BLOW-UP PROBLEMS FOR THE HEAT EQUATION WITH LOCAL NONLINEAR NEUMANN BOUNDARY CONDITIONS

By

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ABSTRACT

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This thesis studies the blow-up problem for the heat equation $u_t = \Delta u$ in a C^2 bounded open subset Ω of $\mathbb{R}^n (n \geq 2)$ with positive initial data u_0 and a local nonlinear Neumann boundary condition: $\frac{\partial u}{\partial n} = u^q$ on partial boundary $\Gamma_1 \subseteq \partial\Omega$ for some $q > 1$ and $\frac{\partial u}{\partial n} = 0$ on the rest of the boundary. The motivation of the study is the partial damage to the insulation on the surface of space shuttles caused by high speed flying subjects.

First, we establish the local existence and uniqueness of the classical solution for such a problem. Secondly, we show the finite-time blowup of the solution and estimate both upper and lower bounds of the blow-up time T^*. In addition, the asymptotic behaviour of T^* on q, M_0 (the maximum of the initial data) and $|\Gamma_1|$ (the surface area of Γ_1) are studied.

- As $q \searrow 1$, the order of T^* is exactly $(q-1)^{-1}$.

- As $M_0 \searrow 0$, the order of T^* is at least $\ln(M_0^{-1})$; if the region near Γ_1 is convex, then the order of T^* is at least $M_0^{-(q-1)}/\ln(M_0^{-1})$; if Ω is convex, then the order of T^* is at least $M_0^{-(q-1)}$. On the other hand, if the initial data u_0 does not oscillate too much, then the order of T^* is at most $M_0^{-(q-1)}$.

- As $|\Gamma_1| \searrow 0$, the order of T^* is at least $\ln(|\Gamma_1|^{-1})$ and at most $|\Gamma_1|^{-1}$. If the region near Γ_1 is convex, then the order of T^* is at least $|\Gamma_1|^{-\frac{1}{n-1}} \Big/ \ln\left(|\Gamma_1|^{-1}\right)$ for $n \geq 3$ and $|\Gamma_1|^{-1}/\left[\ln\left(|\Gamma_1|^{-1}\right)\right]^2$ for $n = 2$. If Ω is convex, then the order of T^* is at least

$$|\Gamma_1|^{-\frac{1}{n-1}} \text{ for } n \geq 3 \text{ and } |\Gamma_1|^{-1}\Big/\ln\left(|\Gamma_1|^{-1}\right) \text{ for } n = 2.$$

Finally, we provide two strategies from engineering point of view (which means by changing the setup of the original problem) to prevent the finite-time blowup. Moreover, if the region near Γ_1 is convex, then one of the strategies is applied to bound the solution from above by M_1 for any $M_1 > M_0$.

For the space shuttle mentioned in the motivation of this thesis, Γ_1 is on its left wing of the shuttle, so the region near Γ_1 is indeed convex. In addition, the relation between T^* and small surface area $|\Gamma_1|$ is of particular interest for this problem. As an application of the above estimates to this problem, let $n = 3$ and $|\Gamma_1| \searrow 0$, then the order of T^* is between $|\Gamma_1|^{-\frac{1}{2}}\Big/\ln\left(|\Gamma_1|^{-1}\right)$ and $|\Gamma_1|^{-1}$. On the other hand, one of the strategies can be applied to prevent the temperature from being too high.

This thesis seems to be the first to systematically study the heat equation with piecewise continuous Neumann boundary conditions. It also seems to be the first to investigate the relation between T^* and $|\Gamma_1|$, especially when $|\Gamma_1| \searrow 0$. The key innovative part of this thesis is Chapter 4. First, the new method developed in Chapter 4 is able to derive a lower bound for T^* without the convexity assumption of the domain which was a common requirement in the historical works. Secondly, even for the convex domains, the lower bound estimate obtained by this new method improves the previous results significantly. Thirdly, this method does not involve any differential inequality argument which was an essential technique in the past on the blow-up time estimate.

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Introduction

1.1 Motivation and mathematical model

This thesis is partially motivated by the Space Shuttle Columbia disaster in 2003. When the space shuttle was launched, a piece of foam broke off from its external tank and struck the left wing damaging the insulation there. As a result, the shuttle disintegrated during its reentry to the atmosphere due to the enormous heat generated near the damaged part. Actually, such damages on the wings were also found in previous shuttles too. But the engineers suspected that the previous damages were so small that the shuttle managed to land safely before the temperature became too high. Motivated by this, this thesis intends to study the relation between the blow-up time of the temperature inside the shuttle and the area of the broken part on the left wing. The goal is to rigorously verify the engineers' conjecture from a mathematical perspective. In addition, some strategies that could prevent the blowup will also be explored.

In Figure 1.1, let u be the inside temperature of the space shuttle. During the re-entry of the shuttle to the atmosphere, the air was compressed at a very high speed. Then many chemical reactions happened and produced enormous radiative heat flux, which was the main source of the heat. In physics, the radiative heat flux is proportional to the fourth power of the temperature. Due to this nonlinear effect, we consider a simplified model as follows. On

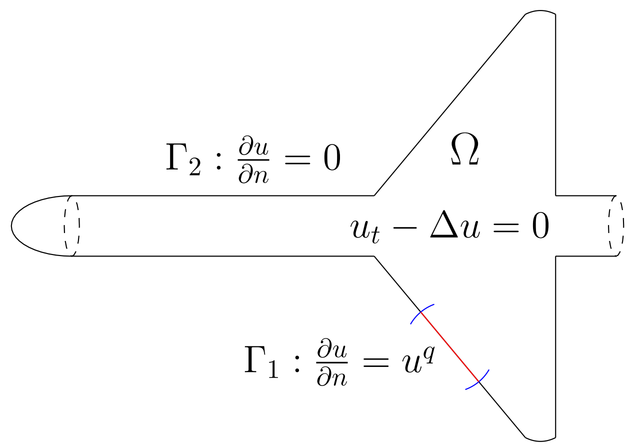

Figure 1.1: Mathematical Model

the broken part Γ_1, $\frac{\partial u}{\partial n} = H(u) \sim u^q$ for some $q > 1$; on the other part Γ_2, $\frac{\partial u}{\partial n} = 0$, since the insulation there is intact. Finally, the inside temperature of the shuttle is supposed to satisfy the heat equation. Thus, the following math model is adopted (see more descriptions in Section 1.4).

$$\begin{cases} u_t(x,t) = \Delta u(x,t) & \text{in} \quad \Omega \times (0,T], \\ \frac{\partial u(x,t)}{\partial n(x)} = u^q(x,t) & \text{on} \quad \Gamma_1 \times (0,T], \\ \frac{\partial u(x,t)}{\partial n(x)} = 0 & \text{on} \quad \Gamma_2 \times (0,T], \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.1.1}$$

where

$$q > 1,\ \Gamma_1 \neq \emptyset,\ u_0 \in C^1(\overline{\Omega}),\ u_0(x) \geq 0,\ u_0(x) \not\equiv 0. \tag{1.1.2}$$

1.2 Historical works

1.2.1 Blow-up phenomenon for Cauchy problems

In the seminal work [9], Fujita studied the Cauchy problem

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = u^p(x,t) & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u(x,0) = \psi(x) & \text{in} \quad \mathbb{R}^n, \end{cases} \tag{1.2.1}$$

where $n \geq 1$, $\psi \in C^2(\mathbb{R}^n)$ is nonnegative and ψ, $D_i\psi$, $D_{ij}\psi$ are all bounded on $\mathbb{R}^n$. It is shown that if $1 < p < 1 + \frac{2}{n}$, then the only nonnegative global solution is $u \equiv 0$, and if $p > 1 + \frac{2}{n}$, then there exist positive global solutions for positive and sufficiently small ψ. Since then, there is vast literature studying the nonlinear blow-up phenomenons. The number $1 + \frac{2}{n}$ is called the critical power in the sense that when $p < 1 + \frac{2}{n}$, any positive solution blows up in finite time; when $p > 1 + \frac{2}{n}$, there exist positive global solutions. The existence of such a critical power is a feature of this kind of blow-up problem. The study of the borderline case is more involved and usually obtained separately after the subcritical and supercritical are established. In the works [14] and [19], it is shown that the critical power $p = 1 + \frac{2}{n}$ case belongs to the blow-up regime.

Similar questions were also asked for the nonlinear wave equations and the situation there is more complicated.

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = |u|^p(x,t) & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u(x,0) = \psi_0(x) & \text{in} \quad \mathbb{R}^n, \\ u_t(x,0) = \psi_1(x) & \text{in} \quad \mathbb{R}^n, \end{cases} \tag{1.2.2}$$

where $n \geq 1$, ψ_0 and ψ_1 are nonnegative, compactly supported and let either of them be positive somewhere. In the pioneering work [17], John showed that when $n = 3$, the critical power for (1.2.2) is $p = 1 + \sqrt{2}$. Again this means if $1 < p < 1 + \sqrt{2}$, then any solution blows up in finite time; if $p > 1 + \sqrt{2}$, then there exist global solutions for suitably small initial data. For general dimensions, the problem is also called the Strauss conjecture and the critical power is conjectured to be the positive root of the quadratic equation below for $n \geq 2$ and infinity for $n = 1$.

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

Now this guess has been confirmed after several decades' work. The subcritical cases can be found in [13,39]. The supercritical cases are dealt with in [10,12,26]. Finally, the borderline cases are also proved to be in the blow-up regime, see [37,46]. The one dimensional case was discussed in [13] and [18].

It is also interesting to notice a problem which combines both nonlinear heat and wave equations.

$$\begin{cases} u_t(x,t) + u_{tt}(x,t) - \Delta u(x,t) = |u|^p(x,t) & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u(x,0) = \psi_0(x) & \text{in} \quad \mathbb{R}^n, \\ u_t(x,0) = \psi_1(x) & \text{in} \quad \mathbb{R}^n, \end{cases}$$

where ψ_0 and ψ_1 are compactly supported. See [42,47] for more details.

1.2.2 Parabolic blow-up problems in bounded domains

Now let us focus on the parabolic type of nonlinear equations. Besides the Cauchy problems, people also study the boundary value problems including both Dirichlet and Neumann types.

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = f\big(x,t,u(x,t)\big) & \text{in} \quad \Omega \times (0,T], \\ F\big(x,t,u(x,t)\big) = 0 & \text{on} \quad \partial\Omega \times (0,T], \\ u(x,0) = \psi(x) & \text{in} \quad \Omega, \end{cases} \tag{1.2.3}$$

or

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = f\big(x,t,u(x,t)\big) & \text{in} \quad \Omega \times (0,T], \\ \frac{\partial u(x,t)}{\partial n(x)} = F\big(x,t,u(x,t)\big) & \text{on} \quad \partial\Omega \times (0,T], \\ u(x,0) = \psi(x) & \text{in} \quad \Omega. \end{cases} \tag{1.2.4}$$

For detailed discussions on the history, we refer the readers to the surveys [5, 24] and the books [7, 15, 35].

The typical examples are

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = u^p(x,t) & \text{in} \quad \Omega \times (0,T], \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T], \\ u(x,0) = \psi(x) & \text{in} \quad \Omega, \end{cases} \tag{1.2.5}$$

and

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = 0 & \text{in} \quad \Omega \times (0,T], \\ \frac{\partial u(x,t)}{\partial n(x)} = u^q(x,t) & \text{on} \quad \partial\Omega \times (0,T], \\ u(x,0) = \psi(x) & \text{in} \quad \Omega, \end{cases} \tag{1.2.6}$$

where $p > 1$, $q > 1$ and the initial data is positive. But the blow-up properties of (1.2.5) and

(1.2.6) are quite different from (1.2.1). More precisely, for the problem (1.2.5), there exists some positive global solution, see [28, 38]. On the other hand, for the problem (1.2.6), any solution blows up in finite time, see [16, 36, 44].

For the more general problems (1.2.3) and (1.2.4), the research topics include the local and global existence and uniqueness of the solutions [1–4, 21, 27, 44]; nonexistence of global solutions and upper bound for the blow-up time [16, 21–23, 25, 27, 31, 36, 44]; lower bound for the blow-up time [31–34, 43]; blow-up sets, blow-up rate and the asymptotic behaviour of the solutions near the blow-up time [8, 11, 16, 21, 27, 29, 36, 45].

When considering the bounds of the blow-up time, the upper bound is usually related to the nonexistence of the global solutions and this area has brought extensive attention over several decades. Various methods on this issue have been developed, such as the comparison method, the concavity method, the Green's function method, the energy method, and the unbounded Fourier coefficient method (see [23]). Most methods have in common that they first consider some nonlinear functional of the solution and try to establish a first order differential inequality for that functional, then it is shown that such a differential inequality can not hold beyond some finite time T which serves as an upper bound.

The lower bound was not studied as much in the past but was paid more attention in recent years. However, the lower bound can be argued to be more useful in practice, since it provides an estimate of the safe time. In contrast to the upper bound case, not many methods have been developed to deal with the lower bound. But the existing methods again have in common that they first consider some nonlinear functional of the solution and try to establish a first order differential inequality for that functional, then it is shown that such differential inequality will hold for at least some time T, which serves as a lower bound.

1.3 Difficulties and main ideas

Generally speaking, there are two main difficulties. First, Although there has been vast literature on the blow-up problem of the parabolic type, few of them deal with discontinuous Neumann boundary conditions. Second, the existing works on the estimate of the lower bound of the blow-up time only work for convex domains. But by examining the graph 1.1, the domain is clearly not convex. In the following, we discuss these difficulties and the corresponding strategies in more details.

First, for the theory on existence and uniqueness of the solution to (1.2.4), the key tool is the jump relation of the single-layer potentials (see Theorem 2.2.1). In order to generalize the theory to the linear problem (2.3.1) with piecewise continuous Neumann boundary conditions, we establish an adapted version of the jump relation in Theorem 2.2.6. Taking advantage of this adapted relation, a classical solution can be constructed to satisfy (2.3.1) pointwise. In addition, such a solution also fits the condition (2.3.2). By imposing the condition (2.3.2) into the definition of the solution (see Definition 2.3.1), the uniqueness follows from the maximum principle and the Hopf lemma. After the theory is established for the linear problem, the existence and uniqueness for the nonlinear case (2.4.1) will be justified by applying the iterative arguments and fixed point theorems.

Secondly, for the estimate of the upper bound for the blow-up time, we adopt the idea in [36] which introduces a suitable energy function and shows the finite blowup of this energy function. But due to the discontinuity of the normal derivative, the original argument does not carry through directly. So we need to additionally introduce a sequence of approximated solutions $\{v_j\}_{j\geq 1}$ to u, see (3.3.5), to justify the argument.

Thirdly, for the estimate of the lower bound for the blow-up time, our method is new in order to deal with more general domains. In Subsection 4.3.2, the lower bound of T^* is derived without any convexity assumption of the domain which is a common requirement in the previous works. Let $M(t)$ denote the supremum of the solution on $\overline{\Omega} \times [0, t]$. The idea is to chop the range of $M(t)$ into suitable pieces and find a lower bound for the time spent in each piece by analysing the representation formula of the solution. Then adding all these lower bounds together yields a lower bound for T^*. This strategy does not introduce any differential inequalities which often appeared in the historical works on the blow-up time estimate. The proofs in Subsection 4.4.3 and Subsection 4.5.2 adopt the similar idea but with some convexity assumptions on the domain. These assumptions make it possible to chop the range of $M(t)$ in a more delicate way to improve the estimate.

Finally, for the strategies to prevent the finite-time blowup, the ideas are similar to those in Chapter 4.

1.4 Notations

In this thesis, unless stated otherwise,

- Ω represents a bounded open subset in $\mathbb{R}^n$ ($n \geq 2$) with C^2 boundary $\partial\Omega$.
- Γ_1 and Γ_2 denote two disjoint relatively open subsets of $\partial\Omega$. $\partial\Gamma_1 = \partial\Gamma_2 \triangleq \widetilde{\Gamma}$ is C^1. Moreover, $\Gamma_1 \neq \emptyset$ and $\partial\Omega = \Gamma_1 \cup \widetilde{\Gamma} \cup \Gamma_2$.
- $|\Omega|$ and $|\Gamma_1|$ represents the volume of Ω and the surface area of Γ_1 respectively. That is,

$$|\Omega| = \int_{\Omega} dx, \quad |\Gamma_1| = \int_{\Gamma_1} dS(x).$$

The normal derivative in (1.1.1) is understood in the following way: for any $(x,t) \in \partial\Omega \times (0,T]$,

$$\frac{\partial u(x,t)}{\partial n(x)} \triangleq \lim_{h \to 0^+} (Du)(x_h, t) \cdot \overrightarrow{n}(x), \tag{1.4.1}$$

where $\overrightarrow{n}(x)$ denotes the exterior unit normal vector at x and $x_h \triangleq x - h\overrightarrow{n}(x)$ for $x \in \partial\Omega$. Since $\partial\Omega$ is C^2, x_h belongs to Ω when h is positive and sufficiently small.

In the above notations, Γ_1 is not allowed to be empty, since otherwise the blowup will not occur. On the other hand, Γ_2 is allowed to be empty and in that case, problem (1.1.1) has been studied extensively in the past. In Section 4.6, the results obtained in this thesis will be compared with the previous results when Γ_2 is empty. Finally, in some extreme situations, $\widetilde{\Gamma}$ may also be empty. For example, let $\Omega = \{x \in \mathbb{R}^n : \frac{1}{2} < |x| < 1\}$, $\Gamma_1 = \{x \in \mathbb{R}^n : |x| = \frac{1}{2}\}$ and $\Gamma_2 = \{x \in \mathbb{R}^n : |x| = 1\}$. It is worth mentioning that all the results in this thesis also apply to this situation.

For any function $f : A \to \mathbb{R}$, we follow the convention to denote the supremum norm of f to be

$$||f||_{\infty,A} = \sup_{x \in A} |f(x)|.$$

When there is no ambiguity, $||f||_\infty$ will be used short for $||f||_{\infty,A}$. For any $T > 0$, define

$$\mathcal{A}_T = C^{2,1}(\Omega \times (0,T]) \cap C(\overline{\Omega} \times [0,T])$$

and

$$\mathcal{B}_T = \{g : (\Gamma_1 \cup \Gamma_2) \times (0,T] \to \mathbb{R} \,\big|\, \text{for } i = 1 \text{ or } 2,\ g|_{\Gamma_i \times (0,T]} \text{ is uniformly continuous}\}.$$

For any $g \in \mathcal{B}_T$, the restriction function $g|_{\Gamma_i \times (0,T]}$ ($i = 1$ or 2) has a unique continuous extension to $\overline{\Gamma}_i \times [0,T]$ which will be denoted as g_i. But one should notice that g may not be able to extend to a continuous function on $\partial\Omega \times (0,T]$, since it may have a jump between Γ_1 and Γ_2. We endow $\mathcal{B}_T$ with the supremum norm:

$$||g||_{\infty, (\Gamma_1 \cup \Gamma_2) \times (0,T]} = \sup_{(x,t) \in (\Gamma_1 \cup \Gamma_2) \times (0,T]} |g(x,t)|.$$

It is readily seen that $\mathcal{B}_T$ is a Banach space.

We write

$$M_0 = \max_{x \in \overline{\Omega}} u_0(x) \tag{1.4.2}$$

and denote $M(t)$ to be the supremum of the solution u to (1.1.1) on $\overline{\Omega} \times [0,t]$:

$$M(t) = \sup_{(x,\tau) \in \overline{\Omega} \times [0,t]} u(x,\tau). \tag{1.4.3}$$

Φ always refers to the fundamental solution to the heat equation:

$$\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} \exp\Big(-\frac{|x|^2}{4t}\Big), \quad \forall\, (x,t) \in \mathbb{R}^n \times (0,\infty). \tag{1.4.4}$$

The surface integral with respect to the variable x will be denoted as $dS(x)$. In addition, $C = C(a,b\dots)$ and $C_i = C_i(a,b\dots)$ represent positive constants which only depend on the parameters $a, b\dots$. One should also note that C and C_i may stand for different constants from line to line.

1.5 Main results

The solution to (1.1.1) is understood in the following way.

Definition 1.5.1. *For any $T > 0$, a solution to (1.1.1) on $\overline{\Omega} \times [0,T]$ means a function $u \in C^{2,1}\big(\Omega \times (0,T]\big) \bigcap C\big(\overline{\Omega} \times [0,T]\big)$ that satisfies (1.1.1) pointwise and for any $(x,t) \in \widetilde{\Gamma} \times (0,T]$, $\frac{\partial u(x,t)}{\partial n(x)}$ exists and*

$$\frac{\partial u(x,t)}{\partial n(x)} = \frac{1}{2}\, u^q(x,t). \tag{1.5.1}$$

Definition 1.5.2. *The maximal existence time T^* for (1.1.1) is defined as*

$$T^* = \sup\{T \geq 0 : \textit{ there exists a solution to (1.1.1) on } \overline{\Omega} \times [0,T]\}.$$

When $T^* > 0$, a function $u \in C^{2,1}\big(\Omega \times (0,T^*)\big) \bigcap C\big(\overline{\Omega} \times [0,T^*)\big)$ is called a maximal solution if $u|_{\overline{\Omega} \times [0,T]}$ is a solution to (1.1.1) on $\overline{\Omega} \times [0,T]$ for any $T \in (0,T^*)$.

Based on these two definitions, the existence and uniqueness theory of the maximal solution u is established in Theorem 2.1.1. In addition, Theorem 2.1.1 also claims the positivity of u at any positive time. For convenience, we will just call u to be solution instead of maximal solution. Actually, we deal with the existence and uniqueness theory in much more general settings. First, the key tool, jump relation of the single-layer potential with piecewise continuous density, is set up in Subsection 2.2.2. Secondly, the theory for linear case is built in Section 2.3. Finally, the theory for the nonlinear case in a very general form is framed in Section 2.4. The targeted problem (1.1.1) is just a special case in Section 2.4.

Theorem 3.1.1 concludes that the solution u does not exist globally. Moreover, it is the supremum norm $M(t)$ that blows up first. As a convention, we just call T^* to be the blow-up time. If the initial data u_0 is strictly positive, then an explicit formula for an upper bound of T^* is given in Theorem 3.1.1.

In Theorem 4.1.1, a lower bound for T^* is obtained for any C^2 domain. If it is locally convex near Γ_1, then Theorem 4.1.3 improves the lower bound estimate significantly. Combining these estimates, the asymptotic behaviour of T^* on q, M_0 and $|\Gamma_1|$ are understood quite well. The following is a summary of the conclusions.

- As $q \searrow 1$, the order of T^* is exactly $(q-1)^{-1}$.
- As $M_0 \searrow 0$, the order of T^* is at least $\ln(M_0^{-1})$; if the region near Γ_1 is convex, then the order of T^* is at least $M_0^{-(q-1)}/\ln(M_0^{-1})$; if Ω is convex, then the order of T^* is at least $M_0^{-(q-1)}$. On the other hand, if the initial data u_0 does not oscillate too much, then the order of T^* is at most $M_0^{-(q-1)}$.
- As $|\Gamma_1| \searrow 0$, the order of T^* is at least $\ln(|\Gamma_1|^{-1})$ and at most $|\Gamma_1|^{-1}$. If the region near Γ_1 is convex, then the order of T^* is at least $|\Gamma_1|^{-\frac{1}{n-1}} \Big/ \ln\left(|\Gamma_1|^{-1}\right)$ for $n \geq 3$ and $|\Gamma_1|^{-1}/\left[\ln\left(|\Gamma_1|^{-1}\right)\right]^2$ for $n = 2$. If Ω is convex, then the order of T^* is at least $|\Gamma_1|^{-\frac{1}{n-1}}$ for $n \geq 3$ and $|\Gamma_1|^{-1}/\ln\left(|\Gamma_1|^{-1}\right)$ for $n = 2$.

In order to compare with the previous results on the lower bound estimate of T^* for the convex domains and $\Gamma_1 = \partial\Omega$, we also derive a result for the convex domains in Section 4.4, see Theorem 4.1.4. Since the results will be compared under the assumption that $\Gamma_1 = \partial\Omega$, the order of the lower bound on $|\Gamma_1|$ as $|\Gamma_1| \searrow 0$ is not of interest. Instead, the order of the lower bound on M_0 as $M_0 \searrow 0$ or $M_0 \to \infty$ is more important. So the lower bound in Theorem 4.1.4 has a better order on M_0, no matter $M_0 \searrow 0$ or $M_0 \to \infty$, than those in Theorem 4.1.3 and Remark 4.5.11, but it loses order on $|\Gamma_1|^{-1}$ as $|\Gamma_1| \searrow 0$.

Finally, we investigate two strategies to prevent the blowup: repairing the broken part and adding a pump. For the method of repairing the broken part, it is shown in Theorem 5.1.2 that if the area of the broken part decreases at an exponential rate, then the solution will not blow up in finite time. Furthermore, let the region near Γ_1 be convex and M_0 denote the maximum of the initial data. Then Theorem 5.1.4 claims that for any $M_1 > M_0$, the solution will be bounded by M_1 ($M_1 > M_0$) if the area of the broken part decreases at a super exponential rate. For the method of adding a pump, it is shown in Theorem 5.1.5 that by adding a suitable pump near the broken part, the solution will not blow up in finite time.

Chapter 2

Existence and Uniqueness

2.1 Main theorem and outline of the approach

The goal of this chapter is to establish the existence and uniqueness of the targeted problem (1.1.1). The main theorem is as follows.

Theorem 2.1.1. *The maximal existence time T^* for (1.1.1) is positive and there exists a unique maximal solution $u \in C^{2,1}\big(\Omega \times (0, T^*)\big) \cap C\big(\overline{\Omega} \times [0, T^*)\big)$ to (1.1.1). Moreover, $u(x,t) > 0$ for any $(x,t) \in \overline{\Omega} \times (0, T^*)$.*

Since (1.1.1) is a nonlinear problem, the strategy is to first deal with the linear case and then build the nonlinear case by fixed point theorems. The arguments for the nonlinear problem is standard, the main difficulty lies in the linear case. The key technique is a generalization of the jump relation for the heat potentials. Previously, the jump relation of the heat potentials with continuous density is well-known, but due to the discontinuity of the normal derivative in (1.1.1), we need to extend the result to the heat potentials with piecewise continuous density.

The organization of this chapter is as follows. Section 2.2 discusses the jump relation of the heat potentials with piecewise continuous density. In Section 2.3, we study the linear problem (2.3.1) and prove the global existence and uniqueness of the classical solution. In Section 2.4, we first set up the local existence and uniqueness for the nonlinear problem

(2.4.1) and then discuss the maximal solution. As a corollary, Theorem 2.1.1 is justified.

2.2 Jump relation

The key tool in the proof of the existence of the solution to the parabolic equations with Neumann boundary conditions is the jump relation of the single-layer and double-layer potentials. Historically, people study these potentials with continuous density, but in order to adapt to the current problem, we generalize the results for the potentials with piecewise continuous density in this section.

2.2.1 Heat potentials with continuous density

Let $g \in C\big(\partial\Omega \times [0,T]\big)$. The single-layer heat potential with density g is given by

$$U(x,t) = \int_0^t \int_{\partial\Omega} \Phi(x-y, t-\tau) g(y,\tau)\, dS(y)\, d\tau, \quad \forall\,(x,t) \in \Omega \times (0,T], \tag{2.2.1}$$

where Φ is defined as in (1.4.4) and dS means the surface integral. The double-layer heat potential is given by

$$V(x,t) = \int_0^t \int_{\partial\Omega} \frac{\partial\Phi(x-y,t-\tau)}{\partial n(y)}\, g(y,\tau)\, dS(y)\, d\tau, \quad \forall\,(x,t) \in \Omega \times (0,T], \tag{2.2.2}$$

where $\overrightarrow{n}(y)$ denotes the exterior unit normal vector at y. The following Theorem 2.2.1 and 2.2.2 are two fundamental properties of the jump relations.

Theorem 2.2.1. *Let $g \in C\big(\partial\Omega \times [0,T]\big)$ and U be the single-layer heat potential defined in*

(2.2.1). Then for any $(x,t) \in \partial\Omega \times (0,T]$,

$$\lim_{h\to 0^+} DU(x_h,t)\cdot \overrightarrow{n}(x) = \int_0^t \int_{\partial\Omega} \frac{\partial\Phi(x-y,t-\tau)}{\partial n(x)}\, g(y,\tau)\, dS(y)\, d\tau + \frac{1}{2}\, g(x,t), \tag{2.2.3}$$

where $x_h = x - h\overrightarrow{n}(x)$. Moreover, the convergence in the above limit is uniform on $\partial\Omega \times [\tau_0, T]$ for any $\tau_0 > 0$.

Proof. The pointwise convergence of (2.2.3) can be found in Theorem 1, page 137 in Section 2, Chapter 5 of [7], note that the author in [7] uses $\overrightarrow{\nu}(x)$ to denote the interior unit normal direction at x, so the jump is $-g(x,t)/2$ which differs a sign with (2.2.3). Actually in that theorem, it proves the pointwise convergence in much more general cases: arbitrary parabolic kernels; $C^{1,\alpha}$ domains; and convergence from interior cones. As a result, the uniform convergence is not clear in that setting. But if restricted to the heat kernel Φ, C^2 domain case and convergence from the normal direction, then the uniform convergence of (2.2.3) can be established through the same proof. □

Theorem 2.2.2. *Let $g \in C\big(\partial\Omega \times [0,T]\big)$ and V be the double-layer heat potential defined in (2.2.2). Then*

$$\lim_{h\to 0^+} V(x_h,t) = V(x,t) - \frac{1}{2}\, g(x,t) \quad \forall\, (x,t) \in \partial\Omega \times (0,T], \tag{2.2.4}$$

where $x_h = x - h\overrightarrow{n}(x)$. Moreover, the convergence in the above limit is uniform on $\partial\Omega \times [\tau_0, T]$ for any $\tau_0 > 0$. In particular, V can be continuously extended from $\Omega \times (0,T]$ into $\overline{\Omega} \times (0,T]$.

Proof. We refer the readers to the proof of Theorem 9.5, page 176, Section 2, Chapter 9 of [20], note that the author in [20] uses $\overrightarrow{\nu}(y)$, rather than $\overrightarrow{n}(y)$, to denote exterior unit normal direction at y. In that proof, it only deals with the dimension $n = 2$ or 3, but the proof for arbitrary dimensions can be carried out almost identically. $\square$

Proposition 2.2.3. *Theorem 2.2.1 and Theorem 2.2.2 are equivalent.*

Proof. By the explicit formulas (2.2.1) and (2.2.2), it suffices to show the uniform convergence on $\partial\Omega \times (0, T]$ of the following limit.

$$\begin{aligned} &\lim_{h\to 0^+} \int_0^t \int_{\partial\Omega} D\Phi\big(x_h - y, t-\tau\big) \cdot \big[\overrightarrow{n}(y) - \overrightarrow{n}(x)\big]\, g(y,\tau)\, dS(y)\, d\tau \\ &= \int_0^t \int_{\partial\Omega} D\Phi(x - y, t-\tau) \cdot \big[\overrightarrow{n}(y) - \overrightarrow{n}(x)\big]\, g(y,\tau)\, dS(y)\, d\tau. \end{aligned} \tag{2.2.5}$$

By Corollary 3.2.2, there exists $\sigma > 0$ such that for any $h < \sigma$ and for any $x \in \partial\Omega$, there exists an interior ball touching x with radius h. As a result, $|x_h - x| \leq |x_h - y|$ for any $y \in \partial\Omega$. Thus,

$$|x - y| \leq |x - x_h| + |x_h - y| \leq 2\,|x_h - y|.$$

Now using the fact $\partial\Omega \in C^2$ again, we get

$$\big|\overrightarrow{n}(y) - \overrightarrow{n}(x)\big| \leq C|y - x| \leq C\,|x_h - y|.$$

Consequently,

$$\begin{aligned} \Big|D\Phi\big(x_h - y, t-\tau\big) \cdot \big[\overrightarrow{n}(y) - \overrightarrow{n}(x)\big]\, g(y,\tau)\Big| &\leq \frac{C\,|x_h - y|^2}{(t-\tau)^{n/2+1}} \exp\left(-\frac{|x_h - y|^2}{4(t-\tau)}\right) \\ &\leq \frac{C}{(t-\tau)^{n/2}} \exp\left(-\frac{|x_h - y|^2}{8(t-\tau)}\right) \\ &\leq \frac{C}{(t-\tau)^{n/2}} \exp\left(-\frac{|x - y|^2}{32(t-\tau)}\right). \end{aligned}$$

Since the right hand side of the above inequality is integrable on $\partial\Omega \times [0, t]$, if we split the integral as following

$$\int_0^t \int_{\partial\Omega} = \int_0^\epsilon \int_{\partial\Omega} + \int_\epsilon^t \int_{\partial\Omega} = I + II,$$

then the integral I is uniformly small and the uniform convergence on II is clear. As a result, the uniform convergence of (2.2.5) is established. $\square$

2.2.2 Heat potentials with piecewise continuous density

The previous subsection introduces jump relation of the single-layer and double-layer potentials with continuous density. But Theorem 2.2.1 and Theorem 2.2.2 are still not enough for our problems. For example, in order to show the existence of the solution to (2.3.1), the boundary functions β and g are only assumed in $\mathcal{B}_T$, not in $C\big(\partial\Omega \times [0, T]\big)$. Thus we need to adapt this jump relation to our case. So in this section, we will establish a similar jump relation for the single-layer heat potential with piecewise continuous density as that in Theorem 2.2.1. This jump relation for the single-layer heat potential with discontinuous density in Theorem 2.2.6 will be mainly applied to show the existence and uniqueness of the solution to (2.3.1) and (2.4.1).

First, we extend the definition of the single-layer heat potential to include the function space $\mathcal{B}_T$ in which the function is only piecewise continuous on the boundary. For any $\varphi \in \mathcal{B}_T$, define the single-layer heat potential with density φ to be

$$U(x,t) = \int_0^t \int_{\partial\Omega} \Phi(x-y, t-\tau)\varphi(y,\tau)\, dS(y)\, d\tau, \quad \forall\, (x,t) \in \Omega \times (0, T]. \tag{2.2.6}$$

Then we introduce a similar definition in which the integration is on Γ_i (i=1 or 2):

$$U_i(x,t) = \int_0^t \int_{\Gamma_i} \Phi(x-y,t-\tau)\varphi(y,\tau)\, dS(y)\, d\tau, \quad \forall\, (x,t) \in \Omega \times (0,T]. \tag{2.2.7}$$

Recalling that the interface between Γ_1 and Γ_2 is defined to be $\widetilde{\Gamma} = \overline{\Gamma}_1 \cap \overline{\Gamma}_2$. Thus, the surface measure of $\widetilde{\Gamma}$ is 0, which implies that

$$U(x,t) = U_1(x,t) + U_2(x,t).$$

Recalling also the notations in Section 1.4: for $i = 1$ or 2, $\varphi|_{\Gamma_i \times (0,T]}$ is uniformly continuous, thus we can extend this function to a continuous function φ_i on $\overline{\Gamma}_i \times [0,T]$. In the following, when $x \in \widetilde{\Gamma}$ and $t > 0$, Lemma 2.2.5 will establish the jump relation for U_i ($i = 1$ or 2) at (x,t) with the jump $\varphi_i(x,t)/4$; when $x \in \Gamma_i$ ($i = 1$ or 2) and $t > 0$, Lemma 2.2.4 state the jump relation for U_i at (x,t) with the jump $\varphi_i(x,t)/2$ (here $\varphi_i(x,t)/2$=$\varphi(x,t)/2$, since $x \in \Gamma_i$).

The essence of the proof is the same as that of Theorem 2.2.1. Before the proof, we introduce some notations that will be used in the proof and later argument. We write $\mathbf{0}$ and $\tilde{\mathbf{0}}$ to be the origins in $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$ respectively and $\mathbf{e_n}$ denotes the point $(0,0,\cdots,0,1)$ in $\mathbb{R}^n$. For any point $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$, we write

$$\tilde{y} = (y_1, y_2, \cdots, y_{n-1}).$$

For any $r > 0$,

$$B_r \triangleq B(\mathbf{0}, r)$$

means the ball in $\mathbb{R}^n$ with radius r and

$$\widetilde{B}_r \triangleq B(\tilde{\mathbf{0}}, r)$$

represents the ball in $\mathbb{R}^{n-1}$ with radius r. Γ is used to denote the Gamma function, i.e. $\Gamma : (0, \infty) \to \mathbb{R}$ defined by

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t}\, dt.$$

It should not be confused with the partial boundaries Γ_1, Γ_2 and $\widetilde{\Gamma}$.

Lemma 2.2.4. *Assume $\varphi \in \mathcal{B}_T$ and $i = 1$ or 2. Define U_i as in (2.2.7). Then for any $x \in \Gamma_i$ and $t \in (0, T]$,*

$$\lim_{h \to 0^+} DU_i(x_h, t) \cdot \overrightarrow{n}(x) = \int_0^t \int_{\Gamma_i} \frac{\partial \Phi(x - y, t - \tau)}{\partial n(x)}\, \varphi(y, \tau)\, dS(y)\, d\tau + \frac{1}{2}\, \varphi(x, t), \qquad (2.2.8)$$

where $x_h = x - h\overrightarrow{n}(x)$.

Proof. The proof is almost the same as that of Theorem 2.2.1, since Γ_i is a relatively open subset of $\partial\Omega$ and φ is continuous on $\Gamma_i \times (0, T]$. $\square$

But the situation when $x \in \widetilde{\Gamma}$ is different from that in Lemma 2.2.4, since $\widetilde{\Gamma}$ is the boundary of Γ_i. The following lemma claims that in this situation, the jump is only half of that in Lemma 2.2.4. The proof for Lemma 2.2.5 is also similar to that of Theorem 2.2.1, but for completeness, we include a detailed proof.

Lemma 2.2.5. *Assume $\varphi \in \mathcal{B}_T$ and $i = 1$ or 2. Define U_i as in (2.2.7). Then for any*

$x \in \widetilde{\Gamma} \triangleq \overline{\Gamma}_1 \cap \overline{\Gamma}_2$ and $t \in (0,T]$,

$$\lim_{h\to 0^+} DU_i(x_h,t)\cdot \overrightarrow{n}(x) = \int_0^t\int_{\Gamma_i} \frac{\partial\Phi(x-y,t-\tau)}{\partial n(x)}\,\varphi(y,\tau)\,dS(y)\,d\tau + \frac{1}{4}\,\varphi_i(x,t), \tag{2.2.9}$$

where $x_h = x - h\overrightarrow{n}(x)$ and φ_i represents the continuous extension of $\varphi|_{\Gamma_i\times(0,T]}$ on $\overline{\Gamma}_i\times[0,T]$.

Proof. We assume $i=1$ (The case $i=2$ is similar). By (1.4.4), (2.2.9) becomes

$$\begin{aligned}&\lim_{h\to 0^+} -\int_0^t\int_{\Gamma_1} \frac{(x_h-y)\cdot\overrightarrow{n}(x)}{(t-\tau)^{n/2+1}}\,\exp\left(-\frac{|x_h-y|^2}{4(t-\tau)}\right)\varphi_1(y,\tau)\,dS(y)\,d\tau\\ &= -\int_0^t\int_{\Gamma_1}\frac{(x-y)\cdot\overrightarrow{n}(x)}{(t-\tau)^{n/2+1}}\,\exp\left(-\frac{|x-y|^2}{4(t-\tau)}\right)\varphi_1(y,\tau)\,dS(y)\,d\tau + \frac{(4\pi)^{n/2}}{2}\,\varphi_1(x,t).\end{aligned}\tag{2.2.10}$$

Without loss of generality, we assume $x=\mathbf{0}$, otherwise we can do a translation. After this, we further assume $\overrightarrow{n}(\mathbf{0}) = -\mathbf{e_n}$, otherwise we can do a rotation which preserves dot product and the distance. By these two simplifications, we have $x=\mathbf{0}$ and $\overrightarrow{n}(x) = -\mathbf{e_n}$, therefore $x_h = h\mathbf{e_n}$ and (2.2.10) is reduced to

$$\begin{aligned}&\lim_{h\to 0^+}\int_0^t\int_{\Gamma_1}\frac{h-y_n}{(t-\tau)^{n/2+1}}\,\exp\left(-\frac{|h\mathbf{e_n}-y|^2}{4(t-\tau)}\right)\varphi_1(y,\tau)\,dS(y)\,d\tau\\ &= -\int_0^t\int_{\Gamma_1}\frac{y_n}{(t-\tau)^{n/2+1}}\,\exp\left(-\frac{|y|^2}{4(t-\tau)}\right)\varphi_1(y,\tau)\,dS(y)\,d\tau + \frac{(4\pi)^{n/2}}{2}\,\varphi_1(\mathbf{0},t).\end{aligned}$$

By a change of variable in τ, it is equivalent to

$$\begin{aligned}&\lim_{h\to 0^+}\int_0^t\int_{\Gamma_1}\frac{h-y_n}{\tau^{n/2+1}}\,\exp\left(-\frac{|h\mathbf{e_n}-y|^2}{4\tau}\right)\varphi_1(y,t-\tau)\,dS(y)\,d\tau\\ &= -\int_0^t\int_{\Gamma_1}\frac{y_n}{\tau^{n/2+1}}\,\exp\left(-\frac{|y|^2}{4\tau}\right)\varphi_1(y,t-\tau)\,dS(y)\,d\tau + \frac{(4\pi)^{n/2}}{2}\,\varphi_1(\mathbf{0},t).\end{aligned}\tag{2.2.11}$$

Because $\partial\Omega \in C^2$ and $\widetilde{\Gamma} = \partial\Gamma_1 \in C^1$, we can straighten the boundary. More specifically, after relabeling the coordinates, there exist $\phi_1 \in C^2 : \mathbb{R}^{n-1} \to \mathbb{R}$, $\phi_2 \in C^1 : \mathbb{R}^{n-2} \to \mathbb{R}$, $\eta_0 > 0$ and a neighborhood $S_{\eta_0} \subset \partial\Omega$ of $\mathbf{0}$ such that S_{η_0} can be parametrized as

$$S_{\eta_0} = \{\big(\tilde{y}, \phi_1(\tilde{y})\big) : \tilde{y} \in \widetilde{B}_{\eta_0}\}$$

and for any $y \in \widetilde{\Gamma} \cap S_{\eta_0}$, we not only have $y_n = \phi_1(\tilde{y})$, but also $y_{n-1} = \phi_2(y_1, y_2, \cdots, y_{n-2})$. Fixing η_0 and then for any $\eta < \eta_0$, we define

$$S_\eta = \{\big(\tilde{y}, \phi_1(\tilde{y})\big) : \tilde{y} \in \widetilde{B}_\eta\},$$

which is a subset of S_{η_0} and a small neighborhood of $\mathbf{0}$. Then we denote

$$S_{\eta,1} = S_\eta \cap \Gamma_1, \quad \widetilde{S}_\eta = S_\eta \cap \widetilde{\Gamma},$$

$$\widetilde{B}_{\eta,1} = \{\tilde{y} \in \widetilde{B}_\eta : \big(\tilde{y}, \phi_1(\tilde{y})\big) \in S_{\eta,1}\}, \quad P_\eta = \{\tilde{y} \in \widetilde{B}_\eta : \big(\tilde{y}, \phi_1(\tilde{y})\big) \in \widetilde{S}_\eta\}.$$

After these preparations, we begin the technical proof. Given any $\epsilon > 0$, we want to find $\delta = \delta(\epsilon) > 0$ such that for any $0 < h < \delta$, the difference between the two sides of (2.2.11) is within $C\varepsilon$ for some constant C.

