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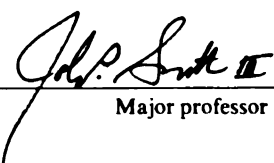
HIGH SCHOOL AND COLLEGE STUDENTS'
KNOWLEDGE OF RATE OF CHANGE

presented by

Garnet S. Hauger

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HIGH SCHOOL AND COLLEGE STUDENTS' KNOWLEDGE OF RATE OF CHANGE

VOLUME I

By

Garnet S. Hauger

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ABSTRACT

HIGH SCHOOL AND COLLEGE STUDENTS' KNOWLEDGE OF RATE OF CHANGE

By

Garnet S. Hauger

Rate of change is important from historical, academic, and everyday points of view. But many students struggle with learning rate of change and especially the derivative in calculus. One way to think about rate of change considers three perspectives and three types of reasoning. This dissertation attempted to discover what knowledge of rate of change precalculus, calculus, and postcalculus students have, how this knowledge differs among these three groups, and if the three perspectives and types of reasoning provide an interpretive frame for understanding students' knowledge of rate of change.

In this study knowledge was taken as strategies and learning as strategy construction. Hence I was interested in what strategies precalculus, calculus, and postcalculus students used to solve tasks involving rate of change, how strategy use differed among these three groups of students, and if these strategies seemed consistent with the three perspectives and types of reasoning.

Twelve precalculus, fifteen calculus, and ten postcalculus students were given the same set of tasks in individual audio and video taped interviews. Students' written work and verbal explanations were used to identify strategies they used to solve the problems, classify strategies in terms of the three perspectives and types of reasoning, and compare and contrast strategies used by the three groups.

Nineteen different strategies were used. Many strategies, like "delta d" and "visual slope", were used by each of the three groups and most students used more than one strategy on each task. Precalculus students' use of many strategies showed they have knowledge that can be used to solve tasks involving rate of change and that may support their learning of the derivative in calculus. Calculus and postcalculus students' use of these same strategies suggests they are based on knowledge that is shared by all three groups of students and may therefore serve an important role in coming to know rate of change.

Some strategies were used primarily by some groups but not others. Calculus and postcalculus students used slope of tangent line but no precalculus student did. Although all three groups used average rate of change to address rate of change at point, precalculus students used long and medium sized intervals before the point, calculus students used short intervals before the point, and postcalculus students used intervals straddling the point. Over the course of the tasks, precalculus students came to use shorter intervals before the point, calculus students used the shortest available intervals before the point and some straddle intervals, and postcalculus students relied more and more on straddle intervals.

Interval based strategies were dominant in addressing rate of change at a point, usually via average rate of change. Some numerical and algebraic strategies were helpful in exploring differences between additive and multiplicative comparison of changes over intervals.

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DEDICATION

To my parents whose respect and thirst for knowledge fueled my own desire to know and whose constant encouragement energized my educational efforts.

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Many people supported me in this effort. First, my husband, children, and grandchildren alternately provided a quiet and restful place and noisy and rambunctious breaks. I shall never be able to completely repay this debt. Second, my colleagues at Spring Arbor College and Christian Fellowship Center released me from some obligations so I would have more time for this work. They also supported me with words of encouragement and, at times, sympathy. I will try to do the same for others there and elsewhere.

My advisor, Jack Smith, deserves an award for his many hours of reading, commenting, and hand holding. He also paused in his busy schedule to laugh and cry with me. He may never know how many times I nearly gave up and what his words of solace and assurance meant to me. They kept me going when I was sure I could not.

Finally, there is a large group of women who have motivated me in ways that only women can. I wish to especially thank Susan, Sandy, Denice, Marjorie, Carol, Lauren, Jan, and Miriam. There were many others, but these women were always there to laugh, cry, and celebrate with me. There were, of course, many men who also encouraged me, especially among the men of the Natural Science Division of Spring Arbor College. But the gift of encouragement from women held me up during particularly tough times and for that I shall always be grateful.

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Chapter 1

INTRODUCTION TO THE PROBLEM

Rate of change is important for at least three main reasons. First, the quantitative study of rate of change and more generally the development of the calculus represents a major intellectual achievement of humankind. Second, the concept of rate of change has tremendous usefulness in mathematics and the physical, natural, and social sciences. Third, we live in a world characterized by change, especially in terms of motion and growth.

But rate of change is difficult to learn. It took humankind over 2000 years to fully work out a quantitative analysis of rate of change. It should therefore not be surprising when people struggle in their attempts to master the concept in a few months or years. Also although we live in a world where motion and growth are prevalent, very few of us engage in the kind of explicit study of change and rate of change that would allow us to easily come to the same insights seen by Galileo and others.

In addition, the formal concept of rate of change, especially the derivative, is a schooled concept, not one easily accessible outside of school. Although rate of change is included in some form in most middle and high school mathematics programs, a more complete and general study of rate of change is usually reserved for calculus. It takes a lot of schooling in mathematics to get to a study of calculus. Hence even for

those who want to, a fairly large investment in time and energy is needed to initiate a serious study of rate of change.

This opening chapter discusses these issues in more detail. The first section presents a more detailed discussion of the importance of rate of change. The second section provides evidence that learning rate of change is difficult. The third section outlines one characterization of what it means for someone to understand rate of change. This chapter closes with a set of questions this dissertation attempted to address.

Rate of change is important

Today mathematicians and scientists have well developed conceptual tools for thinking about change. But that was not always so. Change, and especially motion, was of interest to scholars before the time of Christ. However, a quantitative study of change did not come until the Middle Ages, and precise quantitative characterizations of average and instantaneous rate of change are more recent. The creation of calculus, including a modern study of change, is a significant intellectual achievement in and of its own right and represents one of humankind's greatest efforts to understand the world. The long history of rate of change involves the work of many mathematicians and scientists to understand important classes of phenomena and invent concepts to make sense of these phenomena and solve many related problems.

Rate of change is not only important as a major intellectual achievement but it is also important because of its usefulness in several academic disciplines. Mathematics, physics, chemistry, biology, economics, and other areas of the social sciences deal with change in one way or another. Some, like mathematics and physics, involve quite detailed studies of change as central concepts in the discipline.

Others, like chemistry and biology, use the concept of rate of change to understand certain aspects of the physical and natural world. Still others, like economics, political science, and psychology, use change to explain and predict phenomena in the social sciences. So rate of change is important for a serious study of many academic disciplines and client programs related to those disciplines.

Finally rate of change is a prevalent part of our everyday lives, especially as it relates to motion and growth. Whether moving by walking or running or riding on a bike or in a car, train, or airplane, speed of travel is a primary consideration in our work and leisure activities. Similarly, when considering the growth of our investments, plants in our gardens, or the population of our country, rates can yield valuable information for planning and predicting.

This first section gives a brief history of the study of change. I relied on Boyer (1959), Edwards (1979), Kline (1972), and Kaput (1994) as secondary sources and Aristotle's *Physics* (Book V) and Galileo's *Dialogue concerning two new sciences* as primary sources in this history. The second and third parts of this section deal with the concept of change in several academic disciplines and in everyday life.

A brief history of rate of change.

Greek analysis of motion. Aristotle (*Physics*, Book V, 228b) defined motion to be either uniform or nonuniform. The movement of heavenly bodies was considered uniform while the motion of objects on earth was nonuniform. These descriptions were based on sense perceptions of moving objects. The movement of animals and humans was obviously not uniform. They moved at varying speeds; they could be seen moving faster at some times, slower at others, and slowing down and

speeding up. In contrast, the movement of stars and planets seemed to be uniform. Hence there was no concept of acceleration in astronomy. Furthermore, astronomical motion was seen as circular and could be easily studied using the geometry of the circle (Boyer, 1959).

Motion was not studied in quantitative terms because the concept of number was tied to geometry of quantities. Numbers represented lengths, areas, or volumes. Since motion on the earth was not seen in geometrical terms, it was not within the purview of number (Boyer, 1959). In addition, what we know today as rational number was not the same as the ancient Greeks' concept of rational number. Relationships between geometrical objects were often expressed in terms of proportions, A is to B as C is to D, but ratios of numbers, A/B , were not accepted as numbers in their own right.

Speed would later come to be defined as a ratio with the numerator representing a distance traveled and the denominator representing the time to travel that distance. So a quantitative study of speed would require acceptance of these ratios as numbers in their own right (Kline, 1972). Finally, the world of mathematics at that time was concerned only with discrete quantities. Since motion was considered continuous, it was included in physics but was not in mathematics (Edwards, 1979). Hence the ancient Greeks did not submit motion to the scrutiny of number. Motion was simply described as uniform or nonuniform.

The ancient Greeks' ideas of motion were guided by the notion that objects move because of some external force that not only sets them in motion but also continues to act to keep them moving. The emphasis was on seeing moving objects as moving toward a goal propelled along by

application of some force. This meant, for example, that moving objects did not have speed at a point (Boyer, 1959).

In addition, motion was studied mostly within the context of astronomy. Since, in this case, motion was circular, it could be described completely in terms of the geometry of the circle. Since it was uniform, speed of an object at an instant in time was not a meaningful question (Edwards, 1979). Finally, the ancient Greeks relied on the senses for collection of data about how the world operates. Since motion at an instant is not observable, the concept of speed at a point was not part of Greek mathematics and science (Kline, 1972). All of these factors made the formation of the concept of instantaneous rate of change impossible at this time.

Rate of change in medieval times. In the 13th and 14th centuries the idea that a moving object continues to move because a force continues to act on it was replaced by the notion that a moving object continues to move because of some tendency of its own. While the ancient Greek's emphasized the moving of an object toward a goal under the application of an external force, medieval scholars focused on the movement itself and attributed to moving objects properties related to their own motion.

One of these properties was speed at a point. This made possible the intuitive idea of instantaneous rate of change, that moving objects have a speed at each instant in time, and ushered in a quantitative study of rate of change, including speed, the rate of change of distance with respect to time (Boyer, 1959). Duns Scotus, Thomas Bradwardine, and Richard Suiseth (The Calculator) set the foundation for such a study by defining motion as uniform if equal distances are traveled in equal

times. Uniform acceleration was then defined as equal increments of velocity being acquired in equal intervals of time.

Nicole Oresme described instantaneous velocity in terms of distance traveled if the object continued to move at that same constant speed over some interval of time. Some scholars thought of time as divided into very small intervals so that one hour contained 22,560 instants of time. In this view instantaneous velocity was thought of as the speed needed to travel a given distance in one instant of time. These definitions for uniform motion, uniform acceleration, and instantaneous velocity were expressed in terms of distances traveled over intervals of time and served as the basis for the continuing study of rate of change (Edwards, 1979).

Fourteenth century Scholastics distinguished between uniform motion (equal increments of distance acquired in equal time intervals) and nonuniform motion (unequal increments of distance acquired in equal time intervals). They also distinguished between uniform acceleration (equal increments of speed acquired in equal time intervals - uniform nonuniform motion) and nonuniform acceleration (unequal increments of speed acquired in equal time intervals - nonuniform nonuniform motion).

Scholastics established a number of results regarding uniform speed and acceleration. Richard Suiseth showed that a body moving with constant acceleration over an interval of time has an average speed equal to the mean of the speeds at the beginning and end of the interval. This statement assumes the existence of speed at a point, in particular speed at the beginning and ending times of the interval. Suiseth's argument included a numerical example with a beginning speed of four units of distance per unit of time and an ending speed of eight

and a verbal description of what happens to the speed as the beginning speed is increased uniformly and the ending speed is decreased uniformly. He concluded that they arrive together at a speed of six so that the result is the same as traveling at a constant speed of six the whole time (Boyer, 1959).

Later, Oresme established the same result but expressed it in terms of the distance traveled instead of the average speed: a body starting at rest and moving with constant acceleration over an interval of time travels the same distance as a body moving the same amount of time at a constant speed equal to half of the final velocity of the first body. Oresme provided a geometrical demonstration of this result in which time was represented by a horizontal line and speed at each instant was represented by a line segment of the appropriate length perpendicular to the horizontal time line at the instant. (See Figure 1.)

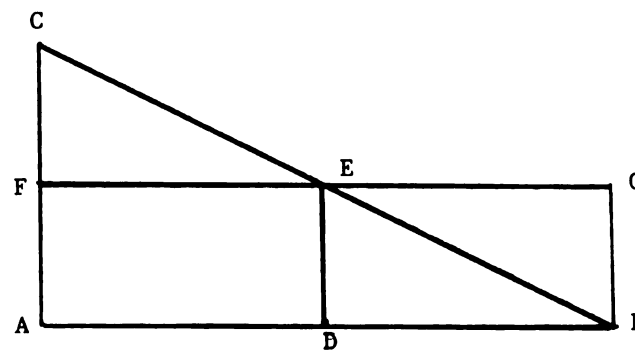


Figure 1 - Diagram showing speed at points in time

The diagonal line from B to C represented speed increasing constantly from rest to final velocity. The horizontal line from G to F represented constant speed equal to half of the final velocity

represented by the line segment AC. Oresme showed that triangles BGE and CFE are congruent and therefore the area of rectangle BGFA is equal to the area of the triangle BCA. He took this as evidence that the distances traveled were therefore the same even though he nowhere explained why distance could be represented by area under the speed-time graph.

In keeping with other scholars of his time, he may have been thinking of the area of the region formed by thin rectangles with each rectangle corresponding to a vertical line representing velocity over a very small time interval on the horizontal time axis. Each rectangle then would have an area equal to the distance traveled by that velocity over that small amount of time (Boyer, 1959). This expresses the inverse relationship between speed and distance as expressed in the fundamental theorem of calculus: the rate of change of the distance is the speed and the area under the speed-time graph yields the distance traveled. Later, Galileo established the same result but expressed it in terms of time rather than average speed or distance. This will be discussed below.

Oresme discovered another interesting result. For a speed that could be represented graphically as a semicircle, he noticed that the speed changed the least when it was near its maximum. In other words, the rate of change of the velocity was least at the maximum velocity (Boyer, 1959). Kaput (1994) describes Oresme's portrayals of rate of change using geometrical objects as an attempt "to represent a total event as a global representation" (p.92) and claims that "Oresme used the shape of the entire figure as the basis for talk about varying intensities" (p. 92), that is, varying speed. Oresme used these global

representations repeatedly to investigate uniform motion and uniform acceleration.

In summary, medieval scholars described motion in terms of distance traveled during intervals of time and thought of speed at a point in terms of speed over a very small interval of time. They discovered many famous results concerning uniform speed and uniform acceleration and they did so by examining relationships among time, distance, and speed and using graphical representations of speed versus time with distance represented by area. Their work in these quantitative beginnings of average and instantaneous rate of change set the stage for the 16th and 17th century considerations of these concepts.

Rate of change in the 16th and 17th centuries. In the 16th and 17th century came many more characterizations of rate of change. Perhaps the best known is the work of Galileo (1638) who completely characterized the motion of falling objects. Galileo began his discussion with a definition of uniform motion stated in terms of distance traveled in intervals of time: "By steady or uniform motion, I mean one in which the distances traversed by the moving particle during any equal intervals of time, are themselves equal." (p. 191). He then stated four axioms and six propositions for a deductive system which characterized relationships among time, distance, and speed under uniform motion. All of these statements were based on distance traveled in intervals of time. (See Appendix A for complete statements of the four axioms and six propositions.) For example, the first axiom stated that under uniform motion, distance traveled in a longer interval of time is greater than distance traveled in a shorter interval of time and

the first proposition established that the time intervals are in the same ratio as the distances.

Galileo next considered what he called naturally accelerated motion, the motion of falling bodies, and derived the results for which he is best known. Just as he defined uniform motion in terms of equal increments of distance acquired in equal increments of time, he defined uniform acceleration in terms of equal increments of speed acquired in equal increments of time and concluded that "the increment of speed is proportional to the increment of time" (p. 198). He continued his discourse with many results, including the same result as Suiseth and Oresme concerning the average speed of a uniformly accelerating object over an interval of time. While Suiseth expressed his result in terms of velocity and Oresme expressed his in terms of distance, Galileo expressed his in terms of time:

"The time in which any space is traversed by a body starting from rest and uniformly accelerated is equal to the time in which that same space would be traversed by the same body moving at a uniform speed whose value is the mean of the highest speed and the speed just before acceleration began." (p.208)

This result led to a definition of average speed for general motion: the average speed to travel a given distance over a specified interval of time is that uniform speed required to travel the same distance in the same time. This lead to a computational formula for average speed as the ratio of the distance traveled to the length of the time interval, the same formula used to compute the speed if it were assumed to be uniform.

Note two things here. First, uniform speed is associated with distance traveled in an interval of time; that is, uniform speed is a way of characterizing speed over intervals. Second, moving objects are

assumed to have speed at a point, in this case at the beginning and ending instants of the time interval; that is, instantaneous speed is a way of characterizing speed at a point. Galileo provided a geometrical demonstration of this result using a figure very similar to Oresme's and, like Oresme, Galileo based his argument on area under the velocity-time graph. He used similar geometrical demonstrations to conclude that the distance traveled is related to the square of the length of the time interval. From this result he also obtained the famous result that distances traveled in consecutive and equal time intervals have the same ratio as the sequence of odd numbers, 1, 3, 5, 7,

Specific contributions of Newton and Leibniz. Newton and Leibniz are given joint credit for the invention of calculus. They were the first to explicitly state the inverse relationship between the two primary concepts of calculus - integration and differentiation. Others had obtained results hinting strongly at this relationship but apparently Newton and Leibniz were the first to recognize its importance and generality. This relationship is known today as the fundamental theorem of calculus. Stated loosely, it is the proposition that the rate of change of the area of the region formed by the graph of a function between a fixed X value and a variable X value x is the value of the function at x . In other words, the derivative of the integral of a function is just the function itself.

Here is how Newton arrived at this relationship. In an effort to complete a catalog of functions whose integration (area) formulas were known, Newton decided to start first by assuming that an integration formula for a function was of a particular form and then see what this meant about the function. So he started with an unknown function $f(X)$

whose graph was a smooth curve lying above the horizontal axis from $X=0$ to $X=c$ where c is some positive number. Let $A(x)$ represent the area of the region bounded by the horizontal axis, the graph of $f(X)$, and the vertical lines $X=0$ and $X=x$. Then $A(X)$ is the integration formula for $f(X)$ and is itself a function which is defined for values of X from $X=0$ to $X=c$. Suppose it is known that $A(X)$ is equal to aX^b . Can we determine an expression for $f(X)$?

To answer this question Newton formed two expressions for $A(x+h)$, the area of the region bounded by the horizontal axis, the graph of $f(X)$, and the vertical lines $X=0$ and $X=x+h$. (See Figure 2.) One expression for $A(x+h)$ was $a(x+h)^b$ which gives the exact value of the area $A(x+h)$. The other expression for $A(x+h)$ was approximately $A(x) + h \cdot f(x) = ax^b + h \cdot f(x)$ which estimates the area $A(x+h)$ by adding to the area $A(x)$ the area of the rectangle determined by the interval on the horizontal axis from $X=x$ to $X=x+h$ and the height of the graph of $f(X)$ at $X=x$.

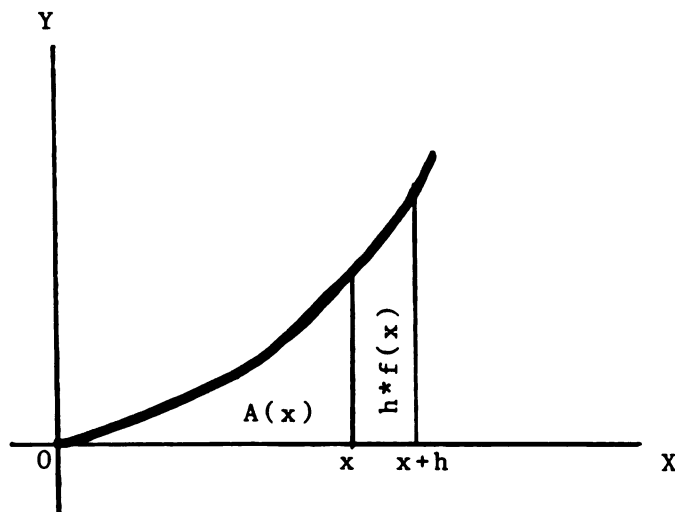


Figure 2 - Graph of a function showing area

Newton set these two expressions equal to each other and applied his binomial theorem to $a(x+h)^b$. He then subtracted ax^b from both sides of the equation and divided by h . On one side of the equation he now had $f(x)$ and on the other side he had $ab \cdot x^{(b-1)}$ and several terms involving powers of h . He reasoned that terms with h in them could be neglected since h was chosen to be close to zero (under the assumption that $x+h$ was chosen to be close to x) and raising h to positive integral powers would only make the terms smaller. This left $f(x)$ on one side of the equation and $ab \cdot x^{(b-1)}$ alone on the other side. Since this result would hold for any value of x in the domain of $f(X)$ (except possibly the right end point, c , of the domain), he concluded that if $A(X) = ax^b$, then $f(X) = ab \cdot X^{(b-1)}$.

His argument also showed that if $f(X) = ab \cdot X^{(b-1)}$, then $A(X) = ax^b$. It was already known that the rate of change of the function given by the expression ax^b is $ab \cdot X^{(b-1)}$. Hence Newton had shown, for this particular class of functions, that the rate of change of the area under a curve defined by a function is just the function itself.

Prior to the discovery of this result, area under the graph of a function given by $f(X)$ was determined by dividing the region into smaller and smaller rectangles and adding up the areas of these rectangles. With this result of Newton's, area under the graph of a function $f(X)$ from $X=0$ to $X=x$ could now be determined by "doing derivatives backwards", that is, by finding an expression for the function $g(X)$ whose derivative is equal to $f(X)$, and then evaluating $g(X)$ at $X=x$. From this it was then possible to obtain a method for finding the area under the graph of function $f(X)$ between any two values of X . This result not only provided a computationally easier way to

find areas but it also connected the extremely useful but before now isolated operations of differentiation and integration. Newton and Leibniz were the first to make this connection (Kline, 1972).

Two other reasons Newton and Leibniz are given credit for the invention of calculus are their ability to see different techniques as fundamentally related to each other and to make techniques more generally applicable. For example, mathematicians and scientists from ancient Greece to the time of Newton and Leibniz devised many methods for finding tangents to specific curves. But they did not apparently have a general technique that allowed them to find tangents to any arbitrary curve. Nor did they associate tangents to curves with rate of change. Newton and Leibniz formed general techniques for finding tangents to any smooth curve and related tangents to rate of change.

Rate of change today. In contemporary mathematics, instantaneous rate of change of a function is defined as the derivative of that function. The derivative of a function $f(X)$ is itself a function, designated by $f'(X)$. For $X = x$, a specific value of the domain of $f'(X)$, the value of the derivative, $f'(x)$, is defined as the limit of $[f(x+h) - f(x)]/h$ as h goes to zero,

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

The number obtained from this limit is the definition of the slope of the line tangent to the graph of $f(X)$ at the point $(x, f(x))$ and the instantaneous rate of change of $f(X)$ at $X=x$. For a value of h is not equal to zero, $[f(x+h) - f(x)]/h$ is the slope of a line containing the two points $(x+h, f(x+h))$ and $(x, f(x))$; the smaller h is in absolute value, the closer that line is to being the line tangent to the graph of $f(X)$ at the specific point $(x, f(x))$. Similarly, $[f(x+h) - f(x)]/h$ is

the average rate of change of $f(x)$ over the interval between x and $x+h$. The smaller h is in absolute value, the closer the average rate of change is to the instantaneous rate of change of $f(X)$ at $X=x$.

More generally the derivative is a function. The derivative of the function $f(X)$ is the function $f'(X)$ obtained by forming for each value of h the function

$$f_h(X) = \frac{f(X+h) - f(X)}{h}$$

and taking the limit of this class of functions as h goes to zero. The function $f_h(X)$ gives for each value of X , the average rate of change of the dependent variable of $f(X)$ over the interval from $X+h$ and X if h is negative and X to $X+h$ if h is positive. The limit of this class of functions, $f'(X)$, is not a numerical value; it is a function. This function can, of course, be evaluated for particular values of X to yield the value of the derivative at specific points. But the function $f'(X)$ has its own place in the study of calculus aside from the function that gives the slope of the line tangent to the graph of $f(X)$ or the instantaneous rate of change of $f(X)$ at specific values of X .

Modern significance.

Rate of change is an important concept in mathematics, other academic disciplines, and everyday life. In mathematics rate of change serves both as a tool and a theoretical underpinning of advanced mathematics. With other academic disciplines, rate of change is a tool used to investigate quantities that change over time. In everyday life, rate of change is evident in subtle but pervasive ways.

Rate of change in mathematics. Calculus is the study of properties of continuous functions including their derivative and integral functions. While the derivative function can be studied

without any specific reference to rate of change, most calculus courses today include some work with rate of change. The amount and type of emphasis on rate of change varies with calculus texts and teachers. While it is generally the case that reform calculus texts give more attention to rate of change as an important instantiation of the derivative, almost all modern calculus texts have a healthy strand of examples, problems, and applications of the derivative as rate of change.¹

For example, one primary application of the derivative as rate of change is the solution of differential equations in which derivative equations are given representing rates of change of quantities with respect to other quantities. Students are then asked to find the functions which have these rates of change. The differential equations course taken after a full year of elementary calculus then has as its main topic the solution of classes of differential equations with varying types of conditions. Many of the examples used in the differential equations course are rate of change problems. That is, the differential equations under study are set in a context of the rate of change of one quantity with respect to another quantity.

¹ Reform calculus is distinguished from traditional calculus by more use of multiple representations of functions, especially graphs and tables in addition to equations, technology, real situations and data, extended projects, laboratory settings, and group work both in and out of class. Daily assignments are more likely to emphasize a smaller number of problems, with problems going into more depth. There is less reliance on lecture and more use of extended conversations in the classroom with students playing an active role in guiding and contributing to the direction of the lesson. Reform calculus is also more likely to have a trimmed syllabus which deletes several topics traditionally included in calculus in favor of more detailed treatment of a fewer number of topics.

Similarly the concept of rate of change appears in vector calculus, numerical analysis, and mathematical modeling courses. In vector calculus, partial derivatives of functions of two or more independent variables are studied and often examples involving the rate of change of quantities represented by the independent variable of one of these functions are dealt with. In numerical analysis, numerical techniques for obtaining solutions to derivatives and differential equations are developed. Many of these examples are set in the context of rate of change.

In mathematical modeling, one typical set of techniques deals with situations where the rate of change of one variable with respect to another variable is known or conjectured. These situations are often represented by derivatives of differential equations which when solved yield important relationships between the variables. It is fair to say that half of the course work in the current undergraduate mathematics curriculum makes use, in some way or another, of knowledge of the derivative in general and rate of change in particular.

Another example of the prevalence of rate of change and differentiation in mathematics is the use of the Mean Value Theorem and Taylor's Theorem in upper division mathematics courses. The Mean Value Theorem states that under appropriate continuity and differentiation conditions the average rate of change of the dependent variable of the function over an interval of the independent variable is equal to the value of the derivative at some value of the independent variable in the interval. This theorem is used in vector calculus to obtain several theoretical results.

Taylor's Theorem states that functions which are differentiable at least N times can be represented by a $(N-1)$ th degree polynomial where each term of the polynomial contains a factor involving a derivative evaluated at a specific value and where the error term is dominated by the N th derivative. This theorem is used repeatedly in numerical analysis as the basis of many approximation methods.

In recognition of its important role in mathematics, rate of change has also become one of the major strands in several reform curricula in K-12 mathematics, including the Connected Mathematics Project for middle school and the Core Plus project for high school. In addition, the Technical Educational Resource Center (TERC) has developed curriculum materials on change and rate of change for kindergarten through fifth grade and Jim Kaput at University of Massachusetts-North Dartmouth has produced software, called SIMCALC, for elementary school students' explorations of change and rate of change (Kaput, 1997). The mathematics education who developed these materials believe that rate of change is an important topic for students at all educational levels to study and that it is possible to present change and rate of change in contexts appropriate for students at these ages. In short, they believe that calculus as rate of change can be and should be made more accessible to students much earlier in the school curriculum (Kaput, 1994).

Rate of change in the sciences. Lynn Steen, in his influential book On the shoulders of giants: New approaches to numeracy that maps out the major strands of mathematics, included a chapter about change. Ian Stewart, the author of this chapter, argued that "every natural phenomenon, from the quantum vibrations of subatomic particles to the

universe itself, is a manifestation of change" (p. 183). He went on to write that while some of these changes seem too complex to submit to analysis, we must look for patterns of change so we can understand and control our world. His chapter investigates change and patterns of change in several areas in mathematics and science and requires a fair amount of college level mathematics and science.

Others have also noted the importance of the ideas and techniques of calculus in the natural, physical, and social sciences (Douglas, 1986). In physics, rate of change appears significantly in kinematics where differentiation is used to find the velocity from position functions and integration is used to find the position from velocity functions. Rate of change is also used in chemistry and biology. For example, in chemistry, points on a titration curve relating time and concentration yield a table of values which is then used to compute differences in successive concentrations. These differences are examined to identify the concentration value which yields the greatest reaction rate (largest difference) and the concentration value before which the concentration was increasing at an increasing rate (differences positive and getting successively larger) and after which the concentration was increasing at a decreasing rate (differences positive and getting successively smaller).

In ecology, numerical data concerning changing patterns in the population of animals and plants are used to support theories concerning ecological balances and imbalances. Classical studies in population yield differential equations which can then be solved to yield information about how plant and animal populations interact to yield growth or decline. Rates and rate of change are also important in

economics where quarterly and annual information concerning inflation, growth of savings and debt, and changes in business and personal income are all expressed as rates. These rates are used both to form and confirm economic theories. In many different fields of science and engineering, dynamical systems play a crucial role. Calculus, as the study of change, is the basis for this study.

Rate of change in everyday life. Ian Stewart's illustrations were primarily from the disciplines of mathematics and science but his remarks also apply to everyday knowledge of the world and how it operates. We live in a world of motion. It is interesting that motion was the first venue for the study of rate of change and at no time is motion more obviously a characteristic of our daily lives than today. We walk and run, ride on bicycles and in-line skates and in cars, planes, and trains. In short, we are daily submitted to a variety of experiences involving moving fast and slow and speeding up and slowing down. Change and rate of change are a constant part of our daily lives.

But motion is not the only example of rate of change in our everyday lives. We also deal with change when we record children's height and weight over the years of their growth or visually over shorter intervals of time. We also note the changing temperature with the seasons and time of day; we watch the growth or decline of our investments in savings and pension plans; and we lament the rise of prices, especially if it is not also accompanied by a similar rise in our incomes. We are also annually reminded of changes in crime, population, charitable giving, and health care costs. Many of these changes are reported in the belief that we can understand, predict, and

control them. They are continual reminders that we live in a world characterized by change.

Learning rate of change is a challenging intellectual task

In the opening paragraphs of this chapter I noted that since it took humankind over 2000 years to arrive at a quantitative analysis of rate of change, we should not be surprised when students struggle with trying to master it in a few months or years. Even though we live in a world characterized by change, few need to investigate rate of change in the same way as Oresme, Galileo, and Newton in order to successfully negotiate this world. So most of us do not. For those who want or need to study rate of change in a detailed way, a fair amount of mathematical background is required and this is not easy to acquire. All of these factors contribute to the challenge of learning rate of change. Here I present additional evidence that many students struggle with learning rate of change. This evidence comes from my own teaching, reports of various organizations concerned with mathematics education, and research on learning rate of change.

Examples from my teaching.

Problems with knowledge of functions. Over my 25 years of teaching calculus I have identified several areas where students have difficulty with rate of change. First, students lack knowledge of functions. In particular, they do not know graphs of families of functions, important properties of those graphs, and connections between graphs and equations. For example, students coming into calculus rarely know the graphs of logarithm or exponential functions, trigonometric functions, or polynomials other than lines; they are not facile at associating x- and y-intercepts on a graph with solutions of

corresponding equations; they do not associate the shape of a graph with corresponding properties of its equation. This and other knowledge of functions is very important in studying calculus.

Most students remember having had experiences in high school with many of these functions but do not remember much about their graphs, their algebraic properties, or connections between the two. They also do not seem to have much knowledge of the role of the domain and range of functions. In particular, they do not generally know that a function is associated with a relationship between two varying quantities and that algebraic, graphical, and tabular properties of a function can reveal the nature of this relationship.

Problems with general knowledge of rate of change. Students usually have some experiences with rates in middle and high school. However, that experience is typically limited to a study of uniform or constant rate of change and is represented almost exclusively by the formula $d = r \cdot t$. There are at least two problems with this. One is that reliance on only one representation, an equation, does not help students see what constant rate might look like in other representations, like graphs and tables of values. For example, students may not see that constant rate is related to a linear function.

The second problem with studying only uniform rates is that it may lead students to believe that all rates are constant and therefore subject to the formula $d = r \cdot t$. This provides two difficulties for students. One is that under conditions of varying rates, students' use of the formula $d = r \cdot t$ is likely to lead to incorrect or very inaccurate answers. Without appropriate experiences with both constant and varying

rates, students may not understand why or that a formula for constant rate does not work for varying rates.

The other difficulty is that if students only study constant rates, they will not have developed ways of dealing with variable rates.

Average rate of change provides a way of simplifying and initially dealing with variable rates. Recall that Oresme and Calculator in the middle ages and Galileo in the 16th century used average rate of change to make sense of a situation involving varying speed under conditions of constant acceleration. If students have knowledge of average rate of change, they might be able to use it to make sense of variable rates. But if their only experiences are with constant rates, they are not likely to know about average rate of change or see the need for it especially since in the case of constant rates, average rate of change over an interval is the same as the uniform rate of change.

Since average rate of change provides the conceptual basis of analyzing instantaneous rate of change, it seems to be good for students to have some experiences with average rate of change before they come to calculus or get extensive experience with it in calculus. Many calculus texts, including some reform texts, do not, in my opinion, provide enough help for students to investigate differences between uniform rates and varying rates or understand how average rate of change can help make sense of variable and instantaneous rate of change.

Traditional calculus texts typically include a page or two devoted to average rate of change with few if any words about differences between uniform and variable rates. Calculus reform texts include more discussion, but limit their discussion of average rate of change to relatively few pages. One calculus reform text makes average rate of

change the focus of its initial consideration of rate of change, but it is not widely used.²

Problems with instantaneous rate of change. If students believe moving objects have speed at each instant of time, then it is often possible to have fruitful conversations about how rate of change at a point could be defined. If students do not believe moving objects have speed at a point, then they may find it difficult to come to such a belief or to set aside their disbelief and take on the intellectual task of thinking about how it may be reasonably defined if there were such a thing. For the most part, calculus text writers and teachers assume that speed at a point makes sense to students and proceed without much preamble to define instantaneous rate of change in terms of the limit of average rate of change over intervals of decreasing width around the point.

My experience is that this has dire consequences. Instantaneous rate of change often does not make sense to students and its connections to average rate of change eludes them. Acceptance of the issue of rate of change at a point does not come easily. It is not an observable physical phenomenon. Nor does it follow the same rules of formation as uniform or varying speed both of which have to do with actual distance

² Calculus in Context, one of the NSF sponsored calculus reform texts features an exceptionally compelling example of the use of rate of change to make sense of a relatively complex situation. It is an epidemiology model of the spread of a disease throughout a population. This example is used to motivate an introduction to rate of change and eventually the derivative. Other examples like this appear in some detail throughout the text, making it a refreshing change from other calculus texts, both traditional and reform. This text was written by The Five Colleges Calculus Project (Smith College, Amherst College, Hampshire College, Mount Holyoke College, and University of Massachusetts-Amherst) and is available from W. H. Freeman and Company.

traveled over a period of time. In the case of instantaneous rate of change, there is no period of time; there is a point in time. How can one travel a real, observable distance in a point of time? It is not unusual for students to say there is no such thing as speed at a point because you have to have an interval of time to have a speed.

In addition, students often have trouble with the limit concept. Instantaneous rate of change is defined in terms of a particular limit.

So students who are struggling with understanding limits may also struggle with understanding instantaneous rate of change. Most calculus texts have extensive work on limits before their treatment of the derivative and instantaneous rate of change. But my observations lead me to believe that understanding the limit concept comes very hard for students and even more so, the connection between limits and instantaneous rate of change.

All of these factors, lack of knowledge of elementary functions and of connections between their graphs and algebraic properties, lack of experience with situations involving varying speed and average rate of change, and lack of sense-making with limits and the concept of rate of change at a point, conspire to make learning rate of change difficult. If students are to grasp rate of change, then at a minimum, they must understand that functions represent relationships between varying quantities, that they can be represented as equations, graphs, and tables of values, and that those representations are related to each other in important ways that reveal rate of change.

Additionally it may be helpful for students to understand differences between constant and varying rates, that average rate of change can provide a way of initially making sense of varying rates, and

that average rate of change and the limit concept provide a way of constructing such a thing as instantaneous rate of change. This means, among other things, that students need, before they get to calculus, experiences with variable and average rate of change in situations involving multiple representations of relationships between varying quantities and that in calculus, students need to be given much more support in seeing relationships among uniform, average, and instantaneous rate of change.

Evidence of difficulty from reports on calculus.

The Mathematical Association of America, National Research Council, and the National Science Foundation have all indicated concern about mathematics education in the United States, with some special concern for calculus. Each year over a million students take calculus in high school or college (National Research Council, 1989). The number of students taking calculus has risen steadily since the mid 1960's (National Research Council, 1991), yet about half of those students either fail calculus (Douglas, 1986a, 1986b) or do so poorly they are discouraged from continuing their study of mathematics (National Research Council, 1991).

In the 1981-82 Second International Mathematics Study (SIMS) less than a third of students taking calculus in high school and only a little more than half who had finished calculus could obtain the value of a definite integral for the area of a region under the graph of a linear function when the region and linear function were represented by a graph. Other international comparisons show US students performing poorly on items about conceptual understanding of calculus and relatively simple standard calculus computations (McKnight, et al,

1987). These reports of calculus students' performances led in the 1980's to a \$7 million calculus reform initiative funded by the National Science Foundation. A major focus of that effort has been to produce curriculum materials and foster instruction that emphasize conceptual understanding of calculus.

Calculus has been described as a gateway to the study of advanced mathematics and natural, physical, and social sciences (Douglas, 1986a).

Its role in preparation for study in these areas has increased in the last thirty years. More uses of calculus concepts and techniques have come into being with an emphasis on dynamical systems in biology, physics, and economics. Computer science, an area of study concerned primarily with discrete mathematics, is asked to deal with continuous change in both theoretical and applied ways. Students who wish to pursue serious study in these academic areas need a good conceptual foundation in the calculus. The calculus reform effort was borne out of evidence that students in traditional calculus classes do not have this conceptual understanding and has worked to develop the means for helping students acquire this understanding.

Research on difficulties learning calculus.

The conceptual difficulties involved in learning calculus have not escaped the attention of mathematics education researchers. In chapter 2, I will give a more detailed discussion of research about students' difficulties with calculus. Here my purpose is to provide some evidence that students have conceptual difficulties with calculus.

Some research studies show calculus students having trouble forming and manipulating algebraic representations for relationships between changing quantities (Monk, 1992a, 1992b; White and Micheltmore,

1996). Others show that calculus students are often unfamiliar with graphs of families of functions (linear, quadratic, rational, exponential, trigonometric) and significant relationships between their equations and their graphs (Hauger, 1991; Evan, 1989, 1990, 1993). Still others concern students' knowledge of functions (Dreyfus and Eisenberg, 1982, 1983, 1987; Ayers, Davis, Dubinsky, and Lewin, 1988; Vinner, 1983; Vinner and Dreyfus, 1989). All of these studies deal with knowledge traditionally considered prerequisite for the study of calculus that teachers of calculus unfortunately often assume students already possess.

Other studies deal more directly with specific calculus concepts and chronicle difficulties students have with these concepts. Some of these studies deal with students' knowledge of integration (Orton, 1983b), limits (Cornu, 1981; Tall and Vinner, 1981; Monaghan, 1991; Williams, 1991; Confrey, 1980; Graham and Ferrini-Mundy, 1989; Tall and Schwarzenberger, 1978; Davis and Vinner, 1986), and continuity (Tall and Vinner, 1981; Karplus, 1979; Vinner, 1987). These, like rate of change, are typical topics in calculus, and these studies show that students have difficulty understanding them.

Most pertinent to this dissertation are studies concerning calculus students' knowledge of derivative and rate of change. Some of these studies will be more extensively reviewed in Chapter 2, but I have chosen one in particular to discuss here because it deals with calculus and postcalculus students' general knowledge of the derivative, a concept they have studied extensively in calculus. In particular, it includes a variety of standard elementary calculus tasks involving rate of change. It was based on individual interviews with a large sampling

of students, 60 students near the end of their elementary calculus course and 50 in mathematics courses beyond calculus.

This study, conducted by Orton (1984, 1983a), showed that these calculus and postcalculus students experienced many difficulties with rate of change tasks. Students had difficulty distinguishing among constant rate of change, average rate of change, and rate of change at a point. In one task students were given a verbal description of two changing quantities symbolized by the variable symbols X and Y . In the verbal description, the rate was identified as a constant rate and its value was given. Students were also given a graph and table of values showing the linear relationship between X and Y including first order differences for Y , differences between two consecutive y values in the table, $y_{i+1} - y_i$. When asked for the rate of increase in Y when $X=2.5$ and when $X=t$, over one-fifth of the students gave the value of Y . Similar results were obtained in another task involving a linear graph and its equation.

When given the equation and graph of a quadratic function, two-thirds of the students could not find the average rate of change of Y for values of X in the interval $[a, a+h]$ and almost 90 percent could not use average rate of change over this interval to discuss how they would find the rate of change at a specific or general value of X . None of the students attempted to draw a line tangent to the curve and estimate its slope or give a verbal description of such a procedure. Given a graph with points labeled and their X and Y coordinates easily visible, over one-third of the students were unable to find the average rate of change of Y between selected values of X . Common errors included forming a ratio with ΔY over one of the X values or one of the Y

values over one of the X values or simply giving one of the Y values as the answer. Even when correct numerical answers were obtained, required negative signs were often omitted.

Students were asked what was meant by the symbols dy/dx and $(\Delta y)/(\Delta x)$. Many could not say nor could they relate the two. Many could not calculate the derivative expression for some given function equations, nor could they successfully solve equations involving setting the derivative expression to zero or say what was meant by the number obtained when evaluating the derivative expression for a given numerical value of X. Many of these students could not relate the process of finding the derivative of a function to the slope of a curve or the line tangent to a curve, nor could they relate rotating secant lines to the tangent line. These are all standard ways of working with rate of change in calculus. This study and others document difficulties calculus and postcalculus students had with rate of change and showed that even after instruction in calculus, these students often could not solve problems involving rate of change. Overall, the Orton study suggests that students are not learning much in our calculus classes.

What does it mean to know rate of change?

This dissertation explores high school and college students' knowledge of rate of change. Rate of change is a multifaceted concept.

It seems reasonable at this point to discuss some of the most important elements. Since rate of change is considered within the context of functions, I first discuss functions and their representations as graphs, tables, and equations. Then I introduce rate of change in terms of three perspectives and three types of reasoning.

Functions and their representations.

Generally, rate of change refers to the character of change in one quantity relative to change in another quantity. We think of rate of change in the context of a function describing a relationship between two variables, one called the independent variable (often referred to as X) and the other called the dependent variable (often referred to by Y or $f(X)$). More precisely then, rate of change refers to change in Y , the dependent variable, relative to change in X , the independent variable.

Figure 3 shows the population of yeast cells in a culture over a period of 18 hours (from midnight to 6 p.m.) as a graph and a table. This is an example of a function with time as the independent variable and number of cells as the dependent variable. Rate of change in this context refers to change in the number of cells relative to change in time. The domain of a function is the set of allowable values of the independent variable and the range is the set of values of the dependent variable. In our example, the domain consists of the real numbers from 0 (representing midnight) to 18 (representing 6 p.m.) and the range is the set of real numbers from 9.6 to 661.8.

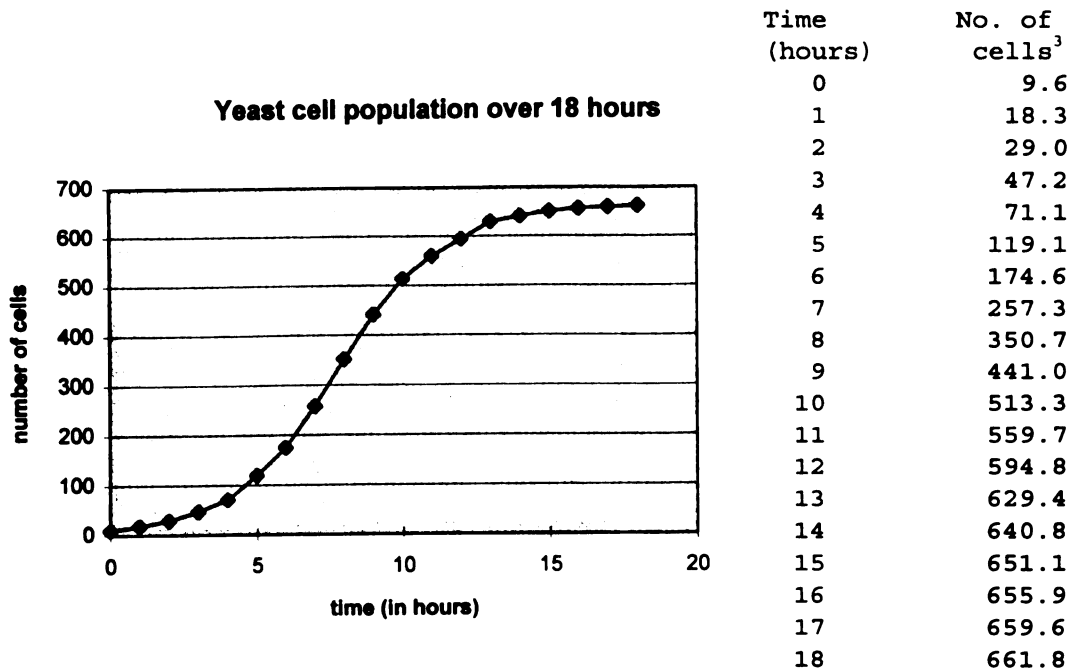


Figure 3 - Graph and table of values for growth of yeast cells over time

Functions are mathematical objects and cannot be directly examined, but they can be represented in several ways. They can be represented as graphs with the independent variable labeled and scaled on the horizontal axis and the dependent variable on the vertical axis. The coordinates of points on the graph are formed by values of the dependent variable and their associated independent variable values. One point on the graph in Figure 3 has coordinates (4, 71.1). It shows that for a time value of 4 the number of cells was 71.1. Graphs of a

³ This data came from Pearl (1927). In this article, Pearl explains how the cell count was obtained: "Because of the minute size of the individual cells, and of other technical difficulties, some sort of indirect method is commonly used to measure the size of the population, in place of direct counts." (p. 533). Apparently the values in the column labeled "No. of cells" represent an estimate of the number of cells determined by laboratory techniques based on weighing the specimen or counting cells or parts of cells in one microscope window. That would explain why the number of cells are not whole numbers but have decimal values.

function are usually formed by connecting points associated with corresponding values of the independent and dependent variables.

Functions can also be represented by tables of numerical values where one column displays values of the independent variable and the other column the corresponding values of the dependent variable. In the table in Figure 3, the left column contains values of the independent variable time and the right column contains values of the dependent variable number of cells. Functions can also be represented by equations. The equation of best fit for the function represented by the data in Figure 3 is

$$f(X) = \frac{665}{1 + e^{4.1896 - 0.5355X}}$$

where X represents the time in hours and f(X) represents the number of cells. The number of cells for a particular time value can be found by evaluating the expression on the right side of this equation for that time value. The time value for a particular number of cells can be found by setting the right side of the equation equal to that number of cells and solving for x. This equation is a logistic curve. Logistic curves have been studied extensively. In fact, it was a consideration of rate of change in situations involving population growth that generated equations for logistic curves (Giordano and Weir, 1985).

Three perspectives and three types of reasoning.

If rate of change refers to the change in one quantity relative to change in another, there are different ways to examine and analyze that change. I call these ways perspectives because they represent different views of looking at rate of change and give, I believe, different information. I refer to these as global, interval, and point perspectives on rate of change. They represent a successive narrowing

of intellectual focus on the nature of change in a situation. A global perspective considers changes in the dependent variable over the entire domain of the independent variable. For a given function $f(X)$, its derivative function, $f'(X)$, yields information about the rate of change of $f(X)$ over its entire domain. This is one way mathematicians might think about a global perspective on rate of change.

An interval perspective refers to change in the dependent variable of a function over intervals of the independent variable smaller than its entire domain. One important mathematical concept associated with an interval perspective is average rate of change. For a given function, the average rate of change in the dependent variable over the interval from $X=a$ to $X=b$ is given by $(f(b) - f(a))/(b - a)$. A point perspective focuses on change in the dependent variable for an individual value of the independent variable. For a function $f(X)$, the value of the its derivative $f'(X)$ at $x=c$, $f'(c)$, yields the rate of change of the dependent variable of $f(X)$ at the point $(c, f(c))$.

Below I will discuss each perspective in more detail. I will also introduce and discuss three types of reasoning about rate of change - qualitative, numerical, and algebraic. These types of reasoning are different means of analyzing what is happening in situations involving rate of change. Qualitative reasoning refers primarily to the use of graphical features to draw conclusions about rate of change, especially comparative judgments like "more", "less", "most", and "least". Numerical reasoning concerns the calculation or use of numerical values to make decisions about rate of change. Algebraic reasoning involves the use of algebraic notation (expressions, formulae, and equations) to make judgments about rate of change. (See Appendix B.)

Global perspective on rate of change. This perspective focuses on change in the dependent variable over the entire domain of the independent variable. For example, deciding for what values of the domain of the independent variable the dependent variable is increasing or decreasing, increasing or decreasing at a fast or slow constant rate, or increasing or decreasing at successively faster or slower rates involve judgments from a global perspective because the entire domain of the independent variable or large sections of it must be considered.

In Figure 3 it is fairly easy to see from the graph that the number of cells increased throughout the entire time from 0 to 18 and the number of cells increased successively faster until about 8 a.m. and then increased successively slower from 8 a.m. to 6 p.m.. This can be seen most easily by noting that the graph rises throughout the entire time and gets successively steeper until 8 a.m. and then successively flatter after that. This is an example of qualitative reasoning since it relies on graphical characteristics to make these conclusions.

One could also use numerical reasoning to make this conclusion. The graph or the table could be used to note that the number of cells increased as time increased. From the graph it can be seen that from $X=0$ to $X=1$ to $X=2$ and so on, the number of cells increased from about 10 to about 20 to about 30 and so forth. Using the table of values, one can see that from $X=0$ to $X=1$ to $X=2$ and so on, the number of cells increased from 9.6 to 18.3 to 29 and so forth. These are examples of numerical reasoning since they rely on calculation of or reference to numerical values.

Algebraic reasoning could also be used to make these decisions. The derivative expression for a function yields information about the

rate of change of the dependent variable of the function relative to its independent variable. Values of the independent variable which make the derivative expression positive are associated with increasing values of the dependent variable. Similarly, values of the independent variable which make the derivative expression negative yield decreasing values of the dependent variable. For the function in Figure 3, the derivative expression is

$$f'(X) = \frac{356.1075 * e^{4.1896 - 0.5355X}}{(1 + e^{4.1896 - 0.5355X})^2}$$

The expression on the right side of this equation is positive for all values of X and in particular for values of X from zero to 18. Hence the dependent variable for this function increases for all values of the independent variable. That is, the number of cells increases over the entire eighteen hours. This is an example of algebraic reasoning in making decisions about global rate of change since it depends on the use of a formula.

Interval perspective on rate of change. In the previous section, we examined the behavior of the dependent variable, number of cells, over the entire domain of the independent variable, the entire 18 hours of time. That helped us understand some facets of rate of change. To understand other facets, it is helpful to divide the domain of the independent variable into smaller sections. This helps us examine more closely the behavior of the dependent variable. We could, for example, look at the growth in the number of cells over one hour intervals. This represents an interval perspective.

To decide for what values of time the number of cells is growing the fastest or the slowest, we could visually compare vertical jumps or drops on the graph over intervals of one hour to see which one is the

greatest or least. In Figure 3, an examination of the graph indicates that the greatest jumps are made between $X=7$ and $X=8$ and between $X=8$ and $X=9$ and the smallest jump appears to be from $X=17$ to $X=18$. This means that the number of cells grew fastest from $X=7$ to $X=8$ and from $X=8$ to $X=9$ and the slowest from $X=17$ to $X=18$. This represents qualitative reasoning since it uses visual comparisons to make this judgment.

We could also decide which intervals show the fastest or slowest growth by computing changes in the number of cells over one hour intervals and noting which changes are largest or smallest. In Figure 3, using either the graph or the table of values, computation of changes in the number of cells each hour reveals that the largest change occurs from $X=7$ to $X=8$ with an addition of about 90 cells and the smallest change occurs from $X=17$ to $X=18$ with an addition of about 2 cells. So the cells grew the fastest from $X=7$ to $X=8$ and the slowest from $X=17$ to $X=18$. This represents numerical reasoning since it involves computing and referring to numerical values.

A special concept associated with an interval perspective is average rate of change. It is a helpful bridge between constant rate and instantaneous rate of change. With constant rate, the change in the dependent variable over every interval of the independent variable of a particular size is the same. With varying rates, the average rate of change in the dependent variable over an interval of the independent variable is that constant rate that yields the same change in the dependent variable over the same size interval of the independent variable. So average rate of change helps us make sense of varying rate of change by relating varying rates to the more easily understood concept of constant rate of change. It is computationally defined as

the ratio of the change in the dependent variable to the corresponding change in the independent variable. For a function $f(X)$, the average rate of change of the dependent variable of $f(X)$ over the interval of the independent variable from $X=a$ to $x=b$ is given by

$$(f(b) - f(a))/(b - a).$$

In Figure 3, the average rate of change of the number of cells over the intervals from $X=3$ to $X=4$, $X=4$ to $X=5$, and $X=5$ to $X=6$ are 23.9 cells per hour, 48 cells per hour, and 55.5 cells per hour, respectively. We can therefore conclude that the number of cells increased at successively faster rates in these hours. If we had information about smaller intervals of time, say 15 minute intervals, we could estimate rates of growth over every quarter hour and thereby get more precise information about the behavior of the cell population. This represents an interval perspective on rate of change that makes use of algebraic reasoning.

Point perspective on rate of change. Whereas a global perspective concerns changes in the dependent variable over the entire domain of the independent variable and an interval perspective considers change in the dependent variable over some interval of the independent variable smaller than the entire domain, a point perspective refers to change in the dependent variable at a particular value of the independent variable. I call this a point perspective to draw attention to the fact that we are thinking about rate of change at a point. It emphasizes the notion that from the overall features of rate of change of a function (global perspective), we zoom in on rate of change over a smaller interval (interval perspective), and then zoom in closer to rate of change at a point (point perspective).

A point perspective is what mathematicians usually call instantaneous rate of change. For a function $f(X)$, the instantaneous rate of change of the dependent variable at $X=c$ is $f'(c)$, the value of the derivative expression for $f'(X)$ at $X=c$. If one has the expression $f'(X)$ for the derivative of a function $f(X)$, then one can find the instantaneous rate of change by evaluating $f'(X)$ at $X=c$. But how can the instantaneous rate of change be found if one does not have an expression for the derivative function?

One can obtain an approximation for the instantaneous rate of change of the dependent variable $f(X)$ at $X=c$ by computing the average rate of change of $f(X)$ for the smallest available interval of the independent variable before, after, or straddling c . The smaller the interval, the closer we would expect that approximation to be to the instantaneous rate of change. Or one could find the slope of a small line segment connecting two points on the graph of the function near $(c, f(c))$. That would give an approximation of the slope of the graph at $(c, f(c))$ and hence should be an approximation of the instantaneous rate of change of $f(X)$ at $X=c$. These represent algebraic reasoning since they both use a formula, one the formula for average rate of change and the other the formula for the slope of a line, to find rate of change at a point.

These approximations work because the instantaneous rate of change of $f(X)$ at $X=c$ is defined as the limit of the average rate of change of $f(X)$ over intervals of X to the immediate left and right of $(c, f(c))$ as the width of these intervals decreases to zero, provided that limit exists. Expressed symbolically, this limit looks like this:

$$\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

The algebraic expression $(f(c+h) - f(c))/h$ is the average rate of change of the dependent variable over the interval of the independent variable from $X=c$ to $X=c+h$ when h is positive and $X=c+h$ to $X=c$ when h is negative. It is also the slope of the line containing points $(c, f(c))$ and $(c+h, f(c+h))$. This line is often called a secant line. Thus, the limit above is the limit of the slopes of secant lines, often called rotating secant lines, and defines the slope of the line tangent to the graph of the function at the point $(c, f(c))$. The tangent line then is defined as the line which has this slope and contains the point $(c, f(c))$.

If one has an equation for $f(X)$, then theoretically, at least, one can use this limit to find an exact value for $f'(c)$, the instantaneous rate of change of $f(X)$ at $X=c$. Otherwise $f'(c)$ can be approximated using one of the methods I noted above or by drawing a line that looks like it might be tangent to the graph at $(c, f(c))$ and then approximating its slope. This may represent both qualitative and algebraic reasoning since visual properties of the graph are being used to make sense of rate of change and the slope formula is being used to translate that visual property into a numerical value for the rate of change of the dependent variable at a particular value of the independent variable.

Questions about students' learning of rate of change

In this opening chapter I have argued that rate of change is important and that is it difficult for some students to learn. The main goal of this dissertation was to investigate the knowledge of rate of change of students at various points in mathematics education and make conjectures about how that knowledge might support learning rate of change and especially learning instantaneous rate of change. This and

the research and analysis discussed above suggest a series of questions about learning rate of change:

1. What knowledge of rate of change do students bring with them to calculus that might support their learning of rate of change?

A systematic examination of precalculus students' knowledge of rate of change in specific situations may help us see how this knowledge supports learning of rate of change.

2. What do students who have had calculus know about rate of change that may provided clues about how rate of change is learned?

A similar examination of calculus and postcalculus students knowledge of rate of change in the same situations may provide information about students learn rate of change and specifically what specific knowledge gets used in that learning and how.

3. What similarities and differences in knowledge of rate of change exist between those who have had calculus and those who have not had calculus and what might this tell us about what knowledge supports learning rate of change and how it supports it?

If precalculus, calculus, and postcalculus students have some knowledge of rate of change that is similar, that may mean that this knowledge is especially useful to assemble more advanced knowledge of rate of change. If precalculus, calculus, and postcalculus students have different knowledge of rate of change, then that may show paths of progression in the learning of rate of change.

4. In what ways do global, interval, and point perspectives on rate of change provide an interpretive frame for students' knowledge and how rate of change is learned?

5. In what ways do students use qualitative, numerical, and algebraic types of reasoning and what might that tell us about how rate of change is learned?

If the perspectives and types of reasoning are useful in analyzing students' work with rate of change, we would like to know in what ways they are useful.

Chapter summary

In this chapter I have argued that rate of change is important by providing a brief history of rate of change and giving some indication of its importance in mathematics, other academic disciplines, and everyday life. I have also shown that rate of change is challenging to learn by discussing examples from my own teaching, reports from various groups concerned about mathematics education, and some of the research on learning calculus.

Finally, I have provided a framework for what it means to understand rate of change by positioning it within the context of functions and their representations and by organizing rate of change in terms of three perspectives - global, interval, and point - and discussing within each perspective three different methods of reasoning - qualitative, numerical, and algebraic. I closed this chapter with a listing of questions about learning rate of change that will frame this study.

In the next two chapters I review literature relevant to student's knowledge of rate of change and what it means to learn within the context of knowledge as strategies and learning as strategy change. I follow that with a chapter on research methodology, three chapters on data analysis, and a final chapter on conclusions and recommendations.

Chapter 2

REVIEW OF LITERATURE ON LEARNING AND USING RATE OF CHANGE

There is a substantial literature on students' thinking and performance with mathematical tasks and problems that involve rate of change, and there are different interpretive lens the analyst can bring to reviewing this literature. One approach is to examine student performance on standard calculus tasks. The Orton (1984, 1983a) study reviewed in Chapter 1 showed difficulties calculus and postcalculus students had with both conceptual and computational aspects of the derivative and rate of change. This study was based on the performance of a large number of students, dealt with tasks considered fairly standard in calculus, and showed that completing a course which has the derivative as a main concept is no guarantee that students are able to apply differentiation appropriately or correctly or that they understand how the derivative is related to rate of change.

Other studies showed similar results. Studies by Peters (1982), Clement (1989), Graham and Ferrini-Mundy (1989), Selden, Mason, and Selden (1989), and White and Mitchelmore (1996) showed that students' performance on tasks involving the derivative and rate of change often showed a lack of expertise we would expect of students who have completed calculus. Researchers often want to examine where students' knowledge does not match accepted knowledge so they can think about why this is the case, what it says about learning, and what implications it has for teaching. The broad finding from this approach is that students

do not do well on these tasks, suggesting they have not learned a great deal from their course work, especially calculus.

Another approach considers research on technological support for students' learning. Tall (1987) showed that calculus students who used computer software to draw secant lines rotating to tangent lines performed significantly better on a test of conceptual understanding of slope, the derivative and rate of change than calculus and postcalculus students who had a traditionally taught calculus course. Speiser and Walter (1994) used time-lapsed photos of the motion of a cat to introduce the concept of rate of change at a point and to discuss issues related to estimating it. Beckmann (1988), Heid (1988), and Ellison (1994) discussed the effectiveness of using the computer as a tool for learning calculus. These studies presented pictures of possibility for using technology to help students come to an appropriate understanding of calculus concepts and especially rate of change as the derivative.

A third approach examines research on how, specifically, students solve problems involving rate of change, makes inferences about the current state of their knowledge, and thinks about how their knowledge of rate of change could then be advanced. This approach emphasizes what students actually did to solve tasks involving rate of change, considers what that may say about their knowledge of rate of change, and explores ways that knowledge may support growth of other knowledge of rate of change. It contrasts with the first approach discussed above where the emphasis is on whether or not students got the right answer and what percent of them did or did not. The research reviewed here showed what students actually did to solve problems involving rate of change. It also showed difficulties precalculus, calculus, and postcalculus

students had with tasks involving rate of change, considered possible reasons for those difficulties and, in some cases, discussed ways to help students learn rate of change. But the primary focus of my review of these studies will be the knowledge students displayed in working tasks with an eye toward seeing how that knowledge may be productive in learning rate of change. If students are to be successful in learning calculus and if we are to be effective in teaching calculus, this view of students' knowledge and learning could be quite helpful.

Research on learning and using rate of change

This review is organized as much as possible around studies dealing primarily with precalculus students followed by studies dealing with calculus and postcalculus students. There are some common themes for each group. For example, several studies show that precalculus, calculus, and postcalculus students have difficulty gleaning information about velocity from position graphs and information about position from velocity graphs. Similarly, in situations involving tables of values or where students can make tables of values, they tend to use changes in the dependent variable over intervals of the independent variable to make conclusions about rate or change. These studies show difficulties students have when working with rate of change. But they also show knowledge they have that helps them make sense of rate of change in situations involving graphs and tables and may serve as a base for their expanding knowledge of rate of change.

Research involving precalculus students and velocity and position graphs.

Several studies (Nemirovsky and Rubin, 1991, 1992; Rubin and Nemirovsky, 1991; Monk and Nemirovsky, 1992; Nemirovsky, 1994) at

Technical Education Research Centers (TERC) concerned high school students' knowledge of rate of change. Students in these studies had taken two years of algebra and some science in high school but had not yet taken calculus. The purpose of these studies was to see what connections students made between functions and their derivatives in the context of predicting the graph of one when given the graph of the other.

Students worked with a device involving a car moving on a track so that both the position and the velocity of the car at each moment in time could be graphed. One end of the track was assigned a position value of zero so that when the car was at that end of the track it had position zero and as it moved away from that point its position value increased and as it moved back toward this point its position value decreased. The position value of the car was always positive. The velocity of the car was positive if the car was moving away from position zero and negative if it was moving toward position zero.

In this context students' concepts of the relationship between a function and its derivative were studied in terms of the relationship between the position and velocity functions as represented by graphs. Both position and velocity were graphed against time so that the graphs showed either the position or the velocity of the car at each point in time.

Nemirovsky and Rubin (1992) made three assumptions about connections people make between a function and its derivative. One is that because of everyday experiences, everyone has some knowledge of the relationship between distance and speed. The second is that no one has complete knowledge of the connections between distance and speed. No

matter how much one knows about functions, their derivatives, and relationships between the two, there are still other things to be learned. The third is that context is important in performance. Two situations may involve the same mathematical concept but if they are presented in different contexts, like different physical devices or different function representations, they may evoke different responses from students. Hence conclusions about what students know must take problem context into account.

Results of the work at TERC. Students were presented with a velocity graph and asked to predict the position graph. Then students were asked to move the car on the track to try to generate the given velocity graph and to compare their predicted position graph with the position graph the device generated.

These students also assumed that graphs of a function and its derivative were similar in certain ways. For example, students often assumed that a function and its derivative increased and decreased together. If the graph of the velocity increased, then, for these students, so did the position function. In actuality a velocity function which is increasing but negative has a corresponding position function which decreases. Similarly a decreasing velocity function can be associated with a position function that increases.

Students also often assumed that the graphs of a function and its derivative had the same shape. If the velocity function was a line, then they predicted the position function would also be a line. If the velocity graph was concave upward, then they predicted the position function would be too. In many cases, these predictions were incorrect. Generally, graphs of a function and its derivative do not have the same

shape although there is a relationship between the shapes of the two functions. For example, if a velocity function is an horizontal line lying above the horizontal axis, then the corresponding position function will be a line with positive slope.

These students also believed that any change in the velocity function meant the same kind of change in the position function. If the velocity function changed from increasing to decreasing, then these students predicted that the position function would too. If the velocity function changed from positive to negative, then according to these students so did the position function. In actuality, when a velocity function is positive and changes from increasing to decreasing, the position function changes from increasing at an increasing rate to increasing at a decreasing rate. When a velocity function changes from positive to negative, the position function changes from increasing to decreasing.

Reasons for assuming functions and their derivatives are alike.

