GREEN'S FUNCTIONS FOR VARIANTS OF THE SCHRAMM-LOEWNER EVOLUTION

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ABSTRACT

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We prove upper bounds for the probability that a radial SLE_κ curve comes within specified radii of n different points in the unit disc. Using this estimate, we then prove a similar upper bound for the probability that a whole-plane SLE_κ passes near any n points in the complex plane. We then use these estimates to show that the lower Minkowski content of both the radial and whole-plane SLE_κ traces has finite moments of any order.

For $\kappa \leq 4$, the reverse flow of the Loewner equation driven by $\sqrt{\kappa}B_t$ generates a random continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ called the conformal welding. In studying backward SLE, this plays the roll of the global random object, rather than the SLE trace. Given any $x, y > 0$ we use the Girsanov theorem to construct a family of probability measures, depending on some parameters, under which the conformal welding satisfies $\phi(x) = y$ almost surely. For one such law, we prove a one-point estimate for the backward SLE welding and show how it coincides with the Green's function. In another case, we decompose the law of the welding conditioned to pass through (x, y) into two pieces. Using this decomposition, we integrate this law over a set $U \subset [0, \infty) \times [0, \infty)$ to get a new measure on weldings which is absolutely continuous with respect to the original backward SLE welding. Moreover, the Radon-Nikodym derivative is given by the capacity time that the graph of ϕ spends in U.

In the last chapter, we study a generalization of the chordal Loewner equation called chordal measure driven Loewner evolution. We show existence of a solution to the equation, and a one-to-one correspondence between the appropriate measures and all continuously

growing families of $\mathbb{H}$-hulls. In [19], the notion of measure driven Loewner evolution was first introduced in the radial setting, and a similar theorem was proven. This result is pure complex analysis, without any reference to probability theory.

I would like to dedicate this thesis to my wonderful wife, Lauren Conway, who has never waivered in her support of and confidence in me. I could not have written this without her standing by my side. "Best of wives and best of women."

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TABLE OF CONTENTS

Chapter 1

Introduction

1.1 Description of SLE

This thesis is concerned with the study of conformally invariant randomly growing processes in subdomains of the complex plane. The main object is the Schramm-Loewner evolution, or SLE, which was introduced by Oded Schramm in [29] as a candidate for several lattice models in statistical physics. The processes all arise from solving a variant of the Loewner differential equation driven by time scaled Brownian motion. In all cases, there is a domain $D \subset \mathbb{C}$ and a random curve γ contained in the closure of D starting and ending at fixed points.

To motivate the definition, we will informally describe the loop erased random walk in a domain from a to b. Let $D \subset \mathbb{C}$ be a simply connected domain, and let $\delta > 0$ be a small number. Consider the lattice $D_\delta = D \cap \delta\mathbb{Z}^2$, and let γ_δ be the path of a simple random walk on D_δ starting at the point (nearest to) $a \in \partial D$, and stopped when it hits a point (nearest to) b in $\overline{D}$, the closure of D. The path γ_δ can be turned into a simple path γ_δ^{LE} by removing all of the loops. Is there is a random path γ so that $\gamma_\delta^{LE} \to \gamma$ in law as $\delta \to 0$? SLE provides an answer, which is that the scaling limit is an SLE curve.

SLE has a parameter $\kappa > 0$ on which the behavior of the SLE_κ path depends. For the loop erased random walk, SLE_2 has been proven to be the scaling limit for the loop erased random walk [13]. Several other lattice models have proven to converge to SLE_κ for other values of κ. SLE_3 and $\text{SLE}_{16/3}$ are the scaling limits for Ising model interfaces [35], SLE_6 is the scaling limit of the critical percolation explorer curve [34], SLE_4 is the scaling limit of the harmonic explorer [30], and SLE_8 is the scaling limit of the uniform spanning tree Peano curve [13].

There are several varieties of SLE which differ based on where they begin and end, and each variety has a standard domain in which it is easier to define and study. The two primary types are called chordal SLE_κ, which begins and ends at fixed boundary ponts, and radial SLE_κ, which begins at a fixed boundary point and ends at a fixed interior point.

Making this more precise, fix a simply connected domain D, and fix a point $a \in \partial D$ (technically, a should be a prime end of D, but we will not define that now), and let $b \in \overline{D}$. If $b \in \partial D$ (or b is a prime end of D), then chordal SLE_κ in D from a to b is a random path in $\overline{D}$ which begins at a and ends at b almost surely. The standard domain for studying chordal SLE is the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$, starting at 0 and ending at ∞, where ∞ can seen as a point in the Riemann sphere. If b is in the interior of D, then radial SLE_κ in D from a to b is a random path in D which begins at a and ends at b almost surely. The standard domain for radial SLE_κ is the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, starting at 1 and ending at 0.

Once the SLE path is defined in its standard domain, in either case, it can be defined in any simply connected domain D. To define chordal SLE_κ in a simply connected domain D from a to b, let $\phi : \mathbb{H} \to D$ be a conformal tranformation with $\phi(0) = a$ and $\phi(\infty) = b$. Then if μ_κ is the probability measure on paths in $\mathbb{H}$ which is the law of the standard chordal SLE_κ, then the law of SLE_κ in D from a to b is given by the pushforward meaure induced by ϕ on μ_κ.

This describes conformal invarience. Schramm proved that the only processes which have conformal invarience and the domain Markov property are SLE processes. These properties are described more precisely as:

- (Conformal invarience) If γ is an SLE$_\kappa$ path in D from a to b, and $\phi : D \to D'$ is a conformal transformation with $\phi(a) = a'$ and $\phi(b) = b'$, then $\phi(\gamma)$ is a time changed SLE$_\kappa$ path in D' from a' to b'.

- (Domain Markov property) Let T be a stopping time with respect to the filtration generated by an SLE$_\kappa$ path in a domain D from a to b. For any $t > 0$, Let D_t be the component of $D \backslash \gamma[0, t]$ which contains the target point b. Then if $\gamma^T(t) = \gamma(T + t)$, after conditioning on $\gamma[0, T]$, the path γ^T is an SLE$_\kappa$ path in D_T from $\gamma(T)$ to b.

The laws of chordal SLE and radial SLE are not absolutely continuous with each other, but there is local absolute continuity [31]. If γ is radial SLE$_\kappa$ in $\mathbb{D}$ from 1 to 0, let $\epsilon > 0$ and $T = \inf\{t > 0 : |\gamma(t)| = \epsilon\}$, which is finite a.s.. Then radial SLE$_\kappa$ restricted to time $t \leq T$ is absolutely continuous with respect to a stopped chordal SLE$_\kappa$ up to a time change, so the paths share many of the same local properties. For example, in each case, the Hausdorff dimension of the SLE$_\kappa$ path is $d = \min\{1 + \kappa/8, 2\}$ [3]. Also, the path has two phase transitions as κ varies [27].

We state the transitions when γ is the chordal SLE$_\kappa$ path in $\mathbb{H}$ from 0 to ∞. For $0 < \kappa \leq 4$, $\gamma(0, \infty)$ is a simple path which stays in $\mathbb{H}$. For $\kappa \geq 8$, γ is a space filling path. That is, $\gamma(0, \infty) = \mathbb{H}$ almost surely. For $4 < \kappa < 8$, γ has both boundary intersections and self intersections. If $\kappa < 8$, for any $z \in \overline{\mathbb{H}}$, we have $\mathbb{P}[z \in \gamma] = 0$. This is proven in [27] reducing the SLE$_\kappa$ maps started on the boundary to the study of Bessel processes.

Note that if $\kappa \leq 4$, the domain D_T, as described in the domain Markov property, is exactly equal to $D\backslash\gamma[0,T]$, since the path is simple and has no boundary intersections. For $\kappa > 4$, D_T the piece of the domain which has not yet been cut off from the endpoint by the path.

Another variety of SLE is called whole-plane SLE, which is a path in $\mathbb{C}$ which grows from 0 to ∞. It can be seen as a limit of radial SLE_κ paths in large discs from the boundary to 0. This can be extended to whole-plane SLE from any $a \in \hat{\mathbb{C}}$ to any other $b \in \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ is the extended complex plane, by applying a Möbius transformation which takes 0 to a and ∞ to b, which gives an SLE path connecting any two interior points of $\hat{\mathbb{C}}$.

The domain Markov property has a slightly different form for whole-plane SLE_κ. If γ^* is the whole-plane SLE_κ path, then γ^* is a path $\gamma^* : (-\infty, \infty) \to \mathbb{C}$. For any stopping time T, the path $\gamma^T(t) = \gamma^*(T+t)$, conditioned on $\gamma^*(-\infty, T]$, is a a radial SLE_κ in $\mathbb{C}\backslash D_T$, where D_T is the unbounded component of $\mathbb{C}\backslash\gamma^*(-\infty, T]$.

1.2 Definitions of variants of SLE

There are several varieties of the Loewner equation. First, we describe the types of hulls which correspond to each Loewner equation. We then briefly discuss the chordal equation, then introduce radial and whole-plane Loewner equations. Complete details can be found in [7]. We also introduce the reverse Loewner equation and define backward SLE.

1.2.1 Hulls in the complex plane

Let $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ be the upper half-plane. A set $K \subset \mathbb{H}$ is called an $\mathbb{H}$-hull if K is relatively compact in $\mathbb{H}$ and $\mathbb{H}\backslash K$ is simply connected. Then there is a unique conformal map $g_K : \mathbb{H}\backslash K \to \mathbb{H}$ with $g_K(z) = z + \frac{c}{z} + O(z^{-2})$ as $z \to \infty$. The constant c is called the half-plane capacity, and is denoted by $\mathrm{hcap}(K)$, and it is equal to 0 if and only if K is empty. We call a domain $H \subset \mathbb{H}$ and $\mathbb{H}$-domain if $H = \mathbb{H}\backslash K$ for some $\mathbb{H}$-hull K.

Half-plane capacity is monotonic, in that if $K_1 \subset K_2$ are both $\mathbb{H}$-hulls, we have $\mathrm{hcap}(K_1) \leq \mathrm{hcap}(K_2)$. Moreover, if $K_1 \subset K_2$ are both $\mathbb{H}$-hulls, we can define a new $\mathbb{H}$-hull $K_2/K_1 := g_{K_1}(K_2\backslash K_1)$, called the quotient hull. In [26], it has been shown that $\mathrm{hcap}(K_2) = \mathrm{hcap}(K_1) + \mathrm{hcap}(K_2/K_1)$. Therefore, if K_1 is properly contained in K_2, it follows that $\mathrm{hcap}(K_1) < \mathrm{hcap}(K_2)$. Half-plane capacity also satisfies the scaling rule $\mathrm{hcap}(rK + a) = r^2\mathrm{hcap}(K)$ for any $a, r \in \mathbb{R}$.

In [26], the authors define the support of an $\mathbb{H}$-hull K and prove the following results. Let $B_K = \overline{K} \cap \mathbb{R}$, and let $K' = \{\overline{z} : z \in K\}$. Define the double of K by $\hat{K} = K \cup K' \cup B_K$. Then $g_K(\mathbb{C}\backslash\hat{K}) = \mathbb{C}\backslash S_K$, for a compact $S_K \subset \mathbb{R}$, which we will call the support of K. Then the inverse $f_K := g_K^{-1}$ defined in $\mathbb{H}$ can be extended conformally to $\mathbb{C}\backslash S_K$ by the Schwarz reflection principle, but no further. If K only has one component, then S_K is an interval, but in general K can consist of several components. Let S_K^* denote the convex hull of S_K, so that S_K^* is a compact interval. If $K_1 \subset K_2$, then $S_{K_1}^* \subset S_{K_2}^*$.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc. A $\mathbb{D}$-hull is a set $K \subset D$ which is relatively closed in $\mathbb{D}$, $0 \notin K$, and $\mathbb{D}\backslash K$ is simply connected. By the Riemann mapping theorem, there is a unique conformal map $g_K : \mathbb{D}\backslash K \to \mathbb{D}$ with $g_K(0) = 0$ and $g_K'(0) > 0$. The $\mathbb{D}$-capacity of K is defined by $\mathrm{dcap}(K) = \ln(g_K'(0))$. A set $D \subset \mathbb{D}$ is called a $\mathbb{D}$-domain if $D = \mathbb{D}\backslash K$ for a $\mathbb{D}$-hull K.

$\mathbb{D}$-capacity can be readily shown to be monotonic with respect to inclusion. If $K_1 \subset K_2$ and the quotient hull $K_2/K_1 := g_{K_1}(K_2\backslash K_1)$, then $g_{K_2} = g_{K_2/K_1} \circ g_{K_1}$, and the chain rule implies that $g_{K_2}'(0) = g_{K_2/K_1}'(0)g_{K_1}'(0)$. Taking the natural logarithm of both sides yields the monotonicity.

A compact hull in $\mathbb{C}$ is a set $K \subset \mathbb{C}$ such that $0 \in K$, K is connected and compact, and $\mathbb{C} \backslash K$ is connected. Then there is a unique conformal map $F_K : \mathbb{C} \backslash \overline{\mathbb{D}} \to \mathbb{C} \backslash K$ with $\lim_{z \to \infty} F_K(z)/z > 0$. Looking at $\mathbb{C} \backslash K$ under the inversion $z \mapsto 1/z$ gives a simply connected domain U containing 0, for which there is a unique conformal Riemann map $f_K : \mathbb{D} \to U$ with $f_K'(0) > 0$. Then $F_K(z) = 1/f_K(1/z)$, and the logarithmic capacity of K is defined by $\mathrm{cap}(K) = \log(f_K'(0))$.

1.2.2 Chordal Loewner evolution in $\mathbb{H}$

The chordal Loewner equation describes the growth from a boundary point to another boundary point. The standardized domain is the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, and the chordal Loewner equation driven by a real valued $\lambda \in C[0, \infty)$ at $z \in \mathbb{H}$ is

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \lambda_t}, \quad g_0(z) = z. \tag{1.1}$$

If $\tau_z = \inf\{t > 0 : \mathrm{Im}(g_t(z)) = 0\}$ is the lifetime of (1.1) at z, define $K_t = \{z \in \mathbb{H} : \tau_z \leq t\}$. Then K_t is an $\mathbb{H}$-hull with half-plane capacity $\mathrm{hcap}(K_t) = 2t$, and $g_t = g_{K_t}$ is the conformal map sending $\mathbb{H} \backslash K_t$ to $\mathbb{H}$.

If $\kappa > 0$ and B_t is a standard one dimensional Brownian motion, then (1.1) driven $\lambda_t = \sqrt{\kappa} B_t$ is called chordal SLE$_\kappa$. In [27], it is proven that there is a path $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ called the chordal SLE$_\kappa$ trace, also called the path or curve, which grows from 0 to ∞. Then for each $t > 0$, $\mathbb{H} \backslash K_t$ is the unbounded component of $\mathbb{H} \backslash \gamma[0, t]$.

1.2.3 Radial Loewner evolution in $\mathbb{D}$

Given any real valued continuous function $\lambda \in C[0,\infty)$, the radial Loewner equation driven by λ at $z \in \mathbb{D}$,

$$\partial_t g_t(z) = g_t(z)\frac{e^{i\lambda(t)} + g_t(z)}{e^{i\lambda(t)} - g_t(z)}, \quad g_0(z) = z. \tag{1.2}$$

If $\tau_z = \inf\{t > 0 : |g_t(z)| = 1\}$ is the lifetime of (1.2) at $z \in \mathbb{D}$ and $K_t = \{z \in \mathbb{D} : \tau_z \le t\}$, then K_t is a $\mathbb{D}$-hull with $\operatorname{dcap}(K_t) = t$.

For $\kappa > 0$, the radial SLE$_\kappa$ process is the solution (1.2) for $\lambda(t) = \sqrt{\kappa}B_t$, where B_t is standard one dimensional Brownian motion. Similarly to the chordal case, there is a a radial trace $\gamma : [0,\infty) \to \overline{\mathbb{D}}$ so that $\mathbb{D}\backslash K_t$ is the component of $\mathbb{D}\backslash\gamma[0,t]$ containing 0 with $\gamma(0) = 1$ and $\gamma(\infty) = 0$. The radial SLE$_\kappa$ trace has the same phrase transitions as the chordal SLE$_\kappa$ and the same dimension. The radial SLE$_\kappa$ process can be studied by looking at the covering space for $\mathbb{D}$ under the exponential map e^{iz}, which is the cylinder $\mathbb{H}^* = \{[z]_\sim : z \in \mathbb{H}\}$, where $z \sim w$ if $z - w \in 2\pi\mathbb{Z}$. This space will be discussed in more detail in Section 2.2.2.

1.2.4 Whole-plane Loewner evolution in $\mathbb{C}$

If $\lambda \in C(-\infty,\infty)$ is real valued and $z \in \mathbb{C}\backslash\{0\}$, the whole-plane Loewner equation driven by λ is

$$\partial_t g_t^*(z) = g_t^*(z)\frac{e^{-i\lambda_t} + g_t^*(z)}{e^{-i\lambda_t} - g_t^*(z)}, \text{ and } \lim_{t\to-\infty} e^t g_t^*(z) = z. \tag{1.3}$$

This can be interpreted as a radial Loewner equation driven by $-\lambda$ started from $t = -\infty$. For each $t \in (-\infty,\infty)$, $K_t = \{z \in \mathbb{C} : \tau_z \le t\}$ is a compact hull in $\mathbb{C}$ with logarithmic capacity $\operatorname{cap}(K_t) = t$, and $g_t^* : \mathbb{C}\backslash K_t \to \mathbb{C}\backslash\overline{\mathbb{D}}$ conformally.

To define whole-plane SLE$_\kappa$, we need to define two sided Brownian motion. Let B_t^1 and

B_t^2 be independent standard one dimensional Brownian motions, and let Y be an independent uniform $[-\pi/\sqrt{\kappa}, \pi/\sqrt{\kappa}]$ random variable. Construct a two sided Brownian motion $B : \mathbb{R} \to \mathbb{R}$ by $B_t = B_t^1 + Y$ if $t \geq 0$, and $B_t = B_{-t}^2 + Y$ if $t < 0$. Then g_t^* is a whole-plane SLE$_\kappa$ if it solves (1.3) with $\lambda_t = \sqrt{\kappa} B_t$.

There is a curve $\gamma^* : (-\infty, \infty) \to \mathbb{C}$ called the whole-plane SLE$_\kappa$ trace. For each $t \in \mathbb{R}$, $\mathbb{C} \backslash K_t$ is the unbounded component of $\mathbb{C} \backslash \gamma^*(-\infty, t]$. Also, $\lim_{t \to -\infty} \gamma^*(t) = 0$ and $\lim_{t \to \infty} \gamma^*(t) = \infty$, and so the whole-plane SLE$_\kappa$ curve grows from 0 to ∞ in the entire complex plane. Since at any time $s \in \mathbb{R}$ the process $(B_t)_{t \geq s}$ is a Brownian motion with a random start time, it follows that γ^* has the same phrase transitions and dimension depending on κ as the chordal and radial processes.

1.2.5 Backward Loewner evolution

The backward Loewner equation started at $z \in \overline{\mathbb{H}}$ driven by $\lambda_t \in C[0, T)$ is

$$\partial_t f_t(z) = \frac{-2}{f_t(z) - \lambda_t}, \quad f_0(z) = z. \tag{1.4}$$

The process $(f_t)_{t \leq T}$ is called the backward Loewner process driven by λ. Suppose solving equation (1.1) with the function $t \to \lambda(t_0 - t)$ generates a forward Loewner trace $t \to \gamma_{t_0}(t_0 - t)$ for each $t_0 \leq T$. Then we say that λ generates a family of backward Loewner traces $(\gamma_{t_0})_{t_0 < T}$. If the paths γ_{t_0} are all simple paths, then $f_{t_0} : \mathbb{H} \to \mathbb{H} \backslash \gamma_{t_0}[0, t_0]$ is conformal for each $t_0 \leq T$. Note that in this case, growth does not occur at the tip as in the forward case. Rather, growth occurs at the base of the curve, which is $\lambda(t)$ at each t, and the rest of the curve is the image of the previous curves under a conformal map.

Suppose $\kappa > 0$, and let $\lambda_t = \sqrt{\kappa} B_t$. Then the process given by solving (1.4) is called the backward SLE_κ process, and will be denoted by $BSLE_\kappa$. For a fixed $T < \infty$, the trace $\gamma_T - \lambda_T$ has the same law as a forward SLE_κ trace on $[0, T]$. So, to study the SLE trace at a finite time, it suffices to study backward SLE, where the Loewner equation may be easier to work with. However, the backward SLE traces are not a good global object to study, since the images $\gamma_T[0, T]$ evolve over time, rather than grow from the tip. If the traces are simple, there is another object called the conformal welding which serves as global object to study.

For $\kappa \leq 4$, the SLE_κ traces are simple, and so the $BSLE_\kappa$ process generates a random function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ called the conformal welding. The conformal maps f_t can be extended to the boundary of $\mathbb{H}$. For any $x \in \mathbb{R}$, let τ_x denote the lifetime of the backward Loewner equation, which is finite almost surely. For $x, y > 0$, the welding is defined by $\phi(x) = y$ if $\tau_x = \tau_{-y}$, which is true if and only if $f_t(x) = f_t(-y)$ for some $t > 0$. Then ϕ is a random monotonic function on $\mathbb{R}^+$. This welding is the main object of study in [26], where the welding is shown to satisfy a reversability property analogous to a fundamental property of the forward SLE trace [38] [20].

We will also need to keep track of which points are welded together at a given capacity time t. To this end, let $b_t = \sup\{x > 0 : \tau_x \leq t\}$ and $a_t = \sup\{y > 0 : \tau_{-y} \leq t\}$, so that $f_t(b_t) = f_t(-a_t) = \lambda_t$ for every t. Define $\Phi(t) = (b_t, a_t)$, which we will call the welding curve. Then if $Q_1 = [0, \infty) \times [0, \infty)$ is the first quadrant, $\Phi : [0, \infty) \to Q_1$ is a random path whose image is the graph of the welding function ϕ.

1.3 Natural parametrization and Green's functions

The lattice models which converge to SLE paths converge not only as sets, but also as continuous functions. To make that conclusion, the lattice model paths and the SLE paths must be parametrized in a particular way. By construction, SLE is parametrized so that the capacity of the path grows as a constant rate. The lattice model paths which converge to SLE must also use the capacity parametrization, which is not the natural way to run the model. It is desirable to run the models so that each discrete segment takes the same length of time. The most intuitive approach is to simulaneously run the SLE path parametrized by length, but since the fractal dimension of SLE_κ is $d = \min\{1 + \kappa/8, 2\} > 1$, the length of the SLE_κ trace is always infinite.

Recently, there has been a program to develop a d-dimensional measurement of length for chordal SLE_κ which can be used to study convergence. In [12], the Doob-Meyer theorem was used to create an increasing process related to the path which was called the natural parametrization, or natural length, for $\kappa < 5.021\ldots$. It was conjectured to coincide with some d-dimensional measurment. In [17], the Doob-Meyer construction was extended to all $\kappa < 8$. In [22], it was proven that the Hausdorff measure of the path is 0 almost surely. Instead, the Minkowski content of the path has been proven to be the correct candidate for the natural parametrization. The d-dimensional Minkowski content of a set $E \subset \mathbb{C}$ is defined by

$$\text{Cont}_d(E) = \lim_{r \to 0} r^{d-2} \text{Area}\{z \in \mathbb{C} : \text{dist}(z, E) < r\},$$

provided that the limit exists. In [9], it was proven that the d-dimensional Minkowski content of the path exists almost surely, and that it differs from the Doob-Meyer construction by a multiplicative constant. The first proof of convergence in the natural parametrization is [14], where it is proven that the loop erased random walk converges to SLE_2 when parametrized by the Minkowski content.

One of the main tools for studying the natural parametrization is called the Green's function, which gives the normalized probability that the SLE path passes through a point. For $\kappa \geq 8$, $\mathbb{P}[z \in \gamma] = 1$ for all z, so the question is not interesting. For $\kappa \in (0, 8)$, $\mathbb{P}[z \in \gamma] = 0$ implies that the probability $\mathbb{P}[\text{dist}(z, \gamma) < r]$ should decay as $r \to 0$. Moreover, if the dimension of the path is d, the probability should decay as r^{2-d}. The Green's function is defined to be

$$G(z) = \lim_{r \to 0} r^{d-2} \mathbb{P}\left[\text{dist}(z, \gamma) < r\right],$$

provided this limit exists. In the chordal case, this was first studied with the event $\{\text{rad}_{\mathbb{H} \setminus \gamma}(z) < r\}$ rather than distance, where $\text{rad}_D(z)$ is defined to be the conformal radius of D at z (defined in Section 2.2.2).

In [27], it was observed that for chordal SLE_κ and any $z \in \mathbb{H}$, the process

$$M_t(z) = |g_t'(z)|^{2-d} G(g_t(z) - \sqrt{\kappa} B_t)$$

should be a local martingale. To construct the natural parametrization in [12], the authors weighted the probability measure by $M_t(z)$ and used the Girsanov theorem to construct what is called two-sided radial SLE_κ through z, which is chordal SLE_κ conditioned to pass through z. In order to further study the Minkowski content, the two point Green's function was proven to exist and analyzed [9] [10] [16]. In [23] and [24], the Green's function for any finite number of points was proven to exist for chordal SLE_κ. That is,

$$G(z_1, \ldots, z_n) = \lim_{r \to 0} \prod_{k=1}^{n} r_k^{d-2} \mathbb{P}\left[\cap_{k=1}^{n} \text{dist}(z_k, \gamma) \leq r\right]$$

was proven to exist, where $r = (r_1, \ldots, r_n)$ and $z_1, \ldots, z_n \in \mathbb{H}$ are arbitrary. The upper bound for the multipoint Green's function found in [23] is used to prove that the Minkowski content has finite moments of all orders.

In [37], the construction of two-sided radial SLE using the Girsanov theorem is generalized. For $\rho \in \mathbb{R}$ and $z \in \mathbb{H}$, functions $G^\rho(z)$ are found so that if

$$M_t^\rho(z) = |g_t'(z)|^p G^\rho(g_t(z) - \sqrt{\kappa} B_t)$$

for an appropriate power $p = p(\rho)$, then $M_t^\rho(z)$ is a local martingale which can be used to create a measure $\mathbb{P}_z^\rho$ under which $\mathbb{P}_z^\rho[z \in \gamma] = 1$. A few special cases of these generalized Green's functions are used to prove decompositions of the SLE_κ path. In each case, the measures $\mathbb{P}_z^\rho$ are averaged over $z \in U$ weighted by $G^\rho(z)$ to generate a new measure $\mathbb{P}_U^\rho$, which is absolutely continuous with respect to the original measure $\mathbb{P}$. The Radon-Nikodym derivatives for three of these constructions are found, and are shown to be the Minkowski content of $\gamma \cap U$, the capacity time spent by γ in U, and the analogue of the natural parametrization for the boundary intersection of γ in $U \subset \mathbb{R}$ for $\kappa \in (4, 8)$. The result for Minkowski content extends the result in [5] which was proven for $\kappa \leq 4$. These decomposition theorems have been used to define SLE loop measures [41].

The Green's function for radial SLE_κ is not as well understood as the Green's function for chordal SLE_κ. In [2], the conformal radius version of the one-point Green's function is proven to exist for $\kappa \in (0, 8)$, though an exact form is only found for $\kappa = 4$. In Chapter 2, we establish an upper bound for the probability that radial SLE_κ in $\mathbb{D}$ from 1 to 0 passes near any finite collection of points in $\overline{\mathbb{D}}$. This should serve as the first step towards proving the multipoint Green's function for radial SLE_κ exists. The strategy is to move from the covering space $\mathbb{H}^*$ of $\mathbb{D}$ under the exponential map e^{iz}, where $\mathbb{H}^*$ is the upper half-plane $\mathbb{H}$ under the equivalence relation $z \sim w$ if $z - w \in 2\pi\mathbb{Z}$. Here, one-point estimates are developed in [2] for both the boundary and interior case, which we adapt into appropriately general and conformally invarient estimates. After establishing a sufficiently robust one-point estimate for radial SLE_κ, we follow the strategy in [23] to extend the estimate to multiple points.

The Minkowski content of radial SLE_κ and whole-plane SLE_κ have not yet been proven to exist, but the lower content

$$\underline{\mathrm{Cont}}_d(E) = \liminf_{r\to 0} r^{d-2}\mathrm{Area}\{z \in \mathbb{C} : \mathrm{dist}(z,\gamma) \le r\}$$

always exists. Using the multipoint estimate for radial SLE_κ we develop, we prove that $\underline{\mathrm{Cont}}_d(\gamma)$ has finite moments of all orders. If the Minkowski content were proven to exist, the same proof would show that $\mathrm{Cont}_d(\gamma)$ has finite moments of all orders.

Using the reversibility of whole-plane SLE_κ [40] [21] and the domain Markov property, we then establish a similar upper bound for the probability that a whole-plane SLE_κ passes near any finite collection of points using the estimate for radial SLE_κ. Using this estimate and a proof similar to the radial case, we then prove that the lower Minkowski content of a whole-plane SLE_κ path in any compact set has finite moments of any order.

In Chapter 3, we establish a theory of Green's functions for the backward SLE_κ welding for $\kappa \le 4$. Given any two points $x, y > 0$ we establish estimates for the probability of the event that x and y are almost welded together in a way analogous to the estimate of the probability that an SLE_κ path passes near a point. Normalizing this estimate yields a function $G(x,y)$ which we call a backward SLE_κ Green's function. Moreover, we observed that the process

$$M_t(x,y) = G(f_t(x) - \sqrt{\kappa}B_t, \sqrt{\kappa}B_t - f_t(-y))$$

is a local martingale, and weighting the original probability measure by $M_t(x,y)$ gives a measure $P_{x,y}$ under which $\phi(x) = y$ almost surely.

Following the strategy in [37], we generalize this construction to get a family of functions $G^{a,b}(x,y)$ which serve as generalized Green's functions for the BSLE$_\kappa$ welding. For any a,b, we find powers p,q so that

$$M_t^{a,b}(x,y) = f_t'(x)^p f_t'(-y)^q G^{a,b}(f_t(x) - \sqrt{\kappa}B_t, \sqrt{\kappa}B_t - f_t(-y))$$

is a local martingale. If $a,b \leq -\dfrac{4+\kappa}{2}$, using this local martingale with the Girsanov theorem yields a measure $\mathbb{P}_{x,y}^{a,b}$ under which $\phi(x) = y$ almost surely.

For $a = b = -4$, we show that this measure gives insight into the BSLE$_\kappa$ welding in the capacity parametrization in a way similar to what has been proven for forward chordal SLE$_\kappa$. Given any $U \subset [0,\infty)\times[0,\infty)$, we average the law of the welding conditioned to pass through U by integrating $\mathbb{P}_{x,y}^{-4,-4}$ against $G^{-4,-4}(x,y)$ over U. This yields a decomposition theorem similar to those proven in [37] for the welding, and gives a measure $\mathbb{P}_U$ which is absolutely continuous with respect to the original probability measure. The resulting Radon-Nikodym derivative is then shown to be the capacity time spent by the graph of the welding ϕ in U.

1.4 Generalizations of the Loewner equation

When Charles Loewner first introduced his differential equation in 1923 [18], it was a tool used to study the Bieberbach conjecture. His version of the equation was the radial Loewner equation, and the setting was entirely deterministic. The strategy was to solve (1.2) using real valued continuous functions to study slit-domains as a way to answer extremal questions about conformal maps in the unit disc. In [29], Oded Schramm recontextualized Loewner's ideas in order to study scaling limits of discrete random processes by using a Brownian motion as the driving function. Along with introducing randomness, Schramm also introduced the chordal Loewner ODE to study growth from a boundary point to another boundary point.

In the text [7], Lawler introduces generalized versions of both the chordal and radial Loewner equation. Rather than being driven by a continuous function λ_t, the equations he introduces are driven by a family of locally bounded Borel measures $\{\mu_t\}_{t\geq 0}$ on $\mathbb{R}$ (or $\partial\mathbb{D}$ in the radial case) so that $t \mapsto \mu_t$ is weakly continuous. In the chordal case, the Loewner equation driven by $\{\mu_t\}_{t\geq 0}$ is

$$\partial_t g_t(z) = \int_{\mathbb{R}} \frac{1}{g_t(z) - u} d\mu_t(u), \quad g_0(z) = z.$$

Then for each $t > 0$, $g_t : \mathbb{H}\backslash K_t \to \mathbb{H}$ is conformal for some increasing family of $\mathbb{H}$-hulls $(K_t)_{t\geq 0}$ with $\text{hcap}(K_t) = \int_0^t \mu_s(\mathbb{R})ds$. This family of measures can be reparametrized so that the corresponding family of hulls K_t satisfy $\text{hcap}(K_t) = 2t$. For a real valued continuous function λ, the regular chordal Loewner equation driven by λ is the above equation with $\mu_t = \delta_{\lambda_t}$, where δ_x is the point mass measure at x.

In [19], Miller and Sheffield introduce a further generalization in the radial case which they called measure driven Loewner evolution. They establish existence and uniqueness in the radial setting, and then prove a one-to-one correspondence between families of $\mathbb{D}$-hulls and solutions to the measure driven Loewner evolution. Moreover, convergence on hulls was shown to be equivalent to convergence of the corresponding measures.

In Chapter 4, we introduce measure driven chordal Loewner evolution. For a measure μ on $[0,\infty) \times \mathbb{R}$ with the appropriate assumptions, we show existence and uniqueness of the solution to the integral equation

$$g_t(z) - z = \int_{[0,t]\times\mathbb{R}} \frac{2}{g_s(z) - u} d\mu(s,u)$$

for all $z \in \mathbb{H}$. For a family of measures $\{\mu_t\}_{t\geq 0}$ as in Lawler's definition, we define a measure μ by $d\mu(t,u) = d\mu_t(u)dt$ so that μ and $\{\mu_t\}_{t\geq 0}$ generate the same process..

If $\mathcal{N}$ is the family of measures which generate a measure driven chordal Loewner equation, any $\mu \in \mathcal{N}$ generates a family of $\mathbb{H}$-hulls $(K_t)_{t\geq 0}$ with $\text{hcap}(K_t) = 2t$ so that $g_t = g_{K_t}$. Conversely, we show that any family of $\mathbb{H}$-hulls $(K_t)_{t\geq 0}$ parametrized by capacity is the resulting family of hulls for some measure $\mu \in \mathcal{N}$. Moreover, we endow $\mathcal{N}$ with a topology and show that a family of measures $\mu^n \to \mu$ in $\mathcal{N}$ if and only if the domains $\mathbb{H}\backslash K_t^n \to \mathbb{H}\backslash K_t$ in the Carathéodory topology (to be defined in Section 4.2) for each t, which is equivalent to the conformal maps $(g_t^n)^{-1}$ converging locally uniformly to g_t^{-1}.

Chapter 2

Multipoint estimates for radial and whole-plane SLE

2.1 Statement of results

The main theorem of this chapter is the multipoint estimate for radial SLE$_\kappa$.

Theorem 1. *Fix $\kappa \in (0,8)$. Let γ be a radial SLE$_\kappa$ in $\mathbb{D}$ from 1 to 0, let $z_k \in \mathbb{D}\backslash\{0\}$ for $k = 1, \ldots, n$, and let $z_0 = 1$. Let $y_k = 1 - |z_k|$ be the distance of each point to the boundary, and define $l_k = \min\{|z_k|, |z_k - 1|, |z_k - z_1|, \ldots, |z_k - z_{k-1}|\}$. Then there exists an absolute constant $C_n < \infty$, depending on κ and n, so that*

$$\mathbb{P}[\cap_{k=1}^{n}\{dist(\gamma, z_k) < r_k\}] \leq C_n \prod_{k=1}^{n} \frac{P_{y_k}(r_k \wedge l_k)}{P_{y_k(l_k)}}. \tag{2.1}$$

The function $P_y(x)$ used in the upper bound in Theorem 1 is defined by

$$P_y(x) = \begin{cases} y^{\alpha-(2-d)}x^{2-d}, & x \leq y \\ x^{\alpha}, & x \geq y \end{cases},$$

where $d = 1 + \kappa/8$ is the Hausdorff dimension of the path, and $\alpha = 8/\kappa - 1$ is related to the boundary exponent for SLE$_\kappa$. This upper bound mixes the estimates for interior points and points near the boundary. Roughly speaking, if the point z_k is far from the boundary, the term on the right hand side of (2.1) corresponding to z_k will be $(r_k/l_k)^{2-d}$. If z_k is near the boundary of the unit disc, then the corresponding term on the right side of (2.1) is $(r_k/l_k)^{\alpha}$. If z_k is close, but not too close, then the corresponding estimate is a mixture of the two.

