

DYNAMIC GEOMETRY TASK DESIGN FOR AXIOMATIC GEOMETRY:
STUDENT ENGAGEMENT WITH AXIOMATIC REASONING

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ABSTRACT

DYNAMIC GEOMETRY TASK DESIGN FOR AXIOMATIC GEOMETRY: STUDENT ENGAGEMENT WITH AXIOMATIC REASONING

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Responding to calls for studies on task design and enactment using technology in geometry classroom, this dissertation connects theoretical and empirical studies to instructional practices by designing, enacting, and revising a sequence of tasks using DGEs for college students in an axiomatic geometry course. First, I discuss a set of mathematical activities using DGEs that consist the core of the task sequence in this study. I illustrate a sequence of instructional tasks designed and enacted in an axiomatic geometry course where a DGE plays a crucial role in students' mathematical activities in class. The illustration of the task sequence consists of the mathematical activities intended in the design of each task as well as student reasoning. Student work collected in the actual classroom provides pedagogical implications to revise the task sequence

Second, I report an empirical study on students' uses of DGEs and their engagement in mathematical reasoning and axiomatic reasoning while enacting three tasks in the sequence. Students used DGEs to communicate their mathematical ideas and to examine mathematical statements describing properties of geometric objects within axiomatic systems and models of hyperbolic geometry. The analyses of this study revealed case themes describing student use of DGEs, engagement in mathematical reasoning and axiomatic reasoning, and relationships thereof. The findings of the analysis provide practical implications to revise the task design as well as theoretical implications to better understand the nature of student engagement in advanced mathematical reasoning in such technology-rich environments.

At last, not the least, I address theoretical consideration on understanding of epistemic aspects of student learning in axiomatic geometry supported by technology and appropriate mathematical activities exploiting pedagogical roles of technology. I address students' epistemological shifts that have been discussed in the existing literature of student learning of advanced geometry in connection with student work collected and analyzed in the empirical study reported above. First, students make a shift in the ontological view of geometric models from Euclidean to non-Euclidean geometry, in which the geometric models are considered conscious artifacts of mathematical design. Second, students make a shift in the epistemological view of mathematical proofs from absolutism to fallibilism, in which proofs can be characterized with a variety of functions and forms. Drawing on the prior literature, I argue that making successful shifts can benefit students in axiomatic geometry and that such shifts can be facilitated by engaging in mathematical activities with supports of dynamic geometry environments. In particular, I highlight examples of student work reported in the empirical study that illustrate those different views of geometric models and mathematical proofs captured observed from students who were on the process of such shifts.

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To my wife Zihyun Lim and my son Jaden Younu Bae, I owe this all to you.

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KEY TO ABBREVIATIONS

DGE	Dynamic Geometry Environment
EPP	Euclidean Parallel Postulate
GEX	Geometry Explorer

CHAPTER 1: Introduction

Dynamic Geometry Environments (DGEs) are powerful tools for learning and teaching of geometry (e.g., Common Core State Standards Initiative, 2012). These tools allow students to investigate geometric phenomena by engaging in experiments and modeling with interactive diagrams using the dragging feature of the tool (Olive, 2000). In particular, the drag feature of DGEs have been highlighted by many researchers and educators for its affordances allowing students to collect empirical evidence, create their own conjectures and construct formal proofs (e.g., Hollebrands & Okumuş, 2018; Komatsu, 2017). Some researchers focused on specific uses of the drag feature when students are engaging in proof-related activities and identified different types of such usages regarding the extant literature of proof and proving (e.g., Armella & Sriraman, 2005; Arzarello et al., 1998; Leung, 2008). Some researchers expanded their discussion on pedagogical use of DGEs to address new aspects of mathematical knowledge, practices, and proof and proving (e.g., de Villiers, 1998; Jones, Mackrell, & Stevenson, 2010). In the extant literature, researchers and practitioners also discussed pedagogical implication of using DGEs in geometry classroom in association with the nature of new aspects of teaching and learning of mathematics with technology. However, using DGEs in practice needs more attentions. Mathematics education research can shed light on effective use of technology grounded on the actual classroom practices of teachers and students.

Since DGEs have been introduced and widely used in mathematics classroom, researchers have stressed that it is important to understand how teachers and students can use DGEs effectively in classroom. Though using DGEs in geometry education has been mainstream, there is not enough research on the role of technology and its specific effects on teaching and learning of geometry (Sinclair et al., 2016). This is attributed to complexities of

changes in ways of students' interaction with geometric objects and discourse in technology-integrated classroom. In this regard, researchers have investigated explorative and multifaceted questions, for instance, designing suitable tasks for exploiting a given technology and establishing theories to better understand pedagogical roles of technology (Sinclair, 2017). In particular, some researchers explored ways of designing mathematical tasks that make appropriate use of pedagogical merits of DGEs (e.g., Komatsu & Jones, 2019; Leung, 2011). They provided theoretical models and principles for designing effective DGE tasks for eliciting student engagement in advanced reasoning in mathematics (e.g., proof by contradiction, generalization). However, the existing literature indicate that there is little research investigating effects of task design on specific aspects of student learning in the actual classroom settings (Hollebrands & Okumuş, 2018; Komatsu & Jones, 2019). Beyond the design of such tasks, there is a need for more studies focused on gaps between students' perceptions of mathematical tasks and intentions of teachers and task designers. Ainley and Margolinas (2015) argued that such gaps between student perception and intended tasks may result in missing opportunities to learn.

Responding to calls for studies on task design and enactment using technology in geometry classroom, this dissertation connects theoretical and empirical studies to instructional practices by designing, enacting, and revising a sequence of tasks using DGEs for college students in an axiomatic geometry course. First, I discuss a set of mathematical activities using DGEs that consist the core of the task sequence in Chapter 2. Then, I report an empirical study investigating ways in which students interacted with DGEs while engaging in advanced mathematical reasoning including axiomatic reasoning in Chapter 3. At last, I address theoretical consideration on understanding of epistemic aspects of student learning in axiomatic geometry

supported by technology and appropriate mathematical activities exploiting pedagogical roles of technology in Chapter 4.

In Chapter 2, I illustrate a sequence of instructional tasks designed and enacted in an axiomatic geometry course where a DGE plays a crucial role in students' mathematical activities in class. The illustration of the task sequence consists of the mathematical activities intended in the design of each task as well as anticipated student reasoning along with supports of the DGE. It is followed by student work collected in the actual classroom that provide pedagogical implications to revise the task sequence. I hope to develop this chapter to publish an article written for practitioners of teaching collegiate mathematics who are interested in pedagogical initiatives and novel ideas that support student learning of geometry. I intend to submit this paper to a journal with such readership, which is dedicated to improving undergraduate mathematics teaching and learning. I believe this paper has a potential in contributing to the community of practitioners by exchanging ideas from various intellectual endeavors centered around practices including survey on particular topics in mathematics, examination of pedagogical approaches, and programmatic initiatives to affect individual students, courses, and institutions.

In Chapter 3, I present an empirical study on students' uses of DGEs and their engagement in mathematical reasoning and axiomatic reasoning while they enacted a task sequence in an axiomatic geometry course. In their enactment of three tasks of the sequence, students used DGEs to communicate their mathematical ideas and to examine mathematical statements describing properties of geometric objects within axiomatic systems and models of hyperbolic geometry. The analysis of this study revealed case themes describing student use of DGEs, engagement in mathematical reasoning and axiomatic reasoning, and relationships thereof. The findings of the analysis provide practical implications to revise the task design as

well as theoretical implications to better understand the nature of student engagement in advanced mathematical reasoning in such technology-rich environments. I intend to improve this chapter and submit it to one of the major mathematics education journals with the readership interested in connecting theories and practices in technology-integrated mathematics education.

In Chapter 4, I address students' epistemological shifts that have been discussed in the existing literature of student learning of advanced geometry in connection with student work collected and analyzed in the empirical study reported in Chapter 3. First, students make a shift in the ontological view of geometric models from Euclidean to non-Euclidean geometry, in which the geometric models are considered conscious artifacts of mathematical design. Second, students make a shift in the epistemological view of mathematical proofs from absolutism to fallibilism, in which proofs can be characterized with a variety of functions and forms. Drawing on the prior literature, I argue that making successful shifts can benefit students in axiomatic geometry and that such shifts can be facilitated by engaging in mathematical activities with supports of dynamic geometry environments. In particular, I highlight examples of student work reported in Chapter 3 that illustrate those different views of geometric models and mathematical proofs captured observed from students who were on the process of such shifts. I hope to develop this chapter to publish an article written for mathematical education researchers who are specifically interested in understanding how students engage in advanced reasoning using dynamic geometry. I intend to submit the paper to a journal with such readership, which aims and scopes specifically emphasize studies to clarify the process of student development in mathematics. This paper might indicate particular aspects of student development in axiomatic geometry using DGEs not commonly recognized as well as clarify potential obstacles to student learning that result from neglect of a need for supports to making successful shifts in their views

of geometric models and mathematical proofs. Note that this chapter is an extension of an article published in the proceedings of the 40th annual meeting of the North American chapter of the international group for the psychology of mathematics education (PME-NA 40) in 2018 (Bae, 2018).

CHAPTER 2: DGE Task Design for Axiomatic Geometry

Abstract

Dynamic Geometry Environments (DGEs) have received attention by mathematic educators for supporting student learning in Euclidean and non-Euclidean geometries. With the attention to DGEs for axiomatic geometry, I illustrate a sequence of instructional tasks designed and enacted in an axiomatic geometry course where a DGE plays a crucial role in students' mathematical activities in class. The illustration of the task sequence consists of the mathematical activities intended in the design of each task as well as anticipated student reasoning along with supports of the DGE.

Introduction

In advanced geometry courses offered in mathematics departments, students are encouraged to take an axiomatic approach to study foundations of geometry including Euclidean geometry and beyond. In this approach to geometry, students are given formal definitions and formal proofs to understand axiomatic systems of geometries with different natures (K. F. Hollebrands, Conner, & Smith, 2010). Learning geometry in this approach allows students to understand how different sets of axioms lead to different or contrasting results in Euclidean and non-Euclidean geometry (National Council of Teachers of Mathematics [NCTM], 1989).

On the other hand, this axiomatic approach to geometry requires a departure from a significant reliance on concrete examples and visual representations in making sense of geometric objects. Making such shift in ways of perceiving and interacting with diagrams took hundreds of years even for past mathematicians in the history of discovering non-Euclidean geometry and establishing foundations of geometry in the modern language of mathematics (de Villiers, 1998). Not to mention, this shift to the axiomatic perspective on geometry has been

considered the highest level in students' cognitive development in geometry (van Hiele, 1986) and was reported in the literature with evidences of students' difficulties in learning non-Euclidean geometry due to this difference in views of geometry (e.g., Guven & Baki, 2010).

Dynamic Geometry Environments (DGEs) have received attention by mathematic educators for supporting student learning in non-Euclidean geometry (Dwyer & Pfiefer, 1999; Guven & Karatas, 2009; K. F. Hollebrands et al., 2010; Hvidsten, 2017; Olive, 2000; Stevenson & Noss, 1999). The technology demonstrates diagrams in non-Euclidean geometry and allows students to interact with dynamic constructions while investigating properties of figures to build mathematical knowledge of non-Euclidean geometry (e.g., Dwyer & Pfiefer, 1999; Guven & Karatas, 2009; Olive, 2000), engaging in formal reasoning about abstract concepts (e.g., Hollebrands et al., 2010). Hvidsten (2017) published a college level textbook for axiomatic geometry in which he introduced *Geometry Explorer*, a DGE that he developed for Euclidean, hyperbolic, elliptic, and spherical geometry. With the attention to DGEs for axiomatic geometry, I illustrate a sequence of instructional tasks designed and implemented in an axiomatic geometry course where a DGE plays a crucial role in students' mathematical activities in class. The illustration of the task sequence consists of the mathematical activities intended in the design of each task as well as anticipated student reasoning along with supports of the DGE. Note that, in this paper, axiomatic geometry means a study of foundations of geometry where various systems of geometry characterized by different sets of axioms.

The Course

In this section, I provide an overview of an axiomatic geometry course taught at a large public university where a DGE was integrated with the course design including the course material, class activities, course assignments, and assessments. The course design is inspired by

an instructional design theory of Realistic Mathematics Education (See Freudenthal, 1991) that emphasizes a role of guided reinvention in developing conceptual understanding. In this course, students built up an axiomatic system of neutral geometry by starting from the primitive axioms for incidence geometry and by introducing additional axioms to develop geometric concepts such as distance, betweenness, plane separation, congruence, and parallelism. In this section, I describe the course objectives and the course materials to provide an overview of the semester-long course design and the context of the task sequence following in the next section.

Course Objectives

There are three major objectives in this course including (a) understanding concepts of axiomatic systems and models, (b) engaging in axiomatic reasoning, and (c) developing knowledge of non-Euclidean geometry.

Understanding concepts of axiomatic systems and models. The concepts of axiomatic systems and models are crucial to establish the foundation of geometry where geometry is described by a set of selected axioms and represented by mathematical model that satisfy those axioms. Based on those concepts, students examine each axiom in a given system of a geometry to understand what it means to have that axiom in the system. Given a system, students construct and verify a model of the system by interpreting undefined terms in a particular way, which may involve sophisticated mathematical structure (e.g., stereographic projection onto disk of Euclidean geometry or Poincaré model for hyperbolic geometry). For instance, students investigate possible models of incidence geometry such as the four-point model or Cartesian space by verifying each axiom in the system holds in those models. In addition, robust understanding of axiomatic systems and models leads students to make a shift in their ways of perceiving diagrams in geometry (Bae, 2018). It allows students to make sense of diagrams in

non-Euclidean geometry where diagrams are parts of consciously designed model of the geometry that need to be interpreted in particular ways to satisfy the axioms. Otherwise, students might not understand why line segments in Poincaré model are *curved* in the Euclidean space.

Engaging in axiomatic reasoning. As described in the previous studies on student understanding of mathematical concepts such as functions (Dubinsky, 1992) or limits (Cottrill et al., 1996), students in advance mathematics are expected to objectify complicated mathematics concepts. One aspect of student understanding in axiomatic geometry is to engage in axiomatic reasoning in which axioms, geometric models, and the whole system of a geometry turn into mathematical objects for investigation. Axiomatic reasoning means to reason about those metamathematical objects including axioms, systems of axioms, and models of systems with respect to their properties. To be specific, mathematical activities in this course involves this axiomatic reasoning when students investigate, modify, and create axioms and models of a geometry. Axiomatic reasoning is necessary to examine properties of axioms including consistency, completeness, and independence, all of which play crucial roles in reconstructing students' existing knowledge of Euclidean geometry and expanding to non-Euclidean geometry in association with the relationship between different geometries. For instance, this approach to the foundations of Euclidean geometry opens up a space for a close examination of existing axioms in the system and possible variations of axioms. Students inquire about questions such as what if the system does not contain the parallel postulate, or what are alternative models of Euclidean geometry. Students also classify different systems of geometry including neutral, Euclidean, hyperbolic, and elliptic geometries and identify relationships between them by comparing axioms in those systems, especially different versions of parallel postulates.

Developing knowledge of non-Euclidean geometry. Students develop knowledge of non-Euclidean geometry by investigating properties of geometric objects in comparison with those in Euclidean geometry. Students identify differences in properties of geometric objects that are familiar with them from school geometry. For example, students investigate properties of quadrilaterals in hyperbolic geometry to see if how the classification of quadrilaterals is different from Euclidean geometry that they know from previous experience in geometry. Students encounter new concepts and geometric objects in non-Euclidean geometry that do not exist in Euclidean geometry. In hyperbolic geometry, for instance, students have infinitely many parallel lines to a given line and a point not on the line. They investigate this infinite set of parallel lines and explore particular parallel lines with special properties distinguished from others (See lines M and N in Figure 2.1). In Euclidean geometry where we have the unique parallel line, different types of parallel lines have never been investigated.

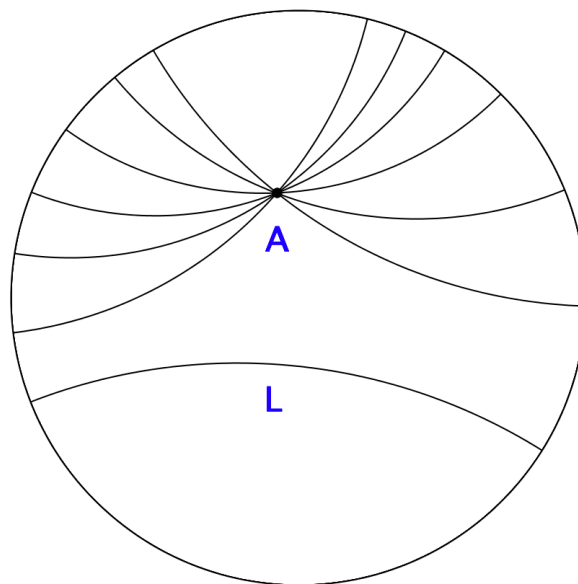


Figure 2.1. Infinitely many lines passing through a point A that are parallel to a line L in Poincaré model of hyperbolic geometry

Course Material

The main course materials include a lecture note and a textbook that is highly integrated with a DGE developed by the author of the textbook. The proof journal consists of undefined terms, postulates, definitions, and theorems to be proved by students. This journal is structured in a particular order to illustrate the development of an axiomatic system for Euclidean and hyperbolic geometries. It starts with a set of undefined terms and primitive postulates to introduce incidence geometry in the system, then introduces basic theorems that one can derive from the current set of axioms. By adding more definitions, postulates, and consequent theorems, the journal introduces the development of major concepts in geometry including distance, betweenness, angle measurements, congruence, and parallelism. Students are instructed to complete and submit their own proof journal by providing proofs for given theorems, working on examples and exercise problems, and filling in gaps between any significant changes in the system they developed throughout the course.

Axiomatic system of Euclidean geometry in the course. Among several versions of axiomatic systems for Euclidean geometry, the proof journal of this course introduces a particular system from a collegiate geometry textbook, *Elementary Geometry from an Advanced Standpoint* (3rd edition) (Moise, 1990). This system was originally proposed by the School Mathematics Study Group (MSG) in the 1960s as a part of the New Math curricula for high school students at that time. The authors wrote the MSG axioms based on Birkhoff's axioms where the number of axioms were significantly reduced from Hilbert's system due to the real number system implanted within the system (e.g., the completeness of the reals). However, the MSG system include a relatively large number of axioms that are not necessary in the sense that some of the axioms are logical consequences of others. This is because the intent of writing

axioms for high school geometry was not to emphasize excessive proofs of basic theorems from a minimal set of axioms in the system but elicit students' intuition and encourage sense-making in learning of geometry even though it causes a redundancy in the system (Venema, 2012). Then, Moise, who was also a member of the MSG, modified this MSG system for college level courses where he used a reduced set of axioms to move faster for covering more of the substance of the theory in geometry than high school geometry (Moise, 1990). Thus, the modified MSG system is considered appropriate for undergraduate mathematics major and minor students in this course to walk through one way to develop foundations of Euclidean geometry.