Nemirovsky and Rubin (1992) posited several reasons students believe position and velocity functions are alike. One is that velocity and position functions often are alike. An increasing velocity function is often associated with an increasing position function. For example, if the velocity function is increasing and positive, then the position function is increasing. Nemirovsky and Rubin concluded that when students' assumptions of resemblance's turn out to be correct, they may be led to generalize that all velocity and position functions resemble each other.

Another reason Nemirovsky and Rubin gave for students' beliefs that a function and its derivative are alike is that it helps them make

sense of complex situations. If in some situations a function and its derivative are similar, then the assumption that a function and its derivative are always similar simplifies situations and helps narrow the possibilities. It also encourages students to notice and take advantage of patterns, something we often explicitly teach mathematics students to do.

A third reason is that our use of everyday language suggests some resemblances between velocity and position. In many everyday contexts with motion, going faster means going farther. Since, in this context, more of one implies more of the other it is easy to generalize that less of one means less of the other. This leads to difficulties when velocity is positive but decreasing while position is still increasing which is exactly what happens in situations involving slowing down as one moves forward.

My own perspective is that it is easy to confuse velocity with speed and position with distance. In most everyday contexts, we most often refer to speed rather than velocity and distance rather than position. When I drive from home to the store and back again I would not generally describe my rate in terms of velocity, 35 miles per hour going from home to the store and -35 miles per hour going from the store to home. I would just say that I traveled a speed of 35 miles per hour the whole way. Similarly, I would refer to the total distance for the trip, about 24 miles with 12 miles from home to store and 12 miles from store to home. I probably would not refer to the trip as starting at position zero before I left home, going in the positive direction to position 12, and then moving in a negative direction back to position zero. That would be a perfectly valid description of the trip but in my

everyday mode, I would not tend to think of it that way or describe my trip to the store in those terms.

A variational approach is needed. Nemirovsky and Rubin (1992) claimed that students need a variational rather than a resemblances approach to functions and their derivatives in order to come to understand relationships between the two. A resemblances approach to functions and their derivatives assumes that the graphs of a function and its derivative resemble each other, that if one is increasing so is the other, if one is positive then so is the other, or if one is a line so is the other. A variational approach relates local variation in a position function to the corresponding value of the velocity function to understand relationships between the two functions. More specifically, a variational approach considers how changes in the position over a small interval of time relates to the velocity over that same interval.

A steep or large slope on the graph of an position function means a large change in position which in turn means that the velocity is also large in absolute value.

Another example of a variational approach is to use the value of the velocity at a point to infer how much the position changes over a small interval of time around that point. If the velocity three seconds after the car started moving, at $t=4$, is -5 feet/second, then we could infer that from $t=3.5$ to $t=4.5$ the car moved about 2.5 feet in the negative direction so that the position of the car decreased about 2.5 feet over that half second. Nemirovsky and Rubin noted that students need to think about these kinds of variational relationships to better understand relationships between velocity and position functions.

One precalculus student's construction of meaning of the sign of the velocity. Nemirovsky (1994) reported on one student's work involving negative velocity when using the car on the track. Initially the student, Laura, believed that velocity was always positive. In her mind, greater and less velocity was associated with the car moving at faster and slower speeds, respectively. She did not at this point consider what the direction of the motion of the car might mean in terms of the velocity.

As she moved the car back and forth the device produced velocity graphs that had some sections above and other sections below the horizontal axis. According to her remembrance, when she moved the car away from the motion detector, the car was going slower and when she moved the car toward the motion detector, the car was going faster. Therefore, she came to associate velocity above the horizontal axis (velocity for when the car was moving away from the motion detector, positive velocity) with going slower and velocity below the horizontal axis (velocity for when the car was moving toward the motion detector, negative velocity) with going faster. In this context zero velocity was associated with the place where the car changed from going slower to going faster.

Later in the interview Laura associated negative velocity with distance closer to the motion detector and positive velocity with distance farther away. So whereas before she associated the sign of the velocity with speed of the car - positive means slower, negative means faster - she now associated sign of the velocity with distance of the car from the motion detector - negative means closer, positive means farther away.

With suggestions from the interviewer, Laura conducted trials where the car moved only one direction, either away from or toward the motion detector. Laura then came to see that positive velocity means moving away from the motion detector and negative velocity means moving toward the motion detector with zero velocity associated with the car changing direction. Laura had come to see the sign of velocity as giving information about the direction of movement.

Nemirovsky noted that Laura's understanding of velocity changed because of experiences during this interview, that as she proceeded with the interview, she radically changed her understanding of the sign of the velocity. Some might think therefore that this particular sequence of experiences holds the key to students learning the sign of the velocity. He cautions us to be careful about making this conclusion for two reasons. First, there may have been other experiences that would have helped Laura come to this understanding of velocity sign. Second, there is no guarantee that this particular sequence of experiences would lead other students to the same understanding of the sign of velocity. According to Nemirovsky, we cannot be sure that a single set or sequence of problems will necessarily yield particular knowledge in students.

Summary of TERC studies. In these studies we saw some high school students' knowledge of rate of change in the context of position and velocity graphs for a car moving on a track. These students assumed that velocity and position graphs were similar. According to these researchers students assume similarities between velocity and position because often they are similar, especially in certain everyday contexts.

Assuming similarity simplifies problem solving, helps students use their previous experience, and encourages them to recognize and use

patterns. Nemirovsky and Rubin call for experiences with rate of change that help students construct a variational approach to make sense of situations. Nemirovsky guided one student through a set of experiences with position and velocity that helped her understand what the sign of the velocity meant.

Research involving precalculus students' study of functions via rate.

Confrey and Smith (1994, 1995) investigated students' thinking about rate of change, specifically in the context of linear and exponential functions. They discussed two ways of thinking about functions that influence how students think about rate of change. One is a correspondence view of functions, what Monk (1992a, 1992b) called a pointwise view of functions. The focus is on relating values of the independent variable to corresponding values of the dependent variable, for example finding a value of the dependent variable for a given value of the independent variable. This places the emphasis on describing a function in terms of the correspondence between the independent and dependent variable. When we express Y as a function of X (for example, as $Y = f(X)$), we are emphasizing a correspondence aspect of functions.

Another view of functions is called a covariation view. This is what Monk (1992a, 1992b) called an across-time view of functions. In this view, the focus is on relating changes in the dependent variable for changes in the corresponding values of the independent variable, for example relating consecutive differences in Y values for consecutive one unit differences in X values. This places the emphasis on describing a function in terms of change or rate of change in the dependent variable relative to change in the independent variable. When we define a function recursively (for example, as $Y_{n+1} = f(Y_n)$), we are emphasizing a

covariation aspect of functions. According to Confrey and Smith and Monk, students need a covariation view of functions in order to understand calculus. Covariation makes rate of change more visible and makes it easier to describe functions in terms of their rate of change.

Confrey and Smith worked with high school students who had taken at least two years of algebra but who had not yet taken calculus. They were given a table of values for a situation involving exponential growth. The particular context was the growth of a cell population over nine hours, from $t=0$ to $t=9$. The table of values showed the number of cells at $t=0$ and at the end of each hour. Students were asked to predict the number of cells at $t=10$. They focused their attention on relating the y -values to each other. This is more consistent with a covariation view of function than with a correspondence view.

Some students related the y -values using the difference between consecutive values of Y , $Y_{n+1} - Y_n$, for one unit differences in the x -values. Confrey and Smith called this an additive rate of change. Students had used additive rate of change earlier in the study to discover that with linear functions these differences were all the same constant. Here with the exponential situation, the additive rate of change did not yield differences that were all the same constant. So that led students to consider other relationships between y_n and y_{n+1} that yielded the same constant. For exponential functions, the ratio y_{n+1}/y_n yields the same constant. This leads to $y_{n+1} = k_1 * y_n$ as a relationship between consecutive y -values for exponential functions. Confrey and Smith called this a multiplicative rate of change.

Another relationship between y_n and y_{n+1} that holds for exponential functions is $y_{n+1} = y_n + k_2 * y_n$ where k_2 is a constant. Confrey and Smith

called this a "proportional new to old" rate of change. It emphasizes the proportion of y_n to be added to y_n (the "old") to get y_{n+1} (the "new"). Because of students description of linear and exponential functions in terms of changes from one y -value to the next, Confrey and Smith conjecture that change and rate of change may be a way that students investigate functions and may therefore be an entry point to their study of functions.

Confrey and Smith refer to work by Kaput and West (1990) in which the concept of rate also involves homogeneity, that is the ratio of changes in Y to changes in X remains the same over different changes in X . Thompson (1994a) refers to this as "reflectively abstracted constant ratio" (p. 192). By that he means that students must come to see constant rate as meaning that no matter how large or small the change in X , the ratio of the change in Y to the change in X is the same. Homogeneity holds for constant rate but does not hold for variable rates. Confrey and Smith advocate expansion of the definition of rate to include consideration of variable as well as constant rates. This, they believe, will ease the transition to calculus, the study of variable rates.

This study by Confrey and Smith involved equations and table of values in students' exploration of functions and their rates of change.

We might also consider the role of graphs. Confrey and Smith referred to Nemirovsky's (1991) six basic shapes of graphs from which graphs of many continuous functions can be constructed. Nemirovsky suggested that many students would find calculus easier if they first understood how these six basic shapes display different rates of change.

Confrey and Smith further wondered if relationships between these six basic shapes and slope represent another way to investigate rate of change. They proposed that many students make connections between slope and rate of change without reference to the numerical value of the slope or of changes in the dependent variable over intervals of the dependent variable. They conjectured that students' knowledge of relationships between slope and shape of graph may serve as a basis for advancing their knowledge of rate of change.

These studies showed that high school students often find it reasonable to study functions in terms of change in the dependent variable for fixed changes in the independent variable rather than as a relationship between the dependent variable and the independent variable. That is, students think about functions in terms of covariation rather than as a correspondence. This suggests that one entry point to the study of functions is via rate of change and that students' understanding of rate of change is enhanced with this approach to functions. Confrey and Smith suggested other ways to study functions and rate of change including consideration of variable as well as constant rate of change and connections between qualitative slope and shape of graph.

Research about children's views of constant rate.

Patrick and Alba Thompson (1992) interviewed middle school students around situations involving constant rate. They were interested in how students come to understand rate as a quantity formed by the ratio of two other quantities, each of which is the change in another quantity. The Thompsons identified four different ways students think about constant rate. Some students think of constant rate in

terms of "speed as a distance" and "time as a ratio". These students think of speed as a distance that is traveled in some unit measure of time. A speed of 15 feet per second means 15 feet are traveled and it takes one second to do so. To find the distance traveled at a given speed over a given amount of time, these students lay out the number of speed-distances corresponding to the number of time units and add up the total distance. For example, to find the distance traveled in five seconds at a speed of 12 feet per second, these students measure out five speed-distances of "12 feet per second" and add up the total distance to get 60 feet.

To find the time it takes to travel 100 feet at a speed of 20 feet per second, a student who thinks about rate in terms of "speed as a distance" will find the number of units of the speed-distance "20 feet per second" that can be measured out in 100 feet; that number will be the number of seconds it takes to travel 100 feet. This presents a view of time as the ratio of the total distance to the speed-distance.

To find the speed it takes to travel a given distance in a given amount of time, students often guess a speed and then check it by laying down an appropriate number of speed-distances corresponding to the given number of time units to see if the resulting distance matches the given distance. To find the speed needed to travel 100 feet in six seconds, a student will guess a speed, say 15 feet per second, and then measure out six speed-distances of "15 feet per second" to see if they add up to 100 feet. In this case six speed-distances of "15 feet per second" do not cover the full 100 feet. So the student may then try a faster speed, say 16 feet per second, and repeat the procedure until a speed-distance is found so that six of them measures out to 100 feet.

Students with this view of constant rate may conclude that there is no speed at which one can travel to cover 100 feet in six seconds since they may not be able to find a speed so that six of the resulting speed-distances adds up to exactly 100 feet. The Thompsons refer to this view of constant rate as additive since it relies on finding out how many speed-distances to add up to yield a certain distance.

Other ways of thinking about speed. A second way students think about constant rate is in terms of speed as a distance with an implied number of time units. Students with this view can find the time associated with a given distance less than 15 feet since they see ratios of increments of distance to the corresponding increments of time as being constant. So a distance of five feet implies a time of one-third of a second. But they cannot find the time associated with a distance more than 15 feet since they do not think of ratios of total accumulations of distance to total accumulations of time as having the same constant ratio. So a distance of 45 feet does not imply to them a time of three seconds. Students with this view also cannot find the distance for a given time since time does not necessarily imply distance. So a time of one-third of a second does not necessarily imply a distance of five feet nor does a time of three seconds automatically yield 45 feet.

A third way students think about constant rate is in terms of increments of time and distance occurring together so that each increment in time implies a corresponding increment in distance and each increment in distance implies a corresponding increment in time. A speed of 15 feet per second is seen as made up of two quantities - 15 feet and one second - so that given a distance less than 15 feet, one

can find the corresponding time and given a time less than one second, one can find the corresponding distance. Students with this view of constant rate see that the ratio of an increment of distance to a corresponding increment of time is constant for all increments. However, they are not able to consider that at each increment the ratio of the total accumulation of distance to the total accumulation of time is also the same constant. They do not associate 45 feet with three seconds nor do they associate three seconds with 45 feet.

A fourth way students think about constant rate is in terms of distance and time accumulating together so that not only is the ratio of an increment of distance to the corresponding increment of time constant for all increments but at each increment the ratio of the total accumulation of distance to the total accumulation of time is the same constant. A speed of 15 feet per second not only implies that five feet means one-third of a second and vice versa but it also implies that 45 feet means three seconds and vice versa. According to the Thompsons, this fourth view of constant rate is what it means to know constant rate.

Developmental stages for rate. The Thompsons imply that these four ways of thinking about constant rate are developmental stages that students work through with the first two stages associated with an additive view of rate and the last two stages associated with a multiplicative view of rate. Patrick Thompson (1994b) gave a detailed account of his work with one 10 year old as she moved through these four stages. Initially her understanding of rate included speed as a distance (how far in one unit of time) and time as a ratio (how many speed-distances in a given distance). She measured out a given distance

in units of speed-distance to yield the time it took to travel a given distance, with one speed-distance corresponding to one unit of time. She used guess and check to determine speed when given a distance to be traveled in a given time. These are all behaviors that are consistent with the first view of rate described above.

Over several sessions her understanding of rate was extended so that a certain number of speed-distances necessarily implied a corresponding number of time units, a view consistent with the second view of rate described above. Later she came to see rate in terms of both distance and time changing together so that the ratio of changes in distance to the ratio of changes in time were constant. This view is consistent with the third view of rate described above.

In the final sessions she worked with other types of rates but initially relied on her earlier conception of "speed as a distance" to solve these new rate problems. As she continued to work with these problems she came to distinguish between two changing quantities in each situation so that their ratio remained constant over increments and accumulations. This view of rate is consistent with the fourth view described above.

Importance of constant rate in coming to know variable rate. In another study (Thompson, 1994b) Patrick claimed that one reason calculus students struggle with the Fundamental Theorem of Calculus is because they have "impoverished images of rate" (p. 229). I take this to mean he believes students have trouble understanding variable rate and its relationship to accumulation because they have not yet grasped the significance of constant rate. They have not come to see constant rate as having something to do with two quantities changing together in such

a way that changes in one necessarily imply changes in the other, that the ratio of any size increment of one quantity to the corresponding increment of the other quantity yields the same constant no matter how large or small the increment, and that the ratio of accumulations of one of the quantities to the corresponding accumulations of the other quantity yields the same constant. In other words, they have not come to see rate as "reflectively abstracted constant ratio" (1994a, p. 192).

As a closing remark to this study (1994a), Thompson noted that it does little good to teach students that speed is distance divided by time, specifically that constant speed can be found by dividing the amount of distance traveled by the amount of time to travel it, if they do not already have an idea that motion involves two quantities - distance and time - and that in the case of constant speed these two quantities change together in such a way that the ratio of changes in each quantity at each iteration is the same from iteration to iteration and is the same as the ratio of total accumulations of each quantity at each iteration. His concern is that students, even college students, do not have an view of constant rate on which they can then build an understanding of variable rate and accumulation.

Patrick and Alba Thompson showed us that a proper understanding of constant rate means being able to coordinate two changing quantities so that the ratios of increments or total accumulations of one to increments or total accumulations of the other are all equal to the same constant. They also showed that elementary school aged children do not come to this understanding easily. Further if students come to calculus without a proper understanding of constant rate than Patrick argues it is little wonder if they struggle with variable rate.

Research about college physics students' work with velocity and position graphs.

McDermott, Rosenquist, and van Zee (1987) studied university physics students' knowledge of position and velocity as represented by graphs. In one task students were given a position-time graph showing the position of two objects moving at different constant rates. (See Figure 4.) Object A started at zero centimeters at $t=0$ seconds and ended at 85 centimeters at $t=10$ seconds. Object B started at 20 centimeters at $t=0$ seconds and ended at 55 centimeters at 10 seconds. Since the rate for object A is faster than for object B, the line for object A has steeper slope. The two lines intersect at $t=4$ seconds with the line for object A below the line for object B from $t=0$ to $t=4$ seconds and the line for object A above the line for object B from $t=4$ to $t=10$ seconds.

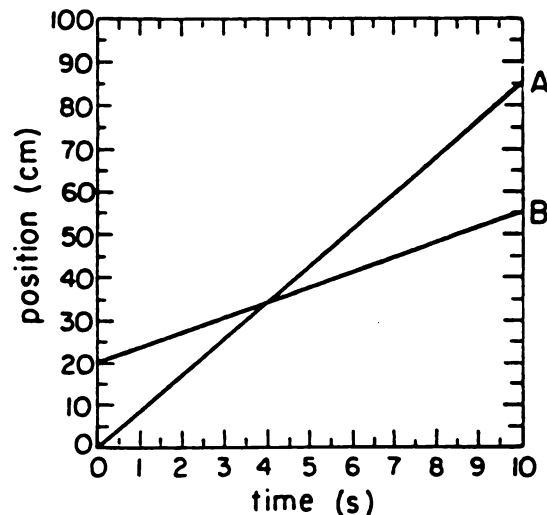


Figure 4 - Position of two moving objects over 10 seconds

Students were asked if the speed of object A was greater to, less than, or equal to the speed of object B at $t=2$ seconds. Many students said that since the graph of object A was below the graph of object B at

$t=2$ seconds, the speed of object A was less than the speed of object B.

Students were using relative heights instead of relative slopes of the position graphs of objects A and B to make conclusions about their speeds. Students were also asked if objects A and B ever had the same speed. Many said that they had the same speed at $t=4$ seconds because that is where the graphs of their positions intersect. They were using the fact that objects A and B have the same position to conclude that they have the same speed.

These responses are similar to responses of students in the TERC studies where students assumed resemblance's between a function and its derivative. Specifically, students in the TERC studies used graphs of velocity functions to make conclusions about graphs of position functions. They assumed that a position function looks similar to its velocity function. In the McDermott et al. study, students used a graph of the position function to make conclusions about the velocity. They may have assumed that the velocity resembles the position function or that there is only one graph for a motion.

Using position-time graphs to interpret speed. Students were also given a position-time graph for one moving object and asked to specify the points where the object was moving the slowest, speeding up, slowing down, and turning around. (See Figure 5.) Many students identified the point with height zero (point D) as the point where the object was moving the slowest. A correct answer is point B or F where a slope of zero indicates a speed of zero, the slowest speed possible. Many students specified a point where the graph was increasing (point A) as the point where the object was speeding up. A correct answer here is point G since the magnitude of the slope is increasing there.

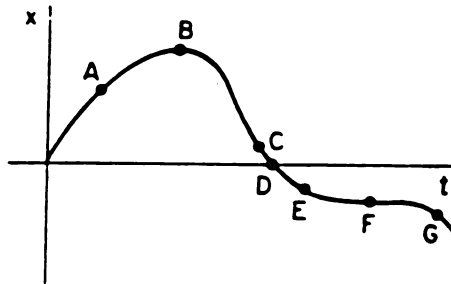


Figure 5 - Position of a moving object over time

In both cases students appeared to confuse position and velocity. That would help explain why point D, with the smallest position, was identified as the point of smallest speed. It would also explain why point A, with an increasing position, was identified as the point where the speed was increasing.

Differences between speed and velocity. Some of the confusion here may be due to the difference between speed and velocity. A velocity of 15 feet per second or -15 feet per second yields a speed of 15 feet per second. Speed is always positive while the sign of the velocity designates direction. Slope of a position-time graph yields information about the velocity of a moving object. If the graph is increasing and getting steeper, then since the slope of the graph is positive and increasing, both the velocity and speed are positive and increasing. But if the graph is decreasing and getting steeper, then the slope is negative and decreasing ("getting more negative") so that the velocity is negative and decreasing while the speed is positive and increasing. So if students saw that point G has a slope that is negative and decreasing so that the velocity was negative and decreasing

at point G, they still may not have seen point G as a place where the speed is increasing.

Slowing down and turning around. To answer the question about where the object was slowing down, some students indicated the point G where the graph achieves its lowest height. This is also a point where the slope of the graph is negative which in the minds of some students may mean that the object is slowing down. The object is slowing down at points A, C, D, and E since the magnitude of the slope is decreasing. Many students associated the point where the graph crossed the x-axis (point D) as the point where the object turned around. Point B is where the object turns around since its position was increasing before that point and decreasing after. Again we see students assuming resemblance's between position and velocity.

In this study students were also given several graphs of position versus time and asked to sketch graphs of the velocity versus time. Many students sketched velocity-time graphs that were incorrect. On a position-time graph, a peak or dip would show when an object reverses direction. The velocity-time graph would cross the horizontal axis at these times. Many students in this study sketched a velocity graph that had a peak or dip to show an object changing directions. This is another example of students assuming resemblance's between a function and its derivative.

Students in this study also had trouble relating velocity-time and acceleration-time graphs. When given the velocity-time graph and asked to find the acceleration, many used the ratio velocity/time instead of the ratio $(\Delta v)/(\Delta t)$. In a situation involving constant acceleration, they obtained a numerical value for that acceleration by

dividing the value of the velocity at a point on the velocity-time graph by the corresponding time value.

Summary of McDermott, Rosenquist, and van Zee study. As in the TERC studies with high school students, these university physics students, all of whom had completed calculus, appeared to assume resemblances between functions and their derivatives. While in the TERC studies students were given a velocity graph and asked to predict the position graph, in this study students were given a position graph and asked to identify points on the position graph satisfying a particular requirement for velocity or sketch a graph of the velocity. In both the TERC studies and this study of physics students we saw students giving answers that imply they were assuming that a velocity and position function are similar.

Students use of the ratio of velocity to time to find the acceleration rather than $(\Delta v)/(\Delta t)$ raises questions about connections students might be making when formulas such as distance = rate*time and velocity = distance/time and acceleration = velocity/time are used for instruction involving rate. These formulas hold in certain situations but in general they do not hold for situations involving variable rates.

Research involving calculus and postcalculus students' work with related rates.

Steve Monk (1992a, 1992b) conducted two studies about calculus and postcalculus students' understanding of rate in the context of related rates problems. One study used a problem involving a six foot man walking away from a 16 foot light pole with a shadow of the man being cast by the light. A physical device made it possible for students to

move a cardboard "man" along a track and to measure distance from the pole to the man, from the pole to the tip of the shadow, and from the man to the tip of the shadow. A flexible wire from the top of the light pole to the head of the man and then on to the track made it possible to decide where the tip of the shadow was at each position of the man along the track. (See Figure 6.) Students were asked to consider that the man was walking at a constant rate of five feet per second and to determine the rate at which the tip of the shadow was moving and the rate at which the length of the shadow was growing when the man was 10 feet from the pole.

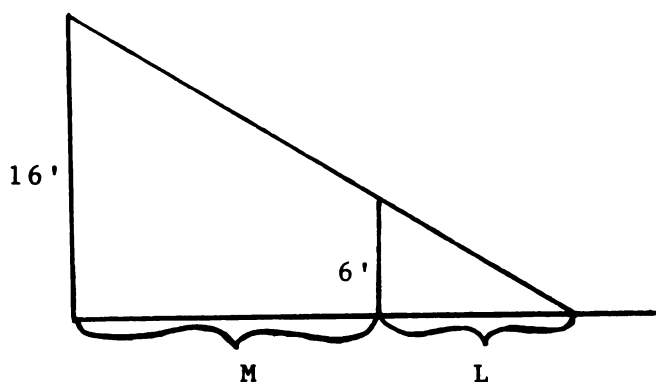


Figure 6 - Diagram of a 6' man walking away from a 16' light pole

Students' intuitive models. Some students used what Monk called intuitive models of the situation to make conclusions about the tip and length of the shadow. One student said that since the tip of the shadow was farther away from the light pole than the man, it must be moving faster. That is, greater distance means greater speed. Monk noted that this student's response was similar to the response of college physics students reported by McDermott, Rosenquist, and van Zee (1987) where students responded in ways that make us think they believe that velocity resembled position, that greater position meant greater velocity.

This is also consistent with what Nermirovsky and Rubin noticed in their studies. Their subjects were thinking that more velocity means more distance. Monk's student was thinking that more distance means more velocity. In many situations in everyday life more velocity does mean more distance traveled in a given amount of time and vice versa . This can lead students to reasonably conclude that more of one necessarily means more of the other.

Monk's student also said that since the length of the shadow was getting longer, then its tip was moving at an increasing rate. This is another example of students thinking that more of one thing necessarily means more of another; in this case more length means increasing speed.

This student also noted that since one object was not moving at the same constant rate as another object, then it was not moving at a constant rate. She was not distinguishing between two objects moving at the same constant rate and two objects moving at different constant rates.

Students' uses of quantities. In this study students made use of quantities to explore the rate of change of the tip and length of the shadow. In this task there were several quantities, each with total lengths (total distance from the light pole to the man, total distance from the man to the tip of the shadow, and total distance from the light pole to the tip of the shadow) and incremental lengths (length of one step of the man, corresponding increase in the length of the shadow, and corresponding increase in the distance the tip of the shadow is from the light pole). Monk called comparisons of increments of the same quantity "within" comparisons and comparisons of increments of one quantity to increments in another quantity "between" comparisons.

Some students used one of these comparisons to support statements about the other type of comparison. For example, one student examined increments of the length of the shadow for one foot increments in the man's distance from the light pole. She found that increments in the length of the shadow were equal to each other but since they were not equal to the increments of the man's distance from the pole, she concluded that the length of the shadow could not be increasing at a constant rate. She compared increments of one quantity to increments of another quantity to make a decision about the rate of change of one of the quantities. She was using a "between" comparison when she should have been using a "within" comparison. Monk called this a "comparison" error.

Some students confused increments of one quantity with total distance or length of the same quantity. Monk called this an "increment/amount" error. Four of the eight students said the rate of change of the length of the shadow was increasing based on observations that the total length of the shadow was increasing. Some students noted that the distance between the pole and the tip of the shadow was greater than the distance between the pole and the man and concluded therefore that the tip of the shadow was increasing at a rate faster than the man was walking. They were comparing total length of one quantity to total length of another quantity when they should have been making "within" comparisons of increments of each of the quantities.

Students' uses of tables of values and average rate of change.

One student recorded the distance from the light pole to the tip of the shadow (L1) for each one foot increment in the man's distance from the light pole. Then for each resulting position of the shadow tip he

divided L_1 by the corresponding total distance from the pole to the man (L_2). He then used these quantities to make conclusions about the rate of change of the tip of the shadow with respect to the position of the man. In effect, he was dividing total change in one quantity by total change in another quantity. This yields the average rate of change in one of the quantities relative to the largest available intervals of the other quantity. This is the analysis Monk gives this students' work.

There is another way to look at the ratio L_1/L_2 . It is the value of the position of the tip of the shadow divided by the value of the position of the man. It is the same thing as finding, when given an equation of y in terms of x , the rate of change of y with respect to x at a particular value of x by dividing the corresponding value of y by the value of x . It happens in the case of the shadow problem and in many other situations that this ratio has an interpretation both as an overall average rate of change and as a ratio of a value of the dependent variable to the corresponding value of the independent variable. The former method is a crude but correct way to think about rate of change. The latter method is not, though in certain cases it often yields the same numerical answers as the former method.

One student made a table of values for the length of the shadow (L_3) for each one foot increment in the position of the man. She then computed increments of L_3 for each increment in the position of the man and looked for an equation that relates these increments. Because of variation in the values of the increments of L_3 , she was unable to find an equation. However, this and other students' use of total and incremental distances and lengths show how students use these quantities to try to make sense of situations involving rate of change.

Students' uses of representations of functions. Several students tried to obtain algebraic representations of this situation. Most could not. One student obtained an equation relating the position of the man (M) to the length of the shadow (L): $M = 5/3 L$. She then used this equation to construct a table of values and plotted the corresponding points to obtain a graph of M versus L. She then used the shape of the graph, a straight line, to conclude correctly that since the rate of change of the position of the man is constant than so is the rate of change of the length of the shadow. She did not attempt to differentiate the equation in terms of time or use it in ways taught in calculus.

The second study by Monk was situated around a problem involving a 14 foot ladder leaning against a wall. The bottom of the ladder begins to move away from the foot of the wall at a constant rate of five feet per second. In the physical model, the ladder was on a ruled track and could be moved so that distances between the foot of the wall and the foot of the ladder and between the top of the ladder and the foot of the wall could be measured. (See Figure 7.)

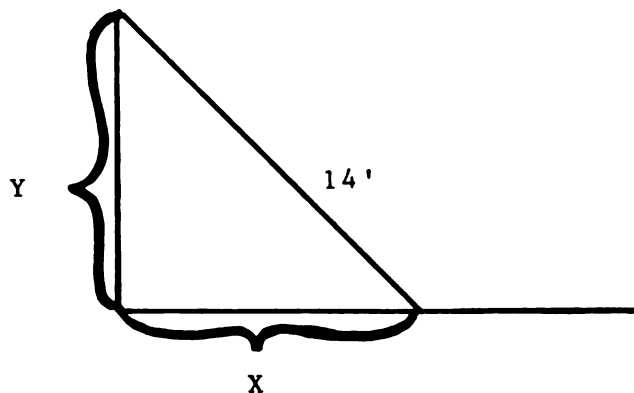


Figure 7 - Diagram of 14' ladder sliding down a wall

Monk asked students to decide if the top of the ladder moved down the wall at an increasing, decreasing, or constant speed and to explain their answers. He also asked them to draw a graph to represent the situation and explain it. He was interested in the variables they chose to represent, the shape of their graphs, and how they used their graphs to decide if the top of the ladder moved down the wall at an increasing, decreasing, or constant speed.

Three students constructed their graphs by making a table of values for X , the distance between the base of the ladder and the foot of the wall, and Y , the distance between the top of the ladder and the foot of the wall, plotting the corresponding points, and sketching a smooth curve through the points. They obtained graphs that were correct. This is a tedious method of obtaining a graph, but it was used successfully to obtain a graph of Y versus X and helped students confirm or deny their earlier answers about the speed of the ladder as it moved down the wall.

All of the other students used a few data points, information previously obtained from the model, and conclusions made from their earlier work to simply strike an arc that seemed to have the right shape. What was surprising to Monk was that five of these students sketched a curve first and then decided how the horizontal and vertical axes should be labeled so that the graph matched information about the situation they already had. This yielded graphs that had X on the horizontal axis and Y or ΔY or the quantity $14 - Y$ on the vertical axis.

Summary of the Monk studies. I find two things particularly interesting about these two studies of calculus and postcalculus

students' knowledge of rate of change. One is that very few of these students worked these related rates problems in ways that have been traditionally taught in calculus. It can be depressing for a calculus teacher to realize that very few students learn these standard techniques and is yet another set of evidence that students struggle with learning rate of change.

The other, more positive, thing is that most of these students had ways of addressing rate of change that helped them make sense of the situation. They used intuition based on other experiences with distance and speed. They used various quantities like total and incremental lengths to investigate rate. They used numerical measurements of X and Y to ascertain relationships between X and Y and dx/dt and dy/dt . They used equations and formulas in several ways to represent relationships among various quantities, especially X and Y . Finally, they were able to use information obtained about the situation to construct graphs of the relationship between two quantities and to describe how their graphs confirmed or denied their earlier conclusions. These represent a variety of knowledge students have for making sense of situations involving rate of change.

Thompson's study of advanced mathematics students.

Thompson (1994b) claimed that one reason students have difficulty with the Fundamental Theorem of Calculus is because they have "impoverished images of rate" (p. 229). As mentioned above, I believe he means that students do not understand constant rate as "reflectively abstracted constant ratio" (1994a, p. 192). They have not come to see constant rate in terms of two changing quantities so that the ratio of changes in one to corresponding changes in the other is constant no

matter how large or small the changes. Many college mathematics students think of rate in terms of one quantity changing or think in terms of two quantities changing without regard to the ratio of corresponding changes in the two quantities maintaining a constant ratio over all such changes. If students do not understand constant rate of change in this way, then it should not be puzzling that they often do not understand variable rate of change.

Thompson's work with eight upper division mathematics students and eleven graduate students revealed one picture of "impoverished images of rate" for a situation involving variable rate of change. He gave these students a function represented by an equation $Y = f(X)$ and asked them to obtain a graph of the function. He then asked them to form functions given by $f_h(X) = [f(X+h) - f(X)]/h$ for values of h like 0.1, 0.01, and 0.001. Many of these students could not describe these functions. For each value of h , $f_h(X)$ represents the function which for a given value of X yields the average rate of change of Y over intervals of the form $[X, X+h]$ if h is positive and $[X+h, X]$ if h is negative. As h goes to zero, this family of functions converges to the function which is called the derivative of $f(X)$. Many of these students could not also say what function was obtained by taking the limit of this class of functions as h goes to zero.

Even if students have, before they come to calculus, the kind of knowledge of constant rate Thompson advocates, they would necessarily recognize what the functions $f_h(X)$ give or what function is obtained when h goes to zero. An important intermediate step between knowledge of constant rate and limit of average rate of change of Y over intervals of X of decreasing width is an understanding of average rate of change

under conditions of variable rate. Knowing constant rate may be a good place to start in that process but my own sense of the growth of understanding of rate is that students also need many varied experiences with average rate of change before and early in calculus in order for them to understand what it is that the functions f_h and the limit of f_h as h goes to zero tell us.

Summary of all studies reviewed here and consideration of holes in the research.

These studies showed some of what precalculus, calculus, and postcalculus students think and believe about rate of change. In the TERC studies precalculus students often made inappropriate conclusions about the position of a moving object given the graph of its velocity function. In the McDermott et al. study physics students who had completed calculus made incorrect conclusions about the velocity of moving objects given a graph of their position functions. In both of these studies students were unable to properly connect a function and its derivative given the graph of one of them. (See Table 1.)

In the Monk studies calculus and postcalculus students were unable to work related rates problems in ways that are taught in calculus. Most of them were unable to form appropriate algebraic representations of the situation and of those who did, none tried to differentiate the equations to address rate. Some who obtained algebraic representations used them to make tables of values, then made graphs from the tables, and then made conclusions about the rate of change of one of the quantities relative to the other by analyzing their graphs. Many students used increments and accumulations of quantities but they often made incorrect conclusions about rate from them.

Table 1 - Summary of literature reviewed

Researchers	Subjects	Results
TERC Studies (Nemirovsky, Rubin, and Monk)	Precalculus students	Given velocity graphs, students were asked to predict position graphs; they often assumed position graphs were similar to velocity graphs.
Confrey and Smith	Precalculus students	Given tables for several functions, students were asked to predict y values for the next x value; they used differences in given y values to study rate of change of linear and exponential functions.
Thompsons' study	Middle school students	Given two of the quantities speed, time, and distance, students were asked to find the third quantity; they thought about speed in four ways.
McDermott, Rosenquist, and van Zee	Univeristy physics students who had had calclus	Given position graphs, students were asked to predict velocity graphs; students often assumed velocity graphs were similar to position graphs.
Monk	Calculus and postcalculus students	Students were given two related rates problems; some used intuitive models, some compared increments and accumulations, and some made and used graphs to make decisions about rates; students often confused increments of one quantity with accumulations of that quantity or increments of another quantity.
Thompson	Postcalculus	Students were asked to form $f_h(x) = [f(x+h) - f(x)]/h$ for values of h approaching zero; many did not know what $f_h(x)$ or its limit as h goes to zero represented.

In the Thompson study, postcalculus students were often unable to correctly describe functions formed from Newton's quotient or to say what the limit of these functions mean. For upper division mathematics majors and graduate students in mathematics, this shows a serious lack of understanding of rate of change. All of these studies show where precalculus, calculus, and postcalculus students' understanding of rate of change is something other than what we would hope for and especially so in the case of students who have had calculus.

So one way to summarize these studies is to highlight deficiencies they show in students' knowledge. The TERC, Monk, and Thompson studies certainly do that. But another way to summarize these studies is to think about how students' knowledge, as incorrect and incomplete as it is, might serve them in coming to a more conventional understanding of rate of change. What knowledge did students in these studies have that may hold such promise? In the TERC studies, Rubin and Nemirovsky pointed out that often a function and its derivative do resemble each other. Students in those and the McDermott studies often assumed resemblances where there were none, but perhaps this tendency on students' part can be used to advantage in helping students come to know where a function and its derivative do properly resemble each other and where they do not. This seems a particularly productive possibility for learning rate of change.

Furthermore, these studies were based on particular situations involving motion. Are there other contexts for motion where students make appropriate connections between a function and its derivative in the context of graphs? What might a set of tasks about motion look like that could explore this further? These seem like important questions to

pose for trying to understand how students make connections between the graphs of a function and its derivative and how these connections could be capitalized on in helping students advance their understanding of rate of change.

In the Monk studies students' use of increments and accumulations of quantities to explore rate seems to me to hold promise for students' coming to know rate of change. Given that increments connect to average rate of change and, through a particular limit, to instantaneous rate of change, perhaps a consideration of students' tendencies to use increments should be further explored. What other tasks might show how students use increments and accumulations to make sense of rate of change? The Confrey and Smith study showed precalculus students using increments to explore properties of exponential and linear functions represented initially as tables of values and distinguishing them on the basis of patterns in these increments. In what other ways do students use increments to make sense of rate of change? This seems like an important question to try to answer.

In the Thompsons' study of middle school students' ways of thinking about constant speed, we saw a particular notion of what it means to come to know rate. Are there ways to investigate older students knowledge of rate in situations involving both constant and varying rates of change? What might a set of tasks look like for that investigation? One comment I made of Patrick Thompson's claim concerning calculus and postcalculus students' "impoverished images of rate" (1994b, p. 229) was that it is a long way from constant rate to a proper understanding of instantaneous rate of change and it must surely involve coming to know average rate of change in situations involving

varying rates and connections between average rate of change and instantaneous rate of change via limits. I think he would agree that research showing how students come to understand average rate of change and make connections to instantaneous rate of change are welcome. A set of tasks for investigating older students' knowledge of rate of change should include opportunities for them to show what they understand about average and instantaneous rate of change.

So a second way to think about these studies is to consider students' knowledge in light of how it may serve them in learning more about rate of change. Taken that way, some holes in the research concerning students' knowledge of rate of change exist. There appears to be no research that investigates how students from different points in mathematics education (for example, precalculus, calculus, and postcalculus) address a common set of tasks concerning rate of change. There are several reasons such a study could be a valuable addition to the literature. One is that it may provide evidence that some of the ways students addressed tasks involving rate of change in the studies reviewed above are not atypical but are representative of what other students at the same or other places in mathematics education do with other situations or function representations. This information would show us how prevalent these methods of addressing rate of change are and should cause us to think even harder about what they say about students' knowledge and how it can be used to advance their understanding of rate of change.

Another reason such a study would be a valuable addition to the literature is that we could see how many different ways students at these levels of mathematics education address rate of change. If there

are lots of different methods, then this both provides a richer picture of students' knowledge and shows how complex learning rate of change can be. It might imply that there is not a single piece of knowledge we can assume all students have and upon which we can plan all instruction in rate of change. At the same time it gives us more ways to think about how to design successful instructional interventions.

A final reason a study of knowledge of rate of change of precalculus, calculus, and postcalculus students would be a helpful addition to the literature is that we could see how similarly or differently students at these different points in mathematics education address the same problems. If precalculus, calculus, and postcalculus students use some common methods for addressing rate of change, then that might show that some knowledge is so useful that students at different points of mathematics education find it helpful in making sense of situations involving rate of change. If students at different points in mathematics education use different methods for addressing rate of change, then we should think about what this might say about learning rate of change. It may suggest that students progress along particular streams in coming to know rate of change and provide clues about how we can help that progression along.

This dissertation attempted to fill this hole in the research literature. This study presented to groups of precalculus, calculus, and postcalculus students a set of tasks about rate of change that gave them opportunities to make connections between functions and their derivatives. Students were asked to construct graphs and tables of values for functions when given information about their rates. They were asked to use given graphs and tables of values for functions to

make conclusions about their rates. They were also given an opportunity to use a variety of methods to address rate of change including use of increments and accumulations, shape and slope of a graph, average rate of change of one quantity over intervals of various sizes of another quantity. They were also given the chance to relate average rate of change to instantaneous rate of change. Four of the six tasks in this study focused on motion in contexts different from the TERC and McDermott et al. studies to see if students have ways of thinking about rate of change in situations involving motion that make appropriate connections between a function and its derivative. The other two tasks dealt with growth, one of a biomass and the other of the area of a circle. It was hoped that these tasks and these function representations would give students ample opportunity to show how they were thinking about rate of change, would show various methods of addressing rate of change, and would provide a view of how students at these levels of mathematics education come to know rate of change.

Relationship to perspectives and types of reasoning

In Chapter 1 I presented a particular view of rate of change in terms of three perspectives - global, interval, and point - and three type of reasoning - qualitative, numerical, and algebraic. Here I would like to discuss these in light of the literature reviewed above. If these perspectives and types of reasoning are reasonable ways of looking at rate of change, then at least some of the literature on students' knowledge of rate of change should show students' thinking in ways that are consistent with them.

Rate of change perspectives.

Recall from Chapter 1 that global rate of change considers changes in the dependent variable of a function over the entire domain of the independent variable, interval rate of change refers to change in the dependent variable over intervals of the independent variable smaller than its entire domain, often intervals of some convenient length like one unit, and point rate of change focuses on change in the dependent variable for an individual value of the independent variable, what mathematicians usually refer to as instantaneous rate of change.

Is there evidence in the studies reviewed above that these perspectives provide a framework for analyzing students' reasoning about rate of change? Did the students in these studies display knowledge that could be interpreted in light of these three perspectives? In the TERC studies, precalculus students created position graphs that resembled given velocity graphs over the entire domain of the independent variable. In the McDermott et al. study university physics students created velocity graphs that resembled given position graphs. In both cases students seemed to be thinking about the functions in a global way matching the increasing/decreasing behavior of one of the functions to the same behavior in the other over all relevant values of time, the independent variable. Clement (1989) called this type of error a global correspondence error since students assumed the two graphs resembled each other over their entire domains.

In the Confrey and Smith study, precalculus students used changes in the dependent variable over intervals of the independent variable of length one to explore the rate of change of exponential functions. In the Monk study calculus and postcalculus students also used changes in

the dependent variable for one unit intervals of the independent variable to make decisions about rate of change. While we might hope that these students would have more sophisticated ways of addressing rate of change based on what was taught in their calculus course, their use of changes over intervals shows that one way students think about rate of change is in terms of intervals.

In Thompson's study of advanced mathematics students, students formed functions that give the average rate of change of the dependent variable over intervals of the independent variable and were asked to consider what happens to those functions as those intervals decrease in length about an individual value. This is at the heart of instantaneous rate of change and the derivative. Students had difficulty thinking about these functions, connecting them to average rate of change, and specifying what their limit was. This would seem to imply that point rate of change is not a meaningful way for students to think about rate of change.

However students in the TERC and McDermott et al. study were asked to make judgments about velocity at individual points in situations involving varying rates and they proceeded to do so without questioning the existence of such a thing. That, of course, does not mean they believed that rate of change at a point makes sense or that they had ways of finding it. However, their everyday experiences with motion and speed may have helped them believe that moving objects have speed at points, that speed can change from instant to instant, and that it has a measurement in terms of distance traveled in a certain amount of time. It is unclear based on the studies reviewed above to what extent students think in terms of rate of change at a point.

Reasoning types.

Recall from Chapter 1 that qualitative reasoning refers primarily to the use of graphical features like slope and shape to make conclusions about rate of change, numerical reasoning concerns the calculation or use of numerical values to make decisions about rate of change, and algebraic reasoning involves the use of formulas or equations to make judgments about rate of change. Is there evidence in the studies reviewed here that these types of reasoning are used by students in addressing rate of change? In the TERC studies students used shape and increasing and decreasing properties of the graphs of velocity functions to make conclusions about the graphs of position functions. In the McDermott et al. study students used similar properties of graphs of position functions to make conclusions about velocity. In the Monk studies students constructed graphs of the relationship between two varying quantities and used those graphs to make and confirm conjectures about the rate of change of one of the quantities with respect to the other. The responses of students in these studies give evidence that students use properties of graphs, and hence qualitative reasoning to make sense of rate of change.

Students in the Confrey and Smith study computed and used first order differences to make sense of rate of change of linear and exponential functions. Students in the Monk studies used increments of one quantity over one unit changes in another quantity to make conclusions about the rate of change of one of the quantities with respect to another. In each of these studies, students calculated some, or in some cases a lot, of numerical values to help them make sense of

rate of change. They were using what I have called numerical reasoning to reason about rate of change.

In the Monk studies, students often tried to find an equation relating variables in the situation. Some were successful but then were not able to differentiate these equations to address rate of change of one of the variables with respect to another. They were however, often able to use their equations in other ways to address rate of change in these situations. Some students transformed the equation into a table of values or a graph and then used these to make conclusions about the rate of change of one quantity with respect to another. In as much as the equations helped these students proceed with their analysis, I saw it as an example of algebraic reasoning.

In the Thompson studies, students were given an equation representing a function and asked to work with equations of the function giving the average rate of change over certain intervals and more generally to try to related this to the function giving the rate of change at a point. Students were often unable to work very successfully with the equations yielding the average rate of change of the dependent variable over intervals of the independent variable. But they were able to obtain the derivative equation for most of the given function equations and to use them to address rate of change. These are examples of algebraic reasoning.

Summary.

The studies reviewed above show students thinking in ways that are consistent with at least two of the three perspectives on rate of change and the three types of reasoning. One of the goals of this dissertation was to see to what extent these three perspectives and types of

reasoning make sense in terms of how students think about rate of change.

Chapter summary

Of the research literature concerning students' knowledge of rate of change, I chose to review studies which showed how students solved problems involving rate of change so that inferences about the current state of their knowledge could be made and conjectures about how that knowledge might serve them in learning more about rate of change could be put forth. All of the literature reviewed here showed difficulties precalculus, calculus, and postcalculus students had and considered possible reasons for those difficulties. Some of it also discussed ways to help students learn rate of change.

Studies conducted at TERC showed precalculus students assumed that position and velocity graphs resembled each other and as a result often made incorrect conclusions about the position graph when given the velocity graph. A study by McDermott et al. found university physics students who had completed calculus made incorrect conclusions about velocity given graphs of position functions and often drew velocity graphs that inappropriately resembled given position graphs. Monk found that calculus and postcalculus students were unable to work standard related rates problems in ways taught in calculus. Thompson found that postcalculus students did not know what functions formed by the Newton Quotient yielded or what their limit meant. This literature showed students struggling with rate of change.

In addition, this literature showed knowledge precalculus, calculus, and postcalculus students have that may support their learning of rate of change. Rubin and Nemirovsky explained that one reason

students assume a function and its derivative resemble each other is that often they do. Students in the Monk studies relied on increments of quantities and shape of graphs to make decisions about the rate of change of one quantity with respect to another. Confrey and Smith reported on precalculus students who used increments to make sense of functions and their rates of change. Patrick and Alba Thompson gave an accounting of how middle school students come to understand constant rate.

These pieces of knowledge about and methods of dealing with rate of change as well as the insights about how younger children learn constant rate gave impetus to the design of this study about older students' learning of rate of change. I wanted to see if precalculus, calculus, and postcalculus students displayed some of this same knowledge in other situations involving rate of change. I also wanted to see what other knowledge students used to solve tasks involving rate of change. Finally, I wanted to think about what role this knowledge might play in learning more about rate of change.

In addition to showing difficulties precalculus, calculus, and postcalculus students have with rate of change, and showing what methods they use to solve rate of change problems, I was interested in the extent to which this literature showed students responding in ways that are consistent with three perspectives on rate of change and three types of reasoning. These studies showed students responding in ways consistent with global and interval perspectives on rate of change and using qualitative, numerical, and algebraic type of reasoning. In this dissertation I further examined students' responses to rate of change tasks in terms of these perspectives and types of reasoning.

Chapter 3

EPISTEMOLOGICAL FRAME

This dissertation concerns knowledge of rate of change in precalculus, calculus, and postcalculus students within the context of a particular set of tasks. The purpose of this study was to investigate these students' knowledge of rate of change, note similarities and differences in this knowledge among the three groups of students, and think about what this means for learning rate of change. In this chapter I describe a particular view of knowledge and learning, briefly review two studies that illustrate this view, and relate this view to this study.

A constructivist view of learning

Basic tenets. Constructivist theories of learning view learners as actively trying to make sense of situations, using their prior knowledge about previous situations, and altering their knowledge to fit the constraints of new situations. Three tenets are at the heart of this statement. One is that individuals actively try to make sense of situations. They are not passive recipients of knowledge about the physical and social world but rather actively try to interpret their interactions with the world so that those interpretations are consistent with other interactions they have had. A second tenet is that individuals use their prior knowledge about other situations to make sense of new situations. People come to situations with knowledge acquired by previous interactions with the world and they use this

knowledge to interpret new situations. A third tenet is that individuals alter their knowledge to take into consideration new interactions with the world. Prior knowledge adjusts so that it accounts for new situations.

Variety and similarity in individual's knowledge. This view of learning accounts for variety in individuals' knowledge. Not all individuals have the same prior experiences. It is therefore likely that people bring to new situations prior knowledge that is different in some way from the prior knowledge of other people. This might mean that people with different prior knowledge interpret a given situation differently. It is expected then that interaction between an individual's prior knowledge and a given situation will not necessarily result in exactly the same knowledge as another individual's experience with the same situation. This means then that in any group of people we can expect variety in their knowledge.

However, in many areas of learning, learners do in fact arrive eventually at similar states of knowledge. They may show variety in the length of time to construct this knowledge or they may show different intermediate states of knowledge on the path to this knowledge. But the arrival of many people at similar knowledge raises questions about how this happens. Perhaps similar states in people's knowledge are due to particular sets of common experiences with a more knowledgeable other. Instruction centered around a particular set of experiences under the guidance of a teacher may account for some of the similarity in people's knowledge. Similar states in individuals' knowledge may also be attributable to some knowledge being so fundamental to a wide variety of experiences both in school and out that everyone simply acquires it in

time. The acquisition of knowledge for performing simple arithmetic may be one type of knowledge. Mechanisms for how people come to different and similar knowledge is an important part of the research agenda for constructivist theories of learning.

In this dissertation I was interested in the variety in students' knowledge about rate of change. I was also interested in similarities in their knowledge and what both the variety and similarities might mean about learning rate of change.

Studying learning rate of change in this study. There are many ways to study learning rate of change. One way is to implement one or more instructional interventions and see how students' knowledge changes over time. Another way is to examine the concept of rate of change, partition it into its several subconcepts, and conjecture about how students might acquire knowledge of these subconcepts. A third way is to give students a particular set of tasks concerned with rate of change, see what it is that they actually do with these tasks, and think about what this says about learning rate of change.

I selected this latter approach. I was interested in what precalculus, calculus, and postcalculus students actually did when presented with specific tasks involving rate of change. I was also interested in how what they did differed from group to group and what that might tell us about how rate of change is learned. These seem important things to find out if one is concerned about students' learning and one's teaching of rate of change before, during, and after calculus.

Knowledge as strategies. In this study I took as knowledge what students actually did to solve a particular set of problems. Others

have taken this same stance about knowledge when studying how student learn. Siegler and Jenkins (1989) define knowledge as strategies, "any procedure that is nonobligatory and goal directed" (p. 11). By "nonobligatory" they mean that a particular strategy must not represent the only way to accomplish a specific task; it must also be one of multiple different ways of completing the task with each way referred to as a strategy. By "goal directed" they mean that strategies are invoked to accomplish a particular goal, usually completion of a given task or solution of a given problem. For Siegler and Jenkins, what students actually do is called strategies and these strategies are a type of knowledge.

Learning as strategy construction. Siegler and Jenkins further define learning as strategy construction, which they further define as strategy acquisition and generalization. Strategy construction depends on the existence of prior knowledge, often in the form of other strategies. When a new strategy is learned, it does not immediately, at least, replace older strategies. It exists along side already existing strategies and competes with them for use in solving problems. People choose which strategies to use on which tasks based on the perceived advantages of strategies in solving those tasks. As strategies are used, their relative successes and failures help influence decisions about how often and on what tasks they will be used in the future (Siegler and Jenkins, 1989).

Strategies can be acquired in a number of ways. One way is through explicit instruction. All of the children in the Siegler and Jenkins study had been taught, prior to the study, at least one strategy for performing simple addition. Presumably that is how they acquired

that strategy. I learned how to find rate of change at a point by differentiating an equation and evaluating the derivative equation at that point because I had explicit instruction in that strategy. Similarly I learned to find rate of change at a point by finding the slope of a tangent line because someone showed me how to do that. In each of these cases someone explicitly taught a strategy.

Another way strategies are acquired is by implicit instruction. Some of the children in the Siegler and Jenkins study had acquired other strategies for simple addition by watching older students, like their siblings, perform addition. I learned how to approximate rate of change at a point by computing the average rate of change over a very small interval around the point because I watched some of my students use that strategy. I learned how to address rate of change at a point by examining properties of the graphs of rates of change because I saw this strategy used in a colleague's advanced physiology text. In each of these cases no one was explicitly trying to teach a strategy; a new strategy was learned by observing others using the strategy.

Strategies can also be acquired by discovery or invention. In the Siegler and Jenkins study, students invented several strategies for simple addition over the weeks of the study. My students discovered on their own the strategy for finding rate of change at a point by computing the average rate of change over a very small interval around the point. I never taught it to them. I once saw a student use the pattern of distances traveled by a falling object over successive quarter second intervals to make a conjecture about how far the object would have traveled in the quarter second after it hit the ground if it had been allowed to continue its fall and then use that information to

decide with what speed the object hit the ground. I believe she invented this strategy on her own. Another student used the acceleration due to gravity to find the speed of a falling object at points in time. Although he may have had some prior instruction concerning the acceleration due to gravity, his strategy represented, I believe, an invented use of this knowledge to find the speed of the falling object at specific points.

Strategy generalization refers to the establishment of conditions of applicability of strategies. For example, with simple addition, students in the Siegler and Jenkins study discovered that the MIN strategy could be applied with great efficiency to particular types of problems when other strategies could not. Similarly, using the slope of the tangent line for answering certain types of questions about rate of change is quite efficient but to find a fairly good approximation for the rate of change at a point, using the slope of the tangent line may not work as well because of inaccuracies in drawing the tangent line or not being able to find exact coordinates for points on the tangent line.

Once a strategy has been acquired, it is not necessarily used to the exclusion of other strategies or even frequently in situations where its use is advantageous. Experience with tasks and strategies helps determine which strategies are useful for which tasks including relative advantages of each strategy for each type of task. Gaining this experience takes time. Once these conditions of applicability have been established, use of newly acquired strategies can be suddenly more frequent and adventuresome (Siegler and Jenkins, 1989).

Strengths of this view of knowledge and learning. Strategies then represent a particular type of knowledge and learning, in this context,

means constructing knowledge of strategies, including both acquiring strategies, by explicit or implicit instruction or discovery, and establishing conditions of applicability. This view of knowledge and learning has several strengths. One is that it is consistent with data showing that individuals use several strategies, rather than a single strategy, to solve the same types of problems and that, even with the same task, a person may use one strategy one day and a different strategy the next day.

Another strength of this view of learning is that it accounts for data which shows that people take varying amounts of time to acquire and generalize strategies and the conditions under which they do so varies. This is consistent with a theory of learning which posits that people construct knowledge through experience with situations and just as people's experiences vary, then so also do the circumstances associated with their acquisition and generalization of strategies to deal with situations.

Finally this view of learning gives a more complete accounting of interaction between prior knowledge in the form of already existing strategies and the acquisition and generalization of new strategies. One of the tenets of constructivist theories of learning is that prior knowledge plays a pivotal role in the construction of new knowledge. However, mechanisms concerning the role of prior knowledge in that construction have not been very well delineated. The view that learning is strategy construction makes more explicit the role of prior knowledge in the discovery and generalization of strategies by showing the order in which students use strategies and exploring what knowledge in a

previous strategy may be useful for construction of a succeeding strategy.

Unfortunate consequences of this view of knowledge and learning.

This view of knowledge as strategies and learning as strategy construction has at least two major consequences that may discourage researchers from adopting it. One is that it calls for a more involved methodology to study knowledge and learning. A pre-test, intervention, post-test study focusing on improvement of performance will not necessarily uncover either the strategies students use or the circumstances leading to their initial use and generalization. A more intensive study involving multiple interviews with students as they work on the same or similar tasks over a longer period of time will more likely reveal both the strategies students use and how those strategies are discovered and generalized. The period of the study needs to be long enough to identify circumstances leading to the first use of a strategy, the instance of its first use, and uses of the new strategy after its discovery. This view of learning calls for more labor intensive studies to give a fuller accounting of both knowledge and learning (Siegler and Jenkins, 1989).

Another unfortunate consequence of this view of learning, in the eyes of some, is that it does not necessarily lead to a set developmental sequence through which all students pass. In fact, a view open to the variety in people's knowledge denies that there is such a developmental sequence. This view will most likely show that learning does not consist of moving from stage to stage with each stage characterized by use of a single strategy more advanced than strategies used in previous stages and less advanced than those used in future

stages. The view of learning discussed here is that students at any given time possess several different strategies for solving a given problem, there is not necessarily a fixed order in the appearance of strategies, the amount of time it takes to discover a strategy is not the same from student to student, and the frequency of use of a discovered strategy varies from person to person and from situation to situation. This view of learning is more complicated than other views and therefore may be rejected in favor of simpler explanations (Siegler and Jenkins, 1989). It has a more complex set of implications for teaching and learning, one that may not be welcome by teachers already overwhelmed by the many competing demands on their time and energy.

Two studies based on this view of knowledge and learning

Several studies have taken knowledge as strategies and learning as strategy construction. I have chosen to briefly review two studies here. Both are about learning in mathematical domains. One is a longitudinal study about strategies used to perform simple addition by young children over about three months. This study examined learning by looking at strategy use and change in the same group of individuals over a period of time. The other is a cross sectional study about strategies for working with rational numbers and involved three groups of students at different points in their mathematical education. This study examined learning by comparing and contrasting strategy use among the three groups of students. These two studies illustrate the usefulness of looking at strategies as knowledge and strategy construction as learning and provide venues for thinking about a similar study of students' knowledge and learning of rate of change.

A study of simple addition. Siegler and Jenkins (1989) studied knowledge of addition of single digit whole numbers in eight 4- and 5-year olds. They interviewed each child two or three times a week for eleven weeks. During the first seven weeks, children were given addition problems with addends from 1 to 5. During the last four weeks, in addition to these small-addend problems, children were given problems with one of addend from 1 to 5 and the other addend between 10 and 30. These latter problems were called challenge problems.

At the beginning of the study these children could add two numbers from 1 to 5 with reasonable accuracy (at least 67 percent correct for each child) but had not yet used the Min strategy (counting on from the larger addend). The Min strategy is an important strategy for simple addition since it yields both accurate and quick answers. The goal of this study was to investigate how these children discovered, without the benefit of instruction, the Min strategy and how they used it once they discovered it. Siegler and Jenkins were particularly interested in the role of other addition strategies and related knowledge in that discovery and the circumstances leading to its generalization.

In this study students used a variety of strategies to solve simple addition problems and the distribution of frequency of use of strategies changed over the course of the study as students invented new strategies and modified and expanded use of both new and old strategies.

Siegler and Jenkins ordered the strategies from less advanced to more advanced based on the prior knowledge, number of steps, and amount of self-regulation required for each strategy. While students tended to switch from less advanced to more advanced strategies over the time

period of the study, they also used more and less advanced strategies throughout the entire eleven weeks.

Seven of the eight children in the study discovered the Min strategy and their discovery occurred primarily on problems which students had already solved correctly and for which they had other strategies for solving. It can therefore be inferred that the presence of new problems which students do not know how to solve is not a requirement for the invention of new strategies.

One strategy seemed to play a significant role in the discovery of the Min strategy. Students discovered this strategy before they discovered the Min strategy and it seemed to bridge the gap between the larger number of steps required for less advanced strategies and the smaller number of steps for the Min strategy and between the need for less self-regulation with less advanced strategies and more self-regulation with the Min strategy.

Once the Min strategy was discovered students did not immediately use it to the exclusion of other strategies. They continued to use other strategies, too. However, when the challenge problems were presented, the frequency of use of the Min strategy increased not only on the challenge problems but also on all problems. The relative advantages of the Min strategy on these problems and its greater efficiency on all problems made it an attractive choice for performing simple addition.

Siegler and Jenkins' study had two major findings. One is that students used several different strategies for solving the same kind of problems. The other is that some strategies seemed to influence the discovery of other strategies and certain types of problems encouraged

more frequent use of certain strategies. If we are concerned about what might lead students to learn strategies for solving particular types of mathematics problems, then this information about what strategies and problems influence the discovery and use of other strategies could be quite helpful.

A study about rational numbers. Smith (1990) studied knowledge of rational numbers in elementary, middle, and high school students. Specifically, he looked at strategies for comparing two fractions, adding two fractions, and finding fractions between two others. For the comparison task, students were given two fractions and asked to decide if one was greater than the other or if they were equal. For adding fractions, students were given a fraction and asked to select, from among four fractions, two whose sum yielded the given fraction. For the betweenness task, students were given two fractions or two consecutive natural numbers and asked to find numbers between them.

The 10 elementary, 10 middle, and 10 high school students in this study were interviewed individually. The goals of this study were to investigate rational number knowledge in students at these three points in their mathematics education and to infer from that how students might acquire knowledge of rational numbers. Smith was particularly interested in describing strategies these students used on these fraction tasks. He was also interested in using similarities and differences in strategy use among three groups of students, most competent, less competent, and least competent, to make claims about learning about rational numbers.

There were four main findings of this study. One is that students used a variety of strategies to address all three types of tasks. The

students in this study used 28 different strategies for comparing fractions, 14 for adding fractions, and six for finding a fraction between two numbers. A second finding is that some of the strategies were applicable to all problems while others were applicable only to problems with specific features, like finding a fraction between two given fractions that straddle $1/2$. Of the 28 strategies for the comparison tasks, 12 were general and 16 were specific. Of the 14 strategies for the addends task, 10 were general and four were specific. All six of the strategies for betweenness were general.

A third finding was that some of the strategies were taught in school while others were apparently invented by these students during the interviews. Some of the invented strategies were alterations of instructed strategies while others seemed to have no clear influence from schooling. With the comparison task, only eight of the 28 strategies students used were instructed strategies. The other 20 were constructed strategies. Of these 20 constructed strategies, 14 represented correct ways to think about fractions and two more were correct for the specific situations to which they were applied. Hence, most of the strategies students invented were correct for the problems on which they were used.

The fourth finding concerned four modes of reasoning that seemed to characterize the strategies. The "parts" mode refers to partitioning quantities into a certain number of equal parts and reasoning about these parts. The "components" mode involves looking at the numerator and denominator of a fraction as numerical quantities and reasoning about the relationship between a numerator and a denominator, two numerators, or two denominators or some combination of these. The

"reference point" mode consists of using relative position of rational numbers to other numbers to make conclusions about which of two fractions is greater. The "transform" mode entails transforming a fraction into an equivalent fraction or decimal representation. Thinking about strategies in terms of these modes helped Smith think about their role in students' construction of rational number knowledge. Similarly, I am wondering if my characterization of knowledge of rate of change in terms of the three perspectives and types of reasoning may provide a way of thinking about the role that particular strategies play in students' construction of knowledge of rate of change.

One of the goals of the Smith study was to make inferences about how students acquire rational number knowledge. He went about this by contrasting the rational number knowledge of the least competent students (the elementary students) and the most competent students (primarily the high school students) where competence was judged by the number of correct solutions obtained. The most competent students used a wider variety of strategies including more strategies in each of the four modes of reasoning. In contrast, the least competent students used fewer strategies and tended to use primarily "parts" strategies. He conjectured that "parts" reasoning supports "components" and "reference point" reasoning. Similarly, in this study I want to see if strategies classified as one perspective or type of reasoning seem to support students' construction of strategies of another perspective or type of reasoning.

Summary of the two studies. In both the Siegler and Jenkins and the Smith studies, knowledge was operationalized in terms of strategies for solving particular problems and learning was taken as strategy

construction. The primary goals of each study were to describe students' knowledge in terms of the strategies they used and to make inferences about learning based on changes in the use of strategies across time (Siegler and Jenkins) or differences in strategy use among groups of students at different points in their mathematics education (Smith).

In both studies students used a variety of strategies and the discovery or invention of new strategies did not replace older strategies. In the Siegler and Jenkins study students came in time to use the Min strategy with greater frequency, but they still used other strategies where appropriate. In the Smith study, students used specific strategies where they were especially effective on problems with particular numerical features, but retained and used general strategies when specific strategies did not fit the given problem.

Concerning learning, in both studies the researchers conjectured that certain strategies play a role in the construction of other strategies. In the Siegler and Jenkins study one particular strategy appeared in most students' solutions before the Min strategy. In the Smith study, strategies based on "parts" reasoning appeared before strategies based on "components", "reference point", and "transform" reasoning. This made these researchers think that these previously learned strategies might be important factors in students' learning.

Finally both studies showed that students' strategies are often tailored to specific properties of problems. In the Siegler and Jenkins study, problems with one small and one larger addend encouraged students to make more frequent use of the Min strategy. In the Smith study, students were quite adept at inventing or adapting strategies to provide

quick and accurate solution of problems with specific numerical features.