The following Lemma about the functions P_y is Lemma 2.1 in [24], and can be proven with a case by case argument.

Lemma 1. *For* $0 \leq x_1 < x_2$, $0 \leq y_1 \leq y_2$, $0 < x$, *and* $0 \leq y$, *we have*

$$\frac{P_{y_1}(x_1)}{P_{y_1}(x_2)} \leq \frac{P_{y_2}(x_1)}{P_{y_2}(x_2)};$$

$$\left(\frac{x_1}{x_2}\right)^{\alpha} \leq \frac{P_y(x_1)}{P_y(x_2)} \leq \left(\frac{x_1}{x_2}\right)^{d-2} = \frac{P_{x_2}(x_1)}{P_{x_2}(x_2)};$$

$$\left(\frac{y_1}{y_2}\right)^{\alpha-(2-d)} \leq \frac{P_{y_1}(x)}{P_{y_2}(x)} \leq 1.$$

Theorem 1 can then be used to prove a similar estimate for whole-plane SLE_κ:

Theorem 2. *Fix* $\kappa \in (0, 8)$, *and let* γ^* *be a whole-plane* SLE_κ *trace from* 0 *to* ∞. *Let* $z_1, \ldots, z_n \in \mathbb{C}\backslash\{0\}$. *For each* $k = 1, \ldots, n$, *let* $0 < r_k < |z_k|$ *and define* $l_k = \min\{|z_k|, |z_k - z_1|, \ldots, |z_k - z_{k-1}|\}$. *Then*

$$\mathbb{P}[\cap_{k=1}^{n}\{dist(\gamma^*, z_k) < r_k\}] \leq C_n \prod_{k=1}^{n} \left(\frac{r_k \wedge l_k}{l_k}\right)^{2-d}. \tag{2.2}$$

Note that the expression of this bound is simpler than Theorem 1, since there are no boundary points with which to be concerned. Both of these estimates are used to prove the following theorem about Minkowski content:

Theorem 3. *Fix* $\kappa \in (0, 8)$*.*

a) Let γ *be a radial* SLE_κ *trace in* $\mathbb{D}$ *from* 1 *to* 0*. Then* $\mathbb{E}[\underline{Cont}_d(\gamma)^n] < \infty$ *for all* $n \in \mathbb{N}$*.*

b) Let γ^* *be a whole-plane* SLE_κ *trace from* 0 *to* ∞*, and suppose* $D \subset \mathbb{C}$ *is compact. Then* $\mathbb{E}[\underline{Cont}_d(\gamma^* \cap D)^n] < \infty$ *for every* $n \in \mathbb{N}$*.*

The chapter will be organized as follows. First we review preliminary information, including information about the covering space $\mathbb{H}^*$. We also review some information about crosscuts, prime ends, and extremal length which will be used. Next, we provide one-point estimates for radial SLE in the forms which will be useful for us. We then use these one-point estimates to prove some key lemmas, followed by the proofs of the main theorems.

2.2 Preliminaries

2.2.1 Crosscuts and prime ends

In later sections, we will be studying the behavior of the radial SLE curve as it crosses many interior curves, creating different components of the initial domain $\mathbb{D}$. We need to introduce some notation which will make it easier to distinguish which component is discussed at any point in time. This is the same framework introduced in [23].

Recall that a crosscut in a simply connected domain $D \subset \mathbb{C}$ is a simple curve $\rho : (a, b) \to D$ such that $\lim_{t\to a^+} \rho(t) := \rho(a^+)$ and $\lim_{t\to b^-} \rho(t) := \rho(b^-)$ both exist and are elements of the boundary of D. Then ρ lies inside D, but the endpoints do not. The endpoints $\rho(a^+)$ and $\rho(b^-)$ determine prime ends for the domain. Note that if $f : D \to D'$ is a conformal map, then $f(\rho)$ is a crosscut in D'. More information about crosscuts and prime ends can be found in [1].

Note that if ρ is a crosscut in a domain D, then ρ divides D into two components. Even more generally, let $K \subset D$ be relatively closed. Let S be either a connected subset of $D\backslash K$ or a prime end of $D\backslash K$. We then define $D(K;S)$ to be the component of $D\backslash K$ which contains S. We also introduce the symbol $D^*(K,S) = D\backslash (K \cup D(K;S))$, which is the union of the remaining components of $D\backslash K$. This notation is useful for expressing whether K seperates points. For example, if $\rho \subset D$ is a crosscut which separates two points $z, w \in D$, then $D(\rho; z) \neq D(\rho; w)$. In fact, in this case, we have $D(\rho; w) = D^*(\rho; z)$, and $D(\rho; z) = D^*(\rho; w)$.

Since we will be working with domains which have 0 as an interior point, and in particular will be concerned with components containing 0, we will use $D(K)$ and $D^*(K)$ to represent $D(K; 0)$ and $D^*(K; 0)$ respectively. Note that this is a departure from the notation in [23], where the point being suppressed was the prime end ∞. Since we are working with radial SLE rather than chordal SLE, the change is to reflect the fact that we are looking at components with respect to the endoint of the SLE curve, which in this work will be the interior point of 0.

For an example of how this notation will be used, let γ be a radial SLE_κ curve in $\mathbb{D}$ from 1 to 0, and define $D_t = \mathbb{D}(\gamma[0,t])$. This is the component of $\mathbb{D}\backslash\gamma[0,t]$ which contains the origin. If ρ is a crosscut in D_t, and $z \in D_t\backslash\rho$, we will ask if $z_0 \in D_t(\rho)$. That is, does ρ seperate z_0 from 0 in D_t? Suppose the circle $\xi = \{|z - z_0| = r\}$ is contained in $D_t\backslash\rho$. If ξ doesn't enclose 0, then $\mathbb{D}^*(\xi) = \{|z - z_0| < r\}$. Then $\mathbb{D}^*(\xi) \subset D_t^*(\rho)$ means the circle ξ is seperated from 0 in D_t by ρ.

The next lemma is Lemma 2.1 in [23]:

Lemma 2. *Let $D \subset \tilde{D}$ be simply connected domains in $\mathbb{C}$. Let ρ either be a Jordan curve in $\tilde{D}$ which intersects ∂D or a crosscut in $\tilde{D}$. Let Z_1, Z_2 be two connected subsets or prime ends of $\tilde{D}$ such that $\tilde{D}(\rho; Z_j)$ is well defined for both $j = 1, 2$, and are nonequal. This means that $\tilde{D} \backslash \rho$ is a neighborhood of both Z_1 and Z_2 in D, and Z_1 is disconnected from Z_2 in $\tilde{D}$ by ρ.*

Suppose that D is a neighborhood of Z_1 and Z_2 in $\tilde{D}$. Let Λ be the set of connected components of $D \cap \rho$. Then there exists a unique $\lambda_1 \in \Lambda$ such that $D(\lambda_1; Z_1) \neq D(\lambda_1; Z_2)$, and if $\lambda \in \Lambda$ such that $D(\lambda; Z_1) \neq D(\lambda; Z_2)$, then $D(\lambda_1; Z_1) \subset D(\lambda; Z_1)$ and $D(\lambda; Z_2) \subset D(\lambda_1; Z_2)$.

The λ_1 obtained in Lemma 2 will be referred to as the first subcrosscut of ρ to seperate Z_1 and Z_2 in D. The conclusion of the lemma states that of all subcrosscuts of ρ in D which disconnect Z_1 and Z_2, λ_1 is closest to Z_1 in the sense that the component containing Z_1 it determines is contained in the component determined by any other such subcrosscut.

2.2.2 The covering space $\mathbb{H}^*$ for $\mathbb{D}$

The upper half-plane can be seen as a covering space for the unit disc under the exponential map. Let $\mathbb{H}^*$ be the cylinder defined by $\mathbb{H}^* = \{[z]_\sim : z \in \mathbb{H}\}$, where $\sim$ is the equivalence relation $z \sim w$ if $z - w \in 2\pi\mathbb{Z}$. Under this equivalence relation, the map $e^i : \mathbb{H}^* \to \mathbb{D}\backslash\{0\}$ defined by $e^i(z) = e^{iz}$ is a conformal map. The boundary of $\mathbb{H}^*$ is $\mathbb{R}^* = \mathbb{R}$ with the same equivalence relation. We can treat 0 as the preimage of ∞ under e^i, and $e^{i0} = 1$. The map $\frac{1}{i}\log : \mathbb{D}\backslash\{0\} \to \mathbb{H}^*$ is also seen as a conformal map.

Any set $K' \subset \mathbb{H}^*$ can be viewed as a 2π-periodic set $K + 2\pi\mathbb{Z} \subset \mathbb{H}$. An $\mathbb{H}^*$-hull is a set $K' \subset \mathbb{H}^*$ so that $e^i(K') \subset \mathbb{D}$ is a $\mathbb{D}$-hull, and an $\mathbb{H}^*$-domain is a set $H \subset \mathbb{H}^*$ so that

$e^i(H) \subset \mathbb{D}$ is a $\mathbb{D}$-domain. Therefore, an $\mathbb{H}^*$-domain is $\mathbb{H}^* \backslash K'$ for an $\mathbb{H}^*$-domain K'.

Radial SLE$_\kappa$ can be studied by looking at the covering space $\mathbb{H}^*$. Let $\lambda \in C[0,\infty)$ and $z \in \mathbb{H}^*$. The covering radial Loewner equation driven by λ is given by

$$\partial_t h_t(z) = \cot_2(h_t(z) - \lambda_t), \quad h_0(z) = z, \tag{2.3}$$

where $\cot_2(z) = \cot(z/2)$. Note that if (g_t) is the radial Loewner evolution driven by λ, then $h_t(z) = -i \log g_t(e^{iz})$ satisfies (2.3). If $\lambda_t = \sqrt{\kappa} B_t$, where B_t is a standard one dimensional Brownian motion, let γ' be the preimage under e^i of the radial SLE$_\kappa$ trace in $\mathbb{D}$ from 1 to 0 driven by λ. Then we call γ' a radial SLE$_\kappa$ trace in $\mathbb{H}^*$ from 0 to ∞, which is a 2π-periodic family of paths in the cylinder $\mathbb{H}^*$. If H is an $\mathbb{H}^*$-domain with a prime end w_0', we can define a radial SLE$_\kappa$ process in H from w_0' to ∞ by mapping conformally from H to $D = e^i(H)$, to $\mathbb{D}$, then back to $\mathbb{H}^*$ with $\frac{1}{i}\log$.

To work in the cylinder $\mathbb{H}^*$, we must carefully define what distance and conformal radius mean in this domain. For $z', w' \in \overline{\mathbb{H}}^*$ which can be represented as $z + 2\pi\mathbb{Z}, w + 2\pi\mathbb{Z}$ respectively for $z, w \in \overline{\mathbb{H}}$, then the distance from z' to w' in $\overline{\mathbb{H}}^*$ is defined to be Euclidean distance between the sets $z + 2\pi\mathbb{Z}$ and $w + 2\pi\mathbb{Z}$ in $\overline{\mathbb{H}}$. It will be written as $|z' - w'|_*$ to distinguish from the distance between points in $\mathbb{C}$. Then $|z' - w'|_*$ is the distance between closest representatives of equivalence classes.

Similarly, if $A', B' \subset \overline{\mathbb{H}}^*$ with $A' = A + 2\pi\mathbb{Z}$ and $B' = B + 2\pi\mathbb{Z}$, then the distance from A' to B' in $\mathbb{H}^*$ is the Euclidean distance from $A + 2\pi\mathbb{Z}$ to $B + 2\pi\mathbb{Z}$, and is denoted by $\text{dist}_{\mathbb{H}^*}(A', B')$. Given any $z' \in \overline{\mathbb{H}}^*$ with $z' = z + 2\pi\mathbb{Z}$, the ball of radius r centered at z' is denoted by $B(z', r)$, and is represented in $\overline{\mathbb{H}}$ by $B(z, r) \cap \overline{\mathbb{H}} + 2\pi\mathbb{Z}$. Note that for $r < \pi$, then the representatives of $B(z', r)$ are nonoverlapping in $\overline{\mathbb{H}}$.

In [2], several estimates are proven about one-point estimates with conformal radius instead of distance, so we need to provide the definition of conformal radius in $\mathbb{H}^*$. Given any simply connected domain $D \subset \mathbb{C}$, and a point $z \in D$, the conformal radius of z in D is defined by $\text{rad}_D(z) = 1/|\phi'(z)|$, where $\phi : D \to \mathbb{D}$ is conformal with $\phi(z) = 0$. It follows readily that $\text{rad}_D(z)$ is a conformal invariant. That is, if $\phi : D \to D'$ is a conformal map and $z \in D$, then $\text{rad}_{\phi(D)}(\phi(z)) = |\phi'(z)|\text{rad}_D(z)$. Since $\phi : \mathbb{H}^* \to \mathbb{D}\backslash\{0\}$ defined by $\phi(z') = e^{iz'}$ is a conformal map in the covering space, we can add ∞ into $\mathbb{H}^*$, thus adding 0 back into the image of $\phi : \mathbb{H}^* \cup \{\infty\} \to \mathbb{D}$ so that ϕ is conformal and the cylinder is simply connected. We can then use the conformal invarience formula to define conformal radius in the covering space $\mathbb{H}^*$, and any other $\mathbb{H}^*$-domain.

If a set D is not connected, we define $\Upsilon_D(z)$ to be the conformal radius of z in the component of D in which z lies. This will be relevant when z becomes cut off from the target of the SLE path by the curve. In this case, even though we define D_t to denote the component of $D\backslash\gamma[0,t]$ containing the target by time t, we will still use the notation $\Upsilon_{D_t}(z)$.

For slightly easier calculations, we introduce the scaling $\Upsilon_D(z) = \text{rad}_D(z)/2$. From the definition, a quick calculation verifies that $\text{rad}_{\mathbb{D}}(z) = 1 - |z|^2$, and so $\Upsilon_{\mathbb{D}}(z) = \dfrac{1-|z|^2}{2}$. For any $z' \in \mathbb{H}^*$, we can show that

$$\Upsilon_{\mathbb{H}^*}(z') = \sinh(y'), \tag{2.4}$$

where $y' = \text{Im}(z')$. To see this, note that the conformal invarience definition shows that

$$\text{rad}_{\mathbb{H}^*}(z') = \frac{1}{|ie^{iz'}|}(1 - |e^{iz'}|^2) = |e^{-iz'}| - |e^{iz'}| = e^{y'} - e^{-y'},$$

and dividing by 2 gives (2.4).

Remark 1: We will need to use applications of Koebe's distortion theorem for the map

e^i on sets in $\mathbb{H}^*$. For this to be valid, the set will have to sit inside $B(z,r)+2\pi$ for some $r<\pi$ so that e^i can be restricted to a conformal map on a subdomain of $\mathbb{C}$.

Remark 2: In [2], the covering space $\mathbb{H}^*$ is defined slightly differently than we do by assuming $z \sim w$ if $z-w \in \pi\mathbb{Z}$. In this case, the associated conformal map is $\phi(z') = e^{2iz'}$, and the same calculation as above gives $\mathrm{rad}_{\mathbb{H}^*}(z') = 2\cosh(z')\sinh(z') = \sinh(2y')$. However, this slight change in coordinates only affects the estimates by a constant factor.

2.2.3 Extremal length and conformal transformations

Let $d_\Omega(A,B)$ denote the extremal distance from A to B in Ω. For the definition of extremal distance, see [1]. Note that this is distinct from the notations $\mathrm{dist}(A,B)$ and $\mathrm{dist}_{\mathbb{H}^*}(A,B)$, both of which represent Euclidean distance. Define $\Lambda(R) = d_{\Omega_R}([-1,0],[R,\infty))$, where $\Omega_R = \mathbb{C}\backslash\{[-1,0]\cup[R,\infty)\}$. By Teichmüller's theorem [1], this is maximal among doubly connected domains in modulus (extrememal distance between boundary components) which separate $\{-1,0\}$ and $\{w,\infty\}$ with $|w|=R$. Moreover, $\Lambda(R) \leq \dfrac{1}{2\pi}\log(16(R+1))$ for all values of $R \geq 1$.

We are going to need to prove some estimates about extremal length which are conformally invariant, and so are more flexible than estimates involving Euclidean distance. Moreover, we are going to look at extremal distance in the cylinder $\mathbb{H}^*$. Since $\mathbb{H}^*$ looks locally like $\mathbb{H}$, if two sets are sufficiently close to each other, the extremal distance between them should behave the same as the extremal distance in $\mathbb{H}$.

Lemma 3. *Let η be a crosscut in $\{z \in \mathbb{H} : Re(z) > 0\}$ with endpoints $0 < a < b$. Define $r = \sup\{|z-a| : z \in \eta\}$. Then there is a constant $C < \infty$ so that, if $r < a$,*

$$\frac{r}{a} \leq Ce^{-\pi d_{\mathbb{H}}(\eta,(-\infty,0])}. \tag{2.5}$$

Proof. Let Ω be the component of $\mathbb{H}\backslash\eta$ whose boundary contains 0, and define $\Omega' = \Omega \cup (0,a) \cup (b,\infty)$. Let $\Omega^{\text{doub}} = \Omega' \cup \{\overline{z} : z \in \Omega\}$, and let η^{doub} be the component of the boundary of Ω^{doub} which does not contain 0, which is the closure of the double of η. Then $d_{\mathbb{H}}((-\infty,0],\eta) = 2d_{\Omega^{\text{doub}}}((-\infty,0],\eta^{\text{doub}})$.

Fix some w in the closure of η with $|w-a| = r$, and let ξ be the argument of $w-a$. Then $\phi(z) = \left((z-a)e^{-i\xi}/r\right) - 1$ is the affine transformation with $\phi(a) = -1$, $\phi(w) = 0$, and $|\phi(0)| = |ae^{-i\xi}/r + 1|$. If $\hat{\Omega} = \phi(\Omega^{\text{doub}})$, then $\hat{\Omega}$ is a doubly connected domain seperating $\{-1,0\}$ from $\{\phi(0),\infty\}$. By the conformal invarience of extremal distance and Teichmüller's theorem, we have

$$d_{\Omega^{\text{doub}}}((-\infty,0],\eta^{\text{doub}}) = d_{\hat{\Omega}}(\phi((-\infty,0],\phi(\eta^{\text{doub}}))) \le \frac{1}{2\pi}\Lambda\left(|ae^{-i\xi}/r+1|\right)$$

$$\le \Lambda\left(\frac{a}{r}+1\right) \le \frac{1}{2\pi}\ln(16(a/r+1)+1) \le \frac{1}{2\pi}\ln\left(33\frac{a}{r}\right).$$

Rearranging this proves (2.5) for r small.

□

The following application of Koebe's distortion theorem will be used repeatedly in the next section to show that interior estimates are comparable after applying conformal maps. We also derive a growth estimate for the inverse of the covering map for the cylinder.

Lemma 4. *a) Let $D \subset \mathbb{C}$ be a domain, and assume that $B(z_0,R) \subset D$. Suppose ϕ is a conformal map defined on D, and let $M =$dist$(z_0,\partial D)$. Suppose that $r < R/7$, and define*

$$\tilde{R} = \frac{R|\phi'(z_0)|}{(1+R/M)^2},\ \ \tilde{r} = \frac{r|\phi'(z_0)|}{(1-r/M)^2}.$$

Then

$$\phi(B(z_0,r)) \subset B(\phi(z_0),\tilde{r}) \subset B(\phi(z_0),\tilde{R}) \subset \phi(B(z_0,R)),$$

and there is an absolute constant $C < \infty$ *so that*

$$\frac{r}{R} \leq \frac{\tilde{r}}{\tilde{R}} \leq C\frac{r}{R}.$$

b) Let D *be a* $\mathbb{D}$*-domain, and let* H *be an* $\mathbb{H}^*$*-domain with* $e^i(H) = D$*. Let* $z_0 \in D$ *and* $z_0' \in H$ *so that* $e^{iz_0'} = z_0$*. Then there is a constant* $C < \infty$ *so that if* $y_0 = 1 - |z_0| \leq 1/2$*, we have* $dist(z_0, \partial D) \leq C\, dist_{\mathbb{H}^*}(z_0', \partial H)$*.*

Proof. Applying the Growth Theorem to the univalent map $f : \mathbb{D} \to \mathbb{C}$ defined by

$$f(w) = \frac{\phi(Mw + z_0) - \phi(z_0)}{M\phi'(z_0)},$$

we get that if $|z - z_0| = \rho \in (0, M)$, then

$$\frac{\rho/M}{(1+\rho/M)^2} \leq \frac{|\phi(z) - \phi(z_0)|}{M|\phi'(z_0)|} \leq \frac{\rho/M}{(1-\rho/M)^2}. \tag{2.6}$$

Then $\tilde{r}$ is $M|\phi'(z_0)|$ times the right hand side of (2.6) for $\rho = r$, and $\tilde{R}$ is $M|\phi'(z_0)|$ times the left hand side for $\rho = R$. Then if

$$\frac{\tilde{r}}{\tilde{R}} = \frac{r}{R}\frac{(1+R/M)^2}{(1-r/M)^2}$$

is smaller than 1, we get the desired containments. If $r < R/7$ and $R < M$, then

$$\frac{\tilde{r}}{\tilde{R}} \leq \frac{1}{7}\frac{4}{(6/7)^2} < 1.$$

Moreover,

$$1 \leq \frac{(1+R/M)^2}{(1-r/M)^2} \leq \frac{4}{(6/7)^2},$$

which proves the comparability.

To prove part *b)* let, $\phi(z) = \frac{1}{i}\log(z)$ be the map from $D\backslash\{0\}$ to H. Let $r =$dist$(z_0, \partial D)$, so that the assumption $y_0 \leq 1/2$ implies that a single valued branch of ϕ is conformal on the open ball $B(z_0, r)$. By Koebe's 1/4 theorem,

$$B(z_0', r/4) \subset \phi(B(z_0, r)) \subset H.$$

If $w' \in \partial H$, this implies that $|z_0' - w'|_* \geq r/4 = \text{dist}(z_0, \partial D)/4.$

□

2.3 Interior and boundary estimates

The following estimate is Proposition 5.1 in [2].

Proposition 1. *(Interior estimate 1) Let γ' be a radial SLE$_\kappa$ curve in $\mathbb{H}^*$ from 0 to ∞. There exists constants $0 < c_1 < c_2 < \infty$ so that if $z' \in \mathbb{H}^*$ with $y' = Im(z') \leq 1$ and $\epsilon \leq 1/2$, then*

$$c_1 \left(\frac{y'}{|z'|_*}\right)^{\alpha} \epsilon^{2-d} \leq \mathbb{P}\left[\Upsilon_\infty(z') \leq \epsilon \Upsilon_0(z')\right] \leq c_2 \left(\frac{y'}{|z'|_*}\right)^{\alpha} \epsilon^{2-d},$$

where $\Upsilon_t(z')$ is the conformal radius of z' in $\mathbb{H}^*\backslash\gamma'[0,t]$.

Lemma 5. *(Interior estimate 2) Let γ' be a radial SLE_κ trace in $\mathbb{H}^*$ from 0 to ∞. Then there is a constant $C < \infty$ so that if $z' \in \mathbb{H}^*$ with $y' = Im(z') \le \ln(2)$ and $r < y'$,*

$$\mathbb{P}[dist_{\mathbb{H}^*}(z', \gamma') \le r] \le C \left(\frac{y'}{|z'|_*} \right)^{\alpha} \left(\frac{r}{y'} \right)^{2-d} = C \frac{P_{y'}(r)}{P_{y'}(|z'|_*)}.$$

Remark: By restricting to $y' \le \ln(2)$, we are restricting slightly further than we are in Proposition 1. We do this restriction so that we can use Lemma 4, part *b*). Later, we will use part *a*) of Lemma 4 applied to a Loewner map to extend to the whole space.

Proof. Let $z \in \mathbb{D}$ so that $e^{iz'} = z$, denote the $\mathbb{H}^*$-hull determined by $\gamma'[0,t]$ by H_t, and let $D_t = e^i(H_t)$ for each t. By the definition of conformal radius of domains in $\mathbb{H}^*$, we have $\Upsilon_{D_t}(z) = |(e^i)'(z')|\Upsilon_t(z') = e^{-y'}\Upsilon_t(z')$ for all $t < \infty$. By Koebe's 1/4 theorem and the assumption $y' \le \log(2)$, we have

$$\Upsilon_t(z') = e^{y'} \Upsilon_{D_t}(z) \le C\text{dist}(z, \partial D_t).$$

Part *b*) of Lemma 4 implies that $\text{dist}(z, \partial D_t) \le C\text{dist}_{\mathbb{H}^*}(z', \partial H_t)$, and therefore

$$\Upsilon_t(z') \le C\text{dist}_{\mathbb{H}^*}(z', \partial H_t) \le C\text{dist}_{\mathbb{H}^*}(z, \gamma'[0,t]).$$

Taking $t \to \infty$ gives $\Upsilon_\infty(z') \le c\text{dist}_{\mathbb{H}^*}(z', \gamma')$ for an absolute constant $c < \infty$.

By the above paragraph, for small ϵ, we have $\{\text{dist}_{\mathbb{H}^*}(z', \gamma') < \Upsilon_0(z')\frac{\epsilon}{c}\} \subset \{\Upsilon_\infty(z') \le \epsilon\Upsilon_0(z')\}$. Therefore,

$$\mathbb{P}[\text{dist}_{\mathbb{H}^*}(z', \gamma') \le \Upsilon_0(z')\frac{\epsilon}{c}] \le C \left(\frac{y'}{|z'|_*} \right)^{\alpha} \epsilon^{2-d}.$$

Renaming variables and picking ϵ sufficiently small depending on z' and r, we can let $r = \Upsilon_0(z')\frac{\epsilon}{c}$. Then

$$\mathbb{P}[\mathrm{dist}_{\mathbb{H}^*}(z', \gamma') \le r] \le C \left(\frac{y'}{|z'|_*}\right)^{\alpha} \left(\frac{cr}{\Upsilon_0(z')}\right)^{2-d} \le C \left(\frac{y'}{|z'|_*}\right)^{\alpha} \left(\frac{r}{y'}\right)^{2-d},$$

where the last inequality follows from (2.4), which says that $\Upsilon_0(z') = \Upsilon_{\mathbb{H}^*}(z') = \sinh(y') \ge y'$. This is equal to $P_{y'}(r)/P_{y'}(|z'|_*)$, since $|z'|_* \ge y'$ implies that $P_{y'}(|z'|_*) = |z'|_*^{\alpha}$ and $r < y'$ implies that $P_{y'}(r) = (y')^{\alpha-(2-d)} r^{2-d}$.

□

The initial boundary estimate we will work with is Lemma 5.1 in [2].

Lemma 6. *(Boundary estimate 0) There is a constant $C < \infty$ so that if γ' is a radial SLE$_\kappa$ curve in $\mathbb{H}^*$, $x' \in \mathbb{R}^*$, and $r < |x'|_*$, then*

$$\mathbb{P}[dist(x', \gamma') \le r] \le C \left(\frac{r}{|x'|_*}\right)^{\alpha}.$$

We want to modify this estimate into a more general and conformally invariant version which can be applied in more general domains. In the next lemma, we will derive an estimate involving the extremal distance between two crosscuts in the cylinder $\mathbb{H}^*$. This will be determined by the domain between the crosscuts, and will be the same as the extremal distance between a representation of each of them in $\mathbb{H}$.

Lemma 7. *Let D be any $\mathbb{D}$ domain, and let γ be radial SLE$_\kappa$ in D from a prime end w_0 of D to 0. Let ρ, η be crosscuts in D with small radius, η contained in the component of $D \backslash \rho$*

distinct from the component containing 0 *and* w_0*. Then*

$$\mathbb{P}[\gamma \cap \eta \neq \emptyset] \leq Ce^{-\alpha\pi d_D(\rho,\eta)}. \tag{2.7}$$

Similarly, the inequality (2.7) holds when γ' *is radial* SLE_κ *in an* $\mathbb{H}^*$ *domain* D *from* w_0' *to* ∞*, and* ρ, η *are non-self-intersecting* 2π*-periodic crosscuts in* $\mathbb{H}^*$ *so that* ρ *separates* w_0' *and* ∞ *from* η*.*

Proof. In either case of the lemma, let ϕ be a conformal map from the domain to $\mathbb{H}^*$ sending w_0 to 0. Then $\rho' = \phi(\rho)$ is a non-self-intersecting crosscut in $\mathbb{H}^*$ which separates $0, \infty$ from $\eta' = \phi(\eta)$, and $\phi(\gamma) = \tilde{\gamma}$ is a time changed radial SLE_κ curve in $\mathbb{H}^*$. Let η_0' be the representative for η' in $\mathbb{H}$ with $\eta_0' \subset \{z \in \mathbb{H} : 0 < \mathrm{Re}(z) \leq 2\pi\}$, and define ρ_0' similarly.

Denote the endpoints of η_0' by $0 < a < b < 2\pi$, and let $x = \min\{a, 2\pi - b\}$. If $x = a$, define $r = \sup\{|z - a| : z \in \eta_0'\}$. If $x = 2\pi - b$, let $r = \sup\{|z - b| : z \in \eta_0'\}$. Either way, $\eta_0' \subset B(x, r)$, and so by Lemma 6, we have

$$\mathbb{P}[\gamma \cap \eta \neq \emptyset] = \mathbb{P}[\tilde{\gamma} \cap \eta' \neq \emptyset] \leq \mathbb{P}[\mathrm{dist}(\tilde{\gamma}, x) \leq r] \leq C\left(\frac{r}{x}\right)^{\alpha}.$$

Applying Lemma 3 to η_0', we have

$$\frac{r}{x} \leq C\left[e^{-\pi d_{\mathbb{H}}(\eta_0', (-\infty, 0])} \vee e^{-\pi d_{\mathbb{H}}(\eta_0', [2\pi, \infty))}\right] \leq Ce^{-\pi d_{\mathbb{H}}(\rho_0', \eta_0')},$$

where the last inequality is due to the comparison principle of extremal length. Combining these two inequalities gives

$$\mathbb{P}[\gamma \cap \eta \neq \emptyset] \leq Ce^{-\alpha\pi d_{\mathbb{H}}(\rho_0', \eta_0')} = Ce^{-\alpha\pi d_D(\rho, \eta)},$$

where the last equality is from the conformal invariance of extremal length, and since it is the domain between the crosscuts which determines the length.

□

Now we can combine the interior and boundary estimates into the general one-point estimate for the cylinder. We now also remove the assumption that $y_0' \leq \ln(2)$ to extend the estimate to the whole space.

Lemma 8 (One-point estimate for cylinder)**.** *Suppose that H is an $\mathbb{H}^*$ domain, and γ' is radial SLE$_\kappa$ in H from a prime end w_0' to ∞. Fix $z_0' \in \mathbb{H}^*$ with imaginary part y_0', and let $\pi > R > r > 0$. Define $\rho = \{z' \in \mathbb{H}^* : |z' - z_0'|_* = R\}, \eta = \{z' \in \mathbb{H}^* : |z' - z_0'|_* = r\}$. Moreover, suppose that $\{z' \in \mathbb{H}^* : |z' - z_0'|_* \leq R\} \subset D$ and that $w_0' \notin \{x' \in \mathbb{R}^* : |z_0' - x'|_* < R\}$. Then*

$$\mathbb{P}[\gamma' \cap \eta \neq \emptyset] \leq C \frac{P_{y_0'}(r)}{P_{y_0'}(R)}.$$

Proof. Observe that, by assumption, the representatives for ρ in $\mathbb{H}$ under the equivalence relation consists of multiple disconnected components, rather than having overlap. This means any of the covering maps are locally conformal at each piece of ρ, and everything inside of ρ, so Lemma 4 will be applicable.

The proof breaks down into three cases, each depending on how far from $\mathbb{R}^*$ is the point z_0'. The first case is the far away case, when $y_0 > R$. Also, we must first assume that $y_0' \leq \ln(2)$. Let $h_H : H \to \mathbb{H}^*$ be the canonical conformal map taking w_0' to 0 and ∞ to ∞, so that $\tilde{\gamma} = h_H(\gamma')$ is a time-changed radial SLE$_\kappa$ curve from 0 to ∞ . Let $\tilde{z}_0 = h_H(z_0')$, and let $\tilde{y}_0 = \text{Im}(\tilde{z}_0)$. Since h_H is conformal on a simply connected neighborhood of any component of ρ, Lemma 4 *a*) implies that, assuming $r < R/7$ without loss of generality (else we modify the constant by a factor of 7),

$$\mathbb{P}[\gamma \cap \eta \neq \emptyset] = \mathbb{P}[\tilde{\gamma} \cap h_H(\eta) \neq \emptyset] \leq \mathbb{P}[\mathrm{dist}_{\mathbb{H}^*}(\tilde{z}_0, \tilde{\gamma}) \leq \tilde{r}]$$

$$\leq C \frac{P_{\tilde{y}_0}(\tilde{r})}{P_{\tilde{y}_0}(|\tilde{z}_0|_*)} \leq C \frac{P_{\tilde{y}_0}(\tilde{r})}{P_{\tilde{y}_0}(\tilde{R})} = C \left(\frac{\tilde{r}}{\tilde{R}}\right)^{2-d} \leq C \left(\frac{r}{R}\right)^{2-d},$$

where $\tilde{r}, \tilde{R}$ are defined as in Lemma 4 with respect to the conformal map r, R, and h_H.

Assume now that $z_0' \in H$ is arbitrary, and let $h_t : H \backslash \gamma'[0,t] \to H$ be the Loewner maps for the radial SLE$_\kappa$ process in H. For any $T > 0$, define $\gamma^T(t) = \gamma'(T+t)$, so that conditioned on $\mathcal{F}_T$, γ^T is a radial SLE$_\kappa$ trace from $\gamma'(T)$ to ∞. Define $z_t' = h_t(z_0')$, and define a stopping time by $\tau = \inf\{t \geq 0 : \mathrm{Im}(z_t) \leq \ln(2)\}$. Define r_τ, R_τ as in Lemma 4 at z_0' with respect to r, R, h_τ. Then Lemma 4 implies that

$$\mathbb{P}[\mathrm{dist}_{\mathbb{H}^*}(z_\tau', \gamma^\tau) < r | \mathcal{F}_\tau, \tau < \infty] \leq C \left(\frac{r_\tau}{R_\tau}\right)^{2-d} \leq C \left(\frac{r}{R}\right)^{2-d}.$$

Since $\mathbb{P}[\tau < \infty] \leq 1$, this proves the lemma for all $z_0' \in H$.

For case 2, we consider the close case, $0 < y_0' < r$. We will use the boundary estimates to derive an upper bound of $\left(\frac{r}{R}\right)^{\alpha}$. By modifying the constant slightly, we can assume that $R > 4r$. Then in order to cross from ρ to η, γ' must also pass through $\rho' = \{z' \in \mathbb{H}^* : |z' - \mathrm{Re}(z_0')|_* = R/2\}$ and $\eta' = \{z' \in \mathbb{H}^* : |z' - \mathrm{Re}(z_0')|_* = 2r\}$, which are two semicircles such that $d_D(\rho', \eta') = (1/\pi)\log(R/4r)$. Using Lemma 7, we get that

$$\mathbb{P}[\gamma' \cap \eta \neq \emptyset] \leq \mathbb{P}[\gamma' \cap \eta' \neq \emptyset] \leq Ce^{-\alpha\pi d_D(\eta', \rho')} = C \left(\frac{r}{R}\right)^{\alpha}.$$

Lastly, we have to mix interior and boundary estimates in the middle distance case when

$0 < r < y_0' < R$. Let $\rho' = \{z' \in \mathbb{H}^* : |z' - z_0'|_* = y_0'\}$ which is a circle tangent to $\mathbb{R}^*$ between η and ρ. Let $T = \inf\{t > 0 : \gamma'(t) \in \rho'\}$, which is a stopping time so that $\{\gamma' \cap \eta \neq \emptyset\} \subset \{T < \infty\}$. Using case 2, we can see that

$$\mathbb{P}[T < \infty] \leq C \frac{P_{y_0'}(y_0')}{P_{y_0'}(R)}.$$

Define $\gamma^T(t) = \gamma'(T + t)$. By the domain Markov property and the first case, we can see that

$$\mathbb{P}[\gamma^T \cap \eta \neq \emptyset | \gamma'[0, T], T < \infty] \leq C \frac{P_{y_0'}(r)}{P_{y_0'}(y_0')}.$$

Combining these two inequalites gives case 3.

□

Using the Growth Theorem estimate again, we can prove the analogous estimate for radial SLE in the disc.

