MODULATIONAL STABILITY OF MULTI-PULSES WITHIN THE FUNCTIONALIZED CAHN-HILLIARD GRADIENT FLOW

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ABSTRACT

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The Functionalized Cahn-Hilliard (FCH) energy is a model describing the interfacial energy in a phase separated mixture of amphiphilic molecules and a solvent. On a bounded domain in $\mathbf{R}$, the Euler-Lagrange equation for the mass constrained Functionalized Cahn-Hilliard(FCH) free energy with zero functionalization terms is derived and a large family of multi-pulse critical points is constructed. We show that the FCH energy with no functionalization terms subject to a mass constraint has global minimizers over a variety of admissible sets. We introduce a multi-pulse ansatz as the extensions of the periodic multi-pulse critical points to $\mathbf{R}$ and establish the H^2-coercivity of the second variation of the energy about multi-pulse ansatz. Modulational stability and the dynamic evolution of the multi-pulse ansatz with respect to the Π_0-gradient flow are also addressed.

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Introduction

1.1 Functionalized Cahn-Hilliard Free Energy

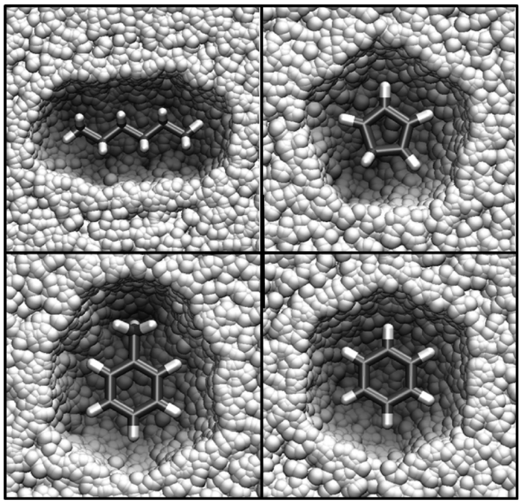
Amphiphilic molecules are chemical compounds consisting of a hydrophobic group and a hydrophilic group, such as lipids and surfactants. When a molecule with an amphiphilic structure is introduced to a solvent(water), the hydrophobic group may alter the structure of the solvent which causes an increase in the free energy of the system. As a respond to this change the system minimizes the contact between the hydrophobic group with formation of bilayer interfaces and pores. Network formations differ from single layer interfaces that occur in binary metals and other purely hydrophobic blends. While single layer interfaces separate two distinct phases from each other, bilayers separate one phase by thin sheets of other phase.

The Cahn-Hilliard(CH) free energy has been used broadly to model single layer interfaces in hydrophobic blends. In 1958, the CH free energy was proposed as

$$\mathcal{E}(u) = \int_\Omega \frac{\varepsilon^2}{2}|(\nabla u)|^2 + W(u)dx, \tag{1.1.1}$$

in [Cahn and Hilliard, 1958] to describe the free energy of an interface occurred by a phase separation in a binary mixture due to spinodal decomposition. Here on a fixed domain $\Omega \subset \mathbf{R}^n$, $u \in H^1(\Omega)$ is the volume fraction of one of the components in the binary mixture,

$W : \mathbf{R} \to \mathbf{R}$ is the free energy density of the mixture and ε is the thickness of the interface.



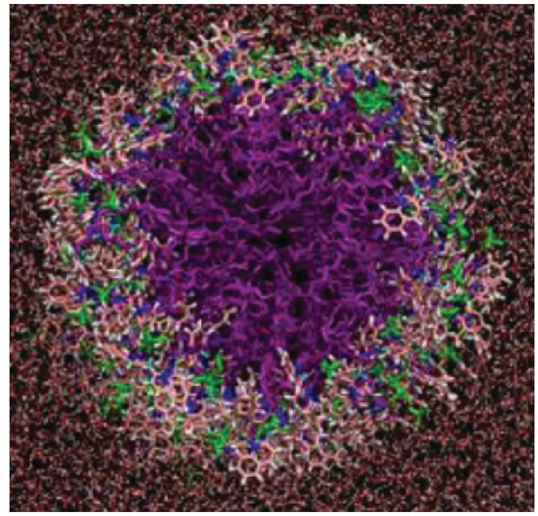


Figure 1.1: When the hydrocarbons in different shapes introduced into a solvent, the solvent particles create a cavity to avoid the solute(left)[Wiebe et al., 2012]. The simulation depicts packing of amphiphilic molecule at interface between external solvent molecules and internal solvent(right).

Since CH was introduced, its minimizers, minimizers subject to a constraint and critical points of the CH have been broadly studied. Although a double well with two unequal depth minima is generic, $W(u)$ is assumed to be a smooth double-well potential with two equal depth minima at $b_\pm$,i.e, $W(b_-) = W(b_+)$ and a maxima at $b_- < b_0 < b_+$ in most studies. This assumption on the form of the potential does not affect the following minimization problem with a mass constraint

$$\min\left\{\int_\Omega \frac{\varepsilon^2}{2}|(\nabla u)|^2 + W(u)dx : u \in H^1(\Omega), \int_\Omega (u - b_-)dx = c\right\}, \tag{1.1.2}$$

since this problem is equivalent to minimizing

$$\begin{aligned}\tilde{\mathcal{E}}(u) &= \int_\Omega \frac{\varepsilon^2}{2}|(\nabla u)|^2 + W(u)dx - \lambda \int_\Omega (u - b_-)dx \\ &= \int_\Omega \frac{\varepsilon^2}{2}|(\nabla u)|^2 + W(u)dx - c\lambda,\end{aligned} \tag{1.1.3}$$

where λ is a Lagrange multiplier and so $c\lambda$ is only a fixed quantity. The functional $\tilde{\mathcal{E}}$ is the same as $\mathcal{E}$ with the well W replaced by $W(u) + \lambda u$. An appropriate choice of λ will render $W(u) + \lambda u$ a double well with equal depth minima. The critical points of the CH are the solutions to $\frac{\delta \mathcal{E}}{\delta u} = 0$ where

$$\frac{\delta \mathcal{E}}{\delta u} = \varepsilon^2 \Delta u - W'(u), \tag{1.1.4}$$

is the variational derivative with respect to L^2 inner-product. Changing variables into the inner variables $z = \frac{z}{\varepsilon}$, at the leading order the critical point equation becomes

$$\partial_z^2 u - W'(u) = \lambda. \tag{1.1.5}$$

From the phase-plane analysis, the critical points of the CH subject to a mass constraint are heteroclinic orbits or single layer interfaces which are the co-dimension 1 interfaces between two distinct phases. Further, the single layer interfaces are the minimizers of the CH subject to a mass constraint. The Γ-convergence as $\varepsilon \to 0$ for the single layers structures of CH free energy to a scaled surface area was established in [Modica, 1987] and [Sternberg, 1988]. A model for the network formation in amphiphilic mixtures motivated by small-angle X-ray scattering (SAXS) data was introduced in [Gompper and Schick, 1990] and [Teubner and Strey, 1987] adding a higher order term in the CH free energy. Based on these models, The FCH free energy was developed in [Promislow and Wetton, 2009] and in

[Gavish et al., 2011] as a mathematical model to describe the interfacial energy in a phase-separated mixture with amphiphilic molecules.

$$F(u) = \int_\Omega \frac{1}{2}\left(\varepsilon^2 \Delta u - W'(u)\right)^2 - \varepsilon^p \left(\frac{\eta_1 \varepsilon^2}{2}|\nabla u|^2 + \eta_2 W(u)\right) dx, \tag{1.1.6}$$

where $\varepsilon \ll 1$ is the ratio of amphiphilic molecule length to domain size and $\eta_1 > 0$,$\eta_2 \in \mathbf{R}$. For $p = 1$, the FCH corresponds to the strong functionalization while for $p = 2$ it is a model for the weak functionalization. It is generic to assume that $W(u)$ is a double-well with two unequal depth minima at $b_\pm$. Further, due to the quadratic term in FCH it is not possible to rewrite the energy in terms of an equal-depth double-well. We also assume that $\alpha_\pm := W''(b_\pm) > 0$. As a consequence, the dominant term in the FCH energy is the square of the first L^2 variational derivative of CH energy. Indeed the FCH energy can be viewed as measuring L^2 distance to the critical points of the associated CH energy. When $\eta_1 = 0$ and $\eta_2 = 0$, all critical points of CH free energy are the global minimizers of the FCH free energy since $\frac{\delta \mathcal{E}}{\delta u} = 0$ asserts that $F(u) = 0$. In [Promislow and Zhang, 2013], the existence of global minimizers was established over a variety of admissible function space for a general class of high order energies such as FCH free energy. Further, the authors showed that the the critical points of CH are the critical points of the higher order energies.

1.2 Description of the Problem

Let a small amount of a polymer(soap) be added to a solvent(water) in a container with impermeable walls and allow the system reach an equilibrium. Suppose that the mass of the soap, m, scales with ε with a relation $m = \varepsilon M$ where $\varepsilon > 0$ is a small parameter and

$M = \mathcal{O}(1)$. We adapt the FCH free energy introduced in the previous section to model the free energy of this system. For fixed $L_2 > 0$, independent of ε, and $\Omega = [0, L_2] \subset \mathbf{R}$, the FCH free energy describing the free energy of the soap-water mixture is

$$I(u) = \int_0^{L_2} \frac{1}{2} \left(\varepsilon^2 \partial_x^2 u - W'(u) \right)^2 dx, \tag{1.2.1}$$

subject to the mass constraint

$$m := \int_0^{L_2} (u - b_-) dx = \varepsilon M, \tag{1.2.2}$$

where $M \in [0, M^*] \subset \mathbf{R}$ for fixed M^* and u satisfying non-flux boundary conditions. Here the density function $u \in H^2(\Omega)$ map Ω into $\mathbf{R}^+$ and the potential $W(u)$ is an unequal double well with two minima at $b_\pm$ for which $W(b_-) = 0 > W(b_+)$ and a maxima at b_M where $b_- < b_M < b_+$. For simplicity, we prefer converting our problem from macroscopic to misroscopic level by converting spacial variables into inner variables. Introducing the inner variable $z = \frac{x}{\varepsilon}$ in (1.2.1), the inner scaling of the FCH free energy takes the form

$$I(u) = \int_0^{\frac{L_2}{\varepsilon}} \frac{1}{2} \left(\partial_z^2 u - W'(u) \right)^2 dz, \tag{1.2.3}$$

subject to the mass constraint

$$\int_0^{\frac{L_2}{\varepsilon}} (u - b_-) dz = M. \tag{1.2.4}$$

In this thesis, we aim constructing a special class of critical points of the inner scaling of the FCH free energy subject to the mass constraint (1.2.4) as the possible minimizers of the free energy. Apparently, $I(u) \geq 0$ for all $u \in H^2(\Omega)$. When $I(u)$ is free of any constraints, the solutions to

$$\begin{cases} \partial_z^2 u = W'(u), \\ \partial_z u(0) = 0, \partial_z u(\frac{L_2}{\varepsilon}) = 0, \end{cases} \tag{1.2.5}$$

are global minimizers of I since they return $I(u) = 0$. To establish the existence of these solutions we write the (1.2.5) as a non-linear system of first order differential equations

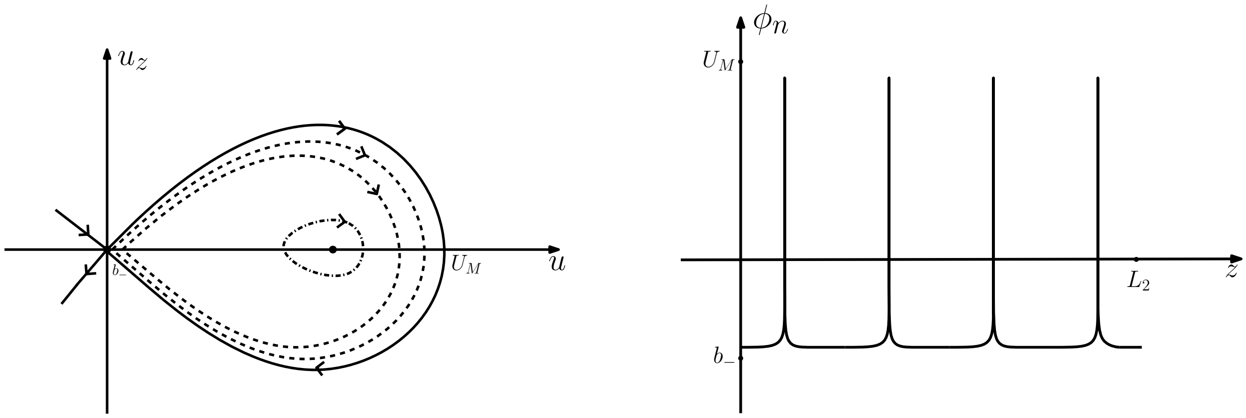
$$\begin{aligned} v_1' &= v_2 \\ v_2' &= W'(v_1) \end{aligned} \tag{1.2.6}$$

and analyze the associated phase-plane diagram. The dynamical system (1.2.6) has two saddle points at $(b_\pm, 0)$ and a center at the equilibrium point $(b_M, 0)$.(See Figure 1.2.) A homoclinic solution is an orbit connecting a saddle point to itself. From the first integral of (1.2.5)

$$\frac{1}{2}(\partial_z u)^2 = W(u) + C_n, \tag{1.2.7}$$

for any constant C_n. For the choice $C_n = 0$, we deduce that we have an orbit starting at b_-, hitting the $u_z = 0$ axis at U_M for which $W(u) = 0$ and joining back to b_- by reversibility. Thus, there exists a homoclinic solution, ϕ_h, converging to b_- as $z \to \infty$. The homoclinic solutions do not satisfy the Neumann boundary conditions, having exponentially small derivatives at $z = 0$ and $z = \frac{L_2}{\varepsilon}$. We can construct periodic solutions of (1.2.5) which do satisfy the boundary conditions. The solutions satisfying the boundary conditions of (1.2.5), ϕ_n, are the periodic solutions at the center starting at a point between b_- and U_M on the axis $u_z = 0$ and ending at a point on the same axis when $z = \frac{L_2}{\varepsilon}$ making $n = \frac{k+1}{2}$,

$k \in \mathbb{N}$ tours. (See Figure 1.2.) By adjusting the value of the minimum of ϕ_n we adjust the period, which we can tune to be an integral multiple of $\frac{L_2}{\varepsilon}$. Translating the n-pulse periodic so that the zero derivative points lie at $z = 0$ and $z = \frac{L_2}{\varepsilon}$ gives an exact solution of (1.2.5).



(a) Phase plane diagram

(b) An illustration for ϕ_4

Figure 1.2: Figure 1.2a is the phase plane for the ODE (1.2.6) which demonstrates the orbit (solid line) homoclinic to b_- attaining its maxima at U_M and the periodic solutions (dotted lines) that are the solutions to the boundary value problem(1.2.5). Figure 1.2b is an example of an $n = 4$ pulse solution to (1.2.5)

Upon this analysis, among many possible critical points of the mass constrained inner scaling of the FCH free energy we desire to obtain a special class of those which are the expansions of the minimizers of the unconstrained inner scaling of the FCH free energy, n-periodic pulses.

Further, motivated by [Promislow, 2002] we survey the modulational stability and dynamical evolution of the n-pulse structure of the inner scaling of the FCH energy. For this purpose, we introduce the mass-preserving projection gradient flow of the FCH energy given in (1.2.3)

$$\begin{aligned} u_t &= -\Pi_0 \frac{\delta I}{\delta u}(u), \\ u(z,0) &= U_0(z), \end{aligned} \tag{1.2.8}$$

where the zero-mass projection, Π_0, is given as $\Pi_0 f := f - \langle f \rangle_\Omega$ with $\langle f \rangle_\Omega := \frac{1}{|\Omega|} \int_\Omega f(x) dx$ and observe that any critical point of the. As the zero-mass projection gradient of the FCH energy evolves the mass of the initial data is preserved,

$$\frac{d}{dt} \langle u \rangle_\Omega = 0, \tag{1.2.9}$$

and the FCH energy given in (1.2.3), I, decreases,

$$\frac{d}{dt} I(u) \leq 0. \tag{1.2.10}$$

The mass-preserving gradient flow of the CH free energy modeling a phase separation process in a binary mixture was analyzed in [Rubinstein and Sternberg, 1992].

1.3 Main Results

In Chapter 2, the main goal is to construct a class of the critical points of the inner scaling of the FCH free energy subject to the mass constraint over the space of functions $u \in H^2(\Omega)$ satisfying the no-flux boundary conditions. In this purpose, we derive the Euler-Lagrange(E-L) equation of the inner scaling of the FCH,

$$\begin{cases} \left(\partial_z^2 - W''(u)\right)\left(\partial_z^2 u - W'(u)\right) = \lambda_\varepsilon 1, \\ \partial_z^3 u(0) = 0, \partial_z^3 u\left(\frac{L_2}{\varepsilon}\right) = 0, \partial_z u(0) = 0, \partial_z u\left(\frac{L_2}{\varepsilon}\right) = 0, \end{cases} \tag{1.3.1}$$

over the admissible space

$$\mathcal{A}_1 := \{u \in H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right) \Big| \int_0^{\frac{L_2}{\varepsilon}} (u - b_-)dz = M, u_z(0) = u_z\left(\frac{L_2}{\varepsilon}\right) = 0\}. \tag{1.3.2}$$

Motivating the corresponding scaling of the Langrange multiplier $\lambda_\varepsilon = \varepsilon\lambda$ to the scaling of the mass constraint, we use an asymptotic expansion to solve the E-L equation and obtain a class of the critical points of the mass-constrained FCH energy I. Another result we present in this chapter is that the mass constraint value has a significant impact on the minimizers belonging to the class of the critical points we construct.

In Chapter 3, utilizing the classical tools from Calculus of Variations we obtain the existence of the global minimizers of the FCH energy, I, subject to the mass-constraint (1.2.4) over the admissible space $\mathcal{A}_1$.

In Chapter 4, we introduce n-pulse ansatz as the corrected extensions of the n-pulse solutions of the E-L equation to the whole line $\mathbf{R}$. We establish the H^2-coercivity of the second variation of the FCH energy I about n-pulse ansatz and further the H^2-coercivity of the second variation of the FCH energy I about periodic multi-pulses ϕ_n. With an application of the result on modulational stability of the steady-state solutions of the gradient system in [Promislow, 2002], we demonstrate that n-pulse ansatz, the steady-state solutions of the mass-preserving projection gradient of I, is stable in the modulational sense. The evolution equations of the pulse-locations and the background are derived as

$$\begin{aligned} &\bar{\lambda}' = 0, \\ &p_i' = -\alpha_-^{3/2}\left(e^{-\sqrt{\alpha_-}(p_{i+1}-p_i)} - e^{-\sqrt{\alpha_-}(p_i-p_{i-1})}\right) + \mathcal{O}(\delta^2). \end{aligned} \tag{1.3.3}$$

With an analysis of the evolution equations of the pulse locations we conclude that the stationary solutions, equally spaced n-pulses(periodic) are spectrally stable. More significantly, for a given initial data in an ε^2-neighborhood of the n-pulse ansatz we recover the pulse dynamics to $\mathcal{O}(\delta^2)$ where δ is exponentially small in ε. Moreover, the solutions remain within a $\mathcal{O}(\delta)$ neighborhood in H^2 of the periodic n-pulses(equally spaced).

Chapter 2

The Euler-Lagrange Equation

In the calculus of variations, the Euler-Lagrange equations are used to construct the critical points of a functional. In this section, we will derive the Euler-Lagrange equation for the problem

$$\min_{u\in\mathcal{A}} \int_0^{\frac{L_2}{\varepsilon}} \frac{1}{2}\left(\partial_z^2 u - W'(u)\right)^2 dz \quad \textit{subject to} \quad \int_0^{\frac{L_2}{\varepsilon}} (u-b_-)dz = M, \tag{2.0.1}$$

where the choice of the space of admissible functions $\mathcal{A} \subset H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$ will be addressed later. As described earlier, the problem is based on an experiment during which a polymer is being added in to a solvent to form a dispersion in a fixed container with an impermeable boundary and then the system is allowed to relax to reach its equilibrium. To model this, we consider $u \in H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$ satisfying the mass constraint and the Neumann boundary conditions due to impermeable membrane. With this description, we define the space of the admissible functions

$$\mathcal{A}_1 := \{u \in H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right) \Big| \int_0^{\frac{L_2}{\varepsilon}} (u-b_-)dz = M, u_z(0) = u_z\left(\frac{L_2}{\varepsilon}\right) = 0\}. \tag{2.0.2}$$

Additionally, the free energy given (1.2.3) is well-defined on the following natural admissible spaces

$$\mathcal{A}_0 := \left\{ u \in H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right) \,\Big|\, \int_0^{\frac{L_2}{\varepsilon}} (u - b_-)dz = M \right\}, \tag{2.0.3}$$

and

$$\mathcal{A}_2 = \left\{ u \in H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right) \cap H_0^1\left(\left[0, \frac{L_2}{\varepsilon}\right]\right) | \int_0^{\frac{L_2}{\varepsilon}} (u - b_-)dz = M, u_z(0) = u_z(\frac{L_2}{\varepsilon}) = 0 \right\}. \tag{2.0.4}$$

It may be easily observed that $\mathcal{A}_2 \subset \mathcal{A}_1 \subset \mathcal{A}_0$.

We will construct the Euler-Lagrange equation over all these spaces and discuss the necessary boundary conditions. In the sequel, we will further see that the value of the mass constraint has a considerable impact on the form of the actual minimizer(s).

2.1 Derivation of the Euler-Lagrange Equation over Various Admissible Sets

The Euler-Lagrange equation characterizes the smooth critical points of the free energy functional $I(\cdot)$. Recall that we consider $u \in H^2$ satisfying the mass constraint $\int_0^{\frac{L_2}{\varepsilon}} udz = M$ and the boundary conditions $u_z(0) = u_z\left(\frac{L_2}{\varepsilon}\right) = 0$ in the problem and our main purpose is to construct the Euler-Lagrange equation for the critical points satisfying these conditions in this section.

We first consider the largest space of admissible functions

$$\mathcal{A}_0 := \left\{ u \in H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right) \,\Big|\, \int_0^{\frac{L_2}{\varepsilon}} (u - b_-)dz = M \right\}. \tag{2.1.1}$$

Let $u \in \mathcal{A}_0$ be any critical point of the energy $I(u)$ subject to the mass constraint and form a curve $u + \tau v \in \mathcal{A}_0$ for $\tau \in \mathbf{R}$ and $v \in \mathcal{A}_0'$ where

$$\mathcal{A}_0' = \left\{ v \in H^2 \left(\left[0, \frac{L_2}{\varepsilon} \right] \right) | v \perp 1 \right\}, \tag{2.1.2}$$

is the tangent plane to $\mathcal{A}_0$. The orthogonality condition is seen to be required by observing that $u + \tau v \in \mathcal{A}_0$ holds if and only if $v \in H^2 \left(\left[0, \frac{L_2}{\varepsilon} \right] \right)$ and $\frac{d}{d\tau} \int_0^{\frac{L_2}{\varepsilon}} u + \tau v dz = 0$, this second requirement implies

$$\begin{aligned} 0 =& \frac{d}{d\tau} \int_0^{\frac{L_2}{\varepsilon}} (u + \tau v)\, dz, \\ =& \int_0^{\frac{L_2}{\varepsilon}} v dz, \end{aligned} \tag{2.1.3}$$

and we deduce that $v \perp 1$.

We denote by $i(\tau)$ the evaluation of I on the curve $u + \tau v$

$$i(\tau) := I(u + \tau v). \tag{2.1.4}$$

For the following calculations, let us assume that $i(\tau)$ is differentiable at $\tau = 0$ which will be established in Theorem 2.1.2 later. Assuming that u is a critical point of I, i has zero derivative at $\tau = 0$, i.e. $i'(0) = 0$. We formally calculate the variation of I

$$\begin{aligned} i'(\tau) &= \int_0^{\frac{L_2}{\varepsilon}} \frac{1}{2} \frac{d}{d\tau} \left(\partial_z^2 (u + \tau v) - W'(u + \tau v) \right)^2 dz, \\ &= \int_0^{\frac{L_2}{\varepsilon}} \left(\partial_z^2 (u + \tau v) - W'(u + \tau v) \right) \left(\partial_z^2 v - W''(u + \tau v) v \right) dz, \\ &= \int_0^{\frac{L_2}{\varepsilon}} \left(\partial_z^2 (u + \tau v) - W'(u + \tau v) \right) \partial_z^2 v - \left(\partial_z^2 (u + \tau v) - W'(u + \tau v) \right) W''(u + \tau v) v dz. \end{aligned} \tag{2.1.5}$$

Let $\tau = 0$ and since $i'(0) = 0$

$$0 = i'(0) = \int_0^{\frac{L_2}{\varepsilon}} \left(\partial_z^2 u - W'(u)\right) \partial_z^2 v - \left(\partial_z^2 u - W'(u)\right) W''(u) v dz, \tag{2.1.6}$$

for all $v \in {\mathcal{A}_0}'$.

Further, if the critical point $u \in H^4\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$ we may twice integrate by parts the second term in the integrand and obtain

$$\begin{aligned} 0 := G_u(v) = &\underbrace{\int_0^{\frac{L_2}{\varepsilon}} \left(\partial_z^2 \left(\partial_z^2 u - W'(u)\right) - \left(\partial_z^2 u - W'(u)\right) W''(u)\right) v dz +}_{A} \\ &\underbrace{-\partial_z \left(\partial_z^2 u - W'(u)\right) v\Big|_0^{\frac{L_2}{\varepsilon}} + \left(\partial_z^2 u - W'(u)\right) \partial_z v\Big|_0^{\frac{L_2}{\varepsilon}}}_{B}, \end{aligned} \tag{2.1.7}$$

for all $v \in \mathcal{A}_0'$. The map $v \mapsto G_u(v)$ is the weak formulation of the variational derivative of the free energy $I(u)$. The equality in (2.1.7) holds for all $v \in \mathcal{A}_0$ but different subspaces $\mathcal{A}_0'$ afford information on different terms in the RHS of (2.1.7). In particular, if we choose v from the subspace $\mathcal{S}_0' = \{v \in C_c^\infty\left(\left[0, \frac{L_2}{\varepsilon}\right]\right) \Big| v \perp 1\} \subset \mathcal{A}_0'$ then the boundary terms B are 0 and we deduce that

$$\int_0^{\frac{L_2}{\varepsilon}} \left(\partial_z^2 - W''(u)\right) \left(\partial_z^2(u) - W'(u)\right) v dz = 0, \tag{2.1.8}$$

for all $v \in \mathcal{S}_0'$.

By the density of S_0' in $\mathcal{A}_0'$ in $L^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$ we infer that

$$\left(\partial_z^2 - W''(u)\right) \left(\partial_z^2 u - W'(u)\right) \perp \mathcal{A}_0', \tag{2.1.9}$$

since $\mathcal{A}_0' = \left\{ v \in H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right) | v \perp 1 \right\}$ we see that $\mathcal{A}_0'^{\perp} = span\{1\}$ and hence

$$\left(\partial_z^2 - W''(u)\right)\left(\partial_z^2 u - W'(u)\right) = \lambda_\varepsilon 1, \tag{2.1.10}$$

for some Lagrange multiplier λ_ε. Consequently, the relation (2.1.8) holds for all $v \in \mathcal{A}_0'$ even for those which are not in S_0'. This information implies that $A = 0$ in (2.1.7) and we conclude that $B = 0$ to preserve the equality in (2.1.7). Since the trace map $v \in \mathcal{A}_0' \mapsto (v(0), v(\frac{L_2}{\varepsilon}), \partial_z v(0), \partial_z v(\frac{L_2}{\varepsilon})) \in \mathbf{R}^4$ is onto, we may choose v_1, v_2, v_3, v_4 for which this trace map yields the canonical basis e_1, e_2, e_3, e_4. These choices of v show that the critical point u from the admissible space $\mathcal{A}_0$ satisfy the following boundary conditions,

$$\begin{cases} \left(\partial_z^3 u - W'(u)\partial_z u\right)|_{z=0} = 0, \quad \left(\partial_z^3 u - W'(u)\partial_z u\right)|_{z=\frac{L_2}{\varepsilon}} = 0, \\ \left(\partial_z^2 u - W(u)\right)|_{z=0} = 0, \quad \left(\partial_z^2 u - W(u)\right)|_{z=\frac{L_2}{\varepsilon}} = 0. \end{cases} \tag{2.1.11}$$

We summarize the results obtained so far in the following proposition.