Learning rate of change

This dissertation examined knowledge of rate of change in students at three points in their mathematics education. The goals of this study were to describe the knowledge these students had, note similarities and differences in this knowledge among these three groups of students, and think about what this means for learning rate of change. In this section I will consider these goals in the context of a constructivist view of learning and specifically in terms of knowledge as strategies and learning as strategy construction.

In the Siegler and Jenkins study 4- and 5-year old students used several strategies to perform simple addition and in the Smith study elementary, middle, and high school students used a large number of strategies when comparing, adding, or finding a number between two fractions. This leads me to believe there may also be a number of different strategies for solving rate of change problems. Students in the Monk studies reviewed in Chapter 2 used several different strategies for addressing rate of change in two related rates problems. Some used changes in one quantity over fixed intervals of another quantity. Others used graphs of the relationship between two quantities to address rate of change of one quantity with respect to the other. Still others formed and manipulated equations for the relationship between varying quantities. Thompson's studies showed young students using changes in one quantity over intervals of another quantity to address rate of change and total change. These are examples of strategies students in those studies used to address rate of change.

Smith's study dealt with rational number knowledge in students near the beginning, in the middle, and near the end of their formal study of rational numbers. His goal was to use similarities and differences in rational number knowledge among these three groups of students to investigate students' learning of rational numbers. Similarities in knowledge, like common use of "parts" reasoning, among the three groups pointed to a possible base rational number knowledge that all students at these points of mathematics education have and which could serve as important prior knowledge for the construction of other knowledge of rational number. Differences in knowledge among the three groups were used to make conjectures about intermediate states of knowledge and possible roles of "parts" reasoning in the construction of other modes of reasoning about rational numbers.

In a similar way this dissertation dealt with knowledge of rate of change in students near the beginning, in the middle, and near the end of their formal study of variable rate of change. More specifically, this study investigated knowledge of rate of change in three groups of students, one before they have had calculus, one near the end of a year of calculus, and one after having had several courses in mathematics beyond calculus. Similarities and differences in rate of change knowledge among these three groups of students were then used to infer how students learn rate of change.

Smith classified many of the rational number strategies his subjects used in terms of four modes of reasoning. This makes me wonder if students' knowledge of rate of change can also be classified in certain ways. In Chapter 6, I discuss students' knowledge of rate of change in terms of the three perspectives of rate of change and three

methods of reasoning I outlined in Chapter 1. More specifically I wanted to see if these three perspectives and methods of reasoning made sense in terms of how students address rate of change.

Both the Siegler and Jenkins study and the Smith study made attempts to study the role of certain strategies in the discovery of other strategies. In the Siegler and Jenkins study one strategy seemed to have played a role in the discovery of the Min strategy. In the Smith study "parts" strategies seemed to have played a role in the use of "components" and "reference point" strategies and may have influenced the use of "transform" strategies. Similarly in this study I wanted to see if students' strategies for addressing rate of change yielded evidence that some strategies may play a role in the discovery or construction of other strategies. In particular, could strategies be classified as consistent with a global, interval, or point rate of change perspective and if so, what role do strategies of one particular perspective play in students' use of strategies of one another perspective?

This dissertation, then, focused on identification of strategies students at three different points in their mathematics education used to solve problems about rate of change (see Chapter 5). In this way it attempted to answer the first two questions at the end of Chapter 1 ("What knowledge of rate of change do students bring with them to calculus that might support their learning of rate of change?" and "What do students who have had calculus know about rate of change that may provide clues about how rate of change is learned?"). It also noted similarities and differences in strategies used by the three groups (see Chapter 7) and used this information to answer the third question at the

end of Chapter 1 ("What similarities and differences in knowledge of rate of change exist between those who have had calculus and those who have not had calculus and what might this tell us about what knowledge supports learning rate of change and how it supports it?").

This dissertation also investigated relationships among strategies to see if use of some strategies seemed to support use of other strategies and thereby play a role in advancing students' knowledge of rate of change. It did this by seeing if the three perspectives on rate of change and the three types of reasoning about rate of change made sense in terms of the strategies students used and provided an appropriate interpretive frame for students' learning of rate of change (see Chapter 6). In this way it attempted to answer questions 4 and 5 at the end of Chapter 1 ("In what ways do global, interval, and point perspectives on rate of change provide an interpretive frame for students' knowledge and how rate of change is learned?" and "In what ways do students use qualitative, numerical, and algebraic types of reasoning and what might that tell us about how rate of change is learned?").

Chapter 4

RESEARCH METHODOLOGY

To study students' knowledge of rate of change, I designed six tasks and presented them to students in individual interviews. I chose students from three points in mathematics education - 12 who were taking a precalculus course, 15 who were in their second semester of calculus, and 10 who had taken at least two mathematics courses beyond calculus. Interviews were audio and video taped and tapes were then transcribed word-for-word. These transcriptions and students' written work were used to identify strategies students used to solve the tasks, to examine similarities and differences in strategy use among the three groups of students, and to classify, if possible, strategies in terms of three perspectives on rate of change and three types of reasoning about rate of change. The primary goal was to see what knowledge these students had about rate of change and to then make conjectures, if possible, about how rate of change is learned.

A brief overview of the design

Students and content. A study which focuses on learning a particular concept should include students who have not yet studied that concept, but who might be considered ready to study that concept. Students who have not yet had formal instruction in rate of change, especially varying rates of change, would, hopefully, engage with tasks involving rate of change in such a way as to reveal knowledge they have. This in turn may help us think about how that knowledge gets used

or could get used in learning more about rate of change. Students who had not yet taken calculus were therefore included in this study. These students were taking a course called Precalculus, typically a mathematics course immediately preceding calculus.

Since a formal study of rate of change, especially varying rates of change, is usually reserved for calculus, it seemed reasonable for this study to include students who were taking calculus. Since the first semester of calculus is usually where rate of change is most explicitly dealt with, this study included students who had finished first semester calculus and were taking second semester calculus. Most of calculus instruction concerns functions represented as equations with support from graphical and tabular representations. The tasks for this study represented functions only as graphs and tables. I assumed these students, because of their instruction in calculus, would have some knowledge of rate of change that the precalculus students may not have.

But I was not sure what that knowledge would look like since it did not make use of the primary function representation used in calculus.

Learning a concept often involves conceptual changes after formal instruction in the concept is finished. This may especially be the case if students have further instruction in related concepts. Students' knowledge of rate of change may continue to grow after they have completed calculus and as they take more mathematics courses even if those courses do not explicitly deal with rate of change. I therefore included in this study students who had taken at least two courses in mathematics beyond calculus. As with the calculus student, their instruction would primarily involve functions represented by equations with some support from graphical and tabular representations. Again I

assumed that these students might have knowledge of rate of change that other two groups of students may not have but was not sure what that knowledge would look like.

Subjects for this study therefore came from three groups of students. One group consisted of high school students taking a Precalculus course; another group was made up of college students in their second semester of calculus; a third group was composed of college mathematics majors who had taken at least two courses beyond calculus.

Tasks. Students were asked to work six tasks dealing with rate of change. Appendix C contains the complete statements of the tasks. The first three tasks dealt with same basic situation: "Two people start at opposite corners of a room, walk toward each other, pass, and then proceed to their respective opposite corners. This takes a total of eight seconds and the opposite corners of the room are 20 feet apart."

In the first task the two people slow down as they get closer to each other, pass, and then speed up as they get farther apart. In the second task the two people maintain the same steady pace the whole way.

In the third task the two people speed up as they get closer to each other, pass, and then slow down as they get farther apart. For each of these three tasks students were asked to construct a graph and table of values that showed the distance between the two people at each point in time and to explain how their graphs and tables showed the two people slowing down, speeding up, and maintaining a steady pace. I refer to these three tasks as the "walking" tasks.

In the other three tasks students were given a situation and accompanying graph and table of values showing the relationship between two quantities in the situation and asked specific questions about the

rate of change of one of the quantities relative to the other quantity.

In Task 4, the situation involved a ball thrown straight up into the air. The accompanying graph and table of values showed the relationship between time and the height of the ball. Students were asked to describe the height of the ball over the entire time, find the speed of the ball over selected intervals of time, and find the speed of the ball at particular points in time. This task is referred to as the "thrown ball" task.

For Task 5, students were given a graph and table of values showing the population of yeast cells in a culture over 18 hours. They were asked to describe the population of yeast cells over the 18 hours, find the rate of growth over selected intervals of time, and find the rate of growth at a particular point in time. Students were then given a more detailed graph and table of values for the population of the yeast cells around this point and once again asked to find the rate of growth at this point. This task is referred to as the "biomass" task.

Task 6 concerned the relationship between the radius and area of concentric circles formed when a pebble is dropped into a calm pond. Students were given a graph and table of values showing the relationship between the radius and area of the circles. They were asked to describe the area of the circles as the radius grows from zero to five feet, find the rate of growth over selected intervals of change in the radius, and specify the rate of growth of the area when the radius was three feet. This task is referred to as the "circles" task.

All of the tasks were given to the students in the same order. The only exception was with five students who had difficulty getting started with the first "walking" task and who therefore completed the

second "walking" tasks first. The second "walking" task appeared to be easier than the first one and working the second one first seemed to give these students confidence and information about the nature of the tasks and what was required of them. Nearly all of the subjects completed all six tasks. The only exception was four calculus and three postcalculus students who did not return for a second interview and so did not complete "circles" task or the second half of the "thrown ball" task. In most cases, these were students who could not be reached later because they had graduated or left the area. In a later section I will give a more complete description and justification for the tasks.

Data. The data for this study consisted of subjects' written work and their explanations of their reasoning. The graphs and tables of values students constructed for the "walking" three tasks were part of the data as were their explanations of how those graphs and tables showed slowing down, speeding up, and steady pace. In the "thrown ball", "biomass", and "circles" tasks, students were given graphs and tables of values for the situations presented and asked several specific questions about the rate of change of one quantity relative to another quantity. Students' verbal and written answers along with any calculations or drawings constituted part of the data for these three tasks.

Interviews. Data were collected in individual interviews which lasted about an hour, were audio and video taped, and were conducted by the author. Most students completed the interview in one sitting. One of the high school students and most of the college students were interviewed at two separate times. This was not due to the interviews taking longer than an hour, but was due to constraints on students'

schedules. It was easier for them to come for two shorter interviews than for one longer interview.

In each interview students were told there would be six tasks and that for each task they would be given some information and asked to answer some questions. It was expected that one hour would be sufficient time to work through the tasks in enough detail to assess students' knowledge of rate of change, and with very few exceptions students finished interviews within a few minutes of an hour. Three students, one in each of the three groups, took more than an hour. They either had trouble getting started on a task and needed more time to understand the task and what was asked of them or did more talking about the task, what they were thinking, and why they were doing what they were doing.

Transcripts. I transcribed the audio and video tapes word for word except for interruptions by others or conversations about topics unrelated to the tasks. This included conversations about the weather, school, students' plans for the future, and inquiries about members of their families or friends whom I might have known. Word-for-word transcriptions were helpful for my analysis of the data because even though I had viewed the tapes several times, I wanted to have students' actual words available to identify strategies they used and support claims about their knowledge.

Description and justification of tasks

I wanted to make sure the tasks I used in this study were set in familiar contexts and used function representations students at these levels of mathematics education would feel comfortable with. In addition, since students' responses to these tasks were going to be

classified in terms of three perspectives on rate of change and three types of reasoning about rate of change, I wanted to make sure the tasks gave students ample opportunity to respond in these ways if they were so inclined. I discuss each of these and results of the pilot work below.

Accessibility of tasks to all subjects.

Focus on tables and graphs. The tasks in this study involved functional relationships between two changing quantities. Functions can be represented as equations, graphs, tables of values, or verbal descriptions. In this study situations were presented verbally with accompanying graphs and tables of values or they requested students to construct graphs and tables of values. For several reasons equations were not used to represent functions nor were students asked to use or construct equations to address rate of change.

The most important reason has to do with how student typically think about rate of change outside of classrooms. Functions are represented in many areas of mathematics and the natural, physical, and social sciences as well as everyday life. They are often introduced first as graphs and tables of values in the middle and high school mathematics curriculum. In other areas of study in high school and college, graphs and tables are frequently used to represent relationships between changing quantities. In biology, for example, the most common way functional relationships are represented is by graphs. Outside of school, functions are most commonly represented as graphs and tables. In banking, tables are used to represent interest and principal for loans and investments. So graphs and tables were used here because they are more typical of how functions are represented both in and out of school.

Another reason has to do with what we might expect students to do with function representations. Presented with an equation, students who have had calculus might try to address rate of change by differentiating the equation. However, precalculus students would not be expected to know how to do this. So it is not clear what information might be obtained about their knowledge of rate of change if situations were represented as equations. Additionally, as we saw in the Orton study in Chapter 1, even when students have had calculus, they often cannot correctly differentiate equations. If students cannot use equations to address tasks involving rate of change, then other function representations should be used. Since students are more likely to be familiar with and correctly use graphs and tables of values, I decided to use these functions representations in the tasks for this study.

Familiar contexts. The six tasks for this study represent contexts which are usually relatively familiar to students at this level of schooling. I thought, therefore, that students may be more able to show their knowledge of rate of change in these contexts. The "walking" tasks and the "thrown ball" task dealt with motion. Most of us would have lots of everyday experiences with walking and throwing. So these contexts should be familiar to students at these ages. The "biomass" task dealt with growth of yeast in a culture dish. Although students would not be expected to necessarily have experience with this particular situation, they all probably have had experience with growth in general - their own changing height, the growth of plants, the accumulation of interest in a savings account. These experiences with growth make the situation in this task relatively accessible to them. The "circles" task involved dropping a pebble into a pond and noticing

the enlarging concentric circles that results. Students at this age could reasonably be expected to have this exact or similar experience, enough to make the situation in this task a familiar one.

Relationship of tasks to global, interval, and point perspectives on rate of change.

One of the goals of this dissertation was to see if global, interval, and point rate of change perspectives relate to strategies students used to solve the rate of change tasks and to think about what this might mean for learning rate of change. I therefore wanted to design tasks that could be addressed using a variety of strategies and that would give students ample opportunity to use strategies consistent with these perspectives if they so desired.

My goal was not to force students to use particular strategies that matched one of the perspectives in a predetermined way. Instead my goal was to give students tasks that could be addressed using a number of different strategies and to see if the strategies students actually used could be classified in terms of the perspectives. The extent to which these three perspectives accounted for students' strategies would then be used to judge the adequacy of these views of rate of change. In the following paragraphs I will discuss the design of the tasks in terms of global, interval, and point rate of change perspectives.

Global rate of change. A global rate of change perspective looks for and describes changes in the dependent variable of a function over the entire domain of the independent variable. In the "walking" tasks, students were asked to construct graphs and tables of values to show the two people slowing down, speeding up, or maintaining a steady pace. Students might address these tasks using strategies consistent with a

global rate of change perspective. For example, if students drew graphs of a particular shape because they believed those shapes showed speeding up, slowing down, or maintaining steady pace, then they were using a strategy that attends to changes in the dependent variable over all or a significant part of the domain of the independent variable.

The "thrown ball", "biomass", and "circles" tasks also have the potential for eliciting strategies consistent with a global rate of change perspective. They ask students to give descriptions of the behavior of the dependent variable over the entire domain of the independent variable. For example, in the "biomass" task, one could use changes in the visible slope of the graph over the domain of the independent variable to conclude that the population of yeast cells increases at successively faster rates until about $t=8$ and then increases successively slower rates after that. Since this strategy focuses on changes in the yeast cell population over the entire 18 hours, it would be classified as representing a global rate of change perspective.

Interval rate of change. An interval rate of change perspective concerns partitioning the domain of the independent variable into convenient intervals and looking at change or average rate of change of the dependent variable over these intervals. In the "walking" tasks, students might explain how their graphs or tables of values show speeding up, slowing down, or steady pace by referring to changes in distance over one second intervals of time. This would represent an interval perspective.

Parts of the "thrown ball" task could be addressed by referring to changes in the height of the ball over one unit time intervals. To

decide when the ball traveled the fastest or the slowest, one could refer to changes in the distance the ball traveled in one quarter second intervals. To find the speed of the ball at a point in time, one could compute the average rate of change over the smallest available interval before the point. These represents an interval perspective on rate of change. The "biomass" and "circles" tasks gave similar opportunities to address rate of change from an interval perspective.

Point rate of change. Point rate of change is what mathematicians most often call instantaneous rate of change and deals with rate of change at a point. The "thrown ball", "biomass", and "circles" tasks contain items that could be addressed using this perspective. For example, in the "biomass" task, students might find how fast the population of yeast cells was changing at $t=4$ by drawing a line tangent to the graph at that point and estimating its slope. This is a point perspective on rate of change.

In the "thrown ball" tasks one could identify when the ball is traveling the fastest or the slowest by specifying the point where the slope of the graph is the steepest. This represents a point perspective. Similar opportunities were available in the "biomass" and "circles" tasks.

Relationship of tasks to qualitative, numerical, and algebraic reasoning.

Another goal of this dissertation was to see how students thinking involved qualitative, numerical, and algebraic types of reasoning. Since graphs and tables of values were used to represent functions in these tasks, it may seem reasonable that students would use qualitative and numerical types of reasoning but may not be obvious that if or how

they might use algebraic reasoning. However, there are ways to use algebraic reasoning to address tasks using only graphs and table of values. This will be addressed below. My main concern was to give students opportunities to use these three types of reasoning if they were so inclined.

Qualitative reasoning. Qualitative reasoning refers to using graphical features to make conclusions about rate of change. Each of the six tasks gave students opportunities to reason this way if they were so inclined. In the "walking" tasks, students were asked to create graphs that showed slowing down, speeding up, and steady pace and to explain how their graphs showed these various rates of change. One could refer to the relative steepness of a graph to explain how it shows slowing down, speeding up, or steady pace. This is an example of qualitative reasoning.

For the "thrown ball" task one could use the given graph to decide when the ball was traveling the fastest and the slowest by identifying the section of the graph that has the steepest or flattest slope. One could use relative shape and direction of the graph to decide when the ball is moving up or down and to make statements about the ball getting successively slower or faster. The "biomass" and "circles" tasks give similar opportunities to use qualitative reasoning.

Numerical reasoning. Numerical reasoning concerns the calculation of or use of numerical values to make decisions about rate of change. Each of the six tasks gave students opportunities to reason numerically.

In the "walking" tasks, students could compute or refer to numerical values like difference between consecutive values of the dependent variable to explain how graphs and tables show slowing down, speeding

up, or steady pace. Similarly in the "thrown ball", "biomass", and "circles" tasks, students might use the given graphs or tables to compute changes in the dependent variable over intervals of the independent variable to decide when the rate of change was the greatest or least.

Algebraic reasoning. Algebraic reasoning involves the use of formulas or equations to make judgments about rate of change. Each of the six tasks gave students opportunities to reason algebraically. Equations for relationships between changing quantities were not given or asked for in any of the six tasks. But the second "walking" task, the "thrown ball" task, and the "circles" task are all associated with relatively simple and familiar equations so that students at these levels of mathematics education may know them. Obtaining such an equation and using it to address rate of change would be considered algebraic reasoning.

There are other ways students could use algebraic reasoning on these tasks. For example, the formula $(y_2 - y_1)/(x_2 - x_1)$ yields the slope of the line containing the points (x_1, y_1) and (x_2, y_2) . It also yields the average rate of change in y over the interval from x_1 to x_2 .

Use of this formula to address rate of change represents algebraic reasoning since it uses a formula or equation to make judgments about rate of change.

Results of pilot work.

To judge the accessibility of the tasks to the students and the range and variety of strategies I interviewed six students (two precalculus, two calculus, and two postcalculus) in a pilot study of the "walking", "thrown ball", and "biomass" tasks. Overall I found that the

students attempted the tasks without hesitation. They generally constructed correct graphs and tables of values for the "walking" tasks and gave appropriate descriptions of the relationship between quantities in the "thrown ball" and "biomass" tasks. I took this as evidence that they found the situations presented in these tasks fairly familiar and could use graphs and tables of values to cipher the nature of relationships between quantities.

The students in the pilot study used strategies that could be classified in terms of global, interval, and point perspectives on rate of change. Some students referred to the relative shape of graphs over the entire domain of the independent variable to explain how graphs in the "walking" tasks showed slowing down, speeding up, or steady pace. This is a strategy that is consistent with a global perspective on rate of change. All of the students referred to changes over one second intervals in the "thrown ball" task to identify intervals of time where the dependent variable changed the fastest and the slowest. This is a strategy in keeping with an interval perspective on rate of change.

In the "thrown ball" and "biomass" tasks, some students used average rate of change over the smallest available intervals before and after points in time to address rate of change at a point and indicated that their answers were estimates for the rate of change at the respective points in time. Although these students used a strategy I would classify as consistent with interval rate of change to get at point rate of change, it seemed clear that these students believed there was such a thing as rate of change at a point. Results of the pilot study indicate that students use strategies consistent with global, interval, and point perspectives on rate of change.

The students in the pilot study also used strategies that could be identified as qualitative, numerical, or algebraic types of reasoning. In the "walking" tasks, some students referred to the visual slope to explain how their graphs showed the two people slowing down, speeding up, or maintaining steady pace. This strategy represents qualitative reasoning since it relies on graphical features to make conclusions about rate of change.

Some students computed and referred to numerical values, especially changes in the dependent variable for one second or one hour intervals of time to explain how their graphs and table in the "walking" tasks showed slowing down, speeding up, or steady pace. Similarly some students used these first order differences to decide over what intervals of time the dependent variable was changing the fastest and the slowest in the "thrown ball" and "biomass" tasks. These are examples of numerical reasoning.

Some students computed the average rate of change in the dependent variable over intervals of time or found the slope of the line segment formed by the points corresponding to endpoints of an interval of the independent variable. These strategies use formulas to inform decision making about rate of change. These are examples of algebraic reasoning since they use a formula or equation to make decisions about rate of change.

Selection of subjects

In this section I will describe how students in each group were recruited and chosen for this study. I will also give characteristics of their mathematics curricula and compare the distribution of the potential population and sample by gender, race and ethnicity, and

ability. I will also discuss limitations of the sample, including the relationship of the subjects to the researcher, use of volunteers, and concerns about the sample in terms of gender, race, ethnicity, and ability.

The three groups.

Three groups of students were chosen from three different points in their mathematics education: One group taking their last mathematics course before calculus; a second group taking their second semester of calculus; and a third group of mathematics majors with at least two courses beyond calculus. The precalculus students came from the same high school. Most of the calculus and all of the postcalculus students came from a small four year church related liberal arts college where I teach. The rest of the calculus students came from a two year publicly supported community college. All three of these schools are located in the same county in southeast Michigan. This county consists of rural communities and small towns with one larger town serving as the county seat.

The precalculus group. The precalculus students were taking a course called Precalculus, typically the last mathematics course before calculus. These students came from two sections of the course taught by the same teacher. The text for their course was Precalculus, Second Edition, by Larson and Hostettler (1985; D.C. Heath). I approached the teacher, one of my mathematics students who had graduated a few years earlier, and asked her if she would allow me to recruit subjects from her Precalculus classes. She invited me to visit her two classes, explain my study, and ask students if they would volunteer to be subjects. My recruitment script went something like this:

"Hi. My name is Garnet Hauger and I teach mathematics at Spring Arbor College. I am working on a Ph.D. at Michigan State University and my dissertation has to do with how students learn rate of change. I need some students who will volunteer for this study. I will interview each of you individually. The interviews will be audio and video taped and will last about an hour. I will ask you some questions about mathematics and something known as rate of change. You do not have to study for these questions before hand. I am just interested in what you already know. You will be paid \$10 for your time and I will arrange to interview you here at your school during the school day. So if you are interested in being a part of this study, just put your name and telephone number on the sheet coming around."

There were a total of about 50 students in these two classes.

Twenty-one volunteered for this study. Six boys and six girls were randomly chosen from the volunteers to participate in the study. Each of these students was given a consent form and asked to sign it and have a parent or guardian sign it if they were under the age of 18. Copies of the consent forms and letters of explanation are in Appendix D.

The number of males and females in the combined classes and the group of volunteers was roughly even. I therefore chose six males and six females for the study. However, the racial mix of the classes, volunteers, and sample was limited. In fact, the only non-white person in the two classes volunteered for the study and was chosen for the sample.

The ability of the students in these two precalculus classes and in the sample as measured by the grades in their Precalculus course was diverse. Grades in this course ranged from D to A. Each of these students was in a college preparation track and most of the students in these two classes went on to college immediately after graduation. Four of the twelve students in the sample were National Merit Finalists or Semi-Finalists during their senior year, the year after they

participated in this study. This is a higher percentage than in the two classes or the group of volunteers.

The calculus group. All of the students in the calculus group were taking second semester calculus from a four year liberal arts college or a two year community college. The calculus students from the four year college were students in my second semester calculus class, the only section taught that school year. The calculus curriculum was an NSF sponsored calculus reform text, Calculus from graphical, numerical, and symbolic points of view, vols. 1 and 2, Preliminary Edition, by Ostebee and Zorn (1994) and published by Saunders. As with the precalculus students I explained my study and asked students if they would like to volunteer for the study. My recruitment script went like this:

"As many of you know, I am working on a Ph.D. at Michigan State University. My dissertation has to do with how students learn rate of change. I need some students who will volunteer for this study. I will interview each of you individually. The interviews will be audio and video taped and will last about an hour. I will ask you some questions about mathematics and rate of change. You do not have to study for this interview. I just want to know what you already know about rate of change. Your performance in this interview will in no way affect your grade for this course or for other courses you might take from me. You will be paid \$5 for your time and given some extra credit for this course if you are selected to be part of this study. I will give another assignment to earn an equal amount of extra credit for those who do not wish to volunteer for the study or for those who volunteer but are not chosen. The interviews will take place here in the science building at a time that is convenient for both of us. So if you are interested, just put your name and telephone number of the sheet coming around."

There were 22 students in this class and 14 of them volunteered for the study. I randomly chose 11 of them to be in the study. Each of the volunteers was asked to sign a consent form. These students were

offered only five dollars instead of ten since they were also offered some extra credit in the course they were taking from me.

The number of males and females in the class and the group of volunteers was roughly even. The sample included six males and five females. The racial/ethnic makeup of the class was limited but did include one black male student from Kenya and one female student from Serbia. Both of these students were chosen for the sample. The class contained two adult college students. Both were selected for the sample.

Grades for the first and second semester calculus classes, the volunteers, and the sample ranged from D to A. The general abilities of the sample varied a bit. One of these students is now in medical school, one is in graduate school in genetics, and one just completed student teaching in mathematics and physics and received three teaching job offers. At least two students have struggled with their college programs. One left school at the end of his freshman year because of low grades, though he showed a fair amount of talent for mathematics. Another student will graduate from college this spring but has had to repeated several courses, some in his major. Compared to college students I have taught, this group was of mixed ability.

The calculus students from the two year community college were in one of two second semester calculus classes using a traditional calculus text, Calculus, Fourth Edition, by Anton (1992) and published by John Wiley and Sons. The instructor invited me to explain my study to the class and ask for volunteers. The recruitment script for these students was nearly identical to the one used for the precalculus group. There

were about 20 students in this class; seven volunteered for the study; four were selected. They were asked to sign a consent form.

The class and group of volunteers were nearly all male. The class contained three females, two of which volunteered for the study. One female and three males were chosen for the sample. The class had no racial or ethnic minorities; so neither did the sample of students selected from this class. The class did however contain one older college student. He volunteered and was selected for the sample.

The range of grades for first and second semester for this class was varied. The grades for the four students chosen for the sample also varied from C to A.

The postcalculus group. All of the postcalculus students came from the four year college and were taking an upper division mathematics course that semester. All of these students were taking or had taken a course from me and had taking calculus from a traditional calculus text.

They received the same recruitment script as the calculus students from the same college. To be eligible for the study, these students had to have completed at least two mathematics courses beyond calculus. Of the 15 eligible students, eleven volunteered for the study and ten were selected. The one who was not chosen had time conflicts that made scheduling an interview difficult.

The group of eligible students contained about the same number of males as females; the sample contained six females and four males. All of the eligible students, and hence the sample, were white. This group of students was however diverse in terms of ability. Three of these students were National Merit Finalists or Semi-Finalists. One of these students is now in a Ph.D. program in mathematics at Cornell University.

Two of these students struggled to finish their mathematics majors, obtaining mostly C's in their mathematics courses. Grades in upper division mathematics courses for students in this sample ranged from C to A.

Limitations of the sample.

Relationship of subjects to researcher. The calculus and postcalculus students at the four year college were or had been, at the time of the interviews, students in one of my classes. This had the potential for providing several difficulties. One is related to the desire of students to appear knowledgeable to their teacher and to feel embarrassed if they feel their answer is wrong or cannot answer a question. I assured students that most of the questions had several different acceptable answers, that I would press them for multiple answers and detailed explanations, and that in any case their answers would not affect their grades in courses they took from me or in recommendations I wrote for them. As a guarantee against this, I transcribed interview tapes after the end of the semester in which the interviews were conducted and in many cases after students had finished their last course with me.

In general, students' responses made me appreciate their knowledge and abilities, not because they gave me the answer I thought they would or ought to give, but because they often showed great thoughtfulness in considering the questions and forming their answers and gave answers that were quite resourceful and ones I had not thought of myself. This relates to the second difficulty that can come from interviewing one's own students. There is a potential to interpret students' responses in particular ways because you think you already know what their knowledge

is and how they are likely to respond to the presented tasks. I wanted to be careful I did not interpret students' answers by superimposing on those answers what I thought they already knew or ought to know about rate of change. I tried very hard to give the same meanings to their actual words as I gave to the same or similar words when given by students I did not already know. Toward this end it was very helpful to have word-for-word transcriptions in front of me and to view the video tapes many times.

I had some relationship with half of the high school students before the interviews. Five of the students were children of colleagues at the college; one was a son of a former student. These students were not chosen because of this relationship; they were randomly chosen from a larger pool of volunteers. So it just happened that they were in the study. These students and I live in a close knit community where it is not unusual to be very well acquainted with many families in the community. As a long standing member of this community and one who maintains active and vital relationships with my colleagues and neighbors, I am fairly well known among families whose children are planning to go to college and especially those who are most interested in mathematics. For this reason almost any collection of potential subjects for this study would have included students whose parents were my friends, colleagues, or former students. Since my prior knowledge of these students' mathematical abilities and knowledge was limited and unrelated to any anticipation of how they might respond to these specific tasks, I do not think this provided a difficulty in collecting or interpreting the data.

Use of volunteers. The use of volunteers in research that aims for results of wide validity can be a problem. Volunteers are likely to be more outgoing and perhaps confident of themselves in areas having to do with the study. High school students who are less outgoing or not very confident of their abilities in mathematics are less likely to volunteer for a study in which the interviewer is a college mathematics teacher. College mathematics students who are reserved or unsure of their mathematical knowledge may be more reticent about volunteering for a study in which the researcher is their own teacher. Such students may not have volunteered for this study and therefore the responses of those types of students, in as much as they might differ from the responses of more outgoing and confident students, would be missing.

Issues of gender, race, ethnicity, and ability. Of the 37 students in the study 18 were female and 19 were male. However, the sample does not contain many non-white students. The group of precalculus students in the study contained one Asian student; the group of calculus students had one black student from Kenya and one student from Serbia. The other 34 students in the study were white American students from predominantly middle class backgrounds. So this sample fairly represents this latter subgroup in the population but it does not represent non-white students or students from other than middle class backgrounds. The sample does however represent the population of college students in the two colleges and the local community from which the high school students were chosen.

The high school students in this study were preparing to go to college. They were all taking college preparation courses and were considered to be in the top 25 percent of their high school student

body. They therefore were not necessarily representative of the general population of all high school students but were considered representative of high school students preparing for college and in particular for taking calculus either in high school or college. This is the population most relevant to this study.

The calculus students in this study were pursuing college programs primarily in mathematics and science. They therefore would not be representative of all college students though they may be representative of all college students required to take calculus for degree requirements in client disciplines, including majors and minors in mathematics. The postcalculus volunteers were mathematics majors and minors and therefore may not be representative of all college students or even of students in many programs requiring calculus. However, because many client disciplines like chemistry, physics, and computer science often require enough mathematics to warrant a minor in mathematics, the postcalculus students in this study may represent students from those academic areas. So while the calculus and postcalculus students in this study certainly were not representative of all college students, they might be considered representative of students required to take calculus for further study in mathematics and some areas of science.

Data collection

In this section I describe the interview process, specifically the interview environment, the probes I used and their purposes. In general this discussion is useful because it helps in understanding how students' responses in these interviews might be similar to or different

from responses of students in classrooms or private conversation where teachers might use the same probes.

Physical environment.

For the most part interviews were conducted at the schools the students attended. The only exceptions were interviews for five of the 12 high school students which were conducted in my office at the college where I teach. These students did not want to miss any of their classes at the high school and so chose to come to my office at the end of their school day for their interviews. The other high school students were interviewed during the school day in a small room in the library at their school. The students from the four year college were interviewed in my office or in an unused biology lab. The students from the two year college were interviewed in a relatively quiet corner of a student lounge. The college students arranged their interviews so they did not conflict with classes, work, or other activities. Some of these interviews were at night.

There were disadvantages to each of these environments. Nearly all of them were subject to some background noise no matter what precautions I took. A note on the door about an interview in progress met with limited success in guaranteeing quiet or preventing interruptions. Even so, the interruptions were few and did not continue for a long time. These interruptions did not seem to bother the students although they did bother me. They did not appear to alter either the quality or quantity of students' responses. After interruptions students were able with very little difficulty to pick up where they left off, continuing their thought patterns and explanations.

Probes.

The specific questions listed in the tasks were for the most part adequate in eliciting appropriate responses from students. (See Appendix B for the complete tasks.) Nonetheless it was helpful to ask students additional questions to get further information about or clarify their answers. I will discuss each probe and the circumstances in which each was used.

The most common probe was "How did you get that?" and was used primarily for questions calling for numerical answers in parts of the "thrown ball", "biomass", and "circles" tasks. Though I often felt I understood how students got their numerical answers, this probe helped to confirm this and gave me a chance to hear students' own explanations of how they got their answers.

Another probe that was helpful in clarifying what students were thinking was "Do you mean [this] or [that]?". I used this probe in cases where I felt the student was giving one of two responses but was not sure which one. So I simply asked which one they meant. For example, when students referred to the distance apart in the "walking" tasks, I wanted to be sure if they were talking about the distance the two people were apart (the d value in the table or on the graph) or the distance between consecutive points on the graph or in the table (Δd). So I simply asked them which of these two they meant.

For the "walking" tasks students occasionally had difficulty describing how their graphs or tables of values showed the two people slowing down, speeding up, or walking steady. When this happened I asked students to compare their graph or table of values in the current task with one of the other two tasks. In this case I used the probe

"How does [this one] compare to [that one]?" My intent was to give students an opportunity to compare and contrast situations in which they had already noted differences with respect to rate of change and to think, if they do desired, about what that might mean in terms of the graphs and the tables of values.

In a few cases when a student could not proceed, I introduced mild interventions like "Would this help?" or "What about this?". In each circumstance, I tried to review with the student what strategies they had used on other tasks and ask them if those same or different strategies might work here. For example, in the "biomass" task, when students were presented with an enlarged graph and a more detailed table of values, a few of them seemed initially flustered. I proceeded by asking them to review what they had done in earlier parts of this same task and consider if they might use the same or a different strategy here. That was often successful in soliciting a response.

Similarly, if a student had used intervals before $t=4$ in earlier parts of the "biomass" task, I might have suggested that the student again use an interval before $t=4$. In that case they could choose to use the same size or larger or smaller interval before $t=4$. Since this part of the "biomass" task provided information about smaller intervals than was provided in earlier parts of this tasks, students had more choices here. When students had been consistently using only intervals before $t=4$, I might ask them if what was happening after $t=4$ had any bearing on the rate of change at $t=4$. Students did not always take to my suggestions, but my encouragement to try the same thing or something similar seemed to be successful in getting them to try something.

Another mild intervention was included in the set of questions in the "biomass" task where students were asked to decide how much and how fast the cell population changed over intervals of various size around $t=4$. In the event that students had not yet used intervals of an hour or less before and after $t=4$ to address the rate of change at $t=4$, I wanted to see how they would respond to suggestions about using these types of intervals to find the average rate of change. So part f of that task contains a number of questions that were intended to walk students through the use of several different intervals to address change near and at $t=4$. If students had by this time in the interview used some or all of these intervals, I skipped most of the questions in this part. That included the vast majority of students in the study. For a few students I pressed for answers to several of the questions in part f to see if they were open to using strategies based on consideration of change over particular kinds of intervals if they had not yet done so.

The last question in part f of the "biomass" task was asked of nearly every student: "How would you get an exact answer?". (The actual wording on the task sheet is "Describe a procedure for finding exactly how fast the population changes at 4am." but in the interviews I often used the wording "How would you get an exact answer?".) This question was designed to see if students had ideas about how they would get the exact value of the rate of change of the cell population at $t=4$. Many of the precalculus and some of the calculus and postcalculus students seemed flustered by this question. For example, many students responded initially to the questions with "What do you mean?". So I responded with two probes here. The probe "Is that [latest answer] an exact or

approximate answer?" was designed to see if they had an opinion about exact versus approximate answers. The probes "How would you get a better approximation?" and "How would you get an exact answer?" were used to see if they made connections between their approximate answers and an exact answer.

Data analysis

This study had three main goals: to see what knowledge of rate of change students had; to see how this knowledge differed; and to see if the three perspectives on rate of change and the three types of reasoning provide an interpretive frame for thinking about students' knowledge of rate of change and how it changes over time. In this section I will describe the methods of data analysis I used to meet these goals. Generally there were three main phases of the analysis: identifying strategies students used to address these tasks; classifying those strategies in terms of global, interval, and point rate of change and qualitative, numerical, and algebraic reasoning; and comparing and contrasting the distribution of use of strategies among the three groups.

Identifying strategies.

In chapter 3 I discussed knowledge as strategies as part of the epistemological frame for this study. To discover knowledge of rate of change that the students in this study had, I identified strategies they used to work the tasks. This process had three parts. One part involved reading literal transcripts and identifying unambiguous statements of different types of responses. Some students were quite clear about what strategies they were using and gave responses that make

it possible to distinguish that strategy from other possible strategies.

Other responses were less clear and needed corroborating data.

A second part of the process of identifying strategies students used involved ciphering responses which were less clear and figuring out the most probable strategy being used. This was most useful when a student's response seemed to be one of two strategies but did not make clear which one. Comparing the student's response to clearer statements given by other students helped to identify the strategy being used. A third part of the process was to closely examine students' unclear responses and construct an argument concerning what strategies the student was using. This required not only comparison with responses of other students but also generating and considering strategies no other students had used.

This process was not linear. All three parts were used on different students' responses in a single pass through the transcripts.

In addition, multiple readings of the transcripts helped me make final decisions about students' strategies. This was especially the case in a few instances when the first reading of a student's response yielded no clear idea about the strategy the student was using. Subsequent readings of the transcripts of all students yielded more information about students' strategies so that multiple comparisons could be made among responses. This helped me decide the most probable strategy each student was using. Sometimes a student used a strategy no other student had used and if the student's response was not very clear, it was extremely difficult to decide what strategy the student used and provide evidence that would convince others. Multiple readings of the transcripts and comparisons with other students' responses helped.

Classifying strategies by perspective and type of reasoning.

In addition to identifying strategies students used to address rate of change, I was interested if they fit global, interval, and point perspectives and qualitative, numerical, and algebraic reasoning. In the above section "Description and justification of tasks", I discussed relationships of the tasks to these perspectives and types of reasoning, indicated specific parts of each task I believed would give students opportunities to use strategies consistent with particular perspectives and types of reasoning if they were so inclined, and gave examples of specific strategies that were consistent with each perspective and type of reasoning. Here I want to generally define how I will classify strategies as consistent with global, interval, or point perspectives and qualitative, numerical, and algebraic reasoning.

Classifying strategies in terms of the three perspectives. Global rate of change considers changes in the dependent variable of a function over the entire domain of the independent variable. Strategies which focus on relative changes in the dependent variable over all or a significant part of the domain of the independent variable will therefore be classified as consistent with a global rate of change perspective. For example, strategies based on the relative slope or shape of the graph over the entire domain of the independent variable are consistent with a global rate of change perspective.

Interval rate of change refers to change in the dependent variable over intervals of the independent variable smaller than the entire domain. Examining changes in the dependent variable over intervals of the independent variable of one unit length, for example, is consistent with an interval rate of change perspective. Point rate of change

focuses on change in the dependent variable for an individual value of the independent variable. Drawing a line tangent to the graph at a point corresponding to a particular value of the independent variable is an example of a strategy that is consistent with a point rate of change perspective.

Some strategies provide interesting connections between interval and point perspectives. One could compute the slope of a line containing point $P(x_0, y_0)$ and another point near P to approximate the rate of change at point P . This is an interval strategy used to address rate of change at a point. Similarly one could approximate rate of change at a point by computing the average rate of change of the dependent variable over an interval of the independent variable around that point. This also makes use of an interval strategy to get at rate of change at a point.

Classifying strategies in terms of the three types of reasoning.

Qualitative reasoning refers primarily to the use of graphical features to make conclusions about rate of change. Strategies which focus on properties of a graph to make conclusions about rate of change of the dependent variable with respect to the independent variable based on those properties will be classified as making use of qualitative reasoning. Noting that the visible slope of the graph gets successively steeper and concluding therefore that the dependent variable is changing successively faster is a strategy that makes use of a graphical feature and therefore would be classified as qualitative reasoning. Visually comparing the rise and fall of a graph over several intervals to decide over which interval of the independent variable the dependent variable

changes the most or fastest is a strategy that represents qualitative reasoning.

Numerical reasoning concerns the calculation or use of numerical values to make decisions about rate of change. Examining the numerical values of the dependent variable of a function via a graph, table, or equation to decide if the dependent variable is increasing or decreasing is a strategy that refers to or calculates numerical values. Therefore, I would classify it as numerical reasoning. Computing and referring to changes in the dependent variable over intervals of the independent variable to specify the interval where the dependent variable changes the most or fastest is an example of numerical reasoning. These are examples of numerical reasoning since they rely on numerical values to make conclusions about rate of change.

Algebraic reasoning involves the use of formulas or equations to compute values to make judgments about rate of change. Obtaining the derivative equation for a function and finding values of the independent variable that make the derivative values positive and using this to make a conclusions about where the function is increasing is one example of algebraic reasoning. Locating the coordinates of two points on a line tangent to the graph at a point $P(x_0, y_0)$ and using the formula for slope of a line to find the slope of this tangent line as an answer for how fast the dependent variable is changing at x_0 is another example of algebraic reasoning. Computing the average rate of change of the dependent variable over an interval of the independent variable using the formula for average rate of change $(y_2 - y_1)/(x_2 - x_1)$ is yet another example of algebraic reasoning. Strategies that make use of a formula or equation will be classified as making use of algebraic reasoning.

Comparing and contrasting the three groups.

One way to investigate how students learn a schooled concept is to compare and contrast the knowledge of students at different points in their educational experience. In this study about students' knowledge of rate of change, students at three different points in their mathematics education were asked to complete several tasks involving rate of change. An assumption was made about the level of competence with rate of change of each of the groups. Since the precalculus group had not yet formally studied rate of change, their level of competence was the same as a traditional "before" group. The calculus group was finishing their year of elementary calculus and could therefore be considered an "after" group. The postcalculus group was several courses beyond calculus and so could be seen as several months beyond the "after" group. These three groups were taken as representing different levels of competence with regard to knowledge of rate of change. Differences in the distribution of use of strategies of these three groups could then be used to make conjectures about how students learn rate of change.

When examining the distribution of strategies among the three groups I looked for three differences. One difference had to do with the strategies themselves. I anticipated that some strategies would be used by some groups but not by others. In this case I wanted to consider possible reasons for this difference and think about what it might mean about students' learning of rate of change. One possibility is that strategies used by the postcalculus group are in some sense more advanced than strategies used by the calculus group which in turn are more advanced than strategies used by the precalculus group. This would

represent an important conclusion about students' knowledge of rate of change and suggest paths for learning rate of change.

A second difference related to strategies used by more than one group but with different frequencies. If calculus and postcalculus students used some of the same strategies but used them with different frequencies, that may say something about the growth of knowledge of rate of change after formal instruction in rate of change is finished. Similarly, if precalculus and calculus students used some of the same strategies but with different frequencies, that may say something about intermediate states of knowledge from before instruction in rate of change to during and after instruction in rate of change. Both of these possibilities provide interesting opportunities to make conjectures about students' learning of rate of change.

A third difference concerned different uses of the same strategies by the groups. If more than one group used the same strategy but used it in different ways, then this may say something about transition of knowledge of rate of change across time. For example, to estimate rate of change at a point, precalculus students may use large intervals before the point, calculus students may use small intervals before the point, and postcalculus students may use intervals both before and after the point. In this case each group would be using intervals but using them in different ways to address rate of change at a point. This would be an important conclusion about differences in knowledge of these three groups and would suggest some possible paths for learning rate of change.

Chapter summary

In this chapter I discussed the cross sectional design of this study, gave a description and justification of the tasks, and described the selection and characteristics of the subjects, data collection procedures, and methods of data analysis. In brief, students from three different points of mathematical experience were asked to work six tasks in individual interviews which were audio and video taped. Word-for-word transcripts of the interviews along with students' written work were used to identify strategies students used, classify those strategies in terms of three perspectives on rate of change and three types of reasoning about rate of change, and compare and contrast the three groups of students with regard to what strategies they used, the frequency with which they used them, and how those strategies were used.

The strategies students used were taken as their knowledge of rate of change. The classification of those strategies in terms of the three perspectives on rate of change and the three types of reasoning about rate of change was used to judge the adequacy of that framework for what it means to know rate of change. Differences in what strategies were used, the frequency of use of strategies, and how strategies were used among the three groups were used to make conjectures about how students learn rate of change. This then is how the three goals of the dissertation were met.

The next three chapters discuss the results of this work. Chapter 5 concerns the identification of strategies students in the study used to work the tasks. Chapter 6 deals with the classification of strategies in terms of the three perspectives and three types of reasoning. In Chapter 7 I compare and contrast the use of strategies

among the three groups. Specifically, in that chapter I look at the strategies each group used and examine similarities and differences among the three groups with regard to frequency of use of strategies and how strategies were used.





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HIGH SCHOOL AND COLLEGE STUDENTS' KNOWLEDGE OF RATE OF CHANGE

VOLUME II

By

Garnet S. Hauger

AN ABSTRACT OF A DISSERTATION

Submitted to
Michigan State University
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Chapter 5

IDENTIFYING STRATEGIES

At the end of Chapter 4 I indicated that the data analysis for this study involved three parts - identifying strategies, classifying strategies in terms of three perspectives and three types of reasoning, and comparing and contrasting the three groups of students with regard to their strategy use. These then help answer the questions posed at the end of Chapter 1 about students' knowledge of rate of change. As indicated in Chapter 3, in this study knowledge was taken to be strategies. Therefore the analysis of students' knowledge was an analysis of the strategies they used.

The part of the analysis reported in this chapter concerns identification of the strategies students used to work the six tasks in this study and has two major sections, one section for the "walking" tasks and one section for the "thrown ball", "biomass", and "circles" tasks. In each section, I give statements of the corresponding tasks and correct answers. Then I focus on identification of strategies students used to work these tasks.

There were several reasons for grouping the "walking" tasks together and the "thrown ball", "biomass", and "circles" tasks together for data analysis. Two reasons had to do with what was given and what was requested in the tasks. In the "walking" tasks students were verbally given information about similar situations and asked to create graphs and tables of values that show different types of rate of change

- slowing down, speeding up, and steady pace - and to explain how their graphs and tables showed these rates of change. In the "thrown ball", "biomass", and "circles" tasks students were given verbal descriptions of situations along with graphs and tables of values showing relationships between two quantities in each situation and asked questions about the rate of change of one of the quantities with respect to the other.

Another reason for grouping certain tasks together had to do with the contexts of the tasks. The context was the same for the "walking" tasks; it involved two people starting at opposite corners of a room, walking toward each other, and proceeding to their respective opposite corners after passing. The contexts for the "thrown ball", "biomass", and "circles" tasks were not as obviously similar. The "biomass" and "circles" tasks had to do with growth - growth of population in one task and growth of area in the other. The "thrown ball" task described a motion, as did the "walking" tasks, but it was the motion of a thrown ball rather than the motion of two people walking.

A final reason tasks were grouped this way had to do with specific questions students were asked to answer. In the "thrown ball", "biomass", and "circles" tasks students were asked questions aimed at discovering strategies they use for addressing rate of change at a point. There are no items in the "walking" tasks which pointedly asked students to address rate of change at a point. Those tasks focused more on global and interval perspectives on rate of change. This was another reason for grouping the "walking" tasks together and the "thrown ball", "biomass", and "circles" tasks together.

The "walking" tasks

Task statements.

Tasks 1, 2, and 3 presented three similar situations:

Task 1 - Two people start at opposite corners of a room and walk toward each other. As they walk, they both slow down as they get closer to each other, pass, and then speed up as they get farther apart. This takes a total of eight seconds. The opposite corners of the room are 20 feet apart.

Task 2 - These two people again start at opposite corners of the room and walk toward each other but this time they maintain the same steady pace the whole way. Again it takes a total of eight seconds for each to walk the 20 feet.

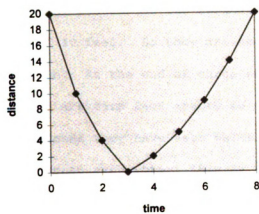
Task 3 - These two people once more start at opposite corners and walk toward each other, but this time as they walk, they both speed up as they get closer to each other, pass, and then slow down as they get farther apart. Once again it takes eight seconds for each to walk the 20 feet.

For each task, students were asked to create a graph showing the distance between the two people (distance apart) at each moment in time and explain how their graph showed the two people slowing down, speeding up, or walking at a steady pace. Then students were asked to create a table of values for their graph and explain how their table showed the two people slowing down, speeding up, or walking at a steady pace.

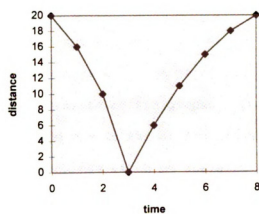
Correct answers.

Correct graphs for Tasks 1 and 3 are shown in Figure 8. Task 2 is the only task that requires a graph that meets the horizontal axis at the point $(4,0)$. Since the two people walk at the same steady pace throughout the entire time and since they each have to walk 20 feet in eight seconds, they each walk two and a half feet each second. Before they start walking, they are 20 feet apart; therefore, the point $(0,20)$ is on the graph. After one second, they have each walked two and a half feet for a total of five feet. So at the end of the first

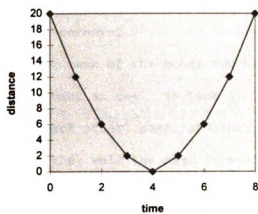
Task 1



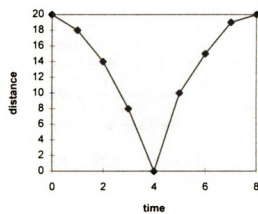
Task 3



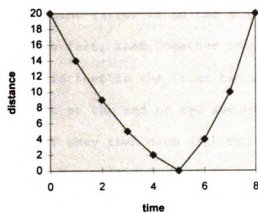
Task 1



Task 3



Task 1



Task 3

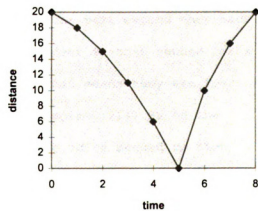


Figure 8 - Some correct graphs for Tasks 1 and 3

second, they are 15 feet apart. That means the point $(1,15)$ is on the graph. At the end of two seconds, they have each walked five feet for a total of 10 feet. So they are ten feet apart and the point $(2,10)$ on the graph. At the end of three seconds, they have walked a total of 15 feet and are five feet apart, so the point $(3,5)$ is on the graph. After four seconds they have each walked 10 feet and are in the middle of the room next to each other; this is represented on the graph by the point $(4,0)$. Similarly, the points $(5,5)$, $(6,10)$, $(7,15)$, and $(8,20)$ are on the graph and represent the distance between the two people at various points in time as they walk away from each other toward their respective opposite corners.

For each of the other two tasks, the two people do not necessarily have to meet at $t=4$. In task 1, the two people slow down as they walk toward each other, pass, and then speed up after they pass each other. Even if they walk the same pace as each other at every moment, it is still possible for them to meet before $t=4$, at $t=4$, or after $t=4$. For example, if they each walk five feet in the first second, together they walk 10 feet so that at the end of that second they are 10 feet apart and the point $(1,10)$ is on the graph. If in the next second they each walk three feet, then together they walk six feet in that second for a total of 16 feet in the first two seconds. That means they are four feet apart at the end of two seconds and the point $(2,4)$ is on the graph. If they then each walk two feet in the third second so that together they walk four feet in that second, then at the end of three seconds they will meet and the point $(3,0)$ is on the graph. They then proceed to the other side of the room making sure they walk more feet in each successive second. A sample graph could go through the points

(4,2), (5,5), (6,9), (7,14), (8,20). This would mean that each person walked one foot, one and a half feet, two feet, two and a half feet, and three feet in each successive second after $t=3$.

To get a table of values for each task, one could simply read points from the graph or construct distance values to match appropriate rate of change conditions. Task 2 is the only task that has a single correct table of values. It must contain points (0,20), (1,15), (2,10), (3,5), (4,0), (5,5), (6,10), (7,15), and (8,20). But Tasks 1 and 3 have many correct tables of values. The only requirement is that they show slowing down and speeding up on the appropriate sections. Sample tables for Tasks 1 and 3 are shown in Figure 9.

Three tables of values for Task 1

<u>Time (t)</u>	<u>Dist (d)</u>	<u>Time (t)</u>	<u>Dist (d)</u>	<u>Time (t)</u>	<u>Dist (d)</u>
0	20	0	20	0	20
1	10	1	12	1	14
2	4	2	6	2	9
3	0	3	2	3	5
4	2	4	0	4	2
5	5	5	2	5	0
6	9	6	6	6	4
7	14	7	12	7	10
8	20	8	20	8	20

Three tables of values for Task 3

<u>Time (t)</u>	<u>Dist (d)</u>	<u>Time (t)</u>	<u>Dist (d)</u>	<u>Time (t)</u>	<u>Dist (d)</u>
0	20	0	20	0	20
1	16	1	18	1	18
2	10	2	14	2	15
3	0	3	8	3	11
4	6	4	0	4	6
5	11	5	10	5	0
6	15	6	15	6	10
7	18	7	19	7	16
8	20	8	20	8	20

Figure 9 - Some correct tables of values for Tasks 1 and 3

Identification of strategies.

Strategies for explaining graphs. Students used several different strategies to explain how their graphs showed the two people speeding up, slowing down, or walking at a steady pace. Table 2 gives each strategy and the number of students using each strategy on each task.

Table 2 - Strategies used to explain graphs

Frequency (percent) of students' explanation strategies

<u>Strategy</u>	<u>Task 1</u>	<u>Task 2</u>	<u>Task 3</u>
Delta d	24 (65%)	31 (84%)	26 (70%)
Visual slope	22 (60%)	16 (43%)	20 (54%)
Delta d and delta t	8 (22%)	0 (0%)	4 (11%)
Reference to a point	5 (14%)	2 (5%)	0 (0%)
Delta t	5 (14%)	1 (3%)	4 (11%)
Meeting point	3 (8%)	2 (5%)	2 (5%)
Velocity equation	0 (0%)	2 (5%)	0 (0%)

Note: Percentages add up to more than 100% in each column since some students gave more than one explanation.

"Delta d". On each of the three tasks, over 60 percent of the students referred to the actual or relative change in distance each second. I called this strategy "delta d". If these changes get successively greater in absolute value the two people are speeding up; if they get successively smaller, they are slowing down; if they are constant, they are walking at a constant pace. Here are students' explanations using the "delta d" strategy":

Task 1 -

"The first second they cover a lot. Then as they gradually slow down, at each progressing second, they cover less." DG - Precalculus

"As they're getting closer, as the time's going into a half, then they're getting closer at a slower rate. (Ok. And that's shown by?) They move a little less each second." BS - Calculus

"Well, they're going to go faster, they're going to cover more feet in the first second 'cause they're going faster. Then as they slow down they're going to cover less feet for each second."
JG - Postcalculus

Task 2 -

"Every second that passes they're going the same amount of distance." LC - Precalculus

"They both [both sides of the graph] change 5 feet for every second before they reach each other and after they reach each other." RF - Calculus

"Because they're going an equal distance for every - between every second, they're going the same distance." JS - Postcalculus

Task 3 -

"(How does your graph show the two people speeding up as they walk toward each other?) Um, well, they only walked 3 feet the first time [from $t=0$ to $t=1$], then 4 [from $t=1$ to $t=2$], then 6 [from $t=2$ to $t=3$], then 7 [from $t=3$ to $t=4$]." AJ - Precalculus

"Because within the time frame they're moving um a fairly small amount of distance [from $t=0$ to $t=1$] compared to the same time frame in which they're moving a larger amount of distance [from $t=3$ to $t=4$]" HL - Calculus

"Um. Basically I guess because the first second [from $t=0$ to $t=1$] I made it so they only covered four feet. And then from the first to the second [from $t=1$ to $t=2$] they covered five, so they're speeding up. You can see they speed up. It goes from 4 here [from $t=1$ to $t=2$] and then it finishes off at 6 down here [from $t=3$ to $t=4$]." RC - Postcalculus

Each of these students referred to distance covered each second.

Slowing down was denoted by covering less distance in succeeding seconds, speeding up by covering more distance, and steady pace by covering the same distance each second. Some students actually computed distances covered each second (RF, AJ, and RC) while others visually compared relative distances traveled each second (DG, BS, JG, LC, JS, and HL). Regardless which way they did it, these students were comparing change in distance each second.

"Visual slope". The second most used strategy for explaining how graphs showed speeding up, slowing down, or steady pace was "visual slope". If the slope gets successively flatter, the two people are slowing down; if the slope gets successively steeper, they are speeding up; if the slope does not change, they are walking at a steady pace. On each task at least 40 percent used visual slope to make decisions about when the two people were slowing down, speeding up, or maintaining a steady pace:

Task 1 -

"When they're going faster, the lines are steeper and they're not as steep when they are going slower." KH - Precalculus

"They started out 20 feet apart and they were walking quickly to begin with so the slope of the line on the graph would be steep and as they slow down, the slope is going to decrease and become more gradual. And then as they meet the slope is going to increase because they're going to speed up and start to move farther apart more quickly." DW - Calculus

"(How does your graph show the two people slowing down?) By flattening it out. (And why does flattening it out show slowing down?) Because as your curve gets flatter it means that it's approaching zero. (It's approaching zero?) Yes. (What's the 'it' in that sentence?) As your slope, um, becomes, um, as it moves more and more towards a horizontal, it's going to slow down [moves pen along curve from $t=2$ to $t=4$]. It'll be your rate of change. (So you're identifying slope?) Uh huh. (Ok. You started to draw something there.) [She was striking the curve with what looked like tangent line segments down the left side of the curve.] Oh. Like if you take like a particular point on the graph and draw like uh the slope at that particular point - like right here - it would be about right here - right here - it would be like - here - and it shows that - it would like here - right here. [As she says this she is drawing tangent line segments at several points down the left side of the curve.] (So you're not drawing the slope.) It's the slope of a tangent line. (What is it about the slope of the tangent line that tells you the speed or the rate of change of that graph?) The slope of the tangent line will just show you the rate of change at the particular part of time at such and such a distance. (So on this side [left side of her graph] you're indicating it's slowing down, the slope is, uh, what did you say? Getting smaller? Decreasing? I can't remember your exact words.) Yeah. I just said it's becoming more horizontal. That's what I said. (And so how does this side

[right side of graph] show speeding up?) Because it's becoming more vertical." SE - Postcalculus

Task 2 -

"It's gotta be straight [draws v-shaped graph]. The slope of the line is the same the whole way so they're covering the same distance in every second." MH - Precalculus

"Because if you picked any two points, it's always the same. The rate of change is the same as the slope of the line, and the slope of the line is just the same the whole way because it's a straight line." AM - Calculus

"Because it's not curved and it's not showing a change of pace. It's keeping the same x, rise over the run. (Ok. Rise over the run is the same.) Same steepness of slope." LB - Postcalculus

Task 3 -

"(How does your graph show the two people speeding up?) Well, all of a sudden they make a sudden drop right here [pointing to the section of the graph between $t=3.5$ and $t=4$ where the graph drops 5 feet] where they're all of a sudden five feet apart and then just drop all the way to zero. (And how does your graph show the two people slowing down?) They - when they get up here [pointing to the upper right part of the curve] it just kind of levels off - not really levels off in a way - but slowly moves up." BG - Precalculus

"They're going slower here [at the beginning] so the slope will be less and then it will gradually increase and get steeper. The steeper the slope is, the faster a person will be traveling because the distance changes the most. The faster they're walking, the most distance they're covering over a period of time." SB - Calculus

"The graph gets steeper as they get closer together which means they're moving faster. And as they go apart, the graph starts out steep and gets less steep. So they're slowing down. (Why does going steeper mean they're going faster?) Because the amount of distance per second is higher." MG - Postcalculus

Responses were classified as making use of the "visual slope"

strategy if they referred to the graph or slope as steep, flat, vertical, horizontal, sudden drop or jump, gradual or level, curved or straight. References to rate of change were also classified as "visual slope" if it was obvious students were attending to visual features of the graph. For example, SE, AM, and LB referred to rate of change in

discussing slope. Some students used both "visual slope" and "delta d" to explain how their graphs showed speeding up, slowing down, or steady pace (MH, SB, and MG).

"Delta t" and "delta d and delta t". Two strategies, "delta t" and "delta d and delta t", are similar to "delta d" in that they involved changes in one quantity over intervals of another quantity. Recall that "delta d" involved changes in distance over fixed intervals of time. The "delta t" strategy involved changes in time for fixed intervals of distance and the "delta t and delta d" strategy involved changes in both distance and time. These three strategies, "delta d", "delta t", and "delta d and delta t" are similar. But there are important differences among them. While "delta d" concerns the amount of distance covered in a fixed time (like one second), "delta t" refers to the amount of time to cover a fixed distance (like five feet). In each of these strategies, one quantity, time or distance, is fixed and speed is expressed in terms of changes in the other quantity. With "delta d and delta t" neither time nor distance is fixed as speed is expressed in terms of more distance in less time or less distance in more time. Here are some examples of students' responses:

Delta t -

"As they're getting closer it takes more seconds than just one second to travel a certain number of feet." KG - Precalculus - Task 1

"It took them say two seconds here [from $t=0$ to $t=2$] to walk these five feet. And in this part [from $t=3.5$ to $t=4$] it took them only half a second to walk five feet." AM - Calculus - Task 3

"They went a very short distance - but they went [pointing to the beginning] five feet here in the first two seconds and then this last five feet [near $t=4$] they went in about one half second." SE - Postcalculus - Task 3

Delta t and delta d -

"So in a shorter time they go more distance [traces pen along curve from $t=0$ to $t=2$] than they go down here [from $t=2$ to $t=4$]."
JW - Precalculus - Task 1

"At the top they travel less distance in a longer time [pointing to the upper left of the curve]." JW - Precalculus - Task 3

"They're going over more distance in less time." LC - Calculus - Task 1

"Well, in the beginning, the slope is much - it's ah - ok - a greater distance is covered - no - a smaller distance is covered in a longer time compared to this other part over here [near $t=4$]." PG - Calculus - Task 3

"It's a longer distance in not as much time." JS - Postcalculus - Task 1

"They're going a very large amount of distance in a very short amount of time." JS - Postcalculus - Task 3

Mathematicians and scientists most often express speed in terms of distance per unit time so that "speeding up" means "more feet per second" and "slowing down" means "fewer feet per second". Responses like this would be classified as "delta d". But there is nothing inherently wrong with expressing speed in terms of time to cover unit distance. In this case "speeding up" would mean "less time to cover a certain distance" and "slowing down" would mean "more time to cover a certain distance". These responses would be classified as "delta t".

It is possible to represent speed in terms of distance per time without measuring one quantity in terms of 1 unit of the other. In this case, "speeding up" would be characterized as "more distance in less time" or "less time to cover more distance" and "slowing down" would be characterized as "less distance in more time" or "more time to cover less distance". There is nothing inherently wrong with this conceptualization of speed. Expressing speed in terms of a unit

measurement of one of the quantities makes comparisons easier, but is not required.

There are several reasons students may not find distance per unit time an obvious way to express speed. One reason concerns how speed is expressed in some everyday situations. Driving a car makes miles per hour (distance per unit time) a common way of referring to speed. But in other situations, speed is expressed in terms of time per unit distance. For example, in foot, bicycle, and car races we declare a winner based on shortest time to cover a fixed distance. Another reason concerns the nature of the question. A numerical answer is not required to decide if the two people are speeding up or slowing down. What is required is some kind of comparison of distance covered in time. That can be done without making either time or distance a unit measurement. So it is not surprising that some students gave those kinds of answers here.