Lemma 9 (One-point estimate for disc)**.** *Let D be a $\mathbb{D}$ domain with a prime end w_0, and let γ be a radial SLE$_\kappa$ curve in D from w_0 to 0. Let $z_0 \in \overline{\mathbb{D}}$, $y_0 = 1 - |z_0|$. Suppose that $0 < r < R < dist(z_0, \{0, w_0\})$. Let $\rho = \{z \in D : |z - z_0| = R\}$, and $\eta = \{z \in D : |z - z_0| = r\}$, and assume that $\{z \in \mathbb{D} : |z - z_0| \leq R\} \subset D$. Then*

$$\mathbb{P}[\gamma \cap \eta \neq \emptyset] \leq C \frac{P_{y_0}(r)}{P_{y_0}(R)}.$$

Proof. Let $\phi : \mathbb{D} \backslash \{0\} \to \mathbb{H}^*$ be defined by $\phi(z) = \frac{1}{i} \log(z)$, and suppose $z_0' \in H := \phi(D)$ so that $e^{iz_0'} = z_0$. The proof will break into the same 3 cases as in the proof of Lemma 8:

$R < y_0$, $r > y_0$, and $r < y_0 < R$. In the first two cases, we want to prove that

$$\mathbb{P}[\mathrm{dist}(z_0, \gamma)] \leq C \left(\frac{r}{R}\right)^p$$

for $p \in \{\alpha, 2-d\}$, so it will suffice to prove an upper bound for r/R which will work for both cases simultaneously. The third case can be handed similarly to the third case in Lemma 8.

By assumption, ϕ is a conformal map on a domain which has positive distance from $B(z_0, R)$. Let γ' be a radial SLE$_\kappa$ trace in H from $\phi(w_0)$ to ∞. Define $\tilde{r}, \tilde{R}$ as in Lemma 4 with respect to ϕ and z_0. If $r < R/7$, the inclusions in Lemma 4, part a), imply that in the interior case, we can apply the interior estimate of Lemma 8 to $z_0', \tilde{r}, \tilde{R}$, and in the boundary case we can apply the boundary estimate. In either case,

$$\mathbb{P}[\mathrm{dist}(z_0, \gamma) \leq r] \leq \mathbb{P}[\mathrm{dist}_{\mathbb{H}^*}(z_0', \gamma') \leq \tilde{r}] \leq C \left(\frac{\tilde{r}}{\tilde{R}}\right)^p \leq C \left(\frac{r}{R}\right)^p,$$

where $p \in \{\alpha, 2-d\}$ is chosen appropriately.

□

Lemma 10. *Let D be a $\mathbb{D}$ domain with a prime end w_0, and let γ be a radial SLE$_\kappa$ curve in D from w_0 to 0. Let ρ be a crosscut in D such that $D^*(\rho)$ is not a neighborhood of w_0 in D, and let $S \subset D^*(\rho)$. Let $\tilde{D}$ be a domain that contains D, and $\tilde{\rho}$ be a subset of $\tilde{D}$ that contains ρ. Let $\tilde{\eta}$ be either a Jordan curve in $\tilde{D}$ which intersects ∂D or a crosscut in $\tilde{D}$. Suppose that $\tilde{\eta}$ disconnects $\tilde{\rho}$ from S in $\tilde{D}$. Then*

$$\mathbb{P}[\gamma \cap S \neq \emptyset] \leq C e^{-\alpha \pi d_{\tilde{D}}(\tilde{\rho}, \tilde{\eta})}.$$

Proof. By Lemma 2, $\tilde{\eta}$ has a subcrosscut η in D which disconnects S from ρ. Since $S \subset$

$D^*(\rho)$, we have $\eta \subset D^*(\rho)$ and $S \subset D^*(\eta)$. Therefore, $D(\rho;\eta) = D^*(\rho)$ is neither a neighborhood of 0 nor w_0 in D. Using the boundary estimate from Lemma 7, we see that

$$\mathbb{P}[\gamma \cap S \neq \emptyset] \leq \mathbb{P}[\gamma \cap D^*(\eta) \neq \emptyset] \leq Ce^{-\alpha\pi d_D(\rho,\eta)} \leq C^{-\alpha\pi d_{\tilde{D}}(\tilde{\rho},\tilde{\eta})}.$$

The last inequality follows from the comparison principle for extremal length, since any path in D connecting ρ and η is a subarc of itself contained in $\tilde{D}$ which connects $\tilde{\rho}$ and $\tilde{\eta}$.

□

2.4 Components of crosscuts

Before we state the main theorem of this section, we will introduce the notation to be used. Let $\mathcal{F}_t$ be the right continuous filtration determined by the radial SLE curve γ. For any set $S \subset \overline{\mathbb{D}}$, let $\tau_S = \inf\{t \geq 0 : \gamma(t) \in S\}$. For any stopping time τ, define $\gamma^\tau(t) = \gamma(\tau + t)$. Define $D_t = \mathbb{D}(\gamma[0,t])$.

Theorem 4. *Let γ be a radial SLE$_\kappa$ curve in $\mathbb{D}$ from 1 to 0. Suppose that $z_0, z_1, \ldots, z_m \in \overline{\mathbb{D}}\backslash\{0,1\}$. For each z_j, let $0 < r_j \leq R_j$, and define the circles $\hat{\xi}_j = \{|z - z_j| = R_j\}$ and $\xi_j = \{|z - z_j| = r_j\}$. Assume that neither 0 nor 1 are contained in $\mathbb{D}^*(\hat{\xi}_j; z_j)$ for each j, and that $\overline{\mathbb{D}^*(\hat{\xi}_j)} \cap \overline{\mathbb{D}^*(\hat{\xi}_i)} = \emptyset$ for $j \neq j$. Let $r_0' \in (0, r_0)$ and define $\xi_0' = \{|z - z_0| = r_0'\}$. Define the event*

$$E = \{\tau_{\xi_0} < \tau_{\hat{\xi}_1} \leq \tau_{\xi_1} < \cdots < \tau_{\hat{\xi}_m} \leq \tau_{\xi_m} < \tau_{\xi_0'} < \infty\}.$$

If $y_j = 1 - |z_j|$, *then*

$$\mathbb{P}[E|\mathcal{F}_{\tau_{\xi_0}}] \leq C^m \left(\frac{r_0}{R_0}\right)^{\alpha/4} \prod_{j=1}^{m} \frac{P_{y_j}(r_j)}{P_{y_j}(R_j)}.$$

Note that the assumptions here imply that neither the start point, 1, nor the endpoint 0, of the SLE curve are enclosed by the discs $\hat{\xi}_j$. We also assume interiors of circles with different centers have no overlap, and that the boundary circles do not meet. The proof is similar to the proof of Theorem 3.1 in [23], but for completeness we include complete details.

Proof. Consider the discs ξ which intersect the boundary. We know that the probability that γ hits the points in $\xi \cap \partial\mathbb{D}$ is equal to 0, and so $\tau_\xi = \tau_{\xi \cap \mathbb{D}}$ a.s. Therefore, we can assume that each ξ is either a Jordan curve or a crosscut in $\mathbb{D}$. For each $j = 0, 1, \ldots, m$, let $\tau_j = \tau_{\xi_j}$ and $\hat{\tau}_j = \tau_{\hat{\xi}_j}$, and define $\tau_{m+1} = \tau_{\xi_0'}$.

By the Domain Markov Property of SLE and Lemma 9, we see that

$$\mathbb{P}[\tau_j < \infty|\mathcal{F}_{\hat{\tau}_j}] \leq C \frac{P_{y_j}(r_j)}{P_{y_j}(R_j)}. \tag{2.8}$$

Combining these together gives

$$\mathbb{P}[E|\mathcal{F}_{\tau_0}] \leq C^m \prod_{j=1}^{m} \frac{P_{y_j}(r_j)}{P_{y_j}(R_j)}.$$

If $r_0 = R_0$, then we are done. Suppose that $R_0 > r_0$. Create a new arc $\rho = \{z \in \mathbb{D} : |z - z_0| = \sqrt{R_0 r_0}\}$, so that ρ is either a Jordan curve or a crosscut in $\mathbb{D}$ between ξ_0 and $\hat{\xi}_0$. Therefore, we know that

$$d_{\mathbb{D}}(\rho, \xi_0), d_{\mathbb{D}}(\rho, \hat{\xi}_0) \geq \frac{\log(R_0/r_0)}{4\pi}. \tag{2.9}$$

Note that ρ seperates ξ_0 from 0. In the following argument, we will need to keep track of how the $\mathbb{D}$-domains D_t are divided by ρ at any particular time. Let $T = \inf\{t \geq 0 : \xi_0' \not\subset D_t\}$. Then if $\tau_0 \leq t < T$, ξ_0' is a connected subset of D_t. In this case, since the starting point 1 is outside of $\hat{\xi}_0$ and γ inteserc ts ξ_0, it must be that γ intersects ρ, and so ρ intersects ∂D_t. By Lemma 2, there is a first subcrosscut of ρ in D_t, to be denoted ρ_t, which seperates ξ_0' from 0 for each $\tau_0 \leq t < T$.

Now, we need to break the event E into several cases based on the behavior of the curve γ as it intersects the circles in the correct order. Let $I = \{(j, j+1) : 0 \leq j \leq m\} \cup \{(j, j) : 1 \leq j \leq m\}$, and define a sequence of events $\{A_i : i \in I\}$ by

1. $A_{(0,1)} = \{T > \tau_0\} \cap \{\mathbb{D}^*(\xi_1) \subset D^*_{\tau_0}(\rho_{\tau_0})\}$

2. $A_{(j,j)} = \{T > \tau_j\} \cap \{\mathbb{D}^*(\xi_j) \subset D_{\tau_{j-1}}(\rho_{\tau_{j-1}})\} \cap \{\mathbb{D}^*(\xi_j) \subset D^*_{\tau_j}(\rho_{\tau_j})\}$, $1 \leq j \leq m$.

3. $A_{(j,j+1)} = \{T > \tau_j\} \cap \{\mathbb{D}^*(\xi_j) \subset D_{\tau_j}(\rho_{\tau_j})\} \cap \{\mathbb{D}^*(\xi_{j+1}) \subset D^*_{\tau_j}(\rho_{\tau_j})\}$, $1 \leq j \leq m-1$.

4. $A_{(m,m+1)} = \{T > \tau_m\} \cap \{\mathbb{D}^*(\xi_m) \subset D_{\tau_m}(\rho_{\tau_m})\}$.

Observe that for each j, the events $A_{(j,j)}, A_{(j,j+1)}$ are $\mathcal{F}_{\tau_j}$ measurable. What I claim is that

$$E \subset \cup_{i \in I} A_i. \tag{2.10}$$

To see this, observe that $A_{(m,m+1)}$ is the event that at time τ_m, ξ_m lies outside of ρ, relative to 0 and the path γ. If that does not happen, then ξ_m must lie inside ρ at time τ_m. Now suppose that $A_{(m,m)}$ does not happen, which is the event that at time τ_{m-1}, ξ_m lies outside of ρ, but at time τ_m, ξ_m lies inside of ρ. Then it must be that ξ_m lies inside of ρ at τ_{m-1}. Proceeding along this way inductively proves (2.10). Complete details can be found in Section 2.7. Now it will suffice to show that

$$\mathbb{P}[E\cap A_i|\mathcal{F}_{\tau_0}]\le C^m\left(\frac{r_0}{R_0}\right)^{\alpha/4}\prod_{j=1}^m\frac{P_{y_j}(r_j)}{P_{y_j}(R_j)} \tag{2.11}$$

for each $i\in I$. We will break it down to the four cases $i=(m,m+1),(j,j),(j,j+1)$, and $(0,1)$. In all of these cases, we will use the convention $\gamma^T(t)=\gamma(T+t)$ for any $T,t\ge 0$.

Case $(0,1)$. Suppose that $A_{(0,1)}$ occurs, and that $\tau_0<\hat{\tau}_1$. First, we claim that $\hat{\xi}_1\subset\mathbb{D}^*_{\tau_0}(\rho_{\tau_0})$. Note that both $\hat{\xi}_1$ and $\mathbb{D}^*(\xi_1)$ are contained in $\mathbb{D}^*(\hat{\xi}_1)\cup\hat{\xi}_1$, which is a connected subset of the disjoint union $D_{\tau_0}(\rho_{\tau_0})\cup D^*_{\tau_0}(\rho_{\tau_0})$. By assuming that the event $A_{(0,1)}$ occurs, however, we can conclude that they must both be contained in $D^*_{\tau_0}(\rho_{\tau_0})$, which proves the claim. Also, note that ρ disconnects $\hat{\xi}_1$ from ξ_0' in $\mathbb{D}$, and ρ must intersect ∂D_{τ_0}. By Lemma 2, there is a subcrosscut ρ'_{τ_0} of ρ which is first to seperate $\hat{\xi}_1$ from ξ_0' in the domain D_{τ_0}. Since both $\hat{\xi}_1$ and ξ_0' lie in $D^*_{\tau_0}(\rho_{\tau_0})$, so does ρ'_{τ_0}. Note that this implies that $\rho'_{\tau_0}\ne\rho_{\tau_0}$, and that $D^*_{\tau_0}(\rho'_{\tau_0})\subset D^*_{\tau_0}(\rho_{\tau_0})$. Since ρ_{τ_0} was defined to be the first subcrosscut of ρ in D_{τ_0} that disconnects ξ_0' from 0, and since $D^*_{\tau_0}(\rho'_{\tau_0})$ is contained in the domain determined by ρ_{τ_0}, it cannot be that ρ'_{τ_0} disconnects ξ_0' from 0. Therefore, we conclude that $\xi_0'\subset D_{\tau_0}(\rho'_{\tau_0})$ and $\hat{\xi}_1\subset D^*_{\tau_0}(\rho'_{\tau_0})$.

Observe that $\mathbb{D}^*(\xi_0)$ is a connected subset of $D_{\tau_0}\backslash\rho'_{\tau_0}$, and contains ξ_0' and a curve which approaches $\gamma(\tau_0)\in\xi_0$. Therefore,

$$D_{\tau_0}(\rho'_{\tau_0};\gamma(\tau_0))=D_{\tau_0}(\rho'_{\tau_0};\xi_0')=D_{\tau_0}(\rho'_{\tau_0}).$$

It follows that $D_{\tau_0}(\rho'_{\tau_0};\hat{\xi}_1)=D^*_{\tau_0}(\rho'_{\tau_0})$ is not a neighborhood of $\gamma(\tau_0)=\gamma^{\tau_0}(0)$, where γ^{τ_0} (conditioned on $\mathcal{F}_{\tau_0}$) is a radial SLE$_\kappa$ curve in the $\mathbb{D}$-domain D_{τ_0}. Since $\tau_0<\hat{\tau}_1$, the event

$\{\hat{\tau}_1 < \infty\}$ implies that the conditioned SLE curve γ^{τ_0} visits $\hat{\xi}_1$. Since $\hat{\xi}_0$ disconnects $\hat{\xi}_1$ from $\rho'_{\tau_0} \subset \rho$ in $\mathbb{D}$, and $\hat{\xi}_0$ intersects ∂D_{τ_0}, we can apply Lemma 10 to conclude that

$$\mathbb{P}[\hat{\tau}_1 < \infty | \mathcal{F}_{\tau_0}, A_{(0,1)}, \tau_0 < \hat{\tau}_1] \leq Ce^{-\alpha d_{\mathbb{D}}(\rho, \hat{\xi}_0)} \leq C \left(\frac{r_0}{R_0} \right)^{\alpha/4}.$$

Note that the second inequality follows from (2.9). Combining the above inequality with inequality (2.4) proves that inequality (2.11) holds for the case $i = (0, 1)$.

Case $(j, j + 1)$, for $1 \leq j \leq m - 1$. Suppose that $A_{(j,j+1)}$ occurs, and $\tau_j < \hat{\tau}_{j+1}$. By the same argument used in the case for $A_{(0,1)}$, we can conclude that there exists a subcrosscut of ρ, which we will call ρ'_{τ_j}, which disconnects $\hat{\xi}_{j+1}$ from ξ'_0 in D_{τ_j}. It follows that $D^*_{\tau_j}(\rho'_{\tau_j}) \subset D^*_{\tau_j}(\rho_{\tau_j})$, and then we can conclude that $\xi'_0 \subset D_{\tau_j}(\rho'_{\tau_j})$ and $\hat{\xi}_{j+1} \subset D^*_{\tau_j}(\rho'_{\tau_j})$. Since $\mathbb{D}^*(\xi_j)$ is a connected subset of $D_{\tau_j}(\rho_{\tau_j})$ and contains a curve approaching $\gamma(\tau_j) \in \xi_j$, we can see that $D_{\tau_j}(\rho_{\tau_j}; \gamma(\tau_j)) = D_{\tau_j}(\rho_{\tau_j}; \mathbb{D}^*(\xi_j)) = D_{\tau_j}(\rho_{\tau_j})$. Therefore, $D^*_{\tau_j}(\rho'_{\tau_j}) \subset D^*_{\tau_j}(\rho_{\tau_j})$ is not a neighborhood of $\gamma(\tau_j) = \gamma^{\tau_j}(0)$.

Since $\tau_j < \hat{\tau}_{j+1}$, the event $\{\hat{\tau}_{j+1} < \infty\}$ implies that γ^{τ_j}, the radial SLE$_\kappa$ curve in D_{τ_j} after conditioning on $\mathcal{F}_{\tau_j}$, visits $\hat{\xi}_{j+1}$. Since $\hat{\xi}_0$ disconnects $\hat{\xi}_{j+1}$ from ρ, and therefore from ρ'_{τ_j}, in $\mathbb{D}$, Lemma 10 implies that

$$\mathbb{P}[\hat{\tau}_{j+1} < \infty | \mathcal{F}_{\tau_j}, A_{(j,j+1)}, \tau_j < \hat{\tau}_{j+1}] \leq Ce^{-\alpha\pi d_{\mathbb{D}}(\rho, \hat{\xi}_0)} \leq C \left(\frac{r_0}{R_0} \right)^{\alpha/4}.$$

As in the above case, this proves inequality (2.11) for the case $i = (j, j + 1)$ when $1 \leq j \leq m - 1$.

Case $(m, m + 1)$. Suppose that $\tau_m < \tau_{m+1}$, and that $A_{(m,m+1)}$ occurs. The set $\mathbb{D}^*(\xi_m)$ is connected and contained in $D_{\tau_m} \backslash \rho_{\tau_m}$, and also $\gamma(\tau_m) \in \xi_m$. This implies that $D_{\tau_m}(\rho_{\tau_m}; \gamma(\tau_m)) =$

$D_{\tau_m}(\rho_{\tau_m}; \mathbb{D}^*(\xi_m)) = D_{\tau_m}(\rho_{\tau_m})$. Therefore, $D^*_{\tau_m}(\rho_{\tau_m})$ is not a neighborhood of $\gamma^{\tau_m}(0) = \gamma(\tau_m)$ in D_{τ_m}. Since we are assuming that $\tau_m < \tau_{m+1}$, the event $\{\tau_{m+1} < \infty\}$ implies that the curve γ^{τ_m}, which after conditioning on $\mathcal{F}_{\tau_m}$ is a radial SLE$_\kappa$ in D_{τ_m} from $\gamma(\tau_m)$ to 0, must visit $\xi_0' \subset D^*_{\tau_m}(\rho_{\tau_m})$. Since ξ_0 disconnects ξ_0' from ρ in $\mathbb{D}$ and intersects ∂D_{τ_m}, we can apply Lemma 10 to show that

$$\mathbb{P}[\tau_{m+1} < \infty | \mathcal{F}_{\tau_m}, A_{(m,m+1)}, \tau_m < \tau_{m+1}] \leq C e^{-\alpha \pi d_{\mathbb{D}}(\xi_0, \rho)} \leq C \left(\frac{r_0}{R_0}\right)^{\alpha/4}.$$

Case (j,j), for $1 \leq j \leq m$. We now prove inequality (2.11) for $A_{(j,j)}$. Fix a j in $\{1, \dots, m\}$, and define $\sigma_j = \inf\{t \geq \tau_{j-1} : \mathbb{D}^*(\xi_j) \subset D^*_t(\rho_t)\}$. This can be seen as the first time after τ_{j-1} that the SLE curve γ hits the crosscut ρ on the "correct" side of $\hat{\xi}_j$, i.e. γ cuts the disc off from 0. There are a few observations to make about σ_j which follow from Lemma 11 (in Section 2.7). First, σ_j is an $\mathcal{F}_t$-stopping time. If $\sigma_j < \infty$, then $\mathbb{D}^*(\xi_j) \subset D^*_{\sigma_j}(\rho_{\sigma_j})$. If the event $A_{(j,j)}$ occurs, then so does $\{\tau_{j-1} < \sigma_j < \tau_j\}$. Finally, if $\tau_{j-1} < \sigma_j < \infty$, then it must be that $\gamma(\sigma_j)$ is an endpoint of ρ_{σ_j}. This implies that $D^*_{\sigma_j}(\rho_{\sigma_j})$ is neither a neighborhood of $\gamma(\sigma_j)$ nor of 0.

We define two events which seperate the event $A_{(j,j)}$. Define

$$F_< = \{\sigma_j < \hat{\tau}_j\}, \text{ and } F_\geq = \{\tau_j > \sigma_j \geq \hat{\tau}_j\}.$$

Notice that $A_{(j,j)} \subset F_< \cup F_\geq$, so if we can prove (2.11) for both $F_<$ and $F_\geq$ instead of $A_{(j,j)}$, the case will be proven.

First, assume that $F_<$ happens. Then $\mathbb{D}^*(\hat{\xi}_j) \cup \hat{\xi}_j$ is a connected subset of $(\mathbb{D} \backslash \gamma[0, \rho_{\sigma_j}]) \backslash \rho$ that contains $\mathbb{D}^*(\xi_j)$, and so $\hat{\xi}_j \subset D^*_{\sigma_j}(\rho_{\sigma_j}; \mathbb{D}^*(\xi_j)) = D^*_{\sigma_j}(\rho_{\sigma_j})$. Since $\hat{\xi}_0$ disconnects ρ from

$\hat{\xi}_j$ in $\mathbb{D}$, Lemma 10 and (2.9) imply that

$$\mathbb{P}[\hat{\tau}_j < \infty | \mathcal{F}_{\sigma_j}, F_<] \leq Ce^{-\alpha\pi d_{\mathbb{D}}(\rho, \hat{\xi}_0)} \leq C\left(\frac{r_0}{R_0}\right)^{\alpha/4}.$$

This implies that

$$\mathbb{P}[\tau_j < \infty, F_< | \mathcal{F}_{\tau_{j-1}}] \leq C\left(\frac{r_0}{R_)}\right)^{\alpha/4} \frac{P_{y_j}(r_j)}{P_{y_j}(R_j))}.$$

Next, we assume that $F_\geq$ occurs, which is the more difficult case. Define $N = \lceil \log(R_j/r_j) \rceil$, where $\lceil \cdot \rceil$ is the integer valued ceiling function. In the event $F_\geq$, the SLE curve passes through the outer circle $\hat{\xi}_j$, then hits the crosscut ρ before returning to hit the inner circle ξ_j. What we need to do is divide the annulus $\{r_j \leq |z - z_j| < R_j\}$ into N subannuli until we can identify the last subcircle to be crossed before the time σ_j.

Define the circles $\zeta_k = \{|z - z_j| = \left(R_j^{N-k} r_j^j\right)^{1/N}\}$ for $0 \leq k \leq N$. Note that $\zeta_0 = \hat{\xi}_j$, $\zeta_N = \xi_j$, and a higher index k indicates that ζ_k is more deeply nested inside $\hat{\xi}_j$. Then $F_\geq \subset \cup_{k=1}^N F_k$, where

$$F_k = \{\tau_{\zeta_{k-1}} \leq \sigma_j < \tau_{\xi_{\zeta_k}}\}.$$

If the event F_k occurs, then $\zeta_k \subset D^*_{\sigma_j}(\rho_{\sigma_j})$. This is because $\mathbb{D}^*(\zeta_k) \cup \zeta_k$ is a connected subset of $\left(\mathbb{D} \backslash \gamma[0, \sigma_j]\right) \backslash \rho$ which contains ζ_k, $\mathbb{D}^*(\xi_j)$, and $D^*_{\sigma_j}(\rho_{\sigma_j})$. By Lemma 10 and inequality (2.9), we conclude that

$$\begin{aligned} \mathbb{P}[\tau_{\zeta_k} < \infty | \mathcal{F}_{\sigma_j}, F_k] &\leq Ce^{-\alpha\pi d_{\mathbb{D}}(\rho, \zeta_{k-1})} \leq Ce^{-\alpha\pi(d_{\mathbb{D}}(\rho, \hat{\xi}_0) + d_{\mathbb{D}}(\zeta_0, \zeta_{k-1}))} \\ &\leq C\left(\frac{r_0}{R_0}\right)^{\alpha/4}\left(\frac{r_j}{R_j}\right)^{\alpha(k-1))/2N}. \end{aligned} \quad (2.12)$$

By Lemma 9, we get that

$$\mathbb{P}[F_k|\mathcal{F}_{\tau_{j-1}}, \tau_{j-1} < \hat{\tau}_j] \leq C \frac{P_{y_j}\left((R_j^{N-k+1} r^{k-1})^{1/N}\right)}{P_{y_j}(R_j)},$$

$$\mathbb{P}[\tau_j < \infty | \mathcal{F}_{\zeta_k}, F_k] \leq C \frac{P_{y_j}(r_j)}{P_{y_j}\left((R_j^{N-k} r_j^k)^{1/N}\right)}.$$

Combining the above three inequalities and the upper bound in Lemma 1 gives

$$\mathbb{P}[\tau_j < \infty, F_k|\mathcal{F}_{\tau_{j-1}}, \tau_{j-1} < \hat{\tau}_j] \leq C \left(\frac{r_0}{R_0}\right)^{\alpha/4} \left(\frac{r_j}{R_j}\right)^{\frac{\alpha(k-1)}{2N}} \left(\frac{r_j}{R_j}\right)^{-\alpha/N} \frac{P_{y_j}(r_j)}{P_{y_j}(R_j)}.$$

Since $F_{\geq} \subset \cup_{k=1}^N F_k$, adding up the above inequality yields

$$\mathbb{P}[\tau_j < \infty, F_{\geq}|\mathcal{F}_{\tau_{j-1}}, \tau_{j-1} < \hat{\tau}_j] \leq C \left(\frac{r_0}{R_0}\right)^{\alpha/4} \frac{P_{y_j}(r_j)}{P_{y_j}(R_j)} \left[\left(\frac{r_j}{R_j}\right)^{-\alpha/N} \frac{1-(r_j/R_j)^{\alpha/2}}{1-(r_j/R_j)^{\alpha/2N}}\right].$$

In Section 2.7, we prove that

$$\left(\frac{r_j}{R_j}\right)^{-\alpha/N} \frac{1-(r_j/R_j)^{\alpha/2}}{1-(r_j/R_j)^{\alpha/2N}} \leq \frac{e^{\alpha}}{1-e^{-\alpha/4}}, \tag{2.13}$$

and so we get

$$\mathbb{P}[\tau_j < \infty, F_{\geq}|\mathcal{F}_{\tau_{j-1}}, \tau_{j-1} < \hat{\tau}_j] \leq C \left(\frac{r_0}{R_0}\right)^{\alpha/4} \frac{P_{y_j}(r_j)}{P_{y_j}(R_j)},$$

which is the same bound as that acheived for $F_<$, though with a different constant. Combining these two inequalities gives

$$\mathbb{P}[\tau_j < \infty, A_{(j,j)} | \mathcal{F}_{\tau_{j-1}}, \tau_{j-1} < \hat{\tau}_j] \leq C \left(\frac{r_0}{R_0}\right)^{-\alpha/4} \frac{P_{y_j}(r_j)}{P_{y_j}(R_j)},$$

which completes the proof of (2.11) in the final case, and so the theorem follows. □

2.5 Concentric circles

Let Ξ be a family of mutually disjoint circles in $\mathbb{C}$ with centers in $\overline{\mathbb{D}}\backslash\{0\}$, none of which pass through or enclose 0 or 1. We can define a partial order on Ξ by $\xi_1 < \xi_2$ if ξ_2 is enclosed by ξ_1. Note that the larger circle has a smaller radius than the larger circle because the order is determined by visiting time. If $\xi_1 < \xi_2$, any continuous path which hits both circles must pass through ξ_1 before it hits ξ_2. When γ is an SLE curve in $\mathbb{D}$ starting from 0, $\xi_1 < \xi_2$ means that $\tau_{\xi_1} < \tau_{\xi_2}$, assuming γ passes through ξ_2. Also, observe that circles in Ξ are not necessarily contained in $\mathbb{D}$. In fact, we want to account for circles in Ξ which have center in the boundary as well.

Further, we assume that Ξ has a partition $\cup_{e\in\mathcal{E}}\Xi_e$ with the following properties:

1. For each $e \in \mathcal{E}$, the elements of Ξ_e are concentric circles whose radii form a geometric sequence with a common ratio of 1/4. For each e, let the common center be z_e. Note that elements of Ξ_e are totally ordered. Let R_e be the radius of the smallest circle (in the ordering on Ξ), and let r_e be the radius of the largest circle. Then there is some integer $M \geq 0$ with $R_e = r_e 4^M$.

2. Let $A_e = \{z \in \mathbb{C} : r_e \leq |z - z_e| \leq R_e\}$ denote the closed annulus containing all of the circles in Ξ_e. Then we assume that the collection of annuli $\{A_e\}_{e\in\mathcal{E}}$ is mutually

disjoint.

We make a couple of quick remarks about the generality of this assumption. It may be that $|\Xi_e| = 1$, in which case $r_e = R_e$, and the annulus A_e is just the single circle contained in Ξ_e. Also, if $e_1 \neq e_2$, that does not necessarily mean that $z_{e_1} \neq z_{e_2}$. There can be multiple sets of concentric circles in Ξ with the same center with gaps between them. In this case, if members of Ξ_{e_2} have the smaller radii, we have $R_{e_2} < r_{e_1}$. Also, there can be two components of the partition Ξ_{e_1}, Ξ_{e_2} so that each element in Ξ_{e_2} is ordered larger than each element of Ξ_{e_1}. In this case, the annulus A_{e_2} is contained in the bounded component of $\mathbb{C}\backslash A_{e_1}$.

Theorem 5. *Let Ξ be a family of circles with the properties listed above, and assume that γ is a radial SLE$_\kappa$ curve in $\mathbb{D}$ from 1 to 0. For each $e \in \mathcal{E}$, let $y_e = 1 - |z_e|$. Then there exists a constant $C_{|\mathcal{E}|} < \infty$, which only depends on κ and the size of the partition $|\mathcal{E}|$, so that*

$$\mathbb{P}[\cap_{\xi \in \Xi}\{\gamma \cap \xi \neq \emptyset\}] \leq C_{|\mathcal{E}|} \prod_{e \in \mathcal{E}} \frac{P_{y_e}(r_e)}{P_{y_e}(R_e)}.$$

The strategy is to consider all possible orders σ that the SLE curve γ can visit all elements of Ξ. Under σ, γ may pass through several elements of a family Ξ_{e_0}, leave and visit other Ξ_e's, before returning to pass through the more inner circles in Ξ_{e_1}. Theorem 4 provides an estimate of the price paid by γ in order to return to the interior circles of Ξ_{e_0}. This gives an estimate for the probability of $\cap_\xi\{\gamma \cap \xi \neq \emptyset\}$ in the prescribed order σ. We then add up over all appropriate orders σ and show that the constant only depends on $|\mathcal{E}|$.

Proof. Define S to be the set of permutations $\sigma : \{1, 2, \dots, |\Xi|\} \to \Xi$ such that $\xi_1 < \xi_2$ implies $\sigma^{-1}(\xi_1) < \sigma^{-1}(\xi_2)$. Then S is the set of viable orders in which γ can visit the elements of Ξ for the first time. For an ordering $\sigma \in S$, $\sigma(j) \in \Xi$ is the j-th circle visited by γ. Define the event $E^\sigma = \{\tau_{\sigma(1)} < \cdots < \tau_{\sigma(|\Xi|)} < \infty\}$. Then $E := \cap_{\xi \in \Xi}\{\gamma \cap \xi \neq \emptyset\} = \cup_{\sigma \in S} E^\sigma$. What we need to do is bound $\mathbb{P}[E^\sigma]$.

Fix some $\sigma \in S$. Our first goal is to create a subpartition of $\{\Xi_e\}_{e \in \mathcal{E}}$ into $\{\Xi_i\}_{i \in I}$, where the elements of Ξ_i receive first visits from γ without interruption in the event E^σ. For each $e \in \mathcal{E}$, let $N_e = |\Xi_e| - 1$, and label the elements of Ξ_e by $\xi_0^e < \cdots < \xi_{N_e}^e$. Define $J_e \subset \{0, 1, \ldots, N_e\}$ by

$$J_e = \{1 \leq n \leq N_e : \sigma^{-1}(\xi_n^e) > \sigma^{-1}(\xi_{n-1}^e) + 1\} \cup \{0\}.$$

Then n is a nonzero element of J_e if, after γ visits ξ_{n-1}^e, the curve visits other new circles in Ξ before ξ_n^e. That is, there is some $\xi \in \cup_{e' \neq e} \Xi_e$ such that $\tau_{\xi_{n-1}^e} < \tau_\xi < \tau_{\xi_n^e}$. Order the elements of J_e by $0 = s_e(0) < s_e(1) < \cdots < s_e(M_e)$, where $M_e = |J_e| - 1$ is the number of times that the progress of γ through Ξ_e is interrupted. Define $s_e(M_e + 1) = N_e + 1$. Using this framework, each Ξ_e can be partitioned into $M_e + 1$ subsets

$$\Xi_{(e,j)} = \{\xi_n^e : s_e(j) \leq n \leq s_e(j+1) - 1\}, \ 0 \leq j \leq M_e.$$

These are the elements of Ξ_e which are visited without interruption. Let $I = \{(e, j) : e \in \mathcal{E}, 0 \leq j \leq M_e\}$. Then $\{\Xi_i\}_{i \in I}$ is a finer partition with the desired properties.

For $i \in I$, let $i = (e_i, j)$. We need to do some relableing. Let

$$z_i = z_{e_i}, \ y_i = 1 - |z_i|, \ \min\{\Xi_i\} = \xi_{s_{e_i}(j)}^{e_i},$$

$$\max\{\Xi_i\} = \xi_{s_{e_i}(j+1)-1}^{e_i}, \ P_i = \frac{P_{y_i}(R_{\max\{\Xi_i\}})}{P_{y_i}(R_{\min\{\Xi_i\}})},$$

where $R_{\max\{\Xi_i\}}$ is the radius of $\max\{\Xi_i\}$, and $R_{\min\{\Xi_i\}}$ is the radius of $\min\{\Xi_i\}$. By Lemma 9, we can see that

$$\mathbb{P}[\tau_{\max\{\Xi_i\}} < \infty | \mathcal{F}_{\tau_{\min\{\Xi_i\}}}] \leq CP_i. \tag{2.14}$$

For $e \in \mathcal{E}$, let $P_e = P_{y_e}(r_e)/P_{y_e}(R_e)$. We claim that for each $e \in \mathcal{E}$, we have

$$\prod_{j=0}^{M_e} P_{(e,j)} \leq 4^{\alpha M_e} P_e. \tag{2.15}$$

This follows from Lemma 1, and the details are provided in Section 2.7.

Observe that $|I| = \sum_{e\in\mathcal{E}}(M_e + 1)$ is the number of uninterrupted sequences of circles visited by γ under σ. Then σ induces a map $\hat{\sigma} : \{1, \ldots, |I|\} \to I$ so that if $n_1 < n_2$, we have $\max\{\sigma^{-1}(\Xi_{\hat{\sigma}(n_1)})\} < \min\{\sigma^{-1}(\Xi_{\hat{\sigma}(n_2)})\}$, and $n_1 = n_2 - 1$ implies $\max\{\sigma^{-1}(\Xi_{\hat{\sigma}(n_1)})\} = \min\{\sigma^{-1}(\Xi_{\hat{\sigma}(n_2)})\} - 1$. In other words, $\hat{\sigma}$ is the order that the families Ξ_i are visited. In particular, we can rewrite the event E^σ as

$$E^\sigma = \{\tau_{\min \Xi_{\hat{\sigma}(1)}} < \tau_{\max \Xi_{\hat{\sigma}(1)}} < \cdots < \tau_{\min \Xi_{\hat{\sigma}(|I|)}} < \tau_{\max \Xi_{\hat{\sigma}(|I|)}} < \infty\}.$$

We will use Theorem 4 to estimate the probability of this event.

