THREE-MANIFOLDS OF HIGHER RANK

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ABSTRACT

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Fixing $\varepsilon = -1, 0$, or 1, a complete Riemannian manifold is said to have higher hyperbolic, Euclidean, or spherical rank if every geodesic admits a normal parallel field making curvature ε with the geodesic. In this thesis, we establish rigidity results for three-manifolds of higher rank without a priori sectional curvature bounds. Complete finite volume three-manifolds have higher hyperbolic rank if and only if they are finite volume hyperbolic space forms. Complete three-manifolds have higher spherical rank if and only if they are spherical space forms.

In addition to the rigidity results, we also provide constructions of non-homogeneous manifolds of higher hyperbolic rank of infinite volume. These examples show the necessity of the finite volume assumption in hyperbolic rank rigidity results.

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Chapter 1

Introduction

Locally symmetric spaces are defined as spaces symmetric with respect to local geodesic involutions, or equivalently, spaces with parallel curvature tensors. Properties of locally symmetric spaces were first studied systematically by Elie Cartan through algebraic classifications. These properties indeed manifest the large symmetries within this special class of Riemannian manifolds. For example, every geodesic on a certain class of locally symmetric spaces (compact rank one symmetric spaces) is contained in a totally geodesic two sphere of constant sectional curvatures one. It is natural to ask if these structures of symmetry characterize locally symmetric spaces.

Rank rigidity results characterize locally symmetric spaces through the following geometric notions of rank. Fixing $\varepsilon = -1, 0$, or 1, a complete Riemannian manifold is said to have higher hyperbolic, Euclidean, or spherical rank if every geodesic admits a normal parallel field making curvature ε with the geodesic.

Manifolds of higher hyperbolic (respectively spherical) rank includes noncompact (respectively compact) rank one symmetric spaces and their Riemannian quotients. Manifolds of higher Euclidean rank includes Riemannian products of manifolds and locally symmetric spaces of higher real rank.

Historically, the notion of Euclidean rank first appeared in the study of nonpositively curved manifolds. Roughly speaking, the Euclidean rank reflects the amount of infinitesimal flat planes in nonpositively curved manifolds. It turns out that structures of nonpositively curved compact manifolds of higher Euclidean rank are quite rigid. Ballmann [1], and independently by Burns and Spatzier [5], proved that a compact, locally irreducible, nonpositively curved manifold of higher Euclidean rank is locally symmetric. The proof of Ballmann also works when the compactness assumption is replaced by the finite volume assumption and a lower bound on sectional curvatures. The Euclidean rank rigidity result was further generalized by Eberlein and Heber in [9] to nonpositively curved manifolds whose fundamental groups satisfy the duality condition, a dynamical assumption on their actions on the universal cover.

On the other hand, nonnegatively curved manifolds of higher Euclidean rank are not rigid. In [17], Spatzier and Strake construct many locally irreducible manifolds of higher Euclidean rank that are not symmetric.

Hamenstädt first generalized the notion of rank into the hyperbolic setting, proving that compact manifolds of higher hyperbolic rank are locally symmetric if $\sec \leq -1$ [10]. The sectional curvature bound was relaxed by Constantine in some cases. He showed that, when the dimension is odd, or when the sectional curvatures are very closely pinched, a compact, nonpositively curved manifold of Euclidean rank one (means not having higher Euclidean rank) has higher hyperbolic rank if and only if it is real hyperbolic [8]. Recently, Connell, Nguyen, and Spatzier showed that a compact manifold of higher hyperbolic rank is locally symmetric if its sectional curvatures are bounded between -1 and $-\frac{1}{4}$ [6].

The first spherical rank rigidity result was given by Shankar, Spatzier, and Wilking. Through a Morse theoretic approach, they showed that manifolds of higher spherical rank are locally symmetric if $\sec \leq 1$ [16]. Partial spherical rank rigidity results were obtained by Schmidt, Shankar, and Spatzier in [14] for manifolds with $\sec \geq 1$.

In contrast, there are fewer rank rigidity results without a priori assumptions on sectional curvatures. Existing results include the works of Molina-Olmos [12], Watkins [18], and Bettiol-Schmidt [3], where Euclidean rank-rigidity results are proved after replacing curvature assumptions with assumptions on having many flats, having no focal points, and dimension, respectively. Specifically, Molina and Olmos showed that a manifold that satisfies a strong irreducibility condition must be locally symmetric if every geodesic is contained in a closed immersed flat of dimension at least 2. The work of Watkins implies that a finite volume, locally irreducible manifold of higher Euclidean rank without focal point is locally symmetric. Bettiol and Schmidt proved that the universal cover of a complete three-manifold splits isometrically if every geodesic in the manifold admits a parallel Jacobi field. Notice that, without a priori assumption on sectional curvature bounds, a parallel field making curvature 0 with the geodesic needs not to be Jacobi.

The main goal of this thesis is to establish hyperbolic and spherical rank rigidity results in dimension three without a priori sectional curvature bounds. In other words, we prove the following two rigidity results.

Theorem 1. A finite volume complete Riemannian three-manifold M has higher hyperbolic rank if and only if M is a finite volume hyperbolic space form.

Theorem 2. A complete Riemannian three-manifold M has higher spherical rank if and only if M is a spherical space form.

In the Euclidean rank case, we show that one can replace the Jacobi assumption with the finite volume assumption in the rank rigidity theorem of Bettiol and Schmidt [3, Theorem A].

Theorem 3. Let M a finite volume three-manifold of higher Euclidean rank. Then the universal cover of M must split isometrically as $N \times \mathbb{R}$.

The finite volume assumption in Theorem 1 is necessary. In this thesis, we give constructions of non-homogeneous manifolds of higher hyperbolic rank with infinite volume. The following theorem is a cleaner version of Theorem 6.1.

Theorem 4. For each dimension $d \geq 3$, there exist a one parameter family of complete, smooth Riemannian metrics g_s on $\mathbb{R}^d$ with infinite volume such that

(1) g_s are not homogenerous with infinite volume,

(2) every g_s geodesic is contained in a totally geodesic hyperbolic plane of curvature -1, and

(3) the sectional curvatures of g_s are between -1 and $-2 + e^s$.

Note that (2) in Theorem 4 implies that these examples have higher hyperbolic rank. To my knowledge, these are the first examples of non-homogeneous manifolds of higher hyperbolic rank. It shows that infinite volume manifolds of higher hyperbolic rank may even fail to be homogeneous, even when the sectional curvatures are arbitrary closely pinched. A more detailed description of properties of these spaces is in Chapter 6.

It should be noted that a non-symmetric manifold with infinite volume and higher hyperbolic rank has already been constructed by Connell in [7]. Homogeneous negatively curved manifolds split as semidirect products of $\mathbb{R}$ and nilpotent Lie groups. The homogeneous examples was constructed on the space obtained by gluing $\mathbb{C}H^2$ and a real hyperbolic plane along the $\mathbb{R}$ factors. Our last theorem show that these homogeneous examples, however, do not exist in dimension three.

Theorem 5. A homogeneous three-manifold of higher hyperbolic rank is a hyperbolic space form.

Our strategy for proving the rigidity theorems is to analyze the local structure of three-manifolds of higher rank in terms of Ricci diagonalizing orthonormal frames. This is roughly done as follows. At non-isotropic points, the Ricci tensor either has two or three distinct eigenvalues. The set of points where the Ricci tensor has three distinct eigenvalues is called the generic set. The majority of work in this paper is to show that on the generic set, the Christoffel symbols of the Ricci diagonalizing frame satisfy an overdetermined system of differential equations. From the system of equations, we deduce that the generic set must be empty for manifolds of higher spherical rank, finite volume manifolds of higher hyperbolic rank, or finite volume three-manifolds of higher Euclidean rank. Hence the rigidity problem for a three-manifold M of higher rank is reduced to the case when M has *extremal curvatures*, that is $\sec_p \geq \varepsilon$ or $\sec_p \leq \varepsilon$ for each $p \in M$.

Three-manifolds with $\mathrm{cvc}(\varepsilon)$, a pointwise notion of having higher rank, and with extremal curvature ε are studied by Schmidt and Wolfson in [15]. Their structural results are strengthened in this thesis by the higher rank assumption, leading to the desired hyperbolic or spherical rigidity results.

In the Euclidean rank case, the parallel fields making curvature 0 with geodesics are actually Jacobi if M has extremal curvatures. Hence the Euclidean rank rigidity result directly follows from the rank rigidity result of Bettiol and Schmidt after showing that the generic set is empty.

We organize the thesis as follows. In Chapter 2, we introduce preliminary results, including a partition of three-manifolds of higher rank into three disjoint sets: isotropic set, extremal set, and generic set. To establish Theorem 1, 2 and 5, it suffices to show that the generic set and the extremal set are empty. To do so, we first study the local structures of generic and extremal sets in Christoffel symbols of Ricci diagonalizing orthonormal frames in Chapter 3. We then derive differential equations of geometric quantities in Chapter 4. In the last section of Chapter 4, we show that the extremal set is empty for the hyperbolic and spherical cases provided that the generic set is empty. Using the differential equations derived in Chapter 4, we prove the rigidity theorems in Chapter 5 by showing that the generic set must be empty. The constructions of non-homogeneous manifolds of higher hyperbolic rank are in Chapter 6. Finally, we give conjectures and possible directions for future works in Chapter 7.

Chapter 2

Preliminary

This section introduces notations and preliminary results. Most of the results in this chapter can be found in [15]. Unless otherwise stated, M denotes a complete Riemannian three-manifold with Levi-Civita connection ∇ and curvature tensor R. For each $p \in M$, let T_pM and U_pM denote the tangent space and unit tangent space at p respectively.

As M is three dimensional, an orthonormal frame $\{e_1, e_2, e_3\}$ of T_pM diagonalizes the Ricci curvature tensor if and only if

$$R(e_1, e_2, e_3, e_1) = R(e_1, e_2, e_2, e_3) = R(e_1, e_3, e_3, e_2) = 0. \tag{2.0.1}$$

Let $\lambda_{ij} = \sec(e_i, e_j)$. Throughout the remainder of the paper, we assume that indices have been chosen such that

$$\lambda_{13} \leq \lambda_{12} \leq \lambda_{23}. \tag{2.0.2}$$

Direct computations using (2.0.1) shows that all sectional curvatures are computable in terms of the λ_{ij} as described in the next lemma.

Lemma 2.1. *Let σ be a 2-plane with unit normal vector $u = c_1e_1 + c_2e_2 + c_3e_3$. Then $\sec(\sigma) = c_1^2\lambda_{23} + c_2^2\lambda_{13} + c_3^2\lambda_{12}$.*

2.1 A Partition of Three-manifolds of Higher Rank

Fix $\varepsilon = -1, 0$, or 1. Recall that M has higher rank if every geodesic $\gamma : \mathbb{R} \to M$ admits a parallel field $V(t)$ perpendicular to $\gamma'(t)$ such that $\sec(V(t), \gamma'(t)) \equiv \varepsilon$ for all $t \in \mathbb{R}$. The following pointwise version of the rank assumption was introduced in [15] to study the local structure of manifolds of higher rank.

Definition 2.2. A Riemannian manifold has constant vector curvature ε, denoted by $\mathrm{cvc}(\varepsilon)$, if every tangent vector is contained in a curvature ε plane.

Clearly, manifolds of higher rank have $\mathrm{cvc}(\varepsilon)$. For $\mathrm{cvc}(\varepsilon)$-manifolds, the following argument using Lemma 2.1 and (2.0.2) implies that λ_{12} equals ε. If $\lambda_{12} < \varepsilon$, then $\lambda_{13} < \varepsilon$ and the vector e_1 is not contained in a curvature ε plane, which is a contradiction. Hence $\lambda_{12} \geq \varepsilon$. A similar argument shows that $\lambda_{12} \leq \varepsilon$, which implies that $\lambda_{12} = \varepsilon$.

Letting $\lambda = \lambda_{13}$ and $\Lambda = \lambda_{23}$, we then have

$$\lambda_{13} = \lambda \leq \varepsilon \leq \Lambda = \lambda_{23}. \tag{2.1.1}$$

Definition 2.3. A point $p \in M$ is said to be

(1) *isotropic* if $\lambda = \Lambda = \varepsilon$,

(2) *extremal* if precisely one of λ or Λ equals ε,

(3) *generic* if $\lambda < \varepsilon < \Lambda$.

Throughout, let $\mathcal{I}, \mathcal{E}, \mathcal{O}$ denote the set of isotropic, extremal, and generic points respectively. A three-manifold M of higher rank is the disjoint union

$$M = \mathcal{I} \cup \mathcal{E} \cup \mathcal{O}.$$

As the sectional curvature is constant on $\mathcal{I}$, hyperbolic or spherical rank rigidity occurs when $\mathcal{E} = \mathcal{O} = \emptyset$.

Note that for each $p \in \mathcal{E}$, either $\lambda < \varepsilon = \Lambda$ or $\lambda = \varepsilon < \Lambda$ at p. This motivates the following partition of $\mathcal{E}$.

Definition 2.4. Define subsets $\mathcal{E}_-$ and $\mathcal{E}_+$ of $\mathcal{E}$ by

$$\mathcal{E}_- = \{p \in \mathcal{E} \mid \lambda = \varepsilon\}$$

and

$$\mathcal{E}_+ = \{p \in \mathcal{E} \mid \Lambda = \varepsilon\}.$$

When p is not an isotropic point, one can apply Lemma 2.1 to characterize the curvature ε planes in T_pM in terms of their normal vectors.

Lemma 2.5. *Let $p \in \mathcal{E} \cup \mathcal{O}$ and let σ be a 2-plane in T_pM with unit normal vector $u = c_1e_1 + c_2e_2 + c_3e_3$. Then $\sec(\sigma) = \varepsilon$ if and only if*

(1) $c_1 = 0$ when $p \in \mathcal{E}_-$,

(2) $c_2 = 0$ when $p \in \mathcal{E}_+$,

(3) $c_1 = \pm mc_2$ when $p \in \mathcal{O}$, where $m = \sqrt{\frac{\varepsilon-\lambda}{\Lambda-\varepsilon}}$.

