared understandings of what was going on in my teaching. The way my teaching was organized, teaching only mathematics, four days per week, gave me time to focus on mathematics teaching that most elementary teachers never have. My schedule allowed me to take advantage of five minute unplanned conversations with the other classroom teacher before and after my teaching. It allowed me to schedule bi-weekly conversations with Jim in conjunction with his observations. It enabled me to observe in Maggie's fifth grade mathematics class at least as often as she observed in mine and talk with her several times a week. Together, these relationships provided a rich context for sustained conversations about mathematics, students, and pedagogy with people, who for a variety of reasons, were closely connected to my practice.[6]

The risks of claiming to be a learner and a teacher. I felt so accepted as a learner by these three different people that year, that I was painfully surprised the first few times I tried to talk about my efforts to make changes in my teaching of mathematics with other teachers, researchers, and teacher educators less familiar with me and my experiences. I found that many of these participants of the larger culture of teaching did not have the same appreciation for the problems of practice I encountered that the other classroom teacher, Jim, Maggie, and I had come to share.

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[6]The intensity of conversation around my teaching experience is not something one curriculum specialist in a district or school could replicate for any mass of people. For an example of the work and challenges confronting someone in a school district trying to help teachers make changes in practice, see McCarthey and Peterson (1993). In this chapter, McCarthey and Peterson contrast the efforts of two teachers in the same school, one making changes in literacy instruction, the other in mathematics, and the supports and learning opportunities provided for each of them.

For example, on several different occasions, shortly after I finished that year of teaching, I told the story of what happened as my students tried to reason about the addition of 3/4 and 2/4. I was intrigued by the complex mathematical ideas and pedagogical issues that this problem and my students' diverse understandings of the problem raised. I tried to initiate a conversation about changing practice around what I thought were several difficult and complex mathematical and pedagogical issues that I saw as key to teaching mathematics for understanding. Much to my surprise, no one wanted to talk about the problems I encountered. Everyone wanted to offer me solutions. One by one, those who had listened to me unveil my problems offered me suggestions for what I should have done differently. My problems were viewed as simple, the solutions were seen as obvious.

My fellow educators wanted to be supportive but we viewed my teaching differently. On the one hand, they saw 5/4 as right and felt it was their responsibility to give me ways to get my students to see that. On the other hand, I was beginning to see the "sense" in 5/8 and wanted to talk about the difficulties with that. What I was concerned about could not be fixed by suggesting that I try this or do that. No one but me seemed to see the teaching or the mathematics I was trying to help students learn as inherently difficult. At the time, I could not understand what was happening. I fought back feelings of inadequacy and frustration. Part of me felt badly for not knowing, thinking that I should have known, while another part of me had spent a year learning to appreciate the value of not knowing, of never being certain, in a practice that was improvised. Why couldn't these people see the complexity I could? I feared I appeared to others as incompetent for having questions about practice, the same questions that I had been praised for recognizing and others had pondered with me during the year I taught. Consequently, on those several occasions, I walked away from these experiences fighting back tears as I questioned who I was, what I knew, and what it was I was supposed to know. I worried that people had just explained away the difficulties I had making changes in my practice thinking that the real problem was that I was an incompetent teacher. I imagined questions people must have been asking themselves. How could she be such a bad teacher and be allowed to become a teacher educator? How could she ever be qualified to earn a Ph.D. in teacher education?

Painful as these experiences were, they helped me to see two additional things that are necessary to make changes in the practice of teaching mathematics. One is to begin to rephrase the questions we ask about teaching, to move from pedagogical questions that have single right answers, imply a perfect method, technique, or strategy to questions that imply a complex practice filled with intellectual, pedagogical, and moral dilemmas.[7] Second, those who teach, teach others to teach, do research on those who teach, or evaluate those who teach need to learn to talk and act while performing each of their roles in ways that acknowledge teaching mathematics for understanding as inherently complex and uncertain. As a community of educators we need to acknowledge to one another and to ourselves that in the case of teaching mathematics for understanding we are inventing a practice that is complex and uncertain for which there is no recipe, model, or prescription. There is much we do not know and can never know for certain about inventing a practice. The challenge, it seems, is to reconstruct a culture of teaching whereby it becomes intellectually possible and emotionally feasible for teachers, administrators, researchers, and teacher educators, those who have traditionally been expected to be authorities on teaching, to appreciate the inherent complexities of teaching mathematics for

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[7]See Lampert (1986) for an in-depth look at dilemma management in teaching. Embracing teaching dilemmas requires a tolerance for and use of uncertainty.

understanding and to view themselves and be viewed by others as certified learners rather than knowers of teaching--lifelong constructors of knowledge of an uncertain practice.