Fix some $e_0 \in \mathcal{E}$, and let $n_j = \hat{\sigma}^{-1}((e_0, j))$ for $0 \leq j \leq M_{e_0}$. In the event E^σ, the family $\Xi_{(e_0,j)}$ is the n_j-th family to be visited by the curve. For $0 \leq j \leq M_e - 1$, $n_{j+1} \geq n_j + 2$ since at least one other family not contained in Ξ_{e_0} must be hit by the curve between them. Fix $0 \leq j \leq M_{e_0} - 1$, and let $m = n_{j+1} - n_j - 1 \geq 1$. We are going to apply Theorem 4 to the family

- $\hat{\xi}_0 = \min \Xi_{e_0}, \xi_0 = \max \Xi_{(e_0,j)} = \max \Xi_{\hat{\sigma}(n_j)}$, and $\xi_0' = \min \Xi_{(e_0,j+1)} = \min \Xi_{\hat{\sigma}(n_{j+1})}$

- $\hat{\xi}_k = \min \Xi_{\hat{\sigma}(n_j+k)}$, and $\xi_k = \max \Xi_{\hat{\sigma}(n_j+k)}$ for $1 \le k \le m$.

In plain words, m is the number of other families $\{\Xi_i\}$ first visited by γ between first visits of the j-th level and the $(j+1)$-st level of the family Ξ_{e_0}. The curves $\xi_k, \hat{\xi}_k$ are the k-th family first visited before returning to $\Xi_{(e_0,j+1)}$.

Define an event by $E^{\sigma}_{[\max \Xi_{\hat{\sigma}(n_j)}, \min \Xi_{\hat{\sigma}(n_{j+1})}]}$ by

$$\{\tau_{\xi_0} < \tau_{\hat{\xi}_1} < \tau_{\xi_1} < \cdots < \tau_{\xi_m} < \tau_{\xi'_0}\} \in \mathcal{F}_{\tau_{\min \Xi_{\hat{\sigma}(n_{j+1})}}}.$$

Theorem 4 implies that

$$\mathbb{P}[E^{\sigma}_{[\max \Xi_{\hat{\sigma}(n_j)}, \min \Xi_{\hat{\sigma}(n_{j+1})}]} | \mathcal{F}_{\max \Xi_{\hat{\sigma}(n_j)}}] \le C^m 4^{-\frac{\alpha}{4}(s_{e_0}(j+1)-1)} \prod_{n=n_j+1}^{n_{j+1}-1} P_{\hat{\sigma}(n)}.$$

Varying $j = 0, 1, \ldots, M_{e_0} - 1$ and using inequality (2.14), we see that

$$\mathbb{P}[E^{\sigma}] \le C^{|I|} 4^{-(\alpha/4)\sum_{j=1}^{M_{e_0}} s_{e_0}(j)-1} \prod_{i \in I} P_i.$$

If we use inequality (2.15) and $|I| = \sum_e M_e + 1$, we can deduce that the right hand side is bounded above by

$$C^{|\mathcal{E}|} C^{\sum_{e \in \mathcal{E}} M_e} 4^{-(\alpha/4)\sum_{j=1}^{M_{e_0}} s_{e_0}(j)} \prod_{e \in \mathcal{E}} P_e.$$

Recall that this estimate was based on a fixed $e_0 \in \mathcal{E}$, but the left hand side does not depend on this choice. Taking the geometric average with respect to $e_0 \in \mathcal{E}$, we can see that

$$\mathbb{P}[E^{\sigma}] \le C^{|\mathcal{E}|} C^{\sum_{e \in \mathcal{E}} M_e} 4^{-\frac{\alpha}{4|\mathcal{E}|}\sum_{e \in \mathcal{E}} \sum_{j=1}^{M_e} s_e(j)} \prod_{e \in \mathcal{E}} P_e.$$

Note that the finer partition I and associated terms $M_e, s_e(j)$ are dependent on our initial choice of order σ. Using the fact that $E = \cup E^\sigma$ and the above inequality, we get

$$\mathbb{P}[E] \le C^{|\mathcal{E}|} \left(\sum_{\left(M_e;(s_e(j))_{j=1}^{M_e}\right)_{e\in\mathcal{E}}} |S_{(M_e,(s_e(j)))}| C^{\sum_{e\in\mathcal{E}} M_e} 4^{-\frac{\alpha}{4|\mathcal{E}|}\sum_{e\in\mathcal{E}}\sum_{j=1}^{M_e} s_e(j)} \right) \prod_{e\in\mathcal{E}} P_e, \tag{2.16}$$

where

$$S_{(M_e,s_e(j)))} = \{\sigma \in S : M_e^\sigma = M_e, s_e^\sigma(j) = s_e(j), \text{ for } 0 \le j \le M_e, e \in \mathcal{E}\}$$

and the first sum in inequality (2.16) is over all possible $\left(M_e; (s_e(j))_{j=1}^{M_e}\right)_{e\in\mathcal{E}}$. That is, for each e, all possible $M_e \ge 0$ and possible orderings $0 = s_e(0) < s_e(1) < \cdots < s_e(M_e) \le N_e$. Recall that $N_e = |\Xi_e| - 1$ is fixed with the initial partition $\mathcal{E}$. To finish the proof of the theorem, it suffices to show that the large term in the parenthesis in (2.16) can be bounded above by some finite constant depending only on $|\mathcal{E}|$ and κ.

We claim that

$$|S(M_e, s_e(j))| \le |\mathcal{E}|^{\sum_{e\in\mathcal{E}} M_e+1}. \tag{2.17}$$

Notice that the pair $(M_e, s_e(j))$ completely determines the partition $\{\Xi_i\}_{i\in I^\sigma}$, and the partition σ can be recovered from the induced partition $\hat{\sigma} : \{1, \dots, |I^\sigma|\} \to I^\sigma$. This is because the order that circles in any given Ξ_i are visited is predetermined by the ordering on Ξ, and so moving from $\hat{\sigma}$ to σ requires no new information. Next, we claim that $\hat{\sigma}$ is determined by knowing $e_{\hat{\sigma}(n)}$, for each $1 \le n \le |I^\sigma|$. If $e_{\hat{\sigma}(n)} = e_0$, then $\hat{\sigma}(n) = (e_0, j_0)$ where j_0 can be determined by $j_0 = \min\{0 \le j \le M_{e_0} : (e_0, j) : (e_0, j) \notin \hat{\sigma}(m), m < n\}$. There are $|I^\sigma| = \sum_{e\in\mathcal{E}}(M_e+1)$ terms to determine for $e_{\hat{\sigma}(n)}$, each of which has at most $|\mathcal{E}|$ possibilities, which proves (2.17). Using this to estimate the constant in (2.16), we see that

$$\sum_{\left(M_e;(s_e(j))_{j=1}^{M_e}\right)_{e\in\mathcal{E}}} |S_{(M_e,(s_e(j)))}| C^{\sum_{e\in\mathcal{E}} M_e} 4^{-\frac{\alpha}{4|\mathcal{E}|}\sum_{e\in\mathcal{E}}\sum_{j=1}^{M_e} s_e(j)}$$

$$\leq \sum_{\left(M_e;(s_e(j))_{j=1}^{M_e}\right)_{e\in\mathcal{E}}} |\mathcal{E}|^{\sum_{e\in\mathcal{E}}(M_e+1)} C^{\sum_{e\in\mathcal{E}} M_e} 4^{-\frac{\alpha}{4|\mathcal{E}|}\sum_{e\in\mathcal{E}}\sum_{j=1}^{M_e} s_e(j)}$$

$$= |\mathcal{E}|^{|\mathcal{E}|} \sum_{\left(M_e;(s_e(j))_{j=1}^{M_e}\right)_{e\in\mathcal{E}}} \prod_{e\in\mathcal{E}} (C|\mathcal{E}|)^{M_e} 4^{-\frac{\alpha}{4|\mathcal{E}|}\sum_{j=1}^{M_e} s_e(j)}$$

$$= |\mathcal{E}|^{|\mathcal{E}|} \prod_{e\in\mathcal{E}} \sum_{M_e=1}^{N_e} (C|\mathcal{E}|)^{M_e} \sum_{0=s_e(0)<\cdots<s_e(M_e)\leq N_e} 4^{-\frac{\alpha}{4|\mathcal{E}|}\sum_{j=1}^{M_e} s_e(j)}.$$

This can be bounded above by removing the upper bound on the variable $M_e = M$, and letting the variables $s_e(j) = s(j)$ start at its lowest possible point j and adding with no upper bound, giving the inquality

$$\leq |\mathcal{E}|^{|\mathcal{E}|} \prod_{e\in\mathcal{E}} \sum_{M=0}^{\infty} (C|\mathcal{E}|)^M \sum_{s(1)=1}^{\infty} \cdots \sum_{s(M)=M}^{\infty} 4^{-\frac{\alpha}{4|\mathcal{E}|}\sum_{j=1}^{M} s(j)}$$

$$\leq |\mathcal{E}|^{|\mathcal{E}|} \prod_{e\in\mathcal{E}} \sum_{M=0}^{\infty} (C|\mathcal{E}|)^M \prod_{j=1}^{M} \sum_{s(j)=j}^{\infty} 4^{-\frac{\alpha}{4|\mathcal{E}|} s(j)}$$

$$= |\mathcal{E}|^{|\mathcal{E}|} \left(\sum_{M=0}^{\infty} \left(\frac{C|\mathcal{E}|}{1-4^{-\frac{\alpha}{4|\mathcal{E}|}}} \right)^M 4^{-\frac{\alpha}{8|\mathcal{E}|}M(M+1)} \right)^{|\mathcal{E}|}.$$

Note that this bound only depends on $|\mathcal{E}|$ and κ. It is finite because inside the exponent it has the form $\sum a^n b^{-n^2} < \infty$ for some $b > 1$.

$\square$

2.6 Main theorems

The strategy for proving Theorem 1 is to construct a family of circles Ξ and a partition $\{\Xi_e\}_{e\in\mathcal{E}}$ satisfying the hypothesis of Theorem 5 from the discs $\{|z-z_j| \le r_j\}$ and the distances l_j. The constant given will depend on the size of the partition $\mathcal{E}$, but then it can be shown that that $|\mathcal{E}|$ can be bounded above in a way which depends only on the number of points n.

Proof of Theorem 1. We can assume without loss of generality that for each $j = 1, \dots, n$, the radius r_j satisfies $r_j = l_j/4^{h_j}$ for some integer $h_j \ge 1$. This is because the ratio must satisfy $4^{-h_j-1} \le r_j/l_j \le 4^{-h_j}$ for some integer h_j, and any increase of r_j by at most a factor of 4 only affects the constant for each term j. Moreover, if $h_j \le 0$, then the j-th term in the inequality (2.1) is equal to 1, so we can assume that that $h_j \ge 1$.

Construct Ξ: for each $1 \le j \le n$, construct a sequence of circles by

$$\xi_j^s = \{|z - z_j| = \frac{l_j}{4^s}\}, \text{ for } 1 \le s \le h_j.$$

The family of circles $\{\xi_j^s\}_{j,s}$ may not be disjoint, so we may have to remove some circles. For any fixed $k \le n$, let $D_k = \{|z - z_k| \le l_k/4\}$, which is a closed disc containing all of the circles centered at z_k. For $j < k \le n$, define $I_{j,k} = \{\xi_j^s : 1 \le s \le h_j, \text{ and } \xi_j^s \cap D_k \neq \emptyset\}$. These are the circles centered at z_j which intersect D_k if z_k is closer to z_j than z_j is to $\{0, 1, z_1, \dots, z_{j-1}\}$.

Define the family

$$\Xi = \{\xi_j^s : 1 \le j \le h_j, 1 \le s \le h_j\} \setminus \cup_{1 \le j < k \le n} I_{j,k}.$$

Then Ξ is composed of mutually disjoint circles by construction. If $\text{dist}(\gamma, z_j) \leq r_j$, then γ intersects each ξ_j^s for $1 \leq s \leq h_j$, and therefore

$$\mathbb{P}[\cap_{j=1}^n \{\text{dist}(z_j, \gamma) \leq r_j\}] \leq \mathbb{P}[\cap_{\xi \in \Xi}\{\gamma \cap \xi \neq \emptyset\}]. \tag{2.18}$$

In order to apply Theorem 5, we need to partition Ξ.

Partitioning Ξ: The family of circles already has a partition $\{\Xi_j\}_{j=1}^n$, where Ξ_j is the set of circles $\xi \in \Xi$ with center z_j, but this may not be sufficiently fine to satisfy Theorem 5. For example, it may be that there are circles centered at z_3 which lie between circles centered at z_2. To make a finer partition, for each Ξ_j, we will construct a graph G_j whose connected components are the uninterrupted circles. Being more precise, the vertex set of G_j is Ξ_j, and $\xi_1, \xi_2 \in \Xi_j$ are connected by an edge if, assuming $\xi_1 < \xi_2$ in the ordering on Ξ introduced in the last section, the radii satisfy $\text{rad}\xi_1 = 4\text{rad}\xi_2$, and the open annulus between ξ_1 and ξ_2 contains no other circles from Ξ. Let $\mathcal{E}_j$ denote the set of connected components in Ξ_j. Then each Ξ_j can be partitioned into $\{\Xi_e\}_{e \in \mathcal{E}_j}$, where Ξ_e is the vertex set of $e \in \mathcal{E}_j$. The circles in Ξ_e, for $e \in \mathcal{E}_j$, are all concentric circles with center z_j whose radii form a geometric sequence with common ratio $1/4$, and the closed annuli A_e are mutually disjoint.

By construction, we see that for any $j < k$ and $e \in \mathcal{E}_j$, the annulus A_e does not intersect D_k, which contains every A_e for $e \in \mathcal{E}_k$. Thus, if we let $\mathcal{E} = \cup_{j=1}^n \mathcal{E}_j$, Theorem 5 implies that we can bound (2.18) by

$$\mathbb{P}[\cap_{\xi \in \mathcal{E}}\{\gamma \cap \xi \neq \emptyset\}] \leq C_{|\mathcal{E}|} \prod_{j=1}^n \prod_{e \in \mathcal{E}_j} \frac{P_{y_j}(r_e)}{P_{y_j}(R_e)}, \tag{2.19}$$

where r_e =rad(max$\{\xi \in \mathcal{E}_e\}$) and R_e =rad(min$\{\xi \in \mathcal{E}_e\}$), where again the ordering for the maximum and minimum is in terms of crossing time. It suffices now to show that $|\mathcal{E}|$ is comparable to some value depending on n, and that $\prod_{e\in\mathcal{E}_j} \dfrac{P_{y_j}(r_e)}{P_{y_j(R_e)}}$ is comparable to $\dfrac{P_{y_j}(r_j)}{P_{y_j}(R_j)}$, where $R_j = l_j/4$.

Bounding $|\mathcal{E}|$: First, we observe a useful fact. If $1 \le j < k \le n$, observe that

$$\frac{\max_{z\in D_k}\{|z-z_j|\}}{\min_{z\in D_k}\{|z-z_j|\}} = \frac{|z_j - z_k| + l_k/4}{|z_j - z_k| - l_k/4} \le \frac{5}{3} < 4. \tag{2.20}$$

To check the inequality, note that $f(x) = (a+x)/(a-x)$ is an increasing function of x for any a, and $l_k \le |z_j - z_k|$ by the definition of l_k. Then $|z_j - z_k|$ cancels in the fraction, leaving 5/3. This inequality has the following two consequences:

a) If $1 \le j < k \le n$, then $|I_{j,k}| \le 1$.

b) If $1 \le j < k \le n$, then $\cup_{\xi\in\Xi_k}\xi \subset D_k$ intersects at most 2 annuli $\{\frac{l_j}{4^r} \le |z - z_j| \le \frac{l_j}{4^{r-1}}\}$.

These follow readily from the definitions and (2.20), but full details are provided in Section 2.7.

These consequences can be used to bound $|\mathcal{E}_j|$. Recall that $|\mathcal{E}_j|$ is the number of connected components made from Ξ_j in the graph G_j, which comes from the simple graph Ξ_j where sequential circles are connected. In order to create a new component, either a vertex or an edge has to be removed. Consequence *a*) says that for each $k > j$, the number of vertices removed is at most 1. Consequence *b*) says that for each $k > j$, the number of edges removed is at most 2. Thus, for each $1 \le j \le n$, the number of components created is bounded above by $\sum_{k>j}(1+2) = 3(n-j)$. It follows that $|\mathcal{E}_j| \leq 1 + 3(n-j)$. Summing over j yields

$$|\mathcal{E}| = \sum_{j=1}^{n} |\mathcal{E}_j| \leq \sum_{j=1}^{n} 1 + 3(n-j) = n + \frac{3n(n-1)}{2}.$$

This completes the proof that $C_{|\mathcal{E}|}$ can be bounded above by some constant $C_n < \infty$ depending only on n and κ.

Final estimate: To finish the proof, we need to show that

$$\prod_{e \in \mathcal{E}_j} \frac{P_{y_j}(r_e)}{P_{y_j}(R_e)} \leq C \frac{P_{y_j}(r_j)}{P_{y_j}(R_j)}.$$

We introduce some new notation. For any annulus $A = \{z : r \leq |z - z_0| \leq R\}$, where $y_0 = 1 - |z_0|$ for $z_0 \in \overline{\mathbb{D}}$ fixed, let $P(A) = P_{y_0}(r)/P_{y_0}(R)$. For $1 \leq s \leq h_j$, let $A_{j,s} = \{l_j/4^s \leq |z - z_j| \leq l_j/4^{s-1}\}$, and let $S_j = \{s \in \{1, \ldots, h_j\} : A_{j,s} \subset \cup_{e \in \Xi_j} A_e\}$. Using this new notation, what we want to show is that

$$\prod_{e \in \mathcal{E}_j} \frac{P_{y_j}(r_e)}{P_{y_j}(R_e)} = \prod_{s \in S_j} P(A_{j,s}) \leq C_n \frac{P_{y_j}(r_j)}{P_{y_j}(R_j)} = C_n \prod_{s=1}^{h_j} P(A_{j,s}).$$

By the estimate in Lemma 1, for each $e \in \mathcal{E}_j$, we have $P_{y_j}(r_e)/P_{y_j}(R_e) \geq (r_e/R_e)^\alpha = 4^{-\alpha}$. Let $S_j^c = \{1, \ldots, h_j\} \backslash S_j$. Then

$$\prod_{s \in S_j} P(A_{j,s}) = \frac{\prod_{s=1}^{h_j} P(A_{j,s})}{\prod_{s \in S_j^c} P(A_{j,s})} \leq \frac{P_{y_j}(r_j)}{P_{y_j}(R_j)} 4^{\alpha |S_j^c|}.$$

We need to estimate $|S_j^c|$. If $s \in S_j^c$, then either $s = 1$ or there is some $k > j$ so that $D_k \cap A_{j,s} \neq \emptyset$. By consequence $b)$ of (2.20), for each $k > j$, this happens at most twice for each $k > j$. Therefore, $|S_j^c| \le 1 + \sum_{k>j} 2 = 1 + 2(n-j)$. By equation (2.19),

$$\mathbb{P}[\cap_{\xi\in\Xi}\{\gamma\cap\xi \neq \emptyset\}] \le C_{(n+(3/2)n(n-1))} \prod_{j=1}^{n} \frac{P_{y_j}(r_j)}{P_{y_j}(R_j)} 4^{\alpha(1+2(n-j))} = C_n' 4^{\alpha(n^2)} \prod_{j=1}^{n} \frac{P_{y_j}(r_j)}{P_{y_j}(R_j)}.$$

□

Using Theorem 1, the analogous estimate for whole-plane SLE can be proven. The strategy will be to appeal to reversibility, reduce to the radial case, and then carefully use the Growth Theorem to show that the estimates from the radial case are comparable to the desired upper bound.

Proof of Theorem 2. By the reversibility of whole-plane SLE$_\kappa$ for $\kappa \le 8$ ([40],[21]), the desired probability is the same as the corresponding probability when γ^* is a whole-plane SLE$_\kappa$ from ∞ to 0, which we will assume for the rest of the proof. For any stopping time T, the path $t \mapsto \gamma^*(T+t)$, conditioned on $\gamma^*(-\infty, T)$, is a radial SLE$_\kappa$ path in $\mathbb{C}\backslash\gamma^*(-\infty, T]$ from $\gamma^*(T)$ to 0.

For $z_1, \ldots, z_n \in \mathbb{C}\backslash\{0\}$, let $z_0 = 0$, and define $R = \max\{|z_j - z_k| : 0 \le j, k \le n\}$. Assume without loss of generality that $r_k \le l_k$ for each k. Otherwise, the corresponding factor on the right hand side of the estimate is 1. Let $T = \inf\{t > -\infty : |\gamma^*(t)| = 4R\}$, which is finite a.s. since the radial SLE$_\kappa$ tip converges to the target point with probability 1. Let ϕ be a conformal map from $\mathbb{C}\backslash\gamma^*(-\infty, T]$ to $\mathbb{D}$ which fixes 0 and sends $\gamma^*(T)$ to 1. Then, conditioned on $\gamma^*(-\infty, T]$, the path $\gamma := \phi(\gamma^*(T + \cdot))$ is a radial SLE$_\kappa$ trace from 0 to 1. Therefore,

$$\mathbb{P}[\cap_{k=1}^{n}\{\text{dist}(z_k, \gamma^*) < r_k\} | \gamma^*(-\infty, T]] = \mathbb{P}[\cap_k\{\gamma \cap \phi(B(z_k, r_k)) \neq \emptyset\} | \gamma^*(-\infty, T]]. \quad (2.21)$$

By the Schwarz lemma, for each k, $z_k^T := \phi(z_k)$ satisfies $|z_k^T| \leq 1/4$, since $|z_k| \leq R$ and ϕ takes $4R\mathbb{D}$ into $\mathbb{D}$. Let $r_k^T = \max\{|\phi(z) - \phi(z_k)| : z \in \partial B(z_k, r_k)\}$, and define $l_k^T = \min\{|z_k - 1|, |z_k^T - z_j^T| : 0 \leq j < k\}$. Since $|z_k^T| \leq 1/4$, l_k^T is not $|z_k^T - 1|$ for any k, and $l_k^T \leq 1/2 < 3/4 \leq 1 - |z_k^T| := y_k^T$. Therefore,

$$\frac{P_{y_k^T}(r_k^T)}{P_{y_k^T}(l_k^T)} = \frac{r_k^T}{l_k^T}$$

for each $k = 1, \ldots, n$. Theorem 1 and (2.6) imply that (2.21) is bounded above by

$$\mathbb{P}[\cap_{k=1}^n \{\text{dist}(z_k^T, \gamma) \leq r_k^T\} | \gamma^*(-\infty, T]] \leq C \prod_{k=1}^n \left(\frac{r_k^T}{l_k^T} \right)^{2-d}. \tag{2.22}$$

It suffices to show that the quotients r_k^T / l_k^T are uniformly comparable to r_k / l_k for each k.

For each k, we will apply the Growth theorem on the ball $B(z_k, 3R)$, since ϕ is conformal on $B(0, 4R)$. There is a small subtlety, in that ϕ may not preserve the order of $z_1, \ldots, z_n$. That is, for a given k, if $k^* \in \{0, \ldots, k-1\}$ so that $l_k = |z_k - z_{k^*}|$, it may be that $l_k^T = |z_k^T - z_j^T|$ for some $j \neq k^*$. This will only affect our estimate by a constant, since if we define $\tilde{l}_k = |z_k^T - z_{k^*}^T|$, we have by (2.6)

$$\frac{\tilde{l}_k}{l_k^T} \leq \frac{|z_k - z_{k^*}|}{|z_k - z_j|} \cdot \frac{\left(1 + \frac{|z_k - z_j|}{3R}\right)^2}{\left(1 - \frac{|z_k - z_{k^*}|}{3R}\right)^2} \leq 1 \cdot \frac{(1 + (1/3))^2}{(1 - 1/3)^2} = C.$$

The second inequality follows by the choice of k^*, and since $|z_k - z_j| \leq R$ for all $0 \leq j, k \leq n$. Therefore, for each k, we have $r_k^T / l_k^T \leq C(r_k^T / \tilde{l}_k)$. By applying (2.6) to z_k and

$w \in \partial B(z_k, r_k)$ with $r_k^T = |\phi(z_k) - \phi(w)|$, and to $l_k = |z_k - z_{k^*}|$, we get

$$\frac{r_k^T}{\tilde{l}_k} \leq \frac{r_k}{l_k} \cdot \frac{\left(1 + \frac{l_k}{3R}\right)^2}{\left(1 - \frac{r_k}{3R}\right)^2} \leq \frac{r_k}{l_k} \frac{(1 + (1/3))^2}{(1 - 1/3)^2} = C\frac{r_k}{l_k}.$$

Therefore, we have $r_k^T / l_k^T \leq C r_k / l_k$ uniformly.

□

Proof of Theorem 3. First, we prove the theorem for radial SLE. Recall that we defined $\text{Cont}_d(E; r) = r^{d-2}\text{Area}\{z \in \mathbb{C} : \text{dist}(z, E) < r\}$ for any $r > 0$. Fixing $r > 0$, we have

$$\mathbb{E}[\text{Cont}_d(\gamma; r)^n] = \mathbb{E}[r^{n(d-2)} \left(\text{Area}\{z \in \mathbb{D} : \text{dist}(z, \gamma) < r\}\right)^n]$$

$$= r^{n(d-2)}\mathbb{E}\left[\left(\int_{\mathbb{D}} \mathbb{I}_{\{z:\text{dist}(z,\gamma)<r\}}(z) dA(z)\right)^n\right]$$

$$= \int_{\mathbb{D}^n} r^{n(d-2)}\mathbb{P}\left[\text{dist}(z_1, \gamma) < r, \ldots, \text{dist}(z_n, \gamma) < r\right] dA(z_1) \ldots dA(z_n).$$

The last equality holds by Fubini's theorem, since all of the terms are positive. By Theorem 1, we have

$$r^{n(d-2)}\mathbb{P}[\text{dist}(z_k, \gamma) < r, \forall k = 1, \ldots, n] \leq r^{n(d-2)} C_n \prod_{k=1}^{n} \frac{P_{y_k}(r \wedge l_k)}{P_{y_k}(l_k)}.$$

If $r \leq l_{i_1}, \ldots, l_{i_k}$ and r is greater than the rest, then the above expression is

$$= r^{n(d-2)} C_n \prod_{j=1}^{k} \frac{P_{y_{i_j}}(r)}{P_{y_{i_j}}(l_{i_j})} \leq C_n r^{n(d-2)} \prod_{j=1}^{k} \frac{r^{2-d}}{l_{i_j}^{2-d}} \leq C_n \prod_{j=1}^{n} l_j^{d-2}.$$

To explain the last inequality, there are two cases. If $r < l_j$, the numerator cancels with one of the r^{d-2}'s outside of the product. If $r > l_j$, then one of the r^{d-2}'s from the outside product is added to the product, and is smaller than l_j^{d-2}.

Thus, if $f(z_1, \ldots, z_n) = \prod_{k=1}^{n} \min\{|z_k|, |z_k - 1|, |z_k - z_1|, \ldots, |z_k - z_{k-1}|\}^{d-2}$ and $r > 0$, then

$$\mathbb{E}[\text{Cont}_d(\gamma; r)^n] \leq C_n \int_{\mathbb{D}^n} f(z_1, \ldots, z_n) dA(z_1) \ldots dA(z_n).$$

By Fatou's Lemma,

$$\mathbb{E}[\underline{\text{Cont}}_d(\gamma)^n] \leq \liminf_{r \to 0} \mathbb{E}[\text{Cont}_d(\gamma; r)^n] \leq C_n \int_{\mathbb{D}^n} f(z_1, \ldots, z_n) dA(z_1) \ldots dA(z_n).$$

If we can show that f is integrable, then we are done.

Fix $k = 1, \ldots, n$, and let $z_1, \ldots z_{k-1} \in \mathbb{D}$ be arbitrary. Let $z_{-1} = 0$ and $z_0 = 1$. Then for any k,

$$\int_{\mathbb{D}} \min\{|z_k|, |z_k - 1|, |z_k - z_1|, \ldots, |z_k - z_{k-1}|\}^{d-2} dA(z_k) \leq \sum_{j=-1}^{k-1} \int_{\mathbb{D}} |z_k - z_j|^{d-2} dA(z_k).$$

Note that for each j, $\mathbb{D} - z_j \subset 2\mathbb{D}$, and so the above inequality satisfies

$$\leq (k+1) \int_{2\mathbb{D}} |z|^{d-2} dA(z) = (k+1) 2\pi \int_0^2 r^{d-1} dr < \infty.$$

By Fubini's theorem, it follows that $\int_{\mathbb{D}^n} f dA \ldots dA < \infty$, and so we are done in the radial case.

The proof for whole-plane SLE$_\kappa$ follows similarly. We start off by writing

$$\mathbb{E}[\text{Cont}_d(\gamma^* \cap D); r)^n] = \int_{D^n} r^{n(d-2)} \mathbb{P}[\text{dist}(z_1, \gamma^*) < r, \ldots, \text{dist}(z_n, \gamma^*) < r] dA(z_1) \ldots dA(z_n).$$

The function $f(z_1, \ldots, z_n)$ excludes the distance from 1 in the whole-plane case. That is,

$$f(z_1, \ldots, z_n) = \prod_{k=1}^{n} \min\{|z_k|, |z_k - z_1|, \ldots, |z_k - z_{k-1}|\}^{d-2}.$$

Then

$$\mathbb{E}[\mathrm{Cont}_d(\gamma \cap D)^n] \leq \int_{D^n} f(z_1, \ldots, z_n) dA(z_1) \ldots dA(z_n).$$

If $D \subset R\mathbb{D}$ for $R < \infty$, then $D - z_j \subset 2R\mathbb{D}$ for each $z_j \in D$, so we can perform the same bound as in the radial case, except integrating over $2R\mathbb{D}$ rather than $2\mathbb{D}$. $\square$

2.7 Technical lemmas

The following lemma is proven in [23], and serves as a technical lemma to prove that a certain random variable is a stopping time.

Lemma 11. *Let $D \subset \mathbb{C}$ be a simply connected domain, and let ρ be a crosscut in D. Let $w_0, w_1,$ and w_∞ be connected subsets or prime ends of D such that $D \backslash \rho$ is a neighborhood of all of them in D. Suppose that ρ disconnects w_0 from w_∞ in D. Let $\gamma(t), 0 \leq t \leq T$, be a continuous curve in $\overline{D}$ with $\gamma(0) \in \partial D$. Suppose for $0 \leq t < T$, $D \backslash \gamma[0, t]$ is a neighborhood of $w_0, w_1,$ and w_∞ in D, and $w_0, w_1 \subset D_t := D(\gamma[0, t]; w_\infty)$. For $0 \leq t < T$, let ρ_t be the first subcrosscut of ρ in D_t that disconnects w_0 from w_∞ as given by Lemma 2. For $0 \leq t < T$, let $f(t) = 1$ if $w_1 \in D_t(\rho_t; w_\infty)$, and $f(t) = 0$ if $w_1 \in D_t^*(\rho_t; w_\infty)$. Then f is right continuous on $[0, T)$, and left continuous at those $t_0 \in (0, T)$ such that $\gamma(t_0)$ is not an end point of ρ_{t_0}.*

We can provide the complete rigorous details for some of the steps in the main theorems, whose arguments would clutter the presentation of the proofs.

Proof of (2.10) in Theorem 4: First, I need to show that, in the event E, if $t \le \tau_j$, then $\mathbb{D}^*(\xi_j)$ is either contained in $D_t(\rho_t)$ or $D_t^*(\rho_t)$ for $j = 1, \dots, m$. If $t \le \tau_j$, and E occurs, then ξ_j cannot be swallowed by the curve, since γ still needs to intersect ξ_j. Therefore, it must be that $\xi_j \subset D_t$. Moreover, since $\mathbb{D}(\xi_j)$ is disjoint from the curve ρ, it cannot be that the subcrosscut ρ_t goes through $\mathbb{D}(\xi_j)$. It follows that either $\mathbb{D}(\xi_j) \subset D_t(\rho_t)$ or $\mathbb{D}(\xi_j) \subset D_t^*(\rho_t)$. Similarly, in the event E, it must be that $\xi_0' \subset D_{\tau_m}$.

Let I be totally ordered by $(0,1) < (1,1) < (1,2) < (2,2) < \cdots < (m,m) < (m, m+1)$. Define a family of events by

$$E_i = E \backslash \cup_{i' > i} A_{i'}.$$

Using a reverse induction argument, we can show that

$$E_i \subset \{\mathbb{D}^*(\xi_{i_2}) \subset D^*_{\tau_{i_1}}(\rho_{\tau_{i_1}})\}$$

for all $i = (i_1, i_2) \neq (m, m+1)$. Note that $E_{(m,m+1)} = \emptyset$. For the base case of the induction, we consider

$$E_{(m,m)} = E \backslash A_{(m,m+1)} = E \cap \Big(\{T \le \tau_m\} \cup \{D_{\tau_m}(\xi_m) \not\subset D_{\tau_m(\rho_{\tau_m})}\} \Big) .$$

Note that $E \cap \{T \le \tau_m\} = \emptyset$, and the preceding paragraph shows that $E \cap \{D_{\tau_m}(\xi_m) \not\subset D_{\tau_m}(\rho_{\tau_m})\}$ is contained in $\{D_{\tau_m}(\xi_m) \subset D^*_{\tau_m}(\rho_{\tau_m})\}$. This proves the base case.

To complete the induction, assume that $E_{(j,j)} \subset \{\mathbb{D}^*(\xi_j) \subset D^*_{\tau_j}(\rho_{\tau_j})\}$. Then

$$E_{(j-1,j)} = E_{(j,j)} \cap A^c_{(j,j)} \subset$$

$$\subset \{\mathbb{D}^*(\xi_j) \subset D^*_{\tau_j}(\rho_{\tau_j})\} \cap \left[\{\mathbb{D}^*(\xi_j) \subset D^*_{\tau_{j-1}}(\rho_{\tau_{j-1}})\} \cup \{\mathbb{D}^*(\xi_j) \subset D_{\tau_j}(\rho_{\tau_j})\}\right].$$

The intersection with the right hand side is empty, and so we get that $E_{(j-1,j)}$ is contained in $\{\mathbb{D}^*(\xi_j) \subset D^*_{\tau_{j-1}}(\rho_{\tau_{j-1}})\}$. This completes the proof that the (j,j) case implies the $(j-1,j)$ case. The argument for $(j,j+1)$ implies (j,j) is identical, and so the the induction is complete.

The final step of the above induction sequence implies that

$$E \setminus \cup_{i>(0,1)} A_i = E_{(0,1)} \subset \{\mathbb{D}^*(\xi_1) \subset D^*_{\tau_0}(\rho_{\tau_0})\} \subset A_{(0,1)},$$

from which claim (2.10) follows.

□

Proof of (2.13) in Theorem 4. Let $x = R_j/r_j$, then $N = \lceil \ln(x) \rceil$. We are trying to show

$$x^{\alpha/N} \frac{1 - x^{-\alpha/2}}{1 - x^{-\alpha/2N}} \leq \frac{e^{\alpha}}{1 - e^{-\alpha/4}}.$$

If $1 < x \leq e$, then $N = 1$, and the left hand side is equal to

$$x^{\alpha} \frac{1 - x^{-\alpha/2}}{1 - x^{-\alpha/2}} = x^{\alpha} \leq e^{\alpha} \leq \frac{e^{\alpha}}{1 - e^{-\alpha/4}}.$$

Suppose that $x > e$, in which case $N \leq \ln(x) + 1$, and so $1 - x^{-\alpha/2(\ln(x)+1)} \leq 1 - x^{-\alpha/2N}$. The function

$$1 - x^{-\alpha/2(\ln(x)+1)} = 1 - e^{-(\alpha/2)\left(\frac{\ln(x)}{\ln(x)+1}\right)}$$

is increasing, and so is bounded from below for $x \in (e, \infty)$ at $x = e$, which yields the bound

$1 - e^{-\alpha/4}$. Therefore, we get

$$\frac{1}{1 - x^{-\alpha/2N}} \leq \frac{1}{1 - e^{-\alpha/4}}.$$

Since $\ln(x) \leq N \neq 0$, we get that $x^{1/N} \leq x^{1/\ln(x)} = e$, and so $x^{\alpha/N} \leq e^{\alpha}$. Combining this with the above inequality completes the proof.

□

Proof of (2.15) in Theorem 5. In the total order on Ξ_e, $\max\{\Xi_(e, j)\} < \min\{\Xi_{(e,j+1)}\}$, and there are no circles between them by construction. By the assumption that sequential radii in Ξ_e form a geometric sequence, we know that $R_{\max\{\Xi_{(e,j)}\}} = 4R_{\min\{\Xi_{(e,j+1)}\}}$ Therefore, expanding the product gives

$$\prod_{j=0}^{M_e} P_{(e,j)} = \frac{P_e(r_e)}{P_e(R_e)} \prod_{j=0}^{M_e-1} \frac{P_{ye}(R_{\max\{\Xi_{(e,j)}\}})}{P_{ye}(R_{\{\min\{\Xi_{(e,j+1)}\}\}})} \leq P_e \prod_{j=0}^{M_e-1} \left(\frac{R_{\max\{\Xi_{(e,j)}\}}}{R_{\min\{\Xi_{(e,j+1)}\}}} \right)^{\alpha} = P_e 4^{\alpha M_e},$$

which proves claim (2.15). Note that the above inequality follows from applying Lemma 1 to each term in the product.

