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AN EXPLORATION OF THE ROLE OF CONTEXT IN HIGH SCHOOL STUDENTS' CONCEPTIONS AND REPRESENTATIONS OF LINEAR, QUADRATIC, AND EXPONENTIAL FUNCTIONS

By

Georgianna Tonne Klein

A DISSERTATION

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ABSTRACT

AN EXPLORATION OF THE ROLE OF CONTEXT IN HIGH SCHOOL STUDENTS' CONCEPTIONS AND REPRESENTATIONS OF LINEAR, QUADRATIC, AND EXPONENTIAL FUNCTIONS

By

Georgianna Tonne Klein

This study examines what sense students made of functions during a summer instructional program that had function as a central concept and that provided rich experiences with multiple representations. The research focuses on whether and how students' conceptions of functions are tied to specific mathematical contexts (linear, quadratic, or exponential functions) or representational contexts (tables, graphs, or symbolic or verbal descriptions of situations). Data consist of interviews of 4 focal students in a class that was part of a Regional Math and Science Center for the Upward Bound Program, along with pre- and posttests, written homework, and fieldnotes and videotapes of classroom observations. The dissertation presents case studies of two of these focal students.

The study found a mismatch between teachers' and students' perspectives--on whether the mathematical content was about functions, on the purpose and focus of student-teacher interactions, and on what can be assumed or ignored in some problem representations. For examining students' thinking, this work proposes a framework for functions that go beyond existing frameworks. It is a two-part framework that separates functions from functional relationships in problems and reflects activities students found salient. In solving and talking about the problems posed, students

attended principally to the representations themselves and only implicitly to functions. They were successful in many tasks, but their thinking was often fragmented. Students found dynamic versions of functions salient, particularly in tables and graphs. Their characterizations of function classes depended on function class, task, and representation. They used shapes of graphs or patterns of change as organizers for both linear and quadratic functions, but had no organizers for exponential functions. Students were facile at fitting general graphs to situations if the independent variable could be thought of temporally. Situated problems supported students' reasoning, but hindered it when problems clashed with the students' informal models.

A major implication is that teachers need to examine what assumptions they bring to teaching and how it shapes how they think about student activity, student understanding, and teaching function concepts. Curricula should be examined for the extent to which they include activities with function that students find salient.

Copyright

By

GEORGIANNA TONNE KLEIN

1995

This work is dedicated with great love and appreciation to

Bruce, my best friend and husband

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CHAPTER 1

THE STUDY

Recently, many groups have called for major reform in mathematics education (AAAS, 1989; National Research Council, 1989; 1989; NCTM, 1991). All point to a need for an increasingly mathematically literate society and the need to educate all students. Together, these groups propose new ways of thinking about teaching and learning mathematics.

The *Curriculum and Evaluation Standards* (NCTM, 1989) focus on mathematics as problem solving, communication, reasoning, and making connections, and advocate the increased use of graphing utilities, by which they mean a computer with appropriate graphing software or a graphing calculator. For grades 9-12 they advocate updating the existing curriculum content to be consistent with that needed in an information society, while retaining a central role for the concept of functions. These new goals and tools for teaching and learning require fundamentally new ways of thinking about instruction and how students learn mathematics. In particular, the *Standards* depict learners as actively engaged in constructing rich connections in their mathematical knowledge. This constructivist view is consistent with current trends in learning and cognition, which Resnick (1989) summarizes:

> Current cognitive theory emphasizes three interrelated aspects of learning that, together, call for forms of instruction theory very different from those that grew out of earlier associationist and behaviorist psychologies. First, learning is a process of knowledge *construction*, not of knowledge recording or absorption. Second, learning is *knowledge-dependent*; people use current knowledge to construct new knowledge. Third, learning is highly tuned to the *situation* in which it takes place. (p. 1, emphasis in original).

While we have new guidelines for curriculum and instruction that are consistent with leading theories of learning and cognition, theses guidelines represent a radical departure from current instructional practice (Grouws, 1992; Romberg & Carpenter, 1986; Welch, 1978). Curriculum materials are only beginning to appear,

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and since graphics calculators have become available relatively recently, much instruction that uses them is in an exploratory state.

This disjuncture between proposed and existing practice poses an additional dilemma for research. In order to consider realistic instructional contexts, researchers have increasingly situated educational research within classrooms rather than the laboratory (Resnick, 1989; Silver, 1985; Wittrock, 1986). Yet, we have had few sites in which to document with research the nature of learning and teaching in classrooms that attempt to follow the *Standards* (Hiebert, 1991).

This study explored mathematics learning by tenth and eleventh grade students who had participated in a classroom where instruction attempted to adhere to the *Curriculum and Evaluation Standards*. Students were participants in a 6-week summer program at an Upward Bound Regional Center for Mathematics and Science. This program focused on increasing the math and science skills of promising high school students who were either economically-disadvantaged or who would be the first in their family to attend college. In addition, the program was intended to encourage students to select careers in these disciplines.

Using the mathematical context of functions, this study aimed to explore the conceptions students develop when they were placed in an instructional setting in which an attempt was made to use talk and visible representations, such as graphs, formulas, and tables, to communicate their mathematical understandings. The study investigated what representations students made or used in solving problems involving functional relationships and how these representations strengthened or limited students' understanding of linear, quadratic, and exponential functions.

The principal question informing the study is:

> *WHAT SENSE DO STUDENTS IN THIS INSTRUCTIONAL SETTING MAKE OF FUNCTIONS AND TO WHAT EXTENT ARE THEIR CONCEPTIONS TIED TO SPECIFIC MATHEMATICAL OR REPRESENTATIONAL CONTEXTS?*

Mathematical context refers to the content topic, which is linear, quadratic, or exponential functions. Representational context refers to the mathematical forms that are used primarily to present these functions in their elementary forms: verbal descriptions, tables, Cartesian graphs, or literal symbolic forms. Media used for these representations will be principally talk, paper and pencil (or chalkboard), and graphics calculators.

In order to answer the principal question, several other subquestions were formed to guide my initial thinking. They are best understood after a review of the literature.

Review of Relevant Literature

The Curriculum and Evaluation Standards

The *Curriculum and Evaluation Standards* for grades 9-12 (NCTM, 1989) call for students to be engaged in actively constructing rich connections in their mathematical knowledge. The summary of changes in instructional practice at the 9-12 level calls for increased attention to getting students actively involving in solving problems, communicating and justifying ideas, making connections, and incorporating a variety of instructional formats (p.127).

Under content topics, the *Standards* call for functions to be integrated across topics at all grade levels. Under the summary of changes in content (p. 126) the *Standards* recommend increased attention to, among other topics, the use of real-world problems to motivate and apply theory, the use of computer graphing utilities to develop conceptual understanding, deductive arguments expressed orally and in sentence or paragraph forms, integration of geometry across topics, and computer-based methods for solving equations and inequalities.

Regarding functions, the *Standards* call for increased emphasis on connections among a problem situation, its model as a function in symbolic form, and the graphs

of that function. Standard 8 sets goals for students to be able to:

> model real-world phenomena with a variety of functions; represent and analyze relationships using tables, verbal rules, equations, and graphs; translate among tabular, symbolic, and graphical representations of functions; recognize that a variety of problem situations can be modeled by the same type of functions, analyze the effects of parameter changes on the graphs of functions; and so that, in addition, college-intending students can understand operations on, and the general properties and behavior of, classes of functions. (p. 154)

Wherever examples are given to exemplify content standards, various levels of activities are suggested. In each case, the levels appear to move from activities grounded in concrete or applications contexts to others in more abstract or mathematically complex contexts. (Abstract is used here in the sense of the degree to which the mathematics is tied to the situational context.)

The View from Cognitive Science and Mathematics Education

Nature of knowledge and learning. Cognitive scientists and mathematics education researchers currently believe that students actively construct knowledge (Davis, 1984; Hiebert & Lefevre, 1986; Resnick, 1989; Resnick & Ford, 1981; von Glaserfeld, 1987). While there is no single, encompassing description of knowledge and learning among cognitive scientists, there is a set of shared assumptions (Putnam, Lampert, & Peterson, 1990). Meaning resides within the individual and is constructed by the individual through interaction with the environment, both physical and social. Knowledge and competence are the products of students' conceptual organization of their individual experiences, not a transferable commodity that can be conveyed by communication.

Experience does not take place in a vacuum, so cognitive theorists and mathematics educators are giving increased attention to the social and situational contexts of learning (Brown, Collins, & Duguid, 1989; Greeno, 1989; Janvier, 1990; Rogoff & Lave, 1984). Greeno (1989) suggests that it is useful to consider thinking, knowing, and learning as a relation involving an agent in a situation, rather than as an activity in an individual's mind. Thinking and learning are situated in contexts of

beliefs and understanding about cognition that differ between people and social groups. Brown et al. (1989) maintain that all knowledge is inextricably a product of activity and situations. They claim:

> A concept, for example, will continually evolve with each new occasion of use, because new situations, negotiations, and activities inevitably recast it in a new, more densely textured form. So a concept, like the meaning of a word, is always under construction. (p 33.)

Since thinking and learning are activities in which students reorganize their previous knowledge, what they already know is important. Further, researchers have begun to recognize that children bring rich intuitive or informal knowledge to the educational setting, and this knowledge plays an important role in cognitive growth (Fischbein, 1982; Glaser, 1984; Greeno, 1989; Schoenfeld, 1985). Educators are concerned with making connections to the informal knowledge that students bring to the educational setting. Studies with young students have shown that given an appropriate environment, even very young students can access informal knowledge before they have skills to represent it formally (Carpenter, Fennema, Peterson, C.Chiang, & Loef, 1988).

Public-private distinctions. Although meaning arises through experience and may be tightly linked to situations and activities, it is nevertheless personal. Janvier (1987b) distinguishes between an individual's private, personal conceptions and the public, shared notions, formal concepts, which make communication possible. Bauersfeld (1991) calls these shared notions, taken-as-shared. Tall and Vinner (1981) make a similar distinction between private conceptions and public, formal concepts through "concept image" and "concept definition" which they define as follows. A concept image is:

> the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and process. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures (p.152).

The formal concept definition is "a form of words used to specify that concept . . . .

which is accepted by the mathematical community at large" (p.152). While their concept definition is slightly more narrowly defined than Janvier's formal concepts since they limit it to verbal definitions, it is essentially the same, since most mathematics concepts could be defined verbally with the help of symbols (both literal and pictorial). While the concept definition must contain only the properties common to all instances under consideration, the concept image is likely to retain other elements associated with those instances that may consciously or unconsciously be used when dealing with the concept. Because not all aspects of the concept image are evoked each time we make use of it (Vinner, 1983; 1992), the concept image can be incoherent without our being aware of it. In a recent paper, Vinner (1992) described this phenomenon as compartmentalization. In particular, he and Dreyfus (1989) found this phenomenon very persistent in students' concept image of functions. Further, they suggest that the concept image is formed at the time of exposure to the concept and related materials, a view consistent with Brown et al.'s (1989) situated cognition.

Teacher/researcher perspectives and student voice. Concept definitions are based on an assumption of shared meaning in a community. It is not unreasonable to assume high school mathematics teachers and this researcher share some level of understanding of elementary mathematical functions with respect to their properties, concepts, and representations. Further, one might expect some reasonable amount of shared expectations about how elementary problem tasks might be approached. In both the curriculum and in the interview setting these same assumptions are likely to be salient for the interviewer, but may or may not be salient for the interview subject. Students are, after all, in the process of attempting to construct meaning for entities the interviewer personally takes for granted.

Another distinction between interviewer's and subject's viewpoint can be made by recognizing that interviews take place in a broader social context. The interviewer

and student interact around a set of tasks, which is itself a social interaction. Confrey and Smith (1994) suggest results need to be interpreted as a particular kind of discourse. Confrey (in press; Confrey & Smith, 1994) describes the interviewer as trying to understand what the student finds problematic and how the student is attempting to resolve it. At the same time, the student is constructing a problem in conjunction with what he or she views as the interviewer's expectations and also views the researcher as more expert. When interviewers analyze the data and attempt to build a model of what the students' actions, words, schemes, and concepts are like, their own understanding of the task influences the model. Confrey (1994) has labeled this as a distinction between perspective and voice:

> Voice is used to describe the model of the child and interviewing is an attempt to give voice to a child's ways of approaching a problem. Perspective surrounds voice, as the context from which the interviewer hears, probes and develops and sets new tasks. (p. 136)

Confrey claims what has been neglected in the past is that what the interviewer learns about the student evolves together with *the interviewer's own changing understanding of the subject matter*. Interviewers tend to describe changes built around their model of the student and neglect their own learning. In analyzing the data, Confrey suggests that the researcher needs to try to separate out these two different strands and to acknowledge the extent to which the researcher's own perspective changes. One's own mathematical knowledge evolves through interactions with the student and through analysis of student voice.

Thus, in approaching analysis of student interviews, it is necessary to assume that students are engaged in meaningful activity constrained by the broader social context, while at the same time trying to understand how the (possibly) unique ways they approach problem tasks (especially when their approaches differ significantly from those shared by mathematics educators) provide an opportunity to confront the

interviewer's perspectives of what constitute the shared assumptions of the mathematics community.

Representations

Definition. Defining representation is a complex issue. Many authors (See for example, Janvier, 1987a; Whitson, 1994) borrow from linguistics and attempt to link and contrast what is being represented, a referent (e.g., a real world situation), from signs used to represent it, and in the process contrast signifier (e.g., graph or table) with what is signified (e.g., rate of change). Vergnaud (1987) points out a difficulty when one attempts to connect these to learning and psychology:

> Symbols used in communication lie at the signifier level, whereas meaning lies at the signified level. This means that symbols are usually public (or social), whereas meanings are private (or individual). Very often they are not the same from one individual to the next one, for instance for the teacher and the student.
>
> The correspondence problem between the signifier and the signified is *the problem of the unicity of meaning*. (p. 229, italics in the original).

While mathematicians can create isomorphisms between a concept, such as a pattern of variation between two variables, and a physical (public) form to display it, such as a graph, as Vergnaud points out, representation is highly idiosyncratic to the individual. We cannot assume that the (public) signifiers do indeed signify the same things (the signified), for different individuals. In addition, meanings have many facets that must be taken into account when defining representations. Davis, Young, and McLoughlin (1982), cited in (Janvier, 1987b) define a representation as "a combination of something written on paper, something existing in the form of physical objects, and a carefully constructed arrangement of idea in one's mind" (Janvier, 1987b, p. 68). Janvier extends this definition by suggesting that a physical object need not be present. He gives verbal or language features equal importance to the public display, the object, and what is in the mind, defining verbal and language features as the links between these other elements. In addition, others point out that

"carefully constructed arrangement of idea in one's mind" is tightly connected to their activities and situations in which they have encountered it (Brown et al., 1989; Davis et al., 1982; Greeno, 1989; Tall & Vinner, 1981; Vinner, 1983). In order to interpret students' conceptions and their presentations of ideas, we need to take this complex framework for representation into account. I will call this broad, complex representation a student's personal representation, even though it includes both public parts as well as private meaning.

In order to communicate one's internal conceptualizations, one also must use external, public forms of representation. In this discussion, I will use the word representations to refer only to those that are in some external, public form. In high school, the primary public forms to communicate mathematical ideas are verbal descriptions of situations, graphs (including diagrams and pictures), literal strings of symbols, and tables. These forms, which could also be called problem representations or external representations, may be presented in various media such as talk, paper and pencil, or the screen of a computing device. In this study, this limited definition will be used for the word representation. I will use the word personal representation to refer to the broader notion of representation discussed in the previous paragraph, connoting both private and public forms of representations employed by an individual.

Understanding and representations. Concepts necessarily exist in some personal form of representation, and each personal representation highlights some aspects of the concept and downplays others. One aspect of the personal representation is likely to be an external representation as defined above. A particular situation may call up an aspect of the concept image associated with that personal representation and no others. For example, for an individual, the zero of a polynomial function (which has a concept definition) may be associated only with the process of setting a polynomial equation to zero and solving it, or it may be

associated with plotting a point on a Cartesian graph, and not be associated at all with the idea that the value sought is one for which the value of a continuous function changes sign.

In addition, at any point in time, even if we consider that which constitutes one's knowledge of a concept (their concept image) as situated in an activity, problem, or external representation, that knowledge must ultimately be sufficiently rich so that it can transcend individual situations to a degree that it is useful as a tool to do or learn mathematics. Indeed, the record of mathematics consists of a long history of the creation of new ideas through abstracting some aspects of a group of ideas.

In the case of elementary functions, if students are to have a good understanding of functions, they must develop understandings in multiple problem representations. They must develop conceptions that are in many ways independent of or go beyond those associated with individual types of representation that they have experienced in learning, and that are beyond characteristics of the specific functions experienced in the first place. For example, they must recognize that if a continuous function is increasing at different rates over two distinct intervals, then in both cases the dependent variable increases as the independent variable increases, but that the steeper of the two segments of the graph indicates a faster rate of change. Finally, they need to recognize that these facts are attributes of many function classes considered. While they are part of the properties of individual functions, they can at the same time be considered independent of individual functions as they are concepts that transcend single functions or function classes.

Further, students must be fluid in their movement between problem representations of the same concept, and be able to use the concepts to solve problems about situations where the functional relationships occur. They must be able to recognize similarities or connections between concepts in multiple representations and

exploit these ideas to create essentially new, more abstract ideas that coordinate and organize the concepts. For example, in advanced mathematics, students must develop a concept of function that Sfard (1987; 1991; 1992) calls structural and others (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky & Harel, 1992) call an object concept, where a function is treated as if it referred to an object-like entity, which can itself be acted on.

Finally, students must be able to recognize individual functions or specific representations as single instantiations of the larger idea. In other words, when students have a rich understanding of functions, they ultimately will be able to decontextualize their thinking about individual functions from specific problem representations and reason about properties of functions in a more general way.

This discussion does *not* imply, however, that students will not use specific instantiations of functions or specific representations to communicate their ideas. Rather, it means they will not be limited to reasoning only within a particular function class or a particular representational form.

Neither does this discussion imply that at any point one can say a student "understands functions" nor that there is some unitary idea that can be said to be the "meaning of functions." Understanding is seen as an ever-deepening, idiosyncratic process, taking many twist and turns. This understanding can be very rich, even if for a particular concept, the student's concept image is not very coherent.

The mathematics community, however, participates in setting generally accepted standards for what constitutes evidence of understanding in a particular set of circumstances. In schooling, these goals are usually expressed in terms of formally defined concepts along with their properties. These formal definition categories, such as quadratic functions, Cartesian graphs, or variables related by tables, also serve as organizing features to set goals for instruction, and in this case, for research.

Experience

The knowledge students bring to instruction is considered important, and this knowledge is related to the experiences they have had. Putnam, Lampert, and Peterson (1990) explain:

> . . . the learner plays an active role in interpreting and structuring the environmental stimuli. Rather than passively receiving and recording incoming information, the learner actively interprets and imposes meaning through the lenses of his or her existing knowledge structure, working to make sense of the world. (p.87)

Part of what we know is intuitive. Fischbein (1982) defines this kind of "affirmatory intuition" as that which consists of representations, notions, interpretations, or statements that are accepted directly as intrinsically meaningful while others are not. Intuitive knowledge serves to affirm the meaningfulness of certain representations or interpretations, that is, it provides a "basis of belief" (Bell, cited in Fischbein, 1982) for mathematical concepts. Fischbein claims that such knowledge is built up through:

> the frame of practical situations as a result of personal involvement of the learner in solving genuine problems raised by these practical situations (p. 12).

He claims that to develop an understanding of a mathematical concept, it is not sufficient to engage in verbal explorations nor blindly practice solving procedures. Rather "you must live the process" (p. 12).

For mathematics education this means that we must seek to provide experiences that allow the learner to build up intuitive knowledge, that is, to develop a "basis of belief" for mathematical concepts. Freudenthal (1983) claims that initially many new ideas, in this case, functions, are perceived as essentially separate things, that is, different phenomena. Only through experience with these different phenomena can we gain an active understanding and begin to see similarities and invent organizing schemes for our ideas, that is, through active engagement with the concepts prior to mathematically formal manipulation of the ideas we can expect to have a meaningful understanding that includes Fischbein's intuition.

The purpose of teaching is to assist in this process. Thus, we should provide experiences that allow students to develop a sense of the concept while developing at the same time the formal aspects of the concept. The curriculum materials chosen for this class, and which formed an experience base for the clinical interviews of the research, were organized to provide experiences for students to develop intuitive understandings of the concept of function.[1]

The Case of Functions

Conceptions of functions. Recently researchers have begun to categorize conceptions of functions that closely parallel understandings desired by mathematicians (Breidenbach et al., 1992; Dubinsky & Harel, 1992; Harel & Kaput, 1991; Kieran, 1992; Sfard, 1987; Sfard, 1991). Many mathematics educators characterize functions by building both on Piagetian distinctions between objects and processes (Piaget, 1970) and the dual mathematical views of functions as processes or, particularly in advanced mathematics, as entities that seem more like objects that can have actions performed on them. Sfard (1987, 1992) distinguishes conceptions of functions as "structural", that is, as static constructs that appear to the thinker as if they were objects, or "operationally" as processes rather than objects. Dubinsky and Harel (1992) summarizing conceptions given in Breidenbach, Dubinsky, Hawks, and Nichols (1992) have characterized four student conceptions of functions:

> *prefunction* . . . the subject does not display very much of a function concept. Whatever the term means to such a subject, this meaning is not very useful in performing the tasks that are called for in mathematical activities related to functions.
>
> An *action* . . . is a repeatable mental or physical manipulation of objects. . . . It is a static conception in that the subject will tend to think about it one step at a time. . . .
>
> A *process* conception of function involves a dynamic transformation of quantities according to some repeatable means that, given the same original

[1]See Appendix C for sample activities.

> quantity, will always produce the same transformed quantity. The subject is able to think about the transformation as a complete activity beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result of what was done.
>
> A function is conceived of as an *object* if it is possible to perform actions on it, in general actions that transform it. (p. 85)

These classifications seem to imply that function is a concept that students either have or do not have. Further, they do not seem to account for the complex nature of the concept of function (Janvier, 1987b). For example, a linear equation in slope-intercept form, a sequence of geometrical transformations, and a sequence of partial sums of an infinite series (a function that maps partial sums onto the natural numbers) are very different kinds of entities. Although in highly advanced mathematics it might be desirable that all these would be seen as functions, it seems that speaking of students as having an object conception of function is too closely related to mathematical goals and too far from student conceptions or actions, particularly those of high school students (See, for example, Leinhardt, Zaslavsky, and Stein, 1990). In addition, labeling some student conceptions as "prefunctional" denies existence of conceptions students do have while making it appear students themselves have somehow not joined a privileged group of those who "understand functions."

Dubinsky and Harel (1992; Schwingendorf & Dubinsky, 1990) do admit that very often in mathematics it is necessary to go back and forth between an object and process conception of the same entity. Yet it would seem better to adopt the language of Moschkovich, Schoenfeld, and Arcavi (1993), who prefer perspectives, rather than conceptions:

> From the *process perspective*, a function is perceived of as linking x and y values: For each value of x, the function has a corresponding y value. From the *object perspective*, a function or relation and any of its representations are thought of as entities--for example . . . in the plane as graphs that, in colloquial language, are thought of as being "picked up whole" and rotated and translated. (p.71)

Perspectives, then, can be applied to specific problems involving functions and bring

to light issues of representation as well. Using the example of two functions x + 3 and 4 + x - 1, Schwartz and Yerushalmy (1992) point out that different representations reveal different perspectives. Viewed as number recipes, the process nature of these two functions is highlighted and they are quite different. Plotted on a Cartesian graph, however, they would appear more entity-like, and they would be indistinguishable.

Other researchers have also employed slightly different dichotomies to characterize thinking about functions, including some closely connected to specific representations. Janvier (1978) contrasts students' understanding of local and global features of graphs (of functions). Goldenberg, Lewis, and O'Keefe (1992) speak of dynamic versus static interpretations of functions and Monk (1992) uses pointwise versus across-time conceptions for categories for students' work with functions. These three versions have characteristics that are both local or global (holistic) and dynamic or static (process/object).

In the process/object approach above, however, both conceptions entail viewing functions as correspondences between values. A process is a procedural means for producing correspondences between two values. A function conceived of as an object is a set of correspondences. For example, in teaching, a rule for a quadratic function could be seen as an algorithm for producing y's from x's or it might be seen as one instance (object) in a family of functions with a particular set of properties, say, of its graph.

An alternative, dynamic approach would be to consider variables covarying, as in across-time conceptions (Janvier, 1978; Monk, 1992). Confrey and Smith's (1994) division of teaching tactics into a *correspondence* approach and a *covariation* approach seems useful. They describe the correspondence approach as one in which:

> one initially builds a rule that allows one to determine a unique y-value from any given x-value. Thus, one builds a correspondence between x and y. This

> approach is emphasized in conventional algebraic notation in which we write: $y = f(x)$.
>
> A covariation approach, on the other hand, entails being able to move operationally from y_m to y_{m+1} coordinating movement from x_m to x_{m+1}. For tables, it involves the coordination of the variation in two or more columns as one moves down (or up) the table. (p. 137)

Thus, even a dynamic process of producing y from x is a correspondence. A covariation approach on the other hand, involves working with the patterns of change in the function as a whole. This approach means the concept of rate of change takes on a more prominent role. In addition, this approach parallels historical roots of functions (Euler) as a dependency between variables, rather than the Dirichlet-Bourbaki definition so prevalent in teaching functions in this century.

Finally, research has shown that even young children can use rate of change as a way to explore functional relationships (Confrey & Smith, 1993; Kaput & West, 1994; Nemirovsky & Rubin, 1991; Thompson, 1994) and so this approach may parallel student problem solving actions as well (Confrey, Smith, Piliero, & Rizzuti, 1991; Rizzuti, 1991). In researching students' thinking and activities in problems involving functional relationships we need to be alert, at a minimum, to all these possibilities to build models of or to describe their work.

Difficulties with learning functions. Mathematical functions have been considered a central concept in high school mathematics education since early in the century (Jones & Coxford, 1970). Current curriculum recommendations (Mathematics Framework for California Public Schools, 1985; Michigan Department of Education, 1988; Michigan Department of Education, draft) consider functions important as well. The NCTM *Curriculum and Evaluation Standards* (NCTM, 1989) recommend that increased attention should be paid to integrating functions across topics at all grade levels, 9-12.

A review of the research indicates that the concept of function is difficult for students (Leinhardt et al., 1990). Even after a fair amount of mathematical study,

students often have very limited views of functions (Artigue, 1992; Dreyfus & Eisenberg, 1983; Even, 1989; Harel & Dubinsky, 1992; Vinner, 1992). Misconceptions abound (Dreyfus & Vinner, 1982; Norman, 1992; Vinner, 1992; Vinner & Dreyfus, 1989). Many studies about functions explore aspects of functions that are in general more advanced mathematically than one would expect to find in the early high school curriculum. For example, the idea that the relationship between elements of the domain and range of a function can be of an arbitrary nature (Even, 1989) is less relevant for 10th graders than for college mathematics majors. Other studies, particularly in undergraduate mathematics, categorize functions either structurally as entities that can serve as mathematical objects to be operated on as in the algebra of functions or operationally as processes (Dubinsky & Harel, 1992; Schwingendorf, Hawks, & Beineke, 1992; Sfard, 1987; 1991; 1992). While these are worthwhile constructs, they seem less useful in the context of this study than in studies of more advanced classes. Some studies on understanding of functions focus primarily on the formal definition of functions (e.g., Markovits, Eylon, & Bruckheimer, 1986; Vinner, 1983).

When we consider only the three classes of elementary functions (linear, quadratic, and exponential), we have eliminated several categories of functions, such as those which are not continuous, those which are periodic in nature, or those for which domains are non-numeric or are defined in a piecemeal fashion. However, even when we consider only the literature associated specifically with these three classes of functions, we find students have a great deal of difficulty. Zaslavsky (1990) studied over 800 10th and 11th grade Israeli students from economically advantaged districts. These students had completed the study of quadratic functions within the previous 6-months. Zaslavsky focused on the relationships between the algebraic form of quadratic functions and its graph in a "naked context" (Monk, 1989), an algebraic or graphical representation without any additional concrete context

associated with it. Using nonstandard problems that required thinking about the abstract qualities of quadratic functions and translation tasks, she found many "conceptual obstacles" to students' understanding.

The five main obstacles were related to:

1. interpretation of graphical information (pictorial entailments); for example, thinking the graph is limited to that which is visible or expecting the graph to have vertical asymptotes.

2. relation between the quadratic function and the quadratic equation.

3. analogies to linear functions, for example, assuming two points are sufficient to determine a quadratic.

4. the seeming change in form of a quadratic function which has a parameter of zero, for example, not considering $y = x^2 + 4$ a quadratic.

5. over-emphasis on only one coordinate of special points such as the vertex.

These obstacles occurred in spite of the fact that a review of the standard curriculum showed that these topics were included.

Other difficulties that students encountered had to do with line of symmetry. In addition, students showed preference for the general form $y = ax^2 + bx + c$ and did algebraic manipulations on other forms ($y = a(x - x_1)(x - x_2)$ and $y = a(x-m)^2 + n$) rather than use graphical information supplied by them, suggesting a procedural approach that lacks an attempt to visualize the graph. Dreyfus and Halevi (1988) also studied the link between algebraic and graphical representations of functions with 10th grade students using a computer environment. They reported that often students had difficulty distinguishing between the vertex and line of symmetry. Some students could not proceed when the teacher asked them to check whether a particular point was on a parabola given by $y = x^2 - 3x + 1$. Although all students knew the parameter a in $y = ax^2 + bx + c$ determined the opening of a parabola, they had difficulties giving any more details. Students also felt a parabola had to have a standard shape and would not agree that a parabola existed between points if the

points did not appear symmetric. Finally, they thought the parabola terminated somewhere, rather than extending indefinitely.

Other studies showed students have difficulty with linear functions (Kaput, 1992), particularly slope, intercepts, connections with the Cartesian plane (Schoenfeld, Smith, & Arcavi, 1990), and the constant function (Markovits et al., 1986). Further, students attribute linearity to nonlinear functions (Dreyfus & Eisenberg, 1983).

There is little research on the understanding of exponential functions. Rizzuti & Confrey (1988) report preliminary results of a case study of one student's construction of exponential functions. Confrey and others (Confrey, 1988; Confrey, 1994; Confrey & Smith, 1993; Confrey & Smith, 1994) have explored exponential functions, and seem principally engaged in connecting notions of multiplication as they relate to exponential functions. They believe that students' difficulties with exponential functions are grounded in a limited view of rate of change as additive, such as that found in slope of a line or that defined for calculus. They found that when students work with exponential functions, that is, in multiplicative situations, they construct rates of change that differ from the additive rate of change. They suggest that rate of change is associated with a unit per unit rate, that multiplicative units differ from additive ones, and that students' construction of number is associated with their conception of unit. Thus, students have difficulty with exponential functions because they have not had appropriate experiences, such as with splitting, sharing, and similarity, that might help to develop multiplicative units and rates of change.

In general, although there are specific small studies (see, for example, Dugdale, 1989; Kaput, 1992; Schoenfeld et al., 1990), other than Zaslavsky (1990) and those just mentioned for exponential functions, there is very little research on specific classes of functions. Further, other than that mentioned, there is little

research in high schools dealing principally with topics usually covered in high school classes at the advanced level.

Functions and representations. The representations we make provide a means of highlighting specific features of a situation, while obscuring others. Thus representations have the potential for aiding problem solving by making critical features salient and displaying them in a way that makes properties more clear. Graphs are a fundamental means of representing functional relationships, particularly of continuous relationships, since they make salient many derived quantities, such as rate of change between variables, or the property that for certain values the function increases without bound.

Interpreting graphs, however, is particularly troublesome for students (Bell, 1989; Bell & Janvier, 1981; Clements, 1989; Janvier, 1978; Kerslake, 1981). For example, students sometimes interpret graphs as pictures of situations or fixate on one part of the graph while ignoring other relevant parts. They confuse height and slope (Janvier, 1978). Students tend to deal with functions piecemeal rather than dealing with them as a whole (Even, 1989).

One reason students cannot easily interpret graphs may be that they simply do not get much experience at it (Janvier, 1978). Eisenberg (1992) claims that students cannot work with visual diagrams because they lack experience. Similarly, Markovits, Eylon, and Bruckheimer (1988) recommend that students have more experiences with graphs.

Translating functions from one representation to another, a skill critical to making use of the power of representations during problem solving, gives students difficulty, too (Zaslavsky, 1990). Typically, students are taught to find pairs of numbers which satisfy a functional equation, plot these as points, and create a graph. They get little practice with paper and pencils in translating representations in the opposite direction. In the past, teachers have assumed that these experiences have

been sufficient (Vinner, 1992). Yet translation between two representations is directional since it involves, for example, a different process to go from a graph to an algebraic function rule than from the rule to a graph. In some cases, as described above, some processes involve intermediate translations as well, such as algebraic form to table to graph (Janvier, 1987c).

A limited number of researchers link difficulties with representations to students' preferences, abilities, or socio-economic status. Dreyfus and Eisenberg (1982; 1984) found while studying formal function concepts that low-ability, low SES, seventh- and eighth-grade students preferred tables, whereas high-ability students preferred graphical representations. Further, low-ability students did better on items that had a concrete situation to frame the task. Markovits et al. (1986) also found students of lower ability had an easier time dealing with tasks that were given in a story, rather than with tasks given mathematically. These studies raise questions about representations and the experiences that students bring from outside school. Some students may find certain representations more useful than others depending on their personal experience. It may be that some representations are inherently more complex than others, and thus there could be a relation between how easy it is to understand or to work with certain kinds of representations and how developed is a student's understanding of functions.

Problem situations have their own complexities, such as complexities in how the problem is stated, that can inhibit understanding within them (Bell & Janvier, 1981; Clarke & Helme, 1994; Janvier, 1981). Janvier (1981), citing an example of children who live near a harbor seeing tide as going up and down while those living near a beach seeing it as going in and out, pointed out the "remarkable diversity of personal perceptions and/or conceptions" that children bring to each situation and which must be taken into account in teaching/learning situations.

Mathematics educators have long been using various representations to present concepts. However, there are several problems associated with assuming that such efforts can be successful (Schoenfeld, 1986). First, the isomorphism between the representation and the concept represented may exist only when both are considered purely as symbol systems, such as in a problem in the decimal system and in a particular concrete representation system[2] such as Dienes blocks. It may also be the case, as will be investigated here, that good representation systems are powerful because they are so coherent in their own right and support reasoning within the representation to such an extent that they resist attempts to decontextualize the concepts or to move them to another representation.

Technological representations. The Curriculum Standards (NCTM, 1989) advocates extensive use of technology in the curriculum. With the increasing availability of computer graphing utilities at affordable prices (DiSessa, 1987), dynamic representations are within reach of the classroom. Graphs, for example, could be experienced as a whole, even prior to plotting of points. The use of these technology-based representations makes possible new instructional approaches that could alleviate some of the difficulties discussed above. Mokros and Tinker (1987), working with middle school students in a micro-computer based laboratory, found that students could learn to communicate through graphs. Through instruction, they successfully dispelled such student misconceptions as interpreting pictures as graphs and the slope-height confusion. Goldenburg, Lewis, and O'Keefe (1992) described using a class of function-visualizing tools called DynaGraphs, in which the user

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[2]Representation system is used here to imply a broader notion of representation, including a collection of representations and possibly operations on them. For example, Dienes blocks as a system can be used to represent arithmetic operations on numbers, whereas one configuration of blocks standing for a single number or being used to reason about a single arithmetic problem would constitute a representation in the limited way it is used in this study. Cartesian graphs would be another representation system, while *a* single Cartesian graph would be a representation.

dynamically manipulates the domain variable. The domain variable and its image under the function are represented each in its own space, for example, two variables moving in two separate lines. This technology allowed students to explore dynamically how variables depend on each other. Two ninth grade students successfully drew the graph of $f(x) = x + 4/(x-3)$ as a means of describing behavior they obtained strictly from observations. (See also Dugdale, 1989; Goldenberg et al., 1992; Rubin, 1991; Schwartz & Yerushalmy, 1992; Thompson, 1987; Yerushalmy & Chazan, 1990.)

While these technologies show great promise, these dynamic representations bring their own difficulties to teaching and learning. Goldenburg (1987), using computer software that dynamically linked symbolic and graphical representations of quadratic functions, showed that perceptual illusions and shifts of attention from one feature to another obscured some of the educational features that graphs were intended to elucidate. Kaput (1992), reporting on high school students who had had one or two years of algebra and who were working with a computer program to guess the function rule for a linear function while getting feedback after each response, found that some students attended only to the last response, rather than make use of the rest of the feedback given earlier and still visible on the screen. In addition, at this point we know little about learning and teaching with graphics calculators.

Context and learning. Another area of potential difficulty is the contextualized nature of learning. As more research is conducted in settings where a small number of subjects are studied in depth, problems associated with contextualization are being uncovered. Recently, Schoenfeld, Smith, and Arcavi (1990) conducted a microanalysis of one student's learning of linear functions on the computer. They found that the student's understandings of graphs of linear functions was much more contextualized than previously thought. Even in a large study, Zaslavsky (1990) found that students adhered to one representational context and did

not use the power of representations available to them. They preferred to stick to algebraic forms even when graphical forms would have provided easier solutions. Janvier (1981) found students' interpretation of graphs of situations depended both on problem statement context and the situation context being represented. Finally, Clarke and Helme (1994), in a small study of 9 adult students and their work and perceptions of both contextualized and decontextualized problems, found on average they were no more successful on one type task than the other, but that individual differences needed to be taken into account to understand the role of context. Students who performed better on contextualized tasks appeared to benefit from being able to understand more precisely the problem requirements and to visualize the situation. Those who had difficulty with contextualized tasks perceived that they did so because context made it more difficult to see the mathematics that was to be done, the task context was not familiar, or they had linguistic difficulties. Finally, although Clarke and Helme reported no clear relationship between solution method and context, in the area of directed numbers and linear equations they found that presenting the problem as an equation stimulated a formal approach, whereas a worded context task triggered an informal approach.

The Study

Purpose

The purpose of this study was to explore the conceptions about functions, functional relationships, and their various representations that students have when given an opportunity to study linear, quadratic, and exponential functions in an instructional setting that was designed to provide experiences recommended by the NCTM *Curriculum and Evaluation Standards* (1989). The setting was chosen to provide access to students of diverse ethnic backgrounds from disadvantaged communities in a such a classroom setting. The materials were organized around the concept of function. They introduced function classes informally by explorations with

situated problems, and emphasized connections between different problem representations (verbal descriptions or situations, symbolic, graphical, and tabular) for functions. Graphical calculators were used throughout. The study sought to determine what conceptions students developed about functions and functional relationships in problem situations and how students worked with the problem representations. It sought to describe connections students made across either functional or representational contexts and to discover if and to what extent problem representation or mathematical contexts contributed or hindered students' understanding of functions or functional relationships. Finally, it attempted to determine if there were other factors that contributed to students' thinking about functions or functional relationships in these representations.

The Question

The principal question informing this study was:

> ***WHAT SENSE DO STUDENTS IN THIS INSTRUCTIONAL SETTING MAKE OF FUNCTIONS AND TO WHAT EXTENT ARE THEIR CONCEPTIONS TIED TO SPECIFIC MATHEMATICAL OR REPRESENTATIONAL CONTEXTS?***

Mathematical context refers to the content topic, which is linear, quadratic, or exponential functions. Representational context refers to the mathematical forms that are used primarily to present these functions in their elementary forms: verbal descriptions, tables, Cartesian graphs, or literal symbolic forms. Media used for these representations were principally talk, paper and pencil (or chalkboard), and graphics calculators.

In order to answer the principal question, several other questions initially framed my thinking. They were:

A. In what kinds of activities and experiences with different representations of these three classes of functions do students engage?

B. What do students understand functions to be?

How do they talk about functions or functional relationships?

C. In what ways and to what extent do students use representations to present their ideas of functions?

Are students able to *create* symbolic, tabular, or graphical representations of problems involving functional relationships?

How do students *interpret* specific properties of functions as they are embodied *within* a particular representational or mathematical context?

Do they understand what constitutes sufficient information to define a given function within a representational context?

D. In what ways and to what extent is their sense of functions similar or different between representational contexts? mathematical contexts?

Do students' facility to translate between representational contexts differ between function classes?

E. In what ways do students' conceptions about functions transcend understandings within a given mathematical or representational context?

What kinds of *connections* do students make between properties and concepts of functions in different representational or mathematical contexts?

To what extent are they aware of the limitations of individual representational contexts in portraying properties of functions?

How are students affected by representational contexts when solving problems? To what extent do students consider alternative representations? Do they prefer using one representation over others?

I began the study with this set of questions, but with the understanding they would become more focused as I continued analysis. In addition, in the course of the study I expected new questions to emerge.

CHAPTER 2

METHODS

This chapter begins with a discussion of the research methodology, followed by data collection and analysis. Data collection is broken into four parts: the setting for the study, the nature of instruction, a description of how subjects were chosen, and data collection procedures, including a description of unexpected events that required adjusting the study. The section on analysis includes a description of the research process and how emerging themes in the data led to a revision of the research subquestions. Finally, the chapter concludes with a discussion of the subjectivity of the researcher.

Methodology

This study combined in-depth exploration of students' thinking in problem situations in a laboratory setting with fieldwork methods (Jacob, 1987) of making detailed observations and recording of students' activities and interactions in the classroom. The primary focus of the study was on data collected in a sequence of clinical interviews, with classroom data providing the backdrop from which to make sense of students' activities in the interviews.

Students are assumed to be diverse in their thinking and in the meanings they give to problems, situations, and events. Research on children's learning of mathematics indicates that developing understanding must take into account *how* learners think about a problem or a concept (Romberg & Carpenter, 1986), so special attention must be paid to the processes that students engage in. Clinical interviews provide in-depth information necessary to explore how individual students think about specific concepts and to access students' prior knowledge. Interviews allow the flexibility to pursue students' understanding of underlying concepts and to explore the processes by which answers are formed.

The interviews presented students with specific mathematical and problem situation contexts that were closely related to activities and mathematical concepts that were part of classroom activities. (See Appendix A for a complete set of questions.) Several of the problems were similar to those given in class. Others were different from class tasks or had a considerably more complex task structure designed to probe understanding. Clinical interviews that use tasks or situations that a student finds problematic because they do not meet expectations as to what should constitute the problem or what the expected path of solution are particularly good at getting at strongly held convictions (Ginsburg, 1977). When a result was surprising or a solution required a struggle, student discussion was expected to reveal which concepts and beliefs students held tenaciously and which they readily relinquished (Confrey, 1980). Such problems revealed both students' ways of thinking and their prior knowledge, which was relevant to this study.

Finally, research has shown that social and instructional contexts contribute to what is being learned (Brown, Collins, & Duguid, 1989; Cobb, 1986; Cooney, Grouws, & Jones, 1988; Greeno, 1987; Steffe & Cobb, 1988). To make sense of what students did in interviews, it was necessary also to explore their activities in the classroom. Since learning is assumed not to be static, studying it over time gave insights into its nature that could not be obtained at a single point in time, and in addition, provided time for me to test budding hypotheses about emerging patterns in the data. It provided an opportunity to observe the constructive processes of interaction between the student as a cognitive agent and the situations in which they acted (Confrey, 1994; Greeno, 1989). Finally, this classroom as a research site provided a social context more like that of a standard classroom than does the laboratory and thus provided a richer basis for forming hypotheses and models of student understanding than could be done from interviews alone.

Data Collection

The Setting

The Program

The study took place at The Regional Center Upward Bound Program in Math and Science, a 6-week summer program at a midwestern university focused on increasing the mathematics and science skills of promising high school students who were either economically disadvantaged or who would be the first in their family to attend college. In addition, the program was intended to encourage students to select careers in mathematics and science. Students came from several different states.

The overall schedule consisted of one week of community and trust building at a camp designed for that purpose, followed by five weeks of academic instruction. The mathematics and writing classes met in 2-hour blocks from Monday through Thursday. Physics and computers met twice a week in 2-hour blocks, while anatomy, biology, chemistry, and microbiology met once a week for one 2-hour block. The final day of class included a Science Olympiad. Interspersed throughout the program were both day and week-end fieldtrips to area businesses, research sites, and cultural events.

Students were in class all day and usually had study hall for an hour and a half after supper. On some evenings study hall was started earlier so students could attend some cultural event. If students did their homework (for four classes), they did not get much sleep. About midway through the program, the director asked teachers to reduce the amount of homework because it was too heavy.

In order to attend the Center, students had to be (with few exceptions) entering the 10th or 11th grade, have completed at least one summer and one academic year in their home Upward Bound Program, be reading at grade level, be interested in or have the potential to do well in an accelerated math and science curriculum, and have already taken at least one year of mathematics at the algebra level or above.

There were 37 students in the program. They were divided into 3 sections of approximately 12 each by the similarity of their mathematical preparation, based on their mathematical backgrounds and a written pretest of mathematical competencies. Each section formed a unit and stayed together as a class for each of the eight subjects studied. This study focused on only one section, from which five focal students were initially selected for intensive study.

The Instructional Context

I had principal responsibility for instruction and provided both the curriculum materials and the broad framework for instruction. Individual decisions about instruction for each week, however, were decided jointly by the two mathematics teachers in the program and me. We met in several planning meetings before the program began and met weekly once classes were under way. A detailed list of assignments can be found in Appendix D.

Philosophy. Broad instructional goals were based on a hierarchical model of what it means to know mathematics (Schram & Wilcox, 1988). At Schram and Wilcox's highest level of knowing are (a) using mathematical knowledge to make sense of new problem situations, (b) engaging in mathematical inquiry, (c) making connections among mathematical ideas, and (d) representing mathematical ideas in meaningful forms. They define authority for knowing as both internal to the individual and collective in that individuals evaluate the results of exploration based on standards of the mathematics community.

Schram and Wilcox's model also includes three hierarchical levels of orientation for students acquiring the record of mathematical knowledge. At the highest level is (a) actively participating in the construction of mathematical knowledge, and (b) exploring, making conjectures, looking for relationships, reasoning and validating arguments about problem situations. Instruction in this class took these activities as goals for tasks in which students should be engaged.

The class. Mathematics class met for five weeks from Monday through Thursday for a 2-hour block, except for the last day, which was given over to Science Olympiad. Each student had access to a Texas Instruments TI-81 graphics calculator during class and shared access during study hall. The teacher used a computer for demonstrating a simulation the first day of class, and for the remainder of class used a projection system for the TI-81 calculator. Homework was assigned daily and study hall was held Monday through Thursday evening for one and a half hours, except in a few instances when students had some activity, such as going to a play. After Program teachers were asked to reduce the homework about midway in the program, much of the work had to be done in class with short homework assignments.

The mathematical content. The mathematical content of the course included linear, exponential, and quadratic families of functions, in that order. Other functions such as absolute value, quotient, or polynomial were included only in a minimal way. The representational forms that were used were verbal descriptions of situations and both paper and calculator versions of symbolic rules, Cartesian graphs, and tables of data. Calculator and paper forms of scattergraphs were used for curvefitting. Situations were verbal descriptions of problems involving functional relationships or verbal descriptions to provide the setting for symbolic, graphical, or tabular representations of functions. Applications were drawn primarily from physics, economics, chemistry, health, and biology. Modeling real world phenomena was a theme running through the materials.

In addition to studying properties and concepts of prototypical functions in each of the three focal function classes, students used various representations of functions to solve equations and inequalities. Students studied families of functions and the role played by parameters. Particular attention was paid to recognizing or interpreting the same function concept in algebraic, tabular, symbolic, and graphical representations.

The class covered large portions of chapters 1 through 5 of the Computer Intensive Algebra (CIA) materials (Fey & Heid, 1989); Algebra Activities 41-43, fitting lines to scattergraphs from UCSMP (1990); and Activities A1-A3, B2, B3 of *Language of Functions and Graphs* (LFG) materials (Swan, 1985) concerning interpreting points, interpreting global features of graphs, matching graphs with situations, finding exponential situations in situations and exploring quadratic functions in situations. The teacher generated several exercises to explore the role of parameters in families of linear and quadratic functions and a table producing program for the calculator. (See Appendix C for examples of lesson tasks.)

Instruction. Instruction was problem centered and conceptually based, with the function concept first introduced through problems in situations in tabular representations. For example, on Day 1 students were introduced to variables related by tables. They were given a situation with a table of values relating age and average height of females. As part of the discussion, they were told:

> As you can see, the average height changes as age increases. In mathematical terms, we call age and height numerical or quantitative variables and say that
>
> height **depends** on age
>
> or
>
> height is a **function** of age.
>
> Because height depends on age it should seem reasonable to think of age as the **input variable** and height as the **output variable** in the relation. (CIA, pp. 1-7, 1-8, emphasis in original)

After exploring functions in general, translating from tables to graphs, and interpreting tables, graphs, and symbolic forms for instances of functional relationships, linear functions were used to introduce the process of producing function rules from tables. All function classes were introduced via situations. For quadratic functions, for example, students were introduced to the function class by exploring a situation using stopping distance on a dry asphalt road of a car of a

particular mass as a function of speed, $d(s) = .005s^2 + .14s$. The second exploration involved Orioles baseball player Cal Ripkin and a high pop-up with height of the ball as a function of time. (See Appendix C, CIA, pp. 4-2 to 4-5.) Function notation was used from the first day of class to express individual correspondences in a relation.

Problems were used to explore attributes of a typical function in a class of functions, such as the shape of its graph or the pattern of change in a table. In addition to the Computer Intensive Algebra activities, both exponential and quadratic functions were explored using activities from *Language of Functions and Graphs* (Looking at Exponential Functions, p. 37, and Finding Functions in Situations, p. 35), respectively. Additional concepts associated with the functions in a given function class were explored through multiple representations of the concepts by creating, interpreting, and translating between representations, both in applied problems and in naked contexts. Guess and test strategies were used to solve equations and inequalities for each function class. Explicit attention was paid to names of function classes.

Students were given problems specifically designed to make links between representations. For example, one question in an exploration involving a situation with a table of Dawn's age in years and her height in centimeters asked, "What information is conveyed by "D(18) = 183"? (CIA, p. 1-9). This problem connected situated tabular data to the symbolic representation for a single instance of the functional relationship. Some problems were exploratory so that more than one correct path could be taken and more than one concept could arise naturally. Students were encouraged to work together to share tentative solutions. They were asked to make conjectures, justify their thinking, and to write about their reasoning. Wherever possible, students were given problems designed to bring to light common misconceptions so that they had an opportunity to puzzle about conflicts and to resolve them (Bell, 1986; 1989).

Modeling. Modeling was a constant theme of instruction. Each function class was introduced as a means to model situations. The teacher emphasized that models were used to make predictions. Students were introduced to the modeling concept and smoothing data in a problem situation involving growing 4 plants in 2 light situations over 7 days. Next, they gathered their own data, comparing rebound height with dropping height for various balls, created their own linear functions to model their data and answered questions from the model. On one other occasion they gathered their own data by timing pine racing cars running down ramps of various inclines. Other times they were given raw data and asked to fit a curve to it. For linear functions they "eyeballed" a line to fit the data, chose two points and created an equation. For quadratic and exponential functions they used guess and test for parameters on prototype functions.

Calculators. Calculators were used daily throughout the class and students seemed to engage in the activities along with the teacher, whose work was displayed by the calculator projection device. Calculator activities included numeric computations (evaluating functions, checking hand computations, and guessing and testing solutions for equations and inequalities), graphing functions, finding graphical solutions to equations involving functions by finding and interpreting curve intersection points, creating scattergraphs of data, guessing and testing curves to fit them, and the use of a program to create a table of values for a function given the beginning and end of an interval on which it was to be evaluated and the step size for incrementing x. When students had explored curvefitting by guess and test, the teacher introduced automatic curvefitting on the calculator, and students used it to compare their own choices for parameters. Finally, the calculator was used to explore vertical and horizontal translations for both linear and quadratic functions in naked context. Except for creating scattergraphs and calculator curvefitting, all

translation activities on the calculator relied on going from a rule to either a graph or a table.

The teacher. The focal class was taught by an experienced high school mathematics teacher, Mr. Bailey (not his real name), who had taught for six years in the standard Upward Bound Program at this same university. He had used the UCSMP (1990) materials in his own school for the first time that year and had taught with the TI-81 calculator. Both the *Computer Intensive Algebra* and *Language of Functions and Graphs* material were new to him, but he was eager to learn new ways of teaching.

Site Selection

I selected this setting because it gave me a unique opportunity to study students who came from ethnically diverse, economically disadvantaged backgrounds and who were intending to go to college. In addition, I could study students in a classroom setting where specific, exemplary materials that attempted to implement the NCTM Standards were being used. Further, the setting gave me an opportunity to influence the mathematical content to focus on three function classes by using existing curriculum materials. Thus, my role was not confounded by the need to create new materials.

The setting choice was ideal in the type of students participating, my access to observing a classroom using particular curriculum materials, and my being able to influence the mathematical content. The setting would have been better had the teacher had experience with the specific curriculum materials, but Mr. Bailey seemed comfortable teaching all segments. Further, the classroom differed from standard classrooms because there was no testing. Even though students received a grade at the end of the program that was sent back to their home Upward Bound Programs, any influence from testing and grading procedures that might occur in standard classrooms was largely absent for most students. The site selection also influenced

my access to information about students' prior knowledge. I knew titles of classes and students' grades on prior work but had little access to the nature of it. Thus, accessing their previous mathematical experience was difficult except through their self-reporting and through inferences I could make when they displayed school-like algorithms in interviews. Finally, the 40 hours of instruction were spread over 5 weeks, rather than what likely would be 8 weeks during a regular school year. This timeframe afforded less time than normal for student reflection and homework between topics of study.

Subject Selection

I collected data at three levels: the program, the section, and five focal students.

Program Level

All students in the program took a written pre-test. (See Appendix B for a copy of the test.) Since students had differing amounts of mathematics and were of different ages, I had planned to use this survey to assist me in placing students in the three program classes (of which I studied only one) to try to minimize diversity of mathematical experience within any one class, while giving me some indication of their prior mathematic knowledge about functions and their representations. Since it was administered to all students at both the beginning and the end of the program, I had hoped to use it for both before and after program comparisons for individuals and to help define typicality of the focal group's responses.

Unfortunately, many students did not take the initial survey seriously. It was originally intended to be given on Friday morning after the students had spent a week in the program, but the director of the program moved the time for the first survey to Monday so that Friday could be used for orientation to classes. I agreed to this switch since it seemed important for the students to have an orientation close to class starting time and this time frame gave me more time to ponder the responses and

potentially make a more careful selection of students into three classes. This decision may have been a mistake, however, since students took the survey on their first morning on campus just before boarding a bus to leave for 3 days at the trust-building camp. They were very excited about meeting new people and about the coming trip; they were focused on the coming day, not the survey. Even though the Academic Program Coordinator and I told the students the survey was important in trying to form classes based on similarities in their mathematics experience, most students did not focus on completing the survey, failing to answer at least half of the questions. Once one student began to make noise and talk (after about 15 minutes), a large number of students seemed to take this as permission not to work on the survey. Most stopped working long before the allotted time both was up. While I had been concerned about how students might react to some of the questions because they were nontraditional, their lack of serious consideration to the survey went beyond what could be explained in this way. The posttest survey was administered by the mathematics teachers, and students in the focal section took it seriously, completing or attempting most items.

Section Level: Assigning Class Section Membership

I was not able to use the initial survey as a measure of typicality, except in perhaps the number of questions that were omitted, nor did the survey provide a general basis of before and after comparison data. I was, however, able to use the results somewhat in class section formation and to some extent in choice of focal students. To form tentative sections I had used information about high school mathematics classes students had taken, along with grades received. I thought of the sections in terms of levels of experience and performance based on grades: novice, moderate, and advanced. After the survey I was able to use students' participation on the survey to confirm or disconfirm tentative assignments to the three sections when I was undecided about where to place a student. The survey was also used in a

minimal way to insure that the focal section had at least some students who had attempted many of survey question, but overall, this strategy turned out not to be as useful as originally planned.

At the same time, I attempted to balance the groups for gender and ethnic make-up. There were 37 students in the program, 12 boys and 25 girls. There were 2 Asian-American, 3 Hispanic, 11 non-Hispanic Caucasian, and 21 African-Americans. The focal section began with 12 students: 2 African-American boys, 4 African-American girls, 2 each non-Hispanic Caucasian girls and boys, and 2 Hispanic girls. By the end of the program the following students had dropped out of the focal section: 1 African-American male, 1 Hispanic female, 1 non-Hispanic Caucasian female, and 1 Caucasian male.[1]

Individual Level: Choosing a Focal Group of Students

During the first week I observed class daily, and I interviewed all students in the focal section for approximately 30 minutes each. At the end of that week, I listened to audiotapes of all interviews, made notes about what students did, and chose focal students based on interviews, my class observations, and to a lesser extent whether they had made a serious attempt on the pretest survey. I aimed for greatest diversity possible by sex, ethnic background, and on heterogeneity of apparent mathematical understanding and experience. I considered whether a student was likely to discuss problems openly with me and attempted to maximize through my choices the variety of kinds of responses that students gave in class, from those responses that were rich mathematically to those that were most puzzling to me.

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[1]Students found the program extremely demanding as they were in class for 8 hours, had study hall at night for only 1.5 hours, and sometimes had other activities at night as well. Yet they were expected to prepare for class. In addition, since students came from geographic areas as dissimilar as inner city and very rural areas, some students had difficulty adjusting to being in such a diverse environment, away from home for six weeks, etc. In addition, some students admitted they were pressured by local Upward Bound directors to participate in this program, which was in its first year of existence. Ken, one of the five focal students dropped out after the third week.

Finally, I chose 2 girls and 3 boys (3 African-Americans and 2 non-Hispanic Caucasian students) in the focal group. I call them Sam, Toni, Edy, Bob, and Ken, not their real names. In addition to the general guidelines given above, I had specific reasons for choosing or not choosing particular students. I rejected one student, who was perhaps the most vocal class participant and ultimately won the teacher's choice for outstanding math student in this class, because my perceptions during her initial interview indicated that she might have trouble trusting me. I felt it might take too long too develop rapport with her to make interviews constructive. I decided against including an Hispanic girl because she had missed class during the first week and since I had no idea whether she was ill and was afraid to risk what might be a recurring pattern. I was particularly interested in Sam because he seemed assertive about his answers in class and in the short time I had observed class, he seemed to think about the mathematics in an unusual way. I chose Edy because she seemed to be the weakest student mathematically. I chose Ken because he seemed to be the strongest mathematically, often giving conceptual answers. Bob and Toni were chosen in part to help create within the focal group a balance of characteristics named above. All students had had two years of high school mathematics, either two years of algebra or algebra and plane geometry.

The Data

Data consists of audio and video tapes and fieldnotes of all class meetings and copies of student homework sheets, as well as all curriculum materials used, memos, announcements, and so forth, and fieldnotes of meetings with staff and other teachers related to the program itself, including preliminary meetings with teachers to plan the curriculum, and of the few times I was able to attend study hall to observe students.

Interview data consists of one audiotaped interview for each member of the focal section, and a combination of audio and video tapes of four additional interviews of approximately 45 minutes each for each of the 4 focal students, and 3 additional

interviews for Ken, who dropped out. Interview data also includes written documents used during problem solving, fieldnotes taken shortly after the interview, and verbatim transcriptions of all interviews for the 4 focal students, including descriptions of actions on videotapes, such as pointing.

During data gathering in class, I videotaped the class myself and wrote fieldnotes. I chose not to have a video operator, as having another "outsider" seemed too intrusive. A college student staff member who was responsible for supervising study hall sometimes attended class, but sat in the corner and students seemed to ignore him completely. Outside of class I made notes of anything that seemed pertinent and interesting, but because I had to interview during evenings, I was unable to expand my classroom fieldnotes on a regular basis. I duplicated homework sheets daily.

During interviews I generally did not make notes to avoid being intrusive and to allow greater concentration on conducting the interviews. After the interviews with focal students, either on the same night or within a day, I watched the tapes and made notes about the sense of the discussion, things I was puzzled about, general observer comments, methodology that needed checking or might influence subsequent interviews, and anything really striking the student had done. I was quite meticulous about this as I was worried about losing activities not visible on tape such as what the student was doing on the calculator. The first interview was audiotaped, and interview 2 for both Toni and Sam was audiotaped, rather than videotaped, because of a miscommunication with the camera delivery person.

During data collection I conducted ongoing analysis through modified analytic induction (Bogdan & Biklen, 1982; Merriam, 1991). As research progressed, I looked for patterns and developed rough definitions and explanations for particular phenomena. Throughout the data collection and analysis, I sought to confirm hypotheses and looked for discrepant cases in an attempt to disprove hypotheses. I

attempted to use each negative case to redefine the phenomenon or reformulate the explanation. As a whole during data collection, analysis was constrained by time available, and much of it took place during transcription of notes and tapes.

Analysis

Process

As discussed above, analysis began during data collection with my expanding fieldnotes, viewing tapes, writing analytic memos to myself of patterns and themes I observed. Most analysis, however, took place during and after the tape transcription phase. I personally transcribed all tapes because I felt it important to record my observations and impressions as I went. Although I employed a typist to make preliminary transcriptions of the audio portion of the several video tapes, I completed the final versions, including detailed descriptions of students' writing and gestures, which entailed several passes at each tape. During that time, which spanned many months, I wrote interpretations, reflections, and summaries of students' actions. Over time, it became clear that I needed to revise my questions because of invalid assumptions built into the questions themselves and because they did not seem adequate to describe student activity. In addition, new issues arose that were not addressed at all.

Evolution of Research Questions

Original Research Questions

I began the research with the following principal research question:

> *WHAT SENSE DO STUDENTS IN THIS INSTRUCTIONAL SETTING MAKE OF FUNCTIONS AND TO WHAT EXTENT ARE THEIR CONCEPTIONS TIED TO SPECIFIC MATHEMATICAL OR REPRESENTATIONAL CONTEXTS?*

Mathematical context referred to the content topic: linear, quadratic, or exponential functions. Representational context referred to the mathematical forms that are used primarily to present these functions in their elementary forms: verbal descriptions, tables, Cartesian graphs, or literal symbolic forms. Media used for these

representations were talk, paper and pencil (or chalkboard), and graphics calculators. I began with several subquestions to frame my thinking. They were:

A. In what kinds of activities and experiences with different representations of these three classes of functions do students engage?

B. What do students understand functions to be?

How do they talk about functions or functional relationships?

C. In what ways and to what extent do students use representations to present their ideas of functions?

Are students able to *create* symbolic, tabular, or graphical representations of problems involving functional relationships?

How do students *interpret* specific properties of functions as they are embodied *within* a particular representational or mathematical context?

Do they understand what constitutes sufficient information to define a given function within a representational context?

D. In what ways and to what extent is their sense of functions similar or different between representational contexts? mathematical contexts?

Do students' facility to translate between representational contexts differ between function classes?

E. In what ways do students' conceptions about functions transcend understandings within a given mathematical or representational context?

What kinds of *connections* do students make between properties and concepts of functions in different representational or mathematical contexts?

To what extent are they aware of the limitations of individual representational contexts in portraying properties of functions?

How are students affected by representational contexts when solving problems? To what extent do students consider alternative representations? Do they prefer using one representation over others?

These questions were organized around content from the curriculum for which it is important for students to gain understanding. At the time the questions were developed, I had created categories that made sense to me for exploring students' understanding and activities. Broadly, Questions A and B were about functions, function concepts, and the three classes of functions. Question A was about students'

understanding of the principal function classes they studied in this program. Question B, where I had hoped to capture ideas connected to those prevalent in the literature such as function as an input-output machine, an equation, a process, or a relationship, was about how students view functions. Question C was intended to get at how students use various representations to represent their ideas of functions. Question D was to compare across representation contexts and function class contexts to describe similarities and differences, thereby providing data for question E, which dealt principally with contextualization issues.

Reshaping Research Questions

During analysis it became clear that I needed to revise the framing questions. They were originally written to provide evidence for how students talked or reasoned about functions and representations that could be compared and contrasted to display evidence of decontextualized or contextualized thinking. Yet during the early stages of analysis, even though the mathematical content of the interview tasks was functions, it was apparent students did not talk much about functions. They seemed actively engaged in trying to make sense of a problem or its representation and in solving the problem in the best way they knew how, but they often seemed focused more on other aspects of the problem solving situation and the representation itself than on functions per se. In addition, students treated situated problems that had realistic contexts quite differently from non-situated problems. Further, there were times during interviews when there were clear mismatches in communication between what I, the interviewer, thought I had asked and the question that students answered. So new issues had come up, and new questions would be needed. While the principal question still seemed appropriate, I needed an alternative framework in order to replace the subsidiary questions.