Proof. Immediate from Lemma 2.1. $\square$

We define the following disjoint partition of U_pM.

Definition 2.6. A vector $X \in U_pM$ is

(1) *isocurved* if every 2-plane in T_pM containing X has curvature ε,

(2) *unicurved* if X is contained in only one curvature ε 2-plane,

(3) *generic* if it is neither isocurved nor unicurved.

Definition 2.7. A geodesic $\gamma : (k, l) \to M$ is isocurved if $\gamma'(t)$ is isocurved for all $t \in (k, l)$.

Since the normal vectors of 2-planes containing X form a two dimensional vector subspace, Lemma 2.5 implies the following characterization of isocurved vectors.

Corollary 2.8. *Let $p \in \mathcal{E} \cup \mathcal{O}$ and let $X = x_1 e_1 + x_2 e_2 + x_3 e_3 \in U_p M$. Then X is isocurved if and only if*

(1) $x_1 = \pm 1$ if $p \in \mathcal{E}_-$,

(2) $x_2 = \pm 1$ if $p \in \mathcal{E}_+$,

(3) $x_3 = 0$ and $x_2 = \pm m x_1$ if $p \in \mathcal{O}$.

On the interior of the generic set or on the interior of the extremal set, it is possible to find a local orthonormal frame that satisfies (2.0.2). Let $\{e_1, e_2, e_3\}$ be such a local orthonormal frame. Throughout, we use the Christoffel symbol Γ_{ij}^k to denote $< \nabla_{e_i} e_j, e_k >$. As the connection is metric, we have that $\Gamma_{ij}^k = -\Gamma_{ik}^j$. In particular, $\Gamma_{ik}^k = 0$.

Lemma 2.9. *For each $p \in \mathcal{O}$,*

$$\Gamma_{11}^3 = m^2 \Gamma_{22}^3 \tag{2.1.2}$$

Proof. The lemma follows from (2.0.1) and the differential Bianchi identity

$$(\nabla_{e_3}) R(e_1, e_2, e_2, e_1) + (\nabla_{e_1}) R(e_2, e_3, e_2, e_1) + (\nabla_{e_2}) R(e_3, e_1, e_2, e_1) = 0.$$

□

Remark 2.10. As the eigenvalues of the Ricci tensor are distinct at points in $\mathcal{O}$, the eigenspaces define three distinct *global* smooth line fields on $\mathcal{O}$. Given a connected component $\mathcal{C}$ of $\mathcal{O}$, each of these line fields may or may not be orientable over $\mathcal{C}$. In the case where one of these line fields is not orientable on $\mathcal{C}$, it resolves to an oriented line field on some double cover of $\mathcal{C}$. As the rank assumption and finite volume assumption passes to finite covers, we may always assume that there exists a *global* Ricci diagonalizing orthonormal frame $\{e_1, e_2, e_3\}$ on $\mathcal{C}$ (the finite volume assumption is required for the proof of hyberbolic rank rigidity).

2.2 Manifolds of Extremal Curvature

Definition 2.11. A three-manifold M of $\mathrm{cvc}(\varepsilon)$ is said to have extremal curvature if for every $p \in M$, $\sec_p \geq \varepsilon$ or $\sec_p \leq \varepsilon$.

The definition is equivalent to $\mathcal{O} = \emptyset$, or equivalently, $M = \mathcal{E} \cup \mathcal{I}$. The following proposition, a structural result used in subsequent sections, is proved in [11] or [15, Corollary 2.10].

Proposition 2.12. *Let M be a three-manifold of* $\mathrm{cvc}(\varepsilon)$ *and with extremal curvature. For each connected component of $\mathcal{C}$ of $\mathcal{E}$, there exists a complete geodesic field E on $\mathcal{C}$ or a double cover of $\mathcal{C}$ whose integral curves are isocurved geodesics.*

The structural result leads to the following rigidity theorems proved in [15, Theorem 1.2] and [15, Theorem 3.5] respectively. A different proof of Theorem 2.13 is given in Chapter 4.

Theorem 2.13. *A finite volume complete Riemannian three-manifold with extremal curvature -1 and higher hyperbolic rank is hyperbolic.*

Theorem 2.14. *Assume that M is a connected, simply-connected, complete and homogeneous three-manifold with extremal curvature -1. If M has higher hyperbolic rank, then M is isometric to the three dimensional hyperbolic space.*

Chapter 3

Local Structures

3.1 Local Structures on Generic Sets

This section studies the local structure of generic points in a three-manifold of higher rank. As the generic set $\mathcal{O}$ is open, we may consider a local orthonormal frame $\{e_1, e_2, e_3\}$ and satisfying (2.1.1). With respect to this local orthonormal frame,

$$\lambda < \varepsilon < \Lambda. \tag{3.1.1}$$

Specifically, this section derives local equations satisfied by the Christoffel symbols with respect to this frame in a three-manifold of higher rank. Roughly speaking, these equations will be derived as follows.

Lemma 3.2 below will show that each generic vector tangent to $\mathcal{O}$ belongs to exactly two curvature ε planes. Generic vectors are characterized by an open condition on tangent vectors. Hence, given a geodesic with generic initial velocity vector, its velocity vectors remain generic for short time, and consequently, also belong to exactly two curvature ε planes. As geodesics have higher rank, they are also contained in a *parallel* family of curvature ε planes. As a result, the parallel field equations apply to at least one of these two families of curvature ε planes along generic geodesics. Essentially, these parallel field equations imply the desired Christoffel symbol relations.

3.1.1 Pointwise Calculations

Given a generic point $p \in \mathcal{O}$, each unit tangent vector $X \in U_pM$ may be expressed in the frame above as $X = x_1e_1 + x_2e_2 + x_3e_3$, where $x_1^2 + x_2^2 + x_3^2 = 1$. For simplicity, we will frequently write $X = (x_1, x_2, x_3)$, omitting the frame vectors from our notation.

Lemma 3.1. *Let $p \in \mathcal{O}$. For $X = (x_1, x_2, x_3) \in U_pM$ a generic or unicurved unit vector,*

$$1 + m^2 - (mx_2 - x_1)^2 > 0$$

and

$$1 + m^2 - (-mx_2 - x_1)^2 > 0.$$

Proof. We prove the first inequality; the second inequality is obtained analogously after replacing m by $-m$. Note that

$$(mx_2 - x_1)^2 = |(x_1, x_2) \cdot (-1, m)|^2 \leq (x_1^2 + x_2^2)(1 + m^2) \leq 1 + m^2, \tag{3.1.2}$$

where the first inequality is from Cauchy-Schwartz and the second inequality is from the fact that X is a unit vector. The statement of the lemma holds if one of these inequalities is strict. The first inequality is strict unless $x_2 = -mx_1$, and the second inequality is strict unless $x_3 = 0$. The lemma now follows from Corollary 2.8.

□

By Lemma 3.1, we may associate to each generic or unicurved unit vector $X = (x_1, x_2, x_3)$ the numbers

$$K = \sqrt{1 + m^2 - (mx_2 - x_1)^2} \tag{3.1.3}$$

and

$$L = \sqrt{1 + m^2 - (-mx_2 - x_1)^2}, \tag{3.1.4}$$

and the following two unit vectors:

$$V^+ = \frac{1}{K}\big(1 + mx_1x_2 - x_1^2, mx_2^2 - x_1x_2 - m, (mx_2 - x_1)x_3\big) \tag{3.1.5}$$

and

$$V^- = \frac{1}{L}\big(1 - mx_1x_2 - x_1^2, -mx_2^2 - x_1x_2 + m, (-mx_2 - x_1)x_3\big). \tag{3.1.6}$$

Direct computations show that $V^+, V^- \in U_pM$, and $V^+, V^- \perp X$. Note that the expression for V^- can be obtained by changing every m in the formula for V^+ to $-m$.

Lemma 3.2. *Let $p \in \mathcal{O}$ and let $X = (x_1, x_2, x_3) \in U_pM$ be a unicurved or generic unit vector and let $V = (v_1, v_2, v_3)$ be a unit vector perpendicular to X.*

(1) If $x_3 = 0$, then $V^+ = \pm V^-$. Moreover, $\sec(X, V) = \varepsilon$ if and only if $V = \pm V^+$.

(2) If $x_3 \neq 0$, then V^+ and V^- are linearly independent. Moreover, $\sec(X, V) = \varepsilon$ if and only if $V = \pm V^+$ or $V = \pm V^-$.

Proof. Let $u = X \times V = (x_2v_3 - x_3v_2, x_3v_1 - x_1v_3, x_1v_2 - x_2v_1)$. As X and V are unit vectors and $X \perp V$, u is a unit normal vector to the plane spanned by X and V. By Lemma

2.5, $\sec(X, V) = \varepsilon$ if and only if

$$x_2 v_3 - x_3 v_2 = \pm m(x_1 v_3 - x_3 v_1).$$

In other words, a unit vector $V = (v_1, v_2, v_3)$ satisfies $V \perp X$ and $\sec(X, V) = \varepsilon$ if and only if it satisfies one of the following two linear systems:

$$\begin{aligned} &x_1 v_1 + x_2 v_2 + x_3 v_3 = 0 \\ &m x_3 v_1 - x_3 v_2 + (x_2 - m x_1) v_3 = 0 \end{aligned} \tag{3.1.7}$$

or

$$\begin{aligned} &x_1 v_1 + x_2 v_2 + x_3 v_3 = 0 \\ &- m x_3 v_1 - x_3 v_2 + (x_2 + m x_1) v_3 = 0. \end{aligned} \tag{3.1.8}$$

To prove the first assertion of (1), use the facts that $x_3 = 0$ and $x_1^2 + x_2^2 = 1$ to show $V^+ = \pm(x_2, -x_1, 0)$ where $\pm$ is positive if and only if $x_2 + m x_1 > 0$. Similarly, $V^- = \pm(x_2, -x_1, 0)$ where $\pm$ is positive if and only if $x_2 - m x_1 > 0$.

To prove the second assertion of (1), note that when $x_3 = 0$ and $x_2 \neq m x_1$, the system (3.1.7) has a one dimensional solution set spanned by V^+. Similarly, when $x_3 = 0$ and $x_2 \neq -m x_1$ the system (3.1.8) has a one dimensional solution set spanned by V^-.

To prove the second assertion of (2), note that system (3.1.7) has a one dimensional solution since, for instance, the determinant

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ mx_3 & -x_3 & x_2 - mx_1 \\ x_3 & 0 & -x_1 \end{vmatrix} = x_3 \neq 0. \tag{3.1.9}$$

Direct verification shows that the solution set is spanned by V^+. The same holds after replacing system (3.1.7) with (3.1.8), m with $-m$, and V^+ with V^-.

Finally, as

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ mx_3 & -x_3 & x_2 - mx_1 \\ -mx_3 & -x_3 & x_2 + mx_1 \end{vmatrix} = -2mx_3 \neq 0,$$

V^+ and V^- are linearly independent, concluding the proof. □

Remark 3.3. Lemma 3.2 implies that a vector $X = (x_1, x_2, x_3) \in U_pM$ is unicurved if and only if $x_3 = 0$ and $x_2 \neq \pm mx_1$, and is generic if and only if $x_3 \neq 0$. Furthermore, each generic vector is contained in exactly two curvature ε planes.

3.1.2 Calculations Along Geodesics

Computations in this subsection are mostly done along geodesics. To be more precise, let $\gamma : [0, \delta] \to \mathcal{O}$ be a geodesic. We may write $\gamma'(t) = (x_1(t), x_2(t), x_3(t))$. Quantities such as V^+ and V^- defined on each tangent space now vary along the geodesic $\gamma(t)$. To simplify the notation, we may not indicate that a certain quantity is a function of t. For example, we write x_i instead of $x_i(t)$ and m instead of $m(\gamma(t))$.

The reader may verify that

$$K' = \frac{mm' - (mx_2 - x_1)(mx_2 - x_1)'}{K} \tag{3.1.10}$$

and

$$L' = \frac{mm' - (-mx_2 - x_1)(-mx_2 - x_1)'}{L} \tag{3.1.11}$$

where K and L are defined in (3.1.3) and (3.1.4).

Lemma 3.4. *Let $\gamma : [0, \delta] \to \mathcal{O}$ be a geodesic such that $\gamma'(t)$ is generic for all $t \in [0, \delta]$. Then either $V^+(t)$ or $V^-(t)$ is a parallel field along γ for all $t \in [0, \delta]$.*

Proof. As M has higher rank, there exists a unit normal parallel field $V(t)$ along $\gamma(t)$ such that

$$sec(\gamma'(t), V(t)) = \varepsilon \ \forall t \in [0, \delta].$$

As $\gamma'(t)$ is generic for each t, part (2) of Lemma 3.2 implies that for each $\tau \in [0, \delta]$, $V(t) = \pm V^+(t)$ or $V(t) = \pm V^-(t)$. By continuity, the lemma follows. $\square$

Lemma 3.5. *Let $\gamma : [0, \delta] \to \mathcal{O}$ be a geodesic whose velocity vectors have components $\gamma'(t) = (x_1(t), x_2(t), x_3(t))$. Assume that $\gamma'(t)$ is generic for all $t \in [0, \delta]$, then $V^+(t)$ is parallel along γ if and only if*

$$\begin{aligned} &\{m\Gamma_{11}^3 - m^2\Gamma_{12}^3\}x_1^2 + \{\Gamma_{21}^3 - m\Gamma_{22}^3\}x_2^2 + \{e_3(m) - (1+m^2)\Gamma_{31}^2\}x_3^2 \\ &+\{\Gamma_{11}^3 - m\Gamma_{12}^3 + m\Gamma_{21}^3 - m^2\Gamma_{22}^3\}x_1x_2 \\ &+\{e_1(m) - (1+m^2)\Gamma_{11}^2 - m\Gamma_{33}^1 + m^2\Gamma_{33}^2\}x_1x_3 \\ &+\{e_2(m) + (1+m^2)\Gamma_{22}^1 - \Gamma_{33}^1 + m\Gamma_{33}^2\}x_2x_3 = 0 \end{aligned} \tag{3.1.12}$$

along $\gamma(t)$.