□

Proof of consequences a) and b) in Theorem 1: To prove a), suppose that it is false. Then there are distinct $\xi_j^1, \xi_j^2 \in I_{j,k}$ with $\dfrac{\mathrm{rad}(\xi_j^1)}{\mathrm{rad}(\xi_j^2)} \geq 4$. By both being in $I_{j,k}$, there is a $z \in \xi_j^1 \cap D_k$ and a $w \in \xi_j^2 \cap D_k$. Then $|z - z_j|/|w - z_j| \geq 4$. Taking the maximum over all $z \in D_k$ then implies that $\max_{z \in D_k}\{|z - z_j|\}/|w - z_j| \geq 4$. Taking the minimum of this in $w \in D_k$ contradicts (2.20).

The proof of b) follows similarly. Suppose b) is false. Then there exist $2 \leq r_1 < r_2 <$

$r_3 \le h_j$ and $w_1, w_2, w_3 \in D_k$ such that

$$\frac{l_j}{4^{r_3}} \le |w_3 - z_j| \le \frac{l_j}{4^{r_3-1}} \le \frac{l_j}{4^{r_2}} \le |w_2 - z_j| \le \frac{l_j}{4^{r_2-1}} \le \frac{l_j}{4^{r_1}} \le |w_1 - z_j| \le \frac{l_j}{4^{r_1-1}}.$$

From this series of inequalities, we get

$$\frac{|w_1 - z_j|}{|w_3 - z_j|} \ge \frac{l_j/4^{r_1}}{l_j/4^{r_3-1}} = 4^{r_3-1-r_1} \ge 4.$$

This contradicts (2.20) in the same way as in a), proving b).

□

Chapter 3

Decomposition of backward SLE in the capacity parametrization

3.1 Introduction

In some cases, the reverse flow of the Loewner equation is easier to study and can be used to answer questions about the regular, or forward, SLE process. Analysis of the reverse flow was used to show existence of the SLE_κ trace for $\kappa \neq 8$ [27]. In [8], a multifractal analysis is used to study moments for the backward SLE flow, which is used to provide a new proof of the Hausdorff dimension of an SLE path. The reversibility of the welding [26] has been used to study ergodic properties of the tip of a forward SLE_κ in [39]. BSLE and the conformal welding have been coupled with the Gaussian free field and what is called the Liouville quantum zipper [33], where the welding of the real line onto the backward SLE_κ traces is seen as the conformal image of gluing random surfaces together. Recently [4], this connection to the Gaussian free field was used to provide a new proof of the existence of the forward SLE trace.

In this chapter, we construct a family of functions $G^{a,b}(x, y)$ with which we use the Girsanov theorem to construct a new measure $\mathbb{P}^{a,b}_{x,y}$ under which $\phi(x) = y$ almost surely. That is, we condition the process so that the graph of the welding function passes through the point (x, y). In order to do this, we need the notion of a BSLE process with force points, which are introduced in [26].

A process (f_t) which solves the chordal Loewner equation with driving function λ which satisfies the SDE:

$$d\lambda_t = \sqrt{\kappa} dB_t + \frac{adt}{f_t(q_1) - \lambda_t} + \frac{bdt}{f_t(q_2) - \lambda_t}, \quad \lambda_0 = x_0$$

is called a $\mathrm{BSLE}_\kappa(a, b)$ process started from $(x_0; q_1, q_2)$, where $a, b \in \mathbb{R}$ are weights at the force points $q_1, q_2 \in \mathbb{R}$. This definition can be extended to include more than two force points, which can also be placed in the interior of $\mathbb{H}$, but this is the amount of generality we will need. This process can be constructed by applying the Girsanov theorem, which will be discussed in a later section, and therefore for $\kappa \leq 4$, the backward SLE process with force points also generates a conformal welding on $\mathbb{R}^+$.

For any set U contained in the first quadrant, we then average the measures of paths passing through U by integrating $P^{a,b}_{x,y}$ against $\mathbb{I}_U(x, y)G^{a,b}(x, y)$. For $\kappa \in (0, 4)$, and for $a = b = -4$, we then prove a decomposition theorem relating this particular case to the amount of capacity time that the graph of ϕ spends in U. In order to establish these results, we review the framework established in [37] to study processes with a random lifetime.

For all $\kappa \leq 4$, and for $a = b = -\kappa - 4$, we also show that the Green's function $G^{a,b}(x, y)$ can be realized as the normalized probability the two points x, y are welded together. This is similar to how the Green's function for the forward SLE_κ process captures the normalized probability that the path passes through a given point.

3.2 Processes with a random lifetime

For this paper, we will need to review the framework introduced in [37] to study stochastic processes with a random lifetime. We will need to introduce the notation which will be used in this paper, and several propositions will be stated without proof. For complete details, the reader can refer to [37].

Define the space

$$\Sigma = \cup_{0<T\leq\infty} C[0,T),$$

where $C[0,T)$ is the set of real valued functions which are continuous on $[0,T)$. Then the law of Brownian motion is a probability measure on Σ. We will have to define a few operations on this space.

- *Lifetime*: First, the lifetime of a function $f \in \Sigma$ is given by $T_f = \sup\{t > 0 : f \in C[0,t)\}$.

- *Killing*: For $0 < \tau \leq \infty$, define $\mathcal{K}_\tau : \Sigma \to \Sigma$ by $\mathcal{K}_\tau(f) = f_{|[0,\tau_f)}$, where $\tau_f = \min\{\tau, T_f\}$.

- *Continuing*: This is an operation which glues two functions together. Define subspaces of Σ by $\Sigma^{\oplus} = \{f \in \Sigma : T_f < \infty, f(T^-) := \lim_{t\to T_f^-} f(t) \in \mathbb{R}\}$ and $\Sigma_{\oplus} = \{f \in \Sigma : f(0) = 0\}$. Then we define $\oplus : \Sigma^{\oplus} \times \Sigma_{\oplus} \to \Sigma$ by

$$f \oplus g(t) = \begin{cases} f(t), & 0 \leq t < T_f \\ f(T_f^-) + g(t - T_f), & T_f \leq t < T_f + T_g \end{cases}.$$

- *Time marked continuation*: Define $\hat{\oplus} : \Sigma^{\oplus} \times \Sigma_{\oplus} \to \Sigma \times [0,\infty)$ by $f\hat{\oplus}g = (f \oplus g, T_f)$.

Then $\hat{\oplus}$ records the information of where the first function ends and the second begins.

Probability measures on Σ will be the laws of the random driving function for the backward Loewner process. To talk about measures, we need to create a sigma algebra on Σ. First, for $t < \infty$, define a filtration by

$$\mathcal{F}_t = \sigma\left(\{f \in \Sigma : s < T_f, f(s) \in U\}, s \leq t, U \subset \mathbb{R} \text{ is measurable}\right).$$

Then define $\mathcal{F} = \vee_{0 \leq t < \infty} \mathcal{F}_t$. Then the operations above are all measurable on $(\Sigma, \mathcal{F})$. Given any two measures μ, ν on $(\Sigma, \mathcal{F})$ let $\mu \otimes \nu$ denote the product measure. We then define the following measures:

- $\mu \oplus \nu$ is defined by the pushforward measure $\oplus_*(\mu \otimes \nu)$ on Σ.
- $\mu \hat{\oplus} \nu$ is the pushforward measure $\hat{\oplus}_*(\mu \otimes \nu)$ defined on $\Sigma \times [0, \infty)$.

We will also have to work with random measures on different spaces, which are called probability kernels. More precisely, suppose $(U, \mathcal{U})$ and $(V, \mathcal{V})$ are measurable spaces. A kernel from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ is a map $\nu : (U, \mathcal{V}) \to [0, \infty)$ such for each $u \in U$, $\nu(u, \cdot) : \mathcal{V} \to [0, \infty)$ is a measure and for each $E \in \mathcal{V}$, the function $\nu(\cdot, E) : U \to [0, \infty)$ is $\mathcal{U}$-measurable. If μ is a measure on $(U, \mathcal{U})$, then ν is called a μ-kernel if it is a kernel on the μ-completion of $(U, \mathcal{U})$. We say that ν is a finite μ-kernel if $\nu(u, V) < \infty$ for μ-a.s. $u \in \mathcal{U}$, and we say ν is a σ-finite μ-kernel if $V = \cup_{n=1}^{\infty} F_n$ such that for each n, for μ-a.s. $u \in U$, we have $\nu(u, F_n) < \infty$.

Combining kernels with measures, we have the following operations for measures:

- If μ is a σ-finite measure on $(U, \mathcal{U})$ and ν is a σ-finite μ-kernel from $(U, \mathcal{U})$ to $(V, \mathcal{V})$, then we define $\mu \otimes \nu$ on $U \times V$ by

$$\mu \otimes \nu(E \times F) = \int_E \nu(u, F) d\mu(u).$$

In this case, $\mu \cdot \nu$ is the marginal measure on V given by $\mu \cdot \nu(F) = \mu \otimes \nu(U \times F)$.

- If ν is a σ-finite measure on V and μ is a σ-finite ν-kernel from $(V, \mathcal{V})$ to $(U, \mathcal{U})$, we define $\mu \overleftarrow{\otimes} \nu$ on $U \times V$ by

$$\mu \overleftarrow{\otimes} \nu(E \times F) = \int_F \mu(v, E) d\nu(v).$$

- If ν is a σ-finite μ-kernel from Σ to $(0, \infty)$, define a measure $\mathcal{K}_\nu(\mu)$ on Σ to be the pushforward measure of $\mu \otimes \nu$ under the map $\mathcal{K} : \Sigma \times (0, \infty) \to \Sigma$ given by $(f, r) \to \mathcal{K}_r(f)$.

To study the law of a random process over time, the notion of absolute continuity can be extended to what we call local absolute continuity. For $0 \leq t < \infty$, let $\Sigma_t = \{f \in \Sigma : T_f > t\}$ be the set of functions defined at least until time t. Note that $\cap_{t>0} \Sigma_t = C[0, \infty)$. For each $t > 0$, let $\mathcal{F}_t \cap \Sigma_t$ denote the restriction of $\mathcal{F}_t$ onto the subspace Σ_t. If μ, ν are measures on $(\Sigma, \mathcal{F})$, then we say that ν is locally absolutely continuous with respect to μ if, for every $t > 0$, $\nu|_{\mathcal{F}_t \cap \Sigma_t}$ is absolutely continuous with respect to $\mu|_{\mathcal{F}_t \cap \Sigma_t}$. We will use the notation $\nu \ll \mu$ to mean absolute continuity and $\nu \lhd \mu$ to mean local absolute continuity.

The following propositions are in [37], and will be stated without proof:

Proposition 2. *Let μ be a measure on $(\Sigma, \mathcal{F})$ which is σ-finite on $\mathcal{F}_0$. Let $(\Upsilon, \mathcal{G})$ be a measurable space. Let $\nu : \Upsilon \times \mathcal{F} \to [0, \infty]$ be such that for every $v \in \Upsilon$, $\nu(v, \cdot)$ is a finite measure on $\mathcal{F}$ that is locally absolutely continuous with respect to μ. Moreover, suppose that* *the local Radon Nikodym derivatives are equal to* $(M_t(v,\cdot))_{t>0}$*, where* $M_t : \Upsilon \times \Sigma \to [0,\infty)$ *is* $\mathcal{G} \times \mathcal{F}_t$ *measurable for every* $t \geq 0$*. The* ν *is a kernel from* $(\Upsilon, \mathcal{G})$ *to* $(\Sigma, \mathcal{F})$*. Moreover, if* ξ *is a* σ*-finite measure on* $(\Upsilon, \mathcal{G})$ *such that* μ*-a.s.,* $\int_\Upsilon M_t(v,\cdot)d\xi(v) < \infty$*, then* $\xi \cdot \nu \lhd \mu$*, and the local Radon-Nikodym derivatives are* $\int_\Upsilon M_t(v,\cdot)d\xi(v)$ *for* $0 \leq t < \infty$*.*

Proposition 3. *Let* μ *be a probability measure on* $(\Sigma, \mathcal{F})$*. Let* ξ *be a* μ*-kernel from* $(\Sigma, \mathcal{F})$ *to* $(0,\infty)$ *that satisfies* $\mathbb{E}_\mu[|\xi|] < \infty$*. Then* $\mathcal{K}_\xi(\mu) \lhd \mu$*, and the local Radon-Nikodym derivatives are* $\mathbb{E}_\mu[\xi((t,\infty))|\mathcal{F}_t]$ *for* $0 \leq t < \infty$*.*

Throughout this chapter, we will use $\mathbb{P}_\kappa$ to denote the measure on Σ which gives the law of $(\sqrt{\kappa}B_t)_{t>0}$, where $(B_t)_{t>0}$ is a standard one dimensional Brownian motion. Then $\mathbb{P}_\kappa$ is supported on $C[0,\infty) \cap \Sigma_\oplus$. We will use $\mathbb{E}_\kappa$ to denote expectation under the measure $\mathbb{P}_\kappa$, and $\mathcal{F}_t^B$ to be the completion of $\mathcal{F}_t$ under $\mathbb{P}_\kappa$ for $0 \leq t$.

We state two more propositions from [37] without proof. The first extends the Girsanov theorem to a statement about local absolute continuity, and the second extends the strong Markov property of Brownian motion.

Proposition 4. *Suppose that* $(X_t)_{0\leq t<T}$ *satisfies the* $\mathcal{F}_t^B$*-adapted SDE:*

$$dX_t = \sqrt{\kappa}dB_t + \sigma_t dt, \quad 0 \leq t < T, \text{ and } X_0 = 0,$$

where T *is a positive* $\mathcal{F}_t^B$*-stopping time. Let* $\mathbb{P}^{\kappa,\sigma}$ *denote the law of* X*. Then* $\mathbb{P}^{\kappa,\sigma} \lhd \mathbb{P}_\kappa$*. Moreover, if* $(M_t)_{0\leq<T}$ *is a* $\mathcal{F}_t^B$*-adapted continuous local martingale that satisfies the SDE*

$$dM_t = M_t \frac{\sigma_t}{\sqrt{\kappa}} dB_t, \quad 0 \leq t < T_0, \text{ and } M_0 = 1, \tag{3.1}$$

then

$$\frac{d\mathbb{P}^{\kappa,\sigma}|_{\mathcal{F}_t \cap \Sigma_t}}{d\mathbb{P}_\kappa|_{\mathcal{F}_t \cap \Sigma_t}} = \mathbb{I}_{\{T>t\}} M_t, \quad 0 \le t < \infty. \tag{3.2}$$

If $(\theta)_t$ is an $\mathcal{F}_t^B$-adapted increasing process, it induces a kernel from Σ to $[0, \infty)$ defined by $d\theta.(f)$, which is the random measure induced by the monotonic function $\theta_t(f)$ on $t \in [0, \infty)$.

Proposition 5. *Let $(\theta_t)_{0 \le t < \infty}$ be a right continuous increasing $\mathcal{F}_t^B$-adapted process that satisfies $\theta_0 = \theta_{0+} = 0$ and $\mathbb{E}_\kappa[\theta_\infty] < \infty$. Then*

$$\mathcal{K}_{d\theta}(\mathbb{P}_\kappa) \hat{\oplus} \mathbb{P}_\kappa = \mathbb{P}_\kappa \otimes d\theta. \tag{3.3}$$

Thus, $\mathcal{K}_{d\theta} \oplus \mathbb{P}_\kappa \ll \mathbb{P}_\kappa$, and θ_∞ is the Radon-Nikodym derivative.

Define $\Sigma^{\mathcal{L}}$ to be the set of $\lambda \in \Sigma$ which generate a Loewner trace. By the existence of the SLE$_\kappa$ trace for any $\kappa > 0$, we know that $\mathbb{P}_\kappa$ is supported in $\Sigma^{\mathcal{L}}$. Let $\Sigma^{\mathbb{C}} = \cup_{0 < t \le \infty} C_{\mathbb{C}}[0, t)$, where $C_{\mathbb{C}}[0, t)$ is the set of complex valued continuous functions with domain $[0, t)$. Define the Loewner operator $\mathcal{L} : \Sigma^{\mathcal{L}} \to \Sigma^{\mathbb{C}}$ to be the map which takes a driving function to its chordal trace. Then the pushforward measure $\mathcal{L}_*(\mathbb{P}_\kappa)$ is the law of the SLE$_\kappa$ curve. More generally, if μ is any law supported in $\Sigma^{\mathcal{L}}$, then $\mathcal{L}_*(\mu)$ is the law of the associated Loewner curve. We can also define the extended Loewner map $\hat{\mathcal{L}} : \{(\lambda, t) : \lambda \in \Sigma^{\mathcal{L}}, t < T_\lambda\} \to \Sigma^{\mathbb{C}} \times \mathbb{H}$ by $\hat{\mathcal{L}}(\lambda, t) = (\mathcal{L}(\lambda), \mathcal{L}(\lambda)(t))$. If ν is any measure on $\{(\lambda, t) : \lambda \in \Sigma^{\mathcal{L}}, t < T_\lambda\}$, then $\hat{\mathcal{L}}$ generates a pushforward measure $\hat{\mathcal{L}}_*(\mu)$ on $\Sigma^{\mathbb{C}} \times \mathbb{H}$. This approach is taken in [37] to study chordal SLE$_\kappa$ in both the capacity parametrization and natural parametrization.

Since our object of study is the conformal welding generated by the driving function, we will consider a different operator. Let $\Sigma^{\mathcal{W}}$ denote the functions in $\Sigma^{\mathcal{L}}$ which generate a continuous conformal welding under the backward Loewner equation. For $\kappa \le 4$, since the

Loewner traces are simple, it follows that $\mathbb{P}_\kappa$ is supported in $\Sigma^{\mathcal{W}}$. We will use Q_1 to denote the first quadrant $[0,\infty)\times[0,\infty)$. The welding operator $\mathcal{W}:\Sigma^{\mathcal{W}}\to\Sigma^{\mathbb{C}}$ can be defined by $\mathcal{W}(\lambda)=$ the welding curve Φ generated by λ. We can also define the extended welding operator $\hat{\mathcal{W}}(\lambda,t)=(\Phi,\Phi(t))\in\Sigma^{\mathbb{C}}\times Q_1$. Then any law μ supported on $\Sigma^{\mathcal{W}}$ generates a pushforward measure $\mathcal{W}_*(\mu)$ on $\Sigma^{\mathbb{C}}$, and any law ν on $\{(\lambda,t):\lambda\in\Sigma^{\mathcal{W}},t<T_\lambda\}$ generates a pushforward measure $\hat{\mathcal{W}}_*(\nu)$ on $\Sigma^{\mathbb{C}}\times Q_1$.

Lemma 12. *If $\kappa\in(0,4)$, then the welding operator $\mathcal{W}$ is injective on the support of $\mathbb{P}_\kappa$.*

Proof. Suppose the backward Loewner processes generated by $\lambda,\tilde{\lambda}$ are $(f_t)_{t\geq 0}$ and $(\tilde{f}_t)_{t\geq 0}$ respectively, and assume they generate the same welding curve Φ. For a fixed $t>0$, define $F_t(z)=f_t(z)-\lambda_t$ and $\tilde{F}_t(z)=\tilde{f}_t(z)-\tilde{\lambda}_t$, and let $G_t(z)=\tilde{F}_t\circ F_t^{-1}(z)$, which maps $\mathbb{H}\backslash\{\gamma_t-\lambda_t\}$ onto $\mathbb{H}\backslash\{\tilde{\gamma}_t-\tilde{\lambda}_t\}$, where $(\gamma_t)_{t\geq 0}$ and $(\tilde{\gamma}_t)_{t\geq 0}$ are the families of backward Loewner traces driven by λ and $\tilde{\lambda}$ respectively.

For $\kappa<4$, since $\gamma_t-\lambda_t$ has the same law under $\mathbb{P}_\kappa$ as a forward SLE$_\kappa$ trace up to time t, it was proven in [27] that $\gamma_t-\lambda_t$ is the boundary of a Hölder domain. In [6], it was shown that this condition is enough to make the image of the trace conformally removable. That is, any conformal map defined on $\mathbb{H}\backslash\{\gamma_t-\lambda_t\}$ which can be extended continuously to the boundary can be conformally extended to $\mathbb{H}$.

Suppose $\Phi(t)=(b_t,a_t)$, and assume $-a_t\leq -y<0<x\leq b_t$ with $\phi(x)=y$. Note that since both processes have the same welding process, they also have the same welding function ϕ. Therefore, $f_t(x)=f_t(-y)$ and $\tilde{f}_t(x)=\tilde{f}_t(-y)$, and so G_t extends continuously from $\gamma_t-\lambda_t$ to $\tilde{\gamma}_t-\tilde{\lambda}_t$. Conformal removability then implies that G_t extends to a conformal

map from $\mathbb{H}$ onto $\mathbb{H}$, and so it must be that

$$G_t(z) = \frac{az+b}{cz+d}$$

for some $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$.

Assuming $\Phi(t) = (b_t, a_t)$, we get $f_t(-a_t) = f_t(b_t) = \lambda_t$ and $\tilde{f}_t(-a_t) = \tilde{f}_t(b_t) = \tilde{\lambda}_t$. This implies that $G_t(0) = 0$, and so $b = 0$. Also, f_t and $\tilde{f}_t$ each fix ∞, and so $c = 0$. Therefore, $G_t(z) = az$ for some constant a. Since $G_t(\gamma_t - \lambda_t) = \tilde{\gamma}_t - \tilde{\lambda}_t$, the scaling rule for half plane capacity implies that $2t = \text{hcap}(\gamma_t - \lambda_t) = |a|^2 \text{hcap}(\tilde{\gamma}_t - \tilde{\lambda}_t) = 2t$, and so $a = 1$ or $a = -1$. Since f_t and $\tilde{f}_t$ both have positive derivative at ∞, we have $a = 1$. Therefore, $\gamma_t - \lambda_t = \tilde{\gamma}_t - \tilde{\lambda}_t$ for each $t > 0$. Since the reverse flow is uniquely determined by the driving function, we get that $\lambda = \tilde{\lambda}$.

□

3.3 Glossary of notation for Ito's formula calculations

We define some terms which will be used for Ito's formula calculations for this chapter in multiple sections.

- Fix $x, y > 0$, and let λ_t be a driving function for the backward Loewner equation (1.4). Let $(f_t)_{t\geq 0}$ be the solution of (1.4) driven by λ. For $z \in \mathbb{R}$, let τ_z be the lifetime of (1.4) started from z. If $x, y > 0$, let $\tau_{x,y} = \min\{\tau_x, \tau_{-y}\}$.
- For $t \leq \tau_x$, define $X_t = X_t(x) = f_t(x) - \lambda_t$.
- For $t \leq \tau_{-y}$, define $Y_t = Y_t(y) = \lambda_t - f_t(-y)$. Note that X_t and Y_t are defined specifically to be nonnegative for all $t \leq \tau_{x,y}$.

- These two formulas imply that

$$\frac{d(X_t + Y_t)}{X_t + Y_t} = \frac{-2}{X_t Y_t} dt. \tag{3.4}$$

- For $t < \tau_{x,y}$, let $W_t = W_t(x, y) = \dfrac{X_t - Y_t}{X_t + Y_t}$. Then $W_t \in (-1, 1)$ for each $t < \tau_{x,y}$. If $\tau_x \neq \tau_{-y}$, then $W_{\tau_{x,y}}$ is well defined and is either 1 or -1.

- For $t \leq \tau_{x,y}$, define $u_t = u_t(x, y) = \int_0^t \frac{\kappa}{X_r Y_r} dr$. By equation (3.4), this is equal to

$$u(t) = -\frac{\kappa}{2} \ln \left(\frac{X_t + Y_t}{x + y} \right). \tag{3.5}$$

  The process u_t will be used to define a random time change.

- For $Z \in \{X, Y, W\}$, we will use $\hat{Z}_t = Z_{u^{-1}(t)}$.

3.4 Generalized Green's functions

In this section, we will be doing Ito's formula calculations using the framework in Section 3.3. For the first lemma, $\lambda_t = \sqrt{\kappa} B_t$ is the driving function for BSLE$_\kappa$. We will then do calculations when λ_t is the driving function for the BSLE$_\kappa(a, b)$ process.

For $\kappa \leq 4$, let (f_t) be the standard backward SLE$_\kappa$ process driven by $\lambda_t = \sqrt{\kappa} B_t$. For $x, y > 0$ define $X_t = X_t(x) = f_t(x) - \lambda_t$ and $Y_t = Y_t(-y) = \lambda_t - f_t(-y)$ as in Section 3.3. Ito's formula then implies that

$$dX_t = -\sqrt{\kappa} dB_t - \frac{2}{X_t} dt, \text{ and } dY_t = \sqrt{\kappa} dB_t - \frac{2}{Y_t} dt.$$

Note that $f_t(x)$ is decreasing in t and $f_t(-y)$ is increasing in t. Since λ_t is unbounded in both directions, then $\tau_x = \inf\{t > 0 : X_t(x) = 0\} < \infty$ a.s. and $\tau_{-y} = \inf\{t > 0 : Y_t(y) = 0\} < \infty$ with probability 1, and X_t, Y_t track the flow of x and $-y$ towards 0. The stopping time $\tau_{x,y} = \tau_x \wedge \tau_{-y}$ stops the process as soon as either x or $-y$ are absorbed by the welding. What we want to do is weight the measure $\mathbb{P}_\kappa$ in such a way that $\tau_{x,y} = \tau_x = \tau_{-y}$ almost surely. That is, we want to change the probability measure so that $\phi(x) = y$ with probability 1. We will find a new family of measures $\mathbb{P}^{a,b}_{x,y}$ on Σ which do this, and we call the process an extended BSLE$_\kappa(a,b)$ process started from $(0; x, -y)$.

Proposition 6. *Fix $a, b \in \mathbb{R}$. Then $M^{a,b}_t = M^{a,b}_t(x,y) := X_t^{-\frac{a}{\kappa}} Y_t^{-\frac{b}{\kappa}} (X_t + Y_t)^\gamma f'_t(x)^p f'_t(-y)^q$ is a local martingale if and only if*

$$p = -\frac{a(a+\kappa+4)}{4\kappa}, \quad q = -\frac{b(b+\kappa+4)}{4\kappa}, \text{ and } \gamma = -\frac{ab}{2\kappa}.$$

In this case, the process $(f_t)_{t>0}$ weighted by this local martingale using the Girsanov theorem is a backward SLE$_\kappa(a,b)$ process started from $(0; x, -y)$.

Proof. For ease of notation, let $\alpha = -a/\kappa$ and $\beta = -b/\kappa$. This follows from an Ito's formula calculation, which will be included for completeness. Observe first that

$$\partial_t f'_t(z) = \frac{d}{dz}\partial_t f_t(z) = \frac{-2}{(f_t(z)-\lambda_t)^2} \cdot (-f'_t(z)) = \frac{2f'_t(z)}{(f_t(z)-\lambda_t)^2}.$$

Therefore,

$$\frac{df'_t(z)^r}{f'_t(z)^r} = r\frac{df'_t(z)}{f'_t(z)} = \frac{2r}{(f_t(z)-\lambda_t)^2}dt.$$

Applying this at both x and $-y$ gives

$$\frac{df_t'(x)^p}{f_t'(x)^p} = \frac{2p}{X_t^2}dt \text{ and } \frac{df_t'(-y)^q}{f_t'(-y)^q} = \frac{2q}{Y_t^2}dt.$$

Also, direct applications of Ito's formula gives

$$\frac{d(X_t+Y_t)^\gamma}{(X_t+Y_t)^\gamma} = \frac{-2\gamma}{X_tY_t}dt, \quad \frac{dX_t^\alpha}{X_t^\alpha} = \frac{-\alpha\sqrt{\kappa}}{X_t}dB_t + \frac{h(\alpha)}{X_t^2}dt,$$

$$\frac{dY_t^\beta}{Y_t^\beta} = \frac{\alpha\sqrt{\kappa}}{Y_t}dB_t + \frac{h(\beta)}{Y_t^2}dt,$$

where $h(t) = -2t + \frac{t(t-1)\kappa}{2}$. By the product rule, we get that

$$\frac{dM_t}{M_t} = \frac{dX_t^\alpha}{X_t^\alpha} + \frac{dY_t^\beta}{Y_t^\beta} + \frac{d\langle X_t^\alpha, Y_t^\beta\rangle}{X_t^\alpha Y_t^\beta} + \frac{d(X_t+Y_t)^\gamma}{(X_t+Y_t)^\gamma} + \frac{df_t'(x)^p}{f_t'(x)^p} + \frac{df_t'(-y)^q}{f_t'(-y)^q}$$

$$= \frac{dX_t^\alpha}{X_t^\alpha} + \frac{dY_t^\beta}{Y_t^\beta} + \frac{d\langle X_t^\alpha, Y_t^\beta\rangle}{X_t^\alpha Y_t^\beta} + \left(\frac{-2\gamma}{X_tY_t} + \frac{2p}{X_t^2} + \frac{2q}{Y_t^2}\right)dt$$

$$= \left(\frac{-\alpha\sqrt{\kappa}}{X_t} + \frac{\beta\sqrt{\kappa}}{Y_t}\right)dB_t + \left(\frac{h(\alpha)+2p}{X_t^2} + \frac{h(\beta)+2q}{Y_t^2} - \frac{\alpha\beta\kappa+2\gamma}{X_tY_t}\right)dt.$$

Notice that $\frac{a}{x^2} + \frac{b}{y^2} - \frac{c}{xy} = 0$ for all $x, y, \in \mathbb{R}$ if and only if $a = b = c = 0$. This occurs if and only if p, q, and γ are chosen as indicated. Once these are chosen, we then get that

$$\frac{dM_t}{M_t} = \left(\frac{-\alpha\sqrt{\kappa}}{X_t} + \frac{-\beta\sqrt{\kappa}}{Y_t}\right)dt.$$

Using this as a drift term and weighting the original measure to obtain a new probability measure $\tilde{\mathbb{P}}$, the Girsanov theorem says that

$$d\lambda_t = \sqrt{\kappa} dB_t = \sqrt{\kappa} d\tilde{B}_t + \left(\frac{-\alpha\kappa}{f_t(x) - \lambda_t} + \frac{-\beta\kappa}{f_t(-y) - \lambda_t} \right) dt,$$

where $\tilde{B}_t$ is a $\tilde{\mathbb{P}}$-Brownian motion. This is precisely the definition of a backward $\mathrm{SLE}_\kappa(-\alpha\kappa, -\beta\kappa)$ process started from $(0; x, -y)$, where $a = -\alpha\kappa$ and $b = -\beta\kappa$.

□

The force points at x and $-y$ serve as magnets. If a and b are chosen correctly, we can force x and $-y$ to be absorbed by the process at the same time, which would imply that $\phi(x) = y$. Along with X_t and Y_t, define W_t as in Section 3.3 by $W_t = \dfrac{X_t - Y_t}{X_t + Y_t}$, which is a process that stays in $(-1, 1)$ for as long as it exists. It only fails to exist if $\tau_x = \tau_{-y}$, where τ_z is the time at which z is swallowed by the BSLE traces. If this does not happen (which is a.s. under $\mathbb{P}_\kappa$), then W_t is constant at ± 1 after $\tau_{x,y}$. The strategy is to perform an appropriate random time change, under which W becomes a diffusion process. We can then analyze the lifetime of this diffusion.

Lemma 13. *A $BSLE_\kappa(a, b)$ process started from $(0; x, -y)$ welds x and y together with probability 1 if $a, b \leq \dfrac{-4-\kappa}{2}$.*

Proof. Consider a backward chordal $\mathrm{SLE}(\kappa; a, b)$ process started from $(0; x, -y)$ with driving function λ. Let X_t, Y_t, W_t, and $u(t)$ be defined as in Section 3.3. We proceed similarly to Proposition 6. By definition,

$$dX_t = -\sqrt{\kappa} dB_t - \left(\frac{a}{X_t} - \frac{b}{Y_t} \right) dt - \frac{2}{X_t} dt, \text{ and } dY_t = \sqrt{\kappa} dB_t + \left(\frac{a}{X_t} - \frac{b}{Y_t} \right) dt - \frac{2}{Y_t} dt.$$

Therefore,

$$dX_t = -\sqrt{\kappa}B_t - \left(\frac{a+2}{X_t} - \frac{b}{Y_t}\right)dt, \text{ and } dY_t = \sqrt{\kappa}dB_t + \left(\frac{a}{X_t} - \frac{b+2}{Y_t}\right)dt. \tag{3.6}$$

Simple Ito's formula calculations show that

$$\frac{dX_t + Y_t}{X_t + Y_t} = \frac{-2}{X_tY_t}dt, \text{ and } \frac{dX_t - Y_t}{X_t - Y_t} = \frac{2}{X_tY_t}dt - \frac{2}{X_t - Y_t}d\lambda_t,$$

and therefore

$$\frac{dW_t}{W_t} = \frac{4}{X_tY_t} - \frac{2}{X_t - Y_t}d\lambda_t \implies dW_t = \frac{-2\sqrt{\kappa}dB_t}{X_t + Y_t} + \left(\frac{4W_t}{X_tY_t} + 2\frac{bX_t - aY_t}{(X_t + Y_t)X_tY_t}\right)dt.$$

Using the basic algebraic equality $2\frac{bx - ay}{x + y} = (b - a) + (b + a)\frac{x - y}{x + y}$, this becomes

$$dW_t = \frac{-2\sqrt{\kappa}dB_t}{X_t + Y_t} + \left(\frac{4W_t}{X_tY_t} + \frac{(b-a) + (b+a)W_t}{X_tY_t}\right)dt.$$

Recall the function $u(t) = \int_0^t \frac{\kappa}{X_rY_r}dr = -\frac{\kappa}{2}\ln((X_t + Y_t)/(x + y))$.Using u to perform a random time change, we get

$$d\tilde{W}_t = -\sqrt{1 - \tilde{W}_t^2}d\tilde{B}_t + \frac{(b-a) + (b+a)\tilde{W}_t}{\kappa}dt, \tag{3.7}$$

where $\tilde{B}_s$ is a Brownian motion and $\tilde{W}_t = W_{u^{-1}(t)}$. Anything satisfying equation (3.7) is associated with a radial Bessel process, and is studied in [39]. Thus, our process W is such a process traversed at a random speed. Observe that $u_{\tau_{x,y}}$ is the lifetime of (3.7). Also, note that $X_{\tau_{x,y}} + Y_{\tau_{x,y}} = 0$ if and only if $u_{\tau_{x,y}} = \infty$. According to the appendix in [39], the lifetime of (3.7) is infinite if $a, b \leq -2 - \frac{\kappa}{2}$ with probability 1. Hence, if $a, b \leq -2 - \frac{\kappa}{2}$, then $X_{\tau_{x,y}} + Y_{\tau_{x,y}} = 0$, and thus x and y are welded together. $\square$

Proposition 6 and Lemma 13 motivate us to define the a, b-backward SLE$_\kappa$ Green's function for $a, b \leq -2 - \kappa/2$ by $G^{a,b}(x, y) = x^{-\frac{a}{\kappa}} y^{-\frac{b}{\kappa}} (x + y)^{-\frac{ab}{2\kappa}}$, so that $M^{a,b}(x, y) = G^{a,b}(X_t, Y_t) f_t'(x)^p f_t'(-y)^q$ is the local martingale whose weighting gives the BSLE$_\kappa(a, b)$ process starting from $(0; x, -y)$. What we want to do now, however, is run the backward SLE$_\kappa(a, b)$ process until it welds x and y together, and then revert the driving function back to a standard Brownian motion after $\tau_{x,y}$.

To make this precise, let $a, b \leq -2 - \kappa/2$. Let $\mathbb{P}^{a,b}_{x,y}$ on Σ be the measure of the driving function for the BSLE$_\kappa(a, b)$ process starting from $(0; x, -y)$ until time $\tau_{x,y}$. Then $\tau_{x,y} < \infty$ means that $\mathbb{P}^{a,b}_{x,y}$ is supported on $\Sigma^{\oplus}$. Since $\mathbb{P}_\kappa$, the law of $\sqrt{\kappa}$ times a Brownian motion, is supported on $\Sigma^{\oplus}$, that means we can define the measure $\mathbb{P}^{a,b}_{x,y} \oplus \mathbb{P}_\kappa$ on $\Sigma^{\mathcal{W}}$, which we will call the law of the driving function for an extended BSLE$_\kappa(a, b)$ process started from $(0; x, -y)$. Then $\mathcal{W}_*(\mathbb{P}^{a,b}_{x,y} \oplus \mathbb{P}_\kappa)$ is the measure on $\Sigma^{\mathbb{C}}$ giving the law of the extended BSLE$_\kappa(a, b)$ started from $(0; x, -y)$ welding curve. Under this measure, the welding curve passes through (x, y) almost surely.