Proposition 2.1.1. *Any critical point of the problem* (2.0.1) *over* $\mathcal{A}_0$ *that lies in* $H^4\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$ *satisfies*

$$\begin{cases} \left(\partial_z^2 - W''(u)\right)\left(\partial_z^2 u - W'(u)\right) = \lambda_\varepsilon 1, \\ \left(\partial_z^3 u - W'(u)\partial_z u\right)|_{z=0} = 0, \quad \left(\partial_z^3 u - W'(u)\partial_z u\right)|_{z=\frac{L_2}{\varepsilon}} = 0, \\ \left(\partial_z^2 u - W(u)\right)|_{z=0} = 0, \quad \left(\partial_z^2 u - W(u)\right)|_{z=\frac{L_2}{\varepsilon}} = 0. \end{cases} \tag{2.1.12}$$

Another admissible space, which is actually our main focus of interest for the minimization problem, is given by

$$\mathcal{A}_1 = \left\{ u \in H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right) \,\middle|\, \int_0^{\frac{L_2}{\varepsilon}} (u - b_-)dz = M, u_z(0) = u_z\left(\frac{L_2}{\varepsilon}\right) = 0 \right\}, \tag{2.1.13}$$

where Neumann-boundary conditions $u_z(0) = u_z\left(\frac{L_2}{\varepsilon}\right) = 0$ emulate no-flux boundary conditions due to the impermeable boundary. Following the same procedure we adopted as deriving the Euler-Lagrange equation satisfied by any $u \in \mathcal{A}_0$, we achieve (2.1.8) for all $v \in \mathcal{A}_1'$ where

$$\mathcal{A}_1' = \left\{ v \in H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right) \middle| v \perp 1, v_z(0) = v_z\left(\frac{L_2}{\varepsilon}\right) = 0 \right\}, \tag{2.1.14}$$

is the tangent plane to the admissible space $\mathcal{A}_1$.

Taking this into consideration and inserting the boundary conditions $u_z(0) = u_z\left(\frac{L_2}{\varepsilon}\right) = 0$ imposed in the admissible space $\mathcal{A}_1$ in (2.1.7) we obtain

$$0 = \partial_z\left(\partial_z^2(u) - W'(u)\right) v\Big|_0^{\frac{L_2}{\varepsilon}}, \tag{2.1.15}$$

for all $v \in \mathcal{A}_1'$. Similar to the previous case, we conclude that any critical point $u \in \mathcal{A}_1 \cap H^4\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$ satisfies

$$\begin{cases} \left(\partial_z^2 - W''(u)\right)\left(\partial_z^2 u - W'(u)\right) = \lambda_\varepsilon 1, \\ \partial_z^3 u(0) = 0, \partial_z^3 u\left(\frac{L_2}{\varepsilon}\right) = 0, \partial_z u(0) = 0, \partial_z u\left(\frac{L_2}{\varepsilon}\right) = 0. \end{cases} \tag{2.1.16}$$

The last natural admissible space we discuss in this section is

$$\mathcal{A}_2 = \left\{ u \in H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right) \cap H_0^1\left(\left[0, \frac{L_2}{\varepsilon}\right]\right) \middle| \int_0^{\frac{L_2}{\varepsilon}} (u - b_-)dz = M, u_z(0) = u_z\left(\frac{L_2}{\varepsilon}\right) = 0 \right\}, \tag{2.1.17}$$

which has the tangent plane

$$\mathcal{A}_2' = \left\{ v \in H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right) \cap H_0^1\left(\left[0, \frac{L_2}{\varepsilon}\right]\right) \middle| v \perp 1, v_z(0) = v_z\left(\frac{L_2}{\varepsilon}\right) = 0 \right\}. \tag{2.1.18}$$

In a similar manner to the previous two cases, it can be easily demonstrated that any minimizer $u \in \mathcal{A}_2 \cap H^4([0, \frac{L_2}{\varepsilon}])$ solves the Euler-Lagrange equations

$$\begin{cases} \left(\partial_z^2 - W''(u)\right)\left(\partial_z^2 u - W'(u)\right) = \lambda_\varepsilon 1, \\ u(0) = 0, u\left(\frac{L_2}{\varepsilon}\right) = 0, \partial_z u(0) = 0, \partial_z u\left(\frac{L_2}{\varepsilon}\right) = 0, \end{cases} \tag{2.1.19}$$

and no new boundary conditions arise. From now on, for simplicity of notation we let $\mathcal{A} := \mathcal{A}_1$.

Our earlier construction of the Euler-Lagrange equation assumed that $i(\tau)$ is differentiable at $\tau = 0$. The following theorem provides a justification for this fact.

Theorem 2.1.2. *Consider $I(u)$ given (1.2.3). Then, for any $u, v \in H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$*

$$i(\tau) = I(u + \tau v), \tag{2.1.20}$$

is differentiable at $\tau = 0$.

Proof. Take any $u, v \in H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$ and set

$$i(\tau) = I(u + \tau v). \tag{2.1.21}$$

First, $i(\tau)$ is finite for all τ since $u \in H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$ and the square of L^2 functions lie in L^1.

Let $\tau \neq 0$. Evaluate the difference quotient for a.e. z

$$\frac{i(\tau) - i(0)}{\tau} = \int_0^{\frac{L_2}{\varepsilon}} \frac{\frac{1}{2}\left(\partial_z^2(u + \tau v) - W'\left(u + \tau v\right)\right)^2 dz - \frac{1}{2}\left(\partial_z^2 u - W'\left(u\right)\right)^2}{\tau} dz. \tag{2.1.22}$$

Inserting the Taylor series expansion of $W'(u + \tau v)$ with integral remainder,

$$W'(u + \tau v) = W'(u) + \tau W''(u)v + \int_u^{u+\tau v} W'''(s)\left(u + \tau v - s\right) ds, \tag{2.1.23}$$

in to (2.1.22), and after some simplifications we find that the quotient reduces to

$$\begin{aligned}
\frac{i(\tau) - i(0)}{\tau} &= \frac{1}{\tau}\int_0^{\frac{L_2}{\varepsilon}} \frac{1}{2}\left(\partial_z^2 u + \tau\partial_z^2 v - W'(u) - \tau W''(u)v - \int_u^{u+\tau v} W'''(s)\left(u + \tau v - s\right) ds\right)^2 dz + \\
&\qquad - \frac{1}{2}\left(\partial_z^2 u - W'\left(u\right)\right)^2 dz, \\
&= \frac{1}{\tau}\int_0^{\frac{L_2}{\varepsilon}} \tau\left(\partial_z^2 u - W'(u)\right)\left(\partial_z^2 v - W''(u)v\right) + \left(\int_u^{u+\tau v} W'''(s)\left(u + \tau v - s\right) ds\right)^2 + \\
&\qquad - \left(\partial_z^2 u - W'(u)\right)\left(\int_u^{u+\tau v} W'''(s)\left(u + \tau v - s\right) ds\right) + \tau^2\left(\partial_z^2 v - W''(u)v\right)^2 + \\
&\qquad - \tau\left(\partial_z^2 v - W''(u)v\right)\left(\int_u^{u+\tau v} W'''(s)\left(u + \tau v - s\right) ds\right) dz, \\
&= \int_0^{\frac{L_2}{\varepsilon}} L^\tau(z) dz,
\end{aligned} \tag{2.1.24}$$

where we have introduced

$$L^\tau := \frac{1}{\tau}\Bigg[\tau\left(\partial_z^2 u - W'(u)\right)\left(\partial_z^2 v - W''(u)v\right) + \left(\int_u^{u+\tau v} W'''(s)\,(u+\tau v - s)\,ds\right)^2 + \\ -\left(\partial_z^2 u - W'(u)\right)\left(\int_u^{u+\tau v} W'''(s)\,(u+\tau v - s)\,ds\right) + \tau^2\left(\partial_z^2 v - W''(u)v\right)^2 + \\ -\tau\left(\partial_z^2 v - W''(u)v\right)\left(\int_u^{u+\tau v} W'''(s)\,(u+\tau v - s)\,ds\right)\Bigg]. \tag{2.1.25}$$

We claim that $L^\tau \in L^1\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$. For the proof we first apply the triangle inequality to the integrand and obtain

$$\int_0^{\frac{L_2}{\varepsilon}} |L^\tau(z)|dz = \frac{1}{\tau}\int_0^{\frac{L_2}{\varepsilon}} \Bigg|\tau\left(\partial_z^2 u - W'(u)\right)\left(\partial_z^2 v - W''(u)v\right) + \\ + \left(\int_u^{u+\tau v} W'''(s)\,(u+\tau v - s)\,ds\right)^2 + \\ -\left(\partial_z^2 u - W'(u)\right)\left(\int_u^{u+\tau v} W'''(s)\,(u+\tau v - s)\,ds\right) \\ + \tau^2\left(\partial_z^2 v - W''(u)v\right)^2 + \\ -\tau\left(\partial_z^2 v - W''(u)v\right)\left(\int_u^{u+\tau v} W'''(s)\,(u+\tau v - s)\,ds\right)\Bigg|dz.$$

Taking $\frac{1}{\tau}$ inside the integral and by the triangle inequality we have

$$
\begin{aligned}
\int_0^{\frac{L_2}{\varepsilon}} |L^\tau(z)| dz \leq \int_0^{\frac{L_2}{\varepsilon}} & \underbrace{\left| \left(\partial_z^2 u - W'(u)\right) \left(\partial_z^2 v - W''(u)v\right) \right|}_{:=A} + \\
& + \underbrace{\left| \tau \left(\frac{1}{\tau} \int_u^{u+\tau v} W'''(s)\,(u+\tau v - s)\, ds \right)^2 \right|}_{:=B} + \\
& + \underbrace{\left| \left(\partial_z^2 u - W'(u)\right) \left(\int_u^{u+\tau v} \frac{1}{\tau} W'''(s)\,(u+\tau v - s)\, ds \right) \right|}_{:=C} + \\
& + \underbrace{\left| \tau \left(\partial_z^2 v - W''(u)v\right)^2 \right|}_{:=D} + \\
& + \underbrace{\left| \left(\partial_z^2 v - W''(u)v\right) \left(\frac{1}{\tau} \int_u^{u+\tau v} W'''(s)\,(u+\tau v - s)\, ds \right) \right|}_{:=E} dz.
\end{aligned} \tag{2.1.26}
$$

By the Lebesgue Dominated Convergence Theorem, it suffices to show that each of the terms A, B, C, D and E are in $L^1\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$. For the first term, by the Holder's inequality

$$
\int_0^{\frac{L_2}{\varepsilon}} A dz \leq \left\| \partial_z^2 u - W'(u) \right\|_{L^2} \left\| \partial_z^2 v - W''(u)\, v \right\|_{L^2}, \tag{2.1.27}
$$

Then, from the triangle inequality we have

$$
\begin{aligned}
\int_0^{\frac{L_2}{\varepsilon}} A dz &\leq \left(\left\| \partial_z^2 u \right\|_{L^2} + \left\| W'(u) \right\|_{L^2} \right) \left(\left\| \partial_z^2 v \right\|_{L^2} + \left\| W''(u) v \right\|_{L^2} \right), \\
&\leq \left(\left\| \partial_z^2 u \right\|_{L^2} + \left\| W'(u) \right\|_{L^2} \right) \left(\left\| \partial_z^2 v \right\|_{L^2} + \left\| W''(u) \right\|_{L^\infty} \left\| v \right\|_{L^2} \right).
\end{aligned} \tag{2.1.28}
$$

By the smoothness of W, for each n there exists an $\alpha_1 > 0$ such that $\left\| W^{(n)}(s) \right\|_{L^\infty} \leq \alpha_1$ for all $s \in \left[-\|u\|_{L^\infty} - \tau \|v\|_{L^\infty}, \|u\|_{L^\infty} + \tau \|v\|_{L^\infty} \right]$. Since $u \in H^2$ implies $\|u\|_{L^\infty}$ is bounded, we conclude that $A \in L^1\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$ for each $u, v \in H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$. The same arguments show that there exists a constant α_1 such that

$$\int_0^{\frac{L_2}{\varepsilon}} D dz \leq \alpha_1 \tau. \tag{2.1.29}$$

On the other hand, bounding B, C and E requires estimates on the integral remainder term. From the smoothness of W and the compact range of u and $u + \tau v$, there exists a $\alpha_2 > 0$ such that $|W'''(s)| \leq \alpha_2$ for s running over u to $u + \tau v$. Then,

$$\left| \frac{1}{\tau} \int_u^{u+\tau v} W'''(s) \left(u + \tau v - s\right) ds \right| \leq \frac{\alpha_2}{\tau} \int_u^{u+\tau v} \left| u + \tau v - s \right| ds. \tag{2.1.30}$$

By the change of variables $t = u + \tau v - s$ we obtain

$$\int_u^{u+\tau v} \left| u + \tau v - s \right| ds = \int_{\tau v}^{0} |t| dt \leq \frac{\tau^2}{2} |v^2|. \tag{2.1.31}$$

Inserting this in (2.1.30) we have

$$\left| \frac{1}{\tau} \int_u^{u+\tau v} W'''(s) \left(u + \tau v - s\right) ds \right| \leq \frac{\alpha_2 \tau}{2} \left| v^2 \right|. \tag{2.1.32}$$

With these estimates we deduce that

$$\begin{aligned} \int_0^{\frac{L_2}{\varepsilon}} B dz &\leq \left\| \int_u^{u+\tau v} W'''(s) \left(u + \tau v - s\right) ds \right\|_{L^2}^2, \\ &\leq \frac{\tau^4}{4} \alpha_2^2 \left\| v^2 \right\|_{L^2}^2, \\ &= \frac{\tau^4}{4} \alpha_2^2 \left\| v \right\|_{L^4}^4. \end{aligned} \tag{2.1.33}$$

Here we controlled the L^4 norm by H^1 and L^∞ norms using the estimate

$$\|v\|_{L^4}^4 = \int_0^{\frac{L_2}{\varepsilon}} v^4 dz \leq \|u\|_{L^\infty}^2 \|u\|_{L^2}^2 \leq \|u\|_{H^1}^2 \|u\|_{L^\infty}^2 \,, \tag{2.1.34}$$

and further L^∞ norm by H^1 norm

$$\|v\|_{L^\infty}^2 = \sup_{z\in\Omega} \left| \int_z^{\frac{L_2}{\varepsilon}} \partial_x(v^2) dx \right| \leq 2\|v\|_{L^2} \|\partial_z v\|_{L^2} \leq \|v\|_{L^2}^2 + \|\partial_z v\|_{L^2}^2 \leq \|v\|_{H^1}^2 \,, \tag{2.1.35}$$

we obtain

$$\|v\|_{L^4}^4 \leq \|v\|_{H^1}^4 \leq C \|v\|_{H^2}^4 \,. \tag{2.1.36}$$

Utilizing this estimate in (2.1.33), we have

$$\int_0^{\frac{L_2}{\varepsilon}} B dz \leq \frac{\tau^4}{4} \alpha_2^2 \|v\|_{H^2}^4 \,. \tag{2.1.37}$$

With the same arguments we obtain upper bounds for C and E such that

$$\begin{aligned}
\int_0^{\frac{L_2}{\varepsilon}} C dz &= \int_0^{\frac{L_2}{\varepsilon}} \left| \left(\partial_z^2 u - W'(u)\right) \left(\frac{1}{\tau} \int_u^{u+\tau v} W'''(s)\,(u + \tau v - s)\, ds\right) \right| dz, \\
&\leq \frac{\tau \alpha_2}{2} \int_0^{\frac{L_2}{\varepsilon}} \left| v^2 \left(\partial_z^2 u - W'\,(u)\right) \right| dz, \\
&\leq \frac{\tau \alpha_2}{2} \|v\|_{L^4}^2 \left(\left\|\partial_z^2 u\right\|_{L^2} + \left\|W'(u)\right\|_{L^2} \right), \\
&\leq \frac{\tau \alpha_2}{2} \|v\|_{H^2}^2 \left(\left\|\partial_z^2 u\right\|_{H^2} + \alpha_1 \right),
\end{aligned} \tag{2.1.38}$$

and

$$\begin{aligned}
\int_0^{\frac{L_2}{\varepsilon}} E dz &= \int_0^{\frac{L_2}{\varepsilon}} \left| \tau \left(\partial_z^2 v - W'(u) v \right) \left(\frac{1}{\tau} \int_u^{u+\tau v} W'''(s) \left(u + \tau v - s \right) ds \right) \right| dz, \\
&\leq \frac{\tau^2 \alpha_2}{2} \int_0^{\frac{L_2}{\varepsilon}} \left| v^2 \left(\partial_z^2 v - W'\left(u \right) v \right) \right| dz, \\
&\leq (1+\alpha_1) \frac{\tau^2 \alpha_2}{2} \left\| v \right\|_{H^2}^3 .
\end{aligned} \tag{2.1.39}$$

Considering all these estimates for A, B, C, D and E and $u, v \in H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$ we conclude that there exists a constant $\gamma := \left(\left\| \partial_z^2 u \right\|_{L^2} + \left\| W'(u) \right\|_{L^2} \right) \left(\left\| \partial_z^2 v \right\|_{L^2} + \left\| W''(u) \right\|_{L^\infty} \left\| v \right\|_{L^2} \right) \in$ $\mathbf{R}$ such that

$$\int_0^{\frac{L_2}{\varepsilon}} |L^\tau(z)| dz \leq \gamma + \mathcal{O}(\tau), \tag{2.1.40}$$

which implies that $L^\tau \in L^1\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$ as $\tau \to 0$.

Indeed our estimates applied to (2.1.25) show that

$$L^\tau \to \left(\partial_z^2 u - W'(u) \right) \left(\partial_z^2 v - W''(u) v \right) \quad as \quad \tau \to 0 \qquad for \quad a.e \quad z, \tag{2.1.41}$$

Applying the Dominated convergence theorem, from (2.1.40) and (2.1.41) we show that the limits

$$\begin{aligned}
i'(0) = \lim_{\tau \to 0} \frac{i(\tau) - i(0)}{\tau} &= \lim_{\tau \to 0} \int_0^{\frac{L_2}{\varepsilon}} L^\tau dz, \\
&= \int_0^{\frac{L_2}{\varepsilon}} \left(\partial_z^2 u - W'(u) \right) \left(\partial_z^2 - W''(u) \right) v dz,
\end{aligned} \tag{2.1.42}$$

exist. This proves that i is differentiable at $\tau = 0$. $\square$

2.2 The Euler-Lagrange Equation with a small mass constraint

Recall that the inner scaling of the Functionalized Cahn-Hilliard free energy with no functionalization terms is

$$I(u) = \int_0^{\frac{L_2}{\varepsilon}} \frac{1}{2}\left(\partial_z^2 u - W'(u)\right)^2 dz.$$

In this section, we address the dependence of the minima of the functional I upon the mass M,

$$\int_0^{\frac{L_2}{\varepsilon}} (u - b_-)dz = M, \qquad (2.2.1)$$

by analyzing an asymptotic expansion of the Euler-Lagrange equation (2.1.10). We construct a class of solutions of the E-L equation which corresponds to a scaling of the Lagrange multiplier, $\lambda_\varepsilon = \varepsilon\lambda$, arising from the ε-dependence of the mass constraint, m. Following this information, specifically, we seek solutions to the perturbed E-L equation,

$$\left(\partial_z^2 - W''(\Phi)\right)\left(\partial_z^2\Phi - W'(\Phi)\right) = \lambda_\varepsilon. \qquad (2.2.2)$$

2.2.1 Notation

Introduce the operator

$$\mathcal{L} := \partial_z^2 - W''(\phi_h), \qquad (2.2.3)$$

that is the linearization of (1.2.5) about ϕ_h, and define the operator

$$\overline{\mathcal{L}}_n(\mathbf{p}) := \partial_z^2 - W''(u_n), \tag{2.2.4}$$

acting on $H^2(\mathbf{R})$ where u_n is the extension ϕ_n to $\mathbf{R}$ defined as

$$u_n := \sum_{j=1}^{n} \overline{\phi}_h(z - p_j) + b_-, \tag{2.2.5}$$

where $\overline{\phi}_h := \phi_h - b_-$ and $\mathbf{p} = (p_1, p_2, ..., p_n)^t \in \mathbf{R}^n$ is the vector of pulse locations. The admissible set of pulse locations is given by

$$\mathcal{P} = \{\mathbf{p} \in \mathbf{R}^n : p_i < p_{i+1} \quad \mathit{for} \quad i = 1, ..., n \quad \mathit{and} \quad \Delta\mathbf{p} \geq l\}, \tag{2.2.6}$$

where $\Delta\mathbf{p} = \min_{i \neq j} |p_i - p_j|$ and $l > 0$ is sufficiently large. Let $\mathcal{B}_j$ denote the functions $\mathcal{L}^{-j} 1 \in L^\infty(\mathbf{R})$ for $j = 1, 2$ that are the solutions to

$$\mathcal{L}^j \mathcal{B}_j = 1, \tag{2.2.7}$$

and orthogonal to the kernel of $\mathcal{L}$. Actually, the function $\mathcal{B}_j$ is in the form

$$\mathcal{B}_j = \hat{\mathcal{B}}_j + \mathcal{B}_{j,\infty}. \tag{2.2.8}$$

Specifically, $\mathcal{B}_1$ takes the form

$$\mathcal{B}_1 = \hat{\mathcal{B}}_1 - \frac{1}{\alpha_-}, \tag{2.2.9}$$

where α_- is the coercivity value of $W(u)$ at b_- and $\hat{\mathcal{B}}_1$ is the solution to the

$$\mathcal{L}\hat{\mathcal{B}}_1 = 1 - \frac{W''(\phi_h)}{\alpha_-}. \tag{2.2.10}$$

Since $\phi_h \to b_-$ at an exponential rate as $z \to \infty$, the RHS of (2.2.10) is in $L^2(\mathbf{R})$ and even about $z = 0$, hence orthogonal to $\phi_h' \in \ker\mathcal{L}$. The existence of $\mathcal{B}_2$ is established similarly. Further, we truncate $\mathcal{B}_j$ to have compact support on the bounded interval $[0, \frac{L_2}{\varepsilon}]$ and introduce the functions

$$\mathcal{B}_{j,n} := \sum_{i=1}^{n} \hat{\mathcal{B}}_j(z - p_i) + \mathcal{B}_{j,\infty}, \tag{2.2.11}$$

so that

$$\overline{\mathcal{L}}^j_{n,b}\mathcal{B}_{j,n} = 1 + o(\delta^2) \tag{2.2.12}$$

where $\overline{\mathcal{L}}_{n,b}$ represents the restriction of the operator $\overline{\mathcal{L}}_n(\mathbf{p})$ to the bounded domain $[0, \frac{L_2}{\varepsilon}]$.

Remark 2.2.1. The kernel of the operator $\mathcal{L} := \partial_z^2 - W''(\phi_h)$, the linearization of (1.2.6) about the homoclinic solution ϕ_h is spanned by ϕ_h'. Indeed, the equation (1.2.6) has a translational symmetry on $\mathbf{R}$ since $\phi_h(z+\gamma)$ also solves the equation (1.2.6). If we insert $\phi_h(z+\gamma)$ in the ODE (1.2.6) and take its derivative with respect to γ we have

$$\mathcal{L}{\phi_h}' = 0, \tag{2.2.13}$$

and conclude that $\ker\mathcal{L} = span\{\phi_h'\}$. Since $\mathcal{L}$ is a second order Sturm-Liouville operator acting on an unbounded domain it has real-valued simple eigenvalues that can be written in a strictly decreasing order. Since $\ker\mathcal{L} = span\{{\phi_h}'\}$, ${\phi_h}'$ is an eigenfunction of $\mathcal{L}$ corresponding to the eigenvalue $\lambda = 0$. Since the eigenfunctions of a Sturm-Liouville operator, $\{\psi_j\}$, has j simple zeros and ${\phi_h}'$ has one node, it is the second largest eigenfunction, ψ_1, corresponding to the second eigenvalue $\lambda_1 = 0$. Then, there exists a ground-state eigenfunction, ψ_0 corresponding to $\lambda_0 > 0$ and the remainder of the spectrum is real and $\mathcal{O}(1)$ distance to the left of 0 by Weyl's essential spectrum theorem.

Remark 2.2.2. By standard perturbation theory, the properties of the point spectrum of $\overline{\mathcal{L}}_n(\mathbf{p})$ presented in Lemma 4.1.4 carry over up to exponentially small terms to the operator on the large bounded domain, $\overline{\mathcal{L}}_{n,b}(\mathbf{p})$. (See Section 9.6 in [Kapitula and Promislow, 2013].) For the purposes of exposition we do not distinguish between the operator $\overline{\mathcal{L}}_n(\mathbf{p})$ acting upon the whole line or $\overline{\mathcal{L}}_{n,b}(\mathbf{p})$ acting upon the large bounded domain when inverting the operator, except where doing so is essential to the argument.

2.2.2 Motivation

Before we proceed to the asymptotic expansion analysis of the E-L equation, we would like to motivate ε-dependence of the Lagrange multiplier, λ_ε, due to the ε-dependence of the total mass, m. Assume u_ε is a solution to equation (2.2.2), or to

$$\partial_z^4 u - 2W'(u)\partial_z^2 u - W'''(u)(\partial_z u)^2 + \big(W''(u)W'(u) - \lambda_\varepsilon\big) = 0, \tag{2.2.14}$$

written explicitly.

Rather than dealing with possible different types of solutions to (2.2.2), we focus on the construction of multipulse solutions as the possible minima of I. For this purpose, we simplify the problem taking $\varepsilon \to 0$ which extends the domain to $\mathbf{R}^+$ and the considering its even extension to $\mathbf{R}^-$. Further, keeping the mass constraint fixed

$$\int_{\mathbf{R}} (u_0 - b_-) dz = M, \tag{2.2.15}$$

we deduce that $u_0 \to b_-$ as $z \to \pm\infty$. Consistent with $u_0 \to b_-$ as $z \to \pm\infty$, the exponential dichotomies of (2.2.2) on $\mathbf{R}$ imply

$$\partial_z^k (u_0 - b_-) \to 0 \quad \textit{for} \quad k = 1, 2, 3 \quad \textit{exponentially as} \quad z \to \pm\infty, \tag{2.2.16}$$

(See Section 2.1.4 in [Kapitula and Promislow, 2013]). In other words, there exist constants $c, \kappa > 0$ such that

$$|\vec{\mathbf{u}}_0 - \vec{\mathbf{b}}_-| \leq c e^{-\kappa |z|}. \tag{2.2.17}$$

where $\vec{\mathbf{b}}_- = (b_-, 0, 0, 0)$, $\vec{\mathbf{u}}_0(\cdot) = \big(u_0(\cdot), u_0'(\cdot), u_0''(\cdot), u_0'''(\cdot)\big)$. Returning to (2.2.14), setting $\varepsilon = 0$ and taking $z \to \pm\infty$ by (2.2.16) we obtain the equality

$$W'(b_-) W''(b_-) = \lambda_0. \tag{2.2.18}$$

Since $W'(b_-) = 0$ we conclude that $\lambda_0 = 0$.