Proof. Writing $V^+(t) = (v_1(t), v_2(t), v_3(t))$ along $\gamma(t)$, we have that

$$\begin{aligned} v_1 &= \frac{1 + mx_1x_2 - x_1^2}{K} \\ v_2 &= \frac{mx_2^2 - x_1x_2 - m}{K} \\ v_3 &= \frac{(mx_2 - x_1)x_3}{K} \end{aligned} \tag{3.1.13}$$

by (3.1.5).

Direct computation using (3.1.3) and (3.1.13) shows that

$$v_1 - mv_2 = K. \tag{3.1.14}$$

The vector field $V^+(t)$ is parallel if and only if it satisfies the parallel field equations:

$$0 = \frac{DV^+}{dt} = \sum_{k=1}^{3}(\frac{dv_k}{dt} + \sum_{i,j}\Gamma_{ij}^k x_i v_j)e_k, \tag{3.1.15}$$

where $i, j \in \{1, 2, 3\}$ and $\Gamma_{ij}^k =< \nabla_{e_i}e_j, e_k >$. In what follows, we prove the lemma by showing that equations (3.1.15) hold if and only if (3.1.12) holds.

As a first step, substitute (3.1.13) into the undifferentiated terms in (3.1.15) to obtain

$$\begin{aligned} v_1' &+ x_1\frac{mx_2^2 - x_1x_2 - m}{K}\Gamma_{12}^1 + x_2\frac{mx_2^2 - x_1x_2 - m}{K}\Gamma_{22}^1 \\ &+ x_3\frac{mx_2^2 - x_1x_2 - m}{K}\Gamma_{32}^1 + x_3\frac{x_3(mx_2 - x_1)}{K}\Gamma_{33}^1 \\ &+ x_2\frac{x_3(mx_2 - x_1)}{K}\Gamma_{23}^1 + x_1\frac{x_3(mx_2 - x_1)}{K}\Gamma_{13}^1 = 0, \end{aligned}$$

$$
\begin{aligned}
v_2' &+ x_2\frac{1+mx_1x_2-x_1^2}{K}\Gamma_{21}^2 + x_2\frac{x_3(mx_2-x_1)}{K}\Gamma_{23}^2 \\
&+ x_1\frac{1+mx_1x_2-x_1^2}{K}\Gamma_{11}^2 + x_1\frac{x_3(mx_2-x_1)}{K}\Gamma_{13}^2 \\
&+ x_3\frac{x_3(mx_2-x_1)}{K}\Gamma_{33}^2 + x_3\frac{1+mx_1x_2-x_1^2}{K}\Gamma_{31}^2 = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
v_3' &+ x_1\frac{1+mx_1x_2-x_1^2}{K}\Gamma_{11}^3 + x_1\frac{mx_2^2-x_1x_2-m}{K}\Gamma_{12}^3 \\
&+ x_2\frac{mx_2^2-x_1x_2-m}{K}\Gamma_{22}^3 + x_2\frac{1+mx_1x_2-x_1^2}{K}\Gamma_{21}^3 \\
&+ x_3\frac{1+mx_1x_2-x_1^2}{K}\Gamma_{31}^3 + x_3\frac{mx_2^2-x_1x_2-m}{K}\Gamma_{32}^3 = 0.
\end{aligned}
$$

Rearrange terms to get

$$
\begin{aligned}
&v_1' + \frac{1}{K}\Big\{(mx_2-x_1)(\sum_{i,j}\Gamma_{ij}^1 x_i x_j) - m\sum_{i=1}^{3} x_i\Gamma_{i2}^1\Big\} = 0, \\
&v_2' + \frac{1}{K}\Big\{(mx_2-x_1)(\sum_{i,j}\Gamma_{ij}^2 x_i x_j) + \sum_{i=1}^{3} x_i\Gamma_{i1}^2\Big\} = 0, \\
&v_3' + \frac{1}{K}\Big\{(mx_2-x_1)(\sum_{i,j}\Gamma_{ij}^3 x_i x_j) + \sum_{i=1}^{3} x_i(\Gamma_{i1}^3 - m\Gamma_{i2}^3)\Big\} = 0.
\end{aligned}
\tag{3.1.16}
$$

As γ is a geodesic, it satisfies the geodesic equations

$$
\sum_{k=1}^{3}(\frac{dx_k}{dt} + \sum_{i,j}\Gamma_{ij}^k x_i x_j)e_k = 0. \tag{3.1.17}
$$

Next, use (3.1.17) in (3.1.16) to obtain

$$v_1' - \frac{m(\sum_{i=1}^3 x_i \Gamma_{i2}^1)}{K} - \frac{mx_2 - x_1}{K} x_1' = 0, \tag{3.1.18}$$

$$v_2' + \frac{(\sum_{i=1}^3 x_i \Gamma_{i1}^2)}{K} - \frac{mx_2 - x_1}{K} x_2' = 0, \tag{3.1.19}$$

and

$$v_3' + \frac{\sum_{i=1}^3 x_i \Gamma_{i1}^3 - m \sum_{i=1}^3 x_i \Gamma_{i2}^3}{K} - \frac{mx_2 - x_1}{K} x_3' = 0. \tag{3.1.20}$$

Now we claim that equations (3.1.18) and (3.1.19) are equivalent. Indeed, subtract m times the left hand side of (3.1.19) from the left hand side of (3.1.18), and use equality (3.1.14) to obtain zero as follows:

$$\begin{aligned}
&(v_1' - mv_2') - \frac{mx_2 - x_1}{K}(x_1' - mx_2') \\
&= (v_1 - mv_2)' + m'v_2 - \frac{mx_2 - x_1}{K}(x_1' - mx_2') \\
&= K' + m'v_2 - \frac{mx_2 - x_1}{K}(x_1' - mx_2') \\
&= \frac{1}{K}\{mm' - (mx_2 - x_1)(mx_2 - x_1)'\} + m'v_2 + \frac{mx_2 - x_1}{K}(mx_2' - x_1') \\
&= 0.
\end{aligned}$$

Since $m \neq 0$, this claim follows.

Next, we will show that (3.1.19) and (3.1.20) are equivalent. To prove this claim, first differentiate the v_i using (3.1.10) and then substitute into (3.1.19) and (3.1.20) to obtain

$$\frac{1}{K^3}\{(x_1^2+x_2^2-1)m' + (mx_2' - x_1')(x_2+mx_1)\}$$
$$+\frac{1}{K}(\sum_{i=1}^{3} x_i\Gamma_{i1}^2) = 0, \tag{3.1.21}$$

and

$$\frac{1}{K^3}\{(x_2+mx_1)x_3m' + (1+m^2)(mx_2' - x_1')x_3\}$$
$$+\frac{1}{K}(\sum_{i=1}^{3} x_i\Gamma_{i1}^3 - m\sum_{i=1}^{3} x_i\Gamma_{i2}^3) = 0. \tag{3.1.22}$$

Use (3.1.17) to show that

$$mx_2' - x_1' = -(x_2+mx_1)(\sum_{i=1}^{3} x_i\Gamma_{i1}^2) - x_3(\sum_{i=1}^{3} x_i\Gamma_{i1}^3 - m\sum_{i=1}^{3} x_i\Gamma_{i2}^3). \tag{3.1.23}$$

Substitute (3.1.23) into (3.1.21) and (3.1.22) to obtain

$$\frac{1}{K^3}\{x_3^2m' - x_3^2(1+m^2)(\sum_{i=1}^{3} x_i\Gamma_{i1}^2)$$
$$+x_3(x_2+mx_1)(\sum_{i=1}^{3} x_i\Gamma_{i1}^3 - m\sum_{i=1}^{3} x_i\Gamma_{i2}^3)\} = 0, \tag{3.1.24}$$

and

$$\frac{1}{K^3}\{x_3(x_2+mx_1)m' - x_3(x_2+mx_1)(1+m^2)(\sum_{i=1}^{3} x_i\Gamma_{i1}^2)$$
$$+(x_2+mx_1)^2(\sum_{i=1}^{3} x_i\Gamma_{i1}^3 - m\sum_{i=1}^{3} x_i\Gamma_{i2}^3)\} = 0. \tag{3.1.25}$$

Observe that equation (3.1.24) and (3.1.25) differ by the nonzero factor $\frac{x_3}{x_2+mx_1}$, concluding the proof of the claim.

Up to now, we have shown that the parallel field equations are equivalent to (3.1.25). Since $K \neq 0$ and $\gamma'(t)$ is generic, (3.1.25) is equivalent to

$$\begin{aligned} & x_3\{(m' - (1+m^2)(x_1\Gamma^2_{11} + x_2\Gamma^2_{21} + x_3\Gamma^2_{31}) \\ & + (x_2 + mx_1)\Gamma^3_{31} - m(x_2 + mx_1)\Gamma^3_{32})\} \\ & + (x_2 + mx_1)(x_1\Gamma^3_{11} + x_2\Gamma^3_{21} - mx_1\Gamma^3_{12} - mx_2\Gamma^3_{22}) = 0. \end{aligned} \tag{3.1.26}$$

To conclude the proof, note that (3.1.12) follows from (3.1.26), since $m' = x_1e_1(m) + x_2e_2(m) + x_3e_3(m)$. □

Remark 3.6. With the same assumption as in Lemma 3.5, an analogous proof replacing m with $-m$ and K with L shows that $V^-(t)$ is parallel if and only if

$$\begin{aligned} & \{-m\Gamma^3_{11} - m^2\Gamma^3_{12}\}x_1^2 + \{\Gamma^3_{21} + m\Gamma^3_{22}\}x_2^2 + \{-e_3(m) - (1+m^2)\Gamma^2_{31}\}x_3^2 \\ & +\{\Gamma^3_{11} + m\Gamma^3_{12} - m\Gamma^3_{21} - m^2\Gamma^3_{22}\}x_1x_2 \\ & +\{-e_1(m) - (1+m^2)\Gamma^2_{11} + m\Gamma^1_{33} + m^2\Gamma^2_{33}\}x_1x_3 \\ & +\{-e_2(m) + (1+m^2)\Gamma^1_{22} - \Gamma^1_{33} - m\Gamma^2_{33}\}x_2x_3 = 0 \end{aligned} \tag{3.1.27}$$

along $\gamma(t)$.

By Remark 3.3, generic tangent vectors form an open subset in the generic set $\mathcal{O}$. Hence, for $p \in \mathcal{O}$ and $X = (x_1, x_2, x_3) \in U_pM$ a generic vector, there exists $\delta > 0$ such that the velocity vectors of the geodesic $\gamma(t) = \exp_p tX$ are generic for all $t \in [0, \delta]$. By Lemmas 3.4 and 3.5, it follows that either (3.1.12) or (3.1.27) hold along $\gamma(t)$ on $[0, \delta]$. In particular, for every $p \in \mathcal{O}$ and generic vector $X \in U_pM$, either (3.1.12) or (3.1.27) holds for X at p. This motivates the following definition.

Definition 3.7. Define subsets B_p^+ and B_p^- of U_pM as follows.

(1) $X \in B_p^+$ if and only if (3.1.12) holds for X at p.

(2) $X \in B_p^-$ if and only if (3.1.27) holds for X at p.

Lemma 3.8. *For each $p \in \mathcal{O}$, $U_pM \subset B_p^+ \cup B_p^-$.*

Proof. Note that the set B_p^+ and B_p^- are closed subsets of U_pM. By Remark 3.3, the generic vectors are dense in U_pM. Hence the remarks preceding Definition 3.7 and the pigeon-hole principle imply the lemma.

□

Proposition 3.9. *For each $p \in \mathcal{O}$, at least one of the following holds:*

$$\begin{aligned}
\Gamma_{11}^3 &= m\Gamma_{12}^3 = m\Gamma_{21}^3 = m^2\Gamma_{22}^3 \\
e_1(m) &= (1+m^2)\Gamma_{11}^2 + m\Gamma_{33}^1 - m^2\Gamma_{33}^2 \\
e_2(m) &= -(1+m^2)\Gamma_{22}^1 + \Gamma_{33}^1 - m\Gamma_{33}^2 \\
e_3(m) &= (1+m^2)\Gamma_{31}^2,
\end{aligned} \tag{3.1.28}$$

or

$$\begin{aligned}
\Gamma_{11}^3 &= -m\Gamma_{12}^3 = -m\Gamma_{21}^3 = m^2\Gamma_{22}^3 \\
e_1(m) &= -(1+m^2)\Gamma_{11}^2 + m\Gamma_{33}^1 + m^2\Gamma_{33}^2 \\
e_2(m) &= (1+m^2)\Gamma_{22}^1 - \Gamma_{33}^1 - m\Gamma_{33}^2 \\
e_3(m) &= -(1+m^2)\Gamma_{31}^2.
\end{aligned} \tag{3.1.29}$$

As B_p^+ and B_p^- are closed subsets of U_pM, Lemma 3.8 implies that at least one of them contains an interior. Hence to prove the Proposition 3.9, it suffices to prove the following lemma.

Lemma 3.10. *Let $p \in \mathcal{O}$. The equation (3.1.28) (respectively (3.1.29)) holds at p if B_p^+ (respectively B_p^-) contains an interior.*

Proof. We prove the case when B_p^+ contains an interior. The proof for the case when B_p^- contains an interior is analogous. If B_p^+ contains an open set in U_pM, then the degree two homogeneous polynomial in variables x_1, x_2, x_3 defined by the left hand side of (3.1.12) vanishes identically on an open subset of T_pM. Hence its coefficients all vanish. Consider the coefficients for x_1x_3, x_2x_3 and x_3^2, we obtain the derivative equations in (3.1.28). As $m \neq 0$, by considering the coefficients for x_1^2 and x_2^2, we have that

$$\Gamma_{11}^3 = m\Gamma_{12}^3 \tag{3.1.30}$$

and

$$\Gamma_{21}^3 = m\Gamma_{22}^3. \tag{3.1.31}$$

Now, (2.1.2), (3.1.30), and (3.1.31) imply that

$$\Gamma_{11}^3 = m\Gamma_{12}^3 = m\Gamma_{21}^3 = m^2\Gamma_{22}^3, \tag{3.1.32}$$

as desired.