For any $\eta \in (0, \eta_0)$ which will be determined later, we split the integral over Γ_1 in (2.2.11) into two parts: $\int_{\Gamma_1} = \int_{S_{\eta,1}} + \int_{\Gamma_1 \setminus S_{\eta,1}}$. Since $\Gamma_1 \setminus S_{\eta,1}$ is away from $\mathbf{0}$, it is easy to see there exists $\delta_1 = \delta_1(\eta, \epsilon)$ such that when $0 < h < \delta_1$, then

$$\begin{aligned} &\Big| \int_0^t \int_{\Gamma_1 \backslash S_{\eta,1}} \frac{h - y_n}{\tau^{n/2+1}} \exp\Big(- \frac{|h\mathbf{e_n} - y|^2}{4\tau}\Big) \varphi_1(y, t-\tau)\, dS(y)\, d\tau \\ &+ \int_0^t \int_{\Gamma_1 \backslash S_{\eta,1}} \frac{y_n}{\tau^{n/2+1}} \exp\Big(- \frac{|y|^2}{4\tau}\Big) \varphi_1(y, t-\tau)\, dS(y)\, d\tau \Big| < \epsilon. \end{aligned} \tag{2.2.12}$$

Next since $\overrightarrow{n}(\mathbf{0}) = -\mathbf{e_n}$, then $D\phi_1(\tilde{\mathbf{0}}) = \tilde{\mathbf{0}}$. As a result, for any $y \in S_{\eta,1}$,

$$\begin{aligned} &|y_n| = |\phi_1(\tilde{y})| = |\phi_1(\tilde{y}) - \phi_1(\tilde{\mathbf{0}})| = |D\phi_1(\theta\tilde{y}) \cdot \tilde{y}| \\ \le& |D\phi_1(\theta\tilde{y}) - D\phi_1(\tilde{\mathbf{0}})|\, |\tilde{y}| \le C\, |\tilde{y}|^2, \end{aligned} \tag{2.2.13}$$

where by the mean value theorem θ is some number between 0 and 1. By (2.2.13), together with the fact $|h\mathbf{e_n} - y| \ge |\tilde{y}|$, we attain

$$|y_n| \exp\Big(- \frac{|h\mathbf{e_n} - y|^2}{4\tau}\Big) \le C\, |\tilde{y}|^2 \exp\Big(- \frac{|\tilde{y}|^2}{4\tau}\Big).$$

Noticing

$$\int_0^t \int_{S_{\eta,1}} \frac{|\tilde{y}|^2}{\tau^{n/2+1}} \exp\Big(- \frac{|\tilde{y}|^2}{4\tau}\Big)\, dS(y)\, d\tau < \infty,$$

then it follows from Lebesgue's dominated convergence theorem that

$$\begin{aligned} &\lim_{h \to 0^+} \int_0^t \int_{S_{\eta,1}} \frac{y_n}{\tau^{n/2+1}} \exp\Big(- \frac{|h\mathbf{e_n} - y|^2}{4\tau}\Big) \varphi_1(y, t-\tau)\, dS(y)\, d\tau \\ =& \int_0^t \int_{S_{\eta,1}} \frac{y_n}{\tau^{n/2+1}} \exp\Big(- \frac{|y|^2}{4\tau}\Big) \varphi_1(y, t-\tau)\, dS(y)\, d\tau. \end{aligned}$$

As a result, there exists $\delta_2 = \delta_2(\eta, \epsilon)$ such that when $0 < h < \delta_2$, then

$$\begin{aligned} \Big| \int_0^t \int_{S_{\eta,1}} \frac{y_n}{\tau^{n/2+1}} \exp\Big(-\frac{|h\mathbf{e_n} - y|^2}{4\tau}\Big) \varphi_1(y, t-\tau)\, dS(y)\, d\tau \\ - \int_0^t \int_{S_{\eta,1}} \frac{y_n}{\tau^{n/2+1}} \exp\Big(-\frac{|y|^2}{4\tau}\Big) \varphi_1(y, t-\tau)\, dS(y)\, d\tau \Big| < \epsilon. \end{aligned} \tag{2.2.14}$$

Now it suffices to verify that $|I_\eta(h,t) - \frac{1}{2}(4\pi)^{n/2}\varphi_1(\mathbf{0}, t)| < C\varepsilon$, where

$$I_\eta(h,t) \triangleq \int_0^t \int_{S_{\eta,1}} \frac{h}{\tau^{n/2+1}} \exp\Big(-\frac{|h\mathbf{e_n} - y|^2}{4\tau}\Big) \varphi_1(y, t-\tau)\, dS(y)\, d\tau. \tag{2.2.15}$$

Recalling that $y_n = \phi(\tilde{y})$, (2.2.15) can be rewritten as

$$\begin{aligned} I_\eta(h,t) &= \int_0^t \int_{\widetilde{B}_{\eta,1}} \frac{h}{\tau^{n/2+1}} \exp\Big(-\frac{|h\mathbf{e_n} - y|^2}{4\tau}\Big) \varphi_1(y, t-\tau)\sqrt{1 + |D\phi_1(\tilde{y})|^2}\, d\tilde{y}\, d\tau \\ &= \int_0^t \int_{\widetilde{B}_{\eta,1}} \frac{h}{\tau^{n/2+1}} \exp\Big(-\frac{|\tilde{y}|^2 + |h - y_n|^2}{4\tau}\Big) \varphi_1(y, t-\tau)\sqrt{1 + |D\phi_1(\tilde{y})|^2}\, d\tilde{y}\, d\tau, \end{aligned} \tag{2.2.16}$$

where $y = \big(\tilde{y}, \phi_1(\tilde{y})\big)$. I_η is hard to compute, so we approximate it by a simpler function. We define $\tilde{I}_\eta(h,t)$ as following

$$\begin{aligned} \tilde{I}_\eta(h,t) &= \int_0^t \int_{\widetilde{B}_{\eta,1}} \frac{h}{\tau^{n/2+1}} \exp\Big(-\frac{|h\mathbf{e_n} - (\tilde{y}, 0)|^2}{4\tau}\Big) \varphi_1(\mathbf{0}, t-\tau)\, d\tilde{y}\, d\tau \\ &= \int_0^t \int_{\widetilde{B}_{\eta,1}} \frac{h}{\tau^{n/2+1}} \exp\Big(-\frac{|\tilde{y}|^2 + h^2}{4\tau}\Big) \varphi_1(\mathbf{0}, t-\tau)\, d\tilde{y}\, d\tau. \end{aligned} \tag{2.2.17}$$

Our strategy is to show that $\tilde{I}_\eta(h,t)$ is close to both $\frac{1}{2}(4\pi)^{n/2}\varphi_1(\mathbf{0}, t)$ and $I_\eta(h,t)$.

Based on (2.2.17) and noticing h is strictly positive, so the integrand is absolutely integrable. Thus, we can reverse the order of integration. Then using the change of variable

$\tau \to \sigma = \left(|\tilde{y}|^2 + h^2\right)/(4\tau)$, we attain

$$\begin{aligned}
\tilde{I}_\eta(h,t) &= \int_{\widetilde{B}_{\eta,1}} \int_0^t \frac{h}{\tau^{n/2+1}} \exp\left(- \frac{|\tilde{y}|^2 + h^2}{4\tau}\right) \varphi_1(\mathbf{0}, t-\tau)\, d\tau\, d\tilde{y} \\
&= \int\limits_{\widetilde{B}_{\eta,1}} \int\limits_{\frac{|\tilde{y}|^2+h^2}{4t}}^{\infty} \frac{4^{n/2}\, \sigma^{n/2-1}\, h}{(|\tilde{y}|^2+h^2)^{n/2}}\, e^{-\sigma}\, \varphi_1\Big(\mathbf{0}, t - \frac{|\tilde{y}|^2+h^2}{4\sigma}\Big)\, d\sigma\, d\tilde{y} \\
&= 4^{n/2} \int\limits_{\widetilde{B}_{\eta,1}} \frac{h}{(|\tilde{y}|^2+h^2)^{n/2}} \int\limits_{\frac{|\tilde{y}|^2+h^2}{4t}}^{\infty} \sigma^{n/2-1} e^{-\sigma}\, \varphi_1\Big(\mathbf{0}, t - \frac{|\tilde{y}|^2+h^2}{4\sigma}\Big)\, d\sigma\, d\tilde{y} \\
&= 4^{n/2} \int_{\widetilde{B}_{\eta,1}} \frac{h}{(|\tilde{y}|^2+h^2)^{n/2}}\, H\big(|\tilde{y}|^2 + h^2, t\big)\, d\tilde{y}, \qquad (2.2.18)
\end{aligned}$$

where

$$H(\lambda, t) \triangleq \int_{\frac{\lambda}{4t}}^{\infty} \sigma^{n/2-1} e^{-\sigma}\, \varphi_1\Big(\mathbf{0}, t - \frac{\lambda}{4\sigma}\Big)\, d\sigma.$$

It is readily to see that

$$\lim_{\lambda \to 0} H(\lambda, t) = \Gamma\Big(\frac{n}{2}\Big)\, \varphi_1(\mathbf{0}, t). \qquad (2.2.19)$$

Consequently, there exists $\delta_3 = \delta_3(\epsilon)$ such that when $\eta < \delta_3$ and $0 < h < \delta_3$, then

$$\Big| H\big(|\tilde{y}|^2 + h^2, t\big) - \Gamma\Big(\frac{n}{2}\Big)\varphi_1(\mathbf{0}, t)\Big| < \epsilon, \quad \forall\, \tilde{y} \in \widetilde{B}_{\eta,1}. \qquad (2.2.20)$$

After taking care of the H term in (2.2.18), let's consider the following integration

$$\int\limits_{\widetilde{B}_{\eta,1}} \frac{h}{(|\tilde{y}|^2+h^2)^{n/2}}\, d\tilde{y},$$

where the integrand $h\left(|\tilde{y}|^2 + h^2\right)^{-n/2}$ is radial in $\tilde{y} \in \mathbb{R}^{n-1}$ and positive when $h > 0$. Since

$\widetilde{\Gamma} = \partial\Gamma_1 \in C^1$, it ensures that P_η almost bisects $\widetilde{B}_\eta$ when η is small, which means $\widetilde{B}_{\eta,1}$ is close to a hemisphere and

$$\lim_{\eta\to 0} \frac{|\widetilde{B}_{\eta,1}|}{|\widetilde{B}_\eta|} = \frac{1}{2}.$$

This limit is the essential reason why the jump is $\frac{1}{2}\varphi(x,t)$ in (2.2.8). As a result, we can find $\delta_4 = \delta_4(\epsilon)$ such that for any $\eta < \delta_4$,

$$1 - \epsilon < \frac{\int\limits_{\widetilde{B}_{\eta,1}} \frac{h}{(|\tilde{y}|^2+h^2)^{n/2}}\, d\tilde{y}}{\frac{1}{2}\int\limits_{\widetilde{B}_\eta} \frac{h}{(|\tilde{y}|^2+h^2)^{n/2}}\, d\tilde{y}} < 1 + \epsilon,$$

i.e.

$$\left| \int\limits_{\widetilde{B}_{\eta,1}} \frac{h}{(|\tilde{y}|^2 + h^2)^{n/2}}\, d\tilde{y} - \frac{1}{2}\int\limits_{\widetilde{B}_\eta} \frac{h}{(|\tilde{y}|^2 + h^2)^{n/2}}\, d\tilde{y} \right| < \frac{\epsilon}{2}\int\limits_{\widetilde{B}_\eta} \frac{h}{(|\tilde{y}|^2 + h^2)^{n/2}}\, d\tilde{y}. \tag{2.2.21}$$

Next, we will estimate $\int\limits_{\widetilde{B}_\eta} \frac{h}{(|\tilde{y}|^2+h^2)^{n/2}}\, d\tilde{y}$. Making the change of variable $\tilde{y} \to \tilde{z} \triangleq \tilde{y}/h$,

$$\int\limits_{\widetilde{B}_\eta} \frac{h}{(|\tilde{y}|^2 + h^2)^{n/2}}\, d\tilde{y} = \int\limits_{\widetilde{B}_{\eta/h}} \frac{1}{(|\tilde{z}|^2 + 1)^{n/2}}\, d\tilde{z}.$$

On one hand,

$$\int\limits_{\widetilde{B}_{\eta/h}} \frac{1}{(|\tilde{z}|^2 + 1)^{n/2}}\, d\tilde{z} \le \int\limits_{\mathbb{R}^{n-1}} \frac{1}{(|\tilde{z}|^2 + 1)^{n/2}}\, d\tilde{z} = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)},$$

while on the other hand,

$$\lim_{h\to 0} \int\limits_{\widetilde{B}_{\eta/h}} \frac{1}{(|\tilde{z}|^2+1)^{n/2}}\, d\tilde{z} = \int\limits_{\mathbb{R}^{n-1}} \frac{1}{(|\tilde{z}|^2+1)^{n/2}}\, d\tilde{z} = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}.$$

Thus, there exists $\delta_5 = \delta_5(\eta, \epsilon)$ such that for any $0 < h < \delta_5$,

$$\left| \int\limits_{\widetilde{B}_{\eta}} \frac{h}{(|\tilde{y}|^2+h^2)^{n/2}}\, d\tilde{y} - \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \right| < \epsilon \tag{2.2.22}$$

and

$$\left| \int\limits_{\widetilde{B}_{\eta,1}} \frac{h}{(|\tilde{y}|^2+h^2)^{n/2}}\, d\tilde{y} - \frac{\pi^{n/2}}{2\Gamma\left(\frac{n}{2}\right)} \right| < C\varepsilon \tag{2.2.23}$$

by noticing (2.2.21). It then follows from (2.2.18), (2.2.20) and (2.2.23) that

$$\left| \tilde{I}_{\eta}(h,t) - \frac{(4\pi)^{n/2}}{2} \varphi_1(\mathbf{0},t) \right| < C\varepsilon. \tag{2.2.24}$$

Now it suffices to show that $\tilde{I}_{\eta}(h,t)$ is close to $I_{\eta}(h,t)$. Firstly, because of (2.2.13), $|\tilde{y}|^2 + |h-y_n|^2$ is comparable to $|\tilde{y}|^2 + h^2$. More precisely, there exist positive constants $m_1 < 1$ and $M_1 > 1$ such that

$$m_1\left(|\tilde{y}|^2+h^2\right) \le |\tilde{y}|^2 + |h-y_n|^2 \le M_1\left(|\tilde{y}|^2+h^2\right). \tag{2.2.25}$$

We can equivalently write it to be

$$m_1\,|h\mathbf{e_n} - (\tilde{y},0)|^2 \le |h\mathbf{e_n} - y|^2 \le M_1\,|h\mathbf{e_n} - (\tilde{y},0)|^2. \tag{2.2.26}$$

Consequently, it follows from (2.2.16) and (2.2.17) that

$$
\begin{aligned}
&|I_\eta(h,t) - \tilde{I}_\eta(h,t)| \\
\le &\int_0^t \frac{h}{\tau^{n/2+1}} \int_{\widetilde{B}_{\eta,1}} \left| e^{-\frac{|h\mathbf{e_n}-y|^2}{4\tau}} - e^{-\frac{|h\mathbf{e_n}-(\tilde{y},0)|^2}{4\tau}} \right| \left|\varphi_1(y,t-\tau)\right| \sqrt{1+|D\phi_1(\tilde{y})|^2}\, d\tilde{y}\, d\tau \\
&+ \int_0^t \frac{h}{\tau^{n/2+1}} \int_{\widetilde{B}_{\eta,1}} e^{-\frac{|h\mathbf{e_n}-(\tilde{y},0)|^2}{4\tau}} \left|\varphi_1(y,t-\tau)\sqrt{1+|D\phi_1(\tilde{y})|^2} - \varphi_1(\mathbf{0},t-\tau)\right| d\tilde{y}\, d\tau \\
\triangleq &I + II. \qquad (2.2.27)
\end{aligned}
$$

For II, since $\varphi_1 \in C\big(\overline{\Gamma}_1 \times [0,T]\big)$, then there exists $\delta_6 = \delta_6(\epsilon)$ such that when $\eta < \delta_6$,

$$
\left|\varphi_1(y,t-\tau)\sqrt{1+|D\phi_1(\tilde{y})|^2} - \varphi_1(\mathbf{0},t-\tau)\right| < \epsilon, \quad \forall\, y \in \widetilde{B}_{\eta,1},\ \tau \in [0,t].
$$

As a result,

$$
\begin{aligned}
II &\le \epsilon \int_0^t \frac{h}{\tau^{n/2+1}} \int_{\widetilde{B}_{\eta,1}} e^{-\frac{|h\mathbf{e_n}-(\tilde{y},0)|^2}{4\tau}}\, d\tilde{y}\, d\tau \\
&= \epsilon \int_0^t \frac{h}{\tau^{n/2+1}} \int_{\widetilde{B}_{\eta,1}} e^{-\frac{|\tilde{y}|^2+h^2}{4\tau}}\, d\tilde{y}\, d\tau \\
&= \epsilon \int_{\widetilde{B}_{\eta,1}} h \int_0^t \frac{1}{\tau^{n/2+1}}\, e^{-\frac{|\tilde{y}|^2+h^2}{4\tau}}\, d\tau\, d\tilde{y} \\
&\le \epsilon \int_{\widetilde{B}_{\eta,1}} h \int_0^\infty \frac{4^{n/2}}{\big(|\tilde{y}|^2+h^2\big)^{n/2}}\, \sigma^{n/2-1}\, e^{-\sigma}\, d\sigma\, d\tilde{y} \\
&= C\,\epsilon \int_{\widetilde{B}_{\eta,1}} \frac{h}{\big(|\tilde{y}|^2+h^2\big)^{n/2}}\, d\tilde{y},
\end{aligned}
$$

where the second inequality is due to the change of variable $\tau \to \sigma \triangleq \frac{|\tilde{y}|^2+h^2}{4\tau}$. Now by another change of variabel $\tilde{y} \to \tilde{z} \triangleq \tilde{y}/h$, we get

$$II \leq C\,\epsilon \int_{\mathbb{R}^{n-1}} \frac{1}{(|\tilde{z}|^2+1)^{n/2}}\, d\tilde{z} = C\varepsilon. \tag{2.2.28}$$

To estimate I, firstly it is easy to see that for any $h > 0$ and $y \in \widetilde{B}_{\eta,1}$,

$$h \leq |h\mathbf{e_n} - (\tilde{y}, 0)|. \tag{2.2.29}$$

Then by (2.2.13),

$$\begin{aligned}
&\Big| |h\mathbf{e_n} - y|^2 - |h\mathbf{e_n} - (\tilde{y},0)|^2 \Big| \\
=& \big| (h-y_n)^2 - h^2 \big| \\
=& |y_n|\,|2h - y_n| \\
\leq& C\,|\tilde{y}|^2 \big(2h + |\tilde{y}|^2 \big) \\
\leq& C\,|h\mathbf{e_n} - (\tilde{y},0)|^3.
\end{aligned}$$

Now it follows from the mean value theorem and (2.2.26) that

$$\begin{aligned}
\Big| e^{-\frac{|h\mathbf{e_n}-y|^2}{4\tau}} - e^{-\frac{|h\mathbf{e_n}-(\tilde{y},0)|^2}{4\tau}} \Big| &\leq \frac{1}{4\tau}\, e^{-\frac{m_1\,|h\mathbf{e_n}-(\tilde{y},0)|^2}{4\tau}} \Big| |h\mathbf{e_n} - y|^2 - |h\mathbf{e_n} - (\tilde{y},0)|^2 \Big| \\
&\leq C\, \frac{|h\mathbf{e_n} - (\tilde{y},0)|^3}{\tau}\, e^{-\frac{m_1\,|h\mathbf{e_n}-(\tilde{y},0)|^2}{4\tau}}
\end{aligned} \tag{2.2.30}$$

Thus, based on (2.2.29) and (2.2.30), we attain

$$I \leq \int_0^t \int_{\widetilde{B}_{\eta,1}} \tau^{-n/2-2}\, e^{-\frac{m_1\,|h\mathbf{e_n}-(\tilde{y},0)|^2}{4\tau}}\, |h\mathbf{e_n} - (\tilde{y},0)|^4\, d\tilde{y}\, d\tau.$$

Again, by reversing the order of integration and the change of variable $\tau \to \sigma \triangleq \frac{|h\mathbf{e_n}-(\tilde{y},0)|^2}{4\tau}$, we get

$$
\begin{aligned}
I &\le \int_{\widetilde{B}_{\eta,1}} |h\mathbf{e_n} - (\tilde{y},0)|^4 \int_0^t \tau^{-n/2-2}\, e^{-\frac{m_1\,|h\mathbf{e_n}-(\tilde{y},0)|^2}{4\tau}}\, d\tau\, d\tilde{y} \\
&\le C \int_{\widetilde{B}_{\eta,1}} \frac{1}{|h\mathbf{e_n} - (\tilde{y},0)|^{n-2}} \int_0^\infty \sigma^{n/2}\, e^{-m_1\sigma}\, d\sigma\, d\tilde{y} \\
&\le C \int_{\widetilde{B}_{\eta,1}} \frac{1}{|\tilde{y}|^{n-2}}\, d\tilde{y}.
\end{aligned}
$$

Hence, there exists $\delta_7 = \delta_7(\epsilon)$ such that when $\eta < \delta_7$, then

$$
I < \epsilon. \tag{2.2.31}
$$

Combining (2.2.31) and (2.2.28), we get

$$
|I_\eta(h,t) - \tilde{I}_\eta(h,t)| < C\varepsilon.
$$

Therefore, we finish the proof.

In summary, for any $\epsilon > 0$, we firstly determine $\delta_3(\epsilon), \delta_4(\epsilon), \delta_6(\epsilon), \delta_7(\epsilon)$ and choose $\eta < \min\{\eta_0, \delta_3, \delta_4, \delta_6, \delta_7\}$. Then we determine $\delta_1(\eta,\epsilon), \delta_2(\eta,\epsilon), \delta_5(\eta,\epsilon)$ and choose $\delta < \min\{\delta_1, \delta_2, \delta_3, \delta_5\}$. $\square$

Theorem 2.2.6. *Let $\varphi \in \mathcal{B}_T$ and define U to be the single-layer heat potential with density φ as (2.2.6).*

(1) If $x \in \Gamma_1 \cup \Gamma_2$ and $t \in (0, T]$, then

$$\lim_{h \to 0^+} DU(x_h, t) \cdot \overrightarrow{n}(x) = \int_0^t \int_{\partial\Omega} \frac{\partial \Phi(x-y, t-\tau)}{\partial n(x)} \varphi(y, \tau)\, dS(y) d\tau + \frac{1}{2} \varphi(x, t). \quad (2.2.32)$$

(2) If $x \in \widetilde{\Gamma}$ and $t \in (0, T]$, then

$$\lim_{h \to 0^+} DU(x_h, t) \cdot \overrightarrow{n}(x) = \int_0^t \int_{\partial\Omega} \frac{\partial \Phi(x-y, t-\tau)}{\partial n(x)} \varphi(y, \tau)\, dS(y)\, d\tau + \frac{1}{4} \big[\varphi_1(x, t) + \varphi_2(x, t)\big], \quad (2.2.33)$$

where φ_i ($i = 1$ or 2) represents the continuous extension of $\varphi|_{\Gamma_i \times (0,T]}$ on $\overline{\Gamma}_i \times [0, T]$.

Proof. (1) Without loss of generality, we suppose $x \in \Gamma_1$, then by Lemma 2.2.4,

$$\lim_{h \to 0^+} DU_1(x_h, t) \cdot \overrightarrow{n}(x) = \int_0^t \int_{\Gamma_1} \frac{\partial \Phi(x-y, t-\tau)}{\partial n(x)} \varphi(y, \tau)\, dS(y)\, d\tau + \frac{1}{2} \varphi(x, t).$$

In addition, since the distance between x and Γ_2 is positive, then it is easy to see that

$$\lim_{h \to 0^+} DU_2(x_h, t) \cdot \overrightarrow{n}(x) = \int_0^t \int_{\Gamma_2} \frac{\partial \Phi(x-y, t-\tau)}{\partial n(x)} \varphi(y, \tau)\, dS(y)\, d\tau.$$

Adding these two equations together, (2.2.32) follows.

(2) (2.2.33) is directly implied by Lemma 2.2.5.

□

2.3 Linear case

2.3.1 Definition of the local solution

In this section, we will show the existence and uniqueness of the solution to the following linear initial-boundary value problem:

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = f(x,t) & \text{in} \quad \Omega \times (0,T], \\ \frac{\partial u(x,t)}{\partial n(x)} + \beta(x,t)u(x,t) = g(x,t) & \text{on} \quad (\Gamma_1 \cup \Gamma_2) \times (0,T], \\ u(x,0) = \psi(x) & \text{on} \quad \Omega, \end{cases} \tag{2.3.1}$$

where $f \in C^{\alpha,\alpha/2}\big(\overline{\Omega} \times [0,T]\big)$, $\beta, g \in \mathcal{B}_T$, $\psi \in C^1(\overline{\Omega})$. We will first show the existence and then apply the existence result to derive the uniqueness.

Definition 2.3.1. *For any $T > 0$, a solution to (2.3.1) on $\overline{\Omega} \times [0,T]$ means a function u in $\mathcal{A}_T$ that satisfies (2.3.1) pointwise and for any $(x,t) \in \widetilde{\Gamma} \times (0,T]$, $\frac{\partial u(x,t)}{\partial n(x)}$ exists and*

$$\frac{\partial u(x,t)}{\partial n(x)} + \frac{1}{2}\big[\beta_1(x,t) + \beta_2(x,t)\big]u(x,t) = \frac{1}{2}\big[g_1(x,t) + g_2(x,t)\big], \tag{2.3.2}$$

where β_i and g_i denote the continuous extensions of $\beta|_{\Gamma_i \times (0,T]}$ and $g|_{\Gamma_i \times (0,T]}$ to $\overline{\Gamma}_i \times [0,T]$ $(i = 1$ or $2)$.

2.3.2 Auxilliary lemmas

Before showing the existence, we state some preliminary results.

Lemma 2.3.2. *Suppose Ω is an open bounded subset in $\mathbb{R}^n$ with $\partial\Omega \in C^2$, then there exists* *a constant $C > 0$ such that for any $x, y \in \partial\Omega$,*

$$|(x-y) \cdot \overrightarrow{n}(x)| \leq C\,|x-y|^2.$$

Proof. This is a standard fact, we omit the proof. $\square$

Lemma 2.3.3. *Suppose Ω is an open, bounded set in $\mathbb{R}^n$ with $\partial\Omega \in C^2$, $0 \leq a < n-1$, $0 \leq b < n-1$, then there exists a constant $C = C(a, b, n, \Omega)$ such that for any $x, z \in \partial\Omega$,*

$$\int_{\partial\Omega} \frac{dS(y)}{|x-y|^a |y-z|^b} \leq \begin{cases} C\,|x-z|^{n-1-a-b} & \text{if} \quad a+b > n-1, \\ C & \text{if} \quad a+b < n-1. \end{cases}$$

Proof. We refer the readers to ([7], Lemma 1, Sec. 2, Chap. 5). $\square$

The following Lemma is mentioned in ([7], Theorem 2, Sec. 2, Chap. 5), but it does not explicitly give the estimate (2.3.6), which will be used in some other places of this thesis. So for the convenience of the readers, we decide to include a complete proof for it. For any $T > 0$, define

$$D_{\Omega,T} = \{(x,t;y,\tau) \,\big|\, x, y \in \Omega, x \neq y, 0 \leq \tau < t \leq T\} \tag{2.3.3}$$

to be the domain of $\{K_j\}_{j \geq 0}$ mentioned in Lemma 2.3.4. The sequence of the functions $\{K_j\}_{j \geq 0}$ will be utilized to find the explicit formula for φ in (2.3.26), where K^* is defined in (2.3.24). K^* is evidently bounded by $\widetilde{K}$ in Lemma 2.3.4.

Lemma 2.3.4. *Given $K_0 : D_{T,\Omega} \to \mathbb{R}$. Let C be a positive constant such that for any $(x,t;y,\tau) \in D_{T,\Omega}$,*

$$|K_0(x,t;y,\tau)| \leq C\,(t-\tau)^{-3/4}\,|x-y|^{-(n-3/2)}. \tag{2.3.4}$$

For any $j \geq 1$, define $K_j : D_{T,\Omega} \to \mathbb{R}$ by

$$K_j(x,t;y,\tau) \triangleq \int_\tau^t \int_{\partial\Omega} K_0(x,t;z,\sigma)\, K_{j-1}(z,\sigma;y,\tau)\, dS(z)\, d\sigma. \tag{2.3.5}$$

Then all the $K_j\,(j \geq 1)$ are well-defined and the series $\sum_{j=0}^{\infty} |K_j|$ converges uniformly to some function $\widetilde{K}$ on $D_{T,\Omega}$. Moreover, there exists some constant $C^* = C^*(n,\Omega,T)$ such that for any $(x,t;y,\tau) \in D_{T,\Omega}$,

$$\widetilde{K}(x,t;y,\tau) \leq C^*(t-\tau)^{-3/4}\, |x-y|^{-(n-3/2)}. \tag{2.3.6}$$

Proof. We first justify that all the $K_j (j \geq 1)$ are well-defined. In fact, by (2.3.4) and (2.3.5), one has

$$\begin{aligned} |K_1(x,t;y,\tau)| &\leq C \int_\tau^t \int_{\partial\Omega} (t-\sigma)^{-3/4} |x-z|^{-(n-3/2)}\, (\sigma-\tau)^{-3/4} |z-y|^{-(n-3/2)}\, dS(z)\, d\sigma \\ &= C \int_\tau^t (t-\sigma)^{-3/4} (\sigma-\tau)^{-3/4}\, d\sigma \int_{\partial\Omega} |x-z|^{-(n-3/2)} |z-y|^{-(n-3/2)}\, dS(z). \end{aligned}$$

Making the change of variable $\rho \triangleq \frac{\sigma-\tau}{t-\tau}$ for σ, then we obtain

$$|K_1(x,t;y,\tau)| \leq C\, (t-\tau)^{-1/2} \int_{\partial\Omega} |x-z|^{-(n-3/2)}\, |z-y|^{-(n-3/2)}\, dS(z).$$

Now there are two cases:

- If $n > 2$, then by Lemma 2.3.3,

$$\int_{\partial\Omega} |x-z|^{-(n-3/2)}\, |z-y|^{-(n-3/2)}\, dS(z) \leq C\, |x-y|^{-(n-2)}.$$

Therefore

$$|K_1(x,t;y,\tau)| \leq C\,(t-\tau)^{-1/2}\,|x-y|^{-(n-2)}. \tag{2.3.7}$$

- If $n = 2$, then we can not use Lemma 2.3.3 directly since $(n-3/2)+(n-3/2) = n-1$ is on the borderline. However, since Ω is bounded, we have $|z-y|^{-(n-3/2)} \leq C\,|z-y|^{-(n-3/2)}\,|z-y|^{-1/4}$. Then Lemma 2.3.3 can be applied to get

$$\begin{aligned}
&\int_{\partial\Omega} |x-z|^{-(n-3/2)}\,|z-y|^{-(n-3/2)}\,dS(z)\\
&\leq C \int_{\partial\Omega} |x-z|^{-(n-3/2)}\,|z-y|^{-(n-5/4)}\,dS(z)\\
&\leq C\,|x-y|^{-1/4}.
\end{aligned}$$

Consequently,

$$|K_1(x,t;y,\tau)| \leq C\,(t-\tau)^{-1/2}\,|x-y|^{-1/4} = C\,(t-\tau)^{-1/2}\,|x-y|^{-(n-7/4)}. \tag{2.3.8}$$

Comparing (2.3.4) with (2.3.7) and (2.3.8), the exponent of $(t-\tau)$ term is added by 1/4 and the exponent of $(x-y)$ term increases by at least 1/4. In other words, the singularity of K_1 in both time and space variables is weaker than K_0 by a certain number 1/4. Thus, after finite steps, we can find j_0 (only depending on n) such that

$$K_{j_0}(x,t;y,\tau) \leq \widetilde{C}, \quad \forall\,(x,t;y,\tau) \in D_{\Omega,T}, \tag{2.3.9}$$

for some constant $\widetilde{C}$. Moreover, for any $j > j_0$, K_j is also well-defined and bounded.

From (2.3.4) and Lemma 2.3.3, there exists a constant C_1 such that

$$|K_0(x,t;y,\tau)| \le C_1\,(t-\tau)^{-3/4}\,|x-y|^{-(n-3/2)}, \quad \forall\,(x,t;y,\tau) \in D_{\Omega,T} \tag{2.3.10}$$

and

$$\int_{\partial\Omega} \frac{dS(y)}{|x-y|^{n-3/2}} \le C_1, \quad \forall\, x \in \partial\Omega.$$

For the rest of the proof, C_1 and $\widetilde{C}$ will be fixed. We start from (2.3.9) to prove by induction that for any $m \ge 0$ and for any $(x,t;y,\tau) \in D_{\Omega,T}$,

$$|K_{j_0+m}(x,t;y,\tau)| \le \frac{\widetilde{C}\,Q^m\,(t-\tau)^{m/4}}{\Gamma\big(1+m/4\big)}, \tag{2.3.11}$$

where $Q \triangleq C_1^2\,\Gamma(1/4)$.

When $m=0$, (2.3.11) is just (2.3.9). Now we suppose (2.3.11) is true for $m=i$ and try to verify it for $m=i+1$. Applying (2.3.11) with $m=i$ and (2.3.4), we obtain

$$\begin{aligned}
|K_{j_0+i+1}(x,t;y,\tau)| &\le \int_\tau^t \int_{\partial\Omega} |K_0(x,t;z,\sigma)\,K_{j_0+i}(z,\sigma;y,\tau)|\,dS(z)\,d\sigma \\
&\le \frac{C_1\,\widetilde{C}\,Q^i}{\Gamma\big(1+i/4\big)} \int_\tau^t (t-\sigma)^{-3/4}\,(\sigma-\tau)^{i/4}\,d\sigma \int_{\partial\Omega} |x-z|^{-(n-3/2)}\,dS(z) \\
&\le \frac{C_1^2\,\widetilde{C}\,Q^i}{\Gamma\big(1+i/4\big)} \int_\tau^t (t-\sigma)^{-3/4}\,(\sigma-\tau)^{i/4}\,d\sigma \\
&= \frac{\widetilde{C}\,Q^{i+1}}{\Gamma(1/4)\,\Gamma\big(1+i/4\big)} \int_\tau^t (t-\sigma)^{-3/4}\,(\sigma-\tau)^{i/4}\,d\sigma,
\end{aligned}$$

where the third inequality and the fourth equality are due to the definitions of C_1 and Q

respectively. By the change of variable $\rho \triangleq \frac{\sigma-\tau}{t-\tau}$ for σ, we attain

$$\begin{aligned}|K_{j_0+i+1}(x,t;y,\tau)| &\le \frac{\widetilde{C}\,Q^{i+1}\,(t-\tau)^{(i+1)/4}}{\Gamma(1/4)\,\Gamma\big(1+i/4\big)}\int_0^1 (1-\rho)^{-3/4}\,\rho^{i/4}\,d\rho \\ &= \frac{\widetilde{C}\,Q^{i+1}\,(t-\tau)^{(i+1)/4}}{\Gamma\big(1+(i+1)/4\big)}.\end{aligned}$$

Consequently, (2.3.11) is true for $m = i+1$ and therefore it is true for any $m \ge 0$.

By (2.3.11), we get that for any $m \ge 0$ and for any $(x,t;y,\tau) \in D_{\Omega,T}$,

$$|K_{j_0+m}(x,t;y,\tau)| \le \frac{\widetilde{C}\,Q^m\,T^{m/4}}{\Gamma(1+m/4)}. \tag{2.3.12}$$

This estimate is significant because the right hand side of (2.3.12) is independent of x, y, t, τ. By applying the ratio test, we can prove

$$\sum_{m=0}^{\infty} \frac{Q^m\,T^{m/4}}{\Gamma(1+m/4)} < \infty. \tag{2.3.13}$$

Next, due to (2.3.4) and the fact that the singularity of K_0 is stronger than any other $K_j (j \ge 1)$, we can find a constant $C_2 = C_2(n,\Omega,T)$ such that for any $(x,t;y,\tau) \in D_{\Omega,T}$,

$$\sum_{j=0}^{j_0} |K_j(x,t;y,\tau)| \le C_2\,(t-\tau)^{-3/4}\,|x-y|^{-(n-3/2)}. \tag{2.3.14}$$

Combining (2.3.14), (2.3.12) and (2.3.13) together, it is readily to see that $\sum\limits_{j=0}^{\infty} |K_j(x,t;y,\tau)|$ converges to $\widetilde{K}(x,t;y,\tau)$ uniformly on $D_{\Omega,T}$ and moreover, there exists some constant C^*, only depending on n, Ω and T such that for any $(x, t; y, \tau) \in D_{T,\Omega}$,

$$\widetilde{K}(x, t; y, \tau) \leq C^*(t-\tau)^{-3/4} \, |x-y|^{-(n-3/2)}.$$

. $\square$

Lemma 2.3.5. *If $f \in C^{\alpha,\alpha/2}\big(\overline{\Omega} \times [0,T]\big)$ and*

$$W(x,t) \triangleq \int_0^t \int_{\Omega} \Phi(x-y, t-\tau) \, f(y,\tau) \, dS(y) \, d\tau, \quad \forall \, (x,t) \in \overline{\Omega} \times [0,T],$$

then $W \in C^{2,1}\big(\Omega \times (0,T]\big)$ and

$$(W_t - \Delta W)(x,t) = f(x,t), \quad \forall \, (x,t) \in \Omega \times (0,T].$$

Proof. This is a standard argument, we refer the readers to ([7], Theorem 9, Sec. 5, Chap. 1). $\square$

2.3.3 Existence

The idea of the proof for the following Theorem 2.3.6 is analogous to ([7], Theorem 2, Sec. 3, Chap. 5).

Theorem 2.3.6. *For any $T > 0$, there exists a solution $u \in \mathcal{A}_T$ to (2.3.1) on $\overline{\Omega} \times [0,T]$.*

Proof. We will construct a solution u to (2.3.1). Firstly, since $\psi \in C^1(\overline{\Omega})$ and $\partial\Omega \in C^2$, one can extend ψ to a larger domain. More precisely, there exists an open set $\Omega_1 \supset \overline{\Omega}$ and $\psi_1 \in C^1(\overline{\Omega}_1)$ such that ψ_1 agrees with ψ on $\overline{\Omega}$. In the rest of the proof, for convenience, we just write ψ for ψ_1. We are looking for a solution u in the following form:

$$
\begin{aligned}
u(x,t) = &\int_{\Omega_1} \Phi(x-y,t)\,\psi(y)\,dy + \int_0^t \int_\Omega \Phi(x-y,t-\tau)\,f(y,\tau)\,dy\,d\tau \\
&+ \int_0^t \int_{\partial\Omega} \Phi(x-y,t-\tau)\,\varphi(y,\tau)\,dS(y)\,d\tau, \quad \forall\,(x,t) \in \overline{\Omega} \times [0,T],
\end{aligned}
\tag{2.3.15}
$$

where $\varphi \in \mathcal{B}_T$ will be determined later.

Due to Lemma 2.3.5, it is easy to see that the function u defined in (2.3.15) belongs to $\mathcal{A}_T$ and satisfies the first and the third equations in (2.3.1), so in order to verify u to be the solution, the only things left to check are

$$
\frac{\partial u(x,t)}{\partial n(x)} + \beta(x,t)u(x,t) = g(x,t), \quad \forall\,(x,t) \in (\Gamma_1 \cup \Gamma_2) \times (0,T] \tag{2.3.16}
$$

and

$$
\frac{\partial u(x,t)}{\partial n(x)} + \frac{1}{2}\big[\beta_1(x,t)+\beta_2(x,t)\big]u(x,t) = \frac{1}{2}\big[g_1(x,t)+g_2(x,t)\big], \quad \forall\,(x,t) \in \widetilde{\Gamma}\times(0,T]. \tag{2.3.17}
$$

The plan is to firstly find a function $\varphi \in \mathcal{B}_T$ such that u defined in (2.3.15) satisfies (2.3.16), then we will prove this u satisfies (2.3.17) as well.

By (1.4.1) and (2.3.15), for any $(x,t) \in (\Gamma_1 \cup \Gamma_2) \times (0,T]$, it follows from the Lebesgue's dominated convergence theorem and Theorem 2.2.6 that

$$
\begin{aligned}
\frac{\partial u(x,t)}{\partial n(x)} = &\int_{\Omega_1} \frac{\partial \Phi(x-y,t)}{\partial n(x)}\,\psi(y)\,dy + \int_0^t \int_\Omega \frac{\partial \Phi(x-y,t-\tau)}{\partial n(x)}\,f(y,\tau)\,dy\,d\tau \\
&+ \int_0^t \int_{\partial\Omega} \frac{\partial \Phi(x-y,t-\tau)}{\partial n(x)}\,\varphi(y,\tau)\,dS(y)\,d\tau + \frac{1}{2}\,\varphi(x,t).
\end{aligned}
\tag{2.3.18}
$$

Therefore, (2.3.16) is reduced to for any $(x,t) \in (\Gamma_1 \cup \Gamma_2) \times (0,T]$,

$$\varphi(x,t) = \int_0^t \int_{\partial\Omega} K(x,t;y,\tau)\,\varphi(y,\tau)\,dS(y)\,d\tau + H(x,t), \tag{2.3.19}$$

where

$$K(x,t;y,\tau) = -2\Big[\frac{\partial\Phi(x-y,t-\tau)}{\partial n(x)} + \beta(x,t)\,\Phi(x-y,t-\tau)\Big]$$

and

$$\begin{aligned} H(x,t) = &- 2\int_{\Omega_1}\Big[\frac{\partial\Phi(x-y,t)}{\partial n(x)} + \beta(x,t)\,\Phi(x-y,t)\Big]\psi(y)\,dy \\ &- 2\int_0^t\int_{\Omega}\Big[\frac{\partial\Phi(x-y,t-\tau)}{\partial n(x)} + \beta(x,t)\,\Phi(x-y,t-\tau)\Big]f(y,\tau)\,dy\,d\tau \\ &+ 2\,g(x,t). \end{aligned}$$

In other words, the proof of (2.3.16) becomes the search for a fixed point $\varphi \in \mathcal{B}_T$ of (2.3.19).