Educating Teachers for Uncertainty[8]

Having examined closely factors that contribute to endemic uncertainties of teaching mathematics for understanding, in this section I look at how teachers might be educated to deal with uncertainty. I contend that the nature of what I was trying to learn--that of trying to invent a practice while learning a practice--is itself uncertain. To teach mathematics for understanding is not about learning a defined method or technique that can first be learned and later implemented. Therefore, it is impossible to say that one has ever "learned" to teach mathematics for understanding. To teach mathematics for understanding is to engage in an activity that is constructive in nature, a process of continuous invention, whereby teaching, itself is defined and redefined, over time, and in interaction with students.

To say that teaching mathematics for understanding is uncertain and continuously under construction might imply to some that I am arguing that there is nothing that can be done to support the education of teachers. It must all be learned by trial and error. Not so. There are things that can be done to educate teachers in support of constructing a practice of teaching mathematics. To understand what, it is helpful to compare the study of teaching mathematics for understanding with learning a field, such as is done in the arts and sciences.

> Learning a field entails not just learning particular ideas and theories, but also learning its characteristic ways of knowing and of constructing new knowledge. This means learning to generate conjectures and seek

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[8]I borrow this idea of preparing for uncertainty from an article by Floden and Clark (1988) where they discuss, in general terms, preparing teachers for uncertainty.

> evidence, to formulate questions and make arguments, to play out ideas, and make revisions. (Lampert, Ball, Rosenberg, & Suzuka, 1993, p. 5)

The ways of knowing and constructing new knowledge in a field are parallel to what it takes to continuously invent a practice of teaching mathematics for understanding. Teachers learn to compose and recompose practices by drawing on particular ideas and theories and working with them in ways that suggest a tentative view of knowledge about teaching. This is not unlike the ways in which students learning mathematics make conjectures, construct arguments, and revise ideas. Just as knowledge of mathematics is socially constructed, so too, I argue, is knowledge about teaching.

How could teachers be helped to construct and reconstruct their own knowledge about teaching? Contemplating how to educate teachers to teach mathematics for understanding entails thinking about how to prepare teachers to perform an uncertain activity. Not only has teaching been redefined through the reforms in mathematics education, as evidenced by the analyses here and in earlier chapters. Ways of helping teachers must also be revisioned to include the uncertain, spontaneous nature of teaching mathematics for understanding. To do this I draw on Yinger's (1988) notion of preparation. It seems an appropriate construct for defining what it would mean to help someone prepare for the uncertainties of continuously inventing a practice of teaching mathematics for understanding. To explain what he means by preparation, Yinger contrasts the more conventional notion of "planning" with that of "preparation":

> Planning seeks to deal with uncertainty by controlling action and outcomes. The goal is to constrain the unpredictable, the random, and the wild. On the other hand, preparation acknowledges our limited ability to predict and the constructive nature of life. Preparation expects diversity, surprise, the random, and the wild. To prepare is to get ready, to become equipped, and to become receptive. The focus of preparation is on

> oneself, not on a framework to constrain possibility. In a sense, preparation enlarges the future. (p.88)

Preparation for teaching mathematics for understanding is about learning what to expect from and during practice and being prepared to act on whatever happens, even the unexpected.

Based on my new understandings of what teaching mathematics for understanding entails and what it demands of an experienced teacher, I examine below what it might mean to educate teachers for its uncertainties. How could one be prepared to teach mathematics for understanding? For example, what could be done to prepare someone to enter the classroom with a wide range of possible options for what to do next and construct new ones in the situation? What could be done to prepare someone with a qualitatively different understanding of mathematics than what they have experienced as teachers or learners? What could be done to alter the culture of teaching to make it a safe place for teachers to talk about the complexities of practice? How could one expand the realm of educating teachers to take into account the need to prepare teachers for the uncertain nature of continuously inventing a practice of teaching mathematics for understanding?