"Reference to a point" and "velocity equation". Some students used the value of the distance at individual points to explain how their graphs showed slowing down, speeding up, or steady pace:

"(How does your graph show the two people slowing down?) By the decrease in the line [running her pen down the left side of her graph]. (Ok. And how does it show the two people speeding up?) By the increase in the line [running her pen up the right side of her graph]. (Ok. You mean the line is going up?) Yeah." KG - Precalculus - Task 1

"(How does your graph show the two people slowing down?) It shows that at say one second they would be approximately 16 feet apart [reading the point (1,16) from her graph] and at two seconds they'd be approximately about 10 feet apart [reading the point (2,10) from her graph], three seconds they'd be about five feet apart [reading the point (3,5) from her graph]. So they're steadily slowing down, approximately five feet per second. And then the same way back speeding up [running her pen along the right side of her graph]." LC - Precalculus - Task 1

"(How does your graph show the two people slowing down?) Well, ok, in like the first second they went about 8, 8 feet - they're 8 feet apart - they're 12 feet apart - so they each go 4 [the point (1,12) is on her graph]. And then as it gets closer to 4 [t=4], they're moving slower so they're getting less apart for each second, kind of. They're getting more apart but at a less rate - slower rate, kind of. So at 2 [t=2], I had them at about 7 feet apart [the point (2,7) is on her graph]. So - but - and then they're going less - they're just moving slower the whole way down because it's feet per second so at here [at t=2] they're at seven feet per two seconds - I don't know what I'm saying." BS - Calculus - Task 1

"(Ok. Now why does that show them going at the same steady pace?) Because when you - let's say they're um - this shows because three seconds into their walking towards each other they're going to be five feet away and then - let's say this is the common ground [the point (4,0)] for right now. And then like three seconds later, after - now this [the point (4,0)] is kind of the starting point over again - they're also - oh - oh, no, they're not! Whoa! Wait a minute! That didn't work." HL - Calculus - Task 2

"(So, for example, over the first second, what's the distance covered?) 15 feet. (Ok. So the difference between zero [t=1] and one [t=1] is five feet.) Five feet [said with the interviewer] (So what would be the distance between 1 and 2 seconds?) It should be uh 10 feet. (The difference between? Not the total distance, but the difference between where they were at at 1 foot [second] and where they were at at 2 feet [seconds]?) It should be five feet." LB - Postcalculus - Task 2

Most of these excerpts have multiple interpretations. But the simplest interpretation is that these students were using the value of the distance between the two people at one or more points to explain speeding up, slowing down, or steady pace. When KG explained that her graph showed slowing down and speeding up by the decrease and increase in the line, respectively, she might have meant that the way the graph is decreasing or increasing shows that the two people are slowing down or speeding up. But the simpler explanation is that she literally meant that the graph shows the two people slowing down and speeding up since distance values at individual points are first decreasing and then increasing.

This is the same interpretation I gave to LC's explanation. She had, at this point in Task 1, the graph that was appropriate for Task 2. So her graph decreased and then increased five feet every second. For her, this decrease and increase in distance values is what showed the two people slowing down and speeding up, respectively. She noticed her error when she got to the table of values for this graph, which is where she then constructed a correct graph for Task 1 and gave a "delta d" explanation for slowing down and speeding up.

The two calculus students (BS and HL) and the postcalculus student (LB) also used distance value at individual points to think about speed. In this excerpt, BS struggled with "delta d" and "distance value". Her final statement "seven feet for two seconds" could mean that since the graph was at height 7 feet at $t=2$, then the two people were traveling at a speed of seven feet for two seconds. Or it could mean that since the graph was at height 7 feet at $t=2$, then over the next two seconds the two people had to go a total of seven feet before they met at $(4,0)$ and that means together they were traveling at an average speed of seven feet per two seconds. The first interpretation focuses on "distance value" while the second focuses on "delta d". She made it clear she did not know what she was thinking. When BS constructed a table of values for this task, she settled on "delta d" for her explanations of slowing down and speeding up.

HL wanted to use the fact that three seconds after they started the two people were the same distance apart as three seconds after they passed each other. This would have been shown by the graph having the same height at $t=3$ and at $t=7$. But her graph did not have the same height at those two points. Her exclamation "Oh, no! Wait a minute!"

That didn't work!" came when she discovered that the height at $t=3$ is not the same as the height at $t=7$. After several other uses of distance value to describe speed she eventually came to describe speed in terms of "delta d".

Like BS, LB struggled with distinguishing between "delta d" and "distance value" as a way of figuring out speed. In the excerpt above she started by giving distance value at individual points and ended by giving delta d values. This excerpt came after a lengthy conversation about what shows speed on the graphs for Tasks 1 and 2. Like BS and HL, she eventually came to describe speed in terms of "delta d" and got more articulate about that as the interview went on.

There are several reasons some students use distance values at individual points as an indicator of speed. One is that these values are is salient. They are there, easily readable; so the temptation to use them is almost overwhelming. Another reason is that many students know that speed is somehow related to distance. The most immediate way of expressing a relationship between speed and distance is to equate the two. It takes specific experiences to learn the proper relationship between speed and distance. Speed is not directly given by distance. Speed is more directly obtained from the change in distance from one time value to another.

A third reason some students focus on distance at individual points when working with speed concerns how speed is represented in many situations in mathematics and science instruction. Most students know that "speed is distance over time". This statement has multiple meanings. If the vertical axis of a graph is labeled "distance", as it is in Tasks 1, 2, and 3, then we should not be surprised that some

students identified speed with distance value at individual points on the graph. Furthermore, since speed is "distance over time" and the horizontal axis is labeled time, some students will find it compelling to think of speed in terms of the distance value at individual points on the graph or in terms of that distance divided by the time.

Two postcalculus students referred to just this kind of velocity equation in their discussions of their graphs for Task 2:

"At one second they are 15 feet apart. At two seconds they are 10 feet apart. And at three seconds they are five feet and at four seconds they meet. But that means that they are going the same speed throughout the whole thing. (And why does that mean they are going the same speed throughout the whole thing?) 'Cause **velocity equals distance divided by time** and in mine, distance divided by time is all going to be the same." TS - Postcalculus

"It could be in a v-shape - straight down and straight up. (What would that represent?) That would represent **velocity, which is distance over time.**" LB - Postcalculus

LB went on to talk about the rate of change properties of her graph in terms of slope while TS calculated velocity by dividing the change in distance by the change in time. The expressions "velocity equals distance divided by time" or "velocity, which is distance over time" have the potential for being interpreted in at least two ways. One way means this: "Velocity has to do with how much distance is traveled over a certain amount of time.". This would imply a computational formula of the form $(\text{distance traveled})/(\text{time to travel that distance})$ or $(\Delta d)/(\Delta t)$.

Another interpretation of these expressions is this: "To get the velocity, just take the distance value and divide it by the time value.". Indeed, this is exactly what another postcalculus student did ~~when~~ she explained how her table of values in Task 1 showed the two people slowing down and then speeding up.

"Meeting point". For each of the three tasks, over 80 percent of the students had final graphs that met the horizontal axis at the point (4,0) which is equivalent to the two people meeting at $t=4$. Only one student had no graphs which met the horizontal axis at $t=4$. All three of his graphs went from (0,0) to (8,20) in various shapes. He was the only student who did not get a correct graph for Task 2. A correct graph for Task 2 must meet the horizontal axis at $t=4$, but graphs for Tasks 1 and 3 can theoretically meet the horizontal axis anywhere between $t=0$ and $t=8$.

Three students had Task 1 graphs that did not meet at $t=4$. They also had Task 3 graphs that did not meet at $t=4$. Two of these students (JF and LC) had the right shape of graphs for both Tasks 1 and 3 and thought that both the meeting point and the shape of the graph said something about the two people slowing down, speeding up, and walking at a constant pace. The other student (JM) had incorrect shapes for both Task 1 and 3 graphs and thought that the meeting point gave complete information about the speed at which the two people were walking:

"(How does your graph show the two people slowing down?) It takes - I figured four seconds would be if they were both going the same speed and it took 8 seconds. Four seconds would make them in the middle. So it would be longer than four seconds if they are going slower at the beginning and faster at the end. So I showed the zero a little past 5 [his graph meets the horizontal axis at (5.25,0)]. So this [pointing to the left side of his graph] takes longer. And then [pointing to the right side of his graph] this last two and three quarters second or whatever is quicker to get back up to 20." JM - Precalculus - Task 1

"covers the same distance in four seconds on both sides of it. So that means that they're going the same speed in the first half [points to $t=0$ to $t=4$ on the horizontal axis] when they meet each other and then they are in the second half [points to $t=4$ to $t=8$ on the horizontal axis]." JM - Precalculus - Task 2

"They cover all 20, they cover 20 feet in three seconds [pointing to $t=0$ to $t=3$ on the horizontal axis] and the remaining 8 [pointing to $t=3$ to $t=8$ on the horizontal axis] seconds is spent

covering the second or going back, I should say [running his finger up the right side of his graph]. (So since it took them only three seconds in the first half, they're speeding up?) Yeah. (And five seconds in the second half, they're slowing down?) Yep." JM - Precalculus - Task 3

"Since as the time goes by, it takes them longer to move - uh - as they move [running her pen down the left side of her graph] closer together, it takes them longer to move more feet toward each other or have less distance between them. And then since they're speeding back up [running her pen up the right side of her graph], it takes less time to get farther apart. [Her graph meets the horizontal axis at $(4.5, 0)$.]" JF - Precalculus - Task 1

"It would take them five seconds to meet because they're slowing down as they're reaching each other. But then once they pass each other, then they're speeding up so they would take less time. So it wouldn't be so balanced [the graph would not meet the horizontal axis at $t=4$; her graph meets the horizontal axis at $t=5$.]" LC - Precalculus - Task 1

"This one's constant speed. [Draws graph.] And on this one they would have to meet at four." LC - Precalculus - Task 2

"The anchor's going to have to be over here [points to $t=2$ on the horizontal axis] because they're both going fast. So it's not going to take them very long to meet but it's going to take them a long time to go sit down again." LC - Precalculus - Task 3

All three of these students used the meeting point to explain how their graphs showed speeding up, slowing down, and steady pace. For Task 1, since the two people were slowing down before they met and speeding up after, then, according to these students, it took them a longer time to meet and a shorter time to get to the opposite corner. For Task 2, since the two people were walking the same pace the whole way, it took the same amount of time to meet as it did to get to the opposite corner. Similarly, for Task 3, since the two people were speeding up before they met and slowing down after, it took less time to meet and more time to get to the opposite corner.

Going slow versus slowing down. One of the confusions about speed is the difference between going slow or fast (magnitude of velocity) and slowing down or speeding up (change in velocity). Going at a slow

constant rate means each second the distance covered is the same, **small** number of feet. However, "slowing down" means each second, the distance covered is less than it was the preceding second. The distance could change a relatively large number of feet each second (indicative of going at a fast pace), but if the number of feet it changes each second is less than the preceding second, then the two people are slowing down.

The responses of JM, JF, and LC above show they were confusing slowing down and speeding up with going at a slow or fast constant rate.

For example, JM's comment "So it would be longer than four seconds if they are going slower at the beginning and faster at the end." indicates he interpreted Task 1 to mean slow on the first part and fast on the second part rather than slowing down until they meet and then speeding up after. In Task 2, his comment further verifies this: "covers the same distance in four seconds on both sides of it. So that means that they're going the same speed in the first half when they meet each other and then they are in the second half." Since they cover the same distance in four seconds before they meet and after they meet, that shows they are going the same speed throughout the whole time.

Of course, the two people can together cover 20 feet in four seconds before meeting but be slowing down, speeding up, or walking at a steady pace. Similarly, covering 20 feet in the second half does not necessarily mean walking at a constant pace. In Task 3, JM had a graph that showed the two people covering 20 feet in three seconds at a relatively constant rate before they met and 20 feet in the remaining five seconds at a relatively constant rate. To him, that showed them speeding up as they walked toward each other and slowing down after they passed. So for him and the other two students quoted here, when the two

people met gave information about how their graphs showed the two people slowing down, speeding up, or walking at a steady pace.

Strategies for explaining tables. Many of the strategies students used to explain how their tables showed the two people speeding up, slowing down, or walking at a steady pace were the same strategies used to explain graphs. Table 3 gives each strategy and the number of students using each strategy on each task.

Table 3 - Strategies used to explain tables of values

Frequency (percent) of students' explanation strategies

<u>Strategy</u>	<u>Task 1</u>	<u>Task 2</u>	<u>Task 3</u>
Delta d	34 (92%)	35 (95%)	35 (95%)
(delta d)/(delta t)	3 (8%)	2 (5%)	2 (5%)
Reference to a point	4 (11%)	0 (0%)	0 (0%)
Meeting point	1 (3%)	2 (5%)	1 (3%)
Delta d and delta t	1 (3%)	0 (0%)	2 (5%)
Delta t	1 (3%)	0 (0%)	1 (3%)
Velocity Equation	1 (3%)	0 (0%)	0 (0%)

Note: Percentages add up to more than 100% in each column since some students gave more than one explanation.

"Delta d". On each of the three tasks over 90 percent of the students referred to the change in distance each second to explain how their tables showed the two people slowing down, speeding up, or maintaining a steady pace. As with graphs, I called this strategy "delta d". Here are some explanations using "delta d":

Task 1 -

"(How does your table of values show the two people slowing down and then speeding up?) There's a greater distance between the numbers at the zero and one seconds. There's a smaller distance between the two numbers between three and four seconds. (Does that mean they're slowing down or speeding up?) Slowing down 'cause the numbers get smaller. And then the space between the numbers gets bigger as the time goes on. (And that means?) They're speeding up." JW - Precalculus

"Well, at time zero, they are 20 feet apart. At time one second, they are 10 feet apart. So in one second, they covered 10 feet. From one second to two seconds, they covered six feet. So they're going a little bit slower 'cause here they covered 10 feet in a second and here they covered six feet in a second. Here [from $t=2$ to $t=3$] they covered three feet in a second and then they covered only one foot [from $t=3$ to $t=4$]. So they're going down until they get in the middle and then from 5 to 6 they go four feet in that second. And they only went one in here [from $t=3$ to $t=4$]. So they're increasing. From 6 to 7 seconds they go five feet so it's increasing. From 7 to 8 seconds they go eight feet which is another increase." RF - Calculus

"They are going the fastest here [pointing to the first part of her table] since they traveled nine feet in one second [from $t=0$ to $t=1$] and then they traveled five feet in one second [from $t=1$ to $t=2$], then four [from $t=2$ to $t=3$] and then two [from $t=3$ to $t=4$], then two [from $t=4$ to $t=5$], then four [from $t=5$ to $t=6$], then five [from $t=6$ to $t=7$], and then nine [from $t=7$ to $t=8$]. (So they're slowing down here [pointing to the first half of the table] and they're speeding up here [pointing to the second half of the table].)" DB - Postcalculus

Task 2 -

"(How does your table show the steady pace of the two people?) It's five feet each time. One second equals five feet." LC - Precalculus

"The intervals of time all have the same distance. They always move the same distance in the same amount of time." LC - Calculus
 "Because the distance changes five feet every second, every consecutive second." MG - Postcalculus

Task 3 -

"(How does your table of values show the two people speeding up?) The difference between per second is greater the closer you get to four. (It's small here [from $t=0$ to $t=1$ to $t=2$]?) Right. (And then it gets bigger here [from $t=2$ to $t=3$ to $t=4$]?) Right. (How does your table of values show the two people slowing down?) The distance increased by quite a bit here [from $t=4$ to $t=5$] and they by less as we go along [from $t=5$ to $t=6$ to $t=7$ to $t=8$]." KG - Precalculus

"It shows that they're going slow at the beginning because from 0 seconds to 1 second, the distance covered is only like one foot. And then from 1 second to 2 seconds is two feet, and from 2 seconds to 3 seconds is three feet, third to fourth is 14 feet. So they started off slow and the distance they covered became bigger. And over here [from $t=4$ to $t=8$] the direct opposite. They started off really fast and then they slowed down because the distance between the seconds gets smaller and smaller." PG - Calculus

"From 0 to 1 second, there's only a difference of one foot. From 1 to 2 seconds, there's a difference of one and a half feet. From 2 to 3 seconds, there's a difference of seven and a half feet. From 3 to 4 seconds, there's a difference of 10 feet. Since differences are going from smaller to larger, that means that they're stating out slower and getting faster. And symmetrically the other side." TS - Postcalculus

These students compared changes in distance each second to explain how their tables showed the two people slowing down, speeding up, or walking at a steady pace. Some students referred directly to differences between consecutive distance values in the table (JW, MG, KG, and TS) while others mentioned the number of feet the two people moved each second (RF, DB, LC-Precalculus, LC-Calculus, and PG). Either way it seems clear these students were using change in distance between the two people to make conclusions about how their tables showed the two people slowing down, speeding up, or walking at a steady pace.

"Delta t" and "delta d and delta t". A few students used the "delta t" or "delta d and delta t" strategy to explain their tables. Students who used the "delta t" strategy referred to speed in terms of the amount of time to cover a fixed distance and students who used the "delta d and delta t" strategy discussed speeding up in terms of "less time to cover more distance" and slowing down in terms of "more time to cover less distance". Two students used each strategy; one of those students (MY) used both:

"Delta t" -

"(How does your table of values show the two people slowing down?) As they're getting closer it takes more seconds than just one second in time to travel a certain number of feet." KG - Precalculus - Task 1

"Delta d and delta t" -

"(Where are the two people walking the fastest?) Down here at the end of your time and again at the beginning of your time the

people are covering more distance in less seconds." MY - Calculus - Task 1

"(How does your table of values show the two people speeding up?) It shows them speeding up 'cause they walk more feet in less time." AJ - Precalculus - Task 3

"Delta t" and "delta d and delta t" -

"(How does your table show the two people speeding up and slowing down?) They start out here zero [at $t=0$], they're both 20 feet apart. Here at the beginning they're losing more time but covering less feet until they get to about the four seconds mark and then they accelerate up until about the five second mark and then they start slowing down again and start losing time as far as distance traveled." MY - Calculus - Task 3

KG and MY used the "delta t" strategy when they referred to variable time to cover a fixed distance. KG said "it takes more seconds than just one second in time to travel a certain number of feet" and MY said "they ... start losing time as far as distance traveled". Both of these statements were taken to refer to more or less time to cover a given distance. MY and AJ used the "delta d and delta t" strategy when they said "covering more distance in less seconds" (MY) and "they walk more feet in less time" (AJ). Both of these statements refer to variable time to cover variable distance.

"(Delta d)/(delta t)". Some students used $(\Delta d)/(\Delta t)$, a formula for slope of a line and average rate of change in distance over intervals of time, to explain their tables:

"(How does your table of values show the two people slowing down?) Well, uh, here [from $t=0$ to $t=1$ to $t=2$] the difference between these is, you know, this is going to reflect approximately their speed because you've got a change of one second here, change in time. And you can divide the change in distance by the change in time. So here you're going to have the pseudo-velocity five here [from $t=1$ to $t=2$], three here [from $t=2$ to $t=3$], and one [from $t=3$ to $t=4$]." AD - Postcalculus - Task 1

"(How does your table of values show the two people walking at a steady pace?) The change in distance over the change in time is the same for every one. It's always five for each second." DR - Calculus - Task 2

"(How does your table of values show the two people speeding up?) [She computes $(\Delta d)/(\Delta t)$ for each one second interval.] They're going the same [in the first two seconds]. They went three feet in one second there [in each of the first two seconds] and then they got a little faster as they went and here [from $t=2$ to $t=3$] they're going five feet per second and here [from $t=3$ to $t=4$] they're going nine feet per second when they were closest to each other." SE - Postcalculus - Task 3

I did not classify these responses as "slope" although the formula the students used is a formula for slope. Nor did I classify these responses as "average rate of change" which has the same formula. These responses are closer to "delta d" since students essentially computed changes in distance over one second intervals. Nonetheless I gave them their own separate strategy classification as " $(\Delta d)/(\Delta t)$ " since these students used the language "change in distance over change in time" (AD and DR) or actually wrote " $(\Delta d)/(\Delta t)$ " and computed $(\Delta d)/(\Delta t)$ values (SE) as part of their argument about how their tables showed rate of change.

Comparing changes in distance over intervals of time of the same size is an appropriate way to discuss slowing down, speeding up, and steady pace. So "delta d" was a popular and successful strategy for these three tasks. " $(\Delta d)/(\Delta t)$ " is a more general strategy that can not only help explain how tables and graphs showing slowing down, speeding up, and steady pace but can also help make more precise statements about the numerical value of the speed over intervals of time. I would therefore consider " $(\Delta d)/(\Delta t)$ " a strategy that gives a more general formulation of rate of change. We will see with the "thrown ball", "biomass", and "circles" tasks the additional usefulness of " $(\Delta d)/(\Delta t)$ " in helping students get at instantaneous rate of change.

"Reference to a point" and "velocity equation". Just as some students used the distance value at points on the graph as a direct measure of rate of change, some students used the distance value at points in the table to talk about speeding up, slowing down, and steady pace:

"(How does your table show the two people slowing down?) Just the numbers go down [pointing to the first half of her table]. (You mean from here to here to here [pointing to the d values in her table]?) Yeah. (And how does your table show the two people speeding up?) 'Cause the numbers go up [pointing to the second half of her table]." KG - Precalculus - Task 1

"(How does your table of values show the two people slowing down and then speeding up?) Well, the numbers are decreasing until this is zero and then they are, the distance is [pointing to the entries in the table, not the distance between entries in the table] increasing as, until it gets to 20. (So when you say they're decreasing, you mean because they're going?) The distance is decreasing as the time is increasing (Ok.) until they get to zero and then the time is increasing as the distance is increasing. (Oh. And that shows them slowing down and speeding up?) Yeah." CH - Postcalculus - Task 1

The first student, KG, used "distance at a point" to explain how her table of values for Task 1 showed the two people slowing down and then speeding up. She used this same strategy to explain how her graph for Task 1 showed slowing down and speeding up. The second student, CH, also used "distance at a point" for each time value to address slowing down and speeding up in Task 1. When discussing her graph for task 1, she fumbled around with her language in trying to describe slowing down and speeding up in terms of slope and "delta d".

Possible reasons students express speed in terms of distance at point were discussed above. The same reasons apply here. A table makes the distance value at individual points readily available; students often know that speed is related to distance; and some students know that a formula for speed is frequently given in terms of distance over

time or distance divided by time. Here the time intervals are one second. So if speed is distance divided by time and time is 1, then speed is simply the value of the distance at the point.

Two other students used distance at a point in discussing how their tables of values showed rate. One of these also referred to an equation for velocity:

"(How does your table of values show the two people slowing down?) Decreasing [moving pen down the first half of her table of values]. (And what is it that's decreasing?) The distance that they are apart. (So since it's going 20, 14, 6, 1.25, 0) You know - oh - that they're slowing down - it would be the velocity - which would be distance over time. [She computed d/t at each time value in the table.] (And that tells you velocity?) Yes. (Over the first interval, they moved?) 14 feet per second [at $t=1$, $d=14$, so that $d/t = 14\text{ft./1sec.}$ for 14 feet per second]. And here they're only moving three feet per second [at $t=2$, $d=6$, so that $d/t = 6\text{ft./2sec}$ for 3 feet per second]. (Over that interval?) Just at the point. (And how does your table show the two people speeding up?) [She computes d/t for the second half of her table and gets .2 at $t=5$, .5 at $t=6$, 1.7 at $t=7$, and 2.5 at $t=8$ - to her this shows speeding up]" SE - Postcalculus - Task 1

Three points can be made about this excerpt. First, it is clear that this student initially used distance at a point, rather than change in distance from one time value to another (" Δd "), to talk about speed. Then she used the decreasing and increasing values of d/t at each time value to explain how her table showed the two people slowing down and speeding up, respectively. She actually performed these computations on her table and labeled them " d/t ". The formula "velocity - which would be distance over time" may have inspired her to use this strategy for addressing speed. A literal interpretation of this formula would directly lead one to form the ratio d/t and use it to make conclusions about speed.

Finally, this student made clear that she was not computing average rate of change over an interval, but rather was, in her mind,

computing rate of change at a point. That is how I interpreted her answer of "Just at the point." to my question "Over that interval?". That does not make her answer more correct; it does however lend credence to the claim that students use distance and distance divided by time to think about speed in general, speed over intervals, and speed at points.

"Meeting point". Two students gave explanations of how their tables showed rate of change based on when the two people met. One of these students (JM) used this strategy on all three tasks for both graphs and tables; SCL used it only with his table for Task 2:

"(How does your table show the two people speeding up?) In the first three seconds, they cover 20 feet - goes from 20 to 10 to 5 to 0 [pointing to the d values in the part of his table for $t=0$ to $t=3$]. And then the remaining five seconds [pointing to the d values in the part of his table for $t=3$ to $t=8$], it covers the same amount of time [distance]. So that means that they slowed down during this part [part from $t=3$ to $t=8$]. And again if four were zero [if the distance value were four when the time value was zero], then they would have covered 10 feet, uh, yeah, each. But since the zero's at three that means that they were going a little faster than average [from $t=0$ to $t=3$]. (So here [from $t=0$ to $t=3$] they're speeding up because it only took them three seconds) Yeah. (and here they're slowing down because it took them five seconds.) Yeah." JM - Precalculus - Task 3

"(How does your table show the steady pace of the two people?) When they get here - for four seconds they meet. So in four seconds they go same distance going across room in same time as they come to this point [pointing to the point in his table with $t=4$ and $d=0$]. (So the fact that it took them four seconds to meet and four seconds to get to the other side, that shows that they were going at a steady pace?) Uh huh." SCL - Precalculus - Task 2

In the section above where I described students' strategies for explaining their graphs, I discussed in detail how JM and two other students used "meeting point" as evidence for how their graphs showed the two people slowing down, speeding up, or walking at a steady pace. Here we have further evidence that JM believed that the meeting point

was the deciding factor in showing speeding up and slowing down. The two people walked a total of 20 feet in three second before they met and they walked another 20 feet in five seconds after they met. To JM this showed that they were speeding up before they met and slowing down after. While it is true that walking 20 feet in three seconds yields a faster average speed than walking 20 feet in five seconds, it is possible for two people to walk 20 feet in three seconds and be slowing down and walk 20 feet in five seconds and be speeding up. In fact, one of the precalculus students (LC) had exactly such a table.

SCL used the "meeting point" strategy only on his table for Task 2. For his graph for Task 2 as well as his graphs and tables for Tasks 1 and 3, he used a variety of strategies finally settling in Task 3 on "delta d". Perhaps in Task 2 he was just experimenting with yet another strategy. Or perhaps he was trying to express in a different way what he had already said about his graph for Task 2. There he said "they take same time going a certain distance" and, after my prompting concerning what this meant for distance traveled each second, made statements about "five feet each second." I classified that as using the "delta d" strategy.

But his wording in the excerpt above makes it clear he is thinking that going 20 feet in four seconds from $t=0$ to $t=4$, meeting at $t=4$, and going 20 feet in four seconds from $t=4$ to $t=8$ by itself shows steady pace. That is, since they covered the same distance, 20 feet, in the same amount of time, four seconds, before they met and after they met, they were going at a steady pace. However, that alone does not show steady pace. It is possible to go 20 feet from $t=0$ to $t=4$ and be slowing down, speeding up, or maintaining a steady pace, meet at $t=4$,

and go 20 feet from $t=4$ to $t=8$ and be slowing down, speeding up, or maintaining a steady pace. Having the two people meet at $t=4$ does not show steady pace. SCL's words here imply that he believed it did.

Summary of strategies used with Tasks 1, 2, and 3. Seven strategies were used to explain how graphs showed slowing down, speeding up, and steady pace. The most frequently used strategies were "delta d" and "visual slope". On each task over 60 percent of the students used "delta d" and over 40 percent used "visual slope" to explain their graphs.

Seven strategies were used to explain how tables of values showed the varying speed of the two people. Over 90 percent of the students used "delta d" to explain their tables. Since slope is a more visible feature of graphs, it is not surprising that students used "visual slope" when working with graphs but not with tables.

Six strategies were used with both graphs and tables and the most frequently used strategy with both was "delta d". I was not surprised that the most common strategy with tables was "delta d" but I was surprised that it was also the most common strategy with graphs. This made me think that students at these levels of mathematics education find relative or actual distance traveled over intervals a meaningful way to make sense of rate of change. This knowledge has, I believe, the potential for being quite helpful in students coming to know average rate of change and instantaneous rate of change, both of which have their basis in changes in the dependent variable over intervals of the independent variable of a function.

Students' use of "visual slope" also points to knowledge that may serve them in learning more about rate of change. The derivative of a

function is the mathematical concept most closely associated with instantaneous rate of change and it is often introduced to students in the context of slope of secant and tangent lines. More generally, derivative and rate of change at a point have to do with slope of curves at points. This makes me think that students' use of "visual slope" represents knowledge they could use to make sense of slope of tangent lines, slope of curves at points, and instantaneous rate of change.

The "thrown ball", "biomass", and "circles" tasks

Task statements.

For each of these tasks students were verbally presented with a situation and a graph of the relationship between two changing quantities and were asked to use the graph to answer some questions about the rate of change of one of the quantities with respect to the other quantity. Then they were given a table of values and asked to answer some of the same questions. I will describe each task in the following paragraphs.

The situation in Task 4, the "thrown ball" task, involved a ball being thrown straight up into the air. Students were given a graph showing the height of the ball at each moment and were asked to describe the height of the ball over time, identify intervals of time when the ball traveled the most and the least number of feet, and find the speed of the ball at the end of one second, two seconds, three seconds, and just before it hit the ground. They were then given a table of values for the height of the ball at each quarter second and asked again to find the speed of the ball at the end of one second, two seconds, three seconds, and just before it hit the ground. My intent in asking some of the same questions twice, once with the graph and once with the table, was to see if students could address rate of change at a point using more than one strategy and specifically if they used some strategies with the graph and other strategies with the table. In each part of the task students were asked to explain their answers. Appendix C contains the complete tasks including graphs and tables of values for each task.

The graph for Task 5, the "biomass" task, showed the population of yeast cells in a culture over a period of 18 hours from midnight to 6 pm. Students were asked to describe the population over this period of time, identify intervals of time when the population changed the most and the least, and find how fast the population was changing at 4 am. They were then given a table of values for this graph and again asked to find how fast the population was changing at 4 am. I chose this point in time since the cell population was increasing slowly before 4 am and more quickly after 4 am. I was interested in how students would treat rate of change at a point in this kind of circumstance.

I then gave the students an enlarged section of the graph near $t=4$ am and a table of values that showed the cell population at each quarter of an hour and asked them once again to find how fast the population was changing at 4 am. My intent in asking the same questions three times, once with a graph, once with a table of values, and once with an enlarged graph and table of values, was to see if students had more than one strategy for addressing rate of change at a point and if a more detailed graph and table of values suggested new strategies to them. I told students I was asking the same questions several times to see if they had more than one way to work the problem and my asking the same question several times did not necessarily mean that their first answers were wrong. For each question students were asked to explain how they got their answers.

In Task 6, the "circles" task, the situation involved a pebble being dropped into a calm pond, causing ripples in the form of concentric circles. A graph showing the relationship between the area

and radius of these circles was given. Students were asked to describe the area of the circles as the radius grew from zero to five feet, identify the one foot increase in the radius for which the area grew the most and the least, find how much the area grew as the radius grew from two to three feet and from three to four feet, and find how fast the area was growing when the radius was three feet.

Here I was interested in what connections students might make between rate of change at a point and changes over intervals before and after the point. If students had not already made some connection by this time in the interviews, I wanted to see what sense this suggestion might make to them. Students were then given a table of values for this graph and asked again to find how much the area grew as the radius grew from two to three feet and from three to four feet and to find how fast the area was growing when the radius was three feet. As with the other tasks, students were asked to explain all of their answers.

Correct answers.

The "thrown ball" task. A correct description of the height of the ball over its four second journey might simply note that the ball went up to a height of about 64 feet over the first two seconds and then came back down over the last two seconds, hitting the ground at $t = 4$. A more complete description would include some mention that the ball slowed down as it went up and sped up as it came down. The ball traveled the most feet the first and last quarter seconds and it traveled the least the two quarter seconds either side of $t = 2$. One way to see this is to note the change in the height of the ball over each quarter second. During the first and last quarter seconds the

height changed the most and during the two quarter seconds to the immediate right and left of $t = 2$, the height changed the least. Another way to see this is to compare the visual slope of each quarter second section of the graph and note that the sections with the steepest slope are the first and last quarter seconds and the sections with the flattest slope are the two middle quarter seconds.

At $t = 1$ the ball was traveling 32 feet per second. If one knows the equation for this graph, then one can differentiate it and evaluate that derivative at $t = 1$. One could draw the line tangent to the graph at the point $(1, 48)$ and estimate the slope. That theoretically yields a correct answer although it may be fraught with drawing and estimation errors. Another way of getting an estimate for the speed of the ball at $t = 1$ is to find the average rate of change of the height over any reasonably small time interval centered at $t = 1$. The intervals from $t = .75$ to $t = 1.25$, $t = .5$ to $t = 1.5$, $t = .25$ to $t = 1.75$, and $t = 0$ to $t = 2$ all yield an average rate of change of 32 feet per second. Because of properties of a parabola (the shape of this graph), the average rate of change of the height over any time intervals centered at a point happens to yield the accepted value for the speed of the ball at that point. So these particular intervals centered around $t = 1$ yield the exact value for the speed of the ball at $t = 1$.

At $t = 2$ the speed of the ball was zero feet per second; that is, the ball was stationary at $t = 2$. One could evaluate the derivative equation at $t = 2$ or could note that the graph reaches a maximum at $t = 2$ and therefore the value of the derivative and the speed of the ball must be zero there. Noting that the line tangent to the graph at $t = 2$

is horizontal and therefore has slope zero also yields a correct answer.

An elegant argument considers the speed of the ball as it goes up and as it comes down. Since the ball slows down as it goes up and then speeds up as it comes down, an argument could be made that at the peak the ball stopped as it changed directions. Since the peak occurs at $t = 2$, the speed of the ball at $t = 2$ has to be zero. One could also compute the average rate of change of the height over a variety of time intervals centered at $t = 2$ and arrive at zero feet per second as the speed of the ball at $t = 2$.

At $t = 3$ the speed of the ball was the same as it was at $t = 1$ except that the ball was moving downward instead of upward. The velocity of the ball at $t = 1$ was 32 feet per second; therefore the velocity of the ball at $t = 3$ was -32 feet per second since velocity refers to both speed and direction. A correct answer for the speed is 32 feet per second. One could get this answer by evaluating the derivative equation at $t=3$ and obtaining -32 feet per second. Or one could argue that since the acceleration due to gravity is (32 feet per second)/second, then since the speed of the ball was zero at $t = 2$, then the speed of the ball at $t = 3$ had to be 32 feet per second. That is, if the starting speed was zero (at $t = 2$) and the speed increased 32 feet per second every second, then at the end of one second (at $t = 3$) the ball was traveling at a speed of 32 feet per second. One could also compute the average rate of change of height over several time intervals centered at $t = 3$ and get -32 feet per second.

At $t = 4$, the ball was stopped, since it had just hit the ground, and therefore the speed of the ball at $t = 4$ was zero feet per second.

But just before the ball hit the ground, it was traveling at a speed of 64 feet per second. The derivative equation evaluated at $t = 4$ yields this. In addition, using an argument based on the acceleration due to gravity, one could argue that at $t = 2$ the speed of the ball was zero feet per second, and since the speed of the ball was increasing 32 feet per second for every second, then not only is the speed of the ball 32 feet per second at $t=3$ but it is also 64 feet per second at $t = 4$.

One could argue that the speed of the ball as it hit the ground was the same as the initial velocity of the ball at $t = 0$ and set about to estimate that by considering the average speed of the ball over the first or last quarter second. A more sophisticated method involves projecting how far the ball would have traveled between $t = 4$ and $t = 4.25$ if it had not hit the ground at $t = 4$ and then computing the average speed over the interval from $t = 3.75$ to $t = 4.25$. Projecting how far the ball would have traveled between $t = 4$ and $t = 4.25$ is pretty easy when one sees the pattern of distance traveled each quarter second from $t = 2$ to $t = 4$. That pattern is this: 1, 3, 5, 7, 9, 11, 13, and 15. If this pattern had continued, then the projected distance traveled from $t = 4$ to $t = 4.25$ would have been 17 and the average speed from $t = 3.75$ to $t = 4.25$ would have been $(15+17)$ feet or 32 feet over a half second or 64 feet per second.

The "biomass" task. A correct description of the growth of the population of yeast cells over the 18 hours would mention that the population increased from less than 10 at midnight ($t = 0$) to a little over 750 at 6 pm ($t = 18$). A more complete description might note that the population grew slowly at first, then rapidly from about 6 am ($t =$

6) to noon ($t = 12$), then continued to increase but at a slower rate. Another description is that the population increased at successively faster rates until about 8 am ($t = 8$) and then it increased at successively slower rates from 8 am to 6 pm ($t = 18$). The population did not grow by the same amount every hour. It changed the most during the middle hours of 7 to 8 am and 8 to 9 am and it changed the least from midnight to 1 am and from 5 to 6 pm. This could be seen by noting the steepness or flatness of the graph or the relative vertical change over these respective time intervals.

To find how fast the population was changing at $t = 4$ am, one could draw the line tangent to the graph at $t = 4$ and estimate its slope. Or one could find the average rate of change of the population over the interval from $t = 3$ to $t = 5$ and use that as an estimate of the rate of change of the population at $t = 4$. The enlarged graph and more detailed table of values could be used to get better estimates of the rate of change of the population at $t = 4$. On the enlarged graph one could draw the tangent line at $t = 4$ and estimate its slope. That might give a better estimate. Using the detailed table of values one could find the average rate of change of the population over the time interval from $t = 3.75$ to $t = 4.25$ and use that as an estimate of the rate of change of the population at $t = 4$ am.

Values for the average rate of change of the population over time intervals of decreasing width bordering on $t = 4$ are shown in Table 4. The pattern formed by these values suggests the rate of change of the population at $t = 4$ is between 31.6 and 34.4 cells per hour. This procedure also suggests one way for finding exactly how fast the

population changes at $t = 4$, as students were asked to do in a later part of Task 5. One could find the average rate of change over intervals of decreasing width bordering at $t = 4$ and note the limiting value of those average rates of change as the width of these intervals goes to zero. Since values for the cell population were not given for time values between $t = 3:45$ and $t = 4:15$ I did not anticipate that students would actually compute a more precise limiting value than what could be obtained from Table 4 although I hoped some students might verbally describe such a procedure.

Table 4

Average rate of change in the number of yeast cells
over intervals of decreasing width bordering $t=4$ am

<u>Interval</u>	<u>Ave. rate of change</u>	<u>Interval</u>	<u>Ave. rate of change</u>
3:00-4:00	23.9 cells per hour	4:00-5:00	48.0 cells per hour
3:15-4:00	26.3 cells per hour	4:00-4:45	42.8 cells per hour
3:30-4:00	28.8 cells per hour	4:00-4:30	38.4 cells per hour
3:45-4:00	31.6 cells per hour	4:00-4:15	34.4 cells per hour
These values suggest the rate of change at $t=4$ is greater than 31.6		These values suggest the rate of change at $t=4$ is less than 34.4	

Together these suggest the rate of change at $t=4$ is
greater than 31.6 and less than 34.4 cells per hour.

Another procedure for finding the exact rate of change at $t = 4$ is to find an equation for the relationship between time and the population, differentiate that equation, and evaluate that derivative at $t = 4$. Since an equation for that relationship was not given and since it was unlikely that students in this study could easily find it, I did not believe students would use this method although I believed that some students would mention such a procedure. Students could, of course,

refer to the slope of the tangent line as an exact answer for the rate of change of the cell population at $t = 4$.

The "circles" task. A correct description of the growth of the area of a circle would note that the area increased as the radius increased. It might also note that the area increased at an increasing rate. The area grew the least as the radius increased from zero to one and it grew the most when the radius increased from four to five. The area grew about 15.7 square feet as the radius grew from 2 to 3 feet and the area grew about 22 square feet as the radius grew from 3 to 4 feet.

The average rates of change over these two intervals are 15.7 and 22 square feet per foot, respectively.

One estimate of the rate of change of the area when the radius is exactly three feet would be the average of 15.7 and 22, the two figures obtained above. That would yield an answer of 18.85 square feet per foot. An exact answer could be obtained by differentiating the area with respect to the radius using the equation $A = \pi r^2$ for the relationship between the radius (r) and the area (A) of a circle. This results in the derivative equation of $A' = 2\pi r$. Evaluating the right side of this equation at $r = 3$ yields 18.85, the same as the estimate above. Another way to get the exact answer would involve drawing the line tangent to the graph at $r = 3$ and finding the slope.

Identification of strategies.

Describing the dependent variable. In Tasks 4, 5, and 6 students were asked to describe what was happening to the dependent variable over the entire domain of the independent variable. Students tended to give one of two descriptions, a less detailed description and a more detailed

description. In the less detailed descriptions, students mentioned only that the dependent variable increased or decreased without indicating the varying rates of increase or decrease. In the more detailed description, students not only mentioned that the dependent variable increased or decreased but also emphasized the varying rates of increase and decrease.

In Task 4 students were asked to "Describe the height of the ball over the four seconds." The less detailed description was "it goes up and then it comes back down" and the more detailed description was "it slows down as it goes up and speeds up as it comes down". Over 60 percent of the students gave the less detailed description.

For Task 5 students were asked to "Describe the population of yeast cells over this period of time." from midnight ($t = 0$) to 6 pm ($t = 18$). All of the students gave one of two sorts of detailed descriptions. In one of the descriptions the change in population over the 18 hours was seen as consisting of three sections. In the first section, the population increased slowly; in the second section, it increased more quickly; in the third section, it increased slowly again.

In the other description the population was seen as consisting of two sections. Over the first section from $t = 0$ to about $t = 8$, the population increased at successively faster rates and over the second section from about $t = 8$ to $t = 18$, the population increased at successively slower rates. This means that in each succeeding hour of the first half, the population increased more than it did the previous hour and in each succeeding hour of the second half, the population increased less than it did the previous hour. About three-fourths of

the subjects gave the first type of description - increasing slowly, increasing quickly, increasing slowly again.

In Task 6 students were asked to "Describe the area of the circle as the radius grows from 0 to 5 feet." The less detailed description was "It increases." The more detailed description was "It increases at an increasing rate." Over 60 percent of the subjects gave the less detailed description.

Identifying intervals of least and most change. In Tasks 4, 5, and 6 students were asked to identify intervals of the independent variable for which the dependent variable changed the most and the least and to explain how they knew. On all three tasks, all of the students correctly identified intervals of the independent variable for which the dependent variable changed the most and the least and primarily three strategies were used to make these decisions.

Task 4 - The "thrown ball" task. Students were asked to decide over which quarter second intervals the ball traveled the most and the least. Table 5 shows the strategies students used.

Table 5

Strategies used to decide when the ball was traveling most and least

<u>Strategy</u>	<u>Frequency (%) of responses</u>
Delta d	37 (100%)
Knowledge of motion	10 (27%)
Visual slope	6 (16%)
Delta t	1 (3%)
Delta d and delta t	1 (3%)

There were 37 students in the study. The totals (number and percent) do not add to 37 (100%) since some students used more than one strategy. All students used at least one strategy. Percents represent the percent of the 37 students to use this strategy.

"Delta d". All of the students referred to distance traveled over quarter second intervals, "delta d", to decide which intervals showed the ball traveling the most and least. Since students were asked to find the quarter second intervals during which the ball traveled the most and the least, it should not be too surprising that they reference relative or actual distances traveled in quarter second intervals to answer this question.

Here are some sample comments:

"(Does the ball travel the same number of feet each quarter second?) No. It's fewer feet as we get closer to the two second point. (How can you tell?) Uh, the little squares [denoting points on the graph for each quarter second] are closer together."
KG - Precalculus

"In some quarters they travel more feet. (Ok. During which quarter seconds does it travel the most?) They travel about 15 feet here [from $t=0$ to $t=.25$]. And here [from $t=1.5$ to $t=1.75$] they travel about five feet." (SCL - Precalculus

"The ball travels say the most, the most feet in like the first second or like in the first couple quarters of a second. And then it keeps decreasing [the distance traveled in successive quarter

second intervals on the way up] and then like as it comes down, in the last couple quarters is has, it makes the most jump or whatever, the biggest difference in feet." AM - Calculus

"(Does the ball travel the same number of feet each quarter of a second?) No. (How can you tell?) Because the distance between this [height at $t=.25$] would be around 15 and this [height at $t=.5$] would be around 29 and then up here [from $t=1.5$ to $t=1.75$] this is about 5." JK - Calculus

"From zero to two seconds, it's going progressively fewer feet per unit of time and as it goes from two to four seconds, it's going progressively more feet per unit of time." AD - Postcalculus

"It covered different amount of feet in different seconds because it's not a slope [read line]. It's a parabola. (Show me a quarter second where it's traveling fast and show me a quarter second where it's traveling slow.) Well, right here [from $t=0$ to $t=.25$] is like 15 feet for a quarter of a second. And like up here [from $t=1.75$ to $t=2$] they only covered like two feet. Which is a different speed than over here [from $t=0$ to $t=.25$]. From this point [$t=2$] to this point [$t=2.5$] they only covered maybe four feet, maybe three feet." LB - Postcalculus

These students referred to actual or relative distance traveled by the ball to decide over which quarter second intervals the ball traveled the most and least. Some students based their decisions on visual comparisons of distance on the graph (KG, AM, and AD) while others referred directly to the actual distances traveled during specific quarter seconds intervals (SCL, JK, and LB).

"Visual slope". The last student quoted above (LB) also used visual slope to support her claim: "It covered different amount of feet in different seconds because it's not a slope." (She consistently used slope throughout this interview when referring to lines.) When pressed for further evidence she referred to the actual number of feet traveled in some quarter second intervals. Other students also made comparisons using visual slope:

"Well, you can see that the slope is decreasing [runs pen up along the left side of the graph]. But I know that the force of gravity is pulling it down and slowing it down so it's traveling less

distance over the time. But it's traveling farther in this quarter second [from $t=0$ to $t=.25$], little bit less, little bit less, little bit less [successively less in successive intervals], lot less, lot less [moves his pen from point to point up along the left section of the graph]." DR - Calculus

"(How can you tell where it's moving the fastest and where it's moving the slowest? What is it about the graph that shows you that?) Well, to me, it's the slope of the graph. Just to look at the slope of the graph. But you can also look at it as a table of values. In the first quarter it's moved almost 15 feet in the air. And in the fifth quarter it moves only like five feet." DW - Calculus

"No. It starts out by moving farther up and then it slows down and doesn't travel as far right in the middle and then it speeds up again. (So you can tell it's slowing down because?) Because for the, the bigger the slope of the line [connects points at $t=1.5$ and $t=1.75$] on the graph] is like this - and it's more time [to cover the same distance] and not as much distance in the same time. (And how can you tell it's going faster?) Because there's more distance in a shorter amount of time." JS - Postcalculus

These students used slope initially to support their claims about when the ball traveled faster and when it traveled slower. The three students quoted above continued to reason about the question. The first student (DR) used knowledge of motion as further evidence that the ball traveled less distance in each successive quarter second on its way up: "the force of gravity is pulling it down and slowing it down so it's traveling less distance over the time". He then walked off comparative distances the ball traveled in each quarter second interval on its way up. He used three different strategies, "delta d", "visual slope", and "knowledge of motion", a strategy to be discussed below, to decide when the ball traveled more and when it traveled less.

The second student (DW) used actual distances traveled in each quarter second to strengthen his claim about when the ball traveled more and when it traveled less. He referred to a table of values. At this point he did not have an actual table of values; he was merely noting

that he could read coordinates of points on the graph and, as he would do if he had a table of values, could then compute the distance the ball traveled each quarter second interval and make his decision that way. He proceeded to do just that. The graphs given in Tasks 4, 5, and 6 were not smooth curves. They were composed of straight line segments joining points. The graph for Task 4 was straight line segments joining points at quarter second intervals. This made it fairly easy to read the coordinates of these points and compute the distance traveled each quarter second.

"Delta t" and "delta d and delta t". The last student quoted above (JS) made a reference to slope but she used the words "bigger slope" while pointing to a section of the graph that has relatively smaller slope. After a brief fumbling around with words, she referred to the distance traveled in an amount of time: "it's more time [to cover the same distance] and not as much distance in the same time" when the ball was traveling slower and "there's more distance in a shorter amount of time" when the ball was traveling faster. She used "delta t" ("it's more time [to cover the same distance]"), "delta d" ("not as much distance in the same time"), and "delta d and delta t" ("there's more distance in a shorter amount of time").

"Knowledge of motion". One of the students discussed above (DR) used knowledge of motion to decide when the ball traveled more and less. There were other students who used this same strategy:

"(Does the ball travel the same number of feet each quarter second?) Pretty much, until it gets to the top when it starts to slow down 'cause of the gravity picking up and when it reaches the peak, it stops. So it doesn't cover as many feet at the top as it does down at the bottom on take off." TW - Precalculus

"Well, obviously when the ball is being thrown, between zero and one quarter of a second, it's going fastest because it just left your hand. It's going faster then say closer to when it's reaching its peak. It slows down. (How does the graph show that?) Here [from $t=0$ to $t=.25$] in a quarter second the distance traveled is about 15 feet. But over here toward its peak in this quarter second [from $t=1.75$ to $t=2$] the ball travels only about just a few feet. It travels fast the closer it gets to the ground. It keeps accelerating due to gravity." MG - Calculus

"Well, it looks like it's covering more some quarter seconds if I just want to look at these little boxes between them [the distance between the points, which are presented by icons that look like little boxes]. (Show a quarter second where it's covering a lot of distance.) Well, anywhere right up through here, anywhere up through the top. I think. It appears that [long pause]. It seems like if you were to toss a ball up, it would have more velocity obviously at the beginning of the toss than it would at the top, than at its maximum. At its maximum it would cover less and it would slow down because it's getting ready for its down fall. So it would go up faster and then it would reach a point where it's slow, where it'd be at its peak and as it'd come down it would go faster and faster. (Is there anything about this graph that substantiates that? Or contradicts it?) No. I guess this supports it. (And how does it support it?) Well, as you throw the ball up, obviously at the beginning it's covering more than it is at the top, I guess, if it is right. And then as it does get right, it hits its maximum or peak. It's obviously, you know, slowed way down. It's not, you know, traveling nearly as fast or covering nearly as much ground 'cause, you know, in between here [at the top of the graph] I guess it's only covering a couple of feet where you can come down here [near $t=.75$] and it's covering about 10, you know, or 11 down here. So, I guess, yeah, ok, so I'm, seems right." RC - Postcalculus

These students used knowledge of motion to decide when the ball traveled most and least. The first student (TW) based his conclusion that "it doesn't cover as many feet at the top as it does down at the bottom on take off" on his knowledge of the effect of gravity in causing the ball to slow down as it went up. The second student (MG) used his knowledge that the ball is going fastest when it first leaves your hand. When I pressed him for further evidence, he cited actual distances the ball traveled in several quarter second intervals. He finished off his

discussion with another reference to motion: "It travels faster the closer it gets to the ground. It keeps accelerating due to gravity."

The third student (RC) gave an initial answer based on "delta d": "if I just want to look at these little boxes between them [the distance between the points, which are represented by icons that look like little boxes]". When asked to identify an interval where the ball traveled more distance, he used his knowledge of motion to satisfy himself that the graph shows what it is supposed to show. In the process he noted that the ball traveled more distance at the bottom than at the top where it was not "covering nearly as much ground". He ended by citing the number of feet the ball traveled in a couple of intervals: "it's only covering a couple of feet [up at the top] where you can come down here [near $t=.75$] and it's covering about 10". He, like several other students, used more than one strategy to reason about when the ball traveled the most and the least.

Task 5 - The "biomass" task. Students were asked to decide over what intervals of time the cell population changed the most and the least. Most of the strategies used in Task 4 were also used here. Table 6 shows the number and percent of the 37 students who used each strategy.

Table 6

Strategies used to decide when the population changed most and least

<u>Strategies</u>	<u>Frequency (%) of responses</u>
Delta p	26 (70%)
Visual slope	22 (59%)
Knowledge of growth of populations	5 (14%)
Delta p and delta t	2 (5%)

Totals (and percents) do not add up to 37 (100 percent) since some students used more than one strategy.

"Delta p". Seventy percent of the students used change in population over one hour intervals of time to make decisions about when the population changed the most and least. They compared distances between points on the graph or number of cells the population grew during one hour time intervals. Some actually computed approximate values for the change in population while others made relative comparisons. Here is a representative sample of comments:

"Looks like over the first four hours there is not a significant amount of reproduction but about 5 hours until 13 hours is when they are going through their major reproductive stage and then 14 through 18 they're more abundant. (And how are you able to tell that from the graph?) Uh. The first four hours, in each of those hours, the number of cells doesn't rise very much. It's still under 100. The number of cells - it's hard to even tell how much it's going up. But then you get up to about 5 hours until 13 hours and it's going up significantly - between 50 and a 100 per hour and back again to insignificant." LC - Precalculus

"It goes less, more, less. (And how do you know?) As you can see here, these time intervals are remaining constant. Now we notice here, let's say this is gonna be one [draws rectangular steps on

the graph from 1 to 2 to 3], that the distance here, I mean the population is growing in here [from $t=0$ to $t=1$] is gonna be say, let's say 10, 15. We go to point 2 [from $t=1$ to $t=2$], the population grows a little more, not much as you can see. The line

{indecipherable}. As you go from here [from $t=2$ to $t=3$] we're seeing a little more increase of people, of cells. Then we notice that within the same time frame, er, the time interval, we're getting a little more people here [from $t=3$ to $t=4$]. If you just wanted to find the rate of change, or the slope of from point to point, you notice that you're getting larger and larger. So, anyways, the time intervals are the same, and you can see here that there's being more cells produced within about the same amount of time [middle section]. And then over here [upper right section of the graph] I see that there's less cells produced within the same amount of time." HL - Calculus

"(How do you know that they're increasing not very much here [from $t=0$ to $t=4$] and a lot here [from $t=5$ to $t=12$]?) Ok, well, just take your cells population per hour. Here [from $t=3$ to $t=4$] you have about 25 cells per hour. And then way right here [from $t=7$ to $t=8$] you have nearly 100 cells per hour. And then up here [from $t=16$ to $t=17$ to $t=18$], it's barely changing at all." RF - Calculus

"It grows a little at first and then it sort of grows a lot more and then it sort of levels off [tracing her pen along the full length of the curve as she talks]. It's growing more in the middle hours and then it tapers off. (How are you able to tell that from the graph?) Just because of the slope of the graph, actually, 'cause there is more of a population growth between these two [$t=6$ to $t=10$] then between these two [$t=0$ to $t=4$]. It's slower here [$t=0$ to $t=4$] because there's not as many yeast cells growing here." DB - Postcalculus

"'Cause here [from $t=0$ to $t=1$ to $t=2$] the difference would be probably 10 and then as you come up from there [near $t=3$ to $t=4$ to $t=5$] it would be about 40 and from there to there to there [from $t=6$ to $t=7$ to $t=8$] it's about 100 cells, not quite, and it increases again and here [near the end] it's about 1 cells, 2 cells." JG - Postcalculus

HL used two strategies. First she used "delta p" by making a rough numerical approximation of the amount of growth in the first hour ("let's say 10, 15") and then for subsequent time intervals she noted that the growth each hour was a little more than the previous hour. She then used "visual slope" by noting that the slope was "getting larger and larger" where the population was growing the fastest. Then she returned to using "delta p" to conclude that the population grew faster in the middle hours and slow again in the later hours.

DB focused first on slope but then turned to comparison of distance between consecutive points to note that the population growth from $t=6$ to $t=10$ was more than the population growth from $t = 0$ to $t = 4$. So she used both visual slope and delta p . LC, RF, and JG used the "delta p " strategy by making approximations of the number of cells that grew in selected hours.

"Visual slope". Over half of the students used visual slope to support their claims about when the population grew the most and the least:

"More some and less others. (Where is it changing the most?) Between 6 and 12. (What is it about the graph that makes you say that?) It's more steep. (When is it changing the least?) The first and the last hours. (Why is that?) It's almost a horizontal line." MH - Precalculus

"They increase more between 5 am and about 12. (How can you tell that?) The slope is much greater there than it is here [from $t=0$ to $t=4$]." JK - Calculus

"During the middle hours, it's greatest rate of change. (And how do you know that?) The slope of the graph is its greatest at the point." AD - Postcalculus

Some students explicitly used the word "slope". Other words students used that made me think they were focusing on visual slope included steep, horizontal, vertical, flatter line, leveled off, curvature, gradual curve, and references to shape of curve. Note that two of the three students cited above used words like "the slope is much greater" (JK) and "the slope of the graph is its greatest" (AD). These are not references to any numerical values; these students did not compute the numerical values for the slopes here. These are qualitative comparisons of the slopes of various sections of the graph. Here I interpreted "greater" to mean "steeper" and "greatest" to mean "steepest".

I asked several calculus and postcalculus students who made explicit reference to slope why slope gave information about rate of growth:

"(Now you used the word 'slope'. Is slope helping you think about how fast it's going?) Yes, because here, the slope, which is how fast, yeah, 'cause slope tells you how much extra cells are growing per hour so here [at the beginning of the graph] the slope's increasing. So more cells are growing per hour." BS - Calculus

"The uh slope is increasing there, this right here [middle section of the graph where he earlier identified the growth as 'going fast']. (And why does an increasing slope mean that?) It means that there's more population in a less amount of time." LC - Calculus

"Horizontal slope means that it's not growing very fast and vertical slope means it's growing very fast. (Why is that?) Because over this amount of time [at the beginning] there's very little difference between this point and this point. But then right here [near $t=6$] it starts growing more for a smaller amount of time." JS - Postcalculus

These students made a connection between slope and change in population per hour which is a key to understanding rate of growth. Increasing or more vertical slope means more change in population per unit time and more horizontal slope means very less change in population per unit time. So a steeper slope means a greater growth rate and a flatter slope means a smaller rate of growth.

"Delta P and delta t". Two of the students quoted above (LC and JS) also used "delta p and delta t" when they said "more population in a less amount of time" (LC) and "it starts growing more for a smaller amount of time" (JS). Recall that this strategy differs from "delta p" in that students who used "delta p" held the change in time constant at one hour while students who used "delta p and delta t" compared changes

in population over varying amounts of time. These were the only two students who used this strategy in Task 5.

"Knowledge of growth of cell populations". Some students used knowledge of growth of populations to make decisions about when the population changed the most and least:

"It's almost like the 2, 4, 8, 16, like a chain reaction kind [tracing his pen along the middle part of the graph] of where they're all reproducing until they get to like a certain number, probably when it fills the petri dish and they start slowing down because of absence of space and then they stop producing right there [right end of graph]." JM - Precalculus

"I assume the population increases because they have plenty of food and probably because waste isn't an aspect. And here [near right end of curve] food decreases and so some of them die plus probably by-products of waste and stuff are affecting them. They are changing more some hours. (Which hours does it change more or the most?) Probably here [near $t=7$] and here [near $t=10$]. (And which hours do they change the least?) To begin with and to end with because population is exponential." AW - Calculus

Students who used this strategy thought of population growth as exponential or told a story about what factors affected population growth to produce so that this graph resulted. It could be argued that students were not really using knowledge of growth of population to make decisions about where the population was growing most and least but instead were simply making use of the shape of the graph to both tell a story about the growth of the population and decide where the population is growing most and least. Either way, it seems clear that students were making sense of the situation using their knowledge of growth of populations.

Task 6 - The "circles" task. Students were asked to decide for what one foot increase in the radius the areas grew the most and the

least and they used primarily the three main strategies used in Tasks 4 and 5. Here are some responses of students using "delta A":

"Distance from here to here [from the point corresponding to $t=0$ to the point corresponding to $t=1$] is a lot shorter and this is longer [distance between the point corresponding to $t=4$ and the point corresponding to $t=5$]." SCL - Precalculus

"It wouldn't be as equivalent as far as, 'cause up here [on the curve between $r=4.75$ and $r=5$] you know, you increase 10 [square] feet and right here [on the horizontal axis from $t=4.75$ to $r=5$] you're only increasing one-fourth foot. (So it's increasing not very much down here) Right (and increasing a lot) Up here you got 10 [square] feet but down here [between $r=0$ and $r=1$ on the horizontal axis] it takes a whole foot right there to go [pointing to the corresponding growth on the curve]." BG - Precalculus

"(For which one foot increase in the radius does the area grow the most?) [Student computes the distance in area between $r=3$ and $r=4$ and gets 22 and between $r=4$ and $r=5$ and gets 33.] I'd say between 4 and 5." MY - Calculus

"At the beginning it grows slower and as the radius increases more the area changes more and more. Like between zero and one, the change isn't as drastic as between 4 and 5." AM - Calculus

"You know it doesn't [grow the same number of square feet for each one foot increase in the radius] because it's kind of like a parabola. And you can tell, like, as it goes, there's more space between them, like, between 4.75 feet to 5 feet they increased an area of 10 square feet." LB - Postcalculus

Some of these students made relative comparisons of change in area over one foot increases in the radius (SCL and AM) while other students gave numerical comparisons (BG and MY). LB used relative ("there's more space between them") and actual ("between 4.75 feet to 5 feet they increased an area of 10 square feet") comparisons of changes in area. She also used "visual slope" by referring to the shape of the graph ("it's kind of like a parabola"). Some other students also used "visual slope":

"It's not constant. If it were constant, it would be a straight line." LC - Precalculus

"(For which one foot increase in the radius does the area grow the most?) Probably right about 3 when it starts to curve way up. (How are you able to tell that?) It goes almost straight up - not quite - there's a greater slant, greater slope, I guess." AJ - Precalculus

"It's an exponential growth [tracing pen along the curve]." MG - Calculus

For Task 6 I classified the "exponential growth" response of MG as referring to slope or shape of curve. But in Task 5 I classified a reference to "population is exponential" as using the strategy "knowledge of growth of populations". It seemed clear here that MG was referring to shape of graph and not to knowledge of the situation and therefore his response was classified as "visual slope". In Task 5 AW's reference to "exponential" was in conjunction with his knowledge of population growth and so his response there was classified as "knowledge of growth of population".

Several students used their knowledge of the situation when they described the area of the circles as the radii grew:

"Well, as your radius increases, you end up covering a bigger area [makes motion with hands indicating bigger and bigger circles]. 'Cause you don't have the radius, you know, at first you've got it going so far around [makes motion of one complete circle using the motion of the pen]. Then you're going this far around [makes a bigger circular motion with his pen]. So you're getting that [makes motion indicating radius of the latter circle he made with his motions] times by, I guess, the circumference or whatever, which you're adding on to." BG - Precalculus

"Initially when the stone first hits the water, the area of the circle starts out small. But as time goes on, the force causing the growing circles diminishes over time and eventually dissipates because there's no more force forcing the circles [makes a motion with hands to indicate enlarging circles]." MY - Calculus

These students used the physical situation to construct a description of the relationship between the radius and area of the circles.

Two postcalculus students used the equation relating the area and radius of circles - $A = (\pi)r^2$. One of these students explicitly mentioned this equation in his description:

"It's going to grow with the square of the radius. So you're going to have a parabolic graph which we have here and you're

going to have an acceleration of $2(\pi)$ feet and velocity or rate of change of growth is going to be $2(\pi)r$ square feet per foot."
AD - Postcalculus

Both students used the equation $A = (\pi)r^2$ to get their answers of $5(\pi)$ and $7(\pi)$ for how much the area of the circles grew as the radius grew from two to three and from three to four feet. They were the only two students who used this equation in Task 6.

Finding rate of change at a point. In Tasks 4, 5, and 6 students were asked to find the rate of change at specific points using the given graph and table of values. Some strategies were used on all three tasks. Other strategies were unique to each task. Most strategies were used with both graphs and tables but the distribution of use of strategies was different between graphs and tables, among the three tasks, and among the four points in task 4 at which students were to find the speed.

Task 4 - The "thrown ball" task. Students were asked to find the speed of the ball at $t = 1, 2, 3,$ and 4 using the given graph and table of values. Appendix E shows the different ways students addressed speed at these time values and possible answers obtained for each. Correct answers were discussed earlier in this chapter.

Speed when using the graph. Table 7 shows the strategies students used to find the speed of the ball at the four time values when using the graph. We note two patterns in Table 7. One is that students used

a variety of strategies. The other is that the distribution of frequency of strategy use is different for each time value.

Table 7

Strategies for finding speed at $t=1$, 2, 3, and 4 with graph

Strategy	Frequency (%) of responses using each strategy			
	$t = 1$	$t = 2$	$t = 3$	$t = 4$
Analysis of moving objects	6 (16%)	20 (54%)	16 (52%)	17 (55%)
Avg. speed over prev. quarter second	12 (32%)	4 (11%)	5 (16%)	10 (32%)
Reference to slope	7 (22%)	11 (30%)	3 (10%)	5 (16%)
Avg. speed over the previous second	15 (41%)	2 (5%)	4 (13%)	2 (6%)
Avg. speed since $t=0$ or $t=2$	5 (14%)	5 (14%)	3 (10%)	4 (13%)
Reference to a point	2 (5%)	3 (8%)	6 (19%)	1 (3%)
Avg. speed over half sec. around pt.	6 (16%)	4 (11%)	2 (6%)	0 (0%)
Desire smaller interval	1 (3%)	0 (0%)	2 (6%)	1 (3%)

For $t=1$ and 2, totals do not add to 37 or 100%, the number of subjects in the study, since many students used more than one strategy. For $t=3$ and 4, percents were computed on the basis of 31 responses since six students did not return for their second interview and so did not answer this question. Totals do not add to 31 since some students used more than one strategy.

Analysis of moving objects. At least half of the students used "analysis of moving objects" to find the speed at $t = 2$, 3, and 4 but relatively few used this strategy at $t = 1$. Representative responses help us see why.

At $t=1$:

"Not as fast as it was down here [first quarter second]. It's starting to slow down 'cause it's not moving as fast as it was. Like in this quarter of a second [from $t=1$ to $t=1.25$] it's only moving close to five feet, six feet per a quarter of a second

whereas down here [from $t=0$ to $t=.25$] that's not even half of what it was going down here." BG - Precalculus

"Um. I don't think it's 48 feet per second. I think it's like maybe - yeah, maybe it is 48 feet per second. No, it's not. I think it's less because it traveled that far [48 feet in the first second] but it was still slowing down. I don't know how much - but less than 48 feet per second." AJ - Precalculus

These students knew that the ball was slowing down as it went up. For BG, that meant that at $t = 1$ "it's not moving as fast as it was" over the first quarter second. He followed that comment with distances traveled over quarter seconds to further make his point that the ball was slowing down. For AJ, since the ball moved 48 feet in one second and was slowing down, then at $t = 1$ the ball was moving at a speed "less than 48 feet per second".

"Analysis of moving objects" helped students know that the ball slowed down but it was not very helpful in yielding a numerical answer to the question "How fast is the ball traveling at the end of the first second?". That may explain why relatively few students used this strategy to answer that question. In contrast was the relatively large number of students who used this strategy at $t = 2, 3$, and 4 .

