STATISTICAL INFERENCE ON SELF-SIMILAR AND INCREMENT STATIONARY PROCESSES AND RANDOM FIELDS

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A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

Statistics – Doctor of Philosophy

2019

ABSTRACT

STATISTICAL INFERENCE ON SELF-SIMILAR AND INCREMENT STATIONARY PROCESSES AND RANDOM FIELDS

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This dissertation is about making statistical inference on self-similar and increment stationary processes/random fields. Self-similarity and, more generally, fractality are seen in many objects in nature which have similar features at different scales, for smaller scale and larger scale. The most well known statistical model that has self-similarity is fractional Brownian motion (fBm). It has been useful for its self-similarity, increment stationarity, and Gaussianity, and it naturally arises as the scaling limits of random walk, having many applications in hydrology, telecommunication network, finance, etc. Some extensions of fBm have been introduced including operator fractional Brownian motion (OFBM)[19][34] and operator scaling Gaussian random field (OSGRF) [8]. OFBM and OSGRF are multivariate processes, random field, respectively, both have operator self-similarity/operator scaling property, Gaussianity, and increment stationarity.

The first topic is about estimating Hurst parameter which is a measure for self-similarity in statistical model. Hurst estimation is examined/developed in OFBM and OSGRF using wavelet transform and discrete variation method, respectively. The asymptotics of the estimators are derived in continuous sample path, discrete sample, and discrete noisy sample in OFBM. In OSGRF, the asymptotics of the estimators are derived in different sampling methods, fixed domain/increasing domain, and/or samples on the exact directions/ samples on the grid lines. The performance of the estimators is examined through simulating OFBM and OSGRF, respectively, and the good choice for scale parameter in wavelet function and discrete variation method is recommended.

The second topic is on measuring dependency between two random fields that are increment stationary. The dependency is measured in spectral domain by defining and estimating coherence in multivariate random field. The concept of coherence is originated from multivariate stationary time series, and it measures correlation between two time series in spectral domain. Recently, the

definition is extended to multivariate random fields by [30]. In this dissertation, the concept of coherence is extended to multivariate random field with stationary increments, its properties are examined and its estimation method is developed. Especially, the concept and the estimator method are applied to OFBM.

ACKNOWLEDGEMENTS

This dissertation research is partially supported by NSF grant DMS-1612885.

I would like to thank my advisor, Dr. Xiao, for his invaluable care about his students, his advice and guidance in my research and future career. I would like to thank my friend Danise for being good company, spending fun and good time with me here in Michigan. I would like to thank all the members in Grad IV for our special time together in prayer and fun. I would also like to thank Sarah, Jessie, Juna and Lori for our good time and our laughs. Finally, I would like to express my deep appreciation to my family, especially my grandpa, for their love that is beyond my understanding.

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CHAPTER 1

INTRODUCTION

Hurst exponent is a measure for self-similarity in time series/stochastic process. It is also called fractal index or long-memory parameter as it measures roughness of path and/or correlatedness among the variables in the path. The most well known statistical model that has Hurst exponent which encompasses all three properties above is fractional Brownian motion. The fractional Brownian motion (fBm) developed by Mandelbrot and Van Ness (1968)[36] has been studied by many authors for its importance in modeling processes that have long memory and/or a statistical self-similarity property.

Fractional Brownian motion (fBm), B_H, is a Gaussian process with stationary increments and covariance function

$$E(B_H(t), B_H(s)) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}),$$

where $H \in (0, 1)$ is the Hurst index. Firstly, it is a self-similar process,

$$\{B_H(ct)\}_{t\in\mathbb{R}} \overset{f.d}{=} \{c^H B_H(t)\}_{t\in\mathbb{R}} \quad \text{every } c > 0,$$

where $\overset{f.d.}{=}$ denotes equality of all finite-dimensional marginal distributions, which can be easily derived from the definition of fBm. Secondly, the larger the Hurst index, $H \in (0, 1)$, is, the smoother the sample path of B_H is, even though for every $H \in (0, 1,)$, the sample function $B_H(t)$ is nowhere differentiable. Thirdly, if $H \in (.5, 1)$, the increments of B_H, $\{B_H(i+1) - B_H(i)\}_{i\in\mathbb{N}}$, is a stationary process with long-memory, i.e.

$$\sum_{i=1}^{\infty} cov(B_H(i+1) - B_H(i), B_H(2) - B_H(1)) = \infty,$$

which means that $\{B_H(i+1)-B_H(i)\}_{i\in\mathbb{N}}$ has strong dependency and its correlation does not decrease as fast as a conventional stationary process which has the absolutely summable autocovariace function like when $H \in (0, .5)$

$$\sum_{i=1}^{\infty} cov(B_H(i+1) - B_H(i), B_H(2) - B_H(1)) < \infty.$$

Because of the usefulness of the Hurst parameter in capturing the above important properties, there have been many applications of fBm and an extensive literature on methodology of estimating Hurst parameter. For example, fBm has been used in image generation and interpolation, texture classification and the modeling of burst errors in communication channels, $1/f$ noise in oscillators and current noise in metal films and semiconductor devices. Some extensions of fBm have been introduced including operator fractional Brownian motions [19][34] and operator scaling Gaussian random fields [8].

A large body of literature studies estimation of the Hurst index, H, or memory parameter, d, for fractional Brownian motion and more general processes. One way is to use discrete variation of sample paths [2][10][13][14], and the other method uses wavelet transform [4][22][26][42][43][48][50]. For stationary processes, Fourier transform can be useful, or if it is non-stationary then one uses this method after differencing the process, $\Delta^k X$, and making it stationary [22][25][46][47]. Overview and comparison of the two methods, wavelet and Fourier methods, can be found in [22].

In this dissertation, Hurst estimation is developed for an operator fractional Brownian motion (OFBM) and operator scaling Gaussian random field (OSGRF) using the wavelet transform and discrete variation method, respectively. In this case, the self-similarity index is a matrix H whose eigenvalues determine many of their statistical properties. An estimator of the Hurst parameters in OFBM using wavelet method is proposed in [4] in an increasing domain. In [4], eigenvalue method was used as the Hurst matrix H was assumed to be diagonizable having eigenvalues h_1, h_2, and only hurst index h_1 was taken into account for the convergence rate of the estimators. And it was mentioned that increasing the value of $h_2 - h_1$ can increase estimator performance. (see Remarks 4.1,4.4 in [4].)

In Chapter 2, the estimator of the Hurst indices of OFBM is studied in a fixed domain using the similar method developed in [4]. The asymptotics of the estimator is derived in a continuous sample path, discrete sample, and discrete noisy sample, respectively. It is revealed that not only h_1, h_2, but the dependence structure within the components, i.e. the covariance matrix $\Gamma(1, 1)$ of $B_H(1)$, and the choice of the scale parameter of wavelet function also affect the performance of the estimator.

In [32], parametric estimation method was developed for estimating Hurst parameters in OSGRFs using a fast Fourier transform approximation to harmonizable representation of OSSRFs. In Chapter 4, a different method is proposed for estimating the Hurst parameters in OSGRF, and its performance is investigated in different sampling methods.

The wavelet and discrete variation method in fBm is introduced in Section 1.1, and the models of OFBM and OSGRF will be explained in Section 1.3. The methodology and the results will be shown in Chapter 2 and Chapter 4 for OFBM and OSGRF, respectively.

The concept of coherence originates from multivariate stationary time series, and it measures correlation between two time series in the spectral domain. Recently, the definition extended to multivariate stationary random fields by [30]. The definition of coherence in stationary time series and random fields are introduced in Section 1.2. In Chapter 3, the concept of coherence is extended to multivariate random fields with stationary increments, its properties are examined and its estimation method is developed. Particularly, the concept and the method are applied to OFBM.

1.1 Estimation methods of Hurst parameter

There are several methods for estimating Hurst parameter. Among them are wavelet method and discrete variation method which will be used in Chapter 2 and Chapter 4, respectively.

1.1.1 Wavelet method

The wavelet transform of B_H is defined as

$$d_{j,k} := \int_{\mathbb{R}} \psi_{j,k}(t) B_H(t) dt,$$

where $\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$, and where ψ is a wavelet function which has the following properties.

(W-1) ψ has compact support and

$$\int_{\mathbb{R}} \psi^2(t) dt = 1$$

(W-2) ψ has M vanishing moments:

$$\int_{\mathbb{R}} t^m \psi(t)dt = 0 \quad \text{for all } m = 0, ..., M-1.$$

Then for any fixed scale j, the wavelet coefficients $\{d_{j,k}\}_k$ is mean-zero stationary Gaussian process with variance

$$E(d_{j,k}^2) = 2^{-j(2H+1)} \int_{\mathbb{R}} \psi(t)\psi(s)|t-s|^{2H} dtds.$$

Note that $\int_{\mathbb{R}} \psi(t)\psi(s)|t-s|^{2H} dtds = E(d_{0,k}^2)$ and the covariance between $d_{0,k}$ and $d_{0,k'}$ decreases as fast as $cov(d_{0,k}, d_{0,k'}) \sim |k-k'|^{2H-2M}$ as $k - k' \to \infty$.

Since

$$\log \overline{d_j^2} \sim -2jH \log 2 + \log E(d_{0,k}^2),$$

the estimator of H is the least square estimate on log-regression of $\overline{d_j^2}$ on $j = J_1, ..., J_2$ where $\overline{d_j^2}$ is the sample mean of $\{d_{j,k}^2; k = 0, 1, ..., n\}$.

$$\hat{H} = -\frac{1}{2} \sum_{j=J_1}^{J_2} w_j \log_2 \overline{d_j^2}$$

where

$$\sum_{j=J_1}^{J_2} w_j = 0, \sum_{j=J_1}^{J_2} jw_j = 1.$$

1.1.2 Discrete variations method

This sections is mainly from [15]. Define $a = \{a_q, q = 0, 1, ..., \ell\}$ as a filter of length $\ell + 1$ with order $p \geq 1$, that is a vector with $\ell + 1$ components satisfying

$$\sum_{q=0}^{\ell} q^j a_q = 0 \quad \text{for } j = 0, ..., p-1 \quad \text{and} \sum_{q=1}^{\ell} q^p a_q \neq 0.$$

Define

$$B_H^a(i) := \sum_{q=0}^{\ell} a_q B_H(i-q).$$

The covariance and correlation functions are given by

$$E(B_H^a(j)B_H^a(i+j)) = -\frac{1}{2}\sum_{q,r=0}^{\ell} a_q a_r |q-r+i|^{2H}$$

for $i, j \in \mathbb{Z}$. By filtering B_H with a, the correlation of B_H^a decreases as $cor(B_H^a(j)B_H^a(j+i)) \sim |i|^{2H-2p}$ as $i \to \infty$. For $m \geq 1$, define dilated filters a^m of a filter a by

$$a_i^m = \begin{cases} a_{i/m} & \text{if } i/m \text{ is integer} \\ 0 & \text{otherwise} \end{cases}$$

for $i = 0, ..., m\ell$. That is, a^m is a filter with length $m\ell + 1$ with order p. Then it is derived that

$$E(B_H^{a^m}(j)B_H^{a^m}(i+j)) = -\frac{1}{2}\sum_{q,r=0}^{m\ell} a_q^m a_r^m |q-r+i|^{2H} = -\frac{1}{2}\sum_{q,r=0}^{\ell} a_q a_r |mq-mr+mi|^{2H},$$

and

$$E\left(\overline{(B_H^{a^m})^2}\right) = m^{2H}\pi_H^a(0),$$

where $\pi_H^a(0) := E((B_H^a(j))^2)$ and $\overline{(B_H^{a^m})^2}$ is the sample mean of $\{(B_H^{a^m}(j))^2, j = 1, ..., n\}$. Then the estimator of H, $\hat{H}$, is the ordinary least square estimate from the regression of $\log \overline{(B_H^{a^m})^2}$ on m. In other words, since

$$\log \overline{(B_H^{a^m})^2} \sim 2H \log m + \log \pi_H^a(0),$$

the estimator $\hat{H}$ is

$$\hat{H} = \frac{A'}{2||A||^2}\left(\log \overline{(B_H^{a^m})^2}\right)_{m=M_1,...,M_2},$$

where $A_m = \log m - \frac{1}{M_2-M_1+1}\sum_{m=M_1}^{M_2} \log m$ and $M_1 \leq m \leq M_2$. In particular, if $m = 2^{M_1}, ..., 2^{M_2}$ then $\hat{H}$ can be expressed as

$$\hat{H} = \frac{1}{2}\sum_{r=M_1}^{M_2} w_r \log_2 \overline{(B_H^{a^{2^r}})^2},$$

where

$$\sum_{r=M_1}^{M_2} w_r = 0, \sum_{r=M_1}^{M_2} r w_r = 1.$$

1.2 Coherence of multivariate stationary processes and multivariate random fields

This section is mainly from 11.6 in [11]. Let $\mathbf{X}_t = (X_{t,1}, X_{t,2})'$ be a bivariate stationary time series with mean zero and covariances $c_{ij}(h) = E(X_{t+h,i}X_{t,j})$ satisfying

$$\sum_{h=-\infty}^{\infty} |c_{ij}(h)| < \infty \quad i, j = 1, 2.$$

Then the matrix

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} C(h) = \begin{pmatrix} f_{11}(h) & f_{12}(h) \\ f_{21}(h) & f_{22}(h) \end{pmatrix}$$

is called the spectral density matrix or spectrum of $\mathbf{X}_t$ where $C(h)$ is the 2×2 matrix with (i, j) element $c_{ij}(h), i, j = 1, 2$. Especially, $f_{ii}(\cdot)$ is called the spectral density of the univariate series $\{X_{t,i}\}, i = 1, 2$, and $f_{12}(\cdot)$ is called the cross spectrum or cross spectral density of $\{X_{t,1}\}$ and $\{X_{t,2}\}$. Since $c_{ii}(\cdot)$ is symmetric around zero, $f_{ii}(\cdot)$ is real-valued and symmetric around zero, but not $f_{ij}(\cdot), i \neq j$. It is followed from the above definition that

$$C(h) = \int_{-\pi}^{\pi} e^{\mathrm{i}h\lambda} f(\lambda) d\lambda$$

and $\mathbf{X}_t$ has spectral representation,

$$X_{t,i} = \int_{(-\pi,\pi]} e^{\mathrm{i}t\lambda} dZ_i(\lambda)$$

where

$$f_{ij}(\lambda) d\lambda = E(dZ_i(\lambda)\overline{dZ_j(\lambda)}), \quad i, j = 1, 2,$$

and

$$E(dZ_i(\lambda)\overline{dZ_j(\mu)}) = 0 \quad \text{for } \lambda \neq \mu \text{ and } \; i, j = 1, 2.$$

It follows that $f_{ij}(\cdot) = \overline{f_{ji}(\cdot)}$ and therefore $f(\lambda) = f^*(\lambda)$ where * denotes complex conjugate transpose. Also, for any $\mathbf{a} = (a_1, a_2)' \in \mathbb{C}^2$, $\mathbf{a}^* f(\lambda)\mathbf{a}$ is the spectral density of $\{\mathbf{a}^*\mathbf{X}_t\}$. Therefore, $\mathbf{a}^* f(\lambda)\mathbf{a} \geq 0$ and the matrix $f(\lambda)$ is non-negative definite.

The correlation between $dZ_1(\lambda)$ and $dZ_2(\lambda)$ is called coherence, $\gamma_{12}(\lambda)$, at frequency λ,

$$\gamma_{12}(\lambda) = f_{12}(\lambda)/[f_{11}(\lambda)f_{22}(\lambda)]^{1/2}.$$

Note that $0 \leq |\gamma_{12}(\lambda)|^2 \leq 1$ for $-\pi \leq \lambda \leq \pi$, and $\gamma_{12}(\lambda)$ close to one indicates a strong linear relationship between $dZ_1(\lambda)$ and $dZ_2(\lambda)$.

The above notion of coherence in multivariate stationary processes was extended to multivariate random fields in [30]. Suppose $\mathbf{X}(t) = (X_1(t), ..., X_p(t))' \in \mathbb{C}^p$ is a mean-zero p-variate weakly statinary random field on $t \in \mathbb{R}^d$ with a matrix-valued covariance function $\mathbf{C}(h) = (C_{ij}(h))_{i,j=1}^p$ where $C_{ij}(h) = Cov(X_i(t+h)X_j(t))$. For complex-valued stationary processes, $Cov(X_i(t)X_j(s)) = E(X_i(t)\overline{X_j(s)})$ so that $C_{ij}(h) = \overline{C_{ji}(-h)}$.

Theorem 1.2.1 (Cramér (1940))**.** *A matrix valued function* $\mathbf{C} : \mathbb{R}^d \to \mathbb{C}^{p\times p}, \mathbf{C} = (C_{ij})_{i,j=1}^p$ *is a non-negative definite if and only if*

$$C_{ij}(h) = \int_{\mathbb{R}^d} e^{\mathrm{i}\omega' h} f_{ij}(\omega) d\omega$$

for $i, j = 1, ..., p$ *such that the matrix* $f(\omega) = (f_{ij}(\omega))_{i,j=1}^p$ *is nonnegative definite for all* $\omega \in \mathbb{R}^d$.

Like before, $f_{ij}(\omega)$ are called spectral and cross-spectral densities for the marginal and cross-covariance functions $C_{ij}(h)$, and $f_{ij}(\omega) = \overline{f_{ji}(\omega)}$. Note that with covariance function $C_{ij}(\cdot)$

$$f_{ij}(\omega) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-\mathrm{i}\omega' h} C_{ij}(h) dh$$

and the coherence function is defined

$$\gamma(\omega) = \frac{f_{12}(\omega)}{\sqrt{f_{11}(\omega) f_{22}(\omega)}}$$

for $\omega \in \mathbb{R}^d$, and $f_{ii}(\omega) > 0$. If $f_{ii}(\omega) = 0$, then $\gamma(\omega) = 0$. The squared coherence satisfies $0 \leq |\gamma(\omega)|^2 \leq 1$ for all ω by Theorem 1.2.1 and $|\gamma(\omega)|^2$ close to one indicates X_1 and X_2 has strong linear relationship at particular frequency bands.

1.3 Operator fractional Brownian motion and operator scaling Gaussian random fields

Operator fractional Brownian motions and operator scaling Gaussian random fields are extensions of fBm with operator self-similar/scaling property in multivariate stochastic processes and random field.

1.3.1 Operator fractional Brownian motion

Operator fractional Brownian motions (OFBM) arise naturally as the scaling limit of various multivariate random walks with (long-range) dependent steps on lattices normalized by linear operators, from asymptotic analysis of nonstationary fractionally integrated multivariate random sequences, from central limit theorem in Hilbert spaces, or from modeling heavy traffic in queuing systems. See Pitt [44], Marinucci and Robinson [37], Rackauskas and Suquet [45], Düker [20, 21], Delgado, R [17], Majewski [35], just to mention a few.

Let $\mathbf{X} = \{\mathbf{X}(t), t \in \mathbb{R}^d\}$ be a p-variate random field. We say that $\mathbf{X}$ is an OFBM with exponent H if it is a Gaussian field with mean zero and stationary increments, and satisfies the following operator scaling (operator self-similar) property: For any $c > 0$,

$$\{\mathbf{X}(ct), t \in \mathbb{R}^d\} \overset{f.d.}{=} \{c^H \mathbf{X}(t)\}, \tag{1.1}$$

where H is a linear operator on $\mathbb{R}^d$ and $c^H = \exp(H \ln c) = \sum_{k=1}^{\infty} \log^k(c) H^k / k!$ for $c > 0$. For convenience, we will not distinguish the operator H from its associated matrix relative to the standard basis of $\mathbb{R}^d$. One can refer to the monographs [38, 41] for more on operator stable distributions.

Denote by λ_H and Λ_H the minimum and the maximum of the real parts of the eigenvalues of H, respectively. Mason and Jurek [38] showed that $0 \le \lambda_H \le \Lambda_H \le 1$ and that $\mathbf{X}$ is non-trivial if and only if $\lambda_H > 0$. It turns out [39] that many sample path properties of OFBM are reflected through the real parts of the eigenvalues of the operator H.

Recently, there has been much interest in studying OFBM in the literature.

Maejima and Mason [34] established a time domain construction of OFBM , while Mason and Xiao [39] gave a spectral domain construction à la Itô-Yaglom, for general $d \geq 1$. When $d = 1$, Didier and Pipiras [19] found both time and spectral domain representations for OFBM which is more general than those given in [34, 39].

An $\mathbb{R}^p$-valued stochastic process $\{\mathbf{X}_H(t)\}_{t\in\mathbb{R}}$ is said to be OFBM if it is an increment stationary, zero-mean, and operator self-similar Gaussian process, i,e. when its law scales according to a matrix exponent H, i.e. for $c > 0$, $\{\mathbf{X}_H(ct)\}_{t\in\mathbb{R}} \overset{f.d}{=} \{c^H \mathbf{X}_H(t)\}_{t\in\mathbb{R}}$. When the operator H is diagonal, it is called the multivariate fractional Brownian motion.

In the spectral domain, with the mild assumption that eigenvalues of H satisfy

$$0 < Re(h_k) < 1, \qquad k = 1, 2, ..., p. \tag{1.2}$$

OFBM admits the following representation:

$$\mathbf{X}_H(t) = \int_{\mathbb{R}} \frac{e^{itx} - 1}{ix} (x_+^{-(H-(1/2)I)} \mathscr{C} + x_-^{-(H-(1/2)I)} \overline{\mathscr{C}}) \tilde{B}(dx) \tag{1.3}$$

for some $p \times p$ complex-valued matrix $\mathscr{C}$, where $x_\pm = \max\{\pm x, 0\}$ and $\tilde{B}(dx)$ is a multivariate complex-valued Gaussian measure such that $\tilde{B}(-dx) = \overline{\tilde{B}(dx)}$ and $E\tilde{B}(dx)\tilde{B}(dx)^* = dx$. In the time domain, in addition to (2), if $Re(h_k) \neq 1/2, k = 1, 2, ..., p$, then OFBM has the following representation:

$$\int_{\mathbb{R}} \left[((t-u)_+^{H-(1/2)I} - (-u)_+^{H-(1/2)I}) \mathscr{M}_+ + ((t-u)_-^{H-(1/2)I} - (-u)_-^{H-(1/2)I}) \mathscr{M}_- \right] B(du), \tag{1.4}$$

where $\mathscr{M}_-, \mathscr{M}_+$ are $p \times p$ matrices with real-valued entries and $B(du)$ is a multivariate real-valued Gaussian measure. The representation (4) is obtained from (3) by taking Fourier transformation of deterministic kernel in (3).

Unlike the univariate case, OFBM does not have the following property in general.

$$E(\mathbf{X}_H(t)\mathbf{X}_H(s)^*) = E(\mathbf{X}_H(s)\mathbf{X}_H(t)^*), \tag{1.5}$$

where * represents Hermitian transposition. In OFBM, time reversibility of the process is equivalent to (1.5). Didier and Pipiras (Theorem 5.1, Corollary 5.1 in [19]) provided some conditions on $\mathscr{C}$

and $\mathscr{M}$ in (1.3) and (1.4) respectively for OFBM to be time-reversible. With this condition, one has the following equation analogous to fBm.

$$E(\mathbf{X}_H(t)\mathbf{X}_H(s)^*) = \frac{1}{2}(|t|^H\Gamma(1,1)|t|^{H^*} + |s|^H\Gamma(1,1)|s|^{H^*} + |t-s|^H\Gamma(1,1)|t-s|^{H^*}), \qquad (1.6)$$

where $\Gamma(1,1) = E\mathbf{X}_H(1)\mathbf{X}_H(1)^*$.

In Chapter 2, OFBM with $d = 1, p = 2$ is used with scaling matrix H, and its eigenvalues (Hurst indices) h_1, h_2. In Chapter 3, OFBM with $d \geq 1, p = 2$ is used.

1.3.2 Operator scaling Gaussian random fields

A scalar valued random field $\{X(t)\}_{t\in\mathbb{R}^d}$ is called operator-scaling if for some $d \times d$ matrix E with positive real parts of the eigenvalues and some $H > 0$ we have

$$\{X(c^E t)\}_{t\in\mathbb{R}^d} \overset{f.d.}{=} \{c^H X(t)\}_{t\in\mathbb{R}^d} \qquad \text{for all } c > 0. \qquad (1.7)$$

If E is the identity matrix, then X has self-similar property with Hurst index H, $\{X(ct)\}_{\mathbb{R}^d} \overset{f.d.}{=} \{c^H X(t)\}_{\mathbb{R}^d}$, and fractional Brownian field is well known class of such field. If E is diagonizable matrix, $Hu_i = \lambda_i u_i, i = 1, ..., d$, then $\{X_i(ct)\}_{t\in\mathbb{R}} \overset{f.d.}{=} \{c^{H/\lambda_i} X_i(t)\}_{t\in\mathbb{R}}$ for any $c > 0$, where $X_i(t) = X(tu_i)$. As this model has different Hurst indices H/λ_i along the different coordinates u_i, it is expected to be useful to capture such features observed in nature object. Among the applications of this model is ground water hydrology, [7].

In [8], harmonizable representation and moving average representation of operator scaling stable random fields (OSSRFs) are defined as below: Harmonizable representation is

$$X_\psi(t) = Re\int_{\mathbb{R}^d}(e^{\mathrm{i}\langle t,\xi\rangle} - 1)\psi(\xi)^{-H-q/\alpha}W_\alpha(d\xi),\ t \in \mathbb{R}^d,$$

where ψ is a continuous, E^t-homogeneous function such that $\psi(\xi) \neq 0$ for $\xi \neq 0$, $0 < \alpha \leq 2$, and $H \in (0, a_1)$, a_1 is the smallest real parts of the eigenvalue of E. Moving average representation is

$$X_\varphi(t) = \int_{\mathbb{R}^d}(\varphi(t-y)^{H-q/\alpha} - \varphi(-y)^{H-q/\alpha})Z_\alpha(dy),\ t \in \mathbb{R}^d,$$

where φ be an E-homogeneous, $(\beta, E)-$admissible function, $0 < \beta, 0 < \alpha \leq 2$, and $0 < H < \beta$. φ is E-homogeneous if $\varphi(c^E t) = c\varphi(t)$, for all $c > 0$ and $t \in \mathbb{R}^d \setminus \{0\}$, and is called $(\beta, E)-$admissible if for any $0 < A < B$, there exists positive constant C such that, for $A \leq ||y|| \leq B, \tau(t) \leq 1 \Rightarrow |\varphi(t+y) - \varphi(y)| \leq C\tau(t)^\beta$. It turns out that the harmonizable representation is more flexible in the class of possible functions φ whereas the moving average representation is more restrictive. However they both satisfy (1), have stationary increments and are continuous in probability.

In [9], explicit covariance functions of operator scaling Gaussian random fields are provided making a fast and exact method of simulation available in these classes. They define a function

$$\tau_E(t) = \Big(\sum_{i=1}^{d} |\langle t, u_i\rangle|^{2a_i}\Big)^{1/2},\ t \in \mathbb{R}^d, \tag{1.8}$$

where $1/a_i = \lambda_i > 0, u_i, i = 1, 2, ..., d$, are eigenvalues and eigenvectors of a diagonizable matrix E, and find out that for $H \in (0, 1]$, $\upsilon_{E,H}(t) = \tau_E(t)^{2H}$ is the semi-variogram funtion of a centered Gaussian random field that has stationary increments and satisfies (1.7). Recall semi-variogram of X is defined by $\upsilon_{E,H}(h) = \frac{1}{2}E(X(t+h) - X(t))^2$.

Throughout this dissertation, var, cov means variance and covariance respectively, vec means vectorization of a matrix, especially for symmetric matrix $vec(\begin{pmatrix} a & b \\ b & d \end{pmatrix}) = (a, b, d)'$, $\rightarrow_d$ indicates convergence in distribution, C and c are a generic constant matrix and scalar respectively, which are not related to j, n, H. For two matrices $A = (a_{ij}), B = (b_{ij})$, $A \leq B$ means $a_{ij} \leq b_{ij}$ for all i, j. $a \sim b$ means $a/b \rightarrow 1$, $a_n = O_p(b_n)$ means a_b/b_n is bounded in probability which also implies a_n converges in probability if $b_n \rightarrow 0$.

CHAPTER 2

ESTIMATING HURST INDICES IN OPERATOR FRACTIONAL BROWNIAN MOTION

2.1 Basic Properties of the Wavelet Coefficients of OFBM

Let $\mathbf{X}_H(t)$ be an operator fractional Brownian motion satisfying (1.1-1.3),(1.5-1.6). For notational convenience, $\mathbf{X}_H(t)$ is denoted as $\mathbf{X}_t = (X_{t,1}, X_{t,2})$ hereafter. Suppose we observe $\mathbf{X}_t$ in a fixed domain $t \in [0, 1]$. The assumptions below are given to wavelet function and OFBM.

ASSUMPTION (W1) :

ψ is a wavelet function with two vanishing moments and has compact support $[0, S]$.

ASSUMPTION (OFBM1):

Time reversibility holds, i.e., $\{\mathbf{X}_t\} \overset{f.d.}{=} \{\mathbf{X}_{-t}\}$.

ASSUMPTION (OFBM 2):

$\mathbf{X}_t$ is a bivariate OFBM with scaling matrix $H = Pdiag(h_1, h_2)P^{-1}, 0 < h_1 \le h_2 < 1$, i.e. H is diagonizable with $P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

For $j \in \mathbb{Z}$ and for some large n greater than 2^j, define the set $S_j = \{k_{i,j} : k_{i,j} = \frac{2^j}{n} i, i = 1, \ldots, n\}$. Denote the random wavelet coefficients of $\mathbf{X}_t$ by

$$\mathbf{d}_{j,k_{i,j}} = \int_{\mathbb{R}} \psi_{j,k_{i,j}}(t)\mathbf{X}_t dt \quad k_{i,j} \in S_j,$$

where $\psi_{j,k_{i,j}}(t) = 2^{j/2}\psi(2^j t - k_{i,j})$. The sequence $\{\mathbf{d}_{j,k_{i,j}}\}$ is centered bivariate Gaussian and stationary with respect to the location parameter i for fixed j. It has the following properties.

(P1)

$$\{\mathbf{d}_{j,k_{i,j}}; i \in [1, \ldots, n]\} \overset{f.d.}{=} \{2^{-j(H+1/2)}\mathbf{d}_{0,k_{i,j}}; i \in [1, \ldots, n]\}.$$

$(P'1)$

$$\{\mathbf{d}_{j,k_{i,j}}, \mathbf{d}_{j+1,k_{i,j+1}}; i \in [1,...,n]\} \overset{f.d.}{=} \{2^{-j(H+1/2)}\mathbf{d}_{0,k_{i,j}}, 2^{-j(H+1/2)}\mathbf{d}_{1,k_{i,j+1}}; i \in [1,...,n]\}.$$

(P2)

$$var(\mathbf{d}_{j,k_{i,j}}) = 2^{-j(H+1/2)}c(\psi, H)2^{-j(H+1/2)},$$

where $c(\psi, H) = 1/2 \int \psi(t)\psi(s)|t-s|^H\Gamma(1,1)|t-s|^H dsdt$.

(P3) $cov(\mathbf{d}_{j,k_{i,j}}, \mathbf{d}_{j,k_{i',j}}) = 2^{-j(H+1/2)}cov(\mathbf{d}_{0,k_{i,j}}, \mathbf{d}_{0,k_{i',j}})2^{-j(H+1/2)}$, and it decays hyperbolically fast (each element of the covariance matrix decays hyperbolically fast) as $|k_{i,j} - k_{i',j}| \to \infty$ due to two vanishing moments of ψ ([13],[49]).

(P4) $|cov(\mathbf{d}_{0,k_{i,j}}, \mathbf{d}_{0,k_{i',j}})_{\ell,p}| \le c(1 + |k_{i,j} - k_{i',j}|)^{2(h_2-2)}, \quad \ell, p = 1, 2, h_1 \le h_2.$

$(P'4)$ $cov(\mathbf{d}_{j,k_{i,j}}, \mathbf{d}_{j+1,k_{i',j+1}}) = 2^{-j(H+1/2)}cov(\mathbf{d}_{0,k_{i,j}}, \mathbf{d}_{1,k_{i',j+1}})2^{-j(H+1/2)}$ and

$$\begin{aligned}
&cov(\mathbf{d}_{0,k_{i,j}}, \mathbf{d}_{1,k_{i',j+1}}) \\
&= \frac{1}{2}\int \psi(t)\psi(s)|2^{-1}t - s + k_{i,j} - 2^{-1}k_{i',j+1}|^H\Gamma(1,1)|2^{-1}t - s + k_{i,j} - 2^{-1}k_{i',j+1}|^H dtds \\
&= \frac{1}{2}\int \psi(t)\psi(s)|2^{-1}t - s + k_{i,j} - k_{i',j}|^H\Gamma(1,1)|2^{-1}t - s + k_{i,j} - k_{i',j}|^H dtds,
\end{aligned}$$

therefore

$$|cov(\mathbf{d}_{0,k_{i,j}}, \mathbf{d}_{1,k_{i',j+1}})_{\ell,p}| \le c(1 + |k_{i,j} - k_{i',j}|)^{2(h_2-2)}, \quad \ell, p = 1, 2, h_1 \le h_2.$$

(P5) By the stationary increments of $\mathbf{X}_t$ and wavelet assumption (W1), $\{(\mathbf{d}_{j,k_{i,j}}, ..., \mathbf{d}_{j+L,k_{i,j+L}}); i = 1, ..., n\}$ is a multivariate stationary process for a fixed $j, L \in \mathbb{Z}$.

2.2 Wavelet estimation of Hurst index in OFBM

2.2.1 Observing sample path of OFBM

Throughout this chapter, assumptions (W1, OFBM1, OFBM2) are assumed. Assume further that a continuous sample path of $\mathbf{X}_t$ is observed for $t = [0, 1]$. Then, the estimation strategy is as follows. Define a 2×2 sample covariance matrix Q_j in the following way:

$$Q_j = \sum_{k \in S_j} \mathbf{d}_{j,k} \mathbf{d'}_{j,k} / n,$$

where $\mathbf{d}_{j,k}$ is a vector of wavelet coefficients of sample path of OFBM,

$$\mathbf{d}_{j,k} = \begin{pmatrix} \int_{\mathbb{R}} X_{s,1} \psi_{j,k}(s) ds \\ \int_{\mathbb{R}} X_{s,2} \psi_{j,k}(s) ds \end{pmatrix}.$$

As n increases, Q_j gets closer to its expectation,

$$Q_j \sim 2^{-j(H+1/2)} c(\psi, H) 2^{-j(H+1/2)}.$$

It is shown later that the eigenvalues of Q_j satisfy $\rho(Q_j)_i \sim c_i 2^{j(2h_i+1)}, i = 1, 2$, for some constants c_i, where $\rho(Q_j)_i$ are the eigenvalues of Q_j. The estimator $h_{i,j}$ of h_i is the log regression of $\rho(Q_j)_i$ on j.

$$h_{i,j} = \frac{1}{2} \sum_{\ell = j_L}^{j} w_\ell \log_2 \rho(Q_\ell)_i - \frac{1}{2}, \tag{2.1}$$

where

$$\sum_{\ell = j_L}^{j} w_\ell = 0, \sum_{\ell = j_L}^{j} \ell w_\ell = 1.$$

Note that the eigenvalues of Q_j are the same as those of $P^{-1} Q_j P$, and

$$P^{-1} 2^{-j(H+1/2)} \mathbf{d}_{0,k} = \begin{pmatrix} 2^{-j(h_1+1/2)} & 0 \\ 0 & 2^{-j(h_2+1/2)} \end{pmatrix} P^{-1} \mathbf{d}_{0,k}.$$

Since our estimators use eigenvalues of the matrix Q_j and $\rho(Q_j) = \rho(P^{-1} Q_j P)$, it is assumed that the matrices P, P^{-1} were premultiplied on both sides of the matrix Q_j and assume that

$H = diag(h_1, h_2)$. i.e. $\rho(Q_j) = \rho(P^{-1}Q_jP)$ so we analyze $P^{-1}Q_jP$ instead. Therefore, from now on, we assume $H = diag(h_1, h_2)$ and

$$var(\mathbf{d}_{j,k}) = \begin{pmatrix} 2^{j(h_1+1/2)} & 0 \\ 0 & 2^{j(h_2+1/2)} \end{pmatrix} P^{-1}\mathbf{c}(\psi, H)P \begin{pmatrix} 2^{j(h_1+1/2)} & 0 \\ 0 & 2^{j(h_2+1/2)} \end{pmatrix}$$

without loss of generality.

Proposition 2.2.1. *Let $2^{j-N} \to m$, where m is a constant, $m \in [0,1]$, as $n \to \infty$. Then as $n \to \infty$,*

$$2^{j/2}\{vec(V_j(Q_j - EQ_j)V_j)\} \to_d N(0, \Sigma_j),$$

where $V_j = \begin{pmatrix} 2^{j(h_1+1/2)} & 0 \\ 0 & 2^{j(h_2+1/2)} \end{pmatrix}$ *and the elements of* Σ_j *are*

$$\lim_{n\to\infty} \sum_{k,k'\in S_j} \frac{2^j}{n}\{E[(\mathbf{d}_{0,k}\mathbf{d'}_{0,k'})_{\ell,\ell'}]E[(\mathbf{d}_{0,k'}\mathbf{d'}_{0,k})_{p,p'}] + E[(\mathbf{d}_{0,k}\mathbf{d'}_{0,k'})_{\ell,p'}]E[(\mathbf{d}_{0,k'}\mathbf{d'}_{0,k})_{p,\ell'}]\},$$

$\ell, p, \ell' p' = 1, 2.$

Proof. Note that $\{\mathbf{d}_{0,k}; k \in S_j\}$ is a stationary Gaussian process for fixed j, and let

$$f_{\ell,p}(\mathbf{d}_{0,k}) := d_{0,k,\ell}d_{0,k,p} - E[d_{0,k,\ell}d_{0,k,p}], \quad \ell, p = 1, 2.$$

Then, $f_{\ell,p}$ has a generalized Hermite rank 2 (see (2.2) in [5]), and

$$2^{j(h_\ell+1/2)}2^{j(h_p+1/2)}(Q_j - EQ_j)_{\ell,p} \overset{f.d.}{=} \sum_{k\in S_j} \frac{d_{0,k,\ell}d_{0,k,p} - E[d_{0,k,\ell}d_{0,k,p}]}{n} := \sum_{k\in S_j} \frac{f_{\ell,p}(\mathbf{d}_{0,k})}{n}.$$

Note that for $\ell, p = 1, 2$,

$$\lim_{n\to\infty} \sum_{k\in S_j} \frac{2^j}{n}|r^{\ell,p}(k)|^2 < \infty,$$

where $r^{\ell,p}(k) = cov(d_{0,0,\ell}, d_{0,k,p})$, by $(P'4)$ and the fact that $S_j = \{\frac{i2^j}{n}, i = 1, 2, ..., n.\}$. Then, by Theorem 4 in [5],

$$n^{1/2} \sum_{k\in S_j} \frac{1}{n}\left[f_{1,1}\left(\frac{2^{j/2}\mathbf{d}_{0,k}}{n^{1/4}}\right), f_{1,2}\left(\frac{2^{j/2}\mathbf{d}_{0,k}}{n^{1/4}}\right), f_{2,2}\left(\frac{2^{j/2}\mathbf{d}_{0,k}}{n^{1/4}}\right)\right]$$

$$= 2^{j/2} \sum_{k\in S_j} \left(\frac{f_{1,1}(\mathbf{d}_{0,k})}{n}, \frac{f_{1,2}(\mathbf{d}_{0,k})}{n}, \frac{f_{2,2}(\mathbf{d}_{0,k})}{n}\right)$$

converges to multivariate normal distribution. □

Corollary 2.2.2. *Let $2^{j-N} \to m$. (see Proposition 2.2.1 for m) Then as $n \to \infty$,*

$$2^{j/2} \begin{pmatrix} vec(V_j(Q_j - EQ_j)V_j) \\ 2^{-1/2} vec(V_{j-1}(Q_{j-1} - EQ_{j-1})V_{j-1}) \\ \cdots \\ 2^{-j_L/2} vec(V_{j-j_L}(Q_{j-j_L} - EQ_{j-j_L})V_{j-j_L}) \end{pmatrix} \to_d N(0, \Sigma_j^*),$$

where Σ_j^* is a block matrix where the (m,m)-th block is Σ_{j-m+1}, and (m,n)-th block, $m > n$, consists of

$$\lim_{n\to\infty} 2^{(m-n)/2} 2^{(h_l+h_p+1)(m-n)} \sum_{k \in S_{j-m+1}, k' \in S_{j-n+1}} \frac{2^j}{n} \{E[(\mathbf{d}_{0,k}\mathbf{d'}_{m-n,k'})_{l,l'}] E[(\mathbf{d}_{0,k'}\mathbf{d'}_{m-n,k})_{p,p'}]$$
$$+ E[(\mathbf{d}_{0,k}\mathbf{d'}_{m-n,k'})_{l,p'}] E[(\mathbf{d}_{0,k'}\mathbf{d'}_{m-n,k})_{p,l'}]\}, \quad l,p,l',p' = 1,2.$$

Proof. Without loss of generality, let $j_L = j - 1$. By $(P'4)$,

$$(d_{j,k,1}, d_{j,k,2}, d_{j-1,k',1}, d_{j-1,k',2})_{k \in S_j, k' \in S_{j-1}}$$

is a vector valued stationary Gaussian process for a fixed j. Define $f_{l,p}^j$ and $f_{l,p}^{j-1}$ as in the above proposition. Since both of their generalized rank are 2, and by $(P'4)$

$$\lim_{n\to\infty} \sum_{k\in S_j} \frac{2^j}{n} cov(d_{0,0,\ell}, d_{0,k,p})^2 < \infty, \lim_{n\to\infty} \sum_{k \in S_j} \frac{2^j}{n} cov(d_{1,0,\ell}, d_{0,k,p})^2 < \infty,$$

for $\ell, p = 1, 2$. Therefore, the result is derived from Theorem 9 in [5].