For the finite domain problem associated to (2.2.2), namely when $\varepsilon \neq 0$ but small, we choose boundary conditions that best approximate the whole line problem. In particular, we assume u becomes asymptotically close to b which solves

$$W'(b) W''(b) = \lambda_\varepsilon, \tag{2.2.19}$$

and impose the exponential dichotomies like boundary conditions

$$\vec{\mathbf{u}}(0) \in \mathcal{W}^u(\vec{\mathbf{b}}) \quad and \quad \vec{\mathbf{u}}\left(\frac{L_2}{\varepsilon}\right) \in \mathcal{W}^s(\vec{\mathbf{b}}),$$

where $\vec{\mathbf{b}} = (b, 0, 0, 0)$, $\vec{\mathbf{u}}(\cdot) = \big(u(\cdot), u'(\cdot), u''(\cdot), u'''(\cdot)\big)$ and $\mathcal{W}^u(\vec{\mathbf{b}})$ and $\mathcal{W}^s(\vec{\mathbf{b}})$ are the unstable and stable manifolds of $\vec{\mathbf{b}}$, respectively. This yields a finite domain problem that best approximates the whole-line problem. This assumption is consistent with the mass constraint, away from the pulses scale like

$$M = \int_0^{\frac{L_2}{\varepsilon}} (u - b_-)dz \approx \frac{L_2}{\varepsilon}(b - b_-) = \mathcal{O}(1), \tag{2.2.20}$$

and we deduce that $b = b_- + \mathcal{O}(\varepsilon)$. From (2.2.19) we deduce that

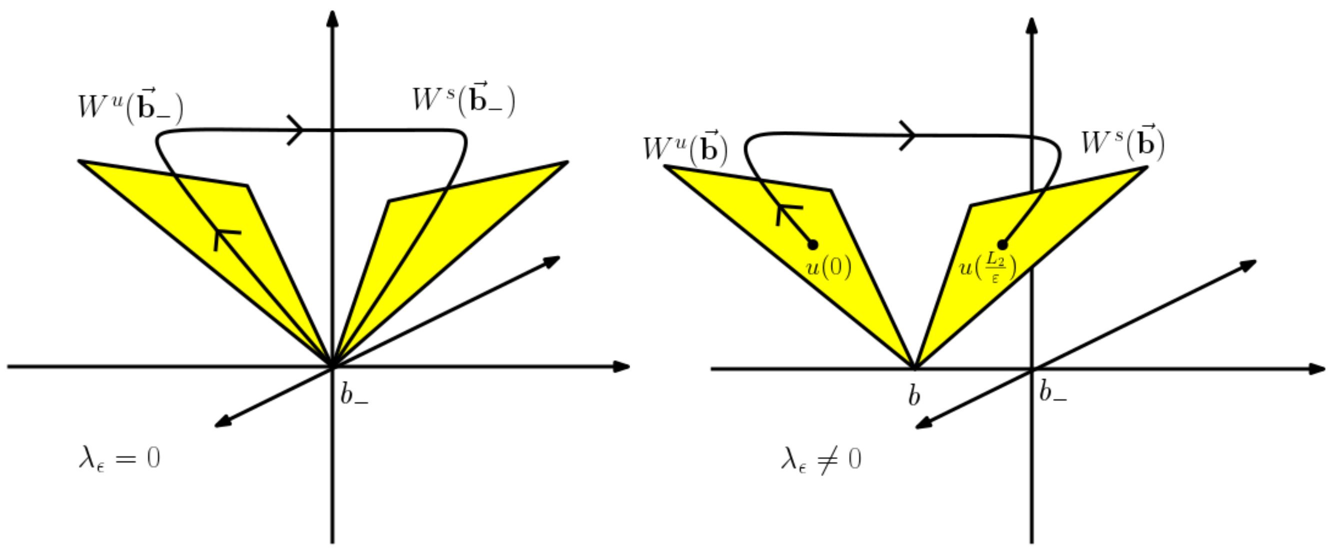
$$\lambda_\varepsilon = \big(W''(b_-)\big)^2(b - b_-) = \mathcal{O}(\varepsilon). \tag{2.2.21}$$

To fix notation, we write $b = b_- + \varepsilon b_1$ and $\lambda_\varepsilon = \varepsilon\lambda$ for $b_1 = \mathcal{O}(1)$ and $\lambda = \mathcal{O}(1)$.

2.2.3 Solutions to the Euler-Lagrange Equation with a small mass constraint

In the light of the comments about our motivation discussed in Section 2.2.2, our main focus of interest is constructing a class of solutions to (2.2.2) when $\lambda_\epsilon = \epsilon\lambda$. We are specifically interested in an asymptotic expansion of these solutions which has an expansion form

$$\Phi = \varphi_0 + \varepsilon\varphi_1 + \mathcal{O}(\varepsilon^2). \tag{2.2.22}$$



(a) Phase plane for when $\lambda_\epsilon = 0$ (b) Phase plane when $\lambda_\epsilon \neq 0$

Figure 2.1: Figure 2.1a represents a phase plane for the dynamical system given in (2.2.2) extended to $\mathbf{R}$ for $\lambda_\epsilon = 0$. Figure 2.1b is a phase plane for the dynamical system when $\lambda_\epsilon \neq 0$. The boundary conditions $\vec{\mathbf{u}}(0) \in W^u(\vec{\mathbf{b}})$ and $\vec{\mathbf{u}}\left(\frac{L_2}{\varepsilon}\right) \in W^s(\vec{\mathbf{b}})$ where $\vec{\mathbf{b}} = (b, 0, 0, 0)$ mimic the behavior of the whole line system. The distance between the fixed points of the dynamical systems is $|b - b_-| = \varepsilon$ while $|u(0) - b| = \mathcal{O}(e^{-\kappa \frac{L_2}{\varepsilon}}) \ll \varepsilon$. However, for the clarity of the graph both distances are depicted in similar lengths.

To find φ_0 and φ_1 we first insert (2.2.22) in (2.2.2).

$$
\begin{aligned}
\varepsilon\lambda &= \left(\partial_z^2 - W''(\Phi)\right)\left(\partial_z^2\Phi - W'(\Phi)\right), \\
&= \left(\partial_z^2 - W''\left(\varphi_0 + \varepsilon\varphi_1 + \mathcal{O}(\varepsilon^2)\right)\right)\left(\partial_z^2\left(\varphi_0 + \varepsilon\varphi_1 + \mathcal{O}(\varepsilon^2)\right) - W'\left(\varphi_0 + \varepsilon\varphi_1 + \mathcal{O}(\varepsilon^2)\right)\right), \\
&= \left(\partial_z^2 - \left(W''(\varphi_0) + \varepsilon\varphi_1 W'''(\varphi_0 +)\right) + \mathcal{O}(\varepsilon^2)\right)\left(\partial_z^2\varphi_0 + \varepsilon\partial_z^2\varphi_1 + \right. \\
&\qquad\qquad \left. - \left(W'(\varphi_0) + \varepsilon\varphi_1 W''(\varphi_0)\right) + \mathcal{O}(\varepsilon^2)\right).
\end{aligned}
\tag{2.2.23}
$$

Matching the terms with the same order of ε, we obtain $O(1)$ and $O(\varepsilon)$ equations

$$
\left(\partial_z^2 - W''(\varphi_0)\right)\left(\partial_z^2\varphi_0 - W'(\varphi_0)\right) = 0, \tag{2.2.24}
$$

$$\left(\partial_z^2 - W''\left(\varphi_0\right)\right)\left(\partial_z^2\varphi_1 - \varphi_1 W''\left(\varphi_0\right)\right) + \varphi_1 W'''\left(\varphi_0\right)\left(\partial_z^2\varphi_0 - W'\left(\varphi_0\right)\right) = \lambda, \tag{2.2.25}$$

As our base profile we choose the solutions ϕ_n for $n \in \{\frac{k+1}{2} : k \in \mathbb{N}\}$ to the problem (1.2.5) as a special classes of solutions of equation (2.2.24). Inserting $\varphi_0 = \phi_n$ in (2.2.25),

$$\begin{aligned}
\lambda &= \left(\partial_z^2 - W''\left(\phi_n\right)\right)\left(\partial_z^2\varphi_1 - \varphi_1 W''\left(\phi_n\right)\right), \\
&= \left(\partial_z^2 - W''\left(\phi_n\right)\right)\left(\partial_z^2 - W''\left(\phi_n\right)\right)\varphi_1, \\
&= \mathcal{L}_n^2\varphi_1,
\end{aligned} \tag{2.2.26}$$

where we have denoted the linearization of (1.2.6) about the periodic solution ϕ_n by

$$\mathcal{L}_n := \partial_z^2 - W''(\phi_n). \tag{2.2.27}$$

For each $n = 1, 2, 3$, the equation

$$\mathcal{L}_n^2\varphi_1 = \lambda, \tag{2.2.28}$$

has a solution $\varphi_1 \in L^2$ if and only if $\lambda \perp \mathit{ker}\mathcal{L}_n$ by the Fredholm Alternative. Indeed, λ is orthogonal to $\ker\mathcal{L}_n$ because we have

$$\int_{\mathbf{R}} \lambda {\phi_n}' dz = 0, \tag{2.2.29}$$

since $\mathit{ker}\mathcal{L}_n = \mathit{span}\{{\phi_n}'\}$ and ${\phi_n}'$ is odd about $z = 0$. Then, the solution to (2.2.28) can be written as

$$\varphi_1 = \lambda\mathcal{B}_{2,n}, \tag{2.2.30}$$

where $\mathcal{B}_{2,n}$ solves

$$\mathcal{L}_n^2 \mathcal{B}_{2,n} = 1. \tag{2.2.31}$$

Therefore, the periodic solutions to the Euler-Lagrange equation (2.1.10) in other words the critical points of I subject to the mass constraint (1.2.4) have the asymptotic expansion

$$\Phi_n = \phi_n + \varepsilon \lambda \mathcal{B}_{2,n} + \mathcal{O}\left(\varepsilon^2\right). \tag{2.2.32}$$

2.2.4 Some Remarks

Recall that the equation (1.2.5) has solutions, ϕ_n, which are global minimizers of the free energy (1.2.3) when $I(u)$ is not subject to any constraints. These have background state, $b = b_-$. In Section 2.2.2 we motivate the construction of the solutions with background. $b = b_n$ where $b_n - b_- = \mathcal{O}(\varepsilon)$ solves

$$W'(b_n)W''(b_n) = \lambda_\varepsilon. \tag{2.2.33}$$

Since $W'(b_-) = 0$ it follows λ_ε is also $\mathcal{O}(\varepsilon)$. Let $\tau(b_n)$ be the period of the orbit ϕ_n. The periodic solutions Φ_n to the Euler-Lagrange equation (2.1.10) which satisfy $\tau(b_n) = \frac{L_2}{n\varepsilon}$ can fit precisely n periods of the orbit into the interval $\left[0, \frac{L_2}{\varepsilon}\right]$ and may be translated to exactly solve the boundary conditions. Moreover, the period scales like $\tau(b_n) = \mathcal{O}\left(ln\left(\frac{1}{b_n - b_-}\right)\right)$ and is monotonically decreasing as $b - b_-$ increases, achieving a minima at the center equilibrium. These considerations suggest $n = \frac{L_2}{\varepsilon\tau(b_n)} = \mathcal{O}\left(\frac{1}{\varepsilon|ln\varepsilon|}\right)$, however we further restrict the size of b_n so that $n = \mathcal{O}(1)$

Remark 2.2.3. If $\tau(b) \neq \frac{L_2}{n\varepsilon}$ then we state without proof that by adjusting the associated ϕ

for which (ϕ, ϕ') passes through $(b, 0)$, we may arrive at a solution of (1.2.5) that satisfies the boundary conditions. Indeed by translating ϕ we may achieve $\phi'(0), \phi'(\frac{L_2}{\varepsilon}) = \mathcal{O}(e^{-\frac{L2}{\varepsilon}})$. The corrections to ϕ are exponentially small and have no impact on the value of the energy or total mass to the order considered here. In particular, we define

$$n := \left\lfloor \frac{L_2}{\tau(b)\varepsilon} - \frac{1}{2} \right\rfloor. \tag{2.2.34}$$

we proceed formally with the construction of Φ_n ignoring the issue of exponentially small boundary mismatch.

2.2.5 Energy values at the critical points

Inserting the critical points (2.2.32) in (1.2.3) and in the mass constraint (1.2.4) we calculate their energy values and their mass constraint M to determine which has minimum energy at prescribed mass.

$$\begin{aligned} I(\Phi_n) &= \int_0^{\frac{L_2}{\varepsilon}} \frac{1}{2}\left(\partial_z^2 \Phi_n - W'(\Phi_n)\right)^2 dz, \\ &= \int_0^{\frac{L_2}{\varepsilon}} \frac{1}{2}\left(\partial_z^2\left(\phi_n + \varepsilon\lambda\mathcal{B}_{2,n} + \mathcal{O}(\varepsilon^2)\right) - W'\left(\phi_n + \varepsilon\lambda\mathcal{B}_{2,n} + \mathcal{O}(\varepsilon^2)\right)\right)^2 dz, \\ &= \int_0^{\frac{L_2}{\varepsilon}} \frac{1}{2}\varepsilon^2\lambda^2\left(\partial_z^2\mathcal{B}_{2,n} - W''(\phi_n)\,\mathcal{B}_{2,n}\right)^2 + \mathcal{O}(\varepsilon^3) dz, \\ &= \int_0^{\frac{L_2}{\varepsilon}} \frac{1}{2}\varepsilon^2\lambda^2\left(\mathcal{L}_n\mathcal{B}_{2,n}\right)^2 + \mathcal{O}(\varepsilon^3) dz, \\ &= \int_0^{\frac{L_2}{\varepsilon}} \frac{1}{2}\varepsilon^2\lambda^2\left(\mathcal{L}_n\mathcal{B}_{2,n}\right)\left(\mathcal{L}_n\mathcal{B}_{2,n}\right) + \mathcal{O}(\varepsilon^3) dz. \end{aligned} \tag{2.2.35}$$

Since $\mathcal{L}_n$'s are self-adjoint and from (2.2.31) we obtain the reduced energy

$$\begin{aligned} I\left(\Phi_n\right) &= \int_0^{\frac{L_2}{\varepsilon}} \frac{1}{2}\varepsilon^2\lambda^2 \left(\mathcal{L}_n^2 \mathcal{B}_{2,n}\right) \mathcal{B}_{2,n} + \mathcal{O}(\varepsilon^3) dz, \\ &= \int_0^{\frac{L_2}{\varepsilon}} \frac{1}{2}\varepsilon^2\lambda^2 \mathcal{B}_{2,n} + \mathcal{O}(\varepsilon^3) dz. \end{aligned} \tag{2.2.36}$$

Let $\bar{\mathcal{B}}_{2,n}$ denote the mass for $\mathcal{B}_{2,n}$ that is

$$\begin{aligned} \overline{\mathcal{B}}_{2,n} &:= \int_0^{\frac{L_2}{\varepsilon}} \mathcal{B}_{2,n} dz, \\ &= \alpha_-^{-2}\frac{L_2}{\varepsilon} + \mathcal{O}(1), \end{aligned} \tag{2.2.37}$$

and M_n be the mass for the critical point (2.2.32)

$$\begin{aligned} M_n : &= \int_0^{\frac{L_2}{\varepsilon}} \Phi_n dz \\ &= \int_0^{\frac{L_2}{\varepsilon}} \left[(\phi_n - b_-) + \varepsilon\lambda\mathcal{B}_{2,n}\right] dz. \end{aligned} \tag{2.2.38}$$

Substituting the mass (2.2.37), we have

$$M_n = L_2\alpha_-^{-2}\lambda + \int_0^{\frac{L_2}{\varepsilon}} (\phi_n - b_-) dz. \tag{2.2.39}$$

For the sake of convenience, we write the mass of ϕ_n in terms of the mass of ϕ_h,

$$\begin{aligned} \int_0^{\frac{L_2}{\varepsilon}} (\phi_n - b_-) dz &= n\int_0^{\frac{L_2}{\varepsilon}} (\phi_h - b_-) dz, \\ &= nM_h, \end{aligned} \tag{2.2.40}$$

where M_h is the mass for the homoclinic solution ϕ_h, namely

$$M_h := \int_0^{\frac{L_2}{\varepsilon}} (\phi_h - b_-)dz. \tag{2.2.41}$$

We insert (2.2.40) in (2.2.38) and obtain

$$M_n = nM_h + \lambda\alpha_-^{-2} L_2. \tag{2.2.42}$$

Setting $M_n = M$ and solving for λ gives an expression for λ in terms of the mass constraint M,

$$\lambda = \frac{\alpha_-^2}{L_2} \left(M - nM_h\right). \tag{2.2.43}$$

Utilizing this formula of λ in (2.2.36), we calculate the reduced energy critical point of I in terms of the mass constraint M

$$\begin{aligned} I\left(\Phi_n\right) &= \int_0^{\frac{L_2}{\varepsilon}} \frac{1}{2}\varepsilon^2\lambda^2\mathcal{B}_{2,n}dz, \\ &= \frac{1}{2}\varepsilon^2\lambda^2\overline{\mathcal{B}}_{2,n}, \\ &= \frac{\varepsilon\alpha_-^2}{2L_2}\left(M - nM_h\right)^2. \end{aligned} \tag{2.2.44}$$

We minimize (2.2.44) for n to find the n-pulses with the minimal free energy. Note that when $M = nM_h$, Φ_n is a global minimizer. Since $I(\Phi_n)$ is a discrete function of n, the closest value of n to $\frac{M}{M_h}$ is the minima of $I\left(\Phi_n\right)$. We conclude that the inner scaling of FCH free energy, $I(u)$, attains its minima, over the n pulse solutions we have constructed, at Φ_n for which n is closest to $\frac{M}{M_h}$.

From Figure 2.2 , it can be observed that among the n-pulse profiles we have constructed

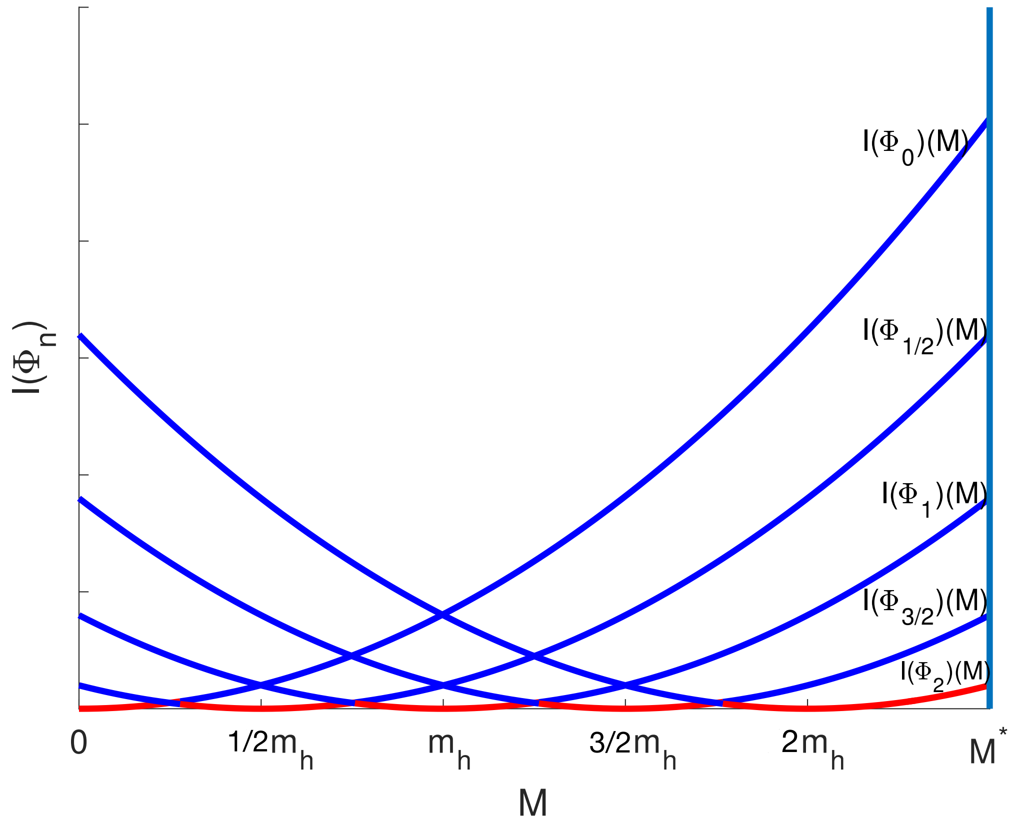


Figure 2.2: The reduced energy, $I(\Phi_n)$, versus mass constraint M. The blue lines demonstrate the energy values at the critical points, Φ_n, for $n = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$. The red lines represent the infimum of the energy values over all the blue lines.

the one with minimal energy sensitively depend on the mass constraint. In the sequel, we show the existence of a global minima and that integral values of n the associated n-pulse Φ_n is a local minima of I.

Chapter 3

Existence Of the Minimizers

3.1 Existence of the Minimizers

In this section, we use classical tools from direct methods of Calculus of Variations to establish the existence of a global minimizer the energy functional (1.2.3) subject to the mass constraint (1.2.4) following the procedures in [Promislow and Zhang, 2013]. Recall that the space of admissible functions is given by

$$\mathcal{A} = \left\{ u \in H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right) \,\middle|\, \int_0^{\frac{L_2}{\varepsilon}} (u - b_-)dz = M, u_z(0) = 0, u_z\left(\frac{L_2}{\varepsilon}\right) = 0 \right\}.$$

We consider a general form for the double-well potential $W(u)$. In addition to the assumptions earlier on, we suppose that $W(u)$ is convex at infinity and satisfies some growth conditions and $b_- = 0$ for the sake of easiness in the calculations. Specifically, there exist $p > 1$, $\bar{c}, c_- > 0$, $u_0 > 0$ and $\beta > 1$ sufficiently large such that

$$\begin{cases} W'(u) \leq \bar{c}|u|^p, & |u| \geq 1, \\ W'(u) \leq c_-, & |u| < 1, \end{cases} \tag{3.1.1}$$

$$W''(u) > \beta, \qquad |u| > u_0. \tag{3.1.2}$$

In the following lemma, we establish an estimate for H^1 norm of $u \in \mathcal{A}$ which is utilized in the proof of the H^2-coercivity of the energy in Theorem 3.1.2.

For simplicity in the proof of existence of a global minima, for fixed $\varepsilon > 0$ and $L_2 \in \mathbf{R}$ we introduce scaled Cahn-Hilliard free energy functional on a large bounded domain $\left[0, \frac{L_2}{\varepsilon}\right] \in \mathbf{R}$

$$E = \int_0^{\frac{L_2}{\varepsilon}} \frac{1}{2}(\partial_z u)^2 + W(u)dz. \tag{3.1.3}$$

The variational derivative of E with respect to L^2 inner product is

$$\frac{\delta E}{\delta u} = -\partial_z^2 u + W'(u), \tag{3.1.4}$$

and note that the FCH free energy, I, can be written in terms of $\frac{\delta E}{\delta u}$

$$\begin{aligned} I(u) &= \int_0^{\frac{L_2}{\varepsilon}} \frac{1}{2}\left(\frac{\delta E}{\delta u}\right)^2 dz, \\ &= \frac{1}{2}\left\|\frac{\delta E}{\delta u}\right\|_{L^2}^2. \end{aligned} \tag{3.1.5}$$

Lemma 3.1.1. *Fix $\varepsilon > 0$. There exists a constant $C > 0$ such that*

$$\|u\|_{H^1}^2 \leq \left\|\frac{\delta E}{\delta u}\right\|_{L^2}^2 + C, \tag{3.1.6}$$

for all $u \in \mathcal{A}$.

Proof. Multiplying $\frac{\delta E}{\delta u}$ by u and integrating by parts, we obtain

$$\int_0^{\frac{L_2}{\varepsilon}} u\frac{\delta E}{\delta u}dz = \int_0^{\frac{L_2}{\varepsilon}} u\Big(-\partial_z^2 u + W'(u)\Big)dz = \int_0^{\frac{L_2}{\varepsilon}} (\partial_z u)^2 + uW'(u)dz. \tag{3.1.7}$$

So,

$$\int_0^{\frac{L_2}{\varepsilon}} (\partial_z u)^2 dz = \int_0^{\frac{L_2}{\varepsilon}} u\frac{\delta E}{\delta u} - uW'(u)dz. \tag{3.1.8}$$

Applying Young's inequality to the first term on the right and adding $\|u\|_{L^2}^2$ to both sides, we have

$$\begin{aligned} \|u\|_{H^1}^2 &\leq \int_0^{\frac{L_2}{\varepsilon}} \frac{u^2}{2} + \frac{1}{2}\left(\frac{\delta E}{\delta u}\right)^2 - uW'(u) + u^2 dz, \\ &= I(u) + \int_0^{\frac{L_2}{\varepsilon}} \frac{3}{2}u^2 - uW'(u)dz. \end{aligned} \tag{3.1.9}$$

We need to get an upper bound for $\int_0^{\frac{L_2}{\varepsilon}} h(z)dz$ where

$$h(z) := \frac{3}{2}u^2 - uW'(u). \tag{3.1.10}$$

From the assumption (3.1.2) there exists β and u_0 such that

$$W'(u) > \beta\,(u - u_0) + W'(u_0) \quad \textit{for} \quad u > u_0, \tag{3.1.11}$$

$$W'(u) < \beta(u_0 + u) + W'(-u_0) \quad \textit{for} \quad u < -u_0. \tag{3.1.12}$$

Multiplying (3.1.11) and (3.1.12) by $-u$ and adding $\frac{3}{2}u^2$, we obtain

$$h(u) < \left(\frac{3}{2} - \beta\right)u^2 + \beta u_0 u - uW'(u_0) \quad \textit{for} \quad u > u_0, \tag{3.1.13}$$

$$h(u) < \left(\frac{3}{2} - \beta\right)u^2 - \beta u_0 u - uW'(-u_0) \quad \textit{for} \quad u < -u_0. \tag{3.1.14}$$

The inequalities (3.1.13) and (3.1.14) imply that on $\mathbf{R}/[-u_0, u_0]$ $h(u)$ is bounded above for sufficiently large β because it is bounded by a function whose dominant term is quadratic with a negative coefficient for sufficiently large β. Further, $h(u)$ is bounded on $[-u_0, u_0]$ since it is continuous. Hence, $h(u)$ is bounded above for sufficiently large β, in other words there exists a constant $C > 0$ independent of ε such that

$$\int_0^{\frac{L_2}{\varepsilon}} h(z)dz < C. \tag{3.1.15}$$

From this and (3.1.9) , the bound we aimed in (3.1.6) has been achieved. □

Theorem 3.1.2. *Fix $\varepsilon > 0$ and L_2, independent of ϵ. The energy functional given in* (1.2.3) *subject to the mass constraint* (1.2.4) *has a global minimizer over the admissible set $\mathcal{A}$. In other words, there exists at least one $u \in \mathcal{A}$ satisfying*

$$I(w) \geq I(u), \tag{3.1.16}$$

for all $w \in \mathcal{A}$.

Proof. Since $I(\cdot)$ is well-defined on $\mathcal{A}$ and bounded below by 0, we define $m := inf_{w\in\mathcal{A}} I(w) \geq 0$.