In the following, we will construct a fixed point of (2.3.19) in $\mathcal{B}_T$. Noticing

$$\begin{aligned} \frac{1}{(t-\tau)^{n/2}}\,e^{-\frac{|x-y|^2}{4(t-\tau)}} &= \Big(\frac{|x-y|^2}{t-\tau}\Big)^{\frac{2n-3}{4}}\,e^{-\frac{|x-y|^2}{4(t-\tau)}}\,(t-\tau)^{-3/4}\,|x-y|^{-(n-3/2)} \\ &\le C(t-\tau)^{-3/4}\,|x-y|^{-(n-3/2)}, \end{aligned}$$

so by the similar argument, we have

$$\frac{|x-y|^2}{(t-\tau)^{n/2+1}}\,e^{-\frac{|x-y|^2}{4(t-\tau)}} \le C(t-\tau)^{-3/4}\,|x-y|^{-(n-3/2)}.$$

Then it follows from Lemma 2.3.2 that for any $x, y \in \partial\Omega$, $0 \leq \tau < t \leq T$,

$$
\begin{aligned}
|K(x,t;y,\tau)| &\leq C\left[\frac{1}{(t-\tau)^{n/2}} + \frac{|x-y|^2}{(t-\tau)^{n/2+1}}\right] e^{-\frac{|x-y|^2}{4(t-\tau)}} \\
&\leq C\,(t-\tau)^{-3/4}\,|x-y|^{-(n-3/2)}. \qquad (2.3.20)
\end{aligned}
$$

Then using the fact $\psi \in C^1(\overline{\Omega}_1)$ and the integration by parts, we obtain

$$
\begin{aligned}
&\int_{\Omega_1} \frac{\partial\Phi(x-y,t)}{\partial n(x)}\,\psi(y)\,dy \\
&= -\int_{\Omega_1} D_y\big[\Phi(x-y,t)\big]\cdot\overrightarrow{n}(x)\,\psi(y)\,dy \\
&= -\int_{\partial\Omega_1} \Phi(x-y,t)\,\psi(y)\,\overrightarrow{n}(y)\cdot\overrightarrow{n}(x)\,dy + \int_{\Omega_1} \Phi(x-y,t)\,D\psi(y)\cdot\overrightarrow{n}(x)\,dy. \qquad (2.3.21)
\end{aligned}
$$

Consequently, as a function in (x,t),

$$
\int_{\Omega_1} \frac{\partial\Phi(x-y,t)}{\partial n(x)}\,\psi(y)\,dy \in C\big(\partial\Omega\times[0,T]\big) \subset \mathcal{B}_T.
$$

Then it is readily to check that $H \in \mathcal{B}_T$.

Next, two sequences of functions $\{K_j\}_{j\geq 0}$ and $\{\varphi_j\}_{j\geq 0}$ will be constructed as following. First, define $K_0 = K$ on $D_{\Omega,T}$ and $\varphi_0 = H$ on $(\Gamma_1\cup\Gamma_2)\times(0,T]$. Then for any $j \geq 1$, define

$$
K_j(x,t;y,\tau) = \int_\tau^t \int_{\partial\Omega} K_0(x,t;z,\sigma)\,K_{j-1}(z,\sigma;y,\tau)\,dS(z)\,d\sigma \qquad (2.3.22)
$$

on $D_{\Omega,T}$ and

$$
\varphi_j(x,t) = \sum_{i=0}^{j-1} \int_0^t \int_{\partial\Omega} K_i(x,t;y,\tau)\,H(y,\tau)\,dS(y)\,d\tau + H(x,t) \qquad (2.3.23)
$$

on $(\Gamma_1 \cup \Gamma_2) \times (0, T]$. Because of (2.3.4), we can prove by induction that for any $j \geq 0$, φ_j is well defined and belongs to $\mathcal{B}_T$. Our goal is to show that $\varphi_j(x,t)$ uniformly converges to some function $\varphi(x,t)$ on $(\Gamma_1 \cup \Gamma_2) \times (0, T]$ as $j \to \infty$, which makes φ to be the fixed point of (2.3.19) in $\mathcal{B}_T$. Writing

$$K^*(x,t;y,\tau) \triangleq \sum_{j=0}^{\infty} K_j(x,t;y,\tau), \tag{2.3.24}$$

by Lemma 2.3.4, we know K^* is well-defined and $\sum_{j=0}^{\infty} K_j$ converges uniformly to K^* on $D_{T,\Omega}$. Moreover, there exists a constant $C^* = C^*(n, \Omega, T)$ such that for any $(x,t;y,\tau) \in D_{\Omega,T}$,

$$|K^*(x,t;y,\tau)| \leq C^*(t-\tau)^{-3/4}\,|x-y|^{-(n-3/2)}. \tag{2.3.25}$$

Consequently, it follows from (2.3.23) and (2.3.24) that φ_j converges uniformly to the function φ on $(\Gamma_1 \cup \Gamma_2) \times (0, T]$, where

$$\varphi(x,t) \triangleq \int_0^t \int_{\partial\Omega} K^*(x,t;y,\tau)\, H(y,\tau)\, dS(y)\, d\tau + H(x,t). \tag{2.3.26}$$

Thus, φ is a fixed point of (2.3.19) in $\mathcal{B}_T$ and therefore the function u defined in (2.3.15) satisfies (2.3.16).

Now as our plan, it only left to show this function u satisfies (2.3.17) as well. Making use of (2.3.15), (1.4.1) and Theorem 2.2.6, we get for any $x \in \widetilde{\Gamma}$, $0 < t \leq T$, $\frac{\partial u(x,t)}{\partial n(x)}$ exists and

$$\begin{aligned}\frac{\partial u(x,t)}{\partial n(x)} &= \int_{\Omega_1} \frac{\partial \Phi(x-y,t)}{\partial n(x)}\, \psi(y)\, dy + \int_0^t \int_{\Omega} \frac{\partial \Phi(x-y,t-\tau)}{\partial n(x)}\, f(y,\tau)\, dy\, d\tau \\ &\quad + \int_0^t \int_{\partial\Omega} \frac{\partial \Phi(x-y,t-\tau)}{\partial n(x)}\, \varphi(y,\tau)\, dS(y)\, d\tau + \frac{1}{4}\,\varphi_1(x,t) + \frac{1}{4}\,\varphi_2(x,t). \end{aligned} \tag{2.3.27}$$

Then we choose two sequences of points $\{\xi_k\}_{k\geq 1} \subset \Gamma_1$ and $\{z_j\}_{j\geq 1} \subset \Gamma_2$ which converge to x, it follows from (2.3.18) that

$$\begin{aligned}\frac{\partial u(\xi_k,t)}{\partial n(\xi_k)} =& \int_{\Omega_1} \frac{\partial \Phi(\xi_k - y, t)}{\partial n(\xi_k)}\, \psi(y)\, dy + \int_0^t \int_\Omega \frac{\partial \Phi(\xi_k - y, t-\tau)}{\partial n(\xi_k)}\, f(y,\tau)\, dy\, d\tau \\ &+ \int_0^t \int_{\partial\Omega} \frac{\partial \Phi(\xi_k - y, t-\tau)}{\partial n(\xi_k)}\, \varphi(y,\tau)\, dS(y)\, d\tau + \frac{1}{2}\, \varphi(\xi_k, t)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial u(z_j,t)}{\partial n(z_j)} =& \int_{\Omega_1} \frac{\partial \Phi(z_j - y, t)}{\partial n(z_j)}\, \psi(y)\, dy + \int_0^t \int_\Omega \frac{\partial \Phi(z_j - y, t-\tau)}{\partial n(z_j)}\, f(y,\tau)\, dy\, d\tau \\ &+ \int_0^t \int_{\partial\Omega} \frac{\partial \Phi(z_j - y, t-\tau)}{\partial n(z_j)}\, \varphi(y,\tau)\, dS(y)\, d\tau + \frac{1}{2}\, \varphi(z_j, t).\end{aligned}$$

Taking $k \to \infty$ and $j \to \infty$, we obtain

$$\begin{aligned}\lim_{k\to\infty} \frac{\partial u(\xi_k,t)}{\partial n(\xi_k)} =& \int_{\Omega_1} \frac{\partial \Phi(x - y, t)}{\partial n(x)}\, \psi(y)\, dy + \int_0^t \int_\Omega \frac{\partial \Phi(x - y, t-\tau)}{\partial n(x)}\, f(y,\tau)\, dy\, d\tau \\ &+ \int_0^t \int_{\partial\Omega} \frac{\partial \Phi(x - y, t-\tau)}{\partial n(x)}\, \varphi(y,\tau)\, dS(y)\, d\tau + \frac{1}{2}\, \varphi_1(x, t)\end{aligned} \tag{2.3.28}$$

and

$$\begin{aligned}\lim_{j\to\infty} \frac{\partial u(z_j,t)}{\partial n(z_j)} =& \int_{\Omega_1} \frac{\partial \Phi(x - y, t)}{\partial n(x)}\, \psi(y)\, dy + \int_0^t \int_\Omega \frac{\partial \Phi(x - y, t-\tau)}{\partial n(x)}\, f(y,\tau)\, dy\, d\tau \\ &+ \int_0^t \int_{\partial\Omega} \frac{\partial \Phi(x - y, t-\tau)}{\partial n(x)}\, \varphi(y,\tau)\, dS(y)\, d\tau + \frac{1}{2}\, \varphi_2(x, t).\end{aligned} \tag{2.3.29}$$

Adding (2.3.28) and (2.3.29) together and noticing (2.3.27), we attain

$$\frac{\partial u(x,t)}{\partial n(x)} = \frac{1}{2}\Big[\lim_{k\to\infty} \frac{\partial u(\xi_k,t)}{\partial n(\xi_k)} + \lim_{j\to\infty} \frac{\partial u(z_j,t)}{\partial n(z_j)}\Big]. \tag{2.3.30}$$

Moreover, since u satisfies (2.3.16), we have

$$\frac{\partial u(\xi_k,t)}{\partial n(\xi_k)} + \beta(\xi_k,t)\,u(\xi_k,t) = g(\xi_k,t),$$
$$\frac{\partial u(z_j,t)}{\partial n(z_j)} + \beta(z_j,t)\,u(z_j,t) = g(z_j,t).$$

Sending $k \to \infty$ and $j \to \infty$, we obtain

$$\lim_{k\to\infty} \frac{\partial u(\xi_k,t)}{\partial n(\xi_k)} = g_1(x,t) - \beta_1(x,t)\,u(x,t) \tag{2.3.31}$$
$$\lim_{j\to\infty} \frac{\partial u(z_j,t)}{\partial n(z_j)} = g_2(x,t) - \beta_2(x,t)\,u(x,t) \tag{2.3.32}$$

Combining (2.3.30), (2.3.31) and (2.3.32) together, (2.3.17) follows. □

2.3.4 Comparison principles, uniqueness and global solution

Next, we will prove the comparison principle and the uniqueness of the solution by applying Theorem 2.3.6. But before that, let's prove the following easier comparison result.

Lemma 2.3.7. *Suppose in (2.3.1), $f \geq 0$ on $\overline{\Omega} \times [0,T]$, $\psi > 0$ on $\overline{\Omega}$ and*

$$\inf_{(\Gamma_1 \cup \Gamma_2)\times[0,T]} g > 0,$$

then the solution $u > 0$ on $\overline{\Omega} \times [0,T]$.

Proof. Since $\psi > 0$ on $\overline{\Omega}$, we have $m \triangleq \min_{\overline{\Omega}} \psi > 0$. Now we claim $u > 0$ on $\overline{\Omega} \times [0,T]$. If not, then there will exist $x_0 \in \overline{\Omega}$ and $t_0 \in (0,T]$ such that

$$u(x_0,t_0) = 0 = \min_{\overline{\Omega}\times[0,t_0]} u.$$

By the strong maximum principle, $x_0 \in \partial\Omega$. If $x_0 \in \Gamma_1 \cup \Gamma_2$, then

$$0 < g(x_0, t_0) = \frac{\partial u(x_0, t_0)}{\partial n(x_0)} + \beta(x_0, t_0)\, u(x_0, t_0) = \frac{\partial u(x_0, t_0)}{\partial n(x_0)} \leq 0,$$

which is impossible. If $x_0 \in \widetilde{\Gamma}$, then

$$\begin{aligned} 0 &< \frac{1}{2}\left[g_1(x_0, t_0) + g_2(x_0, t_0)\right] \\ &= \frac{\partial u(x_0, t_0)}{\partial n(x_0)} + \frac{1}{2}\left[\beta_1(x_0, t_0) + \beta_2(x_0, t_0)\right] u(x_0, t_0) \\ &= \frac{\partial u(x_0, t_0)}{\partial n(x_0)} \leq 0, \end{aligned}$$

which is also a contradiction. Thus, the Lemma follows. $\square$

Corollary 2.3.8. *Suppose in (2.3.1), $f \geq 0$ on $\overline{\Omega} \times [0, T]$, $\psi \geq 0$ on $\overline{\Omega}$ and $g \geq 0$ on $(\Gamma_1 \cup \Gamma_2) \times (0, T]$, then the solution $u \geq 0$ on $\overline{\Omega} \times [0, T]$. In particular, the solution to (2.3.1) on $\overline{\Omega} \times [0, T]$ is unique. If further assuming that $\psi \not\equiv 0$ on $\overline{\Omega}$, then $u(x, t) > 0$ for any $(x, t) \in \overline{\Omega} \times (0, T]$.*

Proof. Due to Theorem 2.3.6, there exists a solution $v \in \mathcal{A}_T$ to the following problem:

$$\begin{cases} v_t(x, t) - \Delta v(x, t) = 1 & \text{in} \quad \Omega \times (0, T], \\ \frac{\partial v(x,t)}{\partial n(x)} + \beta(x, t) v(x, t) = 1 & \text{on} \quad (\Gamma_1 \cup \Gamma_2) \times (0, T], \\ v(x, 0) = 1 & \text{on} \quad \Omega. \end{cases}$$

Now for any $\epsilon > 0$, define $w_\epsilon = u + \epsilon\, v$, then w_ϵ satisfies

$$\begin{cases} (w_\epsilon)_t(x,t) - \Delta w_\epsilon(x,t) = f + \epsilon \geq \varepsilon & \text{in} \quad \Omega \times (0,T], \\ \frac{\partial w_\epsilon(x,t)}{\partial n(x)} + \beta(x,t) w_\epsilon(x,t) = g + \epsilon \geq \varepsilon & \text{on} \quad (\Gamma_1 \cup \Gamma_2) \times (0,T], \\ w_\epsilon(x,0) = \psi + \epsilon \geq \varepsilon & \text{on} \quad \Omega. \end{cases}$$

By invoking Lemma 2.3.7, $w_\epsilon \geq 0$ on $\overline{\Omega} \times [0,T]$. Taking $\epsilon \to 0$, we get $u \geq 0$ on $\overline{\Omega} \times [0,T]$.

Now we further assume that ψ is not identical 0 on $\overline{\Omega}$. If there exists some $(x_0, t_0) \in \overline{\Omega} \times (0,T]$ such that $u(x_0,t_0) = 0$, then

$$u(x_0,t_0) = \min_{\overline{\Omega} \times [0,t_0]} u.$$

By strong maximum principle and $\psi \not\equiv 0$, x_0 has to be on the boundary $\partial\Omega$. In addition, it follows from the boundary condition

$$\frac{\partial u(x_0,t_0)}{\partial n(x_0)} + \beta(x_0,t_0) u(x_0,t_0) = g(x_0,t_0)$$

that $\frac{\partial u(x_0,t_0)}{\partial n(x_0)} \geq 0$. But on the other hand, the Hopf lemma implies that $\frac{\partial u(x_0,t_0)}{\partial n(x_0)} < 0$, which is a contradiction. So $u(x,t) > 0$ for any $(x,t) \in \overline{\Omega} \times (0,T]$. □

Corollary 2.3.9. *Let u be a solution to (2.3.1). Then for any $T > 0$, there exist constants $C = C(\Omega, \beta, T)$ such that for any $x \in \overline{\Omega}$ and $0 \leq t \leq T$,*

$$|u(x,t)| \leq C\big(\sup_{\Omega \times (0,T]} |f| + \sup_{(\Gamma_1 \cup \Gamma_2) \times (0,T]} |g| + \sup_{\overline{\Omega}} |\psi| \big).$$

Proof. Similar to the proof of Corollary 2.3.8, there exists a solution $v \in \mathcal{A}_T$ to the following problem:

$$\begin{cases} v_t(x,t) - \Delta v(x,t) = 1 & \text{in} \quad \Omega \times (0,T], \\ \frac{\partial v(x,t)}{\partial n(x)} + \beta(x,t)v(x,t) = 1 & \text{on} \quad (\Gamma_1 \cup \Gamma_2) \times (0,T], \\ v(x,0) = 1 & \text{on} \quad \Omega. \end{cases}$$

Let $K = \sup_{\Omega\times(0,T]} |f| + \sup_{(\Gamma_1\cup\Gamma_2)\times(0,T]} |g| + \sup_{\overline{\Omega}} |\psi|$. Then it follows from Corollary 2.3.8 that $|u| \le Kv$. Since v is bounded on $\overline{\Omega} \times [0,T]$, the conclusion follows. □

By combining Theorem 2.3.6 and Corollary 2.3.8, the existence and uniqueness of the global solutions is established.

Theorem 2.3.10. *The linear problem (2.3.1) has a unique global solution* $u \in C^{2,1}\big(\Omega \times (0,\infty)\big) \cap C\big(\overline{\Omega} \times [0,\infty)\big)$, *where the global solution means a function that solves (2.3.1) for any* $T > 0$.

2.4 Nonlinear case

2.4.1 Definition of the local solution

This section is devoted to the existence and uniqueness of the solution to the following nonlinear initial-boundary value problem:

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = f(x,t) & \text{in} \quad \Omega \times (0,T], \\ \frac{\partial u(x,t)}{\partial n(x)} = F\big(x, u(x,t)\big) & \text{on} \quad \Gamma_1 \times (0,T], \\ \frac{\partial u(x,t)}{\partial n(x)} = 0 & \text{on} \quad \Gamma_2 \times (0,T], \\ u(x,0) = \psi(x) & \text{on} \quad \Omega, \end{cases} \tag{2.4.1}$$

where $f \in C^{\alpha,\alpha/2}\big(\overline{\Omega} \times [0,T]\big)$, $F \in C^1(\overline{\Gamma}_1 \times \mathbb{R})$ and $\psi \in C^1(\overline{\Omega})$. For any $(x,\sigma) \in \overline{\Gamma}_1 \times \mathbb{R}$, we will use $F_{x_i}(x,\sigma)\,(1 \le i \le n)$ to denote the ith spatial derivative of F and use $F_\sigma(x,\sigma)$ to denote the derivative of its $(n+1)$th variable.

Definition 2.4.1. *For any $T > 0$, a solution to (2.4.1) on $\overline{\Omega} \times [0,T]$ means a function u in $\mathcal{A}_T$ that satisfies (2.4.1) pointwise and for any $(x,t) \in \widetilde{\Gamma} \times (0,T]$, $\frac{\partial u(x,t)}{\partial n(x)}$ exists and*

$$\frac{\partial u(x,t)}{\partial n(x)} = \frac{1}{2}\,F\big(x, u(x,t)\big). \tag{2.4.2}$$

2.4.2 Comparison principles and uniqueness

This time, we will first show some comparison principles and then discuss the existence of the solution.

Theorem 2.4.2. *Suppose $u_i \in \mathcal{A}_T (i = 1, 2)$ is the solution to (2.4.1) on $\overline{\Omega} \times [0,T]$ with right hand side f_i, F_i and ψ_i. If $f_1 \ge f_2$ on $\overline{\Omega} \times [0,T]$, $F_1 \ge F_2$ on $\overline{\Gamma}_1 \times \mathbb{R}$ and $\psi_1 \ge \psi_2$ on $\overline{\Omega}$, then $u_1 \ge u_2$ on $\overline{\Omega} \times [0,T]$. As a consequence, the solution to (2.4.1) is unique.*

Proof. Let $w = u_1 - u_2$, $f = f_1 - f_2$ and $\psi = \psi_1 - \psi_2$. Then $f \ge 0$ on $\overline{\Omega} \times [0,T]$ and $\psi \ge 0$ on $\overline{\Omega}$. In addition, since $F_1 \ge F_2$ on $\overline{\Gamma}_1 \times \mathbb{R}$, we have

$$\begin{aligned} & F_1\big(x, u_1(x,t)\big) - F_2\big(x, u_2(x,t)\big) \\ \ge\ & F_1\big(x, u_1(x,t)\big) - F_1\big(x, u_2(x,t)\big) \\ =\ & \beta(x,t) w(x,t), \end{aligned}$$

where

$$\beta(x,t) = \int_0^1 (F_1)_\sigma\big(x, su_1(x,t) + (1-s)u_2(x,t)\big)\,ds.$$

Thus, w satisfies the equations below

$$
\begin{cases}
w_t(x,t) - \Delta w(x,t) = f(x,t) \geq 0 & \text{in} \quad \Omega \times (0,T], \\
\frac{\partial w(x,t)}{\partial n(x)} - \beta(x,t) w(x,t) \geq 0 & \text{on} \quad \Gamma_1 \times (0,T], \\
\frac{\partial w(x,t)}{\partial n(x)} = 0 & \text{on} \quad \Gamma_2 \times (0,T], \\
w(x,0) = \psi(x) \geq 0 & \text{on} \quad \Omega.
\end{cases}
$$

Now it follows from Corollary 2.3.8 that $w \geq 0$. $\square$

Theorem 2.4.3. *Suppose $u \in \mathcal{A}_T$ is the solution to (2.4.1) with $f \geq 0$ on $\overline{\Omega} \times [0,T]$, $F(\cdot,0) \geq 0$ on $\overline{\Gamma}_1$ and $\psi \geq 0$ on $\overline{\Omega}$, then $u \geq 0$ on $\overline{\Omega} \times [0,T]$. If further assuming $\psi \not\equiv 0$, then $u(x,t) > 0$, $\forall\, x \in \overline{\Omega}$, $0 < t \leq T$.*

Proof. To prove the first statement, we write

$$
F\big(x, u(x,t)\big) = F\big(x, u(x,t)\big) - F(x,0) + F(x,0) = \beta(x,t) u(x,t) + F(x,0),
$$

where

$$
\beta(x,t) = \int_0^1 F_\sigma\big(x, su(x,t)\big)\, ds.
$$

Hence, u satisfies

$$
\begin{cases}
u_t(x,t) - \Delta u(x,t) = f(x,t) \geq 0 & \text{in} \quad \Omega \times (0,T], \\
\frac{\partial u(x,t)}{\partial n(x)} - \beta(x,t)\, u(x,t) = F(x,0) \geq 0 & \text{on} \quad \Gamma_1 \times (0,T], \\
\frac{\partial u(x,t)}{\partial n(x)} = 0 & \text{on} \quad \Gamma_2 \times (0,T], \\
u(x,0) = \psi(x) \geq 0 & \text{on} \quad \Omega.
\end{cases}
$$

It then follows from Corollary 2.3.8 that $u \geq 0$.

Now suppose additionally that $\psi \not\equiv 0$, then similar to the proof of Corollary 2.3.8, by applying the strong maximum principle and the Hopf lemma, we get $u(x,t) > 0, \forall\, x \in \overline{\Omega}$, $0 < t \leq T$. $\square$

2.4.3 Existence

Now we turn to the existence of the solution. As a common process to deal with the nonlinear problem, we will take advantage of the theories for the linear problems and some fixed point theorems. For any $T > 0$ and $R > 0$, define $X_T = C\big(\overline{\Omega} \times [0,T]\big)$ and $X_{T,R} = \{v \in X_T : ||v|| \leq R\}$, where the norm in X_T is the supremum norm as following

$$||u|| = \max_{(x,t)\in\overline{\Omega}\times[0,T]} |u(x,t)|, \quad \forall\, u \in X_T.$$

Then X_T is a Banach space and $X_{T,R}$ is a closed and thus complete subset of X_T. For any $v \in X_{T,R}$, it follows from Theorem 2.3.6 and Corollary 2.3.8 that there exists a unique solution $u \in \mathcal{A}_T$ to the following problem

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = f(x,t) & \text{in} \quad \Omega \times (0,T], \\ \frac{\partial u(x,t)}{\partial n(x)} = F\big(x, v(x,t)\big) & \text{on} \quad \Gamma_1 \times (0,T], \\ \frac{\partial u(x,t)}{\partial n(x)} = 0 & \text{on} \quad \Gamma_2 \times (0,T], \\ u(x,0) = \psi(x) & \text{on} \quad \Omega. \end{cases} \tag{2.4.3}$$

Thus, it determines a mapping $\Psi_T : X_{T,R} \to \mathcal{A}_T$. Our strategy is to pick up suitable T and R such that Ψ_T has a fixed point in $X_{T,R}$, which turns out to be the unique solution to

(2.4.1).

In the proof of Theorem 2.4.5, we will utilize the Schauder fixed point theorem, which requires to verify the following three things:

(i) Ψ_T maps $X_{T,R}$ to $\mathcal{A}_T \cap X_{T,R}$ for some suitably T and R;

(ii) $\Psi_T : X_{T,R} \to X_{T,R}$ is continuous;

(iii) $\Psi_T(X_{T,R})$ is precompact in $X_{T,R}$.

The requirement (iii) is the most technical part, so in order to make the argument more readable, we state the following Lemma 2.4.4 separately which will be used in the proof of (iii). Actually, Lemma 2.4.4 is a fact mentioned without justification in the proof of ([7], Theorem 13, Sec. 5, Chap. 7), but here for the convenience of the readers, we would like to give a complete proof.

Lemma 2.4.4. *Given $T > 0$ and $\{\varphi_j\}_{j\geq 1} \subset L^\infty\big((\Gamma_1 \cup \Gamma_2) \times (0, T]\big)$, we define*

$$w_j(x,t) = \int_0^t \int_{\partial\Omega} \Phi(x-y, t-\tau)\, \varphi_j(y,\tau)\, dS(y)\, d\tau, \quad \forall\, (x,t) \in \overline{\Omega} \times [0,T]. \tag{2.4.4}$$

If $\{\varphi_j\}_{j\geq 1}$ are uniformly bounded on $(\Gamma_1 \cup \Gamma_2) \times (0, T]$, then $\{w_j\}_{j\geq 1}$ are uniformly bounded and equicontinuous on $\overline{\Omega} \times [0,T]$.

Proof. From the assumption, there is a constant C^* such that for any $j \geq 1$ and $(x,t) \in (\Gamma_1 \cup \Gamma_2) \times (0, T]$,

$$|\varphi_j(x,t)| \leq C^*. \tag{2.4.5}$$

In the rest of the proof, we will use C to denote a constant which only depends on n, Ω, T

and C^*. Noticing that

$$
\begin{aligned}
|\Phi(x-y,t-\tau)| &\le C\,(t-\tau)^{-n/2}\, e^{-\frac{|x-y|^2}{4(t-\tau)}} \\
&\le C\,(t-\tau)^{-3/4}|x-y|^{-(n-3/2)},
\end{aligned}
\tag{2.4.6}
$$

it then follows from (2.4.5) and Lemma 2.3.3 that

$$
\begin{aligned}
|w_j(x,t)| &\le C\int_0^t \int_{\partial\Omega} (t-\tau)^{-3/4}\,|x-y|^{-(n-3/2)}\,dS(y)\,d\tau \\
&\le C\int_0^t (t-\tau)^{-3/4}\,d\tau \\
&\le C\,t^{1/4} \le C\,T^{1/4} = C.
\end{aligned}
\tag{2.4.7}
$$

This showed the uniform boundedness. Next, we will prove the equicontinuity, which means for any $\epsilon > 0$, there exists $\delta > 0$ such that the following (1) and (2) are satisfied.

(1) For any $j \ge 1$, $t \in [0,T]$ and x_1, x_2 in $\overline{\Omega}$ with $|x_1 - x_2| < \delta$, we have

$$|w_j(x_1,t) - w_j(x_2,t)| \le C\varepsilon.$$

(2) For any $j \ge 1$, $x \in \overline{\Omega}$, $0 \le t_1 < t_2 \le T$ with $t_2 - t_1 < \delta$, we have

$$|w_j(x,t_1) - w_j(x,t_2)| \le C\varepsilon.$$

By a change of variable in τ, we attain

$$w_j(x,t) = \int_0^t \int_{\partial\Omega} \Phi(x-y,\tau)\,\varphi_j(y,t-\tau)\,dS(y)\,d\tau, \quad \forall\,(x,t) \in \overline{\Omega}\times[0,T]. \tag{2.4.8}$$

Sometimes it is easier to use (2.4.8) to do the estimate.

To prove (1), we split the interval $[0,t]$ into $[0,\epsilon]$ and $[\epsilon,t]$ and take advantage of the uniform boundedness of $\{\varphi_j\}_{j\geq 1}$, it follows from (2.4.8) that

$$\begin{aligned}
|w_j(x_1,t)-w_j(x_2,t)| \leq & C\int_\epsilon^t\int_{\partial\Omega}\tau^{-n/2}\left|e^{-\frac{|x_1-y|^2}{4\tau}}-e^{-\frac{|x_2-y|^2}{4\tau}}\right|dS(y)\,d\tau \\
& + C\int_0^\epsilon\int_{\partial\Omega}\tau^{-n/2}\,e^{-\frac{|x_1-y|^2}{4\tau}}\,dS(y)\,d\tau \\
& + C\int_0^\epsilon\int_{\partial\Omega}\tau^{-n/2}\,e^{-\frac{|x_2-y|^2}{4\tau}}\,dS(y)\,d\tau \\
\triangleq & I+II+III.
\end{aligned}$$

Analogous to the derivation of (2.4.7), we get $II\leq C\,\epsilon^{1/4}$ and $III\leq C\,\epsilon^{1/4}$. To estimate I, we exploit the mean value theorem and find for any $\tau\in(0,T]$, $y\in\partial\Omega$,

$$\left|e^{-\frac{|x_1-y|^2}{4\tau}}-e^{-\frac{|x_2-y|^2}{4\tau}}\right|\leq C\,\tau^{-1/2}\,|x_1-x_2|.$$

As a result,

$$\begin{aligned}
I &\leq C\int_\epsilon^t\int_{\partial\Omega}\tau^{-(n+1)/2}\,|x_1-x_2|\,dS(y)\,d\tau \\
&\leq C\,\epsilon^{-(n+1)/2}\,\delta.
\end{aligned}$$

As long as we take $\delta\leq\epsilon^{(n+3)/2}$, then $I\leq C\varepsilon$. Thus, we finish the proof of (1).

In order to prove (2), we consider two cases.

- $t_1 \le 2\varepsilon$. In this case, we choose $\delta \le \varepsilon$, then again by (2.4.7),

$$\begin{aligned}|w_j(x,t_2) - w_j(x,t_1)| &\le |w_j(x,t_2)| + |w_j(x,t_1)| \\ &\le C\, t_2^{1/4} + C\, t_1^{1/4} \\ &\le C\,(2\varepsilon+\delta)^{1/4} + C\,(2\varepsilon)^{1/4} \le C\,\epsilon^{1/4}.\end{aligned}$$

- $t_1 > 2\varepsilon$. In this case, using (2.4.4), we get

$$\begin{aligned}&|w_j(x,t_2) - w_j(x,t_1)| \\ &\le \int_0^{t_1-\epsilon}\int_{\partial\Omega} \left| \frac{1}{(t_2-\tau)^{n/2}}\, e^{-\frac{|x-y|^2}{4(t_2-\tau)}} - \frac{1}{(t_1-\tau)^{n/2}}\, e^{-\frac{|x-y|^2}{4(t_1-\tau)}} \right| |\varphi_j(y,\tau)|\, dS(y)\, d\tau \\ &\quad + \int_{t_1-\epsilon}^{t_2} \frac{1}{(t_2-\tau)^{n/2}}\, e^{-\frac{|x-y|^2}{4(t_2-\tau)}} |\varphi_j(y,\tau)|\, dS(y)\, d\tau \\ &\quad + \int_{t_1-\epsilon}^{t_1} \frac{1}{(t_1-\tau)^{n/2}}\, e^{-\frac{|x-y|^2}{4(t_1-\tau)}} |\varphi_j(y,\tau)|\, dS(y)\, d\tau \\ &\triangleq I + II + III.\end{aligned}$$

If we choose $\delta \le \varepsilon$, then similar to (2.4.7) and using the mean value theorem,

$$\begin{aligned}II &\le C\big[t_2^{1/4} - (t_1-\epsilon)^{1/4}\big] \\ &\le \frac{C}{(t_1-\epsilon)^{3/4}}\,(t_2 - t_1 + \epsilon) \\ &\le C\,\epsilon^{-3/4}(\delta+\epsilon) \le C\,\epsilon^{1/4}. \qquad (2.4.9)\end{aligned}$$

Similarly,

$$III \le C\big[t_1^{1/4} - (t_1-\epsilon)^{1/4}\big] \le C\,\epsilon^{1/4}. \tag{2.4.10}$$

Employing the mean value theorem again, there exists a constant C such that for any $\tau \in [0,t_1)$, $x \in \overline{\Omega}$, $y \in \partial\Omega$, we have

$$\begin{aligned}\left|\frac{1}{(t_2-\tau)^{n/2}}\,e^{-\frac{|x-y|^2}{4(t_2-\tau)}} - \frac{1}{(t_1-\tau)^{n/2}}\,e^{-\frac{|x-y|^2}{4(t_1-\tau)}}\right| &\le C\,(t_1-\tau)^{-n/2-1}\,|t_2-t_1| \\ &\le C\,(t_1-\tau)^{-n/2-1}\,\delta.\end{aligned}$$

If we choose $\delta \le \varepsilon^{n/2+3}$, then

$$\begin{aligned} I &\le C\int_0^{t_1-\epsilon}\int_{\partial\Omega}(t_1-\tau)^{-n/2-1}\,dS(y)\,d\tau \\ &\le C\,\epsilon^{-n/2-1}\,\delta \le C\,\epsilon. \end{aligned} \tag{2.4.11}$$

Combining (2.4.11), (2.4.9), (2.4.10), we proved (2) and therefore the Lemma follows.

□

Now similar to the arguments in ([7], Theorem 13, Sec. 5, Chap. 7) and ([27], Theorem 1.3), we conclude the following theorem on the existence of the solution.

Theorem 2.4.5. *For the nonlinear equation (2.4.1) with f, F, ψ described there.*

(1) There exists $T_0 > 0$ such that for any $0 < T \le T_0$, there exists a unique solution $u \in \mathcal{A}_T$ to (2.4.1) on $\overline{\Omega} \times [0,T]$.

(2) If F is bounded on $\overline{\Gamma}_1 \times \mathbb{R}$, then for any $T > 0$, there exists a unique solution $u \in \mathcal{A}_T$ to (2.4.1) on $\overline{\Omega} \times [0,T]$.

Proof. Just as the heuristic idea before Lemma 2.4.4, in order to prove the existence of a solution, we will use Schauder fixed point theorem to prove Ψ_T has a fixed point in $X_{T,R}$ for suitable T and R. Thus, we need to verify the following three requirements:

(i) Ψ_T maps $X_{T,R}$ to $\mathcal{A}_T \cap X_{T,R}$;

(ii) $\Psi_T : X_{T,R} \to X_{T,R}$ is continuous;

(iii) $\Psi_T(X_{T,R})$ is precompact in $X_{T,R}$.

In the following, we will prove (1) and (2) in Theorem 2.4.5 together. Actually, the proofs of requirements (ii) and (iii) for (1) and (2) are identically the same, only the proofs of requirement (i) has slightly difference.

Firstly, given $T > 0$, let's recall how we construct $u \triangleq \Psi_T(v)$ for any $v \in X_{T,R}$. We will use the same notations as those in the proof of Theorem 2.3.6 with $\beta = 0$ and $g(x,t) = F\big(x, v(x,t)\big)\mathbb{1}_{\Gamma_1}(x)$, where

$$\mathbb{1}_{\Gamma_1}(x) \triangleq \begin{cases} 1, & x \in \Gamma_1, \\ 0, & x \notin \Gamma_1, \end{cases} \tag{2.4.12}$$

is the indicator function. Thus u has the following expression: for any $(x,t) \in \overline{\Omega} \times [0,T]$,

$$\begin{aligned} u(x,t) \triangleq &\int_{\Omega_1} \Phi(x-y,t)\,\psi(y)\,dy + \int_0^t \int_{\Omega} \Phi(x-y,t-\tau)\,f(y,\tau)\,dy\,d\tau \\ &+ \int_0^t \int_{\partial\Omega} \Phi(x-y,t-\tau)\,\varphi(y,\tau)\,dS(y)\,d\tau. \end{aligned} \tag{2.4.13}$$

Here $\varphi \in \mathcal{B}_T$ satisfies for any $(x,t) \in (\Gamma_1 \cup \Gamma_2) \times (0,T]$,

$$\varphi(x,t) = \int_0^t \int_{\partial\Omega} K(x,t;y,\tau)\,\varphi(y,\tau)\,dS(y)\,d\tau + H(x,t),$$

where

$$K(x,t;y,\tau) = -2\,\frac{\partial\Phi(x-y,t-\tau)}{\partial n(x)} \tag{2.4.14}$$

and

$$\begin{aligned} H(x,t) = &-2\int_{\Omega_1} \frac{\partial\Phi(x-y,t)}{\partial n(x)}\,\psi(y)\,dy \\ &-2\int_0^t\int_{\Omega} \frac{\partial\Phi(x-y,t-\tau)}{\partial n(x)}\,f(y,\tau)\,dy\,d\tau \\ &+2F\big(x,v(x,t)\big)\,\mathbb{1}_{\Gamma_1}(x). \end{aligned} \tag{2.4.15}$$

Since this K also satisfies (2.3.4), we can follow the same way as the derivations of (2.3.26), (2.3.25) to get

$$\varphi(x,t) = \int_0^t\int_{\partial\Omega} K^*(x,t;y,\tau)\,H(y,\tau)\,dS(y)\,d\tau + H(x,t) \tag{2.4.16}$$

for some function K^*. Moreover, there exists a constant $C^* = C^*(n,\Omega,T)$, such that

$$|K^*(x,t;y,\tau)| \le C^*(t-\tau)^{-3/4}\,|x-y|^{-(n-3/2)}. \tag{2.4.17}$$

Next, we will first assume requirement (i) and prove requirements (ii) and (iii), then we will confirm requirement (i) for the Cases (1) and (2) in Theorem 2.4.5 respectively.

Given $T>0$, we assume there exists $R>0$ such that $\Psi_T : X_{T,R} \to \mathcal{A}_T \cap X_{T,R}$. Define $M_F, M_{F_\sigma} : [0,\infty) \to \mathbb{R}$ by

$$M_F(r) = \sup_{\overline{\Gamma}_1\times[-r,r]} |F(x,\sigma)|$$

and

$$M_{F_\sigma}(r) = \sup_{\overline{\Gamma}_1 \times [-r,r]} |F_\sigma(x,\sigma)|.$$

Then for any fixed $r \geq 0$, both $M_F(r)$ and $M_{F'}(r)$ are finite number since $F \in C^1(\mathbb{R})$. In the following, for any $v \in X_{T,R}$, we define u, φ and H as in (2.4.13), (2.4.14) and (2.4.15) respectively. For any $v_j \in X_{T,R}$ $(j \geq 1)$, we define u_j, φ_j and H_j analogously.

- Proof of Requirement (ii). Given $\{v_j\}_{j\geq 1} \subset X_{T,R}$ and $v_j \to v$ in $X_{T,R}$, we want to show $\Psi_T(v_j) \to \Psi_T(v)$ in $X_{T,R}$. Since v and all the $v_j (j \geq 1)$ belong to $X_{T,R}$, then for any $(x,t) \in \overline{\Omega} \times [0,T]$, $|v(x,t)| \leq R$ and $|v_j(x,t)| \leq R$. Thus, by the mean value theorem and the fact $M_{F'}(R) < \infty$, it follows from (2.4.15) that $H_j \rightrightarrows H$ on $(\Gamma_1 \cup \Gamma_2) \times (0,T]$ (here " $\rightrightarrows$ " means uniform convergence). Then by (2.4.16) and (2.4.17), $\varphi_j \rightrightarrows \varphi$ on $(\Gamma_1 \cup \Gamma_2) \times (0,T]$. Finally, due to the expression (2.4.13), we have $u_j \rightrightarrows u$ on $\overline{\Omega} \times [0,T]$, which is equivalent to say $\Psi_T(v_j) \to \Psi_T(v)$ in $X_{T,R}$.

- Proof of Requirement (iii). In this proof, we will use C to denote a constant which is independent of j, x and t, but may depend on n, Ω, Ω_1, T, R, $M_F(R)$, $\sup|f|$, $\sup|\psi|$ and $\sup|D\psi|$. C may be different from line to line. Given any sequence $\{v_j\}_{j\geq 1} \subset X_{T,R}$, we want to show $\{\Psi_T(v_j)\}_{j\geq 1}$ has a subsequence which converges to some function u in $X_{T,R}$. Since $v_j \in X_{T,R}$ for any $j \geq 1$, then for any $j \geq 1$ and for any $(x,t) \in \overline{\Omega} \times [0,T]$, $|v_j(x,t)| \leq R$. Now recalling (2.3.21), we know

$$\Big| \int_{\Omega_1} \frac{\partial \Phi(x-y,t)}{\partial n(x)} \, \psi(y) \, dy \Big|$$

is bounded by some constant C. Consequently by (2.4.15), these exists some constant

C such that for any $j \geq 1$ and for any $(x,t) \in \overline{\Omega} \times [0,T]$,

$$|H_j(x,t)| \leq C.$$

Then due to (2.4.16) and (2.4.17), there exists some constant C such that for any $j \geq 1$ and for any $(x,t) \in \overline{\Omega} \times [0,T]$,

$$|\varphi_j(x,t)| \leq C.$$

Now using (2.4.13) and Lemma 2.4.4, we find $\{u_j\}_{j\geq 1}$ is uniformly bounded and equicontinuous on $\overline{\Omega} \times [0,T]$. Hence it follows from the Arzela-Ascoli theorem that $\{u_j\}_{j\geq 1}$ has a subsequence $\{u_{j_k}\}_{k\geq 1}$ which converges uniformly to some function u on $\overline{\Omega} \times [0,T]$. Since $u_{j_k} \in X_{T,R}$, it is readily to see that u is also in $X_{T,R}$. Thus, $\Psi_T(X_{T,R})$ is precompact in $X_{T,R}$.