□

Proposition 2.2.3. *If H is diagonizable and has real eigenvalues, then for $i = 1, 2$,*

$$|\log \rho(EQ_j)_i + j(2h_i + 1)\log 2 + c_i| = \begin{cases} O(2^{-2j(h_2 - h_1)}), & \text{if } h_1 \neq h_2 \\ 0, & \text{if } h_1 = h_2, \end{cases}$$

where $c_1 = a, c_2 = d - \frac{b^2}{a}$.

Proof. To calculate the eigenvalues of EQ_j, we only need to calculate the eigenvalues of the matrix,

$$\begin{pmatrix} 2^{-j(h_1+1/2)} & 0 \\ 0 & 2^{-j(h_2+1/2)} \end{pmatrix} \begin{pmatrix} a & b \\ b & d \end{pmatrix} \begin{pmatrix} 2^{-j(h_1+1/2)} & 0 \\ 0 & 2^{-j(h_2+1/2)} \end{pmatrix},$$

where

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} = Pc(\psi, H)P^{-1}$$

by Assumption(OFBM2) and (P2).

Let $a_{(j)} = 2^{-j(2h_1+1)}a$, $d_{(j)} = 2^{-j(2h_2+1)}d$, and $b_{(j)} = 2^{-j(h_1+1/2)}2^{-j(h_2+1/2)}b$. Then the eigenvalues $\rho(EQ_j)_i, i = 1, 2$ of EQ_j are

$$\frac{(a_{(j)} + d_{(j)}) \pm \sqrt{(a_{(j)} + d_{(j)})^2 - 4(a_{(j)}d_{(j)} - b^2_{(j)})}}{2}.$$

If $h_1 = h_2 = h$, the result follows since

$$\rho(EQ_j)_i = 2^{-j(2h+1)} \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - b^2)}}{2}. \tag{2.2}$$

Now assume $h_1 \neq h_2$. By Taylor's expansion,

$$\sqrt{(a_{(j)} - d_{(j)})^2 + 4b^2_{(j)}} = \sqrt{(a_{(j)} - d_{(j)})^2} + \frac{4b^2_{(j)}}{2\sqrt{(a_{(j)} - d_{(j)})^2 + \theta_j 4b^2_{(j)}}},$$

where $\theta_j \in [0, 1]$. Therefore

$$\rho(EQ_j)_1 = 2^{-j(2h_1+1)}\Big(a + \frac{4b^2}{4\sqrt{(a2^{-2j(h_1-h_2)} - d)^2 + \theta_j 4b^2 2^{-2j(h_1-h_2)}}}\Big). \tag{2.3}$$

Similarly,

$$\rho(EQ_j)_2 = 2^{-j(2h_2+1)}\Big(d - \frac{4b^2}{4\sqrt{(d2^{-2j(h_2-h_1)} - a)^2 + \theta_j 4b^2 2^{-2j(h_2-h_1)}}}\Big). \tag{2.4}$$

The result is derived by using the Mean value theorem and the fact that $2^{2j(h_2-h_1)} \to \infty$. □

Let

$$g_1(a,b,d) := 2^{-j(2h_1+1)}\Big(a + \frac{4b^2}{4\sqrt{(a2^{-2j(h_1-h_2)} - d)^2 + \theta_j 4b^2 2^{-2j(h_1-h_2)}}}\Big), \tag{2.5}$$

$$g_2(a,b,d) := 2^{-j(2h_2+1)}\Big(d - \frac{4b^2}{4\sqrt{(d2^{-2j(h_2-h_1)} - a)^2 + \theta_j 4b^2 2^{-2j(h_2-h_1)}}}\Big). \tag{2.6}$$

Proposition 2.2.4. *If H is diagonizable and has real eigenvalues, then*

$$2^{j/2}(\log\rho(Q_j)_i - \log\rho(EQ_j)_i) \to_d N(0, \Sigma^\rho)$$

for $i = 1, 2$, where $\Sigma^\rho = g^{(1)\prime}\Sigma g^{(1)}$, and where Σ is from Proposition 2.2.1, and $g^{(1)}$ is the vector of the first derivatives of (g_1, g_2).

Proof. Let

$$V_j E Q_j V_j = \begin{pmatrix} a & b \\ b & d \end{pmatrix}, \qquad V_j Q_j V_j = \begin{pmatrix} a_j & b_j \\ b_j & d_j \end{pmatrix}.$$

Note that

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} = Pc(\psi, H)P^{-1} \quad \text{and} \quad \begin{pmatrix} a_j & b_j \\ b_j & d_j \end{pmatrix} \overset{law}{=} \sum_{k=0}^{2^{j-1}} \mathbf{d}_{0,k}\mathbf{d'}_{0,k}/2^{j-1}.$$

Note that a_j, b_j, d_j are random variables which converge to a, b, d, respectively, with the convergence rate of $2^{(j-1)/2}$ by Proposition 2.2.1. From the previous Proposition 2.2.3 and delta method, the result is derived, since $\log\rho(Q_j)_i - \log\rho(EQ_j)_i = g_i(a_j, b_j, d_j) - g_i(a, b, d)$, which has derivative for each element. By Proposition 2.2.1, the result is derived with $\Sigma^\rho = g^{(1)\prime}\Sigma g^{(1)}$, where Σ is from Proposition 2.2.1, and $g^{(1)}$ is the vector of the first derivatives of (g_1, g_2) with three variables a, b, d. □

Corollary 2.2.5. *For $i = 1, 2$,*

$$2^{j/2}\begin{pmatrix} \log\rho(Q_j)_i - \log\rho(EQ_j)_i \\ \log\rho(Q_{j-1})_i - \log\rho(EQ_{j-1})_i \\ .. \\ \log\rho(Q_{j-j_L})_i - \log\rho(EQ_{j-j_L})_i \end{pmatrix} \to_d N(0, \Sigma^{\rho *}),$$

where $\Sigma^{\rho*}$ is a $(j_L+1)\times(j_L+1)$ matrix whose (ℓ,m)-th element is $\Sigma^{\rho*}_{\ell,m} = g^{(1)\prime}\Sigma^*_{(\ell,m)}g^{(1)}$, and $\Sigma^*_{(\ell,m)}$ is the (ℓ,m)-th block matrix in Corollary 2.2.2.

Proof. The result follows from Propositon 2.2.4 and Corollary 2.2.2. □

Define

$$h_{i,j} = -\frac{1}{2}\sum_{\ell=j_L}^{j} w_\ell \log_2 \rho(Q_\ell)_i - \frac{1}{2}, \tag{2.7}$$

$$h^E_{i,j} = -\frac{1}{2}\sum_{\ell=j_L}^{j} w_\ell \log_2 \rho(EQ_\ell)_i - \frac{1}{2}. \tag{2.8}$$

Proposition 2.2.6. *As $j\to\infty$, for $i=1,2$,*

$$2^{j/2}(h_{i,j} - h^E_{i,j}) \to_d N(0,\sigma^h),$$

where $\sigma^h = W'\Sigma^{\rho*}W$, and $W = (w_j/(2\log 2), ..., w_{j_L}/(2\log 2))'$

Proof. The result is derived from Corollary 2.2.5. □

Proposition 2.2.7. *For $i=1,2$,*

$$|h^E_{i,j} - h_i| = O_p\Big(\frac{c_{i,1}}{c_i}(f_1(j) - f_1(j_L))\Big),$$

where $f_1(j) = 2^{-j(2h_2-2h_1)}$. If $h_2 - h_1 \to 0$,

$$|h^E_{i,j} - h_i| = \frac{c_{i,1}}{c_i}(h_2-h_1)2^{-2j_L(h_2-h_1)} + o((h_2-h_1)2^{-2j_L(h_2-h_1)})$$

for $c_{1,1} = 2b^2/a^2$, $c_{2,1} = 2b^2d/a^3$.

Proof. From (2.3),(2.4) in Proposition 2.2.3,

$$\rho(EQ_j)_i = 2^{-j(2h_i+1)}(c_i + c_{i,1}2^{-2j(h_1-h_2)} + o(2^{-2j(h_1-h_2)})).$$

Therefore

$$\log\rho(EQ_j)_i = -j(2h_i+1)\log 2 + \log c_i + \frac{c_{i,1}}{c_i}2^{-2j(h_2-h_1)} + o(2^{-2j(h_2-h_1)}) \tag{2.9}$$

$$h^E_{i,j} = h_i - \frac{c_{i,1}}{c_i}(f_1(j) - f_1(j_L)). \tag{2.10}$$

If $h_2 - h_1$ is small,

$$2^{-2j(h_2-h_1)} = 2^{-2j_L(h_2-h_1)} + 2^{-2j_L(h_2-h_1)}(-2j(h_2-h_1) + 2j_L(h_2-h_1))\log 2$$
$$+ o(2^{-2j_L(h_2-h_1)}(-2j(h_2-h_1) + 2j_L(h_2-h_1)))$$

and

$$\begin{aligned} h_{i,j}^E - h_i &= -\frac{1}{2}\sum_{\ell=j_L}^{j_U} w_\ell \log_2 \rho(EQ_\ell)_i - .5 - h_i \\ &= -\frac{c_{i,1}}{c_i}(h_2-h_1)2^{-2j_L(h_2-h_1)} + o(-(h_2-h_1)2^{-2j_L(h_2-h_1)}). \end{aligned}$$

□

2.2.2 Observing discrete sample paths from OFBM

With the discrete sample path of OFBM $\mathbf{X}_i^n = (X_{i,1}^n, X_{i,2}^n) = (X_{i/n,1}, X_{i/n,2}), i = 1, ..., n$, define the following:

$$\tilde{\mathbf{d}}_{j,k}^n = \sum_{q=0}^{S2^{N-j}-1} \Big(\int_{k/2^j+q/2^N}^{k/2^j+(q+1)/2^N} \psi_{j,k}(t)dt\Big)\mathbf{X}_{k2^{N-j}+q}^n \tag{2.11}$$

$$\tilde{Q}_j^n = \sum_{k\in S_j} \frac{\tilde{\mathbf{d}}_{j,k}^n \tilde{\mathbf{d}}'^n_{j,k}}{n} \tag{2.12}$$

for $j \leq \log_2 n$ and $2^N = n$. Note that for fixed j, $\{\tilde{\mathbf{d}}_{j,k}^n\}_k$ is a stationary Gaussian process with mean zero and the covariance between $\tilde{\mathbf{d}}_{j,k}^n$ and $\tilde{\mathbf{d}}_{j,k'}^n$ decays hyperbolically fast as $|k - k'| \to \infty$ similarly to (P4). For these reasons, the analogous to Proposition 2.2.1 holds for $\tilde{Q}_j$.

Proposition 2.2.8. *Let $2^{j-N} \to m$. (m in Proposition 2.2.1.) Then as $n \to \infty$,*

$$2^{j/2}\{vec(V_j(\tilde{Q}_j^n - E\tilde{Q}_j^n)V_j)\} \to_d N(0, \tilde{\Sigma}_j),$$

where the elements of $\tilde{\Sigma}_j$ are defined in (2.15).
If $j << \frac{2h_1}{.5+2h_1}\log_2 n$, then

$$2^{j/2}\{vec(V_j(\tilde{Q}_j^n - EQ_j)V_j)\} \to_d N(0, \tilde{\Sigma}_j).$$

Proof. Note that

$$\tilde{\mathbf{d}}^n_{j,k} = \sum_{q=0}^{S2^{N-j}-1} \Big(\int_{k/2^j+q/2^N}^{k/2^j+(q+1)/2^N} \psi_{j,k}(t)dt \Big) \mathbf{X}^n_{k2^{N-j}+q} \tag{2.13}$$

$$\overset{f.d.}{=} 2^{-j(H+1/2)} \sum_{q=0}^{S2^{N-j}-1} \Big(\int_{q2^j/2^N}^{(q+1)2^j/2^N} \psi_{0,0}(t)dt \Big) \mathbf{X}^{2^{N-j}}_{k2^{N-j}+q}. \tag{2.14}$$

Let $\mathbf{d}^{n*}_{0,k} = \sum_{q=0}^{S2^{N-j}-1} \Big(\int_{q2^j/2^N}^{(q+1)2^j/2^N} \psi_{0,0}(t)dt \Big) \mathbf{X}^{2^{N-j}}_{k2^{N-j}+q}$. Then $E(\mathbf{d}^{n*}_{0,k}\mathbf{d}'^{n*}_{0,k'})$ is hyperbolically decaying as $|k-k'| \to \infty$, and the limit of $\mathbf{d}^{n*}_{0,k}$ exists in almost sure sense.

$$\mathbf{d}^*_{0,k} := \lim_n \mathbf{d}^{n*}_{0,k} = \begin{cases} \mathbf{d}_{0,k}, & \text{if } m = 0 \\ \sum_{q=0}^{Sm-1} \Big(\int_{qm}^{(q+1)m} \psi(t)dt \Big) \mathbf{X}^m_{km+q} & \text{if } m \in (0,1]. \end{cases}$$

Therefore, the result follows with the elements of $\tilde{\Sigma}_j$ from

$$\lim_{n\to\infty} \sum_{k,k'\in S_j} \frac{2^j}{n} \{ E[(\mathbf{d}^*_{0,k}\mathbf{d}'^*_{0,k'})_{\ell,\ell'}] E[(\mathbf{d}^*_{0,k'}\mathbf{d}'^*_{0,k})_{p,p'}] + E[(\mathbf{d}^*_{0,k}\mathbf{d}'^*_{0,k'})_{\ell,p'}] E[(\mathbf{d}^*_{0,k'}\mathbf{d}'^*_{0,k})_{p,\ell'}] \}, \tag{2.15}$$

$\ell, p, \ell', p' = 1, 2$. Note that

$$\begin{aligned} V_j E\tilde{Q}^n_j V_j - V_j E Q_j V_j &= E(\mathbf{d}^{n*}_{0,k}\mathbf{d}'^{n*}_{0,k}) - E(\mathbf{d}_{0,k}\mathbf{d}'_{0,k}) \\ &= \sum_{q',q=0}^{S2^{N-j}-1} \Big(\int_{q2^j/2^N}^{(q+1)2^j/2^N} \int_{q'2^j/2^N}^{(q'+1)2^j/2^N} \psi_{0,0}(t)\psi_{0,0}(s) G_{j,n}(t,s,q,q') dt ds \Big) \end{aligned}$$

where

$$G_{j,n} = |t-s|^H \Gamma(1,1) |t-s|^{H'} - |(q-q')2^j/2^N|^H \Gamma(1,1) |(q-q')2^j/2^N|^{H'}$$

and

$$\begin{aligned} (G_{j,n}(t,s,q,q'))_{1,1} &< 2(2^j/2^N)^{2h_1} \\ (G_{j,n}(t,s,q,q'))_{2,2} &< 2(2^j/2^N)^{2h_2} \\ (G_{j,n}(t,s,q,q'))_{1,2} &< 2(2^j/2^N)^{h_1+h_2} \end{aligned}$$

for $q2^j/2^N \le t \le (q+1)2^j/2^N, q'2^j/2^N \le s \le (q'+1)2^j/2^N$. Therefore

$$(V_j E\tilde{Q}_j^n V_j - V_j EQ_j V_j)_{1,1} < c(2^j/2^N)^{2h_1} \tag{2.16}$$

$$(V_j E\tilde{Q}_j^n V_j - V_j EQ_j V_j)_{2,2} < c(2^j/2^N)^{2h_2} \tag{2.17}$$

$$(V_j E\tilde{Q}_j^n V_j - V_j EQ_j V_j)_{1,2} < c(2^j/2^N)^{h_1+h_2}. \tag{2.18}$$

If $j << \frac{2h_1}{.5+2h_1}\log_2 n$, $2^{j/2}(2^j/2^N)^{2h_1} \to 0$, the result is derived. □

Analogous to Corollary 2.2.2 is provided below for discrete case.

Corollary 2.2.9. *Let $2^{j-N} \to m$. (see Proposition 2.2.1 for m) Then as $n \to \infty$,*

$$2^{j/2}\begin{pmatrix} vec(V_j(\tilde{Q}_j^n - E\tilde{Q}_j^n)V_j) \\ 2^{-1/2}vec(V_{j-1}(\tilde{Q}_{j-1}^n - E\tilde{Q}_{j-1}^b)V_{j-1}) \\ \dots \\ 2^{-j_L/2}vec(V_{j-j_L}(\tilde{Q}_{j-j_L}^n - E\tilde{Q}_{j-j_L}^n)V_{j-j_L}) \end{pmatrix} \to_d N(0, \tilde{\Sigma}_j^*),$$

where $\tilde{\Sigma}_j^*$ is a block matrix where the (m,m)-th block is $\tilde{\Sigma}_{j-m+1}$, and (m,n)-th block, $m > n$, consists of

$$\lim_{n\to\infty} 2^{(m-n)/2}2^{(h_\ell+h_p+1)(m-n)} \sum_{k\in S_{j-m+1}, k'\in S_{j-n+1}} \frac{2^j}{n}\{E[(\mathbf{d}_{0,k}^*\mathbf{d'}_{m-n,k'}^*)_{\ell,\ell'}]E[(\mathbf{d}_{0,k'}^*\mathbf{d'}_{m-n,k}^*)_{p,p'}]$$
$$+E[(\mathbf{d}_{0,k}^*\mathbf{d'}_{m-n,k'}^*)_{\ell,p'}]E[(\mathbf{d}_{0,k'}^*\mathbf{d'}_{m-n,k}^*)_{p,\ell'}]\}, \quad \ell, p, \ell', p' = 1, 2.$$

As in the previous case, P^{-1}, P are assumed to be premultiplied by each side of the matrix $\tilde{Q}_j^n$ and see the convergence rate of the estimator. This is the same as multiplying P^{-1} by $\mathbf{X}_i^n$. Therefore it is assumed that $H = diag(h_1, h_2)$. Let

$$\tilde{h}_{i,j} = \frac{1}{2}\sum_{\ell=j_L}^{j} w_\ell \log_2 \rho(\tilde{Q}_\ell^n)_i - \frac{1}{2}, \tag{2.19}$$

$$\tilde{h}_{i,j}^E = \frac{1}{2}\sum_{\ell=j_L}^{j} w_\ell \log_2 \rho(E\tilde{Q}_\ell^n)_i - \frac{1}{2}, \tag{2.20}$$

$i = 1, 2$, where $\sum_{\ell=j_L}^{j} w_\ell = 0, \sum_{\ell=j_L}^{j} \ell w_\ell = 1$. Now the convergence rate of the estimator of discrete data is as follows.

Theorem 2.2.10. *If the domain is fixed interval in $\mathbb{R}$, H is diagonizable where $h_1, h_2 \in (0,1)$ then,*

$$\tilde{h}^E_{i,j} - h_i = O(\frac{c_{i,1}}{c_i}(f_1(j) - f_1(j_L)) + O((\mathbf{c} + \mathbf{c}^*)'(\mathbf{f}_2(j) - \mathbf{f}_2(j_L))) \tag{2.21}$$

for $i = 1, 2$, where $f_1(j) = 2^{-j(2h_2 - 2h_1)}, \mathbf{f}_2(j) = ((\frac{2^j}{n})^{2h_1}, (\frac{2^j}{n})^{2h_1} + (\frac{2^j}{n})^{2h_2}, (\frac{2^j}{n})^{2h_2})'$, $\mathbf{c} = \frac{c'_i}{c_i}$ where $c'_i$is a vector of first derivatives of $c_i(a, b, d)$ with respect to a, b, d, and $\mathbf{c}^*$ is a vector of first derivatives of $\frac{c_{i,1}}{c_i} 2^{-2j_L(h_2 - h_1)}$ with respect to a, b, d.

Proof. Set a_j, b_j, d_j from $\mathbf{v}'_j E Q_j \mathbf{v}_j$, a^n_j, b^n_j, d^n_j from $\mathbf{v}'_j E \tilde{Q}^n_j \mathbf{v}_j$ as in Proposition 2.2.4. The convergence rates of $|a_j - a^n_j|, |d_j - d^n_j|$ and $|b_j - b^n_j|$ are $(2^j/2^N)^{2h_1}, (2^j/2^N)^{2h_2}$ and $(2^j/2^N)^{2h_1} + (2^j/2^N)^{2h_2}$, respectively by (2.16-2.18). Therefore similar methods to Propositions 2.2.3,2.2.7, and by (2.9),(2.10),

$$\tilde{h}^E_{i,j} - h^E_{i,j} = O((\mathbf{c} + \mathbf{c}^*)'(\mathbf{f}_2(j) - \mathbf{f}_2(j_L)).$$

Since $h^E_{i,j} - h_i = O(f_1(j) - f_1(j_L))$ from Proposition 2.2.7, the result follows. □

Theorem 2.2.11. *As $j \to \infty$, for $i = 1, 2$,*

$$2^{j/2}(\tilde{h}_{i,j} - \tilde{h}^E_{i,j}) \to_d N(0, \tilde{\sigma}^h),$$

where $\tilde{\sigma}^h = W'\tilde{\Sigma}^{\rho *} W$, and $\tilde{\Sigma}^{\rho *}$ is analogous with $\Sigma^{\rho *}$ from $\tilde{\Sigma}$ instead of Σ.

Remark 1 *(1) $O_p((f_1(j) - f_1(j_L)))$ is getting smaller as $|h_2 - h_1|$ is getting smaller or j is getting larger.*

(2) $O_p(f_2(j) - f_2(j_L))$ is getting smaller as h_1 is increasing or j is getting smaller.

(3) If $\frac{b^2}{ad}$ in $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ is close to one, then the bias of $\tilde{h}_{2,j}$ is large, especially for larger j, since $f_2(j)$ and c_2^{-1} are large. Since

$$\begin{aligned}\begin{pmatrix} a & b \\ b & d \end{pmatrix} &= P'c(\psi, H)P \\ &= \frac{1}{2}\int \psi(t)\psi(s) \begin{pmatrix} |t-s|^{h_1} & 0 \\ 0 & |t-s|^{h_2} \end{pmatrix} P'E(X_H(1)X_H(1)')P \begin{pmatrix} |t-s|^{h_1} & 0 \\ 0 & |t-s|^{h_2} \end{pmatrix} dtds,\end{aligned}$$

therefore for fixed h_1, h_2, $\frac{b^2}{ad} = c\frac{M_{12}^2}{M_{11}M_{22}}$ *where* $M := P'E(X_H(1)X_H(1)')P$. *This implies that if the determinant of the matrix* $P'E(X_H(1)X_H(1)')P$ *is close to zero, the bias of the estimator* $\tilde{h}_{2,j}$ *will be large.*

2.2.3 Hurst estimator in discrete noisy data from OFBM

Suppose we observe

$$\begin{pmatrix} Y_{i,1}^n \\ Y_{i,2}^n \end{pmatrix} = \begin{pmatrix} X_{i,1}^n \\ X_{i,2}^n \end{pmatrix} + \begin{pmatrix} \xi_{i,1}^n \\ \xi_{i,2}^n \end{pmatrix}, \tag{2.22}$$

$i = 1, ..., n$, where the $\xi_{i,1}^n, \xi_{i,2}^n$ are centered noise terms. $\mathbf{Y}$ is observed in the interval $[0, 1]$, $\mathbf{X}_t = (X_{t,1}, X_{t,2})$ is operator fractional Brownian motion. Define the following terms:

$$\begin{aligned}
\hat{\mathbf{d}}_{j,k}^n &= \sum_{q=0}^{S2^{N-j}-1} \Big(\int_{k/2^j+q/2^N}^{k/2^j+(q+1)/2^N} \psi_{j,k}(t)dt \Big) \mathbf{Y}_{k2^{N-j}+q}^n, \\
\mathbf{e}_{j,k}^n &= \sum_{q=0}^{S2^{N-j}-1} \Big(\int_{k/2^j+q/2^N}^{k/2^j+(q+1)/2^N} \psi_{j,k}(t)dt \Big) \begin{pmatrix} \xi_{k2^{N-j}+q,1}^n \\ \xi_{k2^{N-j}+q,2}^n \end{pmatrix}, \\
\mathbf{v}_{j,k}^n &= \sum_{q=0}^{S2^{N-j}-1} \Big(\int_{k/2^j+q/2^N}^{k/2^j+(q+1)/2^N} \psi_{j,k}(t)dt \Big)^2 \begin{pmatrix} \sigma_1^n & \rho^n \\ \rho^n & \sigma_2^n \end{pmatrix}.
\end{aligned}$$

Note that $\hat{\mathbf{d}}_{j,k}^n = \tilde{\mathbf{d}}_{j,k}^n + \mathbf{e}_{j,k}^n$ and $\mathbf{v}_{j,k}^n$ is the covariance matrix of $\mathbf{e}_{j,k}^n$.

$$\hat{Q}_j^n = \sum_{k \in S_j} \frac{\hat{\mathbf{d}}_{j,k}\hat{\mathbf{d}}'_{j,k} - \mathbf{v}_{j,k}^n}{n}, \tag{2.23}$$

$$\hat{h}_{i,j} = \frac{1}{2} \sum_{\ell=j_L}^{j} w_\ell \log_2 \rho(\hat{Q}_\ell^n)_i - \frac{1}{2} \quad \text{for } i = 1, 2, \tag{2.24}$$

$$\hat{h}_{i,j}^E = \frac{1}{2} \sum_{\ell=j_L}^{j} w_\ell \log_2 \rho(E\hat{Q}_\ell^n)_i - \frac{1}{2} \quad \text{for } i = 1, 2. \tag{2.25}$$

Two different assumptions are made for the noise term.

ASSUMPTION A.

The noise terms $\{\boldsymbol{\xi}_i^n = (\xi_{i,1}^n, \xi_{i,2}^n)\}_i$ are mutually independent and independent of $\mathbf{X}$. Moreover, for $\ell = 1, 2$, $E_{H,\sigma}^n[\xi_{i,\ell}^n] = 0$, $E_{H,\sigma}^n[(\xi_{i,\ell}^n)^2] = \sigma_\ell^n$, $E_{H,\sigma}^n[\xi_{i,1}^n \xi_{i,2}^n] = \rho^n$, $\lim_n \sigma_\ell^n = \sigma_\ell$, $\lim_n \rho_n/(\sigma_1^n \sigma_2^n) = r$, $r \in [-1, 1]$ and $\sup_{i,n} E_{H,\sigma}^n[(\xi_{i,\ell}^n)^4] < \infty$.

ASSUMPTION B.

The noise terms $\{\boldsymbol{\xi}_i^n = (\xi_{i,1}^n, \xi_{i,2}^n)\}_i$ are martingale increments and independent of $\mathbf{X}$, $E_{H,\sigma}^n[\xi_{i,\ell}^n | F_{i/n}^n] = 0$. Moreover, $E_{H,\sigma}^n[(\xi_{i,\ell}^n)^2 | F_{i/n}^n]$, $E_{H,\sigma}^n[(\xi_{i,\ell}^n)^3 | F_{i/n}^n]$ and $E_{H,\sigma}^n[(\xi_{i,\ell}^n)^2 \xi_{i,p}^n | F_{i/n}^n]$ are constants $\sigma_\ell^n, c_{i,\ell}^n$, and $c_i'^n$, respectively. $E_{H,\sigma}^n[\xi_{i,1}^n \xi_{i,2}^n | F_{i/n}^n] = \rho^n$, $\lim_n \sigma_\ell^n = \sigma_\ell$, $\lim_n \rho_n/(\sigma_1^n \sigma_2^n) = r$, $r \in [-1, 1]$ and $\sup_{i,n} E_{H,\sigma}^n[(\xi_{i,\ell}^n)^4] < \infty$ for $\ell = 1, 2$.

Assumption A is a special case of Assumption B. Asymptotic distribution of the estimator will be proved under Assumption B. Note that $\hat{Q}_j^n - \tilde{Q}_j = \sum_{u=1}^3 \mathbf{r}_j^{n,(u)}/n$, with

$$\mathbf{r}_j^{n,(1)} = \sum_k (\mathbf{e}_{j,k}^n)(\mathbf{e}_{j,k}^n)' - \mathbf{v}_{j,k}^n, \qquad \mathbf{r}_j^{n,(2)} = \sum_k \mathbf{e}_{j,k}^n (\mathbf{d}_{j,k})' + \mathbf{d}_{j,k} (\mathbf{e}_{j,k}^n)',$$
$$\mathbf{r}_j^{n,(3)} = \sum_k \mathbf{b}_{j,k}^n (\mathbf{e}_{j,k}^n)' + \mathbf{e}_{j,k}^n (\mathbf{b}_{j,k}^n)'.$$

Since the expectation of $\mathbf{r}_j^{n,(1)}, \mathbf{r}_j^{n,(2)}, \mathbf{r}_j^{n,(3)}$ are zero, $E\hat{Q}_j^n = E\tilde{Q}_j^n$. Therefore for the bias of the estimator $\hat{h}_{i,j}$, we get the same result as in Theorem 2.2.10.

Proposition 2.2.12. *If the domain is a fixed interval in $\mathbb{R}$, H is diagonizable where $h_1, h_2 \in (0, 1)$ then, for $i = 1, 2$,*

$$\hat{h}_{i,j}^E - h_i = O\Big(\frac{c_{i,1}}{c_i}(f_1(j) - f_1(j_L))\Big) + O\Big((\mathbf{c} + \mathbf{c}^*)'(\mathbf{f_2}(j) - \mathbf{f_2}(j_L))\Big). \tag{2.26}$$

To find the variance of the estimator $\hat{h}_{i,j}$, note that

$$\hat{Q}_j^n = \tilde{Q}_j^n + \mathbf{r}_j^{n,(1)}/n + \mathbf{r}_j^{n,(2)}/n + \mathbf{r}_j^{n,(3)}/n. \tag{2.27}$$

The following result is obtained in a similar way to Proposition 2 in [26].

Proposition 2.2.13. *Under Assumption B*

$$|(\hat{Q}_j^n - \tilde{Q}_j^n)_{\ell,\ell}| = O_p(n^{-2}2^{j/2} \vee n^{-1/2}2^{-j(h_\ell+1)}),$$

$$|(\hat{Q}_j^n - \tilde{Q}_j^n)_{\ell,p}| = O_p(n^{-2}2^{j/2} \vee n^{-1/2}2^{-j(h_1+1)}),$$

for $\ell \neq p, \ell, p = 1, 2$.

Proof.

$$\begin{aligned}
E[(\mathbf{r}_j^{n,(1)})^2_{\ell,p}] &\leq c \sum_k E[(\mathbf{e}_{j,k}^n \mathbf{e}_{j,k}^{n\prime} - \mathbf{v}_{j,k}^n)^2_{\ell,p}] \\
&\leq c \sum_k \sum_{q=0}^{S2^{N-j}-1} \Big(\int_{k/2^j+q/2^N}^{k/2^j+(q+1)/2^N} \psi_{j,k}(t)dt \Big)^4 \\
&\times E\Big[\Big(\begin{pmatrix} {\xi^n_{k2^{N-j}+q,1}}^2 & \xi^n_{k2^{N-j}+q,1}\xi^n_{k2^{N-j}+q,2} \\ \xi^n_{k2^{N-j}+q,1}\xi^n_{k2^{N-j}+q,2} & {\xi^n_{k2^{N-j}+q,2}}^2 \end{pmatrix} - \begin{pmatrix} \sigma_1^n & \rho^n \\ \rho^n & \sigma_2^n \end{pmatrix} \Big)^2_{\ell,p} \Big] \\
&\leq cn \frac{n}{2^j} \frac{2^{2j}}{n^4}.
\end{aligned}$$

The first inequality is derived since $(\mathbf{e}_{j,k}^n)(\mathbf{e}_{j,k}^n)' - \mathbf{v}_{j,k}^n$ are uncorrelated when $|k - k'| \geq S$. The second inequality is from the Burkholder-Davis inequality since $(\mathbf{e}_{j,k}^n)(\mathbf{e}_{j,k}^n)' - \mathbf{v}_{j,k}^n$ is a sum of martingale increments. The third inequality is derived by the fact that $\xi_{i,l}^n$ has bounded fourth moment and $\int_{k/2^j+q/2^N}^{k/2^j+(q+1)/2^N} \psi_{j,k}(t)dt \leq c\frac{2^{j/2}}{n}$. Therefore, each element of the matrix $\mathbf{r}_j^{n,(1)}/n$ is of order $n^{-2}2^{j/2}$.

For $\mathbf{r}_j^{n,(2)}/n$ and $\mathbf{r}_j^{n,(3)}/n$, use the Cauchy-Swartz inequality and uncorelatedness of $e_{j,k}^n$ when $|k - k'| > S$, we obtain $\mathbf{r}_j^{n,(3)}/n \leq 2^{-j}n^{-1/2}(n^{-H}C + Cn^{-H})$ and $\mathbf{r}_j^{n,(2)}/n \leq 2^{-j/2}n^{-1/2}(C2^{-j(H+1/2)} + 2^{-j(H+1/2)}C)$. Since $2^{-j}n^{-1/2}(n^{-H}C + Cn^{-H}) \leq 2^{-j/2}n^{-1/2}(C2^{-j(H+1/2)} + 2^{-j(H+1/2)}C)$, the results are derived.

□

Theorem 2.2.14. *If the domain is a fixed interval in $\mathbb{R}$, H is diagonizable then, for $i = 1, 2$,*

$$\hat{h}_{i,j} - \hat{h}_{i,j}^E = O_p(c_{i,j,n}),$$

where

$$c_{i,j,n} = O_p(n^{-2}2^{j(2h_i+3/2)} \vee n^{-1/2}2^{jh_i} \vee 2^{-j/2}). \tag{2.28}$$

Proof. By (2.3),(2.4), the order of $|(\hat{Q}^n_j - E\hat{Q}^n_j)_{1,1}|$ matters for $i = 1$ and the orders of $|(\hat{Q}^n_j - E\hat{Q}^n_j)_{\ell,p}|, \ell, p = 1, 2$, matter for $i = 2$, respectively, since in (2.3) the denominator goes to infinity and in (2.4) it converges to a. Set $\tilde{a}^n_j, \tilde{b}^n_j, \tilde{d}^n_j$ from $\tilde{Q}^n_j$, $\hat{a}^n_j, \hat{b}^n_j, \hat{d}^n_j$ from $\hat{Q}^n_j$ as in Proposition 2.2.4. The convergence rate of $|\tilde{a}^n_j - \hat{a}^n_j|$ is $2^{j(2h_1+1)}|(\hat{Q}^n_j - \tilde{Q}^n_j)_{1,1}|$, the convergence rates of $|\tilde{d}^n_j - \hat{d}^n_j|$ and $|\tilde{b}^n_j - \hat{b}^n_j|$ are $2^{j(2h_2+1)}|(\hat{Q}^n_j - \tilde{Q}^n_j)_{2,2}|$ and $2^{j(h_2+h_1+1)}|(\hat{Q}^n_j - \tilde{Q}^n_j)_{1,2}|$. Since $E\hat{Q}^n_j = E\tilde{Q}^n_j$, and therefore $|\hat{Q}^n_j - E\hat{Q}^n_j| = |\hat{Q}^n_j - \tilde{Q}^n_j| + |\tilde{Q}^n_j - E\tilde{Q}^n_j|$, the result follows from Proposition 2.2.8 and Proposition 2.2.13. □

Remark 2 *Proposition 2.2.12 and Theorem 2.2.14 reveal that with the presence of the noise, the bias of the estimator $\hat{h}_{i,j}$ remains the same as in the case when discrete sample path is given without noise, i.e. the bias of $\hat{h}_{i,j}$ is the same as that of $\tilde{h}_{i,j}$. However, the standard error is much larger as it is given in (2.28), especially when the Hurst indices are big. Theorem 2.2.14 reveals that the bigger the Hurst indices are, the smaller the j should be chosen in order to reduce the variance of the estimator. But even so, the performance of the estimator is not good, since for small j, $2^{-j/2}$ is not small. This means that the larger the Hurst indices are, the worse the estimator performs.*

In practice, it is hard to calculate $\mathbf{v}^n_{j,k}$ since either $\left(\int_{k/2^j+q/2^N}^{k/2^j+(q+1)/2^N} \psi_{j,k}(t)dt\right)^2$ is hard to obtain or $\begin{pmatrix} \sigma^n_1 & \rho^n \\ \rho^n & \sigma^n_2 \end{pmatrix}$ is unknown or both. Next, two ways are provided to replace $\mathbf{v}^n_{j,k}$. Note that $\mathbf{v}^n_{j,k}$ is not changed according to k and it converges to $\frac{1}{n}\int \psi^2_{j,k}(t)dt \begin{pmatrix} \sigma^n_1 & \rho^n \\ \rho^n & \sigma^n_2 \end{pmatrix} = \frac{1}{n}\begin{pmatrix} \sigma^n_1 & \rho^n \\ \rho^n & \sigma^n_2 \end{pmatrix}$ as $n \to \infty$.

Replace $\mathbf{v}^n_{j,k}$ by $\mathbf{v}^{n*} := \frac{1}{n}\int \psi^2_{j,k}(t)dt \begin{pmatrix} \sigma^n_1 & \rho^n \\ \rho^n & \sigma^n_2 \end{pmatrix} = \frac{1}{n}\begin{pmatrix} \sigma^n_1 & \rho^n \\ \rho^n & \sigma^n_2 \end{pmatrix}$, and define

$$\hat{Q}^{n\prime}_j = \sum_{k\in S_j} \frac{\hat{\mathbf{d}}_{j,k}\hat{\mathbf{d}}'_{j,k} - \mathbf{v}^{n*}}{n}, \tag{2.29}$$

$$\hat{h}'_{i,j} = -\frac{1}{2}\sum_{\ell=j_L}^{j} w_\ell \log_2 \rho(\hat{Q}^{n\prime}_\ell)_i - \frac{1}{2} \quad \text{for } i = 1,2, \tag{2.30}$$

$$\hat{h}'^E_{i,j} = -\frac{1}{2}\sum_{\ell=j_L}^{j} w_\ell \log_2 \rho(E\hat{Q}^{n\prime}_\ell)_i - \frac{1}{2} \quad \text{for } i = 1,2. \tag{2.31}$$

Theorem 2.2.15. *If $\mathbf{v}^n_{j,k}$ is replaced by $\mathbf{v}^{n*}$, then for $i = 1,2$,*

$$\begin{aligned} \hat{h}'_{i,j} - \hat{h}'^E_{i,j} &= O_p(c_{i,j,n}) \\ \hat{h}'^E_{i,j} - h_i &= O\Big(\frac{c_{i,1}}{c_i}(f_1(j) - f_1(j_L)\Big) + O\Big((\mathbf{c}+\mathbf{c}^*)(\mathbf{f_2}(j) - \mathbf{f_2}(j_L))\Big) + O(2^j/n^2). \end{aligned}$$

Proof. Since

$$\hat{Q}^{n\prime}_j = \tilde{Q}^n_j + (\mathbf{v}^n_{j,k} - \mathbf{v}^{n*}) + \mathbf{r}^{n,(1)}_j/n + \mathbf{r}^{n,(2)}_j/n + \mathbf{r}^{n,(3)}_j/n \quad \text{and} \tag{2.32}$$

$$\mathbf{v}^n_{j,k} - \mathbf{v}^{n*} = O\Big(\frac{2^j}{n^2}\Big), \tag{2.33}$$

$$E\hat{Q}^{n\prime}_j = E\tilde{Q}^n_j + O(2^j/n^2). \tag{2.34}$$

Therefore $\hat{h}'^E_{i,j} - h_i = \tilde{h}^E_{i,j} - h_i + O(2^j/n)$. Since $\hat{Q}^{n\prime}_j = \hat{Q}^n_j - \mathbf{v}^n_{j,k} + \mathbf{v}^{n*}$, and $\mathbf{v}^n_{j,k} - \mathbf{v}^{n*}$ is not a random variable, the order of $\hat{h}'_{i,j} - \hat{h}'^E_{i,j}$ is the same as the order of $\hat{h}_{i,j} - \hat{h}^E_{i,j}$. □

Next we define different estimator by using the difference between $\hat{Q}^{n\prime}_j - \hat{Q}^{n\prime}_{j-1}$.

$$\hat{Q}^{n\prime\prime}_j = \hat{Q}^{n\prime}_j - \hat{Q}^{n\prime}_{j-1}, \tag{2.35}$$

$$\hat{h}''_{i,j} = -\frac{1}{2}\sum_{\ell=j_L}^{j} w_\ell \log_2 \rho(\hat{Q}^{n\prime\prime}_\ell)_i - \frac{1}{2} \quad \text{for } i = 1,2, \tag{2.36}$$

$$\hat{h}''^E_{i,j} = -\frac{1}{2}\sum_{\ell=j_L}^{j} w_\ell \log_2 \rho(E\hat{Q}^{n\prime\prime}_\ell)_i - \frac{1}{2} \quad \text{for } i = 1,2. \tag{2.37}$$

Note that

$$\hat{Q}^{n\prime}_j - \hat{Q}^{n\prime}_{j-1} = \sum_{k\in S_j} \frac{\hat{\mathbf{d}}_{j,k}\hat{\mathbf{d}}'_{j,k}}{n} - \sum_{k\in S_{j-1}} \frac{\hat{\mathbf{d}}_{j-1,k}\hat{\mathbf{d}}'_{j-1,k}}{n}.$$

Therefore, $\hat{h}''_{i,j}$ is the estimator that can be obtained without knowing $\mathbf{v}^n_{j,k}$ or $\mathbf{v}^{n*}$.