Geometry Explorer. The curriculum of this course is highly integrated with a DGE, Geometry Explorer (GEX). In both the textbook and the proof journal, GEX plays central roles not only in demonstrating examples or working on exercise problems but also in developing key concepts and engaging in problem-solving during class and assignments. GEX provides common features of DGEs including creation and measurement of draggable figures that students can interact with while enacting DGE tasks. In particular, GEX offers multiple models for different geometric systems including Euclidean, hyperbolic, elliptic, and spherical geometries. For instance, there are Poincaré model, Klein model, and upper half-plane model for hyperbolic geometry (See Figure 2.2).

Screencast software. In this course, students bring their laptops with GEX and screencast software installed into the classroom. As a part of class participations in classroom practice, they record their verbal presentations of their findings as well as their interactions with GEX on their computer screens that capture their uses of GEX while engaging in class activities. After each class, students submit their work including summary notes, GEX constructions, and the screencast video from the discussion. In addition, students create screencast presentation to

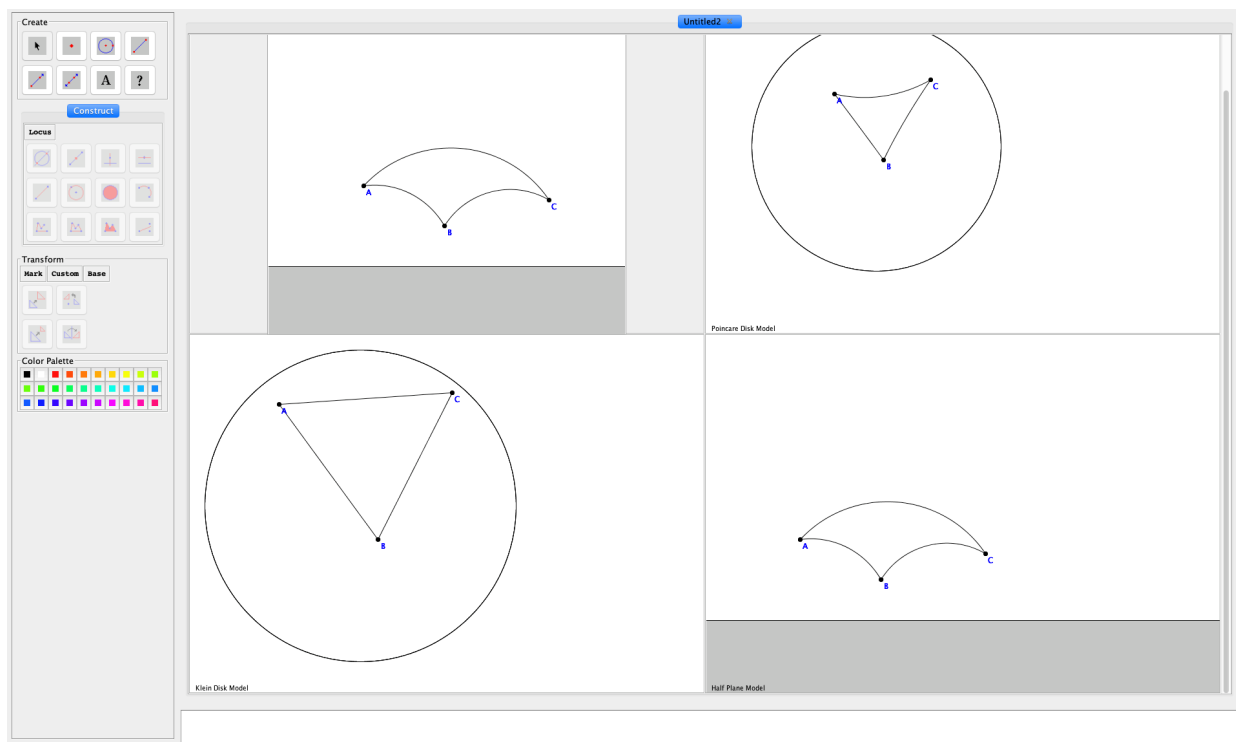


Figure 2.2. A diagram of triangle $\triangle ABC$ in three models of hyperbolic geometry: Upper half-plane model (upper-left), Poincaré model (upper-right), Klein model (left-lower)

respond to homework problems where they are asked to create and manipulate GEX construction. This audiovisual material of student work provides instructors with formative assessment to understand students' work on DGE supported tasks. Screencast video contains students' verbal presentation or group discussion with their uses of GEX, which capture their verbalized reasoning in relation to their observations, construction, and manipulations of dynamic figures in GEX.

The Task Sequence

In this section, I present a sequence of instructional tasks that aims at addressing the independence of parallel postulates and relative consistency of hyperbolic geometry that lead to the fork of Euclidean and hyperbolic geometries transitioning from the neutral geometry. This task sequence provides a snapshot of the course design that shows how DGE tasks are integrated

in the classroom instruction to address the three course objectives described in the previous section. For each task, I discuss mathematical activities intended in the task design along with task objectives in relation with the course objectives of axiomatic geometry.

Tasks in a Sequence

To begin with, I briefly summarize how researchers conceptualize a task sequence in their studies and explain what I meant by a task sequence in this study by addressing for what purposes this task sequence was designed and what constitute the task sequence to serve for such purposes. Researchers have studied students' engagements in task sequences where they focused on student development guided throughout the sequence. In particular, the instructional design theory of Realistic Mathematics Education (RME) emphasizes a role of task sequence in which students first develop their mathematical knowledge and activity, which later serve as tools for sophisticated mathematical reasoning (Zandieh & Rasmussen, 2010). For instance, Zandieh and Rasmussen (2010) reported how students engaged in defining activity regarding planar triangles and spherical triangles in a task sequence. In their report, the authors described how the students defined planar triangles early in the sequence and use those definitions while creating and exploring spherical triangles later in the sequence. They also highlighted that how students' engagement in this defining activity showed different levels of formal reasoning across the sequence. Andrews-Larson, Wawro, & Zandieh (2017) presented a hypothetical learning trajectory (HLT) supporting students in developing reasoning about linear transformations in linear algebra where they conceptualized a task sequence to construct the HLT in association with learning goals, evolution of students' mathematical activities, and roles of instructors. In their report, participants developed local views (e.g., transformation of a vector) and global views (e.g., transformation of a space) of linear transformation across the sequence. In both

studies, researchers conceptualized a task sequence along with specific aspects of students' development in reasoning across the sequence though those development was not necessarily sequential in nature. The researchers also provided detailed reports on students' enactment of the tasks including their specific reference to their prior experience in the task sequences. Their reports on the participants' continued engagements throughout the task sequences indicated that the design of task sequences aimed at supporting students carrying over their work in the previous tasks to the consequent tasks while engaging in the task sequence.

In this study, a task sequence is a set of mathematical tasks designed for particular objectives that can be achieved by students engaging in those tasks in a particular order that reflects an anticipated progress of student development. Each task in the sequence is designed to serve a specific role in supporting the development in relation to the students' work in previous tasks in the sequence. Thus, designing a task sequence involves arranging selected tasks with respect to anticipated students' responses to each task and transitions between those tasks. In sum, what constitute a task sequence in this perspective include a set of mathematical tasks in a particular order, an overarching objective of the entire task sequence, and prompts to make explicit connections in both design and enactment of the task sequence that support students making transitions between those tasks to make progress in the anticipated development. In the following sections, I present the overarching objective of the task sequence with anticipated student development in reasoning. It is followed by descriptions of individual tasks in the sequence to encompass the objective of the task sequence.

The Objective of the Task Sequence

One of the key topics in advanced geometry courses with axiomatic perspective is the independence of Euclidean Parallel Postulates (EPP) that leads students to the fork of geometries

based on different parallelism. To establish the independence of EPP, it is necessary to examine the consistency of a system with all the axioms of the neutral geometry with EPP or the negation of the EPP. As seen in the history of mathematics, one and maybe the only practical way to prove the relative consistency of hyperbolic geometry is to build a model that satisfies all the axioms in the system such as Poincaré model or Klein model. Note that those models were built on Euclidean space, so the hyperbolic geometry is consistent as much as Euclidean geometry. However, verifying a geometric model of an axiomatic system is a lengthy job that has been dismissed in most textbook since it entails proving all the axioms in Euclidean system except its parallel postulate (Moise, 1990).

The objective of the task sequence is to provide students with mathematical experience of investigating independence of EPP and consistency of hyperbolic geometry in a way that is feasible to enact in the classroom without excessive proofs involving significant work drawn from analytic geometry in Euclidean space. In this task sequence, students confirm the relative consistency of hyperbolic geometry by constructing and verifying Klein model in Euclidean plane in GEX and use this model to study parallelism of hyperbolic geometry including limiting parallel lines. Given the affordance of student-generated model of hyperbolic geometry in GEX, the use of technology in this task design exemplifies cognitive technology for classroom teaching and learning that “help transcend the limitations of the mind in thinking, learning, and problem-solving activities” (Pea, 1987, p.91) by both amplifying and reorganizing students’ mental activities.

In addition, this objective of the task sequence contributes to the three course objectives including (a) understanding concepts of axiomatic systems and models, (b) engaging in axiomatic reasoning, and (c) developing knowledge of non-Euclidean geometry. In the broader

context of the entire course, students would engage in this task sequence when they develop an axiomatic system for neutral geometry and investigate the properties of parallel lines in terms of what they can say for sure about parallel lines within the current system without particular parallel postulate. Students are asked to take a step back and probe two possible geometries depending on which parallel postulate they would include in their system. They reason about different axiomatic systems, possible models thereof, and the axioms that distinguish Euclidean and hyperbolic geometries. Later in the sequence, students are expected to use their model of hyperbolic geometry to explore limiting parallel lines and develop the concept in formal way of defining abstract geometric objects independent of particular models. Thus, the aforementioned three course objectives interweave as students engage in this task sequence.

Prior to enacting the task sequence, students are expected to have knowledge of basic Euclidean constructions and experience with GEX features to construct, measure, and interactive with draggable diagrams. In this course, my students used GEX in class activities, homework, take-home midterm exam where they recorded their individual and group work on creating and analyzing draggable diagrams in Euclidean model, spherical model, and Poincaré model in GEX prior to the task sequence enactment in Week 11-13. In those course assignments using GEX, students practiced basic Euclidean constructions (e.g., equilateral triangle, circle passing three non-collinear points, etc.), analyzed a geometric proof by investigating counter-examples in the construction of the proof, and verified that a model satisfies axioms in a system. In particular, students were introduced Poincaré model of hyperbolic geometry in class and explored the model in GEX when verifying given geometric statements (e.g., angle sum of a triangle, existence of rectangles) in Week 5 group project (See Appendix B). For more details about the course design, the weekly schedule of the course with topics is in Appendix A, and DGE tasks

for in-class activities and homework of the course are in Appendix B. In the following sections, I present four tasks in the sequence with emphasis on the objective of individual task in association with its contribution to the overarching objective of the task sequence as well as its connections to other tasks in the sequence.

Task 1: Construction and Verification of Klein Model

The main objective of Task 1 is to establish a mathematical foundation of Klein model for hyperbolic geometry. The task is especially focused on constructing the angle measurement function and verifying postulates in the system that involve angle measurements. This is because the way of measuring angles in Klein model requires complicated geometric construction that students may not be able to investigate without the support of DGEs. In this task, students examine the mathematical structure of the angle measurement function that is consciously designed to satisfy all the postulates in the system of neutral geometry. In this experience, students engage in axiomatic reasoning where they reason about axioms in the system within the context of Klein model by interpreting diagrams in Euclidean constructions in terms of the undefined terms and definitions used in the axioms. The role of technology in the axiomatic reasoning is to provide draggable constructions of Klein angle measurement that instantiate arbitrary figures while students go back and forth between written statements of the postulate and draggable diagrams. In particular, this construction provides evidences of the covariation between given angles and the Klein measurements as students change the draggable diagrams of angles.

Construction of Klein model.

The first task of the sequence involves constructing angle measurement in Klein model and verifying that this construction satisfies the following five angle postulates in the system.

Note that these postulates elaborate plausible assumptions about a concept of angle measurements introduced in axiomatic systems for neutral geometry.

- Postulate 10: There exists a function m from the set of all angles \mathcal{A} to the set of real numbers \mathbb{R} .
- Postulate 11: For every angle in \mathcal{A} , the measurement of the angle is between 0 and 180.
- Postulate 12: (Angle-Construction Postulate) Let \overrightarrow{AB} be a ray on the edge of a half-plane H . For every number r between 0 and 180, there is exactly one ray \overrightarrow{AP} , with a point P in H , such that $m(\angle PAB)$ is r .
- Postulate 13: (Angle-Addition Postulate) If a point D is in the interior of an angle $\angle BAC$, then $m(\angle BAC)$ is the sum of $m(\angle BAD)$ and $m(\angle DAC)$.
- Postulate 14: (Supplement Postulate) If two angles form a linear pair, then they are supplementary.

In this task, students are given a GEX construction for Klein model in Euclidean plane.

This construction contains a unit disk in Euclidean plane where two given lines \overleftrightarrow{AB} and \overleftrightarrow{BC} meet at a point B inside the disk. Those lines with labeled points define an angle $\angle ABC$ inside the disk that is to be measured by Klein angle measurement in this task (See Figure 2.3).

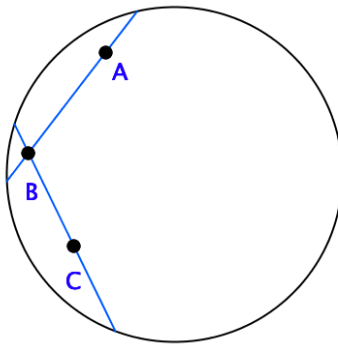


Figure 2.3. Angle $\angle ABC$ in Klein disk defined by two lines \overleftrightarrow{AB} and \overleftrightarrow{BC}

To measure this angle as interpreted in Klein model, students use Euclidean constructions of orthogonal circle for each line that meets Klein disk at right angles on the two intersecting points between the line and the unit disk. Then students construct tangent lines on the intersection of the two orthogonal circles and measure the angle between the tangent lines in terms of Euclidean measurement (See Figure 2.4). Klein angle measurement is not necessarily consistent with students' perceptions of angles that are more attuned to Euclidean geometry. For instance, the angle $\angle ABC$ in Figure 2.4 does not *look* acute in Euclidean geometry but it is measured to be 73.72° by Klein angle measurement that refers the Euclidean measurement of the angle $\angle A'B'C'$.

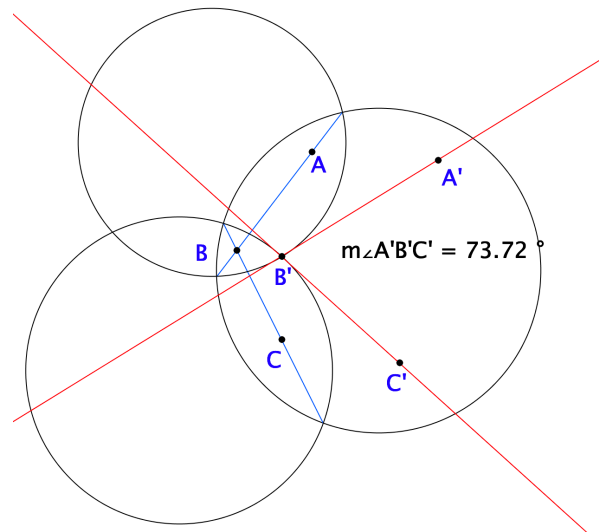


Figure 2.4. Klein measurement of the angle $\angle ABC$ that refers to Euclidean measurement of the angle $\angle A'B'C'$

The construction part of this task demonstrates the one-to-one correspondence between the given angle and corresponding angle formed by the tangent lines of the orthogonal circles so that students ensure the well-definedness of the angle measurement function in Klein model. Note that the existence of a hyperbolic geometry model in Euclidean geometry (e.g., Klein

model or Poincaré model) implies a relative consistency of the hyperbolic system as long as Euclidean geometry is consistent.

Verification of Klein model. In the following verification part of the task, students are asked to verify that their constructions of Klein angle measurement satisfy the five angle postulates. Note that their constructions in GEX are draggable figures so that they can drag the given angle $\angle ABC$ in Klein disk while observing dynamic changes within the whole construction including the corresponding orthogonal circles, tangent angles and the Euclidean measurements thereof. This allows students to change the dynamic diagram of $\angle ABC$ by dragging independent points A , B , and C determining the angle and show each statement of the angle postulates hold no matter how they drag those points. In Postulate 13 and 14, for instance, students construct an arbitrary ray \overrightarrow{BD} where D is in the interior of the angle $\angle ABC$ and compute the sum of the measurements of two angles $\angle ABD$ and $\angle DBC$ (See Figure 2.5).

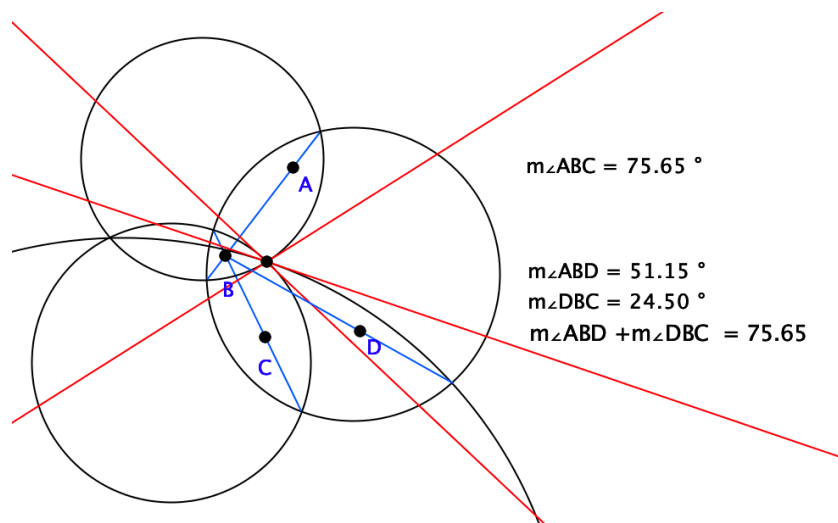


Figure 2.5. The ray \overrightarrow{BD} in Klein disk and the sum of Klein angle measurements $m(\angle ABD)$ and $m(\angle DBC)$ that is equal to $m(\angle ABC)$

Note that the verification of the angle postulates primarily relies on the one-to-one correspondence between arbitrary angles in Klein disk and the angles between tangent lines of orthogonal circles. In other words, a key of this verification part of the task is to know that Klein model borrows its angle measurement from Euclidean geometry which also satisfies all the five postulates. For instance, Postulate 12 requires students' attentions to the surjectivity of the measurement function m in Klein model that relies on the surjectivity of angle measurements in Euclidean geometry. Verifying this postulate is a reverse process of constructing orthogonal circles and tangent lines given the assumption that for any real number r between zero and 180 there exists (or one can construct) a Euclidean angle with its measurement being r .