Functions, functional relationships in problems, and representations. I had to revise how I thought about functions and representations. While I was transcribing

the tapes, I had kept my research questions in the back of my mind without direct reference to them. At the time they were written I had tried to make them, in the spirit of ethnography, as flexible and open-ended as possible to allow students' work to shape what I saw. I was genuinely committed to a belief that students engage in purposeful activity and part of my role as mathematics teacher is to respect that activity and build on whatever conceptions students have in an attempt to help them develop the formal concepts that I understand to be taken-as-shared in the mathematics community. All through the transcribing, I thought I had a strong sense of what the framing questions were, so I did not revisit them until the transcription was almost done. When I reread my research questions, I was shocked since they seemed to have little to do with the students' activities because the questions had a strong underlying assumption that students would somehow be engaged in working on functions per se, that they would have some coherent conception of function or of entities that had enough similarity to compare and contrast across either function or representation contexts. While I expected student activities might be tentative and their knowledge fragile, I was not prepared for the impact of this mismatch with my current thinking. Clearly the questions would need major revision.

I came to think of this disjuncture between my subquestions and what I was seeing in the data as a mismatch between teacher/curriculum developer/researcher perspective and student perspective. A teacher naturally has certain goals for students to develop an understanding of functions and their representations that mirror what he or she takes as shared in a mathematics community. It is natural and expected for teachers to set teaching objectives that will assist students to come to share these assumed meanings. Likewise, it is natural for researchers to begin with their own way of thinking about the content area to frame questions they consider important to answer. Attending only to mathematical concepts and public presentations of them (representations), however, ignores the interpretive activity that students engage in

when constructing meaning (Cobb, Yackel, & Wood, 1992). Confrey (1992) maintains we should recognize that in research "teacher perspective" surrounds "student voice" as models are built about student activity. Since functions and representations were the focal issues of this research, research questions did indeed need to be about functions and function concepts, representation and representation concepts, and issues related to the effect of various contexts on student thinking. Questions needed to address what activities students engaged in and how they talked about or worked with functions and representations. However, the language of the subquestions fit better that aspect of the teacher perspective that is related to setting mathematical goals of instruction and associated, desired activities, rather than focusing on activities, interactions, or ways of reasoning that students actually seemed engaged in. The questions harbored built-in assumptions about student activity that were invalid because they were inconsistent with students' actual activities.

For example, questions such as C, "In what ways and to what extent do students use representations to present their ideas of functions?", implies students had some specific conception of function and that the task they were engaged in was representing that conception. Students did not appear to have such a conception nor to be engaged in such tasks. Although teachers may characterize a problem as about functions, this seemed not to be the case for these students. Therefore I needed to remove the implication in questions that students' focus of attention was on functions or function concepts, rather than on some idea (possibly) related to functions, the representation itself, or something else entirely, such as interaction with the interviewer. So I decided to broaden the scope of the questions to include activities where awareness of function was not required. To distinguish categories for awareness of functions I reserved *functions* to apply to cases where students explicitly referred to functions or function concepts or where I could infer such awareness from their activities. All other activities with functions, I called *functional relationships in*

problems. Finally, there were times when students worked with concepts that I would call function concepts, but when I could not infer that students were aware of these being function concepts. I called these activities *working with functions indirectly* or *having implicit conceptions of functions*.

Question C's subquestion, "Do they understand what constitutes sufficient information to define a given function within a representational context?" implies a function was viewed whole as an entity. Yet students often did not attend to a function or its representation as a whole, but rather attended to individual or isolated aspects of problem representations, the situational context of those problems that had realistic contexts, or on aspects of problem solving that seem more related to interactions with the interviewer (teacher) or expectations of what it means to be in a mathematics class. Clearly the old framing questions were no longer appropriate.

At about the same time, I began to revisit the literature. A principal theme running through the function literature (Harel & Dubinsky, 1992; Kieran, 1992; Moschkovich, Schoenfeld, & Arcavi, 1993; Sfard, 1991). is the distinction between viewing functions operationally (as processes) or structurally (as objects). Since these are significant characteristics of mathematical knowledge, it seemed important to keep them in mind; however they did not seem to match student activity. I needed to find a better way to describe student conceptions.

Because my principal question was about functions and representations and since students seemed to have little awareness of functions, but attended principally to the representations, I was reminded of Mason's (1987) attention *on* symbols and *through* symbols. In discussing what symbols represent, he distinguishes attention *on symbols* themselves from attention *through symbols* to a larger whole. He describes a process in reading a piece of mathematical text whereby attention moves from being fully on each symbol to skipping over details noting merely the shape of the calculations. Thus attention is split between the detail and the shape of the whole,

while you guide yourself toward a goal. I saw student attention as being on the representation, rather than through the representation to function concepts, so it seemed important to characterize how students talked and reasoned about representations themselves. I tried to identify *to what students attended*--what constituted their "pieces" and "wholes". I looked for both processes and objects that could be connected to the mathematics, as well as whether students' attention was on, within, across, or among representations.

Student sense-making or lack thereof had been one of my early organizing categories, but students' sense-making seemed less like that in the function literature and more related to that reported on multiple representations and problem solving in rational number research, where Lesh, Post, and Behr (1987) reported it as being piecemeal, evolving, and plural. They described the acts of representing as plural in two ways. First,

> solutions often are characterized by several partial mappings from parts of the given situation to parts of several (often partly incompatible) representational systems[2]. Each partial mapping represents a "slice" of the problem situation using only part of the available representational system. It is not a mapping from the whole "given" situation to only a single representational system. (p. 38)

They described plural in a second sense, as a student beginning a solution in one representation system and translating it to another and then to yet another. My sense was that in my study students' reasoning and problem solving fit this description well and seemed a long way from the coherence that was implied in my original research subquestions. So I had to find a way to describe their activities in sense-making.

Using what I had seen in the data, I tried to think about categories that would work. The categories: *connections, reasoning, problem solving* and *communication*,

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[2]Lesh et al use the term representation to mean external embodiments of students' internal conceptualizations, specifically referring to static pictures, written symbols, spoken language, real scripts, and manipulative models. Thus, their restricted use of the term is similar to mine.

from the NCTM *Curriculum and Evaluation Standards* (1989) seemed particularly relevant. Evaluation standards in the same document helped me think about what would count for evidence of these four categories of sense-making, particularly in trying to identify what tasks students were engaged in and connecting them to these categories.

Connections seemed important because they could provide evidence of decontextualization of concepts, a major goal of the research. Connections also seemed important to look for since my perception at that time was that students were making so few of them. *Reasoning* was important because some concepts stood out as being particularly salient and useful to students, and there were patterns in reasoning, particularly about graphs, that both contributed to and obstructed students' success in problem solving. *Problem solving* was at the core of student activity. Students brought a variety of strategies and resources to problem solving, including ones not related to mathematics. Students' *communication* with me was primarily through talk, with a lesser amount written or done on the calculator. From their talk I could infer what concepts were salient for them and how they thought about and solved problems about functions, functional relationships, and representations. Yet it was not always possible to separate mathematical communication from that which was nonmathematical. Sometimes students used interaction with me as a problem solving strategy as they tried to connect what they were doing in the interview tasks with nonmathematical tasks and their knowledge of everyday life and of what it means to be in a mathematics class or in school. In addition, there were communication mismatches between the students and me.

Because these issues of student sense-making both cut across functions and representations and introduced non-mathematical topics as well, in an attempt to address what was reflected in student work in interviews, I created a broad,

cross-cutting question about functions and representations:

> **What *conceptions* do students have about functions, functional relationships in problems, and their representations?**

This question was accompanied by subsidiary questions:

> How do they *talk* or *reason* about functions or functional relationships in problems? Which concepts are most *salient*? On what do they focus their *attention*? What resources do students bring to the *problem solving* process? What kinds of *connections* do they make between properties and concepts of functions in different representational or mathematical contexts?

Situations. During analysis it became increasingly obvious that situated problems were quite complex and raised all sorts of issues about their role in the study of functions. Students used situations skillfully to support their reasoning in some cases, but in others their problem solving was severely hampered by them. Their difficulties raised questions about the nature of the task and the concept of modeling.

In addition, my thinking about what constitutes a situated problem got more complicated. I began the study using verbal descriptions synonymously with *situations* or *situational context*. Yet during the study, it became clear there were other kinds of verbal descriptions. For example, one interview problem described a situation where three students were each suggesting a graph to match a given table, but the numbers did not refer to any specific quantity. That problem would not be considered situated, nor would it be considered a verbal description (as a representation). Also, in general, students treated a functional relationship in *any* representation that was situated--whether or not it was described verbally--significantly differently from those that were not. So I needed to clarify definitions for verbal description and situational context.

Even though a problem may include some kind of verbal description, if the description is not of a function or functional relationship, it will not be considered a verbal description (as a representation). Further, a problem will be called situated

only if the mathematics of the problem represents quantities, not just number alone, that is, the mathematics is not set in naked context. References to situations or situational context in connection to representations are used in this limited sense only. Situational context can be provided through labels on tables or graphs, naming quantities for variables in symbolic rules, and through verbal descriptions of situations, or a combination of these.

Situational contexts cut across the other kinds of representation categories. They sometimes helped students and sometimes cause them difficulty in their problem solving. Situations seemed to activate problem solving resources from everyday life and to raise issues about assumptions, both about the situation and about the distinction between perceptions of the researcher and students. To address these issues related to situations I added the question:

> **In what ways and to what extent does the *situational context* contribute to or hinder students' abilities to understand or solve problems involving functions or functional relationships?**

Perspectives. Finally, a major tension that emerged during analysis was the contrast between researcher/teacher and student perspectives. It showed up first in the assumptions of the original research questions and again in a mismatch between student activity and major themes in the literature. In addition, it appeared as mismatches in communication and assumptions in interview problems, particularly in those that were situated. Since that contrast between teacher/researcher perspective and student perspective was an ongoing, overarching theme, I needed a question to address these issues, and thus added a final question (with subsidiary questions):

> **In what ways and to what extent is a *teacher/researcher perspective* different from or consistent with *students' perspectives* about conceptions of functions, functional relationships in problems, and their representations?**
>
> What role do assumptions play? How do perspectives affect communication? How are perspectives related to problem solving activity?

Revised questions. In summary, the principal research question for this study was:

> *WHAT SENSE DO STUDENTS IN THIS INSTRUCTIONAL SETTING MAKE OF FUNCTIONS AND TO WHAT EXTENT ARE THEIR CONCEPTIONS TIED TO SPECIFIC MATHEMATICAL OR REPRESENTATIONAL CONTEXTS?*

Mathematical context referred to the content topic: linear, quadratic, or exponential functions. Representational context referred to the mathematical forms that are used primarily to present these functions in their elementary forms: verbal descriptions or situations, tables, Cartesian graphs, or literal symbolic forms. Media used for these representations were talk, paper and pencil (or chalkboard), and graphics calculators. The revised set of questions that guided analysis are:

A. **What *conceptions* do students have about functions, functional relationships in problems, and their representations?**

> How do they *talk* or *reason* about functions or functional relationships in problems? Which concepts are most *salient*? On what do they focus their *attention*? What resources do students bring to the *problem solving* process? What kinds of *connections* do they make between properties and concepts of functions in different representational or mathematical contexts?

B. **In what ways and to what extent does the *situational context* contribute to or hinder students' abilities to understand or solve problems involving functions or functional relationships?**

C. **In what ways and to what extent is a *teacher/researcher perspective* different from or consistent with *students' perspectives* about conceptions of functions, functional relationships in problems, and their representations?**

> What role do assumptions play? How do perspectives affect communication? How are perspectives related to problem solving activity?

Revisiting Subjects: Narrowing the Focus

Far into the study, as I neared the end of analysis and because of the particular themes that had emerged, I decided to narrow the focus still further from 4 to 2 students. For reporting purposes I concentrated only on Sam and Toni, expecting to periodically bring in snippets of information from the other focal students to support

typicality of or to contrast responses. All four of the focal students had taken the pretest seriously. Most themes that arose during data gathering and analysis cut across all students. For example, all students had fragmented knowledge about functions and thought about them only implicitly. They relied heavily on the situations for reasoning and engaged in tasks that could be interpreted as more closely relating to schooling than to mathematics. However, good examples of themes appeared differentially among the students. I chose Sam and Toni over the other students because writing about them could best illustrate within two students themes that I found cutting across all four students. In addition, one theme, the contrast between researcher perspective and student voice took on prominence. Writing about Sam and Toni allowed me to discuss two dimensions of that theme, mathematics and activities of school, within interview problem examples where students were actively engaged in sense-making either about mathematics or activities of school, but not able to access the mathematics successfully. These choices of Toni and Sam provided me the backdrop for provocative reflection about this theme. In addition, their work provided good examples of contrasts for similar mathematical tasks involving similar representations with subtle, but important, differences in both the mathematics and the representation. Finally, it seemed important to include Toni to illustrate her facility when working with graphs and Sam for the questions he provoked, particularly about his use of situations and social interaction as problem solving strategies. Chapters 3 and 4 will present cases of Toni and Sam, respectively. Chapter 5 will reflect across both cases and present the findings.

Subjectivity of Researcher

Because a major theme underlying the research questions is comparing and contrasting perspectives of students and researcher/teacher, it is important to spell out what professional background I bring to this study and my personal biases about student learning and mathematics teaching.

Professional Background

My view of what mathematics is and what students should learn is shaped by my professional background. I have been a member of a university mathematics department for 20 years. I have:

- a masters degree in mathematics, that is, more mathematics than most high school teachers;
- many years of teaching all levels of high school algebra and trigonometry to college students (particularly to students who have failed in high school mathematics classes), that is, experience teaching similar subjects but to a different audience and in a different setting from most high school mathematics teachers;
- several years of teaching computer science--particularly computer programming, that is, experience reflecting on teaching and learning iterative processes that must come to be thought of as entities, as in understanding the loop concept;
- a great interest in the process of learning from sociological, psychological and philosophical positions both from before and as a result of my doctoral studies;
- experience teaching 9-12 year-olds in a recreational program for the gifted and talented; and
- experience as a researcher for weeks in a high school classroom (other than in this study); that is, considerably less experience with young people of high school age than most teachers.

Although I do not have much experience teaching in a high school classroom, I have experience teaching this subject matter at the same level of mathematics and have thought deeply about both the mathematics and how to help young people understand it. Thus, I am in a unique position to contribute a valid perspective from the point of view of teacher, novice researcher, and mathematician.

View of Student Learning

As implied in the literature review, I see students as creating individual meaning while engaged in purposeful activities in social situations. These meanings I call conceptions. What I call mathematical knowledge are notions presumed to be shared by a community with enough confidence to make communication possible. These notions I call concepts. While those well-versed or acculturated into the

community may be confident that they share similar meanings when working with public representations of functions, meanings reside with the individual and not the representation. What activities an individual engages in and the amount and kinds of experiences they have will influence what they learn. I believe students build meanings closely tied to the situations, experiences, and interactions in which they learned them, but as their experience grows or as they reflect on their knowledge and activities, they will adjust these meanings, sometimes incrementally and sometimes by leaps. I think individuals often have naive, tentative notions about formal concepts that can be explored and utilized informally or intuitively, sometimes without awareness, sometimes long before they can work with formalized symbolic (graphs, tables, rules) representations of the ideas. Learning mathematics is a bit like learning language in the sense that there is a holistic character to how it is acquired, first fragile and naive, sometimes awkward, and then gradually becoming more and more rich and powerful through use and appropriate experience. There are conventions, terms, and a way of speaking (Pimm's (1987) mathematical register) that are shared within the mathematical community and which must be learned, along with concepts, in order to communicate mathematically.

View of Teaching

While my views of learning are grounding in constructivism viewed as a social practice, my views of teaching are that of a guide who sets tasks and interacts with students to acculturate (Bishop, 1988; Cobb et al., 1992) them into a community. I believe the teacher is guided by a set of goals for learning based on her own understanding of the particular group of students, mathematics, teaching/learning, and acceptable social interactions within the particular classroom. The teacher sets tasks and activities for students to engage in, including a wide range of strategies varying from sharing her own understandings to guided explorations, and has the

responsibility to adjust them as her own knowledge about students, student learning, teaching, mathematics, and communication evolves.

My professional background as mathematician, mathematics teacher, and researcher, as well as my views of student learning and of teaching, all provide personal biases that affect my research. How I view student activity, what interpretations I make, and what I fail to see are in great part due to the specific lenses I bring to bear on the research.

CHAPTER 3

THE CASE OF TONI

The cases of two students, Toni and Sam, are presented in chapters 3 and 4, respectively. Toni's case is particularly interesting because there is a great disparity in her facility to work with functional relationships in problems. As will be seen, Toni often approached problems procedurally, yet displayed remarkable facility with some problems involving graphs. She approached some problems by rote even when to do so was not sensible whereas in others she was reluctant to carry out a rote process because the problem made no sense to her.

The first section of this chapter introduces Toni and is followed with three major sections discussing her work. Most of Toni's activities were directed at problems and their representations, with functions generally accessed only implicitly; thus, the first section concentrates on Toni's work with representations. It has two parts, the first discussing those problems in which she appeared to have facility and the second, those in which she had difficulty. Since these categories were not always disjoint, a single problem might include a discussion of both facility and difficulty. Within this section, the problems are organized by problem representation(s) and the nature of the problem task. The discussion includes how Toni went about working with the problem, and whenever possible, describes her problem solving strategies, methods of justification of her reasoning, connections she made across either function classes or representation classes, and finally, any features of her work that were puzzling to the researcher. These subsections suggest, if feasible, possible explanations for some of these difficulties. The second major section presents Toni's conceptions about the three function classes she studied. Finally, the chapter concludes with a section on the special nature of problems Toni encountered when functions were situated in realistic contexts.

Getting to Know Toni

Background

Toni is a soft-spoken, pleasant girl from a small midwestern town. When she graduates from high school, she hopes to attend a state university to pursue a career in the medical field. When Toni was younger, she lived in three states other than her current one, but hopes to move to Chicago as an adult because "it's a bigger place, and there are more jobs." At the time of the data gathering, she was entering eleventh grade. She had completed Algebra I her freshman year, receiving a C-. Though she took Algebra II her sophomore year, she planned to repeat it because, she said she did not have enough time to understand it. She indicated she could do better if she were teaching herself, rather than with teachers explaining math. According to a composite score of 13.6 on the Nelson-Denny reading test, Toni could read above the level of a college freshman.

Toni found this class different from those she had had before and liked it better because "I get to think by myself and someone's not explaining." She usually sat at the back of the classroom, which was very wide, but only about five seats deep. Toni appeared to be a conscientious student, attempting to do her homework even when time was limited, such as after a late night return from a weekend fieldtrip. Her teachers obviously agreed she was a serious student. At the closing banquet for the program, each teacher gave an award to the outstanding student from each section of class they taught. Since there were three sections, three students shared the honor for each subject topic unless a teacher chose to award a tie. Toni was the only outstanding student chosen in her section in Chemistry, Computers, and Microbiology, three of the eight subjects. In addition, during the last week of class a mini-Science Olympiad was held, using standard Science Olympiad events. Toni won first place in both the Water Quality and Metric Estimation events.

Approaches to Problem Solving

Procedures and manipulation. In her mathematics class, Toni often used a procedural, manipulative approach to problem solving, particularly when the problem involved symbolic representations. For example, in the Profit Problem of interview 3[1], students were asked to compare two symbolic function rules relating ticket price to profit for a school event with two different sets of circumstances and to discuss whether or not the second rule actually represented what a student in the situation claimed about it. Toni immediately began to expand the parentheses in the second rule, $P(t) = -100(t - 3)^2 + 700(t - 3) - 200$, saying, "I'm doing this one right here," as if she had an equation to solve.

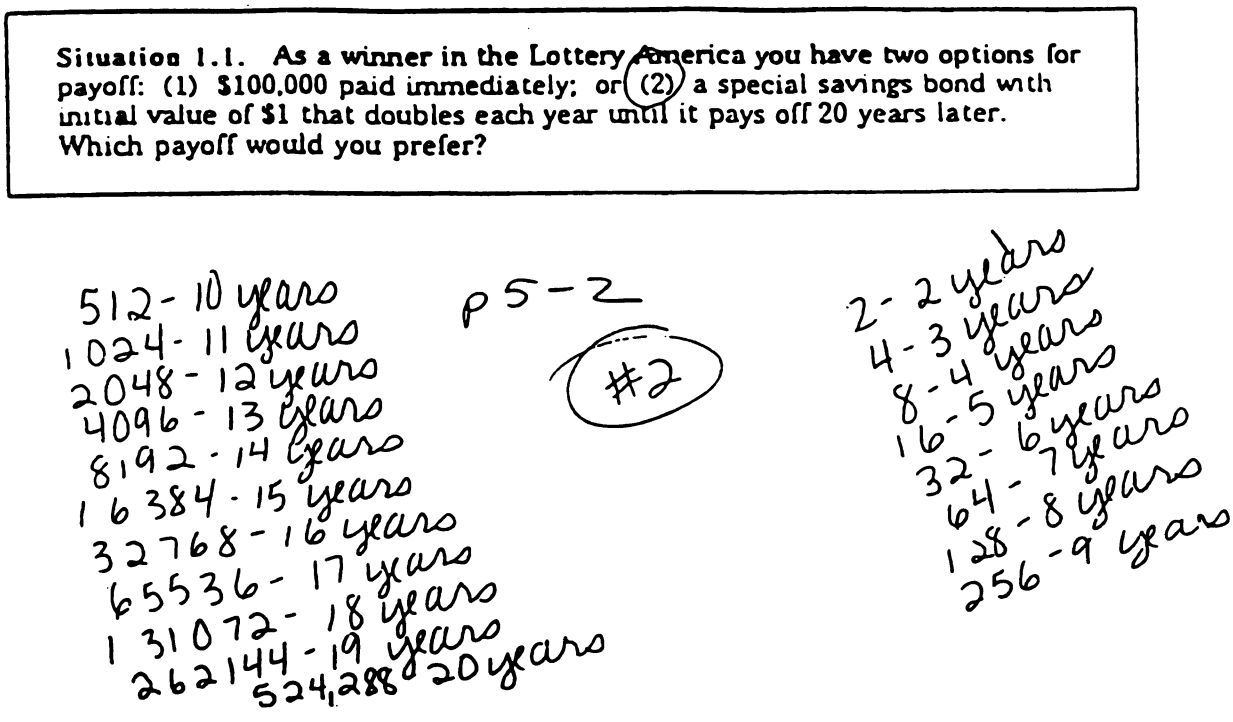


Figure 3.1: Toni's Homework for Situation 1.1 (CIA, p. 5-2)

[1]The full set of interview problems can be found in Appendix A. Problem numbers begin with the interview number, which corresponds to the week of 5-week class in which it was given. For example, problem 4B-1 was given during week 4 of class.

On the homework problem Situation 1.1 for exponential functions, given in Figure 3.1, Toni meticulously wrote out (including the word "years") the value of the bond for each year between 2 and 20, even though a calculator was available. Perhaps she thought it was necessary to document her solution.

A third example of Toni's reliance on procedures as a means of problem solving occurred during problem 4B-1, shown in Figure 3.2 below. Toni was unable

Below is a graph of a function $f(x) = .3(2)^x$. The point on the graph that represents the function values for input b is labeled with a dot. Place a dot on the graph that represents the value of the function for input b + 3.

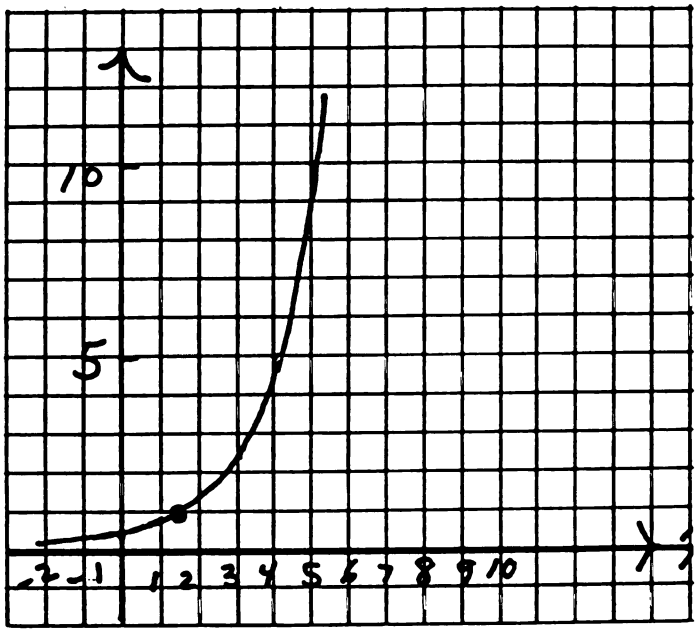


Figure 3.2: Problem 4B-1

to complete the problem because she associated the concept input with a rote procedure for substituting into parentheses. She became confused because it was the 2, not x, that was in parentheses. Toni began by asking if input was "what's in the parentheses." Since she did not seem clear about the definition of input, I mentioned a function that was discussed in class that day--a function about stopping distance as a function of speed--and asked her how the author had used "input" and "output" in that

function rule.[2] She said you had to put input in "both places where there's parentheses." I then explained that input was the value of the variable that you were putting into the function rule and output was the value you were getting out, and as I did so, she echoed my speech, almost simultaneously, as if recalling the meaning. Toni then announced that b + 3 should be going into the parentheses, but when I asked her which parentheses, she indicated those enclosing the 2:

T: These, this one right here [points to (2)][3] because this is the output.

G: In place of the 2?

T: Or, wait. Equals. It should be going in where the x-- I don't know. You can switch it around, so maybe it should be going where the x is over here. [points to f(x)]

Toni seemed to connect her process to one valid part of evaluating the function, substituting for the variable x in f(x). I did not want the discussion to be sidetracked by confusion with the f(x) part of the notation, a common occurrence in classrooms, so I tried to head off further difficulties by writing the function using y, rather than f(x). I reminded Toni that the teacher also sometimes did that. However, she continued to associate input with rotely substituting in parentheses, "The input is 2, um, to the x. You just put a two because [points to (2)] all of them would be to the x and the output is y." I tried to explain the function notation by mentioning that whatever variable was in parentheses on the left named the input variable, but that the whole side, f(x), was considered a name for the output from the function. I explained that it was not the existence of the parentheses, but rather that one substitutes something for the variable, and suggested we could go on to another problem if she

---

[2]After exploring the nature of the function $d(s) = .005s^2 + .14s$ via a table and comparing its table and graph to those for linear functions, students were given exercises that included the instructions (immediately after the rule was repeated), "Remember the input variable, s, is the speed of the vehicle in kilometers per hour and the output, d(s) is stopping distance in meters."

[3]Within transcripts square brackets are descriptions of actions or inserted comments of others. Parentheses connote interpretation.

would like. In spite of my explanations and suggestion to leave the problem, Toni continued with it. She persisted in using her own algorithm to substitute input into parentheses, even when they did not contain a variable, and substituted b + 3 in place of the 2 in the function rule.

Toni's attachment to the process of substitution "in the parentheses," instead of "for the variable" no doubt was related to class activities, where students had had a large number of exercises that introduced them to the function notation and often called attention to different aspects of the notation as they were asked to interpret or solve problems using it. For example, using a table of data of a situation, relating Dawn's age to Dawn's height, students were to fill in blanks in problems such as D(4) = _____ or D( ) = 102 (CIA, p. 1-9). Toni clearly had learned a procedure for rote substitution in expressions with function notation as she exhibited a short time later in problem, 4B-3, where she was asked to give a function value for $f(x) = 10 \cdot 3^x$ for x = b and x = 4.5 without using the calculator. She immediately correctly wrote $10 \cdot 3^b$ and $10 \cdot 3^{\wedge 4.5}$, even though the problem contained no parentheses.

It is unfortunate that I diverted her attention from a decision to substitute for the x in f(x) by eliminating it from the problem and so could not follow up on it. I suggest that she, as do many students in my own algebra classes, adopted an algorithm to substitute everywhere for x when dealing with this notation. In addition, in most classes there is discussion of substituting for "what's in parentheses" on the f(x) side. At one level Toni knew input was connected to "what's in parentheses," but had connected it rotely to existence of parentheses, which caused confusion in this problem. Until she was confronted with an unusual situation, which for her was to connect her knowledge of input to this procedure, but for which input (b + 3) was more complicated than she seemed to have seen, the algorithm itself was not called into question. In any case, she approached this problem algorithmically.

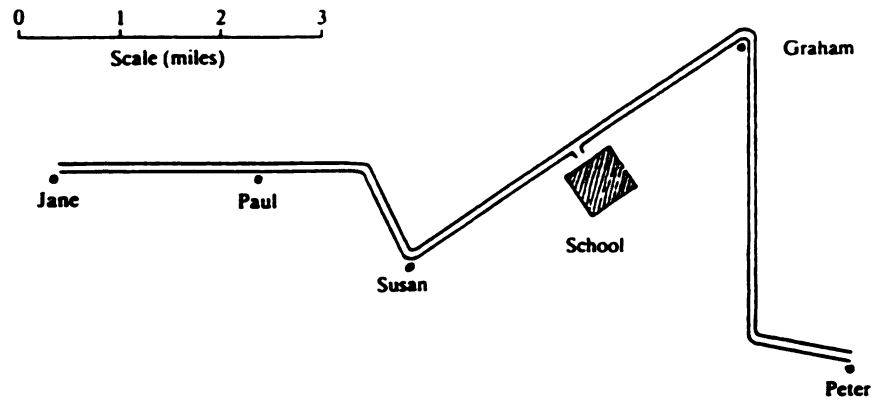
Toni often relied heavily on procedures. If she perceived a procedure should exist, such as in problem 2A-2 to find a rule for a linear function that was given in tabular form, and she could not remember how to do it, she became "stuck." In that problem, she remained stuck in spite of my efforts to help her use the patterns of change that she saw in the table. Further, because of this reliance, Toni was unable to tap the creative problem solving she used at other times, such as in problem 1-1, Going to School, shown on the next page in Figure 3.3.

For Problem 1-1 Toni used a strategy to obtain from the graph a single distance for two points, find two people with the same relative distance from school, consider their travel mode and place the students on the graph by their times. She correctly placed the pair Graham and Susan and the pair Peter and Jane in this way. She placed Paul because he was the only person left for the last dot. She assumed he fit perhaps because it was a school problem and there were five children and five dots. Clearly, Toni expected no manipulative procedure for this problem, and, in addition, she was able to make good use of the visual "order" of the places on the map.

Using patterns among tasks and visual representations. In addition to relying on procedures in problem solving, Toni was often oriented to determining other patterns, such as patterns of change in tables or slopes or shapes of graphs. As will be seen below, she used this to her advantage when working with graphs and tables. She often observed very subtle distinctions in both verbal and pictorial aspects within representations of functions.

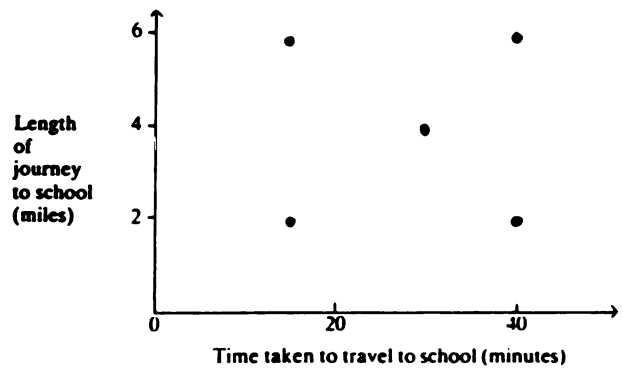
This orientation to patterns was particularly clear during interview 5, where students were given 17 cards--which involved various relations in one of several representations--to sort into categories of their choosing. (See Appendix A for a complete set of cards.) Toni made an extremely quick first sort based on the superficial features of the representations, that is, on the tasks embedded in the verbal

GOING TO SCHOOL



Jane, Graham, Susan, Paul and Peter all travel to school along the same country road every morning. Peter goes in his dad's car, Jane cycles and Susan walks. The other two children vary how they travel from day to day. The map above shows where each person lives.

The following graph describes each pupil's journey to school last Monday.



i) Label each point on the graph with the name of the person it represents.

ii) How did Paul and Graham travel to school on Monday? _____

_____

iii) Describe how you arrived at your answer to part (ii) _____

_____

_____

(continued)

Figure 3.3: Problem 1-1, Going to School (LFG, p.28)

representations and on visual attributes of the cards such as tables, graphs, and word problems. Her first sort included two categories related to task: word problems for which you have to substitute values (M, P, Q, B) and cards on which "they give you values that increase and they want the number (A, S)." She had three pictorial, two graphical (graphs that did or did not look like parabolas), and one tabular categories. All categories were based on superficial characteristics of their representations. Three were principally nonverbal, and the two task categories were principally verbal. In her first task category, Toni described the process of solution and for the second, she added a task that was not there. Cards A and S simply state a relationship. As her first sort took only about 3 minutes to complete, compared to from 7 to 15 minutes for other students, these categories came readily to her and were clearly salient features of the visual representations on these cards.

For her second sort, Toni sorted the cards principally by what type of information was given and the nature of the given task. Her descriptions of tasks often included the specific process needed to solve the problem. Toni's summaries of her sort categories follow the listing of cards in a category given below. Categories are:

(1)	Q, Z:	. . . give you information and then tell you to do something
(2)	J, M, R, T, H, B:	. . . give you diagrams or graphs to look at and then ask you a question about it
(3)	Y, S, A, P:	. . . give you numbers and then just tell you something about it and don't ask anything
(4)	W, G, K, D, E:	They're all graphs. Some of them tell you something about them, but none of them ask questions.

For those cards with a task, Toni had created valid, but unnamed categories: find a function rule (group 1) or find an answer, usually numeric, to the question posed (group 2). She successfully separated out those cards that simply gave a

relationship without a problem solving task, but further classified them by graphs/diagrams and nongraphs. She made one error in this classification. Card H asked for a function rule, but she grouped it with tasks that ask for information (numeric answers), rather than with requesting a function rule. This error probably occurred because, at the time she grouped it with B and T, she had been working with a distinction between the whether the card just gave information or had a task associated with the information that was given. It was not until later that she distinguished the difference between the task of finding a function (Q and Z) and that of finding a numeric value. By that point, card H was buried in a pile of cards and may have been overlooked. On the other hand, H asks, " What kind of function best fits the data?", which Toni might have taken as requiring a different kind of answer than those on cards Q and Z. Card Q asks you to "Find a function, R, relating revenue to the price," and card Z asks you to "Give this function's graph." Both ask you to produce some representation, while H seems more to be seeking an "answer" in much the same way that the more numeric tasks do. In this latter sort, Toni used increasingly subtle distinctions based on categories involving a combination of how words and pictorial information were used on the cards, rather than relying on the mathematical properties that were represented there.

In summary, Toni often focused her attention on what procedure or task was sought in problem solving. She was very observant about visual cues, both verbal and nonverbal. Toni sought clues for how to proceed in problem solving by attending to such features, with uneven success, as will be seen in subsequent sections.

Calculator use. The calculator was used daily in class as a tool to solve problems. While Toni used it in class and periodically in interviews, she did not generally turn to the calculator to assist her with problems even though it was always available and each interview started with an announcement of its availability. Only on problems 4B-3 mentioned above and 4B-4 were students asked not to use it.

Working with Representations

Working with Representations - Facility

Perhaps the most noteworthy feature of Toni's activities with representations in these interviews was her fluid interpretation of graphs of situated variation, a task that is difficult for students at many levels (Bell, Brekke, & Swan, 1987a; Even, 1989; Janvier, 1978; Kerslake, 1981). On three occasions (Problems 2B-1, 2B-2, 5-1) she was given a verbal description of a problem situation (a functional relationship) and asked to select a graph to match it and to justify her conclusion. This section describes Toni's work on these three problems and attempts to characterize patterns in her reasoning about graphs. It continues with a discussion of other ways in which she had facility with graphical representations in tasks involving functional relationships, such as creating graphs from situations or creating situations from graphs. Next it explores tasks in which attention was on the representation as a whole, such as recognizing patterns of change in a table, and finally, discusses Toni's facility with tasks that involve a process or correspondence between single instances of a functional relationship, that is, where the attention is focused on the connection between specific x's and y's.

Fluid Matching of Graphs and Situations

In interview 2, Toni was given a problem from *The Language of Functions and Graphs* (Swan, 1985) asking her to find a graph from the figure below to match the situation given, using variables shown:

> When I started to learn the guitar, I initially made very rapid progress. But I have found that the better you get, the more difficult it is to improve still further. (Proficiency/amount of practice.)

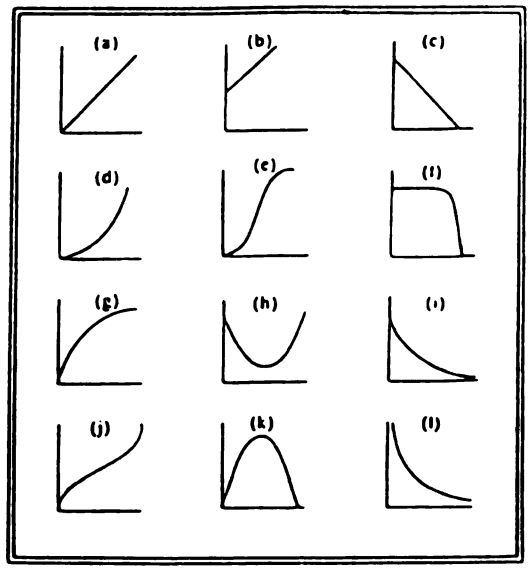


Figure 3.4: Graph Choices for Problems 2B (LFG, p.102)

Initially Toni chose graph e, justified it, and then quickly volunteered why three other candidate graphs failed to match. She described the speed of progress and accurately matched it to the shape of the graph e:

> He or she, or whatever, said that they, um, made a very rapid progress so the line should go upward, uh, really, like steeply. And then uh [pause] it pretty much looks like e because it starts from the bottom and then it goes up pretty fast and then the person says that the better you get, the more difficult it is to improve even still further. So they're already at the level it was pretty much their best, if they're still trying to improve so it's pretty much steady--.

Toni was influenced by a rapid rise at the start of the graph and a flattening out to match the difficulty to improve. She spontaneously rejected j because while it matched a rapid rise, it did not flatten out, "j goes up but then it keeps on going up, but the person said that it gets harder to improve so it would, the line would flatten out." She dismissed k because, "it goes back down and the person would get worse than they were when they began." Finally, she rejected i because its behavior on both sides contradicted the situation by indicating they started well and then got worse. When I asked if it could be g, she felt it was a better choice than e because

earlier she had interpreted rapid progress in proficiency as immediately improving, whereas in e, the graph indicates there would be a short period of slow progress at the beginning.

Toni had effective reasoning strategies. Throughout the problem, she described her choice in terms of the shape of the graph and her justifications in terms of the situation. She was able to segment the graph and match the slopes of the segments with a corresponding speed of improvement. She had equated steep slope with "made very rapid progress," and "more difficult it is to improve" with a flat graph, that is, with a constant proficiency. By choosing this labeling of two distinctive segments of the graph, Toni avoided a potential conflict between the graph and her description of change that might have occurred had she continued to think of "more difficult it is to improve" as proficiency that was changing, rather than no change. She said the proficiency was level, "so they're already at the level it was pretty much their best, if they're still trying to improve so it's pretty much steady--" In reality, by ignoring the subtle change in proficiency, she avoided a decreasing rate of change in proficiency, which might have served as a conflict for how the right segment of the graph appeared. The existence of such distractors in pupils' incorrectly interpreting similar graphs is well-established (Bell et al., 1987a; Even, 1989; Janvier, 1978; Kerslake, 1981). Toni's use of language to coordinate the fit of the situation and graph in this example was enhanced because of her ability to choose a suitable independent variable that could be experienced temporally. In this problem she literally renamed amount of practice, "time." Then increasing horizontal changes were consistent with passage of time when viewing or sketching the graph from left to right. Both change in situation variable and the passage of time could be experienced intuitively and simultaneously, without much overt attention.

Toni exhibited these same reasoning strategies and fluid interpretation, when with little hesitation, she picked out and justified graph k as an appropriate graph for

a second situation from the same problem set, problem 2B-2:

> If schoolwork is too easy, you don't learn anything from doing it. On the other hand, if it is so difficult that you cannot understand it, again you don't learn. That is why it is so important to pitch work at the right level of difficulty. (educational value/difficulty of work)

Toni chose difficulty of work for the horizonal axis and educational value for the vertical axis. This choice of axes made it possible for her to think of difficulty of work as changing over time for some group (of students), and thus, again experience the graph temporally. Further, it allowed her to describe the changes in segments of the graph and connect each to a parallel description of the change in the dependent variable, educational value, while essentially ignoring the independent variable. Again Toni justified her choice by matching a description of change in the shape of the graph with a change in the situation.

In this example, the graph could be viewed as a "picture" of change in educational value and in the previous example, Toni changed the situational description slightly so the graph also could be viewed as a picture of change in proficiency. Toni could, however, interpret graphs where the variation in the situation description differed from a picture of the graph, as in problem 5-1 below in Figure 3.5. A suitable graph for this problem was identical to Toni's choice in the guitar problem, but this time she could not readily revise the situational description as she had done in that problem.

For Donna, after a brief consideration of c, Toni switched to f and gave a reasonable description of the match. She said:

> C is Donna's. [Picks up pen and starts to write]. Oh, wait. [She studies the sheet for a bit.] I think f is Donna's [writes f on sheet] because she started out running and it goes up kind of fast. Then it kind of levels off [traces graph with pen] where she started walking.

Tim and Donna live 1 km from their school. Usually they walk together to school. Yesterday they both left their houses 10 minutes before school started. Tim started to walk but Donna was afraid to be late and started to run. After a while Tim realized that although he tried to walk faster and faster, he had to run if he did not want to be late, and started to run. At the same time, Donna became tired and had to walk instead of run. They both reached school exactly on time. Which of the following graphs is Tim 's ____ and which one is Donna's ____?

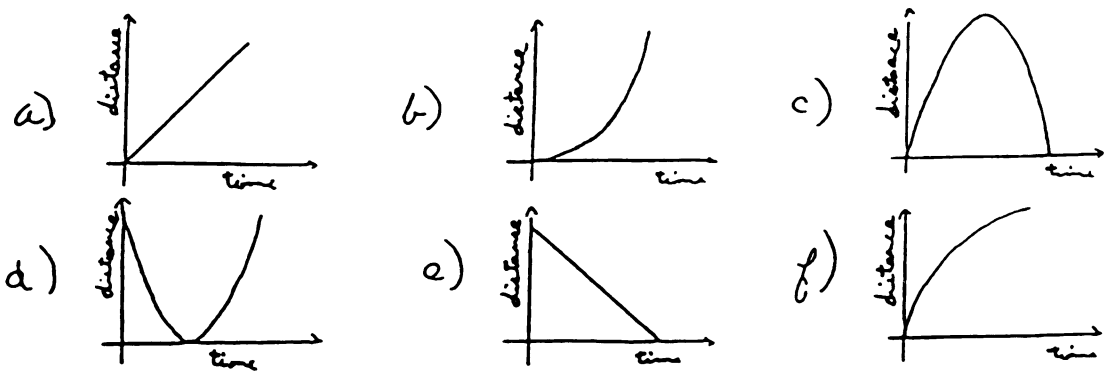


Figure 3.5: Problem 5-1, Tim and Donna Problem (Even, 1989)

After she chose b for Tim's graph and described the match correctly, I asked her how she could tell from the graph that Donna was running. She explained:

T: Because it's going up faster. It's almost going straight up in a little amount of time, and she is covering more distance.

G: What is going up in a littler amount of time?

T: The distance is going up.

G: Oh, okay. So a distance going up in a littler time means that she's running?

T: Yes, because she's covering more distance in less time.

When asked why she had rejected c, Toni correctly explained that the graph's going down to zero on the right would indicate Donna was returning home. As she justified this match, Toni referred to speed only indirectly by mentioning mode of travel. She associated a steep portion of the graph with a fast mode, running, and a flatter portion with walking. Toni described the steep part as covering more distance in less time

(than flatter portions of the graph), a dynamic relation between distance and time, that is, a covariation of the two variables, a subject about which I will say more later.

Initially, when Toni chose c, she was tempted to interpret the graph as a picture of the change in speed, that is, to switch variables when the change in the graph was at odds with the change in speed, a phenomenon similar to that reported in research for pupils of a wide range of ages (Bell et al., 1987a; 1981; Even, 1989; Janvier, 1978). If one reasons about this graph as a description of rate of change (speed), rather than as change in distance, the language describing speed is somewhat consistent with language describing the rate of change for the left part of the graph, assuming that one starts from rest and one's speed goes up. But for the right part of the graph, the rate of change (speed) is decreasing and does not match the graph, which is still going up. Toni's decision to reason directly with the relationship between amount of distance covered over time supported her ability to overcome her initial pull to consider a decreasing graph because speed decreased. She could think of distance itself as going up more slowly, rather than as changing with decreasing speed. Her strategy of reasoning in this fashion meant she could give only tacit attention to time and simply consider only her "motion" as she swept across the graph from left to right.

Her choice of f while describing only changes in speed and her reasons for rejecting c indicate that at some level she understood a subtle distinction between a function (f) that increases over an interval, but which has a decreasing rate of change (speed), and one (c) in which the dependent variable (distance) decreases over some interval. Her interpretation here is considerably more complex than that in the guitar problem where proficiency, the dependent variable, simply leveled off and could be thought of as no change, so that proficiency, rather than its change, could be matched directly with the shape of the graph. The language description in that problem allowed Toni to reason essentially with only the dependent variable (proficiency)

changing over time, which did not seem possible for her in the Tim/Donna problem, where continuous attention to the variable speed was required.

In that problem Toni used mode of travel for indirect speed and relied on being able to coordinate simultaneous variations, that is, a covariation to fit graphs to situation. Time was the independent variable, and Toni used natural language descriptions to connect the kind of change in the dependent variable to the steepness of the graph. Coordinating simultaneous variation in two different representations, a situational description and a graph, clearly is a complex problem. Toni managed quite well because she was able to view the independent variable temporally, to think of the graph in segments, associate steep graphs with fast change and flatter graphs with slow change, recognize that a decreasing graph meant the dependent variable was going down, and to attach natural language descriptors for the change in the situation that fit the descriptors of changes in the steepness of the graph. These strategies seemed critical to her success and were echoed by other students when they were successful in these tasks. In addition, as will be seen in Chapter 4, Sam used the same strategies, but had a great deal of trouble when he chose an independent variable that could not be viewed temporally.

Reasoning about Situated Graphs Directly in the Situation

When a problem was situated, Toni relied on the situation itself to support her reasoning. For example, in problem 1-1, Going to School (see Figure 3.3 above) discussed above, Toni reasoned directly in the situation by coordinating values of data pairs in the relationship. She used the relationships of the situation that she had extracted from the map, graph, and the verbal description of the situation to guide and justify her solution. During the first problem task, which was to match information embedded in a map with points on a Cartesian graph, she recognized from the graph that the two lowest dots had to go with Susan and Graham, who lived closest to school. Toni correctly placed both children on the graph based on knowing only

Susan's mode of travel, walking. She reasoned 40 minutes would be appropriate for two miles, and placed Graham, who varied how he traveled from day to day, on the other dot. Toni had reasoned with one set of numbers in the situation to decide Susan's placement, but without checking that the other would work out well for Graham. She readily placed Peter and Jane because they had the longest distance and must be the top two points on the graph. For their specific placement, Toni said, "And then Jane and Peter live the farthest from the school so both of their points would be the same, uh, in miles." She then placed Peter correctly because he went by car, "so it would take him a shorter amount of time so Peter's would be the first point on the graph, on the top." In naming the bottom two dots, she had reasoned whether the specific data, 2 miles and 40 minutes, was reasonable. In the second case, Toni had reasoned that same distance and faster speed implied a shorter time. She did not compute speed, but had reasoned by comparison of relative travel modes, that is, she relied heavily on the "thinking in the situation" to produce relative orders for quantities to solve this problem.

The second problem task was to extract mode of travel for two of the children. Again Toni reasoned directly in the situation. In deciding travel modes, Toni decided Paul's mode was biking simply by reading off his time as 30 minutes. She had just placed Paul at this spot and had previously announced that he was closer to school than Jane, or she may have noted from the graph that his distance was 4 miles. She had not computed his speed, but again seemed to reason directly about a sensible travel mode for that distance. For Graham, Toni suggested he walked, contradicting an earlier suggestion she had made that he rode a bike. As there was a long pause, I judged her to be somewhat stuck. Since part of the of this first interview was to build rapport, I asked how Susan traveled, and Toni responded by placing the time in opposite order of speed (travel mode) for the same distance.

T: Susan walked, and it took her 40 minutes and Graham, um, oh, I think

> Graham rode his bicycle because Graham and Susan live the same distance. And it took Susan 40 minutes to get to school, and it only took Graham around probably 17 or 18 minutes. So he probably rode his bicycle, too.

When the task of comparing travel mode (speed) for two students whose distance was not the same, she simply reasoned directly in the situation, judging whether a particular time and travel mode seemed sensible.

> G: How do you know that Paul and Graham did not go by car?
>
> T: Because Graham only lived two miles from the school, so it wouldn't take him 18 minutes to get there if he went by car. And Paul lives around 4 miles from the school and it took him about 25 minutes, 25 or 30 minutes to get to school and it wouldn't take that long by car--unless they stopped along the way.

She could readily extract useful information from individual coordinates of the graph and could use that information to reason about a relationship between time and speed given indirectly by travel mode. At no time did she compute or discuss speed directly, but as in the Tim and Donna problem, she spoke of it indirectly through travel mode. She did not seem to reason with speed as an explicit ratio of distance and time, that is, as a single quantity, but seemed rather to reason by coordinating two quantities (that make up such a ratio for speed). She seemed to be making decisions based simply on attributes of a familiar situation.

Toni's fluid reasoning in this problem situation is in contrast to some of the other students[4] and Sam, who attacked this problem in a very school-like, procedural way. Sam made a ruler along the edge of the paper, measured, and accumulated total distances along the map, located on the graph points with that same distance, and then placed two students with the same distance, sometimes totally without regard to either travel mode or time. Toni also often approached problems that were not situated in a

[4]All students in the focal group participated in interview 1 and were given this problem.

procedural fashion, but this problem was both situated and had pictorial or graphical features. She clearly relied on both.

Generating a Graph for a Situation and a Situation for a Graph

In the first section, the problem tasks involved matching variation in two existing representations, a graph and a verbal description of a situation. In each of those cases, Toni displayed facility with coordinating these variations. In this section, where the task differs, Toni displays the same facility with these two representations. Here the task is to generate a new representation for a functional relationship, first, going from a graph to create details of a situation to fit it and then taking a situation depicted mainly pictorially and creating a graph for it.

Toni accurately generated a situation for the graph in problem 10 on the pretest. She wrote:

> I think she got 5 miles from home and then she got lost and was just walking in circles and then she found [t]he way home so then she walked home.

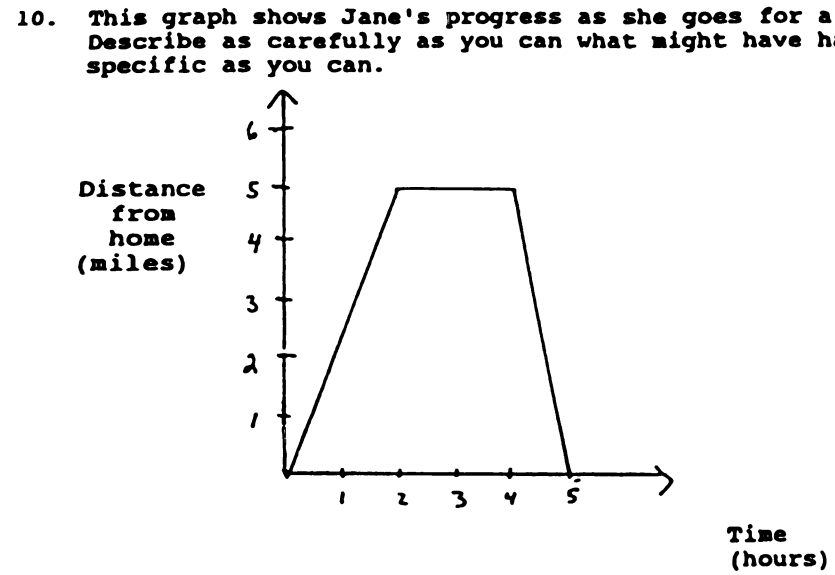


Figure 3.6: Pre/Posttest Problem 10

She gave a similar response to the same question on the posttest:

> Jane could have gone on her country walk and got lost and kept going in

circles or she could have rested out in a field or something and then found her way home.

In both cases, she gave a completely plausible and correct situation to match the graph, in contrast to that often given by many students in other studies (Bell, Brekke, & Swan, 1987b; Brekke, 1987). Toni failed to identify specific quantities such as 5 miles and 5 hours and to distinguish differences in traveling rates between going out and coming back. Each of these omissions could be explained perhaps by her understanding of the level of specificity required. On the other hand, her responses suggest that she attended principally to the dependent variable only, while paying only tacit attention to time, which could be experienced "intuitively" as one sweeps across the graph left to right as she did in previously discussed problems.

In Problem 1-2, the second part of the Going to School Problem, Toni demonstrated that, with minor difficulties, she could create a reasonable graph of speed as a function of distance. After a brief sidetrack in which I clarified that the

iv) Peter's father is able to drive at 30 mph on the straight sections of the road, but he has to slow down for the corners. *Sketch* a graph on the axes below to show how the car's speed varies along the route.

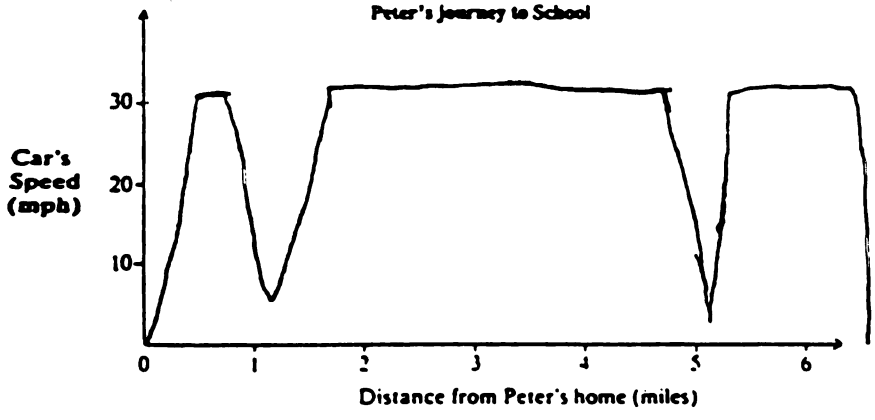


Figure 3.7: Toni's Graph for Problem 1-2; Going to School, Part 2 (LFG, p. 29)

map, but not the graph, given in the first half of the problem (1-1, shown earlier in

Figure 3.3) was pertinent to this problem (1-2), Toni was able to sketch a reasonable rough graph of the situation. She accounted for starting at home from stop (origin), periods of acceleration, slowing down for corners (dips), straightaways making constant speed (level portions on the graph) and coming to a stop at the school at roughly 6.5 miles. While Toni judged the number of miles for the second segment of the journey as 4 instead of 3, she recognized Peter's total mileage from the graph in Problem 1-1 and so adjusted the last segment slightly to attempt end it at around 6 miles. She deduced the value for the speed at the corners from personal experience and when questioned about the dips in the graph, admitted the second corner was sharper and would require a greater slowing down and thus a deeper dip in the graph.

Unlike many other students who had meticulously computed length of journey segments on the map for Problem 1-1, Toni had solved the first part of this problem without assigning measures to individual map segments. So in solving this problem, she just estimated them while she was sketching the graph. Thus, her error of four miles instead of three for the second segment is understandable. Toni checked the distance for the final destination by reading off the second coordinate of the point labeled Peter on the previous graph. While her periods of acceleration were too long, her failure to create a more refined graph could have reflected a lack of knowledge about subtleties of acceleration or that she interpreted the instructions as requiring only a rough sketch. In any case, Toni created the graph in a very quick, fluid way. I remarked in my notes immediately after the interview that she "seemed to do graph by deduction--only cursory glances at scale (map)." She seemed to be imagining driving along that map, keeping in her mind that most of the time the speed was 30 mph, and coordinating the speed with her position.

Note that the independent variable here is distance from home, but since time and distance are proportional when speed is constant, Toni could just as easily have been thinking about distance in a temporal way along the horizontal axis, a technique

that she used on other occasions. Recall that in the guitar problem above, Toni explicitly referred to "amount of practice" as "time," and seemed to experience the graphs in a fluid kind of way that depended on paying little direct attention to the independent variable. Indeed, in the pre/posttest problem 10 about a country walk, she ignored time's interaction with distance, and treated it only as having passed.

This last phenomenon appeared in problems to fit situations to graphs, to create a situation for a graph, and to create a graph from a mainly pictorially presented situation. It suggests that an important component of Toni's successful work with situated graphing problems was being able to view graphs as covariation, with primary attention on the dependent variable and only minimal, possibly intuitive attention to the independent variable. It also seems that being able to reason directly within the situation, without computing certain mathematical quantities such as rate, assisted her in these problems.

Recognizing a Linear Function in a Table

The discussions up to this point have dealt only with those representations that have involved graphs in some way. While the first half of the Going to School problem (1-1) involved only single points or comparison of points, the remaining problems involved perceiving or working with variation over many points, and attention was generally focused on the variation in the problem as a whole, rather than on the relationship between specific x's and y's. Functions defined in tables lend themselves to this kind of variation as well, if one moves up and down the table, rather than across.

Toni was able to take such a covariation approach to tables with some facility. She could recognize a linear function when it was given in table form. In the example about to be discussed, Toni spontaneously justified a table as linear while in the midst of giving an argument to the contrary. During the card sort in interview 5, discussed above, Toni was given 17 cards to sort. The cards represented linear,

quadratic, exponential, and other functions as graphs, tables, symbolic forms, both situated and nonsituated. One table was a nonfunction. After Toni had sorted the cards twice without reference to function classes per se, as described earlier, I asked her to sort them by functions we had studied in class. She separated out and called several that had curved graphs on them (M, T, J)[5] parabolas and next made a new category, quadratics, for card W, a scattergraph. It was clear she believed quadratics were associated with a different function class than parabolas, and when I indicated we needed to straighten out the function class names, she immediately wanted to know if they (quadratics) "were the (names of) ones that changed equally" (in table entries). She then recalled on her own that tables with equal (rate of) change had graphs that were straight lines. Then in the course of trying to straighten out other names of function classes, Toni attempted to find a table with unequal change in output. She selected card R, which contained the table in Figure 3.8. Toni attempted to illustrate unequal change. She said:

> Or the kind of function that isn't equal-- how this one-- [pointing to the table R] Well, this one [pause as she reads card] this one's right. This one would be a linear function. [picks up R] because it's an equal change from 4 to 5 there is 2 (a difference of 2) and if there was a 6 here (in the input list), it would be 48 (another decrease of 2) and 7 is 46 [G:okay] and 8 is 44. So, this one would be linear.

input	output
4	52
5	50
7	46
8	44
10	40
20	20
30	0
35	-10

Figure 3.8: Table from Sort Card R

[5] J and T are actually graphs of exponential functions. M has a quadratic equation as well as a picture on it that is made to appear very graph like.

At first Toni looked only at the output entries and judged it unequal, but soon recovered: "this one's right." She had quickly recognized it as linear. Her process of filling in missing entries showed that she connected equal change in output for equal change in the input with linear graphs. She had greater difficulty sorting out other kinds of change represented in tabular form. As will be seen below under function classes, she seemed to have only two classification categories, equal change and unequal change. Toni found thinking about the table as a covariation useful in recognizing that it represented a linear graph. Whether it was fitting graphs to situations, generating graphs for situations, generating situations for graphs, or working with patterns going down tables, Toni found a covariation approach to functions both useful and salient.

Facile Interpretation of Values from Points on the Graph

Not all tasks involve whole representations. One task students must engage in while solving problems for situated functions represented by graphs is to interpret points on the graph. Even before this class began Toni could interpret points on situated graphs. On the pretest problem 9, shown below in Figure 3.9, she accurately

9. The profit from a rock concert depends on the ticket price. Experience has shown that the graph in Figure 2 gives profit in terms of the ticket price. In each case explain how you got your answer.

a. What profit can be expected when tickets cost $8? _____

b. What is the maximum profit that can be expected? _____

c. What ticket price is predicted to give the maximum profit? _____

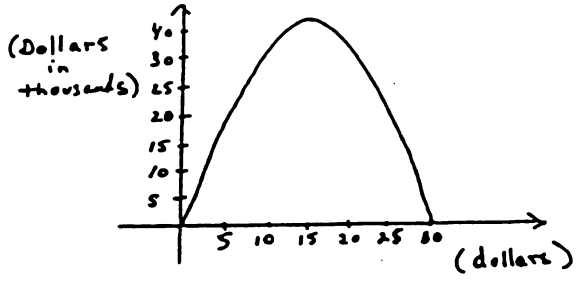


Figure 3.9: Pre/Posttest Problem 9

read off 27 as being associated with a ticket price of 8, identified the point (15, 45) as representing a maximum profit of 45 and occurring when a ticket cost 15 dollars. While she failed to account for "dollars in thousands," she demonstrated that, not only could she read off values, but she could interpret the distinction between a maximum function value and where it occurred. On the posttest, Toni answered problem 9 in essentially the same way, estimating her maximum point slightly more accurately at 42 or 43 and accounting this time for the dollars in thousands.

Toni demonstrated that she had a firm procedure for interpreting points on other problems as well. Once during the sort problem she volunteered that a solution for M (Figure 3.10) would be readily available if you made a graph for it. When I asked how the graph helped, she said:

> T: Because, if you have a graph, and they ask a question, all you have to do is find the information on the graph. Easily just point right to the answer--.

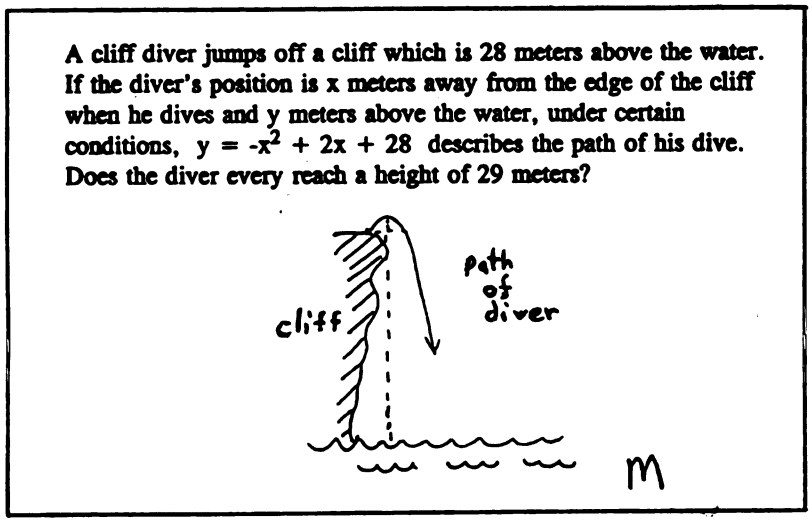


Figure 3.10: Sort Card M

I asked Toni if the process would be the same as that for cards J and T, where a graph was given (see Figure 3.11 below). She insisted it was and went on to explain

a process of picking out the given number on one axis, going either over or up to the graph and then to the other axis to read off the desired value. Note that on cards M and J, the process involves going from y to x and on T, from x to y.

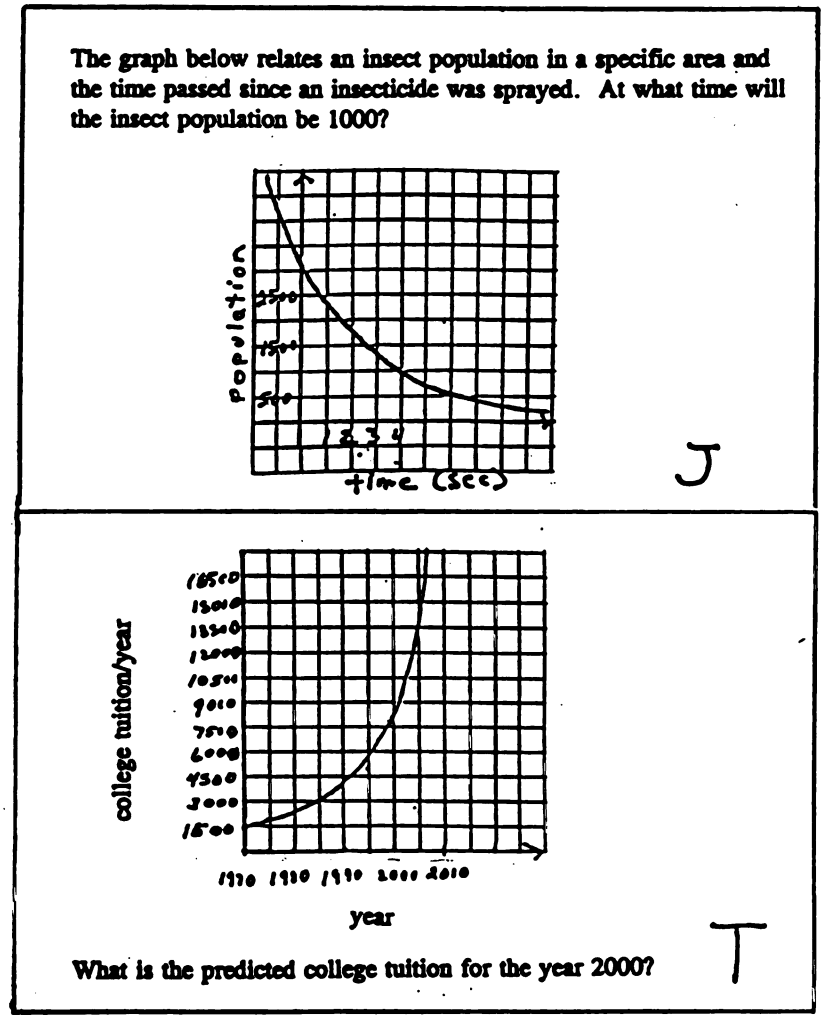


Figure 3.11: Sort Cards J and T

That Toni saw these as instances of the same process, rather than two different processes, and so readily extracted information about maximum function values from a point on a graph in problem 9 of the pretest indicates she was facile with extracting specific information embedded in a graph. In the Going to School problem, discussed

earlier, she readily read off coordinates of points on the graph, could interpret each coordinate independently, and recognized that both coordinates are necessary to define a point. At one level each of these tasks could be viewed as a procedure, but at the interpretive level for the situation they could not, especially in the Going to School problem and the maximization of profit on the pretest. Those problems required additional interpretation to decide what is meant by individual coordinates in the situation, or in the case of the maximization, knowing where to look as well. The inability to interpret the vertex of a parabola correctly as a maximum function value and to determine where the maximum occurs in both naked and situated contexts is quite common among students in my own intermediate and college algebra classes.