At $t=2$:

"Very slowly. (Very slowly. Ok. And why do you say that?) 'Cause it's right in between going up and coming down. It's not moving - um. It's not moving very much." JW - Precalculus

"It's not. It's that split second where it stops and comes back down. (And how can you tell that?) It's right at the very top of the curve." AJ - Precalculus

"Actually, there's kinda no speed right there. (And how do you know that?) Because it reached its maximum point and then it's going to start going down. So it kind of stops there." RS - Calculus

"At the end of two seconds, it's stopped. It's changing from moving up to moving down. So it's at the exact point where it's stopped." DW - Calculus

"Zero. (Zero. Ok. And how do you know that?) Because it's not increasing or decreasing." CH - Postcalculus

"It's not. (It's not. Ok.) For that brief, it just kind of hangs, stuck in the air, you know what I mean, it's hanging in the air. It hits its peak. So that second, just for, it's really fast [the time the ball is not moving is a very short time]." RC - Postcalculus

Some students said that since the ball was between going up and coming down, its speed at $t = 2$ had to be zero (JW, DW, and CH). Other students used the fact that the ball reached its maximum height at $t = 2$ and concluded that its speed had to be zero (AJ, RS, RC). In addition, some students noted that since the ball slowed down as it went up and sped up as it came down, then its speed had to be zero feet per second at $t = 2$. Given the prominence of the point at $t = 2$ on the graph and students' knowledge that moving objects slow down as they go up and speed up as they come down, it is not surprising that over half of the students made "analysis of moving objects" the center of their argument about the speed at $t = 2$.

Similarly half of the students used this strategy to find the speed at $t = 3$:

"I'll say the same again. (The same as?) Well, I was going to say the same as this over here [pointing to the point (1, 48)]. But it's getting faster." KG - Precalculus

"40 feet per second. (And why do you say that?) 'Cause it's equal to this [speed at $t=1$] and that's what I said for this one [speed at $t=1$]." AW - Calculus

"What's the speed at $t=1$? (Is there any difference?) The difference is it's coming down here [at point (3, 48)] and it was going up here [at point (1, 48)]." JS - Postcalculus

At $t = 3$ the ball was coming down and because of the symmetry of the graph and perhaps because of some intuition students have about this

particular situation, these students knew that the speed of the ball at $t=3$ was the same as at $t = 1$ except that the ball was coming down (JS) and speeding up (KG) instead of going up (JS) and slowing down.

Similarly a large number of students used "analysis of moving objects" to find the speed at $t = 4$:

"As fast as it possibly will." BG - Precalculus

"Um. Um. About, well, 60 feet per second. (Why?) 'Cause it was 60 feet over here [pointing to the quarter second from $t=0$ to $t=.25$]." JW - Precalculus

"Same as it was [pointing to the point (0, 0)]. (Same as it was over here [pointing to $t=0$]?) Yeah." LC - Calculus

"Well, it would be traveling the fastest at that point." DW - Calculus

"It's the same as zero. The same as when it started. Negatively." JG - Postcalculus

"Well. Let me think. Common sense tells me it's still going at a good speed. Well, I guess that's what this shows [pointing to the graph near $t=4$]. It shows that it's still going at a good speed. You could figure it out by using the same method [as she used to find the speed at $t=3$, average speed from $t=2.75$ to $t=3.25$]. (Ok.) But we'd have to section if off different or just it shows right here [pointing to the vertical axis] like it traveled, a quarter of a second, it traveled like 15 feet [from $t=3.75$ to $t=4$]. So, by when it's down here, the slope is kind of steeper. (Yes, it is.) So it's going pretty fast and then the ground just boom, just stops it." DB - Postcalculus

Students who used "analysis of moving objects" to address speed at $t = 4$ typically noted that the speed of the ball at $t=4$ was the fastest (BG, DW, DB) or the same as at $t=0$ (LC, JG) or the same as over the interval from $t = 0$ to $t = .25$ (JW). DB used three strategies here - analysis of moving objects ("it's still going at a good speed"), speed over previous quarter second ("a quarter of a second, it traveled like 15 feet"), and slope ("the slope is kind of steeper").

In summary then, "analysis of moving objects" was the most frequently used strategy for finding the speed of the ball at $t = 2, 3$, and 4. Students' knowledge of motion supported by the symmetry and the maximum point of the graph made this strategy particularly useful at these points.

Average speed over previous second and quarter second. While "analysis of moving objects" was the most popular strategy for finding the speed of the ball at $t = 2, 3$, and 4, the most frequently used strategies at $t = 1$ were "average speed over the previous second" and "average speed over the previous quarter second". Over 40 percent of the students used the former and about one-third used the latter to find the speed of the ball at $t = 1$:

Average speed over previous second -

"48 feet per second. (Ok. Why do you say 48 feet per second?) Because it's taken one second and it's between like towards the 50. (Oh. Ok. So it traveled 48 feet in one second?) Yeah." MH
- Precalculus

"It looks like it's traveling between 45 and 50 feet per second. (How did you get that?) If I was to draw a slope straight to this point between zero and one second [draws line segment from the point (0, 0) to the point (1, 48)]. Now according to the slope, it shows that if the ball was at a constant rate of speed, it would be traveling about 48 feet per second. (And so you're saying at that exact point, instant of time, your best estimate is that it's going 48 feet per second?) Yes." MG - Calculus

"[long pause] How fast is it traveling here [points to the point (1, 48)], the first second. [Sigh] 49 feet? per second? (Ok. And how did you get that?) I just looked to see if it's up here at 49 [at $t=1$]. I don't know. (Oh. Since it's traveled 49 feet.) Right. (Over the) Was I supposed to look at the whole, just the first point in general. (Ok. Well, there's two questions here that one might think about. One is how fast has it traveled over the interval from 0 to 1.) Ok. (And the other question is how fast is it traveling at $t=1$. So which question were you answering when you said 49 feet.) Uh huh. (Or is there a difference? Maybe there's no difference between the two questions.) [long pause] Well, if the distance - instead of going - we went by quarters here [between $t=0$ and $t=.25$] (Yeah)

and so I multiplied it by four [referring to his earlier work when he computed the average speed of the ball over the interval from $t=0$ to $t=.25$]. (Ok.) So it would equal like one [So the units would be feet per one second rather than feet per quarter second]. But here it's already [in feet per second], so I don't feel I need to, I guess, multiply anything (Ok.) over that interval? (And

that answers the question how fast is it traveling over the first second? Or does it answer the question how fast is it traveling at the end of the first second? Or does it answer both? Or is there no difference between those questions?) I'm gonna say, uh, [pause] both. It answers both." RC - Postcalculus

These students gave the average speed over the first full second as their answer for the speed at $t = 1$. The first student (MH) simply noted that the ball had traveled 48 feet in that first second. The second student (MG) used the slope of the line containing the points $(0,0)$ to $(1,48)$ to get the average speed over the first second. He then noted that if the ball was traveling at constant speed, the speed of the ball at $t = 1$ would be 48 feet per second. I classified MG's response as using two strategies, "average speed over previous second" and "slope". He did not use slope of tangent line; he used slope of secant line from $(0,0)$ to $(1,48)$. But he did use slope to help him make sense of the situation and obtain his answer. The third student (RC), like the first student, noted how far the ball had traveled in the first second. He went on to explain that since his answer (49 feet per second) was in feet per second and not feet per quarter second, he did not have to convert it to standard units and hence, his answer stood as 49 feet per second. Another strategy was very popular for finding the speed of the ball at $t = 1$:

Average speed over previous quarter second -

"About - ooh - uhm - I don't know, 36 feet per second. (How did you get that?) Between these two points $[(.75, 39)$ and $(1, 48)]$ that's about 9 feet and you multiply that by four." DG - Precalculus

"Um. [long pause] About 40 feet a second. (How did you get that?) Well, I don't know if this is right but I took this last interval right here [from $t=.75$ to $t=1$]. It went approximately 10 feet and then that's 40 feet per second. So for a fraction of one second that's what it was." JK - Calculus

"I would say 8 feet in a quarter of a second. (Oh. 8 feet in - because?) Because it went approximately 8 or 9 feet in one quarter of a second [from $t=.75$ to $t=1$]." CH - Postcalculus

These students used average speed over the previous quarter second interval to get their answer for the speed of the ball at $t = 1$. Two of them (DG and JK) converted their "feet per quarter second" answer to a "feet per second" equivalent. The third student (CH) left her answer in feet per quarter second. One of the issues related to understanding rate is the equivalent nature of constant rate. Eight feet in a quarter of a second is equivalent to a speed of 32 feet per second. This student and six others consistently did not convert their "feet per quarter second" answers to feet per second. (See Appendix H for a discussion of students' responses relative to this issue.)

At $t = 2$ relatively few students used these two strategies.

Representative responses show why.

Average speed over previous one second -

"That one's sixty - ok - uh - I can't figure it out in my head. 60 feet. (Let's see. You're trying to use the height?) Right. (And that's probably around 64 [at $t=2$].) Yeah. But then that's how far it's gone in the whole two seconds. (Yeah. Right.) So then you'd have to subtract the 48 [height at $t=1$] from 64 [the height at $t=2$]. 16." MH - Precalculus

Average speed over previous quarter second -

"About 4 feet per second, uh, 4 or 8 feet per second because from there [$t=1.75$] to there [$t=2$] it's one or two feet and multiply by four and you'll get 4 to 8 feet per second." DG - Precalculus

"Well, I guess I could follow the same premise of what I've been trying to do here. I would say that this [from $t=1.75$ to $t=2$] is probably less than five feet in a quarter second. So I would say

20 feet per second. It doesn't sound right. But what I know of a ball is that it reaches, it's going to stop here [at $t=2$] and then go back down. So obviously (Well, actually I think the distance here is more like one foot in which case) Oh. Ok. So that would be 4 feet per second. (which is closer) But it's gotta be zero from what I know of gravity." AW - Calculus

The first student (MH) used average speed from $t = 1$ to $t = 2$ to find speed at $t = 2$. The other two students (DG and AW) used average speed from $t = 1.75$ to $t = 2$. But for AW this answer did not look right. He then analyzed the situation in terms of "analysis of moving objects" to decide that the speed of the ball at $t = 2$ was zero. His understanding of the situation made him believe more in his second answer than his first.

When we turned to the table of values later in the interview, he again used average rate of change over the quarter second before $t = 2$ to get four feet per second. Then to find the speed of the ball at $t = 3$, he developed a new strategy - the average of the average speeds over the quarter second before and the quarter second after $t = 3$. He then returned to $t = 2$ and used this strategy there. Once he realized that the average speed over the quarter second after $t = 2$ was the negative of the average speed over the quarter second before $t = 2$, he obtained the speed at $t = 2$ as zero feet per second, his final answer when working with the graph.

This last student's response helps us see why more students did not use average rate of change over intervals before $t = 2$ to find the speed at $t = 2$. "Analysis of moving objects" and "slope" were so successful in yielding quick and easy answers and avoided computations that required more effort and often did not yield answers that made much sense.

A few students used "average speed over previous second" or "average speed over previous quarter second" to find the speed at $t = 3$ but over a third of the students used the latter strategy at $t = 4$. For the most part these were the students who used "average speed over half second around point" to find the speed at $t = 1, 2$, and 3 . They would have been prevented from using this strategy here unless they had conjectured the distance the ball would have traveled if it had continued after $t = 4$ and on to $t = 4.25$ seconds. No one did that with the graph or with the table.

In many circumstances in everyday life, what occurs immediately before a point in time is thought to have the most influence on what happens at that point. This would account for students' relatively heavy use here of average speed over intervals before points. Additionally since speed is most often given in terms of distance traveled in one unit of time, the preferred way of referring to speed here would be in terms of feet per second. That would account for some students' use of average speed over the full second before a point rather than average speed over the quarter second before the point to find speed at a point.

Reference to slope. About one-fifth of the students used "slope" to find the speed of the ball at $t = 1$ while about one-third used this strategy at $t = 2$, one-tenth at $t = 3$, and one-sixth at $t = 4$. The following excerpts show how students used this strategy and why frequency of use varied for the four points.

$t=1$:

"Right here to find this velocity you might have to find like the slope of this line or something. Right at this point. The slope of this line. (You want to draw that line in so I know what line

you're talking about?) I can try. [He draws a line segment tangent to the graph at the point (1,48).] I have no clue. But there should be something to do with the slope right here. [Draws vertical and horizontal line segments up and to the right from the point (1,48) on the graph.] Tangent line right there. (How does that tangent line help you answer the question "How fast is the ball traveling at $t=1$?") Well, if I can find the slope of that tangent line here, I would know the rate of change. Do you want me to do that? (If you feel like you can.) Well, I mean, I know you have to have the tangent line there and you have to try to find what the slope is. I don't know exactly how I would go about doing that though." RF - Calculus

"Let's do this graphically. [He draws the line tangent to the curve at the point (1,48), finds the coordinates of two points on that line (.45, 30) and (1.35, 60), and uses these two points to find the slope of the tangent line.] 33.3 feet per second." AD - Postcalculus

"At one second, I would draw a tangent line. [She does that.] And then I would find the slope of that line. (The what?) The slope of the tangent line. (And that would give you?) And that would give me the rate of change at that specific point." SE - Postcalculus

"Hmm. Well. [long pause] That's a good question. At the end of one second? (Yeah. At the end of one second.) [He draws the line tangent to the curve at $t=1$ and identifies points (2,80) and (1,48) on that line and uses these two points to find the slope of the tangent line.] Looks like 32 feet per second. (How did you get that?) Because I drew the tangent to the curve at one second and figured the slope of that tangent." MG - Postcalculus

All four of these students drew a line tangent to the graph at the point (1,48) and indicated the slope of that line would give the speed of the ball at $t = 1$. Two of the four actually found a numerical value for the slope of that line. Both of them located convenient points on their tangent lines and found the slope using those points. More students used "slope" to find the speed of the ball at $t = 2$:

"It's not. Well, it's kinda. It's not going anywhere. It's not moving. The tangent line across there, the slope would be zero. That means it's not going anywhere. There's no slope, so there's no acceleration and no velocity, one of those words." LC - Calculus

"Zero. (And how do you know that?) Because it's stopping - the, well the slope is a horizontal line, er, the line tangent to the

curve is horizontal, so the slope is zero. And when the slope is zero, that means that the velocity is zero because the slope is the derivative and the derivative is the velocity." TS - Postcalculus

Neither of these students actually drew the tangent line or computed its slope. They merely noted that the tangent line was horizontal, a horizontal line has slope zero, slope gives velocity, and therefore the speed of the ball at $t = 2$ was zero feet per second. LC also associated zero slope with no acceleration and no velocity and then indicated he was not sure which one of those he needed here ("one of those words"). TS made her language more precise as she continued talking. She explained that "the slope is a horizontal line" really meant "the line tangent to the curve is horizontal, so the slope is zero". She then went on to associate slope of tangent line with velocity and the derivative.

Only three students used "slope" at $t = 3$ but one-sixth used that strategy to find the speed of the ball at $t = 4$. The low frequency of use of this strategy at $t = 3$ might be explained by student's use of strategies like "analysis of moving objects" and symmetry of the graph that yield answers much more quickly and effortlessly. The high frequency of use of this strategy at $t = 2$ may be for the same reason; it was a quick and easy way to get an answer for the speed of the ball at $t = 2$.

Average speed over half second around point. To find the speed of the ball at $t = 1, 2$, or 3 some students found the average speed over the half second interval around the point. For $t = 1$, this means that students found the average speed over the interval from $t = .75$ to $t = 1.25$; for $t = 2$, students obtained the average speed from $t = 1.75$ to t

= 2.25; and for $t = 3$, students computed the average speed for $t = 2.75$ to $t = 3.25$. I called these "straddle" intervals since in each case they straddle the corresponding point. For example, the interval from $t = .75$ to $t = 1.25$ "straddles" the point at $t = 1$. This strategy was used with highest frequency at $t = 1$:

"Oh. No. I know. I can do it like a rate of change business. (Rate of change business?) Like in this quarter of a second [from $t=.75$ to $t=1$], the rate of change was 10 feet. (Ok.) And in this one [from $t=1$ to $t=1.25$] it was uh about say, would it be like 5? (Yeah. Let's just say five) in a quarter of a second. (in a quarter of a second.) So the average is about 7.5 per quarter. So 7.5 times 4. That'd be 30, about 30. (30. Yeah. 30?) feet per second." AM - Calculus

"Ok. Two ways. From here to there [from $t=.75$ to $t=1$] it would be about 36 feet per second but from there to there [from $t=1$ to $t=1.25$] it would be going about 28 feet per second. So I would probably take the average, about 32 feet per second." JG - Postcalculus

"At the end of one second? (Yeah.) Probably about 35 feet per second. (And how did you get that?) Because if there's a difference of about nine from here to here [from $t=.75$ to $t=1$], then from here [$t=.75$] to here [$t=1$], it's traveling at about 36 feet per second. And if this difference [from $t=1$ to $t=1.25$] is smaller, it looks smaller, then that velocity is going to be a little bit less than say 36 feet per second. So maybe 33 feet per second. Probably somewhere in there. (So you're taking what's happening over this interval [from $t=.75$ to $t=1$] and what's happening over this interval [from $t=1$ to $t=1.25$] to make a decision about what's happening here [at $t=1$].?) Right." TS - Postcalculus

These students used speed over the quarter seconds immediately before and after $t = 1$ to find the speed of the ball at $t = 1$. None of the students made mention of the properties of parabolas that allow this strategy to yield the exact answer. I believe they used this strategy not because they were aware of this property but because it represents a general strategy that can be used with continuous functions to yield reasonable estimates of the rate of change at a point. As we shall see

below, students' responses to tasks 5 and 6, where this quadratic property does not hold, bear this out.

Four students used this strategy at $t = 2$:

"Um. About maybe 7 or 8 feet per second. (Ok. And how are you getting that?) Because it would be approximately I'd say two feet in distance in these intervals [from $t=1.75$ to $t=2$ and from $t=2$ to $t=2.25$]. In this interval [from $t=1.75$ to $t=2$] and the same as in this interval, too [from $t=2$ to $t=2.25$]. And uh per four per

second [a total of four feet over the half second from $t=1.75$ to $t=2.25$] and then times two is eight [for an average speed of eight feet per second]." JK - Calculus

"How fast? It's not traveling at that second. (How can you tell that?) Because from a quarter second before and a quarter second after it didn't travel - or if I did like I did before I'd take the average and that's probably four feet per second there [from $t=1.75$ to $t=2$] and that's negative four feet per second [from $t=2$ to $t=2.25$]. So the average would be zero." JG - postcalculus

JK took into consideration the average speed of the ball immediately before and after $t = 2$. She saw that the ball traveled two feet from $t = 1.75$ to $t = 2$ and another two feet from $t = 2$ to $t = 2.25$ and therefore the ball traveled four feet from $t = 1.75$ to $t = 2.25$ so that the average speed of the ball from $t = 1.75$ to $t = 2.25$ was eight feet per second. Like several students in this study she did not note that the change in height before $t = 2$ was positive while the change in height after $t = 2$ was negative. JG did note this.

Only two students used this strategy at $t = 3$ and none of the students used it at $t = 4$. The low frequency of use at these two points is due, I believe, to students' use of quicker, easier strategies, like "analysis of moving objects" coupled with the symmetry of the graph, where the speed at $t = 3$ was noted to be the same as the speed at $t = 1$ and the speed at $t = 4$ was noted to be the same as at $t = 0$. Also the absence of motion after $t = 4$ makes it less likely that students will

use the "straddle" interval at $t = 4$. The relatively high frequency of use of this strategy at $t = 1$ would conversely be attributable to the lack of knowledge of speed at other points similar to $t = 1$.

Average speed since $t = 0$ or $t = 2$. Five students obtained their answers for the speed of the ball at $t = 1$ and $t = 2$ using the average speed over the interval starting at $t = 0$. At $t = 1$ that strategy yields 48 feet per second since the total distance the ball had traveled since $t = 0$ was 48 feet. This is the same answer obtained from the strategy "average speed over previous second". It was not always easy to distinguish when students were using which of these two strategies. Decisions about which strategy was being used took into consideration the strategies student used at $t = 2$. If it was clear that a student used average speed over the interval from $t = 0$ to get speed at $t = 2$ and either average speed since $t = 0$ or or average speed over the first second for their answer at $t = 1$, then I classified their answer for $t = 1$ as using "average speed since $t=0$ ". The five students who used this strategy at $t = 1$ were the same five students who used this strategy at $t = 2$:

"At that exact point? (Yeah. At that exact point.) It would be easier just to say about 63 feet - well, that's for two seconds. So (If you make it 64, it's easier). Ok. Alright. So I'll say 32 feet per second." KG - Precalculus

"About 31 feet per second. (Ok. And why do you say that?) Because the total distance the ball traveled is about 62 [from $t=0$ to $t=2$]. (Ok.) So in two seconds it traveled 62 feet. So divide by two and that gives 31." SCL - Precalculus

"At the end of the second second? (Yeah.) See, the rate of change changes all the time. But at the end of two seconds I can say that it traveled approximately 64 feet. (So it traveled 64 feet over the first two seconds.) Yes. (And is that what you want to say about how fast it's traveling at the end of two seconds? What is its speed at that position [at the point

(2,64)]?) Its speed at this position - um - yeah - I guess it's 64 feet per two seconds." PG - Calculus

KG and SCL computed the average speed over the interval from $t = 0$ to $t = 2$ and expressed their answers in terms of feet per second. PG also computed the average speed over the interval from $t = 0$ to $t = 2$ but expressed his answer in terms of feet per two seconds. There is nothing inherently wrong with giving an answers in terms of feet per two seconds but the accepted notion of speed usually carries with it the understanding that speed is expressed in terms of distance per unit time. With the table of values, he used slope of the line segment from $t = .75$ to $t = 1$ to get an answer of 32 feet per second for the speed of the ball at $t = 1$. But he then struggled with whether or not that should be 32 feet per second or 32 feet per half second or 32 feet per quarter second. These responses made me think that he was struggling with the idea of rate.

An interesting variation of this strategy considered the average speed from $t = 2$ rather than from $t = 0$ to find the speed at $t = 3$ and $t = 4$:

At $t=3$:

"You're downward slide so you're probably gonna forget about the first half of this graph. Maybe use it in a formula to figure out the downward slide. But I don't know what it is or anything. But right here [pointing to the point (3,48) on the graph], you're talking about? (Yeah. At $t=3$.) It's on a downward trend from 64 feet down to about 48 feet so that's about 16 feet per second." MY - Calculus

At first I thought this student found the speed at $t = 3$ using the average speed over the previous second. But as I further examined his work here and with $t = 1, 2$, and 4 , I decided he was basing his work on average rate of change over intervals starting at some "beginning"

point. When he found the speed of the ball at $t = 1$ and $t = 2$ he used the average speed from $t = 0$. But when he found the speed of the ball at $t = 3$ and $t = 4$ he used the average speed from $t = 2$.

His comment "You're downward slide so you're probably gonna forget about the first half of this graph." made his work and the work of a few other students more understandable. The point at $t = 2$ obviously signals a change in the motion of the ball. So everything that happens after $t = 2$ is considered separate from what happened before $t = 2$. So some students who are inclined to use total distance to find speed at a point will use total distance from the last most significant point where there was an obvious change in motion.

Another student made it clear he was using this version of this strategy at $t = 4$:

"63 feet per second. (Ok. And how did you get that?) Um, because when it reaches here [the point (2,64)] it's 63 feet per second and when it reaches the ground it's equal speed. (Ok. So since it started at 63 feet and then here [at $t=4$] it's gotta be 63 feet per second?) It's gotta be um 31 and a half feet per second [dividing 63 feet by two]. (Ok. Alright. Because? It took two seconds?) Right. (Alright. And that's because it traveled 63 feet) over two seconds." JP - Calculus

JP's words "because when it reaches here [the point (2,64)] it's 63 feet per second and when it reaches the ground it's equal speed" made me think he, like MY, was singling out the section of the graph from $t = 2$ to $t = 4$ as relevant for finding the speed at $t = 4$. He computed the average speed from $t = 2$ to find the speed at $t = 4$ by dividing the distance traveled during that time, 63 feet, by the amount of time to travel that distance, two seconds, to get his final answer of 31.5 feet per second.

I interpreted his initial answer of 63 feet per second as his noticing the height at $t = 2$. He had given this same initial response when he was finding the speed at $t = 2$ but immediately went on there to note that the height was 63 feet. I took this to mean he was saying he did not mean to say 63 feet per second but meant to say the ball was at a height of 63 feet. He went on to give zero feet per second as his answer for the speed at $t = 2$.

Reference to a point. A few students used one or both coordinates of the point at $t = 1, 2, 3$, or 4 to find the speed at each of those points. Most often students divided the second coordinate of the point by the first coordinate or simply attached the phrase "per second" to the value of the second coordinate. Use of this strategy at $t = 1$ yielded the same answer as "average speed over previous second" and "average speed since $t=0$ " and use of this strategy at $t = 2$ yielded the same answer as "average speed since $t=0$ ". So it was not always easy to decide which of these strategies students were using. As with "average speed since $t=0$ ", I used students' responses when finding the speed at other points to help decide if they used "reference to a specific point" at $t = 1$ and $t = 2$. Use of this strategy at $t = 3$ resulted in 48 feet per second if the student referenced only the second coordinate or 16 feet per second if the student divided value of the second coordinate by the value of the first. A couple of students made it clear this is what they were doing:

"By looking at the chart [graph], I would say about 48. (48 feet?) Around there. (48 feet per second?) Yeah. (And how did you get that? You said by looking at the chart.) Uh huh. (Can you elaborate more?) Well, that's the feet and that's 48. So that's about 48. (Ok. 'Cause the ball is at height of 48?) Right." CH - Postcalculus

"48 feet per second. (And why do you say that?) Because it's 48 feet high." JP - Calculus

Both of these students used the value of the second coordinate (height) of the point at $t = 3$ to help them decide the speed of the ball. CH followed her answer above with an even more elaborate strategy based on the height of the ball at $t = 0, 1,$ and 2 to find the speed of the ball at $t = 3$:

"[She adds 63 and 48 and divides that sum by three to obtain 37.] Ok. 37. So the average is 37 feet. (Oh. Because it went 63 in the first two seconds - wait a minute.) From 63 at two seconds - I'm trying to find the average at three. (Ok.) Ok. I don't know why I did that. That's 63 [at $t=2$], and that's 48 [at $t=1$], and that's zero [at $t=0$]. (Uh huh.) Then I divided it by three. And that's 37. (So 63 plus 48 was 111.) Uh huh. (And you divided that by three and that gave you 37 feet per second.) I don't know why. Ok. (Do you want to tell me why you think you did that?) Well, 37, well it's the average. But I don't want the average." CH - Postcalculus

CH used the values of the second coordinate at three previous points in an averaging scheme to obtain the speed at $t = 3$. She expressed considerable doubt that this yielded a reasonable answer. But the point I want to make here is that using the second coordinate of a point or the height of the point on the graph to find speed at the same or other points makes sense to students.

This student used a similarly elaborate algorithm based on the second coordinate of points to find the speed at $t = 4$. She added the heights at $t = 4, 3.5,$ and 3.75 ($0 + 15 + 28$) and then divided by three to get the "average speed" over the last half second and then she multiplied that by two to convert her answer to feet per second. That yielded 28.67 feet per second as her answer for the speed at $t = 4$. One calculus student spoke of using the same strategy by adding up all of

the heights at the quarter second marks from $t = 2$ to $t = 4$ and using that as his answer for the speed at $t = 4$.

It may seem perplexing that students would use the height of the ball as a direct indication of the speed of the ball. However, if students believe that speed has something to do with distance and time and height is represented in feet, a measure of distance, then it may not be such a surprising jump that they think that speed at a point is distance at that point. These students were using the height of the ball, perhaps as distance, at individual points to try to make sense of speed at a point.

Reference to an equation. A few students referred to equations when trying to find the speed of the ball at some of the points. Although the situation in this task is fairly common to mathematics and physics students and the shape of the graph strongly suggests a parabola, none of the students gave evidence that they knew the equation or seemed inclined to try to find it. Since the equation for this situation was not given and since students knew other strategies for finding speed at a point, I was not surprised students did not mention equations. In fact, I was rather surprised when some of them did. Here are the ways equations were referred to at $t = 1$:

"You could take a derivative but I don't really have an equation. (If you had an equation for that [graph] how would you find the speed at that) Just take the derivative and set it equal to one. (Set it equal to one?) For one second. (Do you mean evaluate it at one?) x equal to one." JG - Postcalculus

"But you can do it with equations. (You can?) But I don't know them." TS - Postcalculus

When JG said "you could take a derivative but I don't really have an equation" he probably meant that if he had an equation for this

graph, he would obtain the derivative equation for it and evaluate it at $t = 1$. This strategy yields a correct answer although since he did not have the equation and did not obtain it, he was unable to carry it out. A calculus student similarly said at $t = 3$ that he would differentiate the equation if he had one and plug in three for the t value.

TS was a physics minor who probably encountered equations dealing with this situation in her physics instruction. But she could not immediately recall what those equations were. Indeed most calculus students would most likely have seen a derivation of equations for "falling body" situations. But none of the students in this study gave evidence that they remembered those equations and even though one student tried to use curve-fitting methods to find such an equation, he quickly abandoned the process in favor of easier, quicker strategies.

One might be disappointed that calculus and postcalculus students, some of whom are physics minors, could not easily come up with an equation for this situation. However, the point of this task was not to find out if these students could recall or derive these equations. In addition, even if one has worked with kinematics equations for this particular situation, it is not necessarily easy to recall them here. The context here is not necessarily the same as the context for earlier study of kinematics. Nearly all of us have had the experience of seeing an old concept in a newer context and not been able to make an appropriate connection.

Desire smaller interval. One student expressed a desire for an interval smaller than one quarter second:

"Ok. So take a little interval at the very end here [just before $t=1$ up to $t=1$] per this amount of time and that's how fast it's traveling. That distance per that time." RS - Calculus

This student wanted an interval immediately before $t=1$ which was much smaller than a quarter of a second. She described what she would do with that interval but she did not proceed to obtain a numerical answer. This student and another student similarly wanted a smaller interval either before or around $t = 3$. Yet another student wanted a smaller interval before $t = 4$. This is an important mathematical step between using what is given to approximate rate of change at a point and knowing how to get a better approximation and, through a particular limit, an exact answer.

Other ideas. At $t = 3$ and $t = 4$ a few students noted that the velocity was negative. I did not specifically ask students about the velocity; I asked them about the speed which technically does not require noting the sign. Several other students pointed out that the ball was going down at $t = 3$ in contrast to its motion at $t = 1$ where it was going up. This was usually in concert with their claim that the speed of the ball at $t = 3$ was the same as at $t = 1$ except in the opposite direction.

Finally one student noted that the speed was "changing" at $t = 3$ and $t = 4$. She noted at $t = 3$ that "you don't really know because it isn't going at a steady speed ... It's changing as the gravity pulls it down.". This student knew that the speed was not constant, that the speed of the ball was changing, and that gravity was exerting an effect on the ball that caused its speed to change. It may be that she did not actually believe that the ball has such a thing as speed at a point. Her answer here indicates perhaps a lack of acknowledgment that at each point in time the ball actually does have a speed.

Summary for finding speed at $t = 1$, $t = 2$, $t = 3$, and $t = 4$ when using the graph. Two main stories come out of this analysis of strategies students used to find the speed of the ball at different points in time when using the graph. One is that a variety of strategies were used. The other is that the distribution of strategy use was different at each time value. At $t = 1$, the most prevalent strategies were "average speed over previous second", "average speed over previous quarter second", and "slope". At $t = 2$, $t = 3$, and $t = 4$, over half of the students used "analysis of moving objects" drawing heavily on the symmetry of the graph to decide speed.

At the maximum point, at $t = 2$, thirty percent used "slope", a particularly easy strategy to use there. At the first point after the maximum, when the ball was falling, at $t = 3$, one-fifth of the students referenced the height of the ball at that point while one-sixth used "average speed over previous quarter second", a strategy that was used widely at $t = 1$, the point symmetrical to $t = 3$. Over one-third used "average speed over previous quarter second" at $t = 4$ partly because that was a popular strategy at several points and because students who had used "straddle" intervals at other points could not easily do so here. Figure 10 shows the most frequently used strategies at each of the four time values.

<u>At $t=1$</u>	<u>At $t=2$</u>	<u>At $t=3$</u>	<u>At $t=4$</u>
Avg. speed over prev. sec.	Analysis of moving obj.	Analysis of moving obj.	Analysis of moving obj
Avg. speed over prev. 1/4sec	Reference to slope	Reference to a point	Avg. speed over prev. 1/4 sec.
Reference to slope		Avg. speed over prev. 1/4 sec.	

Figure 10 - Most frequently used strategies for speed at $t=1,2,3$, and 4

Speed of the ball at $t = 1, 2, 3,$ and 4 when using the table of values. Table 8 shows the strategies students used to find the speed of the ball at $t = 1, 2, 3,$ and 4 when using the table of values. There are two patterns in Table 8 that we also saw with the graph. Students used a variety of strategies when working with the table of values, and the distribution of strategy use varied from one time value to another. But at each time value the frequency of strategy use with the table of values differed with the frequency of strategy use with the graph. With the table of values, frequency of use of strategies was more spread out.

Table 8

Strategies for finding speed at $t = 1, 2, 3,$ and 4 with table

Frequency (%) of responses using each strategy

<u>Strategy</u>	<u>$t = 1$</u>	<u>$t = 2$</u>	<u>$t = 3$</u>	<u>$t = 4$</u>
Avg. speed over prev. prev. 1/4 second	8 (22%)	7 (19%)	6 (21%)	14 (48%)
Avg. speed over 1/2 sec. around point	15 (41%)	13 (35%)	9 (31%)	0 (0%)
Analysis of moving objects	3 (8%)	14 (38%)	1 (3%)	4 (14%)
Reference to slope	7 (19%)	4 (11%)	2 (7%)	3 (10%)
Avg. speed over prev. second	8 (22%)	1 (3%)	3 (10%)	3 (10%)
Avg. speed since $t=0$ or $t=2$	3 (8%)	3 (8%)	3 (10%)	4 (14%)
Reference to a point	2 (5%)	2 (5%)	3 (10%)	3 (10%)
Desire smaller interval	2 (5%)	0 (0%)	0 (0%)	2 (7%)

For $t=1$ and 2 , totals do not add to 37 or 100%, the number of subjects in the study, since many students used more than one strategy at each point. For $t=3$ and 4 , percents were computed on the basis of 29 students since eight students did not answer this question. Totals do not add to 29 at $t=3$ and 4 since some students used more than one strategy.

Analysis of moving objects. When using the graph many students used "analysis of moving objects" but relatively few students used this strategy with the table of values except at $t = 2$. Whereas the graph made the maximum point at $t = 2$ visually obvious, the table of values made that maximum numerically obvious with the largest height value occurring at $t = 2$. Students relied on the fact that the ball reached its maximum height at $t = 2$ and their knowledge of motion to make conclusions about the speed of the ball at $t = 2$:

"It's at 64 feet right there and that's the highest it gets. So it's up there stopping and slowing down, and coming back. (So you're saying that it's stopped there because it was going up and then after that it's going down?) Yeah." AJ - Precalculus

Students did not use "analysis of moving objects" very much at $t = 1, 3$, or 4 . One possible reason has to do with how students respond to a graph versus a table of values. With the graph students often traced the curve of the graph from $t = 0$ to $t = 4$ while telling a story about the motion of the ball. So perhaps the graph invoked the "analysis of moving objects" strategy. Or perhaps what looked like "analysis of moving objects" to me was really students' analysis of a graphical shape.

Either way, that strategy did not work so well with the table of values. The table made more obvious the presence of numerical data which might be profitably used to make conclusions about the speed of the ball at $t = 1, 3$, and 4 and students made good use of strategies that took advantage of this data. It is possible to get this numerical data from the graph by using the vertical axis of the graph. But it is much more easily accessible with a table of values.

Reference to slope. Another strategy that was used with greater frequency with the graph but was not used very much with the table of values was "reference to slope". At each of the four time values the percent of students referencing slope with the table of values was less than the percent referencing slope with the graph. Slope is more closely associated with properties of a graph than with a table of values, but even so some students referred to slope when they were using the table of values. At $t = 1$ almost one-fifth of the students referenced slope to find the speed:

"At the first second. But that's not right. (What's not right?) Well. I want to say 48 feet per second but that would give me this line [secant line from the point (0,0) to the point (1,48)] and not this line [tangent line at $t=1$]. (Is there any way you could use this table of values?) Um. That just gives me the different intervals here, right? (It gives you the height at each quarter of a second.) Yeah. (Right.) I would need to have points [on the tangent line], like a point here [on the tangent line near $t=1.5$] and a point near [on the tangent line near $t=.5$]. Or. I could use two really close points [two points on the curve that are really close to two points on the tangent line, like point (.75,39) which is on the graph of the curve and on her drawn tangent line and point (1.25,55) which is on the graph of the curve and nearly on her tangent line]. (Yeah. You could use two really close points. Do you have any points on here that you would classify as really close points?) I could use these two [pointing to points (.75,39) and (1.25,55) in the table of values]. That would give me an approximation [to the slope of the tangent line]." SE - postcalculus

SE was intent on finding the slope of the line tangent to the graph at $t = 1$. She had drawn this tangent line when using the graph and now she used the table of values to get an approximation of the slope of that line. She settled on using values in the table for two points that were the closest to being on her tangent line. This was typical of the students who referenced slope while using the table of values. They said they wanted to find the slope of the tangent line, but since exact

values of d were not available for points on the tangent line, they would use the next best thing, points in the table that were closest to the tangent line. That amounted to using the points $(.75, 39)$ and $(1.25, 55)$ from the table. Most of these students were careful to point out that their answer was only an approximation to the slope of the tangent line and therefore only an approximation to the speed of the ball at $t = 1$.

All four of the students who used slope at $t = 2$ basically ignored the table of values and referenced the horizontal tangent line at $t = 2$ on the graph and gave the slope of that line as the speed of the ball at $t = 2$. At $t = 3$ and $t = 4$ students used the table, as they did at $t = 1$, to help them get a better approximation for the slope of the line tangent to the graph at $t = 3$ and $t = 4$ which in turn gave them the speed of the ball at those two points.

Average speed over previous second and quarter second. Finding the speed over the quarter second before a time value was a popular strategy for approximating the speed of the ball at $t = 1, 2, 3$, and 4 .

About one fifth used this strategy at $t = 1, 2$, and 3 , and almost half used it at $t = 4$.

At $t=1$:

"Um. Ok. Between a quarter of a second, I mean, three quarters of a second and one second it's going about 9 feet per uh, it travels 9 feet in one quarter of a second. So I'd say about 36 feet per second." JM - Precalculus

At $t=2$:

"It's about four feet per second 'cause it went about one [in the quarter second from $t=1.75$ to $t=2$]." DG - Precalculus

At $t=3$:

"I'd say it was between 55 [at $t=2.75$] and 48 [at $t=3$]. That would be 7. so 28 feet per second." JF - Precalculus

"t equals three? (Yeah.) I'll try. [He computes the difference between 55 at $t=2.75$ and 48 at $t=2$, gets 7, then divided 7 by .25; as he does this he points to the graph at those points.] I guess 28 feet per second. (Ok. And that's because you took the interval right before, didn't you?) I was thinking. What if I took the interval after. But that, that would be the next one [pointing to the point (3.25,39) on the graph]." AW - Calculus

At $t=4$:

"Right about 60 feet per second. Right about 48 feet per second, I guess. No, 60 'cause the last interval it goes 15, so that's 60 feet per second and it's still gaining just a little bit." AJ - Precalculus

"Well, you could do the same thing with four and zero, with three point seven five and four. That would give you a reasonable estimate. (Alright. And what would that be?) Well, it'll be 15 over uh point two five or 15 feet per, well minus actually, minus 15 feet per quarter second." DW-Calculus

"The best estimate I could get would be like take 15 divided by point two five which would be 60, negative 60." JG - Postcalculus

At $t = 3$, JF and Aw used the average speed over the previous quarter second but AW then began to think about what information about speed at $t = 3$ was given by the interval immediately following $t = 3$. He went on, in the conversation after that quoted above, to decide that average speed in the quarter second after $t = 3$ was different than average speed in the quarter second before $t=3$ and that the best answer for speed at $t = 3$ was to take the average of these two average speeds. His response was therefore classified as using both average speed over the previous quarter second and average speed over the half second "straddling" $t = 3$ (which included those who averaged the average speed over the quarter second before a point and the average speed over the quarter second after a point).

At $t = 4$, AJ, DW, and JG noted that their answers were reasonable estimates. AJ indicated that his estimate should be adjusted to get a more exact answer since "it's still gaining just a little bit". At $t = 4$, this was the most frequently used strategy. Students who used the half second interval around $t = 1, 2$, and 3 used the quarter second interval before $t = 4$. In order to be classified as using the half second interval around $t = 4$, students had to conjecture how far the ball would have traveled between $t = 4$ and $t = 4.25$ if it had continued to fall and then use the average speed over the interval from $t = 3.75$ to $t = 4.25$ to estimate the speed at $t = 4$. No one in this study did that.

Some students used average speed over the previous full second:

At $t=1$:

"Uhm. 48 feet per second. (Ok. And why do you say that?)
Because the ball traveled 48 feet over the one second interval."
JP - Calculus

At $t=2$:

"So then you'd have to subtract the 48 [at $t=1$] from 64 [at $t=2$].
16." MH - Precalculus

At $t=3$:

"About 15, 16 feet per second [average speed from $t=2$ to $t=3$]."
TW - Precalculus

At $t=4$:

"It's going 48 feet per second [average speed from $t=3$ to $t=4$]."
TW - Precalculus

Average speed over half second around point. Another frequently used strategy was average speed over the half second around the point, a strategy I earlier referred to as using "straddle" intervals. Thirty to forty percent of the students used this strategy at $t = 1, 2$, and 3 but

no one used it at $t = 4$ for the same reasons as discussed above. I will include sample student comments at $t = 1, 2$, and 3 .

At $t=1$:

"Well, this here [from $t=0$ to $t=.25$] is going 15 feet for point two five seconds. This is going 13 in .25 seconds [from $t=.25$ to $t=.5$] and this here is traveling 11 for .25 seconds [from $t=.5$ to $t=.75$]. This is traveling 9 feet for .25 seconds [from $t=.75$ to $t=1$]. So this, so from here to here [from $t=.75$ to $t=1$] it's been going 9 feet per quarter second. And here [from $t=1$ to $t=1.25$] it probably goes, here it goes 7 feet for a quarter second. I mean. If I had to make a wild guess from the data, I'd say it was going 8 feet per quarter second. (And why would you say that?) Ok. Well, because from .75 seconds to one second, you're traveling nine feet per quarter second. (Ok.) Then from one second to 1.25 seconds, you're traveling seven feet per quarter second. (Ok.) And one second is right in between .75 and 1.25. So right around 8. Maybe little bit less or a little bit more." RF - Calculus

"Well, there's time $t=1$ and that's at 48 and I would just do it the same way I did before. So in the quarter second before that it's at 39. So it traveled 9 feet. And then in the next quarter second it travels another 7 feet. So add that up and it's 12, uh, it's 16. So divide by the half second. So it's 32. But it's not exact because it's decreasing as it goes along [the increases in each successive quarter second get less and less]. So it's [the exact answer] not right in the middle. (The speed is not right in the middle?) Yeah. (What would you need to answer that question better or more exactly?) Well, if you had the equation of that then you could take the derivative at that point and then you could answer it exactly. (Any other way?) You could try to draw a tangent line to it. (And what would you need from that tangent line?) You'd have to try to find the slope." BS - Calculus

"What'd I say before [when working with the graph]? I said. I said the same thing [looks at the table of values and notes that his guessed values from the graph for $t=.75$, $t=1$, and $t=1.25$ were exactly right.]. I'd take 32. Going by this. That'd be 9

[change from $t=.75$ to $t=1$]. The difference, that'd be 9 and 7 [change from $t=1$ to $t=1.25$]. And I'd multiply both by four and then take the average and that'd be the same, 32." JG - Postcalculus

"32 feet per second. (How did you get that?) Because in the quarter second before it traveled 9 and in the quarter second after it traveled 7. So the average of those two is 8. So I figured that it must be traveling about 8 feet per quarter second. So 32 feet per second." MG - Postcalculus

All of these students either computed the change or the speed over the intervals from $t = .75$ to $t = 1$ and from $t = 1$ to $t = 1.25$ and then averaged those two answers (RF, JG, MG) or they took the total change from $t = .75$ to $t = 1.25$ and divided by a half (BS). One of the calculus students (BS) was careful to point out that this answer was not necessarily exact. In general she is right; this strategy does not necessarily yield the exact answer. It does yield an exact answer here only because the curve is a parabola. This student seemed to be making a more general point: that taking the average of the change before and after a point does not necessarily yield the exact value for the rate of change at the point.

At $t=2$:

"That would be taking these two [from $t=1.75$ to $t=2.25$] and that would be zero change over five [.5 seconds] which would equal zero. (So what would you say the speed at that point is?) Zero feet per second." SE - Postcalculus

"It goes 63 [at $t=2.25$] minus 63 [at $t=1.75$]. That's zero. (So what's the speed?) Zero feet per second." JS - Postcalculus

At $t=3$:

"So - oh, I know what I gotta, I gotta go before it and after it. So um this is minus 7 [from $t=2.75$ to $t=3$] and this is minus 9 [from $t=3$ to $t=3.25$]. So that's 16. So it falls 16, 16 feet during this - what was that? - 2.75 to 3.25 - during that half second there. So, so five into one sixty [$16/.5 = 160/5$], be 32, so it's going downwards at like 32 feet per second." RG - Calculus

"It's going 32 feet per second. (Wow. And how did you get that?) Because if you subtract 55 minus 39, it's going 16 feet in point 5 seconds. So therefore, it's going 32 feet per second." JS - Postcalculus

At $t = 3$ these students computed average speed over the interval from $t = 2.75$ to $t = 3.25$ and 32 feet per second. Some students noted that the answer should be negative. Technically speaking, a correct

answer for the velocity is -32 feet per second while a correct answer for the speed is 32 feet per second. Velocity takes into consideration direction while speed does not. Several students got answers of 32 feet per second and then noted that while that was the same answer as at $t = 1$, at $t = 1$ the ball was going up and at $t = 3$ the ball was coming down. They noted this difference not by interpreting the negative sign of their answer but by understanding the motion of moving objects in this situation.

Average speed since $t = 0$ or $t = 2$. A few students found the speed at $t = 1, 2, 3$, or 4 by finding the average speed since $t = 0$ or the average speed since $t = 2$:

Average speed since $t=0$:

At $t=2$:

"It's moving 32 feet per second or 64 feet in two seconds." MY - Calculus

At $t=4$:

"[divides 128, the total distance the ball traveled over the four seconds, by 4] 32. (And you got that by adding up the total) Yeah (distance and dividing by four?) Yeah." SCL - Precalculus

Average speed since $t=2$:

At $t=4$:

I'd say - let's see. If it went 64 feet in two seconds, then it would go 32 feet in one second. So 32 feet per second, I guess." JF - Precalculus

Reference to a point. A few students made reference to the coordinates of a point to get the speed of the ball at that time:

At $t=4$:

"It's going zero [looking at the zero entry for the height of the ball at $t=4$ in the table]." KH - Precalculus

"Um. Let's say - it was going, let's see, the height was one [the smallest whole number before zero, the height at $t=4$], but per four seconds. So it's going one foot per four seconds." HL - Calculus

At $t=3$:

"It's going 48 [the height of the ball at $t=3$]." KH - Precalculus

At $t=2$:

"Two seconds. 64. And so I like to do my speed in feet per second. So I'm going. You just can't look here [at the table] and say 64 because that's the height, number one. So you have to kind of do some little math to figure out, if you want to know that. Anyways. 64 feet per two seconds. So it's going to go 32 feet per second." HL - Calculus

HL said you could not just say that the speed was 64 at $t = 2$. You had to convert to feet per second, which she did to get her answer of 32 feet per second. One could argue that HL was computing average speed since $t = 0$. Her work here suggests that since it looks like she was dividing the distance traveled over the first two seconds by two. However, since she consistently used height divided by time at other points, for example, at $t = 4$ as quoted above, I classified her response at $t = 2$ as height divided by time or reference to a specific point. It is clearer HL was using height divided by time at $t = 4$. She could not use the height at $t = 4$ divided by four since that would yield zero feet per second, the speed of the ball after it hit the ground. So she decided to use the height just a little bit before $t = 4$ and divide that by four to get the speed of the ball just before it hit the ground.

Reference to an equation. At $t = 1$ and $t = 3$ a few students referred to an equation:

At $t=1$:

"Well, if you had the equation of that then you could take the derivative at that point and then you could answer it exactly." BS - Calculus

At $t=3$:

"Sure, if you have an equation. (What would you do if you had an equation?) You could plug in time and height probably. (And what would give you?) The speed." TS - Postcalculus

Some students associated an equation with finding its derivative and using it to find the speed at a given time value. Other students wanted to evaluate an equation at the time value to get the speed. If you have an equation for the speed or the derivative of the equation for the distance, then this procedure would give the exact speed at individual points.

Desire smaller interval. At $t = 1$ and $t = 4$ a few students wanted a smaller interval to help them find the speed:

At $t=1$:

"[uses the interval from $t=.75$ to $t=1.25$ to estimate the speed at $t=1$] That's an estimate. (Ok. What would you need to get a better estimate?) Um - maybe smaller time intervals would give me a better estimate." TS - Postcalculus

At $t=4$:

"[uses the interval from $t=3.75$ to $t=4$ to estimate the speed at $t=4$] That would be the average speed over the last quarter second. (Ok.) I don't know how I would get it like at three point nine. I mean, if I had, if I had more numbers here. If I had like say 3.99 to 4 and the height that fell then, then I could, you know, get it exact." RF - Calculus

Both of these students and another who used this strategy first estimated the speed at $t = 1$ or $t = 4$ using the straddle interval from $t = .75$ to $t = 1.25$ or the quarter second interval from $t = 3.75$ to $t = 4$. Then they expressed a desire for a smaller interval near the time value.

Summary for finding speed at $t = 1, 2, 3,$ and 4 using the table of values. As with the graph, students used a variety of strategies to

address speed at a point when using a table of values and the distribution of strategy use was different at each time value. Average speed over the previous quarter second or the half second straddling the time value were heavily used strategies at $t = 1, 2$, and 3 . At $t = 4$ half of the students used average speed over the previous quarter second, but no students used the interval straddling $t = 4$. At $t = 1$ students also used average speed over the previous second and at $t = 2$ students used analysis of moving objects.

Comparison and contrast of strategy use between graphs and tables of values. Table 9 shows the strategies used most often for each time value using the graph and the table of values.

Table 9 - Strategy use by time value for graph and table

<u>Time value</u>	<u>Graph</u>	<u>Table</u>
t=1	Average speed over prev. sec. or qrtr. sec. Reference to slope	Average speed over prev. sec. or qrtr. sec. or half sec. around time
t=2	Analysis of moving obj. Reference to slope	Analysis of moving obj. Avg. spd. prev. 1/4 sec. or 1/2 sec. around time
t=3	Analysis of moving obj. Reference to a point Avg. spd. prev. 1/4 sec.	Average speed over prev. 1/4 sec. or half sec. around time
t=4	Analysis of moving obj. Avg. spd. prev. 1/4 sec.	Average speed over prev. quarter second

Two strategies, reference to slope and analysis of moving objects, were frequently used with graphs but not with tables of values. This is a sensible result since graphs make visual slope more accessible to students. Use of the graph to tell a story about the motion of the ball

may also make analysis of moving objects a more attractive strategy for students. In this particular task the symmetry of the graph and students' everyday knowledge about the motion of thrown ball may have made this strategy more appealing. With the table of values, strategies based on computation of average rate of change over intervals of various size were the most frequently used. This should not be too surprising since tables emphasize numerical data and computation of average rate of change makes good use of that data.

What I find especially interesting is that even with graphs, computation of the average rate of change over one second and one quarter second were two of the most frequently used strategies. Perhaps use of these strategies was enhanced by the positioning of squares at points on the graph for every quarter second. That made reading specific time and height values off the graph a relatively easy thing to do. Even so, students' tendencies to compute average rate of change to address speed at points was both interesting and surprising.

Of additional interest is the use of analysis of moving objects to find the speed at $t = 2$ when using a table of values. This may indicate how much influence a maximum or minimum plays in students' understanding of a situation and in finding the rate of change at that point.

Summary of finding speed at a point in Task 4. Two main results come from an analysis of students efforts to find the speed of the ball at $t = 1, 2, 3$, and 4 . One is that students used a variety of strategies and the other is that the frequency of strategy use was different for each time value. Furthermore, at each time value the distribution of strategy use was different between the graph and the

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table. Generally, with the graph there was more use of strategies that made use of graphical features like slope and shape and symmetry of graph but with the table there was more use of strategies that took advantage of the presence of numerical data, strategies like average speed over intervals of time. Especially prevalent was the use of average speed over intervals of time with both graphs and tables but as students worked through the task, they tended to switch from using average speed over large and intervals before the point to using average speed over small intervals before the point and the smallest available interval straddling the point.

Task 5 - The "biomass" task. Students were asked "How fast is the population (of yeast cells) changing at 4 am?". Appendix F shows the different students addressed rate of change of the population at $t = 4$ and possible answers obtained for each way. Correct answers were discussed earlier in this chapter. As with Task 4 students used a variety of strategies and the distribution of strategy use was different between the graph and the table of values although there was more similarity in Task 5 than in Task 4.

Rate of change at $t = 4$ when using the graph. Table 10 shows the strategies students used to find the rate of growth of the population when using the graph. The three most frequently used strategies were average rate of change in the number of cells over the previous hour, average rate of change over the two hours around $t = 4$, and slope of the line tangent to the graph at $t = 4$. With these three strategies students estimated population values for selected values of time for

points on the curve or the tangent line and then computed the change in number of cells divided by the change in the time.

Table 10

Strategies for finding rate of growth at $t = 4$ when using graph

<u>Strategy</u>	<u>Frequency (%) of responses</u>
Avg. growth over previous hour	11 (30%)
Avg. growth over 2 hours around $t=4$	11 (30%)
Reference to slope	10 (27%)
Reference to a point	6 (16%)
Avg. growth since $t=0$	3 (8%)
"Not quite doubling"	1 (3%)
"It's multiplied by two-thirds."	1 (3%)
Desired smaller interval near $t=4$	1 (3%)

Total does not add to 37, the number of subjects in the study, since some students used more than one strategy.

Here are some sample responses:

Average rate of change in the number of cells over the previous hour -

"About 20 cells an hour because that's how much it changed between 3 and 4." KH - Precalculus

"I would say 25 cells an hour. This looks like 75 [height on the curve at $t=4$]. This looks like 50 [height on the curve at $t=3$]. So difference is 25 in that duration." AW - Calculus

"It's probably about 75 [at $t=4$] and 50 [at $t=3$] over um 4 minus 3 which is 25 over 1 which is 25 cells added per hour." LB - Postcalculus

These students computed the change in the number of cells from $t = 3$ and $t = 4$ and used that as their estimate for how fast the population

is changing at $t = 4$. One precalculus student noted that the rate of change of the cell population at $t = 4$ was more than the average rate of change over the interval from $t = 3$ to $t = 4$:

"A little faster than 25 yeast cells per hour. At 3, it looks about 50 and at 4, it looks like about 75. So that's 25, and it's growing faster. So, a little faster. (And you're saying that because it grew 25 cells but you also) It's still going up." AJ - Precalculus

AJ noticed the sharper rise after $t = 4$ and realized that the average rate of change from $t = 3$ to $t = 4$ needed to be adjusted upward to give a more accurate answer for the rate of change at $t = 4$. Other students capitalized more on the information given about the growth of the population immediately after $t = 4$ to help them estimate the rate of growth at $t = 4$.

Average rate of change in the number of cells from $t=3$ to $t=5$ -

"Well, at 3 the population is 50 and then at 5 it's 125. So that's 75 cells over that two hour period. So if you divide 75 by 2, then at 4, it's about 37.5 cells per hour." BS - Calculus

"What is that? About 50 to 70 [from $t=3$ to $t=4$], 20. And 4 to 5, 110 and 70. I'd say about 40. (How did you get that?) I just found how fast they were growing between these two [$t=3$ and $t=4$]. I just estimated, 70 minus 50, about 20 there. Then 110 minus 70 [from $t=4$ to $t=5$], about 40 there. So between 20 and 40, about 30 cells per hour." DB - Postcalculus

"About 33 cells. (How did you get that?) Well, I'm just kind of estimating - 'cause from here [from $t=3$ to $t=4$], it's about 25 cells per hour and from here [from $t=4$ to $t=5$] it's about 50 cells per hour and about halfway in between is 33." JK - Calculus

BS computed the change from $t = 3$ and $t = 5$ and divided by two whereas DB and JK computed the change from $t = 3$ to $t = 4$, the change from $t = 4$ to $t = 5$, and then averaged those two numbers. JK averaged in an approximate way. Either way, they took into account change in the cell population from $t = 3$ to $t = 5$ to get the rate of change at $t = 4$.

Slope of tangent line -

"[draws line tangent to the curve at $t=4$] Now, let's see. [estimates that the two points (2, 0) and (7.5, 200) are on that tangent line] 200 over 5.5. That means I have to use a calculator. 36 cells per hour." AD - Postcalculus

"[draws a line tangent to the curve at $t=4$] That would show the slope of this line, would show how fast they change there. Do you want numbers numbers? (If you could.) Um. I could say about 100 at 5 [at $t=5$] and about zero at 1.8 [at $t=1.8$] gives me 100 over 3.2, 31.25 cell population per hour." SE - Postcalculus

"It looks like it's changing about 75 cells per 2 hours. It's about - 'cause I'm using the tangent line here. I just took the estimated tangent line over a two hour period [from $t=2$ to $t=4$]. It looks like it grew about 75 cells." MG - Calculus

All three of these students approximated the slope of the tangent line.

Some students offered the information that if they could be guaranteed that they had drawn their tangent line exactly and had obtained exact coordinates of points on that line, their answer for the slope of the tangent line would be exactly how fast the population of yeast cells was changing at $t = 4$.

Several students referenced the specific point at $t = 4$ to find the rate of growth at $t = 4$:

"Um, well, there's 80 cells by four hours. So about 20 cells an hour." JF - Precalculus

"About 17.5. (How did you get that?) 70 [estimated cell value at $t=4$] divided by 4." SCL - Precalculus

"I would say that they're changing at about 75 cells per 4 hours. So would that be 15? (So you're trying to make it per hour?) Yeah. 80 per 4 [decides to use 80 instead of 75] 20 cells per hour." HL - Calculus

These students used the coordinates of the point at $t = 4$ to make decisions about the rate of growth at $t = 4$ most often by dividing the number of cells at $t = 4$ by four. Other students used the average growth since $t = 0$:

"Uh. At that point, we're getting about a, about 20, 20 cells per hour. (And how did you get that?) Uh, because that's around about 80 cells had been developed at that time and it was over a four hour time period." JP - Calculus

"75 in 4, so, uh, it's changing - it's increasing 75 more in 4 hours. So 75 divided by 4 - 18.75 cells per hour. (Ok. And you're saying that because it had actually grown - I mean - at - why are you saying?) It's 18.75 because it went from 0 to 75 more cells in four hours." CH - Postcalculus

JF, SCL, and HL indicated they were thinking solely in terms of number of cells at $t = 4$ while JP and CH were referring to the total growth since $t = 0$. These two strategies are not the same. One considers the value of the dependent variable only at a single point while the other considers the change in the dependent variable over an interval starting at the beginning time value. JF, SCL, and HL were using the former strategy here and used the same strategy when using the table of values as we shall see below. JP and CH used the latter strategy; they referred to the number of cells that had grown over the four hours. Admittedly they did not get it right; they did not notice that the number of cells at $t = 0$ was not zero. But their words clearly indicate they were thinking in terms of amount of growth since $t = 0$.

One student wanted to use a very small interval before $t = 4$:

"At 4 am. (At 4 am.) This much [draws a very small interval before $t=4$ on the curve]. (That much? Could you be a little bit) The amount of cell population of this little interval [she indicates a small part of the curve before $t=4$] over this little amount of time [she indicates a small interval on the horizontal axis with right end point at $t=4$]." RC - Calculus

She was not content to use the one hour interval before $t = 4$ or even the one-quarter hour interval before $t = 4$. She wanted a much smaller interval before $t = 4$.

MY, a calculus student, noted that the speed at $t = 4$ was "not quite doubling". He noted that the number of cells increased 25 from $t = 3$ to $t = 4$ and 45 from $t = 4$ to $t = 5$. Since 45 is almost twice 25, he concluded that "It's about doubling. Just five shy." (MY - Calculus). A precalculus student also noticed a multiplicative relationship:

"[long pause] Ok. It multiplied by two thirds. (Multiplied by two-thirds?) Yeah. I think so. Because it went from 75 [at $t=4$] to 125 [at $t=5$]." MH - Precalculus

This student used the growth from $t = 4$ to $t = 5$ to decide about the rate of growth at $t = 4$. Since the number of cells grew from 75 at $t = 4$ to 125 at $t = 5$, a 66.67% increase, "it multiplied by two thirds".

Actually the increase from $t = 4$ to $t = 5$ is two thirds the number of cells at $t = 4$ so the number of cells at $t = 5$ is equal to the sum of the number of cells at $t = 4$ and two-thirds the number of cells at $t = 4$. The number of cells at $t = 5$ is also the number of cells at $t = 4$ multiplied by one and two-thirds. One student gave a tentative answer of 70 cells in four hours but then decided that did not give us the rate of growth at $t = 4$. She finally responded with "I don't know." (KG - Precalculus). Finally, another precalculus student gave a qualitative answer:

"How fast? Not too fast as far as the course of its life but it's picking up right here [at $t=4$]. I mean, it's starting to pick up." BG - Precalculus

Rate of change at $t=4$ when using a table of values. Table 11 shows the strategies students used to find the rate of growth of the population of yeast cells at $t = 4$ when using the table of values.

Almost half of the students used the average rate of change in the number of cells from $t = 3$ to $t = 5$ to find the rate of growth of the population at $t = 4$. A third used average rate of change from $t = 3$ to $t = 4$. A few students referred to the amount of growth since $t = 0$:

"71.1 (71.1?) 71.1 cells in a four hour period. Oh, but it started at 9.6. So 71.1 minus 9.6 - that's 62.5. So technically it's 62.5 over four hours." CH - Postcalculus

"It's gone from 9.6 [at $t=0$] to 71.1 [at $t=4$]. (So what are you saying about how fast it's changing at 4 am?) Well. (I mean, are you saying it's 71.1 cells per hour or?) Well, you probably - 'cause at zero it's 9.6 - so I would say deduct that from this [71.1] to give you how fast it changes. (So that would be around 61 point something.) Yeah. Somewhere in there. That's how fast I'm saying it's changing." RC - Postcalculus

Table 11

Strategies for finding rate of growth at $t = 4$ using a table of values

<u>Strategy</u>	<u>Frequency (%) of responses</u>
Avg. growth over 2 hours around $t=4$	17 (46%)
Avg. growth over previous hour	12 (32%)
Reference to a point	4 (11%)
Avg. growth since $t=0$	2 (5%)
Reference to slope	1 (3%)
Multiplied by two-thirds	1 (3%)
Increased by one-third	1 (3%)

Total does not add to 37, the number of subjects in the study, since some students used more than one strategy.

Several students referred to the number of cells at $t=4$:

"[Student and interviewer look at table of values and compare values for time and number of cells to values obtained from graph] (Actually it wasn't 80, it was 71.1 [at $t=4$] and actually it didn't start at zero, it started there [pointing to 9.6 as number of cells for $t=0$]. So it didn't grow exactly 80 - more like 62.) Uh huh. (Something like that. Ok. So now does this help you

come up with another answer to the question "How fast is the population changing at 4 am?") Um - [calculates 71.1 divided by 4] 17.8 (And you got that by taking 71.1 and dividing by four?) Yeah. (And that's what? Cells per?) hour (Cells per hour)" JF - Precalculus

"17.775 [71.1 divided by four]" SCL - Precalculus

"71 per at the end of four hours. So per hour that would be what? [divides 71.1 by four] 17 cells per hour. I said 20 [previous answer when using graph]. That's close." HL - Calculus

I pointed out to JF that the number of cells at $t = 0$ was not zero so that the amount of growth from $t = 0$ to $t = 4$ was not the number of cells at $t = 4$; it was the number of cells at $t = 4$ minus the number of cells at $t = 0$. JF did not use this information; instead she divided the number of cells at $t = 4$ by four. I concluded therefore that she was not referring to the amount of growth since $t = 0$ but was referring to the number of cells at $t = 4$ to find the rate of growth of the population at $t = 4$.

One student used slope of tangent line; one maintained her earlier answer of "multiplied by two-thirds" she obtained when using the graph (MH - Precalculus); and one student compared the growth from $t = 3$ to $t = 4$ to the total growth from $t = 0$ to $t = 4$ and declared "it's gotten one-third that amount [the growth from $t = 3$ to $t = 4$ is one third the total growth from $t = 0$ to $t = 4$]" (BG - Precalculus). This last student was the one who when using the graph noted that at $t = 4$ "It's starting to pick up."

Comparison of strategies when using graph and table of values.

With the graph students used intervals either immediately before $t = 4$ or straddling $t = 4$ or used slope of tangent line whereas with the table more students made use of the two hour interval straddling $t = 4$ and

almost no students used slope of tangent line with the table of values. The presentation of numerical data in the table of values might have encouraged students to use more numerical approaches like average rate of change over intervals and less graphical approaches like slope of tangent line.

Using the more detailed graph and table of values. Students were given an enlarged section of the graph from $t = 0$ to $t = 7$ and a table of values containing cell population values for every quarter hour from $t = 3$ to $t = 5$ and asked to again find out how fast the population was changing at 4 am. I told students at the beginning of the interviews that we were trying to see how many different ways they could answer these questions and so we would be asking some questions more than once. Students were generally quite cooperative in answering the same question more than once and especially here in Task 5 where they were asked the same question three times. Table 12 shows the strategies students used and the number and percent of students using each strategy.