Task 2: Analysis of Legendre's Attempted Proof of Euclidean Parallel Postulate

In the previous task of constructing and verifying Klein model, students are expected to confirm that this model satisfies all the postulate in the current system for Euclidean geometry except the EPP. In fact, this indicates the relative consistency of hyperbolic geometry and the independence of EPP in neutral geometry given the existence of a model of the system (e.g., Klein model) that does not satisfy EPP. In addition, this model implies that any attempt to prove EPP in this system must end up with failure such as the one suggested by a past mathematician Adrien-Marie Legendre. This implication leads to Task 2 where students are asked to analyze Legendre's attempt to prove EPP by geometric construction in Klein model of GEX. Students are given a written document describing each step of a geometric construction by which Legendre argued for the uniqueness of the parallel line to a given line and a point not on the line.

Given P not on line l . Drop perpendicular \overleftrightarrow{PQ} from P to l at Q . Let m be the line through P perpendicular to \overleftrightarrow{PQ} . Then m is parallel to l . Let n be any line through P distinct from m and \overleftrightarrow{PQ} . We must show that n meets l . (See Figure 2.7)

Given the existences of parallel lines in neutral geometry system, Legendre starts constructing a line m that is parallel to the given line l and that passes through a point P not on the line l . It is followed by his assertion that any arbitrary line n passing the point P that is not m should meet the line l .

Let \overrightarrow{PR} be a ray of n between \overrightarrow{PQ} and a ray of m with endpoint P . There is a point R' on the opposite side of \overrightarrow{PQ} from R such that $\angle QPR' \cong \angle QPR$. Then Q lies in the interior of $\angle RPR'$. Since line l passes through the point Q , l must intersect one of the sides of this angle. If l meets \overrightarrow{PR} , then certainly l meets n . Suppose l meets side $\overrightarrow{PR'}$ at a point A . Let B be the unique point on \overrightarrow{PR} such that $\overline{PA} \cong \overline{PB}$. Then $\triangle PQA \cong \triangle PQB$. Hence $\angle PQB$ is a right angle, so that B lies on l (and n). Q.E.D.

Given the existence of a model of the system where EPP does not hold, his proof cannot be valid and, in fact, includes a hidden assumption that is logically equivalent to EPP. Students analyze each step of the construction and identify the hidden assumption that causes a circular logical in this proof. Students build the construction in Klein model following each step in the written proof above and explore where the construction fail without an additional assumption. Note that this Klein model is not the Euclidean construction that students created in the previous task but the automated model offered in the software so students can construct basic figures as easy as they could do in Euclidean models. In GEX, students can choose one of hyperbolic geometry models including Poincaré model, Klein model, and half-plane model in which students can create and measure figures.

While constructing their figures following the steps described in the material, students may see that the statement in the fourth step does not necessarily hold for an arbitrary ray \overrightarrow{PR} and the angle $\angle RPR'$ in their construction in Klein model. In this step of his proof, Legendre

assumed that any line passing through a point in the interior of an angle must intersect one of the rays of the angle. However, this assumption is not always true in a system without EPP or any statement logically equivalent to it. For instance, if students drag the angle $\angle RPR'$ to make it wide enough to get close to the line m , neither of the rays \overrightarrow{PR} nor $\overrightarrow{PR'}$ meet the line l . (See Figure 2.7). This provides a counter-example to the statement in the fourth step using the hidden assumption, which, in turns, refutes his argument. Note that there are at least three lines (in red), $m, n, \overleftrightarrow{PR'}$, in this case where all they are parallel to the line l and pass the point P .

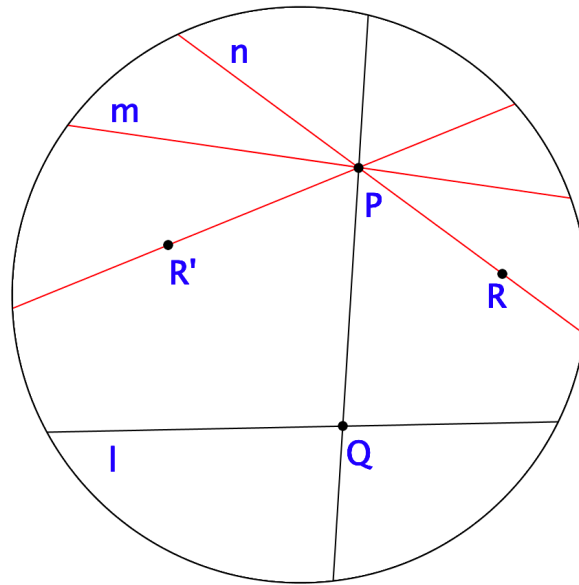


Figure 2.6. Counter-example to Legendre's attempted proof of EPP in Klein model

The objective of this task is to construct a geometric proof in Klein model of GEX and use this dynamic construction to explain why this proof fails in this model. So, an essential mathematical activity in this task is that students identify independent elements of the construction and counter-examples of the construction by dragging those elements. This activity involves reasoning about generalization of a geometric construction that is necessary to check

the validity of the proof constructed. Technology plays a role in constructing the proof in a hyperbolic model and generalize the construction when students drag the independent elements (e.g., the point R of the angle $\angle RPR'$ in Figure 2.7) and look for any changes or invariants in the construction.

Task 3: Exploration of Limiting Parallel Lines in Hyperbolic Geometry

The previous two tasks in the sequence provide students with experiences of close examinations of a hyperbolic geometry model that is necessary to conclude the independence of EPP and the consistency of hyperbolic geometry. This experience involves constructing, verifying, and exploring the sophisticated structure of angle measurements in Klein model that is consciously designed to satisfy all the postulates of neutral geometry but EPP.

In the third task, students use a particular construction feature in hyperbolic geometry mode of GEX that provides a pair of special parallel lines called *limiting parallel lines*. For a given line l and a point P not on l in Klein model, for instance, this feature creates two lines m and n that are pass P and do not meet l (See Figure 2.9).

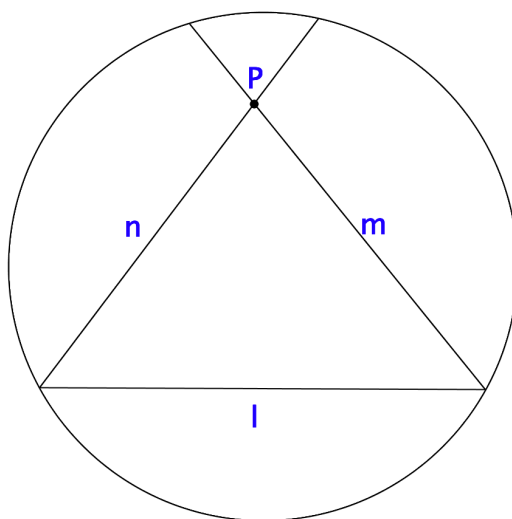


Figure 2.7. Limiting parallel lines to a given line l and a point P not on l in Klein model

Note that the two lines m and n meet l on Klein disk where those intersections are not *interpreted* as points in Klein model. A point is an undefined term in an axiomatic system, so it needs to be interpreted in each model of the system. Since the set of all points in Klein model is limited to the interior of the Klein disk, those two special lines m and n do not meet l in Klein model although there are intersections on the disk. Exploring the limiting parallel lines allows students to attend to this interpretive nature of figures in the axiomatic geometry as well as explore properties of parallel lines in hyperbolic geometry comparing to what they know about parallel lines in Euclidean geometry.

Task 3 is aimed at providing students with opportunities to make an initial sense of limiting parallel lines in Klein model of GEX that are new geometric objects to them. In this task, students are instructed to create limiting parallel lines based upon their examination of this given attempted proof from Task 2 and to respond the following three questions.

- Are the limiting parallel lines examples or counter-examples of $\angle RPR'$ in the Legendre's proof?
- Using the limiting parallel lines, how can you separate examples or counter-examples of $\angle RPR'$ in the Legendre's proof?
- How can you describe the limiting parallel lines in the Klein model? Do you think the description is applicable in any models of hyperbolic geometry?

In the first question, students revisit their work in the previous task and determine if the limiting parallel lines meet the line l or not. They interpret what they create in Klein model using the creation feature for limiting parallel lines that are two lines that seem to meet l only on the Klein disk. Although students may see from the term, *limiting parallel lines*, that those lines are parallel lines, they may also see that those lines are different from other parallel lines from the

term and/or from the diagrams thereof on the screen. It is followed by the last two questions about an essential property of the limiting parallel lines that separate parallel and non-parallel lines (See Figure 2.10).

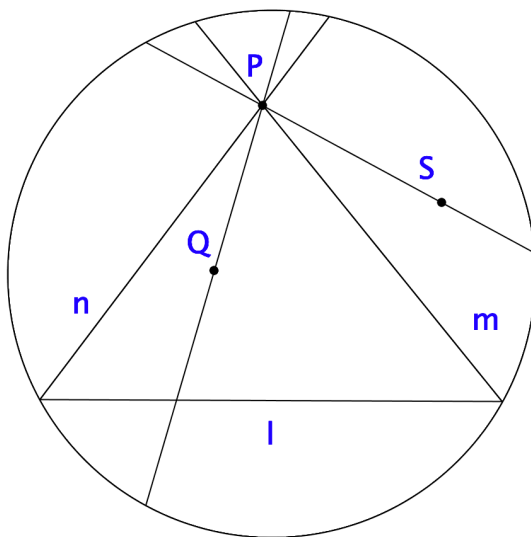


Figure 2.8. Limiting parallel lines m and n with respect to the line l and a point P not on l that separate parallel lines (e.g. \overrightarrow{PS}) and non-parallel lines (e.g., \overrightarrow{PQ}) to l

An arbitrary line \overrightarrow{PQ} intersects l if Q lies in the interior of the angle between m and n where l is also included in the interior of the angle otherwise the line does not intersect l so is parallel to the line (e.g., \overrightarrow{PS}). In other words, the limiting parallel lines separate the entire plane except themselves into two pairs of vertical angles centered at P such that one pair of vertical angles contains non-parallel lines to l thru P (e.g., \overrightarrow{PQ}) and the other pair contains parallel lines (e.g., \overrightarrow{PS}).

The intent of introducing this property in this task is to have students develop a mathematical definition of limiting parallel lines applicable in any models of hyperbolic geometry. The third question specifically challenges students to describe this property or propose a definition in a way that relies only on the system of hyperbolic geometry excluding any other

specification of the model. In the following examples, students used diagrammatic features of Klein model in describing the diagrams of limiting parallel lines that were not necessarily applicable in other models of hyperbolic geometry.

Task 4: Construction of Limiting Parallel Lines in Hyperbolic Geometry

The last question in Task 3 indicates a need for defining limiting parallel lines for all possible models of hyperbolic geometry. In association with the previous task, Task 4 addresses this issue of a universal definition of limiting parallel lines beyond a particular model of the system and asks how to ensure the existence of such lines within the system (See Appendix B). The task is structured with multiple questions that lead students to engage in a mathematization process (Freudenthal, 1991) in which they are guided to reinvent the concept of limiting parallel lines. In this process, students make conjectures from their experience with Klein model in GEX and validate the conjectures by producing a constructive proof for the existence of limiting parallel lines that eventually comes with a rigorous definition.

Students are given worksheets of the task that consists of observations, conjectures, and proof phases. In the observation phase, students revisit one of their homework problems where they proved the existence of a triangle with an arbitrary small interior angle using the following iterative construction of isosceles triangles (See Figure 2.11).

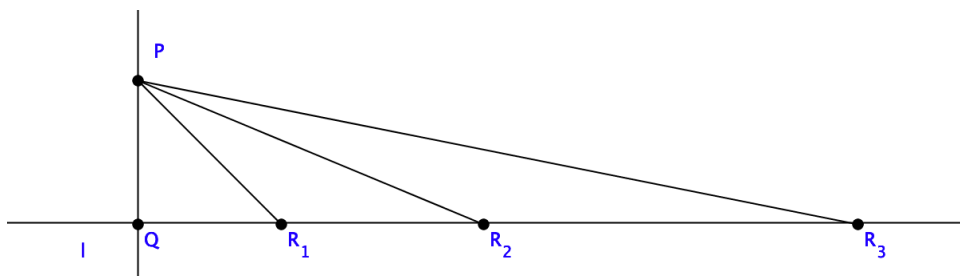


Figure 2.9. Iterative construction of triangles $\Delta R_{n-1}PR_n$ that produces a sequence of lines $\overrightarrow{PR_n}$ in Euclidean geometry

In the diagram above, ΔQPR_1 is an isosceles triangle where \overline{PQ} and $\overline{QR_1}$ are congruent, and $\Delta R_{n-1}PR_n$ are isosceles triangles where $\overline{PR_{n-1}}$ and $\overline{R_{n-1}R_n}$ are congruent. In the homework, students proved that $m(\angle PR_nQ)$ gets small as n increases so they can find a large natural number n where $\angle PR_nQ$ is arbitrarily small. Note that students needed particular axioms and their consequences in the system (e.g., Saccheri-Legendre theorem) as well as the Archimedean Postulate from the real number system. The key message of the task in the observation phase is to gain an insight from the construction of the previous homework problem so that students generate a conjecture to define limiting parallel lines and produce a constructive proof of the existence thereof. In this revisited construction, students are asked to think about a sequence of line $\{\overrightarrow{PR_n}\}_{n=1}^{\infty}$ that looks converging to one of the limiting parallel lines of l at P . They also connect this observation to their work in Task 3 where they observed how limiting parallel lines separate parallel and non-parallel lines in Klein model. Based on all these experience in their homework and the previous task in the sequence, the questions in the observation phase lead students to a conjecture to define limiting parallel lines as limits of non-parallel lines (e.g., $\overrightarrow{PR_n}$ in Figure 2.9) as they go further from the perpendicular line \overrightarrow{PQ} and get close to parallel lines.

This definition of limiting parallel lines would make sense to students given their experience in previous activities including their interaction with the diagrams in Klein model. However, this definition is not written in the language of formal mathematics because this informal idea of the limit of a sequence of figures cannot be articulated within the system in the axiomatic perspective. In the transition from the observation phase to the conjecture phase, students are encouraged to refine their conjectures by using the sequence of angle measurements $\{m(\angle QPR_n)\}_{n=1}^{\infty}$. Since the angle measurement function in the system is a real-valued function,

this is a sequence of real numbers for which students can use the monotone convergence theorem from the theory of real numbers. Given this theorem gives the limit r of the sequence, students argue for the existence of an angle $\angle QPR$ where $m(\angle QPR) = r$ from the angle construction postulate (Postulate 12) in the system. Note that this limit r is called the angle of parallelism with respect to the line l and a point P not on l . One can see the angle of parallelism should always be 90° in Euclidean geometry where there is one and only parallel line, whereas hyperbolic geometry must have the angle of parallelism strictly less than 90° given it allows more than one parallel lines.

In the following proof phase of the task, students provide a complete proof of the existence of the angle $\angle QPR$ where $m(\angle QPR)$ equals the angle of parallelism. To complete the mathematization process in this task connecting their informal observation of limiting parallel lines in Klein model with this formalized definition of $\angle QPR$, especially of \overrightarrow{PR} , students show the definition (or construction) is consistent with their observation on how the limiting parallel lines separate parallel and non-parallel lines in Klein model. The last two questions in the proof phase is to show that any line inside $\angle QPR$ should meet the line l otherwise it should not.

Concluding Remarks

In this paper, I presented a sequence of tasks designed and enacted in an axiomatic geometry course where a DGE plays a crucial role in students' mathematical activities in class. Given the three objectives of the course, the task design aims at providing students with an experience of verifying a hyperbolic model and, based on the experience, to engage in axiomatic reasoning and to develop knowledge of axiomatic systems and hyperbolic geometry.

CHAPTER 3: Student Engagement in Axiomatic Reasoning in DGE Task Sequence

Abstract

In this chapter, I present an empirical study on students' uses of DGEs and their engagement in mathematical reasoning and axiomatic reasoning while they enacted a task sequence in an axiomatic geometry course. The analysis of this study showed that students used DGEs for specific purposes with respect to the mathematical activities they engaged in responding the given tasks. DGEs provided students with experimental tools to model geometric figures and phenomena in hyperbolic geometry. Students reasoned about mathematical and meta-mathematical objects involved in the enacted tasks in which they used DGEs to express their mathematical ideas and to examine mathematical statements describing properties of geometric objects within axiomatic systems and models of hyperbolic geometry. The findings of this study provided detailed portraits of student use of DGEs that were associated with their engagement in mathematical reasoning and axiomatic reasoning.

Introduction

A goal of geometry education at a collegiate level is to provide students with opportunities to experience the axiomatic approach to geometry and appreciate its usefulness for understanding different types of geometries other than Euclidean geometry. Students in college geometry courses are expected to make a shift from high school geometry to advanced geometry where they make meanings of geometric objects using formal definitions and rigorous proofs (K. F. Hollebrands et al., 2010; David Tall, 2008). In expanding their scopes beyond Euclidean geometry and their ways of interacting with diagrams thereof, students are expected to view geometry as an axiomatic structure, which is characterized by its axiomatic system and can be visualized and materialized by different models. This axiomatic perspective on geometry is

necessary to understand how the axioms in the system lead to quite different and often contradictory results in Euclidean and non-Euclidean geometry (National Council of Teachers of Mathematics [NCTM], 1989). In this study, axiomatic geometry means a study of geometry with a focus on this axiomatic approach to understand various geometries with different axiomatic systems including Euclidean and non-Euclidean geometries.

Mathematics departments offer upper-division geometry courses introducing foundations of Euclidean and non-Euclidean geometries in this perspective of axiomatic geometry. As in other upper-division mathematics, students in these courses are usually given formal and abstract definitions of geometric objects. They are expected to produce rigorous proofs when they learn properties of figures in each geometry. On the other hand, they are not able to rely on visual representation as reasoning about those figures in ways the same as how they were able to do in Euclidean geometry. Due to this limited access to those resources supporting student reasoning in axiomatic geometry, educators have made efforts to use Dynamic Geometry Environments (DGEs) for non-Euclidean geometry in class. Comparing to school geometry at K-12 level, however, there is a lack of available tools, class practices, and empirical studies investigating how students reason in axiomatic geometry class using DGEs. The purpose of this chapter is to present an empirical study of students' uses of DGEs and their reasoning in a task sequence using DGEs.

DGEs Support Student Reasoning

In the learning of geometry, DGEs provide students with explorative tools that allow them to create diagrams, make measurements, and drag components of a diagram. Mathematics education researchers and professional organizations have suggested that the contribution of DGE in the teaching and learning of geometry comes from dynamic interaction with manipulable

graphical representations (e.g., Battista, 2007; Common Core State Standards Initiative, 2012; National Council of Teachers of Mathematics, 2000). In particular, researchers have argued for the importance of such interactions with DGEs in facilitating developments of student reasoning that plays a key role in achieving learning goals of axiomatic geometry (Dwyer & Pfeifer, 1999; K. F. Hollebrands et al., 2010; Laborde, Kynigos, Bollebrands, & Strasser, 2006; Olive, 2000; David Tall, 2008) In the following sections, I will address two examples of such developments of student reasoning in axiomatic geometry that had been highlighted along with the role of DGE activities and that are relevant to the task sequence of this study: Mathematization and conceptual embodiment of non-Euclidean geometry.