Theorem 2.2.16. *For $i = 1, 2$,*

$$\begin{aligned}\hat{h}''_{i,j} - \hat{h''}^E_{i,j} &= O_p(c'_{i,j,n}),\\ \hat{h''}^E_{i,j} - h_i &= O\Big(\frac{c_{i,1}}{c_i}(f_1(j) - f_1(j_L)\Big) + O\Big((\mathbf{c} + \mathbf{c}^*)(\mathbf{f_2}(j) - \mathbf{f_2}(j_L))\Big) + O(2^{j+1}/n^2),\end{aligned}$$

where

$$c'_{i,j,n} = O_p(2n^{-2}2^{j(2h_i+3/2)} \vee 2n^{-1/2}2^{jh_i} \vee 2^{-j/2}). \tag{2.38}$$

Proof. Note that

$$\hat{Q}^{n\prime\prime}_j = \tilde{Q}^n_j - \tilde{Q}^n_{j-1} + \mathbf{v}^n_{j,k} - \mathbf{v}^n_{j-1,k} + \mathbf{r}^{n,(1)}_j/n - \mathbf{r}^{n,(1)}_{j-1}/n + \mathbf{r}^{n,(2)}_j/n - \mathbf{r}^{n,(2)}_{j-1}/n + \mathbf{r}^{n,(3)}_j/n - \mathbf{r}^{n,(3)}_{j-1}/n$$

and

$$\begin{aligned}E\hat{Q}^{n\prime\prime}_j &= E\tilde{Q}^n_j - E\tilde{Q}^n_{j-1} + \mathbf{v}^n_{j,k} - \mathbf{v}^{n*} - \mathbf{v}^n_{j-1,k} + \mathbf{v}^{n*}\\ &= E\tilde{Q}^n_j - E\tilde{Q}^n_{j-1} + O(2^{j+1}/n^2)\end{aligned}$$

by (2.33). Since

$$vec(V_j(E\tilde{Q}^n_j - E\tilde{Q}^n_{j-1})V_j - V_j(EQ_j - EQ_{j-1})V_j) = \begin{pmatrix} O((2^j/2^N)^{2h_1}) \\ O((2^j/2^N)^{h_1+h_2}) \\ O((2^j/2^N)^{2h_2}) \end{pmatrix}$$

and

$$EQ_j - EQ_{j-1} = \begin{pmatrix} 2^{-j(h_1+1/2)} & 0 \\ 0 & 2^{-j(h_2+1/2)} \end{pmatrix}\begin{pmatrix} a' & b' \\ b' & d' \end{pmatrix}\begin{pmatrix} 2^{-j(h_1+1/2)} & 0 \\ 0 & 2^{-j(h_2+1/2)} \end{pmatrix},$$

where $a' = a(1 - 2^{2h_1+1})$, $b' = b(1 - 2^{h_1+h_2+1})$, $d' = d(1 - 2^{2h_2+1})$, using the same method as in Proposition 2.2.7 and Theorem 2.2.10, $\hat{h''}^E_{i,j} - h_i$ has the same order as $\tilde{h}^E_{i,j} - h_i$ with $a', b'd'$ instead of a, b, d. Since

$$V_j((\tilde{Q}^n_j - \tilde{Q}^n_{j-1}) - (E\tilde{Q}^n_j - E\tilde{Q}^n_{j-1}))V_j = O_P(2^{j/2})$$

by Corollary 2.2.9, and $\mathbf{r}^{n,(1)}_j/n - \mathbf{r}^{n,(1)}_{j-1}/n + \mathbf{r}^{n,(2)}_j/n - \mathbf{r}^{n,(2)}_{j-1}/n + \mathbf{r}^{n,(3)}_j/n - \mathbf{r}^{n,(3)}_{j-1}/n$ has the order twice as much as the order in Proposition 2.2.13 by (2.27), the result for $\hat{h}''_{i,j} - \hat{h''}^E_{i,j}$ follows the same way as in Theorem 2.2.15. $\square$

Remark 3 *It is revealed from Theorem 2.2.15 that replacing $\mathbf{v}^n_{j,k}$ by $\mathbf{v}^{n*}$ increases the bias of the estimator by $2^j/n^2$ which is small for small j. From $\hat{h}'_{i,j}$ to $\hat{h}''_{i,j}$, by using $\hat{Q}^{n'}_j$, there is a small increase in bias but significant increase in standard error as it is seen by comparing Theorem 2.2.15 and Theorem 2.2.16.*

2.2.4 Estimator for eigenvectors

Assume a discrete sample path of OFBM $\mathbf{X}^n_i = (X^n_{i,1}, X^n_{i,2}) = (X_{i/n,1}, X_{i/n,2})$ is observed for $i = 1, ..., n$. In Sections 2.2.1-2.2.3, to calculate the eigenvalues, it was assumed that eigenvector matrices P, P' were premultiplied both sides of $Q_j, \tilde{Q}^n_j, \hat{Q}^n_j$. Now, to calculate eigenvectors, we no longer have that assumption. Let $(\tilde{\mathbf{u}}^n_{i,j}, \rho(\tilde{Q}^n_j)_i), i = 1, 2$, be the eigenvector and eigenvalue of the matrix $\tilde{Q}^n_j$. By ASSUMPTION (OFBM2), one needs to know θ to obtain eigenvectors. To estimate θ, the eigenvector corresponding to the bigger eigenvalue is chosen, i.e. $\tilde{\mathbf{u}}^n_{1,j} = (\tilde{u}^n_{1,j,1}, \tilde{u}^n_{1,j,2})$. Since $\tilde{\mathbf{u}}^n_{1,j} \approx (\cos\theta, \sin\theta)$, the esitimator of θ is defined as follows:

$$\tilde{\theta}^n_j = \arctan\left(\frac{\tilde{u}^n_{1,j,2}}{\tilde{u}^n_{1,j,1}}\right)$$

Let θ^E_j be the estimator of θ calculated in the same way as above with the matrix $E\tilde{Q}^n_j$.

Theorem 2.2.17. *As $j \to \infty$,*

i)

$$2^{j(h_2-h_1+1/2)}(\tilde{\theta}^n_j - \theta^E_j) \to_d N(0, \sigma).$$

ii)

$$\theta_j^E - \theta = O(\frac{b_j^n}{a_j^n} 2^{j(h_1-h_2)}).$$

Proof. Note that by Proposition 2.2.8,

$$\tilde{Q}_j^n = 2^{-j(2h_1+1)} P \begin{pmatrix} \tilde{a}_j^n & \tilde{b}_j^n 2^{-j(h_2-h_1)} \\ \tilde{b}_j^n 2^{-j(h_2-h_1)} & \tilde{d}_j^n 2^{-2j(h_2-h_1)} \end{pmatrix} P'$$

$$E\tilde{Q}_j^n = 2^{-j(2h_1+1)} P \begin{pmatrix} a_j^n & b_j^n 2^{-j(h_2-h_1)} \\ b_j^n 2^{-j(h_2-h_1)} & d_j^n 2^{-2j(h_2-h_1)} \end{pmatrix} P'$$

and $\tilde{a}_j^n - a_j^n = O_p(2^{-j/2})$, $\tilde{b}_j^n - b_j^n = O_p(2^{-j/2})$, $\tilde{d}_j^n - d_j^n = O_p(2^{-j/2})$. Eigenvectors of matrix does not change even if a constant is multiplied by the matrix. Therefore $\tilde{\theta}_j^n, \theta_j^E$ are the ones calculated from

$$\tilde{Q}_j^n / \tilde{a}_j^n = P \begin{pmatrix} 1 & \tilde{b}_j^n / \tilde{a}_j^n 2^{-j(h_2-h_1)} \\ \tilde{b}_j^n / \tilde{a}_j^n 2^{-j(h_2-h_1)} & \tilde{d}_j^n / \tilde{a}_j^n 2^{-2j(h_2-h_1)} \end{pmatrix} P',$$

$$E\tilde{Q}_j^n / a_j^n = P \begin{pmatrix} 1 & b_j^n / a_j^n 2^{-j(h_2-h_1)} \\ b_j^n / a_j^n 2^{-j(h_2-h_1)} & d_j^n / a_j^n 2^{-2j(h_2-h_1)} \end{pmatrix} P',$$

respectively. Therefore the result for i) follows since $\tilde{Q}_j^n / \tilde{a}_j^n - E\tilde{Q}_j^n / a_j^n \to_d 0$ with

$$\begin{pmatrix} 2^{j(1/2+h_2-h_1)} & 0 \\ 0 & 2^{j(1/2+2(h_2-h_1))} \end{pmatrix} \begin{pmatrix} \tilde{b}_j^n / \tilde{a}_j^n 2^{-j(h_2-h_1)} - b_j^n / a_j^n 2^{-j(h_2-h_1)} \\ \tilde{d}_j^n / \tilde{a}_j^n 2^{-2j(h_2-h_1)} - d_j^n / a_j^n 2^{-2j(h_2-h_1)} \end{pmatrix} \to_d N(0, \Sigma).$$

ii) follows since

$$E\tilde{Q}_j^n / a_j^n - P \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} P' \to 0$$

with

$$\begin{pmatrix} b_j^n / a_j^n 2^{-j(h_2-h_1)} \\ d_j^n / a_j^n 2^{-2j(h_2-h_1)} \end{pmatrix} \to \mathbf{0}.$$

□

Remark 4 *(1) The bigger j is, the better precision $\tilde{\theta}_j^n$ has in terms of both bias and standard error. Therefore $j = \log_2 n$ is the best choice.*
(2) The bigger $h_2 - h_1$ is, the better the estimator $\tilde{\theta}_j^n$ is, for it has smaller bias and smaller standard error.
(3) For the standard error of the estimator $\tilde{\theta}_j$, j and $h_2 - h_1$ play a role, as they increase, the standard error decreases, whereas for the bias of the estimator $\tilde{\theta}_j$, $j, h_2 - h_1$ and b_j^n/a_j^n affect on the performance. Since by (2.13),(2.14)

$$\begin{pmatrix} a_j^n & b_j^n \\ b_j^n & d_j^n \end{pmatrix} = \sum_{q,q'=0}^{S2^{N-j}-1} \int_{q2^j/2^N}^{(q+1)2^j/2^N} \int_{q'2^j/2^N}^{(q'+1)2^j/2^N} \frac{1}{2}\psi(t)\psi(s)D_{q,q'}P'E(X_H(1)X_H(1)')PD_{q,q'}dtds, \tag{2.39}$$

where $D_{q,q'} = \begin{pmatrix} |(q-q')/2^{N-j}|^{h_1} & 0 \\ 0 & |(q-q')2^{N-j}|^{h_2} \end{pmatrix}$.
Therefore the matrix $M := P'E(X_H(1)X_H(1)')P$ also plays a role, since $b_j^n/a_j^n = cM_{1,2}/M_{1,1}$ for fixed H, j, n.

2.3 Simulation Results

Operator fractional Brownian motion was simulated with the circulant embedding method, [12], [28], with $t = i/n, i = 1, ..., n, n = 2^{13}, \theta = .2\pi$. For wavelet function, the second derivative of Gaussian pdf was used, which has the second vanishing moment. The simulation was repeated 60 times, resulting in 60 independent discrete sample paths of OFBM and $(\tilde{h}_{1,j,r}, \tilde{h}_{2,j,r})_{r=1}^{60}$. The R package “wmtsa" was used for wavelet transform of sample paths. In Table 2.1 and Table 2.2, $\mathscr{C} = \begin{pmatrix} .3 & .2 \\ .1 & .4 \end{pmatrix}$ was used, and $\mathscr{C} = \begin{pmatrix} .3 & .1 \\ .1 & .3 \end{pmatrix}$, $\mathscr{C} = \begin{pmatrix} .1 & .3 \\ .2 & .4 \end{pmatrix}$ for Table 2.3 and Table 2.4, respectively. Table 2.1 to Table 2.4 show the results comparing the eigenvalue method in this chapter with different sets of j and element-wise method, which uses the first element and the second element of the paths separately to estimate Hurst indices. The bias and standard error of $(\tilde{h}_{1,j,r}, \tilde{h}_{2,j,r})_{r=1}^{60}$ were recorded for different sets of $h_1, h_2 \in (0, 1)$.

In Table 2.1, it is seen that the estimation works better when $h_1, h_2 \in (.5, 1)$ than when $h_1, h_2 \in (0, .5)$. This was expected from Theorem 2.2.10 since $f_2(j)$ and $f_2(j) - f_2(j_L)$ are smaller when $h_1 > .5$. It is also noticeable that, for smaller h_i, smaller j works better than when larger j was used, which comes from the fact that for smaller h_i, $\mathbf{f}_2(j)$ gets larger dominating other term in (2.2.21), and to make $\mathbf{f}_2(j)$ smaller one needs to use smaller j so that $2^j/n$ becomes smaller. From Table 2.2, it is clear that the eigenvalue method works well even when the difference between h_2 and h_1 is as small as .05.

Table 2.3 shows similar patterns as in Table 2.1 as it shows that, for smaller h_i, $\tilde{h}_j$ with smaller j has smaller bias than when larger j was used. For larger h_i, $\tilde{h}_i$ with larger j performs better, which comes from the fact that, for large h_i, $f_1(j)$ needs to be smaller in (2.2.21). This can be seen as a trade off between $f_1(j)$ and $\mathbf{f}_2(j)$ in (2.2.21) and which term gets larger when h_i becomes smaller or larger.

In Table 2.4, one noticeable difference from previous tables is that the performance of $\tilde{h}_2$ is significantly worse than previous cases especially when h_1 is small. This shows the delicate relation between h_1 and h_2 and the dependency within variables. It arises from the fact that, in this case with $\mathscr{C} = \begin{pmatrix} .1 & .3 \\ .2 & .4 \end{pmatrix}$, c_2 is close to zero, which results in large bias in $\tilde{h}_2$ especially when h_1 is small since it makes both $\mathbf{f}_2(j)$ and c_2^{-1} large. This phenomena was expected from Theorem 2.2.10 and Remark 1. Note that for $\ell, p = 1, 2$,

$$M_{\ell,p} = (P' E[X_H(1) X_H(1)'] P)_{\ell,p} = (P' \mathscr{C} \mathscr{C}' P)_{\ell,p} m_{\ell,p}$$

where

$$m_{\ell,p} = \int \frac{|e^{itx} - 1|}{|x|} |x|^{-(h_\ell + h_p + 1)} dx.$$

Therefore for fixed h_1, h_2, the determinant of M is determined by that of $P'\mathscr{C}\mathscr{C}'P$ and with $\mathscr{C} = \begin{pmatrix} .1 & .3 \\ .2 & .4 \end{pmatrix}$, $det(P'\mathscr{C}\mathscr{C}'P) = 4$, whereas with $\mathscr{C} = \begin{pmatrix} .3 & .2 \\ .1 & .4 \end{pmatrix}, \begin{pmatrix} .3 & .1 \\ .1 & .3 \end{pmatrix}$, $det(P'\mathscr{C}\mathscr{C}'P) = 100, 64$, respectively.

Throughout Tables 2.1- 2.4, it is seen that with $j = 3, j_L = .8$, the estimator $\tilde{h}_{i,j}$ performs well in terms of both bias and standard error for any range of $h_i \in (0, 1)$, whereas for the estimator with element-wise method, the bias for h_1 and standard error for h_1, h_2 are small as those of $\tilde{h}_{i,j}$, but the bias for h_2 is close to $h_2 - h_1$. This implies that for element-wise method, both the estimators for h_1, h_2 are close to h_1.

In Table 2.5, the noise term was added to discrete sample paths with $t = i/n, i = 1, \ldots, n, n = 2^{13}, \mathscr{C} = \begin{pmatrix} .3 & .2 \\ .1 & .4 \end{pmatrix}, \theta = .2\pi$, and $\xi_i^n \overset{iid}{\sim} N(0, v), v = \begin{pmatrix} .2 & .1 \\ .1 & .2 \end{pmatrix}$. The simulation was repeated 60 times to obtain $(\hat{h}'_{i,j,r})_{r=1}^{60}, (\hat{h}''_{i,j,r})_{r=1}^{60}$, and the sample mean and the standard error of the estimators were recorded. Table 2.5 shows the results for $h_i < .5$ and $j = 8, j_L = 6$. Neither the larger set of j nor larger h_i could be used for the estimation of the Hurst indices since in these cases many "NA" were produced for $\hat{h}'_{i,j}, \hat{h}''_{i,j}$. This is from the fact that the variances of $\hat{a}_j^n, \hat{b}_j^n, \hat{d}_j^n$ are large as expected in Proposition 2.2.13, Theorems 2.2.14-2.2.16, making the eigenvalues of $\hat{Q}_j^{n\prime}$ negative which cannot be used for the logarithm for the estimation, i.e. $c_{i,j,n}, c'_{i,j,n}$ are too large in these cases to estimate h_i. From Table 2.5, it is seen that the means of $\hat{h}'_{i,j}, \hat{h}''_{i,j}$ are relatively close to h_i when compared to the estimator $\tilde{h}_{i,j}$ in the presence of noise. With the noise term, the mean of $\tilde{h}_{i,j}$ is far from h_i as seen in the last column in Table 2.5. Also note that the standard error of $\hat{h}'_{i,j}$ is smaller than that of $\hat{h}''_{i,j}$ as it is expected from Theorems 2.2.15-2.2.16.

For Table 2.6, the result was recorded for the eigenvector estimator, $\tilde{\theta}_j$, when $\mathscr{C} = \begin{pmatrix} .3 & .1 \\ .1 & .3 \end{pmatrix}$ with $t = i/n, i = 1, \ldots, n, n = 2^{13}$ as before but varying $\theta \in \{\pi/12, \pi/6, \pi/4\}$. As it was predicted in Theorem 2.2.17, it is seen in Table 2.6 that, the larger the difference between two Hurst indices, $h_2 - h_1$, the better the estimator performs in terms of both bias and standard error. Also, to estimate θ, it is the best to use the largest j possible, in this case it is 13, as it has the smallest bias and smallest standard error among the other possible j. It is also noticeable from Table 2.6 that the standard error is affected by j and $h_2 - h_1$, whereas the bias is affected by $j, h_1 - h_2$, and θ. From Table 2.6, the bias is getting smaller as θ moves from $\pi/12$ to $\pi/4$. This is from the fact that the matrix $M = P'E(X_H(1)X_H(1)')P$ is changed and it affects the bias of the $\tilde{\theta}_j$ as mentioned in Remark 3.

For $\theta = \pi/12, \pi/6, \pi/4$, $M_{1,2}/M_{1,1} = 0.217, 0.107, 510^{-17}$, respectively.

From the results above, the following estimation methods are recommended.

To estimate h_1, h_2:

i) When there is no error and the determinant of $E(X_H(1)X_H(1)^*)$ is not small: The set with larger j is better to estimate h_i when h_i is large (e.g. $h_i \geq .4$) since in this case both bias and standard error are small. The set with wide range of j is recommended when h_i is small, since in this case standard error can be made small and bias is not too large.

ii) When there is no error and the determinant of $E(X_H(1)X_H(1)^*)$ is small: For large h_i, the set of large j is recommended for it makes both bias and standard error small. For small h_i, either the set of small j or the set of wide range of j is suggested, but there is a trade off since the first set has large standard error with smaller bias and the second set has smaller standard error with large bias.

iii) When there is error in the process: It is recommend to use the set of smaller j, but it cannot be too small. As h_i is increasing, the estimation would be worse even with the smaller j, and a different estimation method is needed.

To estimate θ: Pick largest j possible, i.e. $j = \log_2 n$.

Table 2.1: Bias and standard error of $\tilde{h}_{i,j}$

j		$h_1 = .1$ $h_2 = .3$	$h_1 = .2$ $h_2 = .3$	$h_1 = .2$ $h_2 = .4$	$h_1 = .4$ $h_2 = .9$	$h_1 = .25$ $h_2 = .9$	$h_1 = .7$ $h_2 = .8$	$h_1 = .7$ $h_2 = .9$
13–11	$\tilde{h}_1$	.264(.02)	.156(.02)	.164(.02)	.063(.02)	.127(.02)	.019(.02)	.020(.02)
	$\tilde{h}_2$	.109(.02)	.103(.01)	.073(.02)	.019(.02)	.029(.02)	.018(.02)	.013(.02)
11–8	$\tilde{h}_1$	.067(.03)	.021(.03)	.021(.03)	.011 (.03)	.028(.03)	.008(.03)	.004(.03)
	$\tilde{h}_2$	.025(.04)	.025(.03)	.013(.03)	-.001(.03)	.010(.03)	.002(.03)	.012(.04)
8–6	$\tilde{h}_1$	.032(.10)	-.003(.09)	.007(.10)	-.004(.10)	.013(.09)	-.002(.09)	.026(.09)
	$\tilde{h}_2$	.009(.10)	.014(.12)	-.003(.09)	.018(.11)	.040(.10)	.021(.06)	-.014(.09)
6–4	$\tilde{h}_1$	-.003(.11)	.031(.14)	-.005(.11)	-.058(.14)	-.057(.13)	-.260(.21)	-.269(.20)
	$\tilde{h}_2$	.021(.13)	-.012(.13)	.022(.16)	-.375 (.35)	-.484(.33)	-.067(.25)	-.291(.27)
11–6	$\tilde{h}_1$	.050(.03)	.008(.03)	.015(.03)	.009(.04)	.024(.04)	.006(.03)	.013(.03)
	$\tilde{h}_2$	.016(.04)	.020(.04)	.005(.03)	.005(.04)	.020(.03)	.010(.03)	.001(.04)
Element	$\tilde{h}_1$	.054(.03)	.019 (.03)	.029(.04)	.009(.04)	.015(.03)	.007(.04)	.017(.04)
-wise(11-6)	$\tilde{h}_2$	.218(.03)	.097(.04)	.199(.03)	.496(.04)	.657(.03)	.087 (.04)	.184(.04)

Table 2.2: Bias and standard error of $\tilde{h}_{i,j}$

j		$h_1 = .2$ $h_2 = .25$	$h_1 = .35$ $h_2 = .4$	$h_1 = .7$ $h_2 = .75$	$h_1 = .85$ $h_2 = .9$
13–11	$\tilde{h}_1$	.162(.02)	.086(.02)	.024(.01)	.008 (.02)
	$\tilde{h}_2$	.129(.02)	.073(.02)	.016(.02)	.012(.02)
11–8	$\tilde{h}_1$	.020(.04)	.005(.03)	.012(.03)	-.005 (.02)
	$\tilde{h}_2$	.015(.03)	.010(.03)	-.003(.03)	.001 (.03)
8–6	$\tilde{h}_1$	.004(.12)	.011(.10)	-.010(.11)	-.004(.11)
	$\tilde{h}_2$	.072(.11)	.000(.09)	.013(.10)	.057(.09)
6–4	$\tilde{h}_1$	-.028(.10)	-.074(.15)	-.156(.22)	-.388(.25)
	$\tilde{h}_2$	.008(.12)	.029(.16)	-.097(.18)	-.167 (.19)
11–6	$\tilde{h}_1$	.015(.03)	.008(.04)	.003(.03)	-.004(.04)
	$\tilde{h}_2$	.032(.04)	.011(.04)	.003(.03)	.020(.04)
Element	$\tilde{h}_1$	.021(.04)	.006(.03)	.005(.04)	.003(.04)
-wise(11-6)	$\tilde{h}_2$	.052(.03)	.044(.04)	.041(.04)	.040(.04)

Table 2.3: Bias and standard error of $\tilde{h}_{i,j}$

j		$h_1 = .1$ $h_2 = .3$	$h_1 = .2$ $h_2 = .3$	$h_1 = .2$ $h_2 = .4$	$h_1 = .4$ $h_2 = .9$	$h_1 = .25$ $h_2 = .9$	$h_1 = .7$ $h_2 = .8$	$h_1 = .7$ $h_2 = .9$
13–11	$\tilde{h}_1$	.264(.01)	.161(.02)	.163(.02)	.067 (.02)	.132(.02)	.026(.02)	.022 (.02)
	$\tilde{h}_2$	.105(.02)	.108(.02)	.067(.02)	.013(.01)	.013(.02)	.013(.01)	.007(.01)
11–8	$\tilde{h}_1$	.069(.03)	.021(.03)	.021(.04)	.022 (.04)	.027(.04)	-.012(.03)	.009(.03)
	$\tilde{h}_2$	.012(.03)	.006(.04)	.015(.03)	.005(.04)	.007(.03)	.004(.03)	.017(.03)
8–6	$\tilde{h}_1$	.039(.13)	.028(.09)	.028(.11)	.002 (.09)	-.013(.11)	.030(.11)	.009(.10)
	$\tilde{h}_2$	.026(.10)	.021(.10)	.037(.10)	.044(.09)	.012(.12)	.045(.10)	-.009(.09)
6–4	$\tilde{h}_1$	.017(.14)	-.009(.15)	.004(.17)	-.062(.12)	-.049(.15)	-.179(.23)	-.286(.20)
	$\tilde{h}_2$	.007(.11)	-.018(.12)	.032(.16)	-.471 (.36)	-.463(.33)	-.061(.29)	-.241(.29)
11–6	$\tilde{h}_1$	.053(.04)	.021(.03)	.024(.04)	.018(.04)	.012(.04)	-.003(.04)	.009(.04)
	$\tilde{h}_2$	.018(.04)	.011(.04)	.025(.04)	.019(.04)	.009(.05)	.020(.04)	.009(.04)
Element	$\tilde{h}_1$	.040(.03)	.020(.04)	.019(.03)	.011(.04)	.018(.04)	.006(.04)	.003(.05)
-wise(11-6)	$\tilde{h}_2$	.219(.03)	.098(.03)	.198 (.04)	.501(.04)	.664(.03)	.088(.04)	.184 (.04)

Table 2.4: Bias and standard error of $\tilde{h}_{i,j}$

j		h_1 = .1 h_2 = .3	h_1 = .2 h_2 = .3	h_1 = .2 h_2 = .4	h_1 = .4 h_2 = .9	h_1 = .25 h_2 = .9	h_1 = .7 h_2 = .8	h_1 = .7 h_2 = .9
13–11	$\tilde{h}_1$	.261(.02)	.159(.02)	.161(.02)	.071 (.02)	.131(.02)	.022(.02)	.022(.01)
	$\tilde{h}_2$	.268(.02)	.203(.02)	.243(.01)	.159(.02)	.217(.01)	.030(.02)	.053(.02)
11–8	$\tilde{h}_1$	.065(.04)	.013(.03)	.021(.04)	.007 (.03)	.010(.04)	.002(.04)	.012(.04)
	$\tilde{h}_2$	.233(.03)	.076(.03)	.113(.03)	.036(.03)	.062(.04)	-.009(.04)	.003(.03)
8–6	$\tilde{h}_1$	.026(.09)	.029(.10)	.011(.11)	.055 (.12)	.030(.09)	-.017(.10)	-.005(.11)
	$\tilde{h}_2$	.105(.11)	.009(.10)	.032(.09)	.030(.09)	.021(.10)	.035(.10)	.017(.10)
6–4	$\tilde{h}_1$	.025(.11)	-.036(.13)	-.015(.10)	-.060(.15)	.027(.10)	-.229(.26)	-.222(.21)
	$\tilde{h}_2$	-.032(.19)	-.024(.15)	-.090(.18)	-.769 (.51)	-.870(.42)	-.294(.24)	-.752(.42)
11–6	$\tilde{h}_1$	.044(.03)	.015(.04)	.014(.03)	.023(.04)	.011(.04)	-.006(.03)	.008(.03)
	$\tilde{h}_2$	.179(.03)	.051(.04)	.078(.03)	.036(.04)	.044(.03)	.004(.04)	.007(.04)
Element	$\tilde{h}_1$	.060(.04)	.035(.03)	.042(.03)	.014(.04)	.017(.04)	.017(.04)	.015 (.03)
-wise(11-6)	$\tilde{h}_2$	.213(.04)	.096(.03)	.199 (.03)	.493(.04)	.656(.04)	.081(.04)	.172(.03)

Table 2.5: Mean and standard error of $\hat{h}_{i,j}$

	j(6-8)	$Q_j - Q_{j-1}$	with v(6-8)	without v(6-8)
h_1=.1	$\hat{h}_1$	.083(.17)	.080(.09)	.073 (.09)
h_2=.3	$\hat{h}_2$	.238(.20)	.265(.12)	.133(.10)
h_1=.2	$\hat{h}_1$	.179(.17)	.184(.10)	.161 (.10)
h_2=.4	$\hat{h}_2$	.340(.25)	.382(.17)	.069(.09)
h_1=.3	$\hat{h}_1$	.282(.14)	.290(.10)	.219(.09)
h_2=.45	$\hat{h}_2$	.510(.32)	.474(.22)	.025(.11)

Table 2.6: Bias and standard error of $\tilde{\theta}_j$

θ	j	$h_1 = .1$ $h_2 = .3$	$h_1 = .2$ $h_2 = .3$	$h_1 = .2$ $h_2 = .4$	$h_1 = .4$ $h_2 = .9$	$h_1 = .25$ $h_2 = .9$	$h_1 = .7$ $h_2 = .8$	$h_1 = .7$ $h_2 = .9$
$\pi/12$	13	.040(.001)	.147(.003)	.052 (.001)	.004(e-05)	.001(2e-05)	.172(.004)	.067 (.002)
	11	.071(.004)	.189(.010)	.082 (.004)	.009(5e-04)	.003(e-04)	.203(.009)	.091(.004)
	9	.108(.011)	.230(.023)	.113 (.012)	.018(2e-03)	.007(8e-04)	.234(.026)	.121(.013)
$\pi/6$	13	.020(.001)	.072(.003)	.026 (.001)	.002(9e-05)	4e-04(2e-05)	.084(.004)	.033 (.001)
	11	.035(.003)	.092(.009)	.041(.004)	.004(4e-04)	.001(e-04)	.101(.009)	.045 (.004)
	9	.051(.009)	.111(.019)	.056(.009)	.008(2e-03)	.003(6e-04)	.119(.018)	.060 (.010)
$\pi/4$	13	-6e-05(.001)	3e-04(.003)	-e-04 (.001)	5e-06(8e-05)	3e-07(2e-05)	5e-04(.003)	-2e-04 (.001)
	11	-2e-04(.003)	e-03(.008)	-e-04(.003)	3e-05(4e-04)	9e-06(e-04)	-2e-04(.009)	-e-04(.004)
	9	e-03(.009)	.002(.016)	.001(.010)	6e-05(e-03)	-2e-05(6e-04)	-.001(.020)	-.002(.009)

CHAPTER 3

COHERENCE OF MULTIVARIATE RANDOM FIELDS WITH STATIONARY INCREMENTS

3.1 Preliminaries

Let $\mathbf{X} = \{\mathbf{X}(t), t \in \mathbb{R}^d\}$ be a p-variate random field with $\mathbf{X}(t) = (X_1(t), \ldots, X_p(t))' \in \mathbb{C}^p$ as a column vector. Throughout we assume that $\mathbf{X}$ is a second order random field, that is $\mathbb{E}[|X_i(t)|^2] < \infty$ for all $t \in \mathbb{R}^d$ and $1 \le i \le p$. Here $|a|$ denotes the modulus of $a \in \mathbb{C}$.

We say that $\mathbf{X}$ has stationary increments (or $\mathbf{X}$ is intrinsically stationary) in the weak sense, if

$$\mathbb{E}[\mathbf{X}(t) - \mathbf{X}(t-r)] = \mathbf{m}(r)$$

for all $t, r \in \mathbb{R}^d$, and

$$\mathbb{E}[(\mathbf{X}(t+h) - \mathbf{X}(t+h-r_1))(\overline{\mathbf{X}(t) - \mathbf{X}(t-r_2)})'] = \mathscr{D}(h; r_1, r_2)$$

for all $t, h, r_1, r_2 \in \mathbb{R}^d$. Here $\mathbf{m}(r)$ is called the mean vector of the field $\mathbf{X}$ and the matrix-valued function $\mathscr{D}(h; r_1, r_2)$ is called the structure function of the field $\mathbf{X}$, respectively. We refer to [27, 29, 52] for historical accounts of the terminology. Here, we will adapt Yaglom's result for generalized multivariate random fields to our setting.

Theorem 3.1.1 ([52, Thm. 7])**.** *Let $\{\mathbf{X}(t), t \in \mathbb{R}^d\}$ be a second order p-variate random field with stationary increments, mean vector $\mathbf{m}(r)$ and structure function $\mathscr{D}(h; r_1, r_2)$. Then there exist a $p \times d$ matrix $\mathscr{A}$ with complex entries, a $p \times p$ matrix of complex measure $\mathscr{F}$ on $\mathbb{R}^d_* = \mathbb{R}^d \setminus \{0\}$, and a $p \times p$ block matrix $\boldsymbol{\mathcal{U}}$ with $d \times d$ block matrix $\mathscr{U}_{ij}$ such that*

$$\mathbf{m}(r) = \mathscr{A} r,$$

and

$$\mathscr{D}(h; r_1, r_2) = \int_{\mathbb{R}^d_*} e^{\mathrm{i}\lambda \cdot h}(1 - e^{-\mathrm{i}\lambda \cdot r_1})(1 - e^{\mathrm{i}\lambda \cdot r_2}) \mathscr{F}(\mathrm{d}\lambda) + \boldsymbol{\mathcal{U}} r_1 \cdot r_2.$$

Here, each entry $F_{ij}(\mathrm{d}\lambda)$ of the measure $\mathscr{F}(\mathrm{d}\lambda)$ satisfies

$$\int_{\mathbb{R}^d_*} \frac{|\lambda|^2}{1+|\lambda|^2} F_{ij}(\mathrm{d}\lambda) < \infty, \tag{3.1}$$

and $\mathscr{F}(S)$ is Hermitian non-negative definite for all Borel set $S \in \mathbb{R}^d_*$; the $d \times d$ matrices satisfies $\mathscr{U}_{ij} = \mathscr{U}^*_{ji}$, where $*$ denotes the Hermitian conjugate. Let $u_{ij}(k,\ell)$ be the entries of $\mathscr{U}_{ij}$, then $\sum_{i,j=1}^{p} \sum_{k,\ell=1}^{d} u_{ij}(k,\ell)\alpha_{ik}\overline{\alpha}_{j\ell} \geq 0$ for any α_{ik}, $i = 1, ..., p$, $k = 1, ..., d$.

The matrix-valued measure $\mathscr{F}$ is called the spectral measure of $\mathbf{X}$ which plays a crucial role in the sequel. There is a general stochastic representation theorem for $\mathbf{X}$. For the sake of simplicity, we assume throughout that $\mathscr{A} = 0$, $\boldsymbol{\mathcal{U}} = 0$, $\mathbf{X}(0) = 0$. In such a case, the representation of $\mathbf{X}$ is simpler to state.

Theorem 3.1.2 ([52, Thm. 9])**.** *Let $\{\mathbf{X}(t), t \in \mathbb{R}^d\}$ be a second order p-variate random field with stationary increments, mean vector $\mathbf{m}(r) \equiv 0$ and structure function $\mathscr{D}(h; r_1, r_2)$ as in Theorem 3.1.1 such that $\mathscr{A} = 0$, $\boldsymbol{\mathcal{U}} = 0$ and $\mathbf{X}(0) = 0$. Then there exists a vector-valued complex random measure $\mathbf{Z}(\mathrm{d}\lambda) = (Z_1(\mathrm{d}\lambda), \ldots, Z_p(\mathrm{d}\lambda))'$, such that*

$$\mathbf{X}(t) = \int_{\mathbb{R}^d_*} (e^{\mathrm{i}t\cdot\lambda} - 1)\mathbf{Z}(\mathrm{d}\lambda), \tag{3.2}$$

where $\mathbb{E}[Z_i(S_1)\overline{Z_j(S_2)}] = F_{ij}(S_1 \cap S_2)$ for any Borel $S_1, S_2 \subset \mathbb{R}^d_*$.

Reciprocally, if $\mathscr{F}$, the control measure of $\mathbf{Z}$, satisfies (3.1)*, then the random field defined as in* (3.2) *is a second order random field with stationary increments.*

Afterward, $\mathbf{Z}$ is referred to as the random spectral measure of $\mathbf{X}$.

3.2 Coherence: definition and basic properties

Similarly to the stationary case considered by Kleiber [30], the coherence between the component processes of a multivariate random field with stationary increments can be defined as follows.

Let $F_{ij}(\mathrm{d}\lambda)$ be the ij-th entry of the spectral measure $\mathscr{F}(\mathrm{d}\lambda)$. Assume that $F_{ij} \ll \text{Leb}$ with non-vanishing density $f_{ij}(\lambda)$, where Leb is the Lebesgue measure.

Definition 3.2.1. The coherence of component i and component j of the field $\mathbf{X}$ at the frequency λ is defined to be

$$\gamma_{ij}(\lambda) := \frac{f_{ij}(\lambda)}{\sqrt{f_{ii}(\lambda) f_{jj}(\lambda)}}. \tag{3.3}$$

We remark that, if $\{\mathbf{Y}(t), t \in \mathbb{R}^d\}$ is a second order stationary p-variate random field with spectral measure $\mathscr{F}(\mathrm{d}\lambda)$ whose entries are necessarily finite measures, by the multivariate extension of Bochner's theorem (Cramér [16]), then the random field $\{\mathbf{X}(t), t \in \mathbb{R}^d\}$ defined by $\mathbf{X}(t) = \mathbf{Y}(t) - \mathbf{Y}(0)$ has stationary increments with the same spectral measure $\mathscr{F}(\mathrm{d}\lambda)$. It follows from Kleiber [30] that (3.3) is the same as the coherence between the components Y_i and Y_j of $\mathbf{Y}$.

Let us describe some general properties of the coherence function. Since $\mathscr{F}(S)$ is Hermitian non-negative definite, we know $|\gamma(\lambda)| \le 1$. Values of $|\gamma(\lambda)|$ near unity indicates strong linear relationship between X_i and X_j at particular frequency bands.A straightforward way to construct a legitimate spectral density matrix is the following.

Proposition 3.2.1. *Let $f \in L^1(\mathbb{R}^d_*, |\lambda|^2(1+|\lambda|^2)^{-1}\mathrm{d}\lambda)$ be non-vanishing and $\mathscr{C} = (c_{ij})$ be a $p \times p$ Hermitian non-negative definite matrix. Define $f_{ij}(\lambda) = c_{ij} f(\lambda)$. Then the measure $\mathscr{F}(\mathrm{d}\lambda) = (f_{ij}(\lambda)\mathrm{d}\lambda)$ is Hermitian non-negative definite, and*

$$\gamma_{ij}(\lambda) = \frac{c_{ij}}{\sqrt{c_{ii} c_{jj}}}.$$

When the spectral densities f_{ij} are constructed as products of square integrable functions with respect to the measure $|\lambda|^2(1+|\lambda|^2)^{-1}\mathrm{d}\lambda$, the coherence is constant.

Proposition 3.2.2. *For each $1 \le i \le p$, let $f_i : \mathbb{R}^d \to [0, \infty)$ be non-vanishing and satisfy*

$$\int_{\mathbb{R}^d_*} \frac{|\lambda|^2}{1+|\lambda|^2} f_i^2(\lambda)\mathrm{d}\lambda < \infty. \tag{3.4}$$

Define $f_{ij}(\lambda) = f_i(\lambda) f_j(\lambda)$ for $1 \le i, j \le p$. Then the measure $\mathscr{F}(\mathrm{d}\lambda) = (f_{ij}(\lambda)\mathrm{d}\lambda)$ is Hermitian non-negative definite, and

$$\gamma_{ij}(\lambda) = 1.$$

Proof. That $\mathscr{F}$ defined as such is Hermitian is clear. The non-negative definiteness follows from the fact that a matrix defined as $\mathscr{C} = a'a$ for any column vector $a \in \mathbb{R}^p$ is non-negative definite. Finally, the integrability condition (3.1) is satisfied by the Cauchy-Schwartz inequality and the condition (3.4) on f_i. □

Remark 1. Let $d = 1$, $p = 2$ and $f_1(\lambda) = |\lambda|^{-1/2-\alpha}$ and $f_2(\lambda) = |\lambda|^{-1/2-\beta}$, $0 < \alpha, \beta < 1$ then the condition (3.4) is satisfied if and only if $0 < \alpha + \beta < 1$.

Since the Hadamard product of non-negative definite matrices is non-negative definite, we have the following. Recall that the Hadamard product $\mathscr{A} \circ \mathscr{B} = (a_{ij} b_{ij})$ where a_{ij}, b_{ij} are ij-th entry of $\mathscr{A}$ and $\mathscr{B}$.

Proposition 3.2.3. *Let $\mathscr{A}$ be a Hermitian non-negative definite matrix and $\mathscr{F}(\mathrm{d}\lambda)$ be a Hermitian non-negative definite matrix of complex measures with spectral densities f_{ij}. Then $A \circ \mathscr{F}(\mathrm{d}\lambda)$ is non-negative definite and the coherence is*

$$\gamma_{ij}(\lambda) = \frac{a_{ij} f_{ij}(\lambda)}{\sqrt{a_{ii} f_{ii}(\lambda) a_{jj} f_{jj}(\lambda)}}.$$

3.3 Examples

3.3.1 Linear model of coregionalization

In this subsection, we consider the linear model of coregionalization (LMC). Let $\mathbf{W} = \{(W_1(t), W_2(t))', t \in \mathbb{R}^d\}$ be a second order bivariate random field with stationary increments, $\mathscr{A} = (a_{ij})$ be a 2×2 matrix, and define $\mathbf{X} = \mathscr{A}\mathbf{W}$. Denote the spectral density matrix of $\mathbf{W}$ by $\mathbf{g} = (g_{ij})$.

Proposition 3.3.1. *The bivariate random field $\mathbf{X}$, defined as above, has stationary increments and its spectral density matrix $\mathbf{f}$ is equal to $\mathscr{A}\mathbf{g}\mathscr{A}^*$. In particular, if the field $\mathbf{W}$ has uncorrelated components, then*

$$f_{11} = |a_{11}|^2 g_{11} + |a_{12}|^2 g_{22}, \quad f_{22} = |a_{21}|^2 g_{11} + |a_{22}|^2 g_{22}$$

and

$$f_{12} = \overline{f_{21}} = a_{11}\overline{a_{21}}g_{11} + a_{12}\overline{a_{22}}g_{22}.$$

If we assume further that $a_{11} = a_{22} = 1$*, then the coherence between the components of* $\mathbf{X}$ *is of modulus 1 if and only if* $a_{21}a_{12} = 1$.