The key step in the proof of the existence of a minimizer is to show the coercivity of $I(u)$ over the admissible set $\mathcal{A}$. More specifically we show that there exist constants β and γ such that

$$(I(u))^{\frac{3}{2}} \leq -\gamma + \beta \|u\|_{H^2}. \tag{3.1.17}$$

To establish this bound we pursue the preliminary step of bounding the variational derivative of E, given in (3.1.3). By the relation obtained from (3.1.4)

$$\partial_z^2 u = \frac{\delta E}{\delta u} - W'(u). \tag{3.1.18}$$

Taking L^2 norm of both sides and applying the triangle inequality, we obtain

$$\begin{aligned} \left\|\partial_z^2 u\right\|_{L^2} &= \left\|\frac{\delta E}{\delta u} - W'(u)\right\|_{L^2}, \\ &\leq \left\|\frac{\delta E}{\delta u}\right\|_{L^2} + \left\|W'(u)\right\|_{L^2}, \end{aligned} \tag{3.1.19}$$

and by the second relation in (3.1.5),

$$\begin{aligned} \left\|\partial_z^2 u\right\|_{L^2} &\leq \left\|\frac{\delta E}{\delta u}\right\|_{L^2} + \left\|W'(u)\right\|_{L^2}, \\ &= (2I(u))^{\frac{1}{2}} + \left\|W'(u)\right\|_{L^2}. \end{aligned} \tag{3.1.20}$$

Then, we add $\|u\|_{H^1}$ on both sides of (3.1.20) and obtain

$$\|u\|_{H^2} \leq (2I(u))^{\frac{1}{2}} + \left\|W'(u)\right\|_{L^2} + \|u\|_{H^1}. \tag{3.1.21}$$

Here using the bound for H^1 norm of u constructed in Lemma 3.1.1, we have

$$\begin{aligned} \|u\|_{H^2} &\leq (2I(u))^{\frac{1}{2}} + \left\|W'(u)\right\|_{L^2} + \|u\|_{H^1}, \\ &\leq (2I(u))^{\frac{1}{2}} + \left\|W'(u)\right\|_{L^2} + \left(\left\|\frac{\delta E}{\delta u}\right\|_{L^2}^2 + C\right)^{\frac{1}{2}}, \end{aligned} \tag{3.1.22}$$

and by the second relation in (2.1.24), we obtain

$$\|u\|_{H^2} \leq (2I(u))^{\frac{1}{2}} + \left\|W'(u)\right\|_{L^2} + (2I(u) + C)^{\frac{1}{2}}. \tag{3.1.23}$$

Further, by the assumptions (3.1.1) we have

$$\begin{aligned}
\|W'(u)\|_{L^2}^2 &= \bar{c}^2 \int_{|u|>1} |u|^{2p} dz + c_-^2 \int_{|u|<1} |u|^2 dz, \\
&\le \bar{c}^2 \int_0^{\frac{L_2}{\varepsilon}} |u|^{2p} dz + c_-^2 \int_{|u|<1} |u|^2 dz, \\
&\le \bar{c}^2 \|u\|_{L^{2p}}^{2p} + c_-^2 \frac{L_2}{\varepsilon} \\
&\le \bar{c}^2 \|u\|_{H^1}^{2p} + c_-^2 \frac{L_2}{\varepsilon}.
\end{aligned} \tag{3.1.24}$$

Taking square root of both sides of (3.1.24) and by Lemma 3.1.1, we deduce that there exists a $c > 0$ such that

$$\begin{aligned}
\|W'(u)\|_{L^2} &\le c\left(1 + \left\|\frac{\delta E}{\delta u}\right\|_{L^2}^p\right), \\
&= c\left(1 + (2I(u))^{\frac{p}{2}}\right),
\end{aligned} \tag{3.1.25}$$

for $p > 1$. Inserting (3.1.25) in (3.1.23) we conclude that there exist constants $\alpha, \gamma > 0$ such that

$$\|u\|_{H^2} \le \alpha + \gamma \left(I(u)\right)^{\frac{p}{2}}, \tag{3.1.26}$$

for $p > 1$ and this provides the H^2 coercivity of $I(u)$.

The other essential part of the proof is showing the weak lower semi-continuity of the energy. Since the energy functional is bounded below, there exists a minimizing sequence $\{u_k\}_{k=1}^{\infty} \in \mathcal{A}$ and so,

$$I(u_k) \to m. \tag{3.1.27}$$

From the H^2 coercivity of $I(u)$, the sequence $\{u_k\}$ is bounded in H^2 and there exists a subsequence $\{u_{j_k}\}$ and $\bar{u} \in H^2$ such that

$$u_{j_k} \rightharpoonup \bar{u} \quad \textit{weakly in} \quad H^2, \tag{3.1.28}$$

and further, there exists a subsequence $(u_{j_k})_n \to \bar{u}$ in H^1 since $H^1 \subset\subset H^2$. Before the proof of weak lower semi-continuity of $I(u)$, we need to verify that such $\bar{u}$ resides in $\mathcal{A}$. First observe that $\partial_z u_{j_k} \rightharpoonup \partial_z \bar{u}$ weakly in L^2 and $\partial_z^2 u_{j_k} \rightharpoonup \partial_z^2 \bar{u}$ in L^2. From integration by parts, for any $w \in H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$ we have

$$\int_0^{\frac{L_2}{\varepsilon}} \left(\partial_z^2 w\right) v dz = -\int_0^{\frac{L_2}{\varepsilon}} \partial_z w \partial_z v + \left(\partial_z w\right) v\Big|_0^{\frac{L_2}{\varepsilon}}, \tag{3.1.29}$$

for any $v \in C^\infty\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$. We substitute $w = u_{j_k}$ in (3.1.31).Since $u_{j_k} \in \mathcal{A}$ the boundary term is 0 and hence the equation (3.1.31) becomes

$$\int_0^{\frac{L_2}{\varepsilon}} \left(\partial_z^2 u_{j_k}\right) v dz = -\int_0^{\frac{L_2}{\varepsilon}} \partial_z u_{j_k} \partial_z v. \tag{3.1.30}$$

On the other hand, we substitute $w = \bar{u} \in H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$ in (3.1.31) and obtain

$$\int_0^{\frac{L_2}{\varepsilon}} \left(\partial_z^2 \bar{u}\right) v dz = -\int_0^{\frac{L_2}{\varepsilon}} \partial_z \bar{u} \partial_z v + \left(\partial_z \bar{u}\right) v\Big|_0^{\frac{L_2}{\varepsilon}}, \tag{3.1.31}$$

for all $v \in C^\infty\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$. Consequently, since $\partial_z u_{j_k} \rightharpoonup \partial_z \bar{u}$ weakly in L^2 and $\partial_z^2 u_{j_k} \rightharpoonup \partial_z^2 \bar{u}$ in L^2 and comparing the non-zero terms in (3.1.30) and (3.1.31) we conclude that

$$\left(\partial_z \bar{u}\right) v\Big|_0^{\frac{L_2}{\varepsilon}} = 0, \tag{3.1.32}$$

for all $v \in C^\infty\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$. This proves that $\partial_z \bar{u}(0) = \partial_z \bar{u}(\frac{L_2}{\varepsilon})$ and hence, $\bar{u} \in \mathcal{A}$.

For the weak lower-semi continuity of $I(u)$, we utilize the weak convergence of the first variation of $\mathcal{F}$,

$$\frac{\delta \mathcal{F}}{\delta u}\left(Du_{j_k}, u_{j_k}, z\right) \rightharpoonup \frac{\delta \mathcal{F}}{\delta u}\left(D\bar{u}, \bar{u}, z\right) \quad \textit{weakly in} \quad L^2, \tag{3.1.33}$$

which is established in Lemma 3.1.3.

Since weak convergence is lower semi continuous,

$$\liminf \int_0^{\frac{L_2}{\varepsilon}} \left(\frac{\delta \mathcal{F}}{\delta u}\left(Du_{j_k}, u_{j_k}, z\right)\right)^2 dz \geq \int_0^{\frac{L_2}{\varepsilon}} \left(\frac{\delta \mathcal{F}}{\delta u}\left(D\bar{u}, \bar{u}, z\right)\right)^2 dz. \tag{3.1.34}$$

Considering $\bar{u} \in \mathcal{A}$, it follows that

$$m \leq I(\bar{u}) \leq \liminf I(u_{k_j}) = m. \tag{3.1.35}$$

Thus, the energy, I, attains its minima at $\bar{u}$. □

Lemma 3.1.3. *If $u_{j_k} \rightharpoonup \bar{u}$ weakly in H^2 and strongly in L^2, then the variational derivative of $\mathcal{F}$, given in* (3.1.3), *converges*

$$\frac{\delta \mathcal{F}}{\delta u}\left(Du_{j_k}, u_{j_k}, z\right) \rightharpoonup \frac{\delta \mathcal{F}}{\delta u}\left(D\bar{u}, \bar{u}, z\right), \tag{3.1.36}$$

weakly in $L^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$.

Proof. We already know that $\partial_z^2 u_{j_k} \rightharpoonup \partial_z^2 \bar{u}$ in $L^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$ and it suffices to show that $W'\left(u_{j_k}\right) \to W'(\bar{u})$ in $L^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$. By the mean value theorem and since $\bar{u}$, u_{j_k} are uniformly bounded there exists $\xi_{j_k} \in L^\infty\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$ with $\left\|\xi_{j_k}\right\|_{L^\infty}$ uniformly bounded, independent of ε, such that

$$\left|W'\left(u_{j_k}\right) - W'(\bar{u})\right| \leq |W''(\xi_{j_k})||u_{j_k} - \bar{u}|. \tag{3.1.37}$$

Squaring both sides of (3.1.37) and integrating over $\left[0, \frac{L_2}{\varepsilon}\right]$ we obtain

$$\begin{aligned} \left\|W'\left(u_{j_k}\right) - W'(\bar{u})\right\|_{L^2} &\leq \left\|W''(\xi_{j_k})(u_{j_k} - \bar{u})\right\|_{L^2}, \\ &\leq \left\|W''(\xi_{j_k})\right\|_{L^\infty} \left\|u_{j_k} - \bar{u}\ \right\|_{L^2}, \end{aligned} \tag{3.1.38}$$

which converges to zero since $W''(\xi_{j_k})$ is uniformly bounded in $L^\infty\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$ and $u_{j_k} \rightharpoonup \bar{u}$ weakly in $H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$ and also, $u_{j_k} \to \bar{u}$ in $L^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$. Thus, $W'(u_{j_k})$ converges $W'(\bar{u})$ in $L^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$. □

Chapter 4

Modulational Stability of n-pulses

In this chapter, our main interest is the dynamical stability of a manifold of n-pulses given as the graph of an n-pulse ansatz, $\overline{\Phi}_n(z, \mathbf{p}, \bar{\lambda})$. These are related to the periodic multi-pulse solutions, $\{\Phi_n | n \in \mathbf{N}\}$, constructed in Chapter 2 as a class of critical points to the free energy I. The H^2-coercivity of the second variation of I about $\overline{\Phi}_n(z, \mathbf{p}, \bar{\lambda})$, modulational stability and dynamic evolution of the n-pulse ansatz with respect to the Π_0-gradient flow are addressed.

4.1 H^2-coercivity of the second variation of I

In this section, we prove the H^2-coercivity of the second variation of the free energy I about n-pulse ansatz, $\dfrac{\delta^2 I}{\delta u^2}(\overline{\Phi}_n(z, \mathbf{p}), \bar{\lambda})$. The second variation is defined from the Riesz representation theorem

$$\frac{d^2}{d\tau^2} I(u + \tau v)\Big|_{\tau=0} = \Big\langle \frac{\delta^2 I}{\delta u^2}(u)v, v \Big\rangle_{L^2}, \tag{4.1.1}$$

for any $v \in H^2(\mathbf{R})$.

Definition 4.1.1. : Let $\mathcal{D}$ be a subspace of Hilbert space $\mathcal{H}$. A linear operator $A : \mathcal{D} \to H$ satisfying

$$\langle Au, u \rangle \geq \mu \|u\|^2, \quad \forall u \in \mathcal{D} \tag{4.1.2}$$

for some $\mu > 0$ is called a coercive operator and μ is called the coercivity constant.

4.1.1 Introduction: n-pulses

Recall that n-pulse ansatz defined on $\mathbf{R}$ is given in (2.2.5) as

$$u_n := \sum_{j=1}^{n} \overline{\phi}_h \left(z - p_j\right) + b_-, \tag{4.1.3}$$

where $\overline{\phi}_h := \phi_h - b_-$ and $\mathbf{p} = (p_1, p_2, ..., p_n)^t \in \mathbf{R}^n$ is the vector of pulse locations. The admissible set of pulse locations is given by

$$\mathcal{P} = \{\mathbf{p} \in \mathbf{R}^n : p_i < p_{i+1} \quad \textit{for} \quad i = 1, ..., n \quad \textit{and} \quad \Delta\mathbf{p} \geq l\}, \tag{4.1.4}$$

where $\Delta\mathbf{p} = \min\limits_{i \neq j} |p_i - p_j|$ and $l > 0$ is sufficiently large so that the exponential terms $e^{-\sqrt{\alpha_-} l}$ arising in the calculations are negligible. We extend u_n to be defined on all of $\mathbf{R}$, and add a correction term that reduces the size of residual. Recalling the definition of $\mathcal{B}_{2,n}$ given in (2.2.31), we introduce the corrected extension by

$$\overline{\Phi}_n(z, \mathbf{p}, \bar{\lambda}) := u_n + \delta\bar{\lambda}\mathcal{B}_{2,n}, \tag{4.1.5}$$

and let $\mathcal{M}_n = \{\overline{\Phi}_n(\mathbf{p}, \bar{\lambda}) | \mathbf{p} \in \mathcal{P}\}$ be the n-dimensional manifold formed by these solutions. Let $X_n(\mathbf{p})$ represent the tangent plane to the manifold $\mathcal{M}_n$, $X_n(\mathbf{p}) = span\{\frac{\partial \overline{\Phi}_n(\mathbf{p})}{\partial p_i} : \mathbf{p} \in \mathcal{P} \subset \mathbf{R}^n\} = span\{\phi_h'(z - p_i)\}_{i=1}^{n}$.

Recall that in (2.2.4) we introduced the operator

$$\overline{\mathcal{L}}_n(\mathbf{p}) := \partial_z^2 - W''\left(u_n\right), \tag{4.1.6}$$

acting on $H^2(\mathbf{R})$. The second variation of the free energy I about n-pulse ansatz is

$$\mathbb{L} := \left(\partial_z^2 - W''(\overline{\Phi}_n)\right)^2 - \left(\partial_z^2 \overline{\Phi}_n - W'(\overline{\Phi}_n)\right) W'''(\overline{\Phi}_n). \tag{4.1.7}$$

Taylor expanding (4.1.7) about u_n up to $\mathcal{O}(\delta^2)$ terms we obtain

$$\begin{aligned}\mathbb{L} = \Big(\overline{\mathcal{L}}_n - \delta\bar{\lambda} W'''(u_n)\mathcal{B}_{2,n} + \mathcal{O}(\delta^2)\Big)^2 - \Big(&\partial_z^2 u_n - W'(u_n) + \delta\bar{\lambda}\partial_z^2 \mathcal{B}_{2,n} \\ &- \delta\bar{\lambda} W''(u_n)\mathcal{B}_{2,n} + \mathcal{O}(\delta^2)\Big)\Big(W'''(u_n) + \delta\bar{\lambda} W^{(4)}(u_n)\mathcal{B}_{2,n} + \\ &+ \mathcal{O}(\delta^2)\Big).\end{aligned} \tag{4.1.8}$$

Recalling that $\overline{\mathcal{L}}_n \mathcal{B}_{2,n} = \mathcal{B}_{1,n}$ we obtain

$$\begin{aligned}\mathbb{L} = \Big(\overline{\mathcal{L}}_n - \delta\bar{\lambda} W'''(u_n)\mathcal{B}_{2,n} + \mathcal{O}(\delta^2)\Big)^2 - \Big(&\partial_z^2 u_n - W'(u_n) + \\ &+ \delta\bar{\lambda}\mathcal{B}_{1,n} + \mathcal{O}(\delta^2)\Big)\Big(W'''(u_n) + \delta\bar{\lambda} W^{(4)}(u_n)\mathcal{B}_{2,n} + \mathcal{O}(\delta^2)\Big).\end{aligned} \tag{4.1.9}$$

From the definition of u_n, partitioning the domain into sub-intervals about pulse locations as $\left[\frac{p_{i-1}+p_i}{2}, \frac{p_i+p_{i+1}}{2}\right]$ for $i = 1, \ldots, n$ we have $u_n = \phi_i + \mathcal{O}(e^{-\sqrt{\alpha_-}\ell})$ on each sub-interval. We Taylor expand $\partial_z^2 u_n - W'(u_n)$ about $\phi_h(z - p_i)$ and obtain $\partial_z^2 u_n - W'(u_n) = -W''(\phi_h(z - p_i))e^{-\sqrt{\alpha_-}\ell} + \mathcal{O}(e^{-2\sqrt{\alpha_-}\ell})$ on $\left[\frac{p_{i-1}+p_i}{2}, \frac{p_i+p_{i+1}}{2}\right]$ since ϕ_h solves the equation (1.2.6). Summing up over all sub-intervals provides $\partial_z^2 u_n - W'(u_n) = -\sum_{i=1}^{n} W''(\phi_h(z - p_i))e^{-\sqrt{\alpha_-}\ell}$. Here we set $\delta = e^{-\sqrt{\alpha_-}\ell}$ which will be our scaling value of the background parameter through the rest of this thesis.

We insert this expansion into (4.1.9) and calculate the inner product

$$\langle \mathbb{L}v, v\rangle = \Big\langle \Big(\Big(\overline{\mathcal{L}}_n - \delta\bar{\lambda} W'''(u_n)\mathcal{B}_{2,n} + \mathcal{O}(\delta^2) \Big)^2 + \\ + - \Big(-\delta \sum_{i=1}^{n} W''(\phi_h(z-p_i)) + \delta\bar{\lambda}\mathcal{B}_{1,n} \Big) W'''(u_n) + \\ + \mathcal{O}(\delta^2) \Big) v, v \Big\rangle, \tag{4.1.10}$$

for any $v \in H^2$. Here we must be careful when expanding the quadratic term and simplifying further. Precisely, we have

$$\langle \mathbb{L}v, v\rangle = \Big\langle \Big(\overline{\mathcal{L}}_n^2 v - \delta\bar{\lambda}\overline{\mathcal{L}}_n (W'''(u_n)\mathcal{B}_{2,n} v) - \delta\bar{\lambda} W'''(u_n)\mathcal{B}_{2,n}\overline{\mathcal{L}}_n v \Big) + \\ + - \Big(-\delta \sum_{i=1}^{n} W''(\phi_h(z-p_i)) + \delta\bar{\lambda}\mathcal{B}_{1,n} \Big) W'''(u_n) v + \\ v, v \Big\rangle + \mathcal{O}(\delta^2 \left\| v \right\|_{H^2}^2). \tag{4.1.11}$$

Further, for the proof of H^2-coercivity of $\mathbb{L}$ and later to verify H^2-coercivity of $\frac{\delta^2 I}{\delta u^2}(\phi_n) := \mathcal{L}_n^2$ we establish and utilize H^2-coercivity of the second variation of the inner scaling of FCH free energy about n-pulse ansatz at the leading order so it is worth noting that

$$\langle \mathbb{L}v, v\rangle = \langle \overline{\mathcal{L}}_n^2 v, v\rangle + \mathcal{O}(\delta \left\| v \right\|_{H^2}^2). \tag{4.1.12}$$

4.1.2 H^2-coercivity of the second variation of I about n-pulse ansatz

For the stability analysis of the n-pulse ansatz purposes, we establish the H^2-coercivity of the second variation of I about n-pulse ansatz given in (4.1.10).

Theorem 4.1.2. *Consider the inner scaling of FCH free energy, I, given in* (1.2.3) *and*

n-pulse ansatz given in (4.1.5). *Then, the bilinear form induced by* $\mathbb{L}$ *given in* (4.1.10) *is coercive, i.e, there exists a* $\mu > 0$ *independent of* ε *such that*

$$\langle \mathbb{L}v, v \rangle \geq \mu \|v\|_{H^2(\mathbf{R})}, \tag{4.1.13}$$

for all $v \in X_n^{\perp}(\mathbf{p})$.

The essential step in the proof of this theorem is the H^2-coercivity of $\overline{\mathcal{L}}_n^2$ over $X_n^{\perp}(\mathbf{p})$. Before we present the proof of Theorem 4.1.2 we establish the H^2-coercivity of $\overline{\mathcal{L}}_n^2$ over $X_n^{\perp}(\mathbf{p})$.

Theorem 4.1.3. *Consider the operator* $\overline{\mathcal{L}}_n$ *given in* (2.2.4) *on* $H^2(\mathbf{R})$ *which is the linearization of* (1.2.6) *about n-pulses,* u_n, *given in* (2.2.5). *Then, the bilinear form induced by* $\dfrac{\delta^2 I}{\delta u^2}(u_n) = \overline{\mathcal{L}}_n^2$ *is coercive, i.e, there exists a* $\tilde{\mu} > 0$ *independent of* ε *such that*

$$\langle \overline{\mathcal{L}}_n^2 v, v \rangle \geq \tilde{\mu} \|v\|_{H^2(\mathbf{R})}, \tag{4.1.14}$$

for all $v \in X_n^{\perp}(\mathbf{p})$.

The H^2-coercivity of the operator $\overline{\mathcal{L}}_n^2(\mathbf{p})$ arises from the L^2-coercivity of $\overline{\mathcal{L}}_n^2(\mathbf{p})$. We prove the L^2-coercivity of $\overline{\mathcal{L}}_n^2(\mathbf{p})$ in Lemma 4.1.4.

Lemma 4.1.4. *There exists a* $\tilde{\mu} > 0$ *such that*

$$\langle \overline{\mathcal{L}}_n^2(\mathbf{p}) v, v \rangle \geq \tilde{\mu} \, \|v\|^2_{L^2(\mathbf{R})}, \tag{4.1.15}$$

for all $v \in X_n^{\perp}(\mathbf{p})$.

Proof. It suffices to show that there exists a $\tilde{\mu} > 0$ such that

$$\langle (\overline{\mathcal{L}}_n^2(\mathbf{p}) - \tilde{\mu})v, v\rangle \geq 0, \tag{4.1.16}$$

for all $v \in X_n^{\perp}(\mathbf{p})$. ${\overline{\mathcal{L}}_n}^2(\mathbf{p}) - \tilde{\mu} : H^2(\mathbf{R}) \subset L^2(\mathbb{R}) \to H^{-2}(\mathbf{R})$ is a self-adjoint operator on $L^2(\mathbf{R})$. Let

$$b[v, v] := \langle (\overline{\mathcal{L}}_n^2(\mathbf{p}) - \tilde{\mu})v, v\rangle, \tag{4.1.17}$$

be the bilinear form associated to ${\overline{\mathcal{L}}_n}^2(\mathbf{p})$ in the L^2 inner product. We first find the operator induced by the constrained bilinear form defined by the restriction of $b[v, v]$ to $X_n^{\perp}(\mathbf{p})$.

We introduce the orthogonal projection $P : H^2(\mathbf{R}) \to X_n(\mathbf{p})$ and $\Pi := I - P$ which has the range $X_n^{\perp}(\mathbf{p})$. The bilinear form constrained to $X_n^{\perp}(\mathbf{p})$ induces the constrained operator

$$(\overline{\mathcal{L}}_n^2(\mathbf{p}) - \tilde{\mu})_{\Pi} := \Pi(\overline{\mathcal{L}}_n^2(\mathbf{p}) - \tilde{\mu})\Pi : H^2(\mathbf{R}) \cap X_n^{\perp}(\mathbf{p}) \to \Pi H^{-2}(\mathbf{R}). \tag{4.1.18}$$

The operator $\Pi(\overline{\mathcal{L}}_n^2(\mathbf{p}) - \tilde{\mu})\Pi$ is self adjoint, so its spectrum is real-valued. If the point spectra of $\Pi(\overline{\mathcal{L}}_n^2(\mathbf{p}) - \tilde{\mu})\Pi$ is strictly positive for some values of $\tilde{\mu}$ then we obtain (4.1.31).

It remains to show that the point spectra of $\Pi(\overline{\mathcal{L}}_n^2(\mathbf{p}) - \tilde{\mu})\Pi$ is strictly positive. By Proposition 5.3.1 in [Kapitula and Promislow, 2013], number of the negative eigenvalues of the constrained operator is given by the difference between the number of the negative eigenvalues of the operator and the constrained matrix, $D(\tilde{\mu})$,namely,

$$\mathbf{n}((\overline{\mathcal{L}}_n^2(\mathbf{p}) - \tilde{\mu})_{\Pi}) = \mathbf{n}(\overline{\mathcal{L}}_n^2(\mathbf{p}) - \tilde{\mu}) - \mathbf{n}(D(\tilde{\mu})), \tag{4.1.19}$$

where $\mathbf{n}$ represents the count of the negative eigenvalues and $D(\tilde{\mu})$ is given as

$$D_{ij}(\tilde{\mu}) := \langle \phi'(z-p_i), (\overline{\mathcal{L}}_n^2(\mathbf{p}) - \tilde{\mu})^{-1}\phi'(z-p_j)\rangle. \tag{4.1.20}$$

We first calculate the number of negative eigenvalues of the operator $\overline{\mathcal{L}}_n^2(\mathbf{p}) - \tilde{\mu}$ by directly examining the spectrum of the operator.

Since $u_1 = \phi_h(z-p_1)$ and ϕ_h is translation invariant, the linearized operator $\mathcal{L}$ about ϕ_h and $\overline{\mathcal{L}}_1(\mathbf{p})$ are the same operators. Hence, we have that $\sigma(\mathcal{L}) = \sigma(\overline{\mathcal{L}}_1(\mathbf{p}))$. On the other hand, the essential spectrum of $\overline{\mathcal{L}}_1$ is

$$\sigma_{ess}(\overline{\mathcal{L}}_1) = \{-k^2 - \alpha_- : k \in \mathbf{R}\}. \tag{4.1.21}$$

Further, $\sigma_{ess}(\overline{\mathcal{L}}_n(\mathbf{p})) = \sigma_{ess}(\overline{\mathcal{L}}_1(\mathbf{p}))$ by the classical Weyl essential spectrum theorem since the operators have the same limiting states, $lim_{z\to\pm\infty}u_n$. (See Chapter 3 in [Kapitula and Promislow, 2013 On the other hand, to each point spectrum λ_k of $\overline{\mathcal{L}}_1(\mathbf{p})$, there are associated n eigenvalues of $\overline{\mathcal{L}}_n(\mathbf{p})$, $\{\lambda_{k,j}\}_{j=1}^n$ such that $\max_{j=1,\dots,n} |\lambda_k - \lambda_{k,j}|$ decays exponentially with growing pulse separation.(See [Sandstede, 1998].)

Since $\overline{\mathcal{L}}_n(\mathbf{p})$ is a self-adjoint operator, by the spectral mapping theorem $\sigma(\overline{\mathcal{L}}_n^2(\mathbf{p})) = (\sigma(\overline{\mathcal{L}}_n(\mathbf{p})))^2$. Recall that in Remark 2.2.1 we present that the eigenvalues of $\mathcal{L}_n$ has an order $\lambda_0 < 0 = \lambda_1 < \lambda_2 < \dots$. By the choice of $\tilde{\mu} > 0$, the eigenvalues of $\overline{\mathcal{L}}_n^2(\mathbf{p})$ associated to the eigenvalue of $\overline{\mathcal{L}}_1(\mathbf{p})$ at $\lambda_1 = 0$ are shifted to the left by $\tilde{\mu}$(See Figure 4.1). Choosing $\tilde{\mu} > 0$ but less than the minimum of λ_0^2 and λ_2^2 we see that the eigenvalues of $\overline{\mathcal{L}}_n^2(\mathbf{p}) - \tilde{\mu}$ are positive except n eigenvalues of $\overline{\mathcal{L}}_n(\mathbf{p})$ associated to $\lambda_1 = 0$, $\{\lambda_{1,j}\}_{j=1}^n$.

We conclude that

$$\mathbf{n}(\overline{\mathcal{L}}_n^2(\mathbf{p}) - \tilde{\mu}) = n. \tag{4.1.22}$$

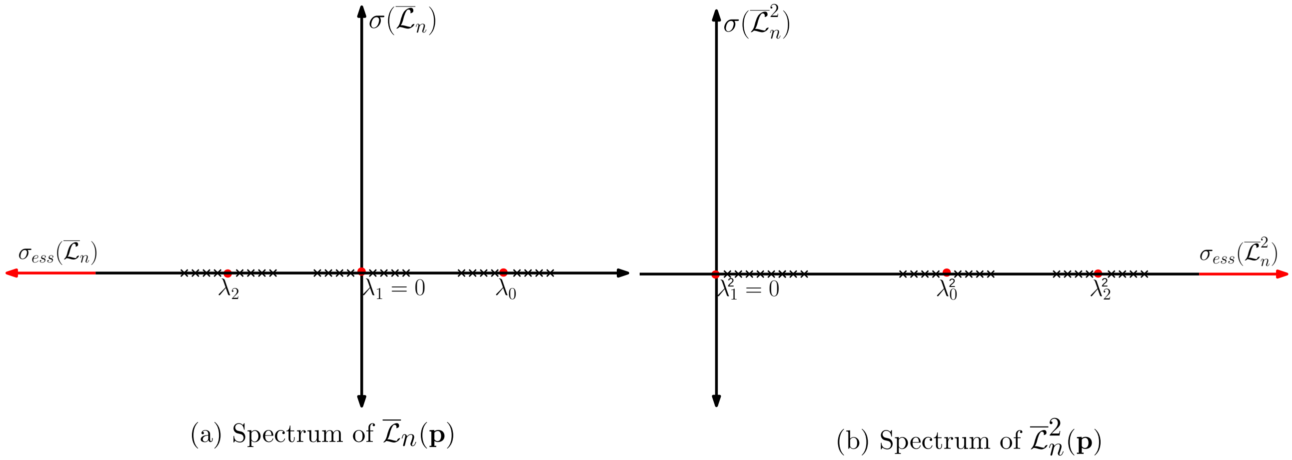


Figure 4.1: Figure (a) is a depiction for the spectrum of $\overline{\mathcal{L}}_n(\mathbf{p})$. In the descending order $\lambda_0 > 0 = \lambda_1 > \lambda_2 > \ldots$ (red disks) are the eigenvalues of $\overline{\mathcal{L}}_1(\mathbf{p}) = \mathcal{L}$. $\overline{\mathcal{L}}_n(\mathbf{p})$ has n associated eigenvalues(black crosses) to each localized eigenvalue of $\overline{\mathcal{L}}_1(\mathbf{p})$ such that $|\lambda_k - \lambda_{k,j}|_{j=1,\ldots,n}$ decays exponentially with growing pulse separation. Figure (b) demonstrates the spectrum for $\overline{\mathcal{L}}_n^2(\mathbf{p})$.