Now we turn to verify Requirement (i).

- Proof of Requirement (i) for (1). We will find $0 < T_0 \leq 1$ such that for any $0 < T \leq T_0$, there exists $R > 0$ such that Ψ_T maps $X_{T,R}$ to $\mathcal{A}_T \cap X_{T,R}$. In this proof, C will denote a constant which is independent of x, t, R and T, but may depend on n, Ω, Ω_1, $\sup|f|$, $\sup|\psi|$ and $\sup|D\psi|$. C may be different from line to line. For the first term of (2.4.15), we recall (2.3.21) again to get for any $(x,t) \in (\Gamma_1 \cup \Gamma_2) \times (0,T]$,

$$\begin{aligned}
& \Big|\int_{\Omega_1} \frac{\partial \Phi(x-y,t)}{\partial n(x)}\,\psi(y)\,dy\Big| \\
\leq\ & \Big|\int_{\partial\Omega_1} \Phi(x-y,t)\,\psi(y)\,\overrightarrow{n}(y)\cdot\overrightarrow{n}(x)\,dy\Big| + \Big|\int_{\Omega_1} \Phi(x-y,t)\,D\psi(y)\cdot\overrightarrow{n}(x)\,dy\Big| \\
\leq\ & C\int_{\partial\Omega_1} |x-y|^{-n}\,dy + C \leq C. \qquad (2.4.18)
\end{aligned}$$

For the second term of (2.4.15), we have for any $(x,t) \in (\Gamma_1 \cup \Gamma_2) \times (0,T]$,

$$\begin{aligned} &\Big| \int_0^t \int_\Omega \frac{\partial \Phi(x-y, t-\tau)}{\partial n(x)} \, f(y,\tau) \, dy \, d\tau \Big| \\ &\leq C \, \sup |f| \int_0^t \int_\Omega (t-\tau)^{-3/4} \, |x-y|^{-(n-1/2)} \, dS(y) \, d\tau \\ &\leq C \, t^{1/4} \leq C \, T^{1/4} \leq C \, T_0^{1/4} \leq C, \end{aligned} \tag{2.4.19}$$

where the last inequality is due to the assumption $T_0 \leq 1$. Then it follows from (2.4.18), (2.4.19) and (2.4.15) that for any $(x,t) \in (\Gamma_1 \cup \Gamma_2) \times (0,T]$,

$$|H(x,t)| \leq C + C \, M_F(R). \tag{2.4.20}$$

Although the constant C^* in (2.4.17) depends on T, it is readily to see that C^* is an increasing function in T. As a result, when T is bounded by 1, C^* will also be bounded by some constant C, which only depends on n and Ω. Based on this observation and (2.4.16), we get for any $(x,t) \in (\Gamma_1 \cup \Gamma_2) \times (0,T]$,

$$\begin{aligned} |\varphi(x,t)| &\leq C^* \, [C + C \, M_F(R)] \int_0^t \int_{\partial\Omega} (t-\tau)^{-3/4} \, |x-y|^{-(n-3/2)} \, dS(y) \, d\tau \\ &\quad + C + C \, M_F(R) \\ &\leq [C + C \, M_F(R)] \, T^{1/4} + C + C \, M_F(R) \\ &\leq C + C \, M_F(R). \end{aligned} \tag{2.4.21}$$

Now by (2.4.13) and (2.4.21), we obtain for any $(x,t) \in \overline{\Omega} \times [0,T]$,

$$\begin{aligned}
|u(x,t)| &\le \sup|\psi| + t\,\sup|f| + \sup|\varphi| \int_0^t \int_{\partial\Omega} (t-\tau)^{-3/4}\,|x-y|^{-(n-3/2)}\,dS(y)\,d\tau \\
&\le C + \big[C + C\,M_F(R)\big]\,C\,T^{1/4} \\
&\le C + C\,M_F(R)\,T_0^{1/4} \\
&\triangleq C_1 + C_1\,M_F(R)\,T_0^{1/4}. \qquad (2.4.22)
\end{aligned}$$

Hence, if we choose R and $T_0 \le 1$ such that

$$R = 2C_1 \quad \text{and} \quad T_0^{1/4}\,M_F(2C_1) < 1, \qquad (2.4.23)$$

then we have $||u|| \le R$ and therefore $u \triangleq \Psi_T(v) \in X_{T,R}$.

- Proof of Requirement (i) for (2). We will prove for any $T > 0$, there exists $R > 0$ such that Ψ_T maps $X_{T,R}$ to $\mathcal{A}_T \cap X_{T,R}$. In this proof, F is a bounded function on $\overline{\Gamma}_1 \times \mathbb{R}$, so $\sup\limits_{\overline{\Gamma}_1\times\mathbb{R}} |F| < \infty$. In the following, we will use C to denote a constant just like that in the proof for (1) but allowing C to depend on T and $\sup\limits_{\overline{\Gamma}_1\times\mathbb{R}} |F|$. As the same derivations as (2.4.18), (2.4.19) and (2.4.20), we attain for any $(x,t) \in (\Gamma_1 \cup \Gamma_2) \times (0,T]$,

$$\begin{aligned}
|H(x,t)| &\le C + C\,M_F(R) \\
&\le C + C\sup_{\mathbb{R}}|F| = C. \qquad (2.4.24)
\end{aligned}$$

Then based on (2.4.16) and (2.4.24), we get for any $(x,t) \in (\Gamma_1 \cup \Gamma_2) \times (0,T]$,

$$
\begin{aligned}
|\varphi(x,t)| &\leq C^* \, C \int_0^t \int_{\partial\Omega} (t-\tau)^{-3/4} \, |x-y|^{-(n-3/2)} \, dS(y) \, d\tau + C \\
&\leq C \, T^{1/4} + C \\
&= C.
\end{aligned}
$$

Therefore, by (2.4.13) and (2.4.21) again, we obtain for any $(x,t) \in \overline{\Omega} \times [0,T]$,

$$
\begin{aligned}
|u(x,t)| &\leq \sup|\psi| + t \, \sup|f| + \sup|\varphi| \int_0^t \int_{\partial\Omega} (t-\tau)^{-3/4} \, |x-y|^{-(n-3/2)} \, dS(y) \, d\tau \\
&\leq C + C \, T^{1/4} \\
&\triangleq C_2.
\end{aligned}
$$

Thus, as long as we take $R > C_2$, we have $||u|| \leq R$ and consequently $u \triangleq \Psi_T(v) \in X_{T,R}$.

$\square$

2.4.4 Maximal solution and application to the target problem

As we can see from Theorem 2.4.2 that the solution to (2.4.1) is only proved to be global when the function F is bounded on $\mathbb{R}$. Thus, we need to study the maximal solution when F is unbounded.

Definition 2.4.6. *We call*

$$
T^* \triangleq \sup\{T \geq 0 : \text{ there exsits a solution to (2.4.1) on } \overline{\Omega} \times [0,T]\}
$$

the maximal existence time for (2.4.1). Moreover, a function $u \in C^{2,1}\big(\Omega \times (0,T^*)\big) \cap C\big(\overline{\Omega} \times [0,T^*)\big)$ *is called a maximal solution if it solves (2.4.1) on* $\overline{\Omega} \times [0,T]$ *for any* $T \in (0,T^*)$.

Remark 2.4.7. *It follows from Theorem 2.4.5 and Theorem 2.4.2 that the maximal solution exists and is unique.*

Just like ([27], Corollary 1.1), when T^* is finite, it coincides with the blow-up time of the L^∞ norm of u. Thus in order to estimate T^*, one only needs to focus on the L^∞ norm of the solution. We state it more precisely in the following theorem.

Theorem 2.4.8. *Let* T^* *be the maximal existence time for (2.4.1) and* u *be the maximal solution. If* $T^* < \infty$*, then*

$$\lim_{t \nearrow T^*} \sup_{(x,\tau)\in\overline{\Omega}\times[0,t]} |u(x,\tau)| = \infty.$$

Proof. We prove by contradiciton. Assume

$$\lim_{t \nearrow T^*} \sup_{(x,\tau)\in\overline{\Omega}\times[0,t]} |u(x,\tau)| < \infty.$$

Then we denote $K = \sup_{\overline{\Omega}\times[0,T^*)} |u| < \infty$. The strategy below is to find a solution that exists beyond T^*, which contradicts to the definition of T^*.

To do this, we firstly construct a bounded and C^1 function $\widetilde{F} : \overline{\Gamma}_1 \times \mathbb{R} \to \mathbb{R}$ such that $\widetilde{F}(x,\sigma) = F(x,\sigma)$ for any $x \in \overline{\Gamma}_1$ and $|\sigma| \le K+1$. Then by Theorem 2.4.5, for any $T > 0$, the problem

$$\begin{cases} \tilde{u}_t(x,t) - \Delta \tilde{u}(x,t) = f(x,t) & \text{in} \quad \Omega \times (0,T], \\ \frac{\partial \tilde{u}(x,t)}{\partial n(x)} = \widetilde{F}\big(x, \tilde{u}(x,t)\big) & \text{on} \quad \Gamma_1 \times (0,T], \\ \frac{\partial \tilde{u}(x,t)}{\partial n(x)} = 0 & \text{on} \quad \Gamma_2 \times (0,T], \\ \tilde{u}(x,0) = \psi(x) & \text{on} \quad \Omega, \end{cases} \tag{2.4.25}$$

has a unique solution $\tilde{u}$. Especially, when $T = T^* + 1$, there exists a solution $\tilde{u} \in \mathcal{A}_{T^*+1}$ to (2.4.25).

Since u also solves (2.4.25) for $T < T^*$, then by uniqueness, $\tilde{u} = u$ on $\overline{\Omega} \times [0, T^*)$. Therefore

$$\sup_{\overline{\Omega} \times [0,T^*)} |\tilde{u}| = K.$$

Now by continuity, there exists $\epsilon > 0$ such that

$$\sup_{\overline{\Omega} \times [0,T^*+\epsilon]} |\tilde{u}| \leq K + 1.$$

Recalling that $\widetilde{F}(x,\sigma)$ and $F(x,\sigma)$ coincide when $|\sigma| \leq K+1$, so $\tilde{u}$ is the solution of (2.4.1) on $\overline{\Omega} \times [0, T^* + \epsilon]$. This contradicts to the definition of T^*. $\square$

As a particular application of the theories established in this section, one can apply Theorem 2.4.5, Theorem 2.4.2, Theorem 2.4.3 with $f = 0$, $F(x,\sigma) = \sigma^q$ and $\psi = u_0$ to our targeted problem (1.1.1) to obtain Theorem 2.1.1

Chapter 3

Upper Bound Estimate of the Blow-up Time

3.1 Main theorem and outline of the approach

The goal of this chapter is to show the finite-time blowup of the solution to (1.1.1) and find an upper bound for the blow-up time. In this chapter, $M(t)$ is defined as in (1.4.3). The next theorem is the main result of this chapter.

Theorem 3.1.1. *Let T^* be the maximal existence time for (1.1.1). Then $T^* < \infty$ and*

$$\lim_{t \nearrow T^*} M(t) = \infty,$$

where $M(t)$ is defined as in (1.4.3). In addition, if $\min_{x\in\overline{\Omega}} u_0(x) > 0$, then

$$T^* \leq \frac{1}{(q-1)|\Gamma_1|} \int_\Omega u_0^{1-q}(x)\, dx. \tag{3.1.1}$$

Inspired by [36], the idea of this proof is very simple, define

$$h(t) = \int_\Omega u^{1-q}(x,t)\, dx.$$

Then by formal computations (assuming u is smooth up to the boundary),

$$
\begin{aligned}
h'(t) &= (1-q)\int_\Omega u^{-q}\, u_t\, dx \\
&= (1-q)\int_\Omega u^{-q}\, \Delta u\, dx \\
&\le (1-q)\int_\Omega \nabla\cdot(u^{-q} Du)\, dx \\
&= (1-q)\int_{\partial\Omega} u^{-q}\, \frac{\partial u}{\partial n}\, dS \\
&= (1-q)|\Gamma_1|. \qquad (3.1.2)
\end{aligned}
$$

(3.1.2) implies that $h(t)$ decreases at a speed which is at least $(q-1)|\Gamma_1|$ every unit time. As a result, (3.1.1) is justified. Although the solution u is smooth inside the domain, it is not C^1 up to the boundary. So we need to consider the integral of u^{1-q} over interior domains of Ω and then take limit as the interior domains approach Ω. In addition, in the limiting process, the uniform limit of the normal derivative is required, so we also have to approximate u by a sequence of smoother functions whose normal derivative is uniformly continuous up to the boundary after any positive time and strictly less than the blow-up time.

The organization of this chapter is as follows. In Section 3.2, we explore some geometric properties of the C^2 domains which make it possible to approximate from inside. All the conclusions in this section should be standard, but for the sake of completeness, we also include the detailed proofs. Section 3.3 discusses how to approximate the solution by some functions with desired regularity of the normal derivative. Finally, Section 3.4 carries out the rigorous proof for the main theorem.

3.2 Approximation for C^2 domain from inside

For any bounded domain $\Omega \subset \mathbb{R}^n$ and for any $h > 0$, we define

$$\Omega_h = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > h\} \tag{3.2.1}$$

and

$$\Omega_h^c = \{x \notin \Omega : \operatorname{dist}(x, \partial\Omega) > h\}. \tag{3.2.2}$$

Lemma 3.2.1. *Let Ω be a bounded, open subset of $\mathbb{R}^n$ with $\partial\Omega \in C^2$. Then for any $x \in \partial\Omega$, there exist positive constants r and σ which only depend on x and Ω such that for any $y \in B(x, r) \cap \partial\Omega$, there is an interior ball touching y with radius σ. Namely,*

$$\overline{B}_\sigma(y - \sigma \overrightarrow{n}(y)) \cap \Omega^c = \{y\}. \tag{3.2.3}$$

Analogously, there is also an exterior ball at y with radius σ. Namely,

$$\overline{B}_\sigma(y + \sigma \overrightarrow{n}(y)) \cap \overline{\Omega} = \{y\}. \tag{3.2.4}$$

Proof. We will only prove (3.2.3), since the proof of (3.2.4) is almost identical. Fix any $x \in \partial\Omega$, by translation and rotation, we can assume x to be the origin and $\overrightarrow{n}(x) = (\tilde{\mathbf{0}}, -1)$, where $\tilde{\mathbf{0}}$ denotes the origin in $\mathbb{R}^{n-1}$.

Since $\partial\Omega \in C^2$, there exist a C^2 function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ and $r \in (0, 1]$ (depending on ϕ and Ω) such that

(1) For any $y \in B_{4r} \cap \partial\Omega$,

$$y_n = \phi(\tilde{y}); \quad \overrightarrow{n}(y) \cdot e_n < 0; \quad |D\phi(\tilde{y})| \leq 1/4. \tag{3.2.5}$$

(2) $B_{4r} \cap \Omega = B_{4r} \cap \{y : y_n > \phi(\tilde{y})\}$.

(3) $B_{4r} \cap \overline{\Omega}^c = B_{4r} \cap \{y : y_n < \phi(\tilde{y})\}$.

Now for any $y \in B_r \cap \partial\Omega$, writing $y = (\tilde{y}, \phi(\tilde{y}))$, then

$$\overrightarrow{n}(y) = \frac{(D\phi(\tilde{y}), -1)}{< D\phi(\tilde{y}) >},$$

where $< \cdot >$ is defined as $< x >= \sqrt{1 + |x|^2}$. We will show that there exists a positive constant σ, which only depends on ϕ and Ω such that (3.2.3) is satisfied.

Let $\sigma \in (0, r]$ be determined later. Define

$$y^0 = y - \sigma \overrightarrow{n}(y).$$

Then

$$\widetilde{y^0} = \tilde{y} - \frac{\sigma D\phi(\tilde{y})}{< D\phi(\tilde{y}) >} \quad \text{and} \quad y_n^0 = \phi(\tilde{y}) + \frac{\sigma}{< D\phi(\tilde{y}) >},$$

where $\widetilde{y^0}$ denotes the first $n-1$ components of y^0 and y_n^0 denotes the last component of y^0.

For any $z = (\tilde{z}, z_n) \in \overline{B}_\sigma(y^0)$, we have $|z - y^0| \leq \sigma$ and

$$|z| \leq |z - y^0| + |y^0| \leq \sigma + (|y| + \sigma) \leq 3r.$$

So in order to show (3.2.3), it suffices to prove $z_n > \phi(\tilde{z})$ for any $z \in \overline{B}_\sigma(y^0) \setminus \{y\}$. Since

$|z - y^0| \le \sigma$, then

$$z_n \ge y_n^0 - \sqrt{\sigma^2 - |\tilde{z} - \widetilde{y^0}|^2}. \tag{3.2.6}$$

Next, the goal is to verify

$$y_n^0 - \sqrt{\sigma^2 - |\tilde{z} - \widetilde{y^0}|^2} \ge \phi(\tilde{z}). \tag{3.2.7}$$

Namely,

$$\phi(\tilde{z}) - \phi(\tilde{y}) \le \frac{\sigma}{< D\phi(\tilde{y}) >} - \sqrt{\sigma^2 - |\tilde{z} - \widetilde{y^0}|}.$$

By Taylor expansion, there exists some η between $\tilde{y}$ and $\tilde{z}$ such that

$$\phi(\tilde{z}) - \phi(\tilde{y}) = D\phi(\tilde{y}) \cdot (\tilde{z} - \tilde{y}) + \frac{1}{2}(\tilde{z} - \tilde{y})^T D^2\phi(\eta)(\tilde{z} - \tilde{y}).$$

Let $\lambda = \max_{|\xi| \le 3r} ||D^2\phi(\xi)||_2$, where $|| \cdot ||_2$ denotes the matrix norm. Then

$$\phi(\tilde{z}) - \phi(\tilde{y}) \le D\phi(\tilde{y}) \cdot (\tilde{z} - \tilde{y}) + \frac{1}{2}\lambda|\tilde{z} - \tilde{y}|^2.$$

So it suffices to show

$$D\phi(\tilde{y}) \cdot (\tilde{z} - \tilde{y}) + \frac{1}{2}\lambda|\tilde{z} - \tilde{y}|^2 \le \frac{\sigma}{< D\phi(\tilde{y}) >} - \sqrt{\sigma^2 - |\tilde{z} - \widetilde{y^0}|}. \tag{3.2.8}$$

Noticing

$$\begin{aligned}\sigma^2 - |\tilde{z} - \widetilde{y^0}| &= \sigma^2 - |\tilde{z} - \tilde{y} + \frac{\sigma D\phi(\tilde{y})}{< D\phi(\tilde{y}) >}|^2 \\ &= \frac{\sigma^2}{< D\phi(\tilde{y}) >^2} - |\tilde{z} - \tilde{y}|^2 - 2\,\frac{\sigma}{< D\phi(\tilde{y}) >}\, D\phi(\tilde{y}) \cdot (\tilde{z} - \tilde{y}).\end{aligned}$$

Let

$$A = D\phi(\tilde{y}) \cdot (\tilde{z} - \tilde{y}); \quad B = \frac{\sigma}{< D\phi(\tilde{y}) >}; \quad E = |\tilde{z} - \tilde{y}|^2.$$

Then (3.2.8) boils down to

$$A + \frac{1}{2}\lambda E \leq B - \sqrt{B^2 - E - 2AB},$$

that is

$$\sqrt{B^2 - E - 2AB} \leq B - A - \frac{1}{2}\lambda E. \tag{3.2.9}$$

Since

$$|\tilde{z} - \tilde{y}| \leq |\tilde{z} - \widetilde{y^0}| + |\widetilde{y^0} - \tilde{y}|$$

$$\leq \sigma + \sigma = 2\sigma,$$

then $|A| \leq |\tilde{z} - \tilde{y}|/4 \leq \sigma/2$ and $E \leq 4\sigma^2$. On the other hand, $|D\phi(\tilde{y})| \leq 1/4$ implies that $B \geq 2\sigma/3$. As a result, as long as we take

$$\sigma \triangleq \min\{r, \frac{1}{12\lambda}\},$$

$B - A - \frac{1}{2}\lambda E \geq 0$. Hence (3.2.9) reduces to

$$B^2 - E - 2AB \leq \big(B - A - \frac{1}{2}\lambda E\big)^2.$$

Simplifying the above inequality, we obtain an equivalent form

$$\lambda E(B - A) \leq E + A^2 + \frac{1}{4}\lambda^2 E^2,$$

which is always true because of the fact that

$$|\lambda E(B - A)| \leq \lambda E(\sigma/2 + \sigma) \leq E/8 \leq E. \tag{3.2.10}$$

Thus, we proved (3.2.7). Then it follows from (3.2.6) that $z_n \geq \phi(\tilde{z})$, which means $z \in \overline{\Omega}$. In addition, if $z_n = \phi(\tilde{z})$, then (3.2.10) should take "equal sign". This implies $E = 0$, which means $\tilde{z} = \tilde{y}$. Moreover, since both (3.2.6) and (3.2.7) should also take "equal sign", we have $z_n = \phi(\tilde{z}) = \phi(\tilde{y}) = y_n$, which implies $z = y$. As a result, (3.2.3) is justified. □

Corollary 3.2.2. *Let Ω be a bounded, open subset of $\mathbb{R}^n$ with $\partial\Omega \in C^2$. Then there exists a positive constant $\sigma = \sigma(\Omega)$ such that for any $x \in \partial\Omega$, there exists an interior ball and also an exterior ball at x with radius σ.*

Proof. It follows from (3.2.1) and the compactness of $\partial\Omega$. □

Corollary 3.2.3. *Let Ω be a bounded, open subset of $\mathbb{R}^n$ with $\partial\Omega \in C^2$. Then there exists a positive constant $\sigma = \sigma(\Omega)$ such that*

(1) for any $h \in (0, \sigma)$, the map $\Psi_h : \partial\Omega \to \partial\Omega_h$ defined by $\Psi_h(x) = x - h\overrightarrow{n}(x)$ is a C^1 diffeomorphism.

(2) for any $h \in (0, \sigma)$, the map $\widetilde{\Psi}_h : \partial\Omega \to \partial\Omega_h^c$ defined by $\widetilde{\Psi}_h(x) = x + h\overrightarrow{n}(x)$ is a C^1 diffeomorphism.

Proof. Again, we will only prove for (1), since the proof for (2) is analogous.

Define σ_1 as that in Corollary 3.2.2. Then for any $h \in (0, \sigma_1)$ and for any $x \in \partial\Omega$, there exists an interior ball touching x with radius h. Namely, $\overline{B}_h(x - h\overrightarrow{n}(x)) \cap \Omega^c = \{x\}$. As a result, $\operatorname{dist}(\Psi_h(x), \partial\Omega) = h$. So Ψ_h is well-defined. Moreover, this also implies that Ψ_h is an injection. To show Ψ_h is surjective, take any $y \in \Omega$ such that $\operatorname{dist}(y, \partial\Omega) = h$. Then there exists $x \in \partial\Omega$ such that $|y - x| = h$. As a result, $y = x - h\overrightarrow{n}(x) = \Psi_h(x)$.

Now Ψ_h has been proved to be a bijection. Next, since $\partial\Omega$ is C^2, then $\overrightarrow{n}$ is C^1 on $\partial\Omega$. Hence, by the inverse function theorem, there exists $\sigma_2 > 0$ such that for any $h \in (0, \sigma_2)$, Ψ_h is a C^1 diffeomorphism. Finally, by taking $\sigma = \min\{\sigma_1, \sigma_2\}$, (1) is justified. $\square$

The following corollary is a simple version of the tubular neighborhood theorem.

Corollary 3.2.4. *Let Ω be a bounded, open subset of $\mathbb{R}^n$ with $\partial\Omega \in C^2$. Then there exists a positive constant $\sigma = \sigma(\Omega)$ such that the map $\Psi : \partial\Omega \times (-\sigma, \sigma) \to \{x \in \mathbb{R}^n : \mathit{dist}\,(x, \partial\Omega) < \sigma\}$ defined by $\Psi(x, h) = x - h\overrightarrow{n}(x)$ is a C^1 diffeomorphism.*

Proof. First, by Corollary 3.2.3, there exists $\sigma > 0$, only depending on Ω, such that $\Psi : \partial\Omega \times (-\sigma, \sigma) \to \{x \in \mathbb{R}^n : \operatorname{dist}(x, \partial\Omega) < \sigma\}$ defined by $\Psi(x, h) = x - h\overrightarrow{n}(x)$ is a bijection. Then again by the inverse function theorem, by choosing σ smaller enough, Ψ is a C^1 diffeomorphism. $\square$

Lemma 3.2.5. *Let Ω be a bounded, open subset of $\mathbb{R}^n$ with $\partial\Omega \in C^2$. Then there exists a positive constant $\sigma = \sigma(\Omega)$ such that for any $h \in (0, \sigma)$ and for any $x \in \partial\Omega$,*

$$\overrightarrow{n_h}\big(\Psi_h(x)\big) = \overrightarrow{n}(x), \tag{3.2.11}$$

where Ψ_h is defined as in Corollary 3.2.3 and $\overrightarrow{n_h}$ denotes the exterior unit normal vector with respect to $\partial\Omega_h$.

Proof. By Corollary 3.2.3, there exists $\sigma = \sigma(\Omega)$ such that for any $h \in (0, \sigma)$, Ψ_h is a C^1 diffeomorphism between $\partial\Omega$ to $\partial\Omega_h$. Next fix $h \in (0, \sigma)$ and $x \in \partial\Omega$, we will show $\overrightarrow{n_h}\big(\Psi_h(x)\big) = \overrightarrow{n}(x)$.

Just like the proof for Theorem 2.2.6, by translating and rotating the coordinates, we can assume that x to be the origin $\mathbf{0}$ and $\overrightarrow{n}(\mathbf{0}) = -\mathbf{e_n} = (0, 0, \cdots, 0, -1)$. As a result, $\Psi_h(x) = h\mathbf{e_n}$ and then it is equivalent to prove that $\overrightarrow{n_h}(h\mathbf{e_n}) = -\mathbf{e_n}$. Suppose the point near $\mathbf{0}$ can be parametrized by

$$y = (\tilde{y}, y_n) = (\tilde{y}, \phi(\tilde{y})).$$

So when y is near $\mathbf{0}$, the exterior unit normal vector at y is

$$\overrightarrow{n}(y) = \frac{(D\phi(\tilde{y}), -1)}{< D\phi(\tilde{y}) >}.$$

Then the point near $h\mathbf{e_n}$ can be parametrized by

$$z = \Psi_h(y) = (\tilde{y}, \phi(\tilde{y})) - h\,\frac{(D\phi(\tilde{y}), -1)}{< D\phi(\tilde{y}) >} \triangleq F(\tilde{y}).$$

As a result, the tangent plane at $h\mathbf{e_n}$ with respect to $\partial\Omega_h$ is spanned by $\{D_i F(\tilde{\mathbf{0}})\}_{i=1}^{n-1}$. Thus in order to prove $\overrightarrow{n_h}(h\mathbf{e_n}) = -\mathbf{e_n}$, it suffices to show $\mathbf{e_n}$ is perpendicular to each $D_i F(\tilde{\mathbf{0}})$, or equivalently the nth component of $D_i F(\tilde{\mathbf{0}})$ is 0. For each $1 \le i \le n-1$, by direct calculations and noticing $D_i\phi(\tilde{\mathbf{0}}) = 0$, we find the nth component of $D_i F(\tilde{\mathbf{0}})$ is indeed 0. $\square$

3.3 Approximation for the solution by smoother functions

By Theorem 2.1.1, there exists a unique nonnegative maximal solution $u \in C^{2,1}\big(\Omega \times (0,T^*)\big) \cap C\big(\overline{\Omega} \times [0,T^*)\big)$ to (1.1.1) such that

$$\begin{cases} u_t(x,t) = \Delta u(x,t) & \text{in} \quad \Omega \times (0,T^*), \\ \frac{\partial u(x,t)}{\partial n(x)} = u^q(x,t) & \text{on} \quad \Gamma_1 \times (0,T^*), \\ \frac{\partial u(x,t)}{\partial n(x)} = 0 & \text{on} \quad \Gamma_2 \times (0,T^*), \\ u(x,0) = u_0(x) & \text{on} \quad \Omega, \end{cases} \tag{3.3.1}$$

where T^* is the maximal existence time to (1.1.1). It is readily seen that the normal derivative of this solution is not continuous along the boundary. But in many discussions, it really requires the continuity of the normal derivative. Thus, the purpose of this section is to construct a sequence of functions that has normal derivatives of better behavior hear he boundary.

Lemma 3.3.1. *Assume g is continuous on $\partial\Omega \times [0,T]$ and $\psi \in C^1(\overline{\Omega})$. Let v be the solution to*

$$\begin{cases} v_t(x,t) = \Delta v(x,t) & \textit{in} \quad \Omega \times (0,T], \\ \frac{\partial v(x,t)}{\partial n(x)} = g(x,t) & \textit{on} \quad \partial\Omega \times (0,T], \\ v(x,0) = \psi(x) & \textit{in} \quad \Omega. \end{cases}$$

Then for any $\tau_0 > 0$, the following limit converges uniformly on $\partial\Omega \times [\tau_0, T]$.

$$\lim_{h \to 0^+} \frac{\partial v(x_h,t)}{\partial n(x)} = \frac{\partial v(x,t)}{\partial n(x)}.$$

Proof. We will mimic the proof of Theorem 2.3.6 with $f = 0$, $\beta = 0$, $\Gamma_1 = \partial\Omega$ and $\Gamma_2 = \emptyset$. First, we extend ψ to a C^1 function to a larger domain $\Omega_1 \Supset \Omega$. Then from that proof, the solution v can be written as

$$\begin{aligned} v(x,t) &= \int_{\Omega_1} \Phi(x-y,t)\psi(y)\,dy + \int_0^t \int_{\partial\Omega} \Phi(x-y,t-\tau)\varphi(y,\tau)\,dS(y)\,d\tau && (3.3.2)\\ &\triangleq v_1(x,t) + v_2(x,t), && (3.3.3) \end{aligned}$$

where φ is a continuous function on $\partial\Omega \times [0,T]$ that is related to g and ψ.

Now fix any $\tau_0 \in (0,T)$, it is readily seen that the following limit converges uniformly on $\partial\Omega \times [\tau_0, T]$.

$$\lim_{h\to 0^+} \frac{\partial v_1(x_h,t)}{\partial n(x)} = \frac{\partial v_1(x,t)}{\partial n(x)}.$$

On the other hand, Theorem 2.2.1 implies the uniform convergence of

$$\lim_{h\to 0^+} \frac{\partial v_2(x_h,t)}{\partial n(x)} = \frac{\partial v_2(x,t)}{\partial n(x)}$$

on $\partial\Omega \times [\tau_0, T]$. □

Next, we take a sequence of boundary pieces $\{\Gamma_{1,j}\}_{j\ge 1}$ such that $\Gamma_{1,j} \subset \Gamma_1$ and $\Gamma_{1,j} \nearrow \Gamma_1$ (see Figure 3.1). Then a sequence of C^∞ cut-off functions $\{\eta_j\}_{j=1}^\infty$ are chosen so that for

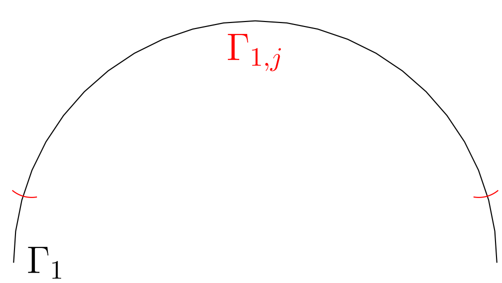

Figure 3.1: $\Gamma_{1,j}$

each $j \geq 1$,

$$\eta_j(x) \begin{cases} = 1, & x \in \Gamma_{1,j}, \\ \in [0,1], & x \in \overline{\Gamma}_1 \setminus \Gamma_{1,j}, \\ = 0, & x \in \partial\Omega \setminus \Gamma_1. \end{cases} \tag{3.3.4}$$

In addition, we require that $\eta_{j+1}(x) \geq \eta_j(x), \forall\, j \geq 1,\ x \in \partial\Omega$.

For any $T \in (0, T^*)$ and for each $j \geq 1$, we consider

$$\begin{cases} (v_j)_t(x,t) = \Delta v_j(x,t) & \text{in} \quad \Omega \times (0,T], \\ \frac{\partial v_j(x,t)}{\partial n(x)} = \eta_j(x)\, u^q(x,t) & \text{on} \quad \partial\Omega \times (0,T], \\ v_j(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{3.3.5}$$

where the function u in the boundary condition $\frac{\partial v_j(x,t)}{\partial n(x)} = \eta_j(x)\, u^q(x,t)$ is just the solution to (3.3.1). Since u is continuous and (3.3.5) is linear in v_j, it follows from Theorem 2.3.6 that there exists a solution $v_j \in C^{2,1}(\Omega \times (0,T]) \cap C(\overline{\Omega} \times [0,T])$ to (3.3.5). In addition, by (3.3.4) and Corollary 2.3.8, $v_j \leq u$ on $\overline{\Omega} \times [0,T]$.

Lemma 3.3.2. *For any* $(x,t) \in \overline{\Omega} \times (0,T]$, $\lim_{j\to\infty} v_j(x,t) = u(x,t)$.

Proof. Define the indicator function $\mathbb{1}_{\Gamma_1}$ as (2.4.12) and let $w_j = u - v_j$, then

$$\begin{cases} (w_j)_t(x,t) = \Delta w_j(x,t) & \text{in} \quad \Omega \times (0,T], \\ \frac{\partial w_j(x,t)}{\partial n(x)} = g_j(x,t) & \text{on} \quad (\Gamma_1 \cup \Gamma_2) \times (0,T], \\ w_j(x,0) = 0 & \text{on} \quad \Omega, \end{cases} \tag{3.3.6}$$

where $g_j(x,t) = [\mathbb{1}_{\Gamma_1}(x) - \eta_j(x)]u^q(x,t)$.

Then similar to the proof of Theorem 2.3.6, for any $(x,t) \in \overline{\Omega} \times (0,T]$,

$$w_j(x,t) = \int_0^t \int_{\partial\Omega} \Phi(x-y, t-\tau)\, \varphi_j(y,\tau)\, dS(y)\, d\tau, \tag{3.3.7}$$

where $\varphi_j \in \mathcal{B}_T$ satisfies for any $x \in \Gamma_1 \cup \Gamma_2$ and $t \in (0,T]$,

$$\varphi_j(x,t) = \int_0^t \int_{\partial\Omega} K(x,t;y,\tau)\, \varphi_j(y,\tau)\, dS(y)\, d\tau + 2g_j(x,t) \tag{3.3.8}$$

with

$$K(x,t;y,\tau) = -2\, \frac{\partial \Phi(x-y, t-\tau)}{\partial n(x)}.$$

Since K satisfies (2.3.4), we can follow the same way as the derivations of (2.3.25) and (2.3.26) to obtain

$$|K^*(x,t;y,\tau)| \le C^*(t-\tau)^{-3/4}\, |x-y|^{-(n-3/2)}. \tag{3.3.9}$$

for some constant $C^* = C^*(n, \Omega, T)$. Moreover,

$$\varphi_j(x,t) = \int_0^t \int_{\partial\Omega} K^*(x,t;y,\tau)\, H_j(y,\tau)\, dS(y)\, d\tau + 2g_j(x,t). \tag{3.3.10}$$

Due to the fact that u is bounded on $\overline{\Omega} \times [0,T]$ and the choice of $\{\eta_j\}_{j\ge 1}$, we know H_j is also bounded on $\overline{\Omega} \times [0,T]$ and

$$\lim_{j\to\infty} H_j(x,t) = 0, \quad \forall\, x \in \Gamma_1 \cup \Gamma_2,\ t \in (0,T].$$

Then it follows from (3.3.9), (3.3.10) and the Lebesgue's dominated convergence theorem that

$$\lim_{j\to\infty} \varphi_j(x,t) = 0, \quad \forall\, x \in \Gamma_1 \cup \Gamma_2, t \in (0,T].$$

In addition, the boundedness of H_j implies the boundedness of φ_j, hence by (3.3.7) and the Lebesgue's dominated convergence theorem, we obtain

$$\lim_{j\to\infty} w_j(x,t) = 0, \quad \forall\, (x,t) \in \overline{\Omega} \times (0,T].$$

□

Lemma 3.3.3. *For any $j \geq 1$ and $T \in (0,T^*)$, let v_j be the solution to (3.3.5) on $\big(\overline{\Omega}\times[0,T]\big)$. Then for any $\phi \in C\big(\overline{\Omega} \times [0,T]\big)$ and for any $0 < t_1 < t_2 \leq T$,*

$$\lim_{k\to\infty} \int_{t_1}^{t_2} \int_{\partial\Omega_{1/k}} \phi(x,t) \, \frac{\partial v_j(x,t)}{\partial n_k(x)} \, dS(x)\, dt = \int_{t_1}^{t_2} \int_{\partial\Omega} \phi(x,t) \, \frac{\partial v_j(x,t)}{\partial n(x)} \, dS(x)\, dt,$$

where $\overrightarrow{n_k}$ denotes the normal derivative with respect to $\partial\Omega_{1/k}$.

Proof. By Corollary 3.2.3 and Lemma 3.2.5, for sufficiently large k, the function $\Psi_k : \partial\Omega \to \partial\Omega_{1/k}$ defined by

$$\Psi_k(\xi) = \xi - \frac{1}{k}\, \overrightarrow{n}(\xi), \quad \forall\, \xi \in \partial\Omega,$$

is a C^1 bijection such that

$$\overrightarrow{n_k}(\Psi_k(\xi)) = \overrightarrow{n}(\xi), \quad \forall\, \xi \in \partial\Omega.$$

As a result, by the change of variable $x = \Psi_k(\xi)$ for x and denoting $dS(x) = F_k(\xi)dS(\xi)$,

$$
\begin{aligned}
&\int_{\partial\Omega_{1/k}} \phi(x,t)Dv_j(x,t)\cdot\overrightarrow{n_k}(x)\,dS(x)\\
=&\int_{\partial\Omega} \phi\big(\Psi_k(\xi),t\big)Dv_j(\Psi_k(\xi),t)\cdot\overrightarrow{n_k}\big(\Psi_k(\xi)\big)F_k(\xi)\,dS(\xi)\\
=&\int_{\partial\Omega} \phi\big(\Psi_k(\xi),t\big)Dv_j(\Psi_k(\xi),t)\cdot\overrightarrow{n}(\xi)F_k(\xi)\,dS(\xi).
\end{aligned}
$$

Integrating t from t_1 to t_2,

$$
\begin{aligned}
&\int_{t_1}^{t_2}\int_{\partial\Omega_{1/k}} \phi(x,t)Dv_j(x,t)\cdot\overrightarrow{n_k}(x)\,dS(x)\,dt\\
=&\int_{t_1}^{t_2}\int_{\partial\Omega} \phi\big(\Psi_k(\xi),t\big)Dv_j(\Psi_k(\xi),t)\cdot\overrightarrow{n}(\xi)F_k(\xi)\,dS(\xi). \qquad (3.3.11)
\end{aligned}
$$

It is readily seen that $\phi\big(\Psi_k(\xi),t\big)$ converges uniformly to $\phi(\xi,t)$ and $F_k(\xi)$ converges to 1 uniformly on $\partial\Omega\times[t_1,t_2]$ as $k\to\infty$. In addition, it follows from Lemma 3.3.3 that $Dv_j(\Psi_k(\xi),t)\cdot\overrightarrow{n}(\xi)$ converges uniformly to $Dv_j(\xi,t)\cdot\overrightarrow{n}(\xi)$. Thus the limit can be taken inside the integral to justify the conclusion. □

3.4 Upper bound on life span: case of C^2 domain

The goal of this section is to prove the unique solution u of (1.1.1) always blows up (i.e. L^∞ norm of u goes to ∞) in finite time. In addition, we will derive an upper bound for the blow-up time in terms of $|\Gamma_1|$, the nonlinearity q and the initial data u_0.

The usual way to prove the blowup of a solution is to introduce a suitable energy function and then derive a differential inequality to show the energy function blows up. This process usually involves integration by parts and therefore requires some continuity of the spatial derivative Du near the boundary. However, u is not smooth, since the normal derivative of u is not continuous along $\widetilde{\Gamma}$. Thus, some approximations are needed to get through this process.

For any k, let $\Omega_{1/k}$ be the same as in (3.2.1). For any $x \in \partial\Omega_{1/k}$, we use $\overrightarrow{n_k}(x)$ to denote the exterior unit normal vector at x with respect to $\partial\Omega_{1/k}$. For any $x \in \partial\Omega$, $\vec{n}(x)$ represents the exterior unit normal vector at x with respect to $\partial\Omega$.