Over half of the students found how fast the population was changing at 4 am by computing the average rate of change in the number of cells from $t = 3.75$ to $t = 4.25$:

"Well, I could do the difference between 3.75 and 4 and 4.25 and 4 and average those. [She does that and gets 33 cells per hour.]"
KH - Precalculus

"Well, I could, let's see. Oh. You do have those broken down farther. I could get a lot closer approximation with this by doing the same method except between 3.75, $t=3.75$ and 4 and then $t=4$ and 4.25 and then - if you want me to, I can do it. [He does and then averages these two answers to get 33 cells per hour.]"
DW - Calculus

"I can do exactly what I did before. It will be a slightly better approximation. in this case you'll get 79.7 [at $t=4.25$] minus 63.2 [at $t=3.75$] equals 16.5 divided by .5 gives me 33." AD - Postcalculus

Table 12

Strategies for finding rate of growth with enlarged graph and table

<u>Strategy</u>	<u>Frequency (%) of responses</u>
Avg. growth over half hour around $t=4$	21 (57%)
Avg. growth over previous quarter hour	11 (30%)
Reference to slope	4 (11%)
Reference to a point	2 (5%)
Avg. growth over two hours around $t=4$	1 (3%)
12 percent increase	1 (3%)

Total does not add to 37, the number of subjects in the study, since some students used more than one strategy.

Four students who used this strategy also referenced the slope of the tangent line in some way. DW quoted above went on to describe his procedure in terms of slopes of certain lines:

"I used the same technique [as on previous parts of this task] of getting the slope of the line between the points $t=3.75$ and $c=63.2$ and $t=4$ and $c=71.1$ and then I got the slope of the line between 4 and 71.1 and 4.25 and 79.7 and then I averaged the two slopes to try and get a better approximation of the exact slope of the point $t=4$ and $c=71.1$." DW - Calculus

Another calculus student said he wanted to find the slope of the tangent line but would approximate that slope by finding the slope of the line between $t = 3.75$ and $t = 4.25$:

"Since it would be kind of hard to find exactly the slope of the tangent line at $t=4$, what I do is I could take - um - I could take the - make an equation of a line between 4 minus a quarter and 4 plus a quarter and then that would be about the slope of the tangent line at 4." MG - Calculus

A postcalculus student did the same thing:

"[draws tangent line at $t=4$] It pretty well goes through these two points [at $t=3.75$ and $t=4.25$] right here [pointing to the corresponding points on the curve]. I would probably use these

two points since they seem to be lying on my [tangent] line. They would probably be pretty close." SE - Postcalculus

SE later said that this answer was better than her previous answers obtained using the points at $t = 3$ and $t = 5$ "because these [the points at $t=3$ and $t=5$] are a little bit further away from my line and I was able to use numbers [at $t=3.75$ and $t=4.25$] that seem to be lying closer to my line." (SE - Postcalculus). She consistently used the slope of the tangent line to find the rate of growth at $t = 4$. She reasoned that since the points on the curve at $t = 3.75$ and $t = 4.25$ were closer to her tangent line than the points at $t = 3$ and $t = 5$, the slope of the line containing the points at $t = 3.75$ and $t = 4.25$ should be closer to the slope of the tangent line than the slope of the line containing the points at $t = 3$ and $t = 5$.

Another postcalculus student found the average rate of change from $t = 3.75$ to $t = 4.25$ and got an answer of 33 cells per hour but then separately found the slope of the tangent line by identifying the coordinates of two points on his tangent line and finding the slope using these two points. His answer was 31.5 cells per hour. He went on to say that he liked his answer 33 cells per hour better because he was not very confident that he had drawn his tangent line accurately. I asked him which answer he would prefer if he could find the exact value for the slope of the tangent line. He said "Yeah. That. Yeah. Oh, yeah. Then if I could get the exact value of the slope of the tangent line then that would be the right answer." (MG - Postcalculus).

Eleven students used average rate of growth from 3.75 to 4.00 to estimate rate of growth at $t = 4$. I asked some of these students if the

rate of change over the interval after $t = 4$ had any bearing on the rate of change at $t = 4$:

"Well, I mean, after 4, they obviously, they made more per hour. But if you're trying to find the rate of increase at 4, I'd use 3.75 to 4. (You wouldn't used these numbers then [pointing to the time values after 4]?) No." DG - Precalculus

"No, because that's the fifth second [hour]. Or that's the way I see it. See that's the fifth second [hour]. So if when you tell me what's happening at 4, I would think all in the fourth second [hour] would be the ones that are included in 4 and then after that would be included in 5." JM - Precalculus

These students made it clear that to find the rate of growth at $t = 4$ they would use only information about the rate of growth before $t = 4$. As noted above, many students used information after $t = 4$ to make decisions about the rate of growth at $t = 4$ but for some students what happened after $t = 4$ had no bearing on the rate of growth at $t = 4$. As we shall see in Chapter 7, it was precalculus rather than calculus and postcalculus students who did not use information after a point to find rate of change at a point. We will discuss possible reasons for their reasoning there.

One student used average rate of growth from $t = 3$ to $t = 5$ and one student said "It changes.". Two students said the enlarged graph and more detailed table of values added no new information:

"It's still going I think about the same. If you blow up the graph it doesn't change the rate." AJ - Precalculus

"It's gonna be the same amount - that just shows - that just goes into more detail - it's not going to change the rate of growth." TW - Precalculus

Both of these students mentioned that the enlarged graph and more detailed table of values did not help them think differently about the rate of change at $t=4$. In fact, both were adamant that the enlarged

graph and more detailed table of values is "not going to change the rate of growth". They are, of course, correct; enlarging the graph and creating a more detailed table of values does not change the rate of growth. It does however give more information which may help one get a more accurate answer for the rate of growth at a point.

Two students referenced used the coordinates of the point at $t = 4$ to get their answers. Both of these students used this same strategy to get their two previous answers for the growth rate at $t = 4$. Two students computed changes from $t = 3.75$ to $t = 4$ and from $t = 4$ to $t = 4.25$ to make qualitative statements about the rate of change at $t = 4$:

"I still say, you know, that's the point of its big increase, I'd say, because from here to here [from $t=4$ to $t=4.25$ to $t=4.5$ to $t=4.75$ to $t=5$ - he points to the cell values for all of these time values as he talks] it starts increasing even more. But right here [from $t=3$ to $t=3.25$ to $t=3.5$ to $t=3.75$ to $t=4$] it's not too much as far as right here at this point. It's almost increased 8 [from $t=3.75$ to $t=4$], but here it's almost up to 10 here [from $t=4$ to $t=4.25$]." BG - Precalculus

One student used percent change from $t = 4$ to $t = 4.25$ to find the rate of growth at $t = 4$:

"[She computes change in number of cells from $t=4$ to $t=4.25$ and gets 8.6.] Is this the number of? (Number of cells) Alright. (at that point in time. So this is the number of cells at 4 am and this is the number of cells at 4:15.) [She then divides 8.6 by 71.1 and gets .12.] So - that didn't really help. (You took the difference) A percent. (And then you divided it by that [71.1]?) Right. And then (And you got 12 percent increase?) Uh huh." MH - Precalculus

This is the same student who in the two previous parts used change from $t = 4$ to $t = 5$ and concluded that "it multiplied by two-thirds". The growth from $t = 4$ to $t = 5$ was a 66 percent increase over the growth from $t = 0$ to $t = 4$. Here, the growth from $t = 4$ to $t = 4.25$ was a 12 percent increase over the growth from $t = 0$ to $t = 4$. This is a way of

looking at rate of change at a point in terms of percent increase immediately after the point in comparison with total change since the beginning time value $t = 0$. This is an interesting way to think about rate of change at a point though not consistent with conventional methods.

Comparison of strategies between original graph and table and enlarged graph and more detailed table. While there were no dramatic differences in strategy use between the original graph and table and the enlarged graph and table, with the latter, more students made use of intervals, especially the half hour interval straddling $t = 4$. For students who had already committed themselves to using intervals with the first table of values, the presence of data about the number of cells at time values closer to $t = 4$ may have encouraged them to use average rate of change over these smaller intervals.

Choosing which answer is best. Twenty-nine of the 37 students had three different answers for the rate of change at $t = 4$ - one obtained using the original graph, one from the original table of values, and one from the enlarged graph and table. I asked these students which of their three answers they liked best and why. All of them said they liked their last answer best. They did not seem bothered that they had three different answers to the same question. As students' comments below indicate, they understood that these answers were estimates for the exact answer and they were being asked to decide which estimate they liked best and why. Table 13 shows the reasons students gave for why they liked their last answer best.

Table 13

Reasons students gave for why they liked their last answer best

<u>Reason</u>	<u>Number (%) of students</u>
Interval was smaller	25 (86%)
Points were closer to tangent line	4 (14%)
Used points on both sides of $t=4$	1 (3%)

(One calculus student gave two answers.)

Only 29 students were asked this question. One calculus student gave two answers. This question was not asked of students who obtained the same answer with the original graph, the original table of values, and the enlarged graph and more detailed table nor was it asked of students who did not obtain numerical answers.

Most of these students said their last answer was best because it was based on a smaller interval, it involved only points closer to $t = 4$:

"(Which of these three answers do you have the most faith in? 20, 35.96, or 33 cells per hour?) 33 (Why?) Because it's going to be more accurate because then you're doing it between 3 and 4, the change in rate over that time is greater than the change in rate just in a quarter of an hour and so there's a bigger margin for error if you're doing 3 and 4 and 4 and 5." KH - Precalculus

"This one's [her last answer] a better estimate because it's over a shorter amount of time." RC - Calculus

"I would probably go with this one [last answer] 'cause it's more defined. I mean, three point seven [pointing to 3.75 to 4], you know, I broke it down a little more, whereas this one [from $t=3$ to $t=4$] is a little more broader, I would say." RC - Postcalculus

Students who gave this type of answer made it clear that considering only points closer to $t = 4$ should give us a better estimate of the rate of change at $t = 4$.

A few students made reference to the tangent line:

"Because this is taking the slope of this line right here [the line through the points with time values 3.75 and 4.25] which is more the slope of the tangent line than is this line right here [the line through the points with time values 3 and 5]. This one is more accurate." JS - Postcalculus

For these students their latest answers were best since the points they used to obtain these answers were closer to the tangent line than the points they used to obtain previous answers and therefore their last answer was closer to being the slope of the tangent line. One student said his last answer was best since it used points on both sides of $t = 4$; his earlier answers used only points before $t = 4$.

Getting a better answer. I then asked these students how they would get yet a better answer for the rate of change at $t = 4$. Table 14 shows their responses.

Table 14

Responses to the question "How would you get a better answer?"

<u>Responses</u>	<u>Number (%) of students</u>
Use smaller intervals	25 (93%)
Take limit of avg. rate of change over smaller and smaller intervals	10 (37%)
Evaluate derivative at $t=4$	10 (37%)
Slope at $t=4$	3 (11%)
Integrate from $t=0$ to $t=4$	1 (4%)
Find an equation for the change	1 (4%)
Evaluate equation for the curve at $t=4$	2 (7%)
Use time after $t=4$ also	1 (7%)

Percents are computed on the basis of 27, the number of students who answered this question.

Most of these students said they would use smaller intervals around $t = 4$ to get a better approximation:

"(What would you have to do to get a better answer?) Take readings that were even closer to 4 - like 3:50 and 4:10. The closer you got to 4, the closer your answer would be." KH - Precalculus

"This one would come out even more accurate [referring to the interval from 3.9 to 4.1]. (Why do you say it would be more accurate?) Because it's giving me about almost - there's less time in between. So we're breaking it down even more." RC - Postcalculus

Other students went on say that to get an exact answer, they would keep taking smaller and smaller intervals and then conjecture what the limiting values was:

"Take them closer and closer and closer [pointing to a table of values for $t=3.9$, $t=3.99$, $t=3.999$, $t=4.001$, $t=4.01$, and $t=4.1$]. Then you could take the limit." HL - Calculus

"Like take the average from there to there [from $t=3.9$ to $t=4.1$] and there to there [from $t=3.99$ and $t=4.01$], take all of the averages and see what the numbers seem to be approaching." JG - Postcalculus

Although only one student (JG) actually mentioned he was computing the limit of the average rates of change over smaller and smaller intervals, it was assumed from the larger context of the students' remarks that they were talking about average rates of change over these intervals and the limit these numbers would be approaching.

Ten students said that if they had an equation for the curve, they would "take the derivative" and evaluate it at $t = 4$. Other students said that if they had an equation for the curve, they would evaluate it at $t = 4$ or integrate it from $t = 0$ to $t = 4$. One student hoped for an equation that would yield the change in population for each t value.

Three students mentioned slope in their procedure to get a better answer:

"Take the tangent to the line. Even with one minute either side of 4 there'd still be a little room between the line and the graph. The degree of accuracy is incredibly different but it still would be just approaching the exact answer." DR - Calculus

"If you took the slope when it was here [from $t=3.75$ to $t=4$], and then you took the slope when it was that [from $t=3.9$ to $t=4.1$] and then you took the slope when it was that [from $t=3.99$ to $t=4.01$]." SB - Calculus

"Well, as you get points closer and closer to 4, there's a value which the slope will approach but not reach. Assuming it's a continuous function. We're assuming it's continuous. So - and differentiable - as they get closer and closer to 4, there'll be a value which they approach but don't actually reach. And that's the limit." AD - Postcalculus

Finally, one student who had until now used only intervals before $t = 4$ now decided that to get a better answer she could also include intervals after $t = 4$.

Summary of students' responses to Task 5. As students moved through the "biomass" task, they made more use of average rate of change over intervals and especially over the smallest available intervals before or around $t = 4$. When asked which of their answers they liked best, they consistently said they liked the one based on the smallest intervals, and when asked how to get an even better answer, they would liked to have had even smaller intervals before or around $t = 4$. This shows a heavy reliance on change in the dependent variable over intervals of the independent variable in thinking about rate of change at a point.

Task 6 - The "circles" task. Students were asked how fast the area of the circle was growing when the radius was exactly three feet. Appendix G shows different ways of addressing this question and answers

obtained for each way. Correct answers were discussed earlier in this chapter. As with Tasks 4 and 5, students used a variety of strategies to find the rate of change of the area when the radius was three and the distribution of strategy between the graph and table differed. Table 15 shows the strategies students used with the graph and table.

Table 15

Strategies for finding the rate of change at $r=3$ using graph and table

<u>Strategy</u>	<u>Graph</u>	<u>Strategy</u>	<u>Table</u>
Refer to slope	8 (27%)	Avg. rate from 2.75 to 3.25	15 (50%)
Avg. rate from 2.75 to 3.00	7 (23%)	Avg. rate from 2.75 to 3.00	6 (20%)
Avg. rate from $r = 0$	7 (23%)	Avg. rate from $r = 0$	4 (13%)
Avg. rate from 2.00 to 4.00	6 (20%)	Reference to a point	2 (7%)
Avg. rate from 2.75 to 3.25	4 (13%)	Refer to slope	2 (7%)
Reference to a point	1 (3%)	Avg. rate from 2.00 to 4.00	1 (3%)
Other strategies	5 (17%)	Other strategies	6 (20%)

Totals do not add to 30, the number of students who were asked this question, since some students used more than one strategy.

The most frequently used strategy with the graph was reference to slope with over one-fourth of the students using this strategy. With the table of values the most frequently used strategy was average rate of change in the area from $r = 2.75$ to $r = 3.25$ with half of the students using this strategy. With the graph other frequently used strategies were average rate of change starting at $r = 0$, average rate of change from $r = 2$ to $r = 4$, and average rate of change from $r = 2.75$ to $r = 3$. Each of these strategies was used by at least one fifth of the students. With the table of values the other frequently used

strategy was average rate of change from $r = 2.75$ to $r = 3$ with one-fifth of the students using this strategy.

Only one strategy was commonly used by students with both the graph and the table of values. With each, about one-fifth of the students used average rate of change from $r = 2.75$ to $r = 3$. Average rate of change from $r = 2.75$ to $r = 3.25$ was used by about one-eighth of the students when using the graph and average rate of change starting at $r = 0$ was used by about one-eighth of the students when using the table of values.

One student recognized that the equation for the relationship between the radius and the area of the circle was $A = (\pi)r^2$, differentiated this equation to get $A' = 2(\pi)r$, then evaluated this at $r = 3$ to get his answer when using the graph. When using the table he decided to use average rate of change of area from $r = 2.75$ to $r = 3.25$.

Another student who had used the equation $A = (\pi)r^2$ to answer the questions about how much the area grew from $r = 2$ to $r = 3$ and from $r = 3$ to $r = 4$ chose to use slope of tangent line to find the rate of change of the area at $r = 3$ when using the graph and the average rate of change from $r = 2.75$ to $r = 3.25$ when using the table of values.

One student said that the area of the circle was not growing when the radius was $r = 3$:

"Well, when it's exactly three feet it's not really growing, is it?" MH - Precalculus

This could be taken to mean that she did not believe that the area had a rate of change at $r = 3$, that rate of change at a point does not make sense. I pressed her further to see what she would do with information about change in the area for one foot and one quarter foot changes in

the radius before and after $r = 3$. We used the table of values to find how much the area grew between $r = 2$ and $r = 3$ (about 15 square feet), between $r = 3$ and $r = 4$ (22 square feet), between $r = 2.75$ and $r = 3$ ("a little less than five"), and between $r = 3$ and $r = 3.25$ ("a little less than five again, but closer to five"). I again asked her how fast the area was growing when the radius was exactly three feet:

"How fast is it growing? It's not growing too fast if it's growing because the difference between the 2.75 and 3 and then the 3 and 3.25 isn't very much." MH - Precalculus

One interpretation of this interchange might be that she had observed that as the intervals for the radius get smaller, the change in area also gets smaller so that at $r = 3$, an interval with zero width, the change had to be zero and therefore the area of the circles was not growing at $r = 3$. Another interpretation is that she compared the change in area from $r = 2.75$ to $r = 3$ to the change in area from $r = 3$ to $r = 3.25$ and noticed that they were nearly the same and therefore there was essentially no change at $r = 3$. That is, she interpreted no change in the change over those two intervals as no change in the area at $r = 3$. Yet another interpretation is that this student simply does not believe that a quantity can be changing at a point.

In Task 4 this student used average rate of change over the previous second to get the speed of the ball at each of the four time values and in Task 5 used percent change over the interval immediately after $t = 4$ relative to change from $t = 0$ to $t = 4$ to find the rate of change of the growth at $t = 4$. This student knew in Task 4 that speed should be measured in feet per second. So she used number of feet traveled over the previous full second to get an answer consistent with

that knowledge. In Task 5 she knew that population growth was dependent on how much population had already grown, so she based her answer there on how much it grew in the interval immediately after $t = 4$ in comparison to how much it had already grown.

In Task 6 she did not use either of those types of strategies to make sense of rate of change at a point. With my guidance she looked at changes in the area over intervals of changes in the radius, a strategy similar to what she did in Task 4. But whereas in Task 4 that strategy yielded an answer that made sense, speed in terms of feet per second, in Task 6 it did not. She did however do something that made pretty good sense. If the change in area over the quarter foot before $r = 3$ is about the same as the change in area over the quarter foot after $r = 3$, then in some sense the area is not changing at $r = 3$ and therefore "it's not really growing, is it?".

In summary, as with Tasks 4 and 5 a significant number of students used slope to find rate of change at a point when working with a graph. But when presented with a table of values, the most frequently used strategy was average rate of change over the half foot interval straddling $r = 3$. Even with the graph a large percent of the students used average rate of change over intervals of varying length - long intervals [from $r = 0$ to $r = 3$], medium intervals [$r = 2$ to $r = 4$], or short intervals [$r = 2.75$ to $r = 3$ or $r = 2.75$ to $r = 3.25$]. In going from the graph to the table, students made less use of longer intervals [from $r = 0$ to $r = 3$ and from $r = 2$ to $r = 4$] and more use of shorter intervals [from $r = 2.75$ to $r = 3$ and from $r = 2.75$ to $r = 3.25$].

The distribution of strategy use in Task 6 had more spread than in Tasks 4 and 5. In Task 4, with both the graph and the table, at one time value only one strategy had a frequency of at least 20 percent of the students, at two time values two strategies had frequencies this large, and at the other time value, three strategies had frequencies this large. In Task 5, with the original graph, three strategies had frequencies of at least 20 percent of the students, and with the original table and the enlarged graph and table, only two strategies had frequencies this large. In contrast, in Task 6, with the graph four strategies had frequencies of at least 20 percent.

One explanation for this is that the situation in Task 6 was less familiar to students than the situations in Task 4 and 5. In Task 4 students made heavy use of "analysis of moving objects". In Task 6 there was no similar strategy. Although students might have had experiences with dropping a rock into a pond and watching the growing circles, it is doubtful they would have had much experience with thinking about the rate of change of the area of the circles. In Task 4, because of everyday experiences with speed, students were more likely to associate speed with the particular situation and therefore use a strategy that connected this situation with speed.

In both Tasks 4 and 5, time was the independent variable and since students are used to thinking about speed and rate of change in terms of time, students were likely to use strategies that took advantage of this. In Task 6, the independent variable was radius, not time, and therefore students could not think about rate of change in terms of change in some quantity over time. This would have prevented them from

relying on one or two strategies that connected rate of change to time. They had to rely on other strategies. That might explain why there was more spread in the use of strategies in Task 6 with the graph.

Another explanation for a larger number of frequently used strategies in Task 6 is that by this time in the interview students had more strategies at their disposal and therefore tended to vary the strategies they used. This pattern of strategy use did not hold however with the table in Task 6. As with Tasks 4 and 5, only two strategies had frequencies of at least 20 percent in Task 6, and as in Tasks 4 and 5, these strategies were among the most appropriate to use with a table of values. So I tend to think that the larger number of high frequency strategies with the graph in Task 6 is more related to the lack of experience with rate of change expressed in terms other than time and a situation where rate of change was a less familiar concept.

Summary of strategies used with Tasks 4, 5, and 6. When describing the function in Tasks 4 and 6, some students gave less detailed descriptions which mentioned only that the dependent variable increased or decreased as the independent variable increased. Other students gave a more detailed description that included some mention of the varying rate of change of the dependent variable. In Task 5 all students gave a more detailed description but there were two such descriptions. In one description the graph was seen as two sections, one where the dependent variable increased at an increasing rate and one where the dependent variable increased at a decreasing rate. In the other description the graph was seen as three sections, where the dependent variable increased slowly, then faster, then slowly again.

To decide which intervals of the independent variable yielded the most and least change in the dependent variable, students used primarily three strategies - "delta d", "knowledge of the situation", and "visual slope". Differences in frequency of use of these strategies seemed to be related to the familiarity of the situation. In Task 4, the most familiar situation, all of the students used "delta d" and over one-fourth used "knowledge of the situation". With Task 5, a less familiar situation, 70 percent of the students used "delta d" and over half used "visual slope". On all three tasks "delta d" was the most frequently used strategy and on tasks 5 and 6, where the situation was less familiar to students, students made more use of "visual slope".

Two main features hold for the strategies students used to find speed at individual points. One is that students used a variety of strategies. The other is that the distribution of strategy use was different among the three tasks, between graphs and tables and, in the case of Task 4, from one time to another. On Task 4, students used a total of eight strategies and all of these strategies were used with both the graph and table of values. With the graph, at $t = 2, 3$, and 4 over half of the students used "analysis of moving objects". At $t = 1$ the most frequently used strategies were average speed over the previous second or quarter second. With the table of values, at $t = 2$ the two most popular strategies were "analysis of moving objects" and average speed over the half second around $t = 2$ with over one-third of the students using each. Average speed over the half second around the point was also the most popular strategy at $t = 1$ and $t = 3$ and average speed over the previous quarter second was the most popular strategy at

$t = 4$. At each time value, "reference to slope" was used with greater frequency on the graph than on the table.

With Task 5, students used a total of 12 strategies and most of them were used with both graph and tables. With the graph, three strategies were each used by more than one-fourth of the students - average growth over the previous hour, average growth over the two hours around $t = 4$, and reference to slope. With the table, two of these strategies - average growth over the previous hour and average growth over the two hours around $t = 4$ - had frequencies of over 30 percent of the students. With the enlarged graph and table of values, two strategies - average growth over the half hour around $t = 4$ and average growth over the previous quarter hour - were the most frequently used strategies with at least 30 percent of the students using each. On this task there was no strategy that played the same role as "analysis of moving objects" played in Task 4. Of special interest is the use of strategies that emphasize change in the population over smaller intervals before and straddling the point.

With Task 6, six main strategies were used with both the graph and the table. With the graph, four of these strategies had a frequency of at least 20 percent of the students - average rate of growth of the area of the circle from $r = 0$, average rate of growth from $r = 2$ to $r = 4$, average rate of growth from $r = 2.75$ to $r = 3$, and reference to slope. With the table, only two strategies had frequencies of at least 20 percent - average rate of growth from $r = 2.75$ to $r = 3$ and average rate of growth from $r = 2.75$ to $r = 3.25$. Half of the students used this last strategy.

Three points are worth mentioning. First, the distribution of use of strategies among the three tasks was different. Second, within each task, the distribution of use of strategies was different between graphs and tables. Third, in Task 4 and within the graph and within the table, the distribution of use of strategies was different among the four time values. These three points seem to say that use of strategies depends on the context of the problem, the function representations given, and the points at which the rate of change is requested. This presents a rich and varied picture of students' knowledge of rate of change and the application of that knowledge to specific situations.

A final point is important to mention about commonalities in the strategies used with each task, with both graphs and tables, and at every point. The most popular class of strategies used in all of these situations was average rate of change of the dependent variable over some interval of the independent variable. This was the case not only with tables of values, where we might expect frequent use of such strategies but it was also the case with graphs. This class of strategies seems therefore to hold a special place in students' knowledge of rate of change.

In Task 5 when students were asked which of their various answers they liked best they said they liked the answer that was based on the average rate of change over the smallest interval, and when they were asked how they would get an even better answer for the rate of growth at $t=4$ overwhelmingly they said they would use still smaller intervals. This reasoning seems to hold an important key for students' knowledge of

rate of change at a point and perhaps points to reasoning that could support students' learning of instantaneous rate of change.

Chapter summary

Two main results come from analysis of strategies students used in these tasks. One is that students used many strategies to address rate of change in these six tasks. When explaining the graphs and tables they made in Tasks 1, 2, and 3 students used a total of eight strategies, six with both graphs and table, one with graphs only, and one with tables only. In Tasks 4, 5, and 6 students used five strategies to decide over what intervals of the independent variable the dependent variable changed the most and least. Four of these strategies were also used in Tasks 1, 2, and 3. To find the rate of change of the dependent variable at a specific value of the independent variable in Tasks 4, 5, and 6, students used 12 different strategies. Most of these strategies were used with both graphs and tables and on all three of the tasks. Appendix H lists all of the strategies.

Generally these strategies can be grouped into three broad categories. One category concerns the various uses of slope and shape of graph. "Visual slope", reference to slope (secant line), and reference to slope (tangent line) fit into this category. "Visual slope" was different from slope of secant and tangent lines in that students who used "visual slope" referred only to the steepness or flatness of a graph whereas students who used slope of secant or tangent lines computed a numerical value of the slope and used that value as their answer for the rate of change of the dependent variable at a specific value of the independent variable. I also included in this

category "knowledge of the situation" since the use of this strategy seemed to be closely tied to interpretation of the graph. Table 16 lists the strategies by category.

Table 16

Strategies by category

Category 1 - Based on slope or shape of graph:

Visual slope	All tasks
Reference to slope (secant line)	Tasks 4, 5, and 6
Reference to slope (tangent line)	Tasks 4, 5, and 6
Knowledge of situation	Tasks 4, 5, and 6

Category 2 - Based on comparison of changes in one or both variables

Subgroup 1 - Based on comparison of change in dependent variable to change in the independent variable

Delta d	All tasks
Delta t	All tasks
Delta d and delta t	All tasks
$(\Delta d)/(\Delta t)$	Tasks 1, 2, and 3
Avg. rate of change over intervals larger than one unit before point	Tasks 4, 5, and 6
Avg. rate of change over intervals of one unit before point	Tasks 4, 5, and 6
Avg. rate of change over intervals less than one unit before point	Tasks 4, 5, and 6
Avg. rate of change over intervals straddling point.	Tasks 4, 5, and 6

Subgroup 2 - Based on comparison of a change in the dependent variable to another change in the dependent variable

"Not quite doubling"	Tasks 4, 5, and 6
"Multiplied by two-thirds"	Tasks 4, 5, and 6
"Increased by one-third"	Tasks 4, 5, and 6
"12 percent increase"	Tasks 4, 5, and 6

Category 3 - Based on a specific point

Reference to a point	Tasks 4, 5, and 6
Meeting point	Tasks 1, 2, and 3
Velocity equation	Tasks 1, 2, and 3

A second category includes strategies based on comparison of changes in one or both variables. This category has two subgroupings. One subgroup contains strategies that compare changes in the dependent variable to changes in the independent variable and includes "delta d", "delta t", "delta d and delta t", and " $(\Delta d)/(\Delta t)$ " which often emphasized change in one variable for unit changes in the other. These strategies were used in Tasks 1, 2, and 3 to explain how graphs and table showed rate of change and in Tasks 4, 5, and 6 to identify intervals of least and most change. This subgrouping also includes average rate of change in the dependent variable over various sizes and types of intervals of the independent variable - long intervals of more than one unit before a point, medium intervals of one unit before a point, short intervals of less than one unit before a point, and intervals straddling a point. These strategies were used in Tasks 4, 5, and 6 to find the rate of change at specific points.

A second subgrouping in this second category of strategies compares changes in the dependent variable to each other. This subgrouping includes "not quite doubling", "multiplied by two-thirds", "increased by one-third", and "12 percent increase". These strategies were based on a comparison of changes in the dependent variable to each other. "Not quite doubling" compares change in the dependent variable over an interval after a point to change in the dependent variable over the same sized interval before the point. "Multiplied by two-thirds" and "12 percent increase" compare change in the dependent variable over an interval after a point to total change in the dependent variable before the point. "Increased by one-third" compares change in the

dependent variable over a one-unit interval before a point to total change in the dependent variable before the point.

The third category consists of strategies that emphasize a specific point to make decisions about rate of change. This includes "reference to a point" used in all six tasks and "meeting point" and "velocity equation" used in Tasks 1, 2, and 3. "Reference to a point" and "velocity equation" use the coordinates of a point to make decisions about rate of change while "meeting point" refers to the positioning of a particular point to convey information about the rate of change.

The other main result of the analysis of strategies students used in these tasks is that some strategies were used with relatively greater frequency than other strategies. In Tasks 1, 2, and 3 students made heavy use of "visual slope" and "delta d" to explain how their graphs and tables showed speeding up, slowing down, and steady pace. In Tasks 4, 5, and 6 students used "delta d", "visual slope", and "knowledge of the situation" to make decisions about when the dependent variable changed the most and least.

To find rate of change of the dependent variable at specific values of the independent variable, students made good use of slope of tangent line, average rate of change over medium and short intervals before the point, and average rate of change over intervals straddling the point. Slope of tangent line was used primarily with graphs but average rate of change was used with both graphs and tables. On Task 4 "knowledge of the situation" was an important strategy that was not used with Tasks 5 and 6.

In conclusion students used a variety of strategies to address rate of change in these six tasks, these strategies fit into one of three broad categories of strategies, and some strategies were consistently used on several tasks and with higher frequency than other strategies.

Chapter 6

CLASSIFYING STRATEGIES BY PERSPECTIVES AND TYPES OF REASONING

In Chapter 1 I discussed three perspectives and three types of reasoning about rate of change. In this chapter I consider each of the strategies identified in Chapter 5 in terms of these perspectives and types of reasoning. My goal here is to see if each strategy can be classified in terms of one or more of these perspectives and one or more of these types of reasoning. The extent to which they can will provide a measure of the adequacy of the perspectives and types of reasoning as an appropriate framework for students' reasoning about rate of change.

Chapter 6 has two major sections. The first section concerns classifying students' strategies in terms of the three perspectives. I review what I mean by global, interval, and point perspectives on rate of change and then examine each strategy to see if it satisfies the definition of at least one of these perspectives. The second section concerns classification of students' strategies in terms of three types of reasoning. I review what I mean by qualitative, numerical, and algebraic types of reasoning and then examine each strategy to see if it satisfies the definition of one or more of these types of reasoning.

I anticipate that it will be possible to classify many strategies in terms of at least one perspective and one type of reasoning. I suspect several strategies can be classified in terms of more than one perspective or type of reasoning. Still some strategies may not fit cleanly into any one of the perspectives or types of reasoning. I

anticipate further that strategies which can be viewed in terms of more than one perspective or type of reasoning may provide pictures of students' reasoning about rate of change that may have implications for coming to know rate of change.

Classifying strategies by perspective

Review of definitions of perspectives.

Recall that global, interval, and point perspectives on rate of change represent a successive narrowing of focus on the nature of change in a situation. A global perspective considers changes in the dependent variable of a function over the entire domain of the independent variable. An interval perspective refers to change in the dependent variable over intervals of the independent variable of some convenient size smaller than its entire domain. A point rate of change perspective focuses on change in the dependent variable for an individual value of the independent variable.

Classifying strategies for the "walking" tasks.

In Tasks 1, 2, and 3, the "walking" tasks, students were asked to create and explain graphs and tables of values that showed the distance between two people as they walked across a room. In Task 1, the two people slowed down, passed, and then sped up; in Task 2, the two people maintained the same steady pace the whole way; in Task 3, the two people sped up, passed, and then slowed down.

Strategies for explaining graphs. Seven strategies were used by students in their explanations of how their graphs showed slowing down, speeding up, or steady pace. Table 17 shows these strategies and how I classified them.

Table 17

Classification of strategies for explaining graphs in terms
of perspectives on rate of change for Tasks 1, 2, and 3

<u>Strategies</u>	<u>Perspectives</u>
Visual slope	Global
Meeting point	Global
Delta d	Interval
Delta t	Interval
Delta d and delta t	Interval
Velocity equation	Interval
Reference to a point	Point

I classified "visual slope" as consistent with a global perspective since students who used this strategy focused on properties of the graph of the dependent variable over the entire domain of the independent variable. Students examined the slope of the graph over its entirety to make claims about how their graphs showed slowing down, speeding up, and steady pace. They did not divide the domain of the independent variable into intervals of some convenient equal size and discuss the relative slope over each of those intervals. For that reason I did not classify this strategy in terms of an interval perspective.

Students did often refer to sections of the graph. But those sections were larger than what I would think of in terms of intervals and were used solely for the purpose of comparison of the visual slope of one large section of the domain to another large section and took into consideration the behavior of the slope of the graph over the entire domain of the independent variable. For this reason I classified it as consistent with a global perspective.

I also classified "meeting point" as consistent with a global perspective since students who used this strategy carefully considered

the behavior of the dependent variable over the entire domain of the independent variable in determining the time when the two people should meet to show slowing down, speeding up, and steady pace. The positioning of the meeting point took into consideration what should be the behavior of the dependent variable from the beginning value of the independent variable to its ending value.

Three strategies, " Δd ", " Δt ", and " Δd and Δt ", were classified as representing an interval perspective since they refer to changes in the dependent variable over intervals of the independent variable in one way or another. " Δd " fixes the size of intervals of the independent variable and looks at changes in the dependent variable over these intervals. " Δt " fixes the size of intervals of the dependent variable and looks at changes in the independent variable over these intervals. " Δd and Δt " looks at changes in the dependent variable over small but variable sized intervals of the independent variable. Since all of these strategies center on changes over intervals I classified them as consistent with an interval rate of change perspective.

"Velocity equation" could be classified as representing a point or interval perspective. As it was used here, it referred to the formula "average velocity equals change in distance divided by change in time". In that context it is consistent with an interval perspective since it refers to changes in the dependent variable, distance, over intervals of the independent variable, time. We will see "velocity equation" used in a way consistent with a point perspective when we classify strategies used with tables in these tasks.

"Reference to a point" was classified in terms of a point perspective since it relies on the behavior of the dependent variable at a single value of the independent variable. Students who used this strategy used the value of the dependent variable at a specific value of the independent variable to give explanations of how the graph showed slowing down, speeding up, and steady pace.

Strategies for explaining tables. Seven strategies were used by students to explain how their tables of values showed slowing down, speeding up, or steady pace. Table 18 shows these strategies and their classification in terms of the three perspectives on rate of change.

Table 18

Classification of strategies for explaining tables of values in terms of perspectives on rate of change for Tasks 1, 2, and 3

<u>Strategies</u>	<u>Perspectives</u>
Meeting point	Global
Delta d	Interval
Delta t	Interval
Delta d and delta t	Interval
(Delta d)/(delta t)	Interval
Reference to a point	Point
Velocity equation	Point

Several of these strategies were the same ones used to explain how graphs showed slowing down, speeding up, or steady pace. These include "meeting point", a global perspective strategy; "delta d", "delta t", and "delta d and delta t", interval perspective strategies; and "reference to a point", a point perspective strategy.

I classified the strategy " $(\Delta d)/(\Delta t)$ " as representing an interval perspective since it concerns changes in the dependent variable over intervals of the independent variable. It is a computing formula

for the slope of a line or the average rate of change over an interval.

Either way it refers to changes in one variable over intervals of the another and is therefore an interval perspective strategy.

With the graph, the strategy "velocity equation" was an interval perspective strategy since it referred to changes in distance over changes in time, that is, changes in the dependent variable over intervals of the independent variable. But with the table of values, one student used the formula "velocity equals distance divided by time" to divide the distance value by the time value at individual points to make conclusions about how her table of values showed the two people slowing down and then speeding up. Since she was making judgments about rate of change by referring to a point, I classified this strategy as consistent with a point perspective on rate of change.

Classifying strategies for the "thrown ball", "biomass", and "circles" tasks.

For Tasks 4, 5, and 6 students were given graphs and tables of values for three different situations and asked to give a description of the dependent variable, identify intervals of the independent variable for which the dependent variable changed the most and least, and find the rate of change of the dependent variable at specific values of the independent variable.

Identifying intervals of least and most change. Students used five different strategies to decide which intervals of the independent variable showed the least and most change in the dependent variable. Three of these strategies - " Δd ", " Δt ", and " Δd and Δt " - are consistent with an interval rate of change perspective for the same reason they were classified that way in Tasks 1, 2, and 3; they

attend to changes in the dependent or independent variable over intervals of one or the other to make decisions about rate of change.

Similarly "visual slope" is classified here, as in Tasks 1, 2, and 3, as consistent with global rate of change since, by comparing slope of the various large sections of the graph, it focuses on properties of the dependent variable over the entire domain of the independent variable.

"Knowledge of the situation", whether it was knowledge of the motion of the ball in Task 4, knowledge of the growth of a population in Task 5, or knowledge of circles generated by dropping a pebble into a pond as in Task 6, was classified as consistent with global rate of change since it relies on an image of how the dependent variable is expected to behave over the entire domain of the independent variable. This strategy was supported by the behavior of the graph over its entirety. This provides further impetus for classifying this strategy in terms of a global perspective. Table 19 summarizes these classifications.

Table 19

Classification of strategies for deciding where the dependent variable changed the most and least in terms of perspectives on rate of change for Tasks 4, 5, and 6

<u>Strategies</u>	<u>Perspectives</u>
Delta d	Interval
Delta t	Interval
Delta d and delta t	Interval
Visual slope	Global
Knowledge of situation	Global

Finding rate of change at a point. When students were asked to find the rate of change of the dependent variable at individual values of the independent variable in Tasks 4, 5, and 6 they used 12

strategies. Table 20 shows these strategies and their classification in terms of global, interval, and point perspectives on rate of change.

Table 20

Classification of strategies for finding rate of change at a point in terms of perspectives on rate of change for Tasks 4, 5, and 6

Strategies	Perspectives
Analysis of moving objects	Global
Avg. rate of change over intervals larger than one unit before point	Interval
Avg. rate of change over intervals of one unit before point	Interval
Avg. rate of change over intervals smaller than one unit before point	Interval
Avg. rate of change over intervals straddling point	Interval
"Not quite doubling."	Interval
"It's multiplied by two-thirds."	Interval
"Increased by one-third"	Interval
12 percent increase	Interval
Reference to slope (secant line)	Interval
Reference to slope (tangent line)	Point
Reference to a point	Point

In Task 4 many students used "analysis of moving objects" to find the speed of the ball at $t = 1, 2, 3$, or 4 , especially when working with the graph. In Tasks 5 and 6 students did not use analysis of the specific situation to address rate of change at a point. When students used "analysis of moving objects" to find the speed of the ball in Task 4, they most often considered the speed of the ball at a specific point in terms of the behavior of the ball over all or a large part of the

domain of time. Therefore, I classified this strategy as referring to a global rate of change perspective.

Four strategies involved computing the average rate of change in the dependent variable over intervals of the independent variable of various sizes - greater than one unit before a point, one unit before a point, and less than one unit before a point - or intervals straddling a point. These strategies were based on comparison of change in the dependent variable to change in the independent variable over some interval of the independent variable and therefore were classified as consistent with an interval perspective on rate of change.

The strategies "not quite doubling", "multiplied by two-thirds", "increased by one-third", and "12 percent increase" were based on a comparison of change in the dependent variable over one interval of the independent variable to change in the dependent variable over another interval of the independent variable. The student who used "not quite doubling" compared change in the number of cells over the interval from $t = 3$ to $t = 4$ to change in the number of cells over the interval from $t = 4$ to $t = 5$. The student who used "multiplied by two-thirds" compared change in the number of cells from $t = 4$ to $t = 5$ to change in the number of cells from $t = 0$ to $t = 4$. The student who used "increased by one-third" compared change in the number of cells from $t = 3$ to $t = 4$ to change in the number of cells from $t = 0$ to $t = 3$. The student who used "12 percent increase" compared change in the number of cells from $t = 4$ to $t = 4.25$ to change in the number of cells from $t = 0$ to $t = 4$. Since these strategies rely on changes in the dependent variable over intervals of the independent variable, I classified them as consistent with an interval perspective.

"Reference to slope" was used in all three of Tasks 4, 5, and 6. When it was used with the graph it most often concerned the slope of the line tangent to the curve at a specific point. In that case I classified it as representing to a point perspective since it concerned what was happening to the dependent variable or the rate of change of the dependent variable at a single value of the independent variable. In a few cases "reference to slope" meant slope of a secant line. In these cases I classified this strategy as consistent with an interval perspective since students who used slope in this way related it to changes in the dependent variable over an interval of the independent variable. This was especially the case in Task 4.

With the table of values "reference to slope" was used in one of two ways, both of which are consistent with a point perspective. One way was to use the table of values to get an approximation for the slope of the tangent line to the curve at a specific point. The other way was to basically ignore the information in the table and to refer to the slope of the tangent line on the graph.

"Reference to a point" was classified as a point perspective strategy since students who used this strategy gave the value of the dependent variable at a specific value of the independent variable or used the value of both the dependent and independent variable to find rate of change at that point.

Summary of classifying strategies in terms of three perspectives on rate of change. In Tasks 1, 2, and 3, the "walking" tasks, the most frequently used strategies were "visual slope" and "delta d". "Visual slope" was classified as representing a global perspective on rate of change and "delta d" was classified as an interval perspective. Given

the nature of the tasks, this is not too surprising. Students were asked to explain how their graphs and tables showed the two people slowing down, speeding up, and maintaining a steady pace. Strategies based on descriptions of graphs over their entire domain, like "visual slope", seem reasonable ones to use with these tasks. Likewise, strategies based on changes in the dependent variable over one unit intervals of the independent variable are sensible ways to think about these tasks.

Similarly in Tasks 4, 5, and 6, "visual slope" and "delta d" as well as "knowledge of situation" were frequently used strategies to identify intervals of the independent variable where the dependent variable changed the most and least. Students' use of "delta d", an interval perspective strategy, is not surprising here. In addition, since students had been given a graph showing the behavior of the dependent variable over the entire domain of the independent variable, it makes sense that they made good use of the global perspective strategies "visual slope" and "knowledge of the situation".

To find the rate of change at a point in Tasks 4, 5, and 6, the most frequently used strategies represented an interval perspective. This was especially the case in Tasks 5 and 6 where students relied heavily on average rate of change in the dependent variable over intervals of various sizes of the independent variable. In Task 4 many students also used "analysis of moving objects", a global perspective strategy and in all three tasks a fair number of students made "reference to slope of tangent line", a point perspective.

So over the six tasks, students made good use of strategies from all three perspectives. To explain their graphs and tables and to

identify where the dependent variable changed the most and least, students made good use of global and interval perspective strategies. To find the rate of change at various points in Tasks 4, 5, and 6 students used primarily interval perspective strategies but also used some global and point perspectives.

Classifying strategies by types of reasoning

Review of definitions of types of reasoning.

Qualitative reasoning refers primarily to the use of graphical features to make conclusions about rate of change. Numerical reasoning concerns calculation or use of numerical values to make decisions about rate of change. Algebraic reasoning involves use of formulas or equations to make judgments about rate of change.

Classifying strategies for the "walking" tasks.

Explaining graphs and tables. Eight strategies were used by students to explain how their graphs and tables of values showed the two people slowing down, speeding up, or maintaining a steady pace. Two of these strategies relied on graphical features to make decisions about rate of change - "visual slope" and "meeting point". "Visual slope" made use of the shape and relative steepness or flatness of the graph and "meeting point" focused on the positioning of a particular point on the graph to show slowing down, speeding up, and steady pace. These strategies were therefore classified as qualitative types of reasoning. Table 21 shows each strategy and its type of reasoning.

Table 21

Classification of strategies for explaining graphs and tables
in terms of types of reasoning for Tasks 1, 2, and 3

<u>Strategy</u>	<u>Types of reasoning</u>
Visual slope	Qualitative
Meeting point	Qualitative
Delta d	Qualitative/numerical
Delta t	Qualitative/numerical
Delta d and delta t	Qualitative/numerical
Reference to a point	Numerical/algebraic
Velocity equation	Algebraic
(Delta d)/(delta t)	Algebraic

The strategies "delta d", "delta t", and "delta d and delta t" can be classified as two types of reasoning, depending on how they were used. When students referred to specific numerical values for changes in the dependent or independent variable over intervals of the other or referred to numerical differences between two values of the dependent or independent variable, then I classified these strategies as numerical reasoning. But when students visually compared rises or falls of the graph without referring to specific numerical values or otherwise identifying those rises and falls in terms of quantities, I classified these as qualitative types of reasoning since students were attending to properties of the graph to make these decisions.

"Reference to a point" used the value of the dependent variable for a specific value of the independent variable or used some combination of the values of the dependent and independent values at a point to make decisions about rate of change. In some cases it was simply the pattern of values of the dependent variable at particular points on the graph or in the table that was used to explain how graphs

and tables showed slowing down, speeding up, and steady pace. In this case I classified this strategy as numerical.

In other cases students used a formula based on the values of the dependent and independent values at particular points. These students divided the value of the dependent variable (distance) at a point by the value of the independent variable (time) at that point. This yielded an answer in terms of "feet per second". Students who used the coordinates of points in this way appeared to be using a formula like "velocity is distance divided by time" to get their answers. Because of this I classified this strategy as algebraic.

Students who used "velocity equation" explicitly mentioned that they were using a formula for finding the speed. The form of the equation used with graphs was "average velocity is change in distance divided by change in time" while the form used with tables was "velocity is distance divided by time". In both cases students indicated they were using a formula that gave them velocity and therefore I classified this as algebraic reasoning.

Similarly, students who used " $(\Delta d)/(\Delta t)$ " were referencing a formula for the slope of a line or for average rate of change to make decisions about rate of change. Therefore this strategy was classified as using algebraic reasoning.

Classifying strategies for the "thrown ball", "biomass", and "circles" tasks.

Identifying intervals of least and most change. Students used five strategies to decide which intervals of the independent variable showed the least and most change in the dependent variable. Table 22

shows these strategies and their classifications in terms of types of reasoning.

Table 22

Classification of strategies for deciding where the dependent variable changed the most and least in terms of types of reasoning for Tasks 4, 5, and 6

<u>Strategies</u>	<u>Types of reasoning</u>
Visual slope	Qualitative
Knowledge of situation	Qualitative
Delta d	Qualitative/numerical
Delta t	Qualitative/numerical
Delta d and delta t	Qualitative/numerical

"Visual slope" made use of graphical features by referring to the steepness or flatness of a graph. Therefore I classified this strategy as qualitative. Students who used "knowledge of situation" were often prompted by the graph to help them tell a story about the motion of a thrown ball, the growth of a population, or the behavior of enlarging circles caused by dropping a pebble into a pond. Since this story-telling feature of their use of this strategy seemed to be based on reasoning supported by the graph, I classified this strategy as qualitative type of reasoning.

As in Tasks 1, 2, and 3 when students used "delta d", "delta t", or "delta d and delta t", they sometimes reached their conclusions by visually comparing rises or falls of the graph without referring to specific numerical values or otherwise associating those rises and falls in terms of quantities. In these cases I classified these strategies as qualitative reasoning since students were basing their decisions on use of graphical features.

But sometimes students used these strategies to explicitly compute and refer to numerical changes in the dependent or independent variable over intervals. In these cases I classified these strategies as numerical reasoning since students were basing their conclusions on computation of numerical values.

Finding rate of change at a point. When students were asked to find the rate of change at a point, they used 12 strategies. Table 23 shows these strategies and their classifications in terms of types of reasoning.

Table 23

Classification of strategies for finding rate
of change at a point in terms of types
of reasoning for Tasks 4, 5, and 6

<u>Strategies</u>	<u>Types of reasoning</u>
Analysis of moving objects	Qualitative
Reference to slope (tangent line)	Qualitative/algebraic
Reference to slope (secant line)	Qualitative/algebraic
"Not quite doubling"	Numerical
Avg. rate of change over intervals larger than one unit before point	Algebraic
Avg. rate of change over intervals of one unit before point	Algebraic
Avg. rate of change over intervals smaller than one unit before point	Algebraic
Avg. rate of change over intervals straddling the point	Algebraic
"It's multiplied by two-thirds"	Algebraic
"Increased by one-third"	Algebraic
12 percent increase	Algebraic
Reference to a point	Algebraic

Since "analysis of moving objects", like "knowledge of the situation" in earlier parts of Tasks 4, 5, and 6, was related to students' use of the graph in Task 4 to reason about the behavior of the thrown ball and especially its height and speed over time, I classified this strategy as using qualitative reasoning.

When students referred to the slope of a tangent line, they most often did so when working with the graph. They drew the line on the graph, identified points on that line, estimated their coordinates, and then used those coordinates and the formula for slope to find the slope of the tangent line. This strategy used features of the graph to initially make sense of rate of change at a point in terms of tangent lines. It also used a formula (for the slope of a line) to arrive at a final answer for the rate of change at a particular point. Because students relied on graphical features to associate rate of change with a particular line I classified this strategy as qualitative. Further because students then applied a formula to find the slope of the line and gave that as the answer for the rate of change at a point, I classified this strategy as also using algebraic reasoning.

Some students used the slope of a secant line on a graph to address rate of change at a point. In these cases students said they were using slope, drew the line, used the formula for slope to find the slope of that secant line and declared that as their answer for speed at a point. Since students were attending to features of the graph and using the formula for slope to find rate of change at a point, I also classified this a qualitative and algebraic reasoning.

One student said in task 5 that the cell population was "not quite doubling". His answer was based on comparison of changes in the cell

population over the interval from $t = 3$ to $t = 4$ and from $t = 4$ to $t = 5$. He numerically compared changes in the dependent variable over two consecutive intervals of the independent variable. Therefore I classified this strategy as numerical.

The four strategies based on average rate of change over intervals involved use of the appropriate formula for computing average rate of change based on change in the dependent variable divided by change in the independent variable. I classified these strategies as algebraic types of reasoning.

"It's multiplied by two-thirds", "increased by one-third", and "12 percent increase" were classified as algebraic since they were based on formulas that multiplicatively compared change in population over an interval immediately before or after $t = 4$ to change in the population from $t = 0$ to $t = 4$ in Task 5. The student who used "it's multiplied by two thirds" and "12 percent increase" divided the change in population over the one hour and one-quarter hour after $t = 4$ by the change over the interval from $t = 0$ to $t = 4$. The student who said "increased by one-third" divided the change in population from $t = 3$ to $t = 4$ by the change in population from $t = 0$ to $t = 4$. While these are not conventional formulas for addressing rate of change, they are ways these students used formulas to make sense of rate of change at a point.

One other strategy was based on a formula for addressing rate of change at a point, "reference to a point". Students who used this strategy to find rate of change at a point most often divided the value of the dependent variable at that point by the value of the independent variable and appeared to be using the formula "velocity is distance

divided by time" to find rate of change at a point. Hence it was classified as algebraic reasoning.

Summary of classifying strategies in terms of three types of reasoning about rate of change. To explain their graphs and tables in Tasks 1, 2, and 3 students used strategies from all three types of reasoning with some strategies representing more than one type of reasoning. "Visual slope", a strategy using qualitative reasoning, was used primarily with graphs and "delta d", a strategy using both qualitative and numerical reasoning, was used with both graph and tables.

To identify where the dependent variable changed the most and least in Tasks 4, 5, and 6 students used strategies like "visual slope" and "knowledge of situation" which used qualitative reasoning and strategies like "delta d" which involved both qualitative and numerical reasoning.

In Tasks 4, 5, and 6 all three types of reasoning were represented in the strategies students used to find rate of change at a point. The most frequently used strategies involved average rate of change over intervals of various sizes, strategies using algebraic reasoning, "reference to slope of tangent line", a strategy involving both qualitative and algebraic reasoning, and, in Task 4, "analysis of moving objects", a strategy using qualitative reasoning.

So over the six tasks, students used strategies from all three types of reasoning. To explain their graphs and tables and to identify where the dependent variable changed the most and least students used strategies relying on qualitative and numerical reasoning. To find rate of change at a point, students made heaviest use of strategies using

algebraic reasoning but they also used strategies relying on qualitative reasoning.

Chapter summary

In this chapter I classified the strategies students used in these six tasks in terms of three perspectives on rate of change and three types of reasoning. Appendix H shows all of the strategies and their classifications in terms of the perspectives and types of reasoning. Students used strategies from all three perspectives and all three types of reasoning. In Tasks 1, 2, and 3, the "walking" tasks, students were asked to explain how their graphs and tables showed the two people slowing down, speeding up, and maintaining a steady pace. The strategies with the highest frequencies in those tasks represented global and interval perspectives and qualitative and numerical types of reasoning.

In Tasks 4, 5, and 6, the "thrown ball", "biomass", and "circles" tasks, students were asked to identify where the dependent variable changed the most and least. As with Tasks 1, 2, and 3, the strategies most often used represented global and interval perspectives and qualitative and numerical types of reasoning. These strategies work very well for addressing questions about rate of change that involve comparisons of the dependent variable over all or large sections of the domain of the independent variable.

However to find rate of change at a point in Tasks 4, 5, and 6, students made more use of algebraic reasoning and interval and point perspective strategies. The most frequently used strategies in these three tasks, and especially in Tasks 5 and 6, were strategies based on average rate of change over various sized intervals, a collection of

interval perspective strategies, and slope of tangent line, a point perspective strategy. These strategies were classified as using algebraic reasoning since they rely on a particular formula for arriving at an answer for rate of change at a point.

Two primary conclusions comes from this part of the analysis. One is that students used strategies from all three perspectives and all three types of reasoning. The second is that strategies based in global and interval perspectives and qualitative and numerical types of reasoning were especially helpful for answering questions about the behavior of the independent variable over all or a significant section of its domain, and strategies based in interval and point perspectives and algebraic reasoning were especially helpful for answering questions about behavior of the dependent variable at individual points.

Perhaps no one is surprised by these results, but I was. I was not sure what strategies students would use to address these tasks. I was fairly sure at least some students would use slope of tangent line to address rate of change at a point. Outside of that I was basically clueless about what students would do. I suspect I am not the only calculus teacher in this boat. One issue to be discussed in Chapter 8 concerns what difference it makes to calculus teachers what students do with problems like these.

Chapter 7

COMPARING AND CONTRASTING THE THREE GROUPS OF STUDENTS

One of the goals of this dissertation was to see how knowledge of rate of change differed among the three groups of students. Since strategies were taken as the unit of knowledge in this study, it made sense to compare and contrast strategy use among the precalculus, calculus, and postcalculus groups. Specifically for each task I noted the strategies used by each group and similarities and differences in frequency of use of strategies among the three groups. Generally I found that the mix of strategy use was not uniform across the three groups.

As in Chapters 5 and 6, here I examine strategies used in Tasks 1, 2, and 3 first and then I look at strategies used in Tasks 4, 5, and 6.

I found that frequency of use of strategies varied by group and by task but that there were clear patterns in what strategies were consistently used by each group. Three patterns emerged. First, certain strategies were used by all three groups with great frequency. For example, in Tasks 1, 2, and 3, all three groups of students made heavy use of "visual slope" and "delta d". Second, some strategies were used by some groups but not by others. Calculus and postcalculus students used slope of tangent line in Tasks 4, 5, and 6 but precalculus students did not. Finally, some strategies were used with greater or less frequency by each group of students as they progressed through the tasks. The

frequency of use of average rate of change over short intervals before a point and straddling a point became more popular as students moved through Tasks 4, 5, and 6.

Comparing strategies of the three groups - "Walking" tasks

In this section I consider similarities and differences among the three groups of students in the frequency of strategies used to explain graphs and tables in each of Tasks 1, 2, and 3. Then I compare and contrast use of strategies by each group for the three tasks combined. Generally I found that all three groups of students made heavy use of two particular strategies, "visual slope" and "delta d", with other strategies used by only a few students in one or two groups.

Strategies for explaining graphs. Table 24 shows strategies students used for explaining how their graphs showed slowing down, speeding up, or steady pace and, for each task, the percent of each group that used each strategy.

One similarity among the three groups and across the three tasks is the predominant use of "delta d" and "visual slope". However, the frequency of use of these and other strategies differed among the three groups and across the three tasks. Precalculus students were the only ones to use "meeting point" and they used it on all three tasks. "Velocity equation" was used by only postcalculus students and then only on Task 2. All other strategies were used by at least two groups on at least one task.

I will examine similarities and differences in strategy use among the three groups by each task. Then for each strategy I will combine frequencies across all three tasks and look at similarities and differences among the three groups.

Table 24

Percent of each group using each strategy to explain
how their graphs showed slowing down, speeding up,
or steady pace in Tasks 1, 2, and 3

(Pre=Precalculus, Cal=Calculus, and Pos=Postcalculus)

<u>Strategy</u>	<u>Task 1</u>			<u>Task 2</u>			<u>Task 3</u>		
	<u>Pre</u>	<u>Cal</u>	<u>Pos</u>	<u>Pre</u>	<u>Cal</u>	<u>Pos</u>	<u>Pre</u>	<u>Cal</u>	<u>Pos</u>
Delta d	58	53	90	83	80	90	75	67	70
Visual slope	42	73	60	25	47	60	33	67	60
Delta d and delta t	17	40	0	0	0	0	8	13	10
Delta t	25	13	0	8	0	0	8	13	10
Meeting point	25	0	0	17	0	0	17	0	0
Reference to a point	33	7	0	0	7	10	0	0	0
Velocity equation	0	0	0	0	0	20	0	0	0

Task 1. Over half of each group used "delta d" and over 40 percent used "visual slope". For the postcalculus group these were the only two strategies used. This is in contrast to the precalculus and calculus groups which used a total of six and five strategies, respectively. Besides "delta d" and "visual slope", calculus students also used "delta d and delta t" and precalculus students used "delta d", "meeting point", and "reference to a point". Perhaps calculus students used "delta d and delta t" because their most recent experiences in calculus suggested that rate of change has something to do with varying distance relative to varying time and this is one way of explicitly referring to rate of change in terms of both.

Precalculus students' use of "delta t", "meeting point", and "reference to a point" shows that these students were thinking about

possible relationships between distance and time in trying to make sense of rate of change. In the case of "meeting point", students were thinking about the need for the two people to cover a total of 20 feet in less time when going fast and more time when going slow. Although not necessarily an accepted way of thinking about slowing down and speeding up, students were nonetheless thinking about the right kinds of relationships between distance and time for addressing rate of change.

Similarly with " Δt ", students interpreted rate of change in terms of how much time to travel a fixed amount of distance. This too shows students making somewhat conventional sense of rate of change. Students who used "reference to a point" focused on a relationship of distance to time but in this case it was the value of the distance at a point in time rather than change in distance over some time period. While this understanding of rate of change is unconventional, it shows that these students were trying to relate distance to time in thinking about rate of change, an important part of learning rate of change.

A larger percent of calculus students used "visual slope" than either precalculus or postcalculus students. It is usually in calculus that students associate slope of a graph with derivative and rate of change. Students who are just finishing their study of calculus then are quite likely to associate rate of change with slope of curve. That may be why calculus students, more than precalculus or postcalculus, used "visual slope" to address rate of change in Task 1.

All but one of the postcalculus students used " Δd " compared to a little over half of the precalculus and calculus students. Courses beyond calculus, like differential equations, mathematical modeling, and numerical analysis, contain strategies for approximating rate of change

using changes over intervals. This would help explain why "delta d" was a more popular strategy with these students than "visual slope" and why it was more popular with postcalculus students than with precalculus or calculus students.

Task 2. At least 80 percent of each group used "delta d" on Task 2. This represents an increase in use from Task 1 for precalculus and calculus students. This contrasts with a decrease in the use of "visual slope" by these same two groups. Precalculus and calculus students used two fewer strategies in Task 2 than in Task 1 while postcalculus students used two more. What accounts for these similarities and differences?

Some precalculus and calculus students who used "visual slope" on Task 1 switched to "delta d" on Task 2. I believe they did this because their graphs for Task 1 were curves which made differences in slope for sections of the curve quite visible. Therefore "visual slope" was a more obvious choice when addressing rate of change in Task 1. But in Task 2 their graphs were linear so that visible change in slope was not as obvious a graphical feature. Furthermore, constant speed, as in Task 2, may have been interpreted as "same number of feet each second" which would account for heavier use of "delta d" by these students.

As in Task 1, some precalculus students used "meeting point" in Task 2. Reasons for use of this strategy in Task 2 are conjectured to be the same as in Task 1.

In Task 2, two postcalculus students used "velocity equation". What would account for this? Notice that "velocity equation" was used only on Task 2. Postcalculus students would most likely have had more experience with relationships between equations and their graphs and

therefore may have been more likely to relate the graph of a line to the constant nature of "change in distance over change in time". This may explain why some of them used "velocity equation" on Task 2 and not on Tasks 1 or 3.

Task 3. At least two-thirds of each group used "delta d" on Task 3. For each group this strategy was used with less frequency on Task 3 than on Task 2 but with greater frequency than on Task 1 for precalculus and calculus students. Two postcalculus students who used "delta d" on Tasks 1 and 2 used either "delta t" or "delta d and delta t" on Task 3.

Students' graphs for Task 3 are less common than their graphs for Tasks 1 and 2. In some sense their graphs in Task 3 were the opposite of their graphs in Task 1. This may have encouraged some students to think in Task 3 about rate of change in terms of change in time relative to change in distance ("delta t") instead of change in distance relative to change in time ("delta d") or to refer to change in both distance and time as in the strategy "delta d and delta t". Similarly less familiar graphs in Task 3 may have been the reason some precalculus and calculus students switched from "delta d" in Task 2 to "delta t" or "delta d and delta t" in Task 3.

The pattern of use of "visual slope" in Task 3 is different from both Tasks 1 and 2. Almost equal percents of calculus and postcalculus groups used "visual slope" in Task 3 while about half that percent of the precalculus group used this strategy. Because calculus and postcalculus students have likely had more experience with relationships between shapes of curves and rate of change, they may have been more likely to use "visual slope" in Task 3 than precalculus students.

As in Tasks 1 and 2, some precalculus students used "meeting point" in Task 3. I suspect their reasons for using it here are the same as on Tasks 1 and 2. Students used this strategy because they knew that rate of change in these tasks had something to do with amount of distance traveled in a certain amount of time and one of the simplest ways to express this is "20 feet in fewer seconds means speeding up and 20 feet in more seconds means slowing down".

Strategies used by each group with Tasks 1, 2, and 3 combined.

Table 25 shows the number and percent of each group using each strategy at least once on Tasks 1, 2, and 3.

Table 25

Number (percent) of each group using each strategy
to explain their graphs in Tasks 1, 2, and 3 combined

<u>Strategy</u>	<u>Precalculus</u>	<u>Calculus</u>	<u>Postcalculus</u>
Delta d	11 (92%)	15 (100%)	10 (100%)
Visual slope	6 (50%)	11 (73%)	8 (80%)
Delta t	4 (33%)	4 (27%)	1 (10%)
Delta d and delta t	2 (17%)	6 (40%)	1 (10%)
Reference to a point	4 (33%)	2 (13%)	1 (10%)
Meeting point	3 (25%)	0 (0%)	0 (0%)
Velocity equation	0 (0%)	0 (0%)	2 (20%)

Over 90 percent of the precalculus students and all of the calculus and postcalculus students used "delta d" at least once in Tasks 1, 2, and 3. Half of the precalculus, almost three-fourths of the calculus, and eighty percent of the postcalculus students used "visual slope" on at least one of the tasks. Notice that as mathematical

background increased so did the percent of students who used each of these two strategies.

Precalculus students were the only ones to use "meeting point" and postcalculus students were the only ones to use "velocity equation". Generally frequency of use of "reference to a point" and "delta t" decreased with more formal mathematics education. Calculus students were more inclined than precalculus and postcalculus students to use "delta d and delta t".

Strategies for explaining tables. Table 26 shows strategies students used to explain how their tables showed slowing down, speeding up, or steady pace and, for each task, the percent of each group using each strategy. For each of the three tasks the strategy most predominantly used by each group was "delta d". At least 90 percent of each group used this strategy on Tasks 1 and 3 and at least 80 percent used it on Task 2. As with graphs, precalculus students were the only ones who used "meeting point" and postcalculus students were the only ones who used "velocity equation".

Task 1. Postcalculus students used two other strategies worthy of note. Two of these students used "reference to a point", a strategy that makes use of the value of the dependent variable, rather than differences in values of the dependent variable, to make decisions about rate of change. One of these postcalculus students used the ratio of the distance value to the time value to address slowing down and speeding up in Task 1.