The next step is calculating the number of negative eigenvalues of the constrained matrix, $\mathbf{n}(D(\tilde{\mu}))$. The eigenfunctions of $\overline{\mathcal{L}}_n^2(\mathbf{p}) - \tilde{\mu}$ corresponding to the eigenvalues $\{\lambda_{1,i}^2\}_{\ell=1}^n$ are in the form $\psi_{1,i} = \sum_{j=1}^n \beta_{ij}\phi_h'(z-p_j)$ up to exponentially small terms. (See [Sandstede, 2001].) Using the definition of D we have the identity

$$\sum_{j=1}^n D_{ij}(\tilde{\mu})\beta_{ij} = \langle \psi_{1,i}, (\overline{\mathcal{L}}_n^2(\mathbf{p}) - \tilde{\mu})^{-1}\phi_h'(z-p_i)\rangle. \tag{4.1.23}$$

Since inverse of a self-adjoint operator is also self-adjoint, transposing $(\overline{\mathcal{L}}_n^2(\mathbf{p}) - \tilde{\mu})^{-1}$ onto $\psi_{1,i}$ provides

$$\sum_{j=1}^n D_{ij}(\tilde{\mu})\beta_{ij} = \Big\langle \frac{\psi_{1,i}}{\lambda_{1,i} - \tilde{\mu}}, \phi_h'(z-p_i)\Big\rangle. \tag{4.1.24}$$

Then, inserting the formula of $\psi_{1,i}$ we obtain

$$\sum_{j=1}^{n} D_{ij}(\tilde{\mu})\beta_{ij} = \Big\langle \frac{\sum_{j=1}^{n} \beta_{ij}\phi_h'(z-p_j)}{\lambda_{1,i}-\tilde{\mu}}, \phi_h'(z-p_i) \Big\rangle + exp.\ small \quad terms$$
$$= \frac{\beta_{ii}}{\lambda_{1,i}-\tilde{\mu}} + exp.\ small \quad terms. \tag{4.1.25}$$

Let A be the matrix of the coefficients β_{ij} of eigenfunction $\psi_{1,i}$ which is comprised of the vector of columns $B_i = (\beta_{i1}, \beta_{i2}, \ldots, \beta_{in})^T$. Then, by (4.1.25) we have

$$DB_i = \frac{1}{\lambda_{1,i}-\tilde{\mu}} B_i, \tag{4.1.26}$$

up to exponentially small terms. Hence, we conclude that $\dfrac{1}{\lambda_{1,i}-\tilde{\mu}} \in \sigma(D)$ and there exists a value of $\tilde{\mu} \in \mathbf{R}$ for which

$$\min_{i=1,\ldots,n} \lambda_{2,i} > \tilde{\mu} > \max_{i=1,\cdots,n} \lambda_{1,i} \quad \forall \quad i = 1, \ldots, n, \tag{4.1.27}$$

that guarantees

$$\mathbf{n}(D(\tilde{\mu})) = n. \tag{4.1.28}$$

If we insert (4.1.22) and (4.1.28) in (4.1.19), we obtain the desired result as

$$\mathbf{n}((\overline{\mathcal{L}}_n^2(\mathbf{p}) - \tilde{\mu})_\Pi) = 0, \tag{4.1.29}$$

which implies the L^2 coercivity of $\overline{\mathcal{L}}_n^2(\mathbf{p})$, i.e. there exists a $\tilde{\mu} > 0$ independent of domain size such that

$$\langle \overline{\mathcal{L}}_n^2(\mathbf{p})v, v\rangle \geq \tilde{\mu}\, \|v\|_{L^2}^2, \tag{4.1.30}$$

for all $v \in X_n^{\perp}(\mathbf{p})$. $\square$

Corollary 4.1.5. *There exists a $\tilde{\mu}_1 > 0$, independent of the domain size, such that*

$$\langle \overline{\mathcal{L}}_n^2(\mathbf{p})v, v\rangle \geq \tilde{\mu}_1 \|v\|_{H^2}^2 , \tag{4.1.31}$$

for all $v \in X_n^{\perp}(\mathbf{p})$.

Proof. We already proved the L^2-coercivity of the bilinear form induced by $\overline{\mathcal{L}}_n^2(\mathbf{p})$ in Lemma 4.1.4. We utilize this result to establish the H^2-coercivity of the bilinear form induced by $\overline{\mathcal{L}}_n^2(\mathbf{p})$. By the L^2-coercivity of $\overline{\mathcal{L}}_n^2$, there exists a constant $\tilde{\mu} > 0$ such that

$$\langle \overline{\mathcal{L}}_n v, \overline{\mathcal{L}}_n v\rangle \geq \tilde{\mu} \|v\|_{L^2(\mathbf{R})}^2 , \tag{4.1.32}$$

for all $v \in {X_n}^{\perp}(\mathbf{p})$. Expanding the inner product, we fix $\theta \in (0, 1)$ and write

$$\begin{aligned} \langle \overline{\mathcal{L}}_n v, \overline{\mathcal{L}}_n v\rangle &= \theta\langle \overline{\mathcal{L}}_n v, \overline{\mathcal{L}}_n v\rangle + (1-\theta)\langle \overline{\mathcal{L}}_n v, \overline{\mathcal{L}}_n v\rangle, \\ &\geq \theta \int_{\mathbf{R}} \left(\partial_z^2 v\right)^2 + P\left(\partial_z^2 v\right) v + Qv^2 dz + (1-\theta)\tilde{\mu} \|v\|_{L^2}^2 , \end{aligned} \tag{4.1.33}$$

where $P := -2W''(u_n)$ and $Q := (W(u_n))^2$. Applying the Holder's inequality to the term with P, $\left\|v\partial_z^2 v\right\|_{L_1} \leq \|v\|_{L_2} \left\|{\partial_z}^2 v\right\|_{L_2}$ we obtain

$$\begin{aligned} \langle \overline{\mathcal{L}}_n v, \overline{\mathcal{L}}_n v\rangle &\geq \theta\left(\left\|\partial_z^2 v\right\|_{L^2}^2 - \|P\|_{L^\infty} \|v\partial_z v\|_{L^1} - \|Q\|_{L^\infty} \|v\|_{L^2}^2\right) + (1-\theta)\tilde{\mu} \|v\|_{L^2}^2 , \\ &\geq \theta\left\|\partial_z^2 v\right\|_{L^2}^2 - \theta \|P\|_{L^\infty} \|\partial_z v\|_{L^2} \|v\|_{L^2} + ((1-\theta)\tilde{\mu} - \theta \|Q\|_{L^\infty}) \|v\|_{L^2}^2 . \end{aligned} \tag{4.1.34}$$

We apply Young's inequality to the second term on the second line of (4.1.34),

$$\begin{aligned}
\langle \overline{\mathcal{L}}_n v, \overline{\mathcal{L}}_n v\rangle &\geq \theta \left\|\partial_z^2 v\right\|_{L^2}^2 - \frac{\theta}{2}\left\|\partial_z^2 v\right\|_{L^2}^2 - \frac{\theta}{2}\left\|P\right\|_{L^\infty}^2 \left\|v\right\|_{L^2}^2 + ((1-\theta)\tilde{\mu} - \theta \left\|Q\right\|_{L^\infty})\left\|v\right\|_{L^2}^2, \\
&= \frac{\theta}{2}\left\|\partial_z^2 v\right\|_{L^2}^2 + \left((1-\theta)\tilde{\mu} - \theta\left\|Q\right\|_{L^\infty} - \frac{\theta}{2}\left\|P\right\|_{L^\infty}^2\right)\left\|v\right\|_{L^2}^2.
\end{aligned} \tag{4.1.35}$$

Choosing $\theta* = \min\left\{\frac{\theta}{2}, \left((1-\theta)\tilde{\mu} - \theta\left\|Q\right\|_{L^\infty} - \frac{\theta}{2}\left\|P\right\|_{L^\infty}^2\right)\right\}$ independent of domain size we obtain

$$\begin{aligned}
\langle \overline{\mathcal{L}}_n v, \overline{\mathcal{L}}_n v\rangle &\geq \theta_* \left(\left\|\partial_z^2 v\right\|_{L^2}^2 + \left\|v\right\|_{L^2}^2\right). \\
&= \theta_* \left\|v\right\|_{H^2}^2.
\end{aligned} \tag{4.1.36}$$

□

Proof of Theorem 4.1.2. We show the H^2-coercivity of the second variation of I about n-pulse ansatz using bilinear form given in (4.1.11).

$$\begin{aligned}
\Big\langle \mathbb{L}v, v\Big\rangle = \Big\langle \Big(&\overline{\mathcal{L}}_n^2 v - \delta\bar{\lambda}\overline{\mathcal{L}}_n(W'''(u_n)\mathcal{B}_{2,n}v) - \delta\bar{\lambda}W'''(u_n)\mathcal{B}_{2,n}\overline{\mathcal{L}}_n v\Big) + \\
&+ -\Big(-\delta\sum_{i=1}^{n} W''(\phi_h(z-p_i)) + \delta\bar{\lambda}\mathcal{B}_{1,n}\Big)W'''(u_n)v, v\Big\rangle + \mathcal{O}(\delta^2\left\|v\right\|_{H^2}^2)
\end{aligned} \tag{4.1.37}$$

$$\begin{aligned}
\langle \mathbb{L}v, v\rangle \geq \Big\langle \overline{\mathcal{L}}_n^2 v, v\Big\rangle &- \delta\bar{\lambda}\Big\langle \overline{\mathcal{L}}_n(W'''(u_n)\mathcal{B}_{2,n}v), v\Big\rangle - \delta\bar{\lambda}\Big\langle W'''(u_n)\mathcal{B}_{2,n}\overline{\mathcal{L}}_n v, v\Big\rangle + \\
&- \delta\Big\langle W'''(u_n)\sum_{i=1}^{n} W''(\phi_h(z-p_i))v, v\Big\rangle - -\delta\bar{\lambda}\Big\langle W'''(u_n)\mathcal{B}_{1,n}v, v\Big\rangle + \\
&+ \mathcal{O}(\delta^2\left\|v\right\|_{H^2}^2).
\end{aligned} \tag{4.1.38}$$

From Corollary 4.1.5, there exists a $\tilde{\mu}_1 > 0$ independent of ε such that $\langle \overline{\mathcal{L}}_n^2 v, v\rangle \geq \tilde{\mu}_1 \left\|v\right\|_{H^2}^2$

and hence

$$
\begin{aligned}
\langle \mathbb{L}v, v\rangle \geq \tilde{\mu}_1 \|v\|_{H^2}^2 &- \delta\bar{\lambda}\Big\langle \overline{\mathcal{L}}_n(W'''(u_n)\mathcal{B}_{2,n}v), v\Big\rangle - \delta\bar{\lambda}\Big\langle W'''(u_n)\mathcal{B}_{2,n}\overline{\mathcal{L}}_n v, v\Big\rangle + \\
&- \delta\Big\langle W'''(u_n)\sum_{i=1}^{n} W''(\phi_h(z-p_i))v, v\Big\rangle - \delta\bar{\lambda}\Big\langle W'''(u_n)\mathcal{B}_{1,n}v, v\Big\rangle + \\
&+ \mathcal{O}(\delta^2 \|v\|_{H^2}^2).
\end{aligned} \tag{4.1.39}
$$

By the smoothness of the functions W over bounded functions and the smoothness of $\mathcal{B}_{1,n}$ and $\mathcal{B}_{2,n}$ and by the definition of $\overline{\mathcal{L}}_n$ we get the following estimates for the inner products in (4.1.39). Since $\mathcal{L}_n$ is self-adjoint, transposing that onto v we have

$$
\begin{aligned}
\Big\langle \overline{\mathcal{L}}_n(W'''(u_n)\mathcal{B}_{2,n}v), v\Big\rangle &= \Big\langle W'''(u_n)\mathcal{B}_{2,n}v, \overline{\mathcal{L}}_n v\Big\rangle \\
&= \left\|W'''(u_n)\mathcal{B}_{2,n}v\overline{\mathcal{L}}_n v\right\|_{L^1} \\
&\leq \left\|W'''(u_n)\right\|_{L^\infty} \left\|\mathcal{B}_{2,n}\right\|_{L^\infty} \left\|v\overline{\mathcal{L}}_n v\right\|_{L^1}
\end{aligned} \tag{4.1.40}
$$

Then, we apply the Holder's inequality to the term with L^1 norm

$$
\Big\langle \overline{\mathcal{L}}_n(W'''(u_n)\mathcal{B}_{2,n}v), v\Big\rangle \leq \left\|W'''(u_n)\right\|_{L^\infty} \left\|\mathcal{B}_{2,n}\right\|_{L^\infty} \left\|\overline{\mathcal{L}}_n v\right\|_{L^2} \|v\|_{L^2}. \tag{4.1.41}
$$

Here we use the fact that there exists a constant $c > 0$ such that $\left\|\overline{\mathcal{L}}_n f\right\|_{L^2} \leq c\|f\|_{H^2}$ for all $f \in H^2(\mathbf{R})$ and get

$$
\begin{aligned}
\Big\langle \overline{\mathcal{L}}_n(W'''(u_n)\mathcal{B}_{2,n}v), v\Big\rangle &\leq c\left\|W'''(u_n)\right\|_{L^\infty} \left\|\mathcal{B}_{2,n}\right\|_{L^\infty} \|v\|_{H^2} \|v\|_{L^2} \\
&\leq \theta_1 \|v\|_{H^2}^2,
\end{aligned} \tag{4.1.42}
$$

for some $\theta_1 > 0$. With a similar calculation, we get an upper bound for the second inner product,

$$
\begin{aligned}
\left\langle W'''(u_n)\mathcal{B}_{2,n}\overline{\mathcal{L}}_n v, v\right\rangle &= \left\| W'''(u_n)\mathcal{B}_{2,n}\overline{\mathcal{L}}_n v^2 \right\|_{L^1} \\
&\le \left\| W'''(u_n) \right\|_{L^\infty} \left\| \mathcal{B}_{2,n} \right\|_{L^\infty} \left\| \overline{\mathcal{L}}_n v^2 \right\|_{L^1} \\
&\le \left\| W'''(u_n) \right\|_{L^\infty} \left\| \mathcal{B}_{2,n} \right\|_{L^\infty} \left\| \overline{\mathcal{L}}_n v \right\|_{L^2} \left\| v \right\|_{L^2} \\
&\le \left\| W'''(u_n) \right\|_{L^\infty} \left\| \mathcal{B}_{2,n} \right\|_{L^\infty} \left\| \overline{\mathcal{L}}_n v \right\|_{H^2}^2 \\
&\le \theta_2 \left\| v \right\|_{H^2}^2 ,
\end{aligned} \tag{4.1.43}
$$

for some $\theta_2 > 0$. For the remaining two inner products, there exist constants $\theta_3 > 0$ and $\theta_4 > 0$ such that

$$
\left\langle W'''(u_n) \sum_{i=1}^{n} W''(\phi_h(z - p_i)v, v \right\rangle \le \theta_3 \left\| v \right\|_{H^2}^2 , \tag{4.1.44}
$$

and

$$
\left\langle W'''(u_n)\mathcal{B}_{1,n} v, v \right\rangle \le \theta_4 \left\| v \right\|_{H^2}^2 . \tag{4.1.45}
$$

Inserting the estimates (4.1.42), (4.1.43), (4.1.44) and (4.1.45) in (4.1.39) provides us

$$
\begin{aligned}
\langle \mathbb{L} v, v \rangle \ge \tilde{\mu}_1 \left\| v \right\|_{H^2}^2 - \delta\bar{\lambda}\theta_1 \left\| v \right\|_{H^2}^2 - \delta\bar{\lambda}\theta_2 \left\| v \right\|_{H^2}^2 + \\
- \delta\theta_3 \left\| v \right\|_{H^2}^2 - \delta\bar{\lambda}\theta_4 \left\| v \right\|_{H^2}^2 + \mathcal{O}(\delta^2 \left\| v \right\|_{H^2}^2).
\end{aligned} \tag{4.1.46}
$$

Choosing the $\tilde{\mu}_2 = \max\{\bar{\lambda}\theta_1, \bar{\lambda}\theta_2, \theta_3, \bar{\lambda}\theta_4\}$, we obtain the H^2-coercivity of $\dfrac{\delta^2 I}{\delta u^2}(\overline{\Phi}_n)$ as

$$
\langle \mathbb{L} v, v \rangle \ge \tilde{\mu}_1 \left\| v \right\|_{H^2}^2 - \delta\tilde{\mu}_2 \left\| v \right\|_{H^2}^2 . \tag{4.1.47}
$$

□

We also show the H^2-coercivity of the $\overline{\mathcal{L}}_n^2(\mathbf{p})$ restricted to the bounded domain $[0, \frac{L_2}{\varepsilon}]$

that is an essential step in the verification of the assumption (H3) in Section 4.2.

Corollary 4.1.6. *Let the operator $\overline{\mathcal{L}}_{n,b}(\mathbf{p})$ acting on $H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$ be the restriction of the $\overline{\mathcal{L}}_n(\mathbf{p})$ given in* (2.2.4) *to the large bounded domain* $[0, \frac{L_2}{\varepsilon}]$*. Then, the bilinear form induced by $\overline{\mathcal{L}}^2_{n,b}(\mathbf{p})$ is coercive, i.e, there exists a $\mu^* > 0$ independent of ε such that*

$$\langle \overline{\mathcal{L}}^2_{n,b}(\mathbf{p})v, v\rangle \geq \mu^* \|v\|_{H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)}, \tag{4.1.48}$$

for all $v \in Y'_n(\mathbf{p})$.

Proof. Let $\tilde{v} \in Y'_n(\mathbf{p})$ have a compact support. Let $v \in H^2(\mathbf{R})$ be the extension of $\tilde{v}$ to $\mathbf{R}$ so that we have

$$\langle \overline{\mathcal{L}}^2_{n,b}(\mathbf{p})v, v\rangle = \langle \overline{\mathcal{L}}^2_{n}(\mathbf{p})v, v\rangle, \tag{4.1.49}$$

for all $v \in X_n^{\perp}(\mathbf{p})$. By Corollary 4.1.5, there exists a $\tilde{\mu}_1$ such that

$$\langle \overline{\mathcal{L}}^2_{n,b}(\mathbf{p})v, v\rangle = \langle \overline{\mathcal{L}}^2_{n}(\mathbf{p})v, v\rangle \geq \tilde{\mu}_1 \|v\|_{H^2(\mathbf{R})}, \tag{4.1.50}$$

for all $v \in X_n^{\perp}(\mathbf{p})$. Since $H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$ with compact support is dense in $H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$ the result follows. □

4.1.3 H^2-coercivity of $\mathcal{L}^2_n$

For the stability of the periodic multi-pulse solutions, Φ_n, given in (2.2.37) of the free energy I, the H^2-coercivity of the second variation of I about Φ_n at leading order is established utilizing the H^2-coercivity of the operator $\overline{\mathcal{L}}^2_n$. See Appendix for the derivation of second variation of I about Φ_n at leading order.

Theorem 4.1.7. *Consider the operator $\mathcal{L}_n$ given in* (2.2.27) *on $H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$ which is the linearization of* (1.2.6) *about periodic multi-pulse solutions, ϕ_n, at the leading order. Then, the bilinear form induced by $\frac{\delta^2 I}{\delta u^2}(u_n) = \mathcal{L}_n^2$ is coercive, i.e, there exists a $\mu^* > 0$ independent of ε such that*

$$\langle \mathcal{L}_n^2 v, v\rangle \geq \mu^* \|v\|_{H^2(\mathbf{R})}, \tag{4.1.51}$$

for all $v \in X_n^{\perp}(\mathbf{p})$.

Lemma 4.1.8. *There exist smooth functions $(\mathbf{p}, \eta) : H^2 \to \mathbf{R}^N \times H^2$ satisfying $\mathbf{p}(0) = \mathbf{p}_*$ such that for all $\|v\|_{H^2} = \mathcal{O}(1)$ sufficiently small*

$$\Phi_n + v = \overline{\Phi}_n(z, \mathbf{p}, \bar{\lambda}) + \eta(v), \tag{4.1.52}$$

where $\eta(v) \in X_n^{\perp}(\mathbf{p})$. Further, there exists $\alpha > 0$ constant such that

$$\|\mathbf{p}(v) - \mathbf{p}_*\|_{L^2} \leq \alpha \|v\|_{H^2}, \tag{4.1.53}$$

where $v \in X_n^{\perp}(\mathbf{p})$.

Proof. Introduce $F = (F_1, ..., F_n)^t$ where

$$F_i(v, \mathbf{p}) := \langle v + \Phi_n(\mathbf{p}) - \overline{\Phi}_n(\mathbf{p}), \phi_n'(z - p_i)\rangle, \tag{4.1.54}$$

for $i = 1, ..., n$ and $\langle\cdot,\cdot\rangle$ is L^2 inner product. $F_i = 0$ for each i since $\eta(v) = v + \Phi_n(\bar{\mathbf{p}}) - \overline{\Phi}_n(\mathbf{p}) \in X_n^{\perp}(\mathbf{p})$. Indeed, F_i for each i attains its minima for some values of v on the compact set of $\mathbf{p} \in \mathcal{P} \subset \mathbf{R}^n$. Assume that F attains its minima at $\mathbf{p}(v_0)$ and so, $\mathbf{p} = \mathbf{p}(v_0)$ is one solution to $F(v, \mathbf{p}) = 0$ for $v = v_0$. We apply the Implicit Function Theorem to $F(v, \mathbf{p})$. The ij th entry of the gradient $\nabla_{\mathbf{p}} F$ is

$$\left(\nabla_{\mathbf{p}} F\Big|_{(v_0,\mathbf{p}(v_0))}\right)_{ij} = \begin{cases} \Big(-\langle \dfrac{\partial \overline{\Phi}_n(\mathbf{p})}{\partial p_i}, \phi_h'(z-p_i)\rangle + & i=j, \\ \quad + \langle v + \Phi_n - \overline{\Phi}_n(\mathbf{p}), -\phi_h''(z-p_i)\rangle\Big)\Big|_{(v_0,\mathbf{p}(v_0))}, & \\ \langle \frac{\partial \overline{\Phi}_n(\mathbf{p})}{\partial p_j}, \phi_h'(z-p_i)\rangle\Big|_{(v_0,\mathbf{p}(v_0))}, & i \neq j, \end{cases}$$
$$= \begin{cases} \Big(-\langle \phi_h'(z-p_i(v(0)), \phi_h'(z-p_i(v_0))\rangle + & i=j, \\ \quad + \langle \mathbf{h}, -\phi_h''(z-p_i(v_0))\rangle\Big), & \\ \qquad 0, & i \neq j, \end{cases} \tag{4.1.55}$$

where $h := v_0 + \Phi_n - \overline{\Phi}_n(\mathbf{p}(v_0))$ is small with $\|h\|_{H^2} = \mathcal{O}(\delta)$. By a proper choice of far apart pulse locations $p \in \mathcal{P}$, $\phi'(z-p_i)$ satisfy

$$(\phi_h'(z-p_i), \phi_h'(z-p_j)_2 = \begin{cases} \kappa \quad \textit{for} \quad i=j, \\ 0 \quad \textit{for} \quad i \neq j, \end{cases} \tag{4.1.56}$$

for constant $\kappa > 0$. From this relation, we write the gradient as

$$\nabla_{\mathbf{p}} F\Big|_{(v_0,\mathbf{p}(v_0)} = -\kappa I + \mathcal{O}(\delta). \tag{4.1.57}$$

We see that $\nabla_{\mathbf{p}} F\Big|_{(v_0,\mathbf{p}(v_0)}$ has non-zero determinant, and from this the Implicit Function Theorem implies that there exists a neighborhood $(v_0, \mathbf{p}(v_0))$ and a unique function $\mathbf{p}(v)$ such that $F(\mathbf{p}(v), v) = 0$. Further, by the Implicit Function Theorem $\mathbf{p}$ is smooth since F_i

is smooth. Thus, the Taylor expansion of $\mathbf{p}$ about $v = 0$ yields

$$\mathbf{p}(v) = \mathbf{p}(0) + \langle \frac{\delta \mathbf{p}}{\delta v}(0), v \rangle_{H^2} + \mathcal{O}(\|v\|_{H^2}^2). \tag{4.1.58}$$

From this, we obtain

$$|\mathbf{p}(v) - \mathbf{p}(0)| \leq \left\| \frac{\delta \mathbf{p}}{\delta v}(0) \right\|_{H^{-2}} \|v\|_{H^2}^2 + c(\|v\|_{H^2}^2) \leq \theta \|v\|_{H^2}^2, \tag{4.1.59}$$

for a constant $\theta > 0$. □

Proof of Theorem 4.1.7. We prove the H^2-coercivity of $\mathcal{L}_n^2(\mathbf{p})$ using H^2-coercivity of $\overline{\mathcal{L}}_n^2(\mathbf{p})$. Expanding the bilinear form induced by $\mathcal{L}_n^2(\mathbf{p})$ in terms of the bilinear form induced by $\overline{\mathcal{L}}_n^2(\mathbf{p})$ and utilizing H^2 coercivity of the bilinear form induced by $\overline{\mathcal{L}}_n^2(\mathbf{p})$ from Corollary 4.1.5, we obtain a lower bound for the first term in the expansion,

$$\begin{aligned} \langle \mathcal{L}_n^2(\mathbf{p})v, v \rangle &= \langle \overline{\mathcal{L}}_n^2(\mathbf{p})v, v \rangle + \langle (\mathcal{L}_n^2(\mathbf{p}) - \overline{\mathcal{L}}_n^2(\mathbf{p}))v, v \rangle, \\ &\geq \tilde{\mu}_1 \|v\|_{H^2}^2 + \langle (\mathcal{L}_n^2(\mathbf{p}) - \overline{\mathcal{L}}_n^2(\mathbf{p}))v, v \rangle, \end{aligned} \tag{4.1.60}$$

for all $v \in X_n^{\perp}(\mathbf{p})$ with $\|v\|_{H^2} \ll 1$. Hence, it suffices to show that the remaining term in the expansion (4.1.60) is small. We attack this term splitting it into

$$\langle (\mathcal{L}_n^2(\mathbf{p}) - \overline{\mathcal{L}}_n^2(\mathbf{p}))v, v \rangle = \langle (\mathcal{L}_n^2(\mathbf{p}) - \overline{\mathcal{L}}_n^2(\mathbf{p}^*))v, v \rangle + \langle (\overline{\mathcal{L}}_n^2(\mathbf{p}^*) - \overline{\mathcal{L}}_n^2(\mathbf{p}))v, v \rangle. \tag{4.1.61}$$

Let $u_n^* := u_n(z, \mathbf{p}^*)$.

$$
\begin{aligned}
\left(\overline{\mathcal{L}}_n^2(\mathbf{p}^*) - \overline{\mathcal{L}}_n^2(\mathbf{p})\right) v &= \left(\partial_z^2 - W''(u_n^*)\right)^2 v - \left(\partial_z^2 - W''(u_n)\right)^2 v, \\
&= \left(\partial_z^4 v - \partial_z^2 \left(W''(u_n^*)\right) v - 2\partial_z^2 v W''(u_n^*) - 2\partial_z v \partial_z (W''(u_n^*)) + \left(W''(u_n^*)\right)^2 v\right), \\
&- \left(\partial_z^4 v - \partial_z^2 \left(W''(u_n)\right) v - 2\partial_z^2 v W''(u_n) - 2\partial_z v \partial_z (W''(u_n)) + \left(W''(u_n)\right)^2 v\right).
\end{aligned} \tag{4.1.62}
$$