□

3.2 Local Structures on Extremal Sets

This section derives relations between Christoffel symbols at interior points to the extremal set $\mathcal{E}$. Let $p \in \mathcal{E}$ and let $\{e_1, e_2, e_3\}$ be a Ricci diagonalizing orthonormal frame at p that satisfies (2.1.1).

Recall that from Definition 2.4, we had a disjoint partition $\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_-$. We first study the case when $p \in \mathcal{E}_-$. The following lemma will show that a vector $X = (x_1, x_2, x_3)$ is unicurved if $x_1 \neq \pm 1$.

Lemma 3.11. *Let $p \in \mathcal{E}_-$ and let $X = (x_1, x_2, x_3) \in U_pM$ with $x_1 \neq \pm 1$. If $V \in U_pM$ is perpendicular to X, then* $\sec(X, V) = \varepsilon$ *if and only if*

$$V = \pm\Big(\frac{1 - x_1^2}{\sqrt{1 - x_1^2}}, \frac{-x_1x_2}{\sqrt{1 - x_1^2}}, \frac{-x_1x_3}{\sqrt{1 - x_1^2}}\Big). \tag{3.2.1}$$

Proof. Write $V = (v_1, v_2, v_3)$. We argue like Lemma 3.2. The unit vector

$$u = X \times V = (x_2v_3 - x_3v_2, x_3v_1 - x_1v_3, x_1v_2 - x_2v_1)$$

is normal to the plane spanned by X and V.

By Lemma 2.5, $\sec(X, V) = \varepsilon$ if and only if

$$x_2v_3 - x_3v_2 = 0.$$

Since $V \perp X$, $\sec(X, V) = \varepsilon$ if and only if the following system of equations hold:

$$(v_1, v_2, v_3) \cdot (0, -x_3, x_2) = 0$$

and

$$(v_1, v_2, v_3) \cdot (x_1, x_2, x_3) = 0,$$

or equivalently, if and only if

$$\begin{aligned} V &= \pm \frac{(x_1, x_2, x_3) \times (0, -x_3, x_2)}{|(x_1, x_2, x_3) \times (0, -x_3, x_2)|} \\ &= \pm \left(\frac{1 - x_1^2}{\sqrt{1 - x_1^2}}, \frac{-x_1 x_2}{\sqrt{1 - x_1^2}}, \frac{-x_1 x_3}{\sqrt{1 - x_1^2}}\right). \end{aligned}$$

$\square$

Remark 3.12. Lemma 3.11 and Corollary 2.8 together show that each vector tangent to $\mathcal{E}_-$ is either isocurved or univurved. A vector $X = (x_1, x_2, x_3)$ is unicurved if and only if $x_1 \neq \pm 1$, and is isocurved if and only if $x_1 = \pm 1$.

Notice that the V in (3.2.1) can be obtained by taking $m = 0$ in (3.1.5). Likewise, setting $m = 0$ in the statement and proof of Lemma 3.5 yields the following:

Lemma 3.13. *Let $\gamma : [0, \delta] \to \mathcal{E}_-$ be a geodesic whose velocity vectors have components $\gamma'(t) = (x_1(t), x_2(t), x_3(t))$. If $\gamma'(t)$ is unicurved for all $t \in [0, \delta]$, then*

$$V(t) = \left(\frac{1 - x_1^2}{\sqrt{1 - x_1^2}}, \frac{-x_1 x_2}{\sqrt{1 - x_1^2}}, \frac{-x_1 x_3}{\sqrt{1 - x_1^2}}\right)$$

is parallel along γ if and only if

$$x_3\{-x_1\Gamma_{11}^2 - x_2\Gamma_{21}^2 - x_3\Gamma_{31}^2 + x_2\Gamma_{31}^3\} + x_2\{x_1\Gamma_{11}^3 + x_2\Gamma_{21}^3\} = 0. \tag{3.2.2}$$

Proposition 3.14. *The following is true for each interior point of $\mathcal{E}_-$:*

$$\Gamma^2_{11} = \Gamma^3_{11} = \Gamma^3_{21} = \Gamma^2_{31} = 0$$

and

$$\Gamma^1_{33} = \Gamma^1_{22}.$$

Proof. Let p be an interior point of $\mathcal{E}_-$. Then each geodesic starting from p remains in $\mathcal{E}_-$ for a short time. Moreover, by Remark 3.12, the unicurved vectors form an open subset of unit vectors tangent to M near p. Hence, a geodesic starting from p whose initial velocity is unicurved, has velocity vectors that remain unicurved for a short time.

Let $X = (x_1, x_2, x_3)$ be a unicurved vector. We claim that

$$x_3\{-x_1\Gamma^2_{11} + x_2(\Gamma^1_{22} - \Gamma^1_{33}) - x_3\Gamma^2_{31}\} + x_2\{x_1\Gamma^3_{11} + x_2\Gamma^3_{21}\} = 0 \tag{3.2.3}$$

at the point p.

To prove the claim, let $\gamma(t) = \exp_p(tX)$. As M has higher rank, there exists a unit normal parallel field $W(t)$ along $\gamma(t)$ with $\sec(\gamma'(t), W(t)) \equiv \varepsilon$. By Lemmas 3.11 and 3.13, $W(t) = \pm V(t)$ for a short time and the claim follows.

As the unicurved vectors form an open subset of U_pM, the degree 2 homogeneous polynomial in variables x_1, x_2, x_3 defined by the left hand side of (3.2.3) vanishes identically on an open subset of T_pM. Hence its coefficients all equal zero, concluding the proof. $\square$

Note that only the assumption that $\Lambda \neq \varepsilon$ was used to prove Proposition 3.14. Hence, for $p \in \mathcal{E}_+$, the same proposition holds after permuting the indices 1 and 2. We record this in the following proposition.

Proposition 3.15. *The following is true for each interior point of $\mathcal{E}_+$:*

$$\Gamma^1_{22} = \Gamma^3_{22} = \Gamma^3_{12} = \Gamma^1_{32} = 0$$

and

$$\Gamma^2_{33} = \Gamma^2_{11}.$$

Chapter 4

From Local to Global

Recall that a three-manifold of higher rank $M = \mathcal{O} \cup \mathcal{E} \cup \mathcal{I}$, and hyperbolic or spherical rank rigidity occurs when $\mathcal{O} = \emptyset = \mathcal{E}$. The goal of this chapter is to study the global structure of $\mathcal{O}$ and $\mathcal{E}$ by the Christoffel symbol relations given in Propositions 3.9, 3.14 and 3.15. These results are the main ingredients in our proof for the rigidity theorems in Chapter 5. More specifically, we first assume that $\mathcal{O}$ is nonempty in section 4.1, and study global structures of $\mathcal{O}$ through evolution equations of along a special family of geodesics. In section 4.2, we study the local structure of $\mathcal{E}$ when $\mathcal{O}$ is empty. In the end of section 4.2, we show that when $\varepsilon = -1$ or 1, $\mathcal{E}$ must be empty provided that $\mathcal{O}$ is empty. To complete the proof of rigidity theorems, we show that $\mathcal{O} = \emptyset$ in Chapter 5.

4.1 Evolution Equations Along Isocurved Geodesics in Generic Sets

The global structure of $\mathcal{O}$ is described in terms of evolution equations of geometric quantities along a specific family of isocurved geodesics. We begin this section with a construction of such a family of isocurved geodesics.

4.1.1 Existence of Isocurved Geodesics

Let $\mathcal{C}$ be a connected component of $\mathcal{O}$. Without loss of generality, we may assume that there exists a global framing $\{e_1, e_2, e_3\}$ over $\mathcal{C}$ satisfying (2.1.1) as in Remark 2.10. With respect to this framing, at each point $p \in \mathcal{C}$ either (3.1.28) or (3.1.29) holds by Proposition 3.9.

Definition 4.1. Define closed subsets $\mathcal{F}^+$ and $\mathcal{F}^-$ of $\mathcal{C}$ by

$$\mathcal{F}^+ = \big\{p \in \mathcal{C} \,|\, \text{ (3.1.28) holds}\big\}$$

and

$$\mathcal{F}^- = \big\{p \in \mathcal{C} \,|\, \text{ (3.1.29) holds}\big\}.$$

Note that $\mathcal{F}^+ \cup \mathcal{F}^- = \mathcal{C}$.

Remark 4.2. The definition of $\mathcal{F}^+$ and $\mathcal{F}^-$ depends on the Ricci diagonalizing orthonormal frame. The reader may verify that points in $\mathcal{F}^+$ and points in $\mathcal{F}^-$ are switched if the orthonormal frame is switched from $\{e_1, e_2, e_3\}$ to $\{e_1, -e_2, e_3\}$.

Lemma 4.3. *After possibly switching the frame* $\{e_1, e_2, e_3\}$ *to* $\{e_1, -e_2, e_3\}$*, we may assume that one of the following holds:*

(1) $\mathcal{F}^+ = \mathcal{F}^- = \mathcal{C}$*, or*

(2) The set $\mathcal{C} \setminus \mathcal{F}^-$ *contains an open set* U *in* $\mathcal{C}$*.*

Proof. Note that the sets $\mathcal{C} \setminus \mathcal{F}^-$ and $\mathcal{C} \setminus \mathcal{F}^+$ are open in $\mathcal{C}$. If both of these sets are empty, then (1) holds. Otherwise, by Remark 4.2, after possibly switching the frame $\{e_1, e_2, e_3\}$ to $\{e_1, -e_2, e_3\}$, (2) holds. $\square$

Convention 4.4. From now on, we always assume that the Ricci diagonalizing frame is selected such that either (1) or (2) in Lemma 4.3 holds.

Definition 4.5. Define two vector fields on $\mathcal{C}$ by

$$E^+ = \frac{1}{\sqrt{1+m^2}} e_1 - \frac{m}{\sqrt{1+m^2}} e_2,$$

$$E^- = \frac{1}{\sqrt{1+m^2}} e_1 + \frac{m}{\sqrt{1+m^2}} e_2.$$

Remark 4.6. Note that these vector fields are isocurved by Corollary 2.8. They are characterized projectively by having the property that every plane containing one of these vectors has curvature ε.

Lemma 4.7. *On $\mathcal{F}^+$, $\nabla_{E^+} E^+ = 0$.*

Proof. On $\mathcal{C}$, direct calculations shows

$$\begin{aligned}< \nabla_{E^+} E^+, e_1 >= &\frac{1}{\sqrt{1+m^2}}\{e_1(\frac{1}{\sqrt{1+m^2}}) - m e_2(\frac{1}{\sqrt{1+m^2}}) \\ &- \frac{m}{\sqrt{1+m^2}}\Gamma^1_{12} + \frac{m^2}{\sqrt{1+m^2}}\Gamma^1_{22}\},\end{aligned}$$

$$\begin{aligned}< \nabla_{E^+} E^+, e_2 >= &\frac{1}{\sqrt{1+m^2}}\{e_1(\frac{-m}{\sqrt{1+m^2}}) + m e_2(\frac{m}{\sqrt{1+m^2}}) \\ &+ \frac{1}{\sqrt{1+m^2}}\Gamma^2_{11} - \frac{m}{\sqrt{1+m^2}}\Gamma^2_{21}\},\end{aligned}$$

and

$$< \nabla_{E^+} E^+, e_3 >= \frac{1}{1+m^2}\{\Gamma^3_{11} - m\Gamma^3_{21} - m\Gamma^3_{12} + m^2\Gamma^3_{22}\}.$$

Direct verification using (3.1.28) shows that each of these components is zero, concluding the proof. □

When $\mathcal{F}^+ = \mathcal{F}^- = \mathcal{C}$, Lemma 4.7 implies that the integral curves of E^+ are geodesics. Moreover, by part (3) of Corollary 2.8, these geodesics are isocurved.

In what follows, we construct a family of isocurved geodesics when there exists an open set $U \subset \mathcal{C} \setminus \mathcal{F}^-$. To prove the desired result, we first prove the lemma below.

Lemma 4.8. *Let $p \in \mathcal{C} \setminus \mathcal{F}^-$ and let $X = (x_1, x_2, x_3) \in U_pM$ be a generic vector. Let V^+ be the vector perpendicular to X defined by (3.1.5). Then along $\gamma_X(t) = \exp_p(tX)$, the parallel vector field $V(t)$ determined by $V(0) = V^+$ satisfies* $\sec(V(t), \gamma_X'(t)) \equiv \varepsilon$.

Proof. As $p \in \mathcal{C} \setminus \mathcal{F}^-$, Lemma 3.10 implies that B_p^- has no interior in U_pM. By continuity, it suffices to prove the lemma for generic vectors $X \in U_pM \setminus B_p^-$.

Let $X = (x_1, x_2, x_3) \in U_pM \setminus B_p^-$ be a generic vector. As M has higher rank, there exists a normal parallel field $V(t)$ along $\gamma_X(t)$ such that $\sec(V(t), \gamma_X'(t)) \equiv \varepsilon$. Since X is generic, Lemma 3.4 implies that $V(t) = V^+(t)$ or $V(t) = V^-(t)$ for a short time. If $V(t) = V^-(t)$ for a short time, then by Remark 3.6, (3.1.27) holds for X at p, contrary to the assumption that $X \notin B_p^-$. □

Proposition 4.9. *Let $p \in \mathcal{C} \setminus \mathcal{F}^-$. Then the geodesic $\alpha(t) = \exp_p(tE^+)$ is isocurved.*

Proof. As generic vectors are dense amongst the unit vectors perpendicular to E^+, it suffices to prove that each generic unit vector $W \perp E^+$ generates a parallel vector field $W(t)$ along $\alpha(t)$ satisfying $\sec(\alpha'(t), W(t)) \equiv \varepsilon$.