But in Task 2 when she tried to use the same strategy she saw that this did not show steady pace since the resulting ratios were not constant. She then used " $(\Delta d)/(\Delta t)$ " to make sense of steady

pace in Task 2 and returned to Task 1 to use this new strategy to discuss how her table there showed slowing down and speeding up. The other postcalculus student who used "reference to a point" on Task 1 made frequent use of the value of the dependent variable in other tasks.

Table 26

Percent of each group using each strategy to explain how their tables showed slowing down, speeding up, or steady pace in Tasks 1, 2, and 3

(Pre=Precalculus, Cal=Calculus, Pos=Postcalculus)

	<u>Task 1</u>			<u>Task 2</u>			<u>Task 3</u>		
<u>Strategy</u>	<u>Pre</u>	<u>Cal</u>	<u>Pos</u>	<u>Pre</u>	<u>Cal</u>	<u>Pos</u>	<u>Pre</u>	<u>Cal</u>	<u>Pos</u>
Delta d	92	93	90	83	100	100	92	93	100
(Delta d)/ (delta t)	0	7	20	0	7	10	0	7	10
Reference to a point	17	0	20	0	0	0	0	0	0
Meeting point	8	0	0	17	0	0	8	0	0
Delta d and delta t	0	7	0	0	0	0	8	7	0
Delta t	8	0	0	0	0	0	0	7	0
Velocity equation	0	0	10	0	0	0	0	0	0

Two precalculus students also used "reference to a point" on Task 1. One used the distance value itself as a measure of speed at individual values of time. The other used the ratio of the distance value to corresponding values of time as a measure of speed. Both of these ways of using values of the dependent variable (distance) at individual values of the independent variable (time) make use of relationships between distance and time to make sense of rate of change. They are not the conventional sense, but, as I have stated elsewhere,

these students assumed that distance and time tell us something about speed and that speed is given by a particular ratio of distance to time.

Two postcalculus students used " $(\Delta d)/(\Delta t)$ ". One of these students also used "distance value" and her use of these two strategies was discussed above. The other student divided the change in distance by the change in time over each one second interval, said that he was doing this, and noticed the decreasing pattern of the resulting answers in the first half of the table and the increasing pattern in the second half and concluded that his table showed slowing down on the first half and speeding up on the second half. One calculus student also used this strategy but no precalculus students did.

Task 2. In addition to " Δd ", the only other strategy used by more than one student in any group was "meeting point". Two precalculus students used this strategy. One of these students used this strategy on all three of these tasks with both the graph and the table. The other student did not use this strategy anywhere else in Tasks 1, 2, and 3 but his use of this strategy here was similar to other students' use of this strategy with graphs. He noted that what showed the steady pace of the two people was that they traveled a total of 20 feet in the four seconds before they met and a total of 20 feet in the four seconds after they met. As with graphs, the notion of amount of distance traveled in amount of time helped these students address slowing down, speeding up, and steady pace in Tasks 1, 2, and 3. This points to a resource that could help these students refine their knowledge of rate of change.

Task 3. " Δd " was used by all but one calculus and one precalculus student and was the only strategy used by more than one student in each group. One precalculus student used "meeting point" and

a different precalculus student used "delta d and t". One calculus and one postcalculus student used " $(\Delta d)/(\Delta t)$ " and even then it was in conjunction with "delta d". Another calculus student used both "delta t" and "delta d and t".

Strategies used by each group with Tasks 1, 2, and 3 combined.

Table 27 shows the number and percent of each group using each strategy at least once on Tasks 1, 2, and 3. Major patterns revealed by this table have already been discussed above.

Table 27

Number (percent) of each group using each strategy
to explain their tables in Tasks 1, 2, and 3 combined

<u>Strategy</u>	<u>Precalculus</u>	<u>Calculus</u>	<u>Postcalculus</u>
Delta d	11 (92%)	15 (100%)	10 (100%)
Reference to a point	2 (17%)	0 (0%)	2 (20%)
(Delta d)/ (delta t)	0 (0%)	1 (7%)	2 (20%)
Meeting point	2 (17%)	0 (0%)	0 (0%)
Delta t	1 (8%)	1 (7%)	0 (0%)
Delta d and delta t	1 (8%)	1 (7%)	0 (0%)
Velocity equation	0 (0%)	0 (0%)	1 (10%)

Summary of strategy use in Tasks 1, 2, and 3.

Two patterns of strategy use emerged. One concerns strategies used by all three groups and one concerns strategies used by some groups but not others. All three groups made heavy use of "delta d" in these tasks. With graphs, calculus and postcalculus students also made heavy use of "visual slope" while precalculus students made medium use of that

strategy. These were the two most commonly used strategies by each of the three groups.

Most of these students would have had some experience with rate in middle school and high school as well as everyday experience with speed expressed in terms of miles per hour or feet per second. These experiences would likely have helped students think of speed in Tasks 1, 2, and 3 in terms of distance traveled each second. That would make " Δd " a candidate as a heavily used strategy.

These students would also have had some school experience with linear graphs and their slopes. In addition the calculus and postcalculus students would have had more experiences with graphs, their slopes and shapes, and connections to rate of change. That would help explain why precalculus students to a lesser extent and calculus and postcalculus students to a greater extent made use of "visual slope" as a way to address rate of change.

Some strategies were used by some groups but not by others. With both graphs and tables the precalculus group was the only group to use "meeting point" and the postcalculus students were the only ones to use "velocity equation". Generally precalculus students made more use of "reference to a point" than the other two groups although all three groups used that strategy with some frequency.

Generally use of " Δt " declined with more formal mathematics education. However, calculus students were more likely than precalculus and postcalculus students to use " Δd and Δt ". Students with more formal mathematics education were more inclined to use " $(\Delta d)/(\Delta t)$ ". These patterns of strategy use are summarized in Table

Table 28

Patterns of strategy use by group
 (Strategies common to all three groups are in **bold**.)
 (Strategies common to two groups are in *italics*.)

<u>Group</u>	<u>Main strategies used</u>
Precalculus	Meeting point, Reference to a point, Delta d, Visual slope , and Delta t
Calculus	Delta d , Visual slope , Delta t, and Delta d and t
Postcalculus	Velocity equation, Delta d , Visual slope , Reference to a point, (Delta d)/(delta t)

Comparing strategies of the three groups -

"thrown ball", "biomass", and "circles" tasks

In this section I examine similarities and differences among the three groups of students in the frequency of strategies used to identify intervals of the independent variable for which the dependent variable changed most and least and to find the rate of change at selected points in Tasks 4, 5, and 6. I look first at frequencies of strategies used by each group for each task and then I note patterns of strategy use for each group across the three tasks.

Identifying intervals of least and most change.

Students used five strategies to identify intervals of the independent variable for which the dependent variable changed most and least. I examine strategies used by each group on each of Tasks 4, 5, and 6. Then I look at strategy use by group across the three tasks.

Task 4. Table 29 shows the number and percent of each group using each strategy to find the intervals of time in which the height of the ball changed the most and the least.

Table 29

Number (percent) of each group using each strategy to identify when the height of the ball changed the most and the least

<u>Strategy</u>	<u>Precalculus</u>	<u>Calculus</u>	<u>Postcalculus</u>
Delta d	12 (100%)	15 (100%)	10 (100%)
Visual slope	0 (0%)	4 (27%)	2 (20%)
Knowledge of motion	2 (17%)	5 (33%)	3 (30%)
Delta t	0 (0%)	0 (0%)	1 (10%)
Delta d and delta t	0 (0%)	0 (0%)	1 (10%)

The most popular strategy by far for each group was "delta d". Every student in the study referred to change in the height of the ball over intervals of time. Calculus and postcalculus students used "visual slope" and all three groups used "knowledge of motion".

Task 5. Table 30 shows the number and percent of each group using each strategy to find intervals of time during which the cell population grew the most and least.

As on Task 4, change in the dependent variable from one time value to the next, what is called "delta p" here, was the most popular strategy for each of the three groups with at least 70 percent of each group using it. "Visual slope" was also frequently used, especially by calculus and postcalculus students. Generally students with more formal mathematics education made more use of "visual slope".

Table 30

Number (percent) of each group using each strategy to identify
when the cell population grew the most and least

<u>Strategy</u>	<u>Precalculus</u>	<u>Calculus</u>	<u>Postcalculus</u>
Delta p	9 (75%)	11 (73%)	7 (70%)
Visual slope	3 (25%)	10 (67%)	8 (80%)
Knowledge of pop. growth	3 (25%)	2 (13%)	0 (0%)
Delta p and delta t	0 (0%)	1 (7%)	1 (10%)

Task 6. Because I failed to ask every student how they knew which intervals of the radius yielded the most and least change in the area, a fair picture of which groups used which strategies and with what frequency is not possible here. But students did use the same basic three strategies as in Tasks 4 and 5 - "delta A", "visual slope", and "knowledge of the situation". From comments made by about half of the students, I know that "delta A" and "visual slope" were used by all three groups. At least one precalculus and one calculus student used "knowledge of the situation" but none of the postcalculus students referred to this. However, a couple of postcalculus students used the equation relating area of a circle to its radius in this and other parts of Task 6. Table 31 summarizes strategy use in this part of Task 6.

Comparison and contrast of responses across Tasks 4, 5, and 6.

When comparing strategies used by each group to identify intervals of the independent variable for which the dependent variable changed the most and least, we see that "delta d" was used by all three groups on all three tasks and it was the most frequently used strategy by each group on each task. "Visual slope" was also used by all three groups on

at least two of the three tasks but calculus and postcalculus students used it more than precalculus students and generally use of "visual slope" increased with more mathematics education.

Table 31

Strategies used to identify intervals of the radius
for which area of circles grew most and least

<u>Strategy</u>	<u>Precalculus</u>	<u>Calculus</u>	<u>Postcalculus</u>
Delta A	Used by some	Used by some	Used by some
Visual slope	Used by some	Used by some	Used by some
Knowledge of situation	Used by one	Used by one	Not used
Equation	Not used	Not used	Used by two

"Knowledge of situation" was used by all three groups on at least two tasks but use of this strategy by each group differed among the tasks. On Task 4 calculus and postcalculus students were more likely to use it than precalculus, on Task 5 precalculus students were more likely to use it than calculus, and on Tasks 5 and 6 no postcalculus student used this strategy. What would explain these patterns of strategy use?

Task 4 was probably a more familiar task than Task 5. That would help explain why "knowledge of motion" was a more popular strategy on Task 4 than "knowledge of growth of populations" was on Task 5. In addition, for students at these levels of mathematics education, feet per second is likely a more familiar way of thinking about rate of change than number of cells per hour. That might be why fewer students used "delta p" on Task 5 than used "delta d" on Task 4. With decreased use of "delta P" and "knowledge of growth of populations" on Task 5, students turned to other strategies, like "visual slope". That would

account for the increased use of "visual slope" on Task 5 compared to Task 4.

Finding rate of change at a point.

Twelve different strategies were used for finding the rate of change of the dependent variable at specific values of the independent variable in Tasks 4, 5, and 6. Below I examine strategies used by each group on each task and then I look at strategy use by group across the three tasks.

Task 4. In Chapter 5 I identified strategies used to find the speed of the ball at each time value $t=1, 2, 3$, and 4 for all three groups combined. I did this first for strategies used with the graph and then for strategies used with the table of values. Here I look at strategies used by each group but with frequencies for the time values combined. I look first at strategies used with the graph and then at strategies used with the table.

Strategies used with the graph. Table 32 shows the number of times each strategy was used by each group to find the speed of the ball at $t=1, 2, 3$, and 4 when using the graph.

For each group "analysis of moving objects" was the most frequently used strategy for finding the speed of the ball at $t=1, 2, 3$, and 4. The situation in Task 4 was no doubt familiar enough that students could rely on knowledge of the movement of the ball to address its speed at various time values.

There were, however, differences in the next two most prevalent strategies used by each group. The precalculus group used average speed over the previous second or quarter second. Both the calculus and postcalculus groups referred to slope of tangent line and calculus

students also used average speed over the previous quarter second while postcalculus students used average speed over the half second around the point.

Table 32

Number of incidences of each strategy by group in finding the speed of the ball at $t=1, 2, 3$, and 4 in Task 4 using the graph

<u>Strategy</u>	<u>Precalc</u>	<u>Calculus</u>	<u>Postcalc</u>
Analysis of moving objects	18 (38%)	26 (48%)	15 (44%)
Avg. speed over previous quarter second	17 (35%)	10 (19%)	4 (12%)
Reference to slope	0 (0%)	14 (26%)	12 (35%)
Avg. speed over the previous second	12 (25%)	6 (11%)	5 (15%)
Avg. speed since $t=0$ or $t=2$	8 (17%)	9 (17%)	0 (0%)
Avg. speed over half sec. around point	0 (0%)	4 (7%)	8 (24%)
Reference to a point	7 (15%)	3 (6%)	2 (6%)
Desire smaller interval	0 (0%)	3 (6%)	1 (3%)

Percents were computed on the basis of the total number of opportunities to answer each item. Precalculus students had 48 opportunities (12 students * 4 time values); calculus students had 54 opportunities (15 students with three students' responses missing for $t=3$ and $t=4$); postcalculus students had 34 opportunities (10 students with three students' responses missing for $t=3$ and $t=4$).

None of the precalculus students referred to slope, a not too surprising result given that they, unlike the calculus and precalculus students, had not yet received instruction on connections between slope of tangent line and rate of change at a point. However, I found interesting the heavy use by each group of average speed over intervals. Precalculus students used medium and short intervals before the point, calculus students used the shortest available intervals before the

point, and postcalculus students used shortest available intervals "straddling" the point.

Postcalculus students' use of "straddle" intervals is more consistent with the definition of instantaneous rate of change for continuous functions as discussed in Chapter 1 and calculus students' use of the smallest one-sided interval before the point would be expected to yield a more accurate answer than using a larger one-sided interval as some precalculus students did.

Generally the more formal mathematics education the group had, the less it used "reference to a point", "average speed since $t=0$ or $t=2$ ", "average speed over the previous second", and "average speed over the previous quarter second" and the more it used "average speed over the half second around the point" and "reference to slope". Average rate of change over the half second around the point and slope of tangent line would be expected to give the most accurate answers for speed at points.

No precalculus students used the slope of the tangent line or average speed over the half second interval "straddling" the point.

Figure 11 summarizes the three most frequently used strategies of each group.

<u>Precalculus</u>	<u>Calculus</u>	<u>Postcalculus</u>
Analysis of moving objects	Analysis of moving objects	Analysis of moving objects
Avg. speed over prev. 1/4 sec.	Reference to slope	Reference to slope
Avg. speed over prev. second	Avg. speed over prev. 1/4 sec.	Avg. spd. over 1/2 sec. around pt.

Figure 11 - Most frequently used strategies of each group when using the graph to find speed at $t=1, 2, 3$, and 4 in Task 4

Strategies used with the table. Table 33 shows the number of times each strategy was used by each group to find the speed of the ball at $t = 1, 2, 3$, and 4 when using the table of values.

Table 33

Number of incidences of each strategy by group in finding the speed of the ball at $t=1, 2, 3$, and 4 in Task 4 using the table of values

<u>Strategy</u>	<u>Precalc</u>	<u>Calculus</u>	<u>Postcalc</u>
Avg. speed over half second around point	0 (0%)	17 (34%)	20 (61%)
Avg. speed over previous quarter second	16 (33%)	16 (32%)	3 (9%)
Analysis of moving objects	6 (13%)	8 (16%)	8 (24%)
Reference to slope	0 (0%)	14 (28%)	2 (6%)
Avg. speed over the previous second	11 (23%)	3 (6%)	1 (3%)
Avg. speed from $t=0$ or $t=2$	9 (19%)	4 (8%)	0 (0%)
Reference to a point	3 (6%)	5 (10%)	2 (6%)
Desire smaller interval	0 (0%)	2 (4%)	2 (6%)

Percents were computed on the basis of the total number of opportunities to answer each item. Precalculus students has 48 opportunities (12 students * 4 time values); calculus students had 50 opportunities (15 students with five students' responses missing for $t=3$ and $t=4$); postcalculus students had 33 opportunities (10 students with three students' responses missing for $t=3$ and $t=4$).

There were differences among the three groups. Precalculus students used average speed over the previous second and quarter second more than any other strategies. Calculus students used average speed over the previous quarter second and average speed over the half second interval around the time value. Postcalculus students overwhelmingly used average speed over the half second interval around the time value.

Precalculus students also used average speed from $t = 0$ or $t = 2$. Calculus students made relatively heavy use of slope of tangent line and postcalculus students used analysis of moving objects. With more mathematics education, students made less use of average speed since $t = 0$ or $t = 2$, average speed over the previous second and quarter second and made more use of average speed over the half second "straddling" the point and analysis of moving objects. Calculus students made more use of slope of tangent line than postcalculus students. Precalculus students did not use at all slope of tangent line or average speed over the half second "straddling" the point.

Two patterns emerge from this data. One is that the more mathematics education students had, the more likely they were to use smaller intervals before the point or intervals straddling the point. The other is that none of the precalculus students used slope of tangent line or intervals straddling the point but both calculus and postcalculus students used these strategies, with calculus students using slope more than postcalculus students, and postcalculus students using "straddle" intervals more than calculus students.

Figure 12 summarizes the most commonly used strategies of each group.

<u>Precalculus</u>	<u>Calculus</u>	<u>Postcalculus</u>
Avg. speed over prev. 1/4 sec.	Avg. speed over 1/2 sec. around point	Avg. speed over 1/2 sec. around point
Avg. speed over prev. sec.	Avg. speed over prev. 1/4 sec.	Analysis of moving objects
Avg. speed since t=0 or t=2	Reference to slope	

Figure 12 - Most frequently used strategies of each group when using the table to find speed at t=1, 2, 3, and 4 in Task 4

Comparison and contrast of strategy use between graph and table on Task 4. All three groups used analysis of moving objects and calculus and postcalculus students used slope with graphs. In contrast, with tables, only the postcalculus students made frequent use of analysis of moving objects and only calculus students frequently used slope. With both graphs and tables, precalculus students used average speed over the previous second and quarter second, calculus students used average speed over the previous quarter second, and postcalculus students used average speed over the half second around the point, what I have called "straddle" intervals. One significant change was calculus students' increased use of "straddle" intervals with tables.

Reasons for decreased use of analysis of moving objects and slope and increased use of average speed over intervals with tables were noted in Chapter 5. However two differences in strategy use between graphs and tables will be discussed here. One is the continued use of analysis of moving objects by the postcalculus students and slope by calculus students. The other is the increased use of straddle intervals by postcalculus and calculus students.

Why would postcalculus students make frequent use of analysis of moving objects with tables of values? One explanation is that these students most likely had the most experience with this particular situation, having encountered it in other mathematics and science classes. Another explanation is that since postcalculus students generally found reasonable numerical estimates for the speed of the ball when using the graph, they turned, when using the table of values, to quick and easy ways to verify their earlier answers. Analysis of moving objects would be such a strategy.

Why would calculus students continue to use slope with tables? Since calculus students had more recently encountered instruction connecting rate of change to slope of tangent line, that strategy was probably more attractive to them for addressing rate of change at a point in all situations, tables as well as graphs. When using the table of values, most of these students said they wanted to find the slope of the tangent line and that they could get a pretty good estimate of that slope using the two points in the table one-quarter of a second on either side of the point of tangency. Thus the table of values made their approximation of slope of tangent line easier than with the graph.

When these calculus students approximated the slope of the tangent line using this technique I did not classify it as average speed over the half second around the point even though numerical it yielded the same answer and algebraically it used the same formula. I classified this strategy as reference to slope, which in these tasks meant slope of tangent line, since these students made it clear that their intent was to find a reasonable estimate of the slope of the tangent line.

Postcalculus students were further removed from slope of tangent line as a primary way of thinking about rate of change at a point and most of them had since learned other techniques based on Newton's quotient, techniques like divided differences and the Newton and secant methods, for addressing rate of change in situations involving a table of values. This may be the reason they made less use of slope and made more use of "straddle" intervals.

But calculus students also made heavy use of "straddle" intervals with the table of values. They made less use of analysis of moving objects with tables than they did with graphs. This means that other strategies had to be used with greater frequency. Generally with tables calculus students made less use of average speed over one and two second intervals before the point and more use of average speed over one-quarter intervals before and half second intervals straddling the point. The table of values may have encouraged more use of average rate of change and the availability of more precise values of the dependent variable closer to the point would have encouraged students to use smaller intervals. It may also have made consideration of information after the point more likely.

Task 5. Here I look first at strategies each group used with the graph, then with the table, and then with the enlarged graph and more detailed table. I follow that with an analysis of each group's reasons for specifying which of their answers was best and how they would get a better answer.

Strategies used with the graph. Table 34 shows the number and percent of each group using each strategy to find the rate of growth of the cell population at 4 am when using the graph.

The most frequently used strategy for precalculus students was average growth over the hour before 4 am while postcalculus and calculus students used slope of line tangent to the graph at $t = 4$ and average growth over the two hour interval straddling $t = 4$. Precalculus and calculus students also made reference to the number of cells at 4 am sometimes dividing it by four, the time value.

As in Task 4 precalculus students used only intervals before the point while calculus and postcalculus students tended to use intervals that "straddled" the point. Also no precalculus student used a "straddle" interval or slope of tangent line. Figure 13 summarizes these trends.

Table 34

Number (percent) of each group using each strategy to find the rate of growth of the cell population at $t=4$ am using the graph

<u>Strategy</u>	<u>Precalc</u>	<u>Calculus</u>	<u>Postcalc</u>
Avg. growth over previous hour	7 (58%)	2 (13%)	2 (20%)
Avg. growth over 2 hours around $t=4$	0 (0%)	6 (40%)	4 (40%)
Reference to slope	0 (0%)	6 (40%)	4 (40%)
Reference to a point	3 (25%)	3 (20%)	0 (0%)
Avg. growth since $t=0$	0 (0%)	1 (7%)	2 (20%)
"Not quite doubling"	0 (0%)	1 (7%)	0 (0%)
"It's multiplied by two-thirds."	1 (8%)	0 (0%)	0 (0%)
Desire smaller interval near $t=4$	0 (0%)	1 (7%)	0 (0%)

<u>Precalculus</u>	<u>Calculus</u>	<u>Postcalculus</u>
Avg. growth over previous hour	Avg. growth over 2 hrs. around pt.	Avg. growth over 2 hrs. near pt.
Reference to a point	Reference to slope Reference to a point	Reference to slope

Figure 13 - Most frequently used strategies of each group to find rate of growth of cell population at $t=4$ in Task 5 when using graph

Strategies used with the table. Table 35 shows the number and percent of each group using each strategy to find the rate of growth of the cell population at 4 am when using the table of values.

Table 35

Number (percent) of each group using each strategy to find the rate of growth of the cell population at 4 am using the table

<u>Strategy</u>	<u>Precalc</u>	<u>Calculus</u>	<u>Postcalc</u>
Avg. growth over the 2 hours around $t=4$	1 (8%)	9 (60%)	7 (70%)
Avg. growth over previous hour	7 (58%)	4 (27%)	1 (10%)
Reference to a point	2 (17%)	2 (13%)	0 (0%)
Avg. growth since $t=0$	0 (0%)	0 (0%)	2 (20%)
Reference to slope	0 (0%)	1 (7%)	0 (0%)
Multiplied by two-thirds	1 (8%)	0 (0%)	0 (0%)
Increased by one-third	1 (8%)	0 (0%)	0 (0%)

Precalculus students used average growth over the hour previous to $t = 4$ while calculus and postcalculus students used average growth over the two hour period from $t = 3$ to $t = 5$. The second most frequently used strategy for calculus students was also average growth over the hour previous to $t = 4$. Some precalculus and calculus students

referenced the number of cells at $t = 4$ and some postcalculus students used the total growth since $t = 0$. Generally with more formal mathematics education, students made less use of average growth over the shortest available interval before the point and more use of average growth over the shortest available interval straddling the point.

In Task 4 and with the graph here in Task 5 no precalculus student used slope of tangent line or "straddle" intervals. But with the table of values in Task 5, one precalculus student used a "straddle" interval.

It may be interesting to see what that student said with the graph and the table here in Task 5:

With the graph - Average growth over the previous hour -

"There it's changing in about 20 cells per hour. (And why do you say that?) Because that's about how much it changed between 3 and 4." KH - Precalculus

With the table of values - Average growth over the two hours from 3 to 5 -

"Well, if I have the exact numbers there [pointing to the section of the table near 4], then I can. [Uses calculator] It's changing about 23.9 cells per hour. (And again you did it by looking at what happens between 3 and 4?) Yeah. Or course, I could do the change from 3 to 4 and the change from 4 to 5 and then average them. (Oh. That's a good idea! Would you like to do that?) I would. [Calculates] Then it comes out to 35.95. (And what's the units on that?) Cells per hour." KH - Precalculus

What caused KH to switch from average growth over the previous hour to average growth over the two hour interval straddling $t = 4$? It is not clear here what prompted her suggestion that this may be a new way of thinking about the problem, although it is clear that once she made the suggestion I strongly encouraged her to continue. Perhaps having the table of values in front of her made her think that the change after $t = 4$ gave information about change at $t = 4$. At any rate here is the

first instance in these tasks of a precalculus student using "straddle" intervals to address rate of change at a point. There are others to come in later parts of Task 5 and in Task 6.

Figure 14 summarizes the trends in strategy use by the three groups in Task 5 with the table of values.

<u>Precalculus</u>	<u>Calculus</u>	<u>Postcalculus</u>
Avg. growth over prev. hour	Avg. growth over 2 hrs. around $t=4$	Avg. growth over 2 2 hrs. near $t=4$
Reference to a point	Avg. growth over prev. hour	Avg. growth since $t=0$
First use of a straddle interval	Reference to a point	

Figure 14 - Most frequently used strategies of each group to find rate of growth of cell population at $t=4$ in Task 5 when using table

Comparison and contrast of strategy use between graph and table on task 5. Precalculus students tended to use the same strategies with both graph and table - average growth over the previous hour and reference to a point, specifically the number of cells at $t = 4$. However, calculus and postcalculus students made frequent use of slope of tangent line when working with the graph but only one of these students used slope with the table. More calculus and postcalculus students used average growth over the two hours straddling $t=4$ with the table than with the graph and twice as many calculus students used average growth over the previous hour with the table than with the graph. Both precalculus and calculus students used reference to a point with the graph and the table of values.

Reasons students might make less use of slope and more use of straddle intervals have been discussed, but why would more calculus

students use average growth over the previous hour with the table than with the graph? Two of the calculus students who used average growth over the previous hour with the table used this same strategy with the graph. One of them offered in both cases that the rate of growth at $t = 4$ would be faster than the average growth over the previous hour but that the latter was a reasonable estimate of the growth rate at $t = 4$. So although this student settled on use of average growth over the previous hour to get her answer, it was clear that she recognized her answer was not an exact answer. It was an estimate that was a bit on the low side. This shows knowledge of rate of change at $t = 4$ that is more specific than simply using the average growth over the previous hour.

One of the calculus students who used average growth over the previous hour with the table but not with the graph noted with the graph that she would compute the average rate of change over a much smaller interval before $t = 4$ if she had the number of cells for those time values. When she got to the table she decided to use the shortest available interval before $t = 4$, the interval from $t = 3$ to $t = 4$. Again this shows knowledge of rate of change at $t = 4$ that is more sophisticated than use of the strategy "average growth over the previous hour" would indicate.

The other student who used this strategy with the table used the number of cells at $t = 4$ divided by 4 to get the rate of growth at $t = 4$ when using the graph. For this student, moving from "reference to a point" to "average growth over the previous hour" represented a move toward more conventional mathematical knowledge. With the enlarged graph and more detailed table this student used a shorter interval

before $t = 4$ and an interval straddling $t = 4$. According to a conventional understanding of rate of change at a point, this student moved toward strategies that yield more accurate answers.

Strategies used with the enlarged graph and more detailed table.

Table 36 shows the number and percent of each group using each strategy to find the rate of growth of the cell population at 4 am when using the enlarged graph and the more detailed table of values.

Table 36

Number (percent) of each group using each strategy to find
the rate of growth of the cell population at 4 am
using the more detailed graph and table

<u>Strategy</u>	<u>Precalc</u>	<u>Calculus</u>	<u>Postcalc</u>
Avg. growth over 1/2 hour around $t=4$	1 (8%)	11 (73%)	9 (90%)
Avg. growth over previous quarter hour	6 (50%)	4 (27%)	1 (10%)
Reference to slope	0 (0%)	2 (13%)	2 (20%)
Reference to a point	1 (8%)	1 (7%)	0 (0%)
Avg. growth over 2 hours around $t=4$	0 (0%)	0 (0%)	1 (10%)
12 percent increase	1 (8%)	0 (0%)	0 (0%)

The most frequently used strategy for the precalculus students was average growth over the previous quarter hour, the shortest available interval before $t = 4$. However, calculus and postcalculus students made heaviest use of average growth over the half hour around $t = 4$, the shortest available interval straddling $t = 4$. Generally use of the former strategy decreased across age and experience while use of the latter increased. The precalculus student who used average growth over a straddle interval here was the precalculus student who used a straddle

interval when working with the original table and was the first precalculus student to use a straddle interval in this study.

A few calculus and postcalculus students said they wanted to find the slope of the line tangent to the graph at $t = 4$ and most of them then estimated that slope using the slope of the secant line through the two points at $t = 3.75$ and $t = 4.25$. I classified this strategy as "reference to slope" rather than "average growth over the half hour around $t = 4$ " since it was clear that students were trying to find the slope of the tangent line rather than the average rate of change over an interval even though the numerical answer obtained and the algebraic formula used were the same. One precalculus and one calculus student used number of cells at $t = 4$ divided by four as their answers for the rate of growth at $t = 4$.

Comparison and contrast of strategy use between original graph and table and more detailed graph and table on Task 5. Across these three parts of Task 5 each group exhibited patterns in their strategy use. Precalculus students tended to use the shortest available intervals before $t=4$, the one hour interval before $t = 4$ with the original graph and table and the one-quarter hour interval before $t = 4$ with the detailed graph and table. They also made less use of reference to a point as they moved through Task 5.

Calculus students used the shortest available interval straddling $t=4$, the two hour interval with the first graph and table and the half hour interval with the second graph and table. Like the precalculus students, they made some use of the smallest intervals before $t=4$ with both sets of graphs and tables and they made less use of reference to a point as they moved through Task 5.

Postcalculus students, like the calculus students, used the shortest available intervals straddling $t = 4$ with a few using the smallest intervals before. Both calculus and postcalculus students used slope of tangent line with the graphs but less so with the tables. None of the precalculus, one of the calculus students, and two postcalculus students used average growth since $t = 0$ with the original graph and table of values but no student used this strategy with the enlarged graph and more detailed table of values. Figure 15 summarizes these trends.

<u>Precalculus</u>	<u>Calculus</u>	<u>Postcalculus</u>
Consistently used shortest interval before $t=4$	Consistently used shortest available straddle interval	Consistently used shortest available straddle interval
Reference to a point decreased across Task 5	Some used shortest available interval before $t=4$	Few used shortest available interval before $t=4$
	Reference to a point decreased across task 5	Reference to slope with graphs but not tables
	Reference to slope with graphs but not tables	Few used average growth since $t=0$

Figure 15 - Trends in the use of strategies by each group across Task 5

Choosing which answer was best in Task 5. Twenty-nine of the 37 students had three different answers for the rate of growth at $t = 4$ based on their work with the original graph, original table of values, and detailed graph and table. Five of the precalculus and three of the calculus students did not have at least two different numerical answers and so I could not ask them which of their answers they preferred and

why. But I did ask the other 29 students these questions. Table 37 shows the responses of each group.

A majority in each of the three groups responded that their last answer was best since it was based on intervals that were smaller. A few calculus and postcalculus students noted that the points they used to estimate the slope of the tangent line with the more detailed graph and table were better answers since they were closer to being on the tangent line.

Table 37

Reasons students gave for why they liked their last answer best and the number (percent) of each group who gave each reason

<u>Reason</u>	<u>Precalc</u>	<u>Calculus</u>	<u>Postcalc</u>
Interval was smaller	7 (58%)	10 (67%)	8 (80%)
Points closer to tangent line	0 (0%)	2 (13%)	2 (20%)
Used points both sides of $t=4$	0 (0%)	1 (7%)	0 (0%)

Percents were computed on the basis of the total number of students in each group even though not all of those students were asked this question.

Getting a better answer in task 5. Twenty-seven of the 37 subjects answered the question "How would you get a better answer?". This included six of the 12 precalculus, 11 of the 15 calculus, and all 10 of the postcalculus students. Table 38 shows the responses of each group to this question.

Of the students asked this question, the majority of each group responded that using smaller intervals would yield a better answer. Calculus and postcalculus students also gave two other answers. They said they would take the limit of the average rate of change in the number of cells over intervals bordering or straddling $t=4$ as the length

of those intervals went to zero. They also said that if they had an equation for the curve they would find the derivative equation and evaluate it at $t = 4$. A few calculus and postcalculus students said they would try to get a more precise estimate of the slope of the tangent line and if they could get the exact value of that slope, then they would have an exact answer for the rate of change in the number of cells at $t = 4$.

Table 38

Responses to the question "How would you get a better answer?" and the number (percent) of each group giving each response.

<u>Responses</u>	<u>Precalc</u>	<u>Calculus</u>	<u>Postcalc</u>
Use smaller intervals	5 (42%)	11 (73%)	9 (90%)
Limit of Avg. ROC over smaller and smaller intervals	0 (0%)	4 (27%)	6 (60%)
Evaluate derivative equation at $t=4$	0 (0%)	4 (27%)	6 (60%)
Slope of tangent line at $t=4$	0 (0%)	2 (13%)	1 (10%)
Integrate from $t=0$ to $t=4$	0 (0%)	0 (0%)	1 (10%)
Find an equation for the change	1 (8%)	0 (0%)	0 (0%)
Evaluate equation for curve at $t=4$	0 (0%)	2 (13%)	0 (0%)
Use time after $t=4$ also	0 (0%)	1 (7%)	0 (0%)

Percents were computed on the basis of the total number of students in each group even though not all of those students were asked this question.

One calculus student thought he could get a better answer by using intervals after $t = 4$ as well as intervals before $t = 4$. In his previous answers in Task 5 he had used only intervals before $t = 4$. Now he thought about combining information about the growth of the population both before and after $t = 4$ to get a better answer for the

rate of change in the number of cells at $t = 4$. This represented an advancement in this student's understanding of rate of change.

Task 6. Here I look at strategies used by each group to find the rate of change of the area when the radius was three. First I examine strategies students used with the graph and then with the table of values.

Strategies used with the graph. Table 39 shows the strategies students used with the graph to find the rate of change of the area when the radius was three and the number and percent of each group that used each strategy.

Table 39

Number (percent) of each group using each strategy when using the graph to find the rate of change of the area when the radius was three

<u>Strategy</u>	<u>Precalc</u>	<u>Calculus</u>	<u>Postcalc</u>
Reference to slope	0 (0%)	6 (55%)	2 (29%)
Avg. growth from $r=2.75$ to $r=3$	5 (42%)	1 (9%)	1 (14%)
Avg. growth from $r=2$ to $r=4$	1 (8%)	3 (27%)	2 (29%)
Avg. growth since $r=0$	4 (33%)	3 (27%)	0 (0%)
Avg. growth $r=2.75$ to $r=3.25$	0 (0%)	2 (18%)	2 (29%)
Reference to a point	0 (0%)	0 (0%)	1 (14%)
	n1 = 12	n2 = 11	n3 = 7

Precalculus students used the average rate of change in the area over the one-quarter foot interval before $r = 3$ and the interval from $r = 0$ to $r = 3$ feet with greatest frequency. Calculus students also used average rate of change in area over the interval from $r = 0$ to $r = 3$ but they also used average rate of change in area over the interval from $r = 2$ to $r = 4$ with equal frequency and slope of tangent line with greater

frequency. Postcalculus students used three strategies with equal frequency - slope of tangent line and average rate of change of the area over the intervals from $r = 2$ to $r = 4$ and from $r = 2.75$ to $r = 3.25$.

Precalculus students used primarily intervals before $r = 3$, calculus used some intervals before $r = 3$ and some intervals straddling $r = 3$, and postcalculus students used intervals straddling $r = 3$. Calculus students made more use of slope than did postcalculus students while precalculus students did not use slope at all. The precalculus student who used the straddle interval from $r = 2$ to $r = 4$ here was the only precalculus student to use straddle intervals thus far in these tasks.

Strategies used with the table. Table 40 shows the strategies used with the table of values to find the rate of change of the area of the circles when the radius was three and the number and percent of each group using each strategy.

The most frequently used strategy for precalculus students was the same as it was with the graph - average rate of change of the area from $r = 2.75$ to $r = 3$. Two precalculus students used average rate of change from $r = 0$ to $r = 3$ and two used average rate of change from $r = 2.75$ to $r = 3.25$. One of these latter students was the same student who used the straddle interval from $r = 2$ to $r = 4$ with the graph and used two different straddle intervals in Task 5. She was the only precalculus student who before now had used a straddle interval in these tasks. She was now joined by another precalculus student.

Table 40

Number (percent) of each group using each strategy when using the table to find the rate of change of the area when the radius was three

<u>Strategy</u>	<u>Precalc</u>	<u>Calculus</u>	<u>Postcalc</u>
Avg. growth $r=2.75$ to $r=3.25$	2 (17%)	8 (73%)	5 (71%)
Avg. growth $r=2.75$ to $r=3$	6 (50%)	0 (0%)	0 (0%)
Avg. growth since $r=0$	2 (17%)	2 (18%)	0 (0%)
Reference to a point	0 (0%)	0 (0%)	2 (29%)
Reference to slope	0 (0%)	2 (18%)	0 (0%)
Avg. growth $r=2$ to $r=4$	0 (0%)	1 (9%)	0 (0%)

For calculus and postcalculus students the most frequently used strategy was average rate of change in the area from $r = 2.75$ to $r = 3.25$. Two calculus students used average rate of change from $r = 0$ and two used slope. Two postcalculus students referenced the point at $r = 2$ by dividing the area at $r = 3$ by three.

So precalculus students used primarily average rate of change in the area over the smallest available intervals before $r = 3$ while calculus and postcalculus students used average rate of change in the area over the smallest available intervals straddling $r = 3$.

Comparison and contrast of strategy use between graph and table in Task 6. The most frequently used strategy for the precalculus students with both the graph and the table of values was average rate of change from $r = 2.75$ to $r = 3$, the smallest available interval before the point. Primarily precalculus students used the same strategies with both graph and table.

However, the distribution of strategy use for calculus and postcalculus students differed between the graph and table. With the

graph, calculus students made predominant use of slope with some use of average rate of change over the intervals from $r = 0$ to $r = 3$ and $r = 2$ to $r = 4$. But with the table, calculus students primarily used average rate of change from $r = 2.75$ to $r = 3.25$, the smallest available straddle interval. Strategy use for the postcalculus group with the graph was spread over five strategies but with the table of values postcalculus students settled primarily on one strategy, average rate of change over the smallest available straddle interval.

Comparison and contrast of responses to Tasks 4, 5, and 6. Two sets of patterns of strategy use in these three tasks are discussed. One concerns patterns of consistent strategy use by each group in the three tasks and the other concerns patterns of decreasing or increasing strategy use for each group across the three tasks.

Patterns of consistent strategy use by each group in the three tasks. One important pattern was the consistent use of slope by calculus and postcalculus students when working with graphs on all three tasks. Specifically this meant that calculus and postcalculus students used slope of tangent line to address rate of change at points. No precalculus student used slope to address rate of change at a point in Tasks 4, 5, and 6.

Another pattern of consistency across the three tasks was the use of average rate of change of the dependent variable over intervals of the independent variable of varying length and type by all three groups of students with both graphs and tables. Precalculus students tended to use medium and short intervals before the point, calculus students used short intervals before the point and intervals straddling the point, and

postcalculus students relied primarily on intervals straddling the point.

Finally, in these three tasks, students in each group persistently used two other strategies. One was the average rate of change in the dependent variable over intervals beginning with the first value of the independent variable and ending at the point. The other was the number obtained by dividing the value of the dependent variable at the point by the value of the independent variable. While precalculus and calculus students were the primary users of these two strategies, the postcalculus group also used both of these strategies.

Figure 16 summarizes these patterns of consistency for each group across the three tasks.

<u>Precalculus</u>	<u>Calculus</u>	<u>Postcalculus</u>
Consistently used medium and short intervals before the point	Consistently used short intervals before the point and intervals straddling the point	Consistently used intervals that straddled the point
	Consistently used slope of tangent line with graphs	Consistently used slope of tangent line with graphs
Persistent use of total change and reference to a point	Persistent use of total change and reference to a point	Some use of total change and reference to a point

Figure 16 - Patterns of consistent use of strategies by group across Tasks 4, 5, and 6

Patterns of changes in strategy use for each group across the three tasks. As we look at strategy use for each group across the three tasks, a couple patterns of decreasing and increasing strategy use emerge. One is that precalculus students made less use of medium

intervals before the point and more use of shorter intervals before the point, calculus students made less use of short intervals before the point and more use of intervals straddling the point, and postcalculus students made more use of the shortest available intervals straddling the point. By the time they got to Task 6 two precalculus students had begun to use straddle intervals and more calculus students had turned to straddle intervals.

Another pattern was decreased use of "reference to a point" which usually meant reference to the value of the dependent variable at a point or the ratio of the value of the dependent variable to the value of the independent variable at a point. This was especially true of precalculus and calculus students although in Task 6 two postcalculus students were still using this strategy.

Figure 17 summarizes these patterns of decreased and increased strategy use by each of the three groups across the three tasks.

<u>Precalculus</u>	<u>Calculus</u>	<u>Postcalculus</u>
Less use of medium and more use of short intervals before point	Less use of short intervals before point and more use straddle intervals	More use of shortest straddle intervals
Decreased use of reference to a specific point	Decreased use of reference to a specific point	
Began to use straddle intervals		

Figure 17 - Patterns of decreased and increased strategy use
by each group across Tasks 3, 4, and 6

Summary of strategy use in Tasks 4, 5, and 6.

As in Tasks 1, 2, and 3 all three groups made heavy use of changes over intervals. When identifying intervals of the independent variable for which the dependent variable changed the most and least, all three groups made heavy use of "delta d" and when finding the rate of change of the dependent variable at individual values of the independent variable, all three groups predominantly used average rate of change in the dependent variable over intervals of the independent variable of varying types and sizes.

Similarly, as in Tasks 1, 2, and 3, calculus and postcalculus students used slope, especially when working with graphs. But unlike Tasks 1, 2, and 3, precalculus students used slope in limited numbers when identifying intervals of least and most change and not at all when finding rate of change at points. Other strategies were used by small numbers of students in each group, but the dominant strategy used by all three groups to address rate of change at a point was average rate of change in the dependent variable over intervals of the independent variable of varying size and type.

Chapter summary

In this chapter I looked at the distribution of strategies used by each group on each of the tasks. I found that frequency of use of strategies varied by group and by task but that there were clear patterns in what strategies were consistently used by each group. Three patterns emerged. First, some strategies were used by all three groups. Second, some strategies were used by some groups but not by others. Finally, some strategies were used with greater or less frequency by

each group as students progressed through the tasks. I deal with these three patterns in turn.

Strategies used by all three groups. In Tasks 1, 2, and 3, all three groups used "delta d" to explain how their graphs and tables showed slowing down, speeding up, and steady pace. With graphs, calculus and postcalculus students also made heavy use of "visual slope" while precalculus students made medium use of this strategy. "Delta d" and "visual slope" were the two most commonly used strategies for each group on Tasks 1, 2, and 3.

In Tasks 4, 5, and 6, as in Tasks 1, 2, and 3, all three groups used "delta d" and "visual slope" to identify intervals of the independent variable for which the dependent variable changed the most and least and calculus and postcalculus students made more use of "visual slope" than precalculus students. To find rate of change in the dependent variable at individual values of the independent variable, all three groups used average rate of change in the dependent variable over intervals of the independent variable of varying type and size. Calculus and postcalculus students also used slope of tangent line with graphs but no precalculus student used this strategy to address rate of change at a point.

Differences in strategy use among the three groups. All three groups used average rate of change in the dependent variable over intervals of the independent variable to address rate of change at a point, but each group tended to use a different type or size of interval. Precalculus students tended to use medium and short intervals before the point, calculus students used short intervals before the

point and intervals straddling the point, and postcalculus students primarily used intervals straddling the point.

In Tasks 1, 2, and 3, precalculus students were the only ones to use "meeting point" and postcalculus students were the only ones to use "velocity equation". Generally precalculus students more than the other two groups used "reference to a point". This strategy made use of the value of the dependent variable or the ratio of the value of the dependent variable to the value of the independent variable at a specific point. Use of "delta t" declined with increased mathematics education.

Strategies that were used with greater or less frequency across the tasks. In Tasks 4, 5, and 6, precalculus students used almost exclusively intervals before a point to address rate of change at the point but as they proceeded through the tasks, they made more use of the shortest available intervals before the point. Calculus students used intervals before the point and intervals straddling the point but as they progressed through the tasks, they made less use of the former and more of the latter. Postcalculus students came more and more to rely on straddle intervals as they moved through Tasks 4, 5, and 6.

As students moved through the six tasks, they made less use of "reference to a point" although all three groups made some use of this strategy throughout all six tasks. The attractiveness of the coordinates of a point as the sole information about rate of change at that point seems to be an irresistible temptation for some students, no matter how much mathematics education they have had.

Chapter 8

SUMMARY AND CONCLUSION

At the end of Chapter 1 I posed several questions about students' knowledge of rate of change. In this final chapter I use the results of this research to suggest answers to those questions. In addition, I relate literature reviewed in Chapter 2 to this dissertation. Finally, I describe how my own teaching of calculus has changed as a result of this research.

Students' knowledge of rate of change

Near the end of Chapter 1, I posed five questions this dissertation would attempt to answer. The first two questions concern students' knowledge and how it supports learning rate of change. These two questions will be discussed below under the heading "What knowledge of rate of change do precalculus, calculus, and postcalculus students have?". The third question in Chapter 1 pertains to similarities and differences in knowledge of rate of change between those who have had calculus and those who have not and what this might tell us about what knowledge supports learning rate of change and how. This question will be addressed under the heading "How does knowledge of rate of change differ among the three groups of students?". The last two questions in Chapter 1 consider the role of the three perspectives and types of reasoning in thinking about students' knowledge of rate of change and how it grows. These questions will be dealt with under the heading "Do the perspectives and types of reasoning provide an interpretive frame for

understanding students' knowledge of rate of change and how it grows over time?".

What knowledge do these three groups have?

A variety of strategies. The most prominent result of this study was that students used many strategies to address rate of change. This was the case with the entire sample, with each of the three groups, and with individual students. The sample of 37 students used a total of nineteen different strategies on the six tasks. Many of these strategies were used by each of the three groups. Most students used more than one strategy to solve each task.

Precalculus students used many of these strategies in the six tasks. We can conclude therefore that they have knowledge that can be used to solve tasks involving rate of change. They used "delta d" and "visual slope" to address slowing down, speeding up, and steady pace; they used average rate of change over medium and short intervals before a point to find rate of change at a point; and they used knowledge of specific situations to answer several questions about rate of change.

These strategies were also used heavily by calculus and postcalculus students. This suggests that this shared knowledge may play a role in supporting learning rate of change before, during, and after calculus. These strategies were used appropriately by all three groups to yield reasonably good estimates for rate of change at points, to correctly describe slowing down, speeding up, and steady pace, and to identify intervals of most and least change. This use of "correct" knowledge suggests that it can be used to build other knowledge of rate of change.

Further, since much of this knowledge has in it the germs of more advanced and conventionally accepted knowledge of rate of change, it makes sense that it can serve as the basis for construction of other knowledge of rate of change. Students' knowledge of "visual slope" and shape of graph and relationship to rate of change of a dependent variable over all or large parts of the domain of the independent variable gives them a base on which it may be possible to build knowledge of rate of change at a point in terms of slopes of rotating secant lines and tangent line. Similarly, students' use of average rate of change over medium and short intervals before a point suggest their receptivity to the definition of instantaneous rate of change in terms of the limit of average rate of change over intervals of decreasing width before and after a point.

General and specific strategies. A second conclusion about students' knowledge of rate of change is that they have some strategies that can be used in nearly every situation. For describing how graphs and tables showed slowing down, speeding up, and steady pace and for deciding where the dependent variable changed the most and least, nearly every student in this study used "delta d" and "visual slope", two strategies that can be applied to many different situations to address rate of change. To find rate of change at a point, calculus and postcalculus students used slope of tangent line and all three groups used average rate of change over intervals of various size and type, strategies which have wide applicability for addressing rate of change at a point. These strategies were general in the sense that they were used in every situation and show that students at these levels of mathematics

education have knowledge of rate of change that can be applied in nearly every situation.

Students in this study also used some strategies that are applicable only to specific situations. Some strategies were used only with certain function representations. For example, slope of tangent line was used extensively with graphs, but not with tables. Some strategies were used only with certain tasks. In Task 4, the "thrown ball" task, students made heavy use of "analysis of moving objects" but in Tasks 5 and 6, the "biomass" and "circles" tasks, students did not use knowledge of the specific situation to address rate of change at a point.

Some strategies were used only at certain points. This was the case in Task 4 where students were asked to find the speed of the ball at more than one point. These strategies were specific in the sense that they were used only in certain situations and not at all in other situations and show that students at these levels of mathematics education have knowledge of rate of change that is specific to particular situations.

How does knowledge differ among the three groups?

Strategies used by all three groups. All three groups used "delta d" and "visual slope" to explain their graphs and tables in Tasks 1, 2, and 3 and identify intervals of the independent variable for which the dependent variable changed the most and least in Tasks 4, 5, and 6. To find rate of change in the dependent variable at individual values of the independent variable, all three groups used average rate of change in the dependent variable over intervals of the independent variable of varying type and size.

Students at these levels of mathematics education then have some common knowledge about rate of change. As students' responses to these tasks showed, this knowledge is very useful both for those who have had calculus and those who have not. This may suggest, as was noted earlier, that this knowledge can serve a significant role in students' construction of knowledge of rate of change across all of these levels of mathematics education.

Strategies used by some groups but not others. Some strategies were used primarily by some groups but not by others. In Tasks 1, 2, and 3 only precalculus students used "meeting point" and only postcalculus students used "velocity equation" to explain their graphs and tables. Generally precalculus students made more use of "reference to a point" than the other two groups, calculus students were more likely than precalculus and postcalculus students to use " Δd and Δt ", and use of " Δt " decreased and use of " $(\Delta d)/(\Delta t)$ " increased with increased mathematics education.

"Meeting point" shows slowing down, speeding up, and steady pace in terms of how much time it takes to cover 20 feet - more time for slowing down, less time for speeding up, and same time for steady pace. Students who used this strategy seemed to be confusing slowing down with going at a steady slow pace and speeding up with going at a steady fast pace. This makes me think that some precalculus students have not yet sorted out these differences. This could help teachers plan appropriately for instruction on rate of change.

Postcalculus students' used "velocity equation" in two different ways in Tasks 1, 2, and 3. One use involved dividing the change in distance by the change in time over selected intervals of time. The

other involved division of the distance value at a point by the corresponding time value, similar to what many students did when they used "reference to a point". I discussed in Chapter 5 why I believe the statement "velocity equals distance over time" provides problems for students at these levels of mathematics education. Here I simply add that teachers need to help students sort out the various meanings of this statement and learn how to appropriately use it in various situations.

Calculus students' use of " Δd and Δt " and decreased use of " Δt " and increased use of " $(\Delta d)/(\Delta t)$ " with increased mathematics education shows, I believe, a sorting out of the appropriate relationship between distance and time to address rate of change as students learn more mathematics. To press the point perhaps beyond what the data supports, precalculus students know that speed has something to do with relationships between distance and time but they have not come to accept the convention of speed as distance per unit of time. Instead they sometimes express speed as time per unit distance, that is, as " Δt ". Calculus students know that speed has to do with changes in distance relative to changes in time but have not settled on speed as distance per unit time. Instead they sometimes express speed as variable distance over variable time, that is, as " Δd and Δt ". Finally, postcalculus students have come to accept that speed is most appropriately expressed as change in distance relative to change in time and hence their use of " $(\Delta d)/(\Delta t)$ ".

In Tasks 4, 5, and 6 calculus and postcalculus students used slope of tangent line to find rate of change at a point, but no precalculus student used this strategy. Since calculus and postcalculus students had instruction in calculus relating rate of change to slope of tangent line,

it should not be surprising that they used it and precalculus students did not. However, given precalculus students' use of "visual slope" in all six tasks, it seems useful to ask how this knowledge of slope might be useful to construction an understanding of rate of change at a point in terms of slope of tangent line.

Although all three groups used average rate of change in the dependent variable over intervals of the independent variable, each group tended to use a different set of intervals. Precalculus students tended to use medium and short intervals before the point, calculus students used short intervals before the point and intervals straddling the point, and postcalculus students relied primarily on intervals straddling the point. The more mathematics education, the smaller the intervals the group tended to use and the more likely the group was to use intervals straddling the point.

Precalculus students' use of average rate of change over some intervals to address rate of change at a point is encouraging for two reasons. One is that they know how to use average rate of change to address rate of change at a point. The other is that since instantaneous rate of change is based on average rate of change over smaller and smaller intervals, it seems reasonable that students who already use average rate of change are primed to consider average rate of change over intervals of decreasing width and the resulting limiting value.

One possible way these students might come to know instantaneous rate of change is by providing several opportunities for them to find rate of change at a point. As precalculus students' responses to these six tasks showed, with each new task, students were more inclined to use shorter intervals before the point and one student began on her own to

also use intervals after the point. One might then hope that some students would begin to think about a limiting value for the average rate of change over smaller and smaller intervals both before and after a point.

Strategy change across tasks. The distribution of strategy use changed across the tasks. In Tasks 4, 5, and 6 precalculus students used intervals before a point to address rate of change at a point but as they proceeded through the tasks, they made more use of the shortest available intervals before the point. Calculus students used intervals before the point and straddle intervals but as they progressed through the tasks, they made less use of the former and more use of the latter. Postcalculus students came more and more to rely on straddle intervals as they moved through Tasks 4, 5, and 6. This pattern complements the pattern discussed above and points even more hopefully to the possibility that students' knowledge of average rate of change over intervals is a primary resource that could help them learn instantaneous rate of change.

Use of "reference to a point" decreased across the tasks although its use had not entirely disappeared by the last task. Some students who used "reference to a point" on earlier tasks turned in later tasks to strategies based on changes in the dependent variable over intervals of the independent variable, like average rate of change. This happened, I believe, because as these students moved through the tasks, they came to see that rate of change is not given by "distance over time", as in dividing the distance value by the time value, a "reference to a point" strategy, but is more appropriately related to "change in distance over change in time".

At least one student, a postcalculus student, made her thinking explicit enough that I can say with assurance that is what she did. In Task 1 where the two people slow down until they meet and then speed up, she divided the distance value at each point by the time value. Since the resulting ratios were decreasing from $t=0$ to $t=4$, she concluded that the two people were slowing down and since the ratios were increasing from $t=4$ to $t=8$, she concluded that the two people were speeding up.

But in Task 2, this same strategy resulted in ratios that were not constant and therefore did not show "steady pace". It was here that she realized that "distance over time" should be interpreted not as distance value divided by time value, but as change in distance divided by change in time. She then used " $(\Delta d)/(\Delta t)$ " on Task 2, got constant ratios and properly concluded that this showed the two people were slowing down. She then returned to Task 1, computed new ratios based on change in distance over change in time and properly used these new ratios to conclude that the two people were slowing down and then speeding up.

This is an account of only one student's experience and can not necessarily be used to make claims about all students. However it is tempting to speculate that more and varied experiences like the ones represented in these tasks can help students come to see that some relationships between distance and time do not yield correct information about rate of change and encourage them to explore other relationships that do.

Do the perspectives and types of reasoning provide an interpretive frame?

The three perspectives. I discuss four issues related to learning rate of change and classification of strategies in terms of the three perspectives: the concept of slope or shape of graph in understanding

rate of change, point perspectives in addressing rate of change at a point, interval perspectives in addressing rate of change at a point, and global perspectives in addressing rate of change at a point.

The concept of slope in understanding rate of change. "Visual slope" was used to explain tables and graphs in Tasks 1, 2, and 3 and identify where the dependent variable changed most and least in Tasks 4, 5, and 6. This strategy was classified as representing a global perspective. "Reference to slope (secant line)" was used to explain tables and graphs in Tasks 1, 2, and 3 and to find rate of change at a point in Tasks 4, 5, and 6. This strategy was classified as consistent with an interval rate of change perspective. "Reference to slope (tangent line)" was used to find rate of change at a point in Tasks 4, 5, and 6 and was classified as portraying a point rate of change perspective. So for each perspective we have a slope based strategy.

Slope is a concept intimately related to rate of change. The formula for slope of a line through two given points (slope of a secant line) yields the average rate of change in the dependent variable over the interval of the independent variable determined by the two points. So slope and average rate of change are computationally equivalent.

Furthermore, just as rate of change at a point can be thought of as the limit of the average rate of change in the dependent variable as the size of the intervals of the independent variable decreases to zero, it can also be thought of as the limit of the slope of secant lines determined by those shrinking intervals. This limit, if it exists, is called the slope of the line tangent to the graph of the function at the point. In fact, this is how the tangent line is defined; it is the line containing the point and having slope equal to this limit.

This limit idea connects the interval perspective of slope (of secant line) to the point perspective of slope (of tangent line) and is relied on heavily in calculus to help students learn instantaneous rate of change. But how does the global perspective of slope relate to the interval and points perspectives? "Visual slope" was used heavily in these tasks by all three groups of students in trying to make sense of rate of change when using graphs. It seems clear that students at these levels of mathematics education relate slope as shape of graph to rate of change. But what connection did students make between this use of slope as rate of change and use of slope of secant and tangent lines as rate of change?

Half of the precalculus students used "visual slope" to explain how their graphs showed slowing down, speeding up, and steady pace in Tasks 1, 2, and 3 and some used it to identify intervals of the independent variable where the dependent variable changed the most and least in Tasks 4, 5, and 6. But none of them used slope of secant line or tangent line in these tasks. I would not have expected them to use slope of tangent line. But I would have expected some of them to use slope of secant line. For example, when using the graph, several precalculus students used average rate of change over an interval. I would have expected some of them to mention that this was also the slope of the line through two particular points. None of them did that.

That makes me think two things. One is that while precalculus students relate slope as shape of graph to rate of change, they do not relate slope of line through two points to rate of change. This seems a crucial point given that instantaneous rate of change has traditionally been introduced in calculus in terms of the limit of the slopes of secant

lines. It might occur to us that students coming to calculus may not easily relate slope of line to rate of change and therefore we should either find a way to help them make that connection or introduce rate of change at a point differently.

Perhaps we should do both. But the other thought I had is that since precalculus students did relate average rate of change to rate of change at a point **even when they were using graphs**, we might ought to think about introducing instantaneous rate of change as the limit of average rate of change over intervals of decreasing width instead of as the limit of the slope of secant lines. Given the prevalent use of average rate of change by the students in this study and most pertinently, by the precalculus students, this would seem to be a very good idea.

Point perspective in addressing rate of change at a point. I discuss here strategies some students used to address rate of change at a point but which are not consistent with accepted ways of thinking about instantaneous rate of change. My purpose is to think about what these strategies say about students' knowledge of rate of change and possible roles they play in students' construction of instantaneous rate of change.

Students used two such strategies - "reference to a point", and "velocity equation (table)". "Reference to a point" was used in Tasks 1, 2, and 3 to explain how graphs and tables showed slowing down, speeding up, and steady pace and in Tasks 4, 5, and 6 to find rate of change at a point. In both cases students used the value of the dependent variable at a particular value of the independent variable to make decisions about rate of change. Sometimes they formed the ratio of the values of the

dependent and independent variables at a point; sometimes they just gave the value of the dependent variable; sometimes they averaged several values of the dependent variable. But they all used in some way or other the value of the dependent variable at a point to address rate of change at that point.

As discussed in Chapter 5, there are several reasons students use this strategy. Values of the dependent variable are readily available with a graph and especially a table and therefore present an almost irresistible temptation to use in some form or other. The simplest way to use them is to directly give, as the rate of change at a point, the value of the dependent variable or the ratio of the value of the dependent variable to the value of the independent variable. Also many students know there is a relationship between speed and distance and the simplest relationship is to equate the two. Finally, sometimes speed is referred to as distance over time. One interpretation of this is that speed is the value of the distance at the time or the value of the distance divided by the value of the time at the point. These factors make the value of the dependent variable or the ratio of the value of the dependent variable to the value of the independent variable a reasonable, though incorrect, choice for addressing rate of change at a point.

Students who use "reference to a point" know that the value of the dependent variable or the ratio of the value of the dependent variable to the value of the independent variable has something to do with rate of change at a point. They do, though not in the way these students think, but rather in a way that makes a conventional understanding of rate of change closer to them than if they did not believe at all that rate of change has anything to do with values of the dependent and independent

variables. Since these students already know this much about rate of change, further experience with rate of change should allow them to experiment with possible relationships between the dependent and independent variables and eventually arrive at a more conventional understanding of rate of change. In fact, over the course of these one hour interviews several students who at one point or another used this strategy, later used strategies based on changes in the dependent variable relative to changes in the independent variable to make sense of rate of change at a point.

"Velocity equation (table)" is a strategy similar to "reference to a point" in that students divided distance values in the table by time values. As they did so they said they were using the formula "velocity is distance over time". They were literally forming the fraction "distance over time" or distance value divided by time value to get an estimate for speed at a point. Here is another instance where students know that rate of change has something to do with a relationship between the dependent and independent variables. As happened in these tasks, students can then use this knowledge to build more conventional knowledge of rate of change.

Interval perspective in addressing rate of change at a point. I consider three types of strategies here. One type consists of strategies based on changes in the dependent or independent variable over intervals of the other variable. " Δd ", " Δt ", and " Δd and Δt " are of this type. All three of these strategies take into consideration coordination of changes in both the dependent and independent variables but they take different stances with regard to standardization. " Δd " considers change in the dependent variable for one unit changes in the

independent variable. "Delta t" considers change in the independent variable for one unit changes in the dependent variable. "Delta d and delta t" considers change in the dependent variable over intervals of the independent variable of varying length.

Students heavy use of "delta d" in this study suggests that students at these levels of mathematics education know that rate of change has something to do with changes in the dependent variable over unit intervals of the independent variable. Decreased use of "delta t" with increased mathematics education suggests that over time students come to realize that focusing on changes in the independent variable for one unit changes in the dependent variable is not the most helpful way to look at rate of change. Calculus students' use of "delta d and t" indicates sorting out the relationship between changes in the dependent variable and changes in the independent variable. These patterns taken together suggest that over time students commit themselves to viewing rate of change as dealing primarily with changes in the dependent variable for fixed changes in the independent variable.

A second type of strategy is based on comparisons of changes in the dependent variable over one interval of the independent variable to changes in the dependent variable over another interval of the independent variable. "Not quite doubling" is based on a comparison of changes in the dependent variable over consecutive one unit intervals of the independent variable. "Multiplied by two-thirds" and "12 percent increase" involve comparisons of change in the dependent variable over an interval of the independent variable immediately after the point with change in the dependent variable over the interval beginning at the first value of the independent variable and ending at the point. "Increased by

one-third" compares change in the dependent variable over an interval of the independent variable immediately before the point with change in the dependent variable over the interval beginning at the first value of the independent variable and ending at the point.

It was primarily the precalculus students who used strategies of the second type. My interpretation for their invention of these strategies is that they knew that rate of change has to do with changes in the dependent variable and were experimenting with relationships between these changes to arrive at conclusions about rate of change that made sense. They, like all of the other students in this study, relied on "delta d" to make judgments about rate of change. To address rate of change at a point they wanted to put their delta d's to good use and one way to do that was to compare delta d's over various intervals of the independent variable. So some precalculus students generalized use of delta d's to include comparison of delta d's to one another as a way of addressing rate of change at a point.

A third type of strategy is based on comparison of change in the dependent variable to change in the independent variable. Strategies based on the average rate of change in the dependent variable over an interval of the independent variable are of this type. These strategies include the four strategies specifically identified as average rate of change and the strategy " $(\Delta d)/(\Delta t)$ ". This latter strategy could have been used in conjunction with slope, specifically slope of secant line. But as it was coded in these tasks, it referred to students who were computing average rate of change in the dependent variable over intervals of the independent variable, who said this is what they were

doing, and who also happened to mention, either verbally or in writing, that they were using the formula " $(\Delta d)/(\Delta t)$ ".

The strategies most heavily used to address rate of change at a point involved average rate of change in the dependent variable over various intervals of the independent variable. There are two stories to be told about use of these strategies. One is that students saw rate of change in terms of comparison of change in the dependent variable to change in the independent variable. For some students this meant seeing rate of change first in terms of a change in the dependent variable, then as comparison of changes in the dependent variable, and finally, as comparison of change in the dependent variable to change in the independent variable. This is one possible path for coming to know rate of change seen in this data.

The other story about strategies based on average rate of change concerns students coming to know instantaneous rate of change. As noted earlier, precalculus students came, over the course of working these tasks, to rely on short intervals before the point, calculus students moved to using the shortest available intervals before the point and intervals straddling the point, and postcalculus students settled decidedly on intervals straddling the point and especially the shortest such intervals. Since instantaneous rate of change, what I have called rate of change at a point, is defined in terms of the limiting value of the average rates of change in the dependent variable over intervals of the independent variable of decreasing width, it is interesting to note that the progression in use of intervals of the independent variable from precalculus to calculus to postcalculus follows this pattern of decreasing width. This seems to say that over time students at these

levels of mathematics education come to see rate of change at a point in terms of average rates of change over smaller and smaller intervals.

We would expect calculus and postcalculus students to more easily make this connection than precalculus students. But what I find encouraging is precalculus students' consistent use of average rates of change over intervals to make sense of rate of change at a point when presumably they had as yet had no instruction about such a thing. This points to the possibility that this knowledge that precalculus students have or can fairly easily construct in a relatively short time, as they did here over a one hour interview, serves a fundamental role in helping these students learn instantaneous rate of change.