Supporting student mathematization. Researchers have argued that DGEs serve as explorative tools for students engaging in mathematization process. In this process, students reconstruct geometry knowledge by conducting experiments, making conjectures, and mathematically explaining or proving their conjectures. In particular, researchers reported that the mathematization in geometry can be supported by the explorative nature of DGE activities (Armella & Sriraman, 2005; Arzarello, Olivero, Paola, & Robutti, 2002; Komatsu, 2017; Komatsu & Jones, 2019; Komatsu, Tsujiyama, Sakamaki, & Koike, 2014; Olive, 2000; Olivero, Paola, & Robutti, 2002; Sinclair et al., 2017). For instance, Laborde et al. (2006) highlighted a duality of reasoning in geometry, meaning the combination of empirical evidence from visual representations and deductive thinking. Problem-solving in geometry requires deductive reasoning, which includes making conjectures and proving properties of geometric objects. On the other hand, it also involves inductive reasoning, which takes place when students observe and interact with diagrams of the geometric objects. Those visual representations provide empirical evidence for students to generate initial conjectures and then revise them. Due to this duality of

deductive and inductive reasoning involved in mathematization of geometry, students can benefit from explorative DGE activities as they conduct their experiments and test their conjectures.

Researchers conducted empirical studies on such DGE activities supporting students' mathematization to reconstruct geometric knowledge in non-Euclidean geometry. In their studies, DGEs allowed participants to get access to empirical evidence to create their own conjectures and construct formal proofs. (Güven & Baki, 2010; K. F. Hollebrands et al., 2010; Stevenson & Noss, 1999). Güven and Baki (2010) reported how preservice teachers explored spherical geometry using a DGE for spherical geometry, *Spherical Easel*, and proved their own conjectures about triangles on sphere. The exploration on diagrams of triangles in spherical model provided the teachers with empirical evidence for their conjectures with considerable conviction as well as supported them in producing a mathematical proof of the conjectures. Hollebrands et al. (2010) investigated how a DGE for non-Euclidean geometry, *NonEuclid*, supported students making their arguments as they solve geometric construction problems in non-Euclidean geometry. In particular, the DGE enabled students to construct accurate diagrams and to abstract general properties and relationship of figures from the construction while making and qualifying their conjectures. Researchers argued that uses of DGEs in explorative activities can support students engaging in the mathematization process that involves both inductive and deductive reasoning.

Conceptual embodiment of non-Euclidean geometry. Another role of DGEs in supporting student reasoning in non-Euclidean geometry is to promote the conceptual embodiment of geometric figures with formal definitions. The conceptual embodiment means a process of developing perceptual representations for abstract concepts of geometry and figures in non-Euclidean geometry given by formal definitions (David Tall, 2008). According to his

description, students “conceptually embody a geometric figure, such as a triangle consisting of three straight line-segments” (p. 7), then are able to access this embodied figure when they reason about the concept. A perceptual representation does not mean a static mental image, rather a dynamic mathematical structure that enables students to “allow a specific triangle to act as a prototype to represent the whole class of triangles” (p. 7). Olive (2000) also addressed the affordance of DGEs in non-Euclidean geometry that allow students to construct and measure diagrams in hyperbolic geometry (e.g., Poincare model constructed in Sketchpad using a script tool). In the learning of axiomatic geometry, students can benefit from developing perceptual representations in non-Euclidean geometry where they lack access to physical models (Junius, 2008).

Tall (2008) argued that the conceptual embodiment for non-Euclidean geometry can be supported by providing a physical embodiment so that students can build perception of geometric objects. In this regard, exploring behaviors of concrete figures in DGEs can facilitate students to develop perceptual representations of the abstract concepts and figures in non-Euclidean geometry. Visual and manipulable diagrams in DGEs allowed students to drag diagrams dependent to each other and observe the dynamic responses the software return to their manipulations. Such experience of interacting with dynamic diagrams enables students to explore properties of abstract figures in non-Euclidean geometry, and in turn, develop their perceptual representations that they can access as engaging in formal reasoning. This idea of student development in DGEs was captured in a similar idea of situated abstraction and situated proofs (Armella & Sriraman, 2005). They argued that students’ explorative experience in DGEs allow them to encapsulate general statements describing mathematical properties they observed from the technology.

In their empirical study, Guven and Karatas (2009) showed that interactive diagrams of a DGE can support students' conceptual embodiments of triangles in spherical geometry. In their study, the participants explored triangles on a sphere of a DGE and abstracted properties of spherical triangles that they did not see in Euclidean triangles. After the DGE exploration, the participants could access their perceptual representations of spherical triangles of which they could refer to properties when they drew paper-and-pencil diagrams of spherical triangles in producing proofs of those properties.

In the above two sections, I addressed two pedagogical benefits of using DGE activities that can support students reasoning involved in learning of axiomatic geometry. The literature indicates that exploring dynamic diagrams of abstract figures in DGEs can facilitate students to reconstruct geometric knowledge thru the process of mathematization and develop perceptual representation of non-Euclidean geometric figures.

Drag Feature of DGEs

Researchers have investigated and argue that the drag feature of DGEs provides students with opportunities to visually experience the variation of conceptual geometric figures in non-Euclidean geometry (Arzarello et al., 1998; Baccaglini-Frank & Mariotti, 2010; Jones et al., 2010; Leung, 2008). In the perspective of mathematical variation in exploration with DGE, students observe emerging variations of figures in dragging activities and discover invariant properties in varying components of the geometric constructions across different geometries (Leung, 2008). Through a lens of variation, therefore, we can describe how students' interactions with DGE including dragging activities contribute to developing their understanding of axiomatic geometry.

Wandering dragging for contrasting. Leung (2008) interpreted students' dragging activities in DGE in relation with different functions of variation. Drag for *contrasting* is to see differences in variations and make conjectures of what can be and cannot be. A primitive dragging strategy illustrating the drag for contrasting is *wandering dragging*, in which students drag around all the varying components of the constructions without strategic exploration, but survey overall changes of the construction. The number of varying components in a geometric construction can be regarded as the dimensions of variation in dragging activities. For example, a circle passing through three distinct points has three dimensions of variation, the three points defining the circle. In wandering dragging, students observe how the circle varies when they drag the three points with shuttling between them. Since students keep switching the dimension of variation, the number of dimensions is not constrained in wandering dragging.

Guided dragging for separation. When a dimension of variation is being dragged while other dimensions are being kept constant, a potential invariant pattern or property might emerge. In this stage, called drag for *separation*, students attempt to separate out hidden geometric patterns or invariant properties. During the drag for separation, students need to decide what to focus on and what to temporally lay aside or keep fixed. Geometrical shape or numerical measurement guide students to determine their strategic dragging activities, which is called *guided dragging*. To visually separate out the hidden patterns and pin down on their conjectures, a transition in dragging strategies is needed. Although the shift to guided dragging can be regarded as refining wandering dragging strategy, students continue to interplay between two different dragging strategies until they finalize their conjectures with converging results from the variations.

Drag test for generalization. To verify the conjecture obtained from the constant interplay between wandering and guided dragging strategies, students need to move on the next stage, *drag for generalization*. It is to test the figure with respect to a given condition to see given diagram keep satisfying the condition while changing the dimensions of variations. In terms of dragging strategy, a generalized diagram should pass this *drag test*. A generalized diagram through the drag test is a *robust construction*, which means a “properly” constructed figure in DGE which is not just a static diagram rather a variable figure that the invariant or defining properties are preserved under dragging. A robust construction created by students in DGE can be an empirical evidence that they verify their conjectures in exploratory activities.

Framing dragging as contrasting, separation, and generalization gives a way to understand pedagogical significance of the drag feature of DGE. The corresponding dragging strategies can be used to describe how students use of DGE and how the strategies shape geometrical knowledge in the context of a specific exploratory activities.

Theoretical Framing

In this section, I provide definitions of constructs and theoretical backgrounds to establish framework for analysis in this study. Then I presented the three research questions of the study.

Students Uses of DGEs

Prior study on students’ uses of DGEs. I investigated how students enacted a sequence of exploratory DGE tasks in an axiomatic geometry course at a large public university in Spring 2016. The study allowed me to establish an analytic framework for investigating students' uses of DGE in the task enactment. In this study, I analyzed what features of DGE (e.g., selecting, creating, dragging, etc.) were used in their group work as well as how the students utilized drag features in their exploration. I created codes to describe the dragging activities with specific

purposes on which students use the drag feature in relation to their reasoning. This analysis allowed me to describe characteristics of DGE usage in groups who were successful in accessing comprehensive empirical evidence to make a generalizable argument. Successful groups spent more time on dragging activities than other features (e.g., point at diagrams or move diagrams, etc.) in their exploration, which provided students with active interactions with dynamic constructions in terms of frequent interchanges between their manipulation and response of the software. In addition, successful groups were strategic in using the drag feature to collect empirical evidence for creating conjectures. They dragged independent elements of given constructions to observe different examples as many as possible while collecting evidence to create their conjectures, then dragged those elements specific to their conjectures on particular example of diagrams. For instance, they would change all three vertices defining an arbitrary angle and see how measurement of the angle changes if the angle is getting close to the boundary of Klein disk.

DGE features and DGE moves. Investigating students' uses of DGEs requires observing all the instant behavioral or physical changes in geometric figures on the screen and to identify their purposes of such instant changes in the global context of the given geometric tasks. In this study, DGE features are technological features provided in the DGE software. Use of a DGE feature includes only the observable physical changes on the screen, not the students' purpose or meaning of the action. A DGE move is a sequence of DGE features used for a specific purpose. For example, a student might move the cursor to a particular place on the DGE screen and stop there for a while or slightly shake or make circles around for the purpose of pointing out a particular figure. A student might drag a figure to place it at a certain position on the screen.

Dragging moves. Among other DGE moves, I identified multiple dragging moves with different purposes informed by an analytic framework of variation in drag mode of dynamic geometry with different modalities (Arzarello et al., 2002; Leung, 2008). Those different purposes were represented by dragging moves including wandering dragging, guided dragging, and dragging by accidents. In this study, wandering dragging means to drag a diagram around all the varying components of the construction without strategic exploration, but survey overall changes of the construction or measurement. Guided dragging happens when geometric diagrams or numerical measurement guide students to determine specific pathways of their dragging in a strategic way. The last dragging move, dragging by accident, was created in the early stage of the analysis of student work to distinguish miscellaneous moves that were not relevant to the local context of students uses of DGEs. Dragging by accident means cases when students select and drag accidentally a diagram right next another component they wanted to drag. In such cases, the student would make another dragging move by dragging the one they wanted to move. Among all the dragging moves identified from student work in this study, dragging by accident accounted for 1.6% (10'' out of 10'35'') of the total duration. This in-depth analysis on the students' different dragging moves enabled me to distinguish how differently students used drag features in relation to their engagement in mathematical reasoning.

Mathematical Reasoning and Axiomathical Reasoning

In mathematics education literature, mathematical reasoning has been defined in different ways as it encompasses various mathematical practices (Conner, Singletary, Smith, Wagner, & Francisco, 2014). Some researchers have focused on practices of making inferences (Thompson, 1996) or drawing conclusions (National Council of Teachers of Mathematics, 2009) from given information about mathematical objects.

Mathematical reasoning. In this study, students enacted mathematical tasks as they were prompted to provide their responses to given problems. That means students' task enactments entailed practices of making inferences that were inherently oriented towards particular purposes—producing acceptable responses to the problems. Adapting Thompson's (1996) definition, I define mathematical reasoning as purposeful inference about mathematical objects grounded on evidence or assumed knowledge. To be specific, mathematical reasoning in this study is situated in the context of task enactments, where they are given instructional tasks with specific interventions and interpret what they are expected to do for successfully enacting the tasks for the course. For example, given a mathematical task of explaining why a mathematical statement holds in a geometric system but not in the other, students make inferences about the statement using empirical evidence or deduction from known theorems or axioms.

Axiomatic reasoning. In this study, axiomatic reasoning is a specific kind of mathematical reasoning of which objects are meta-mathematical objects including axioms, systems of axioms, and models of axioms. When students engage in axiomatic reasoning, they make inferences about those meta-mathematical objects and properties thereof. This involves specific reference to axiomatic systems and models while making inferences about geometric objects in the task enactment. For instance, students would examine if an axiom is independent in a given system of axioms or verify a model of an axiomatic systems by checking if the model satisfies all the axioms in the system. In the sense that students are expected to objectify those meta-mathematical concepts while engaging in axiomatic reasoning, it has been considered an advanced level in student development of geometry (van Hiele, 1986). In addition, reasoning on those meta-mathematical objects inherently entail subsidiary mathematical objects so identifying

student engagement in axiomatic reasoning can be done in a broader scope of the analysis in this study.

Given the context of task enactment using DGE, mathematical or meta-mathematical objects in the definitions above are not only represented in the students' verbal or written responses but also represented in the forms of diagrams and its relationships that the students interacted with DGE. In this regard, students' DGE moves interplay with their mathematical and axiomatic reasoning while students make inference about the geometric figures and axioms involved in their DGE moves with same or related purposes.

Task Design and Enactment

In this study, mathematical tasks are intentionally designed to guide students to engage with intended mathematical activities (e.g., creating geometric constructions, producing mathematical proofs), which in turn, are expected to provide opportunities to attend to the task objectives associated with learning goals of the course. Researchers highlighted that designing effective instructional tasks supporting student learning entails understanding this process from task design to student learning in order for task designers and instructors to refine the task design and improve instructional interventions to facilitate the task enactment in classroom (Ainley & Margolinas, 2015; Komatsu & Jones, 2019; Olive et al., 2010). In particular, I focus on what mathematical activities students actually engaged with during the task enactment in comparison with the intended activities, i.e., how the tasks were enacted as intended in the design.

In this perspective, researchers have investigated mathematical tasks with attentions to multiple phases of tasks being designed, set up, and enacted in the instructional practices. Stein and Lane (1996) conceptualized this process of mathematical tasks represented in written materials, which are set up by teachers in classrooms, implemented by students in classroom, and

eventually contribute to the student learning (Figure 3.1). This framework provides a way to understand how task design may or may not support student learning as intended as well as to investigate effects of task design by comparing features of tasks in each phase and identifying factors that influence each transition between the phases.

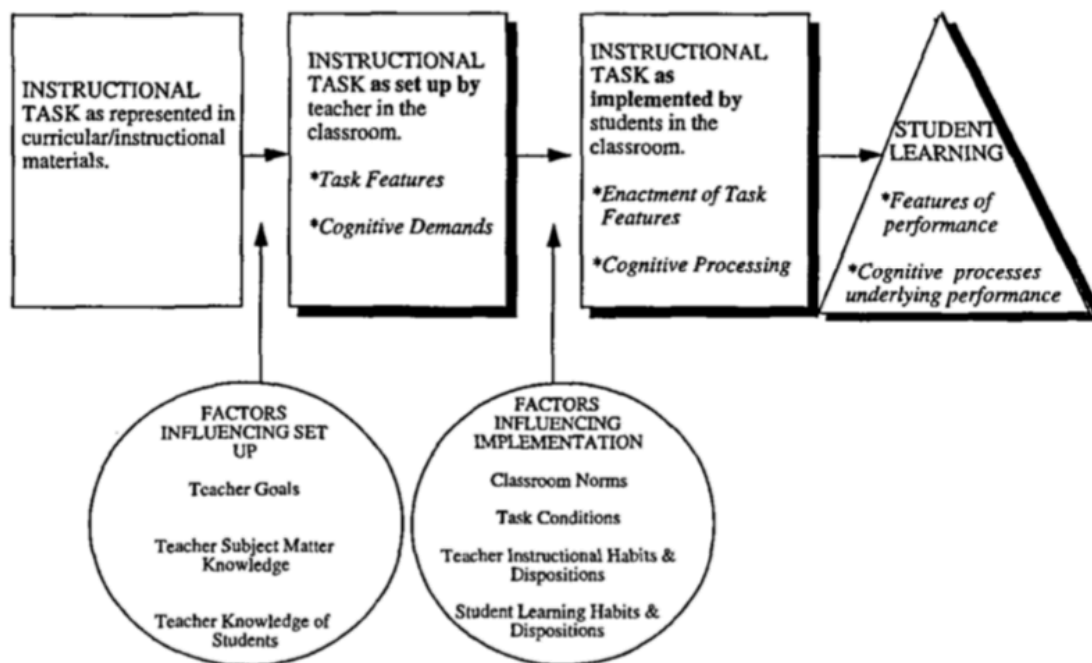


Figure 3.1. Mathematical Task Frame (Stein & Lane, 1996, p.56)

In prior studies using the mathematical task frame (Stein & Lane, 1996), researchers had specific foci on particular aspects of students' mathematical activities—cognitive demand (Stein & Lane, 1996) and proof-related tasks (Bieda, 2010)—in conceptualizing difference between task-as-intended and task-as-enacted. Stein and Lane (1996) found that tasks-as-enacted tended to involve lower cognitive demand than tasks as written or set-up in class by teachers due to various factors including students' intellectual habits, dispositions, or administrative factors like time constraints in class. With a specific focus on proof-related tasks in textbooks, Bieda (2010)

analyzed how tasks were enacted in class involved proof-related mathematical activities including generalization and argumentation as intended in the written curriculum and/or as intended by the teachers who introduced the tasks in the classroom. It is clear in their studies that both foci were not for making direct evidence about student learning from the tasks. Rather their analyses of task design and enactment were focused on examining how the tasks were enacted as intended and on investigating what factors caused such difference, if any.

Task-as-designed and task-as-enacted. In this study, a mathematical task is an activity purposefully designed by an instructor and intended to support students' mathematical learning and development. Through the process of task design, the instructor creates or adopts a set of mathematical activities along with a set of specific task objectives that are expected to be achieved while students enact the task in certain ways. In this regard, the product of this design process is called a *task-as-designed*, which consists of intended mathematical activities, specific task objectives related to learning goals of the course, and instructional interventions for class facilitation. Meanwhile, students do not necessarily enact the tasks in class as intended by the instructor. Task enactment is "the manner in which students actually work on the task" (Stein & Lane, 1996) while interpreting given tasks in their own ways and carrying out the task. This interpretative student work on the task produces a particular student activity that may or may not be as intended in the task-as-designed. The task enacted by the students in response to a given task-as-designed is called a *task-as-enacted*. It is notable that describing task-as-enacted in actual classroom settings can be done by analyzing student work produced in the classroom and by identifying differences between task-as-designed and task-as-enacted. This is because task-as-enacted are results of students' responses to the task-as-designed even though there are lots of

factors that affect those responses other than the task itself (e.g., student knowledge, classroom norm, etc.).