Proof. For any $t, h, r_1, r_2 \in \mathbb{R}^d$, denote $\mathbf{X}(t) = (X_1(t), X_2(t))'$, $\mathbf{W}(t) = (W_1(t), W_2(t))'$, and let $\mathscr{D}_{\mathbf{X}}(h; r_1, r_2)$ and $\mathscr{D}_{\mathbf{W}}(h; r_1, r_2)$ be the structure functions of $\mathbf{X}$ and $\mathbf{W}$ respectively. We have that

$$\begin{aligned}
\mathscr{D}_{\mathbf{X}}(h; r_1, r_2) &= \mathbb{E}\left[(\mathbf{X}(t+h) - \mathbf{X}(t+h-r_1))\overline{(\mathbf{X}(t+h) - \mathbf{X}(t+h-r_1))'}\right] \\
&= \mathscr{A}\mathbb{E}\left[(\mathbf{W}(t+h) - \mathbf{W}(t+h-r_1))\overline{(\mathbf{W}(t+h) - \mathbf{W}(t+h-r_1))'}\right]\mathscr{A}^* \\
&= \mathscr{A}\mathscr{D}_{\mathbf{W}}(h; r_1, r_2)\mathscr{A}^*,
\end{aligned}$$

which completes the proof of the proposition. □

3.3.2 Kernel transform

Now we focus on the effect of convolution to the coherence. Consider a second order complex-valued random field $\{X_1(t), t \in \mathbb{R}^d\}$ that has stationary increments in the weak sense. Denote by f_1 and Z_1 the spectral density and the random spectral measure of X_1, respectively. Assume that $f_1(\lambda)$ is everywhere non-zero for $\lambda \in \mathbb{R}^d_*$ and let $K : \mathbb{R}^d \to \mathbb{R}$ be a continuous symmetric kernel that belongs to $L^2(\mathbb{R}^d, \text{Leb})$. Define

$$X_2(t) := \int_{\mathbb{R}^d} K(u-t)X_1(u)\mathrm{d}u, \qquad t \in \mathbb{R}^d.$$

Proposition 3.3.2. *The random field* $\{X_2(t), t \in \mathbb{R}^d\}$ *is of second order and has stationary increments. Moreover,* $\{\mathbf{X}(t) := (X_1(t), X_2(t))', t \in \mathbb{R}^d\}$ *is a second order bivariate random field with stationary increments, and its spectral density and random spectral measure are respectively written as*

$$\mathbf{f}(\lambda) = \begin{bmatrix} 1 & \widehat{K}(\lambda) \\ \widehat{K}(\lambda) & |\widehat{K}(\lambda)|^2 \end{bmatrix} f_1(\lambda), \quad \mathbf{Z}(\mathrm{d}\lambda) = (1, \widehat{K}(\lambda))' Z_1(\mathrm{d}\lambda), \qquad \lambda \in \mathbb{R}^d_*, \tag{3.5}$$

where the Fourier transform, $\widehat{K}(\lambda) = \int_{\mathbb{R}^d} e^{\mathrm{i}t\cdot\lambda} K(u)\mathrm{d}u$*, is a real valued function. In particular, for* $1 \leq i, j \leq 2$*, the coherence functions* $\gamma_{ij}(\lambda) = 1$ *for all* $\lambda \in \mathbb{R}^d_*$*.*

Proof. We first show that $\{X_2(t), t \in \mathbb{R}^d\}$ has stationary increments. It is enough to calculate the variogram. Note that, for any $s, t \in \mathbb{R}^d$, by Theorem 3.1.2 and the Fubini theorem, we have that

$$\begin{aligned}
&\mathbb{E}\left(|X_2(t+s) - X_2(s)|^2\right)\\
&= \mathbb{E}\left(\left|\int_{\mathbb{R}^d} K(u)(X_1(u+s+t) - X_1(u+s))\mathrm{d}u\right|^2\right)\\
&= \mathbb{E}\left(\left|\int_{\mathbb{R}^d} K(u)\left(\int_{\mathbb{R}^d_*} e^{\mathrm{i}(u+t+s)\cdot\lambda}\left(1 - e^{-\mathrm{i}t\cdot\lambda}\right) Z_1(\mathrm{d}\lambda)\right)\mathrm{d}u\right|^2\right)\\
&= \mathbb{E}\left(\left|\int_{\mathbb{R}^d_*} e^{\mathrm{i}(t+s)\cdot\lambda}\left(1 - e^{\mathrm{i}t\cdot\lambda}\right)\widehat{K}(\lambda) Z_1(\mathrm{d}\lambda)\right|^2\right)\\
&= 2\int_{\mathbb{R}^d_*}(1 - \cos(t\cdot\lambda))\,|\widehat{K}(\lambda)|^2 f_1(\lambda)(\mathrm{d}\lambda),
\end{aligned}$$

where $\widehat{K}$ is the Fourier transform of K. Hence, $\{X_2(t), t \in \mathbb{R}^d\}$ has stationary increments and its spectral density is $f_2(\lambda) := |\widehat{K}(\lambda)|^2 f_1(\lambda)$. Furthermore, we calculate its structure functions as follows. From the definition and Theorem 3.1.1, we get that

$$\begin{aligned}
D_{22}(h; r_1, r_2) &= \mathbb{E}\left[(X_2(t+h) - X_2(t+h-r_1))\overline{(X_2(t) - X_2(t-r_2))}\right]\\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(u)K(v)D_{11}(u+h-v; r_1, r_2)\mathrm{d}u\mathrm{d}v\\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(u)K(v)\left(\int_{\mathbb{R}^d_*} e^{\mathrm{i}\lambda\cdot(u+h-v)}\left(1 - e^{-\mathrm{i}\lambda\cdot r_1}\right)\left(1 - e^{\mathrm{i}\lambda\cdot r_2}\right) f_1(\mathrm{d}\lambda)\right)\mathrm{d}u\mathrm{d}v\\
&= \int_{\mathbb{R}^d_*} e^{\mathrm{i}\lambda\cdot h}\left(1 - e^{-\mathrm{i}\lambda\cdot r_1}\right)\left(1 - e^{\mathrm{i}\lambda\cdot r_2}\right)\widehat{K}(\lambda)\widehat{K}(-\lambda) f_1(\lambda)\mathrm{d}\lambda.
\end{aligned}$$

As for the cross structure functions, $D_{12}(h; r_1, r_2)$ and $D_{21}(h; r_1, r_2)$, using the same methods, we can obtain that,

$$\begin{aligned}
D_{12}(h; r_1, r_2) &= \mathbb{E}\left[(X_1(t+h) - X_1(t+h-r_1))\overline{(X_2(t) - X_2(t-r_2))}\right]\\
&= \int_{\mathbb{R}^d_*} e^{\mathrm{i}\lambda\cdot h}\left(1 - e^{-\mathrm{i}\lambda\cdot r_1}\right)\left(1 - e^{\mathrm{i}\lambda\cdot r_2}\right)\widehat{K}(-\lambda) f_1(\mathrm{d}\lambda),
\end{aligned}$$

and

$$\begin{aligned} D_{21}(h;r_1,r_2) &= \mathbb{E}\left[(X_2(t+h)-X_2(t+h-r_1))\overline{(X_1(t)-X_1(t-r_2))}\right] \\ &= \int_{\mathbb{R}^d_*} e^{\mathrm{i}\lambda\cdot h}\left(1-e^{-\mathrm{i}\lambda\cdot r_1}\right)\left(1-e^{\mathrm{i}\lambda\cdot r_2}\right)\widehat{K}(\lambda)f_1(\mathrm{d}\lambda). \end{aligned}$$

Because the kernel K is real valued symmetric function, we know that $\widehat{K}$ is a real valued function and $\widehat{K}(\lambda) = \widehat{K}(-\lambda)$. Hence, $\{\mathbf{X}(t), t \in \mathbb{R}^d\}$ is a bivariate random field with stationary increments and its spectral density is just the $\mathbf{f}(\lambda)$ in (3.5). Finally, by the definition and Theorem 3.1.2 again, we know that

$$\begin{aligned} X_2(t) &= \int_{\mathbb{R}^d} K(u-t)X_1(u)\mathrm{d}u \\ &= \int_{\mathbb{R}^d} K(u)\int_{\mathbb{R}^d_*}\left(e^{\mathrm{i}(u+t)\cdot\lambda}-1\right)Z_1(\mathrm{d}\lambda) \\ &= \int_{\mathbb{R}^d_*}\left(e^{\mathrm{i}t\cdot\lambda}-1\right)\widehat{K}(\lambda)Z_1(\mathrm{d}\lambda) + \int_{\mathbb{R}^d_*}\left(\widehat{K}(\lambda)-\widehat{K}(0)\right)Z_1(\mathrm{d}\lambda) \\ &= \int_{\mathbb{R}^d_*}\left(e^{\mathrm{i}t\cdot\lambda}-1\right)\widehat{K}(\lambda)Z_1(\mathrm{d}\lambda) + X_2(0), \end{aligned}$$

which implies that the random spectral measure of $\{\mathbf{X}(t), t \in \mathbb{R}^d\}$ is just the $\mathbf{Z}$ in (3.5). □

The following result extends Theorem 2 of Kleiber and Nychka [31] and illustrates the role of coherence in predictive estimation of $\{X_2(t), t \in \mathbb{R}^d\}$ in terms of kernel smoothed process of $\{X_1(t), t \in \mathbb{R}^d\}$.

Theorem 3.3.3. *Suppose that $\{\mathbf{X}(t) = (X_1(t), X_2(t))', t \in \mathbb{R}^d\}$ is a complex-value zero mean bivariate field with stationary increments and its spectral density matrix, $\mathbf{f}(\lambda) = \left(f_{ij}(\lambda)\right)_{i,j=1}^2$, $\lambda \in \mathbb{R}^d_*$, is everywhere non-zero. Let $\mathbf{Z} = (Z_1, Z_2)'$ be the random spectral measure of $\{\mathbf{X}(t), t \in \mathbb{R}^d\}$. Then the continuous symmetric square integrable function, $K : \mathbb{R}^d \to \mathbb{R}$, that minimizes the least squares error, $\mathbb{E}(|X_2(t) - \int_{\mathbb{R}^d} K(u-t)X_1(u)\mathrm{d}u|^2)$, is*

$$\begin{aligned} K(t) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d_*} e^{-\mathrm{i}t\cdot\lambda}\cdot\left(\frac{\Re\mathrm{e}\left((1-e^{\mathrm{i}t\cdot\lambda})f_{12}(\lambda)\right)}{f_{11}(\lambda)} + \widehat{K}(0)\cos(t\cdot\lambda)\right)\mathrm{d}\lambda \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d_*} e^{-\mathrm{i}t\cdot\lambda}\sqrt{\frac{f_{22}(\lambda)}{f_{11}(\lambda)}}\Re\mathrm{e}\left(\left(1-e^{\mathrm{i}t\cdot\lambda}\right)\gamma_{12}(\lambda) + e^{\mathrm{i}t\cdot\lambda}\widehat{K}(0)\sqrt{\frac{f_{11}(\lambda)}{f_{22}(\lambda)}}\right)\mathrm{d}\lambda. \end{aligned} \quad (3.6)$$

Furthermore, the spectral density of the predictor, $\widetilde{X}_2(t) := \int_{\mathbb{R}^d} K(u-t)X_1(u)\mathrm{d}u$, *is*

$$\widetilde{f}_{2|1}(\lambda) = f_{22}(\lambda)\left[\mathfrak{Re}\left(\left(1-e^{\mathrm{i}t\cdot\lambda}\right)\gamma_{12}(\lambda) + e^{\mathrm{i}t\cdot\lambda}\widehat{K}(0)\sqrt{\frac{f_{11}(\lambda)}{f_{22}(\lambda)}}\right)\right]^2, \qquad \lambda \in \mathbb{R}^d_*. \tag{3.7}$$

Proof. By the Fubini theorem, we know that,

$$\mathbb{E}\left(\left|X_2(t) - \int_{\mathbb{R}^d} K(u-t)X_1(u)\mathrm{d}u\right|^2\right) := T_1 + T_2 - T_3 - T_4,$$

where,

$$\begin{aligned}
T_1 &:= \mathbb{E}(X_2(t)\overline{X_2(t)}),\\
T_2 &:= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(u)K(v)\mathbb{E}\left(X_1(u+t)\overline{X_1(v+t)}\right)\mathrm{d}u\mathrm{d}v,\\
T_3 = \overline{T_4} &:= \int_{\mathbb{R}^d} K(u)\mathbb{E}\left(X_2(t)\overline{X_1(u+t)}\right)\mathrm{d}u.
\end{aligned}$$

Theorem 3.1.2 yields that,

$$\begin{aligned}
T_1 &= \mathbb{E}\left(\int_{\mathbb{R}^d_*}\left(e^{\mathrm{i}t\cdot\lambda}-1\right)Z_2(\mathrm{d}\lambda)\cdot\overline{\int_{\mathbb{R}^d_*}\left(e^{\mathrm{i}t\cdot\lambda}-1\right)Z_2(\mathrm{d}\lambda)}\right)\\
&= \int_{\mathbb{R}^d_*}\left(e^{\mathrm{i}t\cdot\lambda}-1\right)\left(e^{-\mathrm{i}t\cdot\lambda}-1\right)f_{22}(\lambda)\mathrm{d}\lambda\\
&= 2\int_{\mathbb{R}^d_*}\left(1-\cos(t\cdot\lambda)\right)f_{22}(\lambda)\mathrm{d}\lambda.
\end{aligned}$$

Similarly, by Theorem 3.1.2 and the properties of $\widehat{K}$, we can obtain that

$$\begin{aligned}
T_2 &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(u)K(v)\mathbb{E}\left(X_1(u+t)\overline{X_1(v+t)}\right)\mathrm{d}u\mathrm{d}v\\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(u)K(v)\mathbb{E}\left(\int_{\mathbb{R}^d_*}\left(e^{\mathrm{i}(u+t)\cdot\lambda}-1\right)Z_1(\mathrm{d}\lambda)\overline{\int_{\mathbb{R}^d_*}\left(e^{\mathrm{i}(v+t)\cdot\lambda}-1\right)Z_1(\mathrm{d}\lambda)}\right)\mathrm{d}u\mathrm{d}v\\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(u)K(v)\cdot\left(\int_{\mathbb{R}^d_*}\left(e^{\mathrm{i}(u+t)\cdot\lambda}-1\right)\left(e^{-\mathrm{i}(v+t)\cdot\lambda}-1\right)f_{11}(\lambda)\mathrm{d}\lambda\right)\mathrm{d}u\mathrm{d}v\\
&= \int_{\mathbb{R}^d_*}\left((\widehat{K}(\lambda))^2 - 2\widehat{K}(\lambda)\widehat{K}(0)\cos(t\cdot\lambda) + (\widehat{K}(0))^2\right)f_{11}(\lambda)\mathrm{d}\lambda,
\end{aligned}$$

and

$$
\begin{aligned}
T_3 = \overline{T_4} &= \int_{\mathbb{R}^d} K(u)\mathbb{E}\left(X_2(t)\overline{X_1(u+t)}\right)\mathrm{d}u \\
&= \int_{\mathbb{R}^d} K(u)\mathbb{E}\left(\int_{\mathbb{R}^d_*}\left(e^{\mathrm{i}t\cdot\lambda}-1\right)Z_2(\mathrm{d}\lambda)\overline{\int_{\mathbb{R}^d_*}\left(e^{\mathrm{i}(u+t)\cdot\lambda}-1\right)Z_1(\mathrm{d}\lambda)}\right)\mathrm{d}u \\
&= \int_{\mathbb{R}^d} K(u)\cdot\left(\int_{\mathbb{R}^d_*}\left(e^{\mathrm{i}t\cdot\lambda}-1\right)\left(e^{-\mathrm{i}(u+t)\cdot\lambda}-1\right)f_{21}(\lambda)\mathrm{d}\lambda\right)\mathrm{d}u \\
&= \int_{\mathbb{R}^d_*}\left(\widehat{K}(\lambda)-\widehat{K}(0)e^{\mathrm{i}t\cdot\lambda}\right)\cdot\left(1-e^{-\mathrm{i}t\cdot\lambda}\right)f_{21}(\lambda)\mathrm{d}\lambda.
\end{aligned}
$$

Note that, the least squares error is a functional of $\widehat{K}$. So, in order to optimize it, considering the functional derivative, we can get the following equation,

$$
\left(1-e^{-\mathrm{i}t\cdot\lambda}\right)f_{21}(\lambda)+\left(1-e^{\mathrm{i}t\cdot\lambda}\right)\overline{f_{21}(\lambda)} = 2\left(\widehat{K}(\lambda)-\widehat{K}(0)\cos(t\cdot\lambda)\right)f_{11}(\lambda), \qquad \lambda\in\mathbb{R}^d_*,
$$

which implies that the minimizer of the least squares error is just

$$
\widehat{K}(\lambda) = \frac{\Re\mathfrak{e}\left((1-e^{\mathrm{i}t\cdot\lambda})f_{12}(\lambda)\right)}{f_{11}(\lambda)} + \widehat{K}(0)\cos(t\cdot\lambda).
$$

Finally, the spectral density of the estimator $\widetilde{X}_2(t)$ follows by Proposition 3.3.2. □

It is natural to consider the p-variate random field related to the kernel transformations. Let $\mathbf{X}$ be a p-variate field with stationary increments admitting a spectral density matrix, $\mathbf{f}(\lambda) = \left(f_{ij}(\lambda)\right)_{i,j=1}^p$, $\lambda\in\mathbb{R}^d_*$, that is everywhere non-zero. Denote its coherence matrix by $\gamma_{\mathbf{f}}(\lambda)$. Let $\{K_i, 1\le i\le p\}$ be a family of real valued symmetric continuous kernels in $L^2(\mathbb{R}^d,\mathrm{Leb})$. For $t\in\mathbb{R}^d$ and $1\le i\le p$, define $Y_i(t)=\int_{\mathbb{R}^d} K_i(u-t)X_i(u)\mathrm{d}u$ and $\mathbf{Y}(t)=(Y_1(t),\ldots,Y_p(t))'$. Then, by the similar calculations as above, we can get the following result,

Proposition 3.3.4. *Under the aforementioned notations, the p-variate random field $\mathbf{Y}=\{\mathbf{Y}(t), t\in\mathbb{R}^d\}$ has stationary increments and its spectral density matrix is given by*

$$
\mathbf{g}(\lambda) = \left[\widehat{K}_i(\lambda)f_{ij}(\lambda)\widehat{K}_j(\lambda)\right]_{i,j=1}^p, \qquad \lambda\in\mathbb{R}^d_*. \tag{3.8}
$$

In particular, $\gamma_{\mathbf{g}}(\lambda)=\gamma_{\mathbf{f}}(\lambda)$ for all $\lambda\in\mathbb{R}^d_*$, here $\gamma_{\mathbf{g}}$ is the coherence matrix of $\mathbf{Y}$.

3.3.3 Estimation of Coherence

To estimate coherence, it is natural to use periodogram. Here we assume that we observe real-valued bivariate intrinsic stationary process on fixed domain, i.e. $\mathbf{X}_n(t) = \mathbf{X}(t/n) = (X_1(t/n), X_2(t/n)), t = 1, 2, ..., n$. Let $\mathbf{X}_n^0(t) := \mathbf{X}_n(t)$, and

$$\mathbf{X}_n^1(t) = \mathbf{X}_n(t) - \mathbf{X}_n(t-1) \tag{3.9}$$

$$\mathbf{X}_n^2(t) = \mathbf{X}_n(t) - 2\mathbf{X}_n(t-1) + \mathbf{X}_n(t-2) \tag{3.10}$$

$$\mathbf{X}_n^j(t) = \mathbf{X}_n^{j-1}(t) - \mathbf{X}_n^{j-1}(t-1) \quad \text{for } j = \{3, ..., k\} \tag{3.11}$$

for some $k < n, k \in \mathbb{Z}$, and

$$D^{j,n}(\lambda) = \sum_{t=1}^{n} \mathbf{X}_n^j(t) e^{-\mathrm{i}t\lambda}$$

$$I^{j,n}(\lambda) = n^{-1} D^{j,n}(\lambda) D^{j,n}(\lambda)^* \qquad \text{for } j = \{1, ..., k\}.$$

Then we can estimate the coherence of $\mathbf{X}_n^j$, $\gamma_{12}^{j,n}(\lambda)$, by

$$\hat{\gamma}_{12}^{j,n}(\lambda) = \frac{I_{12}^{j,n}(\lambda)}{\sqrt{I_{11}^{j,n}(\lambda) I_{22}^{j,n}(\lambda)}}.$$

It is easy to see that

$$I^{j,n}(\lambda) = I^{j-1,n}(\lambda)(1 - e^{-\mathrm{i}\lambda})(1 - e^{\mathrm{i}\lambda})$$

and therefore

$$\hat{\gamma}_{12}^{j,n}(\lambda) = \frac{I_{12}^{j,n}(\lambda)}{\sqrt{I_{11}^{j,n}(\lambda) I_{22}^{j,n}(\lambda)}} = \frac{I_{12}^{j+1,n}(\lambda)}{\sqrt{I_{11}^{j+1,n}(\lambda) I_{22}^{j+1,n}(\lambda)}} = \hat{\gamma}_{12}^{j+1,n}(\lambda) \qquad \text{for } j = \{1, ..., k\}.$$

Lemma 3.3.5.

$$\hat{\gamma}_{12}^{j,n} = \hat{\gamma}_{12}^{j+1,n} \qquad \textit{for } j = \{1, ..., k\}.$$

Denote spectral density of $\{\mathbf{X}_n^j(t), t = 1, ..., n\}$ as $f_n^j(\lambda)$, and its coherence as

$$\gamma_{12}^{j,n}(\lambda) = \frac{f_n^j(\lambda)_{12}}{\sqrt{f_n^j(\lambda)_{11} f_n^j(\lambda)_{22}}}, \quad \lambda \in [-\pi, \pi].$$

Remark 5 *(1) Note that an intrinsic stationary process and the differenced of the process have the same coherence. Let* $\xi_t = \mathbf{X}(t+1) - \mathbf{X}(n)$ *where* $\mathbf{X}(t)$ *satisfies Theorem 3.1.2 with the assumtion that* $F_{ij} \ll$ *Leb with non vanishing density* $f_{ij}(\lambda)$*. Then* $\{\xi_t\}$ *is stationary with spectral density :* $f^{\xi}_{ij}(\lambda) = (1-e^{-\mathrm{i}\lambda})(1-e^{\mathrm{i}\lambda})f_{ij}(\lambda)$*. Therefore coherence of* ξ *is*

$$\gamma^{\xi}_{ij}(\lambda) = \frac{f^{\xi}_{ij}(\lambda)}{\sqrt{f^{\xi}_{ii}(\lambda)f^{\xi}_{jj}(\lambda)}} = \frac{f_{ij}(\lambda)(1-e^{-\mathrm{i}\lambda})(1-e^{\mathrm{i}\lambda})}{\sqrt{f_{ii}(\lambda)f_{jj}(\lambda)(1-e^{-\mathrm{i}\lambda})^2(1-e^{\mathrm{i}\lambda})^2}} = \frac{f_{ij}(\lambda)}{\sqrt{f_{ii}(\lambda)f_{jj}(\lambda)}} = \gamma_{ij}(\lambda)$$

(2) The same thing can be said for discretely sampled series in a fixed domain. Let $\mathbf{X}(t)$ *satisfies Theorem 3.1.2 and assume that* $F_{ij} \ll$ *Leb with non vanishing density* $f_{ij}(\lambda)$*, and* $\mathbf{X}^j_n, \mathbf{X}^{j+1}_n$ *are defined as (3.9-3.11). Then spectral density of* $\mathbf{X}^j_n$ *and* $\mathbf{X}^{j+1}_n$ *are* $f^j_n(\lambda)_{ij} := n(1-e^{-\mathrm{i}\lambda})^j(1-e^{\mathrm{i}\lambda})^j\sum_{\ell=-\infty}^{\infty} f_{ij}(n(\lambda+2\pi\ell))$ *and* $f^{j+1}_n(\lambda)_{ij} := (1-e^{-\mathrm{i}\lambda})^{j+1}(1-e^{\mathrm{i}\lambda})^{j+1} n\sum_{\ell=-\infty}^{\infty} f_{ij}(n(\lambda+2\pi\ell))$ *for* $\lambda \in (-\pi,\pi)$*. Therefore*

$$\gamma^{j,n}_{12}(\lambda) = \frac{f^j_n(\lambda)_{12}}{\sqrt{f^j_n(\lambda)_{11}f^j_n(\lambda)_{22}}} = \frac{f^{j+1}_n(\lambda)_{12}}{\sqrt{f^{j+1}_n(\lambda)_{22}f^{j+1}_n(\lambda)_{22}}} = \gamma^{j+1,n}_{12}(\lambda), \qquad \lambda \in (-\pi,\pi) \tag{3.12}$$

But it is well known that $I^{i,n}$ is not consistent estimator. Therefore we will use smoothed (cross) periodogram. Let $\{\mathbf{X}(t), t \in \mathbb{R}\}$ be a second order bivariate stationary random field satisfying $d = 1$, (3.2) and assume that $F_{ij} \ll$ Leb with non vanishing density $f_{ij}(\lambda)$ with

$$f_{11}(\lambda) \sim d_{11}|\lambda|^{-1-2\alpha_1} \tag{3.13}$$

$$f_{22}(\lambda) \sim d_{22}|\lambda|^{-1-2\alpha_2} \tag{3.14}$$

$$f_{\ell p}(\lambda) \sim d_{\ell p}(e^{\mathrm{i}\theta_1}I_{(\lambda>0)} + e^{\mathrm{i}\theta_2}I_{(\lambda<0)})|\lambda|^{-1-2\alpha_{\ell p}} \tag{3.15}$$

when $|\lambda| \to \infty$ for some constants $d_{\ell p}, \theta_1, \theta_2$ where $0 < \alpha_\ell, \alpha_p < 1$, $\alpha_{\ell p} = (\alpha_\ell + \alpha_p)/2$, $(\alpha_{\ell\ell} = \alpha_\ell)$ for $\ell, p = 1, 2$. For $j \geq 1$

$$\mathbb{E}(I^{j,n}(\lambda)) = n^{-1}\sum_{\ell,m=1}^{n} \mathbb{E}\left(\mathbf{X}^j_n(\ell)\mathbf{X}^j_n(m)^*\right) e^{-\mathrm{i}(\ell-m)\lambda}$$

and

$$\mathbb{E}(\mathbf{X}_n^j(\ell)\mathbf{X}_n^j(m)^*) = \int_{\mathbb{R}} e^{\mathrm{i}(\ell-m)\frac{x}{n}}(e^{-\mathrm{i}x}-1)^j(e^{\mathrm{i}x}-1)^j f(x)dx$$
$$= \int_{[-\pi,\pi]} e^{\mathrm{i}(\ell-m)x} \sum_{q=-\infty}^{\infty} n|e^{-\mathrm{i}x}-1|^{2j} f(n(x+2\pi q))dx.$$

Therefore for any fixed $\lambda \in [-\pi,\pi], \lambda \neq 0$, as $n \to \infty$ the spectral density of $\mathbf{X}_n^j$, $f_n^j(\lambda)$, for $j \geq 1$ has the following limit when $\max\{2\alpha_1, 2\alpha_2\} - 2j < 0$, by [32].

$$f_n^j(\lambda)_{\ell p} \sim d_{\ell p} \sum_{q=-\infty}^{\infty} n \frac{|e^{-\mathrm{i}\lambda}-1|^{2j}}{|n(\lambda+2\pi q)|^{1+2\alpha_{\ell p}}} \quad \text{for } \ell, p = 1, 2.$$

Therefore we have for $\ell, p = 1, 2$,

$$n^{2\alpha_{\ell p}} f_n^j(\lambda)_{\ell p} \to d_{\ell p} \sum_{q=-\infty}^{\infty} \frac{|e^{-\mathrm{i}\lambda}-1|^{2j}}{|(\lambda+2\pi q)|^{1+2\alpha_{\ell p}}} := g(\lambda)_{\ell p}^j \quad \text{for} \quad \max\{2\alpha_1, 2\alpha_2\} - 2j < 0. \quad (3.16)$$

Now let $m_j = \min\{j : \max\{2\alpha_1, 2\alpha_2\} - 2j < 0\}$ and define a set $S_J = \{j : 0 \leq j \leq c + m_j\}$ for some constant c. Note that with $j \in S_J$, $g(\cdot)_{\ell p}^j$ becomes integrable function. (see [32].)

Define the smoothed cross-periodogram as in [32] by

$$\hat{f}^{j,h}(2\pi n^{-1}J) = \sum_{I \in T_n} W_h(I) I^{j,n}(2\pi n^{-1}(J+I)) \quad \text{for} \quad j \in S_J \quad (3.17)$$

where

$$W_h(I) = \frac{K_h(2\pi n^{-1}I)}{\sum_{L \in T_n} K_h(2\pi n^{-1}L)}$$

and K is a symmetric continuous function which satisfies $K_h(x) = K(\frac{x}{h})I_{\{|x| \leq h\}}, K(0) > 0, K(x) \geq 0, J \in T_n, T_n = \{\lfloor -n - 1/2 \rfloor, ..., \lfloor n - n/2 \rfloor\}$. From [32], it is known that

$$n^{\eta} \begin{bmatrix} n^{2\alpha_1} \hat{f}_{11}^{j,h}(2\pi n^{-1}J_1) - g(\lambda_1)_{11}^j \\ n^{2\alpha_{12}} \hat{f}_{12}^{j,h}(2\pi n^{-1}J_1) - g(\lambda_1)_{12}^j \\ n^{2\alpha_2} \hat{f}_{22}^{j,h}(2\pi n^{-1}J_1) - g(\lambda_1)_{22}^j \\ \ldots \\ n^{2\alpha_2} \hat{f}_{22}^{j,h}(2\pi n^{-1}J_r) - g(\lambda_r)_{22}^j \end{bmatrix} \to N(0, \Sigma) \quad (3.18)$$

where

$$h = Cn^{-\nu}, \eta = (1-\nu)/2, \text{ for some } C > 0, 0 < \nu < 1, \tag{3.19}$$

$$\max\{2\alpha_1, 2\alpha_2\} - 2j < 0, \tag{3.20}$$

and $\lim_{n\to\infty} 2\pi n^{-1} J_r = \lambda_r \neq 0$. Elements of Σ are

$$\lim_n n^{2\eta} cov\Big(n^{2\alpha_{\ell p}} \hat{f}^{j,h}_{\ell p}(2\pi n^{-1} J_q), n^{2\alpha_{\ell' p'}} \hat{f}^{j,h}_{\ell' p'}(2\pi n^{-1} J_{q'})\Big)$$
$$= \begin{cases} \{(2\pi/C)\bar{K}_2/\bar{K}_1^2\} g(\lambda_q)^j_{\ell\ell'} g(\lambda_q)^j_{pp'} & \text{if } J_q = J_{q'}, \\ \{(2\pi/C)\bar{K}_2/\bar{K}_1^2\} g(\lambda_q)^j_{\ell p'} g(\lambda_q)^j_{\ell' p} & \text{if } J_q = -J_{q'}, \\ 0 & \text{if } \lambda_q \neq \pm\lambda_{q'} \end{cases}$$

where $\bar{K_p} = \int_{[-1,1]} K(x)^p dx$, and $1 \leq \ell, \ell' \leq p, p' \leq 2$. Define

$$\tilde{\gamma}^{j,n}_{12}(\lambda) = \frac{\hat{f}^{j,h}_{12}(\lambda)}{\sqrt{\hat{f}^{j,h}_{11}(\lambda)\hat{f}^{j,h}_{22}(\lambda)}} \quad \text{for } \; j \in S_J. \tag{3.21}$$

Lemma 3.3.6. *Under (3.19) and* $\lim_{n\to\infty} 2\pi n^{-1} J = \lambda \neq 0$

$$|\tilde{\gamma}^{j_1,n}_{12}(2\pi n^{-1}J) - \tilde{\gamma}^{j_2,n}_{12}(2\pi n^{-1}J)| = O(h) = O(n^{-\nu}), \quad \lambda \neq 0, \lambda \in (-\pi,\pi), \quad j_1, j_2 \in S_J.$$

Proof. Let $j_1 < j_2$ and assume j_2 satisfies (3.20). From (3.17), for $\ell, p = 1, 2$,

$$\hat{f}^{j_2,h}_{\ell p}(2\pi n^{-1}J) = \hat{f}^{j_1,h}_{\ell p}(2\pi n^{-1}J)|1 - e^{-\mathrm{i}2\pi n^{-1}J}|^2 + \hat{f}^{j_2,h}_{\ell p}(2\pi n^{-1}J)\left(1 - \frac{|1 - e^{-\mathrm{i}2\pi n^{-1}J}|^{2(j_2-j_1)}}{|1 - e^{-\mathrm{i}2\pi n^{-1}(J+K)}|^{2(j_2-j_1)}}\right).$$

As $h = Cn^{-\nu}$, $2\pi n^{-1}K \leq Cn^{-\nu}$ and therefore

$$1 - \frac{|1 - e^{-\mathrm{i}2\pi n^{-1}J}|^{2(j_2-j_1)}}{|1 - e^{-\mathrm{i}2\pi n^{-1}(J+K)}|^{2(j_2-j_1)}} = O(n^{-\nu}),$$

which results in

$$n^{2\alpha_{\ell p}} \hat{f}^{j_2,h}_{\ell p}(2\pi n^{-1}J) = n^{2\alpha_{\ell p}} \hat{f}^{j_1,h}_{\ell p}(2\pi n^{-1}J)|1 - e^{-\mathrm{i}2\pi n^{-1}J}|^{2(j_2-j_1)} + O(n^{-\nu}),$$

since $n^{2\alpha_{\ell p}} \hat{f}^{j_2,h}_{\ell p}(2\pi n^{-1}J)$ converges to $g(\lambda)_{\ell p}$ as it was shown in [32]. With (3.21), the result is derived. For the other cases, the result is derived in the similar way. □

In (3.16), $g(\lambda)^j_{12}$ was defined for j satisfying (3.20). Notice that for any j_1, j_2 satisfying (3.20),

$$\frac{g(\lambda)^{j_1}_{12}}{\sqrt{g(\lambda)^{j_1}_{11} g(\lambda)^{j_1}_{22}}} = \frac{g(\lambda)^{j_2}_{12}}{\sqrt{g(\lambda)^{j_2}_{11} g(\lambda)^{j_2}_{22}}},$$

therefore we define

$$\tilde{\gamma}_{12}(\lambda) := \frac{g(\lambda)^j_{12}}{\sqrt{g(\lambda)^j_{11} g(\lambda)^j_{22}}},$$

for any j satisfying (3.20). Then for any $j \in S_J$

$$\gamma^{j,n}_{12}(\lambda) = \frac{f^j_n(\lambda)_{12}}{\sqrt{f^j_n(\lambda)_{11} f^j_n(\lambda)_{22}}} = \frac{n^{\beta_{12}-1} f^{j^*}_n(\lambda)_{12}}{\sqrt{n^{\beta_{11}-1} f^{j^*}_n(\lambda)_{11} n^{\beta_{22}-1} f^{j^*}_n(\lambda)_{22}}} \to \tilde{\gamma}_{12}(\lambda)$$

from (3.12) and (3.16) where j^* satisfies (3.20).

Remark 6 *Note that $\tilde{\gamma}_{12}(\lambda)$ defined above is slightly different than (3.3). This is because we estimate coherence in a fixed domain.*

Theorem 3.3.7. *Let $\{X(t), t \in \mathbb{R}\}$ be a second order bivariate stationary random field with $d = 1$, (3.1),(3.4), (3.19), and $\nu > 1/3$. For $\lim_{n\to\infty} 2\pi n^{-1} J = \lambda \neq 0, \lambda \in (-\pi, \pi)$*

$$n^{\eta}(\tilde{\gamma}^{j,n}_{12}(2\pi n^{-1} J) - \tilde{\gamma}_{12}(\lambda)) \to N(0, \Sigma_\gamma) \qquad j \in S_J$$

where

$$\Sigma_\gamma = \tilde{\gamma}^{(1)\prime}_{12}(\lambda)\Sigma(\lambda)\tilde{\gamma}^{(1)}_{12}(\lambda),$$

$\tilde{\gamma}^{(1)}_{12}(\lambda)$ is a vector of first derivative of $\tilde{\gamma}_{12}$ with respect to $g(\lambda)^j_{11}, g(\lambda)^j_{12}, g(\lambda)^j_{22}$, i.e.

$$\tilde{\gamma}^{(1)}_{12}(\lambda) = \left(-\frac{g(\lambda)^j_{12}}{2\sqrt{g(\lambda)^j_{22}}}\{g(\lambda)^j_{11}\}^{-3/2}, \frac{1}{\sqrt{g(\lambda)^j_{11} g(\lambda)^j_{22}}}, -\frac{g(\lambda)^j_{12}}{2\sqrt{g(\lambda)^j_{11}}}\{g(\lambda)^j_{22}\}^{-3/2} \right)'$$

and $\Sigma(\lambda)$ is from (3.18) for $r = 1, \lambda_1 = \lambda$.

Proof. Assume j satisfies (3.20). By (3.18), asymptotic normality of $\tilde{\gamma}^{j,n}_{12}(\lambda) - \tilde{\gamma}_{12}(\lambda)$ is derived. For j which does not satisfy (3.20), Lemma 3.3.6 gives the result since $n^{-\nu+\eta} \to 0$. □

Remark 7 *The above results show that one doesn't need to know the spectral density on the tail, β_{lp}, to estimate coherence since $\tilde{\gamma}^{0,n}_{12}$ works well as much as $\tilde{\gamma}^{2,n}_{12}$.*

3.3.4 Operator fractional Brownian motion

Let $\mathbf{X} = \{\mathbf{X}(t), t \in \mathbb{R}^d\}$ be a p-variate random field. We say that $\mathbf{X}$ is an OFBM with exponent H if it is a Gaussian field with stationary increments, and satisfies the following operator scaling property: For any $c > 0$,

$$\{\mathbf{X}(ct), t \in \mathbb{R}^d\} \overset{f.d.}{=} \{c^H \mathbf{X}(t)\}, \tag{3.22}$$

where H is a linear operator on $\mathbb{R}^d$ and $c^H = \exp(H \ln c)$ for $c > 0$. Denote by λ_H and Λ_H the minimum and the maximum of the real parts of the eigenvalues of H, respectively.

We need some notations. Let Z be a complex-valued Gaussian measure with the control measure Leb such that, for every Borel set $A \subset \mathbb{R}^d$ with finite Lebesgue measure,

$$\mathfrak{Re} Z(-A) = \mathfrak{Re} Z(A) \text{ and } \mathfrak{Im} Z(-A) = -\mathfrak{Im} Z(A), \text{ a.s.}$$

The construction of such complex valued Gaussian measure is given in Appendix. Let $Z_1, \ldots, Z_p$ be independent copies of Z and $\mathbf{Z} = (Z_1, \ldots, Z_p)'$. Note that $\mathbf{Z}(c\,\cdot) \overset{f.d.}{=} c^{d/2}\mathbf{Z}(\cdot)$ for any $c > 0$. The following Theorem 3.3.9 is essentially Theorem 3 in [44] and Theorem 3.1 in [39]. Since there is a slight difference in the representation of [39] and that of Theorem 3.1.2, we shall prove it for completeness.

We start with a lemma. To simplify the presentation we only consider the case $p = 2$ in this subsection. Without loss of generality we assume that H is in its real canonical form.

Lemma 3.3.8. *Define the random measure*

$$\mathbf{M}(\mathrm{d}\lambda) := \left(\frac{1}{|\lambda|}\right)^{H+dI/2} \mathbf{Z}(\mathrm{d}\lambda), \qquad \lambda \in \mathbb{R}^d_*.$$

i) (H1): If $H = \begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix}$ *with* $0 < h < 1$, *then the spectral density of* $\mathbf{M}$ *is* $|\lambda|^{-(2h+d)} I_2$, *where* I_2 *is the identity operator on* $\mathbb{R}^2$.

ii) (H2): If $H = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix}$ *with* $0 < h_1, h_2 < 1$*, then the spectral density of* $\mathbf{M}$ *is*

$$\begin{bmatrix} |\lambda|^{-(2h_1+d)} & 0 \\ 0 & |\lambda|^{-(2h_2+d)} \end{bmatrix}.$$

iii) (H3): If $H = P \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} P'$ *with* $0 < h_1, h_2 < 1$*, and* $P = (p_{ij})$ *is orthogonal matrix, i.e.* $PP' = I$*, then the spectral density of* $\mathbf{M}$ *is*

$$P \begin{bmatrix} |\lambda|^{-(2h_1+d)} & 0 \\ 0 & |\lambda|^{-(2h_2+d)} \end{bmatrix} P'.$$

iv) (H4): If $H = \begin{bmatrix} h & 1 \\ 0 & h \end{bmatrix}$ *with* $0 < h < 1$*, then the spectral density of* $\mathbf{M}$ *is*

$$\begin{bmatrix} |\lambda|^{-(2h+d)}(1 + \ln^2 \frac{1}{|\lambda|}) & |\lambda|^{-(2h+d)} \ln \frac{1}{|\lambda|} \\ |\lambda|^{-(2h+d)} \ln \frac{1}{|\lambda|} & |\lambda|^{-(2h+d)} \end{bmatrix}, \qquad \lambda \in \mathbb{R}^d_* \setminus \{\lambda \in \mathbb{R}^d : |\lambda| \geq 1\};$$

v) (H5): If $H = \begin{bmatrix} h & -\beta \\ \beta & h \end{bmatrix}$ *with* $0 < h < 1$ *and* $\pm\beta \in \mathbb{R}$ *(which correspond to the conjugate eigenvalues* $h \pm \mathrm{i}\beta$*), then the spectral density of* $\mathbf{M}$ *is* $|\lambda|^{-(2h+d)} I_2$.