Proof of Theorem 3.1.1. Let u be the maximal solution to (1.1.1) as in Theorem 2.1.1. Fix any $0 < \tau_0 < T < T^*$. For any $j \geq 1$, let v_j be the solution to (3.3.5). Then define

$$\epsilon_0 = \min_{(x,t)\in\overline{\Omega}\times[\tau_0,T]} v_1(x,t).$$

It follows from Corollary 2.3.8 that $\epsilon_0 > 0$. Noticing that $\{v_j\}_{j\geq 1}$ is an increasing sequence of functions that converges to u, so

$$\epsilon_0 \leq v_j(x,t) \leq u(x,t) \leq M(T), \quad \forall\, j \geq 1, \forall\,(x,t) \in \overline{\Omega} \times [\tau_0, T]. \tag{3.4.1}$$

Mimicking the idea in [36], for any $j \geq 1$ and $k \geq 1$, define $h_{j,k} : (0,T] \to \mathbb{R}$ and $h_j : (0,T] \to \mathbb{R}$ by

$$h_{j,k}(t) = \int_{\Omega_{1/k}} v_j^{1-q}(x,t)\, dx$$

and

$$h_j(t) = \int_{\Omega} v_j^{1-q}(x,t)\, dx.$$

Then

$$\begin{aligned}
h'_{j,k}(t) &= (1-q)\int_{\Omega_{1/k}} v_j^{-q}\,(v_j)_t\,dx \\
&= (1-q)\int_{\Omega_{1/k}} v_j^{-q}\,\Delta v_j\,dx \\
&= (1-q)\int_{\Omega_{1/k}} \nabla\cdot(v_j^{-q}Dv_j) + q\,v_j^{-q-1}|Dv_j|^2\,dx \\
&\le (1-q)\int_{\Omega_{1/k}} \nabla\cdot(v_j^{-q}Dv_j)\,dx \\
&= (1-q)\int_{\partial\Omega_{1/k}} v_j^{-q}\,\frac{\partial v_j}{\partial n_k}\,dS. \qquad (3.4.2)
\end{aligned}$$

Integrating (3.4.2) for t from τ_0 to T,

$$h_{j,k}(T) - h_{j,k}(\tau_0) \le (1-q)\int_{\tau_0}^{T}\int_{\partial\Omega_{1/k}} v_j^{-q}(x,t)\,\frac{\partial v_j(x,t)}{\partial n_k(x)}\,dS(x)\,dt.$$

Taking $k\to\infty$, by Lebesgue's dominated convergence theorem and Lemma 3.3.3,

$$\begin{aligned}
h_j(T) - h_j(\tau_0) &\le (1-q)\int_{\tau_0}^{T}\int_{\partial\Omega} v_j^{-q}(x,t)\,\frac{\partial v_j(x,t)}{\partial n(x)}\,dS(x)\,dt \\
&= (1-q)\int_{\tau_0}^{T}\int_{\partial\Omega} v_j^{-q}(x,t)\eta_j(x)u^q(x,t)\,dS(x)\,dt. \qquad (3.4.3)
\end{aligned}$$

When $j\to\infty$, it follows from Lemma 3.3.2 that v_j goes to u pointwise. Moreover, η_j converges almost everywhere to $\mathbb{1}_{\Gamma_1}$ on $\partial\Omega$. Thus by taking $j\to\infty$ in (3.4.3) and noticing the bound (3.4.1), it follows from the Lebesgue's dominated convergence theorem that

$$\int_\Omega u^{1-q}(x,T)\,dx - \int_\Omega u^{1-q}(x,\tau_0)\,dx \le (1-q)\int_{\tau_0}^{T}|\Gamma_1|\,dt = (1-q)(T-\tau_0)|\Gamma_1|.$$

So

$$
\begin{aligned}
(q-1)(T-\tau_0)|\Gamma_1| &\leq \int_\Omega u^{1-q}(x,\tau_0)\,dx - \int_\Omega u^{1-q}(x,T)\,dx \\
&\leq \int_\Omega u^{1-q}(x,\tau_0)\,dx.
\end{aligned}
$$

Namely

$$
T \leq \tau_0 + \frac{1}{(q-1)|\Gamma_1|}\int_\Omega u^{1-q}(x,\tau_0)\,dx.
$$

Noticing the right hand side of the above inequality is independent of T, so we can send T to T^*, then

$$
T^* \leq \tau_0 + \frac{1}{(q-1)|\Gamma_1|}\int_\Omega u^{1-q}(x,\tau_0)\,dx. \tag{3.4.4}
$$

Hence, T^* is finite. Then it follows from Theorem 2.4.8 and the positivity of u that

$$
\lim_{t \nearrow T^*} M(t) = \infty.
$$

Now if $\min_{x\in\overline{\Omega}} u_0(x) > 0$, then u has a strictly positive lower bound on $\overline{\Omega} \times [0, T^*)$. As a result, by taking $\tau_0 \to 0$ in (3.4.4), (3.1.1) follows. □

Chapter 4

Lower Bound Estimate of the Blow-up Time

4.1 Main theorems and outline of the approach

The goal of this section is to obtain lower bounds for the blow-up time. Again in this chapter, $M(t)$ is defined as in (1.4.3). The main results consist of three parts.

Part I: C^2 domain Ω.

Theorem 4.1.1. *Assume (1.1.2). Let T^* be the maximal existence time for (1.1.1). Then there exists a constant $C = C(n, \Omega)$ such that*

$$T^* \geq \frac{C}{q-1} \ln \Big(1 + (3M_0)^{-4(q-1)} \, |\Gamma_1|^{-\frac{2}{n-1}} \Big), \tag{4.1.1}$$

where M_0 is given by (1.4.2).

From this theorem, we can study the asymptotic behaviour of T^*.

- $\diamond$ As $q \searrow 1$, T^* is at least of order $(q-1)^{-1}$. Combining with the upper bound in Theorem 3.1.1, the order of T^* is exactly $(q-1)^{-1}$.
- $\diamond$ As $M_0 \searrow 0$, T^* is at least of order $\ln(M_0^{-1})$. As $M_0 \to \infty$, T^* is at least of order

$M_0^{-4(q-1)}$.

◇ As $|\Gamma_1| \searrow 0$, T^* is at least of order $\ln\big(|\Gamma_1|^{-1}\big)$.

Part II: C^2 domain Ω with local convexity near Γ_1.

Definition 4.1.2. *Let Ω be a bounded, open subset in $\mathbb{R}^n$. Then for any $\Gamma \subseteq \partial\Omega$ and $d > 0$, the boundary part of Ω near Γ within distance d is defined to be*

$$[\Gamma]_d \triangleq \{x \in \partial\Omega : dist(x, \Gamma) < d\}. \tag{4.1.2}$$

In the following, as a standard notation, for any set $A \subseteq \mathbb{R}^n$, Conv (A) denotes the convex hull of the set A.

Theorem 4.1.3. *Assume (1.1.2). Let T^* be the maximal existence time for (1.1.1) and M_0 be defined as in (1.4.2). Assume Conv$\big([\Gamma_1]_d\big) \subseteq \overline{\Omega}$ for some $d > 0$. Then there exist constants $Y_0 = Y_0(n, \Omega, d)$ and $C = C(n, \Omega, d)$ such that the following statements hold.*

- *Case 1: $n \geq 3$. Denote*

$$Y = M_0^{q-1}|\Gamma_1|^{1/(n-1)}.$$

  If $Y \leq \frac{Y_0}{q}$, then

$$T^* \geq \frac{C}{(q-1)Y|\ln Y|}. \tag{4.1.3}$$

- *Case 2: $n = 2$. Denote*

$$Y = M_0^{q-1}|\Gamma_1| \ln\Big(\frac{1}{|\Gamma_1|} + 1\Big).$$

  If $Y \leq \frac{Y_0}{q}$, then

$$T^* \geq \frac{C}{(q-1)Y|\ln Y|}. \tag{4.1.4}$$

It is readily seen that the asymptotic behaviour of T^* has been improved a lot. More precisely,

- $\diamond$ As $M_0 \searrow 0$, the order of T^* is at least $M_0^{-(q-1)} \big/ \ln\left(M_0^{-1}\right)$.
- $\diamond$ As $|\Gamma_1| \searrow 0$, the order of T^* is at least $|\Gamma_1|^{-\frac{1}{n-1}} \Big/ \ln\left(|\Gamma_1|^{-1}\right)$ for $n \geq 3$ and $|\Gamma_1|^{-1} \big/ \left[\ln\left(|\Gamma_1|^{-1}\right)\right]^2$ for $n = 2$.

Recalling the upper bound in Theorem 3.1.1, if u_0 does not oscillate too much, which means $\int_\Omega u_0^{1-q}\, dx$ is comparable to M_0^{1-q}, then as $M_0 \searrow 0$, the order of T^* is at most $M_0^{-(q-1)}$. So the lower bound is almost optimal on M_0. In addition, the order of T^* is at most $|\Gamma_1|^{-1}$, so when $n = 2$, the lower bound is also almost sharp.

Part III: Convex C^2 domain Ω.

Theorem 4.1.4. *Assume (1.1.2). Let Ω be convex. Then for any $\alpha \in \left[0, \frac{1}{n-1}\right)$, there exists $C = C(n, \Omega, \alpha)$ such that*

$$T^* \geq \frac{C}{(q-1)M_0^{q-1}\,|\Gamma_1|^\alpha} \left(\min\left\{1, \frac{1}{qM_0^{q-1}|\Gamma_1|^\alpha}\right\}\right)^{\frac{1+(n-1)\alpha}{1-(n-1)\alpha}}, \tag{4.1.5}$$

where T^* is the maximal existence time for (1.1.1) and M_0 is given by (1.4.2). In particular, if α is chosen to be 0 in (4.1.5), then

$$T^* \geq \frac{C_1}{(q-1)M_0^{q-1}} \min\left\{1, \frac{1}{qM_0^{q-1}}\right\}, \tag{4.1.6}$$

for some $C_1 = C_1(n, \Omega)$.

Remark 4.1.5. *As $|\Gamma_1| \searrow 0$, the lower bound (4.1.5) is not the best that we can get. By Remark 4.5.11, the order of T^* is at least $|\Gamma_1|^{-\frac{1}{n-1}}$ for $n \geq 3$ and $|\Gamma_1|^{-1} \big/ \ln\left(|\Gamma_1|^{-1}\right)$ for*

$n = 2$. *Noticing that when* $|\Gamma_1|$ *is not sufficiently small, Theorem 4.1.3 and Remark 4.5.11 are not applicable. So the advantage of (4.1.5) is its validity for any* Γ_1. *This advantage is important when comparing with the previous results for* $\Gamma_1 = \partial\Omega$.

In addition, Theorem 4.1.4 concluded better asymptotic behavior of T^* *as* $M_0 \searrow 0$ *than Theorem 4.1.3 and Remark 4.5.11.*

- ⋄ *As* $M_0 \searrow 0$, *(4.1.6) implies that* T^* *is at least of order* $M_0^{-(q-1)}$. *So by the same discussions after Theorem 4.1.3, if the initial data* u_0 *does not oscillate too much, then the order of* T^* *is exactly* $M_0^{-(q-1)}$.

- ⋄ *As* $M_0 \to \infty$, *(4.1.6) implies that* T^* *is at least of order* $M_0^{-2(q-1)}$.

To obtain a lower bound for T^*, we first exploit the common Gronwall's technique in Subsection 4.3.1. But this estimate is too rough, in order to get better results, we need to be more careful. The idea is to chop the range of $M(t)$ into suitable pieces and find a lower bound for the time spent in each piece by analysing the representation formula. Then adding all these lower bounds yields a lower bound for T^*. In the proof for Theorem 4.1.1, the pieces are just chosen to be $[3^{k-1}M_0, 3^k M_0]$ for $k \geq 1$. In the proofs of Theorem 4.1.3 and Theorem 4.1.4, due to the convexity assumptions and more delicate divisions of the range, the results can be improved significantly. More precisely, we will chop the range to be $[M_{k-1}, M_k]$ for $k \geq 1$, where the sequence $\{M_k\}_{k\geq 0}$ satisfies a nonlinear iterative relation.

The organization of this chapter is as follows. In Section 4.2, we first prove that the classical solution u is also the weak solution which implies the representation formula of u. This representation formula is the fundamental tool in the later sections. In Section 4.3, two methods are presented to obtain the lower bound of T^* for any C^2 domains Ω. The first method is simpler and uses the Gronwall's inequality. The second method performs more delicate estimate and yield Theorem 4.1.1. In Section 4.4, the lower bound is derived for any convex domains Ω and Theorem 4.1.4 is verified. We also explain the main idea in the proof which is also used in the next section. In Section 4.5, we deal with C^2 domains Ω with local convexity near Γ_1 and justify Theorem 4.1.3. We also mention a similar result for the convex domains in Remark 4.5.11. Finally, Section 4.6 compares the results in this chapter with the historical works.

4.2 Weak solution and representation formula

By Theorem 2.1.1, there exists a unique nonnegative maximal solution $u \in C^{2,1}\big(\Omega \times (0,T^*)\big) \cap C\big(\overline{\Omega} \times [0,T^*)\big)$ to (1.1.1), where T^* is the maximal existence time to (1.1.1). In this section, we will first verify that the solution u to (1.1.1) is also a weak solution (See Definition 4.2.1) and then derive representation formulas (4.2.2) and (4.2.3) for u.

4.2.1 Weak solution

Definition 4.2.1. *Let T^* be the maximal existence time for (1.1.1). Then a function $u \in C\big(\overline{\Omega} \times [0,T^*)\big)$ is called a weak solution of (1.1.1) if for any $t \in (0,T^*)$ and for any $\phi \in C^2\big(\overline{\Omega} \times [0,t]\big)$,*

$$\begin{aligned} \int_0^t \int_\Omega u(y,\tau)(\phi_\tau + \Delta\phi)(y,\tau)\, dy\, d\tau &= \int_\Omega u(y,t)\phi(y,t) - u_0(y)\phi(y,0)\, dy \\ - \int_0^t \int_{\Gamma_1} u^q(y,\tau)\phi(y,\tau)\, dS(y)\, d\tau &+ \int_0^t \int_{\partial\Omega} u(y,\tau)\, \frac{\partial \phi(y,\tau)}{\partial n(y)}\, dS(y)\, d\tau. \end{aligned} \tag{4.2.1}$$

In order to prove u satisfies (4.2.1), we will again take advantage of v_j which is the solution to (3.3.5).

Theorem 4.2.2. *The maximal solution u to (1.1.1) is also a weak solution as in Definition 4.2.1.*

Proof. Fix any $t \in (0, T^*)$. Let $\{v_j\}_{j\geq 1}$ be the solution to (3.3.5). For any $k \geq 1$, define $\Omega_{1/k}$ as in (3.2.1).

Then for any $0 < \tau_0 < t,\ k \geq 1,\ j \geq 1$ and $\phi \in C^2\big(\overline{\Omega} \times [0,t]\big)$, we have

$$\int_{\tau_0}^{t} \int_{\Omega_{1/k}} (v_j)_t(y,\tau)\, \phi(y,\tau)\, dy\, d\tau = \int_{\tau_0}^{t} \int_{\Omega_{1/k}} \Delta v_j(y,\tau)\, \phi(y,\tau)\, dy\, d\tau.$$

Integrating by parts and arranging the terms,

$$\begin{aligned} &\int_{\tau_0}^{t} \int_{\Omega_{1/k}} v_j(y,\tau)\, (\phi_\tau + \Delta\phi)(y,\tau)\, dy\, d\tau \\ = &\int_{\Omega_{1/k}} v_j(y,t)\phi(y,t) - v_j(y,\tau_0)\phi(y,\tau_0)\, dy - \int_{\tau_0}^{t} \int_{\partial\Omega_{1/k}} \frac{\partial v_j(y,\tau)}{\partial n_k(y)}\, \phi(y,\tau)\, dS(y)\, d\tau \\ &+ \int_{\tau_0}^{t} \int_{\partial\Omega_{1/k}} v_j(y,\tau)\, \frac{\partial \phi(y,\tau)}{\partial n_k(y)} dS(y)\, d\tau, \end{aligned}$$

where $\overrightarrow{n_k}$ denotes the exterior unit normal vector with respect to $\partial\Omega_{1/k}$.

Sending $k \to \infty$, by Lebesgue's dominated convergence theorem and Lemma 3.3.3, we obtain

$$
\begin{aligned}
&\int_{\tau_0}^{t}\int_{\Omega} v_j(y,\tau)\,(\phi_\tau+\Delta\phi)(y,\tau)\,dy\,d\tau\\
&=\int_{\Omega} v_j(y,t)\phi(y,t)-v_j(y,\tau_0)\phi(y,\tau_0)\,dy-\int_{\tau_0}^{t}\int_{\partial\Omega}\frac{\partial v_j(y,\tau)}{\partial n(y)}\,\phi(y,\tau)\,dS(y)\,d\tau\\
&\quad+\int_{\tau_0}^{t}\int_{\partial\Omega} v_j(y,\tau)\,\frac{\partial\phi(y,\tau)}{\partial n(y)}dS(y)\,d\tau\\
&=\int_{\Omega} v_j(y,t)\phi(y,t)-v_j(y,\tau_0)\phi(y,\tau_0)\,dy-\int_{\tau_0}^{t}\int_{\partial\Omega}\eta_j(y)u^q(y,t)\,\phi(y,\tau)\,dS(y)\,d\tau\\
&\quad+\int_{\tau_0}^{t}\int_{\partial\Omega} v_j(y,\tau)\,\frac{\partial\phi(y,\tau)}{\partial n(y)}dS(y)\,d\tau.
\end{aligned}
$$

Taking $j\to\infty$, then it follows from Lemma 3.3.2 and the Lebesgue's dominated convergence theorem that

$$
\begin{aligned}
&\int_{\tau_0}^{t}\int_{\Omega} u(y,\tau)\,(\phi_\tau+\Delta\phi)(y,\tau)\,dy\,d\tau\\
&=\int_{\Omega} u(y,t)\phi(y,t)-u(y,\tau_0)\phi(y,\tau_0)\,dy-\int_{\tau_0}^{t}\int_{\Gamma_1} u^q(y,\tau)\phi(y,\tau)\,dS(y)\,d\tau\\
&\quad+\int_{\tau_0}^{t}\int_{\partial\Omega} u(y,\tau)\,\frac{\partial\phi(y,\tau)}{\partial n(y)}\,dS(y)\,d\tau.
\end{aligned}
$$

Finally by sending $\tau_0\to 0$, (4.2.1) holds. □

4.2.2 Representation formula

Next by (4.2.1) and some standard steps, we are able to attain the representation formula of u for inside points. Note that this representation formula is different from (2.3.15) which is used in the proof of the existence of the solution.

Lemma 4.2.3. *For the maximal solution u to (1.1.1), it has the representation formula for* *the inside points* $(x,t) \in \Omega \times [0, T^*)$,

$$
\begin{aligned}
u(x,t) = &\int_\Omega \Phi(x-y,t)\, u_0(y)\, dy + \int_0^t \int_{\Gamma_1} \Phi(x-y,t-\tau)\, u^q(y,\tau)\, dS(y)\, d\tau \\
&- \int_0^t \int_{\partial\Omega} \frac{\partial \Phi(x-y,t-\tau)}{\partial n(y)}\, u(y,\tau)\, dS(y)\, d\tau.
\end{aligned} \tag{4.2.2}
$$

Proof. Fix $x \in \Omega$ and $t \in (0, T^*)$. Define $\phi : \overline{\Omega} \times [0,t) \to \mathbb{R}$ by

$$
\phi(y,\tau) = \Phi(x-y,t-\tau) = \frac{1}{(4\pi)^{n/2}} \frac{1}{(t-\tau)^{n/2}}\, e^{-\frac{|x-y|^2}{4(t-\tau)}}.
$$

For any $\epsilon > 0$, define $\phi^\epsilon : \overline{\Omega} \times [0,t] \to \mathbb{R}$ by

$$
\phi^\epsilon(y,\tau) = \Phi(x-y,t+\epsilon-\tau) = \frac{1}{(4\pi)^{n/2}} \frac{1}{(t+\epsilon-\tau)^{n/2}}\, e^{-\frac{|x-y|^2}{4(t+\epsilon-\tau)}}.
$$

From these, one can see that ϕ^ϵ is smooth in its domain and

$$
(\phi^\epsilon)_\tau(y,\tau) + \Delta_y(\phi^\epsilon)(y,\tau) = 0, \quad \forall\, (y,\tau) \in \overline{\Omega} \times [0,t].
$$

Applying (4.2.1) for $\phi = \phi^\epsilon$,

$$
\begin{aligned}
\int_\Omega \phi^\epsilon(y,t) u(y,t)\, dy \;\; = \;\; &\int_\Omega \phi^\epsilon(y,0) u_0(y)\, dy + \int_0^t \int_{\Gamma_1} \phi^\epsilon(y,\tau)\, u^q(y,\tau)\, dS(y)\, d\tau \\
&- \int_0^t \int_{\partial\Omega} \frac{\partial \phi^\epsilon(y,\tau)}{\partial n(y)}\, u(y,\tau)\, dS(y)\, d\tau.
\end{aligned}
$$

Sending $\epsilon \to 0$, it follows from the Lebesgue's dominated convergence theorem that

$$\begin{aligned}
u(x,t) &= \int_{\Omega} \phi(y,0)\, u_0(y)\, dy + \int_0^t \int_{\Gamma_1} \phi(y,\tau)\, u^q(y,\tau)\, dS(y)\, d\tau \\
&\quad - \int_0^t \int_{\partial\Omega} \frac{\partial \phi(y,\tau)}{\partial n(y)}\, u(y,\tau)\, dS(y)\, d\tau \\
&= \int_{\Omega} \Phi(x-y,t)\, u_0(y)\, dy + \int_0^t \int_{\Gamma_1} \Phi(x-y,t-\tau)\, u^q(y,\tau)\, dS(y)\, d\tau \\
&\quad - \int_0^t \int_{\partial\Omega} \frac{\partial \Phi(x-y,t-\tau)}{\partial n(y)}\, u(y,\tau)\, dS(y)\, d\tau.
\end{aligned}$$

The last equality is because $\phi(y,\tau) = \Phi(x-y,t-\tau)$. Now we have proved (4.2.2) for $(x,t) \in \Omega \times (0,T^*)$, next it is obvious to see that (4.2.2) holds for any point $(x,t) \in \Omega \times \{0\}$, thus the Theorem follows. □

Lemma 4.2.3 only gives the formula for the inside points, but we still need the formula for the boundary points $(x,t) \in \partial\Omega \times [0,T^*)$. In order to get that, we combine Lemma 4.2.3 and Theorem 2.2.2.

Corollary 4.2.4. *For the maximal solution u to (1.1.1), it has the representation formula for the boundary points $(x,t) \in \partial\Omega \times [0,T^*)$,*

$$\begin{aligned}
u(x,t) =&2 \int_{\Omega} \Phi(x-y,t)\, u_0(y)\, dy + 2\int_0^t \int_{\Gamma_1} \Phi(x-y,t-\tau)\, u^q(y,\tau)\, dS(y)\, d\tau \\
&- 2\int_0^t \int_{\partial\Omega} \frac{\partial \Phi(x-y,t-\tau)}{\partial n(y)}\, u(y,\tau)\, dS(y)\, d\tau.
\end{aligned} \tag{4.2.3}$$

Proof. For any $h > 0$, we write $x_h = x - h\overrightarrow{n}(x)$ for $x \in \partial\Omega$. As shown in the proof of Theorem 2.2.2, when h is sufficiently small, $x_h \in \Omega$ for any $x \in \partial\Omega$. Consequently we can apply Lemma 4.2.3 to conclude that

$$\begin{aligned}u(x_h,t) = &\int_\Omega \Phi(x_h-y,t)\,u_0(y)\,dy + \int_0^t\int_{\Gamma_1}\Phi(x_h-y,t-\tau)\,u^q(y,\tau)\,dS(y)\,d\tau\\ &-\int_0^t\int_{\partial\Omega}\frac{\partial\Phi(x_h-y,t-\tau)}{\partial n(y)}\,u(y,\tau)\,dS(y)\,d\tau.\end{aligned}$$

Taking $h\to 0^+$, then it follows from Theorem 2.2.2 that

$$\begin{aligned}u(x,t) = &\int_\Omega \Phi(x-y,t)\,u_0(y)\,dy + \int_0^t\int_{\Gamma_1}\Phi(x-y,t-\tau)\,u^q(y,\tau)\,dS(y)\,d\tau\\ &-\int_0^t\int_{\partial\Omega}\frac{\partial\Phi(x-y,t-\tau)}{\partial n(y)}\,u(y,\tau)\,dS(y)\,d\tau+\frac{1}{2}\,u(x,t),\end{aligned}$$

which implies (4.2.3). Now we have proved (4.2.3) for $(x,t)\in\partial\Omega\times(0,T^*)$, next it is obvious to see that (4.2.3) holds for any point $(x,t)\in\partial\Omega\times\{0\}$, hence the Theorem follows. $\square$

4.2.3 Time-shifted representation formula

In Corollary 4.2.4, it derived the representation formula (4.2.3), where the initial time is 0 and the initial data is $u(\cdot,0)=u_0(\cdot)\in C^1(\overline{\Omega})$. Now for any $T\in(0,T^*)$, we are asking that if regarding T to be the initial time and $u(\cdot,T)$ to be the initial data, then is there a representation formula similar to (4.2.3)? It seems trivial, but we should be careful, since u_0 is in $C^1(\overline{\Omega})$ but $u(\cdot,T)$ does not. The next lemma claims that as long as u is the solution to (1.1.1) with the assumption (1.1.2), then for any $T\in[0,T^*)$, there also holds a representation formula which is similar to (4.2.3) but with "initial data $u(\cdot,T)$".

Lemma 4.2.5. *Assume (1.1.2). Let T^* be the maximal existence time and u be the maximal* *solution to (1.1.1). Then for any $x \in \partial\Omega$, $T \in [0, T^*)$ and $t \in [0, T^* - T)$,*

$$u(x, T+t) = 2\int_{\Omega} \Phi(x-y, t)\, u(y, T)\, dy + 2\int_0^t \int_{\Gamma_1} \Phi(x-y, t-\tau)\, u^q(y, T+\tau)\, dS(y)\, d\tau$$
$$- 2\int_0^t \int_{\partial\Omega} \frac{\partial\Phi(x-y, t-\tau)}{\partial n(y)}\, u(y, T+\tau)\, dS(y)\, d\tau \qquad (4.2.4)$$

Proof. When $T = 0$, (4.2.4) is just the representation formula (4.2.3) which has been proven. Next let $T > 0$. We intend to verify (4.2.4) which can be regarded as a representation formula with initial time T and initial data $u(\cdot, T)$.

Define $v : \overline{\Omega} \times [0, T^* - T) \to \mathbb{R}$ by $v(x, t) = u(x, T+t)$. Then $v \in C^{2,1}\big(\Omega \times (0, T^* - T)\big) \bigcap C\big(\overline{\Omega} \times [0, T^* - T)\big)$ and satisfies

$$\begin{cases} v_t(x,t) = \Delta v(x,t) & \text{in} \quad \Omega \times (0, T^* - T), \\ \frac{\partial v(x,t)}{\partial n(x)} = u^q(x, T+t) & \text{on} \quad \Gamma_1 \times (0, T^* - T), \\ \frac{\partial v(x,t)}{\partial n(x)} = 0 & \text{on} \quad \Gamma_2 \times (0, T^* - T), \\ v(x, 0) = u(x, T) & \text{in} \quad \Omega. \end{cases} \qquad (4.2.5)$$

Note that (4.2.5) is a linear problem in v, since u is a fixed function.

We continuously extend $u(\cdot, T)$ to $\mathbb{R}^n$ and still denote it to be $u(\cdot, T)$. Then for any $j \geq 1$, we choose the standard mollifier $\{\eta_\epsilon\}_{\epsilon > 0}$ and define

$$g_j(x) = \big(\eta_{\epsilon_j}(\cdot) * u(\cdot, T)\big)(x),$$

where ϵ_j is chosen to be so small that

$$\max_{x\in\overline{\Omega}} |g_j(x) - u(x,T)| \le 1/j. \tag{4.2.6}$$

Since $g_j \in C^1(\overline{\Omega})$, it follows from Theorem 2.3.10 that there exists $v_j \in C^{2,1}\big(\Omega \times (0, T^* - T)\big) \bigcap C\big(\overline{\Omega} \times [0, T^* - T)\big)$ such that

$$\begin{cases} (v_j)_t(x,t) = \Delta v_j(x,t) & \text{in} \quad \Omega \times (0, T^* - T), \\ \frac{\partial v_j(x,t)}{\partial n(x)} = u^q(x, T+t) & \text{on} \quad \Gamma_1 \times (0, T^* - T), \\ \frac{\partial v_j(x,t)}{\partial n(x)} = 0 & \text{on} \quad \Gamma_2 \times (0, T^* - T), \\ v_j(x,0) = g_j(x) & \text{in} \quad \Omega. \end{cases} \tag{4.2.7}$$

In addition, by an analogous argument as that for Subsections 4.2.1 and 4.2.2, there exists a representation formula for (4.2.7): for any $(x,t) \in \partial\Omega \times [0, T^* - T)$,

$$\begin{aligned} v_j(x,t) = {} & 2\int_\Omega \Phi(x-y,t)\, g_j(y)\, dy + 2\int_0^t \int_{\Gamma_1} \Phi(x-y,t-\tau)\, u^q(y, T+\tau)\, dS(y)\, d\tau \\ & - 2\int_0^t \int_{\partial\Omega} \frac{\partial \Phi(x-y,t-\tau)}{\partial n(y)}\, v_j(y,\tau)\, dS(y)\, d\tau. \end{aligned} \tag{4.2.8}$$

Let $w_j = v_j - v$, then $w_j \in C^{2,1}\big(\Omega \times (0, T^* - T)\big) \bigcap C\big(\overline{\Omega} \times [0, T^* - T)\big)$ and satisfies

$$\begin{cases} (w_j)_t(x,t) = \Delta w_j(x,t) & \text{in} \quad \Omega \times (0, T^* - T), \\ \frac{\partial w_j}{\partial n}(x,t) = 0 & \text{on} \quad \Gamma_1 \times (0, T^* - T), \\ \frac{\partial w_j}{\partial n}(x,t) = 0 & \text{on} \quad \Gamma_2 \times (0, T^* - T), \\ w_j(x,0) = g_j(x) - u(x,T) & \text{in} \quad \Omega. \end{cases}$$

So it follows from the maximum principle and the Hopf lemma that for any $(x,t) \in \overline{\Omega} \times [0, T^* - T)$,

$$|w_j(x,t)| \le \max_{x \in \overline{\Omega}} |g_j(x) - u(x,T)| \le 1/j.$$

Thus

$$|v_j(x,t) - v(x,t)| \le 1/j, \quad \forall\, (x,t) \in \overline{\Omega} \times [0, T^* - T).$$

Now fixing any point $(x,t) \in \partial\Omega \times [0, T^* - T)$ and let $j \to \infty$ in (4.2.8), then it follows from Lebesgue's dominated convergence theorem that

$$\begin{aligned} v(x,t) = {} & 2 \int_\Omega \Phi(x-y,t)\, u(y,T)\, dy + 2 \int_0^t \int_{\Gamma_1} \Phi(x-y,t-\tau)\, u^q(y, T+\tau)\, dS(y)\, d\tau \\ & - 2 \int_0^t \int_{\partial\Omega} \frac{\partial \Phi(x-y,t-\tau)}{\partial n(y)}\, v(y,\tau)\, dS(y)\, d\tau. \end{aligned}$$

Finally noticing that $v(x,t) = u(x, T+t)$ and $v(y,\tau) = u(y, T+\tau)$, we obtain (4.2.4). $\square$

4.3 Lower bound on life span: case of C^2 domain

4.3.1 A traditional way by Gronwall-type technique

Lemma 4.3.1. *Let Ω be a bounded open subset in $\mathbb{R}^n$ with $\partial\Omega \in C^2$. Then there exists a constant $C = C(\Omega, n)$ such that for any $x \in \partial\Omega$ and $t > 0$,*

$$\frac{1}{t^{(n-1)/2}} \int_{\partial\Omega} e^{-\frac{|x-y|^2}{4t}}\, dS(y) \le C.$$

Proof. It is easy to prove this conclusion by simply using the definition of a C^2 boundary and the dimension of $\partial\Omega$ is $n-1$, we omit the proof here. $\square$

Theorem 4.3.2. *Assume (1.1.2). Let T^* be the maximal existence time for (1.1.1). Then there exists a constant $C = C(n, q, \Omega)$ such that*

$$T^* \geq C^{-\frac{2}{n+2}} \left[\ln\left(|\Gamma_1|^{-1} \right) - (n+2)(q-1)\ln M_0 - \ln(q-1) - \ln C \right]^{\frac{2}{n+2}}, \tag{4.3.1}$$

where M_0 is defined as in (1.4.2) and C remains bounded as $q \to 1$. As a result, no matter $|\Gamma_1| \searrow 0$, $M_0 \searrow 0$ or $q \searrow 1$, we have $T^* \to \infty$.

Proof. In the following, C will be used to denote a positive constant which only depends on n, Ω, q and is bounded when $q \searrow 1$. Moreover, C may be different from line to line. We prove by analyzing the representation formula (4.2.3) for u on the boundary points $(x,t) \in \partial\Omega \times [0, T^*)$:

$$\begin{aligned} u(x,t) &= 2\int_\Omega \Phi(x-y,t)\, u_0(y)\, dy + 2\int_0^t \int_{\Gamma_1} \Phi(x-y,t-\tau)\, u^q(y,\tau)\, dS(y)\, d\tau \\ &\quad -2\int_0^t \int_{\partial\Omega} \frac{\partial \Phi(x-y,t-\tau)}{\partial n(y)}\, u(y,\tau)\, dS(y)\, d\tau \\ &= I + II + III. \end{aligned} \tag{4.3.2}$$

Define $\widetilde{M}$, $\overline{M} : [0, T^*) \to \mathbb{R}$ by

$$\widetilde{M}(t) = \max_{y \in \partial\Omega} u(y,t) \tag{4.3.3}$$

and

$$\overline{M}(t) = \max_{\tau \in [0,t]} \widetilde{M}(\tau).$$

It is clear that $\overline{M}$ is increasing and also blows up at T^*. It is also easy to see that

$$I \leq 2M_0, \tag{4.3.4}$$

$$\begin{aligned} II &\leq C \int_0^t \widetilde{M}^q(\tau) \int_{\Gamma_1} (t-\tau)^{-n/2}\, e^{-\frac{|x-y|^2}{4(t-\tau)}}\, dS(y)\, d\tau \\ &= C \int_0^t \widetilde{M}^q(t-\tau) \int_{\Gamma_1} \tau^{-n/2}\, e^{-\frac{|x-y|^2}{4\tau}}\, dS(y)\, d\tau \end{aligned} \tag{4.3.5}$$

and

$$\begin{aligned} III &\leq C \int_0^t \widetilde{M}(\tau) \int_{\partial\Omega} \frac{|x-y|^2}{(t-\tau)^{n/2+1}}\, e^{-\frac{|x-y|^2}{4(t-\tau)}}\, dS(y)\, d\tau \\ &= C \int_0^t \widetilde{M}(t-\tau) \int_{\partial\Omega} \frac{|x-y|^2}{\tau^{n/2+1}}\, e^{-\frac{|x-y|^2}{4\tau}}\, dS(y)\, d\tau \\ &\leq C \int_0^t \widetilde{M}(t-\tau) \int_{\partial\Omega} \frac{1}{\tau^{n/2}}\, e^{-\frac{|x-y|^2}{8\tau}}\, dS(y)\, d\tau \\ &\leq C \int_0^t \widetilde{M}(t-\tau)\, \tau^{-1/2}\, d\tau. \end{aligned} \tag{4.3.6}$$

In (4.3.6), the first inequality is due to Lemma 2.3.2, the last inequality is due to Lemma 4.3.1 and the second inequality is because

$$\frac{|x-y|^2}{\tau} e^{-\frac{|x-y|^2}{8\tau}} \leq \sup_{r \geq 0} r\, e^{-r/8} \leq C.$$

Now we are trying to estimate (4.3.5) and (4.3.6) further. Taking $m = 1 + \frac{1}{n+1}$, it follows from Holder's inequality that

$$\begin{aligned}
\int_{\Gamma_1} \tau^{-n/2} e^{-\frac{|x-y|^2}{4\tau}}\, dS(y) &\leq \tau^{-n/2} \Big(\int_{\Gamma_1} e^{-\frac{m|x-y|^2}{4\tau}}\, dS(y) \Big)^{1/m} |\Gamma_1|^{(m-1)/m} \\
&= \tau^{-\frac{n}{2}+\frac{n-1}{2m}} \Big(\int_{\Gamma_1} \tau^{-\frac{n-1}{2}} e^{-\frac{m|x-y|^2}{4\tau}}\, dS(y) \Big)^{1/m} |\Gamma_1|^{(m-1)/m} \\
&\leq C\, \tau^{-\frac{n}{2}+\frac{n-1}{2m}}\, |\Gamma_1|^{\frac{m-1}{m}} \\
&= C\, \tau^{-\frac{2n+1}{2n+4}} |\Gamma_1|^{\frac{1}{n+2}},
\end{aligned}$$

where the second inequality is because of Lemma 4.3.1. Thus (4.3.5) leads to

$$II \leq C\, |\Gamma_1|^{\frac{1}{n+2}} \int_0^t \widetilde{M}^q(t-\tau)\, \tau^{-\frac{2n+1}{2n+4}}\, d\tau. \tag{4.3.7}$$

By Holder's inequality again,

$$\begin{aligned}
\int_0^t \widetilde{M}^q(t-\tau)\, \tau^{-\frac{2n+1}{2n+4}}\, d\tau &\leq \Big(\int_0^t \widetilde{M}^{\frac{qm}{m-1}}(t-\tau)\, d\tau \Big)^{\frac{m-1}{m}} \Big(\int_0^t \tau^{-\frac{(2n+1)m}{2n+4}}\, d\tau \Big)^{\frac{1}{m}} \\
&= \Big(\int_0^t \widetilde{M}^{q(n+2)}(\tau)\, d\tau \Big)^{\frac{1}{n+2}} \Big(\int_0^t \tau^{-\frac{2n+1}{2n+2}}\, d\tau \Big)^{\frac{n+1}{n+2}} \\
&= C\, t^{\frac{1}{2n+4}} \Big(\int_0^t \widetilde{M}^{q(n+2)}(\tau)\, d\tau \Big)^{\frac{1}{n+2}}.
\end{aligned}$$

Based on this, (4.3.7) becomes

$$II \leq C\, |\Gamma_1|^{\frac{1}{n+2}}\, t^{\frac{1}{2n+4}} \Big(\int_0^t \widetilde{M}^{q(n+2)}(\tau)\, d\tau \Big)^{\frac{1}{n+2}}. \tag{4.3.8}$$

To estimate III, it follows from Holder's inequality that

$$
\begin{aligned}
III &\le C \left(\int_0^t \widetilde{M}^{n+2}(\tau)\, d\tau \right)^{\frac{1}{n+2}} \left(\int_0^t \tau^{-\frac{1}{2}\frac{n+2}{n+1}}\, d\tau \right)^{\frac{n+1}{n+2}} \\
&= C\, t^{\frac{n}{2n+4}} \left(\int_0^t \widetilde{M}^{n+2}(\tau)\, d\tau \right)^{\frac{1}{n+2}}. \qquad (4.3.9)
\end{aligned}
$$

Combining (4.3.2), (4.3.4), (4.3.8), (4.3.9), we obtain

$$
\begin{aligned}
u(x,t) \le & C \Bigg[M_0 + |\Gamma_1|^{\frac{1}{n+2}}\, t^{\frac{1}{2n+4}} \left(\int_0^t \widetilde{M}^{q(n+2)}(\tau)\, d\tau \right)^{\frac{1}{n+2}} \\
& + t^{\frac{n}{2n+4}} \left(\int_0^t \widetilde{M}^{n+2}(\tau)\, d\tau \right)^{\frac{1}{n+2}} \Bigg].
\end{aligned}
$$

Since x is arbitrary on $\partial\Omega$, by raising both sides to the power $n+2$,

$$
\widetilde{M}^{n+2}(t) \le C \left[M_0^{n+2} + |\Gamma_1|\, t^{1/2} \int_0^t \widetilde{M}^{q(n+2)}(\tau)\, d\tau + t^{n/2} \int_0^t \widetilde{M}^{n+2}(\tau)\, d\tau \right].
$$

Thus, due to the definition of $\overline{M}$,

$$
\begin{aligned}
\widetilde{M}^{n+2}(t) &\le C \left[M_0^{n+2} + |\Gamma_1|\, t^{1/2} \int_0^t \overline{M}^{q(n+2)}(\tau)\, d\tau + t^{n/2} \int_0^t \overline{M}^{n+2}(\tau)\, d\tau \right] \\
&\le C\big(1 + t^{n/2}\big) \left[M_0^{n+2} + |\Gamma_1| \int_0^t \overline{M}^{q(n+2)}(\tau)\, d\tau + \int_0^t \overline{M}^{n+2}(\tau)\, d\tau \right].
\end{aligned}
$$

As a consequence,

$$
\overline{M}^{n+2}(t) \le C\big(1 + t^{n/2}\big) \left[M_0^{n+2} + |\Gamma_1| \int_0^t \overline{M}^{q(n+2)}(\tau)\, d\tau + \int_0^t \overline{M}^{n+2}(\tau)\, d\tau \right]. \qquad (4.3.10)
$$

We define

$$E(t) = M_0^{n+2} + |\Gamma_1| \int_0^t \overline{M}^{q(n+2)}(\tau)\, d\tau + \int_0^t \overline{M}^{n+2}(\tau)\, d\tau, \tag{4.3.11}$$

then $E(0) = M_0^{n+2}$ and $E(t)$ is increasing. Now (4.3.10) becomes

$$\overline{M}^{n+2}(t) \le C\big(1 + t^{n/2}\big)E(t)$$

and consequently

$$\begin{aligned} E'(t) &= |\Gamma_1|\, \overline{M}^{q(n+2)}(t) + \overline{M}^{(n+2)}(t) \\ &\le C\,|\Gamma_1|\, \big(1 + t^{n/2}\big)^q E^q(t) + C\big(1 + t^{n/2}\big)E(t). \end{aligned} \tag{4.3.12}$$

Moreover, $E(t)$ also blows up at T^*, since $\overline{M}$ is increasing. (4.3.12) looks like the Bernoulli equation, so we multiply both sides by $E^{-q}(t)$ and define $\Psi(t) \triangleq E^{1-q}(t)$. Then $\Psi(t) \to 0$ as t approaches to T^* and

$$\Psi'(t) + C(q-1)\big(1 + t^{n/2}\big)\Psi(t) \ge -C(q-1)|\Gamma_1|\, \big(1 + t^{n/2}\big)^q. \tag{4.3.13}$$

We introduce the integration factor $\mu(t)$ which is defined as

$$\mu(t) \triangleq \exp\left[C \int_0^t (q-1)\big(1 + \tau^{n/2}\big)\, d\tau\right], \quad \forall\, t \ge 0.$$

It is readily seen that

$$\mu(t) \le C\, \exp\big(C\, t^{1+\frac{n}{2}}\big). \tag{4.3.14}$$

Multiplying (4.3.13) by $\mu(t)$, one gets

$$\big(\mu(t)\Psi(t)\big)' \geq -C(q-1)|\Gamma_1|\left(1+t^{n/2}\right)^q \mu(t).$$

Integrating this inequality and noticing that $\mu(0)\Psi(0) = M_0^{-(n+2)(q-1)}$, one obtains

$$\mu(t)\Psi(t) \geq M_0^{-(n+2)(q-1)} - C(q-1)|\Gamma_1| \int_0^t \left(1+\tau^{n/2}\right)^q \mu(\tau)\, d\tau. \tag{4.3.15}$$

It follows from (4.3.14) that

$$\begin{aligned}
\int_0^t \left(1+\tau^{n/2}\right)^q \mu(\tau)\, d\tau &\leq C \int_0^t \left(1+\tau^{n/2}\right)^q \exp\left(C\,\tau^{1+\frac{n}{2}}\right) d\tau \\
&\leq C\big(1+t^{n/2}\big)^q\, t\, \exp\left(C\,t^{1+\frac{n}{2}}\right) \\
&\leq C \exp\left[(C+1)\,t^{1+\frac{n}{2}}\right].
\end{aligned}$$

Plugging in (4.3.15), we obtain

$$\mu(t)\Psi(t) \geq M_0^{-(n+2)(q-1)} - C(q-1)|\Gamma_1| \exp\left(C\,t^{1+\frac{n}{2}}\right).$$

Taking $t \to T^*$, one obtains

$$\begin{aligned}
C(q-1)|\Gamma_1| \exp\left[C\,(T^*)^{1+\frac{n}{2}}\right] &\geq M_0^{-(n+2)(q-1)} \\
\exp\left[C\,(T^*)^{1+\frac{n}{2}}\right] &\geq C^{-1}(q-1)^{-1}|\Gamma_1|^{-1} M_0^{-(n+2)(q-1)} \\
C\,(T^*)^{\frac{n+2}{2}} &\geq \ln\left(|\Gamma_1|^{-1}\right) - (n+2)(q-1)\ln M_0 - \ln(q-1) - \ln C.
\end{aligned}$$

Hence,

$$T^* \geq C^{-\frac{2}{n+2}} \left[\ln \left(|\Gamma_1|^{-1} \right) - (n+2)(q-1) \ln M_0 - \ln(q-1) - \ln C \right]^{\frac{2}{n+2}}.$$

$\square$

4.3.2 Better estimate by a new method

The aim of this subsection is the same as the last subsection, which is to find a lower bound of T^*. But this subsection will provide a different method which leads to a better lower bound as in (4.1.1).