Point Plotting

Another task involving individual parts of representations of a function that students need to engage in is point plotting to create a graph from either a function rule or a table. Normally, plotting points from a table is seen by school children as a rote process. Toni's facility with this task is discussed here because her understanding of this process and its connection to graphs is uneven. She understood one fairly sophisticated concept while failing to understand others.

Embedded in the process of point plotting are several conventions, such as giving the horizontal coordinate first, placing numbers along axes in ascending order from left to right or from bottom to top, and not in the opposite direction for either axis. Another concerns the connection of axis choice to these conventions. Unlike in advanced mathematics classes, especially where functions might be defined implicitly, school children do not normally encounter much flexibility in axis choice. Thus, if y is a function of x, then x is on the horizontal axis. One fundamental concept that must be understood about translating from tables of functions to Cartesian graphs is that labels for table column headers for coordinate pairs must be consistent with the axes labels along which they are measured. While normally the horizontal coordinate

is given first in an ordered pair of numbers to be plotted, if the first number is labeled to be consistent with the vertical axis, then it must be considered the vertical measure.

Toni exhibited a fairly robust understanding of this concept even though she had great difficulty keeping straight conventions for orienting axes, as will be seen in the next segment of this chapter. Toni could readily plot points for entries in a table any time both its columns were labeled with variable names and the axes were labeled as well, even if they defied the normal convention of putting the horizontal measure in the first column. She understood that there must be a consistency with the names.

For example, after the second day of class, in her homework she created three graphs for data given in horizontal tables. Toni's work on one of the problems is given below. While most students used the horizontal axis for the top line of data

(c) **Situation:** Animal populations tend to rise and fall in cycles. Suppose the following data show how squirrel populations in a central Pennsylvania city varied from 1975 to 1985.

Year	'75	'76	'77	'78	'79	'80	'81	'82	'83	'84	'85
Population	750	700	520	680	730	650	550	625	780	700	530

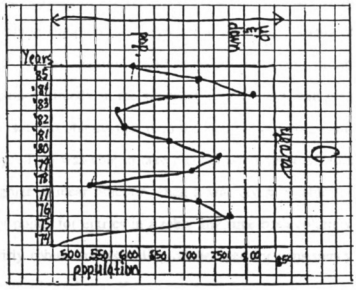


Figure 3.12: Table and Toni's Graph for Problem 5c (CIA, p. 1-22)

and vertical axis for the bottom line, Toni chose the opposite orientation for all three of her graphs. Her point plotting was consistently correct with her choices. Students were learning to plot scattergraphs on their calculators and had been instructed to create a graph on the calculator before plotting it by hand, which could account for Toni's graph orientation on this day since it may reflect simply the order data was entered in the calculator. In addition, it perhaps was too early in the course for the students to think of one variable as dependent on the other or, even if they did, to realize that the independent variable conventionally is graphed on the horizontal axis. Note that Toni apparently found her graph hard to read and added notation so it could be turned and read in an alternative direction. In this problem, the task was to create a graph given a table. Toni was successful as labels were provided for the table, and she was at liberty to choose axis orientation. Her work showed that she understood the necessary consistency between the two sets of labels for this translation task.

In summary, Toni had facility with several representations of functional relationships in problems and associated tasks. She could pick out suitable graphs and fluidly match them to a situation, including those where direction for rate of change was opposite that of change in the dependent variable. She had effective reasoning strategies for matching situations and graphs that included thinking of the independent variable temporally so her attention could be on the dependent variable only. Further, she regularly reasoned directly in the situation. Toni could read off points on a graph and could readily solve problems that required interpreting points on a graph, starting either with x or y. Toni was able to take a covariation approach to tables with some facility and recognize a linear function when it was given in table form. Finally, she could readily plot points for entries in a table any time both columns and axes were labeled, had a fairly robust understanding of the need for consistency in labeling, but often reversed orientation of axes if they were not labeled.

Difficulties Working with Representations

Three areas of difficulty stand out in Toni's ability to work with graph, table, and function rule representations. Two involve graphs or function rules: Toni regularly had trouble with orientation of axes' labels, and she was unable to construct a graph by hand from a function rule or equation. The first difficulty is simply a convention that must be learned. The latter is more complex and somewhat puzzling, since it could be considered by teachers as a fairly simple extension of plotting points from a table. Since Toni could accurately plot and interpret points in the coordinate plane, assuming axes and tables were labeled, it is surprising she was unable to create graphs by hand, that is, by point plotting. The third difficulty is constructing a function rule from a table. A final difficulty, working with situated problems, seems to be of a different nature from these three and will be discussed in a different section.

Conventions for Orientation of Axes

To coordinate or translate between tables or function rules and graphs, one must understand conventions about connections between these representations. As discussed above under facility with representations, Toni understood how to plot points rotely and also understood the convention that consistency must exist between labels on tables and axes for graphs of the tables. Throughout the interviews, however, Toni regularly displayed confusion over conventions for orienting axes. In interview 2, as can be seen in her labels for Alice's graph in problem 2A-1 (Figure 3.13), she had great difficulty keeping straight which axes should be labeled x and y and which should be labeled input and output. This problem gave the table and three graphs as potentially being correct. Students were asked to discuss why they agreed or disagreed with each choice or to create their own more suitable graph.

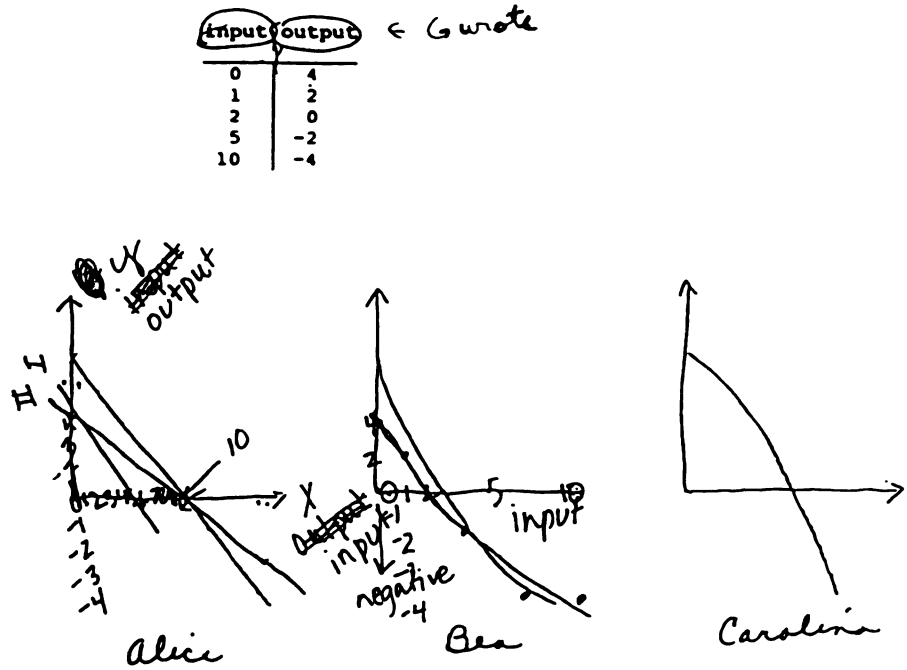


Figure 3.13: Table and Toni's Labels for Alice's Graph in Problem 2A-1

Toni began by explaining why she thought Alice's graph might be a match since she first saw the table as having a constant pattern of change. (The pattern, but not rate, of change was constant.) She called the output column x and then hesitated:

T: But it's got 4 on the x-- This is x, x? [labels vertical axis x]

G: You could put x-- this (horizontal) is the input axis and this (vertical) is the output axis.

T: Ok.

G: So that-- are you using x for input or x for output?

T: X for input.

G: Ok. Then, this would be the, the horizontal would be the x axis.

[T crosses out x and labels x and y axis as directed]

T: Okay. Um, the, or no, it would have to be, x would have to be the [pause] input because it starts with zero. No--[laughing] ok, it would-- x is right down here and then y up here, up the top because it's zero here [pause] and it's going up 4 (vertical).

I had been unsure whether Toni understood the arbitrary nature of naming the independent variable x, so did not want to preclude her naming the output x, but she had needed to know the problem considered input on horizontal, so I told her. In addition, I had needed to know how she intended to use x. Yet even after we had established that x would be horizontal, a few minutes later she again became confused and decided to label the axes input and output to cut down on the confusion of dealing with both x and y and input and output. Again she reversed the names:

T: Um, no. Her y-axis is um, [pause] is, it's better to scale than the x-axis because um, [pause] I'd have to label these input-output, [laughs] I keep getting confused. [starts labeling axes with input on vertical and output on horizontal, which is backwards]

G: Yeah, see this up here at the top? (of the table)

T: Uh-huh.

G: That's why I labeled them input-output.

T: Ok, this is output [pause]

G: Ok.

T: [sigh, laughs] Did I do that wrong again? Um, [pause] let me think. Zero and up four. Output, input. [sigh]

Apparently, Toni had an easier time remembering orientation for x-y labels, but she reversed the labels again and got confused. When she put input on the vertical axis, she used her knowledge of plotting a rote value (0, 4) as horizontal, then vertical, and the concept that labels must be consistent, to determine that input had to be on the horizontal. Thus, she exhibited some understanding of the convention that for a given ordered pair, the first is for horizontal and the second, vertical. The comment, "output, input" at the end was simply her calling out the corrected labels as she wrote them on her paper, not that they were a match to 0 and 4, respectively. In this example, she had two difficulties. She found it somewhat hard to remember x went on the horizontal and in addition, she had no association of "input" with x to help her out when she determined where the label x actually was supposed to go. She did tap

a fairly sophisticated concept of consistency between labels, plus a rote procedure for plotting an isolated pair, to help her reason about the problem.

While part of her confusion in this early, second interview was clearly a lack of associating input with x, as late as the fourth week of class she still had difficulty with orientation, even when x and y were being used rather than input/output. In interview problem 4B-1, where axes of the graph were not labeled in the original problem and she assigned names herself, she assigned y to the horizontal and x to the vertical, but correctly placed them in a table labeled y|x to account for horizontal first. Shortly afterwards in problem 4B-2, she again labeled a table backwards with y first, then read off the point correctly. Once I labeled the horizontal axis as x, she was able to coordinate the data in the table (with a reversed table header) consistently with that in the problem without rewriting the header. (See Appendix A for these problems.)

In these problems Toni had made uneven associations to the mathematical connections between tables, graph axes, and labels. She did not understand the convention that input, or whatever variable is used for an independent variable in a function, is traditionally graphed on the horizontal. She seemed not to associate the word input with its concept as either a value on which some other value depends or as an independent variable in a function. (Functions were treated as dependencies in this class, but the language, independent/dependent variables, was not used.) Part of her difficulty might be attributed to the few, early exercises in which she chose the orientation of axes. It may have just seemed more sensible to her to list the first line of a table as vertical. In each of the problems just discussed here, tables were given vertically, and I have little information about whether this made any difference. While she reversed axes in many problems, she may have understood their correct orientation, but had just adopted some memory cue that itself got distorted. At any rate, the only consistent factor in Toni's labeling of these axes is that she always

labeled the vertical one first, which might be in conflict with vertical tabular representations, in which the horizontal is given first.

Inability to Construct a Graph from an Equation

An example. In class homework and in some interviews, Toni easily created a graph from a table when the table heading was given, particularly when the axes were labeled. Several times, however, Toni displayed difficulties with constructing graphs from an equation or function rule. One difficulty was orientation of the axes discussed above, but there were other puzzling difficulties. At the beginning of interview 4, we discussed graphs she had made in class for $y = .1^x$, $y = .5^x$, $y = 2^x$, and $y = 5^x$. Toni had graphed each as a single point (Figure 3.14). This

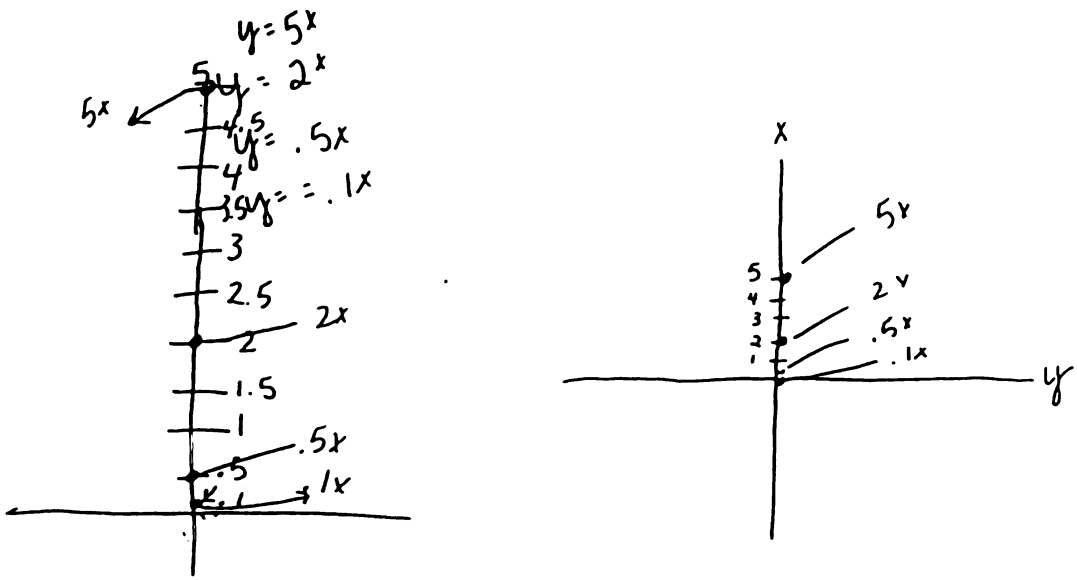


Figure 3.14: Toni's Graphs for Exponential Functions, Original plus Revision

example is significant because it was problematic for the entire focal group. None of the students, except one, graphed any of these functions correctly, and he graphed only the last two correctly. All others graphed either single points on one axis or prominent points on a single axis with straight lines through them.

When first asked to recall what she had thought at the time, Toni thought of x as a vertical axis once more, saying:

> Cause it was point 5 x and this was my x axis [pointing to the vertical] and it had no y so I just put what, I put it on the x-axis because it was just the x.

When I quizzed Toni about what she meant by not having a y, she said:

> Like, uh, a y, 2y-- and like 5x.
>
> [T points to y in top equation, $y = 5^x$, to the x and back to y, then points to a spot on right horizontal axis (2), then to pt above it (5), for the point (2,5)]
>
> I didn't have anything like that. So I didn't think I could go anywhere over on the y-axis [pointing to the horizontal axis] because I didn't have anything for y.

Toni had pointed to $y = 5^x$ indicating the y had no coefficient like a 2 in 2y and then illustrated how having a 2 and 5 (for y and x) would give her a point at (2,5). Recall she had labeled her vertical axis as x. She seemed to be saying a number written with the variable determined its value, and since there had been none in front of the y, she could not move to the right (or left) on the horizontal axis (with label y). The x did have a number with it, 5, and so she just marked a dot at 5 on the x-axis. Toni had assigned an invalid meaning to the constants in the function rule.

I confirmed Toni's thinking that the number in this equation was a value for x to be plotted when I asked her the meaning of the dot at 5 on the vertical axis:

> G: You put a dot here because you-- [pointing to the dot at 5 on the top graph]
>
> T: That was 5 x. Y, but I think maybe-- since this was my y (the horizontal axis), it didn't go over anywhere on the y, so I just thought I'd put it up on the-- [points to the dot at 5].
>
> [a brief interlude where she describes the right graph as a reproduction of the left one for clarity]
>
> G: Okay, so let's look at that one (right graph) for a moment? When-- This dot, then, represented? [pointing to the dot at 5 on vertical axis.]
>
> T: Five x.
>
> G: Five x?

T: Um-huh.

G: So what's the value of x?

T: 5.

G: So, this, uh, I heard two different things. I heard the dot was 5x and I heard that x was 5. Are those the same things or are those different things in how you were thinking about it?

T: I just used the number before x and put it on the graph.

In this exchange, Toni admitted directly that she used 5 for the value of the x coordinate. Again she seemed to consider the y-coordinate as zero, saying her y didn't go anywhere (away from the axis). When I switched to a simpler function (from my perspective), $y = 5x^3$, at first she entered $5x^3$ on the computation screen of the calculator, getting a y value of -5. Perhaps she had intended to produce a calculator graph by entering it under the Y= key (function definition key) and graphing it. Both Toni and I were confused by this output. I did not stop to think that the variable x was a computer variable that had a previous value stored in its memory location. When I asked what the -5 was, Toni said it would be the y because "it" said $y = 5x^3$. So she seemed also to be computing a y value. She went on to say she had no value for x, so she didn't know what to do (about a graph). I then asked her to graph $y = 5x^2$ and $y = 5x + 2$ because I thought they might be more familiar. She had seen many graphs of linear functions and at least one graph of $y = x^2$ in the 3 weeks prior to this class. Toni again started to use the calculator, and unfortunately because of confusion created by her using the variable x on the computation screen and since this was at the beginning of an interview with many problems, I opted not to pursue the issue except to ask in general about her experience with the three polynomials. In spite of her experiences in this class, she said she had never had to graph problems like these.

It appeared from Toni's comments about the exponential functions that she was simply using numbers that were given as part of the equation for values of x and y.

She had seemed not to distinguish the role of 5 in 5^x from that in 5x. Notice that she barely raised the exponent when she wrote her equations, and she had read it as five x. There is other evidence she interpreted constants in function rules given as equations in this way. She used numbers in the equation for values to graph on the posttest, question 6, a question she had left blank on the pretest. In that case, Toni sketched only the point (3,4) for the equation $y = 3x + 4$, left the graph of $y = -x^2 - 2x + 1$ blank, but sketched the point (3,0) for the graph of $y = 3^x$. In these examples from the posttest, apparently she used the same algorithm as she did in this discussion, but perhaps by this time had straightened out the orientation of the axes to put x on the horizontal. In addition, her failure to respond on the pretest supported her claim that she had not had these kinds of problems before. Yet earlier when I had asked her to graph $y = 5x^3$, and she had computed a number, -5, on the calculator, she said it was y and couldn't graph because she had no x. That equation had a 5 right next to the x, so she could not have been thinking *only* that the existence of the 5 near the x in an equation would give the value of x.

There was one other instance when Toni attempted to assign direct meaning to individual numbers in a function rule definition. In interview 3, she attempted to interpret individual terms of a polynomial as separate quantities in the situation. In that problem, profit was given as a function of ticket price. She was not, however, immediately engaged in graphing, and in addition, an attribute of the problem itself suggested this interpretation. This problem will be discussed at length under the section, Special Nature of Situated Problems.

Inexperience as an explanation for graphing difficulty. There could be several explanations for Toni's inability to graph by hand a function given as a rule or an equation. Recall Toni's inclination to look for procedures to solve problems, especially with symbolic forms. Here she was confronted with a task, graphing a relatively new class of functions for which she had no procedure, and made sense of

it the best way she could. When I asked her to graph more familiar equations and she attempted to use the calculator, she might have intended to use the grapher, a procedure she had used many time in class, but had simply chosen the wrong screen in which to enter $5x^3$. The confusing results, including my temporarily not knowing why she got the results she did, might have undermined her ability to produce a graph on the calculator. It is also possible Toni was attempting to create a pair of numbers by using the computation screen, but did not know what to substitute for x.

When the teacher gave the students this task to graph exponential equations, he stressed that he wanted to know what they thought, which they (correctly) took as meaning to graph by hand rather than using the calculator. In Toni's case, I had the strong sense that she did not know she was allowed to choose an arbitrary value for x and then compute the y from it. I felt she did not know how to proceed because she was missing one step in a procedure, the need to create a table. Several times she said she had no value for x, and once no value for y (either they did not exist, or she did not know how to find them).

It is quite possible Toni could not make sense of these translation problems because she lacked the experience necessary to understand she could arbitrarily choose values for x in the domain of the function. Toni had told me during this interview that she had had no previous experience graphing these equations, $y = 5^x$, $y = 5x^3$, $y = 5x^2$ and $y = 5x + 2$. As a teacher I was surprised by this comment, given the amount of experience I thought I had seen in their class. Toni was correct, however, in the sense that students had had experience plotting points for numerous graphs, but had not, in fact, created graphs by hand from equations without guidance from the teacher. Prior to the class in which these exponential functions were graphed, students often plotted points of data relating two variables in situations, but axes were always labeled by the quantities in the situation, not single variable names. Although the students observed the teacher generating arbitrary values in the process

of creating a graph for an equation, when students created graphs by hand, they were either given lists of the first coordinates or told how to extend existing lists. It was not until the week after this interview, when students were introduced to quadratic functions, that they were asked to generate their own lists of pairs. Even then, the text admonished them to choose a value that was sensible in the situation and said what increment to use between numbers. When these exercises were given, they were often done together in class with the teacher, so the teacher or other students would suggest numbers.

Toni often relied on remembering a procedure when working with symbolic representations, and when she could not remember it or did not know it, she often became stuck. Her claiming not to have experience with these kinds of problems may indeed have been accurate because she had not experienced one aspect of this procedure. She might not have realized choosing one's own x's was "allowed." Another possibility is that her difficulties here could be associated with her concept of variable.

Conception of variable as explanation for graphing difficulty. In order to determine if students' difficulty with graphing these exponential functions was related to their conception of variables, in interview 4[6] I added questions on variables modified from the CSMS study (Hart, 1981; Kuchemann, 1981). For students to be able to graph equations, they would need to know that variables can take on many values, that is, serve as a generalized number.

Results from those questions indicate that Toni had a relatively naive, or tentative, conception of variable, but that she understood variables could take on many values. In problem 4A-2, "$x + y + z = x + p + z$ is always, sometimes,

---

[6]The exponential graphing problems were discussed with some students at the end of week 3, which accounts for how the discussion with Toni and the problems on variables both took place in interview 4.

or never true? Explain." She said,

> I think it would be sometimes because it could have the same value. The x and x could be alike, and the z and z would alike, but y and p could still have the same values, but if they didn't, then it wouldn't be the same.

When I asked if the two x's would always have the same value, she replied they would: "If they are in the same problem." So she also understood a constraint on variables within an equation. She gave further evidence that variables could take on many values in problem 4A-7, the classic blue and red pencil problem:

> Blue pencils cost 5 cents each and red pencils cost 6 cents each. I buy some blue pencils and some red pencils, which cost me 90 cents. If b is the number of blue pencils bought and if r is the number of red pencils bought, what can you write down about b and r? (Kuchemann, 1981, p.107)

Toni first treated b and r as specific unknowns, and by testing cases, successfully found a pair of numbers for b and r that would work. When I asked what else she could say, she said b and r could have different values and produced a correct example. Then I asked Toni what she could write down about b and r; she wrote b + r = 90. She explained:

> T: You can have, uh, you could have different numbers for b, and you could have different numbers for r and it would come out. So it would be b + r = 90 [writes it], but you can always have, you can have different numbers in there.
>
> G: And b stands for?
>
> T: Blue pencils and r stands for red pencils.

As this was the last variable question and we needed to go on to the interview questions more directly related to class, I did not pursue what she meant by the last statement. It is clear from this example, however, that Toni realized that variables could take on many values, as well as stand for specific unknowns in equations. Apparently, knowing that variables can take on many values was not sufficient for Toni to generate ordered pairs for graphing.

Another possibility for difficulty in graphing that is related to variables is the ability to see the function rule (equation) as representing a whole set of pairs. Seeing

a function as a whole set requires recognizing that a variable expression stands for a single number. Toni's understanding of this concept was weak.

In problem 4A-3, "If a + 6 = 25, what can you say about what is a + 6 + c?", Toni solved for a, writing "19 + 6 + c" and said, "but I wouldn't know what to put in for c because I didn't have a value for it." It is not clear whether she was simply stating that was all one could write or whether she thought of c only as a specific unknown to which she had no access. She did not, however, consider the expression a + 6 as a single number; otherwise, surely she would have substituted 25 into a + 6 + c and written 25 + c, rather than solving an equation and making a substitution of 19. Being able to think of a + 6 as a legitimate number requires accepting a "lack of closure" (Sutherland, 1989) on the value of a, that is, you can create or use a in symbolic expressions even though you do not know what number it stands for. Accepting lack of closure may be related to seeing the equation as representing a whole set of y-values, one for each x.

Toni's response to problem 4A-4, "Which is larger, 2n or n + 2?" might make it appear that she accepted lack of closure. Her comment:

> Um. 2 n would be larger because if both n's are the same value, then this (2n) is multiplied by n and this [pointing to n + 2] is only added to.

indicates she seemed to attend principally to the existence of two different operations and judged "multiplication makes bigger than addition." To answer this question successfully, one needs to view both 2n and n + 2 as representing variable expressions and then consider a comparison between their values for a particular n. Kuchemann (1981) called a relationship between relationships that could itself be evaluated a second-order relationship (a common term among mathematicians). Such a relationship can be, in general, a fairly sophisticated one. Toni saw this problem as two expressions to compare, but not with the sophistication Kuchemann predicted.

She evidently saw it only as a comparison of two processes, one with multiplication and one with addition, a result consistent with her procedural orientation.

Finally, problem 4A-6 gives fairly firm evidence that Toni was unable to accept lack of closure in variables. The problem stated:

> Sam drew the figure below while babysitting for the neighbor. It has n sides; each has length 3. Unfortunately, the baby spilled paint on the drawing just as Sam was calculating the perimeter of the figure. What is the perimeter?

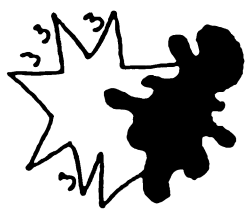


Figure 3.15: Problem 4A-6

Toni counted up the 13 visible sides, added 13 3's together and said that was as much as she could do since she could not see the other sides. Toni saw n at best as a specific unknown. She clearly did not consider it a quantity that could be acted on or written in a relationship.

In summary, although Toni could see a variable as a generalized number in an equation, this knowledge was not sufficient to help her create ordered pairs for the four exponential functions. In addition, that she saw variables as specific unknowns may have contributed to her belief that not knowing a value for x prohibited further activity in the graphing process. Finally, Toni could not accept lack of closure, that is, she could not use a variable to represent a quantity and operate on it or build expressions with it while the value of the variable was unknown. That she could not think of variable expressions as single entities, which themselves could have many values, may have limited her ability to see these equations as representing whole sets of related numbers.

Creating a Function Rule for Table Representation of a Linear Function

A third area where Toni had difficulty with representations of functions was in problem 2A-2, where the task was to create a function rule from a table of input/output values. This problem was similar to those given in class, and I expected Toni to more readily mimic procedures that had been gone over several times. This turned out not to the case.

Toni could readily compute patterns of change for linear functions given in tabular form, but could not translate them into their symbolic form. Problem 2A-2, which gives the table for the function $y = -5x + 15$, is given in Figure 3.16.

> For the table below, what is the function rule? What is the output for input 15? input 30?
>
> | input | output |
> |---|---|
> | 1 | 10 |
> | 2 | 5 |
> | 3 | 0 |
> | 5 | -10 |
> | 10 | -35 |

Figure 3.16: Problem 2A-2

Toni quickly determined the pattern: for input going up by one, output decreased by five. She mentioned the number of decrements between -10 and -35, first as seven and then correctly as five. She was able to fill in the intermediate entries between -10 and -35 and to describe the change as, "the output is decreasing. The farther that you go up in input, the output is decreasing by fives." Toni was stuck, however, when it came to generating a function rule because she could not recall the process done in class. I tried to help her first by calling her attention to input going up by one, as the output decreased by five. After she checked that "function rule" meant "how you find the output," Toni was still stuck. Thinking she may be having trouble with discerning the multiplier because it was negative, I tried

to help her by suggesting a different problem, which I thought was simpler. It went up, rather than down, by fives. Toni first suggested "input plus five," an addition, but rejected it because it did not give 10 for input 1, nor 15 for input 2. Clearly, she had not understood I was proposing a different problem, so I next tried to get her to think about what it meant to go down by five by asking how we usually write going down by five. When she suggested "negative" and I echoed "negative five," she said she still did not understand what to do. She had accurately connected negative five with decreasing, but not to multiplication. Finally, thinking that to focus more directly on the computation that was required, I asked explicitly about what needs to be done to the x:

G: If you call the input x, what do you have to do to x to make it go down five every time you make x go up one.

T: Subtract five?

G: Subtract five. So, how would you write that?

T: X minus 5.

Toni had connected the pattern of change to subtraction, a valid operation for going down the table, but an invalid one for going across the table, about which I was asking. Finally, to help her realize that multiplication, rather than subtraction, was required, I had her create her own outputs for inputs, 1, 2, and 3 to her suggested function, $y = x - 5$. She readily saw that the output was increasing by ones, not decreasing by fives. Toni did not find my exercises useful in finding this function rule. Thinking that an output of 0 for input 1 might be easier for her to see the pattern across the table, I tried once more and gave her another table of values (for $y = -5x + 5$). It was just as hard for her, so I reassured her that it was okay not to solve the problem, and we left it.

Toni readily saw the pattern of change in this table. She was able to connect changes in output with increments of one in the input, which some other students

ignored. Yet she could not connect the additive change going down the table entries with the multiplicative change going across the table, apparently an extremely difficult task unless one has an algorithm to do so. There could be several reasons that made this specific task harder than some others. In class students had had practice with finding function rules from tables, but only one had been for a decreasing linear function. Lack of experience might have contributed to Toni's difficulties, but she seemed to have the most difficulty because she needed both to connect change with multiplication and consider a negative at the same time. While she tentatively considered -5, it is unclear to what operation she intended to connect it. It is possible that she connected decreasing only with subtraction, as text here suggests. When I was discussing a function that increased, she had connected "increase" with addition. In this case, Toni's everyday language associations with specific (and limited) operations inhibited her finding a solution to the problem, as it did for some other students as well. Given Toni's inclination to look for patterns and also to use procedures, it is surprising that she had not developed a rote procedure for this problem task, since several were done in class within the two days preceding this interview.

Part of the difficulty may have been communication problems between Toni and me, partly because we had different points of view. Because this was her first interview as a focal student, I felt it was necessary that she succeed to some extent. So when Toni became stuck, I attempted to help her. Yet each alternative problem I proposed contributed in some way to her difficulties. First, when I decided Toni was having trouble with the negative number and I switched to an alternative problem, she did not switch because I failed to make clear it was an entirely different problem. Then when I asked for an operation for x to make it decrease by 5, I was thinking "across the table." Toni gave a correct answer for evaluating successive table entries, but not for a computation on x to produce y. Finally, I made a bad choice of

a new problem in an attempt to help her more readily see the pattern across the table. In short, I had underestimated the difficulty Toni would have in seeing the patterns when going from covariation down the table to a single symbolic rule that encapsulated the process of computations on x to produce the table. From her point of view, she was actively attempting to make sense of the relationships in the problem, and clearly saw the patterns of change in the two sets as salient features as she responded in terms of them several times. She must have found my suggested problems puzzling at best. In any case, my repeated attempts at getting her to focus on the relationship across the table failed, no doubt partly because this is not the straightforward problem that we mathematics teachers think it is.

That Toni saw this problem so differently from me indicates that there was a mismatch in what features of this problem we saw as easy. Toni found dynamic features, including direction and amount of change, especially salient. Yet to complete this problem, she needed to see change encapsulated into a single quantity, -5, that would become a multiplier in a static representation, the function rule. Apparently, making this transition was quite difficult for her (as it was for other students as well). I did not recognize the magnitude of the difficulty at the time. Our struggle raised questions about what activities I might have given her had I been prepared for her point of view.

Situated Problems Confound Problem Solving

Students were introduced to all functions through situated problems. In previous sections, examples were given that illustrated how the problem situation supported Toni's reasoning. She relied heavily on the situation itself when working with graphs of functional relationships and for providing everyday knowledge to make decisions about relationships in the Going to School problem. Yet situational contexts of problems can invoke informal knowledge that confounds problem solving, as will

be seen in the problem below. This section discusses the problem, the nature of the specific difficulties that Toni encountered, and understanding models in general.

The Profit Problem

The problem in Figure 3.17 below was given during the third interview.

> Carl has been assigned to come up with the price to charge for tickets for a concert the senior class is using for a fundraiser. From previous years, Carl has learned that if they hire a celebrity emcee, the function relating ticket price and profit was
>
> $$P(t) = -100t^2 + 700t - 200.$$
>
> Carl reasoned that if they didn't hire the emcee, they could save his $200 fee. His friend Beth said he should reconsider the problem because he would need to lower the ticket price by at least 3 dollars to get anyone to come if no emcee was hired. She told him he could consider this new situation by using the function
>
> $$P(t) = -100(t - 3)^2 + 700(t - 3)$$
>
> Was she correct? Which plan should he follow?
>
> Does the size of the auditorium make any difference?

Figure 3.17: Problem 3, The Profit Problem

I had intended this interview problem to allow the possibility for students to discuss any of several aspects of their understanding about quadratic functions and representations with which they had gained experience in the class. They had studied quadratic functions, explored the shapes of their graphs, patterns of change in data, and had had several exercises translating quadratic (and linear) functions in naked context horizontally or vertically. Students had seen many examples of functions in the context of school events, which involved profit, attendance, and ticket price. Students had seen only one homework exercise involving profit as a function of ticket price directly, and it is unclear if they did this exercise.

When I wrote the interview problem I had intended to make a complex problem to flush out what ideas they held strongly as suggested by Ginsburg (1977)

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and Confrey (1980); hence, I made it a comparison of two positions. I wanted to give students an opportunity to use several strategies of problem solving: propose computational solutions to compare, make connections between graphs and symbolic forms and translation concepts, discuss properties of quadratic functions in various representations, use the calculator, or something else altogether. For example, I thought someone might create and explore graphs of the two functions on the calculator. I chose values for the polynomial that were similar to some in the text and which would make ticket prices reasonable to what school events might cost. I intended the 200 as a cost to match that for the emcee in the rule because it was recognizable and would allow a discussion of other potential costs. Finally, I was aware that a student might see (t - 3) as a new ticket price, where t was the old ticket price, which would make the use of t in P(t) in the second formula incompatible across the equation. On the left t would be new ticket price, while on the right, t would be old ticket price. But students had not seen P(t - 3) notation, and writing the functions the way I did gave a student the option of raising this issue. I reasoned correctly that, in general, students would ignore the t on the left. My choices did indeed allow Toni to display what issues she held strongly, but my choices, as will be seen, also contributed in some ways to making the problem difficult for her and others. Finally, I added the oblique question about the auditorium size to allow students to discuss its impact, if any, on the function and to allow discussion that the auditorium size was relevant only in the creation of the model and was not an issue here. I wanted to give the students an opportunity to say (although I did not expect it) that the model was based on old data and without new data, one could not change the model. I certainly did not expect the discussion on validity of the model to take the turn that it took for Toni, as outlined below.

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Difficulties in Interpretation

That the problem was situated affected Toni's problem solving in many ways. She operated with an informal model for profit that made it difficult for her to keep straight the meaning of variables and to interpret correctly the functions given.

Informal model and meanings of variables. Toni had difficulty remembering what meaning the variable had. She began the problem by multiplying out the second function rule and, after making several errors, wrote the function rule $P(t) = 600\ t - 1400$. She then attempted "to solve it" by adding the 1400 to both sides. When I asked her what she was doing, she identified p as profit, but while she knew t was associated with tickets, she forgot what t stood for:

T: Fourteen hundred, the profit, the p stands for profit er, [pause]. I think it's profit and ticket or price of ticket. [pause] Umm, I think it's profit and then ticket price or. [pause]

G: You think p is profit? Is that what you said?

T: Yeah.

G: And t is?

T: Tickets. Tickets sold or ticket price or.

G: Tickets sold or ticket price.

T: Yeah. Um, let's see.

She reread the second paragraph and correctly deduced from the verbal description of dropping the ticket price three dollars that t in t - 3 stood for ticket price.

Surprised that she did not look at the phrase in the first paragraph that stated P was a "function relating ticket price and profit," I asked if she could tell in some other way, but she did not understand my question. Finally, I asked her whether the t in the bottom rule was the same t as in the top rule. At first she thought not, but quickly reconsidered, citing similarity of the coefficients in the two rules. For the second time she introduced the possible interpretation of t as the number of tickets:

No, because that t seems, well it would have to be because this is 700t down

> here and then it's 700t up there but, [pause] uh, it wouldn't be that much for a ticket. [pause] It could maybe be total of people, er--.

Here for the first time, Toni considered 700t as a model for either profit or income, considering 700 as ticket price and t implicitly as number of tickets. She rejected this interpretation because 700 was not sensible for a ticket price. Then after reflecting on her answer, she guessed again that t was the number of people attending (number of tickets). At this point, I suggested she reread the problem. She did so and quickly determined again that P was the profit and t was ticket price, but volunteered that "This minus 200 [pointing to top rule] is money that he figures he can save if they don't hire an emcee," a point critical to later discussion.

Toni had trouble sorting out what the variable letters stood for. She knew the problem was about profit, but the meaning of t kept eluding her. First she had said she associated t with tickets or ticket price, then had found out the meaning in the second paragraph of the problem statement. After I asked her whether the t's were the same in both rules, she had decided they were because of the similarity of the rules, but had introduced for a second time that t was number of tickets, admitting she was attempting to interpret 700t as price times tickets. Her interpretation of this quantity implies she was operating with an informal model of profit as price times ticket, a model that showed up over and over in this section. In addition, even though she had identified P, the function name, as standing for profit, she assigned one term of the polynomial, 700t, as profit. In addition, she pointed out the 200 as a cost, implying she was also confusing income with profit. Toni's informal model for profit interfered with her ability to accurately match variables in the function rules with quantities in the situation.

Attaching separate meaning to individual terms in the polynomial. Toni's informal model generated difficulties with interpreting the function rule itself, that is, with the mathematical representation of an aspect of the problem situation. In the

preceding text passage, Toni gave, for the first time, individual terms or coefficients of the polynomial separate meaning as quantities in the situation. This phenomenon seemed to be provoked both by the suggestion in the problem of 200 dollars for the emcee and also by her informal model of profit (or income) as price times number of tickets, which she saw as a reasonable interpretation for the middle term of the polynomial until she realized the 700 was too high for a ticket price.

Toni continued to try to interpret individual terms as separate quantities in the situation. After a 15 minute diversion to pursue questions about her algebra computations that had led to her first function rule, $P(t) = 600\,t - 1400$, I asked Toni what she had been doing before the diversion. She said, "Um, I tried to find how many tickets, or the ticket price and profit but, I'm not real sure how to do it." She again spoke of t as "how many tickets," but quickly corrected herself to repeat an earlier response that the problem was about ticket price and profit. Finally, she said was stuck. Judging that the problem task might be too complex, I decided to narrow the focus to only one polynomial and asked what she could tell from the first rule. She responded:

> Um, [long pause as she looks at problem] this ($-100t^2$) was maybe the cost to them for tickets and this ($+700t$) could have been the profit that they made or [pause] um. I forgot what I was thinking. [pause] this ($+700t$) could be the, uh, profit that they make in tickets and this (-200) would be the 200 dollars you're taking away for the emcee that they hired. [long pause] I don't really know how to put it together to--.

Toni had again attempted to assign situation variables to the individual terms of the polynomial, identifying 700t as profit, 200 as the money to pay the emcee, and $-100t^2$ as perhaps the cost of the tickets. She recognized that costs should be subtracted when computing profit, but seemed to confuse income, 700t in her model, with profit. Her reference to $-100t^2$ probably refers to a simulation about a talent show that had been carried out on the first day of class. In that situation, there were numerous costs, including a fee for printing tickets. In any case, Toni associated terms or

coefficients of the polynomial with quantities from the situation, and she defined 700t as profit, implying a profit model of number of tickets times ticket price. In addition, she seemed to be trying to connect this problem to a problem she had had in class.

Toni's attempts to give each term or coefficient a separate meaning as a situation quantity surely is suggested by the problem statement itself. One distinction between the two situations described, as she herself pointed out, was that the cost of the emcee in the first was 200 dollars. It is the only cost given, and 200 does appear explicitly in the rule. Toni seemed unable to proceed, however, because of her inability to reconcile the rule given with her informal model of profit.

Attaching particular meanings to pieces of symbolic representations was not confined to this interview. Instances occurred, as discussed earlier, when Toni assigned the 5 in $y = 5^x$ to x as a coordinate value for graphing and on the posttest when she graphed $y = 3x + 4$ as a single point (3,4) and $y = 3^x$ as (3,0). Teachers of beginning algebra sometimes see interpretations of coefficients as values for variables in graphing, yet Toni's interpretation of the terms in the Profit Problem here seemed influenced more by the situation and the problem statement than the form of the polynomial. Her activity on these occasions confirms a tendency to (wrongly) interpret the constants separately, but her thinking of 700t as profit may have contributed to or at least supported this erroneous thinking since she had, on occasions prior to this, evaluated or interpreted second degree polynomials correctly in other situated problems.

Recall under the earlier discussion of Toni's facility to interpret points on a graph that on problem 9 on the pretest she had interpreted correctly (as she did on the posttest) three questions asked about the *profit from a rock concert as a function of ticket price*. There the function was given as a graph. On the posttest (only) problem 13 she correctly answered questions about *profit for a school play as a function of ticket price* given in symbolic form, $P(t) = -50t^2 + 800t - 1950$. Of course, she

might have "learned" how to do the latter as a result of this interview question, and in the pretest she was clearly influenced by the graphical form.

In addition, the problem statement may have contributed to Toni's confusion because it was complex. Yet the difficulties produced by her informal model of profit might not have been brought to light without a problem task sufficiently complex to require that attention remain on the situation itself, rather than being temporarily ignored because a problem solution path was immediately clear. In the pre- and posttest problems on profit, the problem could be interpreted directly as a generic problem without regard to whether the problem was about profit and ticket price, cost as a function of widgets produced, or height of a rocket after t seconds. Unlike in problems where Toni expected to have a working procedure and became stuck when she could not remember it, here she tried to understand the complex problem as a whole. In the process, her thinking was more visible because the problem was situated and the task complex.

Not understanding the function dependency. Part of understanding functions is knowing that a function rule represents two quantities, one of which depends on the other. That the problem was situated also contributed to Toni's difficulty understanding the function rule as a relationship or dependency between two variables, ticket price and profit. As can be seen below, I repeatedly failed to get Toni to accept the function rule as a given relationship that could be interpreted independently from its specific situational origin:

G: Are you trying to figure out where this function came from? Or where the numbers in the function came from?

T: Yeah.

G: Ok. Why don't you just assume that they have a long history at this school for making, for having these shows and they know that this function rule is what the profit will be if they're given the ticket price and they hire an emcee for 200 dollars, ok? They know that. That's just information that's given to them. Ok? So, how, can you find out anything about profit from, um, from any ticket price?

T: [T points to the top formula] Could this maybe be from the year before?

G: But like the whole formula could be from the year before. Not the individual pieces but maybe they keep a file to help the people the next year and they say this is the formula you should use if you hire an emcee for 200 dollars. Ok, so if like a ticket costs 10 dollars, if they charged 10 dollars, can you tell me what kind of profit they might make? [points to top rule] If they hire an emcee for 200 dollars and if they charge 10 dollars for a ticket, do you know how much profit they will make?

T: You have to know how many people are going to be there and how much the tickets cost to them to--

G: You have to know how many people are going to be there?

T: To know the profit that they're going to make.

Twice I had tried to explain where the rule might have come from and indicated its generation could be ignored so Toni might accept it. I had tried to focus the task on using the rule by suggesting first that she consider a single ticket price and then later a specific ticket price, but Toni remained uncertain about how to interpret the rule. Even after I had explicitly stated that "they say" to just use this formula, she resisted. Toni contended you could not determine the profit without knowing the number of people and the ticket cost, again implying her informal profit model. Toni's insistence that the rule must refer to the past and that the number of people coming, that is, the number of tickets sold, must be known before you can compute profit indicates she was unable to consider this function as a relationship between profit and ticket price. She seemed to be thinking of it as a formula where values associated with profit must be known before profit could be computed.

I tried a third time to help her understand the function dependency by focusing her attention on the rule itself.

G: If you look at that profit rule, what does that profit function rule do? What does that, what's it a function of?
[long pause as T looks at problem]

T: It's price and profit?

G: It's price and profit, right? Does that mean you have to know how many people are coming?

T: Um, [pause] not really, but you don't, you won't really know if anyone will buy the tickets though, for the ticket price and the profit, or this has already been done so it could be from the year before.

G: Um-huh, so it's like a record of their experience from the year before.

T: I don't know where this is coming from. This [points to middle term] could mean that maybe they sold 700 tickets, but I'm not sure what this is [points to -100], the negative.

Even though she was able to identify the variables and agreed that number of tickets was not among them, Toni still could not accept that it was possible to compute the profit without knowing how many people would come. She had again referred to the rule as relating to the past and had expressed the need to know both how many people came and the ticket price before profit could be computed. Her idea that she could not make sense of the source of the rule, coupled with this informal model for profit, was so strong that, in spite of my suggested directions to proceed with the rule as given and my earlier admonitions to the contrary, she had again attempted to interpret individual terms as separate quantities in the situation. Toni's inability to make sense of the situation inhibited her ability to see this function as a dependency between two quantities, profit and ticket price.

Difficulty understanding the nature of a model. That the problem was situated contributed to Toni's inability to understand the nature of this function. Since she could not understand it as a relationship or dependency between profit and ticket price, she was unable to consider it as a model for predicting profit for a given ticket price. Initially she had treated one of the function rules as an equation to be solved, and her insistence that the problem could not be done except after the event suggests she was treating the function rule as a formula into which one substituted numbers. Since her informal model for profit included number of tickets, the formula could not be used until that number was known. This hypothesis is consistent with Toni's

procedural orientation to solving symbolic problems, as discussed earlier. This function rule made no sense to her because she could not account for the other parts of the formula in terms of the situation in a way consistent with her informal model of profit as tickets times price, and she believed she could not compute profit until she had values for all quantities in her model. In the past, whenever she could not recall a procedure or felt it was incomplete, she became stuck.

An alternative hypothesis is that Toni might have been considering profit as a quantity that could never be computed before an event had taken place and thus might have had difficulty with the idea of a model for profit as a predictor per se, but it seems more likely that she was hindered by her notion that to compute profit, besides ticket price one needed also to know the number of tickets, which was not available until after the event. In any event, she was unable to apply the rule directly or to resolve its role as predictor of future profits with either her understanding of the function rule as a formula or her everyday knowledge that profit for an event cannot be computed a priori, particularly because one needs to know the number of tickets sold, as well as the ticket price, in order to do so. In each case, her everyday knowledge about the situation interfered with her ability to understand this function.

Finally, after I tried again to explain I was not interested in the source of the numbers and three times repeated my request for profit based on a ticket price of $10, she "caught on" that all I was asking her to do was to use that function to compute a profit from a $10 ticket price. Again she announced that the data had to refer to the past and even after I had explicitly told her to trust the formula, she questioned whether or not what she was to do was to substitute 10 for t.

What is so striking about this example is that usually, given an opportunity to compute a value in an expression, Toni would do so. She certainly was capable of it since in other interviews and in class she had demonstrated her ability to compute function values from a function rule given in a similar form. Yet her personal

understanding of this problem--that she needed another intermediary variable, the number of people attending--was so firm that she could not be persuaded for most of the interview to "go into school mode" and just work with the function as she did during the posttest on two profit problems, both functions of ticket price.

Once Toni understood my goal, she continued the problem. She computed on the calculator the profit for several different ticket prices, $6, $4, and $3, but seemed to do so because she perceived correctly that this was what I was suggesting, not because she understood why she was to do it. She correctly interpreted each profit and concluded that at $3 and $4, the profit was the same, saying, "So, maybe that means the maximum people that could go, did go, at those prices." Again she raised the issue of the number of people attending.

Hoping that she might make a connection to a Cartesian graph, I asked Toni if there was some other way to determine the maximum profit. In her response, she displayed several difficulties that were consistent with those discussed earlier. She suggested graphing the points, and when I asked what she would graph, she said, "Um, the price of the tickets and how many people, er, er, the price and the profit." , another reference to people attending. Toni chose the STAT menu to graph discrete data points, but while trying to decide what quantity x and y stood for, she chose labels that were reversed from standard convention and assigned a profit of 1000 to x for the first point. As she struggled to determine the y value, she said:

> And y would be [pause] what, um, I'm not sure what. I understand the other part. [points to the top function rule]. I don't know what they mean in here [pointing back an forth to the terms of the function rule] for people or--. It shouldn't matter, uh. A thousand should be the profit and ticket price was--three.

Once again she attempted to make sense of the coefficients in the polynomial, but caught herself because she had "learned" during this interview that you could compute profit without knowing the number of people attending.

Because we were out of time, I asked her how such a graph could help her find the maximum profit in a way that was different from what she already knew, and she indicated you could look at the points and see where it was maximum. I quickly sketched an x-axis (only), made a graph of the four points above it, calling out the ticket prices as I sketched the points and we jointly recalled each profit value. I asked Toni what she would have done next. She suggested switching the axes so that ticket prices were on the vertical and profit on the horizontal. In addition, she measured profit from right to left, thus making the profit for ticket prices, $3 and $4 appear in a vertical line so she could see they were the same distance (toward the left) from the vertical axis. The remainder of the interview was taken up in our trying to reconcile this orientation of axes with a standard one since she had measured positive values to the left, rather than to the right, of the axis. Toni had answered my question by describing how she would orient a graph to make her results more visible, rather than the question I was asking, which was what new information it would provide. In the process, she switched both axes labels and orientation of numbers.

In summary, Toni's understanding of the situation confounded her ability to solve this problem. Even when the problem task was narrowed to focus on only one of the functions and in spite of profit being defined explicitly as a function of ticket price, she associated profit with the product of price per ticket and number of tickets (number attending). Over and over she interpreted t as the number of tickets in spite of my focusing attention on the meaning of t and her own explicit corrections of its definition. Thinking of t as number of tickets and profit as a product of tickets and price persisted throughout the interview, manifesting itself in Toni's need to interpret terms of the polynomial as quantities from the situational context and making it seem to her impossible to compute profit without knowing the number of people attending.

This confusion made it almost impossible for her to make sense of the predictive nature of the model even after she computed profit values for specific ticket prices.

Connections Between Conceptions of Function Classes and Representations

Function classes have attributes that are peculiar to each class. Recognizing the distinctions between classes is part of understanding functions. Several concepts of functions were implicit in Toni's understanding of representations of the three classes: linear, quadratic, and exponential functions. She had connected aspects of linear and quadratic functions with specific representations or salient features of representations. Some functions were associated with shapes of graphs, others with patterns of change in their output or descriptive phrases about output, and still others with specific symbolic forms.

Function Classes and Shapes of Graphs

Linear and quadratic functions. One way Toni classified linear and quadratic function classes was by their graphs, although she knew quadratic functions only by the term parabola. For Toni parabolas were U-shaped graphs. In interview 5, when she was given the sort problem, the first category she created was about parabolas. After looking at the first seven cards (G, E, R, B, S, J, K) and coming to card D, a parabola, she matched it with G, a general, U-shaped graph. She said, "I'm putting these together because their shapes are kind of like a parabola." As she sorted the remaining cards, she distinguished all cards with graphs, except M, which could have been interpreted as a picture rather than a graph, either as "graphs that look like they're parabolas (T, J, D, G)," or graphs that are not parabolas (W, E, K). Cards T and J are of exponential functions, but Toni explained that both T and J looked like they might be parabolas, presumably because the graphs could be extended into parabolas if another quadrant of the graph were shown.

Later in the same interview, Toni gave both a verbal and motion description of parabolas as U-shaped. I had asked her to try sort the cards in ways that "related to

the kinds of functions that we had studied in class." She again picked out J and T as parabolas, added M, but identified W (Figure 3.18), as a quadratic. After a brief diversion to discuss equal change in a function (see below), I helped her understand that quadratic was the name of the function class and that parabola was the name of the graph associated with it.

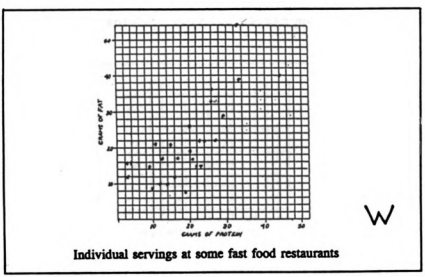

Figure 3.18: Sort Card W

In the process of trying to determine which graphs were connected to the name exponential functions, Toni gave a clear description of how she classified graphs of both linear and quadratic functions. She had just identified W, the scattergraph, as a possible exponential function (non-linear/non-quadratic): "because it (the graph) doesn't go up and come down, it just goes up here. It goes, it just goes all over." As I continued to question her, she contrasted it with both a parabola and the graph of a linear function. She said:

> I think it just changes all over because the linear one stays straight, and the parabola goes up-- [tracing a U-shape in the air]. It can go down; it has a curve, [G: Um-huh], but this one, this one just goes all over. It doesn't have a curve.

Exponential functions. Toni seemed not to have a shape associated with exponential functions. Several times I asked her to recall from class a graph that was different from a line or a parabola. Once she suggested there might be a name for some of the general, irregular graphs that we had studied early in the class. At one point she indicated that graphs with unequal change in output for equal change in input had a name, but in general did not seem to recall a prototypical graph for an exponential function. Toward the very end of the sort problem, after a long

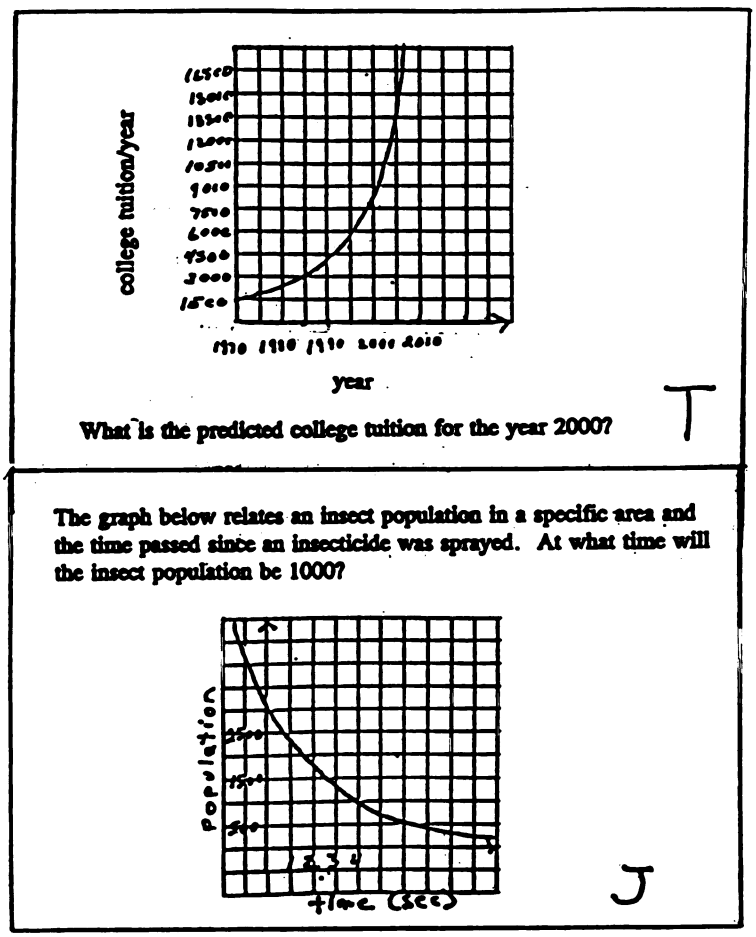

Figure 3.19: Sort Cards T and J

discussion about rates of change and symbolic forms of both exponential and quadratic functions and after I had drawn examples of graphs of exponential functions, I asked her if she could tell whether or not graphs on cards T and J (Figure

3.19) were of quadratic or exponential functions. In that discussion, after first computing examples of rate of change in y for changes in some x's, she talked of J having a flat portion on one side and suggested it might be an exponential, but said the graph on card T looked like it would go back up and thus be a quadratic. It appeared that by this time she might have associated a clear flattening out on one side with the name exponential functions. Since card T showed none of the flat part of the graph, and card J showed only a small portion of it, it is possible there was not enough of the graph on these two cards to suggest they were exponential.

In summary, shape of graph was a salient organizer for Toni for two of the three function classes. Toni used it successfully to classify both linear and quadratic functions. There is little evidence to show that she classified exponential functions by shape.

Patterns or Rates of Change

The function concept, rate of change, shows up readily in representations as slopes of graphs and as patterns of change in tables. Patterns or rates of change, particularly dynamic versions, were also salient concepts for Toni. In the Tim/Donna problem, where the problem required selecting a graph of distance as a function of time, but the words described travel mode (speed), Toni explained the reason for why a steep slope in the graph matched a child running by describing covering more distance in less time. She had informally connected slope to rate of change in a general graph.

Rate of change can be determined in tables from the parallel patterns of change in the two columns, that is, their covariation. For Toni the pattern of change in output for equal input was a salient feature of linear functions. She seemed to have categories "equal change" and "unequal change." As early as the second week of class, Toni used both equal and unequal change in tabular representations to classify the graph as straight or curved respectively. In interview problem 2A-1, the Alice,

Bea, Carolina problem discussed above under the section on orientation of axes, Toni was asked to match an input/output table with entries, (0,4), (1,2), (2,0), (5,-2), (10,-4) to one of three graphs. Recall that Toni first had judged incorrectly that the table represented equal change and had suggested Alice's graph, the only line, could possibly be right because: "it (the table entries) slowly decreases by 2's. So, it would be constant (rate of change); so it would be a pretty straight line." She rejected Alice's graph because of scale, but still thinking of equal change, suggested Bea's graph might be better than Alice's. She continued by explaining how Bea's graph did not fit equal change because the graph should not be curved:

> Bea is pretty much right. . . . Except the bottom of her line right here should be going over more and it shouldn't curve because it's like a constant (change in the table). Um, it like goes right from 2 to 4 so it shouldn't curve at all. It should be a straight line.

Then Toni suddenly discovered that the differences between entries within the input column were not equal and even though the differences within the output column were equal, the rate of change for the function was not equal. However, she judged the differences between the input numbers as slight enough to consider them almost the same, saying:

> Because now I looked at input and it goes pretty much, it goes from 1 to 2, 5 to 10 but those numbers are still pretty much alike.

She plotted points on the graph with a rather crude scale and concluded the graph would be nearly straight:

> So that keeps a pretty much a straight line instead of curving or 10 would be down four over here, so it stays, [laughing] the points are right and everything. It would probably stay in a straight line.

Although she had suggested incorrectly that because the (difference between) input numbers were nearly the same, and thus, the graph would still be pretty straight, she had associated equal change with a linear graph. (At the end of this problem, Toni plotted her own graph, and even though it had an almost identical shape to Bea's

graph, she rejected Bea's as a valid match for the table. She insisted Bea's scale was wrong--even though no scale was given.)

In the sort problem, recall from an earlier discussion on linear functions that Toni at first thought card R, a table of a linear function, had an unequal rate of change because she had looked only at output. She quickly realized it had equal change and spontaneously used this information to justify that the function was linear. In both these problems, 2A-1 and sort card R, Toni at first attended only to the output column, but during problem 2A-1, during the second week, she had a strong enough association of equal change with a line to try to "massage" a table for a nonlinear function into a line graph. By the time she came to the sort problem in week 5, she still had a firm connection between those two concepts, but was much better prepared to justify the match.

This phenomenon could be explained by her attention to procedures and having a procedure to check the output column to make categories of equal/unequal to classify tables. It could also be another manifestation of the inclination to attend to only one variable in functions, that which "the function is about"--the dependent variable. Toni had displayed her ability to think with a dynamic version of graphs by attending principally to the dependent variable. With tables, she displayed the same propensity to think about variation dynamically and perhaps had to consciously force herself to think about the independent variable as well.

Besides having equal change as a salient feature for thinking about linear graphs, Toni also found unequal change salient. She understood it less well, however. In addition, she may have had a category of functions, "those with unequal change," but this is not clear. Recall at first Toni thought "quadratic" and "parabola" were associated with different kinds of functions. She had suggested W, a scattergraph, was a quadratic, but as I started to tell her quadratics and parabolas referred to the same kind of function, she proposed quadratic as a function class that

had unequal change. She had remembered one connection between change and quadratic functions, but did not have names of function classes straight. I think at that time she had perhaps associated unequal change with some specific function class, but just could not remember the name. She described the values as "going up and down and all over," but when I asked what she meant, she described unequal increments in a table:

> T: Like you have the input here [she pulls over a table] and if it always increased the same amount [points along the input column], and the output was always jumping different numbers and didn't have an equal change in the output.
>
> G: You mean between numbers?
>
> T: Um-huh.

At this point she said she had forgotten the names of functions, so I asked her to give me shapes of the graph and said I would help her with the names. Toni then created the straight line in Figure 3.20 below and connected it to constant change. I said they were called linear functions and asked if she knew what quadratics were. She suggested they could go in zigzags, and drew the remaining graphs in Figure 3.20.

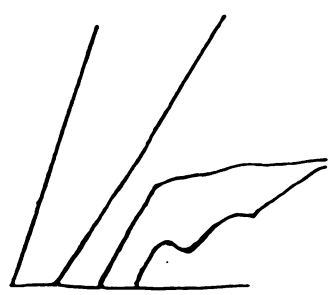

Figure 3.20: Toni's Sketch of Graphs with Unequal Change

I explained that "quadratics and parabolas are the same kind," and asked if she had questions about any other kind. Toni again mentioned "the ones that just kinda go up and down and all over." I asked if she meant the graph on card W, and she agreed, but when I tried to solicit the name scattergraph, she again raised unequal

change. At that point she gave the explanation given above that the table on card, R, was for a linear function. As she finished, Toni said, "But's there's another--. I think there's another function name that goes up and doesn't have a constant (pattern of change)." She clearly was thinking about another graph associated with unequal change. I misunderstood her and thought she was trying to classify all nonlinear functions under one particular function class. Since nonlinear was not a term that had been used anywhere in the class, I went into a digression with her where I tried to understand her thinking, but got involved in trying to get her to see that unequal change was not a sufficient category to distinguish nonlinear functions from each other.

Now, after examining carefully the videotape and transcript, I think she may have been thinking of exponential functions. In class, the teacher had contrasted change in exponentials with the equal change in linear functions. On July 16, students had created a table for $y = 4 \cdot (0.83)^x$, using both the formula and repeated multiplication for $x = 1$ through 10. The class computed the differences between successive y-entries to show that the function was not linear by showing that the differences between entries changed. They associated large differences with steep parts of the graph and small differences with more horizontal parts. On July 22, when rate of change for quadratic functions was being explored for the function $d(s) = .005\ s^2 + .14\ s$, stopping distance as a function of speed, the teacher asked Toni how to tell from a table that the function was not linear and she said, "if the distance (independent variable) doesn't have the same amount between numbers, then we know it is not linear." On her class worksheet that day, although another student had given this response in class, Toni also had written, "It doesn't have one slope; the slope is always different." At any rate, Toni associated at least one function class with unequal change. It was not clear, however, whether she understood this to be an

attribute of many functions, or whether she understood it as a particular one, possibly exponential.