Furthermore, in task 5, when students were asked which of their answers they liked best and how they would get a better answer, precalculus students gave the kind of answers that showed potential for their learning instantaneous rate of change. They said they liked their last answer best because it was based on smaller intervals and to get a better answer they would use even smaller intervals. This is knowledge that could serve these students well in coming to know instantaneous rate of change.

Global perspective in addressing rate of change at a point. One strategy classified as consistent with global rate of change was used to address rate of change at a point - "analysis of moving objects". This strategy was used in Task 4 and was the only task for which students used knowledge of the situation to address rate of change at a point. With the graph this strategy was used equally heavily by all three groups and especially so at $t=2$, 3, and 4. With the table of values, the more

mathematics education the group had, the more likely it was to use this strategy.

An important point to make is that knowledge of the situation helped all three groups of students make sense of rate of change at a point in this task. It seems clear that problem context and students' background knowledge of the situation plays a significant role in how students make sense of rate of change and especially rate of change at a point. Another point is that these students, perhaps because of their prior experiences with moving objects, showed they believed in rate of change at a point, at least in the context of Task 4. They believed the ball had speed at individual points, that instantaneous rate of change exists for this situation. This makes me think that students' background knowledge in other problem contexts can be an important resource in the advancement of their knowledge of rate of change in general and rate of change at a point in particular.

The three types of reasoning about rate of change.

As with the three perspectives I discuss four issues related to learning rate of change and classification of strategies in terms of the three types of reasoning: strategies classified as both qualitative and numerical, strategies classified as both qualitative and algebraic, strategies classified as both numerical and algebraic, and the role of numerical strategies in helping students learn algebraic strategies.

Strategies classified as both qualitative and numerical. Three strategies were given both qualitative and numerical classifications: " Δd ", " Δt ", and " Δd and Δt ". Students whose use of these strategies was based on graphical features made comments like "they traveled less each second", "it grew more in this hour than in this

hour", or "it goes up more here than here". Students who reasoned numerically using these strategies computed the change in the dependent variable for one or more intervals of the independent variable and then used these numerical values to make judgments about rate of change.

What difference does it make if students refer to these changes qualitatively or numerically? For some tasks, it makes no difference at all. These three strategies were used to explain how graphs and tables in Tasks 1, 2, and 3 showed slowing down, speeding up, and steady pace and to identify intervals of the independent variable over which the dependent variable changed the most and least in Tasks 4, 5, and 6. These questions do not necessarily require a numerical answer. So students' use of graphical feature to obtain answers to these questions was quite adequate.

However, in thinking about what knowledge students have which may help them construct other knowledge of rate of change and specifically knowledge of instantaneous rate of change, it seems clear that students need to be able to reason numerically about changes over intervals. Hence students who reason numerically using these three strategies may be further along toward the goal of learning instantaneous rate of change than students who reason qualitatively.

Strategies classified as both qualitative and algebraic. Two strategies were given both qualitative and algebraic classifications: "reference to slope (secant line)" and "reference to slope (tangent line)". When students reasoned using graphical features with these strategies, they associated rate of change with the slope of a line tangent to the graph at a particular point or a secant line through two points on the graph. They referred to a line, drew it on the graph, and

said its slope was the rate of change. When students reasoned algebraically using these two strategies, they related rate of change to the slope of one of these lines, used the formula for slope of a line to obtain a numerical value for the slope of the line, and gave this numerical value as the rate of change.

When working with graphs most students who used one of these strategies reasoned both ways. Most often they reasoned qualitatively first and then algebraically. They drew an appropriate line, said the slope of that line was the rate of change they were hunting for, and then proceeded to find a numerical value for the slope and gave that value as the rate of change. But some students used these strategies to reason only in a qualitative way. They drew a line and said the rate of change was the slope of that line but then did not find the numerical value for that slope. Perhaps they did not believe it was necessary to do so or did not know how.

What do these two ways of reasoning about slope say about students' knowledge of rate of change? When students reason qualitatively using one of these strategies, they are associating the concept of rate of change with the concept of slope with or without actually computing a numerical answer for either. They are making a very important connection between slope and rate of change. However, students who do not know how to get a numerical value for the slope of a line or do not associate that numerical value with the value for the rate of change may not have the same knowledge about rate of change as students who can and do.

Some situations do not require a numerical answer for the rate of change and therefore knowing an algebraic formula for obtaining the slope and using that value for the rate of change may not be important. But

other situations do require a numerical answer for the rate of change and in these instances knowing the formula for slope, being able to use it to find the slope, and giving that answer as the rate of change is very important.

More than that, given the intimate connection between slope and rate of change, it is reasonable to conclude that the more one knows about slope, the more one knows about rate of change. Hence students who know how to compute slope and use the results of that computation to address rate of change are further along in their understanding of rate of change than those who do not.

Strategies classified as both numerical and algebraic. One strategy was given both a numerical and an algebraic classification - "reference to a point". Students who reasoned numerically attended to the pattern of values of the dependent variable on a graph or in a table to address rate of change. This was especially the case in Tasks 1, 2, and 3 when students were asked to explain how their graphs and tables showed the two people slowing down, speeding up, and maintaining a steady pace. Students who reasoned algebraically used a formula based on division of the value of the dependent variable by the value of the independent variable at a point. This usually meant division of the value of the dependent variable by the corresponding value of the independent variable at a point.

Neither of these ways of using values of the dependent and independent variable at a point represents a conventionally accepted view of rate of change. But they both hint at ways students make sense of rate of change and may provide information about learning rate of change. As has been mentioned above, students who know that values of the

dependent and independent variable have something to do with rate of change are further along in their understanding of rate of change than students who do not know this. In addition, students who know that rate of change involves a ratio of two quantities are further along yet. This knowledge should prove to be a primary building block for coming to think of rate of change as the ratio of two differences, a difference between two values of the dependent variable and a difference between two values of the independent variable.

Role of numerical strategies in helping students learn algebraic strategies. One set of numerical strategies plays, I believe, an important role in students' construction of one particular set of algebraic strategies. These numerical strategies are "delta d" and "not quite doubling". The set of algebraic strategies are "increased one-third", "multiplied by two-thirds", "12% increase", " $(\Delta d)/(\Delta t)$ ", and average rate of change in the dependent variable over various size intervals of the independent variable. What possible role can these numerical strategies play in the construction of these algebraic strategies?

The two numerical strategies deal with references to changes in the dependent variable over intervals of the independent variable ("delta d") or additive comparison of changes in the dependent variable over consecutive intervals of the independent variable ("not quite doubling"). The set of algebraic strategies involve multiplicative comparison of change in the dependent variable over one interval of the independent variable to change in the dependent variable over another interval of the independent variable ("increased by one-third", "multiplied by two-thirds", and "12% increase") or multiplicative comparison of change in

the dependent variable to change in the independent variable (" $\Delta d / \Delta t$ ") and average rate of change).

This progression of strategies, from references to changes in the dependent variable to additive comparison of changes in the dependent variable to multiplicative comparison of changes in the dependent variable to multiplicative comparison of change in the dependent variable with change in the independent variable, represents a possible path for construction of knowledge of rate of change at a point. Evidence from responses of students in this study support this claim in at least two ways.

One way is that across the three groups of students from precalculus to calculus to postcalculus, use of strategies progressed in this way. That is, precalculus students were the ones who used strategies based on multiplicative comparison of changes in the dependent variable ("increased one-third", "multiplied by two-thirds", and "12% increase") and a single calculus student was responsible for a strategy based on additive comparison of changes in the dependent variable over two consecutive intervals of the independent variable. So students with less mathematics education were primarily the ones who used strategies based on additive or multiplicative comparisons of changes in the dependent variable. Students with more mathematics education were inclined to rely on strategies based on multiplicative comparisons of change in the dependent variable to change in the independent variable.

The other way this claim is supported by data from this study is the change in strategies across the tasks. For each group, progression through the six tasks saw less use of additive and multiplicative comparison of changes in the dependent variable and more use of

multiplicative comparison of change in the dependent variable to change in the independent variable. Admittedly the data is weak here; only a few students used strategies of the former type and large numbers of students in each group used strategies of the latter type. Nonetheless, there was a pattern of decreased use of the former and increased use of the latter types of strategies by the precalculus and calculus students.

One apparent anomaly needs to be explained. All three groups made heavy use of "delta d", a strategy based on reference to changes in the dependent variable. This would appear to provide contradictory evidence for the possible progression in knowledge described above. But I believe it does not. "Delta d" was used by every student in Tasks 1, 2, and 3 to explain how their graphs and tables showed slowing down, speeding up, and steady pace. It is one of the most appropriate strategies to use there since it provides both quick and accurate information. However, in Tasks 4, 5, and 6, "delta d" is less helpful in addressing rate of change at a point and it was not used there by any student. In other words, where "delta d" was appropriate, it was used heavily by each group of students. Where it was not appropriate, it was not used.

The above discussion about a possible progression of strategies is not meant to imply that strategies based on reference to changes in the dependent variable or additive or multiplicative comparison of changes in the dependent variable are less appropriate or useful. They are quite useful in certain situations. For example, they form the basis of the mathematical concept "percent change". But in thinking about how students come to know instantaneous rate of change, strategies based on additive or multiplicative comparison of changes in the dependent variable are not as helpful as other strategies. They may however

provide a window through which we may view students' knowledge of rate of change and their subsequent learning. In particular, use of these and other strategies seem to imply a progression through intermediate states of knowledge toward knowledge of rate of change at a point.

Summary of three perspectives and types of reasoning. This study showed ways in which the three perspectives on rate of change and the three types of reasoning inform our understanding of students' knowledge of rate of change and how it grows over time. Three major findings come out of this discussion. One is the role that interval strategies play in students' coming to know rate of change. One important part of this finding concerns coming to know instantaneous rate of change and is born out by the pattern of responses in this study. All three groups of students used average rate of change in the dependent variable over intervals of the independent variable to address rate of change at a point. Precalculus students used longer intervals before the point, calculus students used shorter intervals before the point, and postcalculus students used intervals straddling the point. In addition, as students moved through the tasks, precalculus students came to use shorter intervals before the point, calculus students came to use the shortest intervals before the point and some straddle intervals, and postcalculus students came to rely almost exclusively on straddle intervals. This pattern was the single most prominent pattern supported by the data and shows the role that average rate of change plays not only in the formal definition of instantaneous rate of change but also in the thinking of students trying to make sense of instantaneous rate of change.

A second major finding is a hint about how some students progress from thinking in terms of changes in the dependent variable to thinking in terms of the ratio of change in the dependent variable to change in the independent variable. The data here is weak but it provides a window on the thinking of a few students who relied first on changes in the dependent variable, then on additive or multiplicative comparison of changes in the dependent variable, and then on multiplicative comparison of changes in the dependent variable to changes in the independent variable.

The third finding is the various uses of values of the dependent and independent variables to make sense of rate of change. A few students in each of the three groups used the ratio of the value of the dependent variable to the value of the independent variable to make sense of rate of change ("reference to a point"). There are many reasons students are attracted to strategies making use of values of the dependent and independent variable and there may be important ways this knowledge can subsequently be used to help students construct other knowledge of rate of change. My suggestion about this knowledge is that students who use these kinds of strategies know that rate of change has something to do with a relationship between values of the dependent and independent variables. Since rate of change does indeed have something to do with values of these variables, it seems crucial that students who believe that it does, even if the details of that belief are flawed, have knowledge upon which other less flawed knowledge of rate of change can be constructed.

Two minor findings deserve mention here. Context plays a role in how students address rate of change. When students have knowledge of

specific situations they use it make sense of rate of change and often arrive at conclusions more easily and quickly. We saw that especially with Task 4. Also some strategies, like "delta d", may help students come to associate rate of change with number. One important part of coming to know rate of change is realizing that rate of change is a quantity and therefore can be measured in the sense that it can be assigned numerical values that yield information. There is a small hint of that in the various ways students used "delta d". Figure 18 summarizes these themes.

Major findings:

1. Interval rate of change plays a primary role in coming to know rate of change and especially instantaneous rate of change.
2. Interval rate of change plays a primary role in distinguishing between additive/multiplicative comparison of two changes in the dependent variable and multiplicative comparison of change in the dependent variable to change in the independent variable.
3. Values of the dependent and independent variables are attractive in making sense of rate of change. With the right kinds of experiences, proper relationships between rate of change and values of the dependent and independent variables can be constructed.

Minor findings:

1. Context plays a role in how students address rate of change. Often the context of a task can provide quick and easy ways of answering questions about rate of change.
2. Some strategies, like "delta d" and "visual slope" are of such universal use and practicality that they should be considered fundamental building blocks for construction of knowledge of rate of change.

Figure 18 - Findings about students' learning rate of change

Relationship of these findings to other research literature

At the end of Chapter 2, in summarizing the literature reviewed there, I posed several questions about students' knowledge of rate of change. In this section I return to those questions to attempt preliminary answers based on the results of this dissertation. In Chapter 2, I also identified some holes in the research literature and indicated this dissertation would try to fill those holes. Here I consider in what ways it did.

Questions posed about students' knowledge.

What role does students' incorrect knowledge play in learning more correct knowledge of rate of change? Much of the literature reviewed in Chapter 2 showed discrepancies between students' knowledge of rate of change and conventionally accepted knowledge. In the TERC and McDermott et al. studies students often thought that a function and its derivative resemble each other. For example, some students thought that if a function is decreasing, then so its derivative. Other students thought that the value of the dependent variable of a function rather than the slope gives information about the derivative.

In this dissertation, some students made similar conclusions. In Tasks 1, 2, and 3, the "walking" tasks, a few students used the fact that the graph of a function was decreasing to conclude that it showed that the two people were slowing down. This was one way the strategy "reference to a point" was used in these tasks. As these students worked through Tasks 1, 2, and 3 they saw that a decreasing graph did not necessarily show "slowing down". As in the TERC study involving Laura, varied experiences here seemed to help students come to a more

appropriate understanding of the relationship between a function and its derivative.

In Tasks 4, 5, and 6, the "thrown ball", "biomass", and "circles" tasks, some students found the rate of change at a point by referencing the value of the dependent variable at that point. This was another way the strategy "reference to a point" was used in these tasks. As students worked through Tasks 4, 5, and 6 most of them replaced this strategy with more appropriate strategies. Again, I conjecture that their knowledge coupled with more experience led to more acceptable knowledge of rate of change.

Are there other motion contexts where students make appropriate connections between a function and its derivative? Most of the studies reviewed in Chapter 2 were situated in the context of motion and often showed students making inappropriate connections between a function and its derivative. Five of the six tasks in this dissertation concerned motion and showed a variety of ways students think about rate of change. Some of the strategies students used, like "delta d", "visual slope", "average rate of change", and slope of tangent line show appropriate ways of thinking about rate of change. These were heavily used strategies in this study and show some of the rich connections students made between functions represented by graphs and tables and rate of change.

Are there other tasks that show how students use increments and accumulations to make sense of rate of change? In the Monk and Confrey studies students used increments and accumulations of quantities to explore rate. One of the strongest findings of this dissertation is the many and varied ways students used increments to address rate of change.

Strategies like "delta d" and "average rate of change" over intervals of varied size and type show strong tendencies for students at these levels of mathematics education to use increments to address rate of change in tasks involving graphs and tables.

For the most part, students used these strategies in appropriate ways. In Tasks 1, 2, and 3 students used changes in the dependent variable over one unit intervals of the independent variable. This was one of the quickest and most efficient methods of answering questions there. In Tasks 4, 5, and 6 students used average rate of change over smaller and smaller intervals as they worked through the three tasks to find rate of change at a point. When working with graphs and tables this represents one of the most appropriate ways to address rate of change at a point.

Are there ways to investigate older students' knowledge of rate in situations involving both constant and varying rates of change? This question was asked in response to the Thompsons' studies of middle school students' understanding of constant rate and Pat Thompson's study of postcalculus students' poor understanding of the function yielding, for each value of the independent variable, the average rate of change over a fixed sized interval and its limit as the size of the interval decreases to zero.

Thompson conjectured that postcalculus students have "impoverished images of rate". If students do not have an understanding of constant rate as "reflectively abstracted constant ratio", then it makes sense that it would be difficult for them to use a ratio of differences in quantities to study variable rates and to then use average rate of

change over intervals of decreasing width as a way of thinking about rate of change at a point.

In this study most students used "delta t" correctly in Tasks 1, 2, and 3 to argue that their graphs and tables properly showed the two people slowing down, speeding up, and walking at a steady pace. They showed that they understood that "steady pace" meant covering the same distance every second, that "slowing down" meant covering successively less distance each second, and that "speeding up" meant covering successively more distance each second. This shows students able to distinguish between constant and variable rates at least in these contexts.

But it does not show they understood constant rate as "reflectively abstracted constant ratio". Students did not expand their arguments to claim, for example, that five feet per second is the same thing as two and a half feet per half second or a half foot every tenth of a second or ten feet every two seconds. They did not have to do this to work the tasks. So even though it is encouraging that students correctly used "delta d" to address speeding up, slowing down, and steady pace, it is not clear that Tasks 1, 2, and 3 helped students see constant rate in terms of "reflectively abstracted constant ratio" or how that these tasks could have revealed if the students knew constant rate in this way.

Tasks 4, 5, and 6 show a bit more clearly that students did understand average rate of change over intervals of various sizes. Most students, including many of the precalculus students, used average rate of change over intervals and several of them used several different sized intervals, including intervals of less than one unit width.

Furthermore, in Task 5 most students in this study, including many of the precalculus students, said they would use the average rate of change over smaller intervals to get a better approximation to the rate of change at a point and successively smaller intervals to obtain an exact value for that rate of change.

This shows a fair amount of facility in addressing instantaneous rate of change using average rate of change over intervals of decreasing width. But it does not necessarily mean that students recognize that the algebraic expression $(f(x+h) - f(x))/h$ yields the average rate of change in the dependent variable of the function f over an interval of width h between x and $x+h$. Nor does it mean that students recognize that $f_h(X) = (f(X+h) - f(X))/h$ represents a function that yields for every value of X the average rate of change over the interval between X and $X+h$. Nor does it mean that students recognize what the limit of this function is as h goes to zero represents.

A question raised by consideration of the results of this dissertation and the results of others' research. One of the goals of calculus is to help students learn the standard algebraic representation for the derivative as the limit of the average rate of change of the dependent variable of a function as the widths of the intervals of the independent variable decrease to zero, understand what the derivative means in a variety of contexts, and be able to easily and correctly compute it. This dissertation showed that students, even precalculus students, have varied and appropriate knowledge that supports their thinking about rate of change in contexts involving graphs and tables. What is not clear is how this knowledge can then support students' accomplishing the goals of calculus. I will try in the last section of

this chapter to give some ideas about how I have explored this problem in my own teaching.

How this dissertation filled holes in the research.

No other research involving precalculus, calculus, and postcalculus students. This dissertation involved three different groups of students - before, during, and after calculus - around a common set of tasks. As near as I can tell, there are no other studies that have attempted to provide, as this study did, a comparison of the knowledge of rate of change in groups of students from different levels of mathematics education. This fills an important hole in the research since it shows similarities and differences in knowledge among the three groups and gives a more systematic accounting of the knowledge of students at these different places in mathematics learning.

Certain ways of addressing rate of change are typical. The students in this study addressed tasks involving rate of change in many of the same ways as the students in the studies reviewed in Chapter 2. This makes me think that these ways of thinking about rate of change are not atypical but are representative of what other students at these levels of mathematics education do with other situations or function representations. If so, then this knowledge is relatively prevalent in students at these levels of mathematics education, and therefore we should think about how it can be used to help students learn more about rate of change and specifically how it can be used to help students learn calculus.

Many different ways to address rate of change. This systematic study of the knowledge of students before, during, and after calculus showed that there are many ways students at these levels of mathematics

education address rate of change. This provides a rich picture of students' knowledge. But it also shows how complex learning rate of change can be since, in the case of rate of change at a point, there is no single piece of knowledge we can assume all students have and upon which we can therefore base all instruction in rate of change.

Instead there are many pieces of knowledge of rate of change with some students having some pieces and other students in the same class having other pieces. This makes instruction more complex since it should to take into consideration lots of different pieces of rate of change knowledge. It also has the potential of making instruction more fruitful since it implies there is more than one way to teach and learn rate of change.

Common ways of addressing rate of change. This study involving precalculus, calculus, and postcalculus students also showed there are some common ways students at these levels of mathematics education solve tasks involving rate of change. These points of commonality should help us think about how this knowledge serves students at these different levels of mathematics education both in beginning their experience with calculus and continuing to expand their knowledge of rate of change after calculus.

Different ways of addressing rate of change by groups at different points in mathematics education. This study also showed differences in knowledge of rate of change among these three groups of students. These differences suggest, in ways that studies not involving students at all three levels of mathematics education cannot, that there may be particular streams along which knowledge of rate of change proceeds. Earlier in this chapter I made suggestions about some possible streams.

How my teaching has changed as a result of this research

This research has had two broad affects on my teaching of calculus. Before I discuss these affects, I give a brief history of my experience teaching calculus. Then I discuss how my teaching and thinking about learning calculus has changed as a result of this work. I end this chapter and dissertation with some suggestions for future efforts.

A brief history of my experience teaching calculus.

It was all so simple at first. When I started teaching calculus 30 years ago, I worked hard at trying to find the single best way to introduce the derivative so that all students could understand. Calculus texts typically introduce the derivative in terms of the slope of the tangent line which in turn is defined in terms of a particular limit of the slopes of secant lines. So I focused my attention on trying to make this approach to the derivative and rate of change as understandable as possible to students.

Then reality set in. It was not long before I realized that students struggled with the derivative concept and with differentiation in general. I decided that students were having trouble because they did not know enough about the elementary functions. Since the definition of derivative is so married to slope of tangent line and slope in general, I believed that it would be helpful if students were more knowledgeable about the graphs of functions and relationships between their graphs and their equations. Then maybe they could relate better to the definition of the derivative in terms of slope of tangent lines.

My efforts to help. So I took time at the beginning of calculus to review the elementary functions placing special emphasis on their graphs. We used graphs of functions wherever possible to consider concepts and solve problems. This seemed to help. But I was often frustrated when students were not comfortable with the definition of the derivative. They could remember that it had to do with tangent lines and their slopes. But they could not often recall or easily work with Newton's quotient or relate it to slope. It seemed obvious to me that it was. So I was perplexed by their confusion.

Settling in. As years went by I came to accept that students would leave calculus not understanding much about the limit definition of derivative but that they might be able to use the derivative appropriately in particular situations in their science and mathematics classes. I became more sensitive to their need to review the definition and learn more about it as we used it in other mathematics classes. That, too, seemed to help students get a firmer grasp of what the derivative was about.

Then came graduate school. I quickly decided I would focus my research on calculus learning and students' understanding of rate of change. In my first two research projects, I asked individual calculus students to work problems, watched them solve them, and asked them questions about what they were doing. This was helpful to see that students had knowledge they could use on problems but that knowledge was not always what I had predicted or wanted it to be. This both intrigued me and perplexed me. I wanted to know more about students' knowledge and how it might serve them in learning calculus.

I was clueless. When I proposed this dissertation, I admit I had very little idea about what students, even calculus and postcalculus, would do with these tasks. I was pleased and not very surprised that many calculus and postcalculus students used slope of tangent line to address rate of change at a point. But beyond that, I was not sure how students would work the tasks. So I was quite surprised at the variety of approaches they used to solve these problems and was pleased that, for the most part, students displayed knowledge that was appropriate for the tasks. It has understandably left me wondering how this knowledge might then be used to help students learn calculus.

How my thinking about teaching and learning calculus have changed.

Teaching is more complicated now. The first broad effect this dissertation has had on my teaching is that it has made me realize how complicated teaching and learning is. My research showed that students have a variety of ways to work problems involving rate of change. Some of these ways are appropriate and some are not. Some of these ways hold promise for building other knowledge of rate of change and some are specific to particular situations. Knowing this as a teacher makes planning instruction very complicated.

How do I take into consideration these various ways of approaching rate of change? Should I? Should I privilege approaches upon which more advanced knowledge of rate of change can be constructed? If so, how do I do that? Should I allow discussion of approaches that apply only in particular contexts? If I do, how do I relate this specific knowledge to more generally applicable knowledge? Do I need to? These are some of the questions that make teaching calculus more complicated for me now.

Teaching to take into account possible streams of learning rate of change. The second broad effect has come when I have tried to plan instruction that builds on students' knowledge. One concern is how to move students from using average rate of change in numerical and graphical contexts to an understanding of the limit definition of the derivative in its algebraic form. I have tried several approaches with mixed success. I have not attempted a research based investigation of these approaches. I simply offer them here as ones I have tried.

I have used the six tasks of this dissertation and other tasks as an introduction to calculus. Students in my calculus classes have used many of the same strategies as students in this study. In moving from these tasks to a more general discussion of the derivative, I have used two approaches. One approach has been to try to move from numerical representations of the average rate of change over smaller and smaller intervals to algebraic representations of that process.

Moving from the numerical to the algebraic. Students can fairly easily estimate the rate of change at a specific point, like $(1,48)$, by obtaining the average rate of change over intervals like $(.75,1.25)$ and $(.9, 1.1)$ and $(.99,1.01)$ and can even conjecture a limiting value based on this numerical evidence. But representing the endpoints of these intervals in terms of $(1-.25,1+.25)$ and $(1-.1,1+.1)$, and $(1-.01,1+.01)$ and then as $(1-h,1+h)$ for $h = .25, .1, \text{ and } .01$ seems baffling to students. When using the first set of intervals listed above, students may even say something like "I'll just go a little before and a little after." but translating that into $1-h$ and $1+h$ seems to be very difficult.

Once one sees that endpoints of these intervals can be expressed as $1-h$ and $1+h$, there is the more complicated problem of forming the expressions $(f(1+h) - f(1-h))$ and $((1+h) - (1-h))$ and understanding what these mean and then expressing the average rate of change as the ratio of these two expressions. Finally, one has to take the limit of the resulting expression to get one version of the limit definition of derivative for functions continuous throughout an appropriate interval.

My attempts to help students through this have generally not succeeded. There are several issues here that make these transitions hard. One is the general difficulty students have with expressing numerical data in terms of algebraic expressions. Another is the difficulty of algebraically simplifying the ratio of two algebraic expressions. Yet another is the difficulty of working with and understanding limits.

The stream of learning the definition of derivative outlined above is based on my own analysis of what students need to know at each step before they can move on to the next step. This stream seems reasonable in my own mind but for students it represents a huge hurdle to coming to know the derivative in ways mathematicians consider desirable.

Basing instruction on numerical approaches. I have tried a second approach which seems to be more successful in helping students come to an understanding of the derivative formulas for equations and in working rate of change problems when they cannot remember the derivative formulas or how to find them. Unfortunately, it bypasses the algebraic representation of the limit definition of the derivative and would I suspect therefore be unacceptable to many calculus teachers and curriculum writers as an alternative for teaching derivatives.

The approach is based firmly in the camp of using numerical analysis to find derivatives. It goes like this. Students are given a function in equation form, say $y = e^x$. They are asked to find the derivative formula for this function in the following way. We start with a table of values for the given function. Figure 19 shows a sample table.

Then we approximate the value of the derivative at each given value of x by computing the average rate of change over a tiny interval around the x value. For example, to find the derivative at $x = 3$, students might find the average rate of change over the interval $(2.99999, 3.00001)$. This is not necessarily easy even if students have been using average rate of change to find rate of change at a point for several weeks in calculus. But with help most students can see that they have to find, in the case of this function and this value of x , $(e^{3.00001} - e^{2.99999}) / (3.00001 - 2.99999)$.

<u>values of x</u>	<u>values of y = e^x</u>	<u>values of derivative</u>
-3	0.0497871	0.0497871
-2	0.1353353	0.1353353
-1	0.367894	0.367894
0	1.000000	1.000000
1	2.7182818	2.7182818
2	7.3890561	7.3890561
3	20.085537	20.085537

Figure 19 - Finding values of the derivative of $y = e^x$

Students use this strategy to complete the table in Figure 12 and then we discuss what an equation for the derivative function might be

based on the values in the third column of the table. In the case of this function, students often recognize that the values in the third column are equal, or approximately equal, to the values in the second column and therefore the formula for the derivative function is probably $f'(x) = e^x$. Generally this exercise appears to be successful in helping students obtain the derivative formulas for most of the elementary functions.

This approach capitalizes on students' knowledge of rate of change at a point in terms of average rate of change over very small intervals around the point. Many would object to using this procedure because it is applicable only for functions that are continuous on both sides of each value of x . But a more serious objection is that it bypasses the definition of the derivative in its more general form as the limit of $(f(x+h) - f(x))/h$ as h goes to zero. Many who teach college mathematics would certainly find this a serious omission to the teaching of calculus.

But what happens to students under this approach? I will confess here that I have taught calculus this way for several years. So far, students have not seemed to encounter great difficulty by this omission. Admittedly for many students, calculus is their last mathematics class and their use of calculus in courses in their majors relies on knowing how to use shortcut formulas for finding the derivative equations for certain types of functions or understanding conceptually what the derivative gives in specific contexts. In the case of the former, it is possible to learn how to find derivative formulas without knowing much about the derivative as a concept. In the latter case, I suspect, though I have no formal proof for it, that students whose calculus

experience has been based solely on numerical treatments using average rate of change may very well know as much if not more about the conceptual details of the derivative as students who have encountered the more algebraic representation of the limit definition of the derivative.

For mathematics majors and minors and students whose majors require mathematics beyond calculus, it is not clear that students have been harmed by my treatment. The limit definition of derivative expressed in algebraic terms is certainly needed for many advanced mathematics courses, but even students who have had the more formal treatment of the definition of derivative in calculus cannot often recall it and have only fuzzy notions of what it means.

In advanced courses, I review the limit definition of the derivative wherever we need to use it. Students who have experienced my more numerical approach to the derivative in my calculus class do not seem to be at a greater disadvantage in understanding the more algebraic representation of the definition of derivative than students who have had encountered this definition in their calculus course. So at this point it is not clear what harm comes to students in later work in their majors if they have a numerical, rather than algebraic, approach to the definition of limit.

Some suggestions for future efforts

Three issues raised above need to be investigated further. One is how to help students move from the knowledge they bring with them to calculus, especially in contexts involving numerical and graphical representations of functions, to an understanding of the more formal treatment of the derivative and specifically to the algebraic

representation for the limit definition of derivative. Are there ways to bridge this gap that make sense to students? What do they look like?

A second issue concerns numerical treatments of the derivative in calculus instruction. Are such treatments desirable? In what ways do they help students learn calculus and in what ways to they hinder them? Are there better ways of using numerical methods to teach and learn calculus than the ones I have outlined above? Are there ways of capitalizing on numerical methods to help students move sooner or later to a more algebraic consideration of the definition of the derivative? What sense do such methods make to students?

The third issue deals with the consequences of teaching calculus without consideration of the algebraic representation of the limit definition of derivative. It appears that I am coming down on the side of omitting this more formal definition from the first year course in calculus. Admittedly, I have, temporarily, at least, omitted it from my calculus instruction.

But I am not convinced that it should be omitted. My omission has more to do with not knowing how to make it understandable to students when my own experience tells me there are other ways to move students through calculus with less pain and more sense-making and that there is little evidence that this omission really harms students in any serious way. I very much look forward to studies which show that it can be done, that elementary calculus students can make sense out of the more formal definition of the derivative, and that they can do so using the knowledge this study shows they bring with them to calculus.

Chapter Summary

In this closing chapter I discussed students' knowledge of rate of change in the context of this research. This work showed that students at these levels of mathematics education have varied knowledge of rate of change. Some of that knowledge is general in that it can be applied in many different situations. Some of that knowledge is specific in that its use is dependent on the context of the task.

This study also showed that knowledge of rate of change differs among precalculus, calculus, and postcalculus students. Some knowledge, like knowledge based on changed over intervals and shape of graph, is shared by all three groups of students. Some knowledge is used more by some groups than others. Precalculus students were more likely to think about rate of change in terms of changes in the dependent variable over large intervals of the independent variable of a function while calculus and postcalculus students were more likely to use smaller intervals. Calculus and postcalculus students also thought about rate of change in terms of slope of tangent line, while precalculus did not.

This study showed that students' knowledge of rate of change changes with experience. Over the course of working these tasks, all three groups of students used smaller and smaller intervals to address rate of change at a point. Students in all three groups also made less use of incorrect strategies and more use of correct strategies.

The three perspectives and types of reasoning provide an interpretive frame for thinking about students' knowledge of rate of change. Slope can be used to address rate of change from all three perspectives and using more than one type of reasoning. Slope as a global perspective was used by large numbers of all three groups and

involved qualitative reasoning. But slope as an interval or point perspective and based on both qualitative and algebraic reasoning was used only by calculus and postcalculus students. Given the emphasis on slope in the development of the derivative in calculus, efforts should be undertaken to make more explicit the relationship between slope of secant lines and rate of change before students encounter instantaneous rate of change as slope of tangent line. A consideration of slope in terms of both qualitative and algebraic reasoning should be included in these efforts.

Some students used values of one or both of the dependent and independent variable at a point as the sole information in deciding the rate of change at the point. Sometimes their reasoning was numerical, based on patterns formed by the values of the dependent variable for several points and sometimes their reasoning was algebraic, based on a formula for the ratio of the values of the dependent and independent variable. This point perspective knowledge, though incorrect, shows that students are anxious to connect the coordinates of a point to rate of change at that point. Teachers and curriculum writers need to think about ways to help students move from this knowledge to more correct knowledge about rate of change.

Students use changes over interval in many ways to address rate of change at a point. Many of these are based on average rate of change in the dependent variable over interval of the independent variable, appropriate knowledge for addressing rate of change at a point. Other uses of changes over intervals, like numerically or algebraically comparing changes in the dependent variable for two consecutive intervals of the independent variable, are not the most helpful in

thinking about rate of change at a point but provide opportunities for teacher and curriculum writers to help students distinguish between percent change, an under-instructed concept in high school and college mathematics, and rate of change.

Where appropriate many students used knowledge of the given situation to address rate of change at a point. Students thought about the situation in global terms, how the underlying function changed over its entire domain, to then made decisions about rate of change at various points. When students have specific knowledge of particular situations, this knowledge can be quite useful in supporting their thinking about rate of change.

In this closing chapter I also discussed connections between findings in the literature reviewed in Chapter 2 and the results of this work. In both cases, students often used incorrect knowledge but showed progress toward more correct knowledge over the course of the interviews. However, in both cases, students did have some knowledge that was appropriate for addressing rate of change. Students used changes over intervals in powerful ways in this study and in other studies to address rate of change. These ways of using increments seem promising for thinking about how to help students learn rate of change.

In this study, students showed an ability to work with both constant and varying rates of change, in graphical and numerical contexts. But given the difficulties of advanced mathematics students in the Thompson study, an important question to try to answer is how can we advance students' knowledge beyond use of qualitative and numerical approaches to reasoning about rate of change at a point to the algebraic definition of instantaneous rate of change as the limit of the ratio of

a generally expressed change in the dependent variable to change in the dependent variable as this latter goes to zero? For me, this remains the challenge of teaching calculus.

This dissertation filled some holes in the research literature. It showed how precalculus, calculus, and postcalculus students address a common set of tasks involving rate of change. It showed that certain ways of addressing rate of change are fairly typical of students at these levels of mathematics education. It showed that there are many different ways of addressing rate of change, that some ways are more common to some groups than others, and that some ways are more unique to some groups than others.

Suggestions for further research include how students can use their knowledge of rate of change in graphical and numerical contexts to then learn rate of change in algebraic contexts, the value of using more numerical approaches to the introduction of the derivative, and the consideration of postponing the introduction of the algebraically based limit definition of the derivative to later in the calculus sequence.

APPENDICES

APPENDIX A

GALILEO'S AXIOMS AND PROPOSITIONS FOR UNIFORM MOTION

These axioms are taken from pages 190 through 196 of
Dialogues concerning two new sciences (Galileo, 1638)

Axioms

- Axiom I - In the case of one and the same uniform motion, the distance traversed during a longer interval of time is greater than the distance traversed during a shorter interval of time.
- Axiom II - In the case of one and the same uniform motion, the time required to traverse a greater distance is longer than the time required for a less distance.
- Axiom III - In one and the same interval of time, the distance traversed at a greater speed is larger than the distance traversed at a less speed.
- Axiom IV - The speed required to traverse a longer distance is greater than that required to traverse a shorter distance during the same time-interval.

Propositions

- Proposition I - If a moving particle, carried uniformly at a constant speed, traverses two distances the time-intervals required are to each other in the ratio of these distances.
- Proposition II - If a moving particle traverse two distances in equal intervals of time, these distances will bear to each other the same ratio as the speed. And conversely if the distances are as the speeds then the times are equal.
- Proposition III - In the case of unequal speeds, the time-intervals required to traverse a given space are to each other inversely as the speeds.
- Proposition IV - If two particles are carried with uniform motion, but each with a different speed, the distances covered by them during unequal intervals of time bear to each other the compound ratio of the speeds and time intervals.
- Proposition V - If two particles are moved at a uniform rate, but with unequal speeds, through unequal distances, then the ratio of the time-intervals occupied will be the product of the ratio of the distances by the inverse ratio of the speeds.
- Proposition VI - If two particles are carried at a uniform rate, the ratio of their speeds will be the product of the ratio of the distances traversed by the inverse ratio of the time-intervals occupied.

APPENDIX B

METHODS OF ADDRESSING RATE OF CHANGE

Table 41 - Qualitative, numerical, and algebraic
ways of addressing rate of change

<u>Rate of change</u>	<u>Qualitative</u>	<u>Numerical</u>	<u>Algebraic</u>
Increasing/ decreasing	vertical rise/ fall on graph; y values in table get bigger/smaller	differences in y values in table are positive or negative	first derivative is positive or negative
Increasing/ decreasing at a fast or slow rate	graph is steep or flat	absolute value of differences in y values are large or small	values of derivative large or small in absolute value
Increasing/ decreasing at successively faster or slower rates	graph gets successively steeper or flatter	absolute value of differences in y values get bigger or smaller	sign of second derivative
Maximum/ minimum	peaks and dips in graph or highest and lowest points on graph	differences in y values change from positive to negative or vice versa; lowest and highest values of y	first derivative is zero or undefined
Point of inflection	graph changes from successively steeper to successively flatter	absolute value of differences in y change from getting bigger to getting smaller or vice versa.	second derivative is zero or undefined

Table 41 continued

Change and average rate of change over intervals	visual comparison of vertical rise or fall for a given horizontal change	numerical value of vertical change from graph or difference in consecutive y values	ratio of vertical to horizontal changes; ratio of changes in y to changes in x; ratio of $(f(x_{i+1}) - f(x_i))$ to $(x_{i+1} - x_i)$
instantaneous rate of change	visual slope of tangent line	conjecture limiting value based on numerical values of the average rate of change over several intervals; estimate value of slope of tangent line	use of limit definition derivative of equation for fct; evaluate first derivative at given value of x

APPENDIX C

TASKS FOR THIS STUDY

1. Two people start at opposite corners of a room and walk toward each other. As they walk, they both slow down as they get closer to each other, pass, and then they both speed up as they get farther apart. This takes a total of eight seconds. The opposite corners of the room are 20 feet apart.

- a. On the graph paper supplied, draw a graph showing the distance between the two people at each moment in time. Describe your graph.

- b. How does your graph show the two people slowing down?
How does your graph show them speeding up?
How does your graph show when they are walking faster?
How does your graph show when they are walking slower?

- c. Can you make a table of values that goes with your graph?

<u>time (sec)</u>	<u>distance apart (ft)</u>
-------------------	----------------------------

0

1

2

3

4

5

6

7

8

- d. How does your table show the two people slowing down?
How does your table show the two people speeding up?
How does your table show when they are walking faster?
How does your table show when they are walking slower?

2. These same two people decide again to start at opposite corners of the room and walk toward each other but this time they both decide to maintain the same steady pace the whole way. Again it takes a total of eight seconds for each to walk the 20 feet.

a. On the graph paper supplied, draw a graph showing the distance between the two people at each moment in time. Describe your graph.

b. How does your graph show the steady pace of the two people?

c. Can you make a table of values that goes with this graph?

<u>time (sec)</u>	<u>distance apart (ft)</u>
-------------------	----------------------------

0

1

2

3

4

5

6

7

8

d. How does your table show the steady pace of the two people?

3. These same two people decide once more to start at opposite corners and walk toward each other. But this time as they walk, they both speed up as they get closer to each other, pass, and then they both slow down as they get farther apart. Once again it takes eight seconds for each to walk the 20 feet.

a. On the graph paper supplied, draw a graph showing the distance between the two people at each moment in time. Describe your graph.

b. How does your graph show the two people speeding up? passing? slowing down?

c. Can you make a table of values that goes with your graph?

<u>time (sec)</u>	<u>distance apart (ft)</u>
0	
1	
2	
3	
4	
5	
6	
7	
8	

d. How does your table of values show the two people speeding up? passing? slowing down?

4. A ball is thrown straight up into the air. Graph 1 shows the height of the ball at selected points in its four second journey.

a. Describe the height of the ball over the four seconds.

b. Does the ball travel the same number of feet each quarter of a second or does the ball travel more feet some quarter seconds and less feet other quarter seconds?

During which quarter second(s) does it travel the most? How do you know? How fast does the ball travel during these quarter second(s)?

During which quarter second(s) does the ball travel the least? How do you know? How fast does the ball travel during these quarter seconds?

During which quarter seconds does the ball travel the same number of feet? How do you know? How fast does the ball travel during these quarter seconds?

c. How fast is the ball traveling at the end of the first second? How do you know? (How far has the ball traveled in the first second? How fast has the ball traveled in the first second?)

How fast is the ball traveling at the end of two seconds? How do you know?

How fast is the ball traveling at the end of three seconds?
How do you know?

How fast is the ball traveling at the end of four seconds?
How do you know?

How fast is the ball traveling just before it hits the
ground? (How fast is the ball traveling when it hits
the ground?)

Table 1 shows the values for the data points in Graph 1. Let's
use this table to look again at the questions in c. You are
free to use just the table to answer the questions or to use
both the table and the graph.

d. How fast is the ball traveling at the end of the first second?

How fast is the ball traveling at the end of two seconds?

How fast is the ball traveling at the end of three seconds?

How fast is the ball traveling at the end of four seconds?

How fast is the ball traveling just before it hits the
ground?

5. Graph 2 shows the population of yeast cells in a culture over a period of 18 hours (from midnight to 6pm).

a. Describe the population of yeast cells over this period of time.

b. Does the population of yeast cells change about the same each hour or does it change more some hours and less other hours? How do you know?

During which hour(s) does the population change the most?
How do you know? How fast does it change over those hour(s)?

During which hour(s) does the population change the least?
How do you know? How fast does it change over those hour(s)?

During which hours does the population change about the same? How do you know? How fast does it change over each of these hours?

c. How fast is the population changing at 4am? How do you know?

Table 2 shows the population of yeast cells at each hour. Let's use this table to look again at the questions in c. You are free to use just the table to answer the questions or you can use both the table and graph.

d. How fast is the population changing at 4am?

e. Graph 3 is an enlarged section of graph 2 and table 3 gives the population of yeast cells for selected time values in this section. Use these to find out how fast the population is changing at 4am.

f. How much does the population change over the interval from 3am to 4am?

How fast does the population change over the interval from 3am to 4am?

How much does the population change over the interval from 4am to 5am?

How fast does the population change over the interval from 4am to 5am?

How fast does the population change at 4am?

How much does the population change over the interval from 3:30am to 4am?

How fast does the population change over the interval from 3:30am to 4am?

How much does the population change over the interval from 4am to 4:30am?

How fast does the population change over the interval from 4am to 4:30am?

How fast does the population change at 4am?

How much does the population change over the interval from 3:45am to 4am?

How fast does the population change over the interval from 3:45am to 4am?

How much does the population change over the interval from 4am to 4:15am?

How fast does the population change over the interval from 4am to 4:15am?

How fast does the population change at 4am?

Describe a procedure for finding exactly how fast the population changes at 4am.

6. A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. Graph 4 shows the area of the circle as the radius grows from 0 to 5 feet.

a. Describe the area of the circle as the radius grows from 0 to 5 feet.

b. Does the area grow the same number of square feet for each one foot increase in the radius?

For which one foot increase in the radius does the area grow the least? How much does it grow? How fast does it grow?

For which one foot increase in the radius does the area grow the most? How much does it grow? How fast does it grow?

How much does the area grow as the radius grows from 2 to 3 feet? How fast does the area grow as the radius grows from 2 to 3 feet?

How much does the area grow as the radius grows from 3 to 4 feet? How fast does the area grow as the radius grows from 3 to 4 feet?

c. How fast is the area growing when the radius is exactly 3 feet?

Table 4 gives the area for values of the radius from 0 to 5 feet in steps of one-fourth of a foot.

- d. How fast does the area grow as the radius grows from 2 to 3 feet?

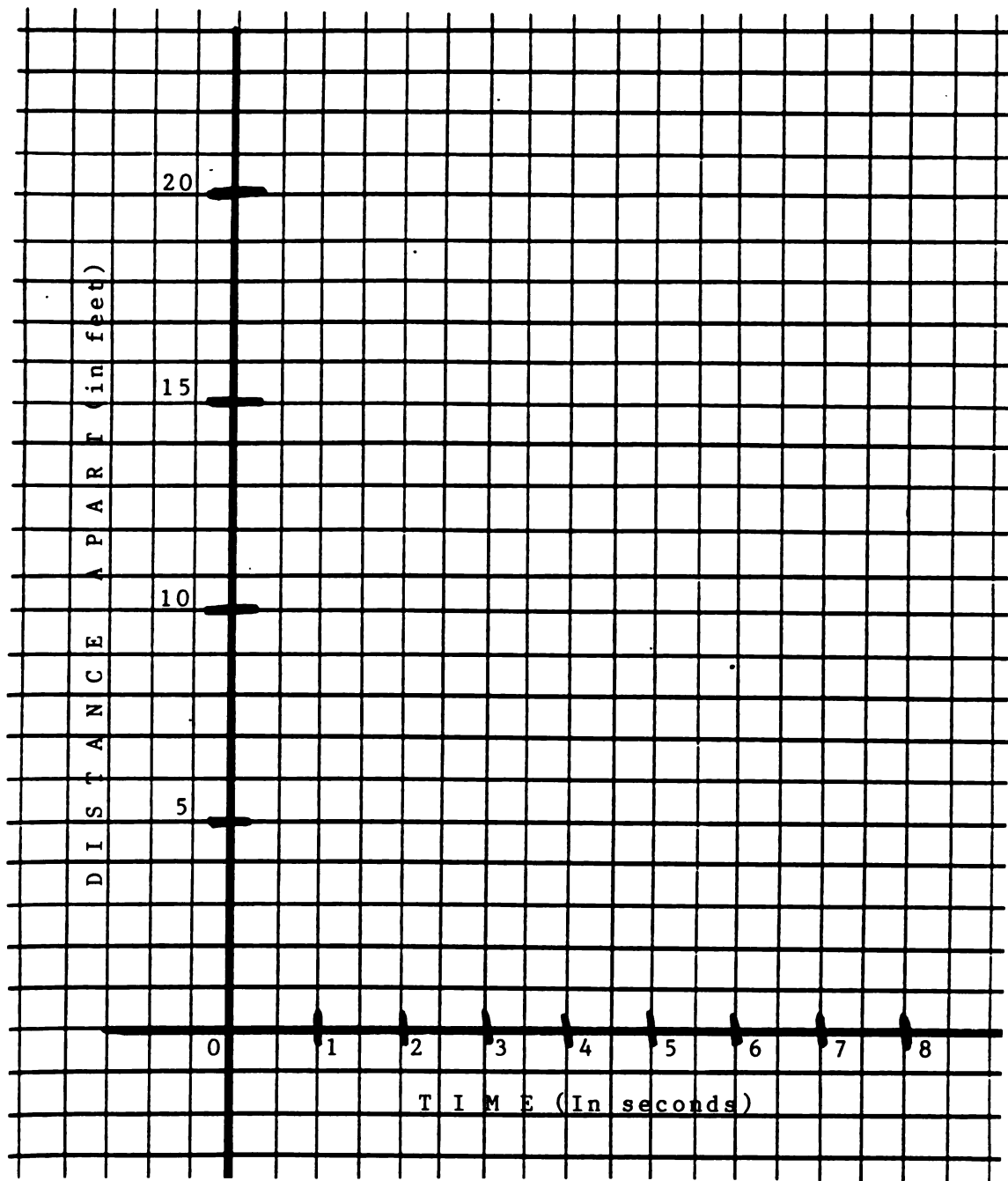
How fast does the area grow as the radius grows from 3 to 4 feet?

How fast is the area growing when the radius is exactly 3 feet?

How fast does the area grow as the radius grows from 2.75 to 3 feet?

How fast does the area grow as the radius grows from 3 to 3.25 feet?

How fast is the area growing when the radius is exactly 3 feet?



Graph 1

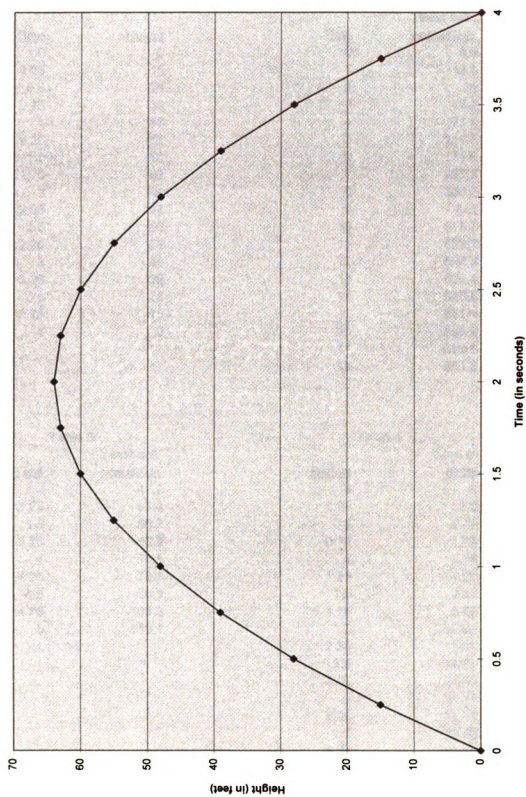


Table 1

<u>Time</u>	<u>Height</u>
0	0
0.25	15
0.5	28
0.75	39
1	48
1.25	55
1.5	60
1.75	63
2	64
2.25	63
2.5	60
2.75	55
3	48
3.25	39
3.5	28
3.75	15
4	0

Table 2

<u>Time</u>	<u>Yeast cell population</u>
0	9.6
1	18.3
2	29
3	47.2
4	71.1
5	119.1
6	174.6
7	257.3
8	350.7
9	441
10	513.3
11	559.7
12	594.8
13	629.4
14	640.8
15	651.1
16	655.9
17	659.6
18	661.8

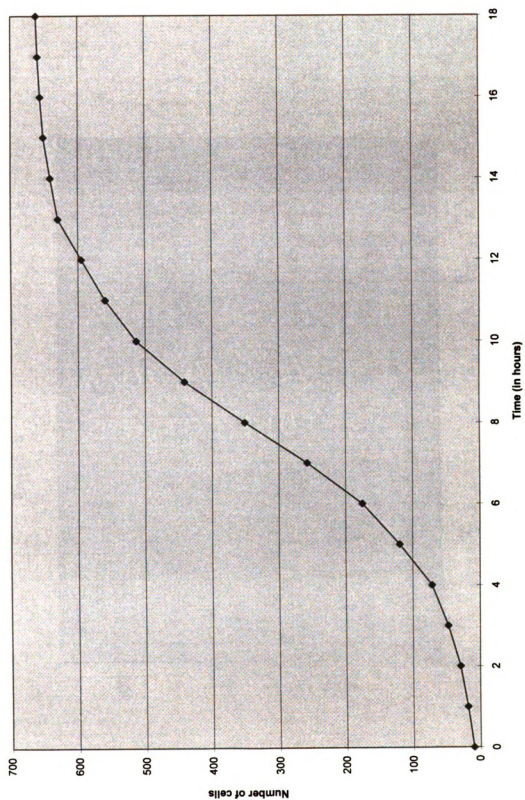
Table 3

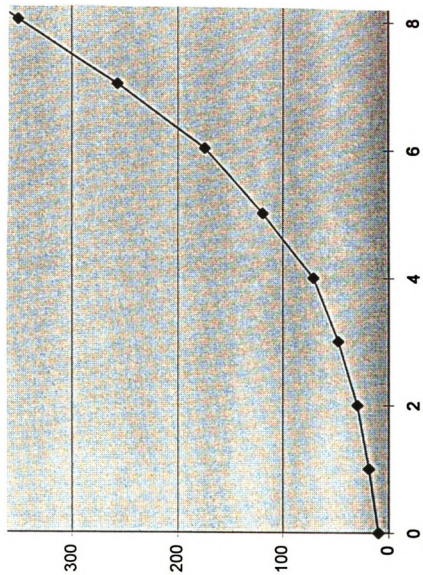
<u>Time</u>	<u>Yeast cell population</u>
3	47.2
3.25	51.4
3.5	56.7
3.75	63.2
4	71.1
4.25	79.7
4.5	90.3
4.75	103.2
5	119.1

Table 4

<u>Radius</u>	<u>Area of circle</u>
0	0
0.25	0.2
0.5	0.79
0.75	1.77
1	3.14
1.25	4.91
1.5	7.07
1.75	9.62
2	12.57
2.25	15.9
2.5	19.63
2.75	23.76
3	28.27
3.25	33.18
3.5	38.48
3.75	44.18
4	50.27
4.25	56.75
4.5	63.62
4.75	70.88
5	78.54

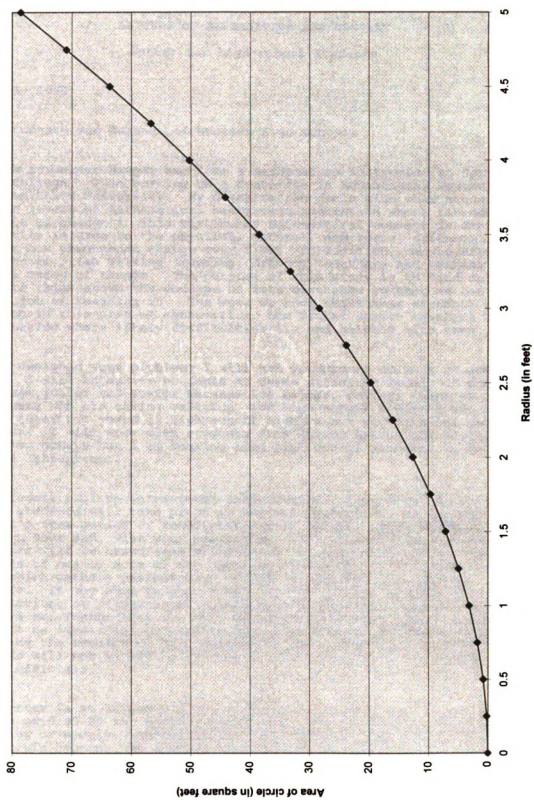
Graph 2





Graph 3

Graph 4



APPENDIX D

LETTERS OF EXPLANATION AND CONSENT

Letter for high school students

April 1, 1994

Dear Students and Parents of Western High School,

My name is Garnet Hauger and I am a mathematics instructor at Spring Arbor College. I am working on a doctorate in Educational Psychology at Michigan State University. My dissertation deals with what students at various levels of mathematical background understand about the concept of 'rate of change.' This particular mathematical concept is important because it is used in the physical, natural, and social sciences to analyze how phenomenon change over time. In addition, many everyday experiences, like walking, running, driving, weather, and banking involve rates of change. The purpose of this study is to find out how students think about the concept of rate of change so that we can do a better job of teaching it. The more we know about what students are thinking in this area of mathematics, the better chance teachers have of anticipating where likely difficulties lie and helping with them.

I have designed some problems I will ask volunteers to work on and talk about. I will interview students at three different levels of schooling - college junior and senior mathematics majors, college freshmen and sophomores who are taking calculus, and high school students who have had at least two years of high school algebra but who have not yet had calculus. I will work with students from Spring Arbor College for the first two groups but I am seeking some high school students to volunteer for the last group.

Each student will be interviewed individually. Interviews with high school students will take place at Western High School during the student's free period or immediately after school. Interviews will last about an hour and, with your permission, will be audio and video taped. The tapes will be heard/seen primarily by me and my advisor. I may show excerpts of one or more of the tapes to some members of a small research group which gathers periodically to help each other with ongoing projects. It may also be that excerpts will be used as part of presentations at professional meetings involving other mathematics teachers and researchers who have an interest in this study. Students will not be identified by name or school when using the tapes or in reporting the results of this research. The performance of individual students will not be made available to any current or future teachers or school officials.

This letter is to invite you to participate in this study. Each student will be paid \$5.00 for participating. If a student wants to stop the interview or simply stop for the day, s/he will be allowed to do so. If this should happen, the student will be paid for the part they have actually completed. If your son or daughter is willing to be a part of this study and if you are willing to give your permission for his/her involvement, simply sign the consent form on the next page and return in the self-addressed stamped envelope. An extra copy of the consent form is on the back of this letter for your information. If you have any

questions about this study, please call me at the office (750-6377) or at home (750-4841). I will return your call as soon as possible. If you would like a short report of the results of this study, I would be happy to make that available even if your son or daughter does not participate. Just check the appropriate blank on the consent form.

Thank you for your time.

Sincerely,

Garnet Hauger,
Associate Professor of Mathematics
Chair, Mathematics Department

Parent Permission Form:

_____ may voluntarily participate in the
(name of student) study described in this letter.

_____ may not participate in the
(name of student) study described in this letter.

Date:

Signature of Parent/Guardian:

Signature of Student:

_____ I would like a short report of the results of this study. When
that report becomes available, please send it to this
address:

Letter for college students

April 1, 1994

Dear College Student,

As many of you know, I am working on a doctorate in Educational Psychology at Michigan State University. My dissertation deals with what students at various levels of mathematical background understand about the concept of 'rate of change.' This particular mathematical concept is important because it is used in the physical, natural, and social sciences to analyze how phenomenon change over time. In addition, many everyday experiences, like walking, running, driving, weather, and banking involve rates of change. The purpose of this study is to find out how students think about the concept of rate of change so that we can do a better job of teaching it. The more we know about what students are thinking in this area of mathematics, the better chance teachers have of anticipating where likely difficulties lie and helping with them.

For this study I have designed some problems which I will ask volunteers to work on and talk about. I will interview students at three different levels of schooling - college juniors and seniors who are mathematics majors, college freshmen and sophomores who are taking calculus, and high school students who have had at least two years of high school algebra but who have not yet had calculus.

Each student will be interviewed individually. The interviews with the college students will take place here on our campus. They will last about an hour and, with your permission, will be audio and video taped. The tapes will be heard/seen primarily by me and my advisor. I may show excerpts of one or more of the tapes to some members of a small research group which gathers periodically to help each other with ongoing projects. It may also be that excerpts will be used as part of presentations at professional meetings involving other mathematics teachers and researchers who have an interest in this study. Students will not be identified by name or school when using the tapes or in reporting the results of this research. Your performance as an individual will not be reported to any current or future teacher or school official.

This letter is to invite your participation in this study. Participation is strictly voluntary. If you choose to participate, your performance will in no way affect your grade in any course you are taking or will take from me. If you choose not to participate, that choice will not in any way affect your grade in any course you are taking or will take from me. If you choose to participate, you will be given extra credit in a course you are taking from me. If you choose not to participate, you will be given the opportunity to earn extra credit in another way. If, for any reason, a participating student wants to stop the interview or simply stop for the day, s/he will be allowed to do so. However, to get the full amount of extra credit, the student needs to complete the entire interview. If you would like to take part in this study, please sign the consent form below and return it to Garnet Hauger, Station 14. The extra copy is for your information. If you have any questions about this research, please call me at extension 1377 or stop by my office, room 102c in the Whiteman-Gibbs Science Center.

If you would like a short report of the results of this study, I would be happy to make that available even if you choose not to participate. Just check the appropriate blank on the consent form.

Sincerely,

Garnet Hauger
Associate Professor of Mathematics
Chair, Mathematics Department

Student Permission Form:

_____ I would like to participate in the research described in this letter.

_____ I would prefer not to participate in this research.

Signature of Student:

Local Address:

Local Phone Number:

_____ I would like a short report of the results of this study. When that report becomes available, please sent it to this address:

APPENDIX E

**TABLE 42 - WAYS STUDENTS FOUND THE SPEED OF THE BALL
AT $t = 1, 2, 3$, AND 4 IN TASK 4**

Strategy	$t=1$	$t=2$	$t=3$	$t=4$
Analysis of moving objects	slowing down	zero;*1 stopped	speeding up; *2	fastest; *3
Reference to a point *4	48/1 48 ft/sec	64/2 32 ft/sec	48/3 16 ft/sec	0/4 0 ft/sec
Avg. speed since $t=0$ or $t=2$ *6	48/1 48 ft/sec	64/2 32 ft/sec	80/3 *5 27 ft/sec	128/4 32 ft/sec
Avg. spd. over prev. sec.	48 ft/sec	16 ft/sec	16 ft/sec	48 ft/sec
Avg. spd. over prev. 1/4 sec.	36 ft/sec	4 ft/sec	28 ft/sec	60 ft/sec
Avg. spd. over 1/2 sec. around pt.	32 ft/sec	0 ft/sec	32 ft/sec	64 ft/sec
Reference to slope *7	32 ft/sec	0 ft/sec	32 ft/sec	64 ft/sec

*1 Likely reasons given; since it's increasing before and decreasing after or going up before and going down after or slowing down before and speeding up after or has a maximum there.

*2 Some students said "same as at $t=1$ ".

*3 Some students said "same as at $t=0$ ".

*4 A variation of this includes giving the height per second, like 64 ft/sec at $t=2$ and 48 ft/sec at $t=3$.

*5 For $t=3$ and $t=4$ students may have used total distance from $t=2$.

*6 A variation of this includes giving the total distance per second, like 64 ft/sec at $t=2$ and 80 ft/sec at $t=3$.

*7 For $t=2$ students may say the speed is zero because the slope is going from positive before $t=2$ to negative after $t=2$.

APPENDIX F

**TABLE 43 - WAYS STUDENTS FOUND THE RATE OF GROWTH OF THE
CELL POPULATION AT T = 4 AM IN TASK 5**

<u>Strategy</u>	<u>Rate of change of population at t=4 am</u>
Reference to a point	71.1 cells per hour or $71.1/4 = 17.8$ cells per hour
Avg. growth rate since t=0	$(71.1 - 9.6) = 61.5$ cells over four hours or $61.6/4 = 15.4$ cells per hour
Avg. growth over the previous hour	$(71.1 - 47.2) = 23.9$ cells per hour
Avg. growth over two hours around t=4 am	$(119.1 - 47.2) = 71.9$ cells over two hours or $71.9/2 = 35.95$ cells per hour
Avg. growth over half hour around t=4 am	$(79.7 - 63.2) = 16.5$ cells over a half or $16.5/.5 = 33$ cells per hour
Reference to slope	The slope of the line tangent to the curve at t=4 am is about 33 cells per hour
Not quite doubling	change from t=3 to t=4 is about 24 and the change from t=4 to t=5 is about 48 so the rate of change is about doubling
Multiplied by two-thirds	The change from t=4 to t=5 is about 50 which is two thirds of the change from t=0 to t=4
Increased by one-third	The change from t=3 to t=4 is one third the change from t=0 to t=4
12 percent increase	The change from t=4 to t=4.25 is about 12 percent the growth from t=0 to t=4

APPENDIX G

TABLE 44 - WAYS STUDENTS FOUND THE RATE OF CHANGE OF THE AREA OF THE CIRCLE AT $r = 3$ FEET IN TASK 6

<u>Strategy</u>	<u>Rate of change of area at $t=3$ feet</u>
Reference to a point	28.27 square feet per foot or $28.27/3 = 9.4$ square feet per foot
Avg. growth rate since $r=0$	28.27 square feet per foot or $28.27/3 = 9.4$ square feet per foot
Avg. growth rate from $r=2$ to $r=4$	$(50.27 - 12.57) = 37.7$ square feet over two feet or $37.7/2 = 18.85$ square feet per foot or $(16\pi - 4\pi) = 12\pi$ over two feet = 6π square feet per foot
Avg. growth rate from $r=2.75$ to $r=3$	$(28.27 - 23.76) = 4.51$ square feet over one fourth of a foot or $4.51/.25 = 18.04$ square feet per foot
Avg. growth rate from $r=2.75$ to $t=3.25$	$(33.18 - 23.76) = 9.42$ square feet over a half foot or $9.42/.5 = 18.84$ square feet per foot
Reference to slope	$2\pi \cdot 3 = 6\pi = 18.85$ square feet per foot

APPENDIX H

**TABLE 45 - STRATEGIES AND THEIR CLASSIFICATIONS
BY PERSPECTIVE AND TYPE OF REASONING**

	Perspective	Reasoning
Category 1 - Based on slope or shape of graph:		
Visual slope	Global	Qualitative
Reference to slope of secant line	Interval	Qualitative & Algebraic
Reference to slope of tangent line	Point	Qualitative & Algebraic
Knowledge of situation	Global	Qualitative
Category 2 - Based on comparison of changes in one or both variables		
Subgroup 1 - Based on comparison of change in the dependent variable to change in the independent variable.		
Delta d	Interval	Qualitative & Numerical
Delta t	Interval	Qualitative & Numerical
Delta d and delta t	Interval	Qualitative & Numerical
$(\Delta d)/(\Delta t)$	Interval	Algebraic
Avg. rate of change over intervals larger than one unit before point	Interval	Algebraic
Avg. rate of change over intervals of one unit before point	Interval	Algebraic
Avg. rate of change over intervals less than one unit before point	Interval	Algebraic
Avg. rate of change over intervals straddling point.	Interval	Algebraic

Table 45 continued

	Perspective	Reasoning
Subgroup 2 - Based on a comparison of a change in the dependent variable to another change in the dependent variable		
"Not quite doubling"	Interval	Numerical
"Multiplied by two-thirds"	Interval	Algebraic
"Increased by one-third"	Interval	Algebraic
"12 percent increase"	Interval	Algebraic
Category 3 - Based on a specific point		
Reference to a point	Point	Numerical & Algebraic
Meeting point	Global	Qualitative
Velocity equation	Interval & Point	Algebraic

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