<u>Educative Curricula</u>

I was thrilled at the onset of my teaching to have the CSMP curriculum at my disposal. It represented mathematics and pedagogy in ways that seemed consonant with the reforms in mathematics education. The mathematics looked different from what I had learned as a student and the pedagogy was designed around a set of problems intended for students to solve that did not look anything like the computation I had had students doing for years. I expected that the teacher's guide would take care of the mathematics I did not understand and the scripted dialogue would serve as a guide for how to respond to my students. At the time, given the content and design of CSMP, these seemed like reasonable expectations.

What I found was that while the mathematics was, indeed, represented in conceptual ways, this deeper meaning of mathematics was not in a form that was accessible to me. I could not understand the mathematics as presented in ways that enabled me to construct my own understandings of particular ideas. While I started out believing that by following the textbook I could carry the mathematics I did not understand, I soon changed my mind. It was insufficient for the understanding of the mathematics to be contained within the textbook and not within *me*. I quickly saw the need to possess an understanding of the mathematics myself and make use of that understanding in teaching.

I saw the same need with respect to understanding students. While having possible student responses represented in a scripted dialogue within the teacher's guide gave me some idea of what to expect from students, my students' ideas were often formulated differently from the responses predicted in the teacher's guide. I quickly found myself at a pedagogical disadvantage as I tried to hear those responses in the language of my students. Without an understanding of the mathematics or the relevance of particular responses, I could not discern the mathematical significance of my students' responses. Without interpretive resources to help me understand the importance of particular students responses of how they might vary, I often floundered as I contemplated what to do next.

While the textbook could not carry the mathematics or be the sole generator of ways to respond to students, my experience raises the question of whether there are ways that curriculum materials could help me understand the mathematics better so that I could rely more on my own understandings in teaching. Are there ways that a textbook could help me to think about how to respond to students? These two responsibilities--understanding the content and responding to students--which I initially saw as residing with the textbook, I now see as inherently the teacher's. Rather than dismiss the notion of a mathematics textbook entirely, however, it is sensible to rethink what it could mean for a textbook to help teachers meet these responsibilities. This is at least a two-part agenda. One aspect involves rethinking *how to make use of the textbook.* What instructive uses might there be for a textbook? Uses that respect the situated nature of teaching but that also offer resources for managing those situations. A second part of the change involves rethinking the *contents of the textbook.* How might textbooks might be revised to become a resource for helping elementary teachers to make sense of the mathematics, what their students understand about the mathematics, how to respond to their students, and which direction to head as they work to construct the curriculum? What would a resource look like that was intended to help teachers determine, follow, and act on their own mathematical and pedagogical sense of what to do next? What would it mean for a curriculum for teachers to be tied to a curriculum for students (Cohen & Barnes, 1993)[9]?

Mathematical content. What is needed is a curriculum for teachers that is based on the premise that mathematical knowledge is socially constructed while at the same time recognizing that there are particular problems, questions, and ideas that are important with which to grapple if one is to learn mathematics for understanding and have some sense of what it means to do mathematics. Problems and activities included in a mathematics textbook should be written for students yet explained to teachers in ways that make explicit this intersection between conventional agreed-upon mathematical knowledge and ideas that are

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[9]See Cohen and Barnes (1993) for discussion in more general terms about what it could mean for curricula to be educative for teachers.

open to interpretation. The text should be written in ways that make math problems accessible sites of exploration and interpretation for students while simultaneously being representations of larger mathematical issues for teachers. Most textbooks are designed to focus on what is important for students to learn. What is needed is a textbook designed to help teachers learn. What is needed is a textbook that helps teachers understand the connections between the problems and activities intended for students and the important mathematical ideas the problems and activities are intended to teach. For it is connections, or the potential for constructing connections between the sense students make of the problem and the ideas they are trying to teach, that teachers must have in mind as they work to help students construct understandings of the mathematics using problems and activities as means, not ends, for understanding.

Given that there is such a strong need in this kind of mathematics teaching to be responsive to students' ideas, it is helpful for curriculum materials to provide likely student responses, as CSMP has done. But in addition, guidance should be provided to allow teachers to interpret what they hear their own students say. Verbatim quotes of likely student responses are not very useful without some understanding of *why* these might be typical responses. What mathematics would someone understand who would say these things? What might someone say who did not have a good understanding of the mathematics? What might be reasonable alternate responses? What such additional information in a text would provide is a context for teachers to assess and make use of their students' responses relative to the mathematics they are trying to teach--an integral part of what is demanded in teaching mathematics for understanding.