Symbolic Connections to Function Classes

Function classes can also be identified by their symbolic forms. Except for quadratics, there is less evidence to suggest Toni connected symbolic forms for functions to the three function classes than there is for connections between function classes and shapes of graphs or patterns of change. In the sort problem, as I continued to probe to see if Toni remembered exponential functions, I reminded her that in class that day we had discussed a function for the racing car experiment. (The discussion included quite a bit of time modeling the data with an exponential function.) When she did not remember the function name, I first sketched the data points that had been gathered in class, then a curve for a decreasing exponential as a model for them. When she still did not remember the name, I asked if she remembered an equation for the graph, but she could only remember one that involved dividing by a square root. When I asked for functions we had studied in class, Toni produced $y = ax^2 + bx + c$ and correctly identified it as quadratic and as having a U-shaped graph. Since the class had spent a long time guessing and testing parameters for a general function, $y = a \cdot b^x$, I continued to elicit an exponential equations by asking if she could recall any other kinds of functions. She first suggested using two terms of a quadratic instead of three and finally, the first term of the quadratic. Not until I had specifically asked if she remembered an exponential had the variable in a different place, and she had made another incorrect guess, did the following occur:

G: Why do you think they call a function exponential?

T: Because it has like to the power of, above it.

G: Um-huh. So, can you give me an example?

T: [drawing a box around ax^2] This is kind of like part of one, I think.

Only after I told her the power in that one was two, did she write ax ^. After further prompting for an example, she wrote 2^3. Finally, after I asked again for a function rule, she wrote $y = 2^x$. Toni's understanding of the symbolic form for an exponential function was weak. She had difficulty remembering it and, although she had associated it with one term, only after leading questioning on my part could she distinguish it from a single term quadratic. Clearly, the number of terms was more salient than the position of the exponent. It is possible that since the focus of attention was on a and b in her search for parameters in the expression, $a \cdot b^x$ in class, it blurred the distinction between exponential and quadratic functions.

The interview questions were insufficient for finding out Toni's conception of the symbolic form of a linear function. As discussed above in problem 2A-2, where Toni was to find a function rule for a table of values, she could find the pattern of change in the table, but could not make progress toward finding the rule, and so we had abandoned the problem. In problem 4B-4 (Figure 3.21) where a function rule for

Without using the calculator determine which function best fits the graph below?

a. $f(x) = 10x$
b. $g(x) = 1 + x$
c. $h(x) = 10 - x$
d. $j(x) = 10^x$
e. $k(x) = 9 + (.1)^x$

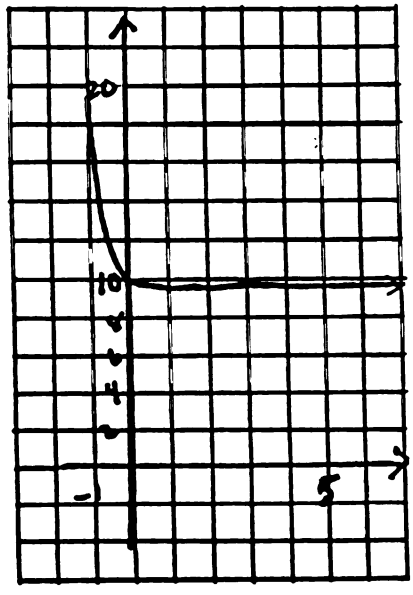

Figure 3.21: Problem 4B-4

a graph was to be chosen from a list of potential rules containing both linear and exponential functions, Toni at first chose part a because she thought x was on the vertical and the rule had y = 10x. She was probably interpreting the coefficient as the value of x as she did in other problems. I clarified the axes orientation and determined that Toni had x on horizontal. Next, she said she chose e because of many y values at 9 but said she was unsure what the $.1^x$ would be. As we were out of time, I could not pursue her thinking. She seemed again to associate numbers in the rule with values of the function and, in addition, it was unclear whether she had a graph shape connected to exponential functions. If not, even if she knew the symbolic form, she would not have been able to connect the two forms. In the two interview tasks for linear functions, table to rule, and rule to graph, Toni had difficulty. On the pre- and posttest she correctly evaluated a linear expression, 3x + 4, for x = 10. On posttest problem 12, where Toni had to select the general symbolic form, y = ax + b from among those for quadratics and exponentials, she saw the table as representing a different linear function for each table entry, work that was inconsistent with her interpreting tables on other occasions. There is little other evidence to suggest one way or the other whether she recognized a function in the form y = mx + b as a linear function.

In conclusion, Toni had strong associations with the symbolic form of quadratic functions, finding the number of terms and the existence of exponents salient. She could not distinguish exponential forms from one-term quadratics, and there was not enough evidence to determine her understanding of symbolic forms of linear functions.

Summary

Toni's approach to problem solving depended on the representation and task. When problems were symbolic or appeared as school problems, she attempted to manipulate symbols or solve equations. She paid close attention to what task was

being suggested by a problem. Toni had a naive conception of variable that allowed her to recognize that variables could take on many values or could stand for unknowns in an equation, but she could not readily represent quantities with variable expressions. When she could not remember a procedure, she got stuck.

On the other hand, when visual or graphical representations were used, she displayed unusual facility on some tasks. She could accurately read off points and interpret them in the situational context. She interpreted global features with ease when fitting graphs to situations, both choosing suitable graphs or modifying the description of the situation so that she could interpret the independent variable temporally and match evolving rates of change in the dependent variable with the slope of the graph.

When problems were situated, Toni relied heavily on the situation to support her reasoning, often thinking in the situation directly, without apparent awareness of the mathematics. In both visually represented problems and situated problems, she actively pursued problem solving even when stuck. Toni had difficulty when her informal model differed with that given in a situated problem. She interpreted terms of polynomials as quantities in the situation, could not keep variable quantities straight, and could not accept that a model could be used to predict values prior to an event.

Toni found shapes of graphs and patterns or rates of change salient concepts when organizing her thinking about linear or quadratic functions. She had a robust understanding of tables that represented linear functions and knew their graph was a line, but could not translate such a table to a function rule. Toni recognized quadratic functions in symbolic form. She did not seem to have parallel organizers, shape of graph or symbolic form, for exponential functions and had difficulty sorting out the conceptual category that she had created "unequal change." Further, Toni had difficulty remembering orientation of axes, but understood consistency of graph axes

labeling conventions. She did not connect the concept of input to its role in graphing and was not able to graph functions by hand.

Generally, Toni had knowledge of some concepts and procedures for functions, but, except for her work with graphs, her knowledge was fragmented. Further, it was generally necessary to imply her understanding of functions as she attended principally to the representations themselves and gave little evidence that she was attending to functions per se.

CHAPTER 4

THE CASE OF SAM

The first section of this chapter introduces Sam and is followed by three major sections discussing his work. The first of these discusses Sam's modes of operating during problem solving, including work with the calculator, interactions with the researcher related to problem solving, and communication difficulties. Like Toni, Sam's work with functions is implicit through his activities with representations, so representations are the focus of the second major section. The section on representations begins with problems where graphs figure prominently and is divided into two segments according to whether the problems take a correspondence or covariational approach to functions. A third segment of working with representations revisits how Sam's reasoning about functions in situated problems caused him difficulties in other representations as well and raises many questions about the discrepancy between his perspective and that of the teacher. Finally, the last major section discusses Sam's conceptions of the three function classes: linear, quadratic, and exponential. The first subsection uses linear functions to display how disjoint and incoherent Sam's work on functions appears to be.

Getting to Know Sam

Sam is a tall, slender boy from a small midwestern city, where, except for brief intervals when he lived in another state, he has lived all his life. When I asked him to talk about himself, he told me for as long as he could remember he had been getting good grades in school. He then reported that he was from the "ghetto" and was not used to "rich stuff." Another teacher had told me that, based on stories Sam had told him, he thought Sam's family was quite poor. In our first interview Sam described himself as being self-motivated:

> The reason I'm doin' what I'm doin' is for myself. I don't like to get rewards for what I'm doin'. It's in me, 'cause I know that I want to do it. I get B's

> in math. Sometimes I get A's dependin' on what the subject is we doin'. So I'd say I was a B student in math.

At the time of data gathering, Sam had completed two years of algebra, receiving an A for the first year. His reading level was slightly higher than the eleventh grade he would be entering in the fall, according to the combined score of 11.5 on the Nelson-Denny Reading test.

During the awards ceremony at the closing banquet for the program, Sam was chosen by his teachers as outstanding student in his section in Biology, Physics, and Anatomy. In Science Olympiad he placed first in the Anatomy event. In addition, he received a Citizenship of Merit Award, given out by the staff to students who received the highest votes in two of three categories: Dorm, Study Hall, and Activities. Apparently, Sam exhibited qualities beyond academics for which he was both respected and liked.

Sam plans to be a zoologist as he has a special interest in ichthyology, the study of fish. He had considered a career in entomology, the study of insects, as well. He said he had been fascinated with animals and stars since childhood and reported being a B student in science. Sam seemed to have a lot of interests since he said he liked ancient history, too.

When I asked Sam during the first interview if he liked the anatomy class (in the summer program) where students had an opportunity to work with a cadaver, he said it was not his favorite class, then went on to display a dry sense of humor with the following story:

> Oh, the human body's okay, except they dried out. They not real, I mean, it's not, like fresh. 'Cause uh, last year in the tenth grade, uh, my teacher-- one of his students came back, and she wanted to dissect the human body while they was still livin'. [laughing] So they asked us do we want to do it. And I wanted to volunteer.

Sam's sense of humor sometimes came out while he worked interview problems.

Once, during the fourth week interview, he worked problem 4A-7:

> Blue pencils cost 5 cents each and red pencils cost 6 cents each. I buy some blue pencils and some red pencils, which cost me 90 cents. If b is the number of blue pencils bought and if r is the number of red pencils bought, what can you write down about b and r?

Sam first interpreted this problem as meaning an equal number of red and blue pencils were required, so he divided 11 into 90 to compute the number of pairs that could be bought. When it did not come out even, he playfully enhanced the situation: "That's 88 and you've got two cents left over, but it's 90 cents. I'm gonna add some tax on there." As will be seen later, Sam sometimes added to or enhanced situated problems as a means of resolving a "glitch" between his work and the problem.

One of the reasons I selected Sam as a focal student was that he often gave unusual answers such as this one about tax and showed independence in his way of working. In my notes after the first interview I had written, "slow, careful thinker--very sure of working his own way in his own time." He often was slower to complete problems than some of the other focal students. For example, because he ran out of time, he did not get to work on interview problems 1-2, 4B-3 and 4B-4 (see Appendix A), all of which were completed by all other focal students. On the sort problem, during interview 5, he was only able to complete one full sort, while all other students completed two and Toni worked on a third. On the other hand, Sam was one of the two focal students who were able to complete the two problems matching graphs and situations from part B of interview 2.

Sam seemed to "do his own thing" and not to care whether others in the class approved. Once when he put a problem on the board, four of the girls giggled and made comments. When I asked him in the next interview how it made him feel, he said he did not mind and that he did not pay any attention to them.

In class, Sam was often a lively participant, offering suggestions, asking questions, or announcing difficulties with his calculator computations. On other days

he was quite quiet, possibly because he was bored. Once on a day he had been quiet for the first hour, he told me at break he thought the topic, compound interest, had been boring. Perhaps when Sam was quiet in class, it was because he was tired, a chronic problem with students in the program. On several days the students had trouble staying awake, in part because the class was immediately after lunch, but also because their workload was so heavy and their schedules were so full. It is quite possible Sam was tired on some of those days that he did not participate much.

Modes of Operating

Sam had several ways of interacting with people, problems, and the technology in this class. This section discusses his use of the graphics calculator, both his positive attitude toward using it and his difficulties with it. The section also discusses his ways of interacting with me, the researcher, and aspects of the discourse that he seemed to use in a strategic way to solve problems, including those that were not effective. Finally, the section discusses communication mismatches between us.

Problem Solving and Technology

Sam's Attitude Toward the Calculator

Sam enjoyed the calculator. Like most of the students, he participated when calculator exercises were done in class. When he was stuck or got surprising results, he would announce it aloud--as did many other students. During the interviews, he used the calculator as much as any of the students and sometimes turned to it to explore the problem at hand when he seemed not to know how else to proceed. Other times he explored the calculator itself. Once during the card sort problem after he had entered data pairs from a table into the calculator using the statistics menu, Sam spontaneously chose the xy-line graph option instead of the scattergraph choice I expected. Xy-line had not been discussed in either class or interviews, whereas students were taught to use the scattergraph option. Of the other three focal students, only one chose to explore the xy-line option in interviews.

Calculator Difficulties as Inhibitor in Problem Solving

Conceptual difficulties: Calculator screen use. Periodically, Sam had difficulties with the calculator. Usually, his difficulties were of a conceptual nature or a combination of conceptual and technical, rather than a technical difficulty only, such as forgetting how to do a sequence of keystrokes. For example, sometimes Sam did not distinguish the conceptual differences between the editing screen used for defining a function to be graphed and that used for evaluating a function for some particular x. In some cases his results were unexpected, did not make sense, and thus caused him confusion. During the third week, the class was exploring exponential functions. Mr. Bailey had just finished leading the class in creating a table of values for x from 0 to 7 for the problem in Figure 4.1. The class had developed the rule $y = 1500 \cdot 2^d$ for the situation by identifying the bacteria at time 0 and by computing

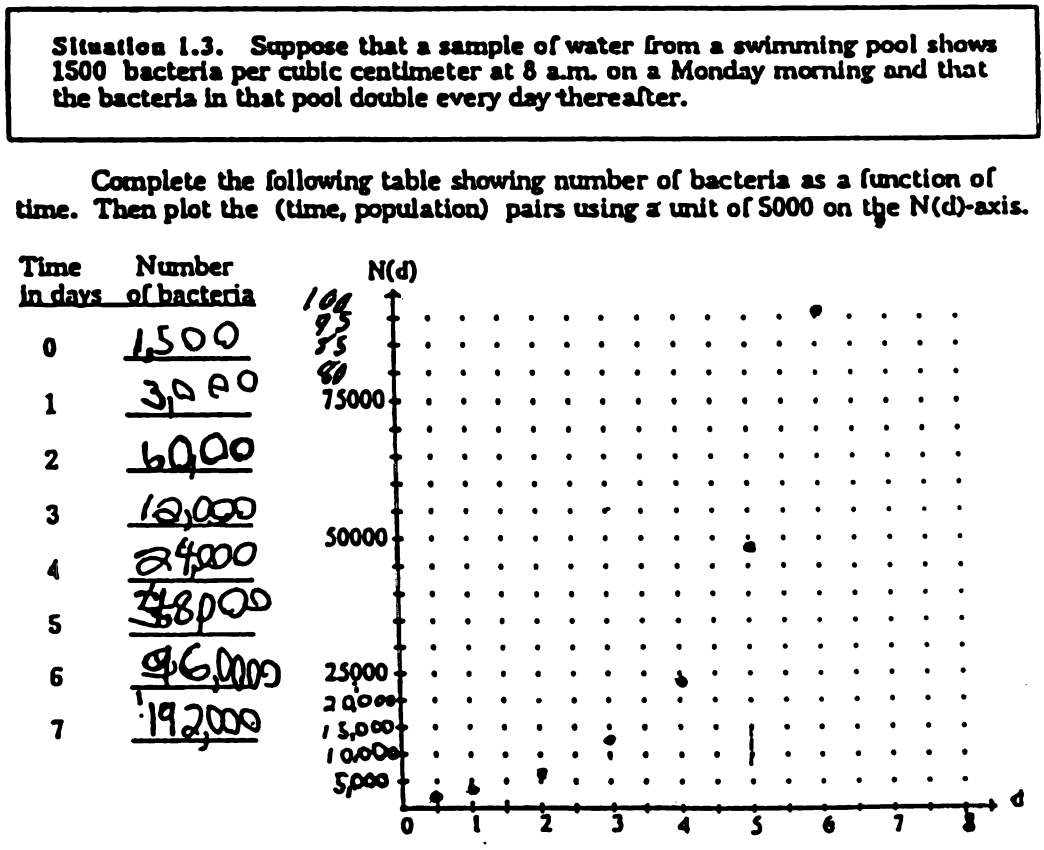

Figure 4.1: Situation 1-3, Swimming Pool Bacteria (CIA, p.5-6)

the value for each new day by doubling the old value. Mr. Bailey then used the idea of "the amount of bacteria in the pool the day before" to introduce negative exponents as going backwards in time. Hence, he introduced an operation opposite from doubling, that is, dividing, by 2. The class extended the table backwards from 0. Next Mr. Bailey asked the students to continue with the problem on their own.

Sam entered the points from the table on the board as data pairs in the calculator, made a scattergraph, and said he wanted to get the graph of the function so he could see how close they came, not realizing if the rule was generated from the pattern in these pairs, the points would indeed be on the graph. Then while Sam tried to enter the rule, he made additional conceptual errors that caused him confusion about the calculator results. First, he used the wrong rule, $N(5) = 1000 \cdot 2^5$ instead of $1500 \cdot 2^d$. Second, he entered the entire rule, including the "N(5) =" on the calculator function definition list (under the key, Y=). Since I was walking around the room, I asked him what the N(5) stood for, and he correctly answered it was the y. Since he was stuck, I suggested that y was already there on the Y= screen[1]. He deleted the "N(5)=," but left the 5 in place of the x and got a horizontal line graph. After I questioned him a bit more, he understood the 5 was a constant and said he needed an x. He then successfully completed the graph he was after. Of course, it did not match the points he had sketched because he had used the wrong rule, one with a constant of 1000 instead of 1500.

Choosing the wrong function may simply have been an oversight, but Sam made several other errors. He attempted to create a Cartesian graph from the function rule to determine if the points the class had just produced for the same rule were, in fact, on the curve. Sam made two conceptual errors in using the calculator,

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[1]Y1=, Y2=, Y3=, and Y4= are given by the calculator when you choose the Y= key to define a function. All function definitions for graphing must use the independent variable, x.

including N(5)= when it was not necessary to include the function name and mistaking $1000 \cdot 2^5$, an instance of the function, for the general function rule, $1000 \cdot 2^x$. Had he understood the conceptual distinctions in these activities with functions, he might have been able to sort out his difficulties himself since he had on other occasions demonstrated that he could use both screens correctly by entering functions to be graphed and by evaluating functions on the computation screen. Sam was not alone in having trouble keeping straight the purposes of specific editing screens. Periodically, other students displayed the same behavior. Recall that at one point Toni also confused screens and entered a rule on the computation screen. She was trying either to compute a function value using the letter x instead of a value for x or to a graph polynomial function by entering the function definition on the computation screen, just the opposite of what Sam was doing here.

Conceptual difficulties: viewing windows and number size. Sometimes Sam's trouble with the calculator inhibited his problem solving ability because it muddled concepts he was attempting to apply in a valid way to the problem at hand. For example, during the sort problem, Sam chose dimensions for the viewing window for the graph of a quadratic function and got results he did not understand. Yet he did not see the dimensions as being a poor choice; rather, he continued to try to make sense of the numbers he saw, which led him to further difficulty.

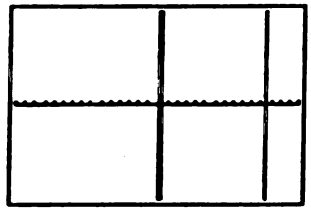

Figure 4.2: Sam's Profit Graph Display

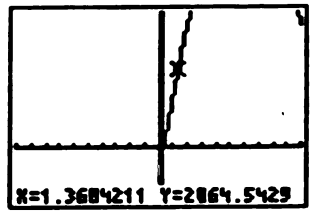

Figure 4.3: A Sample TRACE

Sam had decided the graph of a rule on card B, Profit = $-300p^2 + 3600p - 1500$, was a parabola, but when he graphed the function, he got a vertical line (Figure

4.2) because the y-dimensions for the viewing window were too small relative to the specific x-dimensions chosen[2]:

S: Is it a parabola? [turns on trace[3] and moves the cursor along the graph] It don't look like it's ever going to go down.

G: You're on trace and looking at (values of) the x and y? [S: Um-huh.] What are the numbers doing?

S: Keep going up.

Sam correctly thought that this graph was an inverted parabola, and he expected to find the vertex by finding a place where the y-values reversed directions. This strategy could have helped him find an appropriate viewing window by locating a maximum function value or, through the pattern in the data, confirmed his thinking that the graph was a parabola. However, since the vertex was to the right of the cursor and Sam was moving the cursor to the left, the cursor failed to reach the vertex. As he moved the cursor down the lefthand branch of the parabola, both the x- and y-values continued to go down.

Sam thought because the magnitude of his negative numbers was going up, the cursor on the calculator ought to go up as well and thus ultimately reach the vertex of the parabola. He was trying to coordinate a change in magnitude (size) with an upward direction along the y-axis and thus with an upward change in position of the cursor along the graph. Although these three kinds of change are ordered in the same way for positive numbers, the direction of change in magnitude for negative numbers is exactly backwards from that of both y-values on the axis and the relative vertical position of the cursor.

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[2]If the y-values vary greatly for a small x-interval, the graph will appear vertical.

[3]Trace moves the cursor along the points on a graph (both within and outside the viewing window) and displays the values of the x and y coordinates of the current point. If the point is off the screen, the cursor itself is not displayed. The image scrolls left or right if one traces beyond the x-values of the window currently displayed.

Students and teachers sometimes refer informally to changing negative numbers that have large magnitudes as "getting large," assuming that "order" and "magnitude" are distinguishable from the context of the comments. Clearly, Sam used this informal language, but did not distinguish these concepts here. His comments that the graph was a parabola and that he saw the y's as going up, yet expected them to come down, indicate he understood this function rule to have an inverted parabola as its graph. As he traced left, had he seen the y-values as decreasing, rather than increasing, he might have traced instead to the right to a region where the y-values became positive. At that time the change in magnitude, order of y-values (direction along the y-axis), and order of points (vertical direction of cursor movement) would have been synonymous, and thus easier to interpret. In addition, had he seen his window choice as unsuitable and had a conceptual understanding of the relationship of window size to domain and function values, he might have rectified his problem by changing the dimensions of the window to display a more familiar portion of a parabola. Instead, each step led him further afield from concepts that he knew: that quadratic functions should have graphs that are parabolas, those with negative leading coefficients in the function rule open downward, and y-values reverse direction at the vertex.

In the example just discussed, Sam used informal language that was similar to that often used by teachers to describe invalid conclusions about behavior of numbers for y-coordinates and the position of their associated points on the graph. This example raises questions about what teachers can assume about students' understanding of the ordering of numbers when teachers themselves use everyday language to describe changes in negative numbers. Sam's perceptions and language to describe magnitude only here contributed to conceptual problems. They prevented his making sense of a calculator task with which he was familiar, in which he was using a good strategy, and in which he had valid expectations about the outcome.

As we continued this problem, my efforts to straighten out the number confusion led to continuing difficulties with the viewing window, in part because of Sam's style of responding. I started by deciding to help Sam discover his error and asked if the numbers were positive or negative. He said they were both negative. Then I tried to call his attention to the distinction between the magnitude (absolute value) of a number and the relative order of two signed numbers:

G: Are they going up in absolute value or are they going down? Are they going up if it's negative?

S: If it's absolute value, it's goin' to be positive all the time.

G: I meant the size of the number, is the size of the number going up?

S: Down. [He continues tracing.]

G: It's going down. If I said -200 is going to -300, is it going up or going down?

S: Say that again.

G: If I went from -200 to -300, would you say that's going up or going down?

S: Up.

G: It's going up in size, but the number itself is actually going down, isn't it? Isn't -300 less than -200?

S: Um-huh.

During this exchange, Sam was very busy tracing on the calculator. He appeared to be paying only cursory attention to me. (As will be noted below, it is possible his inattention to me served in part as a problem solving strategy to get me to narrow the focus of my questions to ones he was ready to answer.) When I raised absolute value, he gave a "textbook" answer about absolute value being positive all the time, not attempting to answer either of the questions I asked. Thinking perhaps that the difficulty was my poor questioning, I reconsidered and asked only about the *size* of the number. I hoped he would understand this informal description that is sometimes used with absolute value, but he answered my question as if it were about the order of

signed numbers when I had asked about *unsigned* numbers. I echoed that he had said "down," thinking perhaps he had figured out the discrepancy between magnitude and direction. To confirm which question he was answering, I asked a more focused question about how he perceived change between two specific numbers, from -200 to -300. His answer confirmed that he had not sorted out the concept, but perceived these numbers as increasing, rather than decreasing.

His asking me to repeat the question meant he had paid only partial attention to my questioning, and only when I persisted did he attend to it. Sam often answered questions in this way. He would concentrate on his work and give me answers that seemed only partially connected to the question I had asked. When I persisted, he attended to my question and answered. Sam's use of discourse in this way as a problem solving strategy will be discussed in the next section.

As we continued the problem, Sam again ignored information that he had demonstrated on other occasions. After he answered directly that he saw the change from -200 to -300 as going up, I explained that -300 was less than -200. Sam continued to trace. By this time he had an x-value of -54 and gave a y-value of -703,000. (Since P(-54) = -1,070,700, he either read the number wrong or had entered the problem incorrectly. In either case, the size and position of the numbers is roughly the same.) I asked Sam if it was a low number or a high number, and he said low, but was still confused about why his graph was straight. Finally, he reset the calculator to the standard viewing window of [-10, 10] by [-10, 10], a common strategy to "start over." The resulting graph, shown in Figure 4.4 below, appeared like a ladder made up of the small segment of the graph, the y-axis, and the tick marks for scale. Sam could not understand this graph, yet he did not use any information about the graph or the function rule to choose an appropriate window--information that might have given him a graph he recognized.

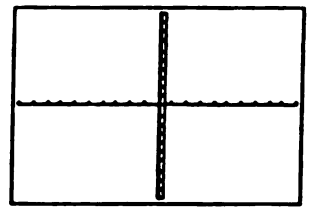

Figure 4.4: Sam's Ladder Graph

Sam's original decision to turn on trace had been a good strategy to explore the nature of the coordinates in the problem and could have helped him choose a suitable viewing window, especially if he had traced in an opposite direction toward 0, getting positive numbers. Instead, his use of everyday language to describe decreasing negative values as getting larger confused him. Even after I had asked several questions to try to help him, he did not use the information he had to get unstuck. When Sam decided to start over with the standard viewing window, he did not attempt to make any connections between the viewing window and the nature of the data involved in the problem, and because of the unusual nature of the graph displayed, he got even more confused. At no time did he use knowledge he had that the graph was an inverted parabola.

Immediately afterwards in the sort problem, Sam ignored knowledge that he had about linear functions and again ran into problems with the calculator viewing window. He failed to connect a straight line graph to a linear function rule, which on several other occasions he had been able to do. When Sam tried to graph the function, $s(p) = -300p + 270$ (on card Q), he also got a vertical line because of the particular window dimensions. He grouped this card with card B, discussed above. Later he grouped them both with card W[4], a scattergraph, in a category he called,

[4]Prior to this, Sam had considered a grouping for both B and W based on references to "fast food restaurant" on W and "company" on B. So when he explored B, W was readily available within view, perhaps influencing his choice of "could be anywhere".

"these dots could be anywhere." (At least three of the four focal students used "it could be anywhere" or "it goes all over" as a descriptor for scattergraphs or non-well-behaved graphs, that is, anything other than lines or parabolas.) Sam grouped these two cards in such a category because he was unable to make sense of the graphs produced, mainly because he could not find a suitable viewing window and did not use a problem solving strategy to step back and reflect on less immediate knowledge he possessed. Often, he made decisions based only on the most recent result or response even when they were not sensible, as will be shown in later passages. Sam did voice concern, however, about how sensible his choice was. At the end of the sort when he read off his sort categories for recording, he said he did not believe the graph of Q. So at least he recognized something was wrong with his thinking.

Interaction as Problem Solving

Throughout the interviews Sam interacted with me in ways that were, in their effect, problem solving strategies. Some were helpful to him, while others hindered his problem solving.

Narrowing the Focus of Questions

In the parabola example above, Sam displayed a mode of interacting in which he paid only lip service to my questions until they were focused on a highly specific piece of information. At the time of the interviews, I felt he was simply concentrating on the problem he was solving and perhaps considered my questions an intrusion or distraction. Several times in interviews when I asked a question to confirm something he had just said and gave two alternatives, Sam agreed with the last one, even if it countered what he had just said. All the while he appeared to pay more attention to the problem than my question. Sometimes I felt he was answering only because he thought I expected an answer. Only when I persisted in questioning would Sam attend to the question and give a thoughtful answer. At one point I wrote in my notes that he may have an attention span problem as there were several

instances in his interviews of his saying one thing and writing another. So my interpretation of this situation seemed reasonable at the time.

On reflection, however, another possibility comes to mind. The question Sam skirted around here on card B and the magnitude of negative numbers was a conceptual question about how numbers are ordered. In the passage above on questioning whether numbers were increasing or decreasing, Sam continued to give me partial or misleading answers. His responses to my questions had an effect on the interviewing process. They caused me to narrow the questions from one about a general concept to one involving the relationship between a single pair of numbers. Although Sam might have been ignoring me somewhat as he concentrated on the problem, the net effect of the interaction was that ultimately I asked a question that he could readily answer. This style of delaying or deflecting responses, if intentional, was a reasonable problem solving strategy on his part--shaping my behavior as an interviewer to simplify the task.

Seeking Clues

Another style of interacting with me that seemed to advance Sam's problem solving was his use of questions to gain clues for how to proceed. Sam began 7 of the 17 problems he worked on during interviews with a sequence of questions that sought additional information about the problem. On two of these occasions (Problems 1-1, 4A-1), his questions were only to find out what I expected of him, much as a cooperative student would ask a teacher in class. For example, at the beginning of the fourth interview he asked if I wanted him to think aloud and whether I wanted him to write his response. On each interview prior to that point, I had asked him to think aloud and had let him give some lengthy responses orally. Clearly, he simply wanted to comply with my wishes.

Sometimes Sam wanted clarification of terms or of the problem statement that might be necessary to solve the problem. For example in problem 2A-1 in Figure

4.5, after he had said he did not understand it, but would try, I asked what he did not understand, inviting a question. He asked about the meaning of input and output, a

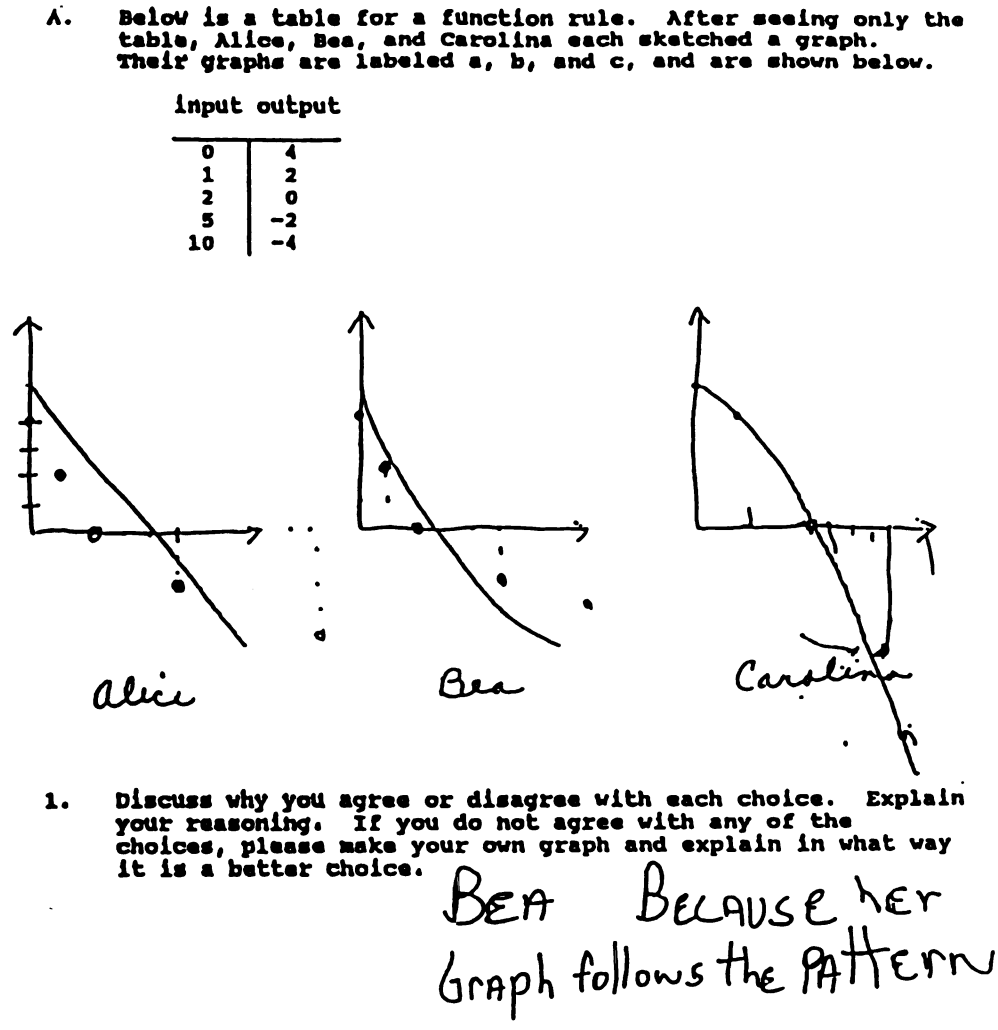

Figure 4.5: Sam's Work on Problem 2A-1, Alice, Bea, and Carolina

clarifying question. After I had explained their meaning, he asked another clarifying question and one to check if he had understood correctly:

S: And I gotta match 'em up with these? (the graphs)

G: What we're saying here is these are three girls. Three girls have made this as a tentative solution. Alice thinks this is the graph. Bea thinks this is the graph, and Carolina thinks this is the graph. So I want you to critique their work [S: Oh, okay.] to say whether you agree or disagree and so on.

S: Which one I agree with?

G: Yes, and why do you think it. So down here is really what I want you to do.

S: Oh, okay.

Since the actual question for the problem was written below the graphs, I reasoned he might not have read the question itself, so I pointed it out to him. His questioning here, both about the meaning of the terms input and output and what task he was to perform, was necessary for him to tackle the problem.

At other times, however, Sam seemed to be seeking hints for how to proceed. In problem 2A-2 (see Figure 4.6 below) he asked a number of questions:

S: You want me to draw this?

G: I want you to read what this problem is and see what you can do with this problem.

S: Now, I'm going to look over this again.

G: Okay. [he reads the problem]

S: This is like that x + 1 stuff equal y.

G: You got it.

S: So that's what the function rule is?

G: The function rule is like what you were doing in class when you were trying to find the function rule.

S: [pause] Okay, now I remember. It'd be like x [pause], two x equal [pause] y. [writes 2x = y] I don't know. I don't know if I'm right. I can't remember.

For the table below, what is the function rule? What is the output for input 15? input 30?

input	output
1	10
2	5
3	0
5	-10
10	-35

Figure 4.6: Problem 2A-2

Sam had seemed to connect this problem to the graphs of the preceding problem (2A-1), discussed above, so he asked if I wanted a graph, a clarifying question. After I told him to just read the problem, he checked his understanding that a function rule was an equation written y equals an expression involving x. When I agreed with him, he asked again about what the function rule was. At the time I sensed he was fishing for a direction to proceed, and so simply mentioned we had done function rules in class and gave completely redundant information. My response no longer added anything useful to the problem solution. Sam then made a guess, but his announcement that he did not really know if his guess was right indicated he was stuck and hinting for additional help. This strategy was wise since when students were stuck, I usually provided some question or bit of information that would help them get going again. We continued the problem with my giving him help by asking if his rule fit the table. By that time Sam had figured out to look for a pattern of change in the data and immediately computed the pattern of change. Our question and answer session had given him enough information, or possibly just time to think, about the problem to get him started.

The written transcript appears to lend itself to an alternative interpretation of my comment, "you got it," as a statement that his answer was correct and thus, he was asking in disbelief that it was the correct function rule. I do not think this is a valid interpretation because of the intonations we both used and because he used the word "stuff," which he often did to refer to a category of things, in this case function rules, rather than a single item. In the Profit Problem from interview 3, a long complex, situated problem involving comparison of two polynomial functions, to be discussed later, Sam also engaged in a long sequence of questions and comments before he chose any plan of action.

At the beginning of interviews, Sam asked a total of 17 questions about the problems themselves and 4 about what I wanted from him. Twice he stated that he

would try to do the problem, and in at least three cases stated either he did not know how to do the problem or thought he would have difficulty doing it. Toni, on the other hand, asked only three questions about the problems themselves in all interview problems, and two of these were about information missing on a diagram, such as which axis was considered the x-axis. She asked what I wanted on three occasions, once said she would try to do the problem, and only once stated she was unsure she could do it at the beginning of an interview problem.

Researcher's Comments and Questions Triggering Responses

Sam seemed attuned to my questions as well, sometimes taking them directly as hints. At other times they seemed to trigger a change in the direction of his problem solving. As we continued problem 2A-2, one of my questions helped him again. He had just described the pattern of change in the table and conjectured a rule, $y = -5x$, which matched the change pattern in the dependent variable, but not the table as a whole. Sam quickly scratched it out. I told him to just put an x through his work so I had a record of his thinking and then asked him why he scratched out the -5x. He explained that when he had substituted 1 for x, the result did not match the table entry. Unbeknownst to me, as will be seen below, this question caused him to reconsider his choice and quickly add the necessary 15 to verbally give the correct rule. As he continued, he made a number of recording errors, but finally wrote the rule correctly as $y = -5x + 15$ and checked its accuracy with $x = 2$ and $x = 3$. When I asked Sam if he was sure about it, he checked the rest of the table entries in his rule. He took my question to indicate the possibility that the rule could be wrong. At the end of the problem he announced that he did the problem with "that hint" from me. When I asked what hint, he said, "When you said why did you scratch out that five, negative five." This question became a hint not to abandon $y = -5x$. Sam had started this problem with several questions that gave him additional information. When he was stuck, he got a suggestion about how to

proceed. He was explicit about how my question caused him to reconsider his first choice. Finally, when I asked if he were sure about a response, it triggered additional checking for a correct solution of the table values. Until I had asked, Sam had omitted checking the remaining table entries in the rule, a step that was necessary unless he gave some other argument to confirm the match between the equation and all table entries.

Sometimes my questions about whether he was sure about a response triggered checks that were not necessary. In working on the Alice, Bea, Carolina problem, discussed above (2A-1 in Figure 4.5), Sam had used the units from a sheet of graph paper to plot the points in the table on the axes for Alice's graph. He then chose Bea's graph as correct, and when I asked why, he explained his choice well:

S: Because the form right there follows the pattern that she got.

G: The pattern of the dots that you drew [T: um-huh] looks more like Bea's pattern?

S: Yeah. So I'm going to say her chart is more accurate than theirs. Now, I'm gonna recheck my stuff to see if I'm right.

Here when I simply echoed his response, substituting "Bea" for "she," he had repeated the point plotting with the same units, this time on Bea's axis. He had judged the match correctly by the shape, so there was no need to repeat this plotting. My question had seemed to trigger the check. Sam's being tuned into my responses was a frequent occurrence, and as a novice interviewer, I had to work very hard to refrain from giving clues to Sam inadvertently.

I assume Sam had learned a strategy of paying careful attention to teachers' responses. It is likely that asking a large number of questions or getting a problem rephrased could result in subtle differences in a problem statement that made it clearer. Answers could also clarify "what the teacher wanted." In addition, it is likely students are familiar with a style of questioning in mathematics classes where, when students give wrong or incomplete results, their teachers echo students'

responses or ask if they are sure of an answer specifically to indicate the students' answers are wrong without saying so directly. Thus, they give students an opportunity to think harder about the problem by extending the time. I had the strong sense in working with Sam that sometimes his questions were "buying time" to think, as well as to get hints, while he thought about how to proceed.

Interaction Difficulties

Living in the Moment or Triggering Your Own Responses

Sam's responses to my comments or questions often seemed as much about our social interaction as the mathematical content. Besides skillfully questioning me to get information for his problem solving and listening for cues among my answers, he also engaged in a style of interacting during interviews that I came to think of as "living in the moment," that is, he seemed sometimes to just react to the last item mentioned, whether it made any sense mathematically or was relevant to the current issue being discussed. For example, his response above, when I was questioning him about magnitude of decreasing negative numbers and used the term, absolute value, was that it was always positive. This answer was nearly correct, but was an answer to a completely different question from the ones I had asked. He had just responded to the words, absolute value, that is, to the language only. Later in interview 2, during a problem to fit a graph to a situation about educational value as a function of difficulty of work (2B-2), and which will be discussed at considerable length below, Sam answered many questions. Many times he simply used his last response about the situation as a trigger for his next action--to create a new aspect of the situation--without apparent regard to the mathematics involved at all. When that example is discussed, emphasis will be on his reasoning within the situation, rather than about the mathematics. One aspect of his reasoning is this living in the moment by "chaining answers," that is, by using a recent or the last response for an action or justification, whether it makes sense to do so or not. Finally, in the sort problem,

where Sam had difficulty coming up with categories, an analysis of his strategies to determine sort classes revealed that sometimes he based new categories on an immediately prior idea. Creative problem solvers often rely on one idea generating the next, and this strategy can be quite productive if each new application is checked for sensibleness. Sam seemed simply to chain together his ideas without sufficient checks. For example, at one point he was trying to pair up cards having graphs with those having either verbal descriptions or tables by matching the same numbers within a problem. This was a good initial strategy, since non-matching numbers could serve as a counterexample for a match between two different representations of the same problem. When Sam did not find any such matches, he abandoned this purpose of two representations for the same data and simply matched cards that had similar numbers. He matched two cards (E and S) that had similar pairs, 500 and 1000, and 5 and 10, but little else in common except that the numbers were double each other in pairs. Card S was a verbal description of a relationship between x and y: For each 5 units x increases, y increases 10. Card E was a graph grid with a single point at 1500 on the y-axis with 500, 1000, and 1500 as y-axis labels. Sam surely was connecting this back to an earlier, tentative match of card S to card A ("As the number of workers doubles, the amount of time to do the work is cut in half"), which he justified by saying, "because it talk about that doubling and stuff." The relationship between numbers within cards E and S were indeed doubles, and both involved 5's and 10's. His justification, however, did not make sense mathematically. He had just chained these ideas together. His sorting contained other examples of this same behavior.

Kaput (1992), in an experimental study with Upward Bound students and using a version of a computer program, Guess My Rule, modified so that students had access to the results of previous guesses, reported similar behavior in a student, KG. KG often used the computer's latest response to her previous guess as a strategy to

form her next guess, ignoring much other information that was immediately available on the screen and which would refute the guess she had just made, had she considered it. Sam's interaction style to live in the moment and chain together responses generally moved the interview along, but did not usually help his problem solving. In general, it hindered his success, as will be seen in subsequent sections.

Talking Past Each Other: Researcher and Student Communication Mismatch

Not all discourse interactions could be connected to problem solving strategies. Some might be more accurately called characterized as communication mismatches. Sometimes during an interview Sam would answer a different question from the one I had asked. This same phenomenon happened with other focal students as well, but not so frequently as with Sam. For example, when we had almost finished problem 2A-2, the problem just discussed above to match a table to a linear function rule, $y = -5x + 15$, I attempted to find out how Sam decided on multiplication, rather than subtraction:

> G: --then you said -5x. What made you multiply rather than, say, subtract, is what I'm thinking about?
>
> S: Because, see when I multiply that 5 [G: um-huh.] it get there faster, than adding. So when you multiply, when I multiplied it, it got me that 10 quicker. [G: um-huh.] And then I could just subtract the 15 from it and it'll make it easier.
>
> G: And you needed this 15 to--? [S: to finish the problem] to finish the problem-- [S: yeah]

Sam had answered my first question by contrasting the relative changes caused by multiplication and subtraction. He seemed to order the operations by some informal rate of change. In discussing the 15, however, he gave two unexpected answers. His first, "to make the problem easier" made no sense to me at the time and sounded like a filler of some sort. It is possible he meant the 15 was necessary to make the rule and table match, but I think not because in his response to what the 15 was needed for, he simply stated his goal, which was to finish the problem. I had expected a

mathematics related answer such as the 15 was necessary so the numbers in the table satisfied the rule, or some portion or variation of this idea.

Other students responded by answering questions in a way that mismatched my intentions, too. Some answered "why" questions with motivation or purpose and "how do you know this" questions with where they had learned it. Clearly, incidences like these, where there is a mismatch between what is asked and how it is heard, can be attributed in part to my novice status as a researcher. Other incidences were surely because I am a math teacher. I assumed the phrase, "how do you know that," would be interpreted as needing a mathematical justification, but the student interpreted the phrase in a valid way as "where did you learn it" or "from whom did you get that information," both of which would likely have been hard to recall. These instances of communication mismatch occurred often enough, however, for both the classroom teacher and me, that I began to wonder how often mathematics teachers are asking mathematical questions and students are hearing questions about motivation to do the problem, goals they have in solving the problem, or other social purposes, for which the language is equally well-suited.

In summary, Sam had several modes of operating that could be considered problem solving strategies: deflecting or ignoring questions until questions were focused enough for him to answer, seeking clues and comments from the interviewer, and using one of his responses to trigger the next. Finally there were communication mismatches between what I, the interviewer, thought I was asking and what Sam and other students apparently heard.

Working with Representations

Like Toni, Sam's work with functions was mainly implied by his work with representations, so I explore his activities with representations in detail. Unlike Toni, however, whose work could generally be divided into two general categories, having facility or having difficulty, Sam's work with representations varied a great deal.

This section has three subsections. The first takes a correspondence approach to functions and the second, a covariational approach. Within the covariation segment, which is principally about fitting graphs to situations, Sam's solutions to three problems are compared and contrasted. The third subsection focuses on how Sam used the situational context in reasoning. In general, whenever possible, this section on working with representations raises issues about teaching, particularly concerning assumptions teachers can make about student understanding.

A Correspondence Approach to Functions

Sam's work with functions depended on the problem task and the representation. This section uses a correspondence approach to functions to illustrate both successes and inconsistencies in Sam's thinking when the general task was the same, but the problem varied slightly. The first subsection discusses Sam's success in finding and interpreting function values for essentially the same function in a similar situational context, but with different representations. The next three subsections use as a unifying theme translations by hand from other representations to graphs. The first two explore translating a table to a graph and the third, Sam's inability to translate from a symbolic form to a graph. These last three subsections describe Sam's activities, mathematical conventions, and apparent familiarity with point-plotting. They posit some explanations for the unevenness of Sam's work. Sam's conception of variable, one explanation, is discussed at length in the final subsection.

Finding and Interpreting Function Values

Sam understood that points that satisfy an equation or function rule should give points on a graph. Once during interview 4, he had miscalculated the coordinates for a function and when the point did not come out on the graph, he recognized this as problematic. During both the pre- and posttest problem 9, he successfully read off coordinates for points on a graph of profit as a function of ticket price for a rock concert (a quadratic function). On problem 13 of the posttest, where a similar rule

for profit for a school play was given as a function of ticket price, Sam successfully calculated both the ticket price that would give the maximum profit, the maximum profit (with minor computation errors), and the profit for another ticket price. On the pretest, Sam interpreted terms of the polynomial as answers for maximum profit and ticket price to give it. In both these problems on the posttest, he correctly interpreted both the maximum profit and the price that would yield that profit. Thus, Sam could not only read off points, but was able to interpret a global feature of a graph both conceptually (maximum function value) and in the situation. Note that in these problems, Sam seemed not to be unduly influenced by the situational contexts as he was when working with a similar function and situation in interview 3, where he was asked to respond to the same task, but where the situational context and the original task complexity overshadowed it. That problem will be discussed later under difficulties in situated problems.

Creating Graphs from Tables: Ignoring Conventions

These next three sections discuss Sam's work with creating graphs when it required using or creating tables. During the first two days of class, students were introduced to creating graphs from tables. As will be seen below, Sam was familiar with point-plotting, but his homework for the second day violated several mathematical conventions for graphing (see Figure 4.7). First, he did not place the origin where the two axes intersected. Zero began on the first scale mark above the origin and on the first tick mark to the right of it. Beyond that, his horizontal units were appropriately labeled with each tick mark representing 15 units. Second, although Sam carefully placed his vertical units in increasing order, they were not consistently spaced. He used intervals 20, 25, 30, 100, 125, and 200. Third, on the horizontal axis he added an extra point at (150,100) and then seemed to plot points for both 150 and 165 one dot too far to the right. Although it is difficult to explain the extra point, the position of the remaining two points could be explained by losing

one's place while plotting points from left to right and misinterpreting which tick mark went with 165.

Situation: Suppose a hot air balloonist takes off at noon and records altitude in meters every 15 minutes into the flight. Typical (time, altitude) data are given in the following table.

time	0	15	30	45	60	75	90	105	120	135	150	165
altitude	0	100	750	875	900	950	900	920	500	625	300	0

(a) Plot these data on the diagram below. You will first have to choose reasonable scales for the horizontal (time) and vertical (altitude) axes. Enter your choices on the diagram.

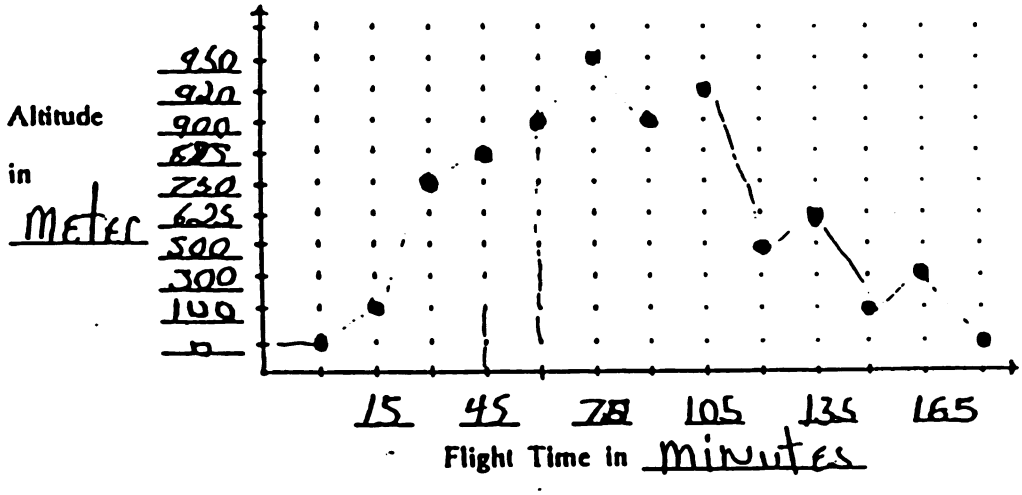

Figure 4.7: Hot Air Balloon Problem (CIA, p. 1-21)

Sam's mathematics here to create axes labels from the table entries seemed heavily influenced by the representations. First, he began with zero on the lowest line, feeling perhaps since lines were provided, it was necessary to use them. He seemed to have chosen his (0,0) point and placed 15 appropriately. He clearly recognized that the vertical labels should be ordered since he had to produce the ordering. Sam also attempted to produce a one to one correspondence between table values and either label lines or tick marks on axes (until he ran out of numbers). If one places 0 on the bottom line and puts the altitude numbers in increasing order up the vertical axis while omitting the second occurrence of 900, one would get exactly

the labeling Sam did. This correspondence between table entries and label lines was stronger than any knowledge about the need for equal units between tick marks. For the time axis, there were fewer label lines than distinct table entries, although the table entries correspond to one tick mark per each table entry. In both labelings, there are extra tick marks at the end. His marking off every other entry in the table indicates he was indeed trying to keep track of which of the numbers to use as labels. Sam had generally created a correct labeling, but had to violate placing (0,0) at the origin and equal units within an axis to do so. His choices seemed influenced by the lines provided next to the graph, that is by the problem presentation.

On the same homework set, when Sam had to create graphs for three situations given in horizontal tables, he ignored conventions again (see Figure 4.8 for one of the problems). In each problem, students were asked to choose reasonable scales for graph axes, to plot the given data using their calculators (scattergraph feature), and on graph paper to make a rough sketch of the graph they got on the calculator. On each graph, Sam either ignored which variable might be considered to depend on the other or the convention that the dependent variable is graphed horizontally or both. He could simply have labeled the vertical axis with the label from the top line and the horizontal with that of the bottom line. Yet, in each of the exercises leading up to this one, students were introduced to graphing and to dependency language for functions. For example, Situation 2.1, a problem about the average height of females by age, explicitly stated "height **depends on age** or height is a **function** of age." (CIA, p. 1-7, emphasis in the original). Sam's apparent lack of attention to dependency was no doubt an artifact of his using the calculator to enter data pairs, assuming he entered the top line first. Recall that Toni also reversed these axes, as did some other students. A graph of a relationship could be given correctly this way and understood mathematically, as long as either the specific dependency given or the convention to put independent variable on the horizontal axis was ignored. Reversing

the dependency for problem 5c gives a relation that is not a function, whereas the first two parts of the problem (5a and 5b, not shown) were one-to-one functions, and thus reversible.

While at this early stage in the class, it seems reasonable that students might not attend heavily to the issue of dependency, Sam ignored other conventions when it seemed unreasonable to do so. In parts a and b (not shown) of this same problem he created graphs with the numbers on the axes in proper order, but with a one to one match between labels for ticks on axes and numbers in the table for both axes. Thus, he again ignored the convention that scale tick marks should stand for equal measures. In problem 5c (Figure 4.8), Sam also created a one to one match between table

Situation: Animal populations tend to rise and fall in cycles. Suppose the following data show how squirrel populations in a central Pennsylvania city varied from 1975 to 1985.

Year	'75	'76	'77	'78	'79	'80	'81	'82	'83	'84	'85
Population	750	700	520	680	730	650	550	625	780	700	580

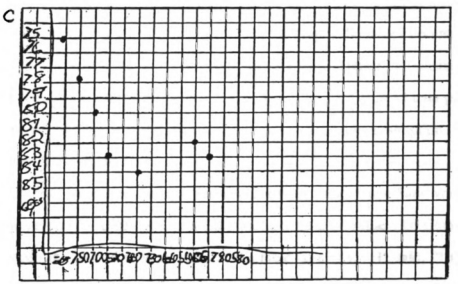

Figure 4.8: Table and Sam's Graph for Problem 5c. (CIA, p. 1-22)

entries and axes tick marks, but numbered his vertical axis with lowest numbers on the top and increasing toward the origin. Although numbers given for the dependent

variables in problems a and b were in increasing order, in part c they were not. These numbers, which Sam placed along the bottom axis in part c were not ordered at all. He simply listed them in the order they were given, supporting the hypothesis above that he used a one to one labeling scheme between table entries and axes labels. Such a zig-zag labeling scheme violates consistent scale units on an axis as well as the ordering of numbers. Further, in this graph, only some of the points plotted match the table entries for problem c. Even if you account for slightly inaccurate recording from the calculator, the pattern of dots on the calculator would not resemble this pattern. Sam's work clearly displayed that his understanding of the task to translate a table representation to a graphical one in these problems was inconsistent.

Sam's difficulties in these problems might have been influenced by the representations and his understanding of task. His conventions for labeling axes were inconsistent between the two problems in Figures 4.7 and 4.8, just discussed. In the hot air balloon problem, the appearance of blank lines for scale labels may have influenced choosing 0 away from the origin. In the animal population problem, since the task required using the calculator to first create a graph, the tables became calculator lists that Sam might have perceived as data lists. Hence he translated them in the order they were entered in the calculator, assuming Sam entered data pairs into the calculator reading the table left to right, not an unreasonable assumption.

Point-plotting: A Familiar Task

In contrast, recall that during interview 2 a week later, when Sam worked on matching the graphs of Alice, Bea, and Carolina to a table of values, he simply took the graph paper and plotted ordered pairs of the table correctly. He said, "I'm gonna use this graph paper and so I can plot some points. See if I can line it up like a real graph. I'm gonna do like regular ordered pairs," indicating that he had some knowledge about point-plotting and ordered pairs. Unlike the animal population problem above, however, in this problem the table was given vertically and the

quantities were not situated in a context. There is no other evidence that the direction of the table might have influenced plotting, but it raises a question about whether this orientation of the table might have had an influence as well.

Besides being able to plot points according to mathematical conventions in Alice's and Bea's graphs, Sam was able to connect the appropriate table entries with the x and y intercepts as well, a more complex task. In discussing Carolina's graph, I had just asked Sam whether the scale could differ on the two axes. He agreed it could and created a scale matching the table entries with zeroes, (2,0) and (0,4), to the intercepts on the graph. This task required accepting differing scales on the axes since the distance from the origin to the y-intercept (4) was less than twice the distance to the x-intercept (2). Only one other student interviewed, Bob, considered this possibility. To hold invariant the connection between table entries with zero and the intercepts, even if it meant accepting different scales on the two axes, is a fairly sophisticated idea. Recall that Toni had rejected one graph as unacceptable strictly on the basis that it would require different scales on the two axes. This task of coordinating table entries of 0 with intercepts on a graph of the table is different from point plotting, which requires a simple procedure translating an attribute of one representation to an attribute of another, one by one. In contrast, this act requires coordinating attributes of two representations connected through the intercept concept.

Sam was familiar with the task of point plotting for tables, that is, translating tables to graphs. His knowledge was unevenly applied, depending on the representation and the task. As these examples show, Sam could mechanically plot points given ordered pairs in a vertical table and had had enough experience at some point to understand a bit about the connections between intercepts and zero entries in coordinate pairs. Yet when he plotted points with data presented in a horizontal table and situated in a real-world context, he did not adhere to the conventions, some of which he had clearly learned previously. Sam had difficulty interpreting points in

some situations as well, as will be shown in the discussion of problem 2B-2 under matching graphs and situations below.

Point-plotting: An Unfamiliar Task

Even though Sam was familiar with point plotting in general, he had difficulty creating a graph from an equation. He did not understand the task included creating ordered pairs of numbers to be used as coordinates on the graph. Like Toni, he was unable to produce suitable graphs for the exponential functions $y = 5^x$, $y = 2^x$, $y = .5^x$, and $y = .1^x$. In class Sam had produced straight line graphs through the y-intercepts 5, 2, .5 and .1 for these rules. (A full discussion of these problems is in the section on understanding function classes.) During interview 3, when we discussed these graphs, Sam could explain how he produced the graphs of linear equations in slope-intercept form by using a procedure of plotting the y-intercept and finding a second point on the line utilizing the slope, but he was not able to graph by hand either a simple quadratic, $y = x^2$, or an exponential equation. As will be discussed in detail later in the section on function classes, Sam was explicit about repeating the teacher's words that "x could be anything," and he demonstrated that he understood this to mean an arbitrary single value. In the graphs he had done in class he had used $x = 1$. In the two additional exponential graphs I asked him to graph ($y = -5 + 2^x$ and $y = 2^x + 4$) during the interview, he chose $x = 2$, probably to show me he understood x could be something other than 1. In $y = 2^x + 4$ he was explicit about the exponent being able to take on only one value:

S: I'm going to make that my number.

G: What do you mean, your number?

S: The number I want it to be, so I'm going to make it equal to a number 2.

G: You are going to make it a number 2? Can it have any other values?

S: Not as an exponent.

At the end of the discussion, I decided to see if he could graph a quadratic, thinking perhaps he was confused because the variable was in the exponent and because he had shown no knowledge of being able to select pairs for the graph. I asked him to graph $y = x^2$:

G: Would you know how to graph this without the calculator? Y equals x squared.

S: That's got an equal sign. I wouldn't know how to graph it.

G: Okay. Ummm, can you tell me some pairs of x and y that would make this equation true? You know a pair of numbers, x and y, that would make this a true equation?

S: One.

G: One? One for what?

S: x.

G: One for x and then y would be?

S: One.

G: One, okay. Can you tell me any other pairs?

S: Any numbers.

G: Like? Like what? Just give me an example?

S: -5 for one.

G: For? -5 for? would be for?

S: y, I mean x.

G: x, okay, if -5 was for x, then what would y be?

S: -25. Wait, it'd be 25.

G: 25? So you said 1 and 1, and you said 5, 25, right? Can you give me a third pair? [G writes these on the sheet. 5 should have been -5.]

S: Hmmm. 1 and 0, 2 and 0, 3 and 0.

Sam clearly did not know what was going on. He was able to produce computations when asked what each value was, but when I asked for more pairs, he first said "any numbers" and in the end just made up numbers in pairs with no connection at all to

the equation. When I asked him if these pairs for x and y had anything to do with the graph of the function rule, he said, "They depend on where it is-- where the point on y is. I don't know if I remember how to graph that. I probably didn't. I don't know if we did it in algebra." He went on to say the calculator knew how to do it, but he did not. In addition, while we were discussing these exponential graphs and his calculator gave a different graph from one he had produced, Sam indicated that the correctness of the graph depended on what the teacher wanted.

Conception of Variable as Explanation for Graphing Difficulty

As with Toni, Sam's difficulty with this graphing problem may have been in part related to his conception of variable. One possible reason he could not create ordered pairs was that he did not see the expressions as representing many pairs of values. Overall, Sam had a naive conception of variable. He recognized that a variable could take on more than one value. Problem 4A-7 was:

> Blue pencils cost 5 cents each and red pencils cost 6 cents each. I buy some blue pencils and some red pencils, which cost me 90 cents. If b is the number of blue pencils bought and if r is the number of red pencils bought, what can you write down about b and r?

Sam found more than one correct set of values for b and r. He was not able to write the relationship, $5b + 6r = 90$, however. Sam also could see that a variable expression could be equivalent to a number, but could not use that relationship in another expression. In problem 4A-5, "If $e + f = 50$, what can you say about $e + f + g$?", he acknowledged that $e + f$ was 50, but seemed to see it as a result or equivalence, not an expression that could be used in other expressions. He said he could do nothing more because he did not know the value of g. Sam was unsure that different variables within one expression could have the same values. Problem 4A-2 was: "$x + y + z = x + p + z$ is always, sometimes, or never true? Explain." Sam first said he thought they could not be equal because the addition process would

lead to different results:

> I don't know if I can believe that. Somethin's gotta equal. [pointing to the two sides of the equation, in turn.] That's gotta equal something other than that. When you add'em together, ain't gonna equal that.

On reflection, he revised his answer to be "sometimes," a correct response. Finally, Sam could not accept lack of closure and create a variable expression as representing a number, while not knowing the variable's value. In problem 4A-6, where the perimeter was sought for a polygon with n sides of length 3, but in which the figure for the polygon was partially covered with paint, Sam demonstrated that he knew a process for finding the perimeter by adding up the lengths of the sides. He could not, however, find the perimeter without knowing the number of sides. Clearly, he saw the n as an unknown, not a way to represent the number of sides in an expression for perimeter.

Although Sam could see variables as generalized numbers and as unknowns in a process or equation, his understanding that distinct variables within an expression could have different values was fragile. Finally, he had difficulty using variables to generate expressions representing numbers, which required seeing the expression as a whole as representing a number.

A Covariation Approach to Functions

Rate and Patterns of Change

As discussed above, Sam was successful in problem 2A-2 to translate a table of a linear function to its symbolic form ($y = -5x + 15$). In that problem Sam connected a pattern of output of decreasing by 5 as input increased by 1 to the -5 multiplier on x because he had ordered the "speed" of the operations. When I asked him why he had chosen multiplication instead of subtraction for going down by 5, he first said it (y) got there faster than (by) adding. He added, "When I multiplied it, it got me that 10 (the y value for $x = 1$) quicker." On one other occasion, Sam attributed an order to the speed of arithmetic operations.

Sometimes Sam attended only to patterns of change in the dependent variable in tables, a non-robust version of covariation. As will be seen below in the next section and under the discussion of function classes and representations, he found patterns of change a salient feature to classify functions as linear or quadratic.

Fitting Graphs and Situations

This section discusses fitting graphs to situations, that is, one version of Sam's covariational approach to functions. After discussing several examples and Sam's reasoning patterns, including resources he brought to problem solving, the section raises questions about the role of situations in solving problems. Like Sam's performance on creating graphs from tables, his work on fitting graphs with situations varied with the problem. In the three interview problems (2B-1, 2B-2, and 5-1) in which Sam had to find and justify a graph to match a given situation, his work was inconsistent. The first two segments present Sam's work with two problems (2B-1, 5-1) that had the same graph, but had different situations. Sam's performance was also quite different. In each segment his reasoning strategies are discussed. In the third segment, Sam's solution and reasoning for problem 2B-2 are discussed. Finally, the third segment also discusses how the situation connoted by the mathematics, along with Sam's strategies, raises interesting questions about teaching graphs of situated problems.

Problem 2B-1: A valid interpretation. Sam was fluid in his interpretation of problem 2B-1. In that problem he was to select a match from several graphs (see Figure 4.9 below) for the situation:

> When I started to learn the guitar, I initially made very rapid progress. But I have found that the better you get, the more difficult it is to improve still further. (Proficiency/amount of practice.)

Sam quickly chose a suitable graph, g, giving the following description to justify his choice:

> Okay, umm, he say he started-- it used to be easy to go when he played. He

> used to go up fast and then he slowed down. Okay, it was very fast and then it went down. He didn't say it decreased, um, he just said it got more difficult to get better. So I'm gonna say that he still is good [G: okay (softly).] and I like, I like g. [pause] Because it went up fast for a while and then it evened it out like it was getting more difficult. It's still not falling. I mean, all he said was it get harder to get better.

When I asked him to describe his axes labels, he placed "time" on the horizontal axis and "the better he got" on the vertical axis. He changed the vertical label to "rapid progress," and asked if progress and getting better were the same thing. When I agreed they were and said the problem had said "proficiency," he changed the horizontal label to "amount of practice," rather than time.