Our goal is to show that backward SLE can be decomposed into the average of weighted extended BSLE$_\kappa(a, b)$ processes. Fix a measurable $U \subset Q_1 = [0, \infty) \times [0, \infty)$, and observe that the measure $\mathbb{P}^{a,b}_{x,y}$ is a probability kernel from Q_1 to $(\Sigma, \mathcal{F})$. Define a new measure on $(\Sigma, \mathcal{F})$ by

$$\mathbb{P}^{a,b}_U = \int \int_U \mathbb{P}^{a,b}_{x,y} G^{a,b}(x, y) dx dy.$$

We say that BSLE$_\kappa$ admits a BSLE$_\kappa(a, b)$ decomposition if for all measurable $U \subset Q_1$

there is some increasing random process $\Theta = \left(\Theta_t^U\right)_{t>0}$ so that

$$\mathbb{P}_U^{a,b}\hat{\oplus}\mathbb{P}_\kappa = \mathbb{P}_\kappa \otimes d\Theta_\cdot^U.$$

Recall then that $d\Theta_\cdot^U$ is a kernel from Σ to $[0,\infty)$.

This is a backward analogue to the definition in [37], which defines what it means for SLE_κ to admit an $\text{SLE}_\kappa(\rho)$ decomposition. In [37], it is proven that SLE_κ admits both an $\text{SLE}_\kappa(\kappa-8)$ decomposition and an $\text{SLE}_\kappa(-8)$ decomposition. In the first case, this corresponds to weighting the SLE_κ curve against the natural parametrization, and in the latter the SLE_κ curve is weighted by the capacity parametrization. In the next section, we will show that BSLE_κ admits a $\text{BSLE}_\kappa(-4,-4)$ decomposition, and see that the corresponding weight is capacity time.

We will need to study the distribution of the stopping time $\tau_{x,y}$ under $\mathbb{P}_{x,y}^{a,b}$.

Lemma 14. *Under $\mathbb{P}_{x,y}^{a,b}$, the stopping time $\tau_{x,y}$ satisfies*

$$\tau_{x,y} = \frac{(x+y)^2}{4\kappa}\int_0^\infty e^{-\frac{4}{\kappa}t}(1-\hat{W}_t^2)dt \tag{3.8}$$

$\mathbb{P}_{x,y}^{a,b}$ a.s., where $\hat{W}$ satisfies (3.7) for $a,b \leq -2-\kappa/2$.

Proof. Recall the time change from before, where $u(t) := \int_0^t \frac{\kappa}{X_sY_s}ds = -\frac{\kappa}{2}\ln((X_t+Y_t)/(x+y))$, and we will use the convention that $\hat{Z}_t = Z_{u^{-1}(t)}$, for $Z \in \{X,Y,W\}$. We claim that $\mathbb{P}_{x,y}^{a,b}$-almost surely,

$$\tau_{x,y} = \int_0^\infty \frac{\hat{X}_s\hat{Y}_s}{\kappa}ds. \tag{3.9}$$

Recall that under $\mathbb{P}_{x,y}^{a,b}$,we have $\tau_x = \tau_y$, and so $u(\tau_{x,y}) = -\frac{2}{\kappa}\ln(X_\tau+Y_\tau) = \infty$ $\mathbb{P}_{x,y}$-almost surely, where $\tau = \tau_{x,y}$. We are using the fact that x and y are welded together here. Then

$$\int_0^\infty \frac{\hat{X}_s \hat{Y}_s}{\kappa} ds = \int_{u^{-1}(0)}^{u^{-1}(\tau)} \frac{\hat{X}_s \hat{Y}_s}{\kappa} \frac{(u^{-1})'(s)}{(u^{-1})'(u(u^{-1}(s)))} ds = \int_0^\tau \frac{X_t Y_t}{\kappa} u'(t) dt = \int_0^\tau 1 dt = \tau_{x,y}.$$

Using $W = \frac{X-Y}{X+Y}$ and the identity $1 - W^2 = \frac{4XY}{(X+Y)^2}$, and the fact that $\hat{X}_t + \hat{Y}_t = (x+y)e^{-\frac{2}{\kappa}t}$, we can rewrite the integrand in equation (3.9) and obtain (3.8).

□

Define a function $\sigma^{a,b}(x, y; t) = \mathbb{P}^{a,b}_{x,y}[\tau_{x,y} \le t]$. By equation (3.8), we can conclude that

$$\sigma^{a,b}(x, y; t) = \sigma^{a,b}\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}; 1\right), \text{ for all } t > 0. \tag{3.10}$$

This holds because

$$\frac{(x+y)^2}{4\kappa} \int_0^\infty e^{-\frac{4}{\kappa}s}(1 - \hat{W}_s^2) ds \le t$$

if and only if

$$\frac{(x/\sqrt{t} + y/\sqrt{t})^2}{4\kappa} \int_0^\infty e^{-\frac{4}{\kappa}s}(1 - \hat{W}_s^2) ds \le 1,$$

where the diffusion $(\hat{W}_t)_{t>0}$ has the same starting point at (x, y) as at $(x/\sqrt{t}, y/\sqrt{t})$.

It is worth observing that the local martingale obtained in Proposition 6 is a local martingale, but not a martingale.

Lemma 15. *If $a, b < 0$, then*

$$\mathbb{E}_\kappa[M_t(x,y)] = G^{a,b}(x,y)\mathbb{P}^{a,b}_{x,y}(\tau_{(x,y)} > t) = x^\alpha y^\beta (x+y)^\gamma (1 - \sigma^{a,b}(x, y; t)), \tag{3.11}$$

where $\alpha = -a/\kappa$, $\beta = -b/\kappa$, and $\gamma = -ab/(2\kappa)$. In particular, M_t is a local martingale, but *a strict supermartingale.*

Proof. For this lemma, since x and y are fixed, we will let $\tau = \tau_{x,y}$. Define $\tau_n = \inf\{t : M_t \geq n\}$. Then for each n, $M_{t\wedge\tau_n}$ is a bounded local martingale, and hence a martingale. Therefore,

$$\mathbb{E}_\kappa[M_t] = \lim_{n\to\infty} \mathbb{E}_\kappa[M_t \mathbb{I}_{\{\tau_n > t\}}] = \lim_{n\to\infty} \mathbb{E}_\kappa[M_{t\wedge\tau_n} \mathbb{I}_{\tau_n > t}]$$

$$= G(x,y) \lim_{n\to\infty} \mathbb{P}^{a,b}_{x,y}(\tau_n > t) = G(x,y)\mathbb{P}^{a,b}_{x,y}\left(\cup_n\{\tau_n > t\}\right).$$

We claim that $\mathbb{P}^{a,b}_{x,y}\left(\cup_n\{\tau_n > t\}\right) = \mathbb{P}^{a,b}_{x,y}(\tau > t)$. It is clear that $\cup_n\{\tau_n > t\} \subset \{\tau > t\}$, since τ is the lifetime of (M_t). It suffices to show $\cap_n\{\tau_n \leq t\} \subset \{\tau \leq t\}$. In the event $\cap_n\{\tau_n \leq t\}$, M_s is unbounded for $s \in [0,t]$. It can easily be seen that if $t \leq \tau$, then $f_t(x) - \lambda_t \leq x - \lambda_t \leq x + y$, and $\lambda_t - f_t(-y) \leq x + y$ similarly. It follows immediately that $X_t^\alpha Y_t^\beta \leq (x+y)^{\alpha+\beta}$, and so this factor of M_t is bounded uniformly for fixed x, y. Therefore, if M_s is unbounded on $[0,t]$, then it must be that either $(X_s + Y_s)^\gamma$ or $f_s'(x)^p f_s'(-y)^q$ is unbounded on $[0,t]$. In the first case, since $\gamma = -ab/(2\kappa) < 0$, this implies that $X_s + Y_s \to 0$ before time t. That is, $\tau_{(x,-y)} \leq t$. In the latter case, the reverse Loewner equation implies

$$f_s'(x)^p f_s'(-y)^q = \exp\{\int_0^s \frac{2p}{X_r^2} + \frac{2q}{Y_r^2} dr\}.$$

This term is unbounded only if p and q are nonnegative and $X_s, Y_s \to 0$ before time t. Therefore, in either case, we have $\tau \leq t$.

□

3.5 Capacity parametrization

The goal of this section is to show that BSLE_κ has a $\mathrm{BSLE}_\kappa(-4,-4)$ decomposition, and to show the relationship between this decomposition and the capacity time for the welding curve. For this section, we will use $\mathbb{P}_{x,y} := \mathbb{P}_{x,y}^{-4,-4}$ for $x, y > 0$.

Lemma 16. $\int_0^\infty \int_0^\infty \mathbb{P}_{x,y}[\tau_{x,y} \le 1] dxdy \in (0, \infty)$.

This lemma states that the probability of two points which are far away being welded together in a short time decays rapidly. By the scaling relation, what matters most is how relatively far x is from 0 in comparison to y. This is reflected in the proof, where we perform a change of variables from (x, y) to $r(t, 1-t)$ and perform the most work integrating the single variable t.

Proof. By equation (3.8), we have $\mathbb{P}_{x,y}$-almost surely that

$$\tau_{x,y} \le \frac{(x+y)^2}{4\kappa} \int_0^\infty e^{-\frac{4}{\kappa}t} dt = \frac{(x+y)^2}{16}.$$

Thus, if $x + y \le 4$, then

$$\mathbb{P}_{x,y}[\tau_{x,y} \le 1] \ge \mathbb{P}_{x,y}[\tau_{x,y} \le (x+y)^2/16] = 1,$$

and therefore

$$\int_0^\infty \int_0^\infty \mathbb{P}_{x,y}[\tau_{x,y} \le 1] \ge \int_{x+y\le 4} 1 dxdy > 0.$$

To use the scaling relation for BSLE to show that $\int_0^\infty \int_0^\infty \mathbb{P}_{x,y}[\tau_{x,y} \le 1] dxdy < \infty$, we perform a change of variables. Observe that the distribution of $\tau_{x,y}$ is the same as $(x+y)^2 \tau_{t,1-t}$, where $t = \dfrac{x}{x+y}$. To simplify notation, we will write $\mathbb{P}_t = \mathbb{P}_{t,1-t}$ and

$\tau_t = \tau_{t,1-t}$ for $0 < t < 1$. Performing the change of coordinates $x = rt, y = r(1-t)$, we get that

$$\int_0^\infty \int_0^\infty \mathbb{P}_{x,y}[\tau_{x,y} \leq 1] dxdy = \int_0^1 \int_0^\infty \mathbb{P}_t[\tau_t \leq \frac{1}{r^2}] rdrdt.$$

For any fixed t, observe that

$$\int_0^\infty \mathbb{P}_t[\tau_t \leq r^{-2}] rdr = \int_0^\infty \mathbb{P}_t\left[\frac{1}{\tau_t} \geq r^2\right] rdr = \frac{1}{2}\mathbb{E}_t\left[\frac{1}{\tau_t}\right],$$

where $\mathbb{E}_t$ is the expectation with respect to the measure $\mathbb{P}_t$. Thus, it suffices to show that $\int_0^1 \mathbb{E}_t\left[\frac{1}{\tau_t}\right] dt$ is finite.

For any small $\delta > 0$, we can break up the integral as

$$\int_0^1 \mathbb{E}_t\left[\frac{1}{t}\right] dt = \int_0^{\delta/2} \mathbb{E}_t\left[\frac{1}{\tau_t}\right] dt + \int_{\delta/2}^{1-\delta/2} \mathbb{E}_t\left[\frac{1}{\tau_t}\right] dt + \int_{1-\delta/2}^1 \mathbb{E}_t\left[\frac{1}{\tau_t}\right] dt.$$

To complete the proof, we will produce a small $\delta > 0$ so that:

a) There is a constant $C_\delta < \infty$ so that if $0 < t \leq \delta/2$,

$$\mathbb{E}_t\left[\frac{1}{\tau_t}\right] \leq C_\delta \mathbb{E}_{\delta/2}\left[\frac{1}{\tau_{\delta/2}}\right] < \infty. \tag{3.12}$$

b) For each $t \in (\delta/2, 1-\delta/2)$,

$$\mathbb{E}_t\left[\frac{1}{\tau_t}\right] \leq \mathbb{E}_{\delta/2}\left[\frac{1}{\tau_{\delta/2}}\right] + \mathbb{E}_{1-\delta/2}\left[\frac{1}{\tau_{1-\delta/2}}\right]. \tag{3.13}$$

Note that if (3.12) is proven, a symmetric argument proves a similar bound for $t \in [1-\delta/2, 1)$. Once a correct δ can be chosen for (3.12), then (3.13) and both boundary estimates imply that $\int_0^1 \mathbb{E}_t\left[1/\tau_t\right] dt < \infty$.

Note that $t_0 = x/(x+y)$ being near 0 is the same as starting the process $W_t = (X_t - Y_t)/(X_t + Y_t)$ at $w = (x-y)/(x+y) = 2t_0 - 1$ near -1, and t_0 being near 1 corresponds to starting the process W_t near 1. Under the measure $\mathbb{P}_{t_0}$, equation (3.7) reduces to

$$d\hat{W}_t = -\sqrt{1-\hat{W}_t^2}dB_t + \frac{-8}{\kappa}\hat{W}_t dt, \quad \hat{W}_0 = w. \tag{3.14}$$

We will use this diffusion to prove (3.13). Suppose $\delta > 0$ is small, and fix any $t^* \in (\delta/2, 1-\delta/2)$. Let $(W_t^*)_{t>0}$ be (3.14) started at $w = 2t^* - 1$, $(W_t^1)_{t>0}$ be (3.14) started at $w = -1+\delta$, and let $(W_t^2)_{t>0}$ be (3.14) started at $w = 1-\delta$. Then for each $t > 0$,

$$1 - (W_t^*)^2 \geq \min\{1 - (W_t^j)^2 : j = 1, 2\}.$$

Thus, by equation (3.8), we can see that there is a coupling so that $\tau_{t^*} \geq \min\{\tau'_{\delta/2}, \tau'_{1-\delta/2}\}$, where τ_{t^*}, $\tau'_{\delta/2}$, and $\tau'_{1-\delta/2}$ are defined on the same probability space, and each τ'_w has the distribution of τ_w under $\mathbb{P}_w$ for $w \in \{\delta, 1-\delta/2\}$. Therefore,

$$\frac{1}{\tau_{t^*}} \leq \max\{\frac{1}{\tau'_{\delta/2}}, \frac{1}{\tau'_{1-\delta/2}}\} \leq \frac{1}{\tau'_{\delta/2}} + \frac{1}{\tau'_{1-\delta/2}}.$$

Applying the expectation gives (3.13).

The rest of the proof is showing that (3.12) holds. Fix (for now) some $\delta > 0$, and let $-1 < w < -1+\delta$. In the t coordinates, suppose $W(0) = w = 2t-1$, so that $-1 < w < -1+\delta$ corresponds to $0 < t < \delta/2$. Fix any $t^* \in (0, \delta/2)$, and let $w = 2t^* - 1$ be the starting point of the process $(\hat{W}_t)_{t>0}$ solving (3.14), where $\hat{W}$ has an infinite lifetime. Define a stopping time by $T_\delta = \inf\{t > 0 : \hat{W}_t = -1 + \delta\}$. Then equation (3.8) implies

$$\tau_{t^*} = \frac{1}{4\kappa}\int_0^\infty e^{-\frac{4}{\kappa}t}(1 - \hat{W}_t^2)dt \geq \frac{1}{4\kappa}\int_{T_\delta}^\infty e^{-\frac{4}{\kappa}t}(1 - \hat{W}_t^2)dt$$

$$= e^{-\frac{4}{\kappa}T_\delta}\frac{1}{4\kappa}\int_0^\infty e^{-\frac{4}{\kappa}t}(1 - \hat{W}_{t+T_\delta}^2)dt.$$

The right hand side is the stopping time for the process started at $t = \delta/2$, so conditioned on $\{T_\delta = s\}$, we have $\tau_{t^*} \geq e^{-\frac{4}{\kappa}s}\tau_{\delta/2}$ in distribution. Therefore, $\mathbb{E}_{t^*}\left[1/\tau_{t^*}|T_\delta = s\right] \leq e^{\frac{4}{\kappa}s}\mathbb{E}_{\delta/2}\left[1/\tau_{\delta/2}\right]$. It follows that

$$\mathbb{E}_{t^*}\left[\frac{1}{\tau_{t^*}}\right] = \sum_{n=1}^\infty \mathbb{E}_{t^*}\left[\frac{1}{\tau_{t^*}}|T_\delta \in [n-1, n)\right]\mathbb{P}_{t^*}[n - 1 \leq T_\delta < n]$$

$$\leq \sum_{n=0}^\infty e^{\frac{4}{\kappa}n+1}\mathbb{E}_{\delta/2}\left[\frac{1}{\tau_{\delta/2}}\right]\mathbb{P}_{t^*}[T_\delta \geq n].$$

We claim that $\mathbb{P}_{t^*}[T_\delta \geq n]$ decays exponentially in n. Observe that

$$\mathbb{P}_{t^*}[T_\delta \geq n] = \mathbb{P}_{t^*}[\{\sup_{0 \leq t \leq 1}\hat{W}_t < -1 + \delta\}, \ldots, \{\sup_{n-1 \leq t \leq n}\hat{W}_t < -1 + \delta\}].$$

In [39], the transition density for $\hat{W}$ is found, and it can be extended via the Kolmogorov consistency theorem to a process which can start at -1. Observe then that during each time interval $k - 1 \leq t < k$, the process lies above a coupling of the same diffusion started at -1. Therefore, if $p_\delta = \mathbb{P}_0[T_\delta \geq 1]$, where $\mathbb{P}_0$ is the law of $\hat{W}$ started at -1, then $\mathbb{P}_{t^*}[\sup_{k-1 \leq t < k}\hat{W}_t \leq \delta - 1] \leq p_\delta$. Moreover, by the Markov property, these couplings are independent of the path before each time, so therefore $\mathbb{P}_{t^*}[T_\delta \geq n] \leq p_\delta^n$.

Thus, we have $E_{t^*}\left[\frac{1}{\tau_{t^*}}\right] \leq \mathbb{E}_{\delta/2}\left[\frac{1}{\tau_{\delta/2}}\right]\sum_n e^{\frac{4}{\kappa}n}p_\delta^n$. Now we need to show that δ can be chosen sufficiently small so that this infinite sum converges. What we need to know is if p_δ can be made sufficiently small as $\delta \to 0$. It is easier to see that this is the case if instead we look at $\theta_t = \arccos(\hat{W}_t)$, which satisfies the SDE

$$d\theta_t = dB_t + \frac{4-\kappa}{\kappa}\cot(\theta_t)dt. \tag{3.15}$$

Then θ is what is called a radial Bessel process. It is a diffusion with range in $[0,\pi]$ represented as a Brownian motion with strong drift directed towards the center of this interval. Also, note that this drift is positive when θ_t is small (we are using that $\kappa \leq 4$). Then

$$\mathbb{P}_0\left[\sup_{0\leq t\leq 1}\theta_t \leq \epsilon\right] \leq \mathbb{P}[\sup_{0\leq t\leq 1} B_t \leq \epsilon] \to 0$$

as $\epsilon \to 0$. It follows then that $\lim_{\delta\to 0} p_\delta = 0$, and a fixed δ can be chosen so that the expectation $\mathbb{E}_{t^*}\left[\frac{1}{\tau_{t^*}}\right]$ can be bounded uniformly in $t^* \in (0,\delta/2)$. Speaking more precisely, δ can be chosen small enough so that $e^{4/\kappa}p_\delta \leq 1/2$, and so for all $t^* \in (0,\delta/2)$

$$e^{\frac{4}{\kappa}n}\mathbb{P}_{t^*}[T_\delta \leq n] \leq e^{\frac{4}{\kappa}n}p_\delta^n \leq 2^{-n}.$$

Therefore, to finish proving (3.12), it suffices to show that $E_{\delta/2}\left[1/\tau_{\delta/2}\right] < \infty$.

Let $\theta_0 = \delta' = \arccos(-1+\delta)$, and define a stopping time by $T = \inf\{t : \theta_t \in \{\delta'/2, \pi - \delta'/2\}\}$. Then equation (3.8) and $1 - \hat{W}_t^2 = \sin^2(\theta_t)$ imply that

$$(4\kappa)\tau_{\delta/2} \geq \int_0^T e^{-\frac{4}{\kappa}s}\sin^2(\theta_s)ds \geq \sin^2(\delta'/2)\int_0^T e^{-\frac{4}{\kappa}s}ds = C_\delta(1-e^{-\frac{4}{\kappa}T}).$$

Therefore,

$$\mathbb{E}_{\delta/2}\left[\frac{1}{\tau_{\delta/2}}\right] \le C_\delta \mathbb{E}_{\delta/2}\left[\frac{1}{1-e^{-\frac{4}{\kappa}T}}\right]$$

$$= C_\delta \sum_{n=0}^{\infty} \mathbb{E}_{\delta/2}\left[\frac{1}{1-e^{-\frac{4}{\kappa}T}} \quad |T \in [2^{-n-1}, 2^{-n})\right] \mathbb{P}_{\delta/2}[T \in [2^{-n-1}, 2^{-n})]$$

$$\le C_\delta \sum_{n=0}^{\infty} \frac{1}{1-e^{-\frac{4}{\kappa}2^{-n-1}}} \mathbb{P}_{\delta/2}[T \le 2^{-n}],$$

where C_δ is some constant which depends only on δ and κ. What we need is a good bound on $\mathbb{P}_{\delta/2}[T \le \epsilon]$ as $\epsilon \to 0$. By the equation (3.15), whenever $t \le T$, we have

$$B_t - At \le \theta_t - \delta' \le B_t + At,$$

where $A = \dfrac{4-\kappa}{\kappa}\cot(\delta'/2) > 0$.

Let $T' = \inf\{t : \theta_t = \dfrac{\delta'}{2}\}$, and $T'' = \inf\{t : \theta_t = \pi - \dfrac{\delta'}{2}\}$, so that $T = \min\{T', T''\}$. Then $\mathbb{P}_{\delta/2}[T \le \epsilon] = \mathbb{P}_{\delta/2}[T' \le \epsilon, T = T'] + \mathbb{P}_{\delta/2}[T'' \le \epsilon, T = T'']$. Both of these probabilities will be estimated similarly, so we will show that the first has sufficiently strong decay. Observe that

$$\mathbb{P}_{\delta/2}[T' \le \epsilon, T = T'] \le \mathbb{P}_{\delta/2}[\inf_{0\le t\le\epsilon} \theta_t \le \delta'/2, \sup_{0\le t\le\epsilon} \theta_t < \pi - \delta'/2]$$

$$= \mathbb{P}_{\delta/2}[\inf_{0\le t\le\epsilon} \theta_t - \delta' \le -\delta'/2, \sup_{0\le t\le\epsilon} \theta_t \le \pi - \delta'/2]$$

$$\le \mathbb{P}[\inf_{0\le t\le\epsilon} B_t - At \le -\delta'/2] = \mathbb{P}[\sup_{0\le t\le\epsilon} B_t + At \ge \delta'/2] \le \mathbb{P}[\sup_{0\le t\le\epsilon} B_t \ge \delta'/2 - A\epsilon],$$

where $\mathbb{P}$ is the law of standard Brownian motion. Note that with δ fixed, ϵ can be chosen small enough so that this probability is nontrivial. Moreover, by the scaling property of

Brownian motion and the reflection principle, we get

$$= \mathbb{P}\left[\sup_{0\le t\le 1} B_t \ge \frac{\delta'/2 - A\epsilon}{\sqrt{\epsilon}}\right] = 2\mathbb{P}\left[B_1 \ge \frac{\delta'/2 - A\epsilon}{\sqrt{\epsilon}}\right].$$

Using the elementary bound $\mathbb{P}[N(0,1) \ge M] \le \frac{C}{M}e^{-M^2/2}$, this implies that

$$\mathbb{P}_{\delta/2}[T' \le \epsilon, T = T'] \le C\sqrt{\epsilon}\exp\{-\frac{((\delta'/2) - A\epsilon)^2}{2\epsilon}\} \le C\sqrt{\epsilon}e^{-K/\epsilon},$$

where C, K are both constants depending only on δ and κ. An identical bound can be found for the event $\{T'' \le \epsilon\}$,so the same bound holds for $\mathbb{P}_{\delta/2}[T \le \epsilon]$.

Combining the inequalities of the previous paragraphs, we can conclude that

$$\mathbb{E}_{\delta/2}\left[\frac{1}{\tau_{\delta/2}}\right] \le C\sum_{n=0}^{\infty}\left(\frac{1}{1 - e^{-\frac{4}{\kappa}2^{-n-1}}}\right)\left(2^{-\frac{n}{2}}e^{-K2^n}\right).$$

The terms $\dfrac{e^{-K2^n}}{1 - e^{-\frac{4}{\kappa}2^{-n-1}}}$ are bounded, and therefore

$$\mathbb{E}_{\delta/2}\left[\frac{1}{\tau_\delta}\right] \le C\sum_{n=0}^{\infty} 2^{-\frac{n}{2}} < \infty.$$

□

We will now make use of the results in Section 3.2, which originate from [37]. For any $N > 0$ and $x, y > 0$, define the measure $P^*_{x,y;N}$ by $P^*_{x,y;N}[E] = \mathbb{P}^*_{x,y}[E\backslash\Sigma_N]$, where $\Sigma_N = \{f \in \Sigma : T_f > N\}$. Note that this is not a probability measure. By Proposition 4, we know that the local Radon Nikodym derivative $\mathbb{P}^*_{x,y}$ with respect to $\mathbb{P}_\kappa$ is given by

$$\frac{d\mathbb{P}^*_{x,y}|_{\mathcal{F}_t\cap\Sigma_t}}{d\mathbb{P}_\kappa|_{\mathcal{F}_t\cap\Sigma_t}} = \mathbb{I}_{\{\tau_{x,y}>t\}}M_t(x,y) = \mathbb{I}_{\{\tau_{x,y}>t\}}G(X_t(x),Y_t(y))f_t'(x)f_t'(-y), \tag{3.16}$$

where $G(x,y) = G^{-4,-4}(x,y) = x^{\frac{4}{\kappa}}y^{\frac{4}{\kappa}}(x+y)^{-\frac{8}{\kappa}}$ is the $(-4,-4)$-backward SLE Green's function in the capacity parametrization. Let $E \in \mathcal{F}_t \cap \Sigma_t$. If $t \geq N$, then $\mathbb{P}^*_{x,y;N}[E] = 0$. If $t < N$, then

$$\mathbb{P}^*_{x,y;N}[E] := \mathbb{P}^*_{x,y}[E] - \mathbb{P}^*_{x,y}[E\cap\Sigma_N] = \int_E \frac{M_t(x,y)}{G(x,y)}d\mathbb{P}_\kappa - \int_{E\cap\Sigma_N}\frac{M_N(x,y)}{G(x,y)}d\mathbb{P}_\kappa$$

$$= \frac{1}{G(x,y)}\int_E M_t(x,y) - \mathbb{E}_\kappa[M_N(x,y)|\mathcal{F}_t]d\mathbb{P}_\kappa.$$

Therefore, we conclude that

$$\frac{d\mathbb{P}^*_{x,y;N}|_{\mathcal{F}_t\cap\Sigma_t}}{d\mathbb{P}_\kappa|_{\mathcal{F}_t\cap\Sigma_t}} = \frac{\mathbb{I}_{\{t<N\}}}{G(x,y)}\left(M_t(x,y) - \mathbb{E}_\kappa[M_N(x,y)|\mathcal{F}_t]\right). \tag{3.17}$$

For each $t \geq 0$, define a function $G_t(x,y)$ by

$$\frac{G_t(x,y)}{G(x,y)} = \frac{M_0(x,y) - \mathbb{E}_\kappa[M_t(x,y)]}{G(x,y)} = \mathbb{P}^*_{x,y}[\tau_{x,y} \leq t].$$

The second equality follows by Proposition 4. By equation (3.10), and the fact that $G(x,y)$ is homogeneous of degree 0, it follows that $G_t(x,y) = G_1\left(\frac{x}{\sqrt{t}},\frac{y}{\sqrt{t}}\right)$. Also, $G_0(x,y) = 0$ for all $x,y > 0$, and $\lim_{t\to\infty} G_t(x,y) = G(x,y)\mathbb{P}^*_{x,y}[\tau_{x,y} < \infty] = G(x,y)$.

Let $C_{\kappa,t} := \int_0^\infty\int_0^\infty G_t(x,y)dxdy = \int_0^\infty\int_0^\infty G(x,y)\mathbb{P}^*(\tau_{x,y} \leq t)dxdy$, and let $C_\kappa = C_{\kappa,1}$. We claim that $C_{\kappa,t} = tC_\kappa$, where $C_\kappa \in (0,\infty)$. The equality follows from scale invarience and the change of variables formula. In particular,

$$C_{\kappa,t} = \int_0^\infty \int_0^\infty G_t(x,y)dxdy = \int_0^\infty \int_0^\infty G_1\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) dxdy$$

$$= (\sqrt{t})^2 \int_0^\infty \int_0^\infty G_1(x,y)dxdy.$$

Since $C_\kappa = \int_0^\infty \int_0^\infty G(x,y)\mathbb{P}^*_{x,y}[\tau_{x,y} < 1]dxdy$, Lemma 16 and the fact that $G(x,y) > 0$ implies that $C_\kappa > 0$. Moreover, G is bounded, which combined with Lemma 16 implies that $C_\kappa < \infty$.

Recall that $Q_1 = [0,\infty) \times [0,\infty)$ denotes the first quadrant. Given a $\lambda \in \Sigma^{\mathcal{W}}$, we denote the welding curve generated by λ by $\Phi : [0,\infty) \to Q_1$, where $\Phi(t) = (x,y)$ if x and $-y$ are absorbed at time t. That is, $\phi(x) = y$ and $\tau_x = \tau_{-y} = t$. For $U \subset Q_1$ measurable, we define an increasing process (Θ_t^U) by

$$\Theta_t^U = C_\kappa m_+\{\Phi^{-1}(U) \cap [0,t]\}, \tag{3.18}$$

where $m_+\{\cdot\}$ is one dimensional Lebesgue measure on $[0,\infty)$. Then Θ_t^U is the capacity time spent by Φ in the set U before t. Note that then $d\Theta_\cdot^U$ is a kernel from $(\Sigma, \mathcal{F})$ to $[0,\infty)$ and can be thought of as an $\mathcal{F}$-measurable random measure on the line. Also, recall that $(x,y) \mapsto \mathbb{P}_{x,y}$ is a probability kernel from Q_1 to Σ, and we can define a measure on Σ for $U \subset Q_1$ measurable by $\mathbb{P}_U = \mathbb{P}_U^{-4,-4} = \int_U \mathbb{P}_{x,y} G(x,y)dxdy$.

Theorem 6. *Fix $\kappa \in (0,4)$. For any measurable $U \subset Q_1$,*

$$\mathbb{P}_U^{-4,-4} \hat{\oplus} \mathbb{P}_\kappa = \mathbb{P}_\kappa \otimes d\Theta_\cdot^U, \tag{3.19}$$

$$\mathcal{W}_*(\mathbb{P}_{x,y}^{-4,-4} \oplus \mathbb{P}_\kappa) \overleftarrow{\otimes} \mathbb{I}_U G^{-4,-4}(x,y)dxdy = \mathcal{W}_*(\mathbb{P}_\kappa) \otimes \mathcal{M}_U^{-4,-4}, \tag{3.20}$$

where $\mathcal{M}_U = \mathcal{M}_U^{-4,-4}$ *is the kernel from the set of welding curves* $\mathcal{W}(\Sigma^{\mathcal{W}})$ *to* Q_1 *defined by* $\mathcal{M}_U(\Phi, E) = \Phi_*(d\Theta^U_\cdot)\{E\} = d\Theta^U_{\{\Phi^{-1}(E)\}}$.

To ensure clarity, we recall that the measures in equation (3.19) are measures on the space $\Sigma^\oplus \times \Sigma_\oplus$, and the measures in equation (3.20) are measures on $\Sigma^{\mathbb{C}} \times Q_1$ supported on $\mathcal{W}(\Sigma^{\mathcal{W}}) \times Q_1$. Throughout the rest of the chapter, for any measure μ on Σ, we will use $\mu|_t$ to denote the restricted measure $\mu|_{\mathcal{F}_t \cap \Sigma_t}$. The proof of the theorem follows the same strategy as the proofs of Theorems 4.1, 5.1, and 6.1 in [37].

Proof. For any $t \leq N$, the renewal property of the backward Loewner equation and the Markov property of $\sqrt{\kappa}B_t$ imply that $M_t(x,y) - \mathbb{E}_\kappa[M_N(x,y)|\mathcal{F}_t] = \mathbb{I}_{\{\tau_{x,y}>t\}}|f_t'(x)||f_t'(-y)|G_{N-t}(X_t,Y_t)$, where $X_t(x) = f_t(x) - \lambda_t$ and $Y_t(y) = \lambda_t - f_t(-y)$. Also, recall the welding curve satisfies $\Phi(t) = (b_t, a_t)$, where $b_t = \sup\{x > 0 : \tau_x \leq t\}$ and $a_t = \sup\{y > 0 : \tau_{-y} \leq t\}$. Therefore,

$$\int_0^\infty \int_0^\infty M_t(x,y) - \mathbb{E}_\kappa[M_N(x,y)|\mathcal{F}_t]dxdy = \int_{a_t}^\infty \int_{b_t}^\infty G_{N-t}(X_t(x), Y_t(y))|f_t'(x)||f_t'(-y)|dxdy$$

$$= \int_0^\infty \int_0^\infty G_{N-t}(x,y)dxdy = C_{\kappa,N-t} = (N-t)C_\kappa. \tag{3.21}$$

The second equality follows from a direct application of the change of variables formula, since $f_t(-a_t) = f_t(b_t) = 0$.

Let ξ be the measure on Q_1 given by $d\xi = G(x,y)dxdy$, and observe that $\nu((x,y),E) = \mathbb{P}_{x,y;N}(E)$ is a kernel from Q_1 to Σ. By Proposition 2, $\xi \cdot \nu = \int_{Q_1} \mathbb{P}_{x,y;N} G(x,y)dxdy$ is absolutely continuous with respect to $\mathbb{P}_\kappa$, and the local Radon-Nikodym derivatives are given by

$$\frac{d\xi\cdot\nu|_t}{d\mathbb{P}_\kappa|_t} = \int_0^\infty\int_0^\infty \frac{d\mathbb{P}_{x,y;N}|_t}{d\mathbb{P}_\kappa|_t} G(x,y)dxdy$$

$$= \int_0^\infty\int_0^\infty \frac{M_t(x,y) - \mathbb{E}_\kappa[M_N(x,y)|\mathcal{F}_t]}{G(x,y)} G(x,y)dxdy = C_\kappa(N-t)\vee 0.$$

The second equality comes from (3.17), and the last inequality comes from (3.21). Applying Proposition 3, we get

$$\int_0^\infty\int_0^\infty \mathbb{P}_{x,y;N} G(x,y)dxdy = \mathcal{K}_{C_\kappa d(t\wedge N)}(\mathbb{P}_\kappa).$$

Taking $N\to\infty$, it follows that

$$\mathbb{P}_{Q_1} = \int_0^\infty\int_0^\infty \mathbb{P}_{x,y} G(x,y)dxdy = \mathcal{K}_{C_\kappa dt}(\mathbb{P}_\kappa).$$

Therefore, $\mathcal{K}_{C_\kappa dt}(\mathbb{P}_\kappa)\hat{\otimes}\mathbb{P}_\kappa = \mathbb{P}_{Q_1}\hat{\otimes}\mathbb{P}_\kappa$, and so Proposition 5 implies that

$$\mathbb{P}_{Q_1}\hat{\otimes}\mathbb{P}_\kappa = \mathbb{P}_\kappa \otimes C_\kappa m_+. \tag{3.22}$$

Note that this is (3.19) in the case $U = Q_1$.