Define

$$
\begin{aligned}
A_0(z, \mathbf{p}) &:= \left(W''(u_n)\right)^2 - \partial_z^2 \left(W''(u_n)\right), \\
A_1(z, \mathbf{p}) &:= -2\partial_z \left(W''(u_n)\right), \\
A_2(z, \mathbf{p}) &:= -2W''(u_n),
\end{aligned} \tag{4.1.63}
$$

which are smooth functions in z. Then,

$$
\begin{aligned}
\left(\overline{\mathcal{L}}_n^2(\mathbf{p}^*) - \overline{\mathcal{L}}_n^2(\mathbf{p})\right) v = &\left(A_0(z, \mathbf{p}^*) - A_0(z, \mathbf{p})\right) v, \\
&+ \left(A_1(z, \mathbf{p}^*)\partial_z v - A_1(z, \mathbf{p})\right) \partial_z v, \\
&+ \left(A_2(z, \mathbf{p}^*)\partial_z^2 v - A_2(z, \mathbf{p})\right) \partial_z^2 v,
\end{aligned} \tag{4.1.64}
$$

and so, by the Mean Value Theorem for several variable functions, there exist $\xi_0, \xi_1, \xi_2 \in [\mathbf{p}(0), \mathbf{p}(v)]$ so that

$$
\begin{aligned}
\left\langle \left(\overline{\mathcal{L}}_n^2(\mathbf{p}^*) - \overline{\mathcal{L}}_n^2(\mathbf{p})\right) v, v \right\rangle &\leq \int_0^{\frac{L_2}{\varepsilon}} \left(A_0(z, \mathbf{p}) - A_0(z, \mathbf{p}^*)\right) v^2 + \left(A_1(z, \mathbf{p}) - A_1(z, \mathbf{p}^*)\right) v \partial_z v \\
&\quad + \left(A_2(z, \mathbf{p}) - A_0(z, \mathbf{p})\right) v \partial_z^2 v dz, \\
&\leq \int_0^{\frac{L_2}{\varepsilon}} |\nabla_{\mathbf{p}} A_0(\xi_0)||\mathbf{p}(v) - \mathbf{p}^*| v^2 + |\nabla_{\mathbf{p}} A_1(\xi_1)||\mathbf{p}(v) - \mathbf{p}^*| v \partial_z v \\
&\quad + |\nabla_{\mathbf{p}} A_2(\xi_2)||\mathbf{p}(v) - \mathbf{p}^*| v \partial_z^2 v dz,
\end{aligned}
$$

$$\leq \left\|\nabla_{\mathbf{p}} A_0(\xi_0)\right\|_{L^\infty} \int_0^{\frac{L_2}{\varepsilon}} |\mathbf{p}(v) - \mathbf{p}^*| v^2 dz + \\ + \left\|\nabla_{\mathbf{p}} A_1(\xi_1)\right\|_{L^\infty} \int_0^{\frac{L_2}{\varepsilon}} |\mathbf{p}(v) - \mathbf{p}^*| v^2 dz + \\ + \left\|\nabla_{\mathbf{p}} A_2(\xi_2)\right\|_{L^\infty} \int_0^{\frac{L_2}{\varepsilon}} |\mathbf{p}(v) - \mathbf{p}^*| v \partial_z^2 v dz. \tag{4.1.65}$$

Utilizing the bound for $|\mathbf{p}(v) - \mathbf{p}^*|$ in Lemma 4.1.8,

$$\left\langle \left(\overline{\mathcal{L}}_n^2(\mathbf{p}) - \overline{\mathcal{L}}_n^2(\mathbf{p}^*)\right) v, v \right\rangle \leq \alpha \|v\|_{H^2} \left(\left\|\nabla_{\mathbf{p}} A_0(\xi_0)\right\|_{L^\infty} \int_0^{\frac{L_2}{\varepsilon}} v^2 dz \right. \\ + \left\|\nabla_{\mathbf{p}} A_1(\xi_1)\right\|_{L^\infty} \int_0^{\frac{L_2}{\varepsilon}} v \partial_z v dz + \\ \left. + \left\|\nabla_{\mathbf{p}} A_2(\xi_2)\right\|_{L^\infty} \int_0^{\frac{L_2}{\varepsilon}} v \partial_z^2 v dz \right). \tag{4.1.66}$$

Now let $C = \max_{1 \leq j \leq 3} \{\left\|\nabla_{\mathbf{p}} A_j(\xi_j)\right\|_{L^\infty}\}$.

$$\left\langle \left(\overline{\mathcal{L}}_n^2(\mathbf{p}) - \overline{\mathcal{L}}_n^2(\mathbf{p}^*)\right) v, v \right\rangle \leq C\alpha \|v\|_{H^2} \left(\|v\|_{H^2}^2 + \|\partial_z v\|_{L^2} \|v\|_{L^2} + \|\partial_z v\|_{L^2} \|v\|_{L^2} \right), \\ \leq C_1 \|v\|_{H^2}^3. \tag{4.1.67}$$

On the other hand, a similar calculation gives a bound for the second term in (4.1.60)

$$\langle (\mathcal{L}_n^2(\mathbf{p}) - \overline{\mathcal{L}}_n^2(\mathbf{p}_*)) v, v \rangle = \int_0^{\frac{L_2}{\varepsilon}} (\left(\partial_z^2 - W''(\phi_n)\right)^2 - \left(\partial_z^2 - W''(u_n^*)\right)^2) v^2 dz, \\ \leq \int_0^{\frac{L_2}{\varepsilon}} \left(A_0(z) - A_0(z, \mathbf{p}^*)\right) v^2 + \left(A_1(z) - A_1(z, \mathbf{p}^*)\right) v \partial_z v, \\ + \left(A_2(z) - A_0(z, \mathbf{p}^*)\right) v \partial_z^2 v dz \\ \leq C_2 \|v\|_{H^2}^2. \tag{4.1.68}$$

Hence, inserting (4.1.67) and (4.1.68) in (4.1.61),

$$\langle (\mathcal{L}_n^2(\mathbf{p}) - \overline{\mathcal{L}}_n^2(\mathbf{p}))v, v\rangle \geq -C_1 \|v\|_{H^2}^3 - C_2 \|v\|_{H^2}^2 , \tag{4.1.69}$$

and (4.1.69) in (4.1.60) we obtain the H^2-coercivity

$$\begin{aligned} \langle \mathcal{L}_n^2(\mathbf{p})v, v\rangle &\geq \tilde{\mu}_1 \|v\|_{H^2}^2 + \langle (\mathcal{L}_n^2(\mathbf{p}) - \overline{\mathcal{L}}_n^2(\mathbf{p}))v, v\rangle, \\ &\geq \alpha \|v\|_{H^2}^2 - C_1 \|v\|_{H^2}^3 . \end{aligned} \tag{4.1.70}$$

□

4.2 Modulational Stability of n-Pulses

We establish the modulational stability for n-pulse ansatz, $\overline{\Phi}_n$, with respect to the Π_0 gradient flow of the inner scaling of the FCH energy given (1.2.3) with an application of Theorem 2.1 in [Promislow, 2002] where modulational stability of manifolds of quasi-stationary solutions to dispersive equations is established.

Introduce the Π_0-gradient flow of the inner scaling of the FCH free energy, I, given (1.2.3)

$$u_t = -\Pi_0 \frac{\delta I}{\delta u}(u), \tag{4.2.1}$$

where $\frac{\delta I}{\delta u}$ is the first variational derivative of I with respect to L^2 inner product and Π_0 is the mass preserving L^2 projection

$$\Pi_0 f := f - \frac{\varepsilon}{L_2} \int_0^{\frac{L_2}{\varepsilon}} f(z)dz. \tag{4.2.2}$$

Consider the family of multi-pulse critical points $\{\overline{\Phi}_n : n \in \mathbb{N}\}$ of the free energy I and the n-dimensional manifold $\mathcal{M}_n = \{\overline{\Phi}_n(\mathbf{p}, \bar{\lambda}) | \mathbf{p} \in \mathcal{P}\}$ where $\mathcal{P} \subset \mathbf{R}^n$ defined in (4.1.4).

For the modulational stability of the n-pulses $\{\overline{\Phi}_n : n \in \mathbb{N}\}$, we apply renormalization techniques from [Promislow, 2002]. We are interested in the evolution of the solutions which lie in a neighborhood of the manifold $\mathcal{M}_n$ which consists of the n-pulse solutions $\overline{\Phi}_n(\mathbf{p}, \bar{\lambda})$ for $p \in \mathcal{P}$. Introduce

$$u(z,t) = \overline{\Phi}_n(z, \mathbf{p}, \bar{\lambda}) + w(z). \tag{4.2.3}$$

We reduce the dynamics of (4.2.1) near the manifold $\mathcal{M}_n$ to a weakly non-linear flow which is predominantly controlled by the terms that are linear in their deviation, w, of U from $\mathcal{M}_n$, and non-linear in the pulse evolution $p = p(t)$ about $\overline{\Phi}_n(\mathbf{p}, \bar{\lambda})$. Taylor expanding the flow (4.2.1) about $\overline{\Phi}_n(\mathbf{p}, \bar{\lambda})$ we obtain

$$-\Pi_0 \frac{\delta I}{\delta u}(u(z,t)) = -\Pi_0 \left(\frac{\delta I}{\delta u}(\overline{\Phi}_n(\mathbf{p}, \bar{\lambda})) \right) - \Pi_0 \overline{\mathcal{L}}_n^2(\mathbf{p}) w + \mathcal{N}(w), \tag{4.2.4}$$

where $\overline{\mathcal{L}}_n(\mathbf{p})$ given (2.2.4) is the linearization about u_n and $\mathcal{N}$ represents the non-linear terms. The linearization about u_n will be weakly time dependent through the slow evolution of the pulse positions, $\mathbf{p}$, and the background state $\bar{\lambda}$. The $n+1$ parameters form the coordinates of the slow n-pulse manifold, one of which is determined by the mass constraint.

In Theorem 2.1 of [Promislow, 2002], there are some assumptions on the linearized operator and the manifold of the steady-state solutions to the gradient flow. Here we present those assumptions adapting to the linearized operator $-\Pi_0 \overline{\mathcal{L}}_n^2(\mathbf{p})$ and the manifold $\mathcal{M}_n$ but defer the verification of each assumption to the proof of Theorem 4.2.1.

(**H0**) The manifold $\mathcal{M}_n$ is quasi-steady, i.e, for $\delta > 0$, the scaling of $\bar{\lambda}$, there exists $M > 0$

such that

$$\left\| -\Pi_0 \frac{\delta I}{\delta u}(\overline{\Phi}_n(\mathbf{p}, \bar{\lambda})) \right\|_{H^2} \leq M\delta. \tag{4.2.5}$$

(**H1**) The spectrum of each operator $-\Pi_0 \overline{\mathcal{L}}_n^2(\mathbf{p})$ consists of a stable part $\sigma_s \subset \{\lambda | \lambda \leq -k_s\}$ for some $k_s > 0$ and a slow part $\sigma_0 \subset \{\lambda | |\lambda| \leq c_0 e^{-\frac{k_0}{\varepsilon}}\}$ for some $c_0, k_0 > 0$.

(**H2**) Each operator $-\Pi_0 \overline{\mathcal{L}}_n^2(\mathbf{p})$ generates a C_0 semigroup S_p which satisfies

$$\left\| S_p(t) u \right\|_{H^2} \leq M e^{-k_s t} \left\| u \right\|_{H^2}, \tag{4.2.6}$$

for all $t \geq 0$, $u \in Y_n' \coloneqq X_n^{\perp}(\mathbf{p}) \cap H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right)$, where $X_n^{\perp}(\mathbf{p})$ is perpendicular to the tangent plane $X_n(\mathbf{p})$ of $\overline{\Phi}_n(\mathbf{p}, \bar{\lambda})$.

Let Y_p represent the slow space of the linearized operator $-\Pi_0 \overline{\mathcal{L}}_n^2(\mathbf{p})$, the $n+1$ dimensional space associated with small eigenvalues of $-\Pi_0 \overline{\mathcal{L}}_n^2(\mathbf{p})$.

Recall that $X_n(\mathbf{p}) = span\{\frac{\partial \overline{\Phi}_n(\mathbf{p})}{\partial p_i} : \mathbf{p} \in \mathcal{P} \subset \mathbf{R}^n\} = span\{\phi_h'(z - p_i)\}_{i=1}^n$ is the tangent plane to the manifold $\mathcal{M}_n$. Introduce $X_{n+1}(\mathbf{p}) = span\Big\{\{\phi_h'(z - p_i)\}_{i=1}^n \cup \{\mathcal{B}_{2,n}\}\Big\}$ and

$$Y_{n+1}'(\mathbf{p}) \coloneqq X_{n+1}^{\perp}(\mathbf{p}) \cap H^2\left(\left[0, \frac{L_2}{\varepsilon}\right]\right), \tag{4.2.7}$$

where $X_{n+1}^{\perp}(\mathbf{p})$ is the orthogonal space to $X_{n+1}(\mathbf{p})$.

(**H3**) Y_p is well-approximated by $X_{n+1}(\mathbf{p})$. In [Promislow, 2002], this assumption is utilized to establish the coercivity of the linearized operator. Instead, here we will establish the H^2-coercivity of $\Pi_0 \overline{\mathcal{L}}_n^2(\mathbf{p})$ over the space $Y_{n+1}'(\mathbf{p})$ which follows from the H^2-coercivity of $\overline{\mathcal{L}}_{n,b}^2(\mathbf{p})$ over $Y_n'(\mathbf{p})$. Recall that H^2-coercivity of $\overline{\mathcal{L}}_{n,b}^2(\mathbf{p})$ over $Y_n'(\mathbf{p})$ was established in Corollary 4.1.6 using Corollary 4.1.5.

We assume that the adjoint of the elements in Y_p and $X_{n+1}(\mathbf{p})$ satisfy the following normalization condition

$$\langle \zeta_i, \zeta_j^\dagger \rangle = \begin{cases} 1 & for \quad i = j, \\ \\ 0 & for \quad i \neq j. \end{cases} \tag{4.2.8}$$

With this condition, the adjoint of $X_{n+1}(\mathbf{p})$ is defined as

$$X_{n+1}^\dagger(\mathbf{p}) := X_n(\mathbf{p}) \cup span\{1\}. \tag{4.2.9}$$

(**H4**) The normalized adjoint eigenvectors $\{\psi_1^\dagger, ..., \psi_{n+1}^\dagger\}$ of the space Y_p satisfy

$$\max_{\substack{i=1,\dots,n \\ \mathbf{p}\in\mathcal{P}}} \left(\left\| \psi_i^\dagger(\mathbf{p}) \right\|_{H^2} + \left\| \nabla_{\mathbf{p}}^2 \psi_i^\dagger(\mathbf{p}) \right\|_{H^2} \right) \leq M, \tag{4.2.10}$$

for some M.

Theorem 4.2.1. *Fix a pulse separation value $\ell = \mathcal{O}(\varepsilon^{-1}) > 0$ in $\mathcal{P} \subset \mathbf{R}^n$. Then, there exists a manifold $\mathcal{M}_n = \{\overline{\Phi}_n(\mathbf{p}, \bar{\lambda}) | \mathbf{p} \in \mathcal{P}\}$ satisfying the hypothesis (H0)-(H4) for some constants M and k and there exist ε_0, M_0 for $\varepsilon \in [0, \varepsilon_0]$ such that for all initial data $u(z, t_0) = u_0(z)$ within ε^2-neighborhood in H^2-norm of $\mathcal{M}_n$ whose mass lies within δ-neighborhood of the mass of $\overline{\Phi}_n(\mathbf{p}, \bar{\lambda})$, the solution u of* (4.2.1) *can be decomposed as*

$$u(z) = \overline{\Phi}_n(z, \mathbf{p}(t), \bar{\lambda}) + w(z, t), \tag{4.2.11}$$

where the deviation $w \in Y'_{n+1}(\mathbf{p}(t))$ satisfies

$$\|w(\cdot, t)\|_{H^2} \leq M_0(\varepsilon^2 e^{-k_s(t-t_0)} + \delta) \quad for \quad t \geq t_0. \tag{4.2.12}$$

The pulse locations $\mathbf{p}(t) = (p_1, \dots, p_n)^t$ may be chosen to lie on a smooth curve in $\mathcal{P}$. After *an initial transient $T^i \sim \frac{1}{|\ln(\varepsilon)|}$, that is, for $t > t_0 + T^i$, the evolution of the pulse locations is governed to leading order by the closed system*

$$p_i' = \Big\langle -\Pi_0 \frac{\delta I}{\delta u}(\overline{\Phi}_n(\mathbf{p}, \bar{\lambda})), \frac{1}{\|\phi_h'\|_{L^2}^2} \phi_h'(z - p_i) \Big\rangle + \mathcal{O}(\delta^2) \qquad for \quad t \geq t_0 + T^i, \tag{4.2.13}$$

for $i = 1, \ldots, n$.

Proof. Here we verify that the $\mathcal{M}_n$ and the linearized operator $-\Pi_0 \overline{\mathcal{L}}_n^2(\mathbf{p})$ satisfy the hypothesis (H0)-(H4) and a direct application of Theorem 2.1 from [Promislow, 2002] provides the result.

(**H0**) We prove that the manifold $\mathcal{M}_n$ is quasi-steady, namely,

$$\left\| -\Pi_0 \frac{\delta I}{\delta u}(\overline{\Phi}_n(\mathbf{p}, \bar{\lambda})) \right\|_{H^2} \leq M\delta, \tag{4.2.14}$$

for some $M > 0$. Recall that $\overline{\Phi}_n(\mathbf{p}, \bar{\lambda}) = u_n + \delta\bar{\lambda}\mathcal{B}_{2,n}$ where $u_n = \sum_{j=1}^n \overline{\phi}_h\left(z - p_j\right) + b_-$ with $\overline{\phi}_h = \phi_h - b_-$. The Taylor series expansion of $\frac{\delta I}{\delta u}$ about u_n provides

$$\begin{aligned}
\frac{\delta I}{\delta u}(\overline{\Phi}_n(\mathbf{p}, \bar{\lambda})) &= \Big(\partial_z^2 - W''(\overline{\Phi}_n(\mathbf{p}))\Big)\Big(\partial_z^2 \overline{\Phi}_n(\mathbf{p}) - W'(\overline{\Phi}_n(\mathbf{p}))\Big) \\
&= \Big(\partial_z^2 - W''(u_n + \delta\bar{\lambda}\mathcal{B}_{2,n}\Big)\Big(\partial_z^2(u_n + \delta\bar{\lambda}\mathcal{B}_{2,n}) - W'(u_n + \delta\bar{\lambda}\mathcal{B}_{2,n})\Big) \\
&= \Big(\partial_z^2 - W''(u_n) - \delta\bar{\lambda}\mathcal{B}_{2,n}W'''(u_n)\Big)\Big(\partial_z^2 u_n + \delta\bar{\lambda}\partial_z^2(\mathcal{B}_{2,n}) + \\
&\qquad - W'(u_n) - \delta\bar{\lambda}\mathcal{B}_{2,n}W''(u_n)) - \frac{1}{2}\delta^2\bar{\lambda}^2(\mathcal{B}_{2,n})^2 W'''(u_n)\Big).
\end{aligned} \tag{4.2.15}$$

Since the tail-tail interaction of the adjacent pulses dominates the value of u_n, on each window $\left[\frac{p_{i-1}+p_i}{2}, \frac{p_i+p_{i+1}}{2}\right]$ we can write $u_n = \overline{\phi}_{i-1} + \phi_i + \overline{\phi}_{i+1}$ where $\phi_i = \phi_h(z - p_i)$ and $\overline{\phi}_i = \phi_i - b_-$. Letting $\phi_{\Delta i} := \overline{\phi}_{i-1} + \overline{\phi}_{i+1}$ and Taylor expanding $\frac{\delta I}{\delta u}(\overline{\Phi}_n(\mathbf{p}, \bar{\lambda}))$ about ϕ_i we obtain

$$
\begin{aligned}
\frac{\delta I}{\delta u}(\overline{\Phi}_n(\mathbf{p},\bar{\lambda})) &= \Big(\partial_z^2 - W''(\overline{\Phi}_n(\mathbf{p},\bar{\lambda}))\Big)\Big(\partial_z^2\overline{\Phi}_n(\mathbf{p},\bar{\lambda}) - W'(\overline{\Phi}_n(\mathbf{p},\bar{\lambda}))\Big) \\
&= \sum_{i=1}^{n}\Big(\partial_z^2 - W''\left(\phi_i + \phi_{\Delta i} + \delta\bar{\lambda}\mathcal{B}_{2,n}\right)\Big)\Big(\partial_z^2(\phi_i + \phi_{\Delta i} + \delta\bar{\lambda}\mathcal{B}_{2,n}) \\
&\qquad - W'\Big(\phi_i + \phi_{\Delta i} + \delta\bar{\lambda}\mathcal{B}_{2,n}\Big) \\
&= \sum_{i=1}^{n}\Big(\partial_z^2 - W''(\phi_i) - W'''(\phi_i)\left(\phi_{\Delta i} + \delta\bar{\lambda}\mathcal{B}_{2,n}\right)\Big)\Big(\partial_z^2\phi_i + \partial_z^2\phi_{\Delta i} + \\
&\qquad + \delta\bar{\lambda}\partial_z^2\mathcal{B}_{2,n} - W'(\phi_i) - W''(\phi_i)\left(\phi_{\Delta i} + \delta\bar{\lambda}\mathcal{B}_{2,n}\right)\Big).
\end{aligned}
\tag{4.2.16}
$$

Letting $\mathcal{L}_i = \partial_z^2 - W''(\phi_i)$ and taking into account that $\phi_i'' = W'(\phi_i)$, $\frac{\delta I}{\delta u}(\overline{\Phi}_n(\mathbf{p},\bar{\lambda}))$ can be written as

$$
\begin{aligned}
\frac{\delta I}{\delta u}(\overline{\Phi}_n(\mathbf{p},\bar{\lambda})) &= \sum_{i=1}^{n}\Big(\mathcal{L}_i - W'''(\phi_i)\Big(\phi_{\Delta i} + \delta\bar{\lambda}\mathcal{B}_{2,n}\Big)\Big)\Big(\mathcal{L}_i\phi_{\Delta i} + \delta\bar{\lambda}\mathcal{L}_i\mathcal{B}_{2,n}\Big) \\
&= \sum_{i=1}^{n}\Big(\mathcal{L}_i - W'''(\phi_i)\phi_{\Delta i}\Big)\mathcal{L}_i\phi_{\Delta i} + \delta\bar{\lambda}\Big(\mathcal{L}_i - W'''(\phi_i)\phi_{\Delta i}\Big)\mathcal{L}_i\mathcal{B}_{2,n} + \\
&\qquad - \delta\bar{\lambda}W'''(\phi_i)\mathcal{B}_{2,n}\mathcal{L}_i\phi_{\Delta i} + \mathcal{O}(\delta^2).
\end{aligned}
\tag{4.2.17}
$$

Applying $-\Pi_0$ onto (4.2.17) and recalling the definition of $\mathcal{B}_{2,n}$ given in (2.2.12) we rearrange the terms

$$
\begin{aligned}
-\Pi_0 \frac{\delta I}{\delta u}(\overline{\Phi}_n(\mathbf{p},\bar{\lambda})) = &-\sum_{i=1}^{n} \Pi_0 \bigg(\Big(\mathcal{L}_i - W'''(\phi_i)\phi_{\Delta i}\Big)\mathcal{L}_i\phi_{\Delta i} + \delta\bar{\lambda}\Big(\mathcal{L}_i - W'''(\phi_i)\phi_{\Delta i}\Big)\mathcal{L}_i\mathcal{B}_{2,n} + \\
&- \delta\bar{\lambda}W'''(\phi_i)\mathcal{B}_{2,n}\mathcal{L}_i\phi_{\Delta i} + \mathcal{O}(\delta^2)\bigg) \\
= &-\sum_{i=1}^{n} \Pi_0 \Big(\mathcal{L}_i^2\phi_{\Delta i} - W'''(\phi_i)\phi_{\Delta i}\mathcal{L}_i\phi_{\Delta i} + \delta\bar{\lambda} - \delta\bar{\lambda}W'''(\phi_i)\phi_{\Delta i}\mathcal{B}_{1,n} + \\
&- \delta\bar{\lambda}W'''(\phi_i)\mathcal{B}_{2,n}\mathcal{L}_i\phi_{\Delta i} + \mathcal{O}(\delta^2)\bigg).
\end{aligned}
\tag{4.2.18}
$$

Since Π_0 onto any constant is 0, $\Pi_0\delta\bar{\lambda} = 0$ and hence,

$$
\begin{aligned}
-\Pi_0 \frac{\delta I}{\delta u}(\overline{\Phi}_n(\mathbf{p},\bar{\lambda})) = &\sum_{i=1}^{n} \Big(-\Pi_0\mathcal{L}_i^2\phi_{\Delta i} + \Pi_0\left(W'''(\phi_i)\phi_{\Delta i}\mathcal{L}_i\phi_{\Delta i}\right) + \delta\bar{\lambda}\Pi_0\left(W'''(\phi_i)\phi_{\Delta i}\mathcal{B}_{1,n}\right) + \\
&+ \delta\bar{\lambda}\Pi_0\left(W'''(\phi_i)\mathcal{B}_{2,n}\mathcal{L}_i\phi_{\Delta i}\right)\Big) + \mathcal{O}(\delta^2).
\end{aligned}
\tag{4.2.19}
$$

Then, we calculate the H^2-norm of (4.3.5). By the triangle inequality we have

$$
\begin{aligned}
\left\| -\Pi_0 \frac{\delta I}{\delta u}(\overline{\Phi}_n(\mathbf{p},\bar{\lambda})) \right\|_{H^2} \leq &\sum_{i=1}^{n} \bigg(\left\| -\Pi_0\mathcal{L}_i^2\phi_{\Delta i} \right\|_{H^2} + \left\|\Pi_0\left(W'''(\phi_i)\phi_{\Delta i}\mathcal{L}_i\phi_{\Delta i}\right)\right\|_{H^2} + \\
&+ \delta\bar{\lambda}\left\|\Pi_0\left(W'''(\phi_i)\phi_{\Delta i}\mathcal{B}_{1,n}\right)\right\|_{H^2} + \\
&+ \delta\bar{\lambda}\left\|\Pi_0\left(W'''(\phi_i)\mathcal{B}_{2,n}\mathcal{L}_i\phi_{\Delta i}\right)\right\|_{H^2}\bigg) + \mathcal{O}(\delta^2).
\end{aligned}
\tag{4.2.20}
$$

Since Π_0 is a H^2-orthogonal projection, for any function $u \in H^2$ we have the estimate

$$
\|\Pi_0 u\|_{H^2} \leq \|u\|_{H^2}, \tag{4.2.21}
$$

and hence the (4.2.20) becomes

$$
\begin{aligned}
\left\|-\Pi_0\frac{\delta I}{\delta u}(\overline{\Phi}_n(\mathbf{p},\bar{\lambda}))\right\|_{H^2} \leq \sum_{i=1}^{n}\Big(&\left\|\mathcal{L}_i^2\phi_{\Delta i}\right\|_{H^2}+\left\|W'''(\phi_i)\phi_{\Delta i}\mathcal{L}_i\phi_{\Delta i}\right\|_{H^2}+\\
&+\delta\bar{\lambda}\left\|\left(W'''(\phi_i)\phi_{\Delta i}\mathcal{B}_{1,n}\right)\right\|_{H^2}+\\
&+\delta\bar{\lambda}\left\|\left(W'''(\phi_i)\mathcal{B}_{2,n}\mathcal{L}_i\phi_{\Delta i}\right)\right\|_{H^2}\Big)+\mathcal{O}(\delta^2).
\end{aligned}
\tag{4.2.22}
$$