Given $W \in U_pM$ as above, consider the great circle in U_pM:

$$c(s) = \cos(s)E^+ + \sin(s)W. \tag{4.1.1}$$

For each small $s \geq 0$, let $\gamma_s(t) = \exp_p(tc(s))$, and note that $\alpha(t) = \gamma_0(t)$ and that $W = c'(0)$. Hence, by continuity, it suffices to prove that $c'(s)$ generates a parallel vector field $W_s(t)$ along $\gamma_s(t)$ satisfying $\sec(\gamma_s'(t), W_s(t)) \equiv \varepsilon$.

Note that direct computation after substituting (4.1.1) into (3.1.5) shows that $c'(s)$ is $-V^+$ associated to $c(s)$. The proposition now follows from Lemma 4.8. $\square$

Corollary 4.10. *Let $p \in \mathcal{C} \setminus \mathcal{F}^-$ and let $\alpha(t) = \exp_p(tE^+)$. Let $I \subset \mathbb{R}$ be the largest connected interval containing zero such that $\alpha(I) \subset \mathcal{C}$. Then on I, $\alpha' = E^+$.*

Proof. Consider the subset $T \subset I$ defined by

$$T = \{t \in I \,|\, \alpha'(t) = E^+(\alpha(t))\}.$$

By construction, $0 \in T$, and T is closed. As I is connected, it remains to prove that T is open.

For $s \in T$, $\alpha'(t)$ is isocurved for each t in a neighborhood of s by Proposition 4.9. The lemma now follows by the continuity of the velocity vectors $\alpha'(t)$ and Remark 4.6. $\square$

We summarize the results obtained in this subsection in the following proposition.

Proposition 4.11. *There exists an open subset $U \subset \mathcal{C}$ with the property that each integral curve of E^+ with initial point in U is an isocurved geodesic.*

Proof. As $\nabla_{E^+} E^+ = 0$ on $\mathcal{F}^+$, the proposition is obvious when (1) holds in Lemma 4.3. When (2) in Lemma 4.3 occurs, take U to be an open subset of $\mathcal{C} \setminus \mathcal{F}^-$, and the proposition follows from Proposition 4.9 and Corollary 4.10. $\square$

4.1.2 Evolution Equations

By Proposition 4.11, there exists a family of isocurved geodesics in $\mathcal{C}$ that are tangent to E^+. In this subsection, let α be one of these geodesics constructed in Proposition 4.11 and let I be the maximal connected open interval in $\mathbb{R}$ containing zero such that $\alpha(I) \subset \mathcal{C}$. Clearly, $I \neq \emptyset$. The goal of this subsection is to derive first order differential equations satisfied by geometric quantities along α on I in Proposition 4.19 below. The geometric quantities are described in terms of a new framing over $\mathcal{C}$.

Definition 4.12. Define another orthonormal frame $\{E_1, E_2, E_3\}$ on $\mathcal{C}$ by $E_1 = E^+$, $E_3 = e_3$, and

$$E_2 = \frac{m}{\sqrt{1+m^2}} e_1 + \frac{1}{\sqrt{1+m^2}} e_2.$$

By Remark 2.10, the orthonormal frame $\{E_1, E_2, E_3\}$ is not Ricci diagonalizing. In what follows below, we reserve the Christoffel symbol notation Γ_{ij}^k for $< \nabla_{e_i} e_j, e_k >$.

Lemma 4.13. *With respect to $\{E_1, E_2, E_3\}$, at least one of $R(E_1, E_2, E_2, E_3)$ or $R(E_1, E_3, E_3, E_2)$ is nonzero, and*

$$R(E_1, E_2, E_3, E_1) = 0. \tag{4.1.2}$$

Proof. As E_1 is isocurved, for every a, b such that $a^2 + b^2 = 1$,

$$\begin{aligned} \varepsilon &= R(E_1, aE_2 + bE_3, aE_2 + bE_3, E_1) \\ &= a^2 R(E_1, E_2, E_2, E_1) + b^2 R(E_1, E_3, E_3, E_1) + 2abR(E_1, E_2, E_3, E_1) \\ &= \varepsilon + 2abR(E_1, E_2, E_3, E_1). \end{aligned}$$

The lemma now follows from (2.0.1), since $\{E_1, E_2, E_3\}$ does not diagonalize Ricci.

□

Note that by Proposition 4.11,

$$\nabla_{E_1} E_1 = 0 \tag{4.1.3}$$

along α on I. By using (4.1.3), the following Christoffel symbols vanish when the geodesic α enters $\mathcal{F}^-$.

Lemma 4.14. *Let $t \in I$. If $\alpha(t) \in \mathcal{F}^-$, then*

$$\Gamma_{22}^3 = \Gamma_{12}^3 = \Gamma_{21}^3 = \Gamma_{11}^3 = 0. \tag{4.1.4}$$

Proof. First calculate

$$< \nabla_{E_1} E_1, E_3 >= \frac{1}{1+m^2}\{\Gamma_{11}^3 - m\Gamma_{21}^3 - m\Gamma_{12}^3 + m^2\Gamma_{22}^3\}. \tag{4.1.5}$$

Next, use (3.1.29) to reduce (4.1.5) to

$$< \nabla_{E_1} E_1, E_3 >= \frac{1}{1+m^2}\{4m^2\Gamma_{22}^3\}. \tag{4.1.6}$$

By (4.1.3) and the fact that m is nonzero, it follows that $\Gamma_{22}^3 = 0$. This, upon substitution into (3.1.29), shows that

$$\Gamma_{22}^3 = 0 = \Gamma_{12}^3 = \Gamma_{21}^3 = \Gamma_{11}^3.$$

□

Lemma 4.15. *Along the geodesic $\alpha(t)$ for all $t \in I$, we have that*

$$\nabla_{E_1} E_2 = \nabla_{E_1} E_3 = 0 \tag{4.1.7}$$

and

$$< \nabla_{E_2} E_1, E_3 >= 0. \tag{4.1.8}$$

Proof. We first prove (4.1.7). By (4.1.3) and the fact that $\{E_1, E_2, E_3\}$ is an orthonormal basis, it suffices to prove that $\nabla_{E_1} E_3 = 0$.

Straightforward computation shows that

$$\nabla_{E_1} E_3 = \frac{1}{1+m^2}\{(\Gamma^1_{13} - m\Gamma^1_{23})e_1 + (\Gamma^2_{13} - m\Gamma^2_{23})e_2\}. \tag{4.1.9}$$

If $\alpha(t) \in \mathcal{F}^+$, substitute (3.1.28) into (4.1.9) to obtain $\nabla_{E_1} E_3 = 0$. If $\alpha(t) \in \mathcal{F}^-$, substitute (4.1.4) into (4.1.9) to obtain $\nabla_{E_1} E_3 = 0$.

To prove (4.1.8), first calculate

$$< \nabla_{E_2} E_1, E_3 >= \frac{1}{1+m^2}\{m\Gamma^3_{11} - m^2\Gamma^3_{12} + \Gamma^3_{21} - m\Gamma^3_{22}\}. \tag{4.1.10}$$

The equation (4.1.8) now follows from (3.1.28) when in $\alpha(t) \in \mathcal{F}^+$, and from (4.1.4) when $\alpha(t) \in \mathcal{F}^-$.

$\square$

Lemma 4.16. *We have the following evolution equations along $\alpha(t)$ for all $t \in I$:*

$$E_1 < \nabla_{E_2} E_1, E_2 >= -\varepsilon - < \nabla_{E_2} E_1, E_2 >^2$$

and

$$E_1 < \nabla_{E_3} E_1, E_3 >= -\varepsilon - < \nabla_{E_3} E_1, E_3 >^2 .$$

Proof. To prove the first equality, first use the fact that E_1 is isocurved and the fact that

$\nabla_{E_1} E_1 = 0$ to show

$$\varepsilon = R(E_2, E_1, E_1, E_2) =< -\nabla_{E_1}\nabla_{E_2} E_1 - \nabla_{[E_2,E_1]} E_1, E_2 > . \tag{4.1.11}$$

By (4.1.7),

$$< \nabla_{E_1}\nabla_{E_2} E_1, E_2 >= E_1 < \nabla_{E_2} E_1, E_2 > . \tag{4.1.12}$$

Use (4.1.7) and (4.1.8) to show

$$\begin{aligned}
& < \nabla_{[E_2,E_1]} E_1, E_2 > \\
&= < \nabla_{\nabla_{E_2} E_1 - \nabla_{E_1} E_2} E_1, E_2 > \\
&= < \nabla_{\nabla_{E_2} E_1} E_1, E_2 > \\
&= < \nabla_{<\nabla_{E_2} E_1, E_2 > E_2 + <\nabla_{E_2} E_1, E_3 > E_3} E_1, E_2 > \\
&= < \nabla_{E_2} E_1, E_2 >< \nabla_{E_2} E_1, E_2 > + < \nabla_{E_2} E_1, E_3 >< \nabla_{E_3} E_1, E_2 > \\
&= < \nabla_{E_2} E_1, E_2 >^2 .
\end{aligned} \tag{4.1.13}$$

The first equality is now obtained by substituting (4.1.12) and (4.1.13) into (4.1.11).

An analogous proof shows the second inequality.

□

Definition 4.17. Define an operator $A : E_1^{\perp} \to E_1^{\perp}$ on $\mathcal{C}$ by $A = \nabla E_1$.

As E_1 has unit length, A is well-defined. In terms of the orthonormal frame $\{E_2, E_3\}$ for $E_1^{\perp}$,

$$tr(A) =< \nabla_{E_2} E_1, E_2 > + < \nabla_{E_3} E_1, E_3 > .$$

In Lemma 4.18, we show that A is given by scalar multiplication along α.

Lemma 4.18. *Along the geodesic $\alpha(t)$ for all $t \in I$, we have that*

(1) $<\nabla_{E_2}E_1, E_3>=<\nabla_{E_3}E_1, E_2>=0$

(2) $<\nabla_{E_2}E_1, E_2>=<\nabla_{E_3}E_1, E_3>$.

Proof. To prove (1), it suffices to prove $<\nabla_{E_3}E_1, E_2>=0$, by Lemma 4.15. To prove that $<\nabla_{E_3}E_1, E_2>=0$, first use (4.1.3) and (4.1.7) to show that

$$\begin{aligned} &R(E_1, E_3, E_2, E_1) \\ &= E_1 <\nabla_{E_3}E_2, E_1> - <\nabla_{[E_1,E_3]}E_2, E_1> \\ &= E_1 <\nabla_{E_3}E_2, E_1> + <\nabla_{<\nabla_{E_3}E_1,E_2>E_2+<\nabla_{E_3}E_1,E_3>E_3}E_2, E_1> \\ &= E_1 <\nabla_{E_3}E_2, E_1> + tr(A) <\nabla_{E_3}E_2, E_1> \end{aligned} \tag{4.1.14}$$

By (4.1.2) and (4.1.14),

$$E_1 <\nabla_{E_3}E_2, E_1>= -tr(A) <\nabla_{E_3}E_2, E_1>.$$

Hence if $<\nabla_{E_3}E_2, E_1>=0$ at $\alpha(0)$, then $<\nabla_{E_3}E_2, E_1>=0$ along α by the uniqueness of solutions to ordinary differential equations.

Direct calculations show that

$$<\nabla_{E_3}E_2, E_1>=\frac{e_3(m)}{1+m^2} - \Gamma^2_{31}.$$

Hence, at points in $\mathcal{F}^+$, the last equality in (3.1.28) implies that $<\nabla_{E_3}E_2, E_1>=0$, concluding the proof of (1).

To prove (2), recall from Lemma 4.16 that $<\nabla_{E_3}E_1, E_3>$ and $<\nabla_{E_2}E_1, E_2>$ satisfy the same first order differential equation, hence it suffices to prove (2) at a point along $\alpha(t)$. Direct calculations show that

$$< \nabla_{E_3} E_1, E_3 >= \frac{1}{\sqrt{1+m^2}}\Gamma^3_{31} - \frac{m}{\sqrt{1+m^2}}\Gamma^3_{32}.$$

Note that

$$\begin{aligned}
\nabla_{E_2} E_1 =&\nabla_{\frac{m}{\sqrt{1+m^2}}e_1+\frac{1}{\sqrt{1+m^2}}e_2} \frac{1}{\sqrt{1+m^2}}e_1 - \frac{m}{\sqrt{1+m^2}}e_2\\
=&\{\frac{m}{\sqrt{1+m^2}}e_1(\frac{1}{\sqrt{1+m^2}}) + \frac{1}{\sqrt{1+m^2}}e_2(\frac{1}{\sqrt{1+m^2}})\\
&-\frac{m^2}{1+m^2}\Gamma^1_{12} - \frac{m}{1+m^2}\Gamma^1_{22}\}e_1\\
&+\{\frac{m}{\sqrt{1+m^2}}e_1(\frac{-m}{\sqrt{1+m^2}}) + \frac{1}{\sqrt{1+m^2}}e_2(\frac{-m}{\sqrt{1+m^2}})\\
&+\frac{m}{1+m^2}\Gamma^2_{11} + \frac{1}{1+m^2}\Gamma^2_{21}\}e_2.
\end{aligned}$$

Hence

$$\begin{aligned}
< \nabla_{E_2} E_1, E_2 >=&\frac{m}{1+m^2}\{me_1(\frac{1}{\sqrt{1+m^2}}) + e_2(\frac{1}{\sqrt{1+m^2}})\}\\
&+\frac{1}{1+m^2}\{me_1(\frac{-m}{\sqrt{1+m^2}}) + e_2(\frac{-m}{\sqrt{1+m^2}})\}\\
&-\frac{m}{\sqrt{1+m^2}}\{\frac{m^2}{1+m^2}\Gamma^1_{12} + \frac{m}{1+m^2}\Gamma^1_{22}\}\\
&+\frac{1}{\sqrt{1+m^2}}\{\frac{m}{1+m^2}\Gamma^2_{11} + \frac{1}{1+m^2}\Gamma^2_{21}\} \qquad (4.1.15)\\
=&\frac{1}{\sqrt{1+m^2}}\frac{1}{1+m^2}(-me_1(m) - e_2(m))\\
&-\frac{m}{1+m^2}\{\frac{m^2}{\sqrt{1+m^2}}\Gamma^1_{12} + \frac{m}{\sqrt{1+m^2}}\Gamma^1_{22}\}\\
&+\frac{1}{1+m^2}\{\frac{m}{\sqrt{1+m^2}}\Gamma^2_{11} + \frac{1}{\sqrt{1+m^2}}\Gamma^2_{21}\}.
\end{aligned}$$

At points in $\mathcal{F}^+$, equality in 2. holds as can be seen after substituting the second and third equalities in (3.1.28) into (4.1.15).