Theorem 3.3.9. *Let* $\mathscr{C}$ *be an invertible linear operator on* $\mathbb{R}^d$ *and* H *be a linear operator on* $\mathbb{R}^d$ *with* $0 < \lambda_H, \Lambda_H < 1$*. Define*

$$\mathbf{X}(t) = \int_{\mathbb{R}^d_*} (e^{\mathrm{i}t\cdot\lambda} - 1) \left(\frac{1}{|\lambda|}\right)^{H+dI/2} \mathscr{C}\mathbf{Z}(\mathrm{d}\lambda), \qquad t \in \mathbb{R}^d. \tag{3.23}$$

Then $\mathbf{X} = \{\mathbf{X}(t),\ t \in \mathbb{R}^d\}$ *is a Gaussian field with stationary increments and satisfies the operator scaling property* (3.22) *with exponent* H*.*

Proof. First, it is not hard to check that Yaglom's condition (3.1) is satisfied for each entry of the spectral density matrix, the form (3.23) thus implies that $\mathbf{X}$ is a second order bivariate field with stationary increments by Theorem 3.1.2.

Second, since $e(\lambda) = e^{\mathrm{i}t\cdot\lambda} - 1$ is Hermitian in the sense that $e(-\lambda) = \overline{e(\lambda)}$, and the matrix $\mathscr{C}$ is real-valued, we get that the stochastic integral (3.23) is real valued. The Gaussianity follows from the fact that $\mathbf{Z}$ is a Gaussian measure.

Finally, the operator-scaling property (3.22) follows from the scaling property of the measure $\mathbf{M}$ defined in Lemma 3.3.8, see [39] for details. □

Combining (3.23) and Lemma 3.3.8, we can obtain easily the coherence of OFBM.

Proposition 3.3.10. *i) Suppose that* $\mathscr{C} = I_2$. *If* $H = \begin{bmatrix} h & 1 \\ 0 & h \end{bmatrix}$ *with* $0 < h < 1$, *then*

$$\gamma_{12}(\lambda) = \frac{\ln \frac{1}{|\lambda|}}{\sqrt{1 + \ln^2 \frac{1}{|\lambda|}}}, \qquad \lambda \in \mathbb{R}^d_* \setminus \{\lambda \in \mathbb{R}^d : |\lambda| \geq 1\};$$

for H *in shape of (H1), (H2) or (H5) in Lemma 3.3.8, we have that* $\gamma(\lambda) = 0$. *If* H *is of (H3), then*

$$\gamma_{12}(\lambda) = \frac{\langle P'(\lambda)e_1, P'(\lambda)e_2\rangle}{\sqrt{\langle P'(\lambda)e_1, P'(\lambda)e_1\rangle\langle P'(\lambda)e_2, P'(\lambda)e_2\rangle}}, \qquad \lambda \in \mathbb{R}^d_*,$$

where

$$P(\lambda) := \begin{bmatrix} p_{11} & p_{12}|\lambda|^{-(h_2-h_1)} \\ p_{21} & p_{22}|\lambda|^{-(h_2-h_1)} \end{bmatrix},$$

and $e_1 = (1,0)'$ *and* $e_2 = (0,1)'$.

ii) Suppose that $\mathscr{C} = (c_{ij})$ *is an invertible* 2×2 *matrix. If* $H = \begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix}$ *with* $0 < h < 1$ *or* $H = \begin{bmatrix} h & -\beta \\ \beta & h \end{bmatrix}$ *with* $0 < h < 1$ *and* $\beta \neq 0$, *then*

$$\gamma_{12}(\lambda) = \frac{\langle \mathscr{C}'e_1, \mathscr{C}'e_2\rangle}{\sqrt{\langle \mathscr{C}'e_1, \mathscr{C}'e_1\rangle\langle \mathscr{C}'e_2, \mathscr{C}'e_2\rangle}}, \qquad \lambda \in \mathbb{R}^d_*;$$

if $H = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix}$ *with* $0 < h_1 < h_2 < 1$, *then*

$$\gamma_{12}(\lambda) = \frac{\langle \mathscr{C}'(\lambda)e_1, \mathscr{C}'(\lambda)e_2\rangle}{\sqrt{\langle \mathscr{C}'(\lambda)e_1, \mathscr{C}'(\lambda)e_1\rangle\langle \mathscr{C}'(\lambda)e_2, \mathscr{C}'(\lambda)e_2\rangle}}, \qquad \lambda \in \mathbb{R}^d_*,$$

where

$$\mathscr{C}(\lambda) := \begin{bmatrix} c_{11} & c_{12} \\ c_{21}|\lambda|^{-(h_2-h_1)} & c_{22}|\lambda|^{-(h_2-h_1)} \end{bmatrix};$$

if $H = P\begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix}P'$ *with* $0 < h_1 < h_2 < 1$*,* $P = (p_{ij})$*, then*

$$\gamma_{12}(\lambda) = \frac{\langle \mathscr{C}_P'(\lambda)e_1, \mathscr{C}_P'(\lambda)e_2\rangle}{\sqrt{\langle \mathscr{C}_P'(\lambda)e_1, \mathscr{C}_P'(\lambda)e_1\rangle\langle \mathscr{C}_P'(\lambda)e_2, \mathscr{C}_P'(\lambda)e_2\rangle}}, \qquad \lambda \in \mathbb{R}^d_*,$$

where

$$\mathscr{C}_P(\lambda) := [P\mathscr{C}(\lambda)P];$$

if $H = \begin{bmatrix} h & 1 \\ 0 & h \end{bmatrix}$ *with* $0 < h < 1$*, then*[1]

$$\gamma_{12}(\lambda) = \frac{c_{11}c_{21}\ln^2|\lambda| - (c_{11}c_{22} + c_{12}c_{21})\ln|\lambda| + c_{11}c_{21} + c_{12}c_{22}}{\sqrt{c_{11}^2 + (c_{11}\ln|\lambda| - c_{12})^2}\cdot\sqrt{c_{21}^2 + (c_{21}\ln|\lambda| - c_{22})^2}}, \quad \lambda \in \mathbb{R}^d_*\backslash\{\lambda \in \mathbb{R}^d : |\lambda| \geq 1\}.$$

Remark 8 $\gamma_{12}(\lambda)$ *tends to* 1 *as* $|\lambda| \to 0$ *or* ∞ *in either* $\{(H3)\}$ *or* $\{(H2, H4)$ *and* $\mathscr{C} \neq I\}$*.*

Let $\{\mathbf{X}(t), t \in [0,1]\}$ be operator fractional Brownian motion observed in a fixed domain whose spectral representation is (3.23) with $d = 1$. For $\mathbf{X}_n^j$ defined as (3.9-3.11), its spectral density is

$$f_n^j(\lambda) = \sum_{q=-\infty}^{\infty} n\frac{|e^{-\mathrm{i}\lambda} - 1|^j}{|n(\lambda + 2\pi q)|^{H+I/2}}\mathscr{C}\mathscr{C}'\frac{|e^{-\mathrm{i}\lambda} - 1|^j}{|n(\lambda + 2\pi q)|^{H+I/2}}, \quad j \in S_J. \tag{3.24}$$

We further restrict H as one of $(H1)$, $(H2)$, and $(H3)$ for which we consider the following cases.

Case I: $\mathscr{C}$ is diagonal matrix and $(H1)$ or $(H2)$

Case II: $\mathscr{C}$ is not diagonal and $(H1)$ or $(H2)$

Case III: $(H3)$

For $(H3)$ and $\mathscr{C} = I$,

$$f_n^j(\lambda) = P\sum_{q=-\infty}^{\infty} n\frac{|e^{-\mathrm{i}\lambda} - 1|^{2j}}{|n(\lambda + 2\pi q)|^{2\Lambda+I}}P', \tag{3.25}$$

[1]Need to be checked again.

where $\Lambda = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix}$. As in proposition 3.3.10 *i*), we have the following:

$$\gamma_{12}^{j,n}(\lambda) = \frac{\langle P^{j,n'}(\lambda)e_1, P^{j,n'}(\lambda)e_2\rangle}{\sqrt{\langle P^{j,n'}(\lambda)e_1, P^{j,n'}(\lambda)e_1\rangle\langle P^{j,n'}(\lambda)e_2, P^{j,n'}(\lambda)e_2\rangle}}, \qquad \lambda \in (-\pi, \pi) \tag{3.26}$$

$$P^{j,n}(\lambda) = \begin{bmatrix} p_{11} & p_{12}q^{j,n}(\lambda) \\ p_{21} & p_{22}q^{j,n}(\lambda) \end{bmatrix} \tag{3.27}$$

$$q^{j,n}(\lambda) = \sqrt{\sum_{q=-\infty}^{\infty} n\frac{|e^{-\mathrm{i}\lambda} - 1|^{2j}}{|n(\lambda + 2\pi q)|^{2h_2+1}} \Big/ \sum_{q=-\infty}^{\infty} n\frac{|e^{-\mathrm{i}\lambda} - 1|^{2j}}{|n(\lambda + 2\pi q)|^{2h_1+1}}} \tag{3.28}$$

Now we are ready to derive the following result. For notational simplicity, $q(\lambda)$ means $q^{j,n}(\lambda)$.

Lemma 3.3.11. *Let* $\mathbf{X}(t)$ *be operator fractional Brownian motion defined as in Theorem 3.3.9 with Case III, observed in a fixed domain. For* $\lambda \neq 0, \lambda \in (-\pi, \pi)$

i) $\mathscr{C} = I$ *and* $(H3)$

$$|\gamma_{12}^{j,n}(\lambda)| - 1 = O(n^{2(h_1-h_2)}), \qquad j \in S_J.$$

ii) $\mathscr{C} \neq I$ *and* $(H3)$

$$|\gamma_{12}^{j,n}(\lambda)| - 1 = O\left(\frac{b_{12}}{b_{11}}n^{h_1-h_2} \vee \frac{b_{22}}{b_{11}}n^{2(h_1-h_2)}\right), \qquad j \in S_J,$$

where $B = (b_{ij}) = P'\mathscr{C}\mathscr{C}'P$.

Proof. i) By remark 5 (2), let $j = 2$ without loss of generality. Since $h_1, h_2 \in (0, 1)$,

$$\sum_{q=-\infty}^{\infty} \frac{|e^{-\mathrm{i}\lambda} - 1|^{2j}}{|(\lambda + 2\pi q)|^{2h_2+1}} \Big/ \sum_{q=-\infty}^{\infty} \frac{|e^{-\mathrm{i}\lambda} - 1|^{2j}}{|(\lambda + 2\pi q)|^{2h_1+1}} \asymp 1, \qquad \lambda \neq 0, \lambda \in (-\pi, \pi).$$

Therefore $q(\lambda) \asymp n^{h_1-h_2}$. By (3.26-3.28),

$$\gamma_{12}^{j,n}(\lambda) = \frac{p_{11}p_{21} + p_{12}p_{22}q^2(\lambda)}{\sqrt{(p_{11}^2 + p_{12}^2q^2(\lambda))(p_{21}^2 + p_{22}^2q^2(\lambda))}}.$$

ii) Similarly, from (3.24)

$$\gamma_{12}^{j,n}(\lambda) = \frac{p_{11}p_{21} + \frac{b_{12}}{b_{11}}(p_{12}p_{21} + p_{11}p_{22})q(\lambda) + \frac{b_{22}}{b_{11}}p_{12}p_{22}q^2(\lambda)}{\sqrt{(p_{11}^2 + 2\frac{b_{12}}{b_{11}}p_{11}p_{12}q(\lambda) + \frac{b_{22}}{b_{11}}p_{12}^2q^2(\lambda))(p_{21}^2 + 2\frac{b_{12}}{b_{11}}p_{21}p_{22}q(\lambda) + \frac{b_{22}}{b_{11}}p_{22}^2q^2(\lambda))}}.$$

□

Remark 9 *(1) In a similar way to proposition 3.3.10,* $\gamma_{12}^{j,n}(\lambda) \equiv 0$ *for Case I.*

(2) For Case II, $\gamma_{12}^{j,n}(\lambda) \asymp c_\gamma$ *where* $c_\gamma = \frac{C_{12}}{\sqrt{C_{11}C_{22}}}$, $C_{\ell,k} = (\mathscr{C}\mathscr{C}')_{\ell,k}, \ell, k = 1, 2$. *This is from (3.24) and the fact that for* $\lambda \neq 0, \lambda \in (-\pi, \pi)$,

$$\sqrt{\sum_{q=-\infty}^{\infty} \frac{|e^{-\mathrm{i}\lambda} - 1|^{2j}}{|(\lambda + 2\pi q)|^{2h_2+1}} \sum_{q=-\infty}^{\infty} \frac{|e^{-\mathrm{i}\lambda} - 1|^{2j}}{|(\lambda + 2\pi q)|^{2h_1+1}}} \asymp \sum_{q=-\infty}^{\infty} \frac{|e^{-\mathrm{i}\lambda} - 1|^{2j}}{|(\lambda + 2\pi q)|^{h_1+h_2+1}},$$

since from (3.24),

$$\gamma_{12}^{j,n}(\lambda) = \frac{C_{12}\sum_{q=-\infty}^{\infty} \frac{|e^{-\mathrm{i}\lambda}-1|^{2j}}{|(\lambda+2\pi q)|^{h_1+h_2+1}}}{\sqrt{C_{11}C_{22}\sum_{q=-\infty}^{\infty} \frac{|e^{-\mathrm{i}\lambda}-1|^{2j}}{|(\lambda+2\pi q)|^{2h_2+1}} \sum_{q=-\infty}^{\infty} \frac{|e^{-\mathrm{i}\lambda}-1|^{2j}}{|(\lambda+2\pi q)|^{2h_1+1}}}}.$$

Especially for (D1), $\gamma_{12}^{j,n}(\lambda) \equiv c_\gamma$.

Let us continue to assume that $(H3)$ and $\mathscr{C} = I$ hold, since results for other cases follow similarly. Let

$$\begin{aligned}
\mathbf{Y}_n^j(t) &= P'\mathbf{X}_n^j(t), \\
D_Y^{j,n}(\lambda) &= \sum_{t=1}^{n} \mathbf{Y}_n^j(t)e^{-\mathrm{i}t\lambda}, \\
I_Y^{j,n}(\lambda) &= n^{-1}D^{j,n}(w_j)D^{j,n}(\lambda)^*, \qquad j \in S_J.
\end{aligned}$$

Define the smoothed cross-periodogram as in (3.17) by

$$\hat{f}_Y^{j,h}(\lambda) = \sum_{K \in T_n} W_h(K)I_Y^{j,n}(\lambda) \qquad \hat{f}^{j,h}(\lambda) = \sum_{K \in T_n} W_h(K)I^{j,n}(\lambda).$$

By (3.25), it is easy to see that the spectral density of Y_n^j is

$$f_Y^j(\lambda) = \sum_{q=-\infty}^{\infty} n\frac{|e^{-\mathrm{i}\lambda} - 1|^{2j}}{|n(\lambda + 2\pi q)|^{2\Lambda+I}}$$

and

$$\hat{f}^{j,h}(\lambda) = P\hat{f}_Y^{j,h}(\lambda)P'. \tag{3.29}$$

Since the spectral density of Y satisfies (3.13-3.15) with $\alpha_1 = h_1, \alpha_2 = h_2, d_{11} = d_{22} = 1, d_{12} = d_{21} = 0$, the result (3.18) is applied for $\hat{f}_Y^{j,h}(\lambda)$,

$$n^{\eta}\begin{bmatrix} n^{h_1}\hat{f}_{Y,11}^{j,h}(\lambda_1) - g_Y(\lambda_1)_{11}^j \\ \dots \\ n^{h_2}\hat{f}_{Y,22}^{j,h}(\lambda_r) - g_Y(\lambda_r)_{22}^j \end{bmatrix} \to N(0, \Sigma_Y), \tag{3.30}$$

where

$$g_Y(\lambda)^j = \sum_{q=-\infty}^{\infty} \frac{|e^{-\mathrm{i}\lambda} - 1|^{2j}}{|(\lambda + 2\pi q)|^{2\Lambda + I}}$$

and Σ_Y is defined as in (3.18) except that g is replaced by g_Y. The coherence estimator of $\mathbf{X}$ is defined as

$$\tilde{\gamma}_{12}^{j,n}(\lambda) := \frac{\hat{f}_{12}^{j,h}(\lambda)}{\sqrt{\hat{f}_{11}^{j,h}(\lambda)\hat{f}_{22}^{j,h}(\lambda)}},$$

and, by (3.29), $\hat{f}^{j,h}(\lambda)$ converges to

$$P\begin{bmatrix} n^{-2h_1} & 0 \\ 0 & n^{-2h_2} \end{bmatrix} g_Y(\lambda)^j \begin{bmatrix} n^{-2h_1} & 0 \\ 0 & n^{-2h_2} \end{bmatrix} P'.$$

Note that by (3.16) and (3.24), in OFBM, $\gamma_{12}^{j,n}(\lambda) = \tilde{\gamma}_{12}$. By (3.25-3.29) and g_Y defined above,

$$\gamma_{12}^{j,n}(\lambda) = \frac{\langle \tilde{P}'(\lambda)e_1, \tilde{P}'(\lambda)e_2\rangle}{\sqrt{\langle \tilde{P}'(\lambda)e_1, \tilde{P}'(\lambda)e_1\rangle\langle \tilde{P}'e_2, \tilde{P}'(\lambda)e_2\rangle}} = \tilde{\gamma}_{12}(\lambda), \tag{3.31}$$

where

$$\tilde{P}(\lambda) = \begin{bmatrix} p_{11} & p_{12}q(\lambda) \\ p_{21} & p_{22}q(\lambda) \end{bmatrix}, \quad q(\lambda) = \frac{n^{-2h_2}g_Y(\lambda)_{22}^j}{n^{-2h_1}g_Y(\lambda)_{11}^j}. \tag{3.32}$$

Theorem 3.3.12. *Let* $\mathbf{X}(t)$ *be operator fractional Brownian motion defined as in Theorem 3.3.9. For* $\lim_{n\to\infty} 2\pi n^{-1}J = \lambda \neq 0, \lambda \in (-\pi, \pi), j \in S_J$ *with*

Case I:

$$n^{\eta}(\tilde{\gamma}_{12}^{j,n}(\lambda) - \tilde{\gamma}_{12}(\lambda)) \to N(0, \Sigma_\gamma),$$

where

$$\tilde{\gamma}_{12}(\lambda) \equiv 0.$$

Case II:

$$n^{\eta}(\tilde{\gamma}_{12}^{j,n}(\lambda) - \tilde{\gamma}_{12}(\lambda)) \to N(0, \Sigma_{\gamma}),$$

where $\tilde{\gamma}_{12}(\lambda) \asymp c_{\gamma}$ *and*

$$c_{\gamma} = \frac{(\mathscr{C}\mathscr{C}')_{1,2}}{\sqrt{(\mathscr{C}\mathscr{C}')_{1,1}(\mathscr{C}\mathscr{C}')_{2,2}}}.$$

Case III: i) $\mathscr{C} = I$ *and* $(H3)$

$$n^{\eta+2(h_2-h_1)}(\tilde{\gamma}_{12}^{j,n}(\lambda) - \tilde{\gamma}_{12}(\lambda)) \to N(0, \Sigma_{\gamma}),$$

where

$$|\tilde{\gamma}_{12}(\lambda)| - 1 = O(n^{2(h_1-h_2)}).$$

ii) $\mathscr{C} \neq I$ *and* $(H3)$

$$n^{\eta+\tau}(\tilde{\gamma}_{12}^{j,n}(\lambda) - \tilde{\gamma}_{12}(\lambda)) \to N(0, \Sigma_{\gamma}),$$

where

$$|\tilde{\gamma}_{12}(\lambda)| - 1 = O(\tau(n, \mathscr{C}, h_2, h_2)) \quad \textit{and} \quad \tau(n, \mathscr{C}, h_2, h_2) = \frac{b_{12}}{b_{11}} n^{h_1-h_2} \vee \frac{b_{22}}{b_{11}} n^{2(h_1-h_2)}.$$

Proof. For $Case I$ and II the results follow from Theorem 3.3.7 and Remark 9.

For $Case III - i)$, by (3.29), $\tilde{\gamma}_{12}^{j,n}(\lambda)$ can be written in the same way as in (3.26) with

$$\tilde{P}^{i,n}(\lambda) = \begin{bmatrix} p_{11} & p_{12}\tilde{q}^{j,n}(\lambda) \\ p_{21} & p_{22}\tilde{q}^{j,n}(\lambda) \end{bmatrix},$$

$$\tilde{q}^{j,n}(\lambda) = \sqrt{\frac{\hat{f}_{Y,22}^{j,h}(\lambda)}{\hat{f}_{Y,11}^{j,h}(\lambda)}}.$$

From (3.30), (3.32), it follows that

$$n^{\eta+2(h_2-h_1)}(\tilde{q}^{j,n}(\lambda) - q(\lambda)) \to N(0, \sigma_q^2),$$

where

$$\sigma_q^2 = q^{(1)\prime}\Sigma_Y(\lambda)q^{(1)},$$

$q^{(1)} = (-1/2\sqrt{g_Y(\lambda)^j_{22}}\{g_Y(\lambda)^j_{11}\}^{-3/2}, 1/2\{g_Y(\lambda)^j_{22}\}^{-1/2}\{g_Y(\lambda)^j_{11}\}^{-1/2})'$, and $\Sigma_Y(\lambda)$ is from (3.30) with $r = 1, \lambda_1 = \lambda$. Therefore by Lemma 3.3.11, and (3.31-3.32), the result is obtained with $\Sigma_\gamma = (\sigma_q \tilde{\gamma}^{(1)}_{12}(\lambda))^2$, where $\tilde{\gamma}^{(1)}_{12}(\lambda)$ is the first derivative of $\tilde{\gamma}_{12}$ with $q(\lambda)$. In $Case III - ii$), the result follows in the same way. □

Remark 10 *(1) As the coherence reflects the correlation between X_1 and X_2 on the spectral domain, depending on $\mathscr{C}$ and H, the dynamics of X_1 and X_2 have different features. In Case I, the coherence function is zero, which implies that it has zero correlation, and in Case II the coherence function depends on $\mathscr{C}\mathscr{C}'$. In Case III, the coherence is getting close to either 1 or -1 as sample size n is increasing on a fixed domain, and the larger the difference in Hurst indices $h_2 - h_1$ is, the closer the coherence is to 1 or -1, which implies that X_1 and X_2 have very strong correlation.*
(2) Note that by (3.16) and (3.24), in OFBM, $\gamma^{j,n}_{12}(\lambda) = \tilde{\gamma}_{12}$. In Case I and II, $\tilde{\gamma}_{12}(= \gamma^{j,n}_{12}(\lambda))$ is a function of j, λ, but not n. However, in Case III, $\tilde{\gamma}_{12}(= \gamma^{j,n}_{12}(\lambda))$ is a function of n, j, λ, and it changes according to sample size n. More specifically, it increases to one as the sample size grows.

3.4 Simulation and estimation of spectra

In this section, we provide simulation results on the squared coherence function of operator fractional Brownian motion discussed in Section 3.3.4. In practice, squared of the coherence function is more often used as it is real-valued in [0, 1], and if its value is near 1 that indicates strong linear relationship between X_1 and X_2 at particular frequency bands.

3.4.1 Independent sample paths: Case I

In this case, where $\mathscr{C}$ and H are diagonal matrices, two processes are independent. As such, the coherence is zero in every frequency. We set H as diagonal matrix with $h_1 = .4, h_2 = .8, \mathscr{C} = I$, and simulate the sample paths with $t = i/n, i = 0, .., n, n = 1000$. As the sample paths are independent,

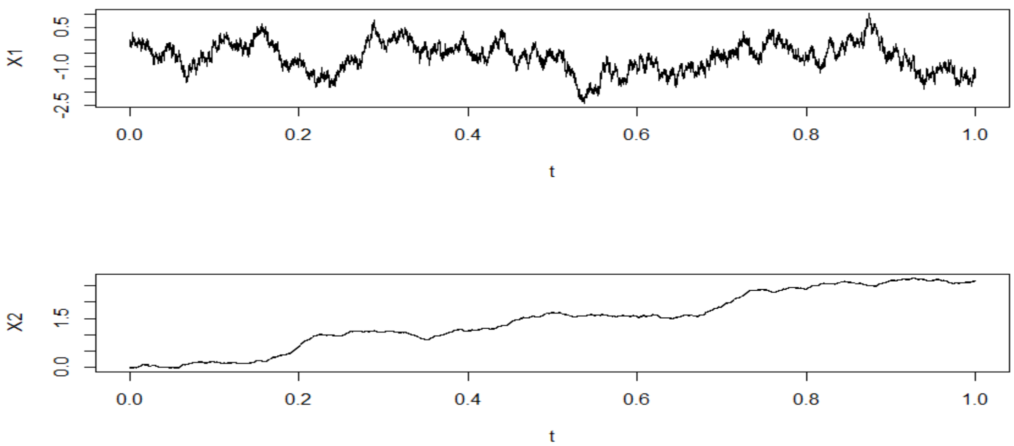


Figure 3.1: sample paths X1 and X2

each of them is governed by the Hurst index h_1, or h_2 as seen in Figure 3.1. Figure 3.2 shows the estimates of squared coherence of X_1 and X_2, $(\tilde{\gamma}_{12}^{0,n})^2, (\tilde{\gamma}_{12}^{2,n})^2$, and its theoretical coherence function $(\tilde{\gamma}_{12})^2$. The result in Figure 3.2 reflects well Theorem 3.3.12, ii). The estimates of squared coherence, $(\tilde{\gamma}_{12}^{0,n})^2, (\tilde{\gamma}_{12}^{2,n})^2$, from original series $\mathbf{X}_n^0$, and two times differenced series $\mathbf{X}_n^2$, respectively, are close to each other except on the frequency near zero and they are close to the theoretical coherence function $(\tilde{\gamma}_{12})^2$, which is zero function.

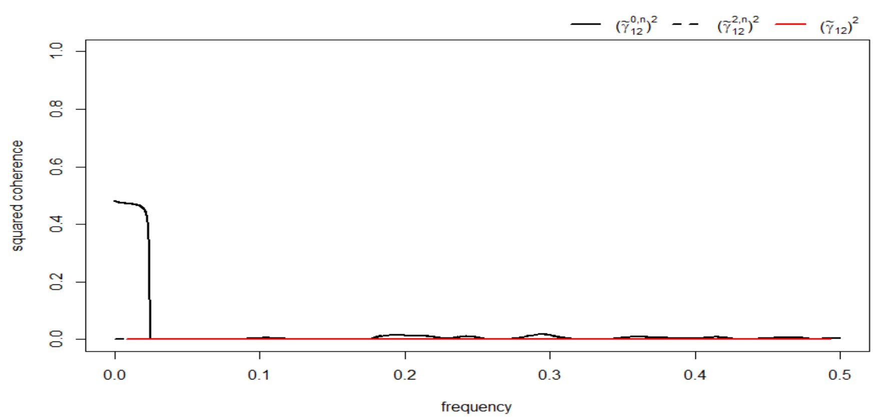


Figure 3.2: The estimates of squared coherence of X1 and X2

3.4.2 Correlated sample paths with diagonal H: Case II

If either $\mathscr{C}$ or H is not diagonal matrix, then sample paths are correlated. Let us first assume that H is diagonal but $\mathscr{C}$ is not, which is Case II. Then, as H is diagonal, the roughness of sample paths are still governed by either h_1 or h_2. But the two processes are not independent anymore. Figure 3.3 shows the simulated sample paths with $\mathscr{C} = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$ and $H = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$, $h_1 = .4, h_2 = .8$ and $t = i/n, i = 0, .., n, n = 1000$.

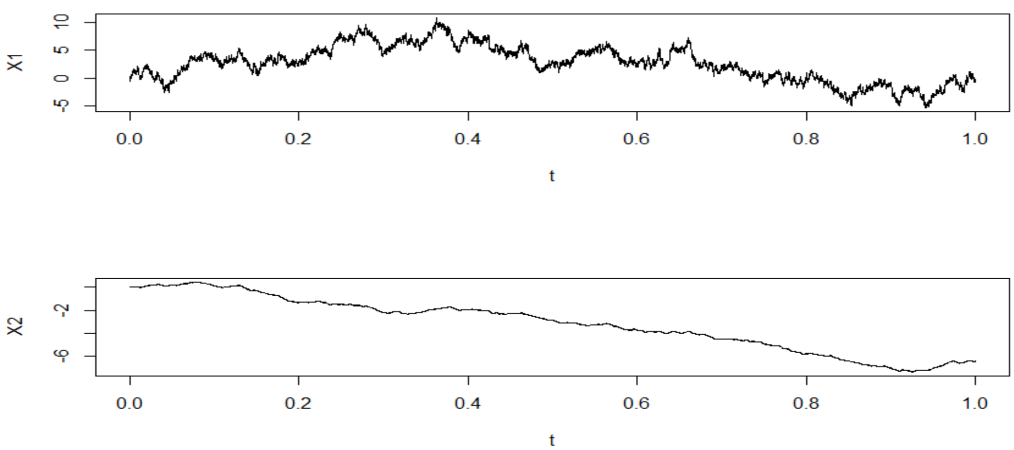


Figure 3.3: sample paths of X1 and X2

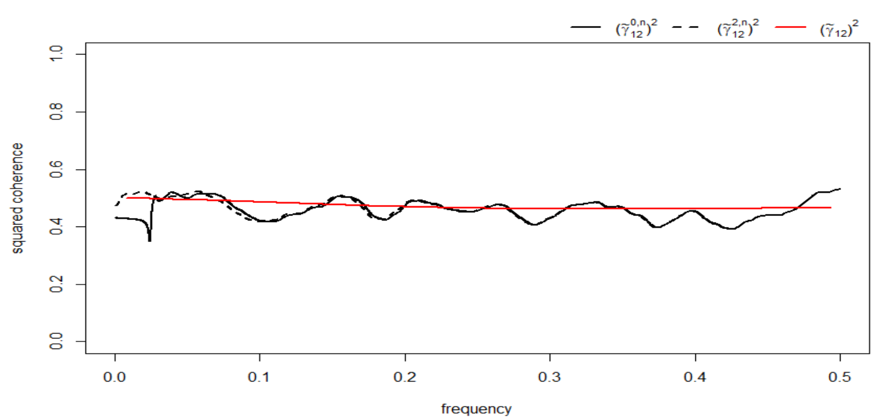


Figure 3.4: The estimates of squared coherence of X1 and X2

The estimates of squared coherence, $(\tilde{\gamma}_{12}^{0,n})^2, (\tilde{\gamma}_{12}^{2,n})^2$, are in Figure 3.4. The estimates of squared coherence are close to each other except on the frequency near zero and are close to the theoretical squared coherence function which is close to $c_\gamma^2 = .5$, since $\mathscr{C}\mathscr{C}' = \begin{pmatrix} 10 & 5 \\ 5 & 5 \end{pmatrix}$, as it was expected from Theorem 3.3.12,iii).

3.4.3 Correlated sample paths with diagonizable H: Case III

Here, we only consider the case where $\mathscr{C} = I$ but H is not diagonal, but diagonizable matrix, $H = P\Lambda P'$, where $PP' = I$ and $\Lambda = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$. The two sample paths are dependent, but their dependency is quite different from Case II and shows different behaviour in sample paths and coherence. Figure 3.5 shows the sample paths with $h_1 = .4, h_2 = .8, \mathscr{C} = I, P = \begin{pmatrix} \cos\theta & \sin -\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \theta = .2 * \pi$, with $t = i/1000, i = 0, 1, .., 1000$.

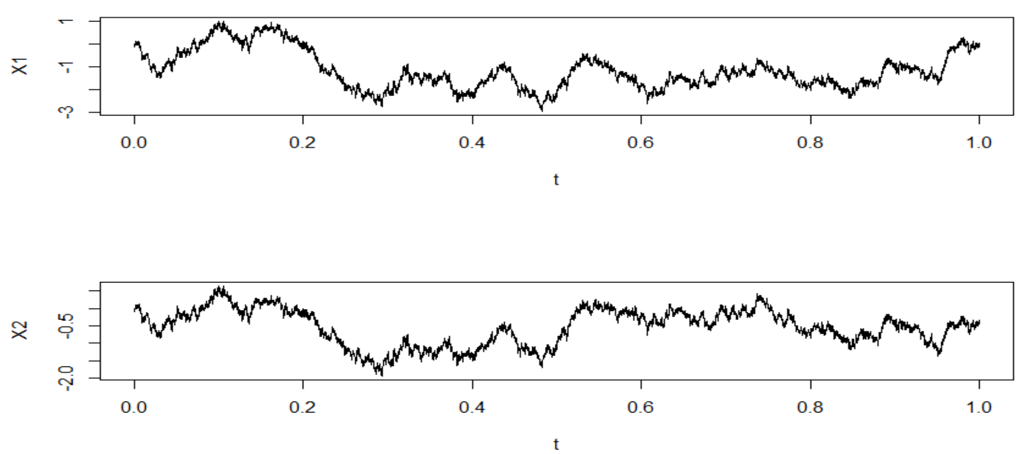


Figure 3.5: sample paths of X1 and X2

The roughness of two sample paths are similar since both paths are governed by $h_1 = .4$. From Figure 3.6, it is seen that the estimates of squared coherence are very close to each other except on frequency zero, and the theoretical coherence function is very close to 1 as it was expected from Theorem 3.3.12. As h_1, and h_2 are getting close to each other, the theoretical coherence function $\tilde{\gamma}_{12}$ and the squared of its estimates, $(\tilde{\gamma}_{12}^{0,n})^2, (\tilde{\gamma}_{12}^{2,n})^2$, are getting smaller. This is observed well in

Figure 3.7, 3.8, and this results were predicted from Theorem 3.3.12.

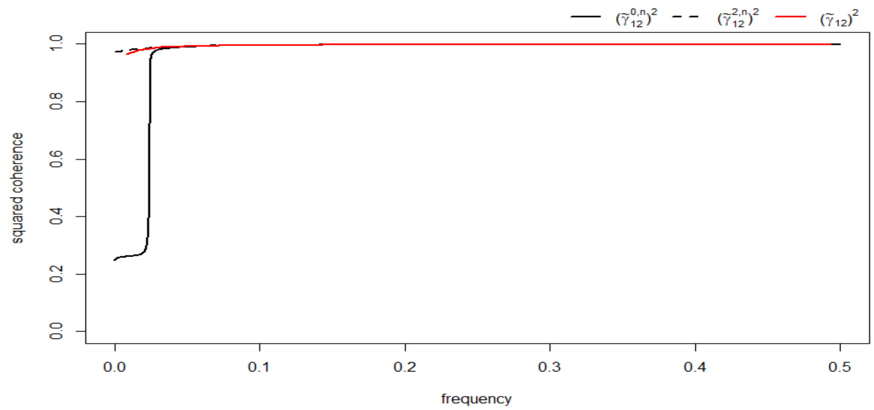


Figure 3.6: $\theta = .2 * \pi, h_1 = .4, h_2 = .8$, coherence of X1,X2

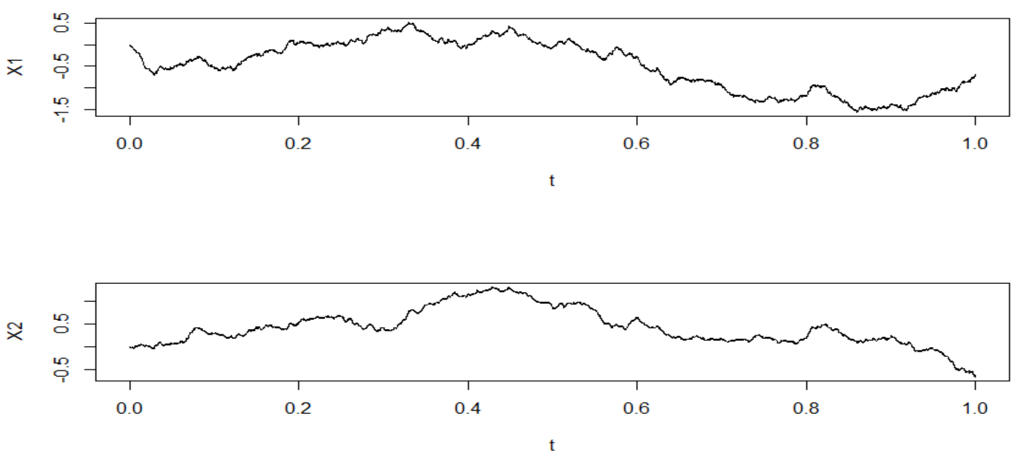


Figure 3.7: $\theta = .2 * \pi, h_1 = .65, h_2 = .8$,

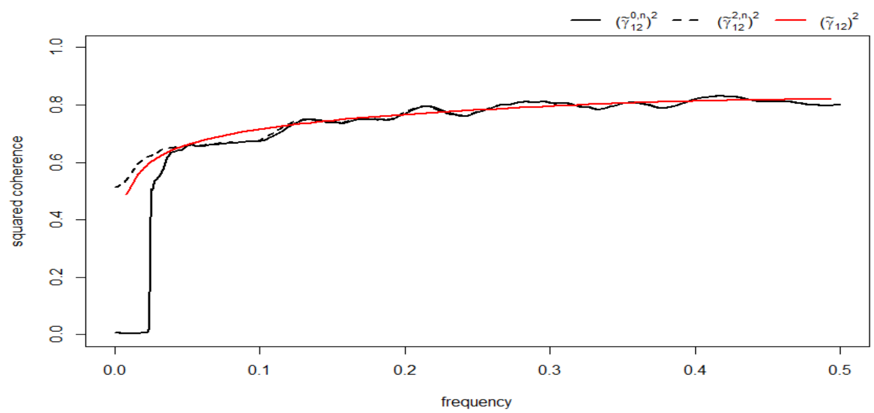


Figure 3.8: $\theta = .2 * \pi, h_1 = .65, h_2 = .8,$

CHAPTER 4

ESTIMATING HURST INDICES IN OPERATOR SCALING GAUSSIAN RANDOM FIELD

4.1 Estimation method

Let X be an operator scaling Gaussian random field that has stationary increment and operator scaling property (1.7). Its semi-variogram is defined as (1.8) and it has the following property: $\upsilon_{E,H}(c^E h) = c^{2H}\upsilon_{E,H}(h)$. Also, note that if X is operator scaling for E and $H > 0$, it is also for E/H and 1 and (E, H) is not uniquely defined. In the following sections, eigenvectors of E, u_1, u_2, and the ratios of H to eigenvalues of E, $H/\lambda_1, H/\lambda_2$, are estimated where $E = U\Lambda U', U = (u_1, u_2), \Lambda = diag(\lambda_1, \lambda_2)$.

Eigenvectors u_1, u_2, are estimated first, and $H/\lambda_1, H/\lambda_2$, are computed by using the relation

$$\{X(c^{\lambda_i}u_i t)\}_{t\in\mathbb{R}} \overset{f.d}{=} \{c^H X(u_i t)\}_{t\in\mathbb{R}} \quad \text{for any } c > 0, \tag{4.1}$$

which means that $\{X(u_i t)\}_{t\in\mathbb{R}}$ is self-similar with Hurst index $H/\lambda_i, i = 1, 2$.

Different sampling regimes are considered in this chapter. In Section 4.1, it is assumed that independent sample paths are obtained from OSGRF in a fixed domain or increasing domain. Later in simulation part, Section 4.2, the performance of the estimators is investigated with X observed on a grid in a fixed domain, $\{X(i/n, j/n), i, j = 0, 1, .., n\}$, in addition to the independent sample paths case. Throughout this chapter, notation E is used either for scaling matrix E or expected value, but with the context, there should be no confusion on what it indicates.

4.1.1 Estimation of eigenvectors of E

We make some assumptions on the matrix E and semi-variogram $\upsilon_{E,H}$.

ASSUMPTION

(A1) $\upsilon_{E,H}(t) = \tau_E(t)^{2H}$ with τ_E in (1.8), $H = 1$ and $a_1 \leq a_2, a_1, a_2 \in (0,1)$.

(A2) The matrix E in (1.7) is diagonizable with eigenvalues $\lambda_2 \neq \lambda_1$.

By (A2), we have the following:

$$E = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \theta \in [0, \pi).$$

Since $H = 1$,

$$\{X(t)\}_{t\in\mathbb{R}^2} \stackrel{f.d.}{=} \{\sum_{i=1}^{2} B_{a_i}(\langle t, u_i \rangle)\}_{t\in\mathbb{R}^2}, \tag{4.2}$$

where $u_1 = (\cos\theta, -\sin\theta)$, $u_2 = (\sin\theta, \cos\theta)$ and B_{a_1}, B_{a_2} are two independent fractional Brownian motion with semi-variogram $||\cdot||^{2a_1}, ||\cdot||^{2a_2}$, respectively.