Case of reasoning and case theme of student engagement in reasoning. In my use of this frame of task-as-designed and task-as-enacted, I focused on articulating student engagement in mathematical reasoning and axiomatic reasoning that appeared in the task-as-enacted. The task-as-designed in this study provided cases of mathematical reasoning and axiomatic reasoning that are descriptions of anticipated student reasoning with respect to the specific prompts and contents of the tasks. The analysis of the task-as-enacted in this study aimed at generating case themes (Creswell & Poth, 2017) that represent issues or specific situations in each case that emerged when the students actually engaged in the reasoning for each case. Thus, the analysis is not for comparing the task-as-designed and the task-as-enacted, rather the task-as-designed provided the initial guideline for analyzing student work and to specify different theme from each case of reasoning from the task-as-enacted.

Research Questions

RQ1. How did students use DGE when enacting the geometric proof tasks?

- a. What DGE features did students use when enacting tasks?
- b. What DGE moves did students make when enacting tasks?

The first research question aims at producing comprehensive descriptions of all the identifiable students' moves in the DGE screen. Students' moves mean interpretations of the students' uses of DGE features that served particular functions in the context of the task enactment. The first sub-question entails identifying instances of DGE software functions that result in observable changes on the screen—at the level of physical actions. The second sub-question focuses on the DGE moves, which involves achieving the students' purposes in the

context of the geometric construction on the screen. These two sub-questions lead to the analysis of student moves, identifying all observable changes made on DGE screen and making interpretations of their functions in terms of geometric meanings of those changes. This analysis provided not only a comprehensive list of particular features of DGE used in the task enactment but also specific functions of the features to better understand how the students used those features. For example, a student may drag (a feature) a vertex of a triangle in the spherical model to see if one can move around the vertex freely on the sphere (a function of the drag feature). Also, the student may drag the vertex specifically toward the equator of the sphere to examine how it affects the angle sum of the triangle (another function of the drag feature).

RQ2. What role did students' DGE moves play in their mathematical reasoning?

- a. What mathematical reasoning did students engage in when enacting the tasks?
- b. How were students' DGE moves involved in their mathematical reasoning?

The second research question focuses on understanding student reasoning and its relationship with the students moves in the task enactment. The first subquestion is to identify inferences about mathematical objects that students made in response to the problems given in the tasks. Based on the identified student reasoning, the second subquestion focuses on how students' DGE moves were included in conclusions or warrants thereof in their inferences. The analysis of students' responses in the written/verbal forms will produce description of student reasoning involved in the task enactment, for instance, explaining why a given proof does not work in a particular geometry or defining a mathematical object. Then, it will be followed by the analysis of bidirectional relationship between the student reasoning and the student moves in DGE. This analysis will reveal how the students' moves supported them to involve in particular

reasoning as well as how the students' reasoning affects their ways of making moves in the task enactment.

RQ3. What role did students' DGE moves play in their axiomatic reasoning?

- a. What axiomatic reasoning did students engage in?
- b. How were students' DGE moves involved in their axiomatic reasoning?

The third research question focuses on identifying types of the student engagements with axiomatic reasoning in relation to the student moves in DGE. The subquestions for RQ3 are based on the previous analysis for RQ2, which will produce types of students' mathematical reasoning and their DGE moves involved. The focus of RQ3 will be on identifying axiomatic aspects of student reasoning in terms of how their inferences entail axioms and models of given geometric systems. Those aspects may include distinguishing axioms from theorems, comparing different geometric systems, and generalizing their findings beyond particular models or systems. In addition, I will also identify particular student moves in DGE that relate such student engagement with axiomatic reasoning.

Method

Background

In Spring 2016, I participated in an axiomatic geometry course at a large public university as a course facilitator and teaching assistant. In this course, I designed and implemented an instructional sequence of DGE-supported tasks for in-class activities. I collected student work from group discussions using a screencast software where the groups recorded their interactions with GEX while enacting the tasks. The analysis of the students' uses of GEX during the task enactment showed how different the groups used DGE features and how the differences were associated with the task enactment. Groups who relatively spent more time on dragging

activities in their exploration than other groups were successful in attending to more various components of the given constructions and exhaustive variations thereof. Dragging activities provided students with more active interactions with dynamic constructions in GEX than other types of actions. In addition, those successful groups were strategic in using the drag feature of GEX in collecting empirical evidence and creating conjectures. They frequently shifted between wandering dragging and guided dragging as they collected data from wandering dragging for creating conjectures and verified the conjectures in guided dragging. The results from this analysis implied that task designers and instructors need to attend students' strategic uses of the drag feature in designing and enacting DGE tasks. In addition, the analytic framework to investigate students' uses of GEX informed the research design of my dissertation study, especially for the data collection and analysis (See more in Bae, 2017).

In Spring 2017, when I taught the same course as an instructor-of-record, the course design and the task sequence were revised by reflecting on prior teaching experience in Spring 2016 and the implications from the analysis (Bae, 2017). In the revised curriculum, students were introduced the basic skills of GEX in the first two weeks with an emphasis on the concepts of robust construction and drag test on dimensions of variations (Leung, 2008). Also, I adopted a textbook published by Michael Hvidsten, who also had developed GEX that have been used in this course. By adopting this textbook for the course, I revised the curriculum to integrate the textbook with the existing lecture notes and the task sequence. Although there was no research activity conducted in this class, I found the revised tasks resulted an improvement in communicating with students about the productive uses of drag features and their performance in DGE activities submitted for their course evaluation. The course design was used for the course in Spring 2018 with an emphasis on the study of hyperbolic and elliptical geometry after

completing the Euclidean system in the middle of the semester where the task sequence of this study was enacted.

Context

This study took place in an axiomatic geometry course offered at a large public university in Spring 2018. The goal of the course is to understand foundations of geometry through the axiomatic approach to develop a system of Euclidean geometry by investigating axioms, theorems, and geometric concepts that resulted throughout this process of developing geometry. In this course, Geometry Explorer (GEX), a DGE developed by Michael Hvidsten, served as a key technological material involved in the instructional sequence to allow students to explore the models of different geometries. The course material includes the textbook, also written by Michael Hvidsten, lecture notes, and the DGE-supported tasks for in-class activities including the task sequence of this study. The tentative weekly schedule of the course including the topics and relevant materials is in Appendix A.

The course was taught in a class of 22 students that meet for three 50-minute class meetings on every Monday, Wednesday, and Friday during the 15 weeks of the semester. In Week 1-2, the students learned the basic concepts and terms related to axiomatic systems and models in the context of Euclidean geometry. Also, they were given a set of homework assignments in which they learn the basic features of GEX and first use this tool to solve geometric construction problems. The concepts of robust construction and drag test of dimensions of variations were introduced in the first two weeks. In Week 3-10, the students were guided to develop an axiomatic system of Euclidean geometry and to explore Euclidean and non-Euclidean models. In this period of the course inspired by the guided reinvention and mathematization (Freudenthal, 1991), students began with introducing a set of undefined terms,

primitive postulates and definitions for incidence geometry, and develop the system by accepting additional postulates and definitions and by investigating consequent theorems. As the development proceeds, the language and concepts in the system gets richer in describing Euclidean geometry. In Week 11-15, students completed the development of axiomatic system for Euclidean geometry and shift to investigate hyperbolic, elliptic, and transformational geometry. The independence of parallel postulates was discussed in Week 11 that led to investigate differences between Euclidean and non-Euclidean geometry. The task sequence of this study was enacted in Week 11-12. See Chapter 2 of this dissertation for further details about the task sequence.

Participants

The research participants were junior and senior college students who enrolled the course and volunteered for the study. Among the 22 enrolled students of the course, 20 students agreed to participate in this study. This study was reviewed by the Institutional Review Board, and the participants were informed of how their rights are protected with respect to the participation in this study. The participants' majors varied across mathematics majors, secondary mathematics education majors, and other majors (e.g., physics, chemistry, economics) with mathematics minor. Secondary mathematics education students were required to take either this course or Higher Geometry which is an advanced analytic Euclidean geometry in junior level. For math minor students, this course accounted for one elective senior level (400-level) mathematics course requirement. The prerequisite for the course is MTH 299 Transitions, an introductory proof course, or compatible course or qualification so that the students were expected to have experience with reading and writing formal mathematical proofs. Besides the prerequisite course,

the course survey showed that all the participants took more than one proof-based advanced courses such as abstract algebra or real analysis.

Table 3.1. Groups of participating students during the task enactment

Group	Student	Gender	Nationality	Stance	Major(s)	Minor(s)
1	Bing	Male	Int'l	Junior	Physics	
	Eric	Male	Domestic	Senior	MathEd	
	Jeremy	Male	Domestic	Sophomore	MathEd & PhysicsEd	
2	Philipp	Male	Domestic	Senior	Math	
	Mike	Male	Domestic	Senior	Computational math	
	NP1	N/A	N/A	N/A	N/A	N/A
3	Deshi	Male	Int'l	Senior	Physics	Math
	Lynn	Female	Domestic	Senior	MathEd	
	Yao	Male	Int'l	Senior	Computational math	
	NP2	N/A	N/A	N/A	N/A	N/A
4	Naomi	Female	Domestic	Senior	Economics	Math
	Emily	Female	Domestic	Senior	Biotech & MathEd	
	Ling	Female	Int'l	Junior	Math	
	Odessa	Female	Domestic	Senior	MathEd	Chemistry
5	Kim	Female	Domestic	Junior	MathEd	Spanish & TESOL
	Yin	Female	Int'l	Senior	MathEd	
	Avery	Male	Domestic	Junior	MathEd	
	Yong	Male	Int'l	Senior	Math	
6	Tina	Female	Domestic	Grad	MathEd	
	Ben	Male	Domestic	Senior	Economics	Math
	Deming	Male	Int'l	Senior	Math	
	Jing	Male	Int'l	Senior	Physics	

Groups. Students participated in group work with three or four peers in each group.

Table 3.1 shows the groups that worked together during the task sequence of this study. Those groups had worked together in class for five weeks prior to this task enactment. Note that NP1 in Group 2 and NP2 in Group 3 did not agree to participate in this study so any student work submitted from a pair or a group including those non-participating students were not used for the

research purpose. During the task sequence enactment in class, students were grouped into pairs or triples with their peers from the existing groups. Each group was told to break into pairs, if possible, and to share a laptop when they enacted the DGE tasks with recording their discussions.

Data Collection

In this study, the primary source of data was student work collected from the four days of class in Week 11-12 where students enacted the task sequence of this study. The student work included technological, verbal, visual, and written products corresponding to the GEX tasks in class that informed of students' uses of GEX in the tasks and their reasoning (See Table 3.2).

Table 3.2. Task Sequence and student work collected in Week 11-12

Day	Task	Collected student work
1	Task 1: Construction and verification of Klein model	Screencast presentation
2	Task 1: Construction and verification of Klein model (cont'd)	Screencast presentation, GEX file, summary note
3	Task 2: Analysis of Legendre's attempted proof	Screencast presentation, GEX file, summary note
4	Task 3: Exploration of limiting parallel lines in Klein model	Screencast presentation, summary note

Screencast presentation. In this course, students brought their laptops with GEX and screencast software installed into the classroom. As a part of class participation, they were frequently asked to record their verbal discussions in groups with their computer screens that show their interactions with diagrams in GEX. After the class, the students would submit their work including summary notes, GEX constructions, and the screencast presentation from the discussion. In addition, students were asked to create screencast presentation for their homework assignment, take-home exam, and group exam at the final. In the presentations, student(s) would

show how they constructed a particular diagram or how they manipulated existing construction to answer non-trivial questions in those assignments.

Screencast presentation provided empirical data to understand relationships between student reasoning and their uses of GEX while enacting the task sequence of this study. This audiovisual material contains video recording with their uses of GEX and audio recordings of students' verbal explanation or group discussion. Those two recordings captured simultaneously during the task enactments inform the students' verbalized reasoning in the discussion that are affected by their observations and manipulations of diagrams in GEX. The screencast presentations collected in the class of this study provided with empirical evidences of how students enacted the task sequence in terms of their uses of GEX that influenced their reasoning.

Student work collected before and after those two focal weeks were also used in the data analysis to support interpretations of the primary data. For example, a particular pattern of a student's use of DGEs or any significant changes identified in the primary data can be evident by referring the students' prior work earlier in the course. In this regard, the student work that are collected before and after the focal weeks were used for the research purpose but limited to support the analysis of the primary data.

Data analysis

The analysis of this study was to address the three research questions investigating students uses of GEX and their engagements in mathematical reasoning and axiomatic reasoning during the task enactment. The stages of the analysis consist of preparation of the data, preliminary exploratory analysis, analysis of DGE features and DGE moves, and analysis of mathematical reasoning and axiomatic reasoning.

Preparation of the data. In the initial preparation of the data for analysis, I created a table to organize the student work collected from the task sequence of this study as well as another student work submitted for their course evaluation. The table includes the weekly schedule of the course with main topics discussed in class and student work collected each week for their course evaluations including in-class activities, homework, and exams with specific types of the student products such as written responses, screencast presentations, GEX files. This table lays out the participants' course work submitted before and after the enactment of the task sequence, which informs anticipated student knowledge and reference to participants' other work that are related to the student work from the task sequence of this study. After developing the table, I transcribed the students' verbal explanation and discussions recorded in the screencast presentations.

Preliminary analysis. At this early stage of the analysis after the preparation of the data, I watched screencast presentation with the transcripts while creating summaries of what each group presented in the recording in response to the given tasks. The summaries include descriptions of what they did on GEX screens and said to answer the questions given in the tasks. This brief summary provides an overview of how students interpreted the tasks and produced their work resulted from their interpretation of the tasks. In addition, the preliminary analysis includes examinations of student-generated GEX construction saved in gex files and submitted with the screencast presentations.

Analysis of DGE features and DGE moves. The analysis of the students' uses of GEX in the screencast presentations for Task 1-3 of the sequence was led by the analytic framework (See Appendix C) developed in my prior study (Bae, 2017). First, I analyzed the screencast presentations to identify all the actions arising in the GEX screen. I created DGE feature codes

which indicate visually identifiable changes on the screen resulted by students manipulating the software. The initial purpose of DGE feature coding is to visually illustrate the students' use of different features of GEX. To diminish the subjectivity of interpretation, I generated those DGE feature codes using verbs to refer physical motions of cursors and figures manipulated by the students. For instance, DGE feature codes included select, move, drag, label, hide, measure visual objects on the screen such as the cursor, diagrams, label, measurements. When a DGE feature was identified from the video recording, I added written description of the feature code application to the segmented video and transcript associated with the use of DGE feature. This descriptive analysis of the video screen of GEX produced written transcripts integrating both visually observable uses of GEX and verbal expressions. At this stage of the analysis, I created a table consisting of columns for timestamp, snapshot, verbal transcripts and DGE feature codes where the associated components are place in the same row (See Appendix D).

In the next step of the analysis, I identified DGE moves consisting of one or more DGE features being used across the video segments where I made an inference of students' purposes of using those features in the global context of the task. Whereas the analysis of DGE features provide objective description of all the instances where students acted on the software, this analysis of DGE moves provides an inferential description of the students' purposeful use of technology within the global purpose of enacting the tasks. I intended to create DGE move codes using words that are self-explanatory while those codes are less explicit than DGE features in the sense that one may not know what the purpose of certain changes on the screen made by students. So, identifying a DGE move is informed by the prior analysis of DGE features, verbal transcripts, and the summary of the screencast presentation produced in the preliminary analysis. For instance, students would use a DGE feature, "select a diagram" or "drag a diagram" to point

at a particular diagram on the screen that represents a figure the student wants to explain. In this case, a DGE move, “point at a diagram” would be identified from this instance with the context in which the student wanted to refer to the diagram in her explanation.

To analyze DGE moves involving drag feature, I used the framework of dragging activities with different purposes adopted from the existing studies (Arzarello et al., 2002; Leung, 2008) and developed in my prior study. The dragging strategies in this framework include wandering dragging, guided dragging, dragging by accident. Note that these codes are not to exclusively explain any cases of drag features used on the screen. Wandering dragging means to drag around all the varying components of the constructions without strategic exploration, but survey overall changes of the construction or measurement. Guided dragging means when diagrams or measurements guide students to determine specific pathways to which they drag a diagram or restrict their dragging. Dragging by accident means when students attempted to drag a non-draggable diagram or drag a diagram in a way to which it cannot be dragged. This in-depth analysis on the students’ different dragging strategies and shifts between them in a segment allowed me to interpret how drag feature affected students’ reasoning in the enactment of the tasks.

Analysis of mathematics reasoning and axiomatic reasoning. Whereas the analysis of DGE features and moves were aimed to capture and codify all instances involving student (purposeful) uses of GEX, the analysis of reasoning is far from producing a thorough list of all the instances where students would reason about mathematical objects or axiomatic systems and determine if and how often DGE affected their reasoning. Rather the goal is to characterize case themes from the actual student work in class that are meaningful to me for understanding affordances and limitations of using DGEs when students enacting the task sequence. Given a

lack of empirical studies on student reasoning in axiomatic geometry using DGEs, it is to understand student reasoning in a particular context of this study from the empirical evidence collected from the actual classroom.

Given the definition of mathematical reasoning in this study, the analysis started from identifying focal mathematical objects to which the students attended in the task enactment. The initial list of mathematical objects included, but not limited to, the focal concepts addressed in the intended mathematical activities instructed in the task ranging from particular geometric figures (e.g., lines, angles, circles) to more broad concepts including axioms, geometric models, and axiomatic systems. In particular, I intended to look for mathematical objects that were involved in the students' purposeful inferences in responses to the given tasks. Then, I created case themes (Creswell & Poth, 2017) that represent issues or specific situations in each case where the cases in this analysis mean the mathematical (axiomatic) reasoning on particular mathematical (meta-mathematical) objects addressed in the task-as-designed. The case themes are to describe and compare how students engaged in mathematical reasoning on those objects that were evidenced in their verbal explanation and that were associated with DGE moves from the codified segments of audiovisual data from the prior stages of the analysis. It is followed by the analysis of axiomatic reasoning in which I address specific focus on students' purposeful inference about meta-mathematical objects that are emphasized in axiomatic perspective of mathematics such as axioms, models, and systems of axioms.

Findings: Analysis of Enacted Task Sequence

In this section, I report the enacted task sequence of this study that consists of various student work collected from the class including screencast recordings, GEX construction files, worksheets and summary notes. The task sequence was enacted by students in class during four

50-min class meetings in Week 11 – 12 that were a part of the regular class meetings for the 15-week course in Spring 2018. The class spent one or two days for each task that were mostly devoted to the student activities except short lectures to review prior classes, introduce and summarize the task of the day.