In short, teachers need to have access to the mathematical knowledge of ideas and connections and relationships in textbooks that curriculum developers possess, who now take responsibility for constructing pedagogical paths through the mathematical terrain which teachers follow. Curricular materials are needed that make visible how pathways through the mathematical terrain are constructed by such things as choices of problems to give students and teachers' next moves. Teachers need to understand the grounds on which decisions about possible problems or activities to do with students are made. They also need to learn what goes into decisions about what to do next so that can learn to make their own decisions and construct their own paths, ones that are responsive to their particular students. As most mathematics textbooks currently exist, the teacher's guides provide a single path through the terrain of mathematics with few of the curricular decisions of the developers made visible to users.

Attention to teachers' learning. These suggestions for what and how mathematical content ought to be represented for teachers in texts would result in a curriculum that is educative for teachers. In earlier reforms of mathematics education, curriculum developers have gone to great lengths to design "teacher proof" materials, curriculum materials intended to keep teachers distanced from teaching, learning, and content (Sarason, 1982). What my work shows is that because of the nature of teaching mathematics for understanding, the teacher must be intimately connected to the decisions that govern the teaching and learning of content. Curriculum developers must find ways to guide teachers' pedagogical and mathematical decisions not make decontextualized decisions for them. One primary way to help teachers to do this is by finding ways to increase teachers' understandings of the content. For example, in the case of rational numbers, curriculum materials could offer a teacher help in thinking through the mathematical issues related to the addition of fractions. Teachers could be alerted to the difficulties students might encounter in identifying and keeping the unit constant by explaining the reasonableness behind such difficulties. They could also be helped to understand the usefulness of particular contexts or representations for understanding particular ideas.

To think of teachers as learners when designing curriculum is something that occurred to Bruner (1977) many years ago, in the context of earlier reforms in education, but few textbook developers have heeded. Bruner (1977), in a preface to a new edition of the *Process of Education*, wrote,

> Let me turn finally to the last of the things that have kept me brooding about this book--the production of curriculum. Whoever has undertaken such an enterprise will probably have learned many things. But with luck, he will also have learned one big thing. A curriculum is more for teachers than it is for pupils. If it cannot change, move, perturb, inform teachers, it will have no effect on those whom they teach. It must be first and foremost a curriculum for teachers. If it has any effect on pupils, it will have it by virtue of having had an effect on teachers. (p. xv)

Perhaps one of the reasons why Bruner's advice has not been followed is that few curriculum developers have understood why it was important or what it might mean to attend to teacher's learning and few educators have taken seriously the need for elementary teachers to understand subject matter--in this case, mathematics. If one thinks teaching is about particular behaviors then telling through textbooks is a sufficient and efficient way to help teachers. Curriculum materials have been directing teachers about what to do for years. Hopefully, what my study does is begin to illustrate the importance of looking at ways to be educative rather than directive in the development of curriculum materials for helping teachers teach mathematics for understanding.[10] As Cohen and Barnes

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[10]Dow (1991), Chapter 4, gives an account of efforts by curriculum developers to construct MACOS materials in ways that were educative for teachers. Dow describes early conversations about what the teacher's guides ought to be like. "A scholar should write it, and it should convey different interpretations and perspectives on the subject as well as suggest useful supplementary materials" (p.140).

(1993) note, teachers need opportunities to learn from new curricula that are similar to the learning reformers intend for students.

Educating Teachers

Besides having a curriculum that is educative for teachers, what else could be done to educate teachers to meet the need to invent a practice of teaching mathematics for understanding while teaching? How does the idea of continuously inventing a practice of teaching mathematics revise current conceptions of what it means to educate teachers? What might it mean to educate teachers while taking seriously the need for them to be continuously learning? A primary challenge is to create learning opportunities that allow teachers to construct tentative knowledge of pedagogy, students, and mathematics[11], to gain an understanding of what it means to know and come to know, to acquire a sense of what it means use what one tentatively knows, and to come up with new ideas. My study implies that the emphasis of helping teachers change their practice should shift away from helping teachers learn new skills or strategies and away from supplying them with new math problems and manipulatives--specifics of what or how to teach--toward learning how to create and recognize choices and make decisions about appropriate math problems, representations, and responses to further students' understandings.