$\square$

We are now ready to prove the evolution equations along the geodesic α.

Proposition 4.19. *Along the geodesic $\alpha(t)$ for all $t \in I$,*

$$E_1(tr(A)) = -2\varepsilon - \frac{1}{2}(tr(A))^2$$

$$\begin{aligned} E_1(R(E_1, E_2, E_2, E_3)) &= -\frac{3}{2} tr(A) R(E_1, E_2, E_2, E_3) \\ E_1(R(E_1, E_3, E_3, E_2)) &= -\frac{3}{2} tr(A) R(E_1, E_3, E_3, E_2). \end{aligned} \tag{4.1.16}$$

Proof. By Lemma 4.16 and Lemma 4.18,

$$\begin{aligned} E_1(tr(A)) &= E_1(< \nabla_{E_2} E_1, E_2 > + < \nabla_{E_3} E_1, E_3 >) \\ &= -2\varepsilon - < \nabla_{E_2} E_1, E_2 >^2 - < \nabla_{E_3} E_1, E_3 >^2 \\ &= -2\varepsilon - \frac{1}{2}(tr(A))^2. \end{aligned}$$

The first equality in (4.1.16) is obtained by expanding the second Bianchi identity

$$(\nabla_{E_3} R)(E_1, E_2, E_2, E_1) + (\nabla_{E_1} R)(E_2, E_3, E_2, E_1) + (\nabla_{E_2} R)(E_3, E_1, E_2, E_1) = 0,$$

and then making simplifications using the Christoffel symbols relations

$$\nabla_{E_1} E_1 = \nabla_{E_1} E_2 = \nabla_{E_1} E_3 = 0$$

$$< \nabla_{E_2} E_1, E_3 > = < \nabla_{E_3} E_1, E_2 > = 0$$

$$< \nabla_{E_2} E_1, E_2 > = < \nabla_{E_3} E_1, E_3 >$$

obtained in Lemma 4.15 and Lemma 4.18.

The second equality in (4.1.16) is similarly obtained starting from the second Bianchi identity

$$(\nabla_{E_3} R)(E_1, E_2, E_3, E_1) + (\nabla_{E_1} R)(E_2, E_3, E_3, E_1) + (\nabla_{E_2} R)(E_3, E_1, E_3, E_1) = 0.$$

$\square$

4.2 Evolution Equations Along Isocurved Geodesics in Extremal Sets

In this section, we prove rigidity results for three-manifolds of higher rank with extremal curvature ε. Recall that a three-manifold M of higher rank has extremal curvature ε if and only if $\mathcal{O} = \emptyset$. Throughout the section, we assume that $\mathcal{O} = \emptyset$ and $\mathcal{E} \neq \emptyset$. Let $\mathcal{C}$ be a connected component of $\mathcal{E}$. By Proposition 2.12, on $\mathcal{C}$ or a double cover of $\mathcal{C}$ (still denoted by $\mathcal{C}$), there exists a complete isocurved geodesic field E. By the definition of $\mathcal{E}_+$ and $\mathcal{E}_-$, $\mathcal{C} \subset \mathcal{E}_-$ or $\mathcal{C} \subset \mathcal{E}_+$.

As the rank assumption passes to covers, we may apply Propositions 3.14 and 3.15 to derive evolution equations along E geodesics on $\mathcal{C}$. The evolution equations are then applied to prove rigidity results for three-manifolds of higher rank with extremal curvature. These results will be used in the proof of the main theorems in the last section.

Definition 4.20. Let E be the isocurved geodesic field on $\mathcal{C}$. Define $\tilde{A} : E^{\perp} \to E^{\perp}$ by $\tilde{A} = \nabla E$.

The trace of $\tilde{A}$ is clearly well-defined and smooth on $\mathcal{C}$.

Proposition 4.21. *On $\mathcal{C}$, we have the following evolution equation for $tr(\tilde{A})$ along the E geodesics:*

$$E(tr\tilde{A}) = -2\varepsilon - \frac{1}{2}(tr\tilde{A})^2.$$

Proof. We prove the case when $\mathcal{C} \subset \mathcal{E}_-$. The proof for the case when $\mathcal{C} \subset \mathcal{E}_+$ is similar. Let $\{e_1, e_2, e_3\}$ be a Ricci diagonalizing orthonormal frame on $\mathcal{E}_-$ satisfying (2.1.1). To prove the proposition, first use Proposition 3.14 to obtain

$$R(e_1, e_3, e_3, e_1) = e_1(\Gamma^1_{33}) - (\Gamma^1_{33})^2.$$

By Corollary 2.8, e_1 is isocurved, so that $R(e_1, e_3, e_3, e_1) = \varepsilon$. Therefore,

$$e_1(\Gamma^3_{31}) = -\varepsilon - (\Gamma^3_{31})^2. \tag{4.2.1}$$

Note that on $\mathcal{E}_-$, we may assume that $E = e_1$. By Proposition 3.14, we have that on $\mathcal{E}_-$,

$$\Gamma^1_{33} = \Gamma^1_{22}. \tag{4.2.2}$$

Hence by (4.2.1) and (4.2.2),

$$\begin{aligned} E(tr\tilde{A}) = e_1(tr\tilde{A}) &= e_1(\Gamma_{31}^3 + \Gamma_{21}^2) \\ &= -2\varepsilon - \frac{1}{2}(tr\tilde{A})^2 \end{aligned}$$

□

The following proposition is a special case for the characterization of three-manifolds of higher spherical rank.

Proposition 4.22. *Let M be a complete Riemannian three-manifold of higher spherical rank and extremal sectional curvature 1, then M has constant sectional curvatures.*

Proof. Let M be a three manifold of higher spherical rank and extremal curvature 1. Then $M = \mathcal{E} \cup \mathcal{I}$. To prove the rigidity result, it suffices to show that $\mathcal{E}$ is empty.

Suppose that $\mathcal{E}$ is nonempty. Let $\alpha : (-\infty, \infty) \to \mathcal{C}$ be a complete isocurved geodesic obtained by integrating the complete geodesic field E on $\mathcal{C}$. By Proposition 4.21,

$$\frac{d}{dt}(tr\tilde{A}) = -2 - \frac{1}{2}(tr\tilde{A})^2 \tag{4.2.3}$$

along $\alpha(t)$. The solution to the differential equation (4.2.3) has a finite time singularity. This contradicts with the fact that $tr\tilde{A}$ is a smooth function along $\alpha(t)$. Hence $\mathcal{E}$ is empty, and M has constant sectional curvature 1. □

The evolution equations also lead to a different proof of Theorem 2.13.

Proof. Let M be a complete Riemannian three-manifold of higher hyperbolic rank with extremal curvature and finite volume.

Suppose that $\mathcal{E}$ is nonempty. Let $\mathcal{C}$ be a connected component of $\mathcal{E}$. Then $\mathcal{C}$ and its double cover have finite volume. By Proposition 4.21, we have the following evolution equation along each isocurved E geodesic:

$$\frac{d}{dt}(tr\tilde{A}) = 2 - \frac{1}{2}(tr\tilde{A})^2. \tag{4.2.4}$$

The solution of the differential equation (4.2.4) has a finite forward time singularity when $tr(\tilde{A})(0) < -2$, and it has a finite backward time singularity when $tr(\tilde{A})(0) > 2$. As $tr(\tilde{A})$ is a smooth function, $tr(\tilde{A})(0) \in [-2, 2]$ along each isocurved E geodesic. Hence there exists an open set U such that either $-2 < tr(\tilde{A})(q) \leq 2$ for all $q \in U$ or $tr(\tilde{A})(q) = -2$ for all $q \in U$.

Let ϕ_t denote the flow generated by the complete geodesic field E. When $-2 < tr(\tilde{A})(q) \leq 2$ for all $q \in U$, (4.2.4) implies that $\mathrm{div}(E) = tr(\tilde{A}) \to 2$ on $\phi_t(U)$ as $t \to \infty$. As

$$\mathrm{div}(E) = \frac{d}{dt}\int_{\phi_t(U)} d\mathrm{vol},$$

the volume of $\phi_t(U)$ is arbitrarily large as $t \to \infty$, contradicting the assumption that $\mathcal{C}$ has finite volume.

When $tr(\tilde{A})(q) = -2$ for all $q \in U$. An analogous argument shows that the volume for $\phi_{-t}(U)$ is arbitrary large as $t \to \infty$, contradicting the assumption that $\mathcal{C}$ has finite volume. Hence $\mathcal{E}$ is empty, and M is hyperbolic. $\square$

Chapter 5

Rigidity Theorems

We prove the main theorems in this chapter. In the proof of each theorem, we first exclude the possibility that $\mathcal{O}$ is nonempty. Then we use Proposition 4.22 or Theorem 2.13 to obtain the desired rigidity result. The proof of Theorem 2 is in section 5.1 and the proof of Theorem 1 is in 5.2.

Let $\mathcal{C}$ be a connected component of $\mathcal{O}$ as in Remark 2.10.

Definition 5.1. On the connected component $\mathcal{C} \subset \mathcal{O}$, define functions R_2 and R_3 by

$$R_2 = R(E_1, E_2, E_2, E_3)$$

and

$$R_3 = R(E_1, E_3, E_3, E_2).$$

Along a geodesic $\alpha(t)$ constructed in Proposition 4.11, we may solve the differential equations (4.1.16) to obtain

$$|R_i(t)| = |R_i(0)| \exp\Big(-\frac{3}{2}\int_0^t tr(A)(s)ds\Big) \tag{5.0.1}$$

for $i = 2, 3$.

Recall from Lemma 4.13 that either $R_2 \neq 0$ or $R_3 \neq 0$ at each point in $\mathcal{C}$. The following lemma is a useful observation for the proof of both theorems.

Lemma 5.2. *Let $p \in \mathcal{C}$ and let $\alpha : (k, l) \to \mathcal{C}$ be a maximal integral curve of E_1 with $\alpha(0) = p$. Furthermore, assume that $\alpha(t)$ is a geodesic.*

(1) If l is finite, then $\lim_{t \to l^-} R_i(\alpha(t)) = 0$, for $i = 2, 3$.

(2) If k is finite, then $\lim_{t \to k^+} R_i(\alpha(t)) = 0$, for $i = 2, 3$.

Proof. We prove (1). The proof of (2) is analogous. Suppose that $l < \infty$, then $\alpha(l) \in \mathcal{E} \cup \mathcal{I}$.

As M is complete, we may extend $\alpha : (k, l) \to \mathcal{C}$ to a complete geodesic $\alpha : \mathbb{R} \to M$. Let $\{E_1(t), E_2(t), E_3(t)\}$ be the orthonormal framing determined by parallel translating the orthonormal frame $\{E_1, E_2, E_3\}$ at $\alpha(0)$ along $\alpha(t)$. By (4.1.7), $\{E_1(t), E_2(t), E_3(t)\}$ agrees with the orthonormal framing $\{E_1, E_2, E_3\}$ of $\mathcal{C}$ as long as $\alpha(t) \in \mathcal{C}$.

If $\alpha(l) \in \mathcal{I}$, then the formula for a curvature tensor at an isotropic point implies that

$$R(E_1(l), E_2(l), E_2(l), E_3(l)) = 0 = R(E_1(l), E_3(l), E_3(l), E_2(l)).$$

The lemma now follows from continuity of parallel translation and the curvature tensor.

Next, consider the case when $\alpha(l) \in \mathcal{E}$. Then exactly one of $\lambda(t)$ or $\Lambda(t)$ converges to ε as $t \to l^-$. If $\Lambda(t) \to \varepsilon$, then $m \to \infty$. The formulae for E_1, E_2 and E_3 in $\mathcal{C}$ then imply that the distances between the ordered orthonormal frames $\{E_1(t), E_2(t), E_3(t)\}$ and $\{-e_2(t), e_1(t), e_3(t)\}$ converge to zero as $t \to l^-$. Continuity of the curvature tensor and (2.0.1) imply (1) in this case. Likewise, if $\lambda \to \varepsilon$, then $m \to 0$. In this case, the distances between the ordered orthonormal frames $\{E_1(t), E_2(t), E_3(t)\}$ and $\{e_1(t), e_2(t), e_3(t)\}$ converge to zero as $t \to l^+$, concluding the proof similarly.

$\square$

5.1 Spherical Rank Rigidity

For three-manifolds of higher spherical rank, we have the following theorem (Theorem 2):

Theorem 5.3. *A complete Riemannian three-manifold M has higher spherical rank if and only if M has constant sectional curvatures one.*

Proof. By Proposition 4.22, it suffices to show that $\mathcal{O}$ is empty. Suppose that $\mathcal{O}$ is nonempty. By Proposition 4.11, there exists a geodesic α such that $\alpha(0) \in \mathcal{F}^+$ and $\alpha' = E_1$ on a connected component $\mathcal{C}$ of $\mathcal{O}$. Let (k, l) be the maximal open interval containing zero such that $\alpha((k, l)) \subset \mathcal{C}$.