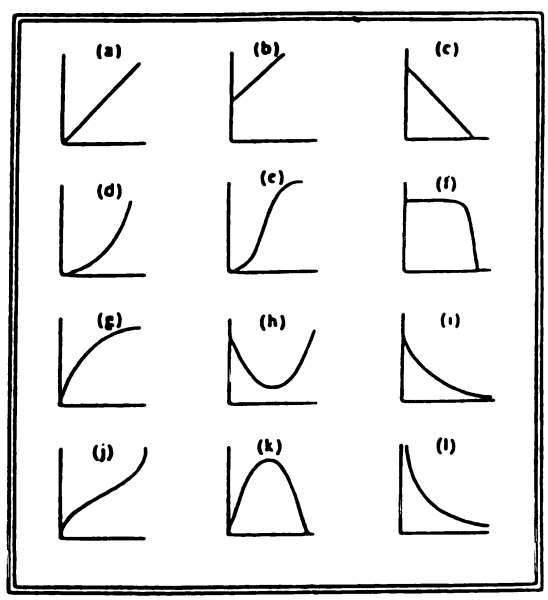

Figure 4.9: Graph Choices for Problem 2B (LFG, p. 102)

Sam had described the change in this graph in terms of only one variable, how much better he got (improvement) and had accurately matched a description of the change in this variable to the slope of the graph. At the beginning where the slope is steep, he said, "go up fast," that is, the rate of improvement was rapid. Then he used "he slowed down" to describe improvement as going up more slowly. "Okay, it was very fast and then it went down" could be interpreted either as a temporary

inclination to think of improvement itself going down or possibly as referring to the rate of change in proficiency as going down. In any case, Sam interpreted it correctly in the next sentence by explaining that the graph went up more slowly, that is, it evened out because improvement was getting more difficult. He had matched a slower change in proficiency (improvement) to a less steep slope. Sam reinforced this justification by describing the graph as not falling because the player was continuing to improve, just at a slower rate. He rejected graph j when I asked if it were a possible choice. Again he described the change in the graph directly: "Look at it. It's going up fast, and then it's evenin' out, and then it's goin' back up. It don't look like it." When I asked him what the tail end of the graph meant, he again described the graph itself in terms only of the rate of change in progress: "That's the end it's increasin' more. Goin' up fast. It's goin' up rapid right there. That rapid progress." In his justifications, Sam spoke only of improvement or progress, which he seemed to use for proficiency. Note that he used similar language to Toni, who spoke of proficiency being "pretty much level." Sam seemed to fluidly move across the graph (increasing time), but with only tacit attention to time. All references to time were indirect, such as through the use of the words, "then" and "for a while." Like Toni, he reasoned with one variable, with a temporal aspect incorporated in the independent variable by sweeping across the graph from left to right.

Tim and Donna: Covariation difficulties. A suitable graph for Donna in the Tim and Donna problem (5-1), given below in Figure 4.10, was identical to that in the guitar problem, as they represented the same pattern of variation. When Sam interpreted this match between a situation and a graph, however, he had more difficulty.

Sam began by reading the problem aloud and mentioned modes of travel and that the children switched modes at the same time. He started with Tim and considered b, c, and a, three of the four graphs that had segments that increased. He

fairly quickly chose b for Tim and justified his choice (while pointing to b and moving his finger upward along the graph): "He started walking, then he started walking faster, then started running." It appeared Sam was illustrating a match in speed with the slope of the graph, but could just as easily have been thinking of the graph as a graph of speed, in which case, it is also a correct match.

> Tim and Donna live 1 km from their school. Usually they walk together to school. Yesterday they both left their houses 10 minutes before school started. Tim started to walk but Donna was afraid to be late and started to run. After a while Tim realized that although he tried to walk faster and faster, he had to run if he did not want to be late, and started to run. At the same time, Donna became tired and had to walk instead of run. They both reached school exactly on time. Which of the following graphs is Tim 's ____ and which one is Donna's ____?

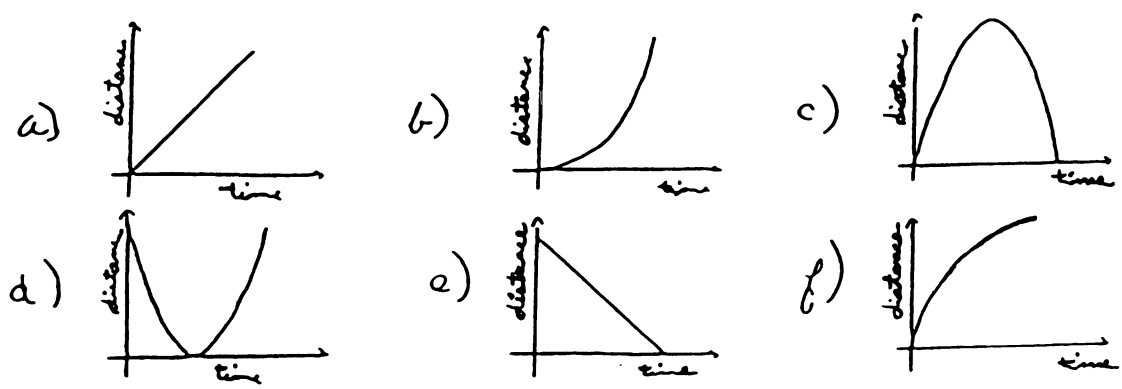

Figure 4.10: Problem 5-1, Tim and Donna Problem

Next, after considering e and c, Sam chose d for Donna:

> Ok, she ran. [points to e, running finger down graph left to right, then moves to c.] She started running then she got to stop. [moves over to d] When he started running, she started walking. [still on d, but looks toward c or f] Hmmm. She didn't exactly stop either; she just- Hmmmm. [He glances at c or f] I think he get b and she get d.

These three graphs, c, e, and d, are the only graphs with decreasing segments. Clearly, by this time, Sam was considering a graph of speed over time, rather than

distance over time, a fact he explicitly admitted later in the interview. In addition, his attention was focused on the need to find a spot on the two graphs where Tim and Donna changed travel modes simultaneously. This need is probably why he tentatively switched to d from e, which had no clear change.

When I asked Sam why he thought b for Tim and d for Donna, he immediately changed his response from d to c for Donna before finally settling on f:

S: She was runnin' and she stopped runnin' [He points briefly to c, running his finger along the graph, before moving to f.] (inaudible) I think it's f now.

G: F? Now could you tell me how you reasoned that Tim would be b and Donna f?

S: Ok. He started walkin', then he got faster 'cause he started to run. Then the speed accelerated. She started runnin' and she-- as she ran, she got tired then she had to stop. Plus it said "at the same time" they started doin' whatever they did, walkin' or somethin' (changing modes). [G: um-huh] So right there [marks graphs b and f with vertical line about halfway across the graph] they both started doin' what they was doin', do the-- do the-- the walkin' or the runnin' that they was doin' at that point.

Here Sam had justified his graphs with a description of speed and acceleration (rate of change of speed) and the coincidence of changing travel modes, which he marked with vertical lines on the two graphs.

I continued by questioning Sam's use of acceleration. He suddenly noticed "distance" on the axis and said he had thought it was speed. Then he quickly described Tim's graph as "getting his distance faster." Sam said, "he covered more distance right there" while pointing to the steep part of the graph b and "he covered less that way [than] when he start goin' faster" while pointing to the less steep part. Again, as in problem 2B-1 above, Sam had associated steepness of the graph with the rate of change (speed) in the dependent variable (distance) and justified his graph by matching a description of the graph with a description of this rate of change while "experiencing" the variable, time, as simply passing.

Being able to match change in the situation and change in shape of the graph (slope) was critical to Sam's reasoning in this problem. In Tim's case, both speed and distance increase. While sweeping across the graph from left to right, that is, in increasing time, one could describe either speed, distance, or the graph, and the description would be essentially the same: in the beginning the ____ went up slowly, and after a while the ____ went up more rapidly. In Donna's case, however, when Sam first tried to coordinate the slope of the graph with speed, he ran into difficulty because he needed a graph that went down. During the first part of the interview, he coordinated the shape of the graph with speed and soon began talking directly about speed and acceleration (rate of change in speed), rather than distance and speed (rate of change in distance). He tentatively chose c, so the graph would match a description of speed first increasing (to go from rest to running) and then decreasing (to slow down for walking), and for much of the time seemed to reconsider c.

Even (1989), who developed the Tim/Donna problem for her Phd dissertation, reported similar responses. Fourteen of the 152 undergraduate prospective teachers who answered this problem chose graph c for Donna. Several of those interviewed attended to superficial characteristics of the graph and either ignored labels or switched the axis label from distance to speed, either initially or midway through where the graph did not match a description of speed.

Sam had started out using speed as his (implicit) label, and even after he successfully justified the graph f above, he had difficulty resisting graph c, which matched speed as going up then down:

> Now this one (f), I'd have to use distance because see she was coverin' more distance-- right there then it started leveling off, and then it kinda came-- and you can tell it's going to come to a drop (beyond what is shown on the graph).

In this problem Sam attended to time directly when he tried to find a place where the graphs changed simultaneously. Yet when he justified the match, as in the guitar problem, he seemed to focus almost exclusively on the dependent variable distance

and experienced the independent variable, time, by attending to it very little, except to consider it as passing as he moved across the graph. Had he not switched to speed, but had considered distance, rather than speed, to begin with, he could have solved this problem immediately exactly like the guitar problem by describing Donna's rate of change in distance (speed) and coordinating the change with the shape of the graph as he moved across the graph from left to right. Note that Sam ultimately used similar language, "evened out" and "leveling off," to account for the far right side of the graph.

It is not clear whether Sam began the problem thinking of the vertical label as distance or speed. Had he misread the problem initially as speed/time, it seems he would have immediately chosen and stuck to either c or e, depending on whether or not he considered time as first measured from when Donna was at rest (c) or when she was already running (e). That he did consider these first indicates he was soon concerned with speed, however. Sam's concern with the simultaneous change and his introduction of Donna stopping, rather than ceasing to run, might have caused him enough confusion that he resorted to reinterpreting the axes as speed/time almost immediately. In any case, Sam used the same strategies as in 2B-1, the guitar problem, to describe the rate of change in the dependent variable directly and coordinate this rate with a description of the slope of the graph, while accessing the independent variable, time, only implicitly. These reasoning strategies clearly contributed to his success on these two problems.

One difference in the two problems is Sam's use of language. In each case, he coordinated a description of change in the dependent variable with a description of change (slope) in the graph, with only tacit attention to time as the independent variable. In the guitar problem, he described change in the dependent variable itself (proficiency or improvement). In the Donna problem, however, he had a name, travel mode (speed), for the rate itself. In the latter case, Sam had more trouble with

his coordination and he struggled not to slip into graphing speed, rather than distance as a function of time. Perhaps the lack of a readily available label for speed of change in proficiency helped him maintain his focus on the dependent variable itself, rather than the rate of change as a variable, which interfered with his matching graph to situation.

Problem 2B-2: Situational confusion. Although Sam was able, with minor trouble, to correctly match graphs with the problems just discussed (2B-1 and 5-1), he had much greater difficulty matching a graph in Figure 4.9 given above with the situation (2B-2):

> If schoolwork is too easy, you don't learn anything from doing it. On the other hand, if it is so difficult that you cannot understand it, again you don't learn. That is why it is so important to pitch work at the right level of difficulty. (educational value/difficulty of work)

As will be seen, Sam misinterpreted the situation statement and confounded his problem solving by an inappropriate graph choice. Further, his choice of educational value, a variable that did not increase over time, for the independent variable made it impossible to tap strategies he had used earlier in the guitar and Tim and Donna problems.

Sam read the problem and had difficulty thinking of it as a description that represented the relationship between two quantities. He confused two *instances* of the relationship for two *relationships* and saw the situation as requiring two graphs because there were two values for difficulty of work that had the same educational value:

> S: Okay, this should be two graphs right there.
>
> G: This should be two graphs?
>
> S: They gave you two different-- well-- [reading aloud] If schoolwork is too easy, you don't learn anything from it, well-- I'm pointing that out because he doin' good, but he ain't gettin' nothing-- or she. [pause] Which other one might work? I'm thinkin' f right now.

Even though he saw this problem as two situations, he chose a single graph, f,

anyway (Figure 4.11 below). Since Sam was carefully attuned to my comments, perhaps he had correctly interpreted the problem instructions given shortly before during the guitar problem, "you have to try to find the best-fitting graph and then say how it fits it," to imply a single graph existed.

Figure 4.11: Graphs f and k from Problem 2B

I asked Sam what he would call his axes. He first labeled the vertical axis as difficulty of work and then switched it to educational value. I asked what the horizontal (axis) label would be, and he said, "the difficulty," but this labeling caused him concern:

> Oh, no. That'd be hard. Let me see, if it's difficult, it would be way down here (on x-intercept). But this sayin'-- Because if he got difficulty, he failed. Naw, I can't say that. [pause] Hmmm. Okay, now, let me think. [pause] This (x-intercept) could be the educational value. It ain't nothin' right there. I like that better.

"Oh, no" meant Sam immediately had second thoughts about labeling the horizontal axis as difficulty of work. His statements, "That'd be hard" and "Let me see, if it's difficult, it would be here (on x-intercept)" indicate he was trying to place a situational value of hard for difficulty of work at the x-intercept. He then checked out his reasoning by referring back to the situation description of work being too hard and not learning it. Sam rejected, or at least doubted, his label choice of difficulty on the horizontal because it did not seem to fit the situation. He had associated a single value of zero with the x-intercept ("It ain't nothin' right there."), and since this value matched the situation value for not learning, he chose educational value as the label for the x-axis on the basis of this one value for the x-intercept.

Although my interpretation of this segment may not be fully evident from Sam's statement here, it is supported in the rest of the interview for this problem. As Sam's work on this problem is difficult to follow, this paragraph is meant to provide a frame or overview of what will be shown below. For much of the rest of this problem, Sam repeatedly acted in the following ways. He interpreted points on each axis as measuring only one quantity. When the point was the y-intercept, he referred to difficulty of work, the label for the y-axis, and when the point was the x-intercept, he used educational value, the label for the x-axis. At the same time, Sam interpreted the values of these quantities only in terms of the vertical measure on the graph, that is, the measure of the second coordinate of the point. Further, he seemed to justify his actions or reason *only* with information about the situation itself, even when it conflicted with the mathematics, what he had said previously, or when information had to be invented to expand the situation that was given.

In the segment of transcript just given above, Sam was trying to resolve a conflict between his axis label, his interpretation of the x-intercept, and his interpretation of the situation. In graph f, if one considers only the vertical measure, then the graph has a high value at the y-axis and a low value at the x-axis. In this segment, I believe Sam saw a conflict between a low value on the graph (at the x-intercept) and a high value for the quantity in the situation (work being hard) he wanted to place there. Since his interpretation of the x-intercept as zero was consistent with a low value (not learning), he resolved his conflict over axis labels by assigning educational value to the horizontal axis. Recall, that not learning was mentioned twice in the problem and was a salient point for Sam. While reading the problem, he had pointed out his concern for how to deal with two values for difficulty of work associated with not learning.

In this problem Sam consistently used only the vertical measure on the graph to make decisions. When he worked with the guitar or Tim/ Donna problems, he

also had relied on his ability to "think with" the dependent variable only so that he could coordinate its change with the graph's going up and down over time. My interpretation of Sam's actions here are consistent with strategies he used to match graphs and situations in the guitar and Tim and Donna problem. Sam had difficulty interpreting this problem because he had chosen his axes in such a way that he could not reason in this same fashion. His inability to label his axes in a reasonable way was probably due to his quick choice for a graph that he saw as fitting the situation easily because it had one high value (at y-axis) and one low value (at x-axis) that he had attended to immediately on reading the description of the situation. Had Sam not made a bad choice of graph to begin with, he might not have found this problem so troublesome. Much later in the interview, when he chose a suitable graph k (Figure 4.11 above), he was able to choose appropriate axes and to justify a fit with the situation--at least for the two x-intercepts and the vertex.

As we continued the problem, I tried to ask questions from which I expected Sam to recognize the mathematical conflict within his representation of this situation and expected him to reconsider his graph choice. He did not, however. Instead, as will be seen in the following segments, Sam created greater problems by introducing new variables to the situation.

Since Sam had focused his attention on the x-intercept, I tried to elicit how he thought about it:

G: The horizontal one would be the educational value? and this point (x-intercept) way over here to the edge would be what then? Would that be a big educational value or a small educational value?

S: I'm gonna say it's-- [pause] I'm gonna say it's both [G: both?] Um-huh, because it give me two stuff, two of 'em. If it's too easy or if it's too hard.

G: Too easy and too hard would be the same spot on the graph?

S: It's dependin' on the person's mind.

When Sam said the x-intercept would represent both a large or small educational

value, I was surprised, not because he had answered a question using values for difficulty of work rather than educational value as I had asked, but because he had put two distinct values for one of the variables on only one spot on the graph. Based on his work with problem 2A-1 earlier in the same interview, I had assumed he understood points had two values (coordinates) associated with them. Yet, here he did not. Sam had seen the x-intercept as zero educational value only, and had simply supplied answers from the situation description to answer my question. When asked to justify his choice of "too easy" and "too hard" being on the same spot on the graph, Sam introduced a third situational variable that was not discussed in the problem at all. When he said it depended on the person's mind, he raised the possibility of more than one person or of more than one perception by a single person (which would depend on the time or situation).

Since Sam's thinking was inconsistent with the representation, I tried to bring the inconsistency to light. As we continued, Sam again referred to the x-intercept as having a low educational value and as being both too easy and too hard. I decided to focus on this discrepancy of two vertical measures at a single point. Since he had called the horizontal axis, educational value, I reminded him that "how hard it is" was the label for the vertical axis and asked whether too easy and too hard would be on the same spot. Sam again raised the issue of multiple people:

S: Hmmm. Let me see. Okay. This might be too hard for somebody, but it might be too easy for the other person.

G: Ohhhh. Okay. Now I understand what you mean by it could be too easy or too hard. Now let's back up a second. Can one graph be-- [pause] for this problem, can one graph be multiple people? Or is it just for one person?

S: Yep, yep, it could be multiple people. A whole bunch of people-- on one graph. I wasn't thinkin' about that. It could be a whole bunch of people, and right there is a high educational value for some people and a low educational value for others.

I had tried to call Sam's attention to his discrepancy in interpreting the x-intercept as

having distinct values for difficulty of work, but he had again introduced a third variable, people. Earlier when he had said, "It's dependin' on the person's mind," I had thought he was talking about different times for one person at a single point on the graph, rather than different people. Here, when he clearly talked about multiple people at each point, I thought the confusion would clear up if I said the graph was for only one person. So I explained directly that we really could not tell the meaning of one point if it referred to more than one person and that he should think of the graph as being only for one person. Note that in his eagerness to respond to my question about multiple people, he introduced multiple levels of educational value as occurring at one point, whereas earlier, he had spoken only of multiple values for work difficulty.

Sam was clearly confused about interpreting points on this graph, and I had tried to make sense of his interpretations and to focus his attention on the difficulty variable in an effort to help him see that he had assigned two different vertical measures to the same point. Twice he answered my questions by introducing new attributes of the situation that could explain what was problematic in the representation, in this case, too easy and too hard at one point on the graph. In situations, certainly things can be too easy or too hard for a person, depending on the day or the some other aspect of the situation, and they can be too easy or too hard, depending on the person. Yet in Sam's explanations, he attended only to these situational aspects, not to what implications they had for the graph. Although I had perceived Sam as having difficulty understanding the distinctions between points on the graph, he was not attending to this issue at all. There was a clear difference in our assumptions about the topic on which we were focusing. I had continued to ask questions to clarify the mathematical constraints on the situation, and he had ignored the constraints of the representation itself. Instead, he simply invented an explanation by expanding the situation (adding people) to match the situational

attributes (both hard and easy with education not valued) that he had assigned to a single point on the graph. When Sam admitted he had not even thought of the possibility of a graph for multiple people, he indirectly admitted that he had not considered the representation itself even though he had used multiple people as a reason to justify his point description.

Presumably in different situations, for one person, something could be too easy at one time and too hard at another, or at one time, something could be too easy or too hard depending on the situation or the person. Generally, such situational variables would be ignored in a graphical relationship between difficulty of work and educational value because the broader context that situates the speaker is either understood or is irrelevant to the mathematics discussion at hand. Each point on the graph is seen as a "choice" or "potential realization" for the relationship, that is, an *instance* of the relationship, representing an educational value for a particular level of difficulty. This instance of the relationship could be perceived as occurring at different times or in different situational contexts, such as the state of a person's mood or as between different persons, but those situational factors are not under discussion at the moment. When one discusses the mathematical relationships of this situation, these other variables are considered irrelevant. Whether a presumed speaker of this situation description is an algebra teacher or the U. S. Secretary of Education has no direct bearing on the mathematical interpretation of this situation, nor does the fact that the speaker might have been talking about algebra, schools, or teaching situations in general. What is assumed, whether or not it is made explicit, is that there exists some sensible context to which the speaker refers, and most of it is invariant during the discussion. There is also an underlying assumption that some subset of this context can vary, making possible the distinctions from instance to instance of the specified relationship, that is, from point to point on the graph. For example, for the same content area the amount of reading or complexity of homework

problems does indeed make work less or more difficult; that is, there exist factors that can account for the existence of different levels of work and thus the existence of distinct points on the graph. Whatever these aspects of the context are, they, too, are irrelevant to the mathematical translation of the situation description to a graph. They need not be made explicit in the discussion.

A student who is trying to make sense of this description may, as Sam did, ponder these factors and conclude from daily life that difficulty of work on a given problem or set of problems could vary from person to person, as could the educational value. Each point on a graph of the relationship between difficulty of work and educational value could indeed represent a different person. If the *same* level of educational value is attained for work that is *easy* for some and *hard* for others because of individual differences among people, the situation could be correctly represented by two distinct points on the graph. Assigning difficulty of work, rather than educational value, to the horizontal axis would allow a correct *function*[5] graph to represent the problem situation, and each point could represent a different person. If on the other hand, one wished to use a single point for this same situation as Sam did, a third variable, person, must be introduced. Such a situation cannot be shown in two dimensions; otherwise we have represented at the same spot, the points (x_1, d_1) and (x_1, d_2), where $d_1 \neq d_2$. A third dimension, p, is required, with points, (x_1,d_1,p_1) and (x_1,d_2,p_2), where $d_1 \neq d_2$ and $p_1 \neq p_2$. In this problem Sam had made explicit two such *instances* of the relationship, difficulty of work being too hard with educational value being low, and difficulty of work being too easy with educational value being low. He had attempted to place them at one point. I do not think Sam was aware of the dimensional issue this action raised, but instead, simply

[5]If axis labels are reversed, a legitimate relation is defined, but the graph would match none of those shown as choices.

had looked to the situation alone for a plausible explanation from everyday life for how these two different "situations" could occur at the same point (simultaneously), without regard for the implications for a graph of a functional relationship between two variables.

In the beginning, since Sam had just completed problem 2A-1 where he had plotted points and had correctly interpreted ordered pairs, I had assumed that he considered each point as having two coordinates. Because of these assumptions, I believed Sam would recognize discrepancies in this knowledge if they were brought to his attention. So after my efforts to help him recognize the difficulty of two vertical measures for one point failed, I continued to quiz Sam about his axis labels and tried to call his attention to a discrepancy in his horizontal measure (educational value) because it had high values for x at the origin and low values some distance from the origin. I continued to probe, and once more he answered in a way that indicated he was not considering ordered pairs, so I switched to the word measure and asked directly about both coordinates of a point:

G: Okay, then what are you measuring along the horizontal?

S: What am I measurin'?

G: Um-huh, what would be the other thing that are you measuring there?

S: This is the difficulty. (y-axis)

G: Um-huh, you put difficulty on the vertical.

S: And this the educational value. (x-axis)

G: Okay. Educational value on the horizontal, difficulty on the vertical. [S writes labels] Yeah, that's a good idea to write it down so then we can remember. Okay, now. This point where the graph touches here (y-intercept). [S: um-huh.] With these two axes, how would you interpret that point for this person?

S: Difficult.

G: It's difficult. What kind of educational value is it?

S: To him it's nothin' because he failed.

G: Educational value is nothing but the work is difficult.

S: Um-huh. See the work is difficult and the educational value, he don't care about it no more.

G: Okay, and how about this point over here where the graph touches (x-intercept).

S: Now that's the educational value.

Sam was surprised by my question about measuring and repeated his earlier contention that difficulty was on the vertical and educational value was on the horizontal, possibly referring to specific points rather than the axes. After I had repeated the axes labels, had explicitly mentioned two axes, and had asked about the y-intercept, he again had interpreted it with only one variable, "difficult." When I asked about its educational value, Sam had answered with words describing the situation, "to him it's nothin' because he failed," even though it contradicted his measure of it along the x-axis. Recall that his scale was reversed, so *high* educational values would be near or towards the origin. Finally, when I asked again about the x-intercept, Sam had again identified it with only one variable, educational value. In both instances here, he had associated only one variable with each point, and when I pushed for additional information at the y-intercept, Sam had supplied it only from the situation description.

Since Sam had at last identified a second value for the y-intercept, low educational value, at the time I thought it possible he had finally corrected his horizontal scale by placing zero at the origin. But as I continued and questioned him about the scale for educational value, he said a high educational value was in the low numbers, 1, 2, or 3. I reminded him that he had told me the y-intercept had no educational value and was very hard work and asked if that was consistent with what he had just said, and he admitted directly that he had been talking about only one variable:

S: Okay, wait, I was talkin' about difficulty of the work. It's too hard.

G: What about the educational value at this point?

S: That's down there, right? (the x-intercept)

G: No, just this point on the graph (y-intercept), not that point on the graph (x-intercept), but this point up here at the very top of the line.

S: That's the difficulty of work.

G: It's only one, it's only measuring one thing?

S: Um-huh.

In these two passages, Sam continuously categorized each intercept by a separate variable. When asked to justify his responses, he used language about the situation from the description even though it contradicted what he had just said about the representation itself-- that at the y-intercept it (the work) was difficult and the educational value was low ("to him it's nothin'").

As the interview went on, several times Sam referred to points as having only one measure, but finally he got the idea from my continued questioning that something was wrong. He seemed not to be able to determine what it was and so began to change the language of the situation to redefine the meaning of educational value. He said hard work made a person not "like school," rather than school not having value. At one point he asked whether educational value meant whether a person values the education. I told him yes, but rephrased it as how much value school was to the person. For most of the rest of the interview Sam continued to talk of liking or valuing education, until near the end where he was trying to justify h (a bowl-shaped parabola) as a possible fit. He had recognized the right most point as difficult work, but high educational value and said, "But he didn't learn. Wait, he didn't; wait if he don't like--. If it difficult understanding, you don't learn anything. But it didn't say he didn't value it."

Sam had resorted to a problem solving strategy to redefine the situation in an attempt to make progress in the problem. In the remainder of the interview, twice

more he raised the possibility of multiple people. Ultimately, when we discussed a point on the graph that was not on an axis, he still assigned a single number, its vertical measure, to it. Only when I explicitly asked for its second measure, did Sam realize that two numbers should be used. He then identified the two as "like an ordered pair," connecting this knowledge to that he had displayed earlier in the interview during problem 2A-1.

Finally, I suggested the possibility of a graph different from f and that Sam reread the problem. He next considered both h (a bowl-shaped parabola) and k (an inverted parabola). He chose k over h because the right part of h was not falling. This time Sam put difficulty on the horizontal and educational value on the vertical and quickly justified the graph by accurately interpreting both x-intercepts and the vertex of the graph. Although he justified graph k by the three extreme points, Sam could have thought about difficulty of work on a continuum of increasing from easy to hard and could have pictured educational value as being low in the beginning and rising and falling as he went across the graph. Over the course of the discussion of this problem, Sam had changed the meaning of educational value to liking school, which he then thought of as changing over time. When he justified his choice of graph k, Sam had started at the origin, but then had placed too hard at the other x-intercept saying, "they don't like it no more," as if time had passed. He used the same language at the end of the passage, repeating "no more" twice, as if he had thought of the problem as a boy's attitude (liking it) about school changing over time.

Sam apparently used his previous problem solving strategies to correctly match graph k and the situation. Even so, as in the Tim/Donna problem, his interpretation of this match was fragile. After he had justified graph k, I went on to ask if the graph could be h, the bowl-shaped parabola that he had tentatively selected earlier. Sam correctly interpreted the upper left point on the graph as difficult, but not valued, and the low point in the middle as easy and somewhat valued. However, when he

came to the upper right point he called it hard work and not valued, words from the description that were inconsistent with the graph, which should have been interpreted both hard work and high value. Sam had again interpreted the two highest points and the vertex correctly in terms of his choice for the vertical measure, difficulty of work, but had trouble interpreting their horizontal measures correctly.

Reasoning strategies. Sam had attempted to use the same strategies for matching graphs to situations in all three of these examples, the guitar problem, Tim and Donna problem, and the schoolwork problem. He was quite successful in the first because the language he used for the graph, "evened out," matched his description of the change in proficiency. Sam ran into more difficulty with the Donna problem because the description of their mode of travel reinforced his disposition to use speed instead of distance as the dependent variable. In the schoolwork problem, he confused the description of two values for the situational variables as determining two separate situations requiring two graphs. He made a poor first choice for a graph. When he justified his decisions by using attributes of the situation while ignoring any mathematical constraints of the problem, the discussion was led far afield as I attempted to understand how he was thinking or to get him to realize his inconsistencies. In each of these difficulties, the problem description seemed to account for his initial difficulties. Since he chose not to use a strategy of checking the representation for consistencies, but reasoned only in the situation, the steps that he took did not appear to him as inconsistencies.

In addition, there is a distinct possibility that he was unable to reason about such graphs unless one variable could be thought of temporally. In both the guitar problem and Tim and Donna problem, Sam either had placed or found time on the horizontal axis, but no such possibility was available for the schoolwork problem. Since Sam consistently interpreted graphs by thinking with the quantity measured vertically, there was no obvious choice for axis labels. His poor choice of

educational value on the horizontal prevented his "massaging" this problem into one in which he could think principally with the dependent variable and coordinate its changes with a description of the shape of the graph. In each case, both the situational context, its description, and the availability of time (or a variable that could be viewed temporally) as one of the variables made a difference to how Sam could match graphs to situations.

Difficulties in Situated Problems

In the example just discussed, Sam made heavy use of situation descriptions in his reasoning, which both supported and hindered his problem solving. He used the situation to support his reasoning in fitting graphs to situations. In the schoolwork problem, his reliance almost exclusively on the situation, rather than mathematics, caused him trouble with his problem solving. Sam's difficulty interpreting problems that were situated continued in the Profit Problem in interview 3 (Figure 4.12) because his informal knowledge interfered with his understanding the mathematics of the problem. Sam's work with the problem also illustrates his questioning techniques to seek clues about how to proceed.

> Carl has been assigned to come up with the price to charge for tickets for a concert the senior class is using for a fundraiser. From previous years, Carl has learned that if they hire a celebrity emcee, the function relating ticket price and profit was
>
> $$P(t) = -100t^2 + 700t - 200.$$
>
> Carl reasoned that if they didn't hire the emcee, they could save his $200 fee. His friend Beth said he should reconsider the problem because he would need to lower the ticket price by at least 3 dollars to get anyone to come if no emcee was hired. She told him he could consider this new situation by using the function
>
> $$P(t) = -100(t - 3)^2 + 700(t - 3)$$
>
> Was she correct? Which plan should he follow?
>
> Does the size of the auditorium make any difference?

Figure 4.12: Problem 3, Profit Problem

Sam began this problem by spending the first seven minutes clarifying its statement and figuring out what he was to do. First, he read the problem, asked for clarification of the word "emcee" and whether the top rule was *the* function rule. He entered the top function rule in variable form in the calculator on the calculator computation screen so he would not "have to do all that writing." Next, because there were two rules, he asked if there were two separate problems. He then focused on the question about the auditorium size. Since he did not know the size and I did not give it when he said he wished he knew what it was, he answered that yes, the size did make a difference because it affected the number of tickets that could be sold. Like Toni, he was concerned about not knowing the number of tickets sold. He announced, "And I don't know how many tickets, how many seats, so I can't tell how many tickets."

Next Sam restated the top formula was with a DJ, but that since the problem wanted to know what happened without a DJ, he needed to use the bottom formula. He typed it in on the computation screen of the calculator. Then he tried to figure out what P[6] and t stood for in the top formula:

> Okay, now, I wonder what the t stands for. [Points to P in top formula.] Okay, this is people? [pause] I'll do this people. Uhh. Or price? and number of tickets?

I asked him to clarify what he said and by reading the problem, he determined profit and ticket price. Sam was again bothered by not knowing the ticket price:

> Okay. Price and number of tickets sold. But I see right now, it's ticket price and profit. Profit and ticket price. [pause] I think. Okay, so the pro--, yeah, yeah, ticket price is right there [pointing to the t in t^2 of the top formula]. Okay, now he gonna lower the tickets $3. I think it says-- [reading] Carl reasoned that if they didn't hire the emcee, they could save $200. [pause] I wish I knew how much the tickets was. Let me look, okay, now.

Next Sam stated that he did not know how much they wanted to raise, and as I

---

[6]P was the name of the function, not a function variable, but several of the students, including Toni, referred to it as a variable.

interpreted his comment as not understanding the assumption they wanted to maximize profit, I suggested they try to raise as much as possible. Finally, without doing any computations and based only on his own ideas about the situation, Sam guessed the top formula was best because of hiring a DJ.

By this time, Sam had spent considerable effort in trying to understand this complex problem. In the process, he had tried to sort out the meaning of variable names, P and t. He had repeatedly raised the issue of needing to know both the number of tickets sold and the ticket price. Like Toni, Sam perhaps had thought of profit as number of tickets times ticket price. Even after he understood the rule had to do with profit and ticket price, he had not seen it as a relationship or model for the relationship. Rather, he seemed to understand the function rules as formulas that required a specific ticket price before any computation could be done.

Sam seemed not to know how to proceed and had just chosen the top rule, so as I had done with Toni, I narrowed the problem to focus on one rule. I asked if he could figure out what ticket price to use in order to make the most profit with the top rule (with the DJ). I hoped at least to learn how he thought about a single quadratic function. This action allowed Sam to momentarily shelve his concerns about not knowing the number of tickets sold or their price since he had a new problem: Which ticket price gave the maximum profit? This problem implied that he was to compare ticket prices, and thus was free to arbitrarily choose ticket prices and compute profit. His inability to distinguish income from profit in his computations, however, soon led to difficulty with his informal model: profit = income - costs.

Sam edited the rule already on the calculator for a ticket price of $5 and calculated P as $800. When I asked what the $800 was, he claimed it was profit, but quickly changed his mind:

> S: That should be the profit, [G: Um-huh.] the profit they made, but now they gotta pay $200 to the DJ. So let's subtract that from the 800 dollars already and uh-- let's-- [subtracts 200 from 800 on the paper

and gets 600.]

Sam said profit here, no doubt because he had just sorted out a meaning for P, but he switched from profit to income by immediately subtracting $200 from profit to pay the DJ. I decided to pursue Sam's understanding of profit and the distinction between income and profit by focusing on *when* expenses are paid, relative to deciding a quantity is considered profit:

G: What's the meaning of the word profit?

S: How much they made.

G: So did they pay the DJ after the profit was calculated or before the profit was calculated?

S: Oh, before it. But see they, when, okay-- If they made 800, they really made 600 no matter what. If they paid him before, and they made 800, no matter what, they made 600. Because they gave him that before they made the profit.

Here, Sam defined profit as "how much they made" and said "they made 800." He had again interpreted 800 as profit, then changed his interpretation from profit to income. In addition, he interpreted my questions about time as being about the time of day the DJ was paid, not when in the course of a computation a quantity was considered profit. I tried again to clarify whether he was thinking of $800 as profit or income, but this time substituted $800 instead of the word, profit.

G: So they gave him the 200 before they calculated the 800 or after they calculated the 800?

S: Umm, I would've said they done it before.

G: They paid him the 200 before they calculated the profit, so their profit would be?

S: Unless he was a generous one. It'd be 800, but look at what happened. They paid 200 to get the DJ, so they might as well have taken that off the 800. [he again computes 800 - 200 and gets 600.]

Sam implied the DJ donated his fee when he said, "Unless he was a generous one." He said in that case, profit would be $800; otherwise it would $600 because the $200 needed to be subtracted. These sentences make sense only if Sam was using his

informal model of profit as income minus costs and considered $800 to be income, which then became profit if there were no expenses. In addition, he apparently interpreted time differently from me. Paying the DJ before they calculated the $800 makes sense in the situational context if you consider $800 income and that you paid the DJ before computing your receipts and profit. Further, each of these statements is contradictory to his earlier identification of P (and $800) as profit or if interpreted as answers to the questions I intended about the relationship between profit, income, and costs. Profit cannot be both $800 and $600. Likewise, costs should not be subtracted both before and after profit is computed.

In spite of identifying P as profit, Sam clearly did not consider this function rule as a rule for profit. Further, he did most of his reasoning in the situation with his own profit model. While at first it looked as if he simply had changed his mind about the amount of profit, twice he saw no contradiction in calling profit $800 and then $600. Sam apparently understood profit and income concepts, but used the terms interchangeably. In addition, he interjected aspects of the situation into his interpretation of my questions about paying the DJ. He considered time only in terms of being before and after the concert, not in terms of the distinction between whether an amount was considered income or profit and not in terms of the function rule given, where it was irrelevant. Again, I had tried to ask Sam a set of questions about the mathematics I saw in the situation, and he had answered me with his understanding of the situation, bringing in his model for profit and income--even though it meant using terms and quantities inconsistently.

We continued the problem, and after more discussion about the meaning of profit, Sam generated an example of his own that confirmed his informal model for profit. I finally explained to him the distinction between his model and the function rule given. Once he had *learned* that the formula stood for profit and that he did not need to compute any costs, he went on to finish the problem by trying out several

ticket prices in both formulas and finally concluding that a $2 ticket without using the DJ gave him the most profit, $1000. His final segment included a number of errors and some confusion about how to deal with the t - 3 in the bottom rule, but these could be explained because he was pretty tired by that point. He did successfully determine the situation that gave him the maximum profit for the ticket prices he chose to explore.

In the Profit Problem, Sam invoked an informal model of profit, which made it difficult to understand the problem given. He repeatedly interpreted and answered my questions in terms of the situation only, which sometimes led to contradictory results. As with the schoolwork problem, the problem's situational context provided Sam with a source of answers to my questions that, though sensible in the situation, ignored mathematical constraints.

Sam's Understanding of Function Classes

The final major section of this chapter explores Sam's understanding of function classes, both within representations and across representations. In general, the classes are discussed separately, but some concepts, such as y-intercepts, are discussed across both linear and quadratic functions. Sam's understanding of linear, quadratic, and exponential functions was fragmented. Although he associated some specific representations with specific kinds of functions, his work with any class of representations or function classes was uneven and often incoherent. This section has three subsections. The first uses Sam's conceptions and activities with linear functions to highlight how fragmented his thinking was. Subsequent short sections focus on quadratic and exponential functions.

Linear Functions

Sam's understanding of function classes was fragmented. He clearly had some understanding of linear functions, but his work with them was uneven, even within the same representation type. During interview 3, in the context of a discussion about

graphs of *exponential* functions, $y = 5^x$, $y = 2^x$, $y = .5^x$, and $y = .1^x$, that he had created during class the day before, Sam displayed knowledge about graphs of *linear* functions. He had drawn each of the four functions (See Figure 4.13) as a straight line passing through or near the origin, each with a different slope. The graphs appeared to be either the linear functions, $y = 5x$, $y = 2x$, $y = .5x$, and $y = .1x$, or as other linear functions with y-intercepts 5, 2, .5 and .1, respectively.

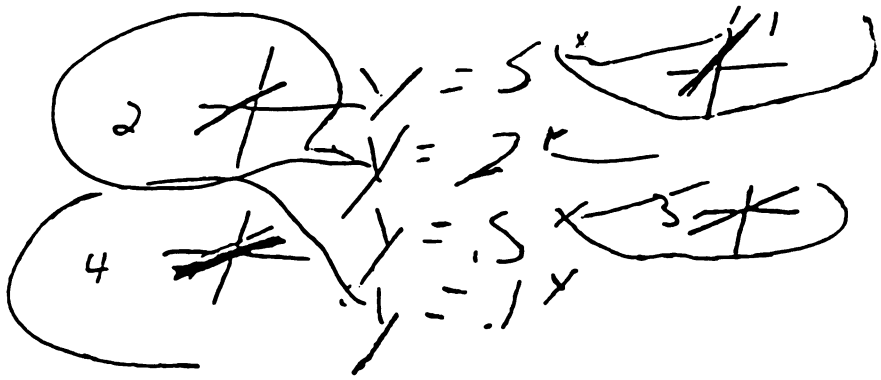

Figure 4.13: Sam's Graphs of Exponential Functions

In the interview, I asked Sam what he had been thinking at the time he made these graphs, even though I did not have his class notes with me. He began by indicating that the graphs were slanted lines through the y-axis at $y = 5$, 2, .5 and .1, and said he was "going by the y-axis." He had treated each constant as the y-intercept and the slopes as arbitrary. When I probed, he said he had looked at the equation and since the teacher had said, "x could be anything," he had made $x = 1$, which in the first equation gave him $y = 5$. When I asked if he might do it differently after the current day's class discussion (where the class had compared graphs of exponential and linear equations), he changed the graph of $y = 5^x$ to a horizontal line ($y = 5$). When I asked how he knew that, he said that after he had made $x = 1$, he was through with it (x), and he did not see any x-coordinates (x variables remaining in the equation).

Because Sam's graph was getting cluttered, I drew and labeled a new set of axes and asked for a fresh graph of $y = 5^x$. He repeated the entire process, including first making the slanted graph, revising it to be horizontal, and justifying the last graph by the absence of "x-values" in the equation. When I asked what he meant by not having any x-values in the equation, he added -3x to the equation, making it $y = 5^x - 3x$. When asked for its graph, as before, he substituted 1 for the exponent and produced a correct graph for $y = 5 - 3x$. He had used a standard school *algorithm* for graphing linear equations in the slope-intercept form: graph the y-intercept at 5, then starting at 5 and using the slope of -3, count down 3 and to the right 1 to locate a second point, and draw a straight line between the two points. I checked his thinking by asking him directly to graph $y = 5 - 3x$, which he did correctly using the same algorithm.

In our discussion Sam had mentioned slope, and when asked for it, wrote a 1 under the 3x and circled all of 3x/1 as constituting the slope. At one point he volunteered that his first line for $y = 5^x$, which had a positive slope, was in error because his equation ($y = 5^x - 3x$) had a minus in the -3x (and thus a negative slope). Although he had forgotten this previous graph actually was of a different equation, $y = 5^x$, his statement indicated some understanding of the connection between the sign of the coefficient of x and slope of the line.[7] I gave Sam four more equations to graph, $y = 2 + 4x$, $y = -2 + 4x$, $y = 2^x + 4$, and $y = -5 + 2^x$. With minor errors, he correctly graphed the first two using the same procedure as above. He chose 2 for the value of x in each of the exponential functions, and graphed $y = 2^x + 4$ and $y = -5 + 2^x$ as horizontal lines, $y = 8$ and $y = -1$, respectively.

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[7]In class that day, Mr. B had used slopes 3 and -3 to illustrate the distinction between positive and negative slope, so it is possible that Sam simply recalled this.

Clearly, Sam knew a procedure for graphing linear equations in slope-intercept form, which is also the symbolic form required for expressing linear functions. He had some understanding of the orientation of lines, having recognized that an equation without an x gave a horizontal line and having distinguished differences between positive and negative slopes from the coefficient of x in the linear term.

Sam was confused, however, about the role of the variable, x, in the exponential equations. He had misinterpreted the variable in his "derived" linear equation when he failed to separate the x from -3 in -3x in discussing slope. In addition, he had interpreted Mr. Bailey's statement that x could be anything to mean a single, arbitrary value, which should be substituted into the equation, rather than as an arbitrary choice of one of many values that x could take on. He seemed to take this statement as permission to turn exponential functions into linear ones. I will say more about his understanding of variables below. All other students in the class except two graphed these equations as dots, or as horizontal or vertical lines through 5, 2, and either plus or minus .5 and .1. One student graphed slanted lines with $y = .1^x$ and $y = .5^x$ below the x-axis. The last student graphed $y = 5^x$ and $y = 2^x$ correctly, but the remaining two graphs incorrectly as increasing.

Surely Sam had learned this linear graphing procedure prior to coming to this class as it had not been discussed explicitly here, even though connections were made between coefficients, intercepts, and slopes. Further, on the pretest, which was taken before classes began, he appeared to use the same algorithm. On problem 6 (Figure 4.14 below) he graphed both $y = 3x + 4$ and $y = x^2 - 2x + 1$ as dots in a straight line. Sam's linear graph for the quadratic equation consisted of points on a graph of $y = -2x + 1$; he had simply dropped the x^2, making it linear and violating the equality. Rather than use the algorithm suggested, it is possible that Sam had substituted values for x, calculated corresponding y's, and then plotted points. I do not think he did this, however, because it seems unlikely he would have chosen

negative x's for substitution, and had he used this method, it seems more probable he would have produced a more accurate graph for the quadratic equation. In any case, Sam used some correct method for graphing linear equations on question 6 of the pretest.

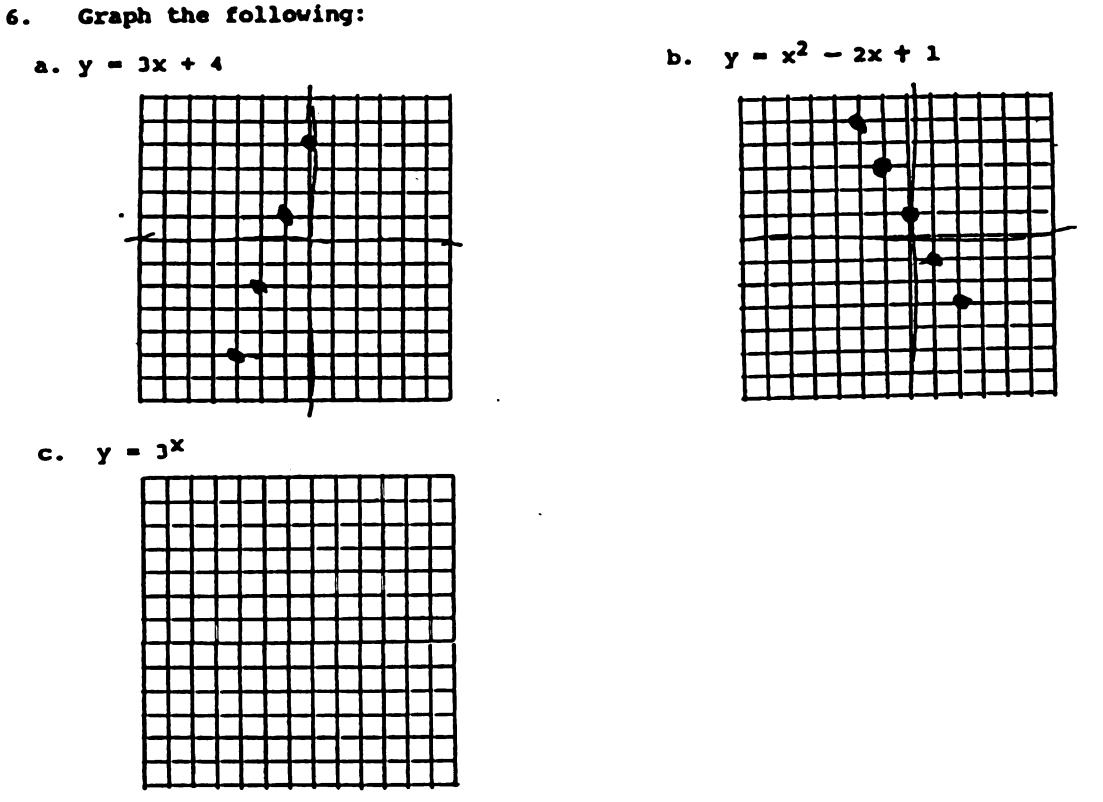

Figure 4.14: Sam's Pretest Problem 6

Sam's thinking about these same two problems changed dramatically on the posttest. There, for both equations he graphed the y-intercept correctly and then made the graphs appear as parabolas opening upward. In $y = 3x + 4$ he had not accessed what he knew about graphing linear functions. Thus, although Sam seemed to know an algorithm for graphing linear equations in this form, he applied his knowledge only intermittently and in addition, had again interpreted the variables in an equation in an invalid way.

Y-intercepts[8]. The y-intercept seemed to be a particularly salient concept for Sam; he was able to use it for different kinds of functions and in several representation contexts. Recall that in the Alice, Bea, Carolina problem (2A-1) to match a table to a graph, after his attention was called to scale, Sam coordinated the table entry for the y-intercept with a graph, even when it required choosing a scale for the y-axis different from the x-axis. As can be seen from the discussion above, he had relied heavily on the y-intercept in graphing functions, even when other parts of the graphing were in error. Question 8 of the pre- and posttests, concerning y-intercepts, was the only question on both tests that Sam got correct in its entirety. Problem 8 (See Appendix B for pre- and post test) required students to find the y-intercept for both quadratic equations in symbolic forms and linear functions given in tabular form. The quadratic equations were given in general form ($y = -3x^2 - 4x + 5$) where the intercept, 5, could be read directly from the equation, and in factored form ($y = 2(x - 7)(x + 1)$), where some transformation or analysis was necessary to determine the intercept, -14. The y-intercept appeared explicitly in one of the tables, but had to be computed for the other. On the pretest, Sam identified the y-intercepts for the quadratics directly in the first case and by multiplying the polynomial out to read off the intercept in the second. He found both the explicit and implicit y-intercepts for the tables by plotting points on a graph. On the posttest, for the quadratics, he repeated his actions from the pretest, and for the linear functions, identified one y-intercept directly from a table entry, (0,1), and plotted points to find the other one.

In addition to identifying or computing y-intercepts, Sam used the y-intercept concept to reason about quadratic functions. In problem 14 of the posttest, students were asked to accept or reject equations of quadratic functions given a graph that

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[8]This discussion overlaps both function linear and quadratic function classes.

showed a quadratic equation, oriented downward, and appearing completely within quadrant IV. Sam used the y-intercept as his reason to reject the first two, $y = x^2 - 4x + 5$ and $y = -x^2 + 4x - 10$, even though for the second, the value of y-intercept was not sufficient reason to do so.

In summary, Sam used the y-intercept concept while graphing linear equations. He recognized it in symbolic forms for two classes of functions, linear and quadratic. He could recognize it in a table of a linear function and could find it by creating a graph of a linear function, given a table without the y-intercept. Sam had successfully used the y-intercept concept in several tasks involving three representations, symbolic, tabular and graphical. As will be seen below under the discussion of slope, however, when Sam had to create the equation of a line from a graph, he was less able to use the y-intercept correctly.

Slope. Sam's understanding of the slope concept was less robust than that of y-intercept. Although he had associated 3 and -3 with a correct graph orientation for linear functions above and had recognized horizontal lines in their symbolic form, he had difficulty identifying a line with a negative slope in problem 4 (Figure 4.15) on

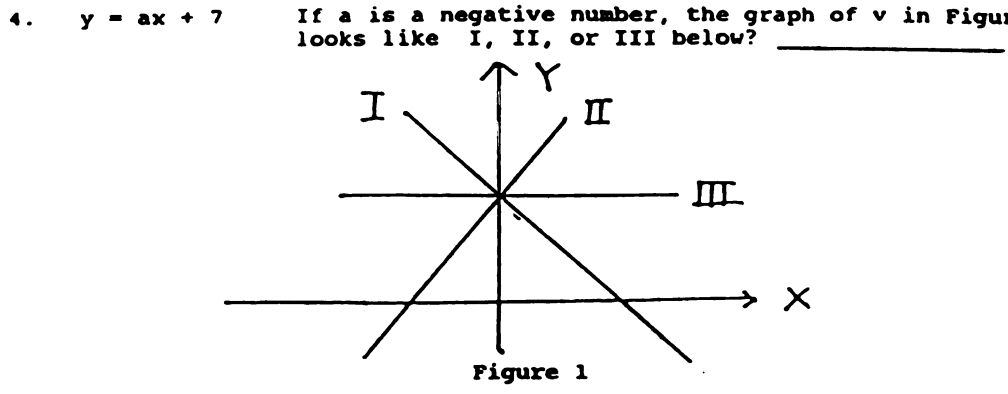

Figure 4.15: Pre/Posttest Problem 4

the pre- and posttests. On the pretest he chose both I and III, and on the posttest, he omitted the problem altogether. In addition, Sam failed to use either y-intercept or

slope correctly when asked to create an equation for a line, given its graph on problem 7 of the pre- and posttests. (Figure 4.16). In both cases, he used the slope-intercept form to get the equation. On the pretest, he calculated the slope as 1/2 x

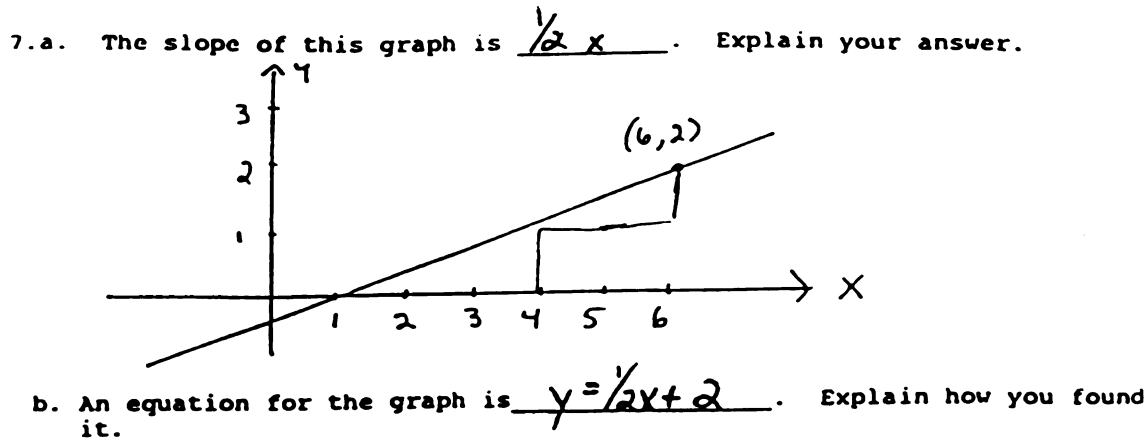

Figure 4.16: Sam's Pretest Problem 7

(including the x) and the y-intercept as 2, the y-value of the only point labeled, (6,2). He had again confused the role of the variable. Since the point (6,2) was labeled on the graph and Sam had drawn lines marking steps along the graph, it appears that he attempted to calculate the relative changes in x and y, using the point (4,1), which was close to, but not on, the graph, to calculate 1/2 for the slope. On the posttest he calculated slope as 1/5 and the equation as $y = 1/5\ x + 1$, using the x, not y, intercept in the equation. He had marked a short vertical line at $x = 5$, which might have influenced his choice of slope. The correct slope was 2/5, and the correct equation is $y = 2/5\ x - 2/5$. In any case, his thinking on both tests was accurate for neither slope nor y-intercept.

This problem task to create a linear equation from a graph is more complex than first identifying slope and y-intercept from an equation in slope-intercept form and then creating a graph from it. In order to create an equation from a graph, one must compute the rate of change from attributes of the graph, determine the

y-intercept, and then use the slope-intercept form of the equation. This task requires identifying attributes of the graph and a computation. The task to graph an equation requires only identifying values that can be seen rather directly in the equation and carrying out an algorithm for graphing. Although many students do in fact learn algorithms for finding an equation from a graph, Sam clearly had not. There was a great deal of emphasis in this class on rate of change, and Sam had earlier used it during interview problem 2A-2, discussed above, to correctly determine the -5 when he found an equation, $y = -5x + 15$, for a different representation, a table of values. In that case, the problem was at least as difficult, if not more complex, than this one. In addition, although Sam had attempted to use rate of change in this problem on the pretest, he seemed not even to try to use it in his solution to the problem 7 on the posttest. Sam's work with the problems just discussed was inconsistent.

The variation in Sam's work here could in part be explained by how he felt that day or by what he perceived the teacher (or interviewer) wanted. He admitted that what he did in school depended on what was required at the moment. During interview 3 at the end of our discussion (reported above) about graphing exponential functions, Sam had discovered there was a discrepancy between what he had graphed, $y = -1$ (for $y = -5 + 2^1$), and the calculator graph for $y = -5 + 2^x$. I asked which was correct, and he said it depended on the teacher, what he was teaching and what he wanted. I continued. While quizzing Sam about what it meant to graph an equation by asking what would he do if I asked him to graph it, he said just do it. So I asked if it would come out straight or curved. He said:

> It dependin' on how I was thinkin'. If I was tired and wanted to get out of it, I do that [pointing to the 2 he had substituted for x]. Add that 2 up in there.

Sam had admitted his work depended on his mood. Perhaps he was tired on the day of the posttest when he omitted problems. These comments also support the hypothesis that Sam was living in the moment.

Rate of change. In the sort problem in interview 5, even though Sam made additional connections between concepts and their representations, his work with linear functions in various representations was as fragmented as in the problems discussed above. Recall that he had connected constant rate of change to linear equations in problem 2A-2. He also associated constant rate of change with graphs of lines, but he was not able to use this information constructively, as will be seen below. Although one of his final sort categories was straight lines, it included only cards R and K, and not the other three cards involving linear functions, P, Q, and S. (See Appendix A for all cards) Even though lines and constant rate of change were salient concepts for Sam, linear functions were not a salient category in his sort process.

Sam had begun by sorting cards into categories by superficial characteristics of their representations. After studying each card and making tentative groupings, he had created stacks of cards that could be characterized as Cartesian graphs with grids (E, J, K, T, W), tables (H, R, Y, Z), verbal description of relationship between two variables (A, S), verbal description of problems to be solved (B, P, Q), and three ungrouped cards (G, D, and M). Actually P is only a verbal statement of a relationship but appears more like a word problem than A or S, which contain only one line each. G and D are Cartesian graphs without grids and M has what looks like a word problem, an equation, and a picture that could be interpreted as a graph.

After Sam had created these piles of cards and without any prompting from me, he adopted a strategy of trying to pair up graphs with grids with either verbal descriptions of problems or tables. He seemed to assume such pairs existed, not an unreasonable judgment since this is often the case in school. He began by comparing numbers on the cards with grids with those on cards with tables, a clever strategy since he was looking for exact matches, but soon gave it up since he found few matches.

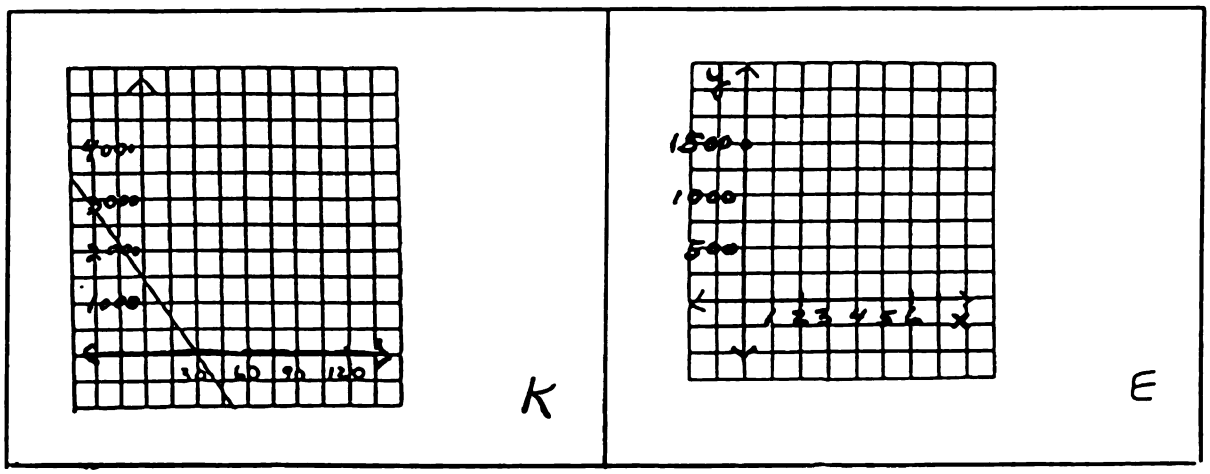

Figure 4.17: Sort Cards K and E

After a short while, I could soon see that Sam thought about linear functions as having constant rate (or pattern) of change. He associated a verbal description of a constant rate of change on card S, "For each 5 units x increases, y increases 10," with a linear graph on card K (See Figure 4.17 above for cards K and E):

S: [Reading from S] For each 5 units x increases, y increases 10– Oh, okay. I probably could find one now. [First looks at J, rejects it and then looks at K. Points to 30, and a couple of places to the left of 30] That (y) increase. Let me think, [looks at calculator] that have to increase 30 times, 30 times 5 [points to 30 again] and you'd have to increase 10 times for every 5. [looks at the calculator again] Hmmm.

[He puts K under the stack and looks card T, then E.]

S: I like this one better for that [groups E (graph with single point) with S and puts them aside.]

G: That's E? So why do you like that one better?

S: Cause it's got those straight numbers right there, [G: um-huh.] and it like, uh, for that 5 [points to x-axis] it increased 500. Maybe for 10 more [points farther out on x-axis], it increased a 1000– so I want it there. So I like them. [Puts S and E together on side.]

Sam had chosen a linear graph, card K, to match to the constant rate of change on card S, but rejected the match either because it did not work or because he could not see how to work it out. At first he had chosen an x-value of 30 on card K and seemed to be counting changes along the x-axis, but in order to match the graph on K

with the word "increase" on S, Sam had counted in the wrong direction (toward, rather than away from the y-axis). As he counted left, he said, "That (y) increase" and repeated the change described on card S. Clearly he was trying to work out the same relative change on the two cards. Sam had, however, described the change in 30 as multiplying by 5 (rather than dividing, if he were counting number of changes) and considered using the calculator, possibly because the numbers were so large. By this time, he might also have seen the graph as decreasing, a conflict with the word increasing on card S, although he said nothing about this. In any case, he rejected card K.

Sam switched to matching card E with card S because he saw greater compatibility between the pairs of numbers on the cards, 5 and 10 (card S), and 500 and 1000 (card E), a strategy he had tried to use earlier. The number 5 appeared on card E, as well as card S. Sam specifically referred to the 5 on card E as a starting x value (from which to describe his change), so it surely influenced his choice. In addition, Sam was able to make a statement about increasing, using these two pairs as a proportion. However, his statement, "Cause it's got those straight numbers right there, [G: um-huh.] and it like, uh, for that 5 (on card E) it increased 500. Maybe for 10 more, it increased a 1000--," made no sense mathematically because he had treated a change from 5 to 10 as a change in x alone, not as a description of rate of change between values of y and values of x. Since card E consisted of the graph of a single point, there was no line or pair of points to represent any kind of change.

Sam had started out with a correct concept: a relationship with a constant rate of change has a graph that is a line. The statement on card S and the graph on card K (his first choice) did not, however, represent exactly the *same* function (his goal). The rate of change differed; therefore, the numbers did not work out. In addition, to match the word "increase," he had to think of change as being in a direction opposite to that for increasing x's, though it is not clear he understood this. Although the 5 on

card E might initially have attracted Sam to it, two other numbers on card E, 500 and 1000, could be put into the same ratio as the five and ten on card S, allowing him to change his focus to a static proportion between number pairs on the two cards (5/10 = 500/1000). To justify his choice, he used the word "increase" from card S to describe a proportion between the pairs of numbers. The proportion was correct, but it was invalid in this problem context because it did not take into account the problem constraints: the numbers, five and ten, in the verbal description represented a rate of change between two variables, not a potential change in a single variable, x. Sam had changed the role of the numbers in the proportion to justify the match. Further, the graph on E had only one point, so did not represent change at all.

A few minutes later Sam again tried to use rates of change to group cards. He placed card A, which contained only the statement, "As the number of workers doubles, the amount of time to do the work is cut in half," in the same group with E and S. Sam repeated a phrase that he had used in the preliminary sort to group A and S and said that because card A was "talking about that doubling stuff" it should be grouped with E. He went on to explain:

> Okay, say they have to do a thousand hours, and they have three workers. They have six and they only got to do 500, so I think that will work out right there. [Puts A in a stack with E and S.]

Sam had interpreted the two pairs of numbers, (3,1000) and (6,500), correctly for the relationship on card A, using two numbers that appeared on card E. Yet neither were coordinates for points on card E.

In this example, when Sam's attempts to use his knowledge of linear functions to match a constant rate of change with a linear graph gave him difficulty because the numbers did not give an exact match, he switched strategies. Returning to an earlier strategy to try to match cards by numbers only, he found pairs of numbers that could be related in a way that made sense to him even though they did not represent corresponding quantities on the cards. He used the word *increase* and a numerically

correct proportion (5/10 = 500/1000) to justify an invalid match between the cards. In doing so, Sam used only a portion of the representation, 500 and 1000 on the y-axis, and ignored the rest of it. He had invented a relationship built out of bits of the representations that made sense to him, a description of change and a proportion between pairs of numbers, rather than attend to the mathematics that was actually represented on card E, namely, a relationship between a single pair of numbers, 0 and 1500. Sam had used this problem solving strategy before. (See particularly the educational value problem (2B-2), that was discussed earlier.) Further, Sam treated a reciprocal function, "As the number of workers doubles, the amount of time to do the work is cut in half" as a linear function because the existence of the word, doubles. He had attempted to find pairs of cards that matched his sense of change or patterns in the data involved.