Preparing for an uncertain and complex practice means strengthening one's capacity to construct, judge, and decide among different options. While teachers need opportunities to learn specifics related to content and pedagogy, a teacher's power to teach--to continuously invent a practice--comes at least as much from what is learned from learning any particular ideas about

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[11]The three knowledge categories of mathematics, students, and pedagogy correspond to David Hawkin's (1974) way of thinking about what goes into teaching: I (pedagogue), Thou (students), It (mathematics).

mathematics, students, and pedagogy. What my thesis implies is a need for teacher education which accomodates both the learning of particular ideas and learning how to come up with new ideas.

What can be learned? Conventional teacher education has long focused on teaching techniques, strategies, and methods for teaching. While my study confirms the need to attend to subject matter in teacher education (e.g., Ball 1988a; Ball & McDiarmid, 1990), which has recently received a lot of attention, it also suggests the need to design ways to layer what it means to know and come to know over any particular domain of learning--mathematics, students, and pedagogy--and learn how to use what one is learning while teaching. A conception of knowledge as unfinished and tentative is necessary to be able to continuously invent a practice of teaching mathematics for understanding and implies the need for inquiry into mathematics, students, and pedagogy as an inherent part of teaching mathematics for understanding. Learning what is entailed in learning in any of these areas, therefore, is as important as learning particulars in each area. Learning to see mathematical knowledge and knowledge about students and pedagogy as constructed and reconstructed is not something that years of traditional mathematics education would have prepared anyone for.

If knowledge is necessarily tentative and changing, what does it mean to *learn to teach*? One can never completely know teaching or declare with any finality that one has "learned to teach." The purpose of educating teachers in the context of teaching mathematics for understanding, therefore, is not to help teachers master a way of teaching. Rather, educating teachers entails, in large part, helping teachers learn ways to think about what it means to construct knowledge and make decisions about action that enable one to improvise responsively in the midst of teaching. When taking a tentative, conditional stance toward knowledge about teaching, one assumes that past teaching experiences--both the particulars of what one has learned to do and what one has learned about coming up with what to do--will inform the construction of knowledge in new teaching experiences.

In what follows, I look at what teacher education should teach. I describe what it might mean to know mathematics, students, and pedagogy in ways that include but move beyond specific understandings in each of these areas and prepare one to teach in continuously new ways that are responsive to students' understandings and one's evolving understandings of mathematical ideas that are important for students to learn.[12] While my thesis has focused on what it might take for an experienced teacher to make changes in practice, in this conclusion I purposefully make no distinctions between the education of preservice and practicing teachers. There are two reasons for this. The first reason is a substantive one. While I remain uncertain about the exact differences that years of teaching experience might make as one attempts to teach in ways aligned with the reforms, I believe that what I have learned would apply equally to preservice teachers. The nature of the teaching one is continuously aiming to invent is the same whether you are a novice or a veteran teacher. Preservice teachers, admittedly, will need additional understandings to compensate for what they might lack without the benefit of past teaching experiences. Just what they might need is beyond the scope of my study. The second reason I do not distinguish here between preservice and inservice teacher education relates to the question of *how* teachers are going to learn what they need to which, admittedly, could vary considerably whether one is referring to preservice or

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[12]Shulman (1986) refers to the knowledge that bridges knowledge of the subject with knowledge of the students as knowledge about teaching or "pedagogical content knowledge." My work represents a way to conceiving of the nature of this knowledge as dynamic--constructed and reconstructed during the act of teaching. Ball (1988a), Chapter 6, makes this same point.

inservice teachers. However, this, too, is not the focus of my thesis. Questions of how teachers are going to learn to teach adventurously are being examined by others (i.e., Lacey & Merseth, 1993; Sykes & Bird, 1992).

To know mathematics. It would be impossible within the confines of any single teacher education experience, whether it be a preservice methods course or a one shot inservice workshop, to help teachers construct a qualitatively different kind of understanding of mathematics in all topics of elementary mathematics to meet the subject matter needs of teaching mathematics for understanding. Therefore, one is faced with issues of breadth versus depth in contemplating how to attend to the need for teachers to learn subject matter. Is the goal to explore a multitude of mathematical topics in limited depth or a few topics in greater depth? Time constraints of preservice or inservice teacher education do not allow for the ideal--to examine many topics, all in depth. Given the past mathematical learning experiences of most teachers and the nature of the teaching they are aiming to do in the context of the current reforms, I argue that it is important for teachers to have the experience of learning a few areas of mathematics in depth. To learn how to teach an area of mathematics in depth means acquiring particular understandings of the topic as well as gaining a perspective on what it means to learn and construct new mathematical ideas. Given the dynamic nature of mathematical knowledge, it makes sense to learn what it means to understand any area of mathematics in depth and apply what was learned from that experience to learning something new.