Set $\varepsilon = 1$ in Proposition 4.19 to obtain

$$\frac{d}{dt}(tr(A)) = -2 - \frac{1}{2}(tr(A))^2. \tag{5.1.1}$$

If $l < \infty$, we obtain a contradiction as follows. By Lemma 4.13, either $|R_2(0)| > 0$ or $|R_3(0)| > 0$. Assume that $|R_2(0)| > 0$. The argument for the case when $|R_3(0)| > 0$ is similar. Note that $-tr(A)$ is well-defined and smooth on (k, l). Furthermore, (5.1.1) implies that $-tr(A)$ is strictly increasing on (k, l). Hence by (5.0.1),

$$\begin{aligned}\lim_{t \to l^-} |R_2(t)| &= \lim_{t \to l^-} |R_2(0)| \exp\Big(-\frac{3}{2}\int_0^t tr(A)(s)ds\Big) \\ &> |R_2(0)| \exp\Big(-\frac{3}{2} tr(A)(0) \cdot l\Big) > 0.\end{aligned}$$

This is a contradiction to Lemma 5.2.

In the case when $l = \infty$, a contradiction is obtained as the solution to (5.1.1) has a finite forward time singularity. $\square$

5.2 Hyperbolic Rank Rigidity

We first prove the following property for the isocurved geodesics constructed in Proposition 4.11.

Lemma 5.4. *Let $\alpha(t)$ be an isocurved geodesic as constructed in Proposition 4.11 and let (k, l) be the maximal open interval containing zero such that $\alpha((k, l)) \subset \mathcal{C}$.*

(1) $l = \infty$ if $tr(A)(0) \geq -2$.

(2) $k = -\infty$ if $tr(A)(0) \leq -2$.

Proof. By taking $\varepsilon = -1$ in Proposition 4.19, we have the following evolution equation along $\alpha(t)$ on (k, l):

$$\frac{d}{dt}(tr(A)) = \frac{1}{2}(2 - tr(A))(2 + tr(A)). \tag{5.2.1}$$

Note that by Lemma 4.13, either $|R_2(0)| > 0$ or $|R_3(0)| > 0$. Assume that $|R_2(0)| > 0$. The argument for the case when $|R_3(0)| > 0$ is similar.

To prove (1), suppose that $l < \infty$ when $tr(A)(0) \geq -2$. A straightforward analysis on the stability of equilibrium points for (5.2.1) shows that $tr(A)(t)$ is bounded by some constants on $[0, l)$ when $tr(A)(0) \geq -2$. Hence by (5.0.1), we have

$$\lim_{t \to l^-} |R_2(t)| = \lim_{t \to l^-} |R_2(0)| \exp\Big(-\frac{3}{2} \int_0^t tr(A)(s) ds \Big) > 0.$$

This is a contradiction to Lemma 5.2, concluding the proof of (1).

To prove (2), suppose that k is finite when $tr(A)(0) \leq -2$. In this case, a straightforward analysis on the stability of equilibrium points for (5.2.1) shows that $tr(A)(t)$ is bounded above by -2 on $(k, 0]$. By (5.0.1),

$$\lim_{t \to k^+} |R_2(t)| = |R_2(0)| \exp\Big(-\frac{3}{2} \int_0^k tr(A)(s) ds \Big) > 0, \tag{5.2.2}$$

contradicting Lemma 5.2. Hence $k = -\infty$, concluding the proof of (2).

$\square$

We now give the proof of Theorem 1, which is restated in the following.

Theorem 5.5. *A complete finite volume Riemannian three-manifold M has higher hyperbolic rank if and only if M is hyperbolic.*

Proof. By Theorem 2.13, it suffices to show that $\mathcal{O}$ is empty. Suppose that $\mathcal{O}$ is nonempty. Let U be the open subset of $\mathcal{C}$ as in Proposition 4.11. For each $p \in U$, let $\varphi_t(p)$ be the integral curve of E_1 starting from p.

Note that after possibly shrinking the open set U, we may assume that either $tr(A)(p) > -2$ for all $p \in U$ or $tr(A)(p) \leq -2$ for all $p \in U$.

If $tr(A)(p) > -2$ for all $p \in U$, then Lemma 5.4 implies that $\varphi_t(U) \subset \mathcal{C}$ for all $t > 0$. Furthermore, (5.2.1) implies that $tr(A)$ approaches to 2 on $\varphi_t(U)$ as t approaches to infinity. As

$$tr(A) = \mathrm{div}(E_1) = \frac{d}{dt} \int_{\varphi_t(U)} d\mathrm{vol}, \tag{5.2.3}$$

the volume of $\mathcal{C}$ is infinite, a contradiction to the assumption that M has finite volume. Hence $\mathcal{O}$ is empty.

Likewise, if $tr(A)(p) \leq -2$ for all $p \in U$, Lemma 5.4 implies that $\varphi_{-t}(U) \subset \mathcal{C}$ for all

$t > 0$. The equation (5.2.1) again shows that

$$\operatorname{div}(-E_1) = -\operatorname{div}(E_1) \to 2$$

on $\varphi_{-t}(U)$ as t approaches to infinity. Hence an analogous argument shows that $\mathcal{C}$ has infinite volume, contradicting to the assumption that M has finite volume. Hence $\mathcal{O}$ is empty. This concludes the proof of the theorem. $\square$

We now prove Theorem 5.

Theorem 5.6. *A homogeneous three-manifold M of higher hyperbolic rank is hyperbolic.*

Proof. By Theorem 2.14, it suffices to rule out the case when $M = \mathcal{O}$. Suppose that $M = \mathcal{O}$. Since M is homogeneous, R_2 and R_3 are constant on $\mathcal{C}$. At each point in $\mathcal{C}$, either $R_2 \neq 0$ or $R_3 \neq 0$. By (4.1.16), $tr(A) = 0$ on $\mathcal{C}$, contradicting with the first equation in (4.19). $\square$

5.3 Euclidean Rank Rigidity

In this section, we show that $\mathcal{O}$ must be empty for finite volume three-manifolds of higher Euclidean rank. This leads to the proof of Theorem 3.

Theorem 5.7. *Let M be a finite volume three-manifold of higher Euclidean rank. Then M has extremal curvatures, i.e., $M = \mathcal{E} \cup \mathcal{I}$.*

To prove Theorem 5.7, we first prove the following lemma.

Lemma 5.8. *Let M be a finite volume three-manifold of higher Euclidean rank and let $\alpha(t)$ be an isocurved geodesic as constructed in Proposition 4.11. Then*

(1) $\alpha((-\infty, \infty)) \subset \mathcal{C}$,

(2) $tr(A) = 0$ along $\alpha(t)$ for all $t \in (-\infty, \infty)$, and

(3) either $< \nabla_{E_3} E_3, E_2 >$ or $< \nabla_{E_2} E_2, E_3 >$ is a non-constant linear function of t along $\alpha(t)$ for all $t \in (-\infty, \infty)$.

Proof. By setting $\varepsilon = 0$ in Proposition 4.19, we have that

$$\frac{d}{dt}(tr(A)) = -\frac{1}{2}(tr(A))^2 \tag{5.3.1}$$

along α on the maximal open interval I containing 0 such that $\alpha(I) \subset \mathcal{C}$. The solution to (5.3.1) goes to $-\infty$ in finite forward time when $trA(0) < 0$, ∞ in finite backward time when $trA(0) > 0$. As $\alpha(0)$ is a generic point, either $R_2(0) \neq 0$ or $R_3(0) \neq 0$. Equation (5.0.1) and Lemma 5.2 together implies that $tr(A)$ develops singularity on I unless $trA(0) = 0$. Therefore, $trA(t) = 0$ on I. The differential equation (4.1.16) then implies that $R_i(t)$ is constant along $\alpha(t)$ for $t \in I$ and $i = 2, 3$. Using Lemma 5.2 again, $I = (-\infty, \infty)$ and $trA(t) \equiv 0$, which implies (1) and (2).

To prove (3), first use Lemmas 4.15 and 4.18 and (2) to calculate

$$R_2 = R(E_1, E_2, E_2, E_3) = E_1(< \nabla_{E_2} E_2, E_3 >)$$

and

$$R_3 = R(E_1, E_3, E_3, E_2) = E_1(< \nabla_{E_3} E_3, E_2 >).$$

As either $R_2(t)$ or $R_3(t)$ is a nonzero constant, (3) follows. $\square$

We now give the proof of Theorem 5.7.

Proof. Let M be a finite volume three-manifold of higher Euclidean rank. Suppose that $\mathcal{O}$

is nonempty. Let U be the open subset of $\mathcal{C}$ as in Proposition 4.11. For each $p \in U$, let $\varphi_t(p)$ be the integral curve of E_1 starting from p. By (1) and (2) of Lemma 5.8, φ_t gives a volume measure preserving $\mathbb{R}$ action on the open set

$$\mathcal{G} = \bigcup_{s \in \mathbb{R}} \varphi_s(U).$$

By assumption, $\mathcal{G}$ has finite measure. The Poincare recurrence Theorem then implies that there exists a small compact set K, a point $x \in K$, and a monotonic sequence of real numbers $\{n_i\}$ to infinity such that $\varphi_{n_i}(x) \in K$. This contradicts with (3) of Lemma 5.8. Hence $\mathcal{O}$ is empty. $\square$

By diagonalizing the Jacobi operator, one can see that when M has extremal curvatures, a normal parallel field making curvature 0 with the geodesic is actually Jacobi. Therefore, Theorem 3 is a corollary of Theorem 5.7 and the rigidity theorem of Bettiol and Schmidt [3, Theorem A]. We would like to remind the reader again that the Euclidean rank in [3] is defined as the least number of independent normal parallel Jacobi fields.

Chapter 6

Examples

We now give constructions of non-homogeneous manifolds of higher hyperbolic rank with infinite volume. Let $\mathbb{H}^n = \{(x_i) \in \mathbb{R}^n | \, x_n > 0\}$ be the upper half-plane model for the hyperbolic space with the metric $h_n = x_n^{-2}(dx_1^2 + \cdots + dx_n^2)$. The examples we construct in this chapter are warped products of $\mathbb{R}$ and the hyperbolic space. Throughout, let

$$M^n = \mathbb{R} \times \mathbb{H}^{n-1} = \{(t, (x_i)) \,|\, t \in \mathbb{R}, \, (x_i) \in \mathbb{H}^{n-1}\}.$$

The main goal of the chapter is to prove a more detailed version of Theorem 4.

Theorem 6.1. *For each $(a, b) \in \mathbb{R} \times [-1, 1]$, M^n admits a complete Riemannian metric $h_{n,a,b}$ satisfying the following conditions.*

(1) For fixed $n \geq 2$, the metrics $h_{n,a,b}$ depend smoothly on (a, b).

(2) There is a surjective homomorphism from $\mathrm{Isom}(\mathbb{H}^{n-1}, h_{n-1}) \to \mathrm{Isom}(M^n, h_{n,a,b})$.

(3) Each tangent 2-plane to M^n containing the tangent vector $\frac{\partial}{\partial t}$ exponentiates to a totally-geodesic surface isometric to the hyperbolic plane.

(4) The sectional curvature of a 2-plane σ at $p = (t, (x_i))$ making angle $\theta \in [0, \frac{\pi}{2}]$ with $\frac{\partial}{\partial t}$ is given by

$$\sec(\sigma) = -1 + \frac{4(1 - b^2 - e^{2a})}{((1 + b)e^t + (1 - b)e^{-t})^2} \sin^2(\theta).$$

(5) For $n \geq 3$, $(M^n, h_{n,a,b})$ Riemannian covers a finite volume manifold if and only if $1 = b^2 + e^{2a}$, or equivalently, if and only if $(M^n, h_{n,a,b})$ is isometric to $(\mathbb{H}^n, h_n)$.

6.1 Warping hyperbolic 1-space over the Euclidean line.

In this section, we describe a way to write hyperbolic space as a warped product over $\mathbb{R}$. Let t denote the Euclidean coordinate on $\mathbb{R}$ and $r > 0$ the Euclidean coordinate on $\mathbb{H}^1$. Consider the foliation of $\mathbb{H}^2$ by the family of (Euclidean) upper-half semicircles with common center the origin in $\mathbb{R}^2$. Each such semicircle, parameterized appropriately, is an h_2-geodesic: For each $r > 0$, the map

$$F_r : \mathbb{R} \to \mathbb{H}^2$$

defined by $F_r(t) = (r \tanh(t), r \operatorname{sech}(t))$ parameterizes the semicircle through $(0, r)$ in the clockwise fashion as a unit-speed h_2-geodesic with the initial point $(0, r)$.

Define a diffeomorphism $F : \mathbb{R} \times \mathbb{H}^1 \to \mathbb{H}^2$ by

$$F(t, r) = F_r(t).$$

In the coordinates (t, r) on $\mathbb{R} \times \mathbb{H}^1$, the metric $F^*(h_2)$ is given by

$$F^*(h_2) = dt^2 + \cosh^2(t) h_1 = dt^2 + e^{2 \ln(\cosh(t))} h_1.$$

Hence, $(\mathbb{H}^2, h_2)$ is isometric to a warped product of the hyperbolic line over the Euclidean line.

In fact, the hyperbolic plane $\mathbb{H}^2$ is isometric to many different warpings of $\mathbb{H}^1$ over $\mathbb{R}$.

The warping functions are described in terms of solutions to the second order differential equation

$$\phi''(t) + (\phi'(t))^2 - 1 = 0. \tag{6.1.1}$$

For $a \in \mathbb{R}$ and $-1 \leq b \leq 1$, the solution to (6.1.1) determined by the initial conditions $\phi(0) = -a$ and $\phi'(0) = b$ is given by

$$\phi_{a,b}(t) = \ln(\frac{(1+b)e^t + (1-b)e^{-t}}{2e^a}). \tag{6.1.2}$$

Let $M^2 = \mathbb{R} \times \mathbb{H}^1$ and let $\pi : M^2 \to \mathbb{R}$ be the first coordinate projection. Ignore the slight abuse of notation in letting $\phi_{a,b}$ also denote the function $\pi^*(\phi_{a,b})$ on M^2.