Comparing (4.1.1) with (4.3.1), for convenience of statement, we use T_1 and T_2 to represent the lower bounds in (4.1.1) and (4.3.1) respectively. The advantage of (4.1.1) is in the following aspects.

- T_1 is always positive, but T_2 will be negative unless $|\Gamma_1|$ is sufficiently small, M_0 is sufficiently small or q is sufficiently close to 1.
- As $q \searrow 1$, the order of T_1 is $\frac{1}{q-1}$, while the order of T_2 is only $\left[\ln\left(\frac{1}{q-1}\right)\right]^{\frac{2}{n+2}}$.
- As $|\Gamma_1| \searrow 0$, the order of T_1 is $\ln\left(\frac{1}{|\Gamma_1|}\right)$, while the order of T_2 is only $\left[\ln\left(\frac{1}{|\Gamma_1|}\right)\right]^{\frac{2}{n+2}}$.
- As $M_0 \searrow 0$, the order of T_1 is $\ln\left(\frac{1}{M_0}\right)$, while the order of T_2 is only $\left[\ln\left(\frac{1}{M_0}\right)\right]^{\frac{2}{n+2}}$.

The problem of the method in Subsection 4.3.1 is that the estimate for the integral terms lose a lot when t is large. For example, in (4.3.6), the term

$$\int_{\partial\Omega} \frac{1}{\tau^{n/2}} e^{-\frac{|x-y|^2}{8\tau}} \, dS(y)$$

is bounded by $C\,\tau^{-1/2}$. When τ is small, this estimate is okay. But when τ is large, the term $\int_{\partial\Omega} \frac{1}{\tau^{n/2}}\, e^{-\frac{|x-y|^2}{8\tau}}\, dS(y)$ obviously decays like $\tau^{-n/2}$, so the bound $\tau^{-1/2}$ is too rough. As a consequence, the main change in this subsection is that we divide the range of $M(t)$ into small pieces and analyse each piece separately. Once we found the lower bound of the time spent in each piece, adding them together yields a lower bound for T^*.

In the rest of this thesis, we define

$$B_1 \triangleq \sup_{\tau>0}\, \sup_{x\in\partial\Omega}\, \tau^{-\frac{n-1}{2}} \int_{\partial\Omega} e^{-|x-y|^2/(4\tau)}\, dS(y). \tag{4.3.16}$$

It is shown in Lemma 4.3.1 that B_1 is a finite positive constant depending only on Ω and n. In addition, for convenience of notation, for any $\alpha \in \big[0, \frac{1}{n-1}\big)$, let

$$n_\alpha \triangleq \frac{1-(n-1)\alpha}{2}. \tag{4.3.17}$$

It is readily seen that $0 < n_\alpha \le \frac{1}{2}$.

Lemma 4.3.3. *Let Ω and Γ_1 be the same as in (1.1.1). Then there exists $C = C(n,\Omega)$ such that for any $\alpha \in \big[0, \frac{1}{n-1}\big)$, $x \in \partial\Omega$ and $t > 0$,*

$$\int_0^t \int_{\Gamma_1} \Phi(x-y, t-\tau)\, dS(y)\, d\tau \le \frac{C}{1-(n-1)\alpha}\, |\Gamma_1|^\alpha\, t^{n_\alpha}. \tag{4.3.18}$$

In particular, if $\alpha = 0$, then

$$\int_0^t \int_{\Gamma_1} \Phi(x-y, t-\tau)\, dS(y)\, d\tau \le C\,\sqrt{t}.$$

Proof. Let $x \in \partial\Omega$, $t > 0$ and $\alpha \in \left[0, \frac{1}{n-1}\right)$. We denote

$$LHS = \int_0^t \int_{\Gamma_1} \Phi(x - y, t - \tau)\, dS(y)\, d\tau.$$

By a change of variable in τ,

$$\begin{aligned} LHS &= \int_0^t \int_{\Gamma_1} \Phi(x - y, \tau)\, dS(y)\, d\tau \\ &= \frac{1}{(4\pi)^{n/2}} \int_0^t \tau^{-n/2} \int_{\Gamma_1} e^{-|x-y|^2/(4\tau)}\, dS(y)\, d\tau. \end{aligned} \tag{4.3.19}$$

For any $m \geq 1$, applying Holder's inequality,

$$\int_{\Gamma_1} e^{-|x-y|^2/(4\tau)}\, dS(y) \leq \left(\int_{\Gamma_1} e^{-m\,|x-y|^2/(4\tau)} \right)^{1/m} |\Gamma_1|^{(m-1)/m}. \tag{4.3.20}$$

Recalling the definition of B_1 in (4.3.16),

$$\begin{aligned} \int_{\Gamma_1} e^{-m\,|x-y|^2/(4\tau)}\, d\tau &= \int_{\Gamma_1} e^{-\,|x-y|^2/[4(\tau/m)]}\, d\tau \\ &\leq \left(\frac{\tau}{m}\right)^{(n-1)/2} B_1 \\ &\leq \tau^{(n-1)/2}\, B_1. \end{aligned}$$

Combining this inequality with (4.3.20),

$$\begin{aligned} \int_{\Gamma_1} e^{-|x-y|^2/(4\tau)}\, dS(y) &\leq B_1^{1/m}\, \tau^{(n-1)/(2m)}\, |\Gamma_1|^{(m-1)/m} \\ &\leq (B_1 + 1)\, \tau^{(n-1)/(2m)}\, |\Gamma_1|^{(m-1)/m}. \end{aligned} \tag{4.3.21}$$

Plugging (4.3.21) into (4.3.19),

$$LHS \leq \frac{B_1+1}{(4\pi)^{n/2}} |\Gamma_1|^{(m-1)/m} \int_0^t \tau^{-\frac{n}{2}+\frac{n-1}{2m}} \, d\tau. \tag{4.3.22}$$

Let

$$m = \frac{1}{1-\alpha}.$$

Then $m \geq 1$ and $(m-1)/m = \alpha$. Therefore (4.3.22) becomes

$$\begin{aligned} LHS &\leq \frac{B_1+1}{(4\pi)^{n/2}} |\Gamma_1|^{\alpha} \int_0^t \tau^{n\alpha - 1} \, d\tau \\ &= \frac{B_1+1}{(4\pi)^{n/2} n_\alpha} |\Gamma_1|^{\alpha} \, t^{n\alpha}, \end{aligned}$$

where the last equality is due to the assumption that $\alpha < \frac{1}{n-1}$. □

Proof of Theorem 4.1.1. In this proof, C denote the constants which only depend on n and Ω, the values of C may be different in different places. But C^* will represent a fixed value which also depends only on n and Ω. Recalling that $M(t)$ is defined as in (1.4.3). For any $k \geq 0$, define

$$M_k = 3^k M_0 \tag{4.3.23}$$

and T_k to be the first time that the function $M(t)$ reaches M_k. Obviously $T_0 = 0$. For any $k \geq 1$, denote

$$t_k \triangleq T_k - T_{k-1} \tag{4.3.24}$$

to be the time spent in the kth step. For any $k \geq 1$, we are trying to find a lower bound t_{k*} for t_k, then summing up all t_{k*} gives a lower bound for T^*.

By the maximum principle and the Hopf lemma, there exists $x^k \in \overline{\Gamma}_1$ such that

$$u(x^k, T_k) = M_k. \tag{4.3.25}$$

Applying the time-shifted representation formula (4.2.4) with $T = T_{k-1}$ and $(x,t) = (x^k, t_k)$, then

$$\begin{aligned} u(x^k, T_k) = 2 &\int_\Omega \Phi(x^k - y, t_k)\, u(y, T_{k-1})\, dy \\ &- 2 \int_0^{t_k} \int_{\partial\Omega} \frac{\partial \Phi(x^k - y, t_k - \tau)}{\partial n(y)}\, u(y, T_{k-1} + \tau)\, dS(y)\, d\tau \\ &+ 2 \int_0^{t_k} \int_{\Gamma_1} \Phi(x^k - y, t_k - \tau)\, u^q(y, T_{k-1} + \tau)\, dS(y)\, d\tau. \end{aligned} \tag{4.3.26}$$

Combining (4.3.25) and (4.3.26),

$$\begin{aligned} M_k \le\ & 2M_{k-1} \int_\Omega \Phi(x^k - y, t_k)\, dy + 2M_k \int_0^{t_k} \int_{\partial\Omega} \left| \frac{\partial \Phi(x^k - y, t_k - \tau)}{\partial n(y)} \right| dS(y)\, d\tau \\ &+ 2M_k^q \int_0^{t_k} \int_{\Gamma_1} \Phi(x^k - y, t_k - \tau)\, dS(y)\, d\tau \\ \triangleq\ & I_k + II_k + III_k. \end{aligned} \tag{4.3.27}$$

Since Φ is the fundamental solution of the heat equation, it is evident that

$$I_k \le 2M_{k-1}. \tag{4.3.28}$$

Secondly it is easy to check that

$$II_k \le C\sqrt{t_k}\, M_k. \tag{4.3.29}$$

Now define

$$r = \frac{1}{2(n-1)},$$

then it follows from (4.3.18) that

$$III_k \leq C\,|\Gamma_1|^r\, t_k^{1/4}\, M_k^q. \tag{4.3.30}$$

Combining (4.3.27), (4.3.28), (4.3.29) and (4.3.30), there exists a constant $C = C(n, \Omega)$ such that

$$M_k \leq 2M_{k-1} + C\,\sqrt{t_k}\,M_k + C\,|\Gamma_1|^r\, t_k^{1/4}\, M_k^q.$$

Recalling that $M_k = 3^k M_0$ and $M_{k-1} = 3^{k-1} M_0$, so

$$3^k M_0 \leq 2 \cdot 3^{k-1} M_0 + C\,\sqrt{t_k}\,3^k M_0 + C\,|\Gamma_1|^r\, t_k^{1/4}\, 3^{qk} M_0^q.$$

Subtracting $2 \cdot 3^{k-1} M_0$ and then dividing by $3^{k-1} M_0$, we obtain the existence of a constant $C = C(n, \Omega)$ such that

$$1 \leq C\sqrt{t_k} + C\,|\Gamma_1|^r\, t_k^{1/4}\, 3^{(q-1)k} M_0^{q-1}. \tag{4.3.31}$$

Rearranging (4.3.31),

$$\left(t_k^{1/4}\right)^2 + |\Gamma_1|^r M_0^{q-1}\, 3^{(q-1)k}\, t_k^{1/4} - \frac{1}{C} \geq 0.$$

Regarding the above inequality to be a quadratic inequality in $t_k^{1/4}$, then it is readily seen that $t_k^{1/4}$ has to be greater than the positive root of the corresponding quadratic equation, that is

$$t_k^{1/4} \geq \frac{1}{2}\left(-|\Gamma_1|^r M_0^{q-1}\, 3^{(q-1)k} + \sqrt{|\Gamma_1|^{2r} M_0^{2(q-1)}\, 3^{2(q-1)k} + \frac{4}{C}}\right).$$

After some algebraic simplification, we obtain

$$t_k^{1/4} \geq \frac{1}{C\sqrt{|\Gamma_1|^{2r} M_0^{2(q-1)}\, 3^{2(q-1)k} + \frac{4}{C}}}.$$

Hence, there exists $C^* = C^*(n, \Omega)$ such that

$$t_k \geq \frac{C^*}{|\Gamma_1|^{4r} M_0^{4(q-1)}\, 3^{4(q-1)k} + 1}.$$

After obtaining the lower bound for each t_k, we can add all of them together to get a lower bound for T^*. Namely,

$$T^* \geq C^* \sum_{k=1}^{\infty} \frac{1}{|\Gamma_1|^{4r} M_0^{4(q-1)}\, 3^{4(q-1)k} + 1}.$$

Therefore,

$$\begin{aligned} T^* &\geq C^* \int_1^{\infty} \frac{1}{|\Gamma_1|^{4r} M_0^{4(q-1)}\, 3^{4(q-1)x} + 1}\, dx \\ &= \frac{C^*}{4(q-1)\ln(3)} \ln\left(1 + \frac{1}{|\Gamma_1|^{4r} M_0^{4(q-1)}\, 3^{4(q-1)}}\right). \end{aligned}$$

Recalling that $r = \frac{1}{2(n-1)}$, then (4.1.1) follows. $\square$

4.4 Lower bound on life span: case of convex C^2 domain

4.4.1 Main idea

First of all, let us recall the method in Subsection 4.3.1. Define $\widetilde{M}$ as in (4.3.3). Then for any $t > 0$, there exists $x^0 \in \partial\Omega$ such that $u(x^0, t) = \widetilde{M}(t)$, so it follows from the representation formula (4.2.3) that

$$\begin{aligned}\widetilde{M}(t) &\le 2M_0 \int_\Omega \Phi(x^0 - y, t)\, dy + 2\int_0^t \widetilde{M}(\tau) \int_{\partial\Omega} \Big| \frac{\partial\Phi(x^0 - y, t-\tau)}{\partial n(y)} \Big| \, dS(y)\, d\tau \\ &\quad + 2\int_0^t \widetilde{M}^q(\tau) \int_{\Gamma_1} \Phi(x^0 - y, t-\tau)\, dS(y)\, d\tau \\ &\triangleq I + II + III. \end{aligned} \tag{4.4.1}$$

I, II and III are called the constant functional, linear functional and nonlinear functional in $\widetilde{M}(t)$ respectively. After estimating

$$\int_{\partial\Omega} \Big| \frac{\partial\Phi(x^0 - y, t-\tau)}{\partial n(y)} \Big| \, dS(y) \quad \text{and} \quad \int_{\Gamma_1} \Phi(x^0 - y, t-\tau)\, dS(y),$$

the lower bound in Theorem 4.3.2 is achieved by applying a Gronwall-type technique to (4.4.1). However, this lower bound is only logarithm of $|\Gamma_1|^{-1}$ as $|\Gamma_1| \searrow 0$. The obstruction that prevents this method obtaining a polynomial order of $|\Gamma_1|^{-1}$ for the lower bound is explained through the following remark.

Remark 4.4.1. *Consider the following two simple integral inequalities. First,*

$$\begin{cases} \phi_1(t) \le A + \int_0^t \phi_1(\tau)\, d\tau + |\Gamma_1| \int_0^t \phi_1^q(\tau)\, d\tau, \quad t > 0, \\ \phi_1(0) = A > 0. \end{cases} \tag{4.4.2}$$

It is easy to see by the Gronwall's inequality that the blow-up time T_1^* satisfies

$$T_1^* \ge \frac{1}{q-1} \ln\left(1 + \frac{1}{A^{q-1}|\Gamma_1|}\right),$$

which is of order $\ln(|\Gamma_1|^{-1})$ as $|\Gamma_1| \to 0$. Second,

$$\begin{cases} \phi_2(t) \le A + |\Gamma_1| \int_0^t \phi_2^q(\tau)\, d\tau, \quad t > 0, \\ \phi_2(0) = A > 0. \end{cases} \tag{4.4.3}$$

It is easy to see by Gronwall's inequality that the blow-up time T_2^* satisfies

$$T_2^* \ge \frac{1}{(q-1)A^{q-1}|\Gamma_1|},$$

which is of order $|\Gamma_1|^{-1}$. From these two differential equations, the obstruction that prevents the lower bound being a polynomial order of $|\Gamma_1|^{-1}$ is the linear term $\int_0^t \phi_1(\tau)\, d\tau$ in (4.4.2). Corresponding to (4.4.1), it is the linear term II:

$$\int_0^t \widetilde{M}(\tau) \int_{\partial\Omega} \left| \frac{\partial \Phi(x^0 - y, t - \tau)}{\partial n(y)} \right| dS(y)\, d\tau.$$

If the linear term II can be eliminated from (4.4.1), then the lower bound is expected to be a polynomial order $|\Gamma_1|^{-\alpha}$ for some $\alpha > 0$. Taking advantage of the convexity of Ω, the identity (4.4.8) in Lemma 4.4.2 can be used to absorb the linear term II into the constant term I in (4.4.1). Let's see how it works.

First, if $t \in (0, T^*)$ satisfies

$$M(t) > M_0 \quad \text{and} \quad \max_{x \in \partial\Omega} u(x,t) = M(t), \tag{4.4.4}$$

then there exists a point $x^0 \in \partial\Omega$ such that $u(x^0,t) = \max_{x \in \partial\Omega} u(x,t) = M(t)$. Thus, it follows from (4.3.3) and (4.4.1) that

$$\begin{aligned} M(t) \le 2M_0 \int_\Omega \Phi(x^0 - y, t)\, dy + 2M(t) \int_0^t \int_{\partial\Omega} \Big| \frac{\partial \Phi(x^0 - y, t-\tau)}{\partial n(y)} \Big|\, dS(y)\, d\tau \\ + 2M^q(t) \int_0^t \int_{\Gamma_1} \Phi(x^0 - y, t-\tau)\, dS(y)\, d\tau. \end{aligned} \tag{4.4.5}$$

Invoking (4.4.8),

$$\int_0^t \int_{\partial\Omega} \Big| \frac{\partial \Phi(x^0 - y, t-\tau)}{\partial n(y)} \Big|\, dS(y)\, d\tau = \frac{1}{2} - \int_\Omega \Phi(x^0 - y, t)\, dy.$$

Plugging this identity into (4.4.5) and simplifying, one has

$$M(t) \int_\Omega \Phi(x^0 - y, t)\, dy \le M_0 \int_\Omega \Phi(x^0 - y, t)\, dy + M^q(t) \int_0^t \int_{\Gamma_1} \Phi(x^0 - y, t-\tau)\, dS(y)\, d\tau. \tag{4.4.6}$$

Now this estimate does not contain the linear term, which should enable us to get a lower bound of polynomial order $|\Gamma_1|^{-\beta}$ for some $\beta > 0$.

To continue from (4.4.6), the Gronwall-type technique will not work, since (4.4.6) is proved to be true only for t satisfying (4.4.4). Then what kind of time t satisfies (4.4.4)? By the maximum principle, if at some $t > 0$, $M(t) > M(t_1)$ for any $0 \le t_1 < t$, then such

t satisfies (4.4.6). As an instance, for any $\lambda_1 > 1$, if we write $M_1 = \lambda_1 M_0$ and denote T_1 to be the first time that $M(t)$ reaches M_1, then T_1 satisfies (4.4.4). Another disadvantage of (4.4.6) is that although it gets rid of the linear functional of $M(t)$, there is an extra term $\int_\Omega \Phi(x^0 - y, t)\, dy$ on the left hand side, which decays like $t^{-n/2}$ when t becomes large. Hence, to avoid the effect of this decay, λ_1 should be kept small. Taking these restrictions into consideration, we need to come up with some delicate strategies. The rough idea is as follows. We will firstly choose a suitably small λ_1 such that there is still a lower bound t_* for T_1, where T_1 is the first time for $M(t)$ to reach $\lambda_1 M_0$. Then we regard $u(\cdot, T_1)$ as the "initial data" and repeat the first step. Finally if such process can proceed for at least L_0 steps, then $L_0 t_*$ is a lower bound for T^*, since the time in each step has a lower bound t_* (the choice of t_* will be the same in each step).

4.4.2 Auxiliary lemmas

The second conclusion of the next lemma is the only place that the convexity is used in this section.

Lemma 4.4.2. *Let Φ be the heat kernel as in (1.4.4). Then*

$$\int_\Omega \Phi(x-y,t)\, dy - \int_0^t \int_{\partial\Omega} \frac{\partial \Phi(x-y,t-\tau)}{\partial n(y)}\, dS(y)\, d\tau = \frac{1}{2}, \quad \forall\, x \in \partial\Omega,\ t > 0. \tag{4.4.7}$$

In addition, if Ω is convex, then

$$\int_\Omega \Phi(x-y,t)\, dy + \int_0^t \int_{\partial\Omega} \left| \frac{\partial \Phi(x-y,t-\tau)}{\partial n(y)} \right| dS(y)\, d\tau = \frac{1}{2}, \quad \forall\, x \in \partial\Omega,\ t > 0. \tag{4.4.8}$$

Proof. The problem

$$\begin{cases} u_t(x,t) = \Delta u(x,t) & \text{in} \quad \Omega \times (0,\infty), \\ \frac{\partial u(x,t)}{\partial n(x)} = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = 1 & \text{in} \quad \Omega, \end{cases} \tag{4.4.9}$$

obviously has the unique solution $u \equiv 1$ on $\overline{\Omega} \times [0,\infty)$. As a result, plugging $u \equiv 1$ into the representation formula (4.2.3) (taking $\Gamma_1 = \emptyset$), (4.4.7) follows.

Now if Ω is convex, then $\frac{\partial \Phi(x-y,t-\tau)}{\partial n(y)} \leq 0$ for any x, $y \in \partial\Omega$. Thus, (4.4.7) implies (4.4.8). □

Lemma 4.4.3. *Define* $F : \partial\Omega \times [0,1] \to \mathbb{R}$ *to be*

$$F(x,t) = \begin{cases} \int_\Omega \Phi(x-y,t)\,dy & \textit{for} \quad x \in \partial\Omega, t \in (0,1], \\ 1/2 & \textit{for} \quad x \in \partial\Omega, t = 0. \end{cases}$$

Then F *is continuous on* $\partial\Omega \times [0,1]$*. As a result,*

$$b_1 \triangleq \min_{\partial\Omega \times [0,1]} F \tag{4.4.10}$$

is a positive constant depending only on Ω *and the dimension* n*.*

Proof. Since $\partial\Omega$ has been assumed to be C^2, the proof can be carried out by standard analysis. We can also prove it by applying (4.4.7) and noticing the uniform decay for $x \in \partial\Omega$ of the following integral

$$\lim_{t\to 0} \int_0^t \int_{\partial\Omega} \left| \frac{\partial \Phi(x-y,t-\tau)}{\partial n(y)} \right| dS(y)\,d\tau = 0.$$

The details are omitted here. $\square$

Next, M_0 and $M(t)$ are still defined as in (1.4.2) and (1.4.3). In addition, we define

$$E_q = (q-1)^{q-1}/q^q, \quad \forall\, q > 1. \tag{4.4.11}$$

By elementary calculus,

$$\frac{1}{3q} < E_q < \min\left\{\frac{1}{q}, \frac{1}{(q-1)\,e}\right\} < 1. \tag{4.4.12}$$

Lemma 4.4.4. *For any $q > 1$ and $m > 0$, write E_q as in (4.4.11) and define $g : (m, \infty) \to \mathbb{R}$ by*

$$g(\lambda) = \frac{\lambda - m}{\lambda^q}, \quad \forall\, \lambda > m. \tag{4.4.13}$$

Then the following two claims hold.

(1) For any $y \in \big(0, m^{1-q}E_q\big]$, there exists unique $\lambda \in \big(m, \frac{q}{q-1}m\big]$ such that $g(\lambda) = y$.

(2) For any $y > m^{1-q}E_q$, there does not exist $\lambda > m$ such that $g(\lambda) = y$.

Proof. Since g is strictly increasing on the interval $\big(m, \frac{q}{q-1}m\big]$ and strictly decreasing on the interval $\big[\frac{q}{q-1}m, \infty\big)$, it reaches the maximum at $\lambda = \frac{q}{q-1}m$. Noticing that

$$g\Big(\frac{q}{q-1}m\Big) = m^{1-q}E_q,$$

then the claims (1) and (2) follow directly. $\square$

4.4.3 Proof of Theorem 4.1.4

Proof. Let $M(t)$ be defined as in (1.4.3). In the following, the first step is that for any $t_* \in (0,1]$, we will find a finite strictly increasing sequence $\{M_k\}_{0\le k\le L}$ such that if T_k denotes the first time that $M(t) = M_k$, then $T_k - T_{k-1} \ge t_*$ for $1 \le k \le L$. The second step is to derive a lower bound for Lt_* as an explicit formula in t_* and then maximize that lower bound for $t_* \in (0,1]$.

Step 1. Let $t_* \in (0,1]$ which will be determined later in **Step 2**. Define M_0 as that in (1.4.2) and $T_0 = 0$. Then for $k \ge 1$, suppose M_{k-1} has been constructed, we are trying to find M_k such that $T_k - T_{k-1} \ge t_*$.

Denote $t_k = T_k - T_{k-1}$. We will first check what happens if $t_k \le 1$. By the maximum principle, there exists $x^k \in \partial\Omega$ such that $u(x^k, T_k) = M_k$, so T_k satisfies (4.4.4). Applying the time-shifted representation formula (4.2.4) with $T = T_{k-1}$ and $(x,t) = (x^k, t_k)$,

$$\begin{aligned} u(x^k, T_k) = {} & 2\int_\Omega \Phi(x^k - y, t_k)\, u(y, T_{k-1})\, dy \\ & - 2\int_0^{t_k}\int_{\partial\Omega} \frac{\partial\Phi(x^k - y, t_k - \tau)}{\partial n(y)}\, u(y, T_{k-1}+\tau)\, dS(y)\, d\tau \\ & + 2\int_0^{t_k}\int_{\Gamma_1} \Phi(x^k - y, t_k - \tau)\, u^q(y, T_{k-1}+\tau)\, dS(y)\, d\tau. \end{aligned} \tag{4.4.14}$$

As a result,

$$\begin{aligned} M_k \le {} & 2M_{k-1}\int_\Omega \Phi(x^k - y, t_k)\, dy \\ & + 2M_k\int_0^{t_k}\int_{\partial\Omega} \left|\frac{\partial\Phi(x^k - y, t_k - \tau)}{\partial n(y)}\right| dS(y)\, d\tau \\ & + 2M_k^q\int_0^{t_k}\int_{\Gamma_1} \Phi(x^k - y, t_k - \tau)\, dS(y)\, d\tau. \end{aligned} \tag{4.4.15}$$

Since Ω is convex, it follows from (4.4.8) that

$$\int_0^{t_k}\int_{\partial\Omega}\left|\frac{\partial\Phi(x^k-y,t_k-\tau)}{\partial n(y)}\right| dS(y)\,d\tau = \frac{1}{2} - \int_\Omega \Phi(x^k-y,t_k)\,dy.$$

Plugging this identity into (4.4.15) and simplifying,

$$(M_k - M_{k-1})\int_\Omega \Phi(x^k-y,t_k)\,dy \le M_k^q \int_0^{t_k}\int_{\Gamma_1}\Phi(x^k-y,t_k-\tau)\,dS(y)\,d\tau. \tag{4.4.16}$$

Due to the assumption that $t_k \le 1$, it follows from Lemma 4.4.3 that

$$\int_\Omega \Phi(x^k-y,t_k)\,dy \ge b_1. \tag{4.4.17}$$

In addition, Lemma 4.3.3 implies the existence of a constant $C = C(n,\Omega)$ such that

$$\int_0^{t_k}\int_{\Gamma_1}\Phi(x^k-y,t_k-\tau)\,dS(y)\,d\tau \le \frac{C\,|\Gamma_1|^\alpha\,t_k^{n\alpha}}{n_\alpha}, \tag{4.4.18}$$

where n_α is defined in (4.3.17). Plugging (4.4.17) and (4.4.18) into (4.4.16),

$$\frac{M_k - M_{k-1}}{M_k^q} \le \frac{C\,|\Gamma_1|^\alpha\,t_k^{n\alpha}}{b_1 n_\alpha}. \tag{4.4.19}$$

In summary, this paragraph claims that if $t_k \le 1$, then M_k will satisfy (4.4.19).

Based on the above observation, denote

$$\delta_1 = \frac{C\,|\Gamma_1|^\alpha\,t_*^{n\alpha}}{b_1\,n_\alpha}, \tag{4.4.20}$$

where the constant C is the same as that in (4.4.19). Then we define M_k to be the solution (if it exists) to

$$\frac{M_k - M_{k-1}}{M_k^q} = \delta_1. \tag{4.4.21}$$

With such a choice for M_k, it is evident that

$$t_k \geq \min\{1, t_*\} = t_*.$$

On the other hand, by applying Lemma 4.4.4, (4.4.21) has a solution $M_k > M_{k-1}$ if and only if $\delta_1 \leq M_{k-1}^{1-q} E_q$. In addition, as long as such a solution exists, M_k can be chosen to satisfy

$$M_{k-1} < M_k \leq \frac{q}{q-1}\, M_{k-1}.$$

Thus, the strategy of constructing $\{M_k\}$ is summarized as following. First, M_0 is defined to be the same as in (1.4.2). Next, for $k \geq 1$, suppose M_{k-1} has been constructed, then based on Lemma 4.4.4, whether defining M_k depends on how large M_{k-1} is.

- ⋄ If $M_{k-1}^{q-1}\,\delta_1 \leq E_q$, then we define $M_k \in \left(M_{k-1}, \frac{q}{q-1}\, M_{k-1}\right]$ to be the solution to (4.4.21).

- ⋄ If $M_{k-1}^{q-1}\,\delta_1 > E_q$, then there does not exist $M_k > M_{k-1}$ which solves (4.4.21). So we do not define M_k and stop the construction.

Based on this construction, if $\{M_k\}_{1 \leq k \leq L_0}$ have been defined, then for any $1 \leq k \leq L_0$, $T_k - T_{k-1} \geq t_*$. Therefore, $T_k \geq k t_*$ for any $1 \leq k \leq L_0$. Applying Theorem 3.1.1, $L_0 \leq T^*/t_* < \infty$, which means the cardinality of $\{M_k\}$ has be to finite. So we can assume the constructed sequence is $\{M_k\}_{0 \leq k \leq L}$ for some finite L.

Step 2. By Lemma 4.4.5,

$$L > \frac{1}{10(q-1)}\Big(\frac{1}{M_0^{q-1}\delta_1} - 3q\Big),$$

so

$$T^* \geq Lt_* > \frac{1}{10(q-1)}\Big(\frac{1}{M_0^{q-1}\delta_1} - 3q\Big)t_*. \tag{4.4.22}$$

Plugging (4.4.20) into (4.4.22),

$$\begin{aligned} T^* &\geq \frac{1}{10(q-1)}\bigg(\frac{b_1 n_\alpha}{M_0^{q-1}\, C\, |\Gamma_1|^\alpha}\, t_*^{1-n_\alpha} - 3q\, t_*\bigg) \\ &= \frac{3q}{10(q-1)}\bigg(\frac{C_1 n_\alpha}{qM_0^{q-1}\, |\Gamma_1|^\alpha}\, t_*^{1-n_\alpha} - t_*\bigg), \end{aligned} \tag{4.4.23}$$

where $C_1 = b_1/(3C)$ is a constant only depending on n and Ω.

In order to maximize the right hand side of (4.4.23), let

$$A = \frac{C_1 n_\alpha}{qM_0^{q-1}\, |\Gamma_1|^\alpha}, \quad \beta = 1 - n_\alpha \in \Big[\frac{1}{2}, 1\Big)$$

and define

$$t_* \triangleq (\min\{1, \beta A\})^{1/(1-\beta)}. \tag{4.4.24}$$

Then it follows from Lemma 4.4.6 that t_* maximizes the right hand side of (4.4.23) on $(0, 1]$ and

$$T^* \geq \frac{3q}{10(q-1)}\,(1-\beta)A\,\big(\min\{1, \beta A\}\big)^{\beta/(1-\beta)}.$$

Noticing that $\beta \geq 1/2$, so

$$
\begin{aligned}
T^* &\geq \frac{3q}{10(q-1)}(1-\beta)A\left(\min\left\{1, \frac{A}{2}\right\}\right)^{\beta/(1-\beta)} \\
&\geq \frac{3C_1 n_\alpha^2}{10(q-1)M_0^{q-1}\,|\Gamma_1|^\alpha}\left(\min\left\{1, \frac{C_1 n_\alpha}{2qM_0^{q-1}\,|\Gamma_1|^\alpha}\right\}\right)^{\frac{1}{n_\alpha}-1} \\
&\geq \frac{C_2}{(q-1)M_0^{q-1}\,|\Gamma_1|^\alpha}\left(\min\left\{1, \frac{1}{qM_0^{q-1}|\Gamma_1|^\alpha}\right\}\right)^{\frac{1}{n_\alpha}-1}, \qquad (4.4.25)
\end{aligned}
$$

where

$$C_2 = \frac{3C_1 n_\alpha^2}{10}\left(\min\left\{1, \frac{C_1 n_\alpha}{2}\right\}\right)^{\frac{1}{n_\alpha}-1} \qquad (4.4.26)$$

is a constant depending on n, Ω and α.

In particular, if we choose $\alpha = 0$ in (4.4.25) and (4.4.26), then it follows from $n_\alpha = \frac{1-(n-1)\alpha}{2} = \frac{1}{2}$ that

$$T^* \geq \frac{C_3}{(q-1)M_0^{q-1}}\min\left\{1, \frac{1}{qM_0^{q-1}}\right\},$$

where C_3 is a positive constant only depending on n and Ω. □

Lemma 4.4.5. *Given $q > 1$, $M_0 > 0$ and $\delta_1 > 0$, denote E_q as (4.4.11) and construct a finite sequence $\{M_k\}_{0\leq k\leq L}$ inductively as follows. For $k \geq 1$, suppose M_{k-1} has been constructed, then based on Lemma 4.4.4, whether defining M_k depends on how large M_{k-1} is.*

- *If $M_{k-1}^{q-1}\,\delta_1 \leq E_q$, then we define $M_k \in \left(M_{k-1}, \frac{q}{q-1}\,M_{k-1}\right]$ to be the solution to*

$$\frac{M_k - M_{k-1}}{M_k^q} = \delta_1. \qquad (4.4.27)$$

- *If $M_{k-1}^{q-1}\,\delta_1 > E_q$, then there does not exist $M_k > M_{k-1}$ which solves (4.4.27). So we* *do not define M_k and stop the construction.*

Denote the last term of this construction to be M_L, then

$$L > \frac{1}{10(q-1)}\Big(\frac{1}{M_0^{q-1}\delta_1} - 3q\Big). \tag{4.4.28}$$

Proof. First, we want to mention that the construction indeed stops in finite steps. In fact, it follows from (4.4.27) that the sequence $\{M_k\}$ is strictly increasing and

$$M_k = M_{k-1} + M_k^q\delta_1 \geq \big(1 + M_0^{q-1}\delta_1\big)M_{k-1}.$$

As a result,

$$M_k \geq \big(1 + \delta_1 M_0^{q-1}\big)^k M_0.$$

Thus, when k is sufficiently large, M_k^{q-1} will exceed E_q/δ_1, which forces the construction to stop.

Next, suppose the construction stops at M_L, that is to say, the constructed sequence is $\{M_k\}_{0\leq k\leq L}$, then the lower bound (4.4.28) for L will be justified based on two situations.

Case 1. $M_0^{q-1}\delta_1 > E_q$. In this case, it follows from (4.4.12) that

$$\frac{1}{M_0^{q-1}\delta_1} < \frac{1}{E_q} < 3q,$$

so the right hand side of (4.4.28) is negative. Thus, (4.4.28) holds since $L \geq 0$.

Case 2. $M_0^{q-1}\delta_1 \leq E_q$. In this case, it is evident that $L \geq 1$. Therefore, since the last term of the sequence is M_L, then

$$M_{L-1}^{q-1}\delta_1 \le E_q \quad \text{and} \quad M_L^{q-1}\delta_1 > E_q.$$

According to the recursive relation (4.4.27),

$$M_{k-1} = M_k\big(1 - M_k^{q-1}\delta_1\big).$$

Raising both sides of the above equality to the power $q-1$ and multiplying by δ_1,

$$M_{k-1}^{q-1}\delta_1 = M_k^{q-1}\delta_1\left(1 - M_k^{q-1}\delta_1\right)^{q-1}.$$

Let $x_k = M_k^{q-1}\delta_1$. Then

$$x_{k-1} = x_k\,(1 - x_k)^{q-1}, \quad \forall\, 1 \le k \le L. \tag{4.4.29}$$

Moreover,

$$x_0 = M_0^{q-1}\delta_1, \quad x_{L-1} \le E_q \quad \text{and} \quad x_L > E_q.$$

Noticing that $M_L \le \frac{q}{q-1}M_{L-1}$, so

$$x_L = \left(\frac{M_L}{M_{L-1}}\right)^{q-1} x_{L-1} \le \left(\frac{q}{q-1}\right)^{q-1} E_q = \frac{1}{q}.$$

Since the right hand side of (4.4.29) is a nonlinear function in x_k, it is better to consider the "reversed" relation of (4.4.29). Thus, we define a new sequence $\{y_k\}_{0\le k\le L}$ in the following way: $y_0 \triangleq \min\{1/2, E_q\}$ and

$$y_k \triangleq y_{k-1}(1-y_{k-1})^{q-1}, \quad \forall\, 1 \le k \le L. \tag{4.4.30}$$

In addition, define $h : (0,1) \to \mathbb{R}$ by

$$h(t) = t\,(1-t)^{q-1}.$$

It is easy to see that h is strictly increasing on $(0, 1/q]$ and strictly decreasing on $[1/q, 1)$. As a result, it follows from $0 < y_0 \le E_q < x_L \le 1/q$ that

$$y_1 = h(y_0) < h(x_L) = x_{L-1}.$$

Keep doing this, we get $y_k < x_{L-k}$ for any $0 \le k \le L$. In particular, when $k = L$, $y_L < x_0 = M_0^{q-1}\delta_1$.

Since $\{y_k\}$ is a decreasing positive sequence and $y_0 \le 1/2$, then $y_k \le 1/2$ for any $0 \le k \le L$. As a result, it follows from (4.4.30) and the mean value theorem that for any $1 \le k \le L$,

$$y_k \ge y_{k-1}\big[1 - 2(q-1)y_{k-1}\big]. \tag{4.4.31}$$

Recalling (4.4.12) again,

$$y_{k-1} \le y_0 \le E_q < \frac{1}{(q-1)e},$$

so

$$1 - 2(q-1)y_{k-1} > 1 - \frac{2}{e} > \frac{1}{5}.$$

Hence, taking the reciprocal in (4.4.31) yields

$$\begin{aligned}\frac{1}{y_k} &\leq \frac{1}{y_{k-1}\big[1-2(q-1)y_{k-1}\big]} \\ &= \frac{1}{y_{k-1}} + \frac{2(q-1)}{1-2(q-1)y_{k-1}} \\ &< \frac{1}{y_{k-1}} + 10(q-1). \end{aligned} \tag{4.4.32}$$

Summing up (4.4.32) for k from 1 to L, then

$$\frac{1}{y_L} < \frac{1}{y_0} + 10(q-1)L. \tag{4.4.33}$$

Since $y_L < M_0^{q-1}\delta_1$ and

$$y_0 = \min\left\{\frac{1}{2}, E_q\right\} > \frac{1}{3q},$$

it follows from (4.4.33) that

$$\frac{1}{M_0^{q-1}\delta_1} < 3q + 10(q-1)L.$$

Thus,

$$L > \frac{1}{10(q-1)}\Big(\frac{1}{M_0^{q-1}\delta_1} - 3q\Big).$$

□

Lemma 4.4.6. *Given two constants $A > 0$ and $\beta \in (0,1)$, define $f : (0,1] \to \mathbb{R}$ by $f(t) = A\,t^\beta - t$. Let*

$$t_0 = (\min\{1, \beta A\})^{1/(1-\beta)}.$$

Then

$$f(t_0) = \sup_{0<t\le 1} f(t) \ge (1-\beta)A\,(\min\{1, \beta A\})^{\beta/(1-\beta)}. \tag{4.4.34}$$

Proof. For any $t \in (0,1]$, $f'(t) = \beta A\, t^{\beta-1} - 1$. So when $0 < t \le t_0$, f is increasing.

- If $\beta A \ge 1$, then

$$\sup_{0<t\le 1} f(t) = f(1) = A - 1 \ge (1-\beta)\,A.$$

- If $0 < \beta A < 1$, then

$$\begin{aligned} \sup_{0<t\le 1} f(t) &= f\left[(\beta A)^{1/(1-\beta)}\right] \\ &= A\,(\beta A)^{\beta/(1-\beta)} - (\beta A)^{1/(1-\beta)} \\ &= (1-\beta)A\,(\beta A)^{\beta/(1-\beta)}. \end{aligned}$$

Combining these two cases, the lemma is justified. □

4.5 Lower bound on life span: case of C^2 domain with local convexity near Γ_1

The global convexity of Ω is not practical in real applications. So in this section, we try to extend the result to locally convex case. Namely, we assume the local convexity near Γ_1 as in Definition 4.1.2.