Define for $i = 2, .., n$,

$$\nabla_1 X(i/n) = X\begin{pmatrix} i/n \\ 0 \end{pmatrix} - 2X\begin{pmatrix} (i-1)/n \\ 0 \end{pmatrix} + X\begin{pmatrix} (i-2)/n \\ 0 \end{pmatrix}, \tag{4.3}$$

$$\nabla_2 X(i/n) = X\begin{pmatrix} 0 \\ i/n \end{pmatrix} - 2X\begin{pmatrix} 0 \\ (i-1)/n \end{pmatrix} + X\begin{pmatrix} 0 \\ (i-2)/n \end{pmatrix}. \tag{4.4}$$

Note that $\{\nabla_1 X(i/n), \nabla_2 X(i/n), i = 2, .., n\}$ are (covariance) stationary processes for fixed n, and

$$\nabla_1 X(i/n) \stackrel{f.d.}{=} \left|\frac{\cos\theta}{n}\right|^{a_1} \nabla B_{a_1}(i) + \left|\frac{\sin\theta}{n}\right|^{a_2} \nabla B_{a_2}(i), \tag{4.5}$$

$$\nabla_2 X(i/n) \stackrel{f.d.}{=} \left|\frac{\sin\theta}{n}\right|^{a_1} \nabla B_{a_1}(i) + \left|\frac{\cos\theta}{n}\right|^{a_2} \nabla B_{a_2}(i), \tag{4.6}$$

where $\nabla B_{a_j}(i) = B_{a_j}(i) - 2B_{a_j}(i-1) + B_{a_j}(i-2)$, $j = 1, 2$.

In an increasing domain, the followings are defined analogous to (4.5-4.6), which is used in later Section 4.1.2. For $i = 2, .., n$,

$$\nabla_1 X(i) = X\begin{pmatrix} i \\ 0 \end{pmatrix} - 2X\begin{pmatrix} i-1 \\ 0 \end{pmatrix} + X\begin{pmatrix} i-2 \\ 0 \end{pmatrix}, \tag{4.7}$$

$$\nabla_2 X(i) = X\begin{pmatrix} 0 \\ i \end{pmatrix} - 2X\begin{pmatrix} 0 \\ i-1 \end{pmatrix} + X\begin{pmatrix} 0 \\ i-2 \end{pmatrix}. \tag{4.8}$$

$\{\nabla_1 X(i), \nabla_2 X(i), i = 2, ..., n\}$ are stationary processes for fixed n, and

$$\nabla_1 X(i) \overset{f.d.}{=} |\cos\theta|^{a_1} \nabla B_{a_1}(i) + |\sin\theta|^{a_2} \nabla B_{a_2}(i), \tag{4.9}$$

$$\nabla_2 X(i) \overset{f.d.}{=} |\sin\theta|^{a_1} \nabla B_{a_1}(i) + |\cos\theta|^{a_2} \nabla B_{a_2}(i). \tag{4.10}$$

For the rest of the chapter, we restrict $\theta \in [0, \pi/2)$, since if $\theta \in [\pi/2, \pi)$, then we can just switch $\{\nabla_1 X(i/n), \nabla_1 X(i)\}$ to $\{\nabla_2 X(i/n), \nabla_2 X(i)\}$, and have the same results.

Lemma 4.1.1. *Under the assumptions* $(A1, A2)$,

i) $\upsilon_{E,H}\begin{pmatrix} h \\ 0 \end{pmatrix} \sim |h\cos\theta|^{2a_1}$ *and* $\upsilon_{E,H}\begin{pmatrix} 0 \\ h \end{pmatrix} \sim |h\sin\theta|^{2a_1}$ *for* $|h| \to 0$ *and for fixed* $\theta \in [0, \pi/2)$.

ii) $E(\nabla_1 X(i/n)^2) = 8\upsilon_{E,H}\begin{pmatrix} 1/n \\ 0 \end{pmatrix} - 2\upsilon_{E,H}\begin{pmatrix} 2/n \\ 0 \end{pmatrix}$, $E(\nabla_2 X(i/n)^2) = 8\upsilon_{E,H}\begin{pmatrix} 0 \\ 1/n \end{pmatrix} - 2\upsilon_{E,H}\begin{pmatrix} 0 \\ 2/n \end{pmatrix}$.

iii) $\lim_n \sum_{i=2}^n cov(\nabla_j X(2)^2, \nabla_j X(i)^2) < \infty,\ j = 1, 2.$

Proof. i) Since

$$\upsilon_{E,H}\begin{pmatrix} h \\ 0 \end{pmatrix} = |h\cos\theta|^{2a_1} + |h\sin\theta|^{2a_2},$$

$$\upsilon_{E,H}\begin{pmatrix} 0 \\ h \end{pmatrix} = |h\sin\theta|^{2a_1} + |h\cos\theta|^{2a_2},$$

the results are derived.

ii)

$$\begin{aligned} E(\nabla_1 X(i/n)^2) = {} & E\left(X\begin{pmatrix} i/n \\ 0 \end{pmatrix} - X\begin{pmatrix} (i-1)/n \\ 0 \end{pmatrix}\right)^2 + E\left(X\begin{pmatrix} (i-1)/n \\ 0 \end{pmatrix} - X\begin{pmatrix} (i-2)/n \\ 0 \end{pmatrix}\right)^2 \\ & + 2E\left[\left(X\begin{pmatrix} i/n \\ 0 \end{pmatrix} - X\begin{pmatrix} (i-1)/n \\ 0 \end{pmatrix}\right)\left(-X\begin{pmatrix} (i-1)/n \\ 0 \end{pmatrix} + X\begin{pmatrix} (i-2)/n \\ 0 \end{pmatrix}\right)\right] \end{aligned}$$

$$= 2\upsilon_{E,H}\begin{pmatrix} 1/n \\ 0 \end{pmatrix} + 2\upsilon_{E,H}\begin{pmatrix} 1/n \\ 0 \end{pmatrix} + 2\upsilon_{E,H}\begin{pmatrix} 1/n \\ 0 \end{pmatrix} + 2\upsilon_{E,H}\begin{pmatrix} 1/n \\ 0 \end{pmatrix} - 2\upsilon_{E,H}\begin{pmatrix} 2/n \\ 0 \end{pmatrix} - 2\upsilon_{E,H}\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

iii) are derived by (4.7-4.10). Let $j = 1$. For $j = 2$, it is derived in the same way.

$$\lim_n \sum_{i=2}^{n} cov(\nabla_j X(2)^2, \nabla_j X(i)^2) = 2 \lim_n \sum_{i=2}^{n} cov(\nabla_j X(2), \nabla_j X(i))^2$$
$$= 2 \lim_n \sum_{i=2}^{n} (|\cos\theta|^{2a_1} cov(\nabla B_{a_1}(2), \nabla B_{a_1}(i)) + |\sin\theta|^{2a_2} cov(\nabla B_{a_2}(2), \nabla B_{a_2}(i)))^2$$

and the fact that $cov(\nabla B_{a_j}(2), \nabla B_{a_j}(i)) \sim c i^{2a_j - 4}$ for some $c \in \mathbb{R}$, thus

$$\lim_n \sum_{i=2}^{n} cov(\nabla B_{a_j}(2), \nabla B_{a_j}(i))^2 < \infty, j = 1, 2.$$

□

Define

$$P_n := \sum_{i=2}^{n} \frac{\nabla_1 X(i/n)^2}{n-1}, \qquad Q_n := \sum_{i=2}^{n} \frac{\nabla_2 X(i/n)^2}{n-1}. \tag{4.11}$$

Lemma 4.1.2. *Assumptions* $(A1, A2)$ *are satisfied. Then,*

$$i) \qquad \frac{P_n}{Q_n} - \frac{EP_n}{EQ_n} = O_p(n^{-.5})$$
$$ii) \qquad \frac{EP_n}{EQ_n} - \left(\frac{\cos\theta}{\sin\theta}\right)^{2a_1} = O(n^{2(a_1 - a_2)} \sin^{2(a_2 - a_1)}\theta \vee n^{2(a_1 - a_2)} \cos^{2(a_2 + a_1)}\theta / \sin^{4a_1}\theta)$$

Proof. i) Using (4.5-4.6), Lemma 4.1.1 iii),

$$\sigma_P^2 := \lim_n var(n^{.5+2a_1} P_n) = \lim_n \sum_{k=-n}^{n} 2(\rho_{a_1}(k) \cos^{2a_1}\theta + n^{a_1 - a_2} \rho_{a_2}(k) \sin^{2a_2}\theta)^2 \tag{4.12}$$
$$\leq \lim_n \sum_{i=2}^{n} cov(\nabla_1 X(2)^2, \nabla_1 X(i)^2) < \infty, \tag{4.13}$$
$$\sigma_Q^2 := \lim_n var(n^{.5+2a_1} Q_n) = \lim_n \sum_{k=-n}^{n} 2(\rho_{a_1}(k) \sin^{2a_1}\theta + n^{a_1 - a_2} \rho_{a_2}(k) \cos^{2a_2}\theta)^2 \tag{4.14}$$
$$\leq \lim_n \sum_{i=2}^{n} cov(\nabla_2 X(2)^2, \nabla_2 X(i)^2) < \infty, \tag{4.15}$$
$$\sigma_{PQ} := \lim_n cov(n^{.5+2a_1} P_n, (n^{.5+2a_1} Q_n) < \infty, \tag{4.16}$$

where $\rho_{a_j}(k) = cov(\nabla B_{a_j}(i), \nabla B_{a_j}(i+k))$, and $\nabla B_{a_j}(i) = B_{a_j}(i) - 2B_{a_j}(i-1) + B_{a_j}(i-2)$, $B_{a_j}(i)$ is a fractional Brownian motion with Hurst index a_j, $j = 1, 2$. Therefore by Theorem 2 in [5],

$$n^{.5+2a_1} \begin{pmatrix} P_n - EP_n \\ Q_n - EQ_n \end{pmatrix} \to_d N(0, \Sigma), \tag{4.17}$$

where $\Sigma_{11} = \sigma_P^2, \Sigma_{22} = \sigma_Q^2, \Sigma_{12} = \sigma_{PQ}$. i) follows using delta method and the fact that EP_n, EQ_n are of order n^{-2a_1} by Lemma 4.1.1 ii).

$$n^{.5}(\frac{P_n}{Q_n} - \frac{EP_n}{EQ_n}) \to_d N(0, \sigma_{P/Q}^2), \tag{4.18}$$

where $\sigma_{P/Q}^2 = \lim_n n^{4a_1}(\frac{1}{(EQ_n)^2}\sigma_P^2 + \frac{(EP_n)^2}{(EQ_n)^4}\sigma_Q^2 - \frac{1}{EQ_n}\frac{EP_n}{(EQ_n)^2}\sigma_{PQ})$.
ii) By Lemma 4.1.1 ii), it is easily derived that

$$\begin{aligned} \frac{EP_n}{EQ_n} &= \left(4\upsilon_{E,H}\begin{pmatrix} 1/n \\ 0 \end{pmatrix} - \upsilon_{E,H}\begin{pmatrix} 2/n \\ 0 \end{pmatrix}\right) \Big/ \left(4\upsilon_{E,H}\begin{pmatrix} 0 \\ 1/n \end{pmatrix} - \upsilon_{E,H}\begin{pmatrix} 0 \\ 2/n \end{pmatrix}\right) \\ &= \frac{\cos^{2a_1}\theta + n^{2(a_1-a_2)}(4-2^{2a_2})(4-2^{2a_1})^{-1}\sin^{2a_2}\theta}{\sin^{2a_1}\theta + n^{2(a_1-a_2)}(4-2^{2a_2})(4-2^{2a_1})^{-1}\cos^{2a_2}\theta}. \end{aligned}$$

□

Define for $i = 4, \ldots, n$,

$$\nabla_1^* X(i/n) = X\begin{pmatrix} i/n \\ 0 \end{pmatrix} - 2X\begin{pmatrix} (i-2)/n \\ 0 \end{pmatrix} + X\begin{pmatrix} (i-4)/n \\ 0 \end{pmatrix}.$$

Let

$$P_n^* := \sum_{i=4}^{n} \frac{\nabla_1^* X(i/n)^2}{n-3}.$$

Corollary 4.1.3. *Under the assumptions* $(A1, A2)$*,*

$$i)\ \frac{P_n^*}{Q_n} - \upsilon_{E,H}\begin{pmatrix}4/n\\0\end{pmatrix}\Big/\upsilon_{E,H}\begin{pmatrix}0\\2/n\end{pmatrix} = O_p(n^{-.5})$$

$$ii)\ \upsilon_{E,H}\begin{pmatrix}4/n\\0\end{pmatrix}\Big/\upsilon_{E,H}\begin{pmatrix}0\\2/n\end{pmatrix} - \left(\frac{2\cos\theta}{\sin\theta}\right)^{2a_1} = O(n^{2(a_1-a_2)}\sin^{2(a_2-a_1)}\theta \vee n^{2(a_1-a_2)}\cos^{2a_2}\theta/\sin^{2a_1}\theta)$$

$$iii)\ \frac{P_n^*}{P_n} - \upsilon_{E,H}\begin{pmatrix}4/n\\0\end{pmatrix}\Big/\upsilon_{E,H}\begin{pmatrix}2/n\\0\end{pmatrix} = O_p(n^{-.5})$$

$$iv)\ \upsilon_{E,H}\begin{pmatrix}4/n\\0\end{pmatrix}\Big/\upsilon_{E,H}\begin{pmatrix}2/n\\0\end{pmatrix} - 2^{2a_1} = O(n^{2(a_1-a_2)}\sin^{2a_2}\theta/\cos^{2a_1}\theta)$$

The estimation method for θ is the following:

Step 1) Estimate $2a_1$ by the ratios of $\frac{P_n^*}{Q_n}$ and $\frac{P_n}{Q_n}$.

$$\widehat{2a_1} = \log\left(\frac{P_n^*}{P_n}\right)/\log 2. \tag{4.19}$$

Step 2) Estimate θ with $\widehat{2a_1}$ and $\frac{P_n}{Q_n}$.

$$\frac{\cos\hat{\theta}}{\sin\hat{\theta}} = \left(\frac{P_n}{Q_n}\right)^{1/\widehat{2a_1}}.$$

Then the estimator of θ is

$$\hat{\theta}_n := \cot^{-1}\left(\left(\frac{P_n}{Q_n}\right)^{1/\widehat{2a_1}}\right). \tag{4.20}$$

Theorem 4.1.4. *Under the assumptions* $(A1, A2)$

$$i)\qquad \widehat{2a_1} - 2a_1 = O_p\left(n^{2(a_1-a_2)}\left(\frac{\sin^{2a_2}\theta}{\sin^{2a_1}\theta} \vee \frac{\cos^{2(a_2+a_1)}\theta}{\sin^{4a_1}\theta}\right) \vee n^{-.5}\right) \tag{4.21}$$

$$ii)\qquad \hat{\theta}_n - \theta = O_p(n^{2(a_1-a_2)}(\sin^{2a_2}\theta \vee \sin^{2-4a_1}\theta\cos^{2(a_2+a_1)}\theta) \vee n^{-.5}\sin^2\theta) \tag{4.22}$$

Proof. By (4.19), $\widehat{2a_1} - 2a_1$ has the same order as $\frac{P_n}{Q_n} - (\frac{\cos\theta}{\sin\theta})^{2a_1}$ and $\frac{P_n^*}{Q_n} - (\frac{2\cos\theta}{\sin\theta})^{2a_1}$, therefore i) follows from Lemma 4.1.2 and its corollary.

For ii), note that $(\frac{P_n}{Q_n})^{1/\hat{2a_1}} - \frac{\cos\theta}{\sin\theta}$ has the same order of $\hat{2a_1} - 2a_1$ and $\frac{P_n}{Q_n} - (\frac{\cos\theta}{\sin\theta})^{2a_1}$. By (4.20) and

$$\frac{d\theta}{d\cot\theta} = -\sin^2\theta,$$

the result follows since

$$\begin{aligned}
\hat{\theta}_n - \theta &= O_p\big(d\theta/d\cot\theta\big(\frac{P_n}{Q_n}^{1/\hat{2a_1}} - \frac{\cos\theta}{\sin\theta}\big)\big)\\
&= O_p(n^{2(a_1-a_2)}\sin^2\theta(\frac{\sin^{2a_2}\theta}{\sin^{2a_1}\theta} \vee \frac{\cos^{2(a_2+a_1)}\theta}{\sin^{4a_1}\theta}) \vee n^{-.5}\sin^2\theta)\\
&= O_p(n^{2(a_1-a_2)}(\sin^{2a_2}\theta \vee \sin^{2-4a_1}\theta\cos^{2(a_2+a_1)}\theta) \vee n^{-.5}\sin^2\theta).
\end{aligned}$$

□

Remark 2. (a). Theorem 4.1.4 implies that the estimates of $2a_1$ and $\cos\theta/\sin\theta$, $\hat{2a_1}$ and $(\frac{P_n}{Q_n})^{1/\hat{2a_1}}$, respectively, are not good when θ is close to zero. But the estimate of θ is uniformly good throughout $\theta \in [0, \pi)$.

(b). In (4.19) (step 1), one can use the ratio of P_n^* and P_n directly, instead of the ratios of P_n^*/Q_n and P_n/Q_n to estimate $2a_1$, and will get the same asymptotic result as in Theorem 4.1.4, which is because of the previous corollary iii),iv).

(c). Theorem 4.1.4 implies consistency of $2\hat{a}_1, \hat{\theta}_n$ in probability. Also note that the bias and variance of $2\hat{a}_1$ are of order $O_p\left(n^{2(a_1-a_2)}(\frac{\sin^{2a_2}\theta}{\sin^{2a_1}\theta} \vee \frac{\cos^{2(a_2+a_1)}\theta}{\sin^{4a_1}\theta})\right), O_p(n^{-1})$, respectively, following the order of the bias and variance of $\frac{P_n}{Q_n}, \frac{P_n^*}{Q_n}$. Likewise, the bias and the variance of $\hat{\theta}_n$ are of order $n^{2(a_1-a_2)}(\sin^{2a_2}\theta \vee \sin^{2-4a_1}\theta\cos^{2(a_2+a_1)}\theta)$, $n^{-1}\sin^4\theta$, respectively.

(d). By (4.12-4.15), it can be seen that the variances $\sigma_P^2, \sigma_Q^2, \sigma_{P/Q}$ in (4.16-4.18) are dependent on a_1, thus the variance of $2\hat{a}_1, \hat{\theta}_n$ are dependent on a_1. More specifically, the smaller a_1 is, the bigger the variances $\sigma_P^2, \sigma_Q^2, \sigma_{P/Q}$ are, as well as the variances of $2\hat{a}_1, \hat{\theta}_n$.

4.1.2 Estimation of ratios of $H(=1)$ to eigenvalues of E

Define $\hat{u_1} = \begin{pmatrix} \cos\hat{\theta} \\ -\sin\hat{\theta} \end{pmatrix}$, and $\hat{u_2} = \begin{pmatrix} \sin\hat{\theta} \\ \cos\hat{\theta} \end{pmatrix}$. Using the relation,

$$\{X(cu_i t)\}_{t\in\mathbb{R}} \overset{f.d.}{=} \{c^{1/\lambda_i} X(u_i t)\}_{t\in\mathbb{R}}, \tag{4.23}$$

we estimate $a_i = 1/\lambda_i, i = 1,2$, which are Hurst indices along the directions $u_i, i = 1,2$, respectively. Among the many well-known Hurst estimation methods, here, we employ discrete variation method.

For a fixed domain, $[0,1]\times[0,1]$, and for $m \in \mathbb{N}\cap\{0\}, 2^m << n$, define

$$\hat{\nabla}_j^m X(i/n) = X\left(\frac{i-2^{m+1}}{n}\hat{u}_j\right) - 2X\left(\frac{i-2^m}{n}\hat{u}_j\right) + X\left(\frac{i}{n}\hat{u}_j\right) \quad j = 1,2, i = 2^{m+1}, ..., n. \tag{4.24}$$

Note that $\{\hat{\nabla}_j^m X(i/n)\}_i$ is a discretized sample path with direction $\hat{u}_j$. If $\hat{\theta} = 0$ then $\hat{\nabla}_j^0 X(i/n) = \nabla_j X(i/n)$, otherwise $\hat{\nabla}_j^0 X(i/n) \approx \nabla_j X(i/n)$. Similar to P_n, Q_n in (4.11), define

$$\hat{P}_n^m := \sum_{i=2^{m+1}}^{n} \frac{\hat{\nabla}_1^m X(i/n)^2}{n-2^{m+1}+1}, \qquad \hat{Q}_n^m := \sum_{i=2^{m+1}}^{n} \frac{\hat{\nabla}_2^m X(i/n)^2}{n-2^{m+1}+1}. \tag{4.25}$$

The estimates of a_1 and a_2 are log-regression of $\{\hat{P}_n^m; m = 1,2,...,\ell_n\}$, $\{\hat{Q}_n^m; m = 1,2,...,\ell_n\}$ on $\{2m\log 2, m = 1,2,...,\ell_n\}$ respectively, i.e.

$$\hat{a}_1 = \frac{1}{2}\sum_{m=1}^{\ell_n} w_m \log_2 \hat{P}_n^m, \tag{4.26}$$

$$\hat{a}_2 = \frac{1}{2}\sum_{m=1}^{\ell_n} w_m \log_2 \hat{Q}_n^m, \tag{4.27}$$

where

$$\sum_{m=1}^{\ell_n} w_m = 0, \sum_{m=1}^{\ell_n} m w_m = 1.$$

Define

$$a_1^E := \frac{1}{2}\sum_{m=1}^{\ell_n} w_m \log_2 E_{X|\hat{u}_1}(\hat{P}_n^m),$$

$$a_2^E := \frac{1}{2}\sum_{m=1}^{\ell_n} w_m \log_2 E_{X|\hat{u}_2}(\hat{Q}_n^m),$$

where the expectation is for X given $\hat{u}_j, j = 1,2$.

Theorem 4.1.5. *In a fixed domain with the assumptions* $(A1, A2)$,

i)

$$n^{.5+2a_1}(\hat{a}_1 - a_1^E) \to_d N(0, \sigma_{a_1}),$$

$$a_1^E - a_1 = \begin{cases} O_P(n^{-2(a_2-a_1)-a_2}) & \text{if } a_2 - a_1 > .25, \\ O_P(n^{-(2+4a_2)(a_2-a_1)}) & \text{if } a_2 - a_1 \le .25. \end{cases}$$

ii) If $a_2 - a_1 \le .25$ *and* $a_1 > .5$,

$$n^{.5+2a_2}(\hat{a}_2 - a_2^E) \to_d N(0, \sigma_{a_2}),$$

$$a_2^E - a_2 = O_P(n^{-(4a_1-2)(a_2-a_1)}).$$

Proof. For a_1^E, by (4.2),

$$E_{X|\hat{u}_1}(\hat{\nabla}_1^m X(i/n)^2) = c_{a_1}^* \left(\frac{2^{2ma_1}}{n^{2a_1}}\right) + c_{a_2}\epsilon_1^{2a_2}\left(\frac{2^{2ma_2}}{n^{2a_2}}\right) \tag{4.28}$$

$$= c_{a_1}^* \left(\frac{2^{2ma_1}}{n^{2a_1}}\right)(1 + c_{a_2}/c_{a_1}^* \epsilon_1^{2a_2} 2^{2m(a_2-a_1)} n^{2a_1-2a_2}), \tag{4.29}$$

where $c_{a_1}^* = E(\nabla B_{a_1}(i)^2)(u_1'\hat{u}_1)^{2a_1}, c_{a_2} = E(\nabla B_{a_2}(i)^2)$, and $\epsilon_1 = u_2'\hat{u}_1$ which has the order of (4.22). Note that $u_1'\hat{u}_1 \sim 1$. Since the order of $\epsilon_1^{2a_2} n^{2a_1-2a_2}$ is either less than $n^{2(a_1-a_2)-a_2}$ or $n^{-(2+4a_2)(a_2-a_1)}$ depending on whether $a_2 - a_1 > .25$ or $a_2 - a_1 \le .25$, the result follows.

For a_2^E,

$$E_{X|\hat{u}_2}(\hat{\nabla}_2^m X(i/n)^2) = c_{a_1}\epsilon_2^{2a_1}\left(\frac{2^{2ma_1}}{n^{2a_1}}\right) + c_{a_2}^*\left(\frac{2^{2ma_2}}{n^{2a_2}}\right) \tag{4.30}$$

$$= c_{a_2}^* \left(\frac{2^{2ma_2}}{n^{2a_2}}\right)(1 + c_{a_1}/c_{a_2}^* \epsilon_2^{2a_1} 2^{2m(a_1-a_2)} n^{2a_2-2a_1}), \tag{4.31}$$

where $c_{a_1} = E(\nabla B_{a_1}(i)^2), c_{a_2}^* = E(\nabla B_{a_2}(i)^2)(u_2'\hat{u}_2)^{2a_2}$ and $\epsilon_2 = u_1'\hat{u}_2$ which has the order of (4.22). Since $\epsilon_2^{2a_1} n^{2a_2-2a_1}$ is of order $n^{-(4a_1-2)(a_2-a_1)}$ when $a_2 - a_1 \le .25$, and $a_1 > .5$ or divergent otherwise, the result follows. Asymptotic normality of $\hat{a}_1, \hat{a}_2$ follows from (4.17) with delta method. $\square$

Remark 3. (a). If the condition of ii) in Theorem 4.1.5 is not met, then $\hat{a}_2$ goes to a_1. This is because $\epsilon_2^{2a_1} n^{2a_2-2a_1}$ in (4.31) diverges and therefore as $n \longrightarrow \infty$

$$\log_2(1 + c_{a_1}/c^*_{a_2}\epsilon_2^{2a_1} 2^{2m(a_1-a_2)} n^{2a_2-2a_1}) \simeq \log_2(c_{a_1}/c^*_{a_2}\epsilon_2^{2a_1} 2^{2m(a_1-a_2)} n^{2a_2-2a_1}),$$

and

$$a_2^E = \frac{1}{2}\sum_{m=1}^{\ell_n} w_m \log_2 E_{X|\hat{u}_2}(\hat{\nabla}_2^m X(i/n)^2) \sim a_2 + (a_1 - a_2) = a_1.$$

(b). Note that here it is assumed that independent sample paths are observed. However, it is expected to have the same bias results for $a_i^E - a_i, i = 1, 2$, even when $\hat{u}_i$ and $\{X(t\hat{u}_i)\}_{t\in\mathbb{R}}$ are dependent (e.g., they are in the same sample surface.) This is because $\hat{P}_n^m$ and $\hat{Q}_n^m$ converge a.s. to its expectation. For the same reason, the bias of the estimators in Theorem 4.1.6 and Theorem 4.1.10 remain true when $\{X\begin{pmatrix} t \\ 0 \end{pmatrix}, X\begin{pmatrix} 0 \\ t \end{pmatrix}, X(t\hat{u}_i)\}_{t\in\mathbb{R}}$ is from one sample surface.

(c). Note that a_1 is estimated two times in the whole estimating procedure: The first time is when θ is estimated in section 4.1.1 (Theorem 4.1.4) and the other is in this section 4.1.2 (Theorem 4.1.5). This is because $\widehat{2a_1}$ was estimated to get the estimate of θ so that (a_1, a_2) are estimated. However, $\widehat{2a_1}$ estimated the first time has both larger bias and variance (Theorem 4.1.4) compared to $\hat{a}_1$ in this section (Theorem 4.1.5).

As the above lemma indicates that the estimator $\hat{a}_2$ is not good when a_1 is small, one can naturally use the sample in a different domain (i.e., an increasing domain.) In an increasing domain, the estimators are obtained in the same way in (4.25-4.27) except that $\hat{\nabla}_j^m X(i/n)$ is changed to $\hat{\nabla}_j^m X(i)$.

Theorem 4.1.6. *For an increasing domain with the assumptions* $(A1, A2)$,

i)

$$\sqrt{n}(\hat{a}_1 - a_1^E) \rightarrow_d N(0, \sigma_{a_1}),$$

$$a_1^E - a_1 = \begin{cases} O_P(n^{-a_2}) & \textit{if } a_2 - a_1 > .25, \\ O_P(n^{-4a_2(a_2-a_1)}) & \textit{if } a_2 - a_1 \leq .25. \end{cases}$$

ii)

$$\sqrt{n}(\hat{a}_2 - a_2^E) \to_d N(0, \sigma_{a_2}),$$

$$a_2^E - a_2 = \begin{cases} O_P(n^{-a_1}) & \text{if } a_2 - a_1 > .25, \\ O_P(n^{-4a_1(a_2-a_1)}) & \text{if } a_2 - a_1 \le .25. \end{cases}$$

Proof.

$$\begin{aligned} E_{X|\hat{u}_1}(\hat{\nabla}_1^m X(i)^2) &= c_{a_1}^* 2^{2ma_1} + c_{a_2}\epsilon_1^{2a_2}(2^{2ma_2}) \\ &= c_{a_1}^* 2^{2ma_1}(1 + c_{a_2}/c_{a_1}^* \epsilon_1^{2a_2} 2^{2m(a_2-a_1)}), \end{aligned}$$

where $c_{a_1}^* = E(\nabla B_{a_1}(i)^2)(u_1'\hat{u}_1)^{2a_1}, c_{a_2} = E(\nabla B_{a_2}(i)^2)$ and $\epsilon_1 = u_2'\hat{u}_1$ which has the order of (4.22). Since the order of ϵ^{2a_2} is either less than n^{-a_2} or $n^{-4a_2(a_2-a_1)}$ depending on whether $a_2 - a_1 > .25$ or $a_2 - a_1 \le .25$, the result follows. For ii),

$$\begin{aligned} E_{X|\hat{u}_2}(\hat{\nabla}_2^m X(i)^2) &= c_{a_1}\epsilon_2^{2a_1} 2^{2ma_1} + c_{a_2}^* 2^{2ma_2} \\ &= c_{a_2}^* 2^{2ma_2}(1 + c_{a_1}/c_{a_2}^* \epsilon_2^{2a_1} 2^{2m(a_1-a_2)}), \end{aligned}$$

where $c_{a_1} = E(\nabla B_{a_1}(i)^2), c_{a_2}^* = E(\nabla B_{a_2}(i)^2)(u_2'\hat{u}_2)^{2a_2}$ and $\epsilon_2 = u_1'\hat{u}_2$ which has the order of (4.22). The result follows for the order of $a_2^E - a_2$. Asymptotic normality of $\hat{a}_1, \hat{a}_2$ follows from the fact that

$$\begin{aligned} \sqrt{n}\Bigg(\sum_{i=2^{m+1}}^{n} \frac{\hat{\nabla}_1^m X(i/n)^2}{n - 2^{m+1} + 1} - E(\hat{\nabla}_1^m X(i)^2)\Bigg) &\to_d N(0, \Sigma_1), \\ \sqrt{n}\Bigg(\sum_{i=2^{m+1}}^{n} \frac{\hat{\nabla}_2^m X(i/n)^2}{n - 2^{m+1} + 1} - E(\hat{\nabla}_2^m X(i)^2)\Bigg) &\to_d N(0, \Sigma_2), \end{aligned}$$

which can be derived in the same way as (4.17). □

4.1.3 For $H < 1$

From now on, the results for $H = 1$ has been shown. This constraint is relaxed in this section, and it will be seen that the results remain the same except the estimators are for $h_i = H/\lambda_i = Ha_i, i = 1, 2$.

Since $H < 1$, (4.5-4.6),(4.9-4.10) no longer hold. Nonetheless, most of the results hold in the similar way.

ASSUMPTION

$(A1')$ $\upsilon_{E,H}(x) = \tau_E(x)^{2H}$ with τ_E in (2), $H < 1$ and $a_1 \le a_2, a_1, a_2 \in (0,1)$.

Lemma 4.1.7. *Under the assumptions* $(A1', A2)$*,*

i) $\upsilon_{E,H}\begin{pmatrix} h \\ 0 \end{pmatrix} \sim |h\cos\theta|^{2h_1}$ *and* $\upsilon_{E,H}\begin{pmatrix} 0 \\ h \end{pmatrix} \sim |h\sin\theta|^{2h_1}$ *for* $|h| \longrightarrow 0$.

ii) $E(\nabla_1 X(i/n)^2) = 8\upsilon_{E,H}\begin{pmatrix} 1/n \\ 0 \end{pmatrix} - 2\upsilon_{E,H}\begin{pmatrix} 2/n \\ 0 \end{pmatrix}$, $E(\nabla_2 X(i/n)^2) = 8\upsilon_{E,H}\begin{pmatrix} 0 \\ 1/n \end{pmatrix} - 2\upsilon_{E,H}\begin{pmatrix} 0 \\ 2/n \end{pmatrix}$.

iii) $\lim_n n^{4h_1} \sum_{i=2}^{n} cov(\nabla_j X(2/n)^2, \nabla_j X(i/n)^2) < \infty, \quad j = 1, 2.$

Proof. i) Since

$$\upsilon_{E,H}\begin{pmatrix} h \\ 0 \end{pmatrix} = (|h\cos\theta|^{2a_1} + |h\sin\theta|^{2a_2})^H,$$
$$\upsilon_{E,H}\begin{pmatrix} 0 \\ h \end{pmatrix} = (|h\sin\theta|^{2a_1} + |h\cos\theta|^{2a_2})^H,$$

the results are derived. ii) It is proved in the same way as in Lemma 4.4.1.

iii) Let $j = 1$. For $j = 2$, the proof goes in the same way. Note that $cov(\nabla_j X(2/n)^2, \nabla_j X(i/n)^2) = 2cov(\nabla_j X(2/n), \nabla_j X(i/n))^2$. For large enough n,

$$cov(\nabla_1 X(2/n), \nabla_1 X(i/n)) \sim c_1 \upsilon_{E,H}^{(4)}\begin{pmatrix} i/n \\ 0 \end{pmatrix}$$

for some constant c_1, where $\upsilon_{E,H}^{(4)}\begin{pmatrix} i/n \\ 0 \end{pmatrix}$ is the fourth derivative of $\upsilon_{E,H}\begin{pmatrix} i/n \\ 0 \end{pmatrix}$ with respect to i. This is because

$$E(X\begin{pmatrix} i+1 \\ 0 \end{pmatrix} X\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = -2\upsilon_{E,H}\begin{pmatrix} i \\ 0 \end{pmatrix} + E\left(X\begin{pmatrix} i+1 \\ 0 \end{pmatrix}^2\right)\Big/2 + E\left(X\begin{pmatrix} 1 \\ 0 \end{pmatrix}^2\right)\Big/2$$

and therefore

$$cov(\nabla_1 X(2/n), \nabla_1 X(i/n)) = -2\Big\{\upsilon_{E,H}\begin{pmatrix} (i+1)/n \\ 0 \end{pmatrix} - 4\upsilon_{E,H}\begin{pmatrix} i/n \\ 0 \end{pmatrix} + 6\upsilon_{E,H}\begin{pmatrix} (i-1)/n \\ 0 \end{pmatrix}$$
$$- 4\upsilon_{E,H}\begin{pmatrix} (i-2)/n \\ 0 \end{pmatrix} + \upsilon_{E,H}\begin{pmatrix} (i-3)/n \\ 0 \end{pmatrix}\Big\}.$$

Since $n^{2h_1}\upsilon_{E,H}^{(4)}\begin{pmatrix} i/n \\ 0 \end{pmatrix}$ is the same as the fourth derivative of $(|i\cos\theta|^{2a_1} + n^{2a_1-2a_2}|i\sin\theta|^{2a_2})^H$, which is

$$\sum_{\substack{k\geq\ell\geq m\geq p\geq 0 \\ k+\ell+m+p=4}} c_{k,\ell,m,p}(|i\cos\theta|^{2a_1} + n^{2(a_1-a_2)}|i\sin\theta|^{2a_2})^{H-k}(|i\cos\theta|^{2a_1-1} + n^{2(a_1-a_2)}|i\sin\theta|^{2a_2-1})^{k-\ell}$$
$$\times (|i\cos\theta|^{2a_1-2} + n^{2(a_1-a_2)}|i\sin\theta|^{2a_2-2})^{\ell-m}(|i\cos\theta|^{2a_1-3} + n^{2(a_1-a_2)}|i\sin\theta|^{2a_2-3})^{m-p}$$
$$\times (|i\cos\theta|^{2a_1-4} + n^{2(a_1-a_2)}|i\sin\theta|^{2a_2-4})^p < c_2 i^{2a_1H-4}$$

for some constants $c_{k,\ell,m,p}, c_2$. Therefore $n^{2h_1}cov(\nabla_1 X(2/n), \nabla_1 X(i/n)) < c_2 i^{2a_1H-4}$, and the results follow since $a_1 H < 1$.

□

Lemma 4.1.8. *Assumptions* $(A1', A2)$ *are satisfied. Then,*

$$i)\ \frac{P_n}{Q_n} - \upsilon_{E,H}\begin{pmatrix} 2/n \\ 0 \end{pmatrix}\Big/\upsilon_{E,H}\begin{pmatrix} 0 \\ 2/n \end{pmatrix} = O_p(n^{-.5})$$

$$ii)\ \upsilon_{E,H}\begin{pmatrix} 2/n \\ 0 \end{pmatrix}\Big/\upsilon_{E,H}\begin{pmatrix} 0 \\ 2/n \end{pmatrix} - \left(\frac{\cos\theta}{\sin\theta}\right)^{2h_1} = O(n^{2(a_1-a_2)}\sin^{2(a_2-a_1)}\theta \vee n^{2(a_1-a_2)}\cos^{2a_2}\theta/\sin^{2a_1}\theta)$$

Proof. i) By Theorem 2 in [5] and

$$\sigma_P^2 := \lim_n var(n^{.5+2h_1} P_n) < 2 \lim_n n^{4h_1} \sum_{i=2}^{n} cov(\nabla_1 X(2/n)^2, \nabla_1 X(i/n)^2) < \infty,$$

$$\sigma_Q^2 := \lim_n var(n^{.5+2h_1} Q_n) < 2 \lim_n n^{4h_1} \sum_{i=2}^{n} cov(\nabla_1 X(2/n)^2, \nabla_1 X(i/n)^2) < \infty,$$

follows

$$n^{.5+2h_1}(P_n - EP_n) \to N(0, \sigma_P), \tag{4.32}$$

$$n^{.5+2h_1}(Q_n - EQ_n) \to N(0, \sigma_Q). \tag{4.33}$$

Therefore

$$n^{.5}\left(\frac{P_n}{Q_n} - \frac{EP_n}{EQ_n}\right) \to N(0, \sigma_{P/Q}),$$

where $\sigma_{P/Q}^2 = \lim_n n^{4h_1}\left(\frac{1}{EQ_n^2}\sigma_P^2 + \frac{EP_n^2}{EQ_n^4}\sigma_Q^2\right)$.
For ii), it is easily derived from the fact that

$$\upsilon_{E,H}\begin{pmatrix} 2/n \\ 0 \end{pmatrix} \Big/ \upsilon_{E,H}\begin{pmatrix} 0 \\ 2/n \end{pmatrix} = \left(\frac{\cos^{2a_1}\theta + 2n^{2a_1-a_2}\sin^{2a_2}\theta}{\sin^{2a_1}\theta + 2n^{2a_1-a_2}\cos^{2a_2}\theta}\right)^H.$$

□

By the above results and the same way as in Theorem 4.1.4, the next theorem follows.

Theorem 4.1.9. *Under the assumptions* $(A1', A2)$,

$$i) \quad \hat{2h_1} - 2h_1 = O_p\left(n^{2(a_1-a_2)}\left(\frac{\sin^{2a_2}\theta}{\sin^{2a_1}\theta} \vee \frac{\cos^{2a_2}\theta}{\sin^{2a_1}\theta}\right) \vee n^{-.5}\right)$$

$$ii) \quad \hat{\theta}_n - \theta = O_p(n^{2(a_1-a_2)} \vee n^{-.5})$$

(4.23) is now changed to the following equation.

$$\{X(cu_i t)\}_{t\in\mathbb{R}} \overset{f.d.}{=} \{c^{H/\lambda_i} X(u_i t)\}_{t\in\mathbb{R}}.$$

Using the same estimator as (4.26-4.27), now one can obtain the estimator of h_i, $h_i = H/\lambda_i, i = 1, 2$.

$$\hat{h}_1 = \frac{1}{2}\sum_{m=1}^{\ell_n} w_m \log_2 \hat{P}_n^m. \tag{4.34}$$

$$\hat{h}_2 = \frac{1}{2}\sum_{m=1}^{\ell_n} w_m \log_2 \hat{Q}_n^m, \tag{4.35}$$

where $\hat{P}_n^m, \hat{Q}_n^m$ defined same as (4.25).

Theorem 4.1.10. *In a fixed domain with the assumptions* $(A1', A2)$,

i)

$$n^{.5+2a_1}(\hat{h}_1 - h_1^E) \to_d N(0, \sigma_{h_1}),$$

$$h_1^E - h_1 = \begin{cases} O_P(Hn^{-2(a_2-a_1)-a_2}) & \text{if } a_2 - a_1 > .25, \\ O_P(Hn^{-(2+4a_2)(a_2-a_1)}) & \text{if } a_2 - a_1 \leq .25. \end{cases}$$

ii) If $a_2 - a_1 \leq .25$ *and* $a_1 > .5$,

$$n^{.5+2a_2}(\hat{h}_2 - h_2^E) \to N(0, \sigma_{h_2}),$$
$$h_2^E - h_2 = O_P(Hn^{-(4a_1-2)(a_2-a_1)}).$$

Proof. For h_1^E, by Lemma 4.1.7 ii), it is derived that

$$\begin{aligned} E_{X|\hat{u}_1}(\hat{\nabla}_1^m X(i/n)^2) &= 8\left(b_{a_1}\left(\frac{2^{2ma_1}}{n^{2a_1}}\right) + \epsilon_1^{2a_2}\left(\frac{2^{2ma_2}}{n^{2a_2}}\right)\right)^H - 2\left(b_{a_1}\left(\frac{2^{2(m+1)a_1}}{n^{2a_1}}\right) + \epsilon_1^{2a_2}\left(\frac{2^{2(m+1)a_2}}{n^{2a_2}}\right)\right)^H \\ &= \left(\frac{2^{2mHa_1}}{n^{2Ha_1}}\right)\Big(8b_{a_1}^H(1 + \epsilon_1^{2a_2}2^{2m(a_2-a_1)}b_{a_1}^{-1}n^{2a_1-2a_2})^H \\ &\qquad - 2b_{a_1}^H 2^{2Ha_1}(1 + 2^{2(a_2-a_1)}\epsilon_1^{2a_2}2^{2m(a_2-a_1)}b_{a_1}^{-1}n^{2a_1-2a_2})^H\Big) \\ &= c_{h_1}^*\left(\frac{2^{2mh_1}}{n^{2h_1}}\right)\Big(1 + \tilde{c}_{a_2}/c_{h_1}^* H\epsilon_1^{2a_2}2^{2m(a_2-a_1)}n^{2a_1-2a_2} + o(\epsilon_1^{2a_2}n^{2a_1-2a_2})\Big), \end{aligned}$$

where $b_{a_1} = (u_1'\hat{u}_1)^{2a_1}$, $c_{h_1}^* = E(\nabla B_{h_1}(i)^2)(u_1'\hat{u}_1)^{2h_1}$, $\tilde{c}_{a_2} = E(\nabla B_{a_2-(1-H)a_1}(i)^2)b_{a_1}^{H-1}$ and $\epsilon_1 = u_2'\hat{u}_1$. The last equality follows by Taylor expansion and the fact that $E(\nabla B_h(i)^2) = 8 - 2 \cdot 2^{2h}$. Since ϵ_1 has the order of (4.22), $\epsilon_1^{2a_2}n^{2a_1-2a_2}$ is either less than $n^{2(a_1-a_2)-a_2}$ or $n^{-(2+4a_2)(a_2-a_1)}$ depending on whether $a_2 - a_1 > .25$ or $a_2 - a_1 \leq .25$, the result follows.