We claim that for any $(x,y)\in Q_1$,

$$\hat{\mathcal{W}}_*(\mathbb{P}_{x,y}\hat{\oplus}\mathbb{P}_\kappa) = \mathcal{W}_*(\mathbb{P}_{x,y}\oplus\mathbb{P}_\kappa)\otimes\delta_{(x,y)}, \tag{3.23}$$

where $\delta_{(x,y)}$ is the point mass measure on Q_1. Observe that $\mathbb{P}_{x,y}$ is supported on $\{f \in \Sigma^{\mathcal{W}} : \mathcal{W}_f(T_f) = (x,y)\}$, and recall that $\hat{\mathcal{W}}_*(\mathbb{P}_{x,y}\hat{\oplus}\mathbb{P}_\kappa)$ is the pushforward of the measure $\mathbb{P}_{x,y}\otimes\mathbb{P}_\kappa$ under the transformation $\hat{\mathcal{W}}\circ\hat{\oplus} : \Sigma^\oplus\times\Sigma_\oplus \to \Sigma^{\mathbb{C}}\times Q_1$ given by $\hat{\mathcal{W}}(\hat{\oplus}(f,g)) =$

$\hat{\mathcal{W}}(f \oplus g, T_f) = (\Phi, \Phi(T_f))$, where Φ is the welding induced by $f \oplus g$. Thus, if $E \subset \Sigma^{\mathbb{C}}$ and $U \subset Q_1$,

$$\hat{\mathcal{W}}_*(\mathbb{P}_{x,y} \hat{\oplus} \mathbb{P}_\kappa)\{E \times U\} = \mathbb{P}_{x,y} \otimes \mathbb{P}_\kappa\{(f,g) : \Phi = \mathcal{W}(f \oplus g) \in E \text{ and } \Phi(T_f) = (x,y) \in U\},$$

which proves the claim.

Integrating (3.23) over all $(x,y) \in Q_1$ with respect to the measure $G(x,y)dxdy$ gives

$$\hat{\mathcal{W}}_*(\mathbb{P}_\kappa \otimes C_\kappa m_+) = \hat{\mathcal{W}}_*(\mathbb{P}_{Q_1} \hat{\oplus} \mathbb{P}_\kappa) = \mathcal{W}_*(\mathbb{P}_{x,y} \oplus \mathbb{P}_\kappa) \overleftarrow{\otimes} G(x,y)dxdy,$$

where the first equality comes from (3.22). It will follow that

$$\mathcal{W}_*(\mathbb{P}_{x,y} \oplus \mathbb{P}_\kappa) \overleftarrow{\otimes} G(x,y)dxdy = \mathcal{W}_*(\mathbb{P}_\kappa) \otimes \mathcal{M}, \tag{3.24}$$

where $\mathcal{M} = \mathcal{M}_{Q_1}$ is the kernel from $\mathcal{W}(\Sigma^{\mathcal{W}})$ to Q_1 given by $\mathcal{M}(\Phi, U) = m_+\{\Phi^{-1}(U)\}$. To see this, observe that $\hat{\mathcal{W}} : \Sigma^{\mathcal{W}} \times [0, \infty) \to \Sigma^{\mathbb{C}} \times Q_1$ is injective by Lemma 12. For $E \in \mathcal{W}(\Sigma^{\mathcal{W}})$ and $U \subset Q_1$, let $(\Phi, z) \in E \times U$, then there is a unique $\lambda \in \Sigma^{\mathcal{W}}$ and $t \in [0, \infty)$ with $\mathcal{W}(\lambda) = \Phi$ and $\Phi(t) = z$. Then

$$\hat{\mathcal{W}}_*(\mathbb{P}_\kappa \otimes C_\kappa m_+)\{E \times U\} = \mathbb{P}_\kappa \otimes C_\kappa m_+\{\hat{\mathcal{W}}^{-1}(E \times U)\} = \int_{\mathcal{W}^{-1}(E)} m_+\{(\mathcal{W}(\lambda))^{-1}(U)\} d\mathbb{P}_\kappa(\lambda)$$

$$= \int_E m_+\{\Phi^{-1}(U)\} d\mathcal{W}_*(\mathbb{P}_\kappa)(\Phi) = \mathcal{W}_*(\mathbb{P}_\kappa) \otimes \mathcal{M}\{E \times U\}.$$

Since the measures agree on all cylinder sets, they must be the same.

For any $U \subset Q_1$, the kernel $\mathcal{M}_U$ is the restriction of $\mathcal{M}$ to $\mathcal{W}(\Sigma^{\mathcal{W}}) \times U$, and so (3.20) follows from restricting both sides of (3.24) to $\mathcal{W}(\Sigma^{\mathcal{W}}) \times U$. Finally, if we apply the inverse map $\hat{\mathcal{W}}^{-1}$ to (3.20), we get equation (3.19).

□

Corollary 1. *Suppose $U \subset [0,\infty) \times [0,\infty)$ with $\int\int_U G^{-4,-4}(x,y)dxdy < \infty$. If the laws of the welding curve of an extended BSLE$_\kappa(-4,-4)$ process started from $(0;x,-y)$ are integrated against $\mathbb{I}_U G^{-4,-4}(x,y)dxdy$, it results in a bounded measure on curves which is absolutely continuous with respect to the law of the BSLE$_\kappa$ welding curve, and the Radon-Nikodym derivative is $|\mathcal{M}_U(\Phi, Q_1)| = C_\kappa m_+(\Phi^{-1}(U))$.*

Proof. For any measurable $E \subset \mathcal{W}(\Sigma^{\mathcal{W}})$, (3.20) implies that

$$\int\int_U \mathcal{W}_*(\mathbb{P}_{x,y} \oplus \mathbb{P}_\kappa)\{E\} G(x,y)dxdy = \mathcal{W}_*(\mathbb{P}_{x,y} \oplus \mathbb{P}_\kappa) \overleftarrow{\otimes} \mathbb{I}_U G(x,y)dxdy\{E \times Q_1\}$$

$$= \mathcal{W}_*(\mathbb{P}_\kappa) \otimes \mathcal{M}_U\{E \times Q_1\} = \int_E \mathcal{M}_U(\Phi, Q_1) d\mathcal{W}_*(\mathbb{P}_\kappa)\{\Phi\}$$

$$= \int_E C_{\kappa,1} m_+\{\Phi^{-1}(U)\} d\mathcal{W}_*(\mathbb{P}_\kappa)\{\Phi\}.$$

□

3.6 One-point estimate

The goal of this section is to establish a one-point estimate for the BSLE welding, similar to the one-point estimate used to establish the Green's function for forward SLE. After doing this, we will see that $G(x,y) = G^{-\kappa-4,-\kappa-4}(x,y) = x^{1+4/\kappa} y^{1+4/\kappa} (x+y)^{-4-\kappa/2-8/\kappa}$ serves as an appropriate Green's function to use to condition welding x and y together. This is the only special case in which we have been able to construct $G^{a,b}$ via a geometrically motivated limit procedure.

Fix any $x, y > 0$. We will again be doing Ito's formula calculations using the definitions from Section 3.3. For clarity, we will review the definitions. Let $X_t = f_t(x) - \sqrt{\kappa}B_t$ and $Y_t = \sqrt{\kappa}B_t - f_t(-y)$ so that both X and Y are positive. Ito's formula implies that

$$dX_t = -\sqrt{\kappa}dB_t - \frac{2}{X_t}dt, \text{ and } dY_t = \sqrt{\kappa}dB_t - \frac{2}{Y_t}dt.$$

Moreover,

$$\frac{d(X_t + Y_t)}{X_t + Y_t} = \frac{-2}{X_t Y_t}dt. \tag{3.25}$$

Define $W_t = \frac{X_t - Y_t}{X_t + Y_t}$ as before, which is a process which stays in $(-1, 1)$ for as long as it exists. It only fails to exist if $\tau_x = \tau_{-y}$, where τ_z is the time so that z is swallowed by the BSLE traces. If this does not happen (which is $\mathbb{P}_\kappa$- a.s.), then W_t is constant at ± 1 after $\tau = \tau_{x,y} := \min\{\tau_x, \tau_{-y}\}$. What we want to estimate is $\mathbb{P}(E_\epsilon)$, where $E_\epsilon = \{X_\tau + Y_\tau \leq \epsilon\}$.

Theorem 7. *For any $x, y > 0$, with the notation above, there is a constant $C < \infty$ depending only on κ so that*

$$\lim_{\epsilon \to 0} \epsilon^{-\frac{1}{2}(\kappa+4)}\mathbb{P}(E_\epsilon) = Cx^{1+4/\kappa}y^{1+4/\kappa}(x+y)^{-4-\kappa/2-8/\kappa} := G(x, y). \tag{3.26}$$

Proof. By Ito's formula,

$$dW_t = \frac{-2\sqrt{\kappa}}{X_t + Y_t}dB_t + \frac{4W_t}{X_t Y_t}dt.$$

Performing the random time change $u(t) = \int_0^t \frac{\kappa}{X_r Y_r}dr$, we get

$$d\tilde{W}_t = -\sqrt{1 - \tilde{W}_t^2}d\tilde{B}_t + \frac{4}{\kappa}\tilde{W}_t dt, \tag{3.27}$$

where $\tilde{B}_s$ is a Brownian motion and $\tilde{W}_t = W_{u^{-1}(t)}$. Recall that $\hat{W}_t$ is a radial Bessel process. Observe that $u_\mathcal{T}$ is the lifetime of a process satisfying (3.7). By [39], there exists a constant C depending only on κ so that for any $t > 0$,

$$\mathbb{P}\{u_\mathcal{T} > t\} \leq C \exp\{\frac{-1}{2}(2 + \frac{8}{\kappa})t\}. \tag{3.28}$$

By equation (3.5), we have $X_t + Y_t = \epsilon$ if and only if $u_t = -\frac{\kappa}{2} \ln \left(\frac{\epsilon}{x+y}\right) := h(\epsilon)$, and $E_\epsilon = \{u_\mathcal{T} > h(\epsilon)\}$. Thus, equation (3.28) implies

$$\mathbb{P}(E_\epsilon) = \mathbb{P}\{u_\mathcal{T} > \frac{-\kappa}{2} \ln \left(\frac{\epsilon}{x+y}\right)\}$$

$$\leq C \exp\{\frac{-1}{2}(2 + 8/\kappa)\frac{-\kappa}{2} \ln \left(\frac{\epsilon}{x+y}\right)\} = C \left(\frac{\epsilon}{x+y}\right)^{\frac{1}{2}(\kappa+4)}. \tag{3.29}$$

In order to find the exact value of equation (3.26), we make use of a more precise bound for $\mathbb{P}(u_\mathcal{T} > t)$. Equation (B.14) in [39] states that

$$|\mathbb{P}\{T > t\} - 2\tilde{p}(t, w)| \leq Ce^{-(6+12/\kappa)t}, \tag{3.30}$$

where $w = (x - y)/(x + y)$ and

$$\tilde{p}(t, w) = C_\kappa (1 - w^2)^{1+4/\kappa} e^{-(1+4/\kappa)t}.$$

Inequality (3.29) implies that

$$|\mathbb{P}(E_\epsilon) - 2\tilde{p}(h(\epsilon), w)| \leq C \exp\{-(6 + 12/\kappa)(-\kappa/2) \ln \left(\frac{\epsilon}{x+y}\right)\} = C \left(\frac{\epsilon}{x+y}\right)^{3\kappa+6}.$$

Multiplying through by $\epsilon^{-1/2(\kappa+4)}$ gives

$$|\epsilon^{-1/2(\kappa+4)}\mathbb{P}(E_\epsilon) - 2\epsilon^{-1/2(\kappa+4)}\tilde{p}(h(\epsilon), w)| \leq C\left(\frac{1}{x+y}\right)^{3\kappa+6} \epsilon^{5/2\kappa+4} \to 0$$

as $\epsilon \to 0$. Thus,

$$\lim_{\epsilon\to 0} \epsilon^{-1/2(\kappa+4)}\mathbb{P}(E_\epsilon) = \lim_{\epsilon\to 0} 2\epsilon^{-1/2(\kappa+4)}\tilde{p}(h(\epsilon), w),$$

if this second limit exists.

By the definition of $\tilde{p}$, we have

$$\tilde{p}(h(\epsilon), w) = C_\kappa(1-w^2)^{1+4/\kappa}\left(\frac{\epsilon}{x+y}\right)^{1/2(\kappa+4)},$$

and therefore, using $w = (x-y)/(x+y)$, we get

$$\epsilon^{-1/2(\kappa+4)}\tilde{p}(h(\epsilon), w) = C_\kappa \frac{(1-w^2)^{1+4/\kappa}}{(x+y)^{1/2(\kappa+4)}} = C_\kappa x^{1+4/\kappa} y^{1+4/\kappa}(x+y)^{-4-\kappa/2-8/\kappa}$$

for every $\epsilon > 0$. □

Chapter 4

Measure driven chordal Loewner evolution

4.1 Theorem statement and definitions

For any $T > 0$, let $\mathcal{N}_T$ denote the set of positive measures μ on $[0, T] \times \mathbb{R}$ whose first coordinate marginal is Lebesgue measure m on the Borel subsets of $[0, T]$. That is, for any Borel set $E \subset [0, T]$, we have $\mu(E \times \mathbb{R}) = m(E)$. We also assume that the support of μ is compact. Define R_T to be the minimal $R > 0$ so that $\mu|_{[0,T]\times\mathbb{R}} = \mu|_{[0,T]\times[-R,R]}$. This set of measures has a natural topology induced by weak convergence of measures. That is, $\mu_n \to \mu$ in $\mathcal{N}_T$ if for every bounded $\phi \in C([0, T] \times \mathbb{R})$, where $C(X)$ = continuous functions on X, we have

$$\lim_{n\to\infty} \int_{[0,T]\times\mathbb{R}} \phi d\mu_n \to \int_{[0,T]\times\mathbb{R}} \phi d\mu.$$

Similarly, define the space $\mathcal{N}$ by $\mu \in \mathcal{N}$ if μ is a measure on $[0, \infty) \times \mathbb{R}$ which has locally compact support and whose first coordinate marginal is Lebesgue measure. Then we define the natural projections $P_T : \mathcal{N} \to \mathcal{N}_T$ by $P_T(\mu) = \mu|_{[0,T]\times\mathbb{R}}$. The topology on $\mathcal{N}$ is determined by $\mu_n \to \mu$ in $\mathcal{N}$ if $P_T(\mu_n) \to P_T(\mu)$ for every positive T.

Let $\mathcal{G}$ be the set of familes of conformal maps (g_t) with $g_t : \mathbb{H}\backslash K_t \to \mathbb{H}$, where (K_t) is a family of growing $\mathbb{H}$-hulls with $\text{hcap}(K_t) = 2t$ for every t. We will always use (f_t) to represent the family of inverse maps $f_t : \mathbb{H} \to \mathbb{H}\backslash K_t$. We put a topology on $\mathcal{G}$ induced by $(g_t^n) \to (g_t)$ if $f_t^n(z) \to f_t(z)$ locally uniformly in (t, z).

For $\mu \in \mathcal{N}$, we say that (g_t) solves the chordal Loewner equation driven by μ if

$$g_t(z) - z = \int_{[0,t]\times\mathbb{R}} \Phi(u, g_s(z)) d\mu(s, u), \quad \Phi(u, z) = \frac{2}{z - u}, \tag{4.1}$$

for every $z \in \mathbb{H}\backslash K_t$, where K_t is the set where the solution blows up by time t.

Theorem 8. *a) There is a one-to-one correspondence between measures $\mu \in \mathcal{N}$ and continuously growing families of $\mathbb{H}$-hulls (K_t) with hcap$(K_t) = 2t$. This correspondence is given by the unique solution (g_t) of (4.1) driven by μ, where $g_t = g_{K_t}$.*

b) Suppose $\{\mu^n\}$ and μ in $\mathcal{N}$ have solutions to (4.1) given by (g_t^n) and (g_t) respectively. Assume further that for all $t > 0$, the associated hulls $\{K_t^n\}_{n\in\mathbb{N}}$ are uniformly bounded. Then $\mu^n \to \mu$ in $\mathcal{N}$ if and only if the associated hulls $K_t^n \to K_t$ in the Carathéodory topology.

The basic argument for the proof is as follows. First, given a measure $\mu \in \mathcal{N}$, we use the general form of the chordal Loewner equation in [7] to approximate a disintegration of μ as $d\mu_t^n dt \to d\mu_t dt = d\mu$. For each n, the family of measures $d\mu_t^n$ define a family of chordal Loewner hulls $(K_t^n)_{t>0}$ with corresponding Loewner maps $(g_t^n)_{t>0}$. We show that subsequences of these converge to a family of conformal maps (g_t) and chordal hulls (K_t) such that g_t solves (4.1). To show the correspondence in the other direction, we start with a family of hulls (K_t). For any hull K with hcap$(K) = 2T$ we show that there is some measure μ_T so that μ_T can be associated with K. For the whole process (K_t), we use this construction to build measures μ_n^δ which account for growth in each interval of length δ.

Taking the limit of this approximation as $\delta \to 0$, we get a measure μ which corresponds to the family of hulls (K_t).

4.2 Preliminary results

Theorem 8 mentions convergence in the Carathéodory sense. The definition of this convergence can be found in [26], and will be restated here. Let $\{D_n\}_{n\in\mathbb{N}}$ and D be domains in $\mathbb{C}$. We say that $D_n \to D$ in the Carathéodory sense if

(i) For any compact $K \subset D$, there is an N so that $K \subset D_n$ for all $n \geq N$.

(ii) For any $z \in \partial D$, there exists a sequence $z_n \in \partial D_n$ so that $z_n \to z$.

Hulls K^n are said to converge in the Carathéodory sense to K if their complements converge. That is, if $\mathbb{H}\backslash K^n \to \mathbb{H}\backslash K$ in the Carathéodory sense. The following lemma can be found in [26].

Lemma 17. *Suppose that $D_n \to D$ as domains, $g_n : D_n \to \mathbb{C}$ are conformal, and $g : D \to \mathbb{C}$ is a function so that $g_n \to g$ locally uniformly on D. Then either g is constant or g is conformal. In the latter case, the following statements hold:*

(a) $g_n(D_n) \to g(D)$ in the Carathéodory topology,

(b) $g_n^{-1} \to g^{-1}$ locally uniformly in $g(D)$.

Here we provide the proof that the assumptions of the classical existence and uniqueness theorem are satisfied for the chordal Loewner ODE driven by a weakly continuous family of measures on $\mathbb{R}$ as introduced in [7].

Proposition 7. *Suppose that* $\{\nu_t\}_{t \leq T}$ *is a family of positive Borel measures on* $\mathbb{R}$ *so that* $t \mapsto \nu_t$ *is continuous in the weak topology. Further, assume that for all* t, *there exists some* $M_t > 0$ *so that* $\nu_s(\mathbb{R}) \leq M_t$ *and* $supp\{\nu_s\} \subset [-M_t, M_t]$ *for all* $s \leq t$. *Then for all* $z \in \mathbb{H}$, *there exists a unique solution to the differential equation*

$$\partial_t g_t(z) = \int_{\mathbb{R}} \frac{2}{g_t(z) - u} d\nu_t(z), \quad g_0(z) = z$$

on some maximal interval $[0, T_z)$.

Proof. Fix $z \in \mathbb{H}$ and $T > 0$. Define $R = \{(t, w) : 0 \leq t \leq T, \mathrm{Im}(w) \geq \mathrm{Im}(z)/2\}$. Let

$$\psi(t, w) = \int_{\mathbb{R}} \frac{2}{w - u} d\nu_t(u),$$

so that $\partial_t g_t(z) = \psi(t, g_t(z))$. By the classical local and maximal existence and uniqueness theorems for ODEs, if both ψ and $\partial_w \psi$ are continuous in some $[0, t] \times B(z, r) \subset R$, then there exists a solution to the ODE in some maximal time interval $[0, T_z)$.

First, we compute $\partial_w \psi(t, w)$. Observe that if w, ξ have imaginary parts at least $\mathrm{Im}(z)/2$, then

$$\frac{\psi(t, w) - \psi(t, \xi)}{w - \xi} = \int_{\mathbb{R}} \frac{1}{(w - \xi)} \left(\frac{2}{w - u} - \frac{2}{\xi - u} \right) d\nu_t(u)$$

$$= \int_{\mathbb{R}} \frac{2}{w - \xi} \cdot \frac{\xi - w}{(w - u)(\xi - u)} d\nu_t(u) \to \int_{\mathbb{R}} \frac{-2}{(w - u)^2} d\nu_t(u),$$

where the limit is as $\xi \to w$. This follows by the dominated convergence theorem, since $|\xi - u|, |w - u| \geq \mathrm{Im}(z)/2$ and ν_t is a finite measure.

Next, we will prove that ψ is continuous in the desired region. Suppose that $(t_n, w_n) \to (t, w)$, with $(t_n, w_n) \in R$. Then

$$|\psi(t_n, w_n) - \psi(t, w)|$$

$$\leq \left| \int_{\mathbb{R}} \frac{2}{w_n - u} d\nu_{t_n}(u) - \int_{\mathbb{R}} \frac{2}{w - u} d\nu_{t_n}(u) \right| + \left| \int_{\mathbb{R}} \frac{2}{w - u} d\nu_{t_n}(u) - \int_{\mathbb{R}} \frac{2}{w - u} d\nu_t(u) \right|.$$

Since Im$(w) \geq$Im$(z)/2$, the function $u \mapsto \dfrac{2}{w-u}$ is uniformly bounded, so weak continuity implies that the second term approaches 0. To estimate the first term, the map $w \mapsto (w-u)^{-1}$ is Lipschitz with constant C depending only on the fixed z, so

$$\left| \int_{\mathbb{R}} \frac{2}{w_n - u} d\nu_{t_n}(u) - \int_{\mathbb{R}} \frac{2}{w - u} d\nu_{t_n}(u) \right| \leq C \int_{\mathbb{R}} |w_n - w| d\nu_{t_n}(u) \leq C M_T |w_n - w| \to 0.$$

Thus, ψ is continuous in the desired region. The proof for $\partial_w \psi$ will be identical based on the above formula, only it will use the Lipschitz constant for $w \mapsto (w-u)^{-2}$.

□

We need to establish an important result about $\mathbb{H}$-hulls. In [36], it is proven that there is a positive measure μ_K with support in S_K^* so that $\|\mu_K\| =$hcap(K), and

$$f_K(z) - z = \int_{S_K^*} \frac{-1}{z-u} d\mu_K(u).$$

Lemma 18. *If K is an $\mathbb{H}$-hull, then for all $z \in \mathbb{H}$,*

$$|f_K(z) - z| \leq 7\,diam(K).$$

Proof. Suppose without loss of generality that $K \subset K_r := \{z \in \mathbb{H} : |z| \leq r\}$, where $r = \max\{|z| : z \in K\}$. In [7], it is proven that hcap$(K_r) = r^2$ and $S_K^* \subset S_{K_r}^* = [-2r, 2r]$.

Fix any $L > 2r$, and suppose that $|z| \geq L$. Then

$$|f_K(z) - z| \leq \int_{S_K^*} \left| \frac{1}{z-u} \right| d\mu_K(u) \leq \frac{\|\mu_K\|}{L-2r} \leq \frac{r^2}{L-2r}.$$

Suppose that $|z| \leq L$. Then the maximum modulus principle implies that

$$|f_K(z)| \leq \sup_{\{|w|=L\} \cup \{-L<w<L\}} |f_K(w)|.$$

Since f_K can be extended to an analytic function on $\mathbb{C} \backslash S_K$ (see [26]), the maximum in $\{|w| \leq L\}$ occurs either on $A_L := \{w \in \mathbb{H} : |w| = L\}$ or on S_K. Extending f_K continuously to $\overline{\mathbb{H}}$, we get that $|f_K(z)| \leq$diam$(K) \leq r$ as $z \to S_K$. For $z \in A_L$, we've already shown that $|f_K(z)| \leq |z| + \dfrac{r^2}{L-2r} = L$. Thus, picking $L = 3r$, we get that

$$|f_K(z) - z| \leq |z| + \max\{r, |z| + \frac{r^2}{3r-2r}\} \leq 3r + \max\{r, 3r + r\} = 7r.$$

□

4.3 A disintegration lemma

We begin by demonstrating how we disintegrate a measure $\mu \in \mathcal{N}$.

Lemma 19. *Let $\mu \in \mathcal{N}$. Then there exists a family of measures $(\mu_t^n)_{t>0, n\in\mathbb{N}}$ on $\mathbb{R}$ so that*

(a) Each μ_t^n is a probability measure on $\mathbb{R}$.

(b) For every $n \in \mathbb{N}$, the map $t \mapsto \mu_t^n$ is continuous with respect to the weak topology on measures on $\mathbb{R}$.

(c) *The measure given by* $d\mu_t^n dt$ *is in* $\mathcal{N}$*, and* $d\mu_t^n dt \to \mu$ *as* $n \to \infty$ *in* $\mathcal{N}$*.*

(d) *For almost every* $t \in [0, \infty)$*, there is a probability measure* μ_t *so that* $\mu_t^n \to \mu_t$ *weakly. Also,* $d\mu_t dt = d\mu$*.*

Proof. First, we must define the measures μ_t^n. For each $n \in \mathbb{N}, t > 0$, define a linear functional $L_{n,t} : C_c(\mathbb{R}) \to \mathbb{C}$ by

$$L_{n,t}(\phi) = n \int_{[t,t+1/n]\times\mathbb{R}} \phi(u) d\mu(s,u). \tag{4.2}$$

To see that this map is well defined, observe that

$$|L_{n,t}(\phi)| \leq n \int_{[t,t+1/n]\times\mathbb{R}} |\phi(u)| d\mu(s,u) \leq \|\phi\|_\infty n\mu([t, t+1/n] \times \mathbb{R}) = \|\phi\|_\infty < \infty.$$

The map $L_{n,t}$ is clearly a positive linear functional. By the Riesz representation theorem, there exists a Radon measure μ_t^n defined on the Borel sets of $\mathbb{R}$ so that equation (4.2)= $\int_\mathbb{R} \phi d\mu_t^n$.

To prove (a), for every $k \in \mathbb{N}$, define $\phi_k \in C_c(\mathbb{R})$ so that $\phi_k(u) = 1$ for $u \in [-k, k]$, and $\phi_k(u) \to 1$ monotonically for every u. By the monotone convergence theorem (applied twice) and equation (4.2),

$$\mu_t^n(\mathbb{R}) = \lim_{k\to\infty} \int_\mathbb{R} \phi_k d\mu_t^n = \lim_{k\to\infty} n \int_{[t,t+1/n]\times\mathbb{R}} \phi_k(u) d\mu(s,u)$$

$$= n\mu([t, t+1/n] \times \mathbb{R}) = 1.$$

We will now make several arguments about the measures converging weakly on $\mathbb{R}$. For weak convergence, we need to test all bounded continuous functions on $\mathbb{R}$. However, we are testing the local properties of the measures μ before some time T. We assume that our measure μ, when restricted to $[0,T]\times\mathbb{R}$, has support contained in $[0,T]\times[-R_T,R_T]$. Therefore, if ϕ is any bounded continuous function, there exists some $\phi^*\in C_c(\mathbb{R})$ so that $\phi^*=\phi$ on $[-R_T,R_T]$. Hence, for all weak convergence arguments below, we will make this reduction automatically and work only with continuous functions with compact support.

To see (b), suppose that $t_k\to t$ in $[0,\infty)$. Then $t_k\to t$ in $[0,T]$ for some T. Let $\phi\in C_c(\mathbb{R})$ be arbitrary. Then

$$\int_{\mathbb{R}}\phi d\mu^n_{t_k}=n\int_{[t_k,t_k+1/n]\times\mathbb{R}}\phi(u)d\mu(s,u)$$

$$=n\int_{[0,T]\times\mathbb{R}}\mathbb{I}_{[t_k,t_k+1/n]\times\mathbb{R}}(s,u)\phi(u)d\mu(s,u). \tag{4.3}$$

Then $\mathbb{I}_{[t_k,t_k+1/n]\times\mathbb{R}}(s,u)\phi(u)\to\mathbb{I}_{\{[t,t+1/n]\times\mathbb{R}\}}(s,u)\phi(u)$ μ-almost everywhere, and is dominated by $\|\phi\|_\infty\in L^1([0,T]\times\mathbb{R},\mu)$. Thus, Lebesgue's dominated convergence theorem implies that

$$\lim_{k\to\infty}(4.3)=n\int_{[0,T]\times\mathbb{R}}\mathbb{I}_{\{[t,t+1/n]\times\mathbb{R}\}}(s,u)\phi(u)d\mu(s,u)=\int_{\mathbb{R}}\phi d\mu^n_t.$$

Now we must show c), starting with $d\mu^n_t dt\in\mathcal{N}$. For any Borel set $E\subset\mathbb{R}$, part (a) implies that

$$\int_E\left(\int_{\mathbb{R}}1d\mu^n_t\right)dt=\int_E dt=m(E).$$

Also, if $t\le T$, then it is clear that the support of μ^n_t is contained in $[-R_{T+1},R_{T+1}]$. Therefore, $d\mu^n_t dt\in\mathcal{N}$. To show convergence, suppose that $\phi\in C_c([0,T]\times\mathbb{R})$. For any $T>0$,

$$\int_{[0,T]\times\mathbb{R}}\phi(t,u)d\mu^n_t(u)dt=\int_{[0,T]}\left(\int_{\mathbb{R}}\phi(t,u)d\mu^n_t(u)\right)dt$$

$$=\int_0^T\left(n\int_{[t,t+1/n]\times\mathbb{R}}\phi(t,u)d\mu(s,u)\right)dt$$

$$= \left(\int_{[0,1/n]\times\mathbb{R}} \int_0^s + \int_{[1/n,T]\times\mathbb{R}} \int_{s-1/n}^s + \int_{[T,T+1/n]\times\mathbb{R}} \int_{s-1/n}^T \right) n\phi(t,u)dtd\mu(s,u).$$

Since ϕ has compact support, fixing n guarantees that this function is $L^1(\mu \times m)$, so the above use of Fubini's theorem is justified. We will show that the first and third terms go to 0, and the middle term converges to $\int_{[0,T]\times\mathbb{R}} \phi(s,u)d\mu(s,u)$.

To bound the first term, observe that

$$\left| \int_{[0,1/n]\times\mathbb{R}} \int_0^s n\phi(t,u)dtd\mu(s,u) \right| \le \|\phi\| n \int_{[0,1/n]\times\mathbb{R}} \int_0^s 1dtd\mu(s,u)$$

$$= \|\phi\| n \int_{[0,1/n]\times\mathbb{R}} sd\mu(s,u) \le \|\phi\| n \int_{[0,1/n]\times\mathbb{R}} \frac{1}{n} d\mu(s,u) = \frac{\|\phi\|}{n},$$

which converges to 0 as n goes to infinity. The third integral converges to 0 by a similar argument. Thus, it suffices to show that

$$\lim_{n\to\infty} \int_{[1/n,T]\times\mathbb{R}} \left(\frac{1}{1/n} \int_{s-1/n}^s \phi(t,u)dt \right) d\mu(s,u) = \int_{[0,T]\times\mathbb{R}} \phi(s,u)d\mu(s,u). \tag{4.4}$$

For each u, the map $t \mapsto \phi(t,u)$ is continuous on $[0,\infty)$, so the fundamental theorem of calculus implies that

$$\lim_{n\to\infty} \frac{1}{1/n} \int_{s-1/n}^s \phi(t,u)dt = \phi(s,u)$$

for each s. Thus, for μ-almost every (s,u),

$$\mathbb{I}_{\{[1/n,T]\times\mathbb{R}\}}(s,u) \left(n \int_{s-1/n}^s \phi(t,u)dt \right) \to \mathbb{I}_{\{[0,T]\times\mathbb{R}\}}(s,u)\phi(s,u)$$

as $n \to \infty$. Also, for every $n \in \mathbb{N}, s > 0$, and $u \in \mathbb{R}$, we have

$$\left| \mathbb{I}_{\{[1/n,T]\times\mathbb{R}\}}(s,u) n \int_{s-1/n}^{s} \phi(t,u)dt \right| \leq \|\phi\|_\infty \mathbb{I}_{[0,T]\times[-N,N]}(s,u) \in L^1(\mu),$$

where the support of ϕ is contained in $[0,T]\times[-N,N]$ for some $N > 0$. Therefore, Lebesgue's dominated convergence theorem implies that (4.4) holds, which completes the proof of (c).

We now prove (d). Since μ_t^n is a collection of Radon probability measures with uniformly compact support for each t, there is a subsequential weak limit μ_t. Let $\{\phi_k\}_{k\in\mathbb{N}}$ be a dense countable subset of $C_c(\mathbb{R})$, which is separable. We claim that for each k, there exists a subset $E_k \subset [0,\infty)$ so that $m(E_k^c) = 0$, and for all $t \in E_k$ we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} \phi_k(u) d\mu_t^n(u) = \int_{\mathbb{R}} \phi_k(u) d\mu_t(u).$$

To see this, define a measure ν_k on $\mathbb{R}$ by $\nu_k(E) := \int_{E\times\mathbb{R}} \phi_k(u) d\mu(s,u)$, which is absolutely continuous with respect to Lebesgue measure. By ([28], Theorem 7.14) we have for m-almost all t (where a.e. depends on ϕ_k, hence the set E_k),

$$\int_{\mathbb{R}} \phi_k d\mu_t^n = \frac{1}{1/n} \int_{[t,t+1/n]} \phi_k(u) d\mu(s,u) = \frac{\nu_k[t,t+1/n]}{m[t,t+1/n]} \to \frac{d\nu_k}{dm}(t).$$

Thus, $\lim_n \int \phi_k d\mu_t^n$ exists, and there is subsequential limit equal to $\int \phi_k d\mu_t$, and therefore $\int \phi_k d\mu_t^n \to \int \phi_k d\mu_t$ for each k.

Let $E = \cap_k E_k$, so that $m(E^c) = 0$, and suppose $t \in E$. Let $\phi \in C_c(\mathbb{R})$. We claim that $\int \phi d\mu_t^n \to \int \phi d\mu_t$ as $n \to \infty$. Let $\epsilon > 0$. Then there exists some k so that $\|\phi_k - \phi\|_\infty < \epsilon$.

Then for sufficiently large n,

$$\left|\int_{\mathbb{R}} \phi(u) d\mu_t^n(u) - \int_{\mathbb{R}} \phi(u) d\mu_t(u)\right|$$

$$\leq \left|\int_{\mathbb{R}} \phi - \phi_k d\mu_t^n\right| + \left|\int_{\mathbb{R}} \phi_k d\mu_t^n - \int_{\mathbb{R}} \phi_k d\mu_t\right| + \left|\int_{\mathbb{R}} \phi_k - \phi d\mu_t\right| \leq \epsilon + \epsilon + \epsilon.$$

Thus, $\mu_t^n \to \mu_t$ weakly for all $t \in E$, and hence for m-almost every t.

The final claim is true because for any $\phi \in C_c([0,T] \times \mathbb{R})$, for any u and almost any s, we have $\int_{\mathbb{R}} \phi(s,u) d\mu_s(u) = \lim_{n\to\infty} \int_{\mathbb{R}} \phi(s,u) d\mu_s^n(u)$. Therefore, the dominated convergence theorem implies that

$$\int_0^t \int_{\mathbb{R}} \phi(s,u) d\mu_s(u) ds = \int_0^t \lim_{n\to\infty} \int_{\mathbb{R}} \phi(s,u) d\mu_s^n(u) ds$$

$$= \lim_{n\to\infty} \int_0^t \int_{\mathbb{R}} \phi(s,u) d\mu_s^n(u) ds = \int_{[0,t]\times\mathbb{R}} \phi(s,u) d\mu(s,u).$$

The last equality follows from part (c), since $d\mu_t^n dt \to d\mu$ in $\mathcal{N}$.

□

Remark: For each $t > 0$ so that μ_t as in part (d) exists, we can exactly repeat this construction with the functional

$$\phi \to n \int_{[t-1/n,t]\times\mathbb{R}} \phi(u) d\mu(s,u)$$

for each n. The resulting measures will converge to the same μ_t for almost all t. Therefore, if $h < 0$, we can use the convention

$$\int_{[t,t+h]\times\mathbb{R}} \phi(s,u)d\mu(s,u) = -\int_{[t+h,t]\times\mathbb{R}} \phi(s,u)d\mu(s,u)$$

as in Riemann integration to discuss differences $g_{t+h}(z) - g_t(z)$ without making reference to whether h is positive or negative.

4.4 Elementary calculations

Next, we will make several calculations assuming that (4.1) driven by μ has a solution (g_t). These calculations will then be applied to the measures $d\mu^n = d\mu_t^n dt$, which have solutions (g_t^n) for each n by Proposition 7. Moreover, the maps g_t^n are all conformal, and a solution to (4.1) will be constructed as a locally uniform limit of g_t^n, and so we can treat any solution as conformal. We state and prove these lemmas in their more general state since we will need them after a solution is proven to exist. First, we have to prove two Lipschitz estimates which will be used repeatedly.