We know that the functions $\mathcal{B}_{j,n}\in H^2$ for $j=1,2$ and by the smoothness of W, for $k\leq 4$ there exists an $\alpha_1>0$ such that $\left\|W^{(k)}(u_n)\right\|_{L^\infty}\leq\alpha_1$. Here the value of α_1 depends upon the uniform bound on $\|u_n\|_{L^\infty}$. Utilizing all these facts we get upper bound for the H^2 norm of -$\Pi_0\frac{\delta I}{\delta u}(\overline{\Phi}_n(\mathbf{p}))$,

$$
\begin{aligned}
\left\|-\Pi_0\frac{\delta I}{\delta u}(\overline{\Phi}_n(\mathbf{p},\bar{\lambda}))\right\|_{H^2} &\leq \sum_{i=1}^{n}\Big(\left\|\mathcal{L}_i^2\phi_{\Delta i}\right\|_{H^2}+\left\|W'''(\phi_i)\right\|_{L^\infty}\left\|\phi_{\Delta i}\mathcal{L}_i\phi_{\Delta i}\right\|_{H^2}+\\
&\qquad+\delta\bar{\lambda}\left\|W'''(\phi_i)\right\|_{L^\infty}\left\|\left(\phi_{\Delta i}\mathcal{B}_{1,n}\right)\right\|_{H^2}+\\
&\qquad+\delta\bar{\lambda}\left\|W'''(\phi_i)\right\|_{L^\infty}\left\|\left(\mathcal{B}_{2,n}\mathcal{L}_i\phi_{\Delta i}\right)\right\|_{H^2}\Big)+\mathcal{O}(\delta^2)\\
&\leq \sum_{i=1}^{n}\Big(\left\|\mathcal{L}_i^2\phi_{\Delta i}\right\|_{H^2}+\alpha_1\left\|\phi_{\Delta i}\mathcal{L}_i\phi_{\Delta i}\right\|_{H^2}+\\
&\qquad+\delta\bar{\lambda}\alpha_1\left\|\left(\phi_{\Delta i}\mathcal{B}_{1,n}\right)\right\|_{H^2}+\\
&\qquad+\delta\bar{\lambda}\alpha_1\left\|\left(\mathcal{B}_{2,n}\mathcal{L}_i\phi_{\Delta i}\right)\right\|_{H^2}\Big)+\mathcal{O}(\delta^2).
\end{aligned}
\tag{4.2.23}
$$

Here we use the fact that the product of two H^2 functions lies in H^2(See Appendix.),

$$\begin{aligned}\left\|-\Pi_0\frac{\delta I}{\delta u}(\overline{\Phi}_n(\mathbf{p},\bar{\lambda}))\right\|_{H^2} \leq \sum_{i=1}^{n}\Big(&\left\|\mathcal{L}_i^2\phi_{\Delta i}\right\|_{H^2} + \alpha_1\left\|\phi_{\Delta i}\right\|_{H^2}\left\|\mathcal{L}_i\phi_{\Delta i}\right\|_{H^2} + \\ &+ \delta\bar{\lambda}\alpha_1\left\|\phi_{\Delta i}\right\|_{H^2}\left\|\mathcal{B}_{1,n}\right\|_{H^2} + \\ &+ \delta\bar{\lambda}\alpha_1\left\|\mathcal{B}_{2,n}\right\|_{H^2}\left\|\mathcal{L}_i\phi_{\Delta i}\right\|_{H^2}\Big) + \mathcal{O}(\delta^2).\end{aligned} \tag{4.2.24}$$

We also use the fact that $\left\|\overline{\mathcal{L}}_n^k f\right\|_{H^2} \leq c_1\left\|f\right\|_{H^{2+2k}}$ for some $c_1 > 0$ and obtain

$$\begin{aligned}\left\|-\Pi_0\frac{\delta I}{\delta u}(\overline{\Phi}_n(\mathbf{p},\bar{\lambda}))\right\|_{H^2} \leq \sum_{i=1}^{n}\Big(&\left\|\phi_{\Delta i}\right\|_{H^6} + \alpha_1\left\|\phi_{\Delta i}\right\|_{H^2}\left\|\mathcal{L}_i\phi_{\Delta i}\right\|_{H^2} + \\ &+ \delta\bar{\lambda}\alpha_1\left\|\phi_{\Delta i}\right\|_{H^2}\left\|\mathcal{B}_{1,n}\right\|_{H^2} + \\ &+ \delta\bar{\lambda}\alpha_1\left\|\mathcal{B}_{2,n}\right\|_{H^2}\left\|\phi_{\Delta i}\right\|_{H^2}\Big) + \mathcal{O}(\delta^2).\end{aligned} \tag{4.2.25}$$

On the window $\left[\frac{p_{i-1}+p_i}{2}, \frac{p_i+p_{i+1}}{2}\right]$, we write

$$\phi_{\Delta i} = \phi_{max}e^{-\sqrt{\alpha_-}(z-p_{i-1})} + \phi_{max}e^{-\sqrt{\alpha_-}(p_{i+1}-z)}, \tag{4.2.26}$$

where ϕ_{max} is the amplitude of the pulse. Using this form of $\phi_{\Delta i}$, we obtain $\left\|\phi_{\Delta i}\right\|_{H^k} = \mathcal{O}(\delta)$ for $k = 1, 2, \ldots$ on $\left[\frac{p_{i-1}+p_i}{2}, \frac{p_i+p_{i+1}}{2}\right]$. Hence, we conclude that there exists a constant $M > 0$ such that

$$\left\|-\Pi_0\frac{\delta I}{\delta u}(\overline{\Phi}_n(\mathbf{p},\bar{\lambda}))\right\|_{H^2} \leq M\delta. \tag{4.2.27}$$

(**H1**) The spectrum of each operator $-\Pi_0\overline{\mathcal{L}}_n^2(\mathbf{p})$ consists of a stable part σ_s and a slow part σ_0. The spectrum of $\overline{\mathcal{L}}_n^2(\mathbf{p})$ has been examined in detail in the proof of Lemma (4.1.4).

It can be easily seen that $\sigma(-\Pi_0\overline{\mathcal{L}}_n^2(\mathbf{p})) = \sigma(-\Pi_0\overline{\mathcal{L}}_n^2(\mathbf{p})\Pi_0) \cup \{0\}$ and further

$$\sigma(-\Pi_0\overline{\mathcal{L}}_n^2(\mathbf{p})\Pi_0) = \sigma_s \cup \sigma_0, \tag{4.2.28}$$

where $\sigma_s \subset \{\lambda|\lambda \leq -k_s\}$ for some $k_s > 0$ and $\sigma_0 \subset \{\lambda||\lambda| \leq c_0 e^{-\frac{k_0}{\varepsilon}}\}$ for some $k_0, c_0 > 0$ and σ_0 consists of $n+1$ eigenvalues.

(**H2**) Each operator $-\Pi_0\overline{\mathcal{L}}_n^2(\mathbf{p})$ generates a C_0 semigroup S_p which satisfies

$$\left\|S_p(t)u\right\|_{H^2} \leq M e^{-k_s t} \left\|u\right\|_{H^2}, \tag{4.2.29}$$

for all $t \geq 0$, $u \in X_{n+1}^{\perp}(\mathbf{p})$. This estimate on the semigroup is a result of Prüss-Gearhart Theoremwhich states that the boundedness of a C_0-semigroup generated by an operator is from boundedness of the resolvent on the right half plane(See [Gearhart, 1978] and [Prüss, 1984]). To complete the verification, we need to show the boundedness of the resolvent of the constrained operator over the space $X_{n+1}^{\perp}(\mathbf{p})$ because the semigroup is constrained to the space $X_{n+1}^{\perp}(\mathbf{p})$. Let $\Pi_p = I - P_p$ be the orthogonal projection where $P_p : H^2(\mathbf{R}) \to X_{n+1}(\mathbf{p})$. Introduce the constrained operator $\Pi_p(-\Pi_0\overline{\mathcal{L}}_n^2(\mathbf{p})\Pi_0)\Pi_p$ that is generating the constrained semigroup onto $X_{n+1}^{\perp}(\mathbf{p})$. Since we already obtained the spectrum of $-\Pi_0\overline{\mathcal{L}}_n^2(\mathbf{p})\Pi_0$, it is easy to see that , $\sigma(\Pi_p(-\Pi_0\overline{\mathcal{L}}_n^2(\mathbf{p})\Pi_0)\Pi_p) = \sigma((-\Pi_0\overline{\mathcal{L}}_n^2(\mathbf{p})\Pi_0)) \setminus \sigma_0$ and hence

$\sigma(\Pi_p(-\Pi_0\overline{\mathcal{L}}_n^2(\mathbf{p})\Pi_0)\Pi_p) \subset \{\lambda|\lambda \leq -k_s\}$. On the other hand, $\Pi_p(-\Pi_0\overline{\mathcal{L}}_n^2(\mathbf{p})\Pi_0)\Pi_p$ is a self-adjoint operator and so, $\zeta \in \mathrm{R}(\Pi_p(-\Pi_0\overline{\mathcal{L}}_n^2(\mathbf{p})\Pi_0)\Pi_p) = \{z \in \mathbb{C}|(\Pi_p(-\Pi_0\overline{\mathcal{L}}_n^2(\mathbf{p})\Pi_0)\Pi_p -$

$\zeta)$ is bijective$\}$ and $\lambda \in \sigma(\Pi_p(-\Pi_0\overline{\mathcal{L}}_n^2(\mathbf{p})\Pi_0)\Pi_p)$,

$$\left\| \left(\Pi_p(-\Pi_0\overline{\mathcal{L}}_n^2(\mathbf{p})\Pi_0)\Pi_p - \zeta\right)^{-1} \right\| \leq \sup_{\lambda} |\lambda - \zeta|^{-1}, \tag{4.2.30}$$

(See [Kato, 1976]).

(**H3**) As mentioned earlier, here we establish the H^2-coercivity of $\Pi_0\overline{\mathcal{L}}_n^2(\mathbf{p})$ over the space $X_{n+1}^{\perp}(\mathbf{p})$ which is a result of the H^2-coercivity of $\overline{\mathcal{L}}_n^2(\mathbf{p})$ over $X_n^{\perp}(\mathbf{p})$. Recall from Corollary (4.1.5) that there exists a $\tilde{\mu}_1 > 0$, independent of domain size, such that

$$\langle \overline{\mathcal{L}}_n^2(\mathbf{p})v, v\rangle \geq \tilde{\mu}_1 \left\|v\right\|_{H^2}^2,$$

for all $v \in X_n^{\perp}(\mathbf{p})$.

Let $v \in X_n^{\perp}(\mathbf{p})$. Then, $w = \Pi_0 v \in X_{n+1}^{\perp}(\mathbf{p}) \subset X_n^{\perp}(\mathbf{p})$ and

$$\begin{aligned}
\langle \Pi_0\overline{\mathcal{L}}_n^2(\mathbf{p})w, w\rangle &= \langle \Pi_0\overline{\mathcal{L}}_n^2(\mathbf{p})\Pi_0 v, \Pi_0 v\rangle \\
&= \langle \overline{\mathcal{L}}_n^2(\mathbf{p})\Pi_0 v, \Pi_0^2 v\rangle \\
&= \langle \overline{\mathcal{L}}_n^2(\mathbf{p})\Pi_0 v, \Pi_0 v\rangle \\
&= \langle \overline{\mathcal{L}}_n^2(\mathbf{p})w, w\rangle \geq \tilde{\mu}_1 \left\|w\right\|_{H^2}^2,
\end{aligned}$$

which proves the H^2-coercivity of $\Pi_0\overline{\mathcal{L}}_n^2(\mathbf{p})$ over the space $X_{n+1}^{\perp}(\mathbf{p})$.

(**H4**) The normalized adjoint eigenvectors $\{\psi_1, ..., \psi_{n+1}\}$ of the space $X_{n+1}(\mathbf{p})$ which are same with the normalized eigenvectors of the space since $\overline{\mathcal{L}}_n^2(\mathbf{p})$ is self-adjoint satisfy

$$\max_{\substack{i=1,\ldots,n \\ \mathbf{p}\in\mathcal{P}}} \left(\left\|\psi_i(\mathbf{p})\right\|_{H^2} + \left\|\nabla_{\mathbf{p}}^2\psi_i(\mathbf{p})\right\|_{H^2}\right) \leq M, \tag{4.2.31}$$

for some M since $\phi_h'(z - p_i) \in C^n$ for some n. Analyticity of the eigenvectors of an unbounded, self-adjoint operator with compact resolvent is proved in details in [Kriegl et al., 2011].

Following the verification of these conditions required in the main theorem in [Promislow, 2002], we apply the theorem to problem 4.2.1 and obtain the desired result. $\square$

4.3 Pulse Dynamics

In this section, we study the ODE's given in (4.2.13) to understand the evolution of an initial data given in a neighborhood of the manifold $\mathcal{M}_n$.

Reformulating the evolution equations of pulse locations given in [Promislow, 2002] we obtain $n+1$ evolution equations, n equations for the pulse locations and an equation for the background parameter $\bar{\lambda}$,

$$p_i' = \left\langle -\Pi_0 \frac{\delta I}{\delta u}(\overline{\Phi}_n(\mathbf{p}, \bar{\lambda})), \frac{1}{\|\phi_h'\|_{L^2}^2} \phi_h'(z - p_i) \right\rangle + \mathcal{O}(\delta^2), \tag{4.3.1}$$

for $i = 1, \ldots, n$ and the evolution of background parameter

$$\bar{\lambda}' = \left\langle -\Pi_0 \frac{\delta I}{\delta u}(\overline{\Phi}_n(\mathbf{p}, \bar{\lambda})), 1 \right\rangle, \tag{4.3.2}$$

where 1 is the adjoint eigenfunction corresponding to $\mathcal{B}_{2,n}$ which is one of the $n+1$ eigenfunctions of $-\Pi_0 \frac{\delta I}{\delta u}(\overline{\Phi}_n(\mathbf{p}, \bar{\lambda}))$.

Recall that the n-pulse ansatz is given as $\overline{\Phi}_n(\mathbf{p}, \bar{\lambda})) = u_n + \delta \bar{\lambda} \mathcal{B}_{2,n}$ with u_n defined in (2.2.5). Note that the mass of an n-pulse configuration is

$$M = n M_h + \frac{b_- L_2}{\varepsilon} + \delta \bar{\lambda} \frac{L_2}{\varepsilon \alpha_-^2}, \tag{4.3.3}$$

where $M_h = \displaystyle\int_0^{\frac{L_2}{\varepsilon}} (\phi_h - b_-) dz$ is the mass of the homoclinic solution.

Remark 4.3.1. We assume that n is a fixed number, independent of ε, and $|p_{i+1} - p_i| \geq \ell$ for all $i = 1, \ldots, n$. These choices provide us $\delta := e^{-\sqrt{\alpha_-}\ell} \ll \varepsilon^p$ for all p.

Let the initial mass of the polymer added to the solvent be

$$M_0 = nM_h + \frac{b_- L_2}{\varepsilon} + \bar{M}_0, \tag{4.3.4}$$

where $\bar{M}_0$ is the excess mass remaining after the formation of n-pulses. Evaluating the inner product given in (4.3.1) and (4.3.2) we would like to construct explicit ODE's for the pulse locations and the background state $\bar{\lambda}$ to study their evolutions. Inserting the expression obtained in (4.2.17) in the inner product in (4.3.1) we have

$$\begin{aligned} p_i' = \Big\langle -\Pi_0\Big(\Big(\mathcal{L}_i - W'''(\phi_i)\phi_{\Delta i}\Big)\mathcal{L}_i\phi_{\Delta i} + \delta\bar{\lambda}\Big(\mathcal{L}_i - W'''(\phi_i)\phi_{\Delta i}\Big)\mathcal{L}_i\mathcal{B}_{2,n} + \\ - \delta\bar{\lambda}W'''(\phi_i)\mathcal{B}_{2,n}\mathcal{L}_i\phi_{\Delta i} + \mathcal{O}(\delta^2)\Big), \frac{1}{\|\phi_i'\|_{L^2}^2}\phi_i'\Big\rangle. \end{aligned} \tag{4.3.5}$$

Remark 4.3.2. Since Π_0 is self-adjoint, we project that onto ϕ_i' and obtain that $\Pi_0{\phi_i}' = \phi_i' + \mathcal{O}(\delta)$. Hence, the higher order terms in p_i' are $\mathcal{O}(\delta^2)$.

$$\begin{aligned} p_i' &= -\Big\langle \Big(\mathcal{L}_i - W'''(\phi_i)\phi_{\Delta i}\Big)\mathcal{L}_i\phi_{\Delta i} + \delta\bar{\lambda}\Big(\mathcal{L}_i - W'''(\phi_i)\phi_{\Delta i}\Big)\mathcal{L}_i\mathcal{B}_{2,n} + \\ &\qquad - \delta\bar{\lambda}W'''(\phi_i)\mathcal{B}_{2,n}\mathcal{L}_i\phi_{\Delta i}, \frac{1}{\|\phi_i'\|_{L^2}^2}\Pi_0\phi_i'\Big\rangle + \mathcal{O}(\delta^2) \\ &= -\Big\langle \mathcal{L}_i^2\phi_{\Delta i} - W'''(\phi_i)\phi_{\Delta i}\mathcal{L}_i\phi_{\Delta i} + \delta\bar{\lambda}\mathcal{L}_i^2\mathcal{B}_{2,n} - \delta\bar{\lambda}W'''(\phi_i)\phi_{\Delta i}\mathcal{L}_i\mathcal{B}_{2,n} + \\ &\qquad - \delta\bar{\lambda}W'''(\phi_i)\mathcal{B}_{2,n}\mathcal{L}_i\phi_{\Delta i}, \frac{1}{\|\phi_i'\|_{L^2}^2}\phi_i'\Big\rangle + \mathcal{O}(\delta^2). \end{aligned} \tag{4.3.6}$$

Here since $\mathcal{L}_i^2$ is self-adjoint and $\phi_i' \in \ker(\mathcal{L}_i)$, we have $\langle \mathcal{L}_i^2(\cdot), \phi_i' \rangle = 0$.

$$\begin{aligned}
p_i' &= -\Big\langle -W'''(\phi_i)\phi_{\Delta i}\mathcal{L}_i\phi_{\Delta i} - \delta\bar{\lambda}W'''(\phi_i)\phi_{\Delta i}\mathcal{L}_i\mathcal{B}_{2,n} - \delta\bar{\lambda}W'''(\phi_i)\mathcal{B}_{2,n}\mathcal{L}_i\phi_{\Delta i}, \frac{1}{\|\phi_i'\|_{L^2}^2}\phi_i' \Big\rangle \\
&\qquad + \mathcal{O}(\delta e^{-\sqrt{\alpha_-}\ell}) \\
&= \Big\langle W'''(\phi_i)\phi_{\Delta i}\mathcal{L}_i\phi_{\Delta i}, \phi_i' \Big\rangle + \delta\bar{\lambda}\Big\langle W'''(\phi_i)(\phi_{\Delta i}\mathcal{L}_i\mathcal{B}_{2,n} + \mathcal{B}_{2,n}\mathcal{L}_i\phi_{\Delta i}), \frac{1}{\|\phi_i'\|_{L^2}^2}\phi_i' \Big\rangle \\
&\qquad + \mathcal{O}(\delta^2).
\end{aligned} \tag{4.3.7}$$

Transposing $W'''(\phi_i)$ to the right of the inner products and writing $W'''(\phi_i)\phi_i' = (W''(\phi_i))_z$, we obtain

$$\begin{aligned}
p_i' = \frac{1}{\|\phi_i'\|_{L^2}^2}\underbrace{\Big\langle \phi_{\Delta i}\mathcal{L}_i\phi_{\Delta i}, \big(W''(\phi_i)\big)_z \Big\rangle}_{A} + \delta\bar{\lambda}\frac{1}{\|\phi_i'\|_{L^2}^2}\underbrace{\Big\langle \phi_{\Delta i}\mathcal{L}_i\mathcal{B}_{2,n} + \mathcal{B}_{2,n}\mathcal{L}_i\phi_{\Delta i}, \big(W''(\phi_i)\big)_z \Big\rangle}_{B} \\
+ \mathcal{O}(\delta^2).
\end{aligned} \tag{4.3.8}$$

For the calculation of A we use the explicit formula of $\mathcal{L}_i$,

$$\begin{aligned}
A &= \Big\langle \phi_{\Delta i}\mathcal{L}_i\phi_{\Delta i}, \big(W''(\phi_i)\big)_z \Big\rangle \\
&= \Big\langle \phi_{\Delta i}\phi_{\Delta i}'', \big(W''(\phi_i)\big)_z \Big\rangle - \Big\langle W''(\phi_i)\phi_{\Delta i}^2, \big(W''(\phi_i)\big)_z \Big\rangle.
\end{aligned} \tag{4.3.9}$$

Taking $W''(\phi_i)$ to the other side in the second inner product and observing that $W''(\phi_i)(W''(\phi_i))_z = \left(\frac{1}{2}\big(W''(\phi_i)\big)^2\right)_z$, we obtain

$$A = \Big\langle \phi_{\Delta i}\phi_{\Delta i}'', \big(W''(\phi_i)\big)_z \Big\rangle - \Big\langle \phi_{\Delta i}^2, \frac{1}{2}\left(\big(W''(\phi_i)\big)^2\right)_z \Big\rangle. \tag{4.3.10}$$

To evaluate these inner products, we write $\phi_i = b_- + \phi_{max}e^{-\sqrt{\alpha_-}|z-p_i|}$ for all $i = 1, \cdots, n$

where ϕ_{max} represents the maximum value of a single pulse in the n-pulse configuration, and on the window $\left[\frac{p_{i-1}+p_i}{2}, \frac{p_i+p_{i+1}}{2}\right]$ we have $\phi_{\Delta i} = \phi_{max}e^{-\sqrt{\alpha_-}(z-p_{i-1})}+\phi_{max}e^{-\sqrt{\alpha_-}(p_{i+1}-z)}$.

$$\begin{aligned} A = \alpha_-\Big\langle(\phi_{max}e^{-\sqrt{\alpha_-}(z-p_{i-1})} + \phi_{max}e^{-\sqrt{\alpha_-}(p_{i+1}-z)})^2, \big(W''(\phi_i)\big)_z\Big\rangle+ \\ -\Big\langle\phi_{max}e^{-\sqrt{\alpha_-}(z-p_{i-1})} + \phi_{max}e^{-\sqrt{\alpha_-}(p_{i+1}-z)}, \frac{1}{2}\Big(\big(W''(\phi_i)\big)^2\Big)_z\Big\rangle. \end{aligned} \tag{4.3.11}$$

We integrate by parts the terms in the last two lines of (4.3.11) and obtain

$$\begin{aligned} A = -2\alpha_-\sqrt{\alpha_-}\langle\phi_{max}^2e^{-2\sqrt{\alpha_-}(p_{i+1}-z)} - \phi_{max}^2e^{-2\sqrt{\alpha_-}(z-p_{i-1})}, W''(\phi_i)\rangle+ \\ +\sqrt{\alpha_-}\langle\phi_{max}^2e^{-2\sqrt{\alpha_-}(p_{i+1}-z)} - \phi_{max}^2e^{-2\sqrt{\alpha_-}(z-p_{i-1})}, \big(W''(\phi_i)\big)^2\rangle. \end{aligned} \tag{4.3.12}$$

Further, writing the explicit formula for ϕ_i and Taylor expanding $W''(\phi_i)$ about b_- we obtain

$$\begin{aligned} A =& -2\alpha_-\sqrt{\alpha_-}\Big\langle\phi_{max}^2e^{-2\sqrt{\alpha_-}(p_{i+1}-z)} - \phi_{max}^2e^{-2\sqrt{\alpha_-}(z-p_{i-1})}, \\ & W''(b_- + \phi_{max}e^{-\sqrt{\alpha_-}|z-p_i|})\Big\rangle+ \\ & +\sqrt{\alpha_-}\Big\langle\phi_{max}^2e^{-2\sqrt{\alpha_-}(p_{i+1}-z)} - \phi_{max}^2e^{-2\sqrt{\alpha_-}(z-p_{i-1})}, \\ & \Big(W''(b_- + \phi_{max}e^{-\sqrt{\alpha_-}|z-p_i|})\Big)^2\Big\rangle \\ =& -2\alpha_-\sqrt{\alpha_-}\Big\langle\phi_{max}^2e^{-2\sqrt{\alpha_-}(p_{i+1}-z)} - \phi_{max}^2e^{-2\sqrt{\alpha_-}(z-p_{i-1})}, \\ & W''(b_-) + W'''(b_-)\phi_{max}e^{-\sqrt{\alpha_-}|z-p_i|}\Big\rangle+ \\ & +\sqrt{\alpha_-}\Big\langle\phi_{max}^2e^{-2\sqrt{\alpha_-}(p_{i+1}-z)} - \phi_{max}^2e^{-2\sqrt{\alpha_-}(z-p_{i-1})}, \\ & \Big(W''(b_-) + W'''(b_-)\phi_{max}e^{-\sqrt{\alpha_-}|z-p_i|}\Big)^2\Big\rangle. \end{aligned} \tag{4.3.13}$$

Recall that $W''(b_-) = \alpha_-$ and define $\gamma_- := W'''(b_-)$. Rearranging (4.3.13) provides

$$\begin{aligned} A = -2\alpha_-\sqrt{\alpha_-}\big\langle &\phi_{max}^2 e^{-2\sqrt{\alpha_-}(p_{i+1}-z)} - \phi_{max}^2 e^{-2\sqrt{\alpha_-}(z-p_{i-1})}, \\ &\alpha_- + \gamma_-\phi_{max}e^{-\sqrt{\alpha_-}|z-p_i|}\big\rangle + \\ &+ \sqrt{\alpha_-}\big\langle \phi_{max}^2 e^{-2\sqrt{\alpha_-}(p_{i+1}-z)} - \phi_{max}^2 e^{-2\sqrt{\alpha_-}(z-p_{i-1})}, \\ &\alpha_-^2 + 2\alpha_-\gamma_-\phi_{max}e^{-\sqrt{\alpha_-}|z-p_i|} + \gamma_-^2\phi_{max}^2 e^{-2\sqrt{\alpha_-}|z-p_i|}\big\rangle. \end{aligned} \tag{4.3.14}$$

For further simplification we move the constants $-2\alpha_-\sqrt{\alpha_-}$ and $\sqrt{\alpha_-}$ to the right side of the inner product.

$$\begin{aligned} A = \big\langle &\phi_{max}^2 e^{-2\sqrt{\alpha_-}(p_{i+1}-z)} - \phi_{max}^2 e^{-2\sqrt{\alpha_-}(z-p_{i-1})}, \\ &- 2\alpha_-^2\sqrt{\alpha_-} - 2\alpha_-\sqrt{\alpha_-}\gamma_-\phi_{max}e^{-\sqrt{\alpha_-}|z-p_i|}\big\rangle + \\ &+ \big\langle \phi_{max}^2 e^{-2\sqrt{\alpha_-}(p_{i+1}-z)} - \phi_{max}^2 e^{-2\sqrt{\alpha_-}(z-p_{i-1})}, \\ &\sqrt{\alpha_-}\alpha_-^2 + 2\sqrt{\alpha_-}\alpha_-\gamma_-\phi_{max}e^{-\sqrt{\alpha_-}|z-p_i|} + \sqrt{\alpha_-}\gamma_-^2\phi_{max}^2 e^{-2\sqrt{\alpha_-}|z-p_i|}\big\rangle. \end{aligned} \tag{4.3.15}$$

Adding up these two inner products and grouping or canceling common terms we obtain

$$\begin{aligned} A &= \big\langle \phi_{max}^2 e^{-2\sqrt{\alpha_-}(p_{i+1}-z)} - \phi_{max}^2 e^{-2\sqrt{\alpha_-}(z-p_{i-1})}, \\ &\quad - \alpha_-^2\sqrt{\alpha_-} + \gamma_-^2\sqrt{\alpha_-}\phi_{max}^2 e^{-2\sqrt{\alpha_-}|z-p_i|}\big\rangle \\ &= \underbrace{\big\langle e^{-2\sqrt{\alpha_-}(p_{i+1}-z)} - e^{-2\sqrt{\alpha_-}(z-p_{i-1})}, -\phi_{max}^2\alpha_-^2\sqrt{\alpha_-}\big\rangle}_{I} \\ &\quad + \underbrace{\big\langle e^{-2\sqrt{\alpha_-}(p_{i+1}-z)} - e^{-2\sqrt{\alpha_-}(z-p_{i-1})}, \gamma_-^2\sqrt{\alpha_-}\phi_{max}^4 e^{-2\sqrt{\alpha_-}|z-p_i|}\big\rangle}_{II}. \end{aligned} \tag{4.3.16}$$