For h_2^E, in the same way as above, it is derived that

$$\begin{aligned}
E_{X|\hat{u}_2}(\hat{\nabla}_2^m X(i/n)^2) &= 8\left(\epsilon_2^{2a_1}\left(\frac{2^{2ma_1}}{n^{2a_1}}\right) + b_{a_2}\left(\frac{2^{2ma_2}}{n^{2a_2}}\right)\right)^H - 2\left(\epsilon_2^{2a_1}\left(\frac{2^{2(m+1)a_1}}{n^{2a_1}}\right) + b_{a_2}\left(\frac{2^{2(m+1)a_2}}{n^{2a_2}}\right)\right)^H \\
&= \left(\frac{2^{2mHa_2}}{n^{2Ha_2}}\right)\Big(8b_{a_2}^H(1+\epsilon_2^{2a_1}2^{2m(a_1-a_2)}b_{a_2}^{-1}n^{2a_2-2a_1})^H \\
&\qquad - 2b_{a_2}^H 2^{2Ha_2}(1+2^{2(a_1-a_2)}\epsilon_2^{2a_2}2^{2m(a_1-a_2)}b_{a_2}^{-1}n^{2a_2-2a_1})^H\Big) \\
&= c_{h_2}^*\left(\frac{2^{2mh_2}}{n^{2h_2}}\right)\Big(1+\tilde{c}_{a_1}/c_{h_2}^* H\epsilon_2^{2a_2}2^{2m(a_1-a_2)}n^{2a_2-2a_1} + o(\epsilon_2^{2a_1}n^{2a_2-2a_1})\Big),
\end{aligned}$$

where $b_{a_2} = (u_2'\hat{u}_2)^{2a_2}$, $c_{h_2}^* = E(\nabla B_{h_2}(i)^2)(u_2'\hat{u}_2)^{2h_2}$, $\tilde{c}_{a_1} = E(\nabla B_{a_1-(1-H)a_2}(i)^2)b_{a_2}^{H-1}$ and $\epsilon_2 = u_1'\hat{u}_2$. The last equality follows by Taylor expansion and the fact that $E(\nabla B_h(i)^2) = 8 - 2\cdot 2^{2h}$. Since ϵ_2 has the order of (4.22), $\epsilon_2^{2a_1}n^{2a_2-2a_1}$ is of order $n^{-(4a_1-2)(a_2-a_1)}$ when $a_2 - a_1 \leq .25$, and $a_1 > .5$ or divergent otherwise, the results follow for the order of $h_2^E - h_2$. Asymptotic normality of $\hat{h}_1, \hat{h}_2$ follows from (4.32-4.33). □

4.2 Simulation results

The simulation of operator scaling Gaussian random field with semi-variogram (1.8) and diagonal matrix E was performed with the algorithm in [9](Figure 4.1, Figure 4.6). As we have diagonizable matrix E in this chapter, the algorithm in [9] was modified. Two different sampling regimes were employed in this section: 1) independent sample paths on exact directions 2) sample surface on grid lines. In Section 4.2.1, independent sample paths were simulated on exact directions. In Section 4.2.2, sample surfaces were simulated on grid lines.

4.2.1 Independent sample paths on exact directions

Here it is assumed that independent sample paths on exact directions are obtained from OSGRF with different parameters (θ, a_1, a_2) when $H = 1$, (θ, h_1, h_2) when $H < 1$. More specifically, two independent sample paths, $\{X\begin{pmatrix} i/n \\ 0 \end{pmatrix}, n = 2^{13}, i = 1, ..., 2^{13}\}$, $\{X\begin{pmatrix} 0 \\ i/n \end{pmatrix}, n = 2^{13}, i = 1, .., 2^{13}\}$, are obtained to estimate θ, and with that directions $\hat{\theta}$, two independent sample paths, $\{X(\frac{i}{n}\hat{u}_1), n =$

$2^{13}, i = 1, ..., 2^{13}\}$, $\{X(\frac{i}{n}\hat{u}_2), n = 2^{13}, i = 1, ..., 2^{13}\}$, are used to estimate (a_1, a_2) for $H = 1$ and (h_1, h_2) for $H < 1$. In an increasing domain regime, $\{X(i\hat{u}_1), i = 1, ..., 2^{13}\}$, $\{X(i\hat{u}_2), i = 1, ..., 2^{13}\}$ are used to estimate (a_1, a_2) for $H = 1$.

4.2.1.1 For $H = 1$

Figure 4.1 shows the simulations of random field X whose semi-variogram is (1.8) with $\theta = 0$ and various a_1, a_2 in a fixed domain $[0, 1] \times [0, 1]$. In the same model with different θ, the picture would look just like Figure 4.1 in a different angle (i.e., the picture rotated by the angle θ from Figure 4.1.)

In Figure 4.2, $\cos\theta/\sin\theta$ was estimated in OSGRF for $\{\theta = i/20 * \pi/2, i = 1, ..., 19\}$, for each of $\{(a_1, a_2)\}$ in Figure 4.1. In other words,

$$\left(\frac{P_n}{Q_n}\right)^{1/\hat{\alpha}_n}$$

are drawn and its convergence rate is in (4.21). It is seen that, for θ close to zero, the estimation is not good as it was expected by (4.21), but as seen in Figure 4.3, the estimate of θ, which one needs ultimately, is good for the whole range of θ, and this result was expected from (4.22).

In Figure 4.3, the estimates of θ, $\theta \in [0, \pi/2]$, were drawn in OSGRF for $\{\theta = i/20 * \pi/2, i = 1, ..., 19\}$ in each case of $\{(a_1, a_2)\}$ as in Figure 4.1. For these estimates, the sample paths of $\{X\begin{pmatrix} i/n \\ 0 \end{pmatrix}; n = 2^{13}, i = 1, .., 2^{13}\}$, $\{X\begin{pmatrix} 0 \\ i/n \end{pmatrix}, n = 2^{13}, i = 1, 2, .., 2^{13}\}$ were used. It is seen from Figure 4.3 that $\hat{\theta}$ works well in any range of set (θ, a_1, a_2) as all other lines are close to the black line.

In Figure 4.4, the estimates of a_1, a_2 were drawn for each case of (a_1, a_2) with varying θ. The different colors represent a different set of (a_1, a_2), and in each color, the solid line is for $\hat{a}_1$, and the dotted line is for $\hat{a}_2$. The estimates $\hat{a}_1, \hat{a}_2$ were obtained in a fixed domain $[0, 1] \times [0, 1]$ with $m = 1, 2, 3, 4 (= \ell_n)$, and sample paths $\{X(\frac{i}{n}\hat{u}_1), n = 2^{13}, i = 1, .., 2^{13}\}$, $\{X(\frac{i}{n}\hat{u}_2), n = 2^{13}, i = 1, .., 2^{13}\}$ for a_1, a_2, respectively, where $\hat{\theta}$ for $\{\hat{u}_j(\hat{\theta}), j = 1, 2\}$ are from the previous step in Figure 4.3. As it was expected in Theorem 4.1.5, Figure 4.4 reveals the fact that $\hat{a}_1$ performs well for any range of the set (θ, a_1, a_2), but $\hat{a}_2$ performs well only when a_1 is large and $a_2 - a_1$ is small– $\{(a_1, a_2); a_1 >$

.5 and $a_2 - a_1 < .25\}$. In Figure 4.4, it is seen that for $\{(a_1, a_2) = (.1, .3), (.2, .3), (.3, .8)\}$, $\hat{a}_1$ and $\hat{a}_2$ are both close to a_1. For $\{(a_1, a_2) = (.6, .8), (.8, .9)\}$, $\hat{a}_1$ is close to a_1 and $\hat{a}_2$ is close to a_2.

In Figure 4.5, the same estimates of θ were used as in Figure 4.3 and Figure 4.4, but a_1, a_2 are estimated in an increasing domain $[0, 2^{13}] \times [0, 2^{13}]$. More specifically, the estimates $\hat{a}_1, \hat{a}_2$ were obtained with $m = 1, 2, 3, 4 (= \ell_n)$, and the sample paths $\{X(i\hat{u}_1), i = 1, .., 2^{13}\}$, $\{X(i\hat{u}_2), i = 1, .., 2^{13}\}$ for a_1, a_2, respectively, where $\hat{\theta}$ for $\{\hat{u}_j(\hat{\theta}), j = 1, 2\}$ are the same as in Figure 4.3, and Figure 4.4. As it was expected in Theorem 4.1.6, it is seen in Figure 4.5 that both $\hat{a}_1, \hat{a}_2$ work well in any set of the range (θ, a_1, a_2), as the solid line which is for $\hat{a}_1$ is close to a_1 and the dotted line which is for $\hat{a}_2$ is close to a_2 in any case of (θ, a_1, a_2).

4.2.1.2 For $H < 1$

Figure 4.6 shows sample surface of OSGRF with various (a_1, a_2) when $H < 1$, and $\theta = 0$. The sample surfaces look different than those in Figure 4.1 as there is no longer clear grid lines in Figure 4.6. Figure 4.7 shows the results for the estimator $\hat{\theta}$, and Figure 4.8 shows the results for the estimator $\hat{h}_1, \hat{h}_2$ with $m = 1, 2, 3, 4 (= \ell_n)$. The estimators $\hat{\theta}, \hat{h}_1 \hat{h}_2$ behave similarly as in the case when $H = 1$. In Figure 4.7, $\hat{\theta}$ was drawn for $\{\theta = i/20 * \pi/2, i = 1, ..., 19\}$, for each (h_1, h_2) with various H in each of the graphs (a)–(f). It is noticeable that the bias of $\hat{\theta}$ increases as $h_2 - h_1$ and therefore $a_2 - a_1$ decreases, and the standard error of $\hat{\theta}$ increases as a_1, a_2 are getting smaller. These results were predicted in Theorem 4.1.9. In Figure 4.8, it is observed that for graphs (a), (b), (c), the estimators $\hat{h}_1, \hat{h}_2$ work well for any range of θ, whereas for graphs (d), (e), only $\hat{h}_1$ works well. For graph (f), $\hat{h}_1$ works well, and $\hat{h}_2$ performs better than that of graphs (d), (e) but not as much as graph (a-c). These results coincide with Theorem 4.1.10, as for (a-c), $(a_1 = h_1/H, a_2 = h_2/H)$ falls into the region $\{(a_1, a_2); a_1 > .5$ and $a_2 - a_1 < .25\}$, whereas in (d), (e), (a_1, a_2) is far from the region, and in F, (a_1, a_2) does not fall into the region but pretty close to the boundary.

4.2.2 sample surface on a grid

Here it is assume that sample surface from OSGRF is available on a grid. More specifically, $\left\{X\begin{pmatrix} i/n \\ j/n \end{pmatrix}, i, j = 1, \ldots, n\right\}$ are simulated and used to estimate (θ, a_1, a_2) for $H = 1$ and (θ, h_1, h_2) for $H < 1$. Since OSGRF with the parameter $(\theta \neq 0, a_1, a_2, H)$ is the same as OSGRF $(0, a_1, a_2, H)$ rotated by the angle θ, $\left\{X\begin{pmatrix} i/n \\ j/n \end{pmatrix}, i, j = 1, \ldots, n\right\}$ were generated from OSGRF $(0, a_1, a_2, H)$ and the sample points

$$S_1 = \left\{X\begin{pmatrix} i/n \\ \lfloor (i - n/2)\tan\theta \rfloor / n + 1/2 \end{pmatrix}, X\begin{pmatrix} \lfloor (i - n/2)\tan\theta \rfloor / n + 1/2 \\ i/n \end{pmatrix}, i = 1, \ldots, n\right\}$$

were chosen to estimate θ, and with $\hat{\theta}$,

$$S_2 = \left\{X\begin{pmatrix} i/n \\ \lfloor (i - n/2)\tan(\hat{\theta} - \theta) \rfloor / n + 1/2 \end{pmatrix}, X\begin{pmatrix} \lfloor (i - n/2)\tan(\hat{\theta} - \theta) \rfloor / n + 1/2 \\ i/n \end{pmatrix}, i = 1, \ldots, n\right\}$$

were chosen to estimate (a_1, a_2) for $H = 1$, (h_1, h_2) for $H < 1$. The estimation method for θ is the same as before except that $\nabla_j X(i/n), j = 1, 2$, in (4.3-4.4) are computed with sample points in S_1. Likewise, the estimation method for $a_j, h_j, j = 1, 2$, are the same as in the previous section except that $\hat{\nabla}_j^m X(i/n), j = 1, 2$, in (4.24) are computed with every 2^m-th points in S_2.

4.2.2.1 For $H = 1$

For each θ from $\{\theta = i/20 * \pi/2, i = 1, \ldots, 19\}$, the set S_1 with $n = 2^{14}$ was chosen from the simulated sample surface to estimate θ. The results are shown in Figure 4.9. With $\hat{\theta}$, and $n = 2^{14}$, the set S_2 was chosen to compute $\hat{\nabla}_j^m X(i/n)$ similarly to (4.24) with every 2^m-th sample point from S_2 to estimate (a_1, a_2). Figure 4.10 and Figure 4.11 show the results for $\hat{a}_1, \hat{a}_2$ with $m = 1, 2, 3, 4 (= \ell_n)$, and $m = 5, 6, 7, 8 (= \ell_n)$, respectively.

Figure 4.10 looks very different than the one obtained in Section 4.2.1.1, especially for the dotted lines which are for $\hat{a}_2$. This is because samples are on grid lines so that for small $\hat{\theta} - \theta$, $\lfloor (i - n/2)\tan(\hat{\theta} - \theta) \rfloor / n + 1/2$ will be the same for many adjacent i values, which results in sample points on the same vertical or horizontal lines in S_2. This affects the estimator of a_2 in a different way for a different value of a_2. This is explained well with (4.30-4.31). For $\hat{a}_2$, as $\hat{\nabla}_2^m X(i/n)$ is computed with S_2,

$$\begin{aligned}\hat{\nabla}_2^m X(i/n) =& X\begin{pmatrix}\lfloor (i-2^{m+1}-n/2)\tan(\hat{\theta}-\theta)\rfloor/n+1/2\\ (i-2^{m+1})/n\end{pmatrix}\\ &-2X\begin{pmatrix}\lfloor (i-2^{m}-n/2)\tan(\hat{\theta}-\theta)\rfloor/n+1/2\\ (i-2^{m})/n\end{pmatrix}\\ &+X\begin{pmatrix}\lfloor (i-n/2)\tan(\hat{\theta}-\theta)\rfloor/n+1/2\\ i/n\end{pmatrix}\quad i=2^{m+1},\ldots,n,\end{aligned}$$

the expectation of its squared (4.30) is changed to

$$\begin{aligned}&E_{X|\hat{u}_2}(\hat{\nabla}_2^m X(i/n)^2) = E\Big((B_{a_1}(\lfloor (i-2^{m+1}-n/2)\tan(\hat{\theta}-\theta)\rfloor/n+1/2)\\ &-2B_{a_1}(\lfloor (i-2^{m}-n/2)\tan(\hat{\theta}-\theta)\rfloor/n+1/2)+B_{a_1}(\lfloor (i-n/2)\tan(\hat{\theta}-\theta)\rfloor/n+1/2))^2\Big)\\ &+E\Big((B_{a_2}((i-2^{m+1})/n)-2B_{a_2}((i-2^{m})/n)+B_{a_2}(i/n))^2\Big)\\ &=\tilde{c}\left(\frac{1}{n^{2a_1}}\right)+c_{a_2}\left(\frac{2^{2ma_2}}{n^{2a_2}}\right),\end{aligned}$$

where $\tilde{c}$ is either zero or $E(B_{a_1}(i)-B_{a_1}(i-1))^2=2$, and $c_{a_2}=E(\nabla B_{a_2}(i)^2)$. This is because for $\hat{\theta}-\theta$ being small, $\lfloor 2^m\tan(\hat{\theta}-\theta)\rfloor \ll 1$, and therefore $\Big\{\lfloor (i-2^{m+1}-n/2)\tan(\hat{\theta}-\theta)\rfloor/n+1/2, \lfloor (i-2^{m}-n/2)\tan(\hat{\theta}-\theta)\rfloor/n+1/2, \lfloor (i-n/2)\tan(\hat{\theta}-\theta)\rfloor/n+1/2\Big\}$ are all the same or at most one of them has an increment of $1/n$ for about $2^{m+1}\lfloor n\tan(\hat{\theta}-\theta)\rfloor$ among $\{i=2^{m+1},\ldots,n\}$. This leads the change in (4.31), replacing $c_{a_1}/c^*_{a_2}\epsilon_2^{2a_1}2^{2m(a_1-a_2)}$ by $\tilde{c}/c_{a_2}2^{-2ma_2}$,

$$\log_2 E_{X|\hat{u}_2}(\hat{\nabla}_2^m X(i/n)^2)=\log_2 c_{a_2}+\log_2\left(\frac{2^{2ma_2}}{n^{2a_2}}\right)+\log_2(1+\tilde{c}/c_{a_2}2^{-2ma_2}n^{2a_2-2a_1}).$$

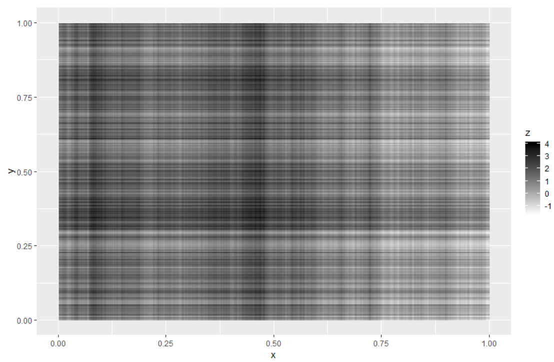
Since $\tilde{c}$ is either 0 or 2, the last term is either 0 or close to $\log_2(\tilde{c}/c_{a_2}2^{-2ma_2}n^{2a_2-2a_1})$. Therefore, $\frac{1}{2}\sum_{m=1}^{\ell_n} w_m\log_2 E(\hat{\nabla}_2^m X(i/n)^2)$ is either a_2 or close to 0, and a_2^E becomes the weighted average between 0 and a_2 with weights around $2^{m+1}\tan(\hat{\theta}-\theta)$, $1-2^{m+1}\tan(\hat{\theta}-\theta)$, respectively. Note that the larger a_2 is, the bigger the bias of $\hat{a}_2$ is, as $\hat{a}_2$ becomes much smaller than a_2. Therefore when

a_2 is small, the estimator $\hat{a}_2$ actually performs better with samples on grid lines than with samples on the exact direction. However, there is not much difference for $\hat{a}_1$, as it always performs well whether samples are on grid or exact directions. This can be explained by (4.28-4.29). Since the term $n^{2a_1-2a_2}$ in (4.29) goes to zero as the sample size grows, it always makes the estimator $\hat{a}_2$ work well.

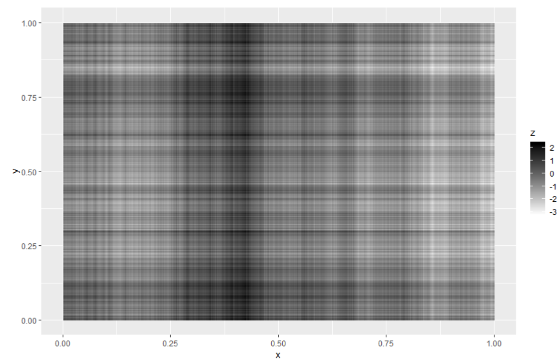
If one increases the scale m that is used for the distance between sample points in estimators $\hat{a}_1, \hat{a}_2$ (4.26-4.27), then the estimators behave similarly to the previous case where sample points are laid exactly along the direction. This can be seen in Figure 4.11. Figure 4.11 looks similar to Figure 4.4 in a way that $\hat{a}_2$ works well only when parameters are in the region $\{(a_1, a_2); a_1 > .5 \text{ and } a_2 - a_1 < .25\}$, and $\hat{a}_1$ works well in general. It is because by choosing larger scale m, the effect of sample points lying on the grid lines is diminished. In other words, the distance between adjacent i, which are used for $\hat{\nabla}_j^m X(i/n)$ in (4.24) and in S_2, is 2^m. Therefore, with the higher m, $\lfloor (i-n/2)\tan(\hat{\theta}-\theta) \rfloor / n + 1/2$ would not overlap for these i (i.e. every 2^m-th sample in S_2 lies on the different vertical and horizontal grid lines.)

4.2.2.2 For $H < 1$

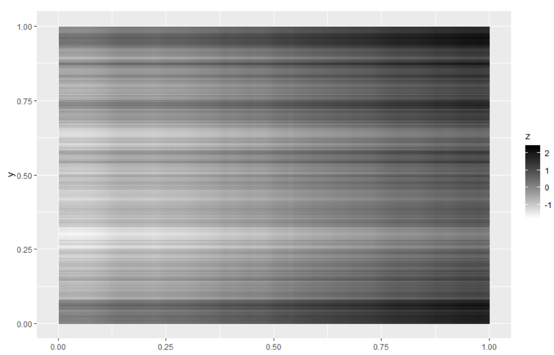
OSGRF with $\theta = 0$ and various a_1, a_2, H were simulated at $\left\{ X \begin{pmatrix} i/n \\ j/n \end{pmatrix}, n = 2^{12}, i, j = 1, \ldots, n \right\}$. The same method was used to estimate θ, h_1, h_2 as in section 4.2.2.1. Here, $\{m = 1, 2, 3, 4 (= \ell_n)\}$ and $\{m = 3, 4, 5, 6 (= \ell_n)\}$ was used for both $h_1 = .6, h_2 = .7$ and $h_1 = .2, h_2 = .4$ with various $H < 1$. The results are shown in Figure 4.12-4.17. The estimators behave similarly to the case when $H = 1$ in Section 4.2.2.1. For $h_1 = .6, h_2 = .7$, the estimators with larger scale $m = 3, 4, 5, 6 (= \ell_n)$ perform well as $\hat{h}_1, \hat{h}_2$ are close to h_1, h_2, respectively, whereas the estimator $\hat{h}_2$ with $m = 1, 2, 3, 4 (= \ell_n)$ is poor since both $\hat{h}_1, \hat{h}_2$ are close to h_1. For $h_1 = .2, h_2 = .4$, the estimators with $m = 1, 2, 3, 4 (= \ell_n)$ performs better as $\hat{h}_1, \hat{h}_2$ are close to h_1, h_2 in a wider range of θ than when $m = 3, 4, 5, 6 (= \ell_n)$.



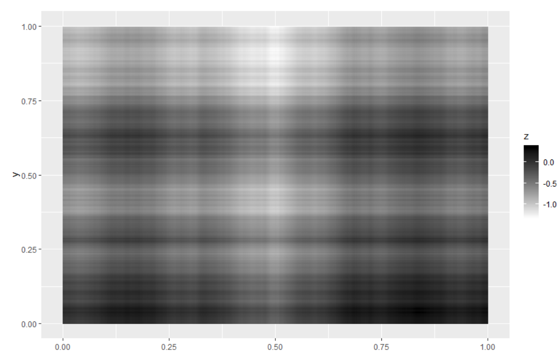
(a)



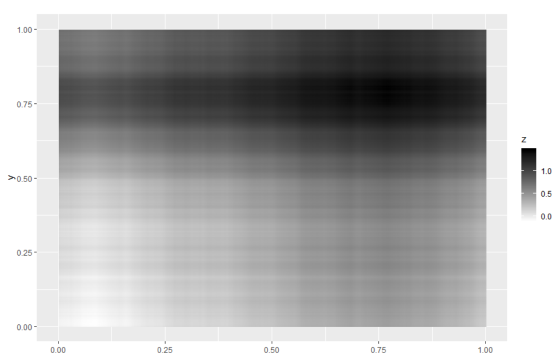
(b)



(c)



(d)



(e)

Figure 4.1: (a): $a_1 = .1, a_2 = .3$, (b): $a_1 = .2, a_2 = .3$, (c):$a_1 = .3, a_2 = .8$, (d):$a_1 = .6, a_2 = .8$, (e): $a_1 = .8, a_2 = .9$

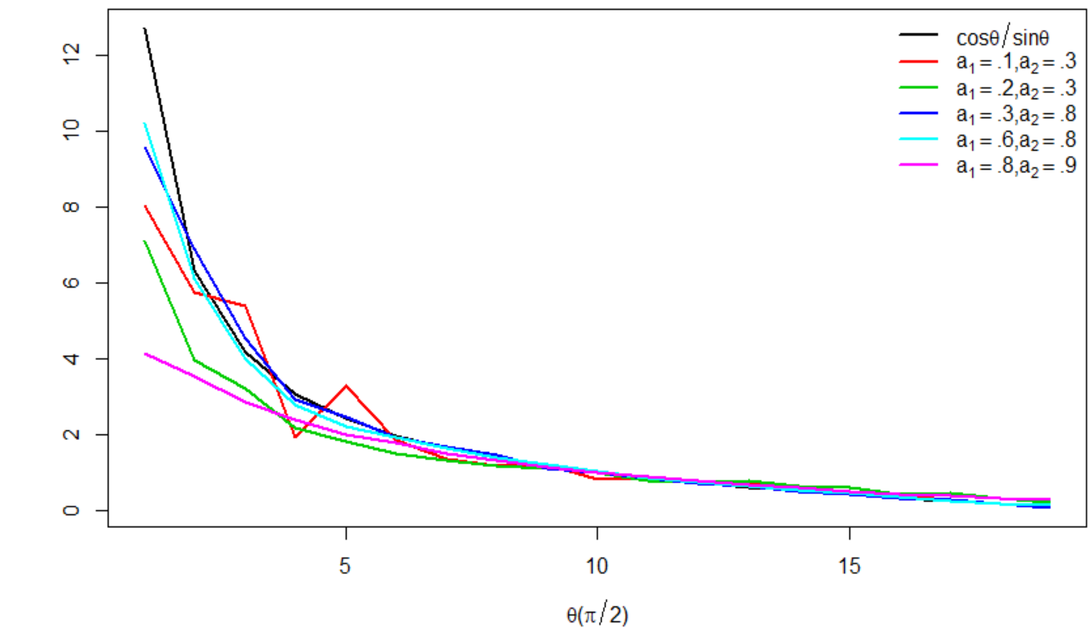


Figure 4.2: Estimates of $\cos\theta/\sin\theta$

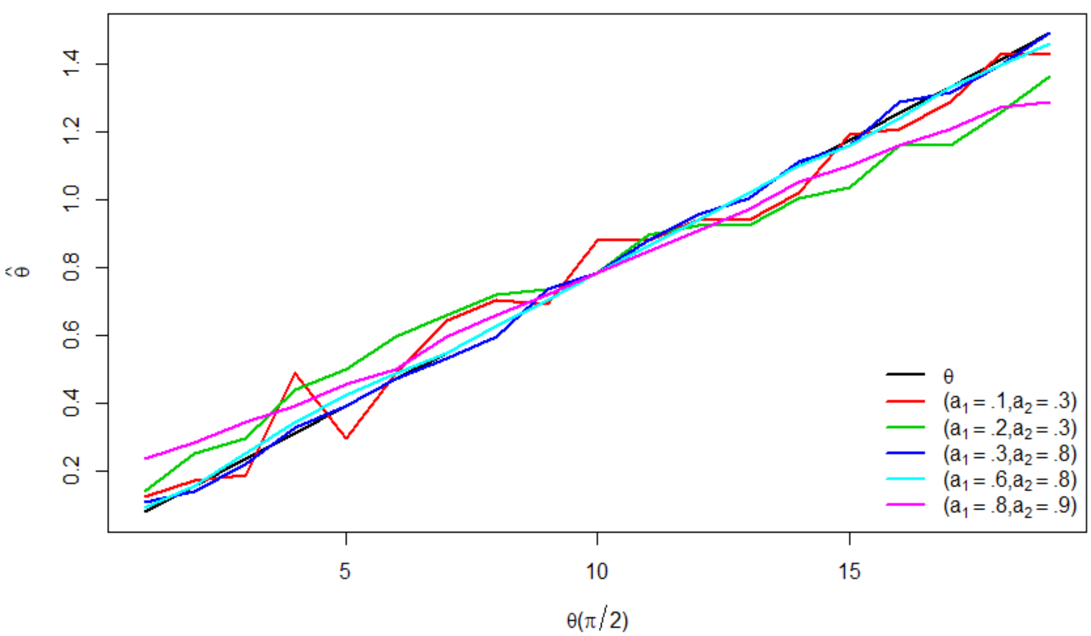


Figure 4.3: Estimates of θ

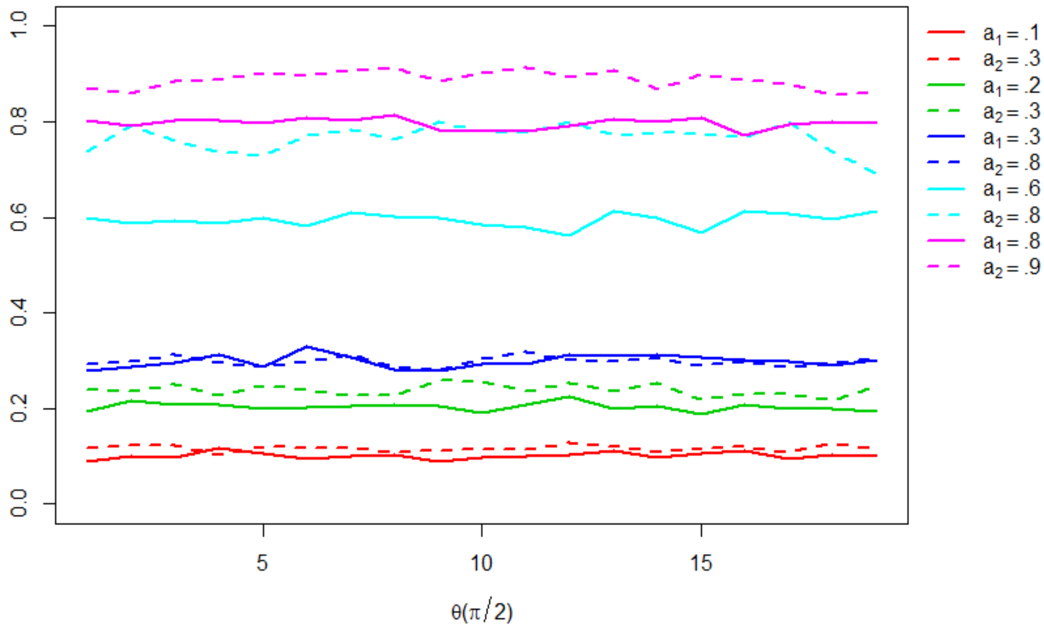


Figure 4.4: Estimates of a_1 and a_2 in a fixed domain

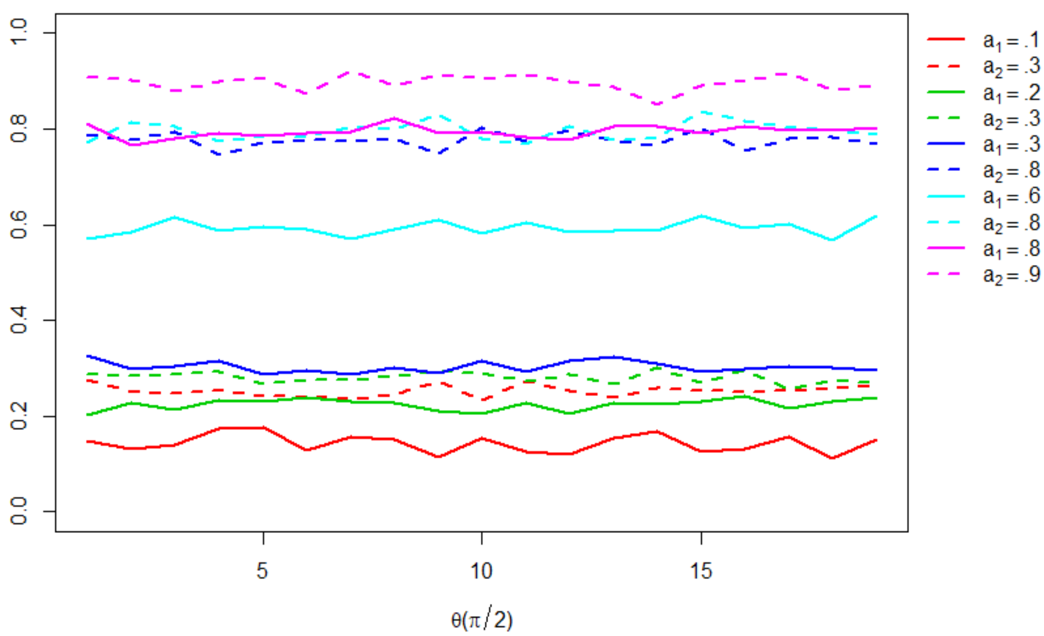
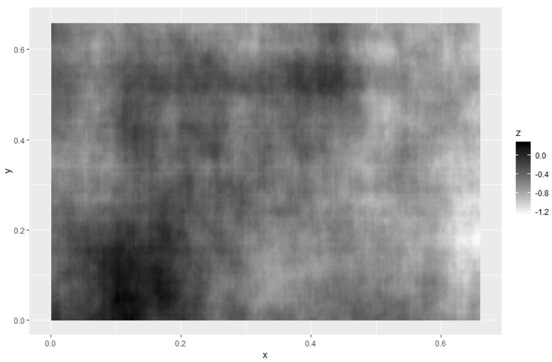
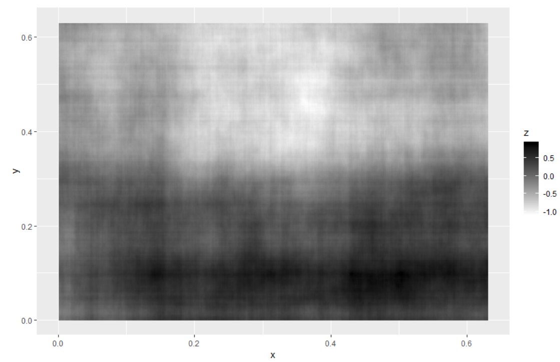


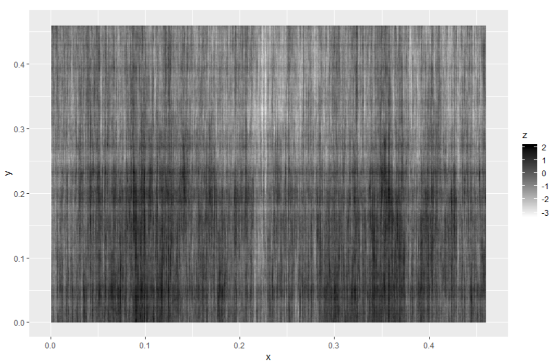
Figure 4.5: Estimates of a_1 and a_2 in an increasing domain



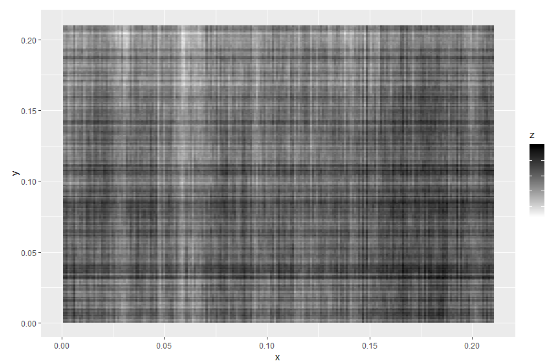
(a)



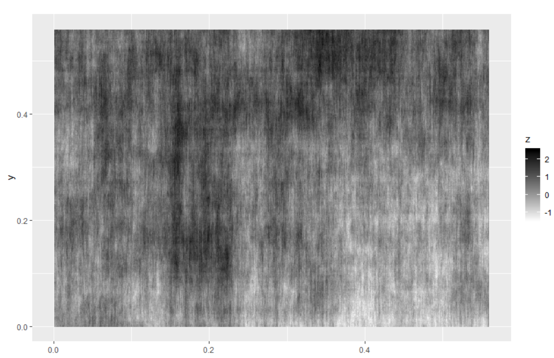
(b)



(c)



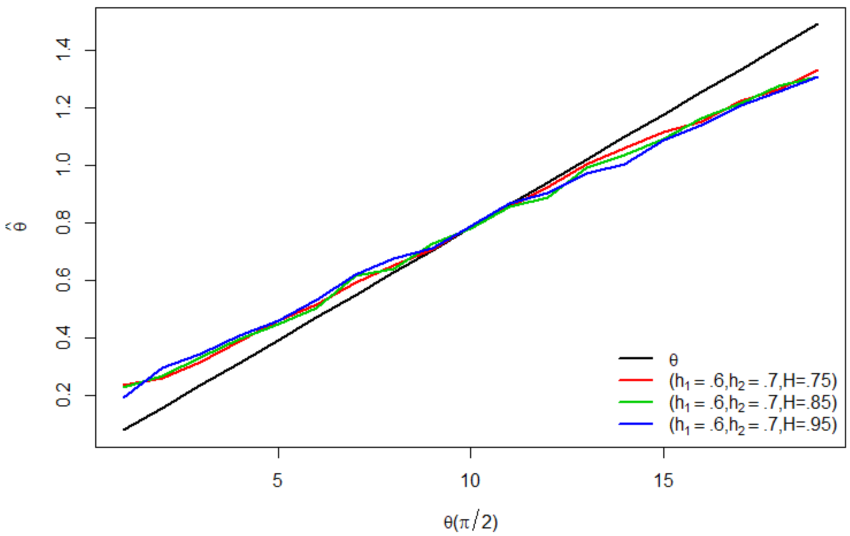
(d)



(e)

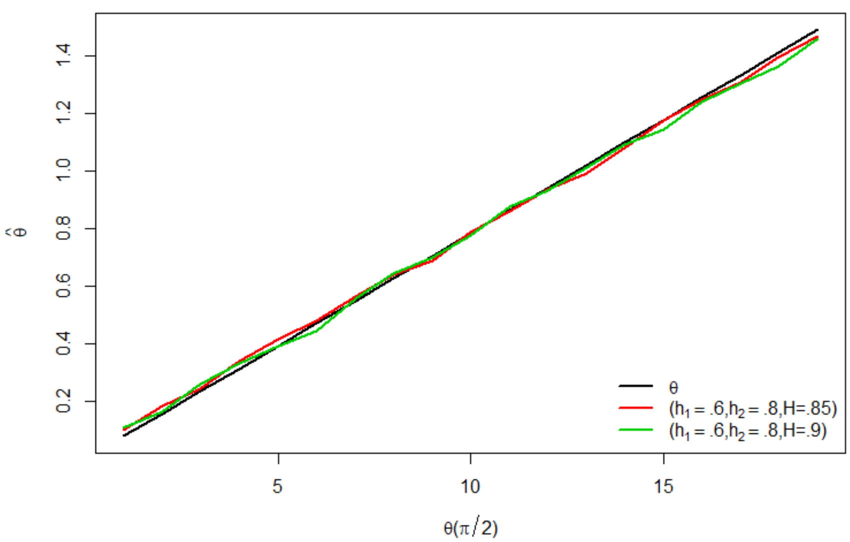
Figure 4.6: (a): $h_1 = .6, h_2 = .7, H = 75$, (b): $h_1 = .6, h_2 = .7, H = .85$, (c):$h_1 = .1, h_2 = .3, H = .4$, (d):$h_1 = .1, h_2 = .3, H = .8$ (e): $h_1 = .2, h_2 = .3, H = .4$

Figure 4.7: Estimates of θ



H =.75(red)
.85(green)
.95(blue)

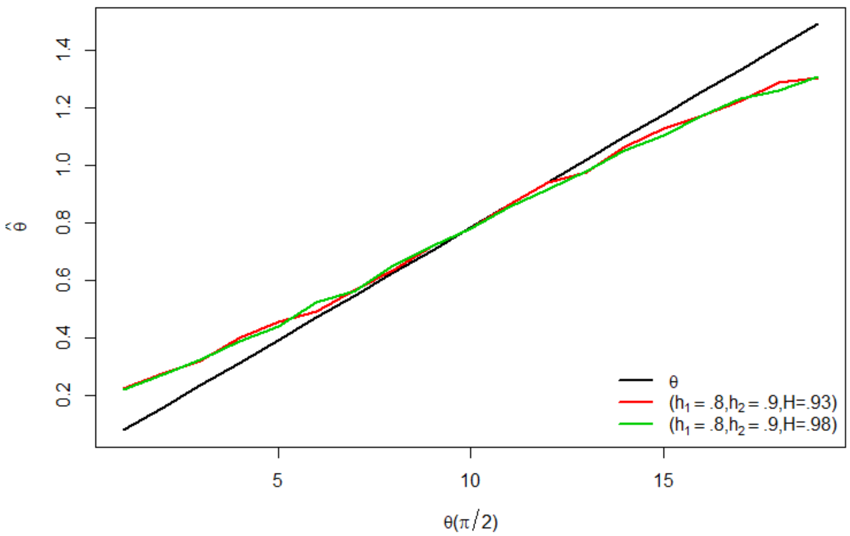
Figure 4.7 (a) $h_1 = .6$, $h_2 = .7$



H =.85(red)
.9(green)