For example, what would it mean to take what one understands about fractions from a study of the topic and apply that experience to efforts to learn another area of mathematics, like probability? How does learning one topic in depth prepare one to learn another? What can one learn to expect the study of a mathematical topic to entail? In my case, what did I learn through my study of fractions, beyond specific concepts of fractions, that would help me to learn to teach probability? First, I learned ways to begin an exploration of an area of mathematics for which I have little understanding. I learned that one way to begin is by seeing what others have learned who have already spent time exploring the territory from other perspectives, for other purposes. This could mean finding people to talk with: mathematicians who understand the subject matter or teachers who have taught the topic in depth. It could also mean reading research in mathematics education done mainly by cognitive psychologists, who do research on how students understand particular mathematical ideas, or what has been written about probability in the history of mathematics. A final strategy could be looking up any explanation of the topic in a mathematics textbook. Given my experiences with textbooks, the sort of understandings I am after might not be accessible in ones that currently exist. However, the content and design of the curriculum materials I described earlier have the potential to be an educative vehicle for teachers. In any case, what one is after in pursuing these sources are some tentative understandings of the difficulties, problems, and main challenges people have confronted in their efforts to make sense of a particular area of mathematics. It is not a matter of finding a book or a person who will tell you how to do it.

As one examines these problems and difficulties others have faced, one is looking for fundamental questions and main ideas around which an understanding of the area hinges. For example, in the area of fractions, questions of the whole and issues related to the unit are the fundamental ideas around which an understanding of fractions depends. In trying to understand these difficulties, one should pay attention to the context in which they arise and the different representations used. What mathematical ideas are accessible through particular representations? What are the mathematical problems with particular representations? Where do questions of probability arise in real life? What information can be gleaned about probability by looking across those contexts? Under what conditions or assumptions do various understandings of probability rest? To have an understanding of a mathematical topic means having a sense, at some fundamental level, of what it is that one understands, and what it might mean to understand, and what one has left to learn.

My study of the mathematics in the context of one lesson--the identification of main ideas and fundamental questions as well as interpretations of and purposes for specific math problems--has helped me explore the mathematics in other lessons. I have not only gained a sense for how to use resources but I have also learned how to ask questions of myself and others to get at the heart of the mathematics found in math problems, representations, and students' responses. Simple questions, like what is a pattern or what does it mean to multiply, have complex solutions, grounded in assumptions and conditions, which, when one tries to understand them, place one at the center of what it means to understand mathematical ideas in a deep and fundamental way.

To know students. How do you prepare teachers for a kind of teaching that is intended to be responsive to particular students? How can what one learns about particular students as they work on a particular mathematical topic help one to continuously invent a practice when one moves on to a new topic or new students? I offer three different and brief examples of situations in which I learned what I should be concerned about in my relationship with students through my relationship with students. I have learned to be concerned about students' understanding in relation to my own. I have learned the need for trusting the reasonableness of all students' intuitions, especially those who for a variety of reasons, I may least expect to have helpful interpretations of mathematical ideas.

First, the interactions with students around patterns, particularly my interactions with Heather, have helped me see that what a teacher can hear in what students say may be as indicative of what a teacher understands about the mathematics as what students understand. That is, what teachers are able to interpret from what students say is highly dependent on what teachers themselves understand about the mathematics. When a teacher has difficulty interpreting what a particular student is trying to communicate, the norm is for the teacher to question the student's understanding. An equally important thing to do is for teachers to question their own mathematical understandings. It is not the students' responsibility to match their understandings with the teacher's. It is the teacher's responsibility to push themselves to make sense of what it is that students understand. Second, the series of lessons around 2/4 + 3/4 is a good example of making an effort to try to understand the assumptions and conditions under which the "obviously wrong answer," 5/8, *could* make sense. For this to happen, it means that a teacher needs to learn to work from the premise that what students say makes sense, and to learn to understand how. Third, my interactions with Sipho help to illustrate what I learned about the potential of all students, even those one may least expect, to make meaningful contributions to their own learning and the learning of others. Expecting to be surprised by what a student understands rather than disappointed by what they do not understand is a useful stance to take toward students during teaching. While knowing how particular students have made sense of ideas will not lead to an exact prediction of the sense future students will make of particular ideas, looking with a critical eye at what can be learned from being in relation to students does help a teacher know what to expect about what might occur while interacting with a new group of students around similar mathematical ideas or the same group of students around a new topic.