4.5.1 Estimates for boundary integrals

Lemma 4.5.1. *Let Ω and Γ_1 be the same as in (1.1.1). Then for any $d > 0$, there exists $C = C(n, \Omega, d)$ such that for any $x \in \overline{\Gamma}_1$ and $t \in (0, 1]$,*

$$\int_0^t \int_{\partial\Omega\setminus[\Gamma_1]_d} \Big|\frac{\partial\Phi(x-y, t-\tau)}{\partial n(y)}\Big|\, dS(y)\, d\tau \le C \exp\Big(-\frac{d^2}{8t}\Big). \tag{4.5.1}$$

Proof. In this proof, C denotes a constant which depends only on n, Ω and d. By a change of variable in τ and the definition of Φ,

$$\begin{aligned}
&\int_0^t \int_{\partial\Omega\setminus[\Gamma_1]_d} \Big|\frac{\partial\Phi(x-y, t-\tau)}{\partial n(y)}\Big|\, dS(y)\, d\tau \\
= &\int_0^t \int_{\partial\Omega\setminus[\Gamma_1]_d} \Big|\frac{\partial\Phi(x-y, t-\tau)}{\partial n(y)}\Big|\, dS(y)\, d\tau \\
\le \ & C\int_0^t \int_{\partial\Omega\setminus[\Gamma_1]_d} \frac{|(x-y)\cdot \overrightarrow{n}(y)|}{\tau^{\frac{n}{2}+1}} \exp\Big(-\frac{|x-y|^2}{4\tau}\Big)\, dS(y)\, d\tau.
\end{aligned} \tag{4.5.2}$$

Since $\partial\Omega$ is assumed to be C^2, then $|(x-y)\cdot\overrightarrow{n}(y)| \le C|x-y|^2$. As a result,

$$\begin{aligned}
\frac{|(x-y)\cdot \overrightarrow{n}(y)|}{\tau^{\frac{n}{2}+1}} \exp\Big(-\frac{|x-y|^2}{4\tau}\Big) &\le C|x-y|^{-n}\Big(\frac{|x-y|^2}{\tau}\Big)^{1+\frac{n}{2}} \exp\Big(-\frac{|x-y|^2}{4\tau}\Big) \\
&\le C|x-y|^{-n} \exp\Big(-\frac{|x-y|^2}{8\tau}\Big) \\
&\le Cd^{-n} \exp\Big(-\frac{d^2}{8\tau}\Big).
\end{aligned} \tag{4.5.3}$$

where the last inequality is due to the fact that $x \in \overline{\Gamma}_1$ and $y \in \partial\Omega \setminus [\Gamma_1]_d$.

Plugging (4.5.3) into (4.5.2),

$$
\begin{aligned}
&\int_0^t \int_{\partial\Omega\setminus[\Gamma_1]_d} \Big|\frac{\partial\Phi(x-y,t-\tau)}{\partial n(y)}\Big|\, dS(y)\, d\tau \\
\leq\ & Cd^{-n} \int_0^t \int_{\partial\Omega\setminus[\Gamma_1]_d} \exp\Big(-\frac{d^2}{8\tau}\Big)\, dS(y)\, d\tau \\
\leq\ & Cd^{-n}|\partial\Omega| \int_0^t \exp\Big(-\frac{d^2}{8\tau}\Big)\, d\tau \\
\leq\ & Cd^{-n}|\partial\Omega|\, t \exp\Big(-\frac{d^2}{8t}\Big) \leq C\exp\Big(-\frac{d^2}{8t}\Big),
\end{aligned}
$$

where the last inequality is due to the assumption that $t \leq 1$. □

By exploiting Lemma 4.5.1, the following is a variant of the identity (4.4.8). So (4.5.4) will play the same role in the proof of Theorem 4.1.3 as (4.4.8) did in the proof of Theorem 4.1.4.

Corollary 4.5.2. *Let Ω and Γ_1 be the same as in (1.1.1). Assume there exists $d > 0$ such that Conv$\big([\Gamma_1]_d\big) \subseteq \overline{\Omega}$. Then there exists $C = C(n, \Omega, d)$ such that for any $x \in \overline{\Gamma}_1$ and $t \in (0, 1]$,*

$$
\int_\Omega \Phi(x-y,t)\, dy + \int_0^t \int_{\partial\Omega} \Big|\frac{\partial\Phi(x-y,t-\tau)}{\partial n(y)}\Big|\, dS(y)\, d\tau \leq \frac{1}{2} + C\exp\Big(-\frac{d^2}{8t}\Big). \tag{4.5.4}
$$

Proof. Splitting the second term on the left hand side of (4.5.4) into two parts:

$$
\begin{aligned}
&\int_0^t \int_{\partial\Omega} \Big|\frac{\partial\Phi(x-y,t-\tau)}{\partial n(y)}\Big|\, dS(y)\, d\tau \\
=\ & \int_0^t \int_{[\Gamma_1]_d} \Big|\frac{\partial\Phi(x-y,t-\tau)}{\partial n(y)}\Big|\, dS(y)\, d\tau + \int_0^t \int_{\partial\Omega\setminus[\Gamma_1]_d} \Big|\frac{\partial\Phi(x-y,t-\tau)}{\partial n(y)}\Big|\, dS(y)\, d\tau
\end{aligned}
$$

It follows from $x \in \overline{\Gamma}_1$ and $\mathrm{Conv}\big([\Gamma_1]_d\big) \subseteq \overline{\Omega}$ that

$$\frac{\partial \Phi(x-y,t-\tau)}{\partial n(y)} \le 0, \quad \forall\, y \in [\Gamma_1]_d.$$

As a result,

$$\begin{aligned}
&\int_0^t \int_{\partial\Omega} \Big| \frac{\partial \Phi(x-y,t-\tau)}{\partial n(y)} \Big| \, dS(y) d\tau \\
&= -\int_0^t \int_{[\Gamma_1]_d} \frac{\partial \Phi(x-y,t-\tau)}{\partial n(y)} \, dS(y) d\tau + \int_0^t \int_{\partial\Omega \setminus [\Gamma_1]_d} \Big| \frac{\partial \Phi(x-y,t-\tau)}{\partial n(y)} \Big| \, dS(y) d\tau \\
&\le -\int_0^t \int_{\partial\Omega} \frac{\partial \Phi(x-y,t-\tau)}{\partial n(y)} \, dS(y) d\tau + 2\int_0^t \int_{\partial\Omega \setminus [\Gamma_1]_d} \Big| \frac{\partial \Phi(x-y,t-\tau)}{\partial n(y)} \Big| \, dS(y) d\tau. \quad (4.5.5)
\end{aligned}$$

Combining (4.5.5) with Lemma 4.5.1, there exists a constant $C = C(n, \Omega, d)$ such that

$$\begin{aligned}
&\int_0^t \int_{\partial\Omega} \Big| \frac{\partial \Phi(x-y,t-\tau)}{\partial n(y)} \Big| \, dS(y)\, d\tau \\
\le\ & -\int_0^t \int_{\partial\Omega} \frac{\partial \Phi(x-y,t-\tau)}{\partial n(y)} \, dS(y)\, d\tau + C \exp\Big(-\frac{d^2}{8t} \Big). \qquad (4.5.6)
\end{aligned}$$

Hence it follows from (4.4.7) and (4.5.6) that

$$\begin{aligned}
&\int_\Omega \Phi(x-y,t)\, dy + \int_0^t \int_{\partial\Omega} \Big| \frac{\partial \Phi(x-y,t-\tau)}{\partial n(y)} \Big| \, dS(y)\, d\tau \\
\le\ & \int_\Omega \Phi(x-y,t)\, dy - \int_0^t \int_{\partial\Omega} \frac{\partial \Phi(x-y,t-\tau)}{\partial n(y)} \, dS(y)\, d\tau + C \exp\Big(-\frac{d^2}{8t} \Big) \\
=\ & \frac{1}{2} + C \exp\Big(-\frac{d^2}{8t} \Big).
\end{aligned}$$

□

Next, we introduce a simple fact which can be regarded as a rearrangement result.

Lemma 4.5.3. *Let $n \geq 1$ and $f : (0, \infty) \to [0, \infty)$ be a decreasing function. Then for any bounded, open subset U of $\mathbb{R}^n$ and for any $x \in \mathbb{R}^n$,*

$$\int_U f(|x-y|)\,dy \leq \int_{B_R(0)} f(|z|)\,dz \tag{4.5.7}$$

where R satisfies $|B_R(0)| = |U|$ (namely the volume of $B_R(0)$ equals the volume of U).

Proof. Define

$$U_1 = U - \{x\}.$$

Then by a change of variable $z = y - x$,

$$\begin{aligned} \int_U f(|x-y|)\,dy &= \int_{U_1} f(|z|)\,dz \\ &= \int_{U_1 \cap B_R(0)} f(|z|)\,dz + \int_{U_1 \setminus B_R(0)} f(|z|)\,dz \\ &\triangleq I_1 + I_2, \end{aligned} \tag{4.5.8}$$

Since f is decreasing, then

$$I_2 \leq f(R)|U_1 \setminus B_R(0)|.$$

Since R is chosen such that $|B_R(0)| = |U| = |U_1|$, then we have $|B_R(0) \setminus U_1| = |U_1 \setminus B_R(0)|$. As a result,

$$I_2 \leq f(R)|B_R(0) \setminus U_1| \leq \int_{B_R(0) \setminus U_1} f(|z|)\,dz, \tag{4.5.9}$$

where the last inequality is again due to the decay of f. Combining (4.5.8) and (4.5.9), we finish the proof. □

Definition 4.5.4. *Let Ω be a bounded, open subset of $\mathbb{R}^n$ with C^1 boundary. Let Γ be a relative open subset of $\partial\Omega$. We say Γ is given by a graph if (upon relabelling and reorienting the coordinates axes), there exists a bounded, open subset $U \subseteq \mathbb{R}^{n-1}$ and a C^1 function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that*

$$\Gamma = \{(\tilde{y}, \phi(\tilde{y})) : \tilde{y} \in U\}.$$

In the following, for any $x \in \mathbb{R}^n$, we will decompose it to be $x = (\tilde{x}, x_n)$, where $\tilde{x}$ denotes the first $n-1$ components of x.

Lemma 4.5.5. *Let Ω be a bounded, open subset of $\mathbb{R}^n (n \geq 3)$ with C^1 boundary. Let Γ be a relatively open subset of $\partial\Omega$ that is given by a graph as defined in Definition 4.5.4. Then there exists a constant $C = C(n, ||\nabla\phi||_{L^\infty(U)})$, where ϕ and U are the same as in Definition 4.5.4, such that for any $x \in \mathbb{R}^n$,*

$$\int_\Gamma \frac{1}{|x-y|^{n-2}}\, dS(y) \leq C|\Gamma|^{1/(n-1)},$$

where $|\Gamma| \triangleq \int_\Gamma dS$.

Proof. By Definition 4.5.4, without loss of generality, we can assume there exists a C^1 function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ and a bounded, open subset U of $\mathbb{R}^{n-1}$ such that

$$\Gamma = \{(\tilde{y}, \phi(\tilde{y})) : \tilde{y} \in U\}. \tag{4.5.10}$$

Thus,

$$\begin{aligned}\int_\Gamma \frac{1}{|x-y|^{n-2}}\,dS(y) &= \int_U \frac{\sqrt{1+|\nabla\phi(\tilde y)|^2}}{|(\tilde x, x_n)-(\tilde y,\phi(\tilde y))|^{n-2}}\,d\tilde y\\ &\le \int_U \frac{\sqrt{1+|\nabla\phi(\tilde y)|^2}}{|\tilde x-\tilde y|^{n-2}}\,d\tilde y\\ &\le C\int_U \frac{1}{|\tilde x-\tilde y|^{n-2}}\,d\tilde y.\end{aligned}$$

Define

$$f(r)=\frac{1}{r^{n-2}},\quad \forall\, r>0.$$

Then it follows from Lemma 4.5.3 that

$$\begin{aligned}\int_U \frac{1}{|\tilde x-\tilde y|^{n-2}}\,d\tilde y &= \int_U f(|\tilde x-\tilde y|)\,d\tilde y\\ &\le \int_{B_R(0)} f(|\tilde z|)\,d\tilde z\\ &= CR = C|U|^{1/(n-1)}.\end{aligned}$$

Again by the parametrization (4.5.10), it is readily seen that $|U|\le|\Gamma|$. Hence,

$$\int_\Gamma \frac{1}{|x-y|^{n-2}}\,dS(y)\le C|U|^{1/(n-1)}\le C|\Gamma|^{1/(n-1)}.$$

$\square$

Corollary 4.5.6. *Let Ω be a bounded, open subset of $\mathbb{R}^n (n\ge 3)$ with C^1 boundary. Then there exists a constant $C=C(n,\Omega)$ such that for any relative open subset Γ of $\partial\Omega$ and for any $x\in\mathbb{R}^n$,*

$$\int_\Gamma \frac{1}{|x-y|^{n-2}}\,dS(y)\le C|\Gamma|^{1/(n-1)}.$$

Proof. Since $\partial\Omega$ is C^1, for any point $x_0 \in \partial\Omega$, the boundary part of Ω near x_0 can be straightened out (thus can be given by a graph as in Definition 4.5.4). As a result, we can split $\partial\Omega$ into finite pieces:

$$\partial\Omega = \bigcup_{i=1}^{K} A_i, \tag{4.5.11}$$

where each $A_i (1 \leq i \leq K)$ is given by the graph of some C^1 function ϕ_i on some open set $U_i \subseteq \mathbb{R}^{n-1}$. The number of total pieces K and $||\nabla \phi_i||_{L^\infty(U_i)}$ only depend on Ω.

So for any $1 \leq i \leq K$, $\Gamma \cap A_i$ is also a boundary part this is given by a graph. Therefore by Lemma 4.5.5, there exists a constant $C = C(n, \Omega)$ such that for any $1 \leq i \leq K$,

$$\int_{\Gamma \cap A_i} \frac{1}{|x-y|^{n-2}} dS(y) \leq C|\Gamma \cap A_i|^{1/(n-1)}.$$

Hence,

$$\begin{aligned}
\int_{\Gamma} \frac{1}{|x-y|^{n-2}} dS(y) &\leq \sum_{i=1}^{K} \int_{\Gamma \cap A_i} \frac{1}{|x-y|^{n-2}} dS(y) \\
&\leq C \sum_{i=1}^{K} |\Gamma \cap A_i|^{1/(n-1)} \\
&\leq CK|\Gamma|^{1/(n-1)} = C|\Gamma|^{1/(n-1)}.
\end{aligned}$$

□

Lemma 4.5.5 and Corollary 4.5.6 will be applied to show our desired Lemma 4.5.7 which is an improvement of Lemma 4.3.3 when $n \geq 3$.

Lemma 4.5.7. *Let $n \geq 3$. Let Ω and Γ_1 be the same as in (1.1.1). Then there exists*

$C = C(n,\Omega)$ such that for any $x \in \mathbb{R}^n$ and $t \geq 0$,

$$\int_0^t \int_{\Gamma_1} \Phi(x-y, t-\tau)\, dS(y)\, d\tau \leq C|\Gamma_1|^{1/(n-1)}. \tag{4.5.12}$$

Proof. In this proof, unless otherwise stated, C represents constants which only depend on n and Ω. First, by the explicit formula (1.4.4) of Φ and a change of variable in τ, we have

$$\int_0^t \int_{\Gamma_1} \Phi(x-y, t-\tau)\, dS(y)\, d\tau = C \int_{\Gamma_1} \int_0^t \tau^{-n/2} e^{-|x-y|^2/(4\tau)}\, d\tau\, dS(y).$$

Then by the change of variable $s = |x-y|^2/(4\tau)$ for τ,

$$\int_{\Gamma_1} \int_0^t \tau^{-n/2} e^{-|x-y|^2/(4\tau)}\, d\tau\, dS(y) \leq C \int_{\Gamma_1} \frac{1}{|x-y|^{n-2}} \int_{|x-y|^2/(4t)}^{\infty} s^{\frac{n}{2}-2} e^{-s}\, ds\, dS(y). \tag{4.5.13}$$

Since $n \geq 3$, $s^{\frac{n}{2}-2}e^{-s}$ is integrable on $(0,\infty)$. As a result,

$$\begin{aligned} \int_{\Gamma_1} \frac{1}{|x-y|^{n-2}} \int_{|x-y|^2/(4t)}^{\infty} s^{\frac{n}{2}-2} e^{-s}\, ds\, dS(y) &\leq \int_{\Gamma_1} \frac{1}{|x-y|^{n-2}} \int_0^{\infty} s^{\frac{n}{2}-2} e^{-s}\, ds\, dS(y) \\ &= C \int_{\Gamma_1} \frac{1}{|x-y|^{n-2}}\, dS(y). \end{aligned}$$

Now applying Corollary 4.5.6,

$$\int_{\Gamma_1} \frac{1}{|x-y|^{n-2}}\, dS(y) \leq C|\Gamma_1|^{1/(n-1)}.$$

$\square$

The following Lemma 4.5.8, Corollary 4.5.9 and Lemma 4.5.10 are parallel results as Lemma 4.5.5, Corollary 4.5.6 and Lemma 4.5.7, but they deal with dimension $n = 2$ rather than $n \geq 3$.

Lemma 4.5.8. *Let Ω be a bounded, open subset of $\mathbb{R}^2$ with C^1 boundary. Let Γ be a relatively open subset of $\partial\Omega$ that is given by a graph as defined in Definition 4.5.4. Then there exists a constant $C = C(\Omega, ||\nabla\phi||_{L^\infty(U)})$, where ϕ and U are the same as in Definition 4.5.4, such that for any $x \in \overline{\Omega}$,*

$$\int_\Gamma \ln\Big(\frac{d_\Omega}{|x-y|}\Big)\, dS(y) \leq C|\Gamma| \ln\Big(\frac{1}{|\Gamma|}+1\Big),$$

where d_Ω denotes the diameter of Ω, namely $d_\Omega = \sup\{|u-v| : u, v \in \Omega\}$.

Proof. By Definition 4.5.4, without loss of generality, we can assume there exists a C^1 function $\phi : \mathbb{R} \to \mathbb{R}$ and a bounded, open set $U \subseteq \mathbb{R}$ such that

$$\Gamma = \{(\tilde{y}, \phi(\tilde{y})) : \tilde{y} \in U\}. \tag{4.5.14}$$

In addition, we define

$$f(r) = \begin{cases} \ln\big(\frac{d_\Omega}{r}\big), & 0 < r \leq d_\Omega, \\ 0, & r > d_\Omega. \end{cases} \tag{4.5.15}$$

Since $x = (\tilde{x}, x_n) \in \overline{\Omega}$, then for any $(\tilde{y}, \phi(\tilde{y})) \in \Gamma$,

$$|\tilde{x} - \tilde{y}| \leq |(\tilde{x}, x_n) - (\tilde{y}, \phi(\tilde{y}))| \leq d_\Omega.$$

As a result,

$$\begin{aligned}\int_\Gamma \ln\Big(\frac{d_\Omega}{|x-y|}\Big)\,dS(y) &= \int_U \ln\Big(\frac{d_\Omega}{|(\tilde{x},x_{n)}-(\tilde{y},\phi(\tilde{y}))|}\Big)\sqrt{1+|\nabla\phi(\tilde{y})|^2}\,d\tilde{y}\\ &\le C\int_U \ln\Big(\frac{d_\Omega}{|\tilde{x}-\tilde{y}|}\Big)\,d\tilde{y}\\ &= C\int_U f(|\tilde{x}-\tilde{y}|)\,d\tilde{y}. \end{aligned} \tag{4.5.16}$$

Now it follows from Lemma 4.5.3 that

$$\begin{aligned}\int_U f(|\tilde{x}-\tilde{y}|)\,d\tilde{y} &\le \int_{B_R(0)} f(|\tilde{z}|)\,d\tilde{z}\\ &= 2\int_0^R f(r)\,dr, \end{aligned} \tag{4.5.17}$$

where $|B_R(0))| = |U|$, namely $2R = |U|$.

For any $\tilde{y}_1, \tilde{y}_2 \in U$,

$$|\tilde{y}_1-\tilde{y}_2| \le |(\tilde{y}_1,\phi(\tilde{y}_1))-(\tilde{y}_2,\phi(\tilde{y}_2))| \le d_\Omega,$$

which implies $\mathrm{diam}(U) \le d_\Omega$. Moreover, since $U \subseteq \mathbb{R}$, then $|U| \le \mathrm{diam}(U) \le d_\Omega$. Thus, $R = |U|/2 \le d_\Omega/2$. So it follows from (4.5.15) that

$$\begin{aligned}\int_0^R f(r)\,dr &= \int_0^R \ln\Big(\frac{d_\Omega}{r}\Big)\,dr\\ &= R\Big[\ln\Big(\frac{d_\Omega}{R}\Big)+1\Big]. \end{aligned} \tag{4.5.18}$$

Again by the parametrization 4.5.14, it is readily seen that $|U| \le |\Gamma|$. Therefore,

$$R \le \min\left\{\frac{|\Gamma|}{2}, \frac{d_\Omega}{2}\right\}.$$

Define

$$g(r) = r\Big[\ln\Big(\frac{d_\Omega}{r}\Big) + 1\Big], \quad \forall\, r > 0.$$

Then g is increasing when $r \in (0, d_\Omega]$. In addition, (4.5.18) implies that

$$\int_0^R f(r)\,dr = g(R).$$

Next, we will estimate $g(R)$ in the following two situations.

- $|\Gamma| \le d_\Omega$.

$$\begin{aligned} g(R) \le g(|\Gamma|) &= |\Gamma|\Big[\ln\Big(\frac{d_\Omega}{|\Gamma|}\Big) + 1\Big] \\ &= |\Gamma|\Big[\ln\Big(\frac{1}{|\Gamma|}\Big) + \ln(d_\Omega) + 1\Big] \\ &\le C|\Gamma|\ln\Big(\frac{1}{|\Gamma|} + 1\Big) \end{aligned} \tag{4.5.19}$$

  for some constant C only depending on Ω.

- $|\Gamma| > d_\Omega$.

$$g(R) \le g(d_\Omega) = d_\Omega.$$

  Define

$$h(r) = r\ln\Big(\frac{1}{r} + 1\Big), \quad \forall\, r > 0. \tag{4.5.20}$$

Then

$$h''(r) = -\frac{1}{r(1+r)^2} < 0, \quad \forall r > 0.$$

This implies $h'(r) > 0$ for any $r > 0$, since $\lim_{r\to\infty} h'(r) = 0$. Hence, h is an increasing function and

$$|\Gamma| \ln\Big(\frac{1}{|\Gamma|} + 1\Big) = h(|\Gamma|) \geq h(d_\Omega) = d_\Omega \ln\Big(\frac{1}{d_\Omega} + 1\Big).$$

Thus, there exists a constant only depending on Ω such that

$$g(R) \leq C|\Gamma| \ln\Big(\frac{1}{|\Gamma|} + 1\Big). \tag{4.5.21}$$

Combining (4.5.16) through (4.5.21), the conclusion follows. $\square$

Corollary 4.5.9. *Let Ω be a bounded, open subset of $\mathbb{R}^2$ with C^1 boundary. Then there exists a constant $C = C(\Omega)$ such that for any relative open subset Γ of $\partial\Omega$ and for any $x \in \overline{\Omega}$,*

$$\int_\Gamma \ln\Big(\frac{d_\Omega}{|x-y|}\Big)\, dS(y) \leq C|\Gamma| \ln\Big(\frac{1}{|\Gamma|} + 1\Big),$$

where d_Ω denotes the diameter of Ω, namely $d_\Omega = \sup\{|u-v| : u, v \in \Omega\}$.

Proof. Similar to the proof of Corollary 4.5.6, we first decompose $\partial\Omega$ as that in (4.5.11). Then

$$\int_\Gamma \ln\Big(\frac{d_\Omega}{|x-y|}\Big)\, dS(y) \leq \sum_{i=1}^{K} \int_{\Gamma\cap A_i} \ln\Big(\frac{d_\Omega}{|x-y|}\Big)\, dS(y).$$

Now since each $\Gamma \cap A_i$ is given by a graph, we can apply Lemma 4.5.8 to conclude there exists a constant $C = C(\Omega)$ such that for each $1 \le i \le K$,

$$\int_{\Gamma \cap A_i} \ln\Big(\frac{d_\Omega}{|x-y|}\Big)\, dS(y) \le C|\Gamma \cap A_i| \ln\Big(\frac{1}{|\Gamma \cap A_i|} + 1\Big).$$

Recalling the function h defined in (4.5.20) is an increasing function, so

$$|\Gamma \cap A_i| \ln\Big(\frac{1}{|\Gamma \cap A_i|} + 1\Big) \le |\Gamma| \ln\Big(\frac{1}{|\Gamma|} + 1\Big).$$

As a result,

$$\int_{\Gamma} \ln\Big(\frac{d_\Omega}{|x-y|}\Big)\, dS(y) \le C|\Gamma| \ln\Big(\frac{1}{|\Gamma|} + 1\Big).$$

□

Next, Lemma 4.5.8 and Corollary 4.5.9 will be applied to show our desired Lemma 4.5.10 which is an improvement of Lemma 4.3.3 when $n = 2$.

Lemma 4.5.10. *Let $n = 2$. Let Ω and Γ_1 be the same as in (1.1.1). Then there exists $C = C(\Omega)$ such that for any $x \in \overline{\Omega}$ and $0 < t \le 1$,*

$$\int_0^t \int_{\Gamma_1} \Phi(x-y, t-\tau)\, dS(y)\, d\tau \le C|\Gamma_1| \ln\Big(\frac{1}{|\Gamma_1|} + 1\Big). \tag{4.5.22}$$

Proof. We proceed similarly as that in the proof of Lemma 4.5.7 until (4.5.13). Next, the situation is different since $s^{n/2-2}e^{-s}$ is not integrable near $s = 0$ when $n = 2$. For convenience, we rewrite (4.5.13) when $n = 2$ as following:

$$\int_{\Gamma_1} \int_0^t \tau^{-1} e^{-|x-y|^2/(4\tau)}\, d\tau\, dS(y) \le C \int_{\Gamma_1} \int_{|x-y|^2/(4t)}^{\infty} s^{-1} e^{-s}\, ds\, dS(y). \tag{4.5.23}$$

Since $t \le 1$ and $x \in \overline{\Omega}$, $|x-y|^2/(4t) \ge |x-y|^2/4$. Thus,

$$\begin{aligned}
\int_{|x-y|^2/(4t)}^{\infty} s^{-1}e^{-s}\,ds &\le \int_{|x-y|^2/4}^{\infty} s^{-1}e^{-s}\,ds \\
&= \int_{|x-y|^2/4}^{d_\Omega^2} s^{-1}e^{-s}\,ds + \int_{d_\Omega^2}^{\infty} s^{-1}e^{-s}\,ds \\
&\le \int_{|x-y|^2/4}^{d_\Omega^2} s^{-1}\,ds + \frac{1}{d_\Omega^2}\int_{d_\Omega^2}^{\infty} e^{-s}\,ds \\
&= 2\ln\left(\frac{d_\Omega}{|x-y|}\right) + C.
\end{aligned}$$

As a result,

$$\int_{\Gamma_1}\int_{|x-y|^2/(4t)}^{\infty} s^{-1}e^{-s}\,ds\,dS(y) \le C|\Gamma_1| + 2\int_{\Gamma_1} \ln\left(\frac{d_\Omega}{|x-y|}\right) dS(y). \tag{4.5.24}$$

Now applying Corollary 4.5.9,

$$\int_{\Gamma_1} \ln\left(\frac{d_\Omega}{|x-y|}\right) dS(y) \le C|\Gamma_1| \ln\left(\frac{1}{|\Gamma_1|}+1\right).$$

Finally noticing that

$$\begin{aligned}
|\Gamma_1| &\le \frac{1}{\ln\left(\frac{1}{|\partial\Omega|}+1\right)} |\Gamma_1| \ln\left(\frac{1}{|\Gamma_1|}+1\right) \\
&= C|\Gamma_1| \ln\left(\frac{1}{|\Gamma_1|}+1\right),
\end{aligned}$$

the lemma is proved. $\square$

4.5.2 Proof of Theorem 4.1.3

The idea of the proof in this subsection is similar to that in Subsection 4.4.1. The main difference is that in the proof of Theorem 4.1.4, we treat t_* as a variable in $(0, 1]$ and the choice of $\{M_k\}$ depends on t_*; however, in the proof below, the lower bound t_* will be a fixed number in $(0, 1]$ and the choice of $\{M_k\}$ does not depend on t_*. On the technical aspects,

- First, Lemma 4.5.4 will be used instead of Lemma 4.4.8 to overcome the lack of global convexity.
- Secondly, Lemma 4.5.7 and Lemma 4.5.10 will be exploited in place of Lemma 4.3.3 to obtain higher order of $|\Gamma_1|^{-1}$ for the lower bound of T^* as $|\Gamma_1| \searrow 0$. The price for obtaining this higher order is that the results in this section only work for small $|\Gamma_1|$.

Proof of Theorem 4.1.3. We will only give the proof for the case $n \geq 3$, since the proof for $n = 2$ follows the same argument except applying Lemma 4.5.10 instead of Lemma 4.5.7 to estimate the last term III_k in (4.5.27).

First, without loss of generality, we can assume $d \leq 1$ for convenience. In the proof below, $C_i (1 \leq i \leq 3)$ and $C_j^* (1 \leq j \leq 2)$ denote constants which only depend on n, Ω and d. Let $M(t)$ be defined as in (1.4.3). The first step of the proof is to find a constant $t_* \in (0, 1]$ and a finite strictly increasing sequence $\{M_k\}_{0 \leq k \leq L}$ such that if T_k denotes the first time that $M(t) = M_k$, then $T_k - T_{k-1} \geq t_*$ for $1 \leq k \leq L$. The second step is to derive a lower bound for Lt_*.

Step 1. Let $t_* \in (0, 1]$ which will be determined later in this step. Define M_0 as that in (1.4.2) and $T_0 = 0$. Then for $k \geq q$, suppose M_{k-1} has been constructed, we are trying to define M_k such that $T_k - T_{k-1} \geq t_*$.

Denote $t_k = T_k - T_{k-1}$. We will first check what happens if $t_k \leq 1$. By the maximum principle and the Hopf lemma, there exists $x^k \in \overline{\Gamma}_1$ such that

$$u(x^k, T_k) = M_k. \tag{4.5.25}$$

Applying the representation formula (4.2.4) with $T = T_{k-1}$ and $(x,t) = (x^k, t_k)$,

$$\begin{aligned} u(x^k, T_k) = {} & 2\int_\Omega \Phi(x^k - y, t_k)\, u(y, T_{k-1})\, dy \\ & - 2\int_0^{t_k}\int_{\partial\Omega} \frac{\partial\Phi(x^k - y, t-\tau)}{\partial n(y)}\, u(y, T_{k-1}+\tau)\, dS(y)\, d\tau \\ & + 2\int_0^{t_k}\int_{\Gamma_1} \Phi(x^k - y, t_k - \tau)\, u^q(y, T_{k-1}+\tau)\, dS(y)\, d\tau. \end{aligned} \tag{4.5.26}$$

Noticing (4.5.25), the above equality implies that

$$\begin{aligned} M_k \leq {} & 2M_{k-1}\int_\Omega \Phi(x^k - y, t_k)\, dy + 2M_k \int_0^{t_k}\int_{\partial\Omega} \Big| \frac{\partial\Phi(x^k - y, t_k-\tau)}{\partial n(y)} \Big|\, dS(y)\, d\tau \\ & + 2M_k^q \int_0^{t_k}\int_{\Gamma_1} \Phi(x^k - y, t_k - \tau)\, dS(y)\, d\tau \\ = {} & 2(M_{k-1} - M_k)\int_\Omega \Phi(x^k - y, t_k)\, dy \\ & + 2M_k\Big(\int_\Omega \Phi(x^k - y, t_k)\, dy + \int_0^{t_k}\int_{\partial\Omega} \Big| \frac{\partial\Phi(x^k - y, t_k-\tau)}{\partial n(y)} \Big|\, dS(y)\, d\tau \Big) \\ & + 2M_k^q \int_0^{t_k}\int_{\Gamma_1} \Phi(x^k - y, t_k - \tau)\, dS(y)\, d\tau \\ \triangleq {} & I_k + II_k + III_k. \end{aligned} \tag{4.5.27}$$

First, since $t_k \leq 1$, Lemma 4.4.3 yields that

$$I_k \leq 2(M_{k-1} - M_k) b_1, \tag{4.5.28}$$

where b_1 is the same as in (4.4.10). Secondly it follows from Corollary 4.5.2 that there exists a constant C_1 such that

$$II_k \leq M_k + C_1 M_k \exp\Big(-\frac{d^2}{8t_k}\Big). \tag{4.5.29}$$

Thirdly, Lemma 4.5.7 implies the existence of a constant C_2 such that

$$III_k \leq C_2 M_k^q |\Gamma_1|^{1/(n-1)}. \tag{4.5.30}$$

Combining (4.5.27), (4.5.28), (4.5.29) and (4.5.30),

$$M_k \leq 2b_1(M_{k-1} - M_k) + M_k + C_1 M_k \exp\Big(-\frac{d^2}{8t_k}\Big) + C_2 M_k^q |\Gamma_1|^{1/(n-1)}.$$

Subtracting M_k from both sides,

$$2b_1(M_k - M_{k-1}) \leq C_1 M_k \exp\Big(-\frac{d^2}{8t_k}\Big) + C_2 M_k^q |\Gamma_1|^{1/(n-1)}.$$

Dividing by $2b_1$ and rearranging the equation,

$$\Big[1 - \frac{C_1}{2b_1} \exp\Big(-\frac{d^2}{8t_k}\Big)\Big] M_k - M_{k-1} \leq \frac{C_2 M_k^q |\Gamma_1|^{1/(n-1)}}{2b_1}.$$

Define

$$C_1^* = \max\Big\{\frac{C_1}{2b_1}, 1\Big\}, \quad C_2^* = \frac{C_2}{2b_1}, \tag{4.5.31}$$

then

$$\Big[1 - C_1^* \exp\Big(-\frac{d^2}{8t_k}\Big)\Big] M_k - M_{k-1} \leq C_2^* M_k^q |\Gamma_1|^{1/(n-1)}. \tag{4.5.32}$$

Let us further temporarily assume t_k is so small that

$$\exp\Big(-\frac{d^2}{8t_k}\Big) \le \frac{1}{2C_1^*}\,\frac{M_k - M_{k-1}}{M_k}, \tag{4.5.33}$$

which is equivalent to

$$\Big[1 - C_1^* \exp\Big(-\frac{d^2}{8t_k}\Big)\Big] M_k - M_{k-1} \ge \frac{1}{2}(M_k - M_{k-1}).$$

Then it follows from (4.5.32) that

$$\frac{M_k - M_{k-1}}{M_k^q} \le 2C_2^* |\Gamma_1|^{1/(n-1)}. \tag{4.5.34}$$

As a summary, this paragraph claims that if $t_k \le 1$ and (4.5.33) holds, then M_k will satisfy (4.5.34).

Based on this observation, denote

$$\delta_1 = 4C_2^* |\Gamma_1|^{1/(n-1)}. \tag{4.5.35}$$

Then we define M_k to be the solution (if it exists) to

$$\frac{M_k - M_{k-1}}{M_k^q} = \delta_1. \tag{4.5.36}$$

With this choice of M_k, it is evident from (4.5.34) that either $t_k > 1$ or t_k violates (4.5.33).

Due to (4.5.36), t_k violating (4.5.33) implies

$$\exp\Big(-\frac{d^2}{8t_k}\Big) > \frac{1}{2C_1^*}\,\frac{M_k - M_{k-1}}{M_k} = \frac{M_k^{q-1}\delta_1}{2C_1^*} \geq \frac{M_0^{q-1}\delta_1}{2C_1^*}. \tag{4.5.37}$$

Now if (the following requirement will be clear later)

$$M_0^{q-1}\delta_1 \leq \frac{1}{6q}, \tag{4.5.38}$$

then the right hand side of (4.5.37) is smaller than 1. Therefore, (4.5.37) is equivalent to

$$t_k > \frac{d^2}{8}\Big[\ln\Big(\frac{2C_1^*}{M_0^{q-1}\delta_1}\Big)\Big]^{-1}.$$

Define

$$t_* = \frac{d^2}{8}\Big[\ln\Big(\frac{2C_1^*}{M_0^{q-1}\delta_1}\Big)\Big]^{-1}. \tag{4.5.39}$$

Since $d \leq 1$ and $C_1^* \geq 1$, it is obvious that $t_* \in (0, 1]$. Moreover, we can conclude that

$$t_k \geq \min\{1, t_*\} = t_*.$$

On the other hand, by applying Lemma 4.4.4, (4.5.36) has a solution $M_k > M_{k-1}$ if and only if $\delta_1 \leq M_{k-1}^{1-q}E_q$. In addition, as long as such a solution exists, M_k can be chosen to satisfy

$$M_{k-1} < M_k \leq \frac{q}{q-1}\,M_{k-1}.$$

Thus, the strategy of constructing $\{M_k\}$ is summarized as following. First, M_0 is defined to be the same as in (1.4.2). Next, for $k \geq 1$, suppose M_{k-1} has been constructed, then based on Lemma 4.4.4, whether defining M_k depends on how large M_{k-1} is.

- $\diamond$ If $M_{k-1}^{q-1}\,\delta_1 \le E_q$, then we define $M_k \in \big(M_{k-1}, \frac{q}{q-1}\, M_{k-1}\big]$ to be the solution to (4.5.36).

- $\diamond$ If $M_{k-1}^{q-1}\,\delta_1 > E_q$, then there does not exist $M_k > M_{k-1}$ which solves (4.5.36). So we do not define M_k and stop the construction.

Based on this construction, if $\{M_k\}_{1\le k\le L_0}$ have been defined, then for any $1 \le k \le L_0$, $T_k - T_{k-1} \ge t_*$. Therefore, $T_k \ge kt_*$ for any $1 \le k \le L_0$. Applying Theorem 3.1.1, $L_0 \le T^*/t_* < \infty$, which means the cardinality of $\{M_k\}$ has be to finite. So we can assume the constructed sequence is $\{M_k\}_{0\le k\le L}$ for some finite L.

Step 2. By Lemma 4.4.5,

$$L > \frac{1}{10(q-1)}\Big(\frac{1}{M_0^{q-1}\delta_1} - 3q\Big).$$

Taking advantage of the requirement (4.5.38),

$$T^* \ge Lt_* > \frac{t_*}{20(q-1)M_0^{q-1}\delta_1}.$$

Writing

$$Y = M_0^{q-1}|\Gamma_1|^{1/(n-1)},$$

then (4.5.38) reduces to

$$Y \le \frac{1}{24C_2^* q}.$$

In addition, recalling the definition (4.5.39) of t_*, then

$$T^* \geq \frac{C_3}{(q-1)Y}\Big[\ln\Big(\frac{C_1^*}{2C_2^*Y}\Big)\Big]^{-1}$$

for some constant C_3. Define

$$Y_0 = \min\Big\{\frac{1}{24C_2^*}, \frac{2C_2^*}{C_1^*}, 1\Big\}.$$

Then for any $Y \leq \frac{Y_0}{q}$,

$$\ln\Big(\frac{C_1^*}{2C_2^*Y}\Big) \leq 2\ln\Big(\frac{1}{Y}\Big) = 2|\ln Y|.$$

Hence,

$$T^* \geq \frac{C_3}{2(q-1)Y|\ln Y|}.$$

$\square$

Remark 4.5.11. *If the whole domain Ω is convex, then due to Lemma 4.4.2, (4.5.29) becomes*

$$II_k \leq M_k.$$

Based on this change, all the exponential terms in the proof of Theorem 4.1.3 will disappear. As a result, t_* can be just chosen as 1 instead of the expression (4.5.39) which contains the logarithm term in the denominator. Consequently, the logarithm term which is in the denominator of (4.1.3) and (4.1.4) will also disappear. Namely, the lower bound in (4.1.3) and (4.1.4) can be improved to be

$$T^* \geq \frac{C}{(q-1)Y}.$$

4.6 Comparison with previous works

As mentioned in the introduction, there are vast literature on the blow-up problems of the parabolic type equations. But few of them deal with the lower bound estimate of the blow-up time. A popular method dealing with the lower bound is established in [32–34]. After that, the similar idea is also applied to some more generalized problems, see [6, 30, 40, 41, 43]. In this section, we will compare Theorem 4.1.4 with the result in [34].

In [34], it studied the problem

$$\begin{cases} u_t(x,t) = \Delta u(x,t) & \text{in} \quad \Omega \times (0,T], \\ \frac{\partial u(x,t)}{\partial n(x)} = F\big(u(x,t)\big) & \text{on} \quad \partial\Omega \times (0,T], \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{4.6.1}$$

where Ω is a convex, bounded open subset in $\mathbb{R}^3$ with smooth boundary and

$$0 \le F(s) \le ks^m \tag{4.6.2}$$

for some $k > 0$ and $m \ge 3/2$. By introducing the energy function

$$\varphi(t) = \int_\Omega u^{4(m-1)}(x,t)\,dx$$

and adopting a Sobolev-type inequality developed in [32], they derive a first order differential inequality for $\varphi(t)$ and then obtain a lower bound for T^*:

$$T^* \ge \int_{\varphi(0)}^{\infty} \frac{d\eta}{K_1\eta + K_2\eta^{3/2} + K_3\eta^3}, \tag{4.6.3}$$

where K_1, K_2 and K_3 are some positive constants depending on Ω and m.

Let us compare Theorem 4.1.4 with (4.6.3) for the problem (4.6.1) where $F(s) = s^q$. For convenience of statement, the lower bounds in Theorem 4.1.4 and (4.6.3) are denoted as T_1 and T_2 respectively.

- First, T_1 works for any $q > 1$ and also give the exact asymptotic rate of T^* as $q \searrow 1$. However, T_2 is valid only for $q \geq 3/2$, due to the restriction (4.6.2) and $m \geq 3/2$ (let $k = 1$ and $m = q$).