We evaluate these inner products on the interval $\left[\frac{p_{i-1}+p_i}{2}, \frac{p_i+p_{i+1}}{2}\right]$. Let $m_i = \frac{p_{i-1}+p_i}{2}$ and

$m_{i+1} = \frac{p_i+p_{i+1}}{2}$. We start by calculating the first inner product I,

$$\begin{aligned} I &= -\alpha_-^2\sqrt{\alpha_-}\phi_{max}^2 \int_{m_i}^{m_{i+1}} (e^{-2\sqrt{\alpha_-}(p_{i+1}-z)} - e^{-2\sqrt{\alpha_-}(z-p_{i-1})})dz \\ &= -\frac{\alpha_-^2\phi_{max}^2}{2}(e^{-2\sqrt{\alpha_-}(p_{i+1}-z)} + e^{-2\sqrt{\alpha_-}(z-p_{i-1})})\Big|_{m_i}^{m_{i+1}} \\ &= -\frac{\alpha_-^2\phi_{max}^2}{2}\Big(e^{-\sqrt{\alpha_-}(p_{i+1}-p_i)} - e^{-\sqrt{\alpha_-}(2p_{i+1}-p_i-p_{i-1})} \\ &\quad + e^{-\sqrt{\alpha_-}(p_{i+1}+p_i-2p_{i-1})} - e^{-\sqrt{\alpha_-}(p_i-p_{i-1})}\Big). \end{aligned} \tag{4.3.17}$$

Similarly, we calculate the second inner product and obtain

$$\begin{aligned} II = &\quad \gamma_-^2\sqrt{\alpha_-}\phi_{max}^4 \int_{m_i}^{m_{i+1}} (e^{-2\sqrt{\alpha_-}(p_{i+1}-z)} - e^{-2\sqrt{\alpha_-}(z-p_{i-1})})e^{-2\sqrt{\alpha_-}|z-p_i|})dz \\ &= \gamma_-^2\phi_{max}^4\Big(\frac{1}{4}(e^{-2\sqrt{\alpha_-}(p_{i+1}-p_i)} - e^{-2\sqrt{\alpha_-}(p_{i+1}-p_{i-1})}+ \\ &\quad + \frac{1}{2}(p_{i+1}-p_i)e^{-2\sqrt{\alpha_-}(p_{i+1}-p_i)}\Big)+ \\ &\quad - \gamma_-^2\phi_{max}^4\Big(\frac{1}{2}(p_i-p_{i-1})e^{-2\sqrt{\alpha_-}(p_i-p_{i-1})}+ \\ &\quad + \frac{1}{4}(e^{-2\sqrt{\alpha_-}(p_i-p_{i-1})} - e^{-2\sqrt{\alpha_-}(p_{i+1}-p_{i-1})}\Big). \end{aligned} \tag{4.3.18}$$

Then, adding I and II together and simplifying we obtain

$$\begin{aligned} A = &-\frac{\alpha_-^2\phi_{max}^2}{2}\Big(e^{-\sqrt{\alpha_-}(p_{i+1}-p_i)} - e^{-\sqrt{\alpha_-}(2p_{i+1}-p_i-p_{i-1})} \\ &+ e^{-\sqrt{\alpha_-}(p_{i+1}+p_i-2p_{i-1})} - e^{-\sqrt{\alpha_-}(p_i-p_{i-1})}\Big)+ \\ &+ \gamma_-^2\phi_{max}^4\Big(\frac{1}{4}e^{-2\sqrt{\alpha_-}(p_{i+1}-p_i)} - \frac{1}{4}e^{-2\sqrt{\alpha_-}(p_i-p_{i-1})}+ \\ &+ \frac{1}{2}(p_{i+1}-p_i)e^{-2\sqrt{\alpha_-}(p_{i+1}-p_i)} - \frac{1}{2}(p_i-p_{i-1})e^{-2\sqrt{\alpha_-}(p_i-p_{i-1})}\Big). \end{aligned} \tag{4.3.19}$$

With similar calculations we evaluate the last inner product B. Taking $\mathcal{B}_{2,n} = \alpha_-^{-2}$ away from pulses provides $\phi_{\Delta i}\mathcal{L}_i\mathcal{B}_{2,n} = -\alpha_-^{-2}W''(\phi_i)\phi_{\Delta i}$ and $\mathcal{B}_{2,n}\mathcal{L}_i\phi_{\Delta i} = \alpha_-^{-2}\phi''_{\Delta i} - \alpha_-^{-2}W''(\phi_i)\phi_{\Delta i}$. Combining these two in the inner product B making further simplifications we obtain

$$\begin{aligned} B &= \Big\langle \phi_{\Delta i}\mathcal{L}_i\mathcal{B}_{2,n} + \mathcal{B}_{2,n}\mathcal{L}_i\phi_{\Delta i}, \big(W''(\phi_i)\big)_z \Big\rangle \\ &= -2\alpha_-^{-2}\Big\langle W''(\phi_i)\phi_{\Delta i}, \big(W''(\phi_i)\big)_z \Big\rangle + \alpha_-^{-2}\Big\langle \phi''_{\Delta i}, \big(W''(\phi_i)\big)_z \Big\rangle \\ &= -\alpha_-^{-2}\Big\langle \phi_{\Delta i}, \Big((W''(\phi_i))^2\Big)_z \Big\rangle + \alpha_-^{-2}\Big\langle \phi''_{\Delta i}, \big(W''(\phi_i)\big)_z \Big\rangle. \end{aligned} \tag{4.3.20}$$

Here recall that $\phi'''_{\Delta i} = \alpha_-\phi'_{\Delta i}$. Expanding $W''(\phi_i)$ about b_- and grouping common terms in these two inner products give

$$\begin{aligned} B &= \alpha_-^{-2}\Big\langle \phi'_{\Delta i}, (W''(\phi_i))^2 \Big\rangle - \alpha_-^{-1}\Big\langle \phi'_{\Delta i}, \big(W''(\phi_i)\big) \Big\rangle \\ &= \alpha_-^{-2}\Big\langle \phi'_{\Delta i}, \alpha_-^2 + 2\gamma_-\alpha_-\phi_{max}e^{-\sqrt{\alpha_-}|z-p_i|} + \gamma_-^2\phi_{max}^2 e^{-2\sqrt{\alpha_-}|z-p_i|} \Big\rangle \\ &\quad - \alpha_-^{-1}\Big\langle \phi'_{\Delta i}, \alpha_- + \gamma_e^{-\sqrt{\alpha_-}|z-p_i|} \Big\rangle \\ &= \alpha_-^{-1}\gamma_-\phi_{max}\Big\langle \phi'_{\Delta i}, e^{-\sqrt{\alpha_-}|z-p_i|} \Big\rangle + \alpha_-^{-2}\gamma_-^2\phi_{max}^2\Big\langle \phi'_{\Delta i}, e^{-2\sqrt{\alpha_-}|z-p_i|} \Big\rangle. \end{aligned} \tag{4.3.21}$$

Then, we calculate the inner products in B.

$$\begin{aligned} B = \int_{m_i}^{m_{i+1}} & \alpha_-^{-1}\gamma_-\phi_{max}\sqrt{\alpha_-}(e^{-\sqrt{\alpha_-}(p_{i+1}-z)} - e^{-\sqrt{\alpha_-}(z-p_{i-1})})e^{-\sqrt{\alpha_-}|z-p_i|} + \\ & + \alpha_-^{-2}\gamma_-^2\phi_{max}^2\sqrt{\alpha_-}(e^{-\sqrt{\alpha_-}(p_{i+1}-z)} - e^{-\sqrt{\alpha_-}(z-p_{i-1})})e^{-2\sqrt{\alpha_-}|z-p_i|}dz \end{aligned} \tag{4.3.22}$$

$$
\begin{aligned}
&= \frac{1}{2}\alpha_-^{-1}\gamma_-\phi_{max}^2(e^{-\sqrt{\alpha_-}(p_{i+1}-p_i)} - e^{-\sqrt{\alpha_-}(p_{i+1}-p_{i-1})}) \\
&\quad - \frac{1}{2}\alpha_-^{-1}\gamma_-\phi_{max}^2\sqrt{\alpha_-}(p_i - p_{i-1})e^{-\sqrt{\alpha_-}(p_i-p_{i-1})})+ \\
&\quad + \alpha_-^{-2}\gamma_-^2\phi_{max}^3\Big(\frac{1}{3}(e^{-\sqrt{\alpha_-}(p_{i+1}-p_i)} - e^{-\frac{1}{2}\sqrt{\alpha_-}(2p_{i+1}+p_i-3p_{i-1})})+ \\
&\quad - (e^{-\sqrt{\alpha_-}(p_i-p_{i-1})} - e^{-\frac{3}{2}\sqrt{\alpha_-}(p_i-p_{i-1})})\Big)+ \\
&+ \frac{1}{2}\alpha_-^{-1}\gamma_-\phi_{max}^2(e^{-\sqrt{\alpha_-}(p_{i+1}-p_{i-1})} - e^{-\sqrt{\alpha_-}(p_i-p_{i-1})}) \\
&\quad + \frac{1}{2}\alpha_-^{-1}\gamma_-\phi_{max}^2\sqrt{\alpha_-}(p_{i+1} - p_i)e^{-\sqrt{\alpha_-}(p_{i+1}-p_i)})+ \\
&\quad + \alpha_-^{-2}\gamma_-^2\phi_{max}^3\Big(\frac{1}{3}(e^{-\frac{1}{2}\sqrt{\alpha_-}(3p_{i+1}-p_i-2p_{i-1})} - e^{-\sqrt{\alpha_-}(p_i-p_{i-1})})+ \\
&\quad - (e^{-\frac{3}{2}\sqrt{\alpha_-}(p_{i+1}-p_i)} - e^{-\sqrt{\alpha_-}(p_{i+1}-p_i)})\Big).
\end{aligned}
\tag{4.3.23}
$$

Inserting (4.3.19), and (4.3.23) in (4.3.8) we obtain the evolution equations of pulse locations,

$$
\begin{aligned}
p_i' = \frac{1}{\|\phi_i'\|_{L^2}^2}\Big(&-\frac{\alpha_-^2\phi_{max}^2}{2}\Big(e^{-\sqrt{\alpha_-}(p_{i+1}-p_i)} - e^{-\sqrt{\alpha_-}(2p_{i+1}-p_i-p_{i-1})} \\
&\quad + e^{-\sqrt{\alpha_-}(p_{i+1}+p_i-2p_{i-1})} - e^{-\sqrt{\alpha_-}(p_i-p_{i-1})}\Big)+ \\
&+ \gamma_-^2\phi_{max}^4\Big(\frac{1}{4}e^{-2\sqrt{\alpha_-}(p_{i+1}-p_i)} - \frac{1}{4}e^{-2\sqrt{\alpha_-}(p_i-p_{i-1})}+ \\
&\quad + \frac{1}{2}(p_{i+1} - p_i)e^{-2\sqrt{\alpha_-}(p_{i+1}-p_i)} - \frac{1}{2}(p_i - p_{i-1})e^{-2\sqrt{\alpha_-}(p_i-p_{i-1})}\Big)\Big)+ \\
&+ \frac{\delta\bar{\lambda}}{\|\phi_i'\|_{L^2}^2}\Big(\frac{1}{2}\alpha_-^{-1}\gamma_-\phi_{max}^2\Big((e^{-\sqrt{\alpha_-}(p_{i+1}-p_i)} - e^{-\sqrt{\alpha_-}(p_i-p_{i-1})})+ \\
&\quad + \sqrt{\alpha_-}((p_{i+1} - p_i)e^{-\sqrt{\alpha_-}(p_{i+1}-p_i)} - (p_i - p_{i-1})e^{-\sqrt{\alpha_-}(p_i-p_{i-1})}\Big)+ \\
&\quad + \alpha_-^{-2}\gamma_-^2\phi_{max}^3\Big(\frac{1}{3}(e^{-\sqrt{\alpha_-}(p_{i+1}-p_i)} - e^{-\sqrt{\alpha_-}(p_i-p_{i-1})}) + (e^{-\sqrt{\alpha_-}(p_{i+1}-p_i)}+ \\
&\qquad - e^{-\sqrt{\alpha_-}(p_i-p_{i-1})}) - (e^{-\frac{3}{2}\sqrt{\alpha_-}(p_{i+1}-p_i)} - e^{-\frac{3}{2}\sqrt{\alpha_-}(p_i-p_{i-1})})+ \\
&\qquad\quad + \frac{1}{3}(e^{-\frac{1}{2}\sqrt{\alpha_-}(3p_{i+1}-p_i-2p_{i-1})} - e^{-\frac{1}{2}\sqrt{\alpha_-}(2p_{i+1}+p_i-3p_{i-1})})\Big).
\end{aligned}
\tag{4.3.24}
$$

Observing that $\|\phi_i'\|_{L^2}^2 = \frac{\phi_{max}^2\sqrt{\alpha_-}}{2}$ and neglecting exponentially small terms comparing to the big terms $e^{-\sqrt{\alpha_-}\ell_i}$ where ℓ_i represents the distance between the centers of two adjacent pulses we conclude that

$$p_i' = -\alpha_-^{3/2}\Big(e^{-\sqrt{\alpha_-}(p_{i+1}-p_i)} - e^{-\sqrt{\alpha_-}(p_i-p_{i-1})}\Big) + \mathcal{O}(\delta^2), \tag{4.3.25}$$

assuming $p_0 = -p_1$ and $p_{n+1} = 2\frac{L_2}{\varepsilon} - p_n$.

After we derived the ODE's for pulse locations we also construct the ODE for the background parameter $\bar{\lambda}$ evaluating the inner product in (4.3.2).

$$\bar{\lambda}' = \big\langle -\Pi_0 \frac{\delta I}{\delta u}(\overline{\Phi}_n(\mathbf{p}, \bar{\lambda})), 1\big\rangle. \tag{4.3.26}$$

Since Π_0 is self-adjoint and $Ker(\Pi_0) = span\{1\}$, we move Π_0 to the right side of the inner product and obtain

$$\bar{\lambda}' = 0. \tag{4.3.27}$$

From this we conclude that there are no background dynamics. Hence, the ODE system, consisting of $n+1$ evolution equations for n pulse positions and $\bar{\lambda}$, is

$$\begin{aligned} &\bar{\lambda}' = 0, \\ &p_i' = -\alpha_-^{3/2}\Big(e^{-\sqrt{\alpha_-}(p_{i+1}-p_i)} - e^{-\sqrt{\alpha_-}(p_i-p_{i-1})}\Big) + \mathcal{O}(\delta^2). \end{aligned} \tag{4.3.28}$$

It is easy to see that if the pulses are equally separated, $p_{i+1} - p_i = p_i - p_{i-1} = \ell$, for $\ell > 0$ big enough for all $i = 1, \ldots, n$ then, these pulses are stationary, namely,

$$p_i' = 0, \quad \forall i = 1, \cdots, n. \tag{4.3.29}$$

Further, we would like to study the stability of these stationary n-pulse configurations examining the Jacobian matrix of this ODE system.

$$J = \begin{pmatrix} \gamma & -\frac{\gamma}{2} & 0 & 0 & \dots & 0 \\ -\frac{\gamma}{2} & \gamma & -\frac{\gamma}{2} & 0 & \dots & 0 \\ 0 & -\frac{\gamma}{2} & \gamma & -\frac{\gamma}{2} & 0 & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & -\frac{\gamma}{2} \\ 0 & 0 & \dots & 0 & -\frac{\gamma}{2} & \gamma \end{pmatrix} \tag{4.3.30}$$

where $\gamma := -\alpha_-^2 e^{-\sqrt{\alpha_-}\ell}$. The matrix J has n eigenvalues

$$\lambda_k = -\alpha_-^2 \left(1 + \cos\left(\frac{k}{n+1}\right)\right) e^{-\sqrt{\alpha_-}\ell} < 0, \tag{4.3.31}$$

for all $k = 1, \dots, n$. From this, we conclude that the stationary solutions are spectrally stable.

Moreover, assuming that the initial data is given for two-pulse configuration $U_0 = \phi_1 + \phi_2$ and setting $\ell_i := p_i - p_{i-1}$ we obtain the evolution equations for pulse locations as

$$\begin{aligned} p_1' &= -\alpha_-^{3/2} \phi_{max}^2 \left(e^{-\sqrt{\alpha_-}\ell_2} - e^{-\sqrt{\alpha_-}\ell_1}\right) + \mathcal{O}(\delta^2), \\ p_2' &= -\alpha_-^{3/2} \left(e^{-\sqrt{\alpha_-}\ell_3} - e^{-\sqrt{\alpha_-}\ell_2}\right) + \mathcal{O}(\delta^2). \end{aligned} \tag{4.3.32}$$

If the distance between ϕ_1 and ϕ_2 is smaller than their distance to adjacent pulses, $\ell_3 > \ell_1 > \ell_2$ or $\ell_1 > \ell_3 > \ell_2$, then $p_1' < 0$ and $p_2' > 0$. From this analysis, we conclude that if two adjacent pulses are closer to each other than other neighbor pulses, then they repel each other. If two pulses more far apart comparing to other neighbor pulses than they attract each other.

4.4 Conclusion

We have shown that the pulses are attracted into and remain within an $O(\delta^2)$ window of the equally spaced (periodic) distribution. Moreover, the full solution remains within an $O(\delta)$ neigborhood in H^2 of the periodic n-pulse. By compactness, a subsequence of times t_n with $t_n \to \infty$ as $n \to \infty$ exists such that $u(t_n)$ converges to u_* in the ball of radius δ of the n-periodic solution. As this is a gradient flow, u_* must be an equilibrium and then we must have $u(t) \to u_*$ for the whole sequence as traversing the distance between u_* and a distinct equilibrium value infinitely many times would expend infinite energy. In particular, the flow converges to an equilibrium which is exponentially close to the periodic n-pulse.

APPENDIX

Appendix

A bound for H^2 norm of the product of two H^2 functions

Lemma A.0.1. *Let $f, g \in H^2([\mathbf{R}])$. Then, there exists a constant $C > 0$ such that*

$$\|fg\|_{H^2} \leq C \|f\|_{H^2} \|g\|_{H^2}. \tag{A.0.1}$$

Proof.

$$\begin{aligned}
\|fg\|_{H^2} &= \int_{\mathbf{R}} (1+k^2)^2 |\widehat{fg}(k)|^2 dk, \\
&= \int_{\mathbf{R}} (1+k^2)^2 \left(\int_{\mathbf{R}} |\hat{f}(k-\tilde{k})\hat{g}(\tilde{k})| d\tilde{k} \right)^2 dk, \qquad \text{(A.0.2)} \\
&= \int_{\mathbf{R}} \left(\int_{\mathbf{R}} (1+k^2) |\hat{f}(k-\tilde{k})||\hat{g}(\tilde{k})| d\tilde{k} \right)^2 dk.
\end{aligned}$$

Writing $k = k - \tilde{k} + \tilde{k}$ and then by the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, we have

$$\begin{aligned}
\|fg\|_{H^2} &\leq \int_{\mathbf{R}} \left(\int_{\mathbf{R}} \left(2^2(1+(k-\tilde{k})^2) + 2^2(1+\tilde{k}^2) \right) |\hat{f}(k-\tilde{k})||\hat{g}(\tilde{k})| d\tilde{k} \right)^2 dk, \\
&= 2^4 \int_{\mathbf{R}} \int_{\mathbf{R}} \left((1+(k-\tilde{k})^2)\hat{f}(k-\tilde{k})||\hat{g}(\tilde{k})| d\tilde{k} + (1+\tilde{k}^2)|\hat{f}(k-\tilde{k})||\hat{g}(\tilde{k})| d\tilde{k} \right)^2 dk, \\
&= 2^4 \int_{\mathbf{R}} \left(|(1+k^2)\hat{f}(k)| * |\hat{g}(k))| + |(1+k^2)\hat{g}| * |\hat{f}| \right)^2 dk, \\
&\leq 2^6 \int_R \left(|(1+k^2)\hat{f}(k)| * |\hat{g}(k))| \right)^2 dk + 2^6 \int_R \left(|(1+k^2)\hat{g}| * |\hat{f})| \right)^2 dk, \\
&= 2^6 \left(\left\| |(1+k^2)\hat{f}| * |\hat{g}) \right\|_{L^2}^2 + \left\| |(1+k^2)\hat{g}| * |\hat{f})| \right\|_{L^2}^2 \right).
\end{aligned} \tag{A.0.3}$$

By the Young's inequality for convolutions that is $\|f * g\|_r \leq \|f\|_p \|g\|_q$ for $f \in L^p$ and $g \in L^q$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$,

$$\begin{aligned}
\|fg\|_{H^2} &\leq 2^6 \left(\left\| (1+k^2)\hat{f} \right\|_{L^2}^2 \|\hat{g}\|_{L^1}^2 + \left\| (1+k^2)\hat{g} \right\|_{L^2}^2 \left\| \hat{f} \right\|_{L^1}^2 \right), \\
&\leq 2^6 \left(\|f\|_{H^2} \|\hat{g}\|_{L^1}^2 + \|g\|_{H^2}^2 \left\| \hat{f} \right\|_{L_1}^2 \right), \qquad \text{(A.0.4)} \\
&\leq C \|f\|_{H^2}^2 \|g\|_{H^2}^2,
\end{aligned}$$

since $\|g\|_{L^1} \leq c \|g\|_{H^2}$ for a constant $c > 0$. $\square$

Second variation of I

The second variation of I with respect to L^2 inner product is calculated taking derivative of $i(\tau)$ twice,

$$
\begin{aligned}
i''(\tau) =& \int_0^{\frac{L_2}{\varepsilon}} \frac{d^2}{d\tau^2} \Big(\partial_z^2 \left(\Phi_n + \tau v\right) - W'(\Phi_n + \tau v)\Big)^2 dz, \\
=& \int_0^{\frac{L_2}{\varepsilon}} \frac{d}{d\tau} \Big(\partial_z^2 \left(\Phi_n + \tau v\right) - W' \left(\Phi_n + \tau v\right)\Big) \Big(\partial_z^2 v - W'' \left(\Phi_n + \tau v\right) v\Big) dz, \\
=& 2 \int_0^{\frac{L_2}{\varepsilon}} \Big(\partial_z^2 v - W'' \left(\Phi_n + \tau v\right) v\Big) \Big(\partial_z^2 v - W'' \left(\Phi_n + \tau v\right) v\Big) \\
&+ \Big(\partial_z^2 \left(\Phi_n + \tau v\right) - W' \left(\Phi_n + \tau v\right)\Big) \Big(-W''' \left(\Phi_n + \tau v\right) v^2\Big) dz, \\
=& 2 \int_0^{\frac{L_2}{\varepsilon}} \Big(\Big(\partial_z^2 v - W'' \left(\Phi_n + \tau v\right) v\Big)\Big)^2 \\
&+ \Big(\partial_z^2 \left(\Phi_n + \tau v\right) - W' \left(\Phi_n + \tau v\right)\Big) \Big(-W''' \left(\Phi_n + \tau v\right) v^2\Big) dz.
\end{aligned}
\tag{A.0.5}
$$

Evaluating this derivative at $\tau = 0$,

$$
\begin{aligned}
i''(0) &= 2 \int_0^{\frac{L_2}{\varepsilon}} \Big(\partial_z^2 v - W'' \left(\Phi_n\right) v\Big)^2 + \Big(\partial_z^2 \Phi_n - W' \left(\Phi_n\right)\Big) \Big(-W''' \left(\Phi_n\right) v^2\Big) dz, \\
&= 2 \int_0^{\frac{L_2}{\varepsilon}} \Big(\partial_z^2 - W'' \left(\Phi_n\right)\Big)^2 v^2 + \Big(\partial_z^2 \Phi_n - W' \left(\Phi_n\right)\Big) \Big(-W''' \left(\Phi_n\right) v^2\Big) dz.
\end{aligned}
\tag{A.0.6}
$$

When we insert the expansion for Φ_n in (A.0.6) and expand it about ϕ_n, we have

$$
\begin{aligned}
i''(0) =& 2\int_0^{\frac{L_2}{\varepsilon}} \left(\partial_z^2 - W''\left(\Phi_n\right)\right)^2 v^2 + \left(\partial_z^2\Phi_n - W'\left(\Phi_n\right)\right)\left(-W'''\left(\Phi_n\right)v^2\right)dz,\\
=& 2\int_0^{\frac{L_2}{\varepsilon}} \left(\partial_z^2 - W''\left(\phi_n + \varepsilon\lambda\mathcal{B}_{2,n}\right)\right)^2 v^2 + \\
& \quad + \left(\partial_z^2\left(\phi_n + \varepsilon\lambda\mathcal{B}_{2,n}\right) - W'\left(\phi_n + \varepsilon\lambda\mathcal{B}_{2,n}\right)\right)\left(-W'''\left(\phi_n + \varepsilon\lambda\mathcal{B}_{2,n}\right)\right)v^2 dz,\\
=& 2\int_0^{\frac{L_2}{\varepsilon}} \left(\partial_z^2 - \left(W''\left(\phi_n\right) + \varepsilon\lambda W'''\left(\phi_n\right)\mathcal{B}_{2,n}\right)\right)^2 v^2 - \\
& \left(\partial_z^2\phi_n + \varepsilon\lambda\partial_z^2\mathcal{B}_{2,n} - \left(W'\left(\phi_n\right) + \varepsilon\lambda W''\left(\phi_n\right)\mathcal{B}_{2,n}\right)\right)\left(W'''\left(\phi_n\right) + \varepsilon\lambda W^{(4)}\left(\phi_n\right)\mathcal{B}_{2,n}\right)v^2 dz,\\
=& 2\int_0^{\frac{L_2}{\varepsilon}} \left(\left(\partial_z^2 - W''\left(\phi_n\right)\right)^2 - \left(\partial_z^2\phi_n - W'\left(\phi_n\right)\right)\left(W'''\left(\phi_n\right)\right) + O(\varepsilon)\right)v^2 dz.
\end{aligned}
\tag{A.0.7}
$$

Since ϕ_n solves $\partial_z^2 u - W'(u) = 0$, at the leading order we obtain

$$
\begin{aligned}
i''(0) &= 2\int_0^{\frac{L_2}{\varepsilon}} \left(\partial_z^2 - W''\left(\phi_n\right)\right)^2 v^2 dz,\\
&= 2\int_0^{\frac{L_2}{\varepsilon}} \left(\mathcal{L}_n v\right)^2 dz,\\
&= 2\langle {\mathcal{L}_n}^2 v, v\rangle.
\end{aligned}
\tag{A.0.8}
$$

Thus, the second variation of I at the leading order is

$$
\frac{\delta^2 I}{\delta U^2}(\Phi_n) := \mathcal{L}_n^2. \tag{A.0.9}
$$

H^2 norm of the projection Π_0

Lemma A.0.2. *Let Π_0 be the orthogonal projection of H^2 onto $U \subset H^2$ given in (4.2.2). Then, for any $u \in H^2$*

$$\|\Pi_0 u\|_{H^2} = \|u\|_{H^2}. \tag{A.0.10}$$

Proof. For any $u \in H^2$,

$$\begin{aligned} \|\Pi_0 u\|_{H^2}^2 &= \langle \Pi_0 u, \Pi_0 u \rangle_{H^2} \\ &= \langle \Pi_0^2 u, u \rangle_{H^2} \\ &\leq \|\Pi_0 u\|_{H^2} \|u\|_{H^2}, \end{aligned} \tag{A.0.11}$$

and therefore, $\|\Pi_0 u\|_{H^2} \leq \|u\|_{H^2}$. On the other hand, since Π_0 is a projection we can write $u = \Pi_0 u + u'$ where $u' \in U^{\perp}$ and obtain

$$\|u\|_{H^2}^2 = \langle \Pi_0 u + u', \Pi_0 u + u' \rangle = \|\Pi_0 u\|_{H^2}^2 + \|u'\|_{H^2}^2, \tag{A.0.12}$$

which proves that $\|\Pi_0 u\|_{H^2} \geq \|u\|_{H^2}$. □

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