Ultimately, Sam called the category into which he had placed A, E, and S, "all used to increase or decrease." When I questioned him about how there could be change on card E when it had only one point, Sam admitted he had not consider this fact. He said:

> Ooh, I didn't look at that. I was just using this like, if I went 5 times this way (decreasing x) [G: um-huh] I would've went up 10 times for every 5 and got 500 (for y).

Sam had not used any attributes of the graph on card E other than the axes and their scale markings to match these two cards, E and S. By talking of possibilities of change instead of change itself, he ignored a fundamental aspect of representations, that they do represent *something* and are constrained by it. Without realizing it, he had made his sort class, possible or potential relationships, so general that it was trivial. This pattern of attending only to a small portion of the representation, while ignoring critical aspects, occurred frequently during Sam's interviews. It often led to his being innovative in enhancing the stated problem to make it fit his problem solving goals.

Sam had seemed to associate constant rate of change with linear equations as stated above. In addition, he successfully used shape of graph as an organizer for linear functions on one occasion. He correctly matched R, a table of a linear function, with a linear graph on card K later in the interview after he had used the calculator to create a scattergraph from R and had seen that the points were in a line. At other times, he recognized symbolic forms of linear expressions, equations, or function rules. For example, in problem 4B-1 (Figure 4.18) Sam had trouble understanding what was required of him because he could see b + 3 only as a linear function itself, rather than as an expression for a number that is three more than b.

Below is a graph of a function $f(x) = .3(2)^x$. The point on the graph that represents the function values for input b is labeled with a dot. Place a dot on the graph that represents the value of the function for input b + 3.

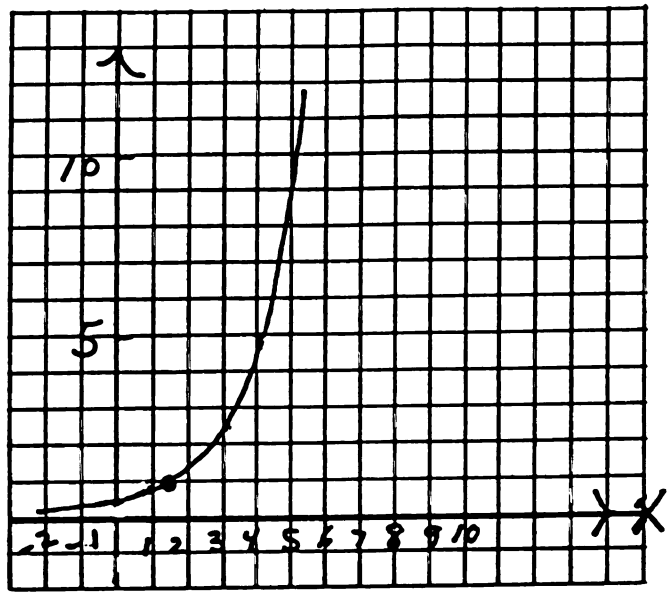

Figure 4.18: Problem 4B-1

Yet, in the sort problem Sam did not use linear functions as a class to classify cards. Cards Q and P, the remaining cards that involved linear functions, got classified as "graphs could be anywhere" and the "strange pile," respectively. Recall Sam had made a calculator graph for Q that looked like a ladder and had placed the card with

B and W. He placed P, a verbal description of a linear relationship, with the strange pile, a group that he considered as defying grouping because they were "all different," from each other and all other cards.

In the course of the sort, Sam had accessed the pattern of change in the data, the nature of the data as increasing or decreasing, and the shape of the graph. He seemed to have associated each concept with linear functions, and in previous interviews, had associated linear function rules with the graph of a straight line. In the sort problem, however, he was, in general, not able to use this information in any particularly constructive way. His thinking was fragmented, and he was easily distracted by other attributes of the cards, such as the situational context or the numbers themselves, proportions between scale labels on axes, or the existence of behavior he did not understand, such as output on a calculator screen.

Quadratic Functions

Sam had some understanding of quadratic functions, but it, too, was fragmented. Like Toni, he did not mention quadratic, but talked mainly of graphs and their symbolic forms as parabolas. As just discussed, he could find the y-intercept for the graph given the equation either in general or factored form. He knew the pattern of data for a quadratic function reversed directions. Recall that during the sort, he entered the quadratic function, Profit = $-300p^2 + 3600p - 1500$ on card B into the calculator and became confused by the behavior of the negative numbers with large magnitudes, as he expected them to change directions, and they did not. When he picked up card R (discussed above under linear functions), he first said he was looking for a parabola, but rejected it because the numbers continued to decrease and did not go back up.

Sam saw quadratic functions as having U-shaped graphs and classified T, an increasing exponential function, as a parabola, saying it could continue going up on the left. Most of the students volunteered this same reasoning. The differences

between "halves" of exponential graphs and "halves" of quadratics appear very slight, and students would not be expected to be able to distinguish them easily[9] unless they were able to see the side of the graph that was asymptotic to the x-axis, which did not appear on card T. Sam matched table Z, a quadratic, with T, because he had entered Z's data into the calculator, made a scattergraph, and saw a shape like a parabola. Other times Sam considered putting U-shaped graphs together, he considered G and M, but rejected them without saying why. That graph G was a bumpy, not a smooth, line graph may have influenced his rejection.

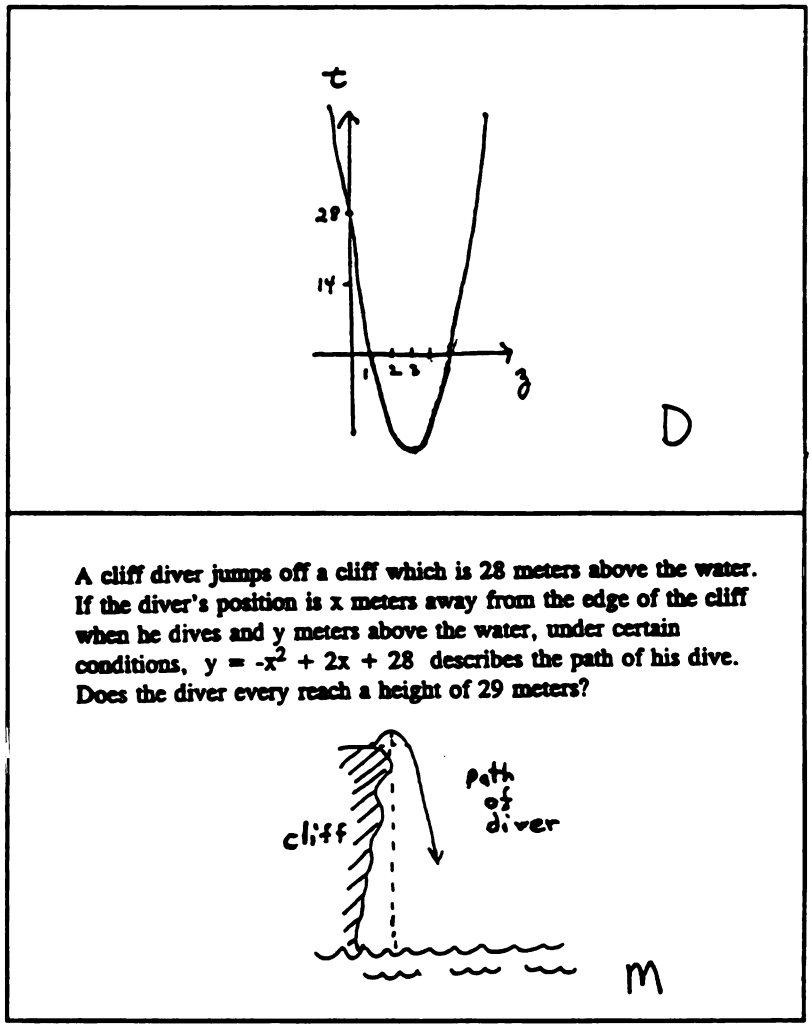

Figure 4.19: Cards M and D of Sort Problem

[9]They had not been exposed to computing second differences to determine quadratic functions.

Sam showed evidence of at least being familiar with an algorithm for finding the vertex of a parabola, when he attempted to match card D, a parabolic graph, with card M. As he read M, he said:

> I was just thinkin' about puttin' this one with this one. (M with D) [picks up D again], but the reason I'm not is because that right there, you're supposed to divide that by 2 [pointing to 2X on M]. It won't give him a one, so the vertex (of D) should have been on 1 and it's not. If it wouldn't put that, I would have picked that one. [points to vertex on graph on D, of which x coordinate is at 2.5]

Sam was still attempting to sort cards into pairs and rejected this match because the vertices were not identical. His reference to dividing by 2 is the first step of an algorithm to complete the square and put the quadratic function into the standard form, $y = a(x - h)^2 + k$, from which the x coordinate of the vertex appears as h. In this case, h would be 1. He grouped together cards H, a decreasing exponential function (table), and J, a decreasing exponential function (graph), by the pattern of data and situational context, saying they both decreased with time. Card H had a variable, years in storage, and J had an axis label "time (sec.)."

Finally, in spite of recognizing the form of a quadratic equation or function rule, and several attributes of it, such as orientation and y-intercept, Sam was unable to translate an equation to a graph by hand. At the end of our discussion above on graphing exponential equations that turned into a discussion of linear equations, I asked Sam to graph $y = x^2$. He said he could only graph it with the calculator. So Sam's difficulties with graphing equations that required generating points cut across function classes. In conclusion, Sam had some knowledge of individual concepts associated with quadratic functions and their representations, but they did not seem very coherent.

Exponential functions

Sam used attributes of the different function classes to try to match pairs of cards, but only for linear and quadratic functions. He did not mention exponential

functions at all during the sort. He did not distinguish exponential graphs from those of quadratics, even though at least as much class time was devoted to exponential functions as to the other two types. During the sort he announced that in class they had done a problem like card H, which contained tabular data about radioactive decay (an exponential function). The data for the interview problem was almost identical to the one given in class, and the situational context was identical. On his homework that day, Sam had correctly plotted points for the table that was given and had made a tentative guess of $100 \cdot .86$. (It is likely he meant $100 \cdot .86^x$.) When asked to use this rule to produce a new table of values to be compared to the raw data in the original table, Sam produced a correct table for the rule $y = 100 \cdot (.875)^x$. He did not, however, record this rule. In the margin of his paper was a 97 and eighty something, too faint to see. These numbers are similar to those that Sam would have found (97.5 and .871) had he used the automatic curvefitting feature on the calculator for exponential functions that Mr. Bailey had shown them in class that day to compare with the results of their trial and error method. Sam had at least explored using the calculator to fit exponential curves automatically. Recall that Sam matched card H with a graph of another exponential function by shape and situational context. Both cards had a combination of decreasing change and time as a context for a variable. Sam had recognized that he had seen a similar problem, but he did not associate it with an exponential function.

When Sam used the calculator for graphing exponential equations in class, he had an unusual amount of difficulty. Recall that when he attempted to graph the function using the function grapher, he first entered the entire expression, including the Y= and then entered the equation with a constant, rather than x, on the function list. Above, when Sam turned equations of exponential functions into graphs of linear functions, he clearly had some difficulty interpreting the variable, as did all of the other focal students. At the time, I tried to find out if he knew how to graph any

equations for which he was required to choose an arbitrary x and find the corresponding y. Recall that he thought the correct graph depended on what the teacher wanted.

Sam did not have enough time during interview 4 to complete two of the interview problems that dealt with exponential problems, so it is possible that he was not given the opportunity to adequately display what he knew. However, on the other two problems, he did not focus on exponential functions. In the first, 4B-1 discussed briefly above under linear functions, he was so fixed on b + 3 as representing a linear function that he did not make much progress interpreting the graph of the exponential function given. In problem 4B-2, from a graph of $g(x) = a \cdot 2^x$, he was asked to identify a. After questioning me about whether he was to find a number for a, he rephrased his task to be finding the number for "that point," (the y-intercept). Thus, he understood that a was the y-intercept, a topic discussed in class that day. Sam then entered $y = x \cdot 1.5^x$ on the calculator and spent most of the time adjusting viewing windows in an attempt to get the shape of the graphs to agree. After some period of time, I finally told him that there was a problem with how he entered the function definition because a was a specific number, not a variable. So Sam determined it was 200 by reading it off the graph, then checked his thinking by graphing $y = 200 \cdot 1.5^x$ and compared the graphs. Overall, it appears Sam's understanding of exponential functions in any representation was limited.

In conclusion, Sam's understanding of function class depended on the function class, the representation, and the task. In general, Sam had more knowledge of linear and quadratic functions than he did for exponential functions, but his knowledge of each function class was fragmented.

Summary

Sam's problem solving involved more than mathematics. He used aspects of the discourse in a strategic way during problem solving. He sought clues through

extensive questioning or by attuning to my questions and responses to refine the problem statement, signal an expected response, or determine a direction in which to proceed. He deflected or ignored questions until the question asked was sufficiently focused so that he could answer it. Sam chained together his responses, going from one to the next, often disregarding previous, relevant responses. Sometimes he interpreted questions that I saw as mathematical in a valid, but non-mathematical, way. Thus, there were communication mismatches between us.

Sam approached functions from both correspondence and covariation perspectives, finding dynamic versions salient. Like Toni, his work with functions was implied through his work with representations. He did not talk about functions directly, did not use function class as an organizing concept, and seemed not to be aware (with minor exceptions) that concepts he talked about were function concepts.

Sam's work with representations was uneven. He could plot points from a table and understood the concept, ordered pairs in naked context. When plotting or interpreting points on a graph of a situated problem, however, he was influenced to ignore mathematical constraints by the situation, the problem representation, and the task description. Except for linear equations in slope-intercept form, Sam could not plot a graph by hand when given an equation.

Sam was facile at fitting graphs to situations when the graph could be viewed temporally and either the change in the dependent variable matched the rise and fall of the graph directly or a description of the change matched a description of the changing slope of the graph. On several occasions, particularly with tables and graphs, Sam reasoned only with the dependent variable.

Sam's understanding of function classes was fragmented. He understood to some extent algorithms for translating a linear function into a graph, a table for a linear function into its symbolic form, and to a lesser extent, how to find the x-coordinate of the vertex of a quadratic function in general form. He recognized the

general form of a quadratic and the slope-intercept form of a linear function. He recognized orientation of both their graphs. Sam had a robust understanding of recognizing or producing the y-intercept for both linear and quadratic functions, but was not able to use the y-intercept successfully in translating from graphs to symbolic form for linear functions. Slope was also a salient attribute of a graph for Sam. His success in using these concepts, however, depended on the task and the representation. He associated equal change with linear graphs, and relied on patterns in the dependent variable and shape of graph as organizer for linear and quadratic function classes. Sam had no such organizers for exponential functions.

If a problem was situated, Sam relied heavily on the situation. He repeatedly interpreted and answered my questions in terms of the situation only, which sometimes led to contradictory results. Sam reasoned directly in the situation, and when it was useful, enhanced the situation to accommodate or justify a next step in his problem solving, even when the enhancement made no sense mathematically. His informal model for profit interfered with his understanding a model of profit as a function of ticket price. He understood a model, which was given as a function rule, to be a formula for computing, rather than as a relationship between variables.

Overall, Sam's knowledge was fragmented, depending on the function class, the representation, and the task within or between representations. His heavy reliance on the situation and his reasoning with the dependent variable only interfered with his accessing knowledge or mathematical constraints that he was able to access on other occasions. Further, many of his problem solving strategies relied heavily on aspects of the discourse or what was required in school.

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CHAPTER 5

CROSS CASE REFLECTIONS: FINDINGS

The purpose of this study was to explore the conceptions about functions, functional relationships, and their various representations that students have when given an opportunity to study linear, quadratic, and exponential functions in an instructional setting that was designed to provide experiences recommended by the NCTM *Curriculum and Evaluation Standards* (1989). Function was the central organizing topic of the curriculum materials. The study sought to determine what conceptions students developed about functions and functional relationships in problem situations and how students worked with the problem representations. It sought to describe connections students made across either functional or representational contexts and to discover if and to what extent problem representation or mathematical contexts contributed to or hindered students' understanding of functions or functional relationships. Finally, it attempted to determine if there were other factors that contributed to students' thinking about functions or functional relationships in these representations.

This chapter steps back from the cases of Toni and Sam presented in the previous two chapters to reflect on them and, guided by the research questions, presents the findings. The principal research question was:

> **What sense do students in this instructional setting make of functions and to what extent are their conceptions tied to specific mathematical or representational contexts?**

There are seven major sections in the chapter. The first five are: Functions as Mathematical Content, Proposing an Alternative Framework for Functions, Students' Conceptions of Functions, Coherence and Connections in Functions and Functional Relationships in Problems, and the Special Nature of Situated Problems. These five sections address the first two research subquestions and their subsidiary

questions:

A. What *conceptions* do students have about functions, functional relationships in problems, and their representations?

How do they *talk* or *reason* about functions or functional relationships in problems? Which concepts are most *salient*? On what do they focus their *attention*? What resources do students bring to the *problem solving* process? What kinds of *connections* do they make between properties and concepts of functions in different representational or mathematical contexts?

B. In what ways and to what extent does the *situational context* contribute to or hinder students' abilities to understand or solve problems involving functions or functional relationships?

The third subquestion is:

C. In what ways and to what extent is a *teacher/researcher perspective* different from or consistent with *students' perspectives* about conceptions of functions, functional relationships in problems, and their representations?

What role do assumptions play? How do perspectives affect communication? How are perspectives related to problem solving activity?

Discussion of issues related to this last subquestion cuts across all sections of the chapter.

The sixth chapter section is Other Factors Affecting Student Understanding of Functional Relationships. Some questions in this section, such as what resources students bring to problem solving, are discussed in previous sections, but only from a mathematical perspective. These responses are expanded to include factors other than mathematics, such as issues related to schooling and communication. Finally, the last section in this chapter summarizes the findings.

Functions as Mathematical Content

In much of Sam's and Toni's thinking during interview problems, they seemed not to be aware of functions. Usually, they focused their attention on attributes or properties of the representation itself or on the problem solving process. Sometimes functions were more recognizable in their activities, but in general, the students did not talk about functions directly. They also seemed not to think about functions often, although there were occasions when thinking about functions could be inferred

from their talk and activities. Since both the classroom curriculum and interview questions had functions as a major part of their content, it is important to understand why students paid so little explicit attention to functions. Although some interview problems did not focus explicitly on functions, whether the problem was about functions was a matter of perspective. There was a large difference between the interviewer's perspective of the content of these problems and the students'. Throughout this chapter the use of "a teacher perspective" is intended to convey both the researcher's personal perspective as a practicing teacher and a belief that it is reasonable to assume the perspective is widely shared among mathematics teachers. It is not, unless specified, a statement about Mr. Bailey's (the classroom teacher) views.

Teacher Perception of Content

From a teacher perspective, function was a central organizing concept of the curriculum. Mathematical function was a fundamental aspect of the curriculum materials and each of the homework problems or interview questions.

Both a conception of function as a relationship between two variables and patterns of change in data were important organizing concepts in *Computer-Intensive Algebra* (CIA). *Language of Functions and Graphs* (LFG) and *UCSMP* curriculum materials also declared themselves to be about functions. Even though the CIA materials took an intuitive approach to working with functions, the materials introduced and explicitly used function terminology, such as input, output, dependency, and names of function classes. Several sections had specific function classes as part of their titles. For example, two of the sections were titled, "Applying Exponential Functions" and "Recognizing Quadratic Functions." Exercises were provided specifically to call students' attention to function terminology and to connecting notation used for the same concepts in different representations. Properties and attributes of linear, quadratic, and exponential functions were explicitly

referred to in exercises that introduced those function classes. Within exercises students were asked to compare and contrast function classes by comparing their behavior in tabular and graphical representations. Mr. Bailey, the classroom teacher, used explicit function terminology, both in announcing goals for instruction and in referring to exercises presented or discussed in class. Thus, a teacher would see homework problems and class discussions and materials as organized around functions and function concepts.

When viewed from a teacher perspective, the interview tasks, too, were about functions and their attributes or properties. An exception was the first interview task, 1-1, Going to School, where students had to match places on a map with points on a point graph. Even in problem 1-1 when students had to compare information embedded in two points on the graph, they could have thought of the points as being on the graph of a linear function. Sometimes the interview task that was set referred directly to input, output, functions, function rules, or a function class. Other times, such as in the Tim and Donna problem, there was no direct reference to functions. Although an interview problem task (See especially interviews 3, the profit problem, or 5, the sort.) might be more complex or slightly different from those given in class, in general the representations of functions were the same, and the mathematical terms used were the same. For example, table headers were labeled input/output, as they often were in class. The f(x) notation for function rules appeared regularly in text materials and was used by the teacher. In summary, from a teacher perspective, curriculum materials, class discussion, and interview questions were about functions and explicitly referred to functions or function attributes, properties, or concepts.

A teacher would expect that if students were engaged in the study of functions, students would use function concepts to organize their thinking. For example, when students in this study connected patterns of equal change in a table with a graph of a line, a teacher would hope to see evidence that students were thinking beyond an

algorithm for matching two representations to seeing each as being a member of a single function class. Further, a teacher would expect that not all talk or use of function concepts would be implicit. Even if such talk or use were mainly implicit, a teacher would expect it to occur with some amount of regularity. For example, a student might use rate of change without reference to it as Toni did in the Tim and Donna problem. When I asked her how she could tell from the graph that Donna was running, she said, "Because it's (the graph) going up faster. It's almost going straight up in a little amount of time, and she is covering more distance." Toni's response to my question described relative rates of change in two variables, thus providing evidence that in this problem she had an intuitive understanding of the concept of rate of change as a covariation of two sets of variables. Whether or not Toni was explicitly aware that her description was of a function concept, rate of change, in addition to a means of matching a situation to an attribute of a graph, however, is impossible to tell. As will be seen below, these kinds of responses where students exhibited function concepts that could be recognized by a teacher as such, were rare. When they did occur, it often was impossible to tell whether the student had any explicit awareness they were about function concepts. A teacher would consider that if students referred to functions or function concepts only implicitly, that is, without any evidence of explicit awareness, that their understandings of functions would be less robust than if they had named or had used function concepts explicitly. If, on the other hand, a student did not talk directly about functions or function concepts, did not seem to use function concepts in their problem solving, and seemed not to attend to the attributes or properties of functions (rather than representations only), a teacher would likely assume that the student was neither aware of nor understood functions and function concepts.

Student Perception of Content

Awareness of functions. Although a teacher might perceive functions to be the principal mathematical content of this class and these interview problems, students did not share this perception. Sam and Toni rarely seemed aware of functions in the interview tasks. Generally they did not talk about functions, but rather talked about the details or attributes of the representation itself or something unrelated to mathematics. For example, they talked about items such as labels on an axis, a problem solving process, or an attribute of the situational context. They asked whether they could write on the paper stating the problem. They did not usually initiate talk about functions or function concepts. They seldom mentioned functions directly unless the problem explicitly referred to functions. In those cases, they referred to function terms only to seek clarification of their meaning or to determine what was expected in the problem. For example, Toni asked for clarification when words involving functions were used in the problem statement. In problem 4B-1, where a dot was placed on the graph of $f(x) = .3\ (2)^x$ for input b, and students were asked to place a dot on the graph that represented the function value for input $b + 3$, Toni checked to see if her understanding of input was correct: "What's in parentheses is the input, isn't it?" She was checking for a fit between the problem statement and her thinking about input, which she saw as a name to identify substitutions in a procedure, not as a function concept.

Students sometimes used function concepts, but it was unclear whether they had any awareness that they were connected to functions. For example, when students used a function attribute, such as rate of change, they did not refer to it directly as a rate, but rather just used it. It was unclear whether they were aware of the attribute as a rate and in addition that it was a function concept. Generally, when Sam and Toni talked about functions or function concepts, it was only implicitly.

From a teacher perspective the class was about the study of functions in general and about linear, quadratic, and exponential functions in particular, so in the sort problem a teacher would expect a student with a robust understanding of these function classes to use them, at least at some point in the process, as organizers in their sorting. Neither Sam nor Toni initiated the use of function class as a sort category. In Toni's case, only when I specifically asked her to sort in some way connected to the function classes we had studied did she use terms associated with functions per se or consider function classes as possible organizers. She could recognize and identify linear functions and she associated parabolas with the symbolic form of quadratics, but she did not mention these as classifications until asked to sort by function class. Sam used attributes of lines or parabolas to try to pair cards, ultimately ending up with two categories having one pair each: "(T, an exponential) looked like it formed a parabola and Z did" and "(R, K) made straight lines." Yet he placed other cards that he had identified as linear or quadratic functions into other groups. So while Sam had used shapes of graphs (of function classes) as a way to group cards in a limited way, he seemed more to be grouping "graph shapes," rather than sorting cards into function classes on the basis of an attribute of the function class, the shape of its graph. Because Toni and Sam displayed using function classes as organizers in a minor way, we can only infer that these classes were salient for them.

In summary, Sam and Toni did not usually initiate talk about functions per se, and when they used function concepts in their problem solving, they did not usually refer to them directly. They did not normally initiate using function classes as organizers for their thinking. In general, Toni and Sam did not seem to consider the concept of function to be a central idea among the problems; indeed, most of the time, from what can be inferred from their talk and activities, they seemed unaware

that the problems were about functions at all. Thus their perception of the content of this class was different from a teacher's.

Nonawareness may not be absence of understanding. Students' not attending explicitly to functions and function concepts per se does not necessarily mean they have no understanding of those concepts. Even though the students were not generally *aware* of function concepts, they *used* function concepts, such as rate of change, to solve problems--although that use had to be inferred from their problem solving activity. For example, in matching graphs to situations, Sam and Toni made extensive use of coordinating a description of the slope of the graph with a description of the speed of change in the dependent variable. It appeared that they thought of these rates as being connected with either the graph or the situation, not with functions; that is, they associated the rate with the two representations, not with functions per se. They treated rate of change in a table representation similarly. When they used rates of change in tables, they described them as patterns of change, which can be described as rates, but do not have to be. Nevertheless, even though they used terms of the representations, rather than functions, students were using function concepts--whether or not they displayed any awareness of functions.

While Toni and Sam did not use function classes as organizers in an overarching way, they did use functions as organizers in a contextualized manner. During the sort problem, the most salient feature for them was the representation. At first, each had sorted the cards by superficial characteristics of their representations, using categories like word problems, tables, or graphs with grids. Sam and Toni did not limit their sorts to these categories, however, but made some connections between specific representations and specific function classes. For example, a U-shaped graph was seen as related to a table of values that reversed directions in output column (while input column increased), and a straight line graph was related to a card verbally describing a linear relationship. Sam and Toni talked about these

characteristics of the representations, implying they had at least a tentative underlying, implicit connection to the function classes.

Teacher and student perspective mismatches. Because of the nature of the interview problems, one might not always expect students to attend to functions per se. Several exercises, however, were similar to function problems given in class and functions were explicitly referred to in all interview problems except 1-1, Going to School, problems 4A to determine students conceptions of variables, and the Tim and Donna problem. Thus, students might be expected to mention functions more frequently. Certainly during the sort problem, after Toni had had an opportunity to generate her own sort categories and she was specifically asked to sort in connection with the function classes that were given in class, one would expect her to be able to do so. Yet she had trouble with this task, as did Bob[1] when asked the same thing. Although Edy started the sort using function class graphs as a guide for categories, she was unsuccessful as she included card E in a category with graphs of lines, but the only line that was on card E was an axis. Sam was not asked to sort by function classes because he had time for only one sort. From a teacher perspective, if each of these students had much coherence to their function knowledge, they could be expected to have at least mentioned functions or function concepts and to make greater connections to function classes. What connections students displayed between function classes, however, was tightly bound to specific representations of functions and depended on the function class itself as will be discussed below.

This phenomenon of students not talking explicitly about functions or talking about functions only indirectly might be explained, however, by their perspective being different from that of the teacher/researcher. From the student perspective, these problems may not be principally about functions, but rather are just problems

[1]Bob and Edy were two other focal students in the study.

that happen to use some of the terminology of functions as one of many other features. Another possibility is related to task purpose, solving the problem. Toni and Sam may have indeed been aware the problems could be thought of as being about functions, but to solve the problems at hand or to discuss them, attention had to be focused on the representation of the function, the nature of the problem solving task, or elsewhere. Recall Mason (1987) distinguished attention "on symbols" themselves from attention "through symbols" to a larger whole. He described a process of reading a piece of mathematical text where attention is split between the detail of the symbols and the shape of the whole argument, while guiding yourself toward a goal. One could make a parallel distinction for attention when solving a problem. Students in this study seemed for the most part to focus their attention "on the symbols," that is, the characteristics or attributes of the representations themselves, including attributes of situations. When their attention shifted to a whole, that is, "through the symbols," it was on whatever was necessary to solve the problem. Generally their attention was on determining the nature of the task, choosing a strategy, or seeking clues about how to proceed from their mathematical repertoire or from interaction with the interviewer. In neither case was students' attention on the nature of functions or function concepts as a teacher might hope.

Proposing An Alternative Framework for Functions

In spite of attention to attributes of representations, to individual computational or problem solving processes, whether or not they were aware of it, both Sam and Toni did display specific conceptions of functions that could be inferred from their talk and activities. Their implicit definitions of functions were characterized as involving some kind of connected change or relationship between two variables. Their perspectives on functions involved both static and dynamic versions, as well as global and point-wise versions. They involved attention on both pieces and wholes. To better fit the conceptions or perspectives of functions taken by students in this

study, a new framework that synthesizes and modifies those proposed in the literature is developed below.

This study suggests that it is important to try to separate the language of a teacher perspective from that of the student's perspective on functions. In order to do so, the language used to talk about student understanding needs revision. The word *conception* of function seems to imply that students' attention is focused on functions as a mathematical concept and that there is some kind of entity that can be spoken of as a whole. *Perspective*, in contrast, implies a point of view students take when engaged in working with functions and problems involving functional relationships. Conceptions of functions, by which I mean the personal meaning students have of functions, seems still to imply a curricular goal that teachers themselves take as shared by the mathematics community and toward which they would like to guide students. Perspectives seems more closely related to describing student activity, so I will use that term to describe students' thinking.

Student Perspectives on Functions

Students in this study take perspectives on functions that can be defined by two broad approaches, correspondence and covariation (Confrey & Smith, 1994). Each of these approaches has both a dynamic version and a static version. That is, a student takes one of four perspectives: a correspondence-dynamic, a correspondence-static, a covariation-dynamic, or a covariation-static (Figure 5.1 below).

A *correspondence approach* encompasses Moschkovich et al.'s (1993) process perspective, where a function is perceived as linking x and y values, and Confrey and Smith's (1994) correspondence approach. The dynamic version can be thought of procedurally as an input of x producing a (unique) y. For example, an x can be substituted into a rule and a y determined. A value for y can be read from a corresponding table entry for x, that is, across the table, or a y value can be read from a graph by first finding x, moving up to the graph, and then over to the y-axis

	Dynamic	Static
Correspondence	Function perceived as linking x and y values through a process.	Function perceived as linking x and y values through a relationship, that is, the relationship is recognized.
Covariation	Function perceived as operationally moving from one y to the next, while coordinating parallel movement from one x to the next.	Function perceived as encapsulating the result or relationship of moving from one y to the next, while coordinating parallel movement from one x to the next.

Figure 5.1: Perspectives Students Take with Functional Relationships in Problems

and reading it off. Each of these correspondences can (for the content in this class) be reversed.[2] By its nature, the dynamic version is procedural and often computational. Static versions of correspondences would be recognizing that an (x,y) pair forms a correspondence (in advanced terminology, they would be "in relation"). For example, in working a problem, when students recognize that an (x,y) pair satisfies a function rule or are coordinates of a point that is on the graph of a particular function, their thinking could be considered taking a correspondence-static perspective. Attention is principally on one instance of the relationship, whether a dynamic or static version is taken. The correspondence perspectives overlap what Sfard (1991) calls operational, and others (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky & Harel, 1992; Moschkovich, et al., 1993) call a process conception. It is, however, not equivalent to these because it includes both operational and structural aspects (Sfard, 1991). Further, it is more local and point-wise oriented than Breidenbach et al. (1992) intend by their process conception, which includes attention to functions as wholes. It is more closely related to

[2]Students do not need to perceive these as functions per se, but merely as relations, and thus they are reversible.

Breidenbach et al.'s action conception, where emphasis in on a single activity within a process. The correspondence-dynamic perspective is consistent with Sfard's operational orientation, and the correspondence-static perspective is consistent with Sfard's structural characterization, but differs from it in that this definition attends to single instances of the relationship, rather than the whole, as is her intention.

Covariation is the second approach. Following Confrey and Smith (1994), a covariation approach involves moving operationally from one y to the next, coordinating parallel movement from one x to the next. For tables, it involves coordination of the variation in two or more columns as one moves up or down the table. On a graph, it means coordinating the changes in y with changes in x as one moves across the graph. While keeping a symbolic form of an increasing function in mind, for example, one might note that y's are increasing as x's are increasing as one traces along its graph on the calculator. A covariation approach is more holistic than a correspondence approach as defined here in that it involves working with patterns and rates of change in, say, a table or graph as a whole. While this approach is principally dynamic, it has a static version. For example, in tables, when the attention is focused on pattern of change as a whole and it is recognized as having some attribute, such as there being equal change between table entries for y per unit change for x, then the table can be perceived statically as having an attribute that encapsulates this change. The student's attention is on the pattern of change as a whole, which is distinct from focusing attention on the movement coordinating the two sets of numbers. This dimension, covariation-static would be included under Sfard's structural category.

Student Awareness of Function and of Function Wholeness

Finally, in addition to the distinction between correspondences between specific x-values and y-values and coordinating simultaneous change in two sets of numbers, either as processes or as relationships, another kind of perspective is

necessary to describe student activity. If one is to have a connected understanding of functions, ultimately one must come to see functions structurally as a whole, as objects or entities that can themselves be operated on (Sfard, 1991). Yet problem solving with functions is complex and within student activity with functions one can discern many levels of wholeness.

Functions can be distinguished along three levels of wholeness: instance, entity, and object (Figure 5.2 below). If only portions of the function are attended to, such as an x producing a y, the level of wholeness will be considered a *instance* level. When a function is seen as a whole, that is, perceived as a single entity representing a single function without reference to details, rather than as an instance of a correspondence between an x-y pair, it will be assigned an *entity* level. It is not, however, seen as an object that can be operated on in its own right. For example, a graph can be thought of as a whole without any understanding that its whole could be operated on as an object. In addition, a function rule can be thought of both as a whole entity *and* also as an object that can be transformed through translations or in combination with other functions to create (possibly new) other functions via arithmetic or composition. For this latter level of wholeness, I reserve the word *object*. All levels of wholeness but the latter involve only one function; whereas taking an object approach involves the possibility of creating new functions by transforming in some way the function under consideration.

Students in this study often used function concepts implicitly, but without apparent awareness they were dealing with functions. Student understanding that includes awareness of functions would be considered by teachers to be a deeper understanding of functions and would be a necessary part of curriculum goals for many mathematics classes, particularly those taken by college intending students. But to limit judgment of understanding of functions only to including such awareness discounts the rich, intuitive knowledge that students bring to solving problems

involving functional relationships. Further, the current mathematics education reform movement in algebra advocates activities from which even young students can construct a sense of functions, while not being bogged down by mathematical representations completely devoid of situational context. In those cases, students might be expected to make use of function concepts without explicit awareness of functions. If awareness were made an essential criterion for understanding, students' knowledge would then likely be discounted. Thus, to both recognize naive, tentative student understanding, while still distinguishing a robust level of understanding about functions, I have added another dimension to this framework: *aware of function-not aware of function*. Three levels of wholeness crossed with two levels of awareness results in six new perspectives: instance-aware, instance-not aware, entity-aware, entity-not aware, object-aware, and object-not aware.

	Aware of Function	Not Aware of Function
Instance	An aspect of the representation is recognized as an instance of a function, but attention is on a part, not the function as a whole.	An aspect of the representation is not recognized as an instance of a function, and attention is on a part, not the representation as a whole.
Entity	A representation for a function is perceived as a whole, that is, a single unit, and explicitly as representing a function.	A representation for a function is perceived as a whole, but not explicitly as representing a function, nor as an object that can be operated on.
Object	A representation for a function is perceived as a whole, explicitly as representing a function, and in addition, as an object that can be operated on to (usually) produce new functions.	A representation is perceived as a whole and as an object that can be operated on, but is not perceived as representing a function.

Figure 5.2: Wholeness and Explicit Awareness of Functions

Nature of Perspectives

What is distinctive in each of these perspectives, student perspectives of function and wholeness/awareness perspectives, is where the focus of attention lies. From both perspectives of the correspondence approach, it is between pairs of x's and y's, that is, an instance of the functional relationship. For those in the covariation perspectives, it is on coordinating changes in two sets of numbers or recognizing the change relationship. Finally, through the wholeness/awareness perspectives, perspectives taken are characterized by the extent to which a function is seen as a whole and whether its representation is seen as a function.

Further, it should be noted that the two kinds of perspectives are complementary. Normally, at any one time, a student's view in solving problems involving functional relationships would include one from each kind, that is, one perspective of function and one perspective of wholeness/awareness. For example, seeing a table as a static entity with equal change as representing a linear function would be considered taking a covariation-static perspective of function, as well as seeing the table as an entity that represents a function (entity-aware). On the other hand, recognizing a table of values as an entity, but not one that constitutes a function and without attention to patterns of change, would be considered taking a correspondence-static perspective and an entity-not aware perspective.

Although the six awareness/level of wholeness perspectives are mutually exclusive, pairs within each of the four student perspectives of function, can form multiple aspects of the same function as when students "see" that x must be multiplied by 5 in a rule to produce a pattern of increments by 5 while running down a table column. It would be expected that students could take more than one perspective of each kind within a single problem. Indeed, students would be expected to move fluidly and flexibly between and among perspectives in solving problems, just as it would be expected their attention moves between being on symbols or through

symbols within a single problem. Further, at times, some perspectives may appear to overlap with boundaries blurred.

Domain and Range

Within this framework no attention is given to the more complete definition of function which includes all three aspects of a function, its domain, range, and the corresponding relationship or dependency between them. These concepts were not included because for the function classes that were the focus of this study, linear, quadratic, and exponential functions, domains are all real numbers, and little direct attention was given to domain and range in the classroom and interviews. The framework could readily be extended, if necessary, to add a level of wholeness between entity and object to include understanding of a single function as constituted of the three parts, domain, range, and correspondence.

Students' Conceptions of Functions

This section begins with a discussion of the students' understanding of functions from both correspondence and covariation perspectives, including examples of how they talk about functions and representations, what concepts they found salient, how they reasoned about the concepts (including levels of wholeness), what problem solving strategies they used, and particular difficulties they had. Whenever possible, explanations are posited. Subsequent related sections address various kinds of coherence that can be associated with functions, both from a teacher perspective and from students' work.

Correspondence Perspectives

Except during specific problem tasks, such as fitting situations to graphs, and some of the time when working with tables, students generally took a correspondence approach to functions. They readily invoked both dynamic and static versions and easily moved between the two perspectives. Students were successful in a wide variety of tasks in many representations, but had difficulty with some individual tasks.

Examples

Usually students could carry out procedures such as evaluating a function rule or reading off coordinates of a point on a graph (dynamic). Given a table of values, they associated correct x and y values by reading across table. Students could translate a table to a graph by plotting points, thus connecting pairs of correspondences in two different representations. Usually they recognized that a table entry of zero would give a point on an axis of its graph (static). Sam could easily find y-intercepts in several representations. Toni, however, seemed able to find them only sometimes in tabular representations and to some extent in graphs, but had difficulty connecting the concept with other representations such as rules. In the Alice, Bea, Carolina problem (2A-1), Toni connected the last entry in a table (10,-4), with the x-intercept, rather than using the table entry (2,0). When students checked the correctness of tentative function rules in problem 2A-2 by substituting inputs from the table to see if the correct output was produced, they used a correspondence-static perspective to recognize the concept that the x and y must be related by their rule and a dynamic version to carry out the checking. Both Sam and Toni understood that an entry in a table or an input/output pair for the function should be associated with a point on the graph of the function, a static version of correspondence. When Sam miscalculated values for input and output for a function in problem 4B-1 (dynamic) and the resulting point (static) was not on the graph of the function, he recognized this immediately as problematic. Sam did not always use this concept consistently. In problem 2B-2, the educational value/difficulty of work problem, he connected only one quantity, not two, to each point on a graph.

Sam and Toni were successful on pre/posttest problem 9, where students had to read an ordered pair from a graph of a quadratic function (dynamic), then interpret it in the situation to give a maximum function value for profit and the ticket price that would produce that amount of profit (dynamic and static). In both cases, their

attention was focused on the correspondence between two specific numbers. In addition, many exercises in class intended for students to make connections between notation of the same correspondence in different representations. These exercises, such as asking for the missing number in D( ) = 102, given a table of values for the situated function D, asking for an interpretation in the situation of D(18) = 183, or finding which of two x-y pairs of values in a table had a larger change in function value (y), required a focus on individual instances of the function, a correspondence.

Generally, students were comfortable with the correspondence perspectives, with difficulties confined to specific, isolated instances. All students, however, had trouble with one task involving these perspectives, creating graphs of exponential functions by hand. This problem is discussed at length under a later section on coherence.

Levels of Wholeness

Most activities involving both correspondence perspectives lent themselves by definition to attending principally to single instances of a functional relationship, that is, an instance level of wholeness. Yet within certain problems, students sometimes connected correspondence perspectives to viewing functions as entities, particularly rules and graphs. This generally happened when either the entity was information that was stated as given in the problem or was the target of the problem task. In problem 9 above, students' recognizing that the vertex of a parabola represented a maximum function value for profit and correctly interpreting it in the situation meant that they had to view the given information, a graph or a rule, as an entity for the function from which they must pick out a particular point. Sometimes the goal of a task, such as creating a graph or seeking a rule for a table of values could be viewed as an entity. For example, Sam had a firm grasp of an algorithm for graphing linear functions in slope-intercept form. In that task his goal was to determine a graph, an entity, and he used the concepts slope and intercept to find specific points (instances)

to produce the graph. Object levels of wholeness will be discussed later under coherence of the function concept.

In summary, students often took a correspondence approach in each of the representations, rules, tables, and graphs. These correspondence perspectives are inherent in tasks such as evaluating functions, interpreting single points, processes that involve translating single instances between representations, and recognizing concepts that relate individual x's and y's. In general, correspondence perspectives involved instance levels of wholeness and involved entities only as either starting points or goals for problem tasks. Generally, students were successful with activities when they used correspondence perspectives, but there were exceptions, most notably graphing equations by hand.

Covariation Perspectives

Students also found the covariation approach to functions useful. In particular, they found both patterns of change in data and rate of change to be salient concepts. Toni, Sam, and other students relied on thinking about dynamic changes in variables to reason about problems involving functional relationships. How they used these ideas depended on the representation of the problem, the function class, and the task at hand.

Patterns of Change

Patterns of change in the dependent variable were particularly salient for students in table representations, whether on paper or the calculator, and in shapes of graphs. Students employed both dynamic and static versions of this kind of covariation as a concept to reason with, as part of a problem solving strategy, to identify linear and quadratic functions, and to make connections across representations within both linear and quadratic function classes.

When working with paper versions of tables, students often computed or described a pattern of change, especially in the dependent variable. During the sort

problem both Sam and Toni used patterns of change voluntarily to classify tables or graphs as either quadratic or linear. They described patterns of data for quadratic functions in tables or graphs as going down and then up (or vice versa) or as "turning around" and associated them with other representations for quadratic or linear functions. Recall that Sam rejected quadratic for the kind of function for a table on card R because the data did not reverse itself and go back up. Sam described changes in the dependent variable for linear functions in tables as either continuing to increase or continuing to decrease. Periodically, he incorrectly attributed linearity to nonlinear functions based on this feature only. Toni also used pattern of change for linear functions to connect representations. Early in the course, when solving the Alice, Bea, Carolina Problem in interview 2, Toni briefly was convinced (incorrectly) that the table's graph should be a straight line because the pattern in the table entries for output (only) had equal increments between entries. While Sam was more likely to attend only to the pattern of the dependent variable, Toni was more likely to actually calculate the amount of change (in tables) in both variables and correctly attend to them as a rate of change, not just the pattern of change in one variable. Once during the sort, while trying to illustrate a nonlinear function, Toni picked out card R with a pattern of non-equal change in the dependent variable. She attempted to justify her choice by calculating the rate of change. When it turned out to be equal, she quickly revised her classification to linear.

Sometimes students used pattern of change in function values as part of a problem solving strategy, particularly in calculator representations of graphs. Sam used the TRACE feature on the calculator graph of a quadratic (card B), to explore the data pattern in the y-coordinate because he found the graph displayed puzzling. Bob, another focal student, twice used the TRACE feature to investigate the pattern of data (in y) on the graph of a quadratic function in the profit problem of interview 3. Once he used the pattern to find a suitable viewing window for his current graph.

Another time he used it as part of a problem solving strategy when he was stuck, saying he thought if he looked at the pattern of the numbers, it might give him an idea for how to proceed.

Rates of Change

Rate of change was also a salient concept for students, especially in tables and graphs. When the pattern of change in the dependent variable is extended to include both sets of variables in a function, it becomes rate of change. Students accessed rate of change in columns of tables or in slopes of graphs. As with pattern of change, students employed both dynamic and static versions of rate of change as a concept to reason with or as part of a problem solving strategy. They used it to make connections across representations within both linear and quadratic function classes, to connect linear or quadratic function classes to their representations (especially tables and graphs), and, in addition, to fit situations to graphs for general functions.

When working with tables, Toni and Sam initiated a covariation-static perspective of rate of change to reason with. They created categories of equal change, unequal change, and it "goes all over" or "does strange stuff," to organize functions into the classes linear, quadratic, or "other," and to connect tables to graphs. (Sometimes Sam and other focal students considered only pattern of change, not rate of change, in these situations.) In class, it was noted that linear functions have equal (rate of) change whereas quadratic and exponential functions have unequal (rate of) change. Even though students had insufficient information to distinguish the two kinds of unequal change in quadratic and exponential functions and were unsure about how these terms should be applied to other functions, they had at least encapsulated a concept, the rate (or pattern) of change represented in a table, and thought of the table as a whole, that is, an entity that represented a particular kind of function or graph. More will be said about these categories of equal-unequal change under discussion of students' conceptions of function classes.

Some tasks involving tables, such as problem 2A-2, seeking a function rule for a linear function represented by a table, lend themselves naturally to using rate of change. Rate of change can be found by computing vertical change patterns within each column of the table and comparing them, either by noting changes in y per unit change in x or by computing a ratio of change between pairs of table entries. Students attempted to use rate of change in problem 2A-2, but had mixed success. In that problem the output decreased as input increased, requiring a negative multiplier (-5) on x in the rule for the rate of change. Sam, Toni, and Bob all began by describing changes in output relative to changes of one in input, that is, an additive rate of change given verbally as dynamic-covariation. Both Toni and Sam described it correctly. Sam immediately connected -5 to its role as a multiplier on the x in the desired rule. Later when asked why he had chosen multiplication, rather than subtraction, he said, "it get there faster." His response suggests he was thinking beyond an algorithm for producing a linear function rule from the table to thinking explicitly about a rate of change, that is, rate as a quantity. Toni, however, got stuck and could not correctly connect the rate of change she had just described to a function rule. Both she and Edy chose to subtract 5 instead of multiplying by 5 in their tentative rules, probably because of the decreasing pattern of change. Bob had difficulties with the problem that were associated with distinctions between informal, everyday words and mathematical use of the same terms. He seemed to understand the algorithm for finding a rule given in class since he described parallel patterns of change in both columns and correctly chose to multiply x by *his* rate, 5. He had trouble, however, because he had described the pattern in the table as "increasing" (magnitude only) by a factor of 5, rather than "decreasing" (magnitude and direction) by a factor of 5. His everyday language was at odds with the mathematical use because it did not include direction.

In class, students had been introduced to a procedure for computing rate of change, which they found particularly salient from a covariation-dynamic perspective. Yet they had difficulty connecting rate of change to its role in this function rule. Why? One possibility is the presence of the negative multiplier, while most of their practice had been with positive numbers. Further, there was interference from everyday language use of "increasing" with the restricted mathematical use of the term. Teachers often use "increasing" in the same way that Bob did to refer to increase in magnitude, but with a brief disclaimer to point out the distinctions between direction and magnitude. Once Bob's attention was called to this distinction by my asking whether the numbers were increasing or decreasing, he changed his multiplier to -5. So he knew the difference. Bob's early comments indicate that teachers cannot assume students will attend to the distinction between magnitude and direction even when they know the difference. Toni and Edy found their intuitive notion of "going down by five" strongly bound to subtraction. Each girl translated directly the operation embedded in the pattern of change in the table, subtract five, into the same operation in their proposed rule. Even though they knew there should be a connection between going down by five and the rule, they were unable to connect the additive pattern of change in the table to the multiplicative change in the rule. This result is consistent with that found by Kaput (1992), where he reported students working with a modified version of a computer program to guess function rules, when given output for input guesses (table entries), tended to translate directly from natural language descriptions of an operation directly to its algebraic form. In this case, Edy and Toni translated "going down by five" to subtract five. Further, Kaput's detailed description of KG, a student, indicated that negative multipliers gave her more difficulty than positive ones. While all students found rate of change salient and attempted to use it in the problem solution, and both Sam and Bob were able to connect it to a multiplier in the proposed function rule, only Sam implied rate of

change was actually a specific rate when he ascribed a speed to the change and applied his order for various rates of change to arithmetic operations.

Although there were specific attributes of this problem, such as multiplication by a negative, that might have contributed to students' trouble, another possible explanation for their difficulty is that students were attempting to go from a tabular representation that they saw dynamically as a covariation of two sets of numbers to a rule representation, a static entity, that they generally viewed from a correspondence perspective. Thus, to make this transition, they would need to make a transition across perspectives from covariation-dynamic to correspondence-static, apparently not an easy transition.

In addition to accessing rate of change in tables, students frequently used it when their task was to choose and justify the fit of a graph (of a general function) to a verbal description of a situation. Recall that Toni was quite fluid in her ability to coordinate the shape of a graph with a verbal description of a situation in all such problems she was given. In the Tim and Donna problem, where the graph was distance over time, and speed was given indirectly through a description of mode of travel, either walking or running, Sam, Toni, and Bob all relied on being able to coordinate a description of the change in rate (speed) of change of the dependent variable (distance over time) with a description of the change in slope of the graph to justify their matches. Being able to verbalize this match seemed to help them fend off distractors in the graph itself, namely, that the graph continued to rise while the speed (rate of change in distance/time) went down, a common reaction in students when attempting the same task or a similar task (Bell, Brekke, & Swan, 1987; Even, 1989; Janvier, 1978; Kerslake, 1981). More will be said about the role of language later.

Although students recognized that a steep slope was connected to a fast rate of change, a static version of covariation, they thought about rate of change as dynamic,

rather than static. When asked why a steep slope was associated with running (in Tim and Donna Problem), both Toni and Bob depicted rate of change dynamically as covering more distance in shorter time. Rather than describing change in distance *per unit change* in time or as a fixed ratio, as is usually given in definitions of rate of change, they described the coordination of two separate variations, changes in distance relative to changes in time. Recall even when students computed change in output per unit change in tables, they described it dynamically in terms of each variable: output changes by so much as input changes by so much. They seemed not to see rate of change as a single quantity.

In addition to using descriptions to coordinate the same rate of change in the two representations, verbal descriptions of situations and graphs, both Sam and Toni attended principally to only one variable when reasoning about graphs in a dynamic way. Also, when they were successful, they relied on experiencing the independent variable temporally. For example, in each interview problem, either time was the independent variable (Tim and Donna problem), or as in both problems of interview 2, Toni chose to interpret the independent variable as passage of time. In the guitar problem, she literally renamed "amount of practice," time. In the educational value/difficulty of work problem, Toni relied on describing difficulty of work as changing over time for some group (of students) in justifying her choice. Her facility at fitting graphs to situations so fluidly may have been due in part to her ability to choose a suitable independent variable that could be experienced temporarily. In that case, increasing horizontal changes are consistent with passage of time when viewing or sketching a graph from left to right. Both change in situation variable and the passage of time can be experienced intuitively and simultaneously, without much overt attention. Thus attention is freed up to deal with other aspects of the problem, such as coordinating the rate of change in the dependent variable with the slope of the graph without getting confused or distracted by characteristics of the either the

situation or the graph. This result, time as independent variable facilitating graphing, is consistent with that reported by Leinhardt, Zaslavsky, and Stein (1990).

With slightly more difficulty, Sam could fit graphs to situations as well, as long as he could visualize the independent variable as increasing over time. His difficulty in the educational value/difficulty of work problem came about in part because he first chose educational value for the x-axis. Yet educational value did not continuously increase as difficulty of work changed. It went up and down, so Sam could not use this strategy of "ignoring" the independent variable, educational value, while charting changes in the dependent variable, difficulty of work, a strategy that he used in tables and other patterns of data. Once he decided to select another graph putting difficulty of work on the horizontal so that variable could be viewed intuitively or experienced tacitly as increasing, he was successful with his match. He could then think of educational value as going up and coming down again as it did in both the situation and the graph that he had chosen while giving only tacit attention to difficulty of work, which he described as changing over time.

Attending to either patterns of change in the data or rate of change seemed natural to these students. These concepts are essential features of a covariation approach to functions. Both Sam and Toni were successful in part because they were able to focus their attention principally on the dependent variable, while giving only cursory attention to the independent variable. This pattern of reasoning is similar to what those familiar with these concepts might invoke when fitting situations to graphs in a fluid way.

When working with tables, students used this same intuitive, dynamic approach to functions, where most of their attention was on the variation of the dependent variable with little on the independent variable. In such cases it is easy to see why students might confuse linear and nonlinear functions. The variation itself draws most of their attention, and they might tacitly assume the other variable in the

table is increasing steadily by single units. Since regular increments in x are necessary to see the regularity of change in y, for y = f(x), students likely see far more examples in class where it is unnecessary to attend to the independent variable than otherwise. Given the curriculum materials, particularly the heavy use of tabular representations where change could be readily computed, it is not surprising then that certain concepts, such as rate of change in variables or pattern of change in data, were salient for all focal students.

Language as Mediator in Reasoning about Change

Natural language figured prominently in students' reasoning with a covariation approach. They relied on it when working with both patterns or rate of change. In addition, natural language sometimes confounded problem solving.

Students depended on descriptions of patterns in tables to distinguish linear and quadratic functions. In problem 2A-2 (discussed above), to find a rule for a linear function given in a table, the task could be thought of procedurally, but it had become complex for students because they had either an incomplete or no working algorithm for it. In attempting to coordinate change here in more than one representation, they relied on the amount of change and a verbal description of the sense of the change, both speed and direction. Sam had an informal notion about the rate of change as "speed" of reaching a destination, that is, how fast numbers changed. He seemed to have an ordering for the rate of change connected to operations subtraction, multiplication, and exponentiation, going from slow to rapid change, respectively. Sam thought of data in tables as "getting there faster" and described the change aloud: "every time this one rise one, that one rise--. It go down. It go down five." The combination allowed him to quickly choose -5 as the correct coefficient for x in the equation. While he might simply have been describing these changes, he seemed to use the language as a means of ordering. Recall Bob confounded his ability to find a correct multiplier by using everyday language "increasing" rather than mathematical

language "decreasing" that is necessary to avoid confusion with negative numbers increasing in magnitude as they get smaller. He remained stuck on +5 until I helped him realize that his table had to be thought of as decreasing, thus creating a closer match between the mathematical rate of change and his everyday description of it.

When students' tasks were fitting situations to graphs, as discussed above, their reasoning was also mediated by use of natural language descriptions of changes to coordinate those in the graph with those in the situation. They had to coordinate change in a quantity that represented rate of change in variables with those on the graph, and using descriptions seemed essential to avoid distractors. In the Donna problem, Even (1989) reported that college mathematics majors who were prospective teachers were distracted by the superficial features of the graph and switched to a graph of speed. Recall that Sam began this way, but once he realized he was describing a graph of speed, he correctly switched to distance and described a match--albeit fragile--by coordinating descriptions. His reasoning, too, was mediated by language.

These results for translating situations into graphs are consistent with those found by Janvier (1978), who reported language as a mediator in several complex translation tasks, particularly where students had to select from among several instances of a graph and needed to coordinate changes. He found in his classic racing car task (to choose an appropriate racing car track from among several for a graph of the car's speed over time) that students could not translate based only on visual cues, but needed to both verbally summarize the tracks and speak them aloud while attempting the coordination. He pointed out that since the mind cannot focus on both representations--the graph and the picture--simultaneously, some strategy had to be invented. Verbal descriptions was one. Clearly, matching a graph with a race track is a different translation task than the one here, but both this task and the racing car task involve two separate representations. In Donna's case (Tim and Donna

Problem), there is a graph of distance over time, and a description of traveling mode (speed given indirectly), that is, rate of change for distance over time. There are similarities, in each, however. In Donna's, a position (distance) is changing and must be connected indirectly to its rate of change (speed) by coordinating two representations of it, an indirect verbal description and the slope of a graph. In the race car problem, a speed is changing and its change (acceleration/deceleration) must be connected indirectly to its rate through its position and visual cues about sharpness of corners and lengths of straightaways. In any case, Janvier's work gives support for the task being complex enough to require mediators and strategies to deal with the problem complexity. Language is one, and as we have seen above, a strategy to narrow the focus of attention to only one variable is another.

Levels of Wholeness

Accessing dynamic patterns of change requires attending to a pattern in the representation as a whole. Even though it may require individual computations between pairs of y's or reading off individual y's, it is focused on a trend in the whole table. Similarly, when using dynamic covariation to fit a situation to a graph, attention is on moving across the graph as a whole. Thus, in each case, attention is, at least implicitly, on entities rather than instances of a function. Object levels of wholeness will be discussed later under coherence within the function concept.

Summary

Students found covariation perspectives, especially dynamic versions, useful in tables and graphs. They used both dynamic and static versions of patterns or rate of change to reason with, as a problem solving strategy, to identify function classes and to connect more than one representation of a function. They used overall patterns of change in data in tables to connect tables and shape of graph for linear and quadratic functions and to explore the behavior of calculator graphs. They used a static version of equal rate of change to separate linear from other kinds of functions. The concept

of unequal rate (patterns) of change was problematic for students, probably because of how it was taught where insufficient distinctions were made between its characteristics in quadratic and exponential functions. Sometimes students' understanding of pattern of change was robust and connected to rate of change and other times it was not. Students had mixed success in connecting dynamic rate of change in a table with static rate of change in a correspondence (rule), in part because of conflict between everyday and mathematical use of language, but possibly because of the presence of a negative multiplier.

In both tables and graphs, students saw rate of change dynamically as two sets of numbers varying. When students were successful fitting graphs to situations, they depended on being able to think principally in terms of variation of the dependent variable while experiencing the independent variable temporally.

Language served an important role in these covariation perspectives. Students used verbal descriptions to coordinate changes in the steepness of the graph with changes in the situation to justify the fit. They named static categories of change, used language to describe patterns (sometimes in a problematic way), and relied heavily on language to coordinate the two representations when fitting situations to graphs. Finally, when students take covariation perspectives, generally their attention is on either static or dynamic patterns in a table as a whole or changes in slope across a graph as a whole, thus, attending to entities is inherent in these perspectives of functions.

Coherence and Connections in Functions and Functional Relationships in Problems

Many problem tasks, in combination with representations given, lend themselves to taking either a correspondence or covariation perspective where students might be expected to attend to individual instances or to the function or representation as a whole, respectively. Yet in any problem involving functional relationships, a successful problem solver moves fluidly between detailed, isolated pieces and larger,

coherent wholes. A teacher would expect students to connect function concepts to both pieces and wholes of representations. For example, the change between two pairs of y-values in a table (for constant change in x-values) must be equal for the function represented to be linear, and this pattern must exist for the table as a whole. In addition, a teacher would like students to perform consistently across both representations and function classes and would like students to perform similar tasks consistently, especially within the same function class or representation class. Understanding functions thus involves coherence among and between ideas and making connections between, among, and within function classes, representation classes, function concepts, and problem tasks.

This section, broken into four segments, discusses how coherent or connected was student thinking. The first segment discusses mainly coherence in problem solving, that is, consistency between, among, and across problems involving similar tasks, concepts, or representations. It addresses connections between concepts and representations. The second segment discusses a specific example about graphing exponential functions by hand where connections are clearly missing. The third segment addresses coherence within a single function class between and among representations, concepts, and tasks. The fourth segment, using the levels of wholeness perspective to frame discussion, examines coherence within the function concept and, for completeness, discusses the concepts domain and range and their connection to the function concept.

Piecemeal Thinking

From a teacher perspective, if students are to have a connected understanding of functions, they need to make connections in their problem solving between, among, and across similar problem tasks, concepts, or representations, and between representations and concepts. For students in this study, much of their thinking about functions was piecemeal and disconnected. Although in some interview problems,

such as 1-1, Going to School, and the problems involving fitting graphs to situations where Toni had good overall strategies for her problem solving, generally her thinking focused on single procedures, computations, or some detailed aspect of a representation. For example, in the profit problem, which took more than half the page to state and involved two separate function rules, Toni started simply by multiplying out the polynomial, $(t - 3)^2 + 700 (t - 3) - 200$, saying, "I'm doing this one right here," as if she had an equation to solve. Recall that in the Alice, Bea, Carolina problem of interview 2, Toni was so focused on the scale being equal for each axis that she rejected Bea's graph as a valid match to a graph with an almost identical shape that she herself had just created by plotting points from the table given. She insisted Bea's scale was wrong--even though no scale was given.

When tasks differed, Toni was unable to connect concepts for the same type function in the same representations. For example, when she was asked in posttest problem 12 to select the general symbolic form, $y = ax + b$ (from among those for quadratics and exponentials) for a table representing a linear function, she saw the table as representing a different function for each table entry. She did not see it as representing a single function as she had done repeatedly during interviews. In addition, Toni had previously used constant rate of change as an organizing concept both for identifying and justifying tables as representing single linear functions. Here in problem 12, the existence of the parameters in the problem statement might have triggered multiple linear function rules, but in any case, her response is inconsistent with the clear understanding she exhibited in her work with tables of linear functions at other times.

Besides recognizing the same concepts for a particular function class in the same representations, a teacher would expect students to make connections between the same concepts in different representations and across function classes. Yet Toni's understanding of connections between other representations and graphs, or between

hand graphs, calculator graphs, and graph concepts lacked coherence. She understood many processes and specific concepts connected to specific representations, but often did not connect them. For example, Toni recognized that a table with equal change had a line graph, and during the sort problem, she had explained that a graph for a rule of the form $y = ax^2 + bx + c$ would be a parabola and would be U-shaped. She could plot points accurately from tables if both axes and tables were labeled. About midway through the class, however, she was unable to plot by hand the graphs of either exponential or one-term polynomial function rules. Further, on posttest problem 6, Toni was unable to graph correctly any of three function classes given in rule form. She graphed the linear and exponential equations as single points using constants in the function rules for coordinates. Toni omitted the graph for the quadratic even though previously she had identified similar rules as having U-shaped graphs. Students had used the calculator repeatedly during the class to create graphs for all three types of functions, and Toni had used graph shape as an organizer for function class for both linear and quadratic functions. Using function rules, students had extended tables in class prior to creating graphs by hand. Yet Toni was unable to access or seemed not to have made connections between calculator graphs and paper ones or between hand plotting graphs of functions given in symbolic form and translating tables to graphs.

Toni's understanding of the concept, input to a function, was also fragmented. She connected input to a process in one representation (what is substituted in parentheses in a rule) and knew input could be a label on a table or graph. While she understood there should be consistency in labeling, she did not know the convention for graphing input on the horizontal. Further, Toni did not understand that input was considered the value that was given first in evaluating a function or a quantity that other values depended on (the independent variable).

A possible explanation for Toni's limited understanding of input may be in exercises students were given to help them understand the notation of various representations. Problems were constructed to request interpretations going from both input to output and output to input in representations: tables, function rules with f(x) notation, and graphs. So students attended as much to the correspondence as to the dependency direction. Recall that when Toni had to create graphs from tables given horizontally, she, as did several others, assigned a variable for the horizontal independently from which variable depended on the other. So Toni's difficulty here might have been an artifact of the class exercises to emphasize connections. It might also have been connected to her propensity to look for visual or procedural cues, which in this case got input for a function (given as a symbolic rule) connected to a procedure for substituting into parentheses, rather than to the function concept of independent variable, which, by convention, is graphed on the horizontal.