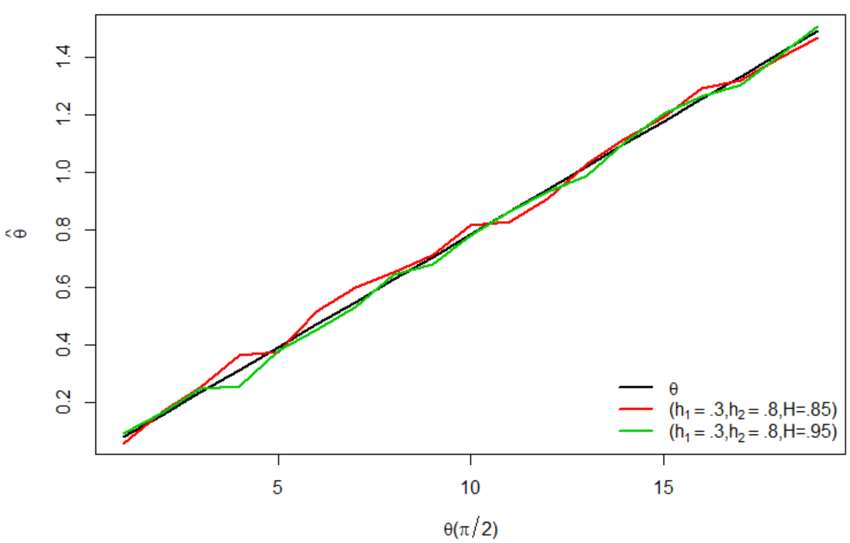
Figure 4.7 (b) $h_1 = .6$, $h_2 = .8$

Figure 4.7 (cont'd)



H =.93(red)
.98(green)

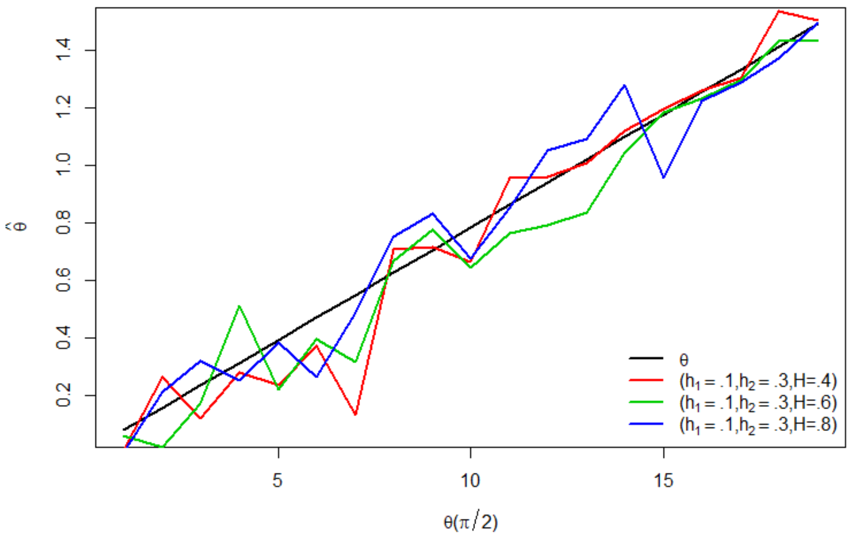
Figure 4.7 (c) $h_1 = .8, h_2 = .9$



H =.85(red)
.95(green)

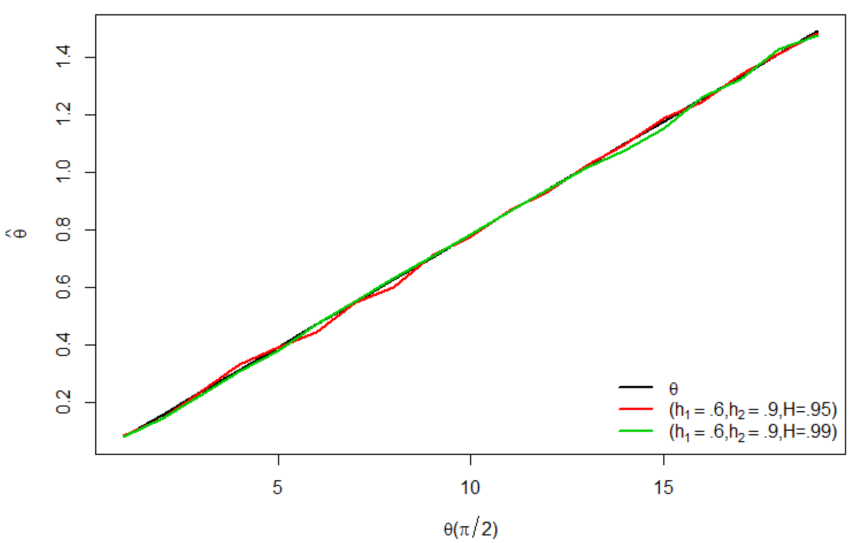
Figure 4.7 (d) $h_1 = .3, h_2 = .8$

Figure 4.7 (cont'd)



H =.4(red)
.6(green)
.8(blue)

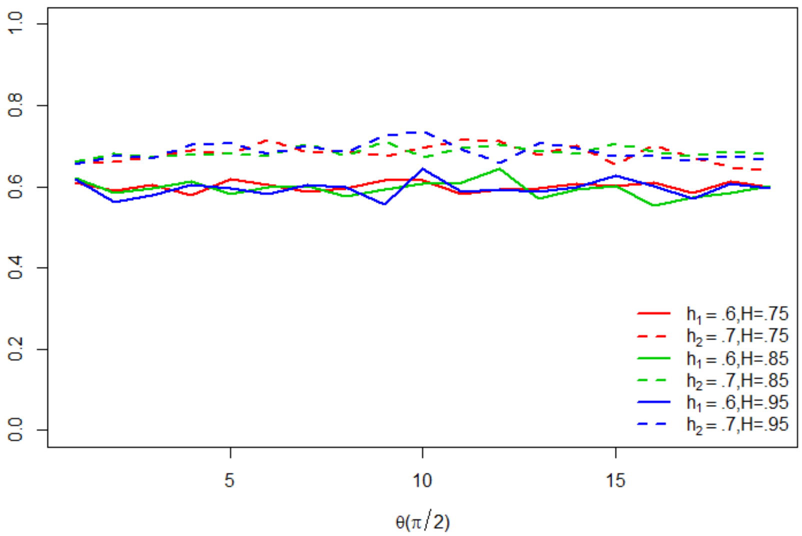
Figure 4.7 (e) $h_1 = .1, h_2 = .3$



H =.95(red)
.99(green)

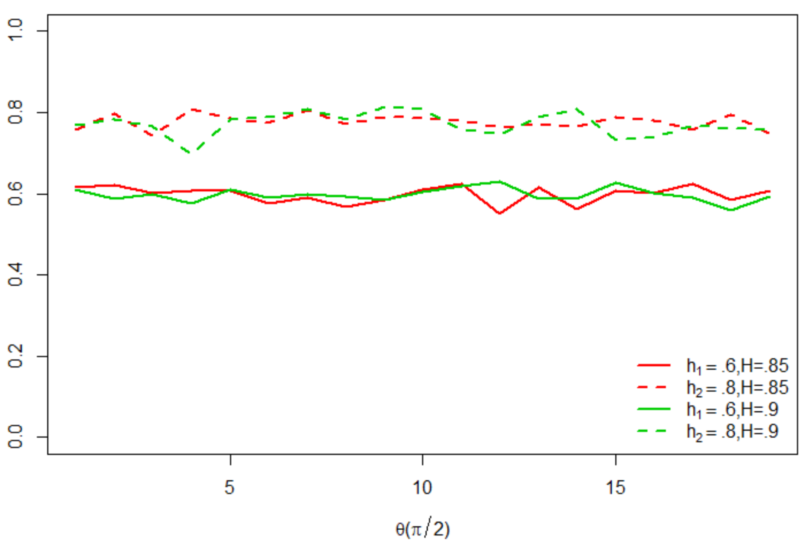
Figure 4.7 (f) $h_1 = .6, h_2 = .9$

Figure 4.8: Estimates of h_1, h_2



H =.75(red)
.85(green)
.95(blue)

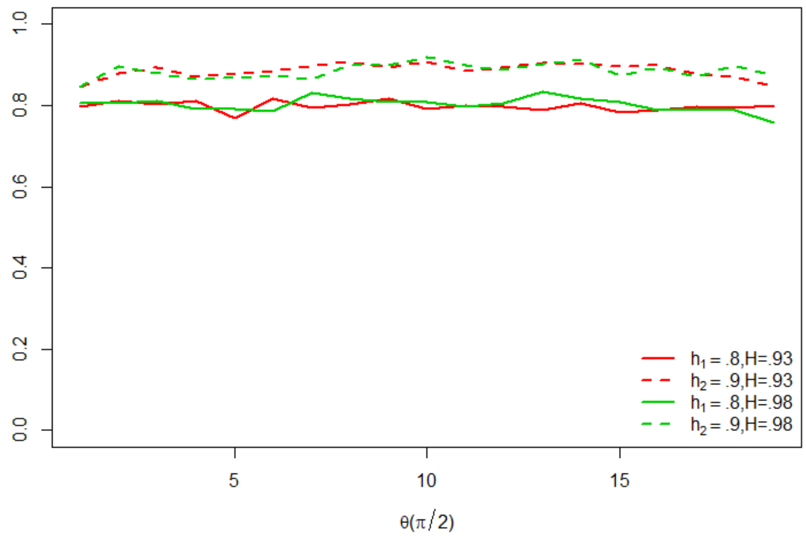
Figure 4.8 (a) $h_1 = .6, h_2 = .7$



H =.85(red)
.9(green)

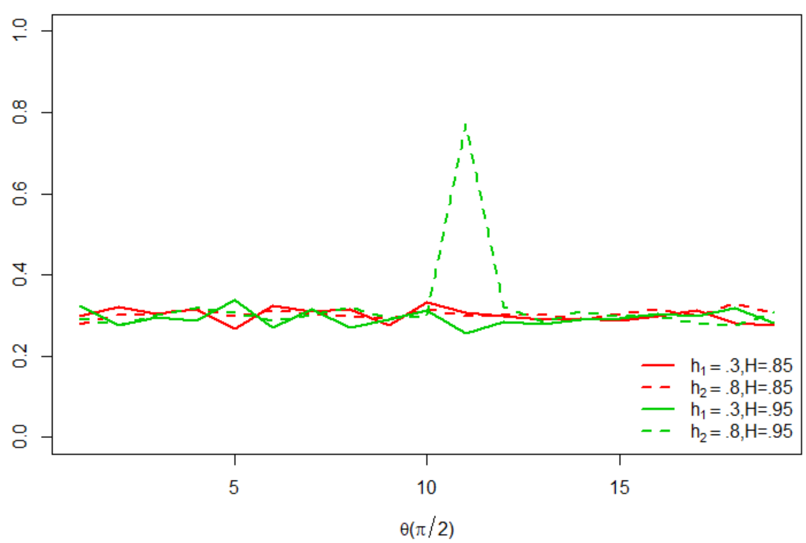
Figure 4.8 (b) $h_1 = .6, h_2 = .8$

Figure 4.8 (cont'd)



H =.93(red)
.98(green)

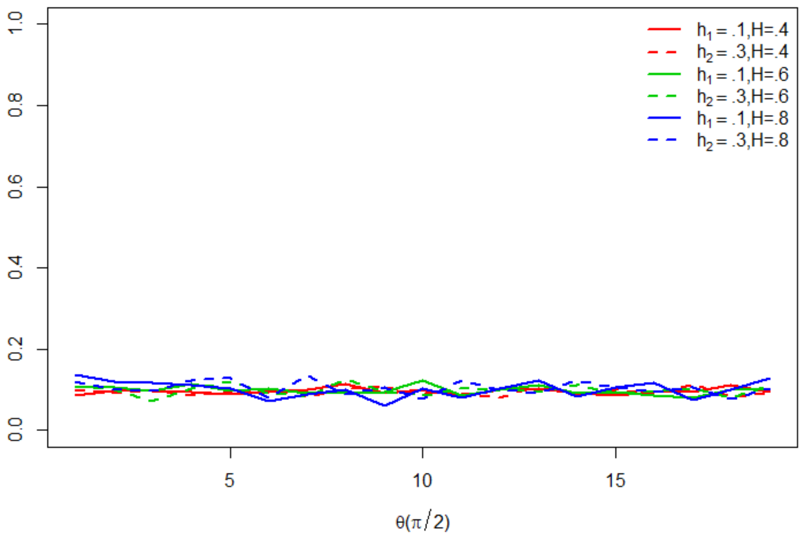
Figure 4.8 (c) $h_1 = .8, h_2 = .9$



H =.85(red)
.95(green)

Figure 4.8 (d) $h_1 = .3, h_2 = .8$

Figure 4.8 (cont'd)

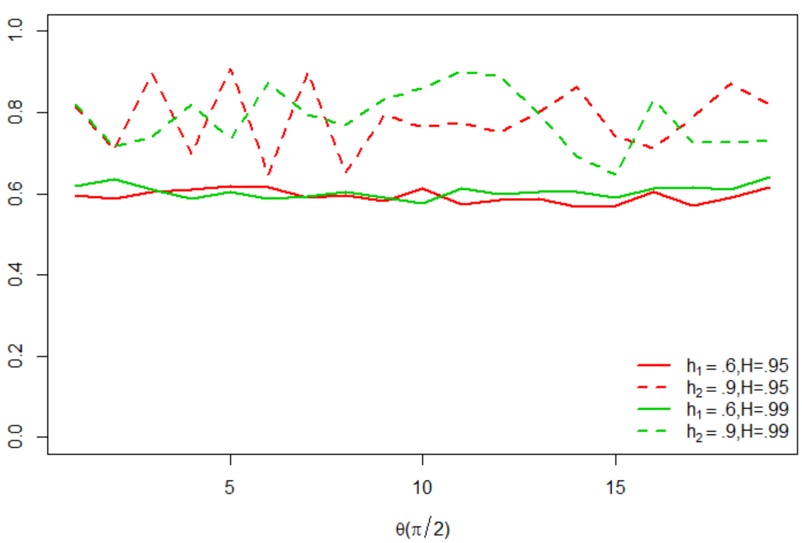


H =.4(red)

.6(green)

.8(blue)

Figure 4.8 (e) $h_1 = .1, h_2 = .3$



H =.95(red)

.99(green)

Figure 4.8 (f) $h_1 = .6, h_2 = .9$

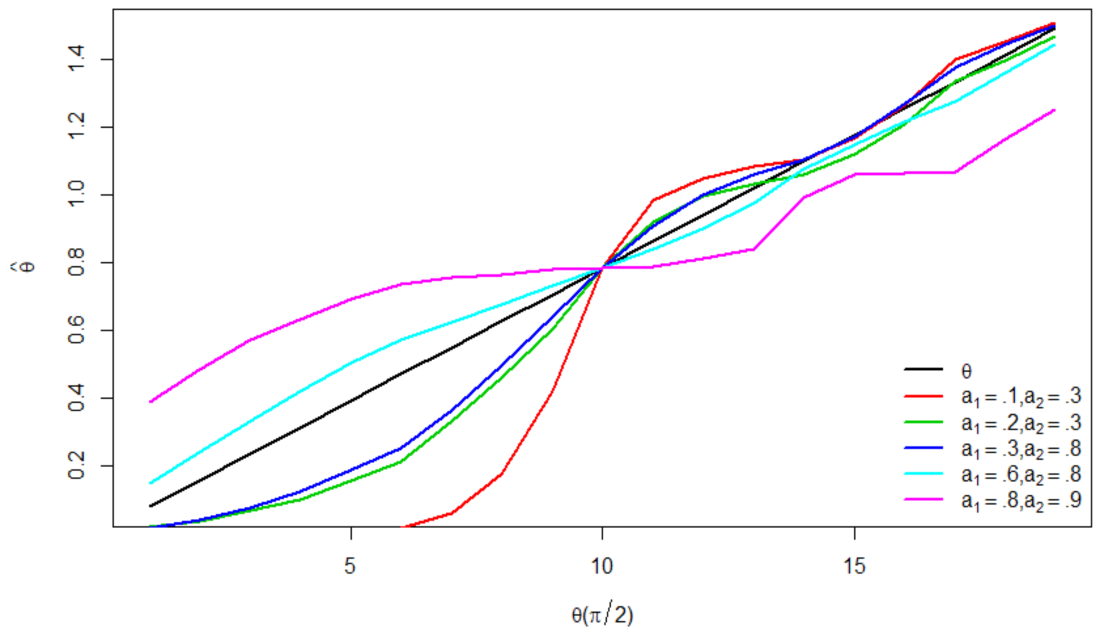


Figure 4.9: Estimates of θ in a sample surface

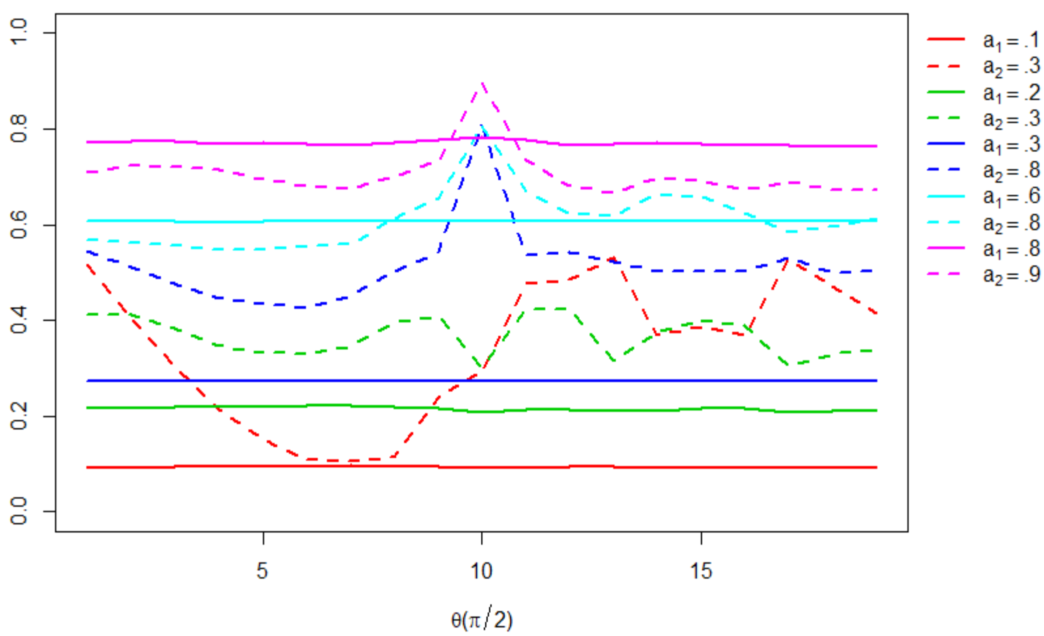


Figure 4.10: Estimates of a_1 and a_2 in a sample surface with $m = 1, 2, 3, 4$

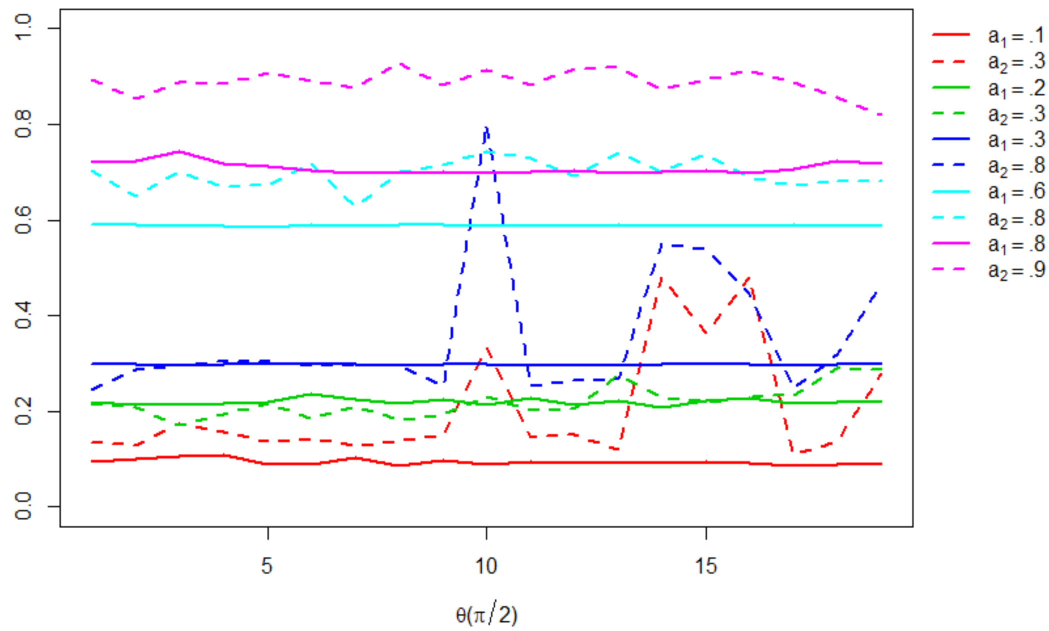


Figure 4.11: Estimates of a_1 and a_2 in a sample surface with $m = 5, 6, 7, 8$

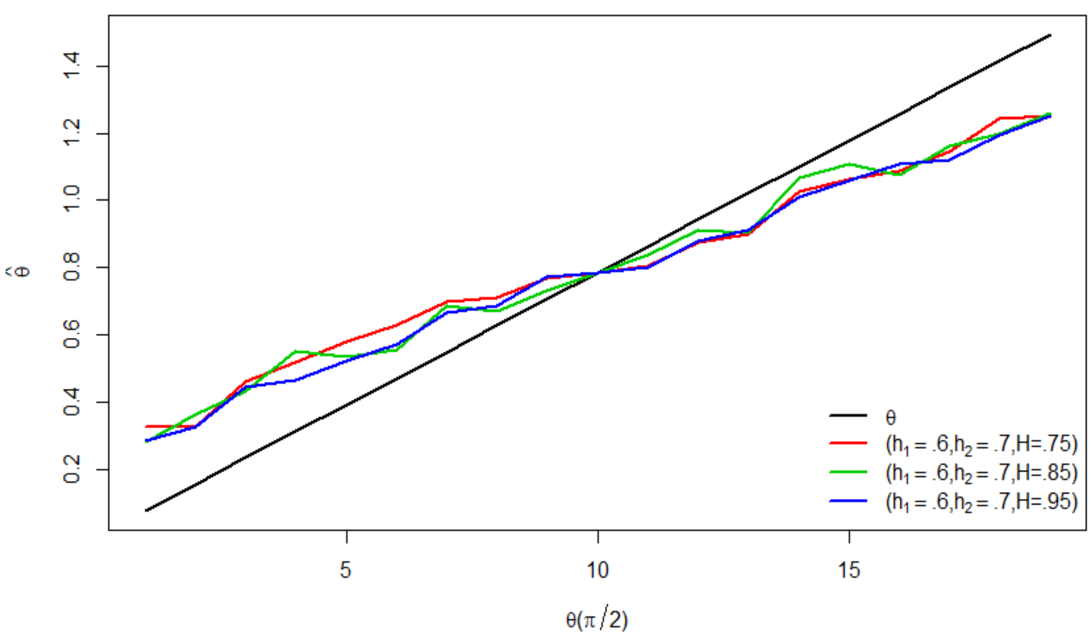


Figure 4.12: Estimates of θ

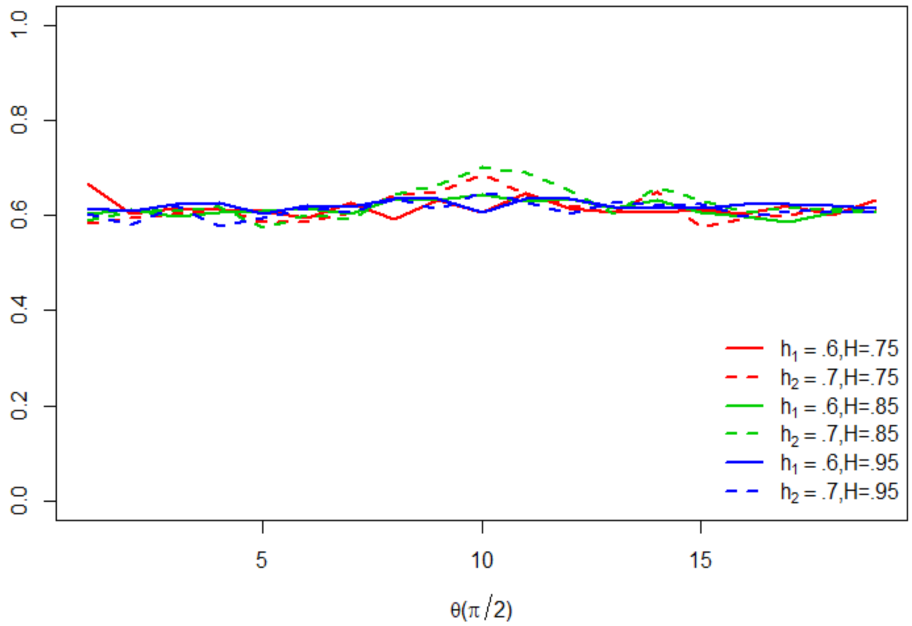


Figure 4.13: Estimates of h_1 and h_2 in an fixed domain with $m = 1, 2, 3, 4$

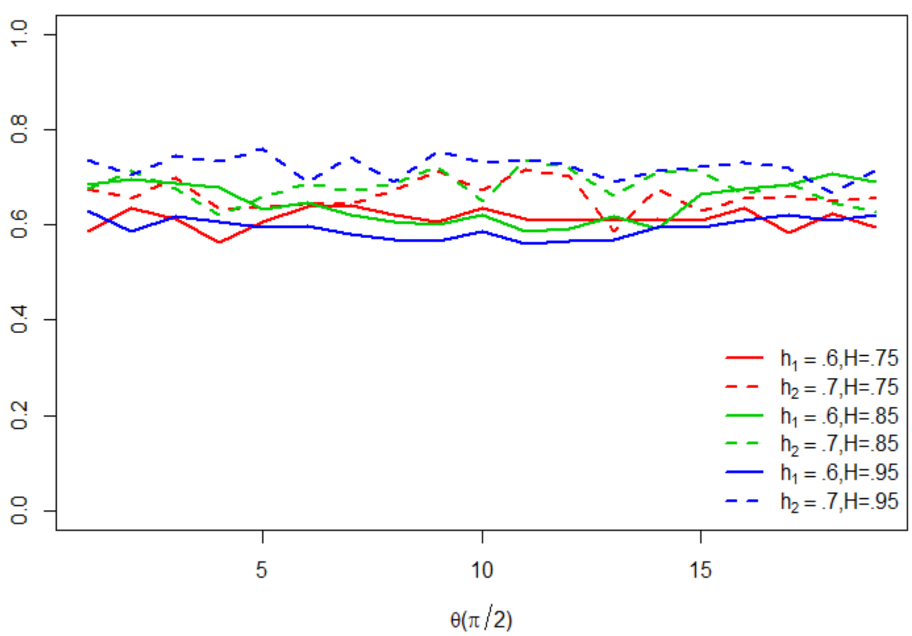


Figure 4.14: Estimates of h_1 and h_2 in an fixed domain with $m = 3, 4, 5, 6$

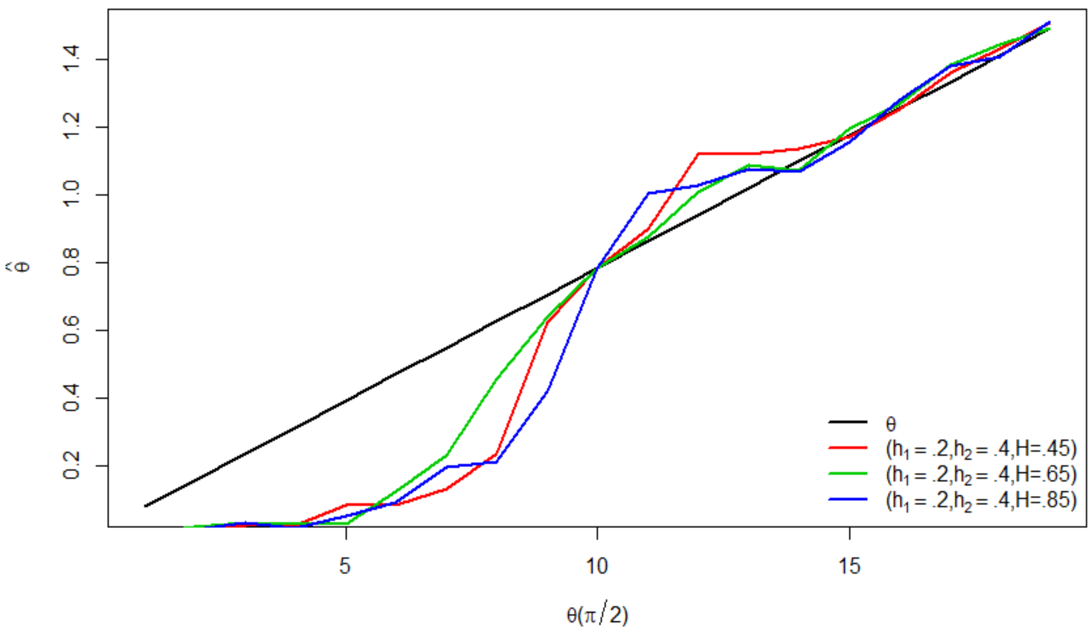


Figure 4.15: Estimates of θ

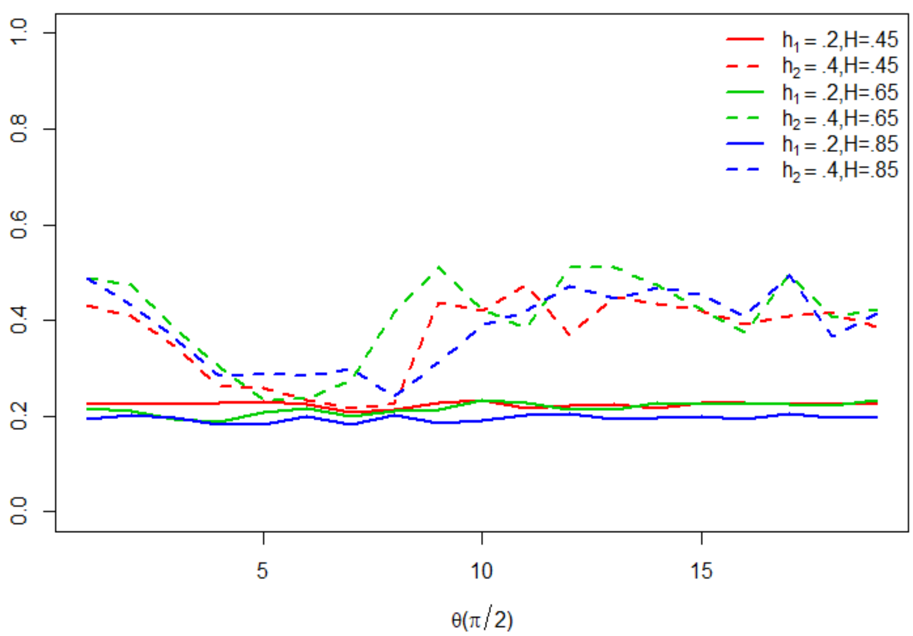


Figure 4.16: Estimates of h_1 and h_2 in an fixed domain with $m = 1, 2, 3, 4$

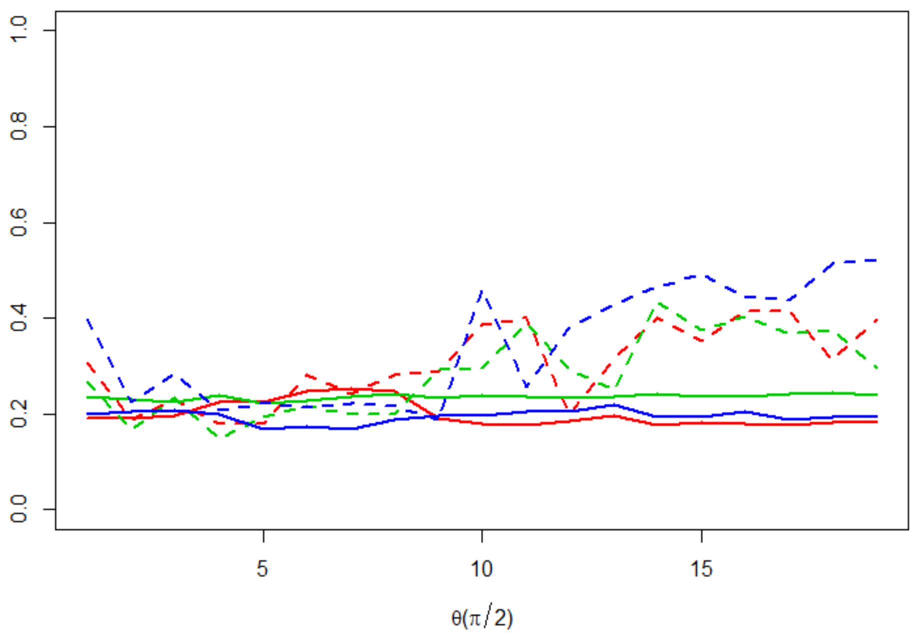


Figure 4.17: Estimates of h_1 and h_2 in an fixed domain with $m = 3, 4, 5, 6$

CHAPTER 5

CONCLUSION AND DISCUSSION

In this dissertation, statistical inference was made on self-similar processes/random fields with stationary increments. First, Hurst parameter, which captures self-similarity, was estimated in multivariate self-similar stochastic processes and random field, OFBM and OSGRF, respectively. Second, the dependency within multivariate random fields with stationary increments was measured in the spectral domain.

Hurst estimation method with wavelet transform and eigen decompositon was examined thoroughly in a OFBM in continuous sample path, discrete sample path, and discrete noisy data. It was revealed both theoretically and empirically that there is an interplay of Hurst parameters, h_1, h_2, the covariance matrix $\Gamma(1,1)$ of $\mathbf{X}_H(1)$, and the choice of the scale parameter j of wavelet function on the performance of the estimator. If continuous sample paths are observed, then the bigger j is, the better the estimators of h_2, h_1 perform. If discrete sample paths are observed, then it is better to choose slightly smaller j than the maximum $j = \log_2 n$. Especially if h_i is small e.g. $h_i < .5$, or determinant of $\Gamma(1,1)$ is small, then the estimators of h_1, h_2 have large bias for j close to its maximum. However, the larger j is, the smaller the standard error of the estimators is. Considering both bias and standard error, it is best to choose a set of j that is slightly smaller than $\log_2 n$ or wide range of j except the very large or small j when discrete sample paths are given. If noise is present in the processes, then it is better to choose much smaller j than in the previous cases, but with the noise term, the estimator does not work well unless both h_1, h_2 are small. For the estimator of θ, it is always the best to choose the largest $j = \log_2 n$, since it has the both smallest bias and standard error among all possible j in any set of (θ, h_1, h_2).

Hurst estimator was developed in OSGRF, and its performance was analyzed for different sampling regimes. The performance of the estimator is affected by the several factors such as values of the parameters, whether it is sampled in grid lines or it is sampled along the exact direction, and whether samples are on a fixed domain or an increasing domain. It was observed that, in a fixed domain setting, the estimator of a_1(or h_1) always performs well, whereas the estimator of a_2(or h_2) performs well if the parameters are in the region of $\{(a_1, a_2); a_1 > .5, a_2 - a_2 < .25\}$ provided that samples are obtained on the exact directions. In a fixed domain, if samples are on grid lines, then the different choice of scale parameter m used in discrete variation method affects the performance of the estimator $\hat{a}_2$(or $\hat{h}_2$) differently for a different set of (a_1, a_2). If the parameters fall in the region $\{(a_1, a_2); a_1 > .5, a_2 - a_2 < .25\}$, higher values of m works better for $\hat{a}_2$(or $\hat{h}_2$), whereas lower values of m work better for $\hat{a}_2$(or $\hat{h}_2$) if the parameter a_2 is less than .5. In an increasing domain with $H = 1$, the estimators $\hat{a}_1, \hat{a}_2$ perform well in any range of (a_1, a_2).

The concept of coherence was extended and defined in multivariate random fields with stationary increments, and its estimator was developed. The estimator was applied to OFBM with sample paths observed in a fixed domain, and its behavior was observed both theoretically and empirically. It was revealed that OFBM in a fixed domain has different dependence structure for different matrices $D(H), \mathscr{C}$ in (3.23). Particularly, if the scaling matrix $D(H)$ is diagonizable, not diagonal, then the squared coherence function converges to constant function of 1 as the sample size grows in a fixed domain, which implies that there exists strong correlation between two stochastic processes. Moreover, in that case, the squared coherence converges to 1 faster if the two Hurst parameters have a bigger difference (i.e., the bigger $\alpha_2 - \alpha_1(h_1 - h_2)$ is, the stronger correlation exists in two series.) If both $\mathscr{C}$ and D are diagonal matrix, then the squared coherence is zero function which means that the two series are independent, having no correlation. If D is diagonal matrix, but not $\mathscr{C}$, then the squared coherence is close to constant function c_γ, whose value is in between 0 and 1, $c_\gamma = \frac{C_{12}}{\sqrt{C_{11}C_{22}}}, C_{\ell,k} = (\mathscr{C}\mathscr{C}')_{\ell,k}, \ell, k = 1, 2.$

Hurst estimation in multivariate stochastic processes and random field is not only an interesting topic but also has practical value as self-similarity and, more generally, fractality are seen in many objects in nature. For future work, a better Hurst estimation method needs to be developed in OFBM when noise is present, since the current method in this dissertation works well only for a small range of parameters when noise is present. Hurst estimation in OFBM with various form of Hurst matrix H is also of interest, especially OFBM with Jordan form H. A better Hurst estimation method for h_2 in OSGRF is needed as the method in this dissertation works well only for some range of parameters. The coherence function defined in this dissertation is a useful measure in capturing dependence structure in two random fields that are increment stationary. In future work, further applications of the coherence in various random fields will be of interest as it will reveal more of its practical value.

APPENDIX

APPENDIX

A.1 Appendix

Theorem 11 (Complex-valued Gaussian random measure) *Let μ be a symmetric measure on $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$ and let $\mathcal{G} = \{A \in \mathcal{B}(\mathbb{R}^N) : \mu(A) < \infty\}$. Then there exists a complex-valued Gaussian process indexed by $\mathcal{G}$, i.e. $\widetilde{W}_\mu = \{\widetilde{W}_\mu(A) : A \in \mathcal{G}\}$, written as $\widetilde{W}_\mu(A) = W_1(A) + \mathrm{i}W_2(A)$ that satisfies the following conditions,*

- $\mathbb{E}\left(\widetilde{W}_\mu(A)\overline{\widetilde{W}_\mu(B)}\right) = \mu(A \cap B)$;
- *for any* $A \in \mathcal{G}$, $\widetilde{W}_\mu(-A) =_{a.s.} \overline{\widetilde{W}_\mu(A)} \Longleftrightarrow W_1(A) =_{a.s.} W_1(-A)$ *and* $W_2(A) =_{a.s.} -W_2(-A)$.

In general, we denote the Gaussian process $\widetilde{W}_\mu$ by $\widetilde{W}$.

Proof. We only give the sketch of the proof. Put

$$E_+ = \{x \in \mathbb{R}^N : x_1 \geq 0,\ x_i \in \mathbb{R}, i = 2, \ldots, N\},$$
$$E_- = \{x \in \mathbb{R}^N : x_1 \leq 0,\ x_i \in \mathbb{R}, i = 2, \ldots, N\},$$

then $(-E_-) = E_+$ and $\mathbb{R}^N = E_+ \cup E_-$. Without loss of the generality, we assume $\mu(E_+ \cap E_-) = 0$. As in the real case, we can take two independently scattered Gaussian random measures W_1 and W_2 on $(E_+, \mathcal{E}_+, \frac{1}{4}\mu)$. Define, for $A \in \mathcal{G}$,

$$\widetilde{W}_1(A) := W_1(A \cap E_+) + W_1(-(A \cap E_-)),$$
$$\widetilde{W}_2(A) := W_2(A \cap E_+) + W_2(-(A \cap E_-)),$$

and

$$\widetilde{W}(A) := \widetilde{W}_1(A) + \mathrm{i}\,\widetilde{W}_2(A).$$

Then $\{\widetilde{W}(A), A \in \mathcal{G}\}$ is a complex-valued Gaussian process with zero mean. We now verify the two conditions. Notice that the distinction of the symbols between the conditions and constructions, then the second condition is obvious. It is easy to see that

$$\begin{aligned}\mathbb{E}(\widetilde{W}(A)\overline{\widetilde{W}(B)}) =&\mathbb{E}(\widetilde{W}_1(A)\widetilde{W}_1(B) + \widetilde{W}_2(A)\widetilde{W}_2(B)) \\ &+ \mathrm{i}\,\mathbb{E}(\widetilde{W}_1(B)\widetilde{W}_2(A) - \widetilde{W}_1(A)\widetilde{W}_2(B)) \\ =&\frac{1}{2}\mu(AB) + \frac{1}{2}\mu(AB) = \mu(AB),\end{aligned}$$

which yields the first condition. □

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