In the following reports for each task, I provide the task design, the description of the class of the day when the task was enacted, and the analysis of their uses of DGEs and engagements in mathematical reasoning. First, I provide the detailed information about the task in terms of the task-as-designed that is focused on the intended mathematical activities and the ways in which students are guided to enact the task in class. This information also includes anticipated student knowledge and skills from the prior class. Second, I describe how the class was implemented and how students participated in the class. In this description, I took an instructor perspective on the overall class to provide readers with information to know what actually happened in the classroom. Lastly, I present the enacted task from student groups in class with specific attentions to students' uses of DGE and their engagement in mathematical reasoning during the task enactment. After reporting the analyses of student uses of DGE and their engagement' in mathematical reasoning, I present characteristics of students' engagement in axiomatic reasoning that identified throughout the task sequence. This provides characteristics of students' enacted tasks with respect to the relationship between their uses of DGE and engagement in axiomatic reasoning.

Task 1: Construction and Verification of Klein Model

Task design. The objective of this task is to establish a mathematical foundation of Klein model for hyperbolic geometry, especially focused on the angle measurement function. The first part of this task is to measure a given angle in Klein model using Euclidean constructions and

measurements with respect to how the angle measurement function is interpreted in this model. In this part, students are given a GEX construction for Klein model in Euclidean plane. This construction contains a unit disk in Euclidean plane where two given lines \overleftrightarrow{AB} and \overleftrightarrow{BC} meet at a point B inside the disk. Those lines with labeled points define an angle $\angle ABC$ inside the disk that is to be measured in this task by using Euclidean constructions of orthogonal circles and tangent lines at their intersection. The Euclidean measurement of the angle between the tangent lines determines the Klein angle measurement of the given angle $\angle ABC$ (See Figure 3.2).

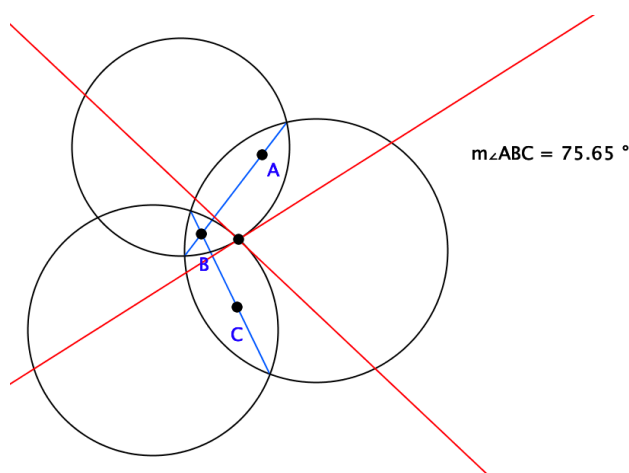


Figure 3.2. Measurement of the angle $\angle ABC$ in Klein model using Euclidean constructions of the corresponding orthogonal circles and tangent lines

In the second part of the task, students verify the soundness of this particular interpretation of the function in Klein model by examining the constructed function in GEX satisfies all the five postulates containing angle measurements (See Figure 3.3). To verify Postulate 11, for instance, students may explore the range of the measurement function while they drag the angle $\angle ABC$. Student are asked to submit a GEX file of their construction and a screencast recording to present their verification of those postulates.

Postulate 10: There exists a function m from \mathcal{A} to \mathbb{R} where \mathcal{A} is the set of all angles and \mathbb{R} is the set of real numbers.

Postulate 11: For every angle in \mathcal{A} , the measurement of the angle is between 0 and 180.

Postulate 12: (Angle-Construction Postulate) Let \overrightarrow{AB} be a ray on the edge of a half-plane H . For every number r between 0 and 180, there is exactly one ray \overrightarrow{AP} , with a point P in H , such that $m(\angle PAB)$ is r .

Postulate 13: (Angle-Addition Postulate) If a point D is in the interior of an angle $\angle BAC$, then $m(\angle BAC)$ is the sum of $m(\angle BAD)$ and $m(\angle DAC)$.

Postulate 14: (Supplement Postulate) If two angles form a linear pair, then they are supplementary.

Figure 3.3. Angle postulates that include the definition or the properties of the angle measurement function in the neutral geometry system (Postulate 10-14)

While enacting the task, students are expected to reason about each of the angle postulates that are interpreted in the context of Klein model and to make sense of draggable diagrams of Euclidean constructions in GEX in terms of representations of arbitrary figures in hyperbolic geometry. To support student engagement in the reasoning, GEX allows students to construct a draggable diagram of an arbitrary angle so that they can examine how the measurement and the corresponding constructions of orthogonal circles and tangent lines co-vary as they drag the angle. In this regard, students need knowledge the concepts of axiomatic systems and models, especially, understanding of what it means to be a model of a system. In particular, students need to understand how undefined terms of the system are interpreted in Klein model. Basic skills of using GEX to create and explore robust constructions are required to construct orthogonal circles and tangent lines in a given Euclidean plane and use it to verify the postulates.

Description of the class. The class began with the review of the previous class about the Saccheri-Legendre theorem in the neutral geometry that says the sum of the measure of the three angles in any triangle is less than or equal to 180 degrees. This was the last theorem that the class

discussed and proved within the neutral geometry system without accepting any particular parallel postulates. Students also discussed consequences including properties of parallel lines that are consequences of the system for the last two weeks. The topic of the day started with introducing Euclidean and hyperbolic parallel postulates and showing the two possible geometries featured by those postulates of which the angle sum of an arbitrary triangle is always exactly equal to 180 degrees in one and strictly less than 180 degrees in the other. Regarding those parallel postulates, two overarching goals of the class and the task sequence for the next two weeks were presented in class: (a) independence of the parallel postulates, and (b) consequences of the parallel postulates. Before starting the task, students were introduced Klein model of hyperbolic geometry and its construction in Euclidean plane including the way of interpreting points, lines, planes, distance, and angles in this model. Students started the task following the instructions presented on the projection screen in the classroom (See Figure 3.4).

- Open the given GEX template (Klein.gex) including the unit disk and two lines \overleftrightarrow{AB} and \overleftrightarrow{BC} .
- Create orthogonal circles for two lines \overleftrightarrow{AB} and \overleftrightarrow{BC} and the angle consisting of the two tangent lines of the chords, which define the Klein angle measurement of $\angle ABC$.
- Using this GEX construction of Klein model, justify that this model satisfies Postulate 10-13 in our system.
- Create and submit screencast video of your group discussion including your justification.

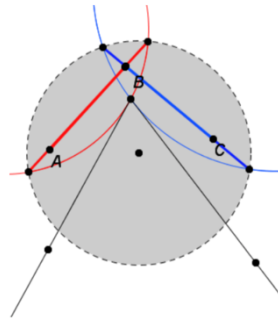


Figure 3.4. Instructions of the Task 1

First, students downloaded the GEX construction file that includes a unit disk in Euclidean plane with two lines \overleftrightarrow{AB} and \overleftrightarrow{BC} intersecting at a point B inside the disk. Students first attempted to create orthogonal circles of the given lines \overleftrightarrow{AB} and \overleftrightarrow{BC} , then tangent lines of the

circles at their intersection inside the disk. Since most groups had difficulties in the basic constructions of the orthogonal circles, I went over the procedure of those constructions in the whole-class for the first five minutes during the task enactment. All groups created the construction of the Klein angle measurement and were encouraged to verify Postulate 10 and 13 first, then the remaining postulates as time permits. In the last five minutes of the class, students were told to wrap up their activities and create a screencast recording. Except one group who completed all the postulates, most groups were still revising their constructions or verifying either Postulate 10 or 13. Groups submitted their screencast recording and GEX files to the class webpage as much as they completed in class. In the following day (Day 2), groups continued to work on the task and were asked to verify any of the remaining postulates. At the end, students submitted their screencast recordings, GEX construction file, and a written answer to the following reflective question asking what they think of this GEX construction activity helpful in understanding the Klein model and our axiomatic systems.

Students' uses of DGEs. The participating groups used a variety of DGE features in their enactment of Task 1 for different purposes as described in DGE moves associated with one or more of those features. As described in Method, DGE features mean students' conscious uses of particular features in GEX that resulted changes on the screen in the recording. Based on those DGE features with transcribed students' verbal explanation, DGE moves were identified and associated with one or more DGE features to interpret students' purposes of using the DGE features. Table 3.3 shows 18 DGE features and 16 DGE moves are identified from the screencast presentations created by the participating student groups (see the Appendix E for all the DGE features and DGE moves identified from the entire student data in this study).

Table 3.3. DGE features and DGE moves identified in the enactment of Task 1

DGE features	Number of occurrences	DGE moves	Number of occurrences
Move the cursor over diagrams	132	Point at a diagram	106
Drag a diagram	84	Point at a measurement	34
Select a diagram	35	Wandering dragging	33
Select a measurement	12	Guided dragging	25
Drag a measurement	7	Label a diagram	8
Create a diagram	5	Fit diagrams to the screen	5
De-select diagrams	4	Create a diagram	4
Label a diagram	3	Point at a computation	4
Measure a diagram	3	Dragging by accident	3
Hide a diagram	2	Measure a diagram	3
Rescale the screen	2	Place a diagram	3
Scroll the screen	2	Place a measurement	1
Un-hide a diagram	2	Undo last steps	1
Drag a label	1		
Undo	1		
Total	295		231

In the screencast presentations created in the enacted Task 1, there were 295 DGE features and 231 DGE moves identified from 25 minutes 18 seconds of the recordings in total. There were 29 DGE features identified in the presentation that didn't have enough evidence to infer a purpose, so they were not associated with DGE moves. Also, some DGE moves were associated with more than one consecutive instances of DGE features. So, the numbers of DGE moves are less than or equal to the number of DGE features identified in each presentation because of those DGE features associated with no DGE moves and multiple DGE features associated with a single DGE move. For instance, groups used the drag feature in GEX for different purposes that include place a diagram in a particular spot on the screen, pointing at a diagram, surveying overall changes in diagrams and measurements, showing particular invariants in varying diagrams, searching for a particular diagram or measurement. On the other hand, students used different DGE features for the same purpose, for instance, they pointed at a

diagram in their presentations by moving the cursor on the diagram, selecting or dragging the diagram.

The frequent DGE features include “move the cursor over diagrams”, “drag a diagram”, “select a diagram”, “select a measurement”. The frequent DGE moves include “point at a diagram”, “point at a measurement”, “wandering dragging”, “guided dragging”. In most cases when students moved the cursor over diagrams or select diagrams and measurements, those DGE features were associated with the frequent DGE moves for pointing at a particular component of the construction (e.g., diagrams, measurements). Sometimes, students dragged diagrams to point at those diagrams, but most cases of the feature were associated with dragging moves (e.g., wandering dragging, guided dragging).

Given the number of pointing moves (point at a diagram, point at a measurement), this analysis indicates that the majority of students’ uses of GEX in Task 1 were to refer to particular diagrams or measurements thereof for their verbal explanation to verify the postulates in Klein model. Students pointed at a diagram or a measurement of a diagram by moving the cursor over them or by dragging and selecting (highlighting) them.

Snapshot (00:24)	Group	Avery and Kim
<p>$m_{\angle}(T,P,U) = 81.69$ angle: b $m_{\angle}(T,P,V) = 42.05$ $m_{\angle}(V,P,U) = 39.64$</p> <p>$m_{\angle}(T,P,V) + m_{\angle}(V,P,U) = 81.69$</p>	DGE move	Point at a measurement
	DGE feature(s)	Move the cursor over diagrams
	Transcript (00:15-00:30)	
	<p>Avery: (00:15) You can see that right here that goes from this angle TPU to</p> <p>Avery: (00:24) a real number. So, postulate 10 holds up that function m does exist.</p>	

Figure 3.5. Example of pointing at a measurement by moving the cursor over diagrams

For instance, Avery and Kim pointed at the real-value 81.69 of $m\angle TPU$ by moving the cursor over that on the screen when they argued for Postulate 10 in their presentation (See Figure 3.5). It is due to the nature of the task where the students were expected to verify their Klein models satisfy the angle postulates given that they were told Klein model satisfies the postulates of hyperbolic geometry. Also, the presentation only captured the verification part of the task enactment after given some time to construct their models before start recording. This confirmatory nature of the task, at least for the verification part, affected the students' task enactments in which students didn't have critical view of the model enough for a close examination.

Despite the overall description of students' uses of GEX responding to Task 1 with a confirmatory nature, student groups showed differences in their interpretation of the task and the ways to respond by presenting their work using GEX (See Table 3.4).

Table 3.4. Summary of DGE features and DGE moves identified in Task 1 (Day 1 & 2)

Day	Pairs or Triples	Postulates	Duration (mm:ss)	Number of DGE features	Number of DGE moves
1	Avery, Kim	10, 13	0:55	9	9
	Bing, Eric, Jeremy	10, 11, 12, 13	3:56	62	55
	Ben, Deming, Tina	10, 13	0:38	4	4
	Emily, Ling	10, 13	1:14	7	7
	Naomi, Odessa	10, 13	1:08	16	14
	Yin, Yong	10, 11, 13	2:40	42	17
2	Avery, Kim	11, 12, 14	1:15	14	9
	Bing, Eric, Jeremy	14	1:00	14	9
	Ben, Tina	14	0:39	5	3
	Deming, Jing	14	0:38	14	13
	Deshi, Yao	14	1:15	13	9
	Emily, Ling	11, 12, 14	1:58	8	7
	Naomi, Odessa	11, 12, 14	1:53	15	14
	Yin, Yong	11, 12, 14	5:30	72	61
Total			25:18	295	231

In particular, some groups presented Table 3.4 shows a summary of the data and the analysis in terms of the length of screencast presentations, postulates addressed in each presentation on Day 1 and 2, and the numbers of DGE features and DGE moves identified in each group. In the six and eight presentations submitted on each Day 1 and Day 2, the participating groups showed differences in the duration of their presentations. In those recordings with different durations, the numbers of DGE features and DGE moves were approximately proportional to the length of duration in each group's recording. This indicates that all groups used DGE features and made DGE moves at similar paces in the sense that there was no group who used GEX significantly more often in a shorter presentation or rarely used in a longer presentation.

The durations of the screencast presentations were different across the groups due to their progress on the task enactment and the nature of their interpretations of the task and their responses. Groups who verified more postulates on that day often created longer presentations such as Bing, Eric, and Jeremy who produced the longest presentation (3'56") on Day 1 to verify Postulate 10 to 13. In their presentation (5'30") on Day 2, Yin and Yong revisited some of the postulates verified on Day 1 again so more postulates were presented in this recording. The durations of recordings also depended on what DGE moves students made in response to their interpretations of the tasks. For instance, Bing, Eric, and Jeremy was the only group who used drag feature for guided dragging on Day 1 to show specifically when the range of the angle measurement is between 0 and 180. In addition, Yin and Yong was the only group who spent significantly more time in their presentations on constructing, labeling, measuring, and placing diagrams whereas other groups with shorter length of recordings often presented with completed constructions and used GEX for reference of their verbal responses to the task. Comparing to

other groups who seemed to present their work without delay or mistake in manipulating GEX, this difference indicates that students may interpret differently to what extent they have to show a perfect work in their presentation.

Students' engagement in mathematical reasoning on Klein angle measurement function. In Task 1, participating groups engaged in mathematical reasoning about geometric objects as they interact with GEX constructions for verifying the angle postulates. One of the primary mathematical objects they reasoned about in this task is the Klein angle measurement function that is interpreted in a particular way of using Euclidean construction. This function is based on the 1-1 correspondence between a given angle in Klein model, Euclidean constructions of corresponding orthogonal circles and tangent lines, and Euclidean angle measurement of the angle between the tangent lines. It is crucial in verifying Postulate 10 thru 14 that students interact with the GEX construction of those diagram dependent each other while reasoning about the existence (or well-definedness), range, and injectivity of the Klein angle measurement function.

In this section, I present how the participating groups differently engaged in mathematical reasoning about the Klein angle measurement function with their emphasis on the following properties, (a) existence, (b) range, (c) injectivity and surjectivity. The case of student reasoning in Task 1 is their inference about one or more of the aforementioned properties of Klein angle measurement function. Students engagement with this case of student reasoning is elaborated with case themes that emerged in the analysis (See Table 3.5). I discuss roles of DGEs that were evidenced in the previous analysis of DGE features and DGE moves.

Table 3.5. Case themes of students' engagement in mathematical reasoning about Klein angle measurement function in Task 1

Case themes
<p>Theme 1.1. Students used the drag feature used when successfully matching input angles and output measurements of the Klein angle measurement function.</p> <ul style="list-style-type: none"> • Some students demonstrated dynamic changes in the output measurement by dragging the input angle. They were successful in connecting the input and output of the function as described in Postulate 10 and used the drag feature to show the dependent relationships between axillary diagrams including orthogonal circles and tangent lines. • Other students presented a static diagram with a measurement without using drag feature in the recordings. They mismatched the input angles and output measurements of the Klein angle measurement function or didn't have the diagrams dependent to each other. <p>Theme 1.2. Students adapted their responses to dynamic changes of the diagram.</p> <ul style="list-style-type: none"> • Some students presented overall changes in the measurement of an angle whereas other students additionally examined when the measurements are close to the end points of the range including 0 and 180. • Some students showed adaptive responses to dynamic changes of diagrams in the recording. They modified DGE moves and verbal explanation according to their interpretations of those changes in diagrams (e.g., disregarding, modifying their constructions, modifying their moves, attributing to technological limitation of the software, explaining technical feature of the software. <p>Theme 1.3. Students used different features to demonstrate the 1-1 correspondence between input angles and output measurements.</p> <ul style="list-style-type: none"> • Some students referred to the unique diagram of angle measurement functions as they verify the injectivity and surjectivity of the function. • Some students demonstrated continuous covariation between the input angles and output measurements to verify the surjectivity

Theme 1.1. Students used the drag feature used when successfully matching input angles and output measurements of the Klein angle measurement function. In their verification of Postulate 10, most students presented their constructions including the given angle $\angle ABC$ and corresponding orthogonal circles and tangent lines with the angle between tangent lines $\angle A'B'C'$. Note that students used different labels for the points in their constructions. Postulate 10 states the existence of the Klein angle measurement function from a specific domain, the set of all angles, to the codomain, the set of real numbers. In most cases of presentations, verifying the existence of the function was done by pointing at or by dragging diagrams of angles and

measurements angles included in their constructions of Klein model although those input and output are not necessarily matched as described in the postulate. Except the two groups (Avery, Kim; Naomi, Odessa), four of the participating groups on Day 1 showed that their constructions kept working successfully to map angles to real numbers while changing the angles (wandering dragging). In the rest of the verification, groups specified the domain and codomain of the function by referring to different angles and measurements (See Table 3.6).

Table 3.6. Different diagrams and measurements used to specify the domain and codomain of the Klein angle measurement function in Postulate 10

Diagrams and measurements used for verifying Postulate 10	Groups
Klein angle and Euclidean measurement of tangent angles (e.g., $\angle ABC$ and $m\angle UQV$ in Figure 3.7)	Bing, Eric, Jeremy Deming, Ben, Tina Emily, Ling Yin, Yong
Tangent angles and Euclidean measurement of tangent angles (e.g., $\angle TPU$ and $m\angle TPU$ in Figure 3.6)	Avery, Kim Naomi, Odessa

Interestingly, two groups who didn't use the drag feature described the Klein angle measurement function mapping tangent angles to its Euclidean measurements.