Sam, too, thought about representations and functions in a piecemeal, disconnected manner. He could access bits and pieces of concepts and procedures, but was unable to use them very constructively or consistently. Recall his disjoint thinking about linear functions across different problems. For example, he could create an equation for a linear function from a table, but not from a graph. He could use slope to graph, but could not compute slope accurately given a graph. In some cases, he gave an incorrect answer on a posttest item when his answer on the same item on the pretest had been correct. The most vivid example of piecemeal thinking within a single problem was in 2B-2, the educational value/difficulty of work problem. Recall he had made a poor choice for the graph, and then interpreted each intercept in terms of only one variable. His thinking in this problem could at best be labeled an attempt to match individual instances of a functional relationship between two representations, a situation and a graph, but it was restricted to considering the two quantities being related only one at a time without reference to the other. He did

not access knowledge that he had demonstrated at a prior time on other problems fitting situations and graphs. He ignored mathematical constraints and processes that he had correctly used on other occasions. Previously, he had interpreted points as having two values associated with them and had worked accurately and flexibly with ordered pairs for points on graphs, both in point-plotting and in interpreting points on graphs correctly in terms of the situational context.

Some of this inconsistent problem solving might be explained by Sam's problem solving strategies. He regularly attended to the dependent variable, especially in tables and graphs, while seeming to ignore the independent variable. In many problems, this caused little difficulty. In other problems, particularly those with situated context, he responded to interviewer questions or a difficulty in the problem solving process by answering with some element in the situation, by inventing aspects of the situation to fit his answer, or simply moving from one response to the next, sometimes ignoring previous correct responses. Sam invoked a strategy, which I call "living in the moment," using whatever was most recently at hand--either things associated with the mathematical problem being discussed or quite often with whatever was taking place in the social interaction. This strategy to cope with mathematics and school that will be discussed further under non-mathematical issues.

Missing Connections, a Puzzlement

One of the most puzzling findings about students' work was their failure to connect knowledge and skills that they had exhibited on other occasions to a task given in class during the third week. Eight of the nine students attending could not sketch (by hand) a correct graph for the equations $y = 5^x$, $y = 2^x$, $y = .5^x$ and $y = .1^x$. Ken, a focal student who dropped out, sketched the first two correctly, but made similar, incorrect, increasing graphs for the last two. The rest of the class graphed

either four dots along one of the axes or prominent dots along an axis at 5, 2, .5, and .1 with straight lines through them.

From a teacher perspective, students should have been able to do this task. Students had had many exercises for other types of functions in which they created graphs by plotting points by hand or using the calculator grapher. They had extended tables themselves for many kinds of functions and then plotted the points on graphs. The class had interpreted values of coordinates and created function rules for several kinds of elementary functions from patterns in tables by guessing and testing rules. Prior to this graphing exercise students had quite a bit of experience with exponential functions, especially with their graphs. In class they had explored $y = 4 \cdot (.84)^x$ in a situation about drug absorption, had created a table for $x = 0$ to 10, and had made a scattergraph for the table. They had contrasted the differences between patterns of change in this function (per unit change in x) and that for a linear function and had discussed the connection between the pattern of change in its table and curvature of the graph. The class had created a table for $y = 4 \cdot (0.5)^x$ and had discussed the situational distinction that $y = 4 \cdot (1.15)^x$ would be a formula for a drug amount that was increasing rather than being absorbed as had previous graphs in the activity. Students had done two situated homework problems that involved doubling, an exponential function. In one, the context was money doubling per unit of time, and students had to make an investment choice. The other involved bacteria in a swimming pool, and students had to extend a table of values after discovering the pattern of doubling. The class had gone over these two problems and had related the first problem to compound interest. They had spent a half hour discussing this problem and its extensions, which included comparing different compounding periods.

Even though students had seen graphs created by hand from rules, had discussed patterns of change in the table and their connection to shape of the graph, had seen several graphs of exponential functions, and had had their attention called to

examples with bases either less than 1 or greater than 1, students could not graph these four functions when asked. No one but Ken made a connection between these equations and the tables and graphs that they had just spent two days discussing. Further, no one seemed to connect these graphing problems to those of other functions previously done in class, homework, and interviews. Why?

One possibility is that this task requires an additional step of needing to arbitrarily choose x-values to create a table of pairs from which to graph. Creating a table involves knowing that it is necessary to do so and an understanding of variables. While Mr. Bailey often created tables for functions in class prior to this time during the third week of class, and students frequently were asked to create graphs from tables, students were never given an equation *only* and told to create a graph by hand. When students had to create a graph from a table in class or for homework, either the graph was to be done on the calculator or else explicit instructions were given for how to proceed. Before plotting points, students were either given a completed table or told how to continue a pattern in an existing table. So creating a table "from scratch" from a rule with the purpose to graph the ordered pairs was not part of students' experience.

Students may not have developed these skills because they were not required of students during classroom graphing activities that translated rules to graphs. Bereiter and Scardamalia (1989) call a process, such as this, where teacher and students create graphs together a "joint cognitive process." They define a joint cognitive process as a single process that has different parts carried out by different people, with the process partitioned among different agents, control passing from one agent to another, and with the "possibility that different goals may be operating simultaneously in different parts of the process" (p. 381). Further, Bereiter and Scardamalia hypothesize that skills a student acquires in an instructional interaction are those that are required by their role in the joint cognitive process. In this activity

the classroom teacher and students jointly created graphs by hand on the board. For students to be active in this process, they needed only to generate appropriate y-values for x-values chosen by the teacher. It was only the teacher's role to generate x's and to direct the steps in creating the graph. Students encountered the requirement to generate x's for the first time during the activity to sketch these four exponential equations. Their failure to choose x's is consistent with their roles in this joint cognitive process, translating rules for exponentional functions to graphs by hand.

The hypothesis that students had not learned to create a table was supported during interviews as students discussed their recollections about graphing these four functions in class. I had a strong sense at the time that they did not know they that the graphing task included first creating a set of pairs, that is, some form of table. None of the four focal students created a T-chart (table) as their teacher had done in class. While each tried to connect a single x with a single y, except for Bob, there was no evidence that they knew they should choose additional values for x to get additional pairs. For $y = 2^x$, Bob described finding x-y pairs, $y = 2^1 = 2$ for $x = 1$ and $y = 2^2 = 4$ for $x = 2$, but graphed these as points at (0,2) and (0,4), ignoring the values of x he had just used. Whereas Bob knew to select more than one x and compute its corresponding y, he failed to use the x's correctly. During the interview he did say he would complete the graphs by drawing horizontal lines through the points just plotted on the y-axis, indicating he knew the graph should not consist of only one point. Recall that Sam reported in class he had made $x = 1$, a fixed value, for each equation, evaluated the function and used it for the y-intercept for a straight line graph in each case. As with Bob, his making a line indicates he knew the graph should consist of more than one point, but he did not understand the process of first creating a table. Later in the interview, when pressed to create a table for $y = x^2$, Sam could only produce y-values for x-values that *I* chose, and then only after I showed him that I wanted him to compute y-values. Even though he said the y gave

"where the point was on the graph," Sam said he did not know how to make the graph, claiming only the calculator "knew how." Thus he did not connect the table to a means of getting from a rule to a graph. Perhaps he expected some function-specific algorithm for graphing like the one he understood for graphing a linear function in slope-intercept form.

In addition, neither Edy nor Toni used more than one x-value. Both girls graphed only one point for each function. At one point in the discussion I gave Edy another equation, $y = x^2$, along with a table with several x-y pairs for this function and asked if she could make the graph. Even though she had plotted points to create a graph from a table in interview 2, she said she had no idea how to proceed and began to look in her notes for help. So Edy did not know the graphing process included making a table of pairs and was unable to use a table she was given. Recall that Toni also did not connect graphing in these problems to the point plotting that she had done earlier on homework and in interview 2. She referred to the 5^x as $5x$ and took 5 as the value for x in $y = 5^x$ and said since she had no value for y, she did not move her point off the axis. Toni's work is consistent with that on other occasions in both interviews and pre- and posttests where she failed to create graphs from rules and where she interpreted constants in the problem as values of variables. Later in this interview, Toni tried to calculate a y value for another rule that I gave her, $y = 5x^3$, but she confused the role of x as she tried finding y values on the computation screen of the calculator. She entered the expression with the variable x, instead of a constant value for x. Since there actually was some previous value of x stored in the calculator's memory, the computation was done with that value, which gave a puzzling y value. In any case, Toni did not seem to understand a process of choosing an x value, calculating the corresponding y, plotting the value, and repeating this procedure. In addition, she had some difficulty with the role of variable in this context, as did Bob, Edy, and Sam.

To produce a table of pairs for a function, one would have to recognize that variables can take on many values. To explore the possibility that students had trouble with these problems because they had naive conceptions about variables, I added several questions from the classic CSMS study (Kuchemann, 1981) to interview 4. (See problems 4A-1 to 4A-7.) From these questions, I concluded that all four focal students understood that variables could take on many values, although Sam's understanding of this may have been fragile. All but Sam understood that different names could stand for the same number in the same problem. All four saw variables as unknowns in equations. None, however, was able to accept lack of closure, that is to accept that a variable has a yet to be determined value, but can be used in expressions as if it is known. For example, recall problem 4A-6 was:

> Sam drew the figure below while babysitting for the neighbor. It has n sides; each has length 3. Unfortunately, the baby spilled paint on the drawing just as Sam was calculating the perimeter of the figure. What is the perimeter? (The figure showed 13 sides of a star-shaped figure with the rest of obscured by a blob of paint.)

None of the students was able to write an algebraic expression, 3n, for the perimeter, even though they all explained a process to find the perimeter. Each said they could not do so without knowing the number of sides. While Edy guessed n + 3, she seemed simply to be trying to use the variable and number given in the problem statement.

While acceptance of closure is necessary to understand algebra, in order to create a table of values in the exponential graphing problem, a conception of variable that included a variable being able to take on many values should have been sufficient. Yet something prevented students from computing such values. It is possible that students did not see the expression $y = 5^x$ as a whole, that is, they did not view it as standing for a whole set of x-y-pairs, an entity seen from a correspondence-static perspective. Thus, they did not see a need to generate a table of values to plot.

It is also possible that students misinterpreted what "freedom" they had with variables. When asked about their graphs, both Sam and Edy reported that the teacher had said "x could be anything".[3] In their recall of this exercise, both reported incorrect interpretations of "x could be anything." In $y = 5^x$, Edy had made x zero and treated it as the value of the x-coordinate, but had evaluated 5^0 as 5 and graphed the point (0,5). For the same equation, Sam had made x an arbitrary constant that remained fixed while he had graphed the resulting equation as a line, first with arbitrary slope through the point, (0,5), and then during the interview corrected it to be a horizontal line, $y = 5$. Both students had interpreted "x could be anything" as if the variable represented a single arbitrary constant, rather than a variable, where the specific choice for x of the moment could be arbitrary.

Sutherland (1987), in reporting a study of the potential for programming in Logo to enhance algebraic understanding of variable use, reported similar behavior in a student who in a problem involving variables M and N, asked whether M meant any number. The researcher replied, "M is any number and N is any number." In a later problem about John having J marbles and Peter having P marbles, the student was asked how many they had altogether, and he wrote 9 because John began with J and had 4 letters and Peter began with P and had 5 letters. The "x can be any number" got translated into a form that made sense to the student, in that case, literally "x can be anything." Edy and Sam justified their interpretations with a similar translation. They claimed that the teacher had said, "x can be anything," and simply reinterpreted this phrase in a way that was meaningful to them, but unlike the students in the Sutherland study, restricted their meaning to the domain of numbers.

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[3]The teacher actually made the statement, "X can be any number," after students had made their graphs.

During his interview Sam displayed additional inconsistent thinking about variables in his discussion of the problem, $y = 5^x$. Besides giving the exponent an arbitrary, fixed value in this and other exponential equations, he treated two distinct x's within a single equation as having different properties or purposes. Recall that at one point while Sam was explaining his way of graphing a line for $y = 5^x$, he had added a 3x to the equation to illustrate the connection between the 3x and the slope of a graph. When he graphed the resulting equation, $y = 5^x + 3x$, he treated the exponent as the constant, 1, but implied the x in 3x was a variable and created a correct graph for $y = 5 + 3x$. Since Sam referred to 3x as the slope, he may actually not have seen x as a variable, but the term 3x simply as being part of a symbolic form that must be present before he could apply the slope-intercept graphing algorithm. In any case, Sam had treated the two x's within one equation differently.

Another possibility for students' piecemeal thinking is that there was something about exponential functions that somehow contributed to students' inability to make connections to experiences they had in class. But except for Sam's graphing linear functions in slope-intercept form, none of these students, when asked, could graph correctly simple polynomial functions. Only Bob was able, after questioning, to make reasonably valid connections among creating tables, plotting points, and graphing exponential functions.

So far I have considered several hypotheses for students' difficulties with this graphing activity: that it did not occur to students to create their own tables, that their conceptions of variable were fragile, that their experience with exponential functions was too limited, or that there was something about the nature of the exponential function itself that was inherently difficult. Another conjecture for their trouble is related to students' approaches to problem solving. Toni's general approach to symbolic problems was to try to recall some procedure or do some manipulation on the symbolic form; she was unable to graph any of the functions

because there was nothing to manipulate and she had nothing to recall. In addition, Toni had some difficulties with connections between symbolic and graphical forms in general and often resorted to interpreting the numbers within the symbolic string as directly meaningful as points in the problem. In the exponent problems she might have been simply resorting to this problem solving strategy. Sam had fragile conceptions of variables, but clearly knew an algorithm for graphing linear functions and might have expected other algorithms should exist.

A final conjecture relates to the use of the calculator. Since data gathering, I have noted behavior similar to Sam's and Toni's in my own algebra classes that use the graphics calculator. My classes have extensive table exploration, including exercises in which I, with input from some students in the class for both x and y values, create tables from an equation on the board and sketch graphs by hand. When students create graphs only on the calculator, they often have no idea how to create a graph by hand until they are specifically told to choose values for x and compute corresponding y's. So student behavior might be an artifact of calculator use. When students clearly have at least one good means, a calculator, to find a graph, they may see little need for hand methods. Thus, the technology may reduce their need to remember a process or to be able to duplicate a teacher's board methods.

I struggled with trying to determine why students in this study had such difficulty with graphing exponential functions from an equation when they had had so much experience with tables and graphs, generally were successful with correspondence perspectives, and did understand that variables could take on multiple values. Yet clearly connections were missing in students' understanding of these graphing problems, and in spite of the experiences students had, they were unable to utilize skills and knowledge that they had demonstrated previously. It appears that individual students may have had different difficulties. It is likely, however, that students did not understand creating a table was part of a graphing process, had

somewhat naive conceptions of variables, and that possibly the functions being exponential might have contributed to their problems. It is also possible that students tended only to learn procedures they were taught or were required to exercise in a joint cognitive process. They may not have had a disposition to attempt to make connections unless required to--even when connections seemed obvious to a teacher. None of the data here is conclusive, however, and I can only speculate on what specific difficulties for students might have been.

Coherence within Function Class

Understanding characteristics and concepts uniquely associated with a particular class of functions is another kind of coherence. Students' characterization of the three function classes--linear, quadratic, and exponential--were contextualized and depended on the function class and representation. Students had classification ideas and made connections across representations for linear and quadratic functions, but had few such connections for exponential functions. What connections they made across representations for exponential functions were weak. Both Sam and Toni used shape of graph, pattern or rate of change in data in tables, a verbal description of change, and the form of the function rule as classifiers for function class membership.

Linear functions. Students held strong connections between some concepts across representations for linear functions. As has been shown, students associated graphs of straight lines, either equal pattern of change in the dependent variable or constant rate of change in a table, and patterns of data that either increased only or decreased only, with linear functions. Sam recognized and correctly interpreted equations (function rules) of linear functions in slope-intercept form during several tasks. He correctly translated from a table of values to a symbolic form. Yet on other translation tasks, such as creating an equation from a graph or associating a line with a negative slope with an equation, $y = ax + 7$, where a had a negative value,

neither he nor Toni was able to access this information on the posttest. Toni was not able to create by hand a graph of a linear function.

Quadratic functions. Students also had strong connections across representations for quadratic functions. Graphs were powerful organizers for quadratic functions. Even the name, parabola, superseded quadratic, when students discussed this class of functions. They saw the graphs as U-shaped. During the sort problem all focal students referred to portions of graphs of exponential functions as halves of parabolas. Because the one graph increased as a function of time ($t \geq 0$) and one showed only a small portion of the part that was asymptotic to the x-axis, both graphs showed mainly the part of the graph that was fairly steep, that is, bowl-shaped. In all cases, students suggested extending the graph into a parabola, rather than that it might flatten out like the graph of an exponential function.

Sam and Toni also connected patterns of data and symbolic forms with quadratic functions. (Toni referred only to parabolas and did not use the term quadratic for this class of functions.) They associated quadratics with patterns of data in tables that reversed directions. Students were familiar with the general form $y = ax^2 + bx + c$ as giving a parabola and associated U-shaped graphs with equations or functions rules of this sort.

Some connections between quadratics, representations, and concepts were more fragile. Sam's work with the symbolic form indicated he understood the sign of the leading coefficient produced a graph oriented as opening downward and suggested he had some familiarity with an algorithm for finding the vertex of a parabola from the symbolic form (not taught in this class). Yet neither Sam nor Toni was able to sketch by hand a graph of $y = x^2 - 2x + 1$ on the posttest. Toni and Bob were unable to distinguish a one term quadratic from an exponential.

In general, each of the above associations depended on being able to see the quadratic in nonsituated form. Both Sam and Toni were able to successfully work

with quadratic functions in some situated problems such as certain cards from the sort problem and pre/posttest problems, 9 (graph) and 13 (rule), involving profit and ticket price, but which appeared more like school problems. Toni was unable to evaluate or interpret a quadratic function model with identical variables, when it occurred in a more complex situated problem in interview 3. Because of interference of her everyday understanding of the situation, she was unable even to compute function values in the profit problem until I had spent considerable effort getting her to ignore the situation that generated the model in the first place. Both Sam and Edy were able to work immediately with the quadratic given in the profit problem, but also had interference from the situation when they tried to interpret the results they computed. Bob was able initially to work with the two quadratic functions given and understood they were to be compared, but he also had trouble with the situation. At one point he began to ignore what he seemed to "know" about quadratic functions in naked context and attributed situation quantities to the coefficients in one of the rules.

In summary, while some representations, such as shape of graph, the general symbolic form ($y = ax^2 + bx + c$), or one in which there was a squared term, were strong connections for students to the class of quadratic functions, other connections were fragile. Students could not distinguish one term quadratics from exponential functions. They also could not distinguish between quadratics or exponentials when working with (unequal) rates of change. Finally, situated problems that did not appear as school problems caused considerable difficulty. Toni had separated the name, quadratic, from "parabola," *her* name for both the graph and the class of quadratic functions.

Exponential functions. Unlike linear and quadratic functions, Sam and Toni had no successful organizers such as graph shape, pattern or rate of change, or symbolic form for exponential functions. Toni had an inaccurate conception of the rule form, but there is not enough data to say whether Sam had any symbolic form

associated with exponential functions. During the sort problem they both viewed graphs of exponential functions as half-parabolas. Only after a long passage when I helped Toni recall what an exponential function was and I had shown her a graph of a decreasing exponential function to try to jog her memory, did she say that card J (graph) could possibly have been an exponential function, rather than a quadratic because the graph looked "like it might straighten out" on the right side. Toni, in particular, had a strong sense that equal change was associated with linear and unequal went with the name quadratic, which she incorrectly thought was different from parabolas. While she may have thought the name quadratic went with exponential functions, it is unclear since she had another category, "goes all over." She was confused, however, about whether this latter category went with exponential functions, with scattergraphs, more general continuous graphs, or possibly more than one kind. Both Toni and Bob indicated for part of the time that they saw the symbolic form of an exponential function as a single term quadratic, that is, $y = ax^2$.

Toni's and Sam's connections between function classes, graphs, and symbolic forms, while reasonably intact for linear and quadratic functions (given Toni's label parabola), completely broke down for exponential functions, in spite of many exercises and computations in all representations of these functions. Students had been introduced to exponential functions as repeated multiplication, where the exponent was the number of multiplies. For symbolic forms, the number of terms in an exponential function seemed more important to them than the fundamental difference, which is the position of the variable.

While I do not have sufficient direct evidence for this claim, it may be that part of this particular confusion is related to activities of curve-fitting that were necessary for teaching modeling. During an experiment where students timed pine racing cars running down a ramp, they had to search by trial and error for suitable parameter values for a and b in $y = a \cdot b^x$, and saw a, b, as well as x and y as

variables. Other times during class students studied the effect of changes in certain parameters, and I recall at least one of the students referred directly to all three, a, b and x as variables. Thus, relative position of variable would cease to be a sufficient distinction between quadratics and single term exponential functions.

In addition, students were unable to distinguish exponential functions from quadratics using either patterns of change or shapes of graphs, salient concepts for them. Toni, in particular, was confused about what functions should be attached to unequal change. One possible explanation for this confusion about categories for unequal change and shapes of graphs is that students were first introduced to connections between unequal change in a table and curvature of a graph using a quadratic example, $y = x^2$. On July 10, during the second week of class, students were learning to run a table-producing program[4] on the calculator. They had not yet studied quadratic or exponential functions as classes of functions, but had seen simple quadratics being used in a few exercises when they were being introduced to this program. On this day Mr. Bailey used $y = x^2$ to introduce the table program for x between -5 and 5. He then used the table to introduce unequal change in y per unit change in x and connected it curvature in the graph by associating large changes (per unit change) with a steep graph segments and small changes with flatter segments. When the teacher introduced exponential functions as a class of functions, unequal change was associated in the same way with portions of the graph, without making sufficient distinctions between exponential and quadratic functions.

A second possible explanation for confusing unequal change and these two functions classes is that, prior to the study of either quadratics or exponentials as functions classes, the class had explored the role of changing parameters (symbolic representation) in effecting horizontal and vertical transformations in the graphs of

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[4]The table program takes as input a lower and upper bound for x's to be evaluated, along with the size of increment in x. It produces x,y pairs for the function listed in the program for these x's.

both linear and quadratic functions. These exercises may have had the effect of associating a U-shaped graph with quadratics. The exercises had been part of a broad strategy to introduce students to function concepts and gain practice with graphing on the calculator across a wide variety of functions prior to studying the attributes of the function classes themselves. Students' informal introduction during the second week of class to patterns of change in quadratic functions and their connections there to shape of the graph for an equation $y = x^2$ that was familiar to them, coupled with additional familiarity with parabolas gained from the transformation explorations, might have been enough to fix the association of unequal change, U-shaped graphs, and $y = x^2$ firmly in their minds. When unequal change in exponential graphs was discussed in the same way as for quadratics, their attention might have been on noting they had seen this phenomenon before. By the time they talked to me about these functions in the sort, they may have forgotten exponentials or they might have had a strong connection between the ideas of unequal change as a result of their early introduction with a familiar function, $y = x^2$.

Further, students had no way of distinguishing exponential and quadratic functions in tabular form by their rate of change, another salient concept. When exponential functions were introduced and change was computed in the table, as mentioned above, Mr. Bailey used an almost identical scenario as with quadratics to discuss informally their rate of change being unequal. When they considered the intervals for one exponential function, he pointed out the amount of change itself was going up, a topic that was not discussed with quadratics. The class did not, however, contrast the nature of the unequal change between the two function classes. Thus, students had no means to distinguish the quadratic and exponential function classes other than that an exponential might go up (or down) a lot faster, a feature that gets blurred depending on the specific constants in exponential functions. In that case, if first differences between function values *only* are computed, it is hard to compare

their relative sizes; that is, the change between these function differences may not readily appear as quadratic or exponential unless the numbers are very "nice." The students did not have access to the one feature, second differences in the dependent variable, that would allow them to distinguish these two kinds of functions (second differences for quadratics are constant, while for exponentials they are not). I had specifically asked Mr. Bailey not to teach second differences[5] because I thought that since students could not understand the origin of the concept, they could use it only as a memorized algorithm. Another feature of class that contributed to students not being able to distinguish these two function classes was that they were introduced in reverse order of the text. The exercises contrasting the two classes were omitted when the first function class was studied because they involved a function class yet to be studied. When the second function class was studied, exercises needed to be added to existing ones, which Mr. Bailey and I did not think to do.

In summary, students might not have been able to access concepts they found salient, patterns of change in a table and shapes of graphs, because of the order these concepts were introduced in class, the similarity between their presentations, and the fact that students were not taught the principal means of distinguishing these two kinds of functions using rates of change in tables. They were not able to distinguish half-graphs of exponentials from quadratics because of similar appearances when exponentials are missing the "tail" of the graph that is asymptotic to the x-axis that allows one to readily distinguish the two. Finally, exercises involving parameter explorations may have helped obscure the fundamental difference in the rule form of quadratics and exponentials, namely, the position of the variable.

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[5]This concept usually requires calculus.

Coherence within the Function Concept

Levels of Wholeness. Another kind of coherence for the function concept concerns levels of wholeness for functions. For students to have a solid understanding of functions, they must be able to think of functions beyond individual or sequences of instances of functional relationships. Functions need to be seen as whole entities, and ultimately as objects that themselves that can be operated on in their own right. As discussed above, Toni and Sam often dealt with representations and problems involving functional relationships in a piecemeal fashion. Yet at times they were successful with individual problems at both instance and entity levels of wholeness. They could adopt both correspondence perspectives for functional relationships, which lent themselves to an instance level of wholeness. Further, sometimes they saw the functional relationships more holistically as entities, particularly through graphs and tables from covariation perspectives. Sam and Toni took both instance and entity perspectives as important organizers, especially for linear and quadratic functions. Students saw graphs and tables as a whole, but only rarely did they take an object perspective.

This absence was to be expected as the *Computer Intensive Algebra* curriculum was designed for beginning algebra students and, in addition, the class covered only about half of the material in the text. Except for a small number of exercises in *CIA* and a few generated by Mr. Bailey to explore linear transformations of graphs, the students did not have much experience with perspectives on functions that viewed them as objects that can themselves be manipulated. Yet twice during the interviews, Bob managed this level of wholeness. Each time he connected appropriate symbolic forms to effect a horizontal, linear transformation of a graph. Once it occurred in problem 2A-2, to find the function $y = -5x + 15$ for a table, and once in the profit problem. In the first case Bob had used an automatic line grapher through the data points from the table to get a graph of the line. He then saw that his own guess, $y =$

-5x, was parallel to it, estimated the distance between the two lines along the x-axis as 3, and correctly substituted (x-3) for x, arriving at the correct equation for the table. In the profit problem, he used a similar strategy. In that problem students were to critique two quadratic functions, given the suggestion that the first paid out 200 dollars to an emcee and the second had a ticket price three dollars lower than the first. Bob graphed both functions and compared different values for the same profit value (profit = 0) to see if the ticket price he expected on the first was three dollars lower than the second. When he saw the x-intercept on the second was approximately three higher (rather than lower) than the first, he decided to translate the function in the opposite direction by substituting t + 3 in place of t - 3. While he failed to account for the effect of a vertical translation of -200 in one function, he nevertheless thought of these functions as entities on which actions could be taken. In both these examples Bob used both an instance level of wholeness to compare x-intercepts and an object level of wholeness as he attempted to pin down and correct differences in the relationship between the two graphs as wholes. This kind of thinking did not occur with any other focal students, except Ken, who dropped out of the program. One could argue that perhaps Bob was simply applying a translation algorithm, but when you consider the extreme difficulty Toni, Sam, and Edy (as will be seen) had with this problem, it seems quite sophisticated thinking about functions. Bob clearly saw the graphs as representing whole entities and operated on them as objects by trying to compare their differences as graphs by looking at a pair of points. In the case of each perspective, however, Bob utilized the non-aware dimension.

Domain and range. A complete definition for the function concept includes the concepts domain and range, as well as a relationship or dependency between variables. So domain and range must at least be mentioned under coherence. When students in this study used domains and ranges, it was consistent with how they were taught. During class students dealt informally with domains and ranges when they

had to find a suitable viewing window on the calculator to display a graph. When problems were situated, it was necessary to attend to domains and ranges to guarantee sensible interpretations of results. For example, in the profit problem both Sam and Toni implicitly discussed domain when they suggested a suitable ticket price for a talent show and discussed range informally when they concluded there was a limit to profit earned because the number of people attending was limited. They were discussing the situation, however, without any apparent consciousness that it was connected to the concept of domain of a function. This informal attention to domains and ranges in interview problems was entirely consistent with how domains and ranges were dealt with in class. Although students worked with domains and ranges, the terms were not used and were never discussed as function concepts, and thus, are not relevant to the coherence discussion.

Special Nature of Situated Problems

This final section related to students' conceptions of functions discusses special issues that arose when problems were situated in a realistic context. The use of realistic situations and applications to motivate meaningfulness in problems and to provide the basis for problem formulation, problem solving, and concept development is integral to the mathematics reform movement. In the grades 9-12 section of the NCTM *Curriculum and Evaluation Standards* (1989) each of the sections--algebra, geometry, trigonometry, and functions--listed realistic situations or applications (within or outside mathematics), real world problems, or modeling as topics to receive increased attention. In the curriculum materials used in this class, modeling was an integral part and all concepts were introduced through situations.

From a teacher perspective the use of situations can give meaning to computations and at the same time lend support to students' reasoning about a problem, from which mathematical problems and understanding of concepts can arise. In addition, the use of real situations may allow the student to see connections

between mathematics and applications in other fields. Although use of realistic situations is seen as necessary to students' developing mathematical understanding of functions, use of realistic situations is not sufficient. If students are to develop mathematical power, either within or outside mathematics, they must be able to move away from a specific real world situation and understand the mathematical concepts without reference to the specific situation in which it was learned and ultimately without reference to any realistic situated context.

Situations, Informal Models, and Reasoning

In this study problems that were situated had mixed value for students. Sometimes students' reasoning was bound to the particular situational context so strongly that the mathematics might have been invisible to them, that is, they were using mathematical concepts that they were unaware of. For example, both Toni and Sam were successful with fitting graphs to situations for problems that even advanced students had difficulty with (Even, 1989). Both relied heavily on coordinating descriptions of situations with descriptions of changes in graphs over time in order to produce a fit, making the mathematics appear intuitive and almost completely invisible, especially to an observer. It was not clear what mathematics, if any, students consciously thought about other than shape of graph.

On other occasions, students' reasoning was bound to the particular situation so strongly that mathematics they understood in other contexts was inaccessible to them. Recall that Sam regularly responded to questions about a problem by simply answering with a description of the real world situational context in the problem or an enhancement to the situation. This strategy worked well until the description conflicted with the mathematical representation as it did in the educational value/difficulty of work problem when Sam choose an incorrect graph. His reliance on the situation obscured mathematical knowledge he had successfully used on other occasions.

The profit problem, a complex, situated problem about profit from a school talent show, gave four of the focal students difficulty. Because the situation connoted by the problem conflicted with students' informal models of the situation, they were unable to make sense of the problem without a great deal of help. The problem statement itself asked for a comparison of two proposed function rules, an inherently difficult task. Yet even when I asked students to focus on only one function rule, they had trouble. I had written profit as a direct function of ticket price, which also complicated the problem for students. The problem might have been too complex for students with their backgrounds since it involved sophisticated aspects of modeling. However, it may be that students' naive understandings became visible precisely because they had a problem that could not readily be extracted from the situation.

From students' comments, they clearly understood that profit depended indirectly on ticket price because it affected the number of people that would come. While Toni understood this, she could not accept, and was unable to use, profit as a function of ticket price without also knowing the number of tickets sold. She could not move beyond the situation to operate on the function rule computationally, even after I repeatedly suggested it. Instead, Toni continued to try to make sense of the source of the rule itself.

Other students, like Sam and Edy, could compute profit for specific ticket prices using the rule, identify their answers as profit, but could not correctly interpret their results because of confusion over income and profit. Each alternately referred to income as profit. Both Sam and Edy struggled with the issue of whether to subtract $200 after they had computed the profit. From their prior experience they understood the concept that money was not considered profit until expenses were paid. However, even though they had indicated P(t) was profit, they insisted on subtracting $200 to pay the DJ from the value they had just computed for profit.

Perhaps in school experiences profit and income are actually much closer to equality than they are in business dealings outside of school. In functions such as a class event, which provided the context for this problem, expenses might be absorbed by the school with the result that ticket sales (income) and profit are the same. In that case, income would always be thought of as ticket price times number attending and cost would be thought of only as those quantities that are explicitly announced, such as the cost of the DJ in this profit problem. If there were no other factors in their experiences that could produce different models for profit, students' informal models for profit legitimately would be aligned with income models.

Situations and the Model Concept

The concept of model was prominent among CIA curriculum materials. Yet in classroom exercises, except during curve-fitting problems, students did not need to actively attend to the model concept for profit and income because the function model presented could be thought of as representing a mathematical process (rule) or relationship (table or graph) that was a direct analog to students' informal models from everyday experience. For example, that profit is a function of number of tickets sold or attendance (CIA, p. 2-26, 2-27) and that attendance is a function of ticket price (CIA, p. 2-16) mirrors their understanding and possibly their personal experience. Thus, tickets sold or attendance were readily accepted as predictors, and students computed answers to problems in which they were asked to make predictions. With no conflict between the model and their everyday sense of the problem, students could shift their attention away from the situation to the mathematics itself, that is, to the representations (other than situational context) and to the problem solving process. They did not need to attend to or even acknowledge that the given representation was a model; attention could be "through" the model "to" the computation or process of determining an answer.

Shifting attention immediately to the representation or solution process could explain how Toni and Sam were able to solve pre/posttest profit problems 9 and 13, which also involved profit as quadratic functions of ticket price with a public entertainment function (rock concert or class play) for a situational context. Both Toni and Sam answered questions about the graph (9) on both pre- and posttest with only minor errors. Toni omitted and Sam gave incorrect answers for the function rule (13) on the pretest, but both computed satisfactory answers on the posttest. While on the posttest they might have "learned" how to do this problem as a result of interview 3, the profit problem, in neither case was there a confounding of the problem by the situational context as there had been in interview 3.

Why might this be? Perhaps Toni and Sam saw these exercises as school problems on a survey or test and thus expected them to have computational answers. These problems did not raise the question of validity of the model or even attend to the idea of a model. Students were asked to compute or interpret values for the situation to get an answer, and so their attention could be immediately focused mainly on the graph or equation, rather than situation if they expected an answer were possible, as is usually the case in school problems. In the profit problem, however, the task was much more complex and open-ended. The problem asked students to compare two models. In addition, since the cost of the DJ was $200 and the constant term in one polynomial was -200, the problem statement itself suggested that individual coefficients in the problem mapped onto quantities in the situation. Students could not ignore attending to the model(s) itself; thus, the situation remained in the foreground and was impossible to ignore. Since the model conflicted with students' informal models, Toni, especially, was initially unable or unwilling to carry out even simple computations on a single polynomial.

Embedded in the interview profit problem are some inherently difficult concepts about modeling. One such concept is that once a model is established, even though it

is based on past behavior of variables, the process of generating the model is no longer relevant. Except for attending to what constitutes a suitable domain, the model must be accepted as valid without further reference to how the model was generated in the first place. This is a sophisticated concept, and while students had gathered data to create their own models and had routinely used models given in the CIA curriculum to make predictions about real situations, they had great difficulty in the profit problem perhaps because they could not ignore the model concept itself.

Situations and Assumptions

Besides informal models and the model concept, other situated problems caused difficulty. Sam, in particular, could not sort out what he was allowed to assume. In working the educational value/difficulty of work problem, Sam continuously invented aspects of or added to the situation to try to make it fit his graph choice. His insistence that one point could have two values for difficulty of work for one educational value led me to question him about the nature of how to interpret a single point on the graph. What Sam needed to do was to reduce the constraints in how he saw the situation to make a two-dimensional graph possible, but what he effectively did (without apparent awareness) was add to the situation so that a third dimension would actually have been necessary to graph his interpretation of the situation. Further, I, the interviewer, saw the graph as a relationship between two variables from which information about the two quantities in the situation could be described independent of information about a broader situational context, such as whether the speaker referred to situations for two different classes, one student's experience in a class over time, or some other aspect of a situation that was completely immaterial. I knew these were just "noise" and could be ignored because information was not given for them. As a teacher, I knew I needed only to think of each point as a different instance of the relationship and interpret it only in terms of high and low numbers (values) with labels (identifying quantities). Sam did not.

Once he had introduced time into the discussion as a reason for why two values for difficulty of work were associated with one educational value, the broader interpretation of the situational distinction between points on the graph became an issue. For a function to represent a situation, there must be something in the broader situational context that accounts for the existence of distinct points on the graph. Usually by the time mathematicians have generated a graph, they need no longer attend to these (causal or explanatory) situational distinctions between points on the graph to analyze the mathematical distinctions between the points. Yet students rightly recognize that these broader contexts are what make the particular graph representing a situation useful in applications, even though they are not relevant in these situated school problems. Because it is sensible to do so, students may attend to these broader situational distinctions until they learn what can be ignored. They need our help in learning what can be assumed and what can be ignored in particular problem contexts. Moschkovich et al. (1993) state this well:

> People familiar with a domain know what to ignore and what to focus on, newcomers do not. [sic] Whatever is obvious to us may not even be apparent to them. The large part of our task is to learn to see the domain as students do, and to help them develop the understandings that allow them to perceive and understand the objects in it in a manner consistent with our perceptions and understandings. (p.78)

It is also possible that in this example it was not just a matter of failing to sort out what assumptions were appropriate, but rather that Sam was exercising a problem solving strategy to enhance a situation to fit an attribute of the graph in order to progress successfully, as will be discussed below under student resources. He might not have attended much to the mathematics. But even in the event that he was simply pushing the definition of the situational context as far as he could because he was dealing with a problem considerably less well-defined than those ordinarily given in school, it is still unlikely that he was clear about what he could assume about the situation. Several times Edy also added to situations to attempt to justify some

problem solving action or to simplify a problem. Other times, she would ask directly what she was allowed to assume. This confusion over what can be assumed when dealing with descriptions of situations is familiar to me as I have observed it repeatedly in my own algebra classes at the university when students are given similar problems.

Besides focal students, other students in the class had difficulty moving beyond assumptions that seemed linked to their previous mathematics experience. During the first week of class students were asked to sketch graphs for the following situation :

> Suppose you were to choose, at random, 100 people and measure how heavy they are. You then ask them to perform in 3 sports: High Jumping, Weight Lifting and Darts. (Three axes are shown, one for each graph, with body weight on the horizontal)

At first students made dots in lines, and only after a lot of prodding by Mr. Bailey, did they consider the possibility that dots could be placed above each other, that is, more than one person could have the same weight. Even then, some groups made graphs shaped generally like parabolas with only a few dots below the arch. Apparently students had few guidelines from standard schooling to handle such problems. Though they were reluctant at the beginning to consider a broad range of possibilities for situations, later both Edy and Sam seemed to expand what assumptions they could make to the extent that they included invalid ones as well.

In summary, from a teacher's perspective, situated problems and modeling are advocated because they support reasoning and provide a meaningful context for the mathematics being learned. Yet situated problems caused students difficulty, particularly when their informal models conflicted with a function being modeled. Students relied on their everyday knowledge to make sense of the situation and the mathematics. When the two were not compatible, students were unable to cope with even familiar mathematical representations. In addition, when working with situations, the situational context that was no longer relevant in the mathematical

representation continued to "intrude" on their mathematical interpretation, in part, because students relied on the situation for support in their reasoning. Students were generally able to cope with situated problems when the concept of a model was transparent, but had difficulty when by the nature of the problem their attention was called to the concept of model as predictor for future situations. Finally, when working with situated problems, students had difficulty understanding which aspects of the situation could be assumed as valid and which could be ignored.

Other Factors Affecting Student Understanding of Functional Relationships

Student Resources

This section recaps what resources students bring to their problem solving from mathematics, and then explores those they bring from other sources. The section discusses knowledge of school mathematics problems and the role of the calculator. Finally, it examines interaction issues connected to what it means to be in school and those connected to communication.

From mathematics. As has been mentioned under sections related to student conceptions of functions and functional relationships in problems, students brought many resources to their problem solving. They relied on their everyday experiences with the mathematics to make sense of problems and were troubled when problems were at odds with this knowledge. Many strategies, such as attention to patterns of change in tables, seemed related fairly directly to the mathematical ideas and skills that were taught or experienced in class. Students found these ideas salient and used them to build on in their sense-making, particularly for linear and quadratic functions. Other student actions were indicative of mathematical knowledge they had learned prior to this class or outside of class. For example, Sam had an algorithm for graphing linear functions and for trying to find coordinates of a vertex of a parabola from the middle term of the polynomial. Neither was taught in this class.

Knowledge of school mathematics. Still other knowledge students used for problem solving was less directly related to mathematical concepts and more closely related to what it means to do mathematics in school. Sometimes students simply manipulated symbols or tried to remember a relevant procedure. On more than one occasion when either Toni or Edy was stuck, each mentioned that they were trying to remember a procedure. Recall Toni began the complex profit problem by removing parentheses in one function rule, that is, by simply manipulating the symbolic form. Other times students seemed to expect that a single, reasonable answer existed for a problem and that they should be able to compute it. Both Sam and Edy began the profit problem by calculating profit for a specific ticket price for each category (with and without a DJ) and made a judgment about the problem based on only one instance, rather than comparing multiple possibilities for ticket prices. Recall Sam's and Toni's ability to cope with a similar problem, profit as a function of ticket price, on the posttest, where the task was school-like. However, in interview 3's profit problem, at first when I asked them to find the ticket price that would maximize profit using a specific rule, the same task, they had great difficulty. The mathematics of that verbal problem, finding the maximum function value in a similar quadratic polynomial function rule with a similar situational context, was almost identical to the mathematics on a similar problem on the posttest. The problem setting of interview 3, however, was more complex and less school-like.

Role of technology. From a teacher perspective the graphics calculator was another resource for solving mathematical problems. It was used daily in class and students used it for homework. Students' use of this resource in interviews was uneven. Toni did not generally turn to the calculator as a tool to solve problems except for computing values that she could not do by hand, such as evaluating 5^0 or evaluating a polynomial. In the paint/perimeter problem (4A-6), she meticulously wrote down and added up 13 3's instead of using the calculator. Even though Toni

seemed visually oriented, she seldom used calculator graphs when I would have expected, such as finding the maximum function value in interview 3. In that problem she relied on computed function values on the calculator. Edy and Sam relied on discrete computations in that problem as well. None of the focal students but Bob used graphs to solve that problem. Sam and Bob, however, often used the calculator for explorations of a problem when not sure of a next step. Both explored the nature of the calculator itself, choosing the xy-line for graphing data points, even though it had not been taught.

Calculator use likely contributed to students' learning, particularly about shapes of graphs, which both Sam and Toni found to be an important organizing tool for function class. Calculators may also have contributed to some of their troubles, however. Neither Toni nor Sam saw clear distinctions between the form of symbolic rule to be entered for doing computations and that on the function definition screen (for graphing). Both Sam and Toni sometimes entered computations with x's rather than numbers, and more than once Sam entered function definitions with constants substituted for the x's. Students had an opportunity to graph data points on the calculator and were able to use guess and test methods to explore curvefitting of an appropriate function. From a teacher perspective these exercises should have been useful because they could give students a better sense of what an algebraic model is. Yet these activities might have contributed to students' failure to understand the distinction between exponential functions in symbolic form and a one-term quadratic as the position of variables (variables and parameters) were not a distinguishing characteristic. Calculators may have hindered as well as helped in understanding of curvefitting. More than once students traced along a linear graph that they had just created from two points (chosen from a line they had drawn to fit a scattergraph) to see if the two points were on the graph. Finally, that they principally used the

calculator for graphing may have contributed to their inability to graph function rules by hand.

These results could be explained in part by the short period of instruction (5 weeks) and students' inaccessibility to calculators. Students had to share calculators with two other sections during study hall and did not have access to calculators overnight. So they had little opportunity to explore or practice with the machines. In addition, except for curvefitting, most use of the calculator required beginning with the rule representation of functions and translating to other representational forms. Except for recognizing certain symbolic forms by function class, for students in this study, the most salient characteristics of functional relationships were dynamic--as in tables and graphs. They were not associated with rule representations, except for computing specific values. So it is possible the dominance of the rule form in calculator use might have made certain concepts less accessible. However, because of the nature of the interview problems where student use of the calculator was not the primary focus, I do not have enough data to substantiate these claims, but rather only to suggest possible ways the calculator was a useful or potentially less useful resource for these students.

Interaction

Interaction and school activities. Some student activities during problem solving were more related to what it means to be in school than to students' understanding of mathematics. Students were "model" students in all interviews. For example, they were always very cooperative with me when I had to adjust interview schedules. They showed up on time and seemed to make a serious effort at doing the problems. Sometimes their behavior was very "classroom-like." They asked for permission to write on the papers that stated the problem or asked if they could use the calculator on specific problems even though I began each interview by pointing out that there was paper, graph paper, the calculator, and a pen that they were

welcome to use. On only one problem did it state they were not to use the calculator, and once or twice I asked students to use the pen, rather than the pencils they brought with them, so their writing would show up on the videotape. Since I was taping and sometimes indicated verbal answers were sufficient, but other times were not, students often asked if they needed to record their answer on the problem sheet. These behaviors served us both by creating a good rapport between us. Their behaviors matched my expectations of what good students are and removed any need for us to spend a lot of energy to build rapport. I was grateful for their cooperation and their actions, through communication, helped us sort out what was expected from them and what would meet my needs as a researcher in much the same way that it would if I were their teacher.

Interaction and problem solving. Other student activity involved interactions with me and included "school behavior" that seemed more directly related to problem solving. Often students took cues for how to proceed from my statements during interviews. Edy, especially, took my questions to probe or confirm her responses as announcements that her previous answer was incorrect and on this basis, changed her answer. At least part of the time, all students took continued questioning on a problem as a sign that their response was to some degree insufficient. A researcher's probing is very similar to a questioning technique that is often used by classroom teachers for a very different purpose: to avoid telling a student their response is wrong, while at the same time giving the student a hint about how to proceed. At least part of the time each of these focal students responded to my interview questions as if they understood this school questioning style and as if it matched my intentions as an interviewer, even though it did not. This behavior sometimes made interviewing more difficult as I had to expend more energy avoiding giving clues than I had hoped.

In addition, students used their own questions to learn how to proceed. Sam began several interview problems with a long sequence of questions about the problem that sometimes seemed more like a search for clues to the solution process than for clarification of the problem statement. Other times he responded to any questions where I gave him an either-or choice by simply agreeing to the last alternative. On careful analysis after the interviews, I discovered that most of these times I used a questioning style such that when I gave two alternatives, the latter was the correct one. Sam might have realized this or simply have adopted a style of responding to the latter alternative. Recall also that in one interview Sam called one of my probing questions a hint that made solving the problem possible. Sam's strategies are reminiscent of those reported in a different context by Anderson, Brubaker, Alleman-Brooks, and Duffy (1983). They studied young children doing seatwork during reading instruction and contrasted poor responders (low attention and poor performance) with adequate responders (high attention and high performance). The low achievers often were aware that work was problematic or did not make sense to them, but the awareness did not lead to strategies for clarifying their understanding. Rather, students frequently devised strategies that involved short-term goals, such as to get an answer or to complete the assignment, not necessarily to understand the work. Similarly, Sam's responses often appeared as if he were simply trying to make progress in the interview rather than with the mathematics.

Mismatches in communication. On several occasions, there was a mismatch in communication between focal students and me. I would ask one question, and students answered another. Generally when this happened, I thought my question was about the mathematics, but students answered about the situational context, their motivation, or something else. For example, recall I asked Sam why he needed the 15 (that he had just added to -5x to create an accurate function rule $y = -5x + 15$ in problem 2A-2). I expected an answer about table entries satisfying one equation but

not another or some argument justifying the mathematical necessity of adding the 15. He said he did it to finish the problem as if it were the task purpose, rather than a mathematical reason why 15, in particular, was necessary. Sam's answer might have meant that 15 was necessary to make the equation correct, a legitimate answer, but one that was different from what I was asking. I have noted the same phenomenon in my own teaching and have had other teachers express their concern over how to phrase questions when attempting to communicate to students that you are looking for mathematical justification.

Several things could explain this phenomenon. First, perhaps I displayed ineptness as an interviewer by wording a question too poorly for students to understand its intent. Surely this happened now and then, as I am a novice researcher. Second, students might not know the answer to the justification question I was asking and answered in a way so as to deflect the question or to disguise this fact, possibly to not lose face. While in general I think this was not the case, Sam, in particular, sometimes seemed to use this strategy in his interactions. A third possibility is that students either do not interpret a question as being about justifying or might know it is about justifying an answer, but do not understand what that entails mathematically. They do not have facility with what David Pimm (1987), quoting Halliday, calls the "mathematical register." A register is a linguistic term that describes a set of meanings that is appropriate for a particular function of the language. In this case, mathematics register refers to those meanings that belong to the mathematical use of natural language and which the language must express if it is used for mathematical purposes. In this case, students do not know what would be the mathematical interpretation or expectation of the question, "why (Sam added a + 15)." Pimm claims that in order for students to communicate mathematically, they must develop language competence with the mathematical register. If teachers use the mathematical register and students do not, then accurate communication is not

possible. Students in this case may not have developed sufficient sense of the mathematics register to understand my questions, and likewise, I was surely not aware that I was even using a "language" that was "foreign" to them until after they have responded.

Summary

This section summarizes the findings in three broad areas: mismatches between student and teacher (researcher) perspectives; students' conceptions of functions and functional relationships, including issues of contextualization and resources students bring to problem solving; and issues surrounding problems situated in everyday contexts.

Mismatches: Teacher and Student Perspectives Revisited

Wherever possible throughout this chapter, findings were compared and contrasted from teacher and student perspectives. The major points were:

- the mathematical content of the curriculum as being about functions depends on teacher/student perspective
- existing frameworks for discussing functions that are limited to procedural (operational) or structural (object), while closely related to curricular goals, are insufficient to describe student activity with functions or functional relationships in problems
- lack of explicit awareness of functions should not be confused with absence of understanding of functional relationships in problems
- both the purpose and content of student/researcher questions and answers were a matter of student/teacher perspective
- whether tasks were about school activities or mathematics was sometimes a matter of student/teacher perspective
- understandings of what can be assumed and what can be ignored in problems about everyday situations differs by student/teacher perspective

That students and teacher take different perspectives across many aspects of teaching, learning and communication is perhaps the most important general finding of this study. Trying to understand students' perspectives, what are legitimate perspectives for knowledgeable, caring teachers, and what impact teachers'

perspectives might have on how they interpret students' work, their teaching, and mathematics forms the basis of discussion of implications in the next chapter.

Functions and Representations

Student thinking about functions could generally be characterized as being about functional relationships in problems with attention principally on the representations themselves and only implicitly about functions per se. Except during specific problem tasks, such as fitting situations to graphs and computing patterns of change in tables, students generally took a correspondence approach to functions with each of the representations rules, table, and graphs. They readily invoked both dynamic and static versions and easily moved between the two perspectives. Students were successful in a wide variety of tasks in many representations, but had difficulty with some individual tasks, such as graphing a rule of an exponential function by hand. Most activities involving both correspondence perspectives lent themselves by definition to attending principally to single instances of a functional relationship, that is, an instance level of wholeness. Yet within certain problems, students sometimes connected correspondence perspectives to viewing functions as entities, particularly rules and graphs when they constituted what was given or the problem goal.

Students also found the covariation approach to functions useful. In both tables and graphs, students found both patterns of change in data and rate of change to be salient concepts. They used these concepts to reason about problems involving functional relationships, as a problem solving strategy, to identify function classes, and to connect more than one representation of a function. How they used these ideas depended on the representation of the problem, the function class, and the task at hand. They used overall patterns of change in data in tables to connect tables and shape of graph for linear and quadratic functions and to explore the behavior of calculator graphs. They used a static version of equal rate of change to separate linear from other kinds of functions. The concept of unequal rate (patterns) of change

was problematic for students, probably because of how it was taught, with insufficient distinctions being made between its characteristics in quadratic and exponential functions. Sometimes students' understanding of pattern of change was robust and connected to rate of change, and other times it was not. Students had mixed success in connecting dynamic rate of change in a table with static rate of change in a correspondence (rule), in part because of conflict between everyday and mathematical use of language, possibly because of the presence of a negative multiplier, or because it required a translation between perspectives on functions. When students were successful fitting graphs to situations, they depended on being able to think principally in terms of variation of the dependent variable while experiencing the independent variable temporally. Language served an important role in these covariation perspectives to name static categories of change, to describe patterns of data, and to coordinate two representations when fitting situations to graphs. Finally, attending to functional relationships as entities is inherent in these covariation perspectives of functions.

Student thinking about functional relationships in problems across tasks, concepts, representation classes, and function classes was generally piecemeal and fragmented. Students exhibited some coherence within the function concept, thinking about functions as entities when taking a covariation perspective. They rarely took object perspectives, which is consistent with the curriculum. Students' characterization of the three function classes--linear, quadratic, and exponential--were contextualized and depended on the function class and representation. Students made connections across representations for linear and quadratic functions, but had few such connections for exponential functions. Those they made across representations for exponential functions were weak. Both Sam and Toni used shape of graph, pattern or rate of change in data in tables, a verbal description of change, and the form of the function rule as classifiers for function class membership.

Students brought many resources to problem solving: knowledge of mathematics learned in class or previously, everyday knowledge, knowledge of what constitutes a school mathematics problem or what kinds of activities working on school problems entail. To a limited extent the calculator was a resource. Students brought communication skills: how questioning in math classes is usually interpreted and how their questions could be used to gain information about how to proceed with the mathematics. There were mismatches in communication between the researcher and students perhaps because of different assumptions about students' understanding of the mathematical register.

Situated Problems

Situated problems had mixed value for students, both supporting and hindering reasoning. Sometimes students' reasoning was bound to the particular situational context so strongly that the mathematics might have been invisible to them, particularly when they used it to fluidly fit situations and graphs. Other times it confounded their problem solving when either they continued to reason in the situation, ignoring mathematical constraints, when they were unsure what could be assumed, or when the mathematics appeared to conflict with their informal models from everyday life. There may be an interaction between student difficulties with situated problems and the complexity of the task or the extent to which the model concept can be ignored.

Students were generally able to cope with situated problems when the concept of a model was transparent, but had difficulty when, by the nature of the problem, their attention was called to the concept of model as predictor for future situations. Finally, when working with situated problems, students had difficulty understanding which aspects of the situation could be assumed as valid and which could be ignored.

CHAPTER 6

IMPLICATIONS

This chapter has three general sections: situations, conceptions of functions, and mismatches in teacher and student perspectives. Whenever possible, the chapter explores how a teacher perspective and assumptions that are inherent in it can impact the teaching and learning of functions. It suggests future directions for research.

Situations

The use of situations is advocated by the current reform movement, and many curriculum projects at the middle school level or above are being developed around modeling, applications, or problems that are grounded in situations. (See, e.g., The Connected School Mathematics Project of Michigan State University or The Middle-school Mathematics through Applications Project of the Institute for Research on Learning.) In the research literature, however, the word situated is often used to refer to many kinds of contexts: the nature of knowing and learning (Brown, Collins, & Duguid, 1989; Lave, Murtaugh, & Rocha, 1984), the social aspects of learning (Saxe, 1991; Wertsch, 1985) or a kind of problem. Debate about a definition of what "situated" means is at a nascent stage.

In studies of representations of functions, however, "situations" defined as a kind of problem are often considered, as in my original research question, simply one representation in a category that includes verbal descriptions of situations, tables, symbols, and graphs (Janvier, 1987; Romberg, Fennema, & Carpenter, 1993). This study has shown that situated problems need to be thought of as a different type of representation from these others. The definition of situational context as a kind of problem (or representation) needs refinement.

One distinction that needs to be clarified within the definition of "situated" problems is the role situations play during the problem solution process in how and

when attention is focused on the situation itself. In this study attention to the situation varied with the problem task. In pre/posttest problems on profit or the sort problem involving situational contexts, students could momentarily suspend the situation, do the mathematics, and return to the situation to interpret the mathematics. This style is typical of application problems in traditional algebra classes. While success with such problems requires more interpretation than with those in naked context, situations here mainly provide quantities for the variable names and possibly some help in remembering generic problem types. For example, students may associate projectile contexts with quadratics and population problems with exponentials. Situations themselves are in the background. In the Profit Problem, however, for Toni especially, the situation remained conspicuous--in part because a successful solution to the problem required it. The situation was critical to formulating the problem solution, as two models based on the situation were being proposed, and choices had to be evaluated against the situation. Toni could not ignore the situation momentarily, treat the problem as a computation, as in a "school problem," and return to it later, as did some other students, because she was actively engaged in sense-making, and the model did not make sense to her. Other students, who treated the task first as a school problem and computed values, ultimately had difficulty because the interpretation required more than assigning quantities to variables. How activated the situation must remain throughout the problem solving task apparently must be taken into account when defining what constitutes a situated problem.

Another issue related to task is its complexity. The Profit Problem was a lengthy problem that was considerably more complex than class or homework tasks. It required students to attend continuously to both the problem statement and situational contexts. Perhaps this task was too complex for students so that dealing with the complexity of the problem task undermined their confidence to deal with the problem in ways that otherwise might have been accessible to them. On the other

hand, a task of this complexity might be necessary to bring to light issues about situated problems that could be missed in problems where students can succeed by attending principally to the naked context.

Being able to move fluidly between situation and the "naked" mathematical representation of the situation is certainly a goal of instruction for problems at many levels of complexity, including some like those in interview 3. But if teachers give no problems of sufficient complexity to raise issues about modeling or situations and students can massage situated problems into school-like problems, it is highly possible that assessment of student knowledge will lead to misinterpretation. For example, Sam's early interpretations of problems fitting a graph to a situation were both fluid and correct. Until he was given a problem in which a description of the graph did *not* match a description of the changing rate of the dependent variable and in which he had difficulty, there was nothing in his work to refute the assumption that he had a good understanding of interpreting global features of graphs. Further, both active sense-making and the disposition to reject aspects of problems that do not make sense are characteristics reform-minded math educators are trying to help students develop. Toni's tenaciously insisting that the model should make sense in the Profit Problem might have been missed altogether had she not been given a problem of this complexity. Further, without nontraditional problems such as fitting graphs to situations for nonstandard functions, where she displayed an exquisite fluidness in interpreting the graphs, Toni's ability and progress in a mathematics class might be judged only on her piecemeal activity with more school-like problems. Thus, this study suggests that problems of sufficient complexity involving situations might give students an opportunity to display their dispositions to sense-making and both successful and problematic areas of problem solving. It suggests complex problems should be included in curricula.

Another question related to task complexity was raised by the Profit Problem and its connection to school tasks: What depth of understanding about the concept of modeling is necessary to do this problem? In this problem, one could ignore the situation except as backdrop and begin computing. Yet to actually complete the problem one needed to compare models, to understand and accept not only constraints such as what might constitute a suitable domain, but also the fundamental idea of what a model is and why it is legitimate to use a model. Were these concepts too advanced for these students?

A fundamental aspect of a model is that predictions about unknown or future cases can be made from the model. In the Profit Problem Toni could not accept that one could compute profit unless it was information from the past. One possible reason was that she had an implicit function model that actually involved composition of functions: income as a function of ticket price and number of tickets sold, followed by profit as a function of income and cost. Including these computations in the model entails knowing the number of tickets *already* sold. Another possibility is that Toni believed profit could not be computed except after the concert, a belief that needs to be suspended in the model concept. Again, there is not enough data in this study to test her understanding of model, and curricula materials were not intended to explore directly this aspect of the model concept. In this class, students had gathered their own data or were given data and used curvefitting to create models. In addition, students were presented with some full-blown models to provide a situational context when certain function classes were introduced. In all cases, students were asked to make predictions from some representation of each model, but the validity of predicting did not become an issue. If students work only problems where they do not question the predictive aspect of the model concept will their attention be *through* the model concept, rather than *on* it? And if it is not on the concept of model, what can be said about their understanding of it?

In this study the idea of students' understanding of a model "accepted on faith" as a predictor in certain situations arose quite by accident. I had not intended to raise it and was surprised by the students' responses to interview 3, particularly Toni's. One reason the issue came up was because Toni could not accept profit directly as a function of ticket price; the problem's model of profit did not fit her informal model. Sam and Edy had similar difficulties. So another question arises about where attention lies: If problem tasks are set so the problem modeled conforms to students' informal models, then this aspect of the model concept might be invisible to students and not attended to at all. Thus, it might not ever be thought about. If problem tasks are set so that it becomes an issue, does that make the task problematic in other ways as it did in interview 3?

Modeling is a major part of the reform movement, and we need to know more about ways students think of models. The Profit Problem raised questions about students' understandings of models, their connections to situations, students' informal models, and problem tasks. These issues should be explored in additional research.

Situations confounded problem solving in other ways for Sam during the educational value/difficulty of work problem (2B-2). He relied on the situation to answer questions while ignoring mathematical connections of ordered pairs, axes, and coordinates of points on a graph of a function that he accessed successfully at other times. His difficulties arose in part because of his strategy to answer questions with information more readily available in the problem description than that potentially available in the mathematics and possibly because he consistently reasoned only about the variation of the dependent variable. His work, however, forced the discussion to turn to the meanings associated with a single point or pairs of points on the graph in the situational setting for the relationship. This setting was irrelevant for the relationship being graphed and should have been ignored. Whether the graph depicted how different people viewed the same lesson, that is, how good a fit was

difficulty of work for different people, whether it depicted different situations for one person over many days, or something else altogether, became an issue. While such points would be of interest if the problem were set in a larger context where such interpretations could be made, no such setting was provided here. Sam did not understand what should be ignored. That he could imagine *many* such contexts only made matters worse. Further, I, the researcher, in attempting to direct Sam's attention to the mathematical difficulties, at first misinterpreted how he was using the situation and contributed as well to this growth of confusing, "potential" contexts. Students need help in learning what to ignore when problems are situated. Teachers cannot assume that what they themselves see as irrelevant will be seen as such by students.

Because of the direction discussion took in the educational value/difficulty of work problem, the meaning of a single point on a graph or the distinction between pairs of points became relevant. Teachers may be prepared to explore and contrast these meanings in mathematical relationships in naked context, yet it is clear from Sam's work that teachers cannot assume that students can just as easily relinquish the situation. Further, teachers need to be prepared to distinguish interpretations in the situation for two distinct points on a graph in cases such as those provoked by Sam's interpretation. Students' lack of experience in sorting out this level of interpretation and teachers' potential lack of awareness of it as an issue may be at the heart of a discrepancy between students' ability to work well with mathematics class applications and their inability to interpret them in applications, say in physics classes, where understanding must by necessity account for another level of fundamental differences in interpretation of situation, that is, distinction between points in a broader situational context. Distinctions between points in the larger situational context often are not addressed at all in algebra classes and rarely in problems where a situation is used to

create an algebraic model, followed simply by an interpretation of some distinction between quantities (variables).

Teachers and mathematics educators need to look hard at assumptions they themselves hold about specific representations, in this case graphical, and what they have learned to ignore. This study supports Moschkovich et al.'s (1993) contention that people familiar with a domain know what to focus on and what to ignore while newcomers do not. Yet Moschkovich et al. were talking about problems in naked context. This study suggests that there are issues about what needs to be addressed and ignored when problems are situated that may not even have been thought about in this way. Mathematics educators need to examine the assumptions they themselves make about representations and examine carefully what they ignore. Further, they need to examine the assumptions they make about students' assumptions about these same issues.

Finally, this study implies that not only do mathematics educators need to address issues potentially absent from the curriculum and their own assumptions about the mathematics, but that they cannot continue to think of students' assumptions as being faulty and of their task as only being to help students gain the teachers' perspective. These students' puzzlements were rich sources of information about subtle nuances of both models and situations that must be taken into account in instruction. The study suggests mathematics educators need to attend to how student difficulties inform what their own assumptions about the mathematics we take-as-shared.

Although curricula need to take assumptions into account, there is a dilemma. Problems that are complex enough to raise the issues may confound it in other ways. In addition, one difficulty with using situated problems to teach about both the assumptions necessary to constrain the mathematics and about what to ignore is how to formulate problems that give this experience without confounding it. One ongoing

struggle in my own teaching is how to write problem instructions with enough specificity so that students understand instructions to include assumptions they make about a situation without being so specific that one "gives away" what students are to explore about the relationship. One problem that I regularly give to university students in a college algebra class is to graph the number of boxes that fit in a van as a function of box size (LFG, 1985). Students frequently introduce multiple vans, while varying box size and number of boxes within a given van. They have difficulty thinking that they need to assume a single van and to somehow fix a definition of box size. Sam's work implies that what can or cannot be assumed about a situation and how it relates to the mathematical constraints of the representation of the relationship at hand must be addressed directly in problems used for instruction, especially when the problems are about situations.