Having a sense of what to expect and being able to interpret students' understandings helps a teacher to build on students' ideas in teaching. For example, if I had had a better understanding of the meaning of multiplication and its relationship to the use of the Minicomputer, I might have been able to make use of students' mental strategies for computing the multiplication problems I gave them. At the time, I dismissed students' responses and grew frustrated with students who followed their own sense making rather than producing right answers in by making use of the Minicomputer in the ways defined by the textbook. Only in retrospect, did I appreciate the ways in which students composed and decomposed numbers to compute multiplication problems. Had I been attuned to their ways of sense making, I might have found ways to represent and make use of some of their mental strategies as alternative ways to understand multiplication and the composition of functions.

To know pedagogy. What does it mean to know pedagogy? Understanding means having a sense of pedagogical problems or dilemmas to expect in the context of teaching and, over time, accruing insight and experience that enable one to construct and reconstruct tentative solutions. I offer two examples of things that I have learned about pedagogy that have helped me learn what to expect. One focuses on things I have learned to be concerned about in holding a discussion. The other centers on what I have learned about what it means to "cover" curriculum. It is not that I have mastered leading a discussion or that I have learned how to construct the perfect curriculum. Rather, I have learned things I need to be aware of, questions I need to ask as I try to lead a discussion and reconsider what it means to cover the curriculum.

Through my efforts to hold a discussion, I have learned some important ideas about what discussions demand of teachers. For example, a teacher needs an understanding of the mathematics she is trying to teach students so that she has some sense of what there is to talk about. This knowledge of what there is to talk about comes from a study of the mathematical territory, as described in an earlier section, where one begins to learn what fundamental questions and ideas are worth spending time talking about. Related to this, one needs an on going sense of purpose. What is it that is important for students to learn? Had I understood this, I might have done a better job sustaining a worthwhile discussion of patterns.

I have also learned to think differently about pedagogy in the context of curriculum. I have learned to reconsider the question of what it means to "cover the curriculum." I have learned to think about curriculum more as layers of concepts or a web of relationships with single problems cutting through layers or sitting in the center of a web of ideas than a scope and sequence of behavioral objectives. If I had thought about curriculum like this at the time of my teaching, I would have started out with a different purpose for doing the problem 2/4 + 3/4 with my students. I would have seen The Muffin Problem as a rich context for students to explore the issues of understanding and representing the unit rather than merely an occasion to reason about the rule of not adding the denominators.

To know oneself. Learning to continuously invent a practice means learning to regard oneself as the inventor of a practice. What it takes to be an inventor of practice encompasses learning all of these things--mathematics, students, and pedagogy--simultaneously, none of which can be known by someone telling you. This means facing oneself as a thinker, someone capable of figuring out what makes sense. It entails considering what one knows and trusting oneself to make decisions based on what one knows, however tentative. It means learning that it is one's role to think for oneself as a teacher. If one comes to see oneself in this light, then learning all of these things takes on a different character. Expectations for what it means to be educated as a teacher are altered.

For example, had I entered this year of teaching seeing myself as the inventor of a practice, I would have expected mathematical questions to arise for which I did not have an answer. A question like, what is a pattern, would have been viewed as the source of a lively discussion rather than as a simple question for which there is an identifiable right answer that I just did not know. I would have expected to wonder about students' intuitions when solving problems. For example, I would have capitalized on the mathematical sense behind students' use of a ruler to try to add fractions rather than brushing off their efforts as confusing nonsense or a venture off the track. And, finally, I would have appreciated that continuously asking the question of what to do next is what propels my teaching. It does not represent a shortcoming or a question for which there will ever be a single right answer. It represents my understanding that knowledge about teaching mathematics for understanding is tentative and uncertain. It is something I am responsible for and capable of constructing and reconstructing in the act of teaching and lies at the heart of my claim that what I have learned is to continuously invent a practice.

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