For a and b as above, consider the warped product Riemannian metric

$$h_{2,a,b} = dt^2 + e^{2\phi_{a,b}} h_1$$

on M^2. In this notation, $h_{2,0,0} = F^*(h_2)$.

Lemma 6.2. *For each $(a, b) \in \mathbb{R} \times [-1, 1]$, $(M^2, h_{2,a,b})$ is isometric to the hyperbolic plane $(\mathbb{H}^2, h_2)$.*

Proof. The proof only uses the fact that $\phi_{a,b}$ is a solution to (6.1.1). For this reason, and to simplify notation, set $h = h_{2,a,b}$ and $\phi = \phi_{a,b}$ in the remainder of this proof. As $(\mathbb{R}, dt^2)$ and $(\mathbb{H}^1, h_1)$ are complete, so too is their warped product (M^2, h) [13, Lemma 40, pg. 209]. Therefore, it suffices to prove that h has constant curvature -1.

The curvature at each point equals -1 as a consequence of (6.1.1) since by [13, Proposition 42 (4), pg. 210], the sectional curvature at a point is given by $-\phi'' - (\phi')^2$. □

6.2 Main Construction

The main construction is a higher dimensional generalization of the analogue in the previous section. However, for dimension greater than two, the metrics we obtain are no longer hyperbolic.

Fix $n \geq 2$ and $(a, b) \in \mathbb{R} \times [-1, 1]$. Let $\phi_{a,b}$ be as in (6.1.2). Let $h_{n,a,b}$ denote the warped product Riemannian metric on $M^n = \mathbb{R} \times \mathbb{H}^{n-1}$ defined by

$$h_{n,a,b} = dt^2 + e^{2\phi_{a,b}} h_{n-1}.$$

The metric $h_{n,a,b}$ is complete by [13, Lemma 40, pg. 209]. We now consider the properties (1)-(5) stated in Theorem 6.1.

Property (1): Property (1) is immediate since the warping functions $\phi_{a,b}$ depend smoothly on $(a, b) \in \mathbb{R} \times [-1, 1]$.

Property (2): Given $F \in \mathrm{Isom}(\mathbb{H}^{n-1}, h_{n-1})$, define $\bar{F} : M^n \to M^n$ by

$$\bar{F}(t, (x_i)) = (t, F(x_i)).$$

Then $\bar{F} \in \mathrm{Isom}(M^n, h_{n,a,b})$ and the map

$$\mathrm{Isom}(\mathbb{H}^{n-1}, h_{n-1}) \to \mathrm{Isom}(M^n, h_{n,a,b})$$

defined by $F \mapsto \bar{F}$ is a monomorphism.

Property (3): To find one such totally-geodesic hyperbolic plane, define $G \in \mathrm{Isom}(\mathbb{H}^{n-1}, h_{n-1})$ by

$$G(x_1, \ldots, x_{n-2}, x_{n-1}) = (-x_1, \ldots, -x_{n-2}, x_{n-1}).$$

As $\bar{G}$ is an isometry, its fixed point set

$$\Sigma = \{(t, (0, \ldots, 0, x_{n-1}))\} \subset M^n$$

is a totally-geodesic surface. The map $(t, (0, \ldots, 0, x_{n-1})) \mapsto (t, x_{n-1})$ defines an isometry between the induced metric on Σ and $(M^2, h_{2,a,b})$. By Lemma 6.2, Σ is isometric to the hyperbolic plane. We conclude by showing that given an arbitrary tangent plane σ as in Property (3) there is an isometry of M^n that carries a tangent plane to Σ to σ.

Let $\pi : M^n \to \mathbb{R}$ denote the first coordinate projection and $\mathcal{F}_{\bar{t}}$ the fiber above $\bar{t} \in \mathbb{R}$. For each $p \in \mathcal{F}_{\bar{t}}$, let $\mathcal{V}_p$ denote the set of tangent 2-planes to M^n at p that contain the vector $\frac{\partial}{\partial t}(p)$. Let

$$\mathcal{X}_{\bar{t}} = \bigcup_{p \in \mathcal{F}_{\bar{t}}} \mathcal{V}_p$$

and let σ_0 denote the tangent space to Σ at the point $(\bar{t}, (0, \ldots, 0, 1))$ and note that $\sigma_0 \in X_{\bar{t}}$. As $\mathrm{Isom}(\mathbb{H}^{n-1}, h_{n-1})$ acts transitively on unit-tangent vectors to $\mathbb{H}^{n-1}$, $\mathrm{Isom}(M^n, h_{n,a,b})$ acts transitively on the set $\mathcal{X}_{\bar{t}}$. Hence, there exists an isometry $\bar{I}$ of M^n that carries σ_0 to σ.

As

$$\exp(\sigma) = \exp(d\bar{I}(\sigma_0)) = \bar{I}(\exp(\sigma_0)) = \bar{I}(\Sigma),$$

$\exp(\sigma)$ is a totally geodesic surface isometric to the hyperbolic plane.

Property (4): Let $p = (t, (x_i)) \in M$ and let $v, w, \frac{\partial}{\partial t}$ be orthonormal vectors at p.

Property (3) implies that

$$R(\frac{\partial}{\partial t}, w, w, \frac{\partial}{\partial t}) = -1. \tag{6.2.1}$$

By [13, Proposition 42 (5), pg. 210],

$$R(v, w, w, v) = -1 + \frac{4(1 - b^2 - e^{2a})}{((1+b)e^t + (1-b)e^{-t})^2}. \tag{6.2.2}$$

By [13, Proposition 42 (3), pg.210],

$$R(\frac{\partial}{\partial t}, w)v = 0. \tag{6.2.3}$$

Now fix a two dimensional subspace $\sigma \subset T_pM$. Assume that σ makes angle $\theta \in [0, \frac{\pi}{2}]$ with $\frac{\partial}{\partial t}$. Then there exist unit length vectors v and w perpendicular to $\frac{\partial}{\partial t}$ such that

$$\{\bar{v} = \cos(\theta)\frac{\partial}{\partial t} + \sin(\theta)v, w\}$$

is an orthonormal basis of σ. By (6.2.1)-(6.2.3) and the symmetries of the curvature tensor,

$$\begin{aligned}
\sec(\sigma) &= R(\bar{v}, w, w, \bar{v}) \\
&= \cos^2(\theta)R(\frac{\partial}{\partial t}, w, w, \frac{\partial}{\partial t}) + \sin^2(\theta)R(v, w, w, v) \\
&= -1 + \frac{4(1 - b^2 - e^{2a})}{((1+b)e^t + (1-b)e^{-t})^2}\sin^2(\theta).
\end{aligned}$$

Property (5): As $(M^n, h_{n,a,b})$ is simply connected and complete, it is isometric to $(\mathbb{H}^n, h_n)$ if and only if it has constant sectional curvatures -1. By (4), this is equivalent to the parameters satisfying $1 = b^2 + e^{2a}$. As $(\mathbb{H}^n, h_n)$ Riemannian covers finite volume manifolds, it remains to prove that when $n \geq 3$ and $1 \neq b^2 + e^{2a}$, $(M^n, h_{n,a,b})$ does not Riemannian cover a finite volume manifold.

Assume that $1 \neq b^2 + e^{2a}$. As isometries preserve sectional curvatures, Property (4) implies that each isometry $F \in \mathrm{Isom}(M^n, h_{n,a,b})$ splits as $F((t, x_i)) = (F_1(t), F_2(x_i))$ for some $F_1 \in \mathrm{Isom}(\mathbb{R}, dt^2)$ and $F_2 \in \mathrm{Isom}(\mathbb{H}^{n-1}, h_{n-1})$. If $b = \pm 1$, then F_1 is the identity. If $b \neq \pm 1$, then F_1 is either the identity or reflection about the critical point $t_0 = \ln(\sqrt{1-b}) - \ln(\sqrt{1+b})$ of $((1+b)e^t + (1-b)e^{-t})^2$.

Let (N, g) be a Riemannian manifold with universal covering $(M^n, h_{n,a,b})$. Its fundamental group $\pi_1(N)$ is identified with a subgroup Γ of $\mathrm{Isom}(M, h_{n,a,b})$ that acts properly discontinuously and fixed point freely on M^n. The association $\gamma \mapsto \gamma_1$ described above defines a homomorphism $\Phi : \Gamma \to \mathrm{Isom}(\mathbb{R}, dt^2)$ whose kernel has index at most two in Γ. Let $(\bar{N}, \bar{g})$ be the Riemannian covering of (N, g) associated to the subgroup $\ker \Phi$ of Γ.

As each isometry in $\ker \Phi$ acts trivially on the first factor of $M^n = \mathbb{R} \times \mathbb{H}^{n-1}$, the metric $\bar{g}$ is also a warped product metric with the same warping function $e^{\phi_{n,a,b}}$. By (4), $\bar{g}$ has negative sectional curvatures. By [4, Lemma 7.6], $e^{\phi_{n,a,b}}$ is a non-constant convex function on $(\bar{N}, \bar{g})$. By [4, Proposition 2.2], $(\bar{N}, \bar{g})$ has infinite volume. Therefore, (N, g) has infinite volume, as required.

Chapter 7

Concluding Remarks

Our ultimate goal is to generalize Theorem 1 and 2 into higher dimensions.

Conjecture 7.1. *A complete finite volume Riemannian manifold has higher hyperbolic rank if and only it is locally symmetric. A complete Riemannian manifold has higher spherical rank if and only if it is locally symmetric.*

The full conjecture seems to be hard. Without additional assumptions on the curvature tensors, it is unlikely that our approach for the three dimensional case can be applied to the higher dimensional cases. Here in this chapter, we mention several possible directions for future work.

7.1 Constructing Examples

In Chapter 6, we perturbed umbilic foliations in the real hyperbolic space to obtain a one parameter family of non-homogeneous manifolds of higher rank. It is natural to ask if one can generalize this construction to other noncompact rank one symmetric spaces.

Problem 7.2. *Is it possible to perturb complex, quaternionic, or octonionic hyperbolic metrics to obtain a one parameter family of non-symmetric metrics of higher hyperbolic rank?*

7.2 Curvature Assumption

Fixing $\varepsilon = -1, 0$, or 1, it is less known if geometric rank characterizes symmetric spaces when $\sec \geq \varepsilon$. Spatzier and Strake showed that manifolds of higher rank are not rigid when $\sec \geq 0$ by constructing many non-symmetric examples in [17]. Partial rigidity results were given by Schmidt, Shankar, and Spatzier in [14]. To explain their result, we introduce the following definitions.

Fix $\varepsilon = -1, 0$, or 1. Let M^d be a complete d dimensional Riemannian manifold with $\sec \geq \varepsilon$. The rank (hyperbolic, Euclidean, or spherical, depending on $\varepsilon = -1, 0$, or 1) of a geodesic $\gamma : \mathbb{R} \to M$ is defined as the number of linearly independent normal parallel fields making curvature ε with the geodesic. As we assume that the sectional curvatures are bounded below from ε, these notions of rank are well-defined. We say that the rank (hyperbolic, Euclidean, or spherical) of M is at least k if the rank of every geodesic in M is at least k.

Schmidt, Shankar, and Spatzier showed that, when d is odd, if $\sec \geq 1$ and the spherical rank of M^d is at least $d-2$, then M^d must have constant sectional curvatures. It is natural to ask if one can prove a similar result in the hyperbolic rank setting.

Conjecture 7.3. *Let M^d be a finite volume, complete, d dimensional Riemannian manifold with* $\sec \geq -1$. *Then the hyperbolic rank of M is at least $d-2$ if and only if M is real hyperbolic.*

It should be noted that when $\sec \geq -1$, the hyperbolic rank of the other negatively curved d dimensional symmetric spaces are less than $d-2$.

7.3 Stronger Rank assumption

Let M be a complete Riemannian manifold such that every geodesic is contained in a closed, immersed flat of dimension at least two. Molina and Olmos proved that M is locally symmetric if it is locally irreducible at any point in an open nonempty subset. Motivated by their results, we give the following definition.

Definition 7.4. A complete Riemannian manifold is said to have higher *integrable* hyperbolic (respectively spherical) rank if every geodesic in the manifold is contained in an immersed totally geodesic submanifold of dimension greater than 2 and constant sectional curvature -1 (respectively 1).

It is natural to ask if the results given by Molina and Olmos can be generalized into the hyperbolic and spherical setting.

Problem 7.5. *Classify manifolds of higher integrable rank.*

Here is one small step towards integral spherical rank rigidity.

Theorem 7.6. *Let M^n be a complete n dimensional Riemannian manifold with $n \geq 3$ and let k be an integer satisfying $2k > n$. Then M^n is isometric to the sphere of constant sectional curvatures* 1 *if and only if every geodesic in M^n is contained in an embedded totally geodesic k dimensional sphere of constant sectional curvatures* 1*.*

Proof. Note that all geodesics in M are simple closed, and prime geodesics have length 2π. Since the index of each geodesic in M is at least k with k greater than $n/2$, the Bott-Samelson Theorem [2, Theorem 7.23] implies that M is a homotopy sphere.

Let $p \in M$ and γ a unit speed geodesic starting from p. By assumption, γ is contained in an embedded totally geodesic k dimensional sphere S_γ of constant sectional curvatures 1.

Let $A(p)$ be the antipodal point of p in S_γ and let η be any other unit speed geodesic starting from p. Then η is also contained in an embedded totally geodesic k dimensional sphere S_η of constant sectional curvatures 1. As $2k > n$, the tangent spaces T_pS_γ and T_pS_η share a line. Hence η also passes through $A(p)$ at time π. Therefore, all geodesics starting from p passes through $A(p)$ at time π and $\mathrm{inj}(M) = \mathrm{diam}(M) = \pi$. Hence M is a Blaschke manifold homotopic to a sphere. By the solution to the Blaschke conjecture for spheres [2, Appendix D], M has constant sectional curvatures 1. $\square$

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