Finally, each of these issues about teaching with situations suggests another, which is similar to that above about the model concept. While the reform movement is grounded in providing students with experience with realistic problems so they can attach meaning to the mathematics, part of the power of mathematics is being able to abstract mathematical concepts away from specific situated contexts. The past has shown us that if teachers begin instruction with full-blown abstract forms of mathematics and applications, students often end up manipulating meaningless symbols on generic problems and confusing the mathematics with the symbols (Schoenfeld, 1986). On the other hand if teachers start with situations so that students can attach meaning to the symbols as they are introduced to concepts, students may continue to reason about the situation itself and not develop meaning for the symbols or the concepts the symbols are intended to represent. If students rely heavily on the situations to support reasoning, there is some danger that they may not be able to abstract away from the situations and work with the same functions in naked context or reason about functions themselves, not just functional relationships

in problems or only about specific attributes of their representations. This study supports Janvier's (1978; 1981) claim that there are inherent difficulties in teaching with situations and Bell and Janvier's (1981) call for including in instruction analyzing complex graphs without reference to situations.

This study followed only five weeks of instruction with textual material that was geared toward an informal, intuitive approach and which dealt primarily with situated data. There was less focus on data in naked context than that which would have come later in the remaining text, had students studied it. Both Toni and Sam had had two years of high school algebra and were older than the intended audience for the materials. However, their propensity to use the situations as support, and in Sam's case, to cling tenaciously to the situation--even when it wasn't sensible to do so--at least raises the question about whether they will ultimately be able to abstract away from the situation. Additional study is needed to follow individual students through curricula such as this one, which is rich in situations and where attention is focused on interpreting the various connections between representations, on to the same mathematical concepts in nonsituated contexts. Research is needed to determine in what ways and to what extent students are successful in building overarching concepts, such as those related to a single function class, that come through abstraction.

In reforming mathematics education, "mathematics for all" is considered important; thus, one question that must be addressed is at what age or level of schooling is it appropriate for us to expect students to deal with this kind of abstraction? Is it sufficient for most students to be able to recognize and apply patterns in similar, but different situations without abstracting out a generic pattern for all similar situations? Is this abstraction necessary for all students? The students in this study, however, were college-intending students about to enter their junior year, and thus, mathematical abstraction should, by necessity, be part of their goals.

Student Conceptions of Functions

Models of student understanding. In general, models for student understanding of functions have been defined principally by mathematical goals. The literature (Harel & Dubinsky, 1992; Kieran, 1992; Sfard, 1991) characterizes conceptions of functions as operational (processes) and structural (objects), as well as variations or expansions of these themes. These characterizations fit well those of mathematicians and curricular goals of algebra reform and, thus, should guide teachers in instruction. However, this study has shown them to be inadequate for describing students' activity with functions or functional relationships in problems and that models for students' understanding of functions need to be refined to more accurately reflect student activity.

One principal issue that arose was to distinguish student activity with functional relationships in problems from that in which students are aware of functions per se. Because of assumptions I, the researcher, was making during analysis, critical aspects of student activity with functions were at first invisible to me. The wording of the original research subquestions had built into them assumptions that students would be explicitly attending to functions. These assumptions are consistent with curricular goals within the mathematics community for understanding of functions. Yet for a long time during the analysis, I did not think the students attended to functions at all. The difficulty was not solely with student activity, but rather in how I thought about their activity. The mismatch involved confusing students' awareness of functions per se with presence or absence of understanding functional relationships in problems. Although I was actively looking for ways to characterize their sense-making and connect it to functions, I underestimated what they were doing because it was not *explicitly* about functions and because they seemed not to be aware of functions. There is some danger that others taking a teacher perspective who observe apparent disjointedness and incoherence in

students' thinking about functions will, as I did, let this perspective overshadow the active problem solving and sense-making activities that students were engaged in because it seems so far from curricular goals of a robust understanding of functions. If we are to understand how students come to develop rich understandings of functions, we need a vehicle to describe their emerging understandings and naive activity. This study suggests that there are several legitimate kinds of understandings about functions, some of which are only implied by student activity. It also suggests that an awareness-nonawareness dimension be included in describing students' activity with functions to recognize reasonable student activity while distinguishing robustness of understanding.

In addition to legitimating implicit function conceptions by including an awareness component, models reflecting student activity must include ways of describing instances, tentative wholes, and robust understandings of functions. Otherwise, assessing student understanding of levels of wholeness might result in a dichotomy--understanding or not understanding. This judgment could have ramifications for students' attitudes toward mathematics and for teachers' assessment of students' work. Further, teachers would lose a resource for informing instruction that builds on student conceptions if students are judged to have nothing to contribute.

In addition, many models of student understanding of functions distinguish only correspondence perspectives of functions. Yet the students in this study found covariation perspectives on functions particularly salient. Thus, models need to include such perspectives. Besides being salient to students, these perspectives illustrate that sometimes students' conceptions of functions--albeit naive--are on entities rather than on instances of functions, and thus consistent with one curricular goal.

Further, helping students to develop dynamic versions of functional relationships in problems could contribute to bridging the gap between operational and

structural perspectives on functions as has been suggested by Kieran (1992). In reviewing the teaching and learning of algebra, she points out that much of school algebra is presented as full-blown structures to be manipulated. Using the algebra research literature as her basis, Kieran calls for more attention to operational (dynamic) aspects of algebra and a search for ways to provide a move from operational to structural (static/relational/object) so students can become fluid with both the operational and structural conceptions of functions.

The specific curriculum that the students in this study engaged in gave students an opportunity to fit situations to graphs and made heavy use of tables, which lend themselves to patterns of change. These kinds of activities no doubt contributed to students taking a covariation-dynamic perspective. Yet, research is needed to analyze this particular curriculum for tasks that are inherently related to each perspective to determine how much experience students had with each. Such results could perhaps indicate whether students were simply producing what they were taught.

In addition, there was limited evidence that students had difficulty with tasks that required cutting across perspectives. For example, when students used rate of change (beyond using a memorized algorithm) to find a rule for a table representation of a function, they had difficulty moving between covariation-dynamic and correspondence-static perspectives. So research is needed to analyze student activity in tasks that cut across or move (translate) between perspectives, particularly covariational approaches and correspondence approaches. Curricula need to be analyzed along these same lines to determine whether the findings of this study were an artifact of this specific curriculum or are more widespread among students.

Role of technology: Graphing by hand. The most puzzling finding of the study was the students' inability to graph functions by hand. One potential reason for their difficulties was that the functions were exponential, but another possible reason was associated with the graphics calculator. The calculator was used in this study for

almost all graphing, for table construction, and both guess and test and graphical solutions to algebraic problems. Each of these uses was principally as a support tool that extended or replaced traditional representations and representation tasks. Only the calculator's use to create scattergraphs was nontraditional. There is limited evidence, however, that students relied too heavily on the graphing component of the calculator and thus failed to participate in the full process of creating graphs by hand, which required an additional translation step of creating ordered pairs. Although there is not sufficient evidence for this conclusion, it nevertheless raises questions about assumptions teachers can make about which tasks will seem obvious to students when technology is introduced as a tool in the classroom. Research is needed to explore effects on learning and instruction when technology replaces activities that were previously done only by hand.

Mismatch between Teacher and Student Perspectives

Throughout this study mismatches between teacher and student perspectives were an issue. When perspectives differ, conflicting assumptions can operate. As stated above, assumptions were a major concern in dealing with situated problems. They also were an issue in defining whether or not work with functional relationships in problems constituted work with functions. This study raised a number of other points related to assumptions. Each suggests further actions.

Curricular tasks need to be examined for assumptions built into them to determine whether they build on conceptions that students find salient and intuitive. For example, Chazan (1993) has taken this position and suggests building a conception of equality based on functions ($f(x) = g(x)$), rather than one built on equations. He claims the former allows students to maintain connections and (potential) coherence among variation and quantities they find salient, but which are lost almost immediately after an equation solution process begins with a quantity added to each side. Students in this study found dynamic dimensions of both

correspondence and covariation approaches salient; yet, little attention is paid in traditional curricula to covariational approaches. In addition, students found patterns of change and rates of change as two dynamically changing sets salient features, but almost never displayed any awareness that they considered "rate of change" as a single quantity even when their descriptions included "per unit changes" in the dependent variable. When rate of change is discussed in the classroom, it is usually defined as a quotient of differences and then simply used as a single quantity, such as slope of a line. Students are asked to move immediately to a static form of rate, quickly leaving behind their dynamic versions.

Students in this study had difficulty with exponential functions. They did not recall the function class name, had no organizers for the class, such as shape of graph, pattern of change in the table, or position of the variable in the symbolic form. Students apparently did not develop an intuitive "feel" for exponential functions, perhaps because dynamic variation, a feature students find salient, in an exponential function is multiplicative, but rate of change is generally defined as additive. Confrey and Smith (1994) suggest that one solution to helping students understand exponential functions is to move away from the narrow definition of rate of change as additive rate of change as defined in calculus and create two new kinds of rates of change for exponential functions. The first is a multiplicative rate of change, that which is found by calculating the ratios between succeeding y-values. In that case, an exponential function can be seen as a multiplicative function with a constant (multiplicative) rate of change. The second rate of change is "a proportional new to old" rate of change, which entails viewing the amount being added on each time as a function itself multiplied by a constant. They give an example that in a table with constant multiplier 9, the amount being added each time between each entry is eight times as large as the previous entry. (If the first three table entries were 2, 18, 162, then the amount added on would be 16 and 144, respectively, and $144 = 8 \cdot 18$.) This

computation is exactly analogous to the role of rate of interest, r, in the formula for compounding interest annually[1], $P_f = P_i(1 + r)^t$. Their suggestions stem from observations in their research with students building these rate concepts while working with problems about exponential functions. To accommodate these definitions, Confrey and Smith (1994, p. 141) redefine rate of change as a unit per unit comparison, thus allowing flexibility in units in both numerator and denominator. They go on to explain that what constitutes a unit here is constructed by the student. Not adhering strictly to additive rates of changes allows students to construct rates of change they find intuitive. In exponential functions, for example, a rate of multiplicative unit per additive unit results in a constant function. For example, in the function $y = 8\ (5)^x$, as each year is *added* there is a constant *multiplier* of 5. An additive unit per additive unit change, however, results in varying rate of change. For example, as x increases from 1 to 2, y increases 160 (from 40 to 200), but as x increases from 2 to 3, y increases 800 (from 200 to 1000).

Students in this study had studied patterns of change only in terms of additive rates of change. Like Confrey and Smith's students, they found dynamic-covariational approaches salient. Perhaps their difficulties with exponential functions could be attributed in part to the mismatch between a narrow, additive rate of change, which constituted their entire "formal" experience of rate of change in this class, and the more intuitive rates of change proposed by Confrey and Smith as being salient in students' construction of exponential functions.

Math educators need to examine tasks in curriculum materials for matches or mismatches between student perspectives and teacher perspectives to determine if

---

[1]If interest is 0.08%, and P dollars in the account were invested for a year, the amount *added on* each year, 0.08 P, is 0.08 times the amount in the account at the beginning of the year, P, assuming no money is withdrawn. At the end of any one year, the amount in the account would be (P + 0.08 P) and at the end of the next year the amount would be: (P + 0.08 P) + (P + 0.08 P)(0.08). The amount added on is always a multiple of 0.08.

taken-as-shared definitions of "correct" mathematics are too confining and thus fail to enhance or build on students' conceptions. Further, they need to examine their own perspectives as teachers of functions (and representations of them) to see in what ways they could adjust instruction to build on what students find intuitive or salient about functions and their representations. In addition, math educators need to attend to how student difficulties and facilities inform their own assumptions regarding student assumptions about what tasks students are engaged in.

Finally, in this study some activities during problem solving were more related to what it means to be in school than to students' understanding of mathematics. Sam used discourse as a problem solving strategy. At times mismatches occurred in communication between students and me, particularly when I would ask a mathematical question and students would answer a nonmathematical one. These findings imply that teachers must take into account the possibility that students have different purposes than they themselves have and that they cannot assume that students understand their comments and questions as being about the topic they have in mind.

Conclusion

I began this study with certain assumptions from my understanding of mathematics, my teaching and the research literature in mathematics education that student thinking could be characterized in terms of how contextualized or decontextualized student thinking was in the study of elementary mathematical functions. I had an open mind to try to hear student voice as I believed that students had rich, but naive, ideas of the concepts that I understand to be shared by the mathematical community.

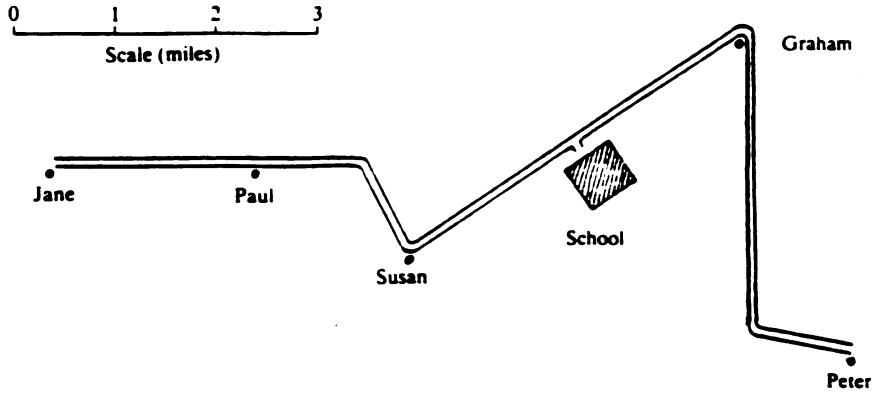
Nevertheless, I was not prepared for how pervasive were my assumptions about their activities, and how difficult it was to turn off the "mathematics master's voice." In addition, while I had witnessed my own students' difficulty justifying mathematics with situational arguments, I was surprised by the tenacity of the

situational context and the informal models in these students' thinking. Further, since students found dynamic perspectives so salient, but much of algebra teaching has involved working with static representations, mathematics educators must recognize that their own view of mathematics may be limited. Thus, math educators must take responsibility for some of students' difficulty in understanding mathematics because their way of thinking about mathematics may not build on what students know and find salient. Finally, since students and teachers are sometimes engaged with different content and for different purposes, teachers must be ever vigilant about what assumptions they are making in their interactions with students.

APPENDICES

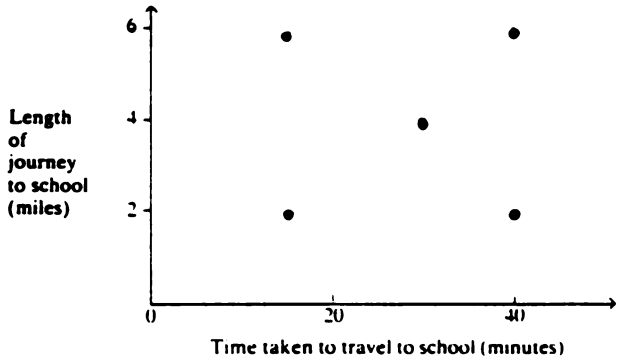
APPENDIX A

GOING TO SCHOOL



Jane, Graham, Susan, Paul and Peter all travel to school along the same country road every morning. Peter goes in his dad's car. Jane cycles and Susan walks. The other two children vary how they travel from day to day. The map above shows where each person lives.

The following graph describes each pupil's journey to school last Monday.



i) Label each point on the graph with the name of the person it represents.

ii) How did Paul and Graham travel to school on Monday? _____

_____

iii) Describe how you arrived at your answer to part (ii) _____

_____

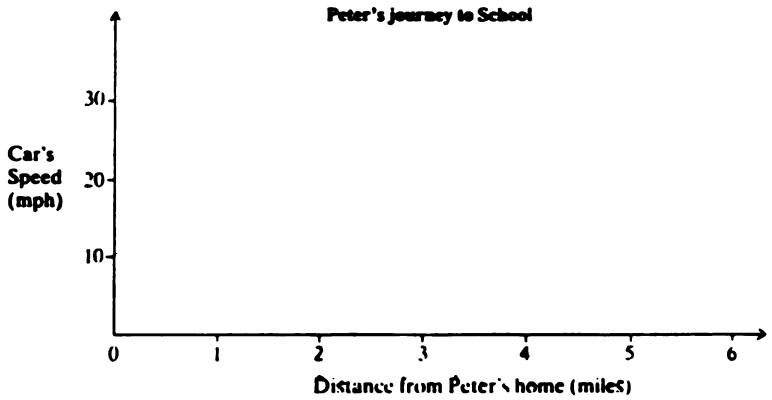
_____

(LF6, p.28) (continued)

Problem 1-2

GOING TO SCHOOL (continued)

iv) Peter's father is able to drive at 30 mph on the straight sections of the road, but he has to slow down for the corners. *Sketch* a graph on the axes below to show how the car's speed varies along the route.



(LFG, p.29)

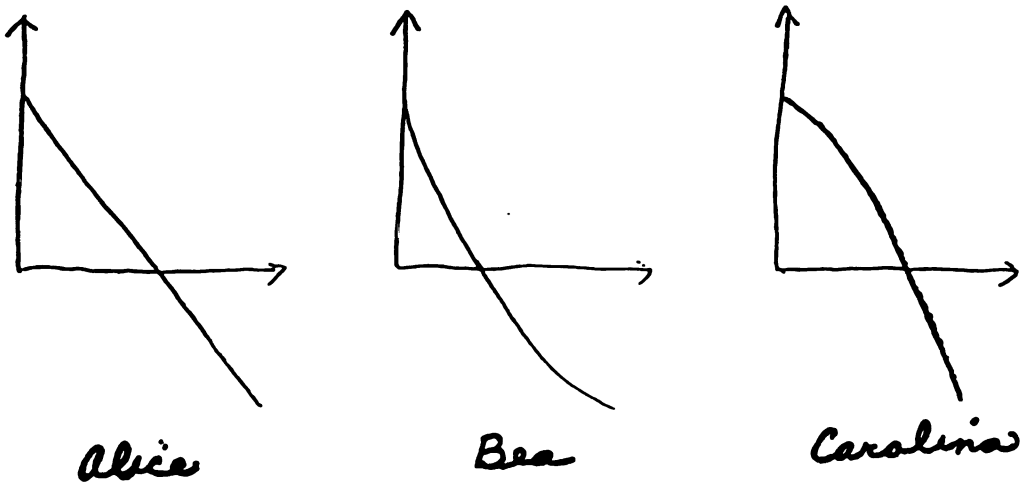
Interview 2 - Part A

Problem 2A-1

Below is a table for a function rule. After seeing only the table, Alice, Bea, and Carolina each sketched a graph. Their graphs are labeled a, b, and c, and are show below.

input	output
0	4
1	2
2	0
5	-2
10	-4

Discuss why you agree or disagree with each choice. Explain your reasoning. If you do not agree with any of the choices, please make you own graph and explain in what way it is a better choice.



Problem 2A-2

For the table below, what is the function rule? What is the output for input 15? input 30?

input	output
1	10
2	5
3	0
5	-10
10	-35

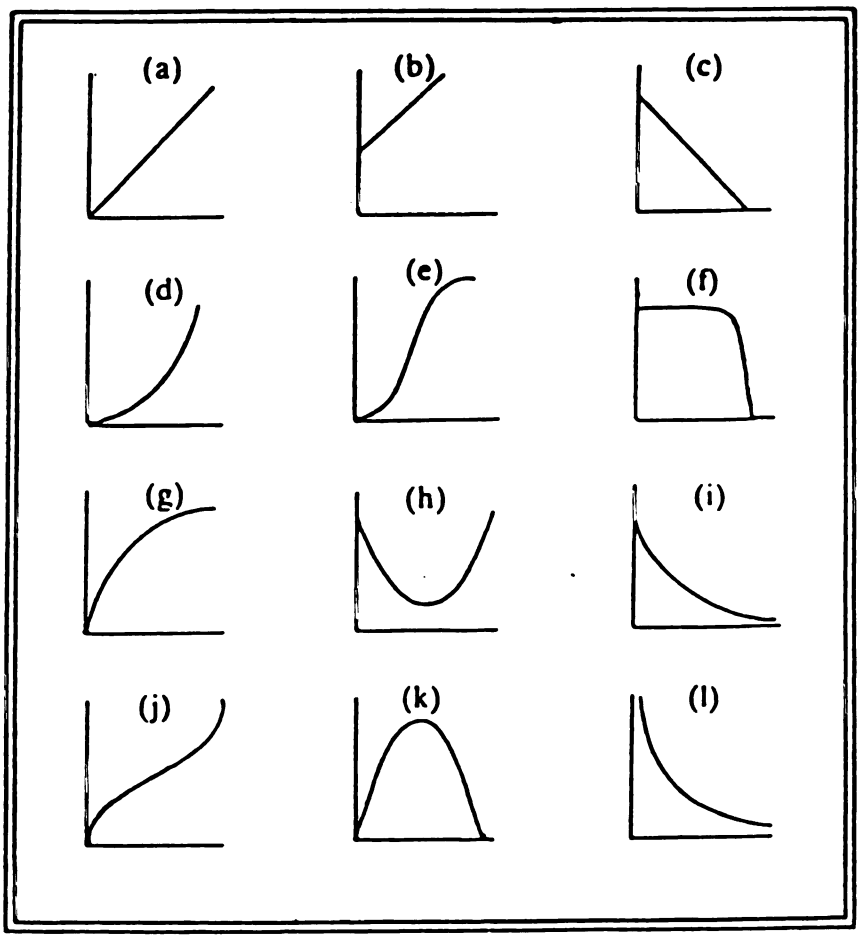
Interview 2 - Part B

Problem 2B-1

When I started to learn the guitar, I initially made very rapid progress. But I have found that the better you get, the more difficult it is to improve still further. (proficiency/amount of practice.)

Problem 2B-2

If schoolwork is too easy, you don't learn anything from doing it. On the other hand, if it is so difficult that you cannot understand it, again you don't learn. That is why it is so important to pitch work at the right level of difficulty. (educational value/difficulty of work)



(LF6)

Interview 3

The Profit Problem

Carl has been assigned to come up with the price to charge for tickets for a concert the senior class is using for a fund-raiser. From previous years, Carl has learned that if they hire a celebrity emcee, the function relating ticket price and profit was

$$P(t) = -100t^2 + 700t - 200.$$

Carl reasoned that if they didn't hire the emcee, they could save his $200 fee. His friend Beth said he should reconsider the problem because he would need to lower the ticket price by at least 3 dollars to get anyone to come if no emcee was hired. She told him he could consider this new situation by using the function

$$P(t) = -100(t - 3)^2 + 700(t - 3)$$

Was she correct? Which plan should he follow?

Does the size of the auditorium make any difference?

Interview 4 - Part A (modified from Hart, 1981)

Problem 4A-1: What can you say about c if $c + d < 10$? (written)
What can you say if d is 5? (given orally)

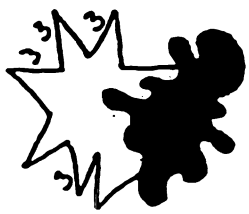
Problem 4A-2: $x + y + z = x + p + z$ is always, sometimes, or never true? Explain.

Problem 4A-3: If $a + 6 = 25$, what can you say about what is $a + 6 + c$?

Problem 4A-4: Which is larger, $2n$ or $n + 2$?

Problem 4A-5: If $e + f = 50$, what can you say about $e + f + g$?

Problem 4A-6: Sam drew the figure below while babysitting for the neighbor. It has n sides; each has length 3. Unfortunately, the baby spilled paint on the drawing just as Sam was calculating the perimeter of the figure. What is the perimeter?

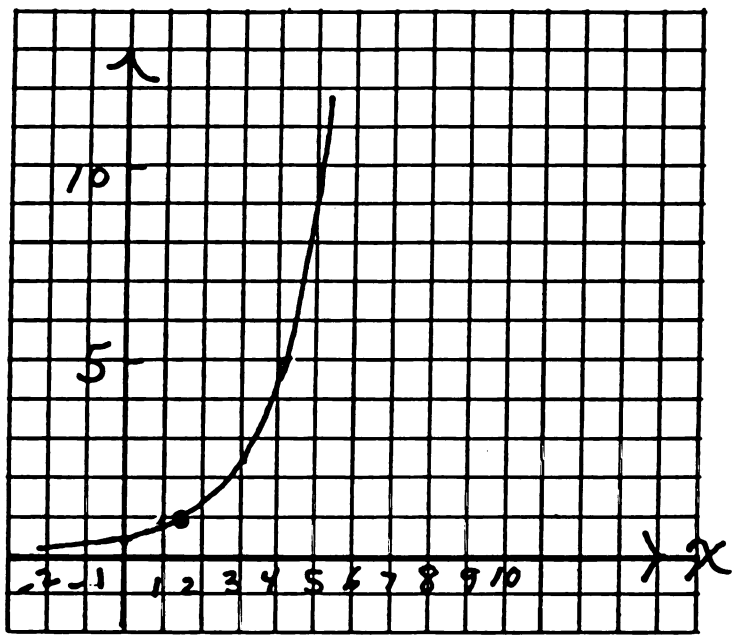


Problem 4A-7: Blue pencils cost 5 cents each and red pencils cost 6 cents each. I buy some blue pencils and some red pencils, which cost me 90 cents. If b is the number of blue pencils bought and if r is the number of red pencils bought, what can you write down about b and r?

Interview 4 - Part B

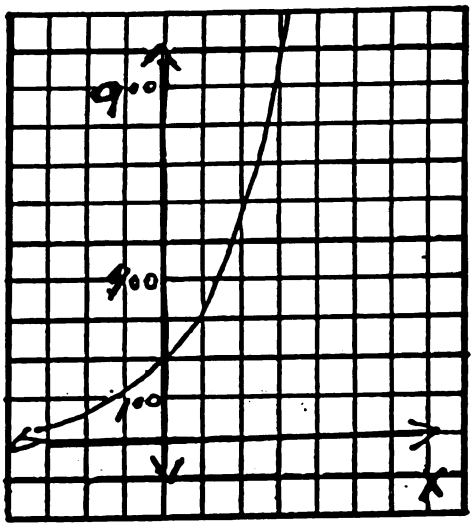
Problem 4B-1

Below is a graph of a function $f(x) = .3(2)^x$. The point on the graph that represents the function values for input b is labeled with a dot. Place a dot on the graph that represents the value of the function for input b + 3.



Problem 4B-2

Below is a graph for an exponential function of the form $g(x) = a\ 1.5^x$. What is a?



Problem 4B-3

$f(x) = 10 \cdot 3^x$ Without using the calculator, give the function value if

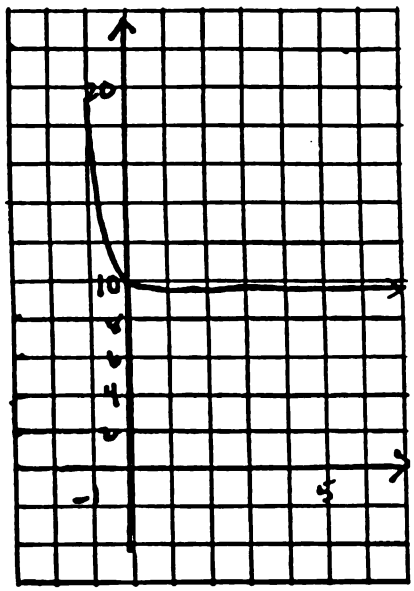
i) x is b?

ii) x is 4.5?

Problem 4B-4

Without using the calculator determine which function best fits the graph below?

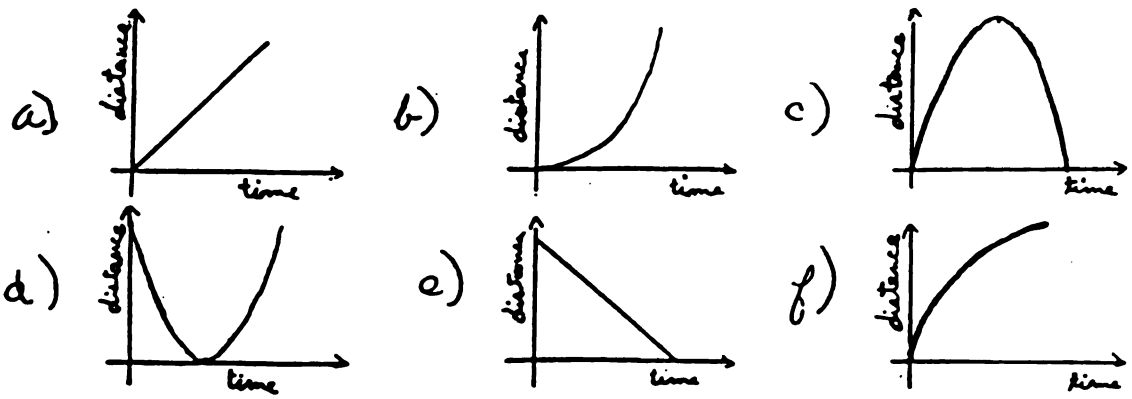
a. $f(x) = 10x$
b. $g(x) = 1 + x$
c. $h(x) = 10 - x$
d. $j(x) = 10^x$
e. $k(x) = 9 + (.1)^x$



Interview 5

Problem 5-1: Tim and Donna Problem (taken from Even, 1989)

Tim and Donna live 1 km from their school. Usually they walk together to school. Yesterday they both left their houses 10 minutes before school started. Tim started to walk but Donna was afraid to be late and started to run. After a while Tim realized that although he tried to walk faster and faster, he had to run if he did not want to be late, and started to run. At the same time, Donna became tired and had to walk instead of run. They both reached school exactly on time. Which of the following graphs is Tim's ____ and which one is Donna's ____?



As the number of workers doubles, the amount of time to do the work is cut in half.

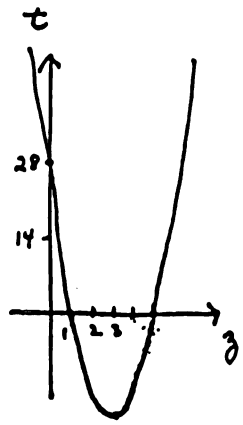
A

The profit a company earns, in dollars, is related to the price it charges, in dollars, for its main product by the rule

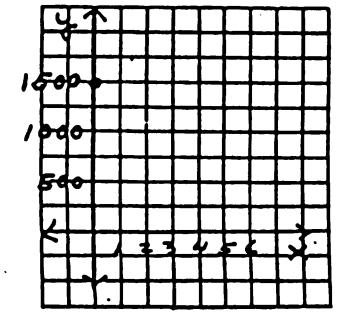
$$\text{Profit} = -300p^2 + 3600p - 1500$$

What profit is made when the company charges $.50?

B



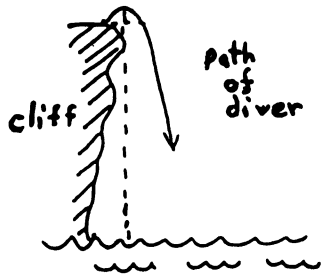
D



E

Interview 5: Sort Cards A, B, D, and E

A cliff diver jumps off a cliff which is 28 meters above the water. If the diver's position is x meters away from the edge of the cliff when he dives and y meters above the water, under certain conditions, $y = -x^2 + 2x + 28$ describes the path of his dive. Does the diver every reach a height of 29 meters?



M

A video rental store charges 3 dollars to rent the first videotape and 1 dollar for each additional tape. The total charge, c, is a function of the number, n, of videotapes rented.

P

A vending company operating a soft drink machine in a school finds the number of bottles sold, s, is related to the price, p, by the function

$$s(p) = -300\,p + 270.$$

The company's daily income (revenue) from the sale of soft drinks in school is related by daily

revenue = number sold * price

Find a function, R, relating revenue to the price.

Q

A table for a function rule

input	output
4	52
5	50
7	46
8	44
10	40
20	20
30	0
35	-10

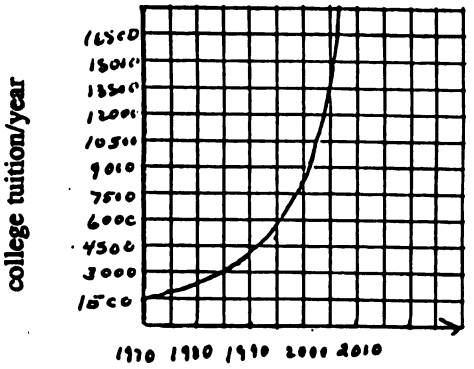
An input of 0 gives what output?

R

Interview 5: Sort Cards M, P, Q, and R

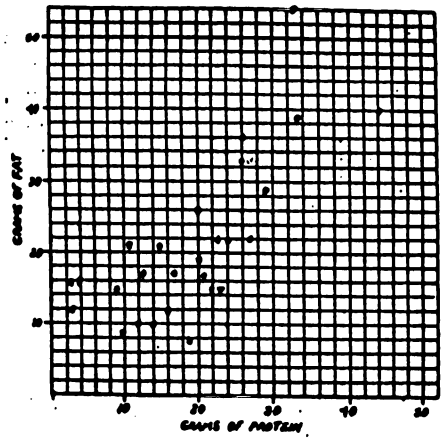
For each 5 units x increases, y increases 10.

S



What is the predicted college tuition for the year 2000?

T



Individual servings at some fast food restaurants

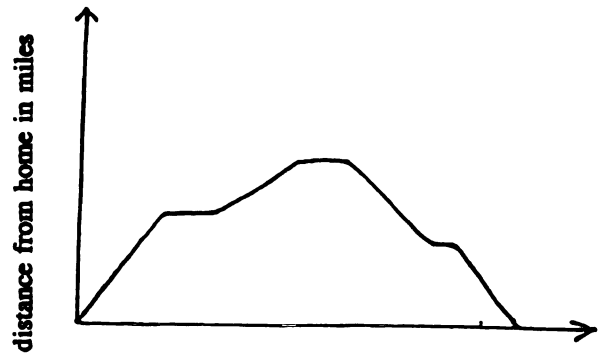
W

A table for a function rule

input	output
5	3
4	1
3	-1
2	-3
1	-1
0	1
1	3

Y

Interview 5: Sort Cards S, T, W, and Y



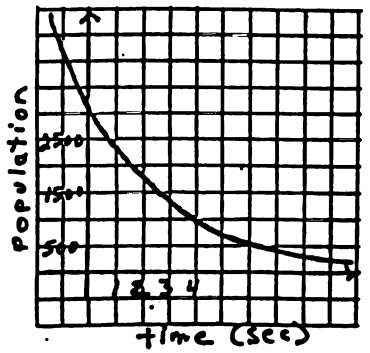
G

The amount of radioactive cobalt in a substance decays (decreases) over time. The table below gives the amount of radioactive cobalt left after several years in storage as measured by a scientist. What kind of function best fits the data?

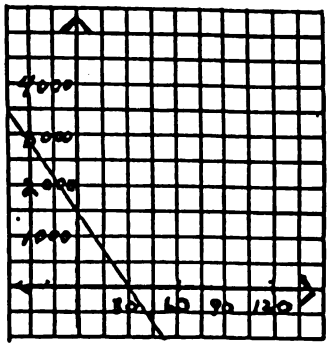
Year in storage	Radioactive cobalt in grams
0	100
2	76
4	56
6	42
8	31
10	24
12	19
14	14
16	11

H

The graph below relates an insect population in a specific area and the time passed since an insecticide was sprayed. At what time will the insect population be 1000?



J



K

Interview 5: Sort Cards G, H, J, and K

A table for a function rule

input	output
-5	50
-3	18
-1	2
1	2
3	18
5	50
7	98

Give this function's graph.

APPENDIX B

Pretest with Cover Sheet
(Posttest Identical except for Cover)

Name:_____ No:_____ Section:_____

SURVEY OF MATHEMATICS BACKGROUND

The purpose of this survey is to learn something about your mathematics background. This information is important because you come from many different schools where classes may be organized quite differently from each other.

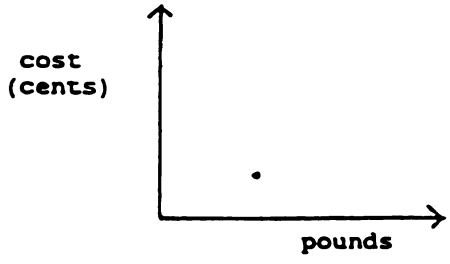
Some of the problems on the survey may be on topics that you have not yet had, and some may appear in forms that are not familiar. We ask that you try to work all of the problems even if some may be new to you. It is fine to guess what you think something means.

Do not worry about not knowing the "correct" answer. This is not a test of your ability, but a survey of what kind of math experiences you have had.
Show us as much of your work as possible, especially in problems you think you cannot finish.

PLEASE FILL IN YOUR NAME above. (Ignore No. and Section).

Mathematics Survey

1. The point shown in this graph represents 2 pounds of sugar costing $.40. MARK and LABEL two points, A, representing 5 pounds costing $.90 and, B, representing 10 pounds costing $1.50.

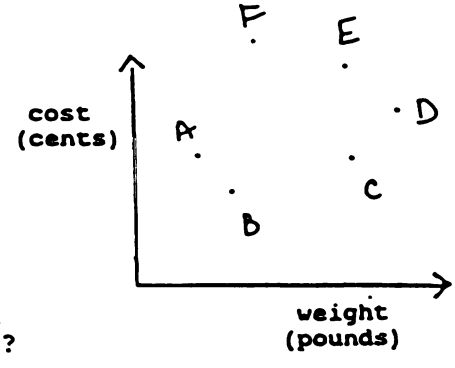


2. a. Multiply out: $(x + 7)(2x + 1)$ _____

   b. If $x = -10$, what is $3x + 4$? _____

   c. Write 2^{-2} as a decimal. _____

3. Each point in this graph represents a bag of fruit.



   a. Which point (or points) represents the heaviest bag(s)? _____

   b. Which point (or points) represents bag(s) that cost least? _____

   c. Which points represent bags of the same weight? _____

   d. Which points represent the same costs? _____

   e. Which of F or C would give the best value for the money?_____ How can you tell?

   f. Which of B or C would give the best value for the money? _____ How can you tell?

   g. Which two packets would give the same value for the money?_____ How can you tell?

4. $y = ax + 7$ If a is a negative number, the graph of y in Figure 1 looks like I, II, or III below? _____

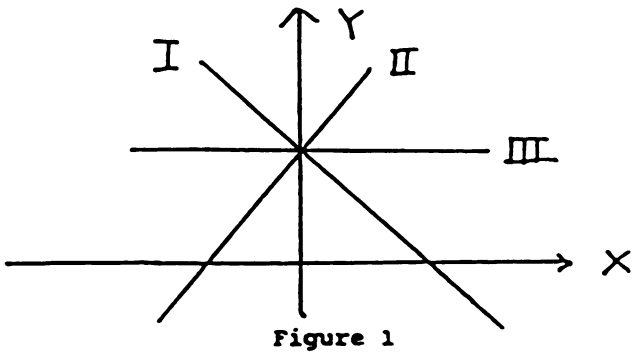


Figure 1

5. Find values for the following:

a. $y = 2(x - 1)(x + 4)$ if $x = 3$ $y =$ _____

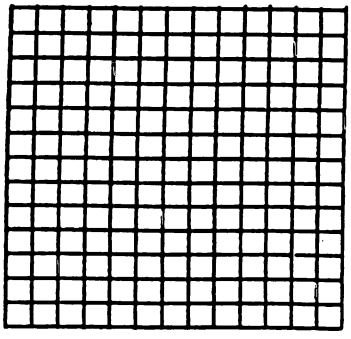
b. $y = x^{0.5}$ if $x = 9$ $y =$ _____

c. $y = f(x) = -2x^2 + 5$ if $x = 2$ $y =$ _____

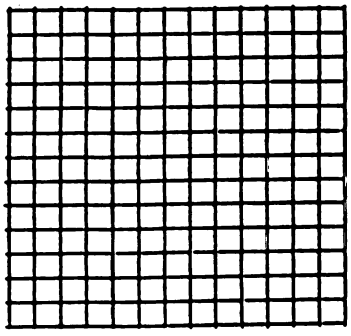
d. $f(a)$ if $f(x) = 4x - 16$ $f(a) =$ _____

6. Graph the following:

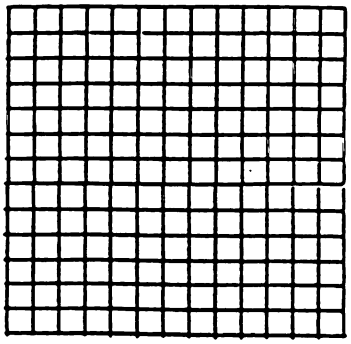
a. $y = 3x + 4$



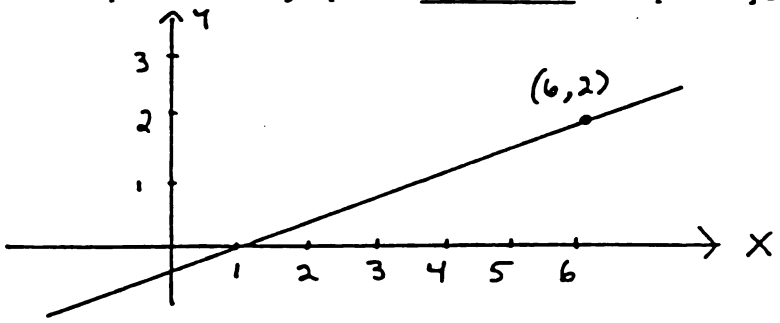
b. $y = x^2 - 2x + 1$



c. $y = 3^x$



7.a. The slope of this graph is _____. Explain your answer.



b. An equation for the graph is_____. Explain how you found it.

8. A graph could be made for the relations given below. In each case, what would be the y-intercept? Explain how you found it.

a. $y = -3x^2 - 4x + 5$ The y-intercept is _____. I found it by _____.

b. $y = 2(x - 7)(x + 1)$ The y-intercept is _____. I found it by _____.

c.

x	y
-2	6
0.5	-2
0	1
1.5	0

The y-intercept is _____. I found it by _____.

d.

x	y
-1	1
1	3
3	5
4	6

The y-intercept is _____. I found it by _____.

9. The profit from a rock concert depends on the ticket price. Experience has shown that the graph in Figure 2 gives profit in terms of the ticket price. In each case explain how you got your answer.

   a. What profit can be expected when tickets cost $8? _____

   b. What is the maximum profit that can be expected? _____

   c. What ticket price is predicted to give the maximum profit? _____

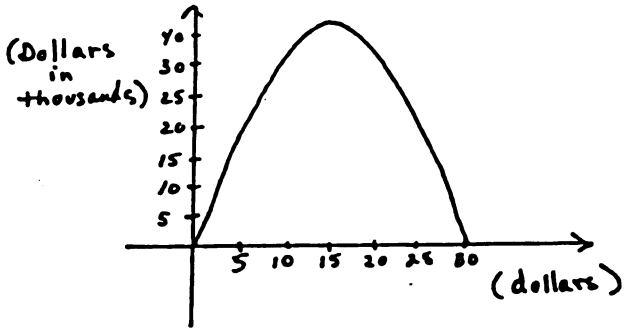
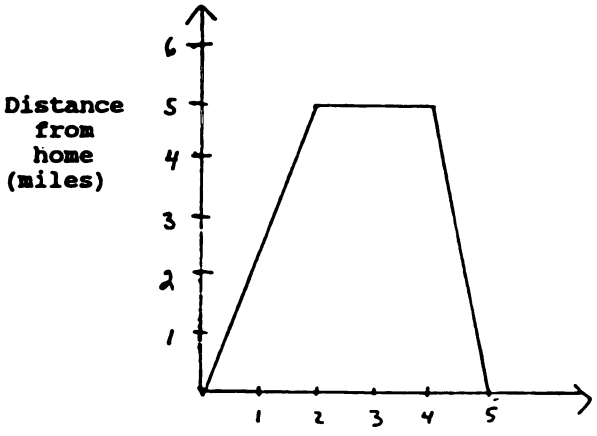
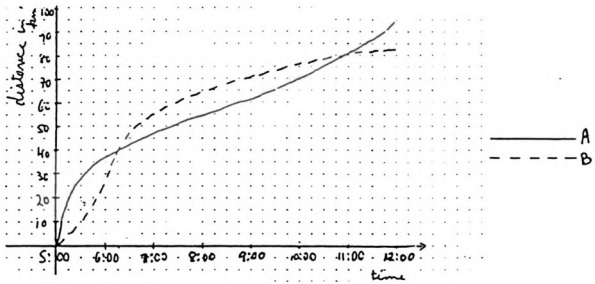


Figure 2

10. This graph shows Jane's progress as she goes for a country walk. Describe as carefully as you can what might have happened. Be as specific as you can.



11. This graph describes the distances A and B traveled.



a. For what times had A traveled further than B? Explain.

b. At what time did A travel the fastest? Explain.

c. Who traveled faster between

i. 5:30 and 6:00?

ii. 8:00 and 9:00?

d. Who travel more kilometers between

i. 6:30 and 8:00?

ii. 10:00 and 11:00?

12. **Some Types of Equations**

I. $y = ax^2 + bx + c$

II. $y = ax + b$

III. $y = C \cdot (a)^x$, $C > 0$

IV. Any equation not of Type I, II, or III.

If a,b,c and C are real numbers, then Types I, II, and III can each represent a whole set of equations. For example,

$y = 2x^2 + 3x - 7$ and $y = -x^2 - 1.4 + 5$ are both of Type I,

$y = -5x + 18$ and $y = 2x$ are of Type II,

$y = 30 \cdot (5)^x$ and $y = (0.4)^x$ are of Type III, and

$y = \frac{1}{5}x^6 + 13x^4 - 12$ is of Type IV.

In each part below, a table relates x and y for an equation. Determine which Type of equation (I, II, III, or IV) could give the table. Only one type is correct. Check its box, but be sure to explain for EACH type why you chose or rejected it.

a.

x	y
-2	2
-1	2.5
0	3
1	3.5
2	4
3	4.5
4	5

[ ] Type I. I chose/did not choose this answer because _____

_____

_____

[ ] Type II. I chose/did not choose this answer because _____

_____

_____

[ ] Type III. I chose/did not choose this answer because _____

_____

_____

[ ] Type IV. I chose/did not choose Type IV because _____

_____

_____

12. (continued)

Equation Types

I. $y = ax^2 + bx + c$
II. $y = ax + b$
III. $y = C \cdot (a)^x, \quad C > 0$
IV. Any equation not of Type I, II, or III.

b.

x	y
-2	-1
-1	-2
0	-1
1	1
2	7
3	14
4	23

[ ] Type I. I chose/did not choose this answer because_____

[ ] Type II. I chose/did not choose this answer because _____

[ ] Type III. I chose/did not choose this answer because _____

[ ] Type IV. I chose/did not choose Type IV because_____

13. The profit from sales for a school play depends on the ticket price. Experience has shown that when an emcee is hired, profit is related to the ticket price by $y = P(t) = -50t^2 + 800t - 1950$. In each case explain how you got your answer.

a. What is the ticket price that gives the maximum profit?

b. What is the maximum profit?

c. What profit can be expected if tickets sell for $10?

14. Which of the following equations could be the equation for the graph in Figure 3. Only one answer is correct. Designate it by checking the box, and explain for EACH equation why you chose or rejected it.

a. [ ] $y = x^2 - 4x + 5$
I chose/did not choose this answer because_____

b. [ ] $y = -x^2 + 4x - 10$
I chose/did not choose this answer because_____

c. [ ] $y = -x^2 + 4x$
I chose/did not choose this answer because_____

d. [ ] $y = -x^2 + 4x - 3$
I chose/did not choose this answer because_____

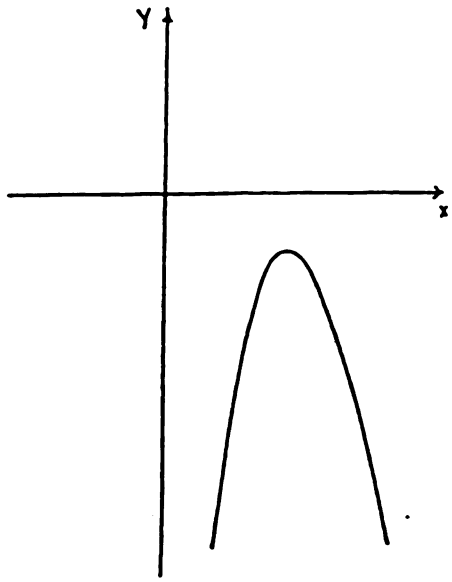


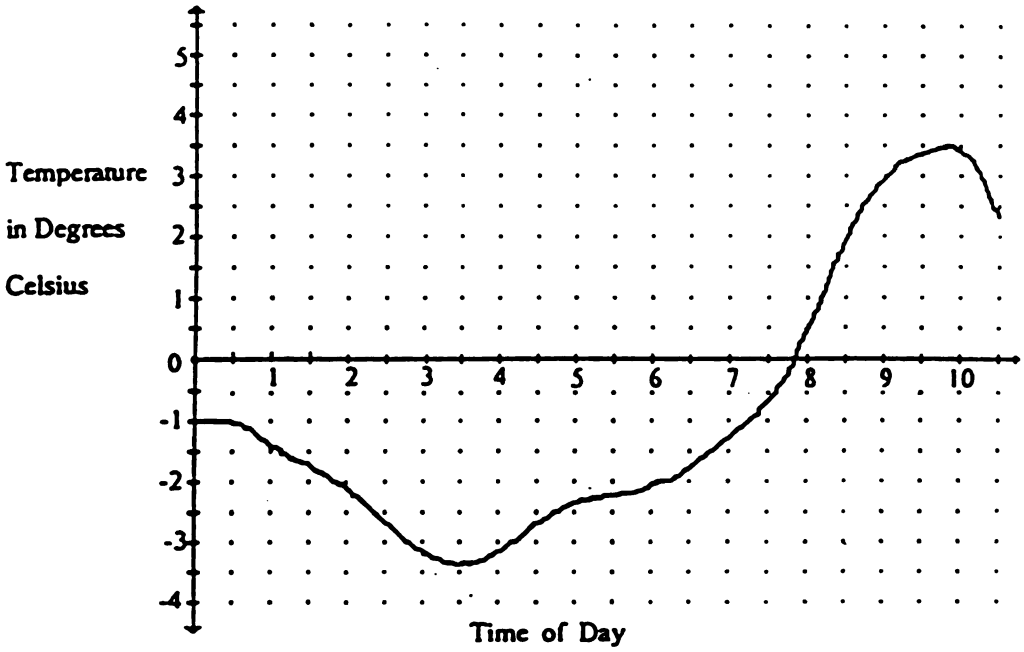
Figure 3

e. [ ] None of the above equations can be the equation of the graph in Figure 3, because_____

APPENDIX C

HOMEWORK

1. **Situation:** The diagram below shows the temperature in Frostburg, Maryland between midnight and 10:30 a.m. on a typical December day. The temperature is a function of the time of day.



For each of the following questions, give coordinates of point(s) that give the answer. Then write your answer in a sentence. Estimate coordinates where necessary.

(a) What was the temperature at 4 a.m.?

(b) What was the temperature at 10:30 a.m.?

(c) When was the temperature -2°C ?

(d) When was the temperature 3°C ?

(e) What was the maximum temperature and when did it occur?

(f) What was the minimum temperature and when did it occur?

(g) When was the temperature rising?

(h) When was the temperature falling?

6. In what two-year interval did Dawn grow the fastest? By how much did she grow and how does this compare with the increase in average height for females between the same ages?

_____

_____

HOMEWORK

1. **Situation:** The table below gives the average heights of American *males* at specific ages.

Age in years	Height in centimeters
0 (Birth)	52
2	87
4	103
6	116
8	127
10	138
12	150
14	163
16	174
18	177

(a) At what age is the average height of males 150 cm?

(b) Use function notation to write the sentence,
"The average height of males at age 14 is 163 cm."
Hint: M(___) = ____.

(c) M(2) = _____.

(d) Write in words the meaning of M(8) = 127.

(e) In what two-year period, shown by the table for males, does the average height increase the most? What is that increase?

(f) How does the pattern of change in average height for males compare to that for females?

2. **Situation:** The Talent Show advisor found the following record of profits from previous Talent Shows.

Year	Profit (in dollars)
1980	-221
1981	-155
1982	-18
1983	233
1984	847
1985	1838
1986	1750
1987	2133

(Note: Negative profit means the show lost money.)

(a) In what year did the Talent Show make the greatest profit?

(b) The amount of profit is a function of the year. What does P(1981) = -155 tell about Talent Show profit?

(c) In 1985 the Talent Show had a profit of $1838. Express that idea using function notation.

(d) In which year did Talent Show profit increase the most from the previous year? Write the calculations needed to find each year-to-year change as follows:

$$P(1981) - P(1980) = (-155) - (-221)$$
$$= 66$$

Then use your calculator to find the results.

(e) Between which consecutive years did the profit decrease?

(f) Describe the overall trend in Talent Show profits shown in the table and make a prediction about profit that could be expected from the 1988 show.

- Check your sketch graphs by plotting *a few* points accurately on graph paper. Share this work out with your neighbour so that it doesn't take too long.

- Do just *one* of the two investigations shown below:

Draw an accurate graph to show how the effect of Triazolam wears off.

After how many hours has the amount of drug in the blood halved?

How does this "Half life" depend on the size of the initial dose?

Write down and explain your findings.

Investigate the effect of taking a 4μg dose of Methohexitone every hour.

Draw an accurate graph and write about its implications.

B3 LOOKING AT EXPONENTIAL FUNCTIONS

Hypnotic Drugs

Sometimes, doctors prescribe 'hypnotic drugs' (e.g. sleeping pills) to patients who, either through physical pain or emotional tension, find that they cannot sleep. (Others are used as mild sedatives or for anaesthetics during operations). There are many different kinds of drugs which can be prescribed. One important requirement is that the effect of the drug should wear off by the following morning, otherwise the patient will find himself drowsy all through the next day. This could be dangerous if, for example, he has to drive to work! Of course, for someone confined to a hospital bed this wouldn't matter so much.

Imagine that a doctor prescribed a drug called Triazolam. (Halcion®).

After taking some pills, the drug eventually reaches a level* of 4μg/*l* in the blood plasma.

How quickly will the drug wear off?

Look at the table shown below:

Drug name (and Brand name)	Approximate formula
Triazolam (Halcion®)	$y = A \times (0.84)^x$
Nitrazepam (Mogadon®)	$y = A \times (0.97)^x$
Pentobombitone (Sonitan®)	$y = A \times (1.15)^x$
Methohexitone (Brietal®)	$y = A \times (0.5)^x$

KEY A = size of the initial dose in the blood
y = amount of drug in the blood
x = time in hours after the drug reaches the blood.

For Triazolam, the formula is $y = A \times (0.84)^x$

In our problem the initial dose is 4 μg/*l*, so this becomes

$$y = 4 \times (0.84)^x$$

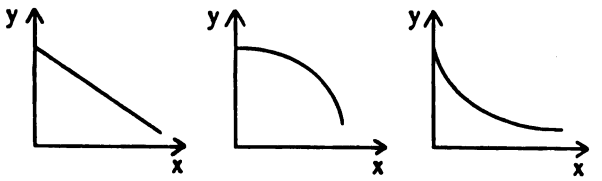
---

* Please note that in this worksheet, doses and blood concentrations are not the same as those used in clinical practice, and the formulae may vary considerably owing to physiological differences between patients.

* Continue the table below, using a calculator, to show how the drug wears off during the first 10 hours.
You do not need to plot a graph.

Time (hours)	Amount of drug left in the blood
x	y
0	4
1	3.36 (= 4 × 0.84)
2	2.82 (= 3.36 × 0.84)
.	.
.	.
.	.

* Which of the following graphs best describes your data?
Explain how you can tell *without plotting*



* *On the same pair of axes*, sketch four graphs to compare how a 4μg dose* of each of the drugs will wear off.
(*Guess* the graphs—*do not* draw them accurately)

* Only three of the drugs are real. The other was intended as a joke! Which is it? Explain how you can tell.
What would happen if you took this drug?

COMPUTER EXPLORATION II

Now use a computer tables program to answer questions in the following situations. For each question you may wish to look at more than one table in order to "zoom in" on an answer that seems accurate enough.

Situation 4.2. The profit for a play is the difference between revenue (total amount of money received) and expenses (total amount of money paid out). Profit generally depends on the ticket sales for the play. Suppose profit is related to ticket sales t by the rule

$$P(t) = -0.02t^2 + 9t - 200$$

Use this rule and tables of values (t, P(t)) for the following problems:

1. First describe the overall pattern in the relation between ticket sales and profit. Then record the information you entered for the table that best showed you the pattern.

   Pattern: _____

   _____

   _____

   Table data: Smallest t = _____
   Largest t = _____
   Step size = _____

2. For which numbers of ticket sales does the play make a positive profit and for which does it lose money?

   Profit: _____

   Loss: _____

3. What number of ticket sales gives the greatest profit?

4. What numbers of ticket sales give at least $500 profit?

5. What is P(0), and how do you explain the fact that this is a negative number?

Situation 4.3. The daily profit function for the Budget Box Office theater is $P(a) = 2.50a - 300$, where a is the attendance.

Use this rule to answer the following questions.

1. Choose starting and ending values and step size for a table that offers a good overall view of this profit function.

   Smallest a = _____
   Largest a = _____
   Step size = _____

2. Describe the overall pattern and any especially important values in the table you produced in (1).

   Pattern: _____

   _____

   _____

   Key values: _____

   _____

   _____

3. For which attendance numbers does the theater make a (positive) profit? For which does it lose money? For which does it break even?

   Profit: _____

   Loss: _____

   Break even: _____

4. What attendance seems likely to give maximum profit?

5. How is the profit function for the Budget Box Office different from that of the play? Does one of them, as a model for its situation, seem to be more reasonable than the other? If so, which one? Why?

Chapter 4

QUADRATIC FUNCTIONS

The distance required to stop a moving car depends on the speed, mass, and braking system of the car and the condition of the road on which the car is travelling. The path of a basketball shot or an artillery shell depends on the initial velocity and elevation of the shot and the force of gravity. The profit of a business depends on the prices, sales, and costs of its products.

In each of those situations, the important variables are often related by a rule of the form

$$f(x) = ax^2 + bx + c \qquad (a \neq 0),$$

where the numbers a, b, and c are constants determined by specific conditions in the situation that the function models. Functions like this are called **quadratic functions.** By the end of this chapter you should be good at recognizing situations in which quadratic functions occur and at using various methods for answering questions about those situations.

As you study any family of functions, it is important to find properties common to all functions in that family. For instance, in Chapter 3, you studied functions with rules of the form $f(x) = mx + b$. You learned that a function of that type always has a graph with constant slope--the graph is a straight line. From tables of values for such linear functions you observed that output values always change at constant rates as input values change.

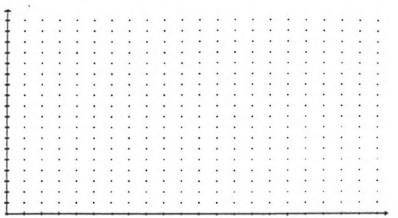
For **linear functions** the values of the constants m and b have special importance in determining the patterns of graphs and tables of (input, output) values. The graph always **crosses the** $f(x)$ **axis at** $(0,b)$. The **constant slope is** m. For each increase of 1 in the value of x, the value of $f(x)$ increases by m.

The addition of the **quadratic or square term** (x^2) in a quadratic function rule gives quite different patterns in the graphs and tables of values. To find properties of quadratic functions it will be helpful to compare them to properties of functions you have studied earlier.

As you encounter a new quadratic function rule, it is generally a good idea to start by generating a graph and a table of value pairs $(x, f(x))$. Use the graphs and tables to answer specific questions of interest, but look also for connections between patterns in those graphs and tables and the constants a, b, and c that determine the function rule.

Next plot the data from your table to make a graph for d(s) on the grid provided next to the table. Make sure you mark your scales on the grid. Check your graph by using a computer program for graphing.

s	d(s)



Speed in kilometers per hour

Use the table and/or graph to answer the following questions.

(a) Describe the trend in stopping distance values as the speed increases.

(b) What properties of the relation between speed and stopping distance can be learned best from the table? And what properties from the graph?

(c) How does the graph of this function seem to differ from those of linear functions you have studied?

(d) How does the table of values seem to differ from those of linear functions you have studied?

(e) Use the rule to calculate d(-50) and d(3000). What would these results mean in this situation?

With the aid of computer table, graph, or symbol manipulation programs or your hand calculator, answer the following questions about the stopping distance function:

$$d(s) = .005s^2 + .14s$$

Remember the input variable, s, is the speed of the vehicle in kilometers per hour and the output, d(s), is stopping distance in meters.

For each question:

(i) Use function notation to write an algebraic equation or inequality matching the given question.

(ii) Decide on the best tool or tools to solve that algebra problem.

(iii) Solve the problem. Then write a sentence stating your answer to the original question. Give your answer to the nearest meter (for stopping distance) or to the nearest 5 km/hr (for speed).

1. What is the stopping distance for a car traveling 50 kilometers per hour?

Equation or inequality: _____

Tool or tools used: _____

Answer: _____

_____

2. What speed requires exactly 35 meters stopping distance?

Equation or inequality: _____

Tool or tools used: _____

Answer: _____

_____

3. What is the stopping distance for a speed of 150 kilometers per hour?

Equation or inequality: _____

Tool or tools used: _____

Answer: _____

_____

4. What is the stopping distance for a speed of 0 kilometers per hour?

Equation or inequality: _____

Tool or tools used: _____

Answer: _____

_____

5. What speed(s) permit a car to stop in less than 90 meters?

Equation or inequality: _____

Tool or tools used: _____

Answer: _____

_____

6. A car stopped in 125 meters. How fast was it going?

Equation or inequality: _____

Tool or tools used: _____

Answer: _____

_____

7. What are the stopping distances for speeds greater than 100 kilometers per hour?

Equation or inequality: _____

Tool or tools used: _____

Answer: _____

_____

APPENDIX D

Upward Bound Regional Science and Mathematics Center, Summer 1991
Syllabus and Schedule of Lessons for Mathematics

Materials used come primarily from:

Computer Intensive Algebra (CIA) published by University of Maryland, College Park, MD (1991).

Languages of Functions and Graphs (LFG), published by Shell Centre for Mathematical Education and the Joint Matriculation Board, Nottingham, England (1985).

Limited materials are taken from *Algebra*, developed by the University of Chicago School Mathematics Project (UCSMP), (1990). The remaining are teacher-generated.

Key to Abbreviations:

All numbers of the form 2-30 represent page numbers from *Computer Intensive Algebra* (CIA);

LFG A3 p. 82 stands for worksheet A3 from *Language of Functions and Graphs*, page 82 of the duplicating masters.

Other notations such as B-1 refer to page numbers for teacher-generated materials unless specifically labeled otherwise. Materials from UCSMP are labeled as such.

Schedule of Lessons

Day	Topic	Materials	Homework
1	Math modeling, Variables related by tables; Introduction to calculator	1-2 to 1-8;	1-9 to 1-13
2	Interpreting points; Scattergraphs; Constructing graphs from tables (calc); Scale/range	LFG A1 1,2,4, p. 64 1-16 to 1-19	Finish LFG, A1, 4; 1-21 to 1-22
3	Interpreting global Features of graphs; Scattergraphs	LFG A2, p.74 1-23	Finish LFG, A2, 5; 1-24 to 1-31
4	Matching graphs with situations; Variables related by Rules	LFG A3, p. 82 1-33 to 1-35 1-55	Finish LFG, A3, 2-3; 1-39 to 1-42
5	Using the calculator to solve problems; Introduction to modeling data	2-15 to 2-18; 1-61 to 1-63	2-19 to 2-20
6	Variables related by tables; Programming the calc. to produce a table; Modeling data: Ball drop experiment	Program Handout; 1-66 to 1-69	2-24 to 2-26
7	Curve-fitting: linear functions; Creating and exploring trends in tables for functions; Linear translations	2-24 to 2-26; B-1, B-2	2-27 to 2-30 B-3, B-4
8	Using tables to solve problems; Exploring algebraic forms of functions: Relating tables, symbols, and graphs	Homework and teacher generated examples	B-5 to B-6
9	Fitting lines to scattergraphs; Creating graphs from data points on calc.	UCSMP p. 41, 43	UCSMP p.42

10	Finding exponential functions in situations; The nature of exponential functions	LFG B3, p. 120	5-4 to 5-5
11	Exponential functions in situations: Compound interest; Recursive and closed form definitions; Negative exponents	Discussion of homework	5-6 to 5-10
12	Properties of linear and exponential functions and their graphs; Curve-fitting: exponential	C-1 to C-3 (modified from CIA)	Finish classwork
13	Exploring quadratic functions in situations; Maximizing a function	LFG, B2 p. 116 4-2 to 4-5	4-6 to 4-9
14	Properties of quadratic functions; Effects of parameters a and c in $y = ax^2 + bx = c$ on graph	Teacher generated examples to explore together	4-56,58,59, 61; Experiment with parameters for cubic and quartic functions
15	Exploring properties of quadratic functions; Effects of parameter b	Handout for group explore.; 4-15 to 4-19	4-20 to 4-25
16	Properties of quadratics; maxima, minima, symmetry, vertex	4-27 to 4-30	Finish classwork
17	Post-Assessment; Modeling Data: Skateboard experiment	6-45 to 6-46	Fit curve of your choice to the data
18	Curve-fitting for skateboard problem: Finding the best fit from several function choices	6-47 to 6-49	4-44 to 4-46
19	Further work on curve-fitting quadratics	4-44 to 4-46 (modified)	Last day of class, no homework
20	Science Olympiad		

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