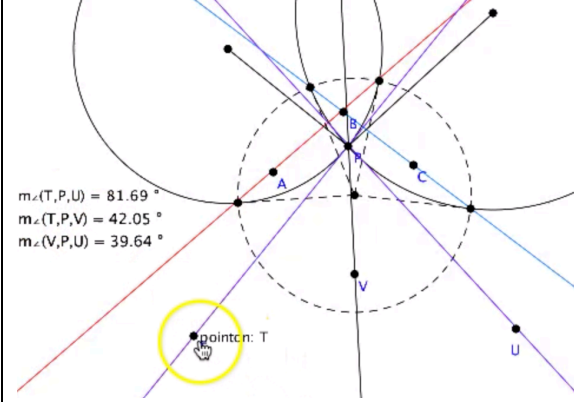
Snapshot (00:19)	Group	Avery, Kim
	DGE move	Point at a diagram
	DGE feature(s)	Move the cursor over diagrams
	Transcript (00:15-00:30)	
	<p>Avery: (00:15) You can see that right here that goes from this angle</p> <p>Avery: (00:19) T</p> <p>Avery: (00:20) PU to a real number. So, postulate 10 holds up that function m does exist.</p>	

Figure 3.6. Avery and Kim's verification of Postulate 10

For instance, Avery and Kim referred to $\angle TPU$ in their construction, the angles between tangent lines corresponding to the given lines \overline{AB} and \overline{BC} and its Euclidean measurement (Figure 3.6). In their presentation, Avery followed each vertex of the $\angle TPU$ by pointing at the diagram using the cursor while explaining that the construction shows that m “goes from” an angle $\angle TPU$ “to a real number”, 81.69 degrees. The function described in this verification maps an angle to its Euclidean measurement $m\angle TPU$ that is essentially identical to Euclidean measurement except the particular angle $\angle TPU$ used instead of an arbitrary angle on Euclidean plane. Students did not engage with the dependent relationships between the given angle $\angle ABC$, the orthogonal circles, and tangent lines that they previously created and already had on their screen. If he attempted to drag points T or U to show how $m\angle TPU$ changes corresponding to the dragging during the presentation or even before, he could have noticed that those points are fixed on the tangent lines and that he could not change the angle and its measurement at all. This unexpected response from GEX construction encouraged the students to examine what caused that response and provide a chance to revisit the postulate, construction, or what they were explaining in the presentation.

On the other hand, the other four groups explained the Klein angle measurement function that maps Klein angle, $\angle ABC$, to Euclidean measurement of the corresponding tangent angles, (e.g., $\angle UQV$ in Yin and Yong’s construction in Figure 3.7 below). The four groups showed how the diagrams and the measurement dynamically responded to their dragging of the point A , B or C when explaining the domain and codomain of the function. For instance, Yin and Yong were explicit in addressing the existence and uniqueness of the orthogonal circles, tangent lines and the angle between them, $\angle UQV$. They explained how those sequence of figures ensures the well-

definedness of the measurement function and assign a measurement to the Klein angle $\angle ABC$ while dragging the point C on the screen as shown in Figure 3.7 above.

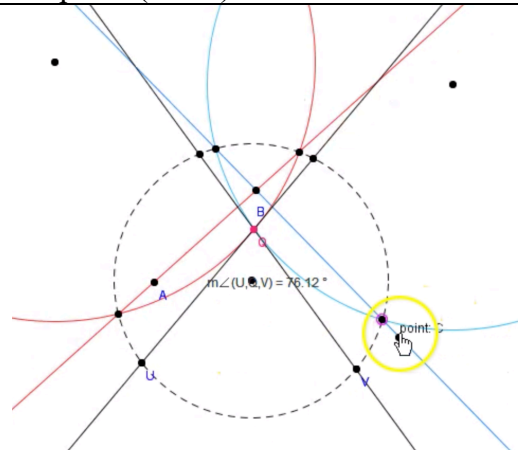
Snapshot (00:35)	Group	Yin, Yong
	DGE move	Wandering dragging
	DGE feature(s)	Drag a diagram
	Transcript (00:15-00:46)	
	<p>Yin: (00:07) for angle ABC, we can only construct the two orthogonal circles and they only intersect at one point and so based on that point we can only construct the two lines. So that's basically the 1-1 function to the angle for the, for the, to assign measurement to ABC. So, we can</p> <p>Yin: (00:35) drag to point C... And there are no other angles, measurements like for another angle.</p>	

Figure 3.7. Yin and Yong's verification of Postulate 10

In sum, students reasoned about the existence of the Klein angle measurement function by showing their construction including input angle and output measurements of the function. The complicated structure of the function consisting of multiple diagrams in this task was not always fully addressed in all the groups' presentations. In particular, the elements of the domain and codomain of the function were mismatched in presentations where the groups did not use drag feature to show the mechanism of the construction assign a measurement to the varying angle even though they constructed those axillary diagrams. Though the data does not indicate student knowledge of the function or a lack thereof, the analysis revealed that their use of drag feature provided with a way to reason about the covarying relationships between dependent diagrams in their constructions of Klein angle measurement function.

Theme 1.2. Students adapted their responses to dynamic changes of the diagram . In their verification of Postulate 11, students examined the range of the Klein angle measurement function. In their presentations, they demonstrated the angle measurements that fall into the interval between 0° and 180° by showing examples of angles using DGE features with pointing or dragging diagrams. Students pointed at a diagram of the given angle and its measurement, dragged the diagram to show overall changes in the measurement that always less than 0° and greater ether 180° .

Among the four groups who submitted their verification of Postulate 11, two groups showed guided dragging to specify particular examples of angles with certain measurement values. For instance, Bing, Eric, and Jeremy used drag feature for guided dragging to show specifically when the angle measurement gets to 0° and 180° but never reach those values. (Figure 3.8).

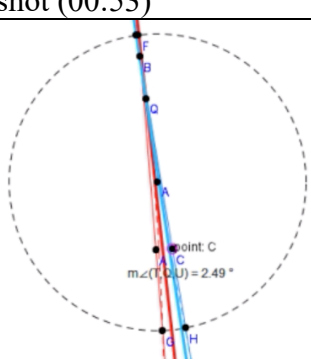
Snapshot (00:53)	Group	Bing, Eric, Jeremy
	DGE move	Guided dragging
	DGE feature(s)	Drag a diagram
	Transcript (00:42-00:57)	
	<p>Eric: (00:42) And then postulate 11 is very similar. We always get an angle between 0 and 180, never 0 or 180. So, as you can see, we can try to get really close to 0 here</p> <p>Eric: (00:53) and it will never reach it</p>	

Figure 3.8. Bing, Eric, and Jeremy's verification of Postulate 11

In their examination on the range of the angle measurement function, Bing, Eric, and Jeremy tried to specifically show that $m\angle TQU$, the Klein measurement of $\angle ABC$, gets close to 0 when \overrightarrow{AB} and \overrightarrow{BC} overlap such that the points A and C are on the same side. What they

emphasized in this verification is that one cannot find an angle in Klein model which measurement is exactly 0° or 180° . It contrasts to other instances of dragging to demonstrate how the measurement of the angle runs thru the interval between 0° and 180° no matter how they dragged it within a limited space. In this case, however, Bing, Eric, and Jeremy tried to show that Postulate 11 holds when they specifically tried to find a minimum and maximum of the angle measurement function, not only when the angle looks normal. In this regard, the DGE feature and DGE move analysis revealed that this group used the drag feature of GEX for guided dragging and that their use of GEX supported a deep engagement with the range of the function and corresponding diagrams (e.g., two lines getting close to each other) where they closely examined a questions they generated in the context of the task (e.g., does this angle measurement never reach 0° ?, when does this angle measurement get close to 0 as much as possible?). The dynamic construction and drag feature of the software allowed students to initiate and continue their investigation on the particular examples of diagrams to address what they think critical in this verification of the range of a function. Similarly, Yin and Yong demonstrated when the diagram of angle $\angle ABC$ gave certain values of their choice (e.g., 0° , 60° , 180°) and relationship between corresponding orthogonal circles and tangent lines at those values (Figure 3.9).

Snapshot (01:20)	Group	Yin, Yong
	DGE move	Guided dragging
	DGE feature(s)	Drag a diagram
	Transcript (01:06-01:29)	
	<p>Yin: (01:06) And then it moves from 0 degree, so it moves from 0 degrees on the left when the blue circle is on the left to then</p> <p>Yin: (01:18) to 60 degrees with this measurement. And actually, it keeps increasing to 180 degrees.</p>	

Figure 3.9. Yin and Yong's verification of Postulate 11

This group placed the point C on the left side of the line \overleftrightarrow{AB} where $m\angle YQZ$, the Klein measurement of $\angle ABC$, is close to 0° and dragged C to the right so that the line \overleftrightarrow{BC} passing the center of the Klein disk when the corresponding orthogonal circle is extremely large. They noted the measurement $m\angle YQZ$ about 60° and commented that the orthogonal circle flipped from the left to the right side. This indicates that this group investigated the mechanism of assigning a measurement value to the angle. It is also notable that this group voluntarily revisited Postulate 11 after they turned in their first verification where they simply showed wandering dragging to demonstrate the overall changes in the measurement.

Observing changes in dynamic constructions while dragging one side of given angles in their presentations, students encountered certain cases of diagrams that showed unexpected or unwanted results. In such cases, students were adaptive to those dynamic changes in GEX diagrams, so they tried to explain the situation, modified their constructions or modified DGE moves according to their interpretations of what they saw on the screen. Sometimes, they disregarded such instances and continued their explanation, attributed to technological limitation of the software, or tried to explain technical feature of the software that caused what was seen on the screen.

In the previous example from Yin and Yong, for instance, the students explained what happened when one of the sides of $\angle ABC$ passes thru the center of Klein disk—its orthogonal circle was flipped from the left to the right on the screen. As it comes close to the center of Klein disk, the orthogonal circle gets larger and flipped when the line \overleftrightarrow{AB} passes the center. This phenomenon caused a technical issue with the tangent line $\overleftrightarrow{QA_1}$ at the intersection of the two orthogonal circles. $\overleftrightarrow{QA_1}$ is defined on the one intersection of the circles inside the Klein disk when the blue orthogonal circle is on the right, but this intersection is switched with the other

intersection outside the Klein disk when the blue orthogonal circle is flipped. So tangent line $\overleftrightarrow{QA_1}$ lies outside the disk when the blue orthogonal circle is on the left (See Figure 3.10).

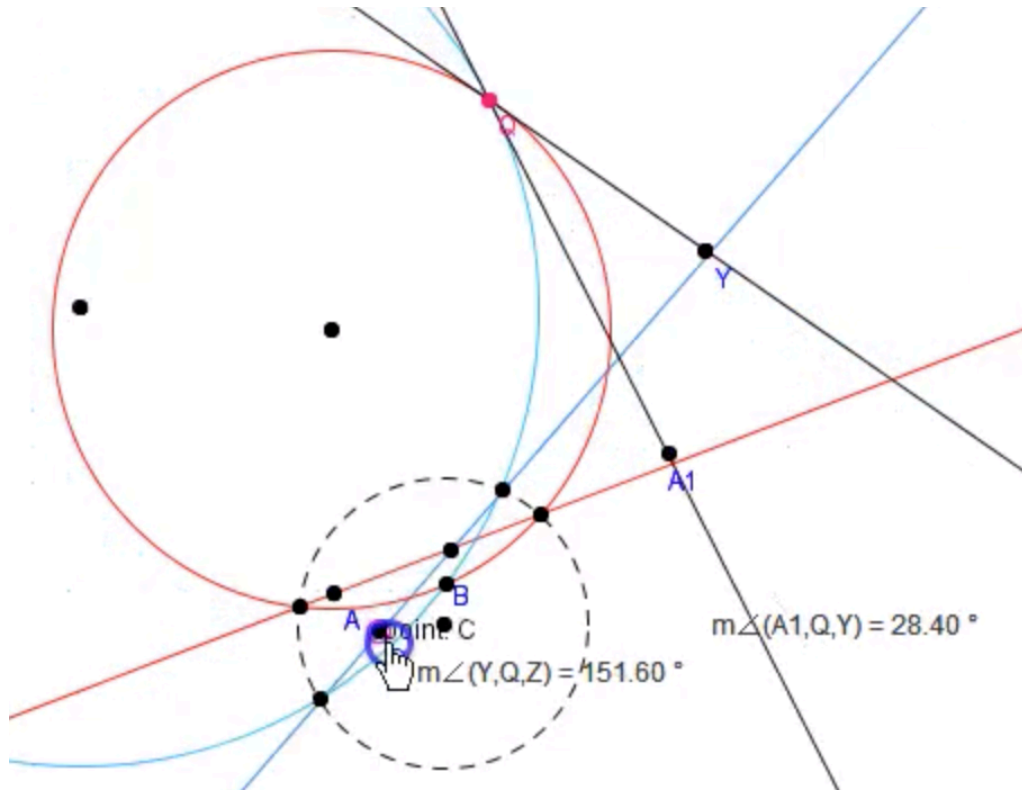


Figure 3.10. Tangent line $\overleftrightarrow{QA_1}$ outside the Klein disk when its corresponding orthogonal circle (blue) is on the left side of the center of Klein disk

To resolve this issue, Yin and Yong used measurements of two supplementary angles on the tangent line $\angle A_1QY$ and $\angle YQZ$ so that they can refer to either of one no matter where the orthogonal circle is placed on the screen. In their presentation captured in Figure 3.9, they referred $m\angle A_1QY$ when the orthogonal circle is on the left and switched their reference to $m\angle YQZ$ once the circle was flipped to the right (See Figure 3.11).

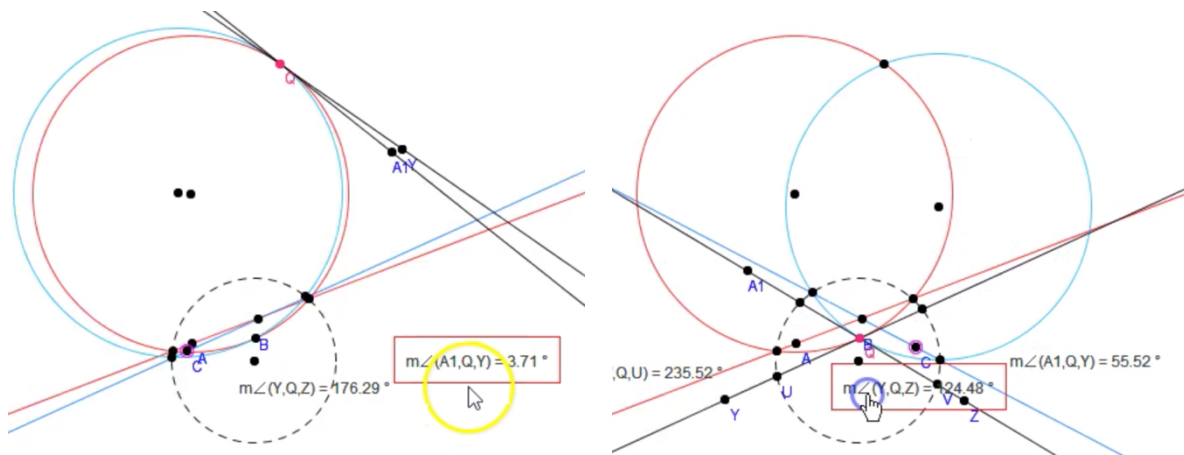


Figure 3.11. Students adjusted uses of two measurements $\angle A_1 Q Y$ and $\angle Y Q Z$ according to the relationship between diagrams

Another example of students' response to the issue of flipping orthogonal circles is to restrict the examples they demonstrated in the presentations. In the following example from Naomi and Odessa's verification of the range, they demonstrated the changing angle and measurement but only on the right side of Klein disk (Figure 3.12).

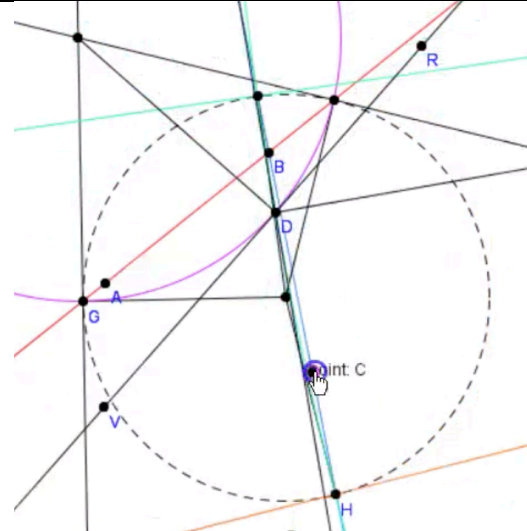
Snapshot (00:13)	Group	Naomi, Odessa
	DGE move	Guided dragging
	DGE feature(s)	Drag a diagram
	Transcript (00:00-00:19)	
	<p>Naomi: (00:00) For postulate 11, we can see that if we move this point C over here, the angle VDW, um, will always be between 0 and 180</p> <p>Naomi: (00:13) no matter how we move it. If we move it too much... it's just because of how the software was made.</p>	

Figure 3.12. Naomi and Odessa's verification of Postulate 11 in Task 1

Although she said “[the angle] will always be between 0 and 180 no matter how we move it”, Naomi’s dragging didn’t cross the center of the Klein disk and moved back to the right side of the disk. Her dragging was guided by the relationship between the center of the Klein disk and \overline{BC} so that the line does not cross the center. Then she said, “if we move it too much... it’s just because of how the software was made”. Her following comment indicated that she knew what would happen when \overline{BC} moved too much across the center of the disk and that she wanted to exclude those unwanted examples in their presentation. To justify that decision which contradicted what she just said, she attributed the situation to the software but didn’t mention what the situation is and what specific feature of the software caused it. However, it didn’t affect their assertion that this demonstration verifies the range of the function and confirmed Postulate 11.

In sum, the analysis of student reasoning on the range of the Klein angle measurement function showed differences in the participating students’ ways of demonstrating the range of the function and of responding to technical issues they encountered. All the groups used the drag feature to demonstrate the overall changes in the angle measurement that falls between 0° and 180° , but two of them were more explicit than others in examining the end points of the range. When the model didn’t work smoothly as they expected, students modified their constructions and adjusted moves to present examples that work well for verifying the range of the function.

Theme 1.3. Different features used to demonstrate the 1-1 correspondence between input angles and output measurements. Postulate 12 states the injectivity and surjectivity of the Klein angle measurement function which can always specify a particular angle with a base point and a ray in Klein model for any given real number between 0 and 180. Students verified this postulate

by demonstrating continuous co-variation of the diagram and its measurement and by specifying the uniqueness of the diagram or the measurement in the construction.

In the following example from Avery and Kim, the continuous motion of the diagram and increasing value of the measurement confirmed that the existence and uniqueness of those angles specific to particular real number between 0 and 180 (Figure 3.13).