- Secondly, as $M_0 \searrow 0$, T_1 is of order $M_0^{-(q-1)}$; however, if the initial data u_0 does not oscillate too much, that is

  $$\varphi(0) = \int_\Omega u_0^{4(q-1)}(x)\, dx \sim M_0^{4(q-1)}\, |\Omega|,$$

  then

  $$T_2 \sim \ln\left(\frac{1}{\varphi(0)}\right) \sim 4(q-1)\ln(M_0^{-1}),$$

  which is only a logarithm order of M_0^{-1}.

- Thirdly, as $M_0 \to \infty$, T_1 is of order $M_0^{-2(q-1)}$; however, if the initial data u_0 again does not oscillate too much, then T_2 grows like

  $$T_2 \sim [\varphi(0)]^{-2} \sim M_0^{-8(q-1)}.$$

  Since M_0 is large,

  $$M_0^{-2(q-1)} >> M_0^{-8(q-1)}.$$

Chapter 5

Prevention of Blowup

5.1 Main theorems and outline of the approach

In this chapter, we provide two methods to prevent the finite-time blowup. The first strategy is to repair the damaged part. The second way is by adding a suitable pump near the damaged part. In this chapter, for any $\alpha \in \left[0, \frac{1}{n-1}\right)$, n_α is defined as in (4.3.17).

Part I: Repairing the broken part

The first strategy is to repair the broken part Γ_1 in the original problem (1.1.1). We are trying to find the repairing rate at which the temperature can be prevented from blowing up in finite time. The setup below will be studied.

$$\begin{cases} u_t(x,t) = \Delta u(x,t) & \text{in} \quad \Omega \times (0,T], \\ \frac{\partial u(x,t)}{\partial n(x)} = u^q(x,t) & \text{on} \quad \Gamma_{1,t} \times (0,T], \\ \frac{\partial u(x,t)}{\partial n(x)} = 0 & \text{on} \quad \Gamma_{2,t} \times (0,T], \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{5.1.1}$$

where $\Gamma_{1,t}$ represents the broken part at time t and $\Gamma_{1,0} = \Gamma_1$. However, since the broken boundary in problem (5.1.1) is changing, the existence of the solution is harder to justify. So in this part, we will assume the existence of the weak solution to (5.1.1) as in Definition

5.1.1 and then show that the weak solution does not blow up in finite time.

Definition 5.1.1. *Given $T > 0$, a continuous function u on $\overline{\Omega} \times [0, T]$ is called a weak solution to (5.1.1) if for any $T_0 \in [0, T)$, $t \in (0, T - T_0]$ and for any $\phi \in C^2(\overline{\Omega} \times [0, t])$,*

$$
\begin{aligned}
& \int_0^t \int_\Omega u(y, T_0 + \tau)(\phi_\tau + \Delta\phi)(y, \tau)\, dy\, d\tau \\
= \; & \int_\Omega u(y, T_0 + t)\phi(y, t) - u(y, T_0)\phi(y, 0)\, dy - \int_0^t \int_{\Gamma_{1,\tau}} u^q(y, T_0 + \tau)\phi(y, \tau)\, dS(y)\, d\tau \\
& + \int_0^t \int_{\partial\Omega} u(y, T_0 + \tau)\, \frac{\partial\phi(y, \tau)}{\partial n(y)}\, dS(y)\, d\tau. \qquad (5.1.2)
\end{aligned}
$$

It is readily seen that if u is a smooth solution, then it is a weak solution. In the rest of this section, we will only deal with the weak solution. The first result below works for any C^2 domain Ω and it says that as long as the area of $|\Gamma_{1,t}|$ decreases at some exponential rate, the temperature will not blow up in finite time.

Theorem 5.1.2. *Let M_0 be defined as in (1.4.2). Then there exists a constant $C = C(n, \Omega, q, M_0)$ such that if*

$$|\Gamma_{1,t}| \le e^{-Ct}|\Gamma_1|,$$

then for any weak solution u to (5.1.1) on $\overline{\Omega} \times [0, T]$,

$$u(x, t) \le 3M_0 e^{Ct}, \quad \forall\, (x, t) \in \overline{\Omega} \times [0, T].$$

The second result deals with convex domains and it says that as long as $|\Gamma_{1,t}|$ decreases at a suitable exponential rate, the temperature can be bounded by any data that is larger than the initial maximum.

Theorem 5.1.3. *Assume Ω is convex. Let M_0 be defined as in (1.4.2). Then for any $B > M_0$, there exists a constant $C = C(n, \Omega, q, M_0, B)$ such that if*

$$|\Gamma_{1,t}| \leq e^{-Ct} |\Gamma_1|,$$

then for any weak solution u to (5.1.1) on $\overline{\Omega} \times [0, T]$,

$$u(x,t) \leq B, \quad \forall\, x \in \overline{\Omega},\, 0 \leq t \leq T.$$

Again, the global convexity may not be practical in real applications. So we want to obtain a similar result by only assuming local convexity near Γ_1. But this time, we can only prove the boundedness of the temperature with double-exponential decay rate of $|\Gamma_{1,t}|$.

Theorem 5.1.4. *Assume Conv$\left([\Gamma_1]_d\right) \subseteq \overline{\Omega}$ for some $d > 0$. Let M_0 be defined as in (1.4.2). Then for any $B > M_0$, there exists a constant $C = C(n, \Omega, d, q, M_0, B)$ such that if*

$$|\Gamma_{1,t}| \leq \exp\left[2\,(n-1)\left(1 - e^{Ct}\right)\right] |\Gamma_1|,$$

then for any weak solution u to (5.1.1) on $\overline{\Omega} \times [0, T]$,

$$u(x,t) \leq B, \quad \forall\, x \in \overline{\Omega},\, 0 \leq t \leq T.$$

Part II: Adding a pump In this part, we consider adding a negative source locally to prevent the finite-time blowup. Now the problem becomes

$$\begin{cases} u_t(x,t) = \Delta u(x,t) - \psi_1(x)\, u^p(x,t) & \text{in} \quad \Omega \times (0,T] \\ \frac{\partial u(x,t)}{\partial n(x)} = u^q(x,t) & \text{on} \quad \Gamma_1 \times (0,T] \\ \frac{\partial u(x,t)}{\partial n(x)} = 0 & \text{on} \quad \Gamma_2 \times (0,T] \\ u(x,0) = u_0(x) & \text{in} \quad \Omega \end{cases} \tag{5.1.3}$$

where (see Figure 5.1) $\Omega_1 \Subset \Omega_2 \Subset \Omega$, $\Gamma_1 \Subset \Gamma$, $p > 1, q > 1$ and

$$\psi_1(x) \begin{cases} = 1, & x \in \Omega_1, \\ = 0, & x \in \Omega \setminus \Omega_2, \\ \in (0,1), & x \in \Omega_2 \setminus \Omega_1 \end{cases}$$

is a smooth function.

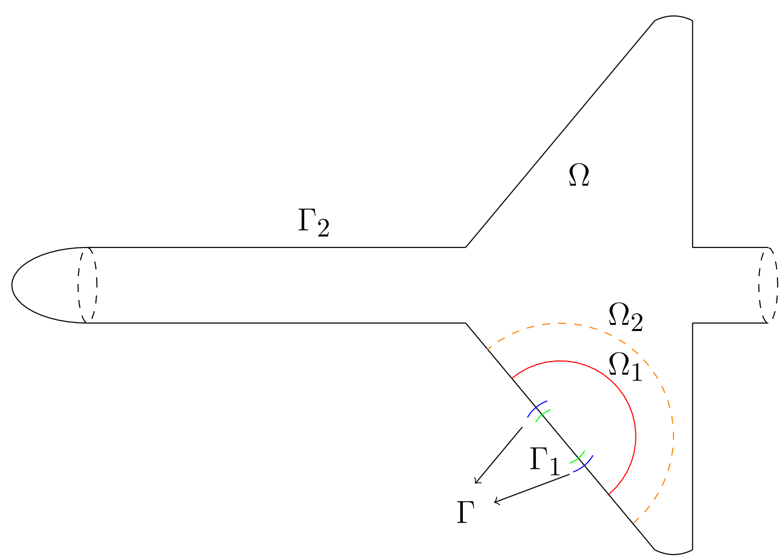

Figure 5.1: Model with a pump

For problem (5.1.3), by the similar outline as that in Section 2.4, we can show it has a local nonnegative solution and the solution can extend as long as the L^∞ norm of u is finite.

We want to demonstrate that by choosing suitable p, the solution will not blow up, which implies the solution is global.

Now we consider the following problem

$$\begin{cases} v_t(x,t) = \Delta v(x,t) - \psi_1(x)\, v^p(x,t) & \text{in} \quad \Omega \times (0,T], \\ \frac{\partial v(x,t)}{\partial n(x)} = \eta(x)\, v^q(x,t) & \text{on} \quad \partial\Omega \times (0,T], \\ v(x,0) = u_0(x), & \text{in} \quad \Omega, \end{cases} \tag{5.1.4}$$

where

$$\eta(x) \begin{cases} = 1, & x \in \Gamma_1, \\ = 0, & x \in \partial\Omega \setminus \Gamma, \\ \in (0,1), & x \in \Gamma \setminus \Gamma_1. \end{cases}$$

By comparison principle, $v \geq u$. So if we can prove that v is always finite, then u will not blow up. The following conclusion is valid for any C^2 domain Ω.

Theorem 5.1.5. *If $p > 1$, $q > 1$ and $p > 2q - 1$, then the solution v to (5.1.4) does not blow up in finite time. As a result, the solution u to (5.1.3) exists globally.*

The organization of this chapter is that in Section 5.2, we discuss how to prevent the finite-time blowup by repairing the broken part. In Section 5.3, we provide another way to control the solution by adding a suitable pump.

5.2 Repairing the broken part

In this section, we study the problem (5.1.1). The results are divided into three subsections, due to different geometric properties of Ω. The Subsection 5.2.1 deals with any C^2 domains but the conclusion only prevents blowup in finite time without providing any specific bound. Next, both Subsection 5.2.2 and Subsection 5.2.3 try to control the temperature under any value that is larger than the initial maximum. Subsection 5.2.2 assumes the global convexity of Ω while Subsection 5.2.3 only requires the local convexity near Γ_1.

5.2.1 Prevention of finite-time blowup

The idea in this subsection is similar to that in Subsection 4.3.1. The difference is that we will use the decay of $|\Gamma_{1,t}|$ to eliminate the nonlinear effect on the boundary. First, we need the analogous representation formulas for the weak solution. The first lemma is the representation formula of the solution for the inside points. Then its corollary is the representation formula for the boundary points.

Lemma 5.2.1. *Assume u is a weak solution to (5.1.1) on $\overline{\Omega} \times [0,T]$ for some $T > 0$. Then for any $x \in \Omega$, $T_0 \in [0,T)$ and $t \in (0, T - T_0]$,*

$$\begin{aligned} u(x, T_0 + t) \;=\; & \int_\Omega \Phi(x-y,t)\, u(y,T_0)\, dy + \int_0^t \int_{\Gamma_{1,\tau}} \Phi(x-y,t-\tau)\, u^q(y, T_0+\tau)\, dS(y)\, d\tau \\ & - \int_0^t \int_{\partial\Omega} \frac{\partial \Phi(x-y,t-\tau)}{\partial n(y)}\, u(y, T_0+\tau)\, dS(y)\, d\tau \end{aligned} \tag{5.2.1}$$

Proof. This proof is similar to that of Lemma 4.2.3. Fix any $x \in \Omega$, $T_0 \in [0,T)$ and $t \in (0, T-T_0]$. For $\epsilon > 0$, define

$$\phi_\epsilon(y,\tau) = \Phi(x-y, t-\tau+\epsilon).$$

Plugging ϕ_ϵ into (5.1.2) and sending $\epsilon \to 0$, (5.2.1) will be justified. $\square$

Corollary 5.2.2. *Assume u is a weak solution to (5.1.1) on $\overline{\Omega} \times [0, T]$ for some $T > 0$. Then for any $x \in \partial\Omega$, $T_0 \in [0, T)$ and $t \in (0, T - T_0]$,*

$$u(x, T_0 + t) = 2 \int_\Omega \Phi(x - y, t) u(y, T_0)\, dy + 2 \int_0^t \int_{\Gamma_{1,\tau}} \Phi(x - y, t - \tau) u^q(y, T_0 + \tau) dS(y) d\tau$$

$$- 2 \int_0^t \int_{\partial\Omega} \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} u(y, T_0 + \tau) dS(y) d\tau. \tag{5.2.2}$$

Proof. Similar to the proof for Corollary 4.2.4, by Lemma 5.2.1 and the jump relation of the double-layer heat potential, (5.2.2) can be proved. □

In the proof below as well as the proofs for Theorem 5.1.3 and Theorem 5.1.4, Corollary 5.2.2 play an essential role.

Proof of Theorem 5.1.2. Applying the representation formula (5.2.2) to (5.1.1) with $T_0 = 0$, we get for any $x \in \partial\Omega$ and $t \in (0, T)$,

$$u(x, t) = 2 \int_\Omega \Phi(x - y, t)\, u_0(y)\, dy - 2 \int_0^t \int_{\partial\Omega} \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)}\, u(y, \tau)\, dS(y)\, d\tau$$
$$+ 2 \int_0^t \int_{\Gamma_{1,\tau}} \Phi(x - y, t - \tau)\, u^q(y, \tau)\, dS(y)\, d\tau.$$

Define $\widetilde{M}$ as (4.3.3). Then for any $m > 1$,

$$\begin{aligned}
\widetilde{M}(t) &\le 2M_0 + 2 \int_0^t \widetilde{M}(\tau) \int_{\partial\Omega} \Big| \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} \Big|\, dS(y)\, d\tau \\
&\quad + 2 \int_0^t \widetilde{M}^q(\tau) \int_{\Gamma_{1,\tau}} \Phi(x - y, t - \tau)\, dS(y)\, d\tau \\
&\le 2M_0 + C \int_0^t (t - \tau)^{-\frac{1}{2}} \widetilde{M}(\tau)\, d\tau + C \int_0^t \widetilde{M}^q(\tau)\, |\Gamma_{1,\tau}|^{\frac{m-1}{m}}\, (t - \tau)^{-\frac{n}{2} + \frac{n-1}{2m}}\, d\tau.
\end{aligned}$$

In order for the integrability of the last term of the above inequality, the power $-\frac{n}{2}+\frac{n-1}{2m}$ should be greater than -1, which means $m<\frac{n-1}{n-2}$. Define $A:[0,T]\to\mathbb{R}$ by

$$A(t)=|\Gamma_{1,t}|. \tag{5.2.3}$$

In addition, let

$$\alpha=(m-1)/m \quad \text{and} \quad \beta=1-n_{\alpha}.$$

Then $\alpha\in(0,\frac{1}{n-1})$, $\beta\in(\frac{1}{2},1)$ and

$$\widetilde{M}(t)\le 2M_0+C\int_0^t(t-\tau)^{-\frac{1}{2}}\widetilde{M}(\tau)\,d\tau+C\int_0^t(t-\tau)^{-\beta}\,A^{\alpha}(\tau)\,\widetilde{M}^{q}(\tau)\,d\tau. \tag{5.2.4}$$

If

$$A^{\alpha}(\tau)\widetilde{M}^{q-1}(\tau)\le 3^{q-1}|\Gamma_1|^{\alpha}M_0^{q-1},$$

then

$$\widetilde{M}(t)\le 2M_0+C\int_0^t(t-\tau)^{-\frac{1}{2}}\widetilde{M}(\tau)\,d\tau+C\int_0^t(t-\tau)^{-\beta}\,3^{q-1}|\Gamma_1|^{\alpha}M_0^{q-1}\,\widetilde{M}(\tau)\,d\tau.$$

Now we are looking for a function

$$v(t)=3M_0\,e^{kt} \tag{5.2.5}$$

for some constant k determined later such that

$$v(t)\ge 2M_0+C\int_0^t(t-\tau)^{-\frac{1}{2}}v(\tau)\,d\tau+C\int_0^t(t-\tau)^{-\beta}\,3^{q-1}|\Gamma_1|^{\alpha}M_0^{q-1}\,v(\tau)\,d\tau.$$

Plugging (5.2.5) into the above inequality, we obtain an equivalent form

$$3M_0 e^{kt} \geq 2M_0 + 3CM_0 \int_0^t (t-\tau)^{-1/2} e^{k\tau}\, d\tau + 3^q C M_0^q |\Gamma_1|^\alpha \int_0^t (t-\tau)^{-\beta} e^{k\tau}\, d\tau. \quad (5.2.6)$$

Notice

$$\begin{aligned}
e^{-kt} \int_0^t (t-\tau)^{-\beta} e^{k\tau}\, d\tau &= \int_0^t (t-\tau)^{-\beta} e^{-k(t-\tau)}\, d\tau \\
&= \int_0^t \tau^{-\beta} e^{-k\tau}\, d\tau \\
&= \frac{1}{k} \int_0^{kt} \left(\frac{s}{k}\right)^{-\beta} e^{-s}\, ds \\
&\leq k^{-(1-\beta)} \Gamma(\beta) \\
&= k^{-n_\alpha} \Gamma(1-n_\alpha),
\end{aligned}$$

where Γ is the standard Gamma function. So in order to satisfy (5.2.6), it suffices to have

$$1 \geq \frac{2}{3} e^{-kt} + Ck^{-1/2} + C\, 3^{q-1} M_0^{q-1} |\Gamma_1|^\alpha k^{-n_\alpha}$$

for some constant $C = C(n, \Omega, \alpha)$. Noticing $\frac{2}{3} e^{-kt} \leq 2/3$, so by taking $k \geq 36C^2$, then it suffices to have

$$\frac{1}{6} \geq C\, 3^{q-1} M_0^{q-1} |\Gamma_1|^\alpha k^{-n_\alpha}.$$

The above inequality is equivalent to

$$k \geq (C\, 3^{q-1} M_0^{q-1} |\Gamma_1|^\alpha)^{1/n_\alpha}.$$

Therefore, it suffices to have

$$k = C \max\left\{1, \left(3^{q-1} M_0^{q-1} |\partial\Omega|^{\alpha}\right)^{1/n\alpha}\right\} \tag{5.2.7}$$

for some constant $C = C(n, \Omega, \alpha)$.

By this choice of k, if A satisfies

$$A^{\alpha}(\tau)\, v^{q-1}(\tau) \leq 3^{q-1} M_0^{q-1} |\Gamma_1|^{\alpha}. \tag{5.2.8}$$

Then it follows from (5.2.6) and (5.2.8) that

$$v(t) \geq 2M_0 + C \int_0^t (t-\tau)^{-\frac{1}{2}} v(\tau)\, d\tau + C \int_0^t (t-\tau)^{-\beta} A^{\alpha}(\tau) v^q(\tau)\, d\tau. \tag{5.2.9}$$

Since $v(0) = 3M_0 > M(0)$, then it follows from (5.2.4) and (5.2.9) that

$$\widetilde{M}(t) \leq v(t), \quad \forall\, 0 \leq t \leq T.$$

According to (5.2.5) and (5.2.8),

$$A^{\alpha}(t) \leq e^{-(q-1)kt} |\Gamma_1|^{\alpha},$$

namely

$$A(t) \leq \exp\left[-\frac{(q-1)kt}{\alpha}\right] |\Gamma_1|.$$

Finally, by choosing $\alpha = \frac{1}{2(n-1)}$, or equivalently by choosing $m = \frac{2n-2}{2n-3}$, the proof is finished.

□

5.2.2 Control of the solution under a given value for convex domains

In Subsection 5.2.1, we have discussed the strategy which can prevent the finite-time blowup. But for the practical problems, it is more useful to control the temperature by a certain value. This section provides a way to control the temperature by any value that is larger than the maximum of the initial data under the assumption that Ω is convex. The idea of this subsection is similar to that in Section 4.4.

Proof of Theorem 5.1.3. Define $A(t)$ as in (5.2.3). The goal is to bound u above by B for any $B > M_0$. Let $\epsilon > 0$ such that $B = e^{\epsilon} M_0$ and let $t_* \in (0, 1]$ be a constant which will be determined later. Define $T_0 = 0$ and $p_0 = 1$. For any $k \geq 1$ and $\epsilon > 0$, let

$$M_k = \exp\left[(1 - 2^{-k})\epsilon\right] M_0. \tag{5.2.10}$$

Let T_k be the first time that $M(t)$ reaches M_k, then the maximum principle implies the existence of $x^k \in \partial\Omega$ such that $u(x^k, T_k) = M_k$. In the following, we will use induction to show that $T_k \geq kt_*$. When $k = 0$, it is obviously true.

Step k ($\mathbf{k \geq 1}$): Suppose $T_{k-1} \geq (k-1)t_*$. Let $t_k \triangleq T_k - T_{k-1}$ be the time spent in the kth step. We intend to show $t_k \geq t_*$, which implies $T_k \geq kt_*$. First, if $t_k \geq 1$, then it has already implied that $t_k \geq t_*$. So in the following, we assume $t_k < 1$. By representation formula (5.2.2), we have

$$
\begin{aligned}
u(x^k, T_k) = 2 \int_{\Omega} \Phi(x^k - y, t_k)\, u(y, T_{k-1})\, dy \\
+ 2 \int_0^{t_k} \int_{\partial\Omega} \frac{\partial \Phi(x^k - y, t_k - \tau)}{\partial n(y)}\, u(y, T_{k-1} + \tau)\, dS(y)\, d\tau \\
+ 2 \int_0^{t_k} \int_{\Gamma_{1, T_{k-1}+\tau}} \Phi(x^k - y, t_k - \tau)\, u^q(y, T_{k-1} + \tau)\, dS(y)\, d\tau.
\end{aligned}
$$

According to the definition of T_{k-1} and T_k,

$$
\begin{aligned}
M_k \le 2M_{k-1} \int_{\Omega} \Phi(x^k - y, t_k)\, dy + 2M_k \int_0^{t_k} \int_{\partial\Omega} \Big| \frac{\partial \Phi(x^k - y, t_k - \tau)}{\partial n(y)} \Big|\, dS(y)\, d\tau \\
+ 2M_k^q \int_0^{t_k} \int_{\Gamma_{1, T_{k-1}+\tau}} \Phi(x^k - y, t_k - \tau)\, dS(y)\, d\tau. \qquad (5.2.11)
\end{aligned}
$$

Since Ω is assumed to be convex, by applying Lemma 4.4.2 and Lemma 4.4.3 to the above inequality, we obtain

$$
\begin{aligned}
b_1(M_k - M_{k-1}) &\le 2M_k^q \int_0^{t_k} \int_{\Gamma_{1, T_{k-1}+\tau}} \Phi(x^k - y, t_k - \tau)\, dS(y)\, d\tau \\
&\le 2M_k^q \int_0^{t_k} \int_{\Gamma_{1, T_{k-1}}} \Phi(x^k - y, t_k - \tau)\, dS(y)\, d\tau.
\end{aligned}
$$

Now Lemma 4.3.3 implies the existence of a constant $C = C(n, \Omega)$ such that

$$
(M_k - M_{k-1}) b_1 \le C M_k^q A^{\alpha}(T_{k-1}) t_k^{n\alpha}, \qquad (5.2.12)
$$

for any $\alpha \in (0, \frac{1}{n-1})$. Recalling the expression (5.2.10), then

$$
M_k - M_{k-1} > 2^{-k} \epsilon\, M_0
$$

and

$$M_k < e^{\epsilon} M_0.$$

In addition, by the assumption that $T_{k-1} \geq (k-1)t_*$ and the fact that A is a decreasing function, we know $A(T_{k-1}) \leq A\big((k-1)t_*\big)$. Therefore, it follows from (5.2.12) that

$$2^{-k}\epsilon < Ce^{\epsilon q} M_0^{q-1} A^{\alpha}\big((k-1)t_*\big)\, t_k^{n\alpha}. \tag{5.2.13}$$

Now if

$$A(t) \leq 2^{-\frac{t}{\alpha t_*}} |\Gamma_1|, \tag{5.2.14}$$

which implies

$$A^{\alpha}\big((k-1)t_*\big) \leq |\Gamma_1|^{\alpha}\, 2^{-(k-1)},$$

then it follows from (5.2.13) that

$$t_k \geq \left(\frac{1}{2C} \frac{\epsilon}{e^{\epsilon q} M_0^{q-1} |\Gamma_1|^{\alpha}}\right)^{1/n\alpha}.$$

Define

$$t_* = \min\left\{1, \left(\frac{1}{2C} \frac{\epsilon}{e^{\epsilon q} M_0^{q-1} |\Gamma_1|^{\alpha}}\right)^{1/n\alpha}\right\}. \tag{5.2.15}$$

Then $t_k \geq t_*$ and this finishes **Step k**.

In summary, if t_* is defined as in (5.2.15) and $A(t)$ satisfies (5.2.14), then the above induction process is valid and can proceed forever. Thus, u will be bounded by B for all the time. Noticing that

$$t_* \geq \min\left\{1, \left(\frac{1}{2C}\frac{\epsilon}{e^{\epsilon q}M_0^{q-1}|\partial\Omega|^\alpha}\right)^{1/n\alpha}\right\} \triangleq K,$$

where K is a constant which only depends on n, Ω, q, M_0, α and B. In order to satisfy (5.2.14), it suffices to have

$$A(t) \leq 2^{-\frac{t}{\alpha K}}|\Gamma_1|.$$

Finally, taking $\alpha = \frac{1}{2(n-1)}$, the proof is finished. □

5.2.3 Control of the solution under a given value for the domain with local convexity near Γ_1

Again, since the global convexity is not practical in real applications. In this subsection, we try to extend the result to locally convex case. But the decreasing rate will be required to be much faster than the convex case. The idea of this subsection is similar to the previous subsection. But different lower bounds will be chosen for each piece, since only the local convexity is available instead of the global convexity.

Proof of Theorem 5.1.4. In the following, unless stated otherwise, C and C_1 will represent constants which may depend on n, Ω, d, α and B. The value of C may change from line to line. The goal is to bound u above by B. Let $\epsilon > 0$ such that $B = e^\epsilon M_0$ and let $t_* \in (0, 1]$ be a constant which will be determined later. Define $T_0 = 0$ and $p_0 = 1$. For any $k \geq 1$, define

$$M_k = \exp\left[(1 - 2^{-k})\epsilon\right]M_0. \tag{5.2.16}$$

Let T_k be the first time that $M(t)$ reaches M_k. Then the maximum principle implies the existence of $x^k \in \partial\Omega$ such that $u(x^k, T_k) = M_k$. We will use induction and choose suitable t_* to show that for any $k \geq 0$,

$$T_k \geq \sum_{i=1}^{k} \frac{t_*}{i}, \tag{5.2.17}$$

with the convention that $\sum\limits_{i=1}^{0} \frac{t_*}{i} = 0$. Since $\lim_{k\to\infty} T_k = \infty$ and $M_k \leq B$, the solution u will be bounded by B for all the time. When $k = 0$, (5.2.17) is obviously true.

Step k (k $\geq$ 1): Suppose (5.2.17) is true for $k-1$, we will show it also holds for k. Let $t_k \triangleq T_k - T_{k-1}$ be the time spent in the kth step. Then it suffices to show that $t_k \geq \frac{t_*}{k}$. If $t_k \geq 1$, then it has already satisfied $t_k \geq \frac{t_*}{k}$, so we assume $t_k < 1$ below. By the representation formula (5.2.2), we have

$$\begin{aligned} u(x^k, T_k) = {}& 2\int_\Omega \Phi(x^k - y, t_k)\, u(y, T_{k-1})\, dy \\ & - 2\int_0^{t_k}\int_{\partial\Omega} \frac{\partial\Phi(x^k - y, t_k - \tau)}{\partial n(y)}\, u(y, T_{k-1} + \tau)\, dS(y)\, d\tau \\ & + 2\int_0^{t_k}\int_{\Gamma_{1,T_{k-1}+\tau}} \Phi(x^k - y, t_k - \tau)\, u^q(y, T_{k-1} + \tau)\, dS(y)\, d\tau. \end{aligned}$$

According to the definition of T_{k-1} and T_k,

$$\begin{aligned} M_k \leq {}& 2M_{k-1}\int_\Omega \Phi(x^k - y, t_k)\, dy + 2M_k\int_0^{t_k}\int_{\partial\Omega} \Big|\frac{\partial\Phi(x^k - y, t_k - \tau)}{\partial n(y)}\Big|\, dS(y)\, d\tau \\ & + 2M_k^q\int_0^{t_k}\int_{\Gamma_{1,T_{k-1}+\tau}} \Phi(x^k - y, t_k - \tau)\, dS(y)\, d\tau. \end{aligned} \tag{5.2.18}$$

Since $\mathrm{Conv}\left([\Gamma_1]_d\right) \subseteq \overline{\Omega}$, by applying Lemma 4.4.3 and Corollary 4.5.2 to the above inequality, we obtain

$$2b_1(M_k - M_{k-1}) \le CM_k \exp\Big(-\frac{d^2}{8t_k}\Big) + 2M_k^q \int_0^{t_k} \int_{\Gamma_{1,T_{k-1}+\tau}} \Phi(x^k - y, t_k - \tau)\, dS(y)\, d\tau.$$

Define $A(t)$ as in (5.2.3). Noticing $|\Gamma_{1,T_{k-1}+\tau}| \le |\Gamma_{1,T_{k-1}}|$, Lemma 4.3.3 implies

$$M_k - M_{k-1} \le CM_k \exp\Big(-\frac{d^2}{8t_k}\Big) + CM_k^q A^\alpha(T_{k-1}) t_k^{n\alpha},$$

for some $\alpha \in \big(0, \frac{1}{n-1}\big)$. Again, noticing that

$$M_k - M_{k-1} > 2^{-k}\epsilon M_0$$

and

$$M_k < e^\epsilon M_0,$$

then

$$2^{-k}\epsilon M_0 \le Ce^\epsilon M_0 \exp\Big(-\frac{d^2}{8t_k}\Big) + Ce^{q\epsilon} M_0^q A^\alpha(T_{k-1}) t_k^{n\alpha}.$$

Dividing by ϵM_0, then there exists a constant $C_1 = C_1(n, \Omega, d, q, M_0, B)$ such that

$$2^{-k} \le C_1 \exp\Big(-\frac{d^2}{8t_k}\Big) + C_1 A^\alpha(T_{k-1}) t_k^{n\alpha}.$$

By induction,

$$T_{k-1} \ge \sum_{i=1}^{k-1} \frac{t_*}{i} \ge t_* \ln k.$$

Hence,

$$2^{-k} \leq C_1 \exp\Big(-\frac{d^2}{8t_k}\Big) + C_1 A^{\alpha}(t_* \ln k) t_k^{n\alpha}. \tag{5.2.19}$$

Since the right hand side of (5.2.19) is an increasing function in t_k, if t_* and $A(t)$ are chosen to satisfy both

$$C_1 \exp\Bigg[-\frac{d^2}{8(\frac{t_*}{k})}\Bigg] \leq 2^{-k-1} \tag{5.2.20}$$

and

$$C_1 A^{\alpha}(t_* \ln k)\Big(\frac{t_*}{k}\Big)^{n\alpha} \leq 2^{-k-1}, \tag{5.2.21}$$

then $t_k \geq \frac{t_*}{k}$ and this finishes **Step k**.

In summary, if t_* and $A(t)$ are chosen to satisfy both (5.2.20) and (5.2.21), then the above induction process is valid and can proceed forever. Thus, u will be bounded by B for all the time. By elementary calculations, if

$$t_* \leq \frac{d^2}{8[\ln C_1 + 2\ln 2]}, \tag{5.2.22}$$

then (5.2.20) is satisfied. Next, in order to realize (5.2.21), it suffices to have

$$A^{\alpha}(t_* \ln k) \leq 2^{1-k} |\Gamma_1|^{\alpha} \tag{5.2.23}$$

and

$$C_1 \Big(\frac{t_*}{k}\Big)^{n\alpha} \leq \frac{1}{4|\Gamma_1|^{\alpha}}. \tag{5.2.24}$$

It is readily to check that if

$$t_* \leq \left(\frac{1}{4C_1 |\partial\Omega|^{\alpha}}\right)^{1/n\alpha}, \tag{5.2.25}$$

then (5.2.24) is satisfied. According to (5.2.22) and (5.2.25), we define

$$t_* = \min\left\{1,\ \frac{d^2}{8[\ln C_1 + 2\ln 2]},\ \left(\frac{1}{4C_1|\partial\Omega|^\alpha}\right)^{1/n\alpha}\right\}. \tag{5.2.26}$$

It is readily seen that t_* is a constant which only depends on n, Ω, d, q, M_0, B and α.

With this choice of t_*, if $A(t)$ satisfies

$$A(t) \le \exp\left[\frac{\ln 2}{\alpha}\big(1 - e^{t/t_*}\big)\right]|\Gamma_1|. \tag{5.2.27}$$

Then it is readily seen that (5.2.23) holds. Finally, taking $\alpha = \frac{1}{2(n-1)}$ and requiring

$$A(t) \le \exp\big[2(n-1)\big(1 - e^{t/t_*}\big)\big]|\Gamma_1|,$$

then $A(t)$ satisfies (5.2.27) automatically. Hence, the proof is finished. □

5.3 Adding a pump

In this section, we provide another way to prevent the finite-time blowup. Let v be the solution to (5.1.4). If

$$m \ge \frac{(p-1)N}{2} \quad \text{and} \quad m \ge (q-1)N,$$

then [3] shows that if the L^m norm of v does not blow up in finite time, then the L^∞ norm of v will not blow up in finite time either. Thus, it is equivalent to bound the L^m norm of v. The idea of the proof of Theorem 5.1.5 is similar to that in [31] where the domain Ω is assumed to be star-shaped. By using Corollary 5.3.2, we are able to generalize the proof for any C^2 domain Ω.

Lemma 5.3.1. *Let $[a, b]$ be a bounded interval and f be a nonnegative C^1 function on $[a, b]$. Then for any $r > 1$ and $\epsilon > 0$,*

$$||f||^r_{L^\infty[a,b]} \leq \epsilon \int_a^b |f'(t)|^2 \, dt + \frac{1}{b-a} \int_a^b f^r(t) \, dt + \frac{r^2}{\epsilon} \int_a^b f^{2r-2}(t) \, dt. \tag{5.3.1}$$

Proof. $\forall\, t_1, t_2 \in [a, b]$,

$$\begin{aligned} f^r(t_1) - f^r(t_2) &= \int_{t_2}^{t_1} r f^{r-1}(t) f'(t) \, dt \\ &\leq r\bigg(\epsilon_1 \int_a^b |f'(t)|^2 \, dt + \frac{1}{\epsilon_1} \int_a^b f^{2r-2}(t) \, dt\bigg), \end{aligned}$$

where ϵ_1 is some positive constant to be determined. Integrating t_2 from a to b,

$$\begin{aligned} (b-a) f^r(t_1) \leq& \int_a^b f^r(t) \, dt + (b-a) r \epsilon_1 \int_a^b |f'(t)|^2 \, dt \\ &+ \frac{(b-a) r}{\epsilon_1} \int_a^b f^{2r-2}(t) \, dt. \end{aligned}$$

Let $\epsilon_1 = \epsilon / r$, then

$$\begin{aligned} (b-a) f^r(t_1) \leq& \int_a^b f^r(t) \, dt + (b-a) \epsilon \int_a^b |f'(t)|^2 \, dt \\ &+ \frac{(b-a) r^2}{\epsilon} \int_a^b f^{2r-2}(t) \, dt. \end{aligned}$$

Since t_1 is arbitrary, then

$$||f||^r_{L^\infty[a,b]} \leq \epsilon \int_a^b |f'(t)|^2 \, dt + \frac{1}{b-a} \int_a^b f^r(t) \, dt + \frac{r^2}{\epsilon} \int_a^b f^{2r-2}(t) \, dt.$$

□

Corollary 5.3.2. *Let $U = \widetilde{U} \times [a,b]$, where $\widetilde{U}$ is a bounded set in $\mathbb{R}^{n-1}$. Let w be a nonnegative C^1 function on U. Then for any $s \in [a,b]$,*

$$\int_{\widetilde{U}} |w(\tilde{x}, s)|^r \, d\tilde{x} \leq \epsilon \int_U |D_{x_n} w(x)|^2 \, dx + \frac{1}{b-a} \int_U w^r(x) \, dx + \frac{r^2}{\epsilon} \int_U w^{2r-2}(x) \, dx.$$

Proof. For any $\tilde{x} \in \widetilde{U}$, define $f : [a,b] \to \mathbb{R}$ by

$$f(t) = w(\tilde{x}, t), \quad \forall \, t \in [a,b].$$

Then by Lemma 5.3.1, for any $s \in [a,b]$,

$$|f(s)|^r \leq \epsilon \int_a^b |f'(t)|^2 \, dt + \frac{1}{b-a} \int_a^b f^r(t) \, dt + \frac{r^2}{\epsilon} \int_a^b f^{2r-2}(t) \, dt.$$

That is,

$$|w(\tilde{x}, s)|^r \leq \epsilon \int_a^b |D_{x_n} w(\tilde{x}, t)|^2 \, dt + \frac{1}{b-a} \int_a^b w^r(\tilde{x}, t) \, dt + \frac{r^2}{\epsilon} \int_a^b w^{2r-2}(\tilde{x}, t) \, dt.$$

Integrating $\tilde{x}$ on $\widetilde{U}$, the conclusion follows. $\square$

Proof of Theorem 5.1.5. Let $m \geq \max\left\{\frac{(p-1)N}{2}, 2\right\}$ and define

$$E(t) = \int_\Omega v^m(x,t) \, dx.$$

Then

$$\begin{aligned} E'(t) &= m\int_\Omega v^{m-1} v_t\,dx \\ &= m\int_\Omega v^{m-1}(\Delta v - \psi_1\, v^p)\,dx \\ &= m\int_\Omega \nabla\cdot(v^{m-1}\nabla v) - (m-1)\,v^{m-2}\,|\nabla v|^2\,dx - m\int_\Omega \psi_1\, v^{m+p-1}\,dx \\ &\le m\int_\Gamma v^{m+q-1}\,dS - m(m-1)\int_\Omega v^{m-2}\,|\nabla v|^2\,dx - m\int_{\Omega_1} v^{m+p-1}\,dx. \end{aligned} \tag{5.3.2}$$

Next it will be shown that for any $\epsilon > 0$, there exists a constant $C = C(\Gamma, \Omega_1, q, m)$ such that

$$\int_\Gamma v^{m+q-1}\,dS \le \epsilon\int_{\Omega_1} v^{m-2}|\nabla v|^2\,dx + C\int_{\Omega_1} v^{m+q-1}\,dx + \frac{C}{\epsilon}\int_{\Omega_1} v^{m+2q-2}\,dx. \tag{5.3.3}$$

Noticing that $v^{m-2}|\nabla v|^2 = |\nabla(v^{m/2})|^2$, so by writing $w = v^{m/2}$ and $r = 2 + 2(q-1)/m$, then $r > 2$ and (5.3.3) is equivalent to

$$\int_\Gamma w^r\,dS \le \epsilon\int_{\Omega_1} |\nabla w|^2\,dx + C\int_{\Omega_1} w^r\,dx + \frac{C}{\epsilon}\int_{\Omega_1} w^{2r-2}\,dx. \tag{5.3.4}$$

Fix any point $x_0 \in \Gamma$. Since Γ is C^2, there exists a neighborhood V of x_0 in Ω_1 that can be straightened by a C^2 bijection $\Psi : B_1 \to V$. Denote $\widetilde{B}_r$ to be the ball in $\mathbb{R}^{n-1}$ with radius r. Define

$$U = \widetilde{B}_{1/2} \times \left(0, \frac{1}{2}\right) \subset B_1.$$

Then $V_0 \triangleq \Psi(U) \subset V$. Define

$$U_0 = \widetilde{B}_{1/2} \times \{0\}, \quad \text{and} \quad \Gamma_0 = \Psi(U_0).$$

By change of variable $x = \Psi(y)$, we obtain

$$\int_{\Gamma_0} w^r(x)\,dS(x) \le C\int_{U_0} (w\circ\Psi)^r(y)\,dS(y)$$
$$\le C\int_{\widetilde{B}_{1/2}} (w\circ\Psi)^r(\tilde{y},0)\,d\tilde{y}.$$

Now applying Corollary 5.3.2 to the function $w\circ\Psi$,

$$\int_{\widetilde{B}_{1/2}} (w\circ\Psi)^r(\tilde{y},0)\,d\tilde{y}$$
$$\le \epsilon\int_U |D_{y_n}(w\circ\Psi)(y)|^2\,dy + 2\int_U (w\circ\Psi)^r(y)\,dy + \frac{r^2}{\epsilon}\int_U (w\circ\Psi)^{2r-2}(y)\,dy.$$

Then using the change of variable $y = \Psi^{-1}(x)$,

$$\epsilon\int_U |D_{y_n}(w\circ\Psi)(y)|^2\,dy + 2\int_U (w\circ\Psi)^r(y)\,dy + \frac{r^2}{\epsilon}\int_U (w\circ\Psi)^{2r-2}(y)\,dy$$
$$\le C\epsilon\int_U \big|(\nabla w)\big(\Psi(y)\big)\big|^2\,dy + 2\int_U w^r\big(\Psi(y)\big)\,dy + \frac{r^2}{\epsilon}\int_U w^{2r-2}\big(\Psi(y)\big)\,dy$$
$$= C\epsilon\int_{V_0} \big|\nabla w(x)\big|^2\,dx + C\int_{V_0} w^r(x)\,dx + \frac{C}{\epsilon}\int_{V_0} w^{2r-2}(x)\,dx$$

In summary, we obtain

$$\int_{\Gamma_0} w^r(x)\,dS(x) \le C\epsilon\int_{V_0} \big|\nabla w(x)\big|^2\,dx + C\int_{V_0} w^r(x)\,dx + \frac{C}{\epsilon}\int_{V_0} w^{2r-2}(x)\,dx \qquad (5.3.5)$$

Finally by a finite cover argument, (5.3.4) is justified, which also means (5.3.3) is verified.

Now combining (5.3.2) and (5.3.3) with $\epsilon = m-1$, we obtain that

$$E'(t) \le C\int_{\Omega_1} v^{m+q-1}(x,t)\,dx + C\int_{\Omega_1} v^{m+2q-2}(x,t)\,dx - m\int_{\Omega_1} v^{m+p-1}(x,t)\,dx,$$

for some constant $C = C(\Gamma, \Omega_1, q, m)$. Since $p > 2q - 1$ from the assumption, then

$$E'(t) \leq C.$$

This implies that $E(t)$ will not blow up in finite time. By applying Theorem 1.1 in [3] and the assumption that $p > 2q - 1$, we can conclude that u will exist globally. $\square$

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