Lemma 20. *a) If $z, w \in \mathbb{H}$ with imaginary parts at least $h > 0$ and $u \in \mathbb{R}$, then*

$$|\Phi(u,z) - \Phi(u,w)| \leq \frac{2}{h^2}|z-w|.$$

b) Suppose (g_t) is a solution to the chordal Loewner equation driven by $\mu \in \mathcal{N}$. Then for any $z \in \mathbb{H}$ and $t_1 < t_2 \leq T < \tau_z$,

$$|g_{t_2}(z) - g_{t_1}(z)| \leq \frac{2}{Im(g_T(z))}|t_2 - t_1|.$$

Proof. To prove a), observe that $|z-u| \geq h$ and $|w-u| \geq h$. Therefore,

$$|\Phi(u,z) - \Phi(u,w)| = \left| \frac{2}{z-u} - \frac{2}{w-u} \right| = 2 \left| \frac{z-w}{(z-u)(w-u)} \right| \leq \frac{2}{h^2}|z-w|.$$

To prove b), observe that (4.1) implies that the imaginary part of $g_t(z)$ is a decreasing function of t. Therefore,

$$|g_{t_2}(z) - g_{t_1}(z)| = \left| \int_{[t_1,t_2]\times\mathbb{R}} \frac{2}{g_s(z)-u} d\mu(s,u) \right| \leq \int_{[t_1,t_2]\times\mathbb{R}} \frac{2}{|g_s(z)-u|} d\mu(s,u)$$

$$\leq \int_{[t_1,t_2]\times\mathbb{R}} \frac{2}{\mathrm{Im}(g_s(z))} d\mu(s,u) \leq \int_{[t_1,t_2]\times\mathbb{R}} \frac{2}{\mathrm{Im}(g_T(z))} d\mu(s,u) = \frac{2}{\mathrm{Im}(g_T(z))}|t_2 - t_1|.$$

□

Lemma 21. *If (g_t) is a solution to the chordal Loewner equation driven by $\mu \in \mathcal{N}$ and $z \in \mathbb{H}$. Then for almost any $t < \tau_z$,*

$$\partial_t g_t(z) = \int_{\mathbb{R}} \frac{2}{g_t(z)-u} d\mu_t(u),$$

where μ_t is the measure from Lemma 19 part d).

Proof. This follows from Lemma 19 part d), which says that $d\mu_t(u)dt = d\mu(t,u)$. In particular, if $T < \tau_z$, Lemma 19 part d) implies that for $t < T$,

$$g_t(z) - z = \int_{[0,t]\times\mathbb{R}} \Phi(u, g_s(z)) d\mu(s,u) = \int_0^t \int_{\mathbb{R}} \frac{2}{g_s(z)-u} d\mu_s(u) ds.$$

Since

$$\left| \int_{\mathbb{R}} \frac{2}{g_s(z)-u} d\mu_s(u) \right| \leq \frac{2}{\mathrm{Im}(g_T(z))} \in L^1([0,T]),$$

the fundamental theorem of calculus implies that $\partial_t g_t(z) = \int_{\mathbb{R}} \frac{2}{g_t(z)-z} d\mu_t(z)$ for almost every t. □

We can now prove uniqueness.

Proposition 8. *A solution of the chordal Loewner equation driven by $\mu \in \mathcal{N}$ must be unique.*

Proof. Fix $z \in \mathbb{H}$, and suppose that g_t and $\tilde{g}_t$ are two solutions of the integral equation

$$g_t(z) = z + \int_{[0,t]\times\mathbb{R}} \frac{2}{g_s(z) - u} d\mu(s, u)$$

up to some time $T + \epsilon$. Then $\{\text{Im}(g_t)\}_{t \le T}$ is decreasing and bounded below by $\text{Im}(g_T)$. The same is true for $\tilde{g}_t$, so we can assume that there is a constant C so that for all $t \le T$, we have $\text{Im}(g_t)$ and $\text{Im}(\tilde{g}_t)$ are bounded below by C. By Lemma 21, for almost all $t \le T$ (depending on z, which is fixed), we know that

$$\partial_t(g_t - \tilde{g}_t) = 2 \int_{\mathbb{R}} \frac{\tilde{g}_t - g_t}{(g_t - u)(\tilde{g}_t - u)} d\mu_t(u).$$

Therefore,

$$|\partial_t(g_t - \tilde{g}_t)| \le 2 \int_{\mathbb{R}} \frac{|\tilde{g}_t - g_t|}{|(g_t - u)(\tilde{g}_t - u)|} d\mu_t \le \frac{2}{C^2} \int_{\mathbb{R}} |\tilde{g}_t - g_t| d\mu_t(u) = \frac{2}{C^2} |g_t - \tilde{g}_t|.$$

Since g_t and $\tilde{g}_t$ are absolutely continuous by Lemma 20 , we get that

$$|g_t - \tilde{g}_t| \le \int_0^t |\partial_s(g_s - \tilde{g}_s)| ds \le \frac{2}{C^2} \int_0^t |g_s - \tilde{g}_s| ds.$$

By Gronwall's inequality applied to $|g_t - \tilde{g}_t|$, it follows that $g_t = \tilde{g}_t$ for all $t \le T$. Since T was an arbitrary number below the lower blow-up time, it follows that $g_t = \tilde{g}_t$ for all t.

□

Next, we need to derive some basic continuity and differentiability properties of the inverse function $f = f_t(z) = g_t^{-1}(z)$.

Lemma 22. *Suppose that $g(t,z) = g_t(z)$ solves the chordal Loewner equation driven by $\mu \in \mathcal{N}$, and let $f_t(z) = g_t(z)^{-1}$. Then both f and f' are continuous in the variables (t,z).*

Proof. The main part of this lemma is that if (K_t) is an increasing family of hulls with $\text{hcap}(K_t) = 2t$, then the map $t \mapsto \mathbb{H} \backslash K_t$ is continuous in the Carathéodory topology. Then Lemma 17, part (b) implies that if $t_n \to t$, then $f_{t_n} \to f_t$ locally uniformly in $\mathbb{H} \backslash K_t$, and therefore $f'_{t_n} \to f'_t$ locally uniformly in $\mathbb{H} \backslash K_t$.

It suffices to prove that $\mathbb{H} \backslash K_{t_n} \to \mathbb{H} \backslash K_t$ in the cases where t_n increases to t and t_n decreases to t separately. First, assume that t_n decreases to t. Then $K_t \subset K_{t_n}$ for all n. I claim that $\cap_n K_{t_n} = K_t$. Let $K' = \cap_n K_{t_n}$, which is an $\mathbb{H}$-hull with $K_t \subset K'$, and assume that $K' \neq K_t$. Then $\text{hcap}(K') > \text{hcap}(K_t)$. However, $2t_n = \text{hcap}(K_n) \geq \text{hcap}(K') > \text{hcap}(K_t) = 2t$, which contradicts the assumption that $t_n \to t$. Thus, $K_t = \cap_n K_{t_n}$. The properties for Carathéodory convergence are readily verified

Now, suppose that t_n increases to t. Note that we do not necessarily have $\cup_n K_{t_n} = K_t$, so we must argue differently. Observe that $\mathbb{H} \backslash K_t \subset \mathbb{H} \backslash K_{t_n}$ for each n, so g_{t_n} is defined on $\mathbb{H} \backslash K_{t_n}$ for each n. Let $D_n = D = \mathbb{H} \backslash K_t$. Then $D_n \to D$ clearly. Also, $g_{t_n} \to g_t$ locally uniformly, and g_t is not constant. By Lemma 17 (a), we have that $g_{t_n}(\mathbb{H} \backslash K_t) \to \mathbb{H}$. Applying it again to the inverse maps yields $\mathbb{H} \backslash K_{t_n} \to \mathbb{H} \backslash K_t$.

□

Lemma 23. *Let $\mu \in \mathcal{N}$, and let $d\mu^n = d\mu_t^n(u)dt$, where μ_t^n is as in Lemma 19. Then there* *are functions* (g_t^n) *which solve (4.1) driven by* μ^n*. If* $f_t^n = (g_t^n)^{-1}$*, then*

$$f_t^n(z) - z = -\int_{[0,t]\times\mathbb{R}} \frac{2(f_s^n)'(z)}{z-u} d\mu^n(s,u) \tag{4.5}$$

for each $z \in \mathbb{H}, t > 0, n \in \mathbb{N}$*.*

Proof. For each n, Lemma 19 part b) and Proposition 7 imply that there is a solution of

$$\partial_t g_t^n(z) = \int_{\mathbb{R}} \frac{2}{g_t^n(z) - u} d\mu_t^n(z), \quad g_0^n(z) = z,$$

which is the solution of (4.1) driven by μ^n. This equation holds for all $t < \tau_z$. If $f_t^n = (g_t^n)^{-1}$, the chain rule then implies that

$$\partial_t f_t^n(z) = \int_{\mathbb{R}} \frac{-2(f_t^n)'(z)}{z-u} d\mu_t^n(u), \quad f_0^n(z) = z.$$

Integrating this gives with respect to t yields (4.5).

□

We are now almost ready to prove existence of a solution to the chordal measure driven Loewner equation. In order to use a limiting argument, we need some sort of control on the size of the hulls $\{K_t^n\}_{t\leq t_0}$ for any fixed t_0. This control will be provided by the following two lemmas.

Lemma 24. *Suppose* (g_t) *is a solution of the chordal Loewner equation driven by* $\mu \in \mathcal{N}$*. For each* t*, define* $M_t = \max\{\sqrt{t}, R_t\}$*. Then for all* $z \in K_t$*, we have* $|z| \leq 4M_t$*. Moreover, if* $|z| > 4M_t$*, then* $|g_s(z) - z| \leq M_t$ *for all* $0 \leq s \leq t$*.*

Remark: This lemma is an extension of Lemma 4.13 in [7], where it is proven for the chordal Loewner equation driven by a continuous function. The proof is different, however. The argument follows the standard proof for the existence of a solution of an ODE.

Proof. Fix $t > 0$ and $z \in \mathbb{H}$ with $|z| \geq 4M_t$. Recall that R_t is defined to be $R_t = \inf\{R > 0 : \mu|_{[0,t]\times\mathbb{R}}$ is supported in $[0,t] \times [-R,R]\}$. We will use a Picard iteration argument to show that $\tau_z \geq t$, where τ_z is the lifetime of (4.1) at z. Let $w \in \overline{B(z, M_t)}$, which implies $|w| \geq 3M_t$. For any $u \in [-R_t, R_t]$, we have $|w - u| \geq |w| - |u| \geq 3M_t - M_t = 2M_t$. Therefore, if $\Phi(u,z) = \dfrac{2}{z-u}$, we have for any $w, w' \in \overline{B(z, M_t)}$,

$$|\Phi(u,w) - \Phi(u,w')| = \left|\frac{2(w-w')}{(w-u)(w'-u)}\right| \leq \frac{2|w-w'|}{4M_t^2} = \frac{1}{2M_t^2}|w-w'|.$$

This implies that $\Phi(u,w)$ has Lipschitz constant $\dfrac{1}{2M_t^2}$ for $w \in \overline{B(z, M_t)}$. Also, if $w \in \overline{B(z, M_t)}$ and $u \in [-R_t, R_t]$,

$$|\Phi(u,w)| \leq \frac{2}{2M_t} = \frac{1}{M_t}.$$

Then the restriction of Φ to $[-R_t, R_t] \times \overline{B(z, M_t)}$ satisfies $\|\Phi\|_\infty \leq 1/M_t$.

To perform the Picard iteration, fix an $a > 0$ to be determined later, and define an operator Λ on the set of continuous functions from $[0,a]$ to $\overline{B(z, M_t)}$ by

$$\Lambda(\phi)(r) = z + \int_{[0,r]\times\mathbb{R}} \Phi(u, \phi(s))d\mu(s,u), \quad 0 \leq r \leq a.$$

We will show that for each $a < t$, $\Lambda(\phi)$ will be a continuous map from $[0,a]$ to $\overline{B(z, M_t)}$. If $a < t$, then $a < M_t^2$ by construction of M_t. For $r \leq a$,

$$|\Lambda(\phi)(r) - z| = \left|\int_{[0,r]\times\mathbb{R}} \Phi(u, \phi(s))d\mu(s,u)\right|$$

$$\leq \int_{[0,r]\times\mathbb{R}} |\Phi(u,\phi(s))| d\mu(s,u) \leq \frac{a}{M_t} \leq M_t. \tag{4.6}$$

Therefore, $\Lambda(\phi)$ sends $[0,a]$ into $\overline{B(z,M_t)}$.

Next, we need to show that Λ is a contraction. For any continuous $\phi_1, \phi_2 : [0,a] \to \overline{B(z,M_t)}$, the Lipschitz estimate implies that for any $r \in [0,a]$,

$$|\Lambda(\phi_1)(r) - \Lambda(\phi_2)(r)| \leq \int_{[0,r]\times\mathbb{R}} |\Phi(u,\phi_1(s)) - \Phi(u,\phi_2(s))| \, d\mu(s,u)$$

$$\leq \int_{[0,a]\times\mathbb{R}} \frac{1}{2M_t^2} \|\phi_1 - \phi_2\|_\infty d\mu(s,u) \leq \frac{a}{M_t^2} \|\phi_1 - \phi_2\|_\infty,$$

where $a/M_t^2 < 1$. Thus, Λ is a contraction between Banach spaces, and the Banach fixed point theorem implies that there is a unique continuous $\phi : [0,a] \to \overline{B(z,M_t)}$ so that

$$\phi(r) = \Lambda(\phi)(r) = z + \int_{[0,r]\times\mathbb{R}} \frac{2}{\phi(s) - u} d\mu(s,u), \quad 0 \leq r \leq a.$$

By Proposition 8, $\phi(r) = g_r(z)$ for all $r \leq a < t$. Thus, $\tau_z \geq t$. Moreover, if $|z| > 4M_t$, this construction can be stretched to $a = t$, and so applying (4.6) to $g_t(z)$ proves $|g_s(z) - z| \leq M_t$ for all $s \leq t$.

□

Lemma 25. *If $\mu \in \mathcal{N}$, let μ_t^n be as in Lemma 19. For each n, let (g_t^n) be the Loewner maps driven by $\{\mu_t^n\}_{t\leq T}$ with corresponding $\mathbb{H}$-hulls $(K_t^n)_{t\leq T}$. Then for every $t_0 > 0$, there exists some $M_{t_0} > 0$ so that $K_{t_0}^n \subset \{z \in \mathbb{H} : |z| \leq M_{t_0}\}$.*

Proof. By Lemma 24, it suffices to show that the support of the measures $\{\mu_t^n\}_{t\leq t_0, n\in\mathbb{N}}$ are uniformly bounded. We have assumed that μ restricted to $[0,t_0+1]\times\mathbb{R}$ is supported in $[0,t_0+1]\times[-R,R]$ for some $R > 0$. Suppose that $x \in \mathbb{R}$ and $\epsilon > 0$ so that $(x-2\epsilon, x+2\epsilon)\times$

$[t, t+1/n) \subset ([0, t_0+1] \times [-R, R])^c$. In particular, $\mu((x-\epsilon, x+\epsilon) \times [t, t+1/n]) = 0$. Let $\phi_k \in C_c(\mathbb{R})$ be a monotonic sequence converging to $\mathbb{I}_{(x-\epsilon, x+\epsilon)}$. Then

$$\mu_t^n(x-\epsilon, x+\epsilon) = \lim_{k\to\infty} \int_{\mathbb{R}} \phi_k(u) d\mu_t^n(u) = \lim_{k\to\infty} n \int_{[t,t+1/n]\times\mathbb{R}} \phi_k(u) d\mu(s, u)$$

$$= n\mu((x-\epsilon, x+\epsilon) \times [t, t+1/n]) = 0.$$

Since the supports of the measures μ_t^n are uniformly bounded, it follows that the hulls K_t^n are bounded independently of n.

□

4.5 Proof of existence

Using the boundedness of hulls from Lemma 25, we can prove equicontinuity of some subsequence of any family f^n of inverse functions associated with a sequence of measures $\mu^n \in \mathcal{N}$ associated with μ.

Lemma 26. *Let $\mu \in \mathcal{N}$, and suppose $(g_t^n)_{t\geq 0}$ is the family of conformal maps which solve the chordal Loewner equations driven by the measures $d\mu^n(t, u) = d\mu_t^n(u)dt$ for each n. Then there exists a family of conformal maps (g_t) so that a subsequence of (g_t^n) converges to (g_t) in $\mathcal{G}$.*

Remark: The proof of this lemma will hold for any $\mu^n \in \mathcal{N}$ with $\mu^n \to \mathcal{N}$ as long as (4.5) holds at each n, and if the hulls (K_t^n) are uniformly bounded for each t. We will do this to prove that Theorem 8 part b) holds.

Proof. Fix $T < \infty$. Applying Lemma 18 to each member of the family $(K_t^n)_{t\leq T, n\in\mathbb{N}}$, and using the assumption that the hulls are uniformly bounded, we get that the family $(f_t^n)_{t\leq T, n\in\mathbb{N}}$ is normal in $\mathbb{H}$. We will use this normality and equation (4.5) to show that the functions f^n are equicontinuous in both t and z on any compact set $[0,N]\times R_N \subset [0,\infty)\times\mathbb{H}$, where $R_N = [-N,N]\times[1/N,N]$. Since $(f_t^n)_{t\leq T,n\in\mathbb{N}}$ is a normal family on $\mathbb{H}$, so are the families of derivatives $(f_t^n)'_{t\leq N,n\in\mathbb{N}}$ and $(f_t^n)''_{t\leq N,n\in\mathbb{N}}$. Since $R_N\subset\mathbb{H}$ is compact, this implies that there are constants $C'_N, C''_N < \infty$ with $|(f_s^n)'(z)|\leq C'_N$ and $|(f_s^n)''(z)|\leq C''_N$ for all $s\leq N$ and $z\in R_N$. Therefore,

$$|(f_s^n)'(z_1) - (f_s^n)'(z_2)| \leq C''_N |z_1 - z_2|, \quad \text{for all } s\leq N, z\in R_N. \tag{4.7}$$

If $0\leq t_1\leq t_2\leq N$ and $z_1, z_2\in R_N$, equation (4.5) implies that

$$|f_{t_1}^n(z_1) - f_{t_2}^n(z_2)| \leq |z_1 - z_2| + 2\left|\int_{[0,t_1]\times\mathbb{R}} \frac{(f_s^n)'(z_1)}{z_1-u} - \frac{(f_s^n)'(z_2)}{z_2-u} d\mu^n(s,u)\right|$$
$$+2\left|\int_{[t_1,t_2]\times\mathbb{R}} \frac{(f_s^n)'(z_2)}{z_2-u} d\mu^n(s,u)\right| = |z_1-z_2| + 2(\mathrm{I}) + 2(\mathrm{II}).$$

If we can bound (I) and (II) uniformly in N, $|t_2-t_1|$, and $|z_1-z_2|$, we will have that f^n is equicontinuous in $[0,N]\times R_N$.

For any $u\in\mathbb{R}$ and $z\in R_N$, we have $|z-u|\geq 1/N$. Therefore, (II) can be bounded by

$$(\mathrm{II}) \leq C'_N \int_{[t_1,t_2]\times\mathbb{R}} \frac{2}{|z-u|} d\mu(s,u) \leq C'_N(2N)|t_2-t_1|.$$

To estimate (I), observe that for any $u\in\mathbb{R}$, we have

$$\left|\frac{(f_s^n)'(z_1)}{z_1-u} - \frac{(f_s^n)'(z_2)}{z_2-u}\right| \leq \left|\frac{(f_s^n)'(z_1)}{z_1-u} - \frac{(f_s^n)'(z_1)}{z_2-u}\right| + \left|\frac{(f_s^n)'(z_1)}{z_2-u} - \frac{(f_s^n)'(z_2)}{z_2-u}\right|$$

$$\leq C'_N \left| \frac{z_2 - z_1}{(z_1 - u)(z_2 - u)} \right| + N \left| (f_s^n)'(z_1) - (f_s^n)'(z_2) \right|$$

$$\leq N^2 C'_N |z_1 - z_2| + N C''_N |z_1 - z_2|,$$

where the last inequality follows from (4.7). Therefore, since the marginal of μ^n is Lebesgue measure and $t_1 \leq N$, we have

$$\text{(I)} \leq \int_{[0,t_1]\times\mathbb{R}} N^2 C'_N |z_1 - z_2| + N C''_N |z_1 - z_2| d\mu(s,u) \leq t_1 \left(N^2 C'_N + N C''_N \right) |z_1 - z_2|$$

$$\leq \left(N^3 C'_N + N^2 C''_N \right) |z_1 - z_2|.$$

Therefore, we have equicontinuity of f^n on $[0, N] \times K_N$ for every N. By the Arzela-Ascoli theorem, there is a subsequence so that $f^{n_k} \to f^N$ for each N locally uniformly on $[0, N] \times K_N$. Taking $N \to \infty$ and applying a diagonal argument gives the desired limit function $f : [0, \infty) \times \mathbb{H}$.

□

Lemma 27. *If (g_t^n) is a solution of the chordal Loewner equation driven by μ^n, $(g_t^n) \to (g_t)$ in $\mathcal{G}$, and $\mu^n \to \mu$ in $\mathcal{N}$, then (g_t) is a solution to the chordal Loewner equation driven by μ.*

Proof. The chordal equation (4.1) implies that

$$g_t^n(z) = \int_{[0,t]\times\mathbb{R}} \Phi(u, g_s^n(z)) - \Phi(u, g_s(z)) d\mu^n(s,u) + \int_{[0,t]\times\mathbb{R}} \Phi(u, g_s(z)) d\mu^n(s,u) + z. \quad (4.8)$$

To estimate this, we will need to show that

$$\lim_{n\to\infty} \sup_{0\le s\le t} \{|g_s^n(z) - g_s(z)|\} = 0. \tag{4.9}$$

This will be justified at the end of the proof. It is easy to show that $\text{Im}(g_s^n(z))$ is decreasing in s for each n, so taking the limit as $n \to \infty$ shows that $\text{Im}(g_s(z))$ is also a decreasing function of s. It also follows that $\text{Im}(g_s^n(z)) \ge \min\{\text{Im}(g_t(z)), \text{Im}(g_t^1(z)), \cdots\} = h > 0$ for each $n \in \mathbb{N}$ and $s \le t$. Strict inequality follows since $\{g_t^n(z)\}_{n=1}^{\infty}$ is a sequence of numbers in $\mathbb{H}$ converging to $g_t(z) \in \mathbb{H}$. By Lemma 20, if $L = 2/h^2$,

$$\int_{[0,t]\times\mathbb{R}} |\Phi(u, g_s^n(z)) - \Phi(u, g_s(z))| d\mu^n(s,u) \le \int_{[0,t]\times\mathbb{R}} L|g_s^n(z) - g_s(z)| d\mu^n(s,u)$$

$$\le L \sup_{s\le t}\{|g_s^n(z) - g_s(z)|\}\mu([0,t]\times\mathbb{R}) = Lt \sup_{s\le t}\{|g_s^n(z) - g_s(z)|\} \to 0$$

as $n \to \infty$ by (4.9).

To estimate the second term in equation (4.8), observe that for a fixed z, since $s \mapsto g_s(z)$ is continuous, the mapping $(s,u) \mapsto \Phi(u, g_s(z))$ is a continuous function. Since $\mu^n \to \mu$ weakly on $[0,t]\times\mathbb{R}$, it follows that

$$\int_{[0,t]\times\mathbb{R}} \Phi(u, g_s(z)) d\mu^n(s,u) \to \int_{[0,t]\times\mathbb{R}} \Phi(u, g_s(z)) d\mu(s,u).$$

Therefore, the above and (4.9) imply that

$$g_t(z) = \lim_{n\to\infty} g_t^n(z) = \lim_{n\to\infty} \text{ equation (4.8)} = \int_{[0,t]\times\mathbb{R}} \Phi(u, g_s(z)) d\mu(s,u) + z.$$

Now we prove (4.9). Suppose otherwise, so that there is some $\epsilon > 0$, some subsequence n_k, and some sequence $s_k \in [0,t]$ with

$$|g_{s_k}^{n_k}(z) - g_{s_k}(z)| \geq \epsilon. \tag{4.10}$$

Passing to a new subsequence if necessary, we can assume that there is some $s^* \in [0,t]$ with $s_k \to s^*$. Since $g_t(z)$ is continuous in t, $g_{s_k}(z)$ converges to $g_{s^*}(z)$ as $k \to \infty$. We also claim that $g_{s_k}^{n_k}(z)$ is bounded. To see this, if h is defined as above,

$$|g_{s_k}^{n_k}(z) - z| \leq \int_{[0,s_k]\times\mathbb{R}} \frac{2}{|g_s^{n_k}(z) - u|} d\mu^{n_k}(s,u) \leq \frac{2s_k}{h} \leq \frac{2t}{h}.$$

Therefore, passing to a subsequence if necessary, we can assume that $g_{s_k}^{n_k}(z)$ converges to some $w \in \mathbb{H}$ as $k \to \infty$. Since $f_t^{n_k}(z)$ converges to $f_t(z)$ locally uniformly in (t,z), it follows that $z = f_{s_k}^{n_k}(g_{s_k}^{n_k}(z)) \to f_{s^*}(w)$, and so $g_{s^*}(z) = w$. Since $g_{s_k}(z) \to g_{s^*}(z)$, we get that

$$\lim_{k\to\infty} g_{s_k}^{n_k}(z) - g_{s_k}(z) = g_{s^*}(z) - g_{s^*}(z) = 0,$$

which contradicts (4.10).

□

Finally, we can prove existence.

Proposition 9. *a) If $\mu \in \mathcal{N}$, then the solution to the chordal Loewner equation driven by μ exists.*

b) If (g_t) is the solution of the chordal Loewner equation driven by $\mu \in \mathcal{N}$, and if $f_t = g_t^{-1}$, *then (4.5) holds for f. That is,*

$$f_t(z) - z = \int_{[0,t]\times\mathbb{R}} \frac{-2f_s'(z)}{z-u} d\mu(s,u).$$

c) *Assume $\mu^n \to \mu$ in $\mathcal{N}$, and suppose the respective corresponding Loewner maps are (g_t^n) and (g_t). Assume further that the associated hulls $\{K_t^n\}_{n\in\mathbb{N}}$ are uniformly bounded for each $t > 0$. Then $g_t^n \to g_t$ in $\mathcal{G}$.*

Proof. Lemma 26 can be applied to the sequence $d\mu_t^n dt \in \mathcal{N}$ to conclude that $(g_t^n) \to (g_t)$ subsequentially in $\mathcal{G}$ to some family of functions (g_t). By Lemma 27 it follows that (g_t) solves the chordal Loewner equation driven by μ, which proves a).

To prove b), Lemma 26 can be applied to f^n (after passing to a subsequence if necessary), to conclude that $|f^n(z) - f(z)| \to 0$. Therefore, it suffices to show that the difference of (4.5) for $f_t^n(z)$ and $f_t(z)$ goes to 0. To see this, observe that

$$\left| \int_{[0,t]\times\mathbb{R}} \frac{-2(f_s^n)'(z)}{z-u} d\mu^n(s,u) - \int_{[0,t]\times\mathbb{R}} \frac{-2f_s'(z)}{z-u} d\mu(s,u) \right|$$

$$\leq 2\int_{[0,t]\times\mathbb{R}} \left| \frac{f_s'(z)}{z-u} - \frac{(f_s^n)'(z)}{z-u} \right| d\mu^n(s,u) + 2\left| \int_{[0,t]\times\mathbb{R}} \frac{f_s'(z)}{z-u} d\mu(s,u) - \int_{[0,t]\times\mathbb{R}} \frac{f_s'(z)}{z-u} d\mu^n(s,u) \right|.$$

The first term goes to 0 by the dominated convergence theorem, since $(f_s^n)_{s\leq t}$ is a normal family, so is the family of derivatives. Therefore, $|(f^n)_s'(z)|$ can be bounded uniformly in $s \leq t$. Since $f_s^n(z) \to f_s(z)$ locally uniformly in z for all $s \leq t$, and so does $(f^n)_s'$. The second term goes to 0 because $\mu^n \to \mu$.

The last assertion follows from Lemmas 25, the proof of 26 with b), and 27. In particular, by b) and Lemma 25, the remark after Lemma 26 implies that there is a subsequential limit

g_t for which f^n converges to f uniformly in both space and time. Lemma 27 implies that g_t is a solution to the Loewner equation driven by μ. Also, Lemma 27 implies that any subsequential limit of g_t^n solves the Loewner equation driven by μ, which must be unique by Proposition 8, and therefore the sequence (g_t^n) converges to (g_t). □

4.6 Growing Hulls

In this section, we prove the reverse direction of Theorem 8. We start with a continuously growing family of hulls (K_t) and construct a measure $\mu \in \mathcal{N}$ whose solution to (4.1) generates the family (K_t).

Proposition 10. *Let K_t be growing hulls with $hcap K_t = 2t$ for every t. Then there is a measure $\mu \in \mathcal{N}$ so that $g_t : \mathbb{H} \backslash K_t \to \mathbb{H}$, where g_t solves the Loewner equation driven by μ.*

We first prove the following lemma. The proof is identical to that of Lemma 6.5 in [19].

Lemma 28. *Let K be an $\mathbb{H}$-hull, and let $2T =$hcap(K). Then there exists some $\mu \in \mathcal{N}_T$ so that if g_t is the solution to the Loewner equation driven by μ, then $g_T : \mathbb{H} \backslash K \to \mathbb{H}$.*

Proof. Let $\epsilon > 0$, and let $\gamma^\epsilon : [0, T_\epsilon] \to \overline{\mathbb{H}}$ be a simple curve so that γ^ϵ begins in $\mathbb{R}$, encloses K, and has Hausdorff distance from $\hat{K}$ less than ϵ, where $\hat{K}$ is the union of K and the convex hull of $\overline{K} \cap \mathbb{R}$. By ([7], Proposition 4.4), there exists a continuous function λ^ϵ so that if g_t^ϵ is the solution of the chordal Loewner equation driven by λ^ϵ, then $g_t^\epsilon : \mathbb{H} \backslash \gamma^\epsilon[0, t] \to \mathbb{H}$ for all t. If we define $\mu^\epsilon = d\delta_{\lambda^\epsilon(t)} dt$ and take a subsequential weak limit as $\epsilon \to 0$, Proposition 9 implies that there is a measure $\mu \in \mathcal{N}$ whose solution g_t satisfies $g_T : \mathbb{H} \backslash K \to \mathbb{H}$.

□

Proof of Proposition 10. Suppose for now that $g_t = g_{K_t}$ for each t. Let $\delta > 0$, and let $l \in \{0, 1, 2, \cdots\}$. Then $\mathrm{hcap}(K_{\delta(l+1)}/K_{\delta l}) = 2\delta$. By Lemma 28, there exists a measure $\tilde{\mu}^{\delta,l}$ on $[0,\delta] \times \mathbb{R}$ so that if $\tilde{g}_t^{\delta,l}$ solves the Loewner equation driven by $\tilde{\mu}^{\delta,l}$, then $\tilde{g}_\delta^{\delta,l} : \mathbb{H}\backslash(K_{\delta(l+1)}/K_{\delta l}) \to \mathbb{H}$. Define a new measure $\mu^{\delta,l}$ on $[\delta l, \delta(l+1)] \times \mathbb{R}$ by shifting the previous measure. That is,

$$\mu^{\delta,l}(E) = \tilde{\mu}^{\delta,l}(\{(x - \delta l, y) : (x,y) \in E\}).$$

Finally, define a measure $\mu^\delta \in \mathcal{N}$ by $\mu^\delta = \sum_{l=0}^{\infty} \mu^{\delta,l}$, and let g_t^δ solve the Loewner equation driven by μ^δ with $g_t^\delta : \mathbb{H}\backslash K_t^\delta \to \mathbb{H}$.

We will show that $K_{\delta l} = K_{\delta l}^\delta$ for every l and δ. We prove this by induction. The case $l = 0$ is obvious, since if $t \leq \delta$, then

$$g_t^\delta(z) = z + \int_{[0,t]\times\mathbb{R}} \Phi(u, g_s^\delta(z)) d\tilde{\mu}^{\delta,0}(s,u),$$

and therefore $g_\delta^\delta = \tilde{g}_t^{\delta,0}(z)$ for each $t \leq \delta$, and $K_\delta/K_0 = K_\delta$.

Suppose this is true for l. Then $f_{\delta l}^\delta : \mathbb{H} \to \mathbb{H}\backslash K_{\delta l}$. For any $t \in [0,\delta]$,

$$G_t^{\delta,l}(z) := g_{\delta l+t}^\delta(f_{\delta l}^\delta(z)) = f_{\delta l}^\delta(z) + \int_{[0,\delta l]\times\mathbb{R}} \Phi(u, g_s^\delta(f_{\delta l}^\delta(z))) d\mu^\delta(s,u)$$

$$+ \int_{[\delta l, \delta l+t]\times\mathbb{R}} \Phi(u, g_s^\delta(f_{\delta l}^\delta(z))) d\mu^\delta(s,u)$$

$$= f_{\delta l}^\delta(z) + (g_{\delta l}^\delta(f_{\delta l}^\delta(z)) - f_{\delta l}^\delta(z)) + \int_{[\delta l, \delta l+t]} \Phi(u, g_s^\delta(f_{\delta l}^\delta(z))) d\mu^{\delta,l}(s,u)$$

$$= z + \int_{[0,t]\times\mathbb{R}} \Phi(u, g_{\delta l+s}^\delta(f_{\delta l}^\delta(z))) d\tilde{\mu}^{\delta,l}(s,u)$$

$$= z + \int_{[0,t]\times\mathbb{R}} \Phi(u, G_s^{\delta,l}(z)) d\tilde{\mu}^{\delta,l}(s,u).$$

Thus, $G_t^{\delta,l}$ solves the Loewner equation driven by $\tilde{\mu}^{\delta,l}$, and so $G_\delta^{\delta,l} = \tilde{g}_\delta^\delta : \mathbb{H}\backslash(K_{\delta(l+1)}/K_{\delta l}) \to \mathbb{H}$. It follows that

$$K^\delta_{\delta(l+1)}/K^\delta_{\delta l} = K_{\delta(l+1)}/K_{\delta l}.$$

Therefore, since $g^\delta_{\delta l} = g_{K_{\delta l}}$ by the induction hypothesis,

$$g^\delta_{\delta(l+1)} = g_{K^\delta_{\delta(l+1)}/K^\delta_{\delta l}} \circ g^\delta_{\delta l} = g_{K_{\delta(l+1)}/K_{\delta l}} \circ g_{K_{\delta l}} = g_{K_{\delta(l+1)}}.$$

If $\mu^k = \mu^{\delta_k}$ is a sequence with a weak limit $\mu \in \mathcal{N}$, which exists because the hulls K_t^δ are uniformly bounded for each $t < \infty$, then Proposition 9 implies that g_t^k converges to g_t, the solution of the Loewner equation driven by μ. Since $f^k \to f$ locally uniformly in space and time, it follows that $K_t^k \to K_t$ in the Carathéodory topology for every t. Hence, g_t solves the Loewner equation driven by μ, and $g_t : \mathbb{H}\backslash K_t \to \mathbb{H}$.

□

The last part of Theorem 8 which remains to be proven is the convergence fact. We need to prove that convergence of a family of hulls implies convergence of the corresponding measures.

Proposition 11. *Suppose that $\mu^n \in \mathcal{N}$, and that $g_t^n : \mathbb{H}\backslash K_t^n \to \mathbb{H}$ is the associated solution of the chordal Loewner equation. Then if $K_t^n \to K_t$ in the Carathéodory topology for some increasing family of hulls K_t and the family $\{K_t^n\}_{n\in\mathbb{N}}$ is uniformly bounded for each t, then there is a measure $\mu \in \mathcal{N}$ so that (K_t) are the hulls associated with μ and $\mu^n \to \mu$ in $\mathcal{N}$.*

Proof. By Proposition 9, we know that there is a measure μ which drives the family of hulls K_t. Let $\tilde{\mu}$ be any subsequential limit of μ^n, which exists because the hulls K_t^n are bounded for each $t < \infty$. Also by Proposition 9, $\mu^{n_k} \to \tilde{\mu}$ implies that $K_t^{n_k}$ converges in the Carathéodory topology to the hulls driven by $\tilde{\mu}$. Since we know that $K_t^{n_k}$ converges to K_t, and the hulls are uniquely determined by the measures, it follows that $\tilde{\mu} = \mu$. Since $\tilde{\mu}$ was an arbitrary subsequential limit, it follows that $\mu^n \to \mu$ in $\mathcal{N}$.

$\square$

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