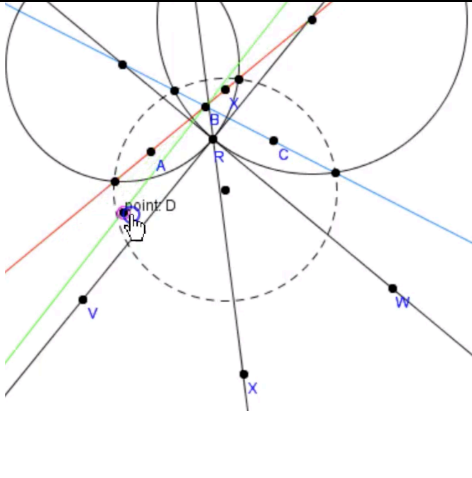
Snapshot (00:46)	Group	Avery, Kim
	DGE move	Guided dragging
	DGE feature(s)	Drag a diagram
Transcript (00:22-00:52)		
<p>Avery: (00:22) Sure. Alright looking at postulate 12. Again, we're saying that this green line is a, another tangent line and you can see that if AB is the edge of the half plane as I'm moving around on this side,</p> <p>Avery: (00:46) it gets every single measure between zero and 180. here gets zero all the way up to 180. So, postulate 12 is justified.</p>		

Figure 3.13. Avery and Kim's verification of Postulate 12

In the presentation, they dragged the point D to show how the diagram of \overline{DB} and its measurement continuously covary while the measurement goes from 0° to 180° . Avery emphasized that this demonstration shows that the line “gets every single measure between 0 and 180” assuming that the continuous motion he showed implied its completeness in covering all possible real numbers from 0 to 180. Another example of using dragging in verifying the injectivity is to argue that there’s no repeating numbers or angles while dragging a ray of the angle. In their presentation, for instance, Naomi said “wherever we move this point C , we can see that the angle VDW there, um, there's no repeating numbers it's all between zero and 180 and the angles just don't repeat itself.” She dragged the point C several times to indicate that the

values in measurement $m\angle VDW$, the Klein measurement of $\angle ABC$ in their construction, showed monotone increase and decrease between 0 and 180 without repetition. Yin and Yong also specified the uniqueness of the angle for a given real number, but she used a particular example to restate the postulate (Figure 3.14).

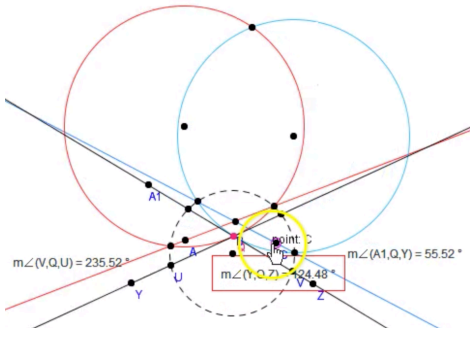
Snapshot (01:44)	Group	Yin, Yong
	DGE move	Wandering dragging
	DGE feature(s)	Drag a diagram
	Transcript (01:32-01:56)	
	<p>Yin: (01:32) And also, we can say about postulate 12. Like for A, for given ray \overrightarrow{BA} and for given angle measurement, let's say, 120, Yin: (01:44) we can only find a ray \overrightarrow{BC} where the measurement is 120 and nothing else like no ray.</p>	

Figure 3.14. Yin and Yong's verification of Postulate 12

Yin and Yong selected a measurement, $m\angle YQZ = 124.48^\circ$ and dragged the point C to highlight that this ray \overrightarrow{BC} and the angle $\angle ABC$ are unique diagrams on the screen. Their argument followed the statement in Postulate 12—For every number r between 0 and 180, there is exactly one ray \overrightarrow{AP} , with a point P in H , such that $m(\angle PAB)$ is r . Then they indicated the only diagram shown on the screen that got assigned the real number they specified by showing how the measurement changes when they dragged the ray around.

In sum, students' verification of injectivity and surjectivity of Klein angle measurement function relied on the unique diagram and its measurement. Students showed continuous covariation and emphasized the fact that they have only one diagram of the measurement that covaried together while dragging the diagram. When students reasoned about the existence and

uniqueness of the angle for an arbitrary real number, GEX provided a way to instantiate a particular number (e.g., 120° in Figure 3.14) and corresponding angle. Also, it allowed them to visualize 1-1 correspondence between covarying diagram and measurement by continuous motion of dragged diagrams.

Task 2: Analysis of Legendre's Attempted Proof of Euclidean Parallel Postulate

Task design. The objective of this task is to create a geometric construction in Klein model of GEX and use this dynamic construction to explain why a proof for EPP fails in this model. In the previous task of constructing and verifying Klein model, it was confirmed that this model satisfies all the postulate in the current system for Euclidean geometry except the EPP. This implies that any attempt to prove EPP in this system must end up with failure such as the one suggested by a past mathematician Adrien-Marie Legendre. Task 2 is to analyze Legendre's attempt to prove EPP by geometric construction in Klein model of GEX. Students are given a written document describing each step of a geometric construction by which Legendre argued for the uniqueness of the parallel line to a given line and a point not on the line (See Figure 3.15).

Given P not on line l . Drop perpendicular \overrightarrow{PQ} from P to l at Q . Let m be the line through P perpendicular to \overrightarrow{PQ} . Then m is parallel to l . Let n be any line through P distinct from m and \overrightarrow{PQ} . We must show that n meets l .

Let \overrightarrow{PR} be a ray of n between \overrightarrow{PQ} and a ray of m with endpoint P . There is a point R' on the opposite side of \overrightarrow{PQ} from R such that $\angle QPR' \cong \angle QPR$. Then Q lies in the interior of $\angle RPR'$. Since line l passes through the point Q , l must intersect one of the sides of this angle. If l meets \overrightarrow{PR} , then certainly l meets n . Suppose l meets side $\overrightarrow{PR'}$ at a point A . Let B be the unique point on \overrightarrow{PR} such that $\overline{PA} \cong \overline{PB}$. Then $\triangle PQA \cong \triangle PQB$. Hence $\angle PQB$ is a right angle, so that B lies on l (and n). Q.E.D.

Figure 3.15. Legendre's attempt to prove EPP in neutral geometry

The essential mathematical activity in this task is to identify independent elements of the construction and demonstrate counter-examples by dragging those elements. This activity involves creating a robust construction of this faulty proof and reasoning on parallel lines in Klein model. Students are expected to analyze each step of the construction and identify the hidden assumption that causes a circular logical in this proof. In this task, students are allowed to use Klein model provide in the software so students can construct basic figures as easy as they do in Euclidean models. In GEX, students can choose one of hyperbolic geometry models including Poincaré model, Klein model, and half-plane model in which students can create and measure figures. In this task, students were asked to work on Klein model that they verified in the previous task of the sequence.

While constructing their figures following the steps described in the material, students may see that the statement in the fourth step does not necessarily hold for an arbitrary ray \overrightarrow{PR} and the angle $\angle RPR'$ in their construction in Klein model. In this step of his proof, Legendre assumed that any line passing through a point in the interior of an angle must intersect one of the rays of the angle. However, this assumption is not always true in a system without EPP or any statement logically equivalent to it. For instance, if students drag the angle $\angle RPR'$ to make it wide enough to get close to the line m , neither of the rays \overrightarrow{PR} nor $\overrightarrow{PR'}$ meet the line l . (See Figure 3.7). This provides a counter-example to the statement in the fourth step using the hidden assumption, which, in turns, refutes his argument. Note that there are at least three lines (in red), $m, n, \overleftrightarrow{PR'}$, in this case where all they are parallel to the line l and pass the point P .

Description of the class. The class started with a short review lecture for the previous two class meetings that summarized student work on Task 1. In this lecture, I recapped the independence of EPP in neutral geometry, which is the implication of student work on verifying

Klein model of hyperbolic geometry. It was followed by the introduction to the Legendre's attempted proof of the uniqueness of parallel lines. I emphasized in class that his proof should be faulty as students already showed that there exists a valid model of hyperbolic geometry that satisfies all the postulates of neutral geometry but not EPP. Note that the existence of parallel lines in neutral geometry was discussed in class a few weeks ago.

At the beginning of the task enactment, I introduced the task of the day and handed out printed copies of Legendre's attempted proof (See Appendix F). In the given task, students were asked to construct this proof in Klein model of GEX and identify a flaw in this written proof that fails in Klein model. To respond this task, they were instructed to submit (a) GEX construction of the proof in Klein model, (b) screencast recording of their construction, and (c) written responses to the following prompt: Write a mathematical statement to describe the hidden assumption and explain why this assumption holds in Euclidean but not in hyperbolic geometry. In particular, the students were instructed to start the screencast recording from the beginning of the activity to capture the process of constructing and analyzing the proof as well as to let them focus on the process rather than producing final presentation as a complete product for the task. Students were also given a printout explaining how to copy an angle in GEX using reflection feature that is a skill needed in the construction.

While students enacted the tasks for the rest of the class, I visited some groups who were confused about what they were supposed to do in the task at first and clarified the instructions in the task. Some groups did not start recording from the beginning and looked at the printouts first. Most groups had no troubles with using DGE features except the reflection feature, so I visited those groups to assist them when creating reflections.

Towards the end of the class, I invited Lynn and asked her to share what she discussed with her groups. She drew their diagram and presented their findings to the class. She reported that there are lines that do not have a common perpendicular but are still parallel to each other in Klein model and that Legendre seems to miss those lines. She also said that the given parallel lines l and m “are not parallel” because the diagram in Klein model does not look like parallel lines in Euclidean geometry but, based on the definition of parallel lines, they are parallel.

Students’ uses of DGEs. The participating groups used diverse DGE features in their enactment of Task 2 for constructing of Legendre’s attempted proof and exploring counter examples. Table 3.7 shows 10 DGE features and 13 DGE moves that were identified from the screencast presentations created by the five participating groups (see the Appendix E for all the DGE features and DGE moves identified from the entire student data in this study).

Table 3.7. DGE features and DGE moves identified in the enactment of Task 2

DGE features	Number of occurrences	DGE moves	Number of occurrences
Move the cursor over diagrams	184	Point at a diagram	162
Select a diagram	54	Create a diagram	33
Drag a diagram	49	Label a diagram	26
Create a diagram	33	Guided dragging	23
Label a diagram	26	Wandering dragging	16
Measure a diagram	15	Measure a diagram	15
Mirror a diagram	7	Reflect a diagram	8
Hide a label	2	Hide a diagram	2
De-select diagrams	1	Point at a feature icon	2
Hide a diagram	1	Dragging by accident	1
		Hide a label	1
		Place a diagram	1
		Point at a measurement	1
Total	372		281

In the screencast presentations created in the enacted Task 2, there were 372 DGE features and 276 DGE moves identified from 40 minutes 20 seconds of the recordings in total. There were 19 DGE features identified in the presentation that didn't have enough evidence to infer a purpose, so they were not associated with DGE moves. The frequent DGE features include "move the cursor over diagrams", "select a diagram", "drag a diagram", "create a diagram", "label a diagram. The frequent DGE moves include "point at a diagram", "create a diagram", "label a diagram", "guided dragging", "wandering dragging", "measure a diagram".

The numbers of occurrences indicate that the majority of students' uses of GEX were to point at diagrams by moving the cursor over diagrams, selecting or dragging diagrams while they collaborate to create and label diagrams following Legendre's constructions. Given that most of the student work recorded in Task 2 were collaborative, the dominance of pointing moves among other DGE moves indicates that students used DGE features to communicate with their peers. The data evidences that GEX provides an environment supporting collaborative group work in which students can refer to diagrams on the screen.

Among five participating groups submitted their work on Task 2, four groups included processes of constructing Legendre's attempted proof in GEX in their screencast recordings. Students created diagrams for arbitrary figures, identified intersections of existing figures, and constructed particular diagrams that are dependent to existing diagrams (e.g., a perpendicular to an existing line, reflecting a point over an existing line, etc.). They also labeled diagram they created and placed diagrams so that they can refer to those diagrams in their conversations.

Drag feature was not necessary in the construction part of this task, but students used it to make DGE moves including wandering dragging and guided dragging during the construction

and, mostly, after completing the construction when they identified the counter example of Legendre's attempted proof.

Table 3.8 shows the summary of student work in terms of the number of DGE features and DGE moves identified from their presentation. Since the groups recorded their work on construction, most groups produced significantly longer screencast presentations in Task 2 than in Task 1 except Deming and Tian who started recording after they completed their construction.

Table 3.8. Summary of DGE features and DGE moves identified in each group in Task 2

Pairs or Triples	Construction or Presentation	Duration (mm:ss)	Number of DGE features	Number of DGE moves
Avery, Kim	Both	11:44	81	56
Bing, Eric	Both	10:32	83	55
Deming, Tina	Presentation	1:49	14	13
Deshi, Yao	Both	8:00	86	66
Naomi, Odessa	Both	8:15	108	91
Total		40:20	372	281

Students' engagement in mathematical reasoning on figures in hyperbolic geometry.

In Task 2, students were instructed to record the entire process of constructing Legendre's attempted proof in GEX and to respond the questions of the task for identifying a hidden assumption in this proof with a counter example in their construction. Two groups (Avery, Kim; Bing, Eric) created two separate recordings for each construction and presentation responding the questions of the task. There were groups (Deshi, Yao; Naomi, Odessa) who created a single recording that captured the construction and showed what they identified as a counter example and a hidden assumption without producing a separate recording for that. Naomi and Odessa recorded right from when they started the construction whereas Deshi and Yao presented a completed construction and demonstrated each step they followed to make the construction in

detail. As noted in the earlier section, Deming and Tina produced a significantly shorter presentation that didn't capture or explain their process of construction. The difference in the extent of student work captured in student-generated recordings, the analysis of the enacted task contained an inherent limitation in identifying students' engagement in mathematical reasoning that are comparable across groups. In much the same way as the analysis of student work in other task, the focus in this report of the analysis is, therefore, not to compare groups but to highlight different ways of their engagement in the intended activities given in the task. On the other hand, it implied other aspects of students' enactment of the task in terms of how students interpreted expectations of their work to be submitted, how much autonomy they can take control in the task, and to what extent they were willing to share their incomplete work as their final products responding this task.

Table 3.9. Case themes of students' engagement in mathematical reasoning about Legendre's attempted proof of EPP in Task 2

Case themes
<p>Theme 2.1. Students demonstrated different ways of using DGEs in the step-by-step examination of statements in the attempted proof.</p> <ul style="list-style-type: none"> • Students examined statements in the attempted proof and determined the validity of each step while interacting with DGE diagrams in different ways. • Students examined statements involving incidence (intersections of lines) or inclusion-exclusion relationship (e.g., a point in the interior of an angle) of figures by pointing at the diagrams on the screen. • Students examined statements involving measurements by using measuring tool in GEX. <p>Theme 2.2. Students used the drag feature used to identify counter examples.</p> <ul style="list-style-type: none"> • Students created independent diagrams representing arbitrary figures and dragged those diagrams to examine if the construction gives them a counter example. • Some students used the drag feature to generate and test a conjecture in order to confirm that there always exist counter examples no matter how they changed the independent diagrams (a given line and a point not on the line). <p>Theme 2.3. Students compared diagrams to figures in Euclidean geometry</p> <ul style="list-style-type: none"> • Some students compared diagrams of lines in this model to those in Euclidean geometry where lines would not be limited by a boundary. They explained that the boundary of the disk restricts lines from extending further and makes this attempted proof fails.

Table 3.9 shows the case themes identified in this analysis of students' engagement in mathematical reasoning about the geometric construction suggested in Legendre's attempted proof of EPP. The focal mathematical objects that were identifiers of student engagement in reasoning in Task 2 include geometric figures represented by GEX diagrams in their construction (e.g. points, lines, angles).

Theme 2.1. Students demonstrated different ways of using DGEs in the step-by-step examination of statements in the attempted proof. In students' recordings including their processes of constructions, they created diagrams of figures following each step of the attempted proof. While executing those step-by-step constructions, students interpreted the meaning of the statements in those steps as well as corresponding diagrams they created on the screen. When they attended to a statement declaring a figure (e.g., let n be a line such that), students created a diagram for the figure that was expected to satisfy the condition defining the figure in the statement. For instance, Avery and Kim discussed whether they want to create a diagram with respect to the following statement—Let n be any line through P distinct from m and \overleftrightarrow{PQ} . Avery thought they didn't need to create a diagram for the line n in their construction that was supposed to represent *any line* as described in the material. Kim suggested to create a diagram of a line in GEX that is defined by the point P and an arbitrary point G . Note that a line needs two distinct points to be created in GEX. She showed that they can move this line freely but fixed at P on the screen by dragging the other point G and claimed that this can represent the arbitrary line n in the statement. They interpreted the meaning of the line n declared in the written statement and represented the arbitrary line with an independent diagram in GEX they can freely drag around. This construction of the line n thru P was generalized in this recording in the sense that the arbitrariness of the diagram was confirmed by the drag feature of GEX (Figure 3.16).

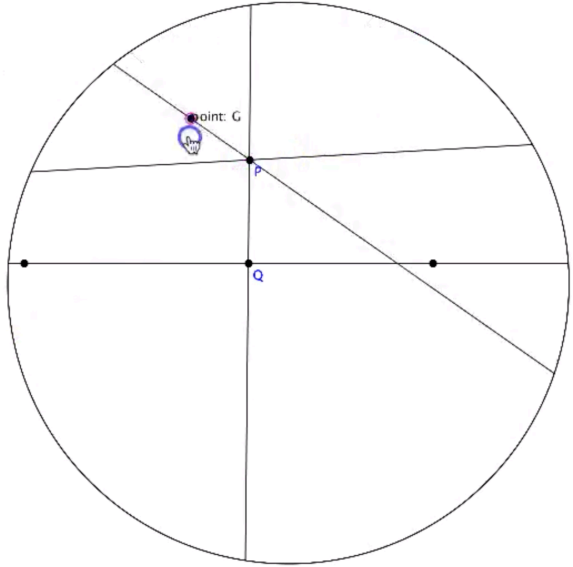
Snapshot (02:23)	Group	Avery, Kim
	DGE move	Wandering dragging
	DGE feature(s)	Drag a diagram
	Transcript (01:55-02:25)	
	<p>Avery: (01:55) Then we're just calling n, any line through P that isn't m. And that isn't a PQ.</p> <p>Kim: Okay</p> <p>Avery: I don't know if we actually have to make that line though. Maybe we just start doing this [the next step].</p> <p>Kim: Okay</p> <p>Avery: Let PR be a ray of it.</p> <p>Kim: Why can't I just make a line that's any line through P.</p> <p>Avery: You can.</p> <p>Kim: (2:23) Now we can just move that in circle wherever we want it.</p> <p>Avery: Yeah. Good point.</p>	

Figure 3.16. Avery and Kim's diagram of an arbitrary line n in Task 2

As they proceeded in the process of constructing the attempted proof, the students examined diagrams they created following each statement of the attempted proof that describes properties of figures. They determined such statements involving intersections of figures or inclusion-exclusion relationships of figures (e.g., a point in the interior of an angle) by looking at and referring to diagrams on the screen. To examine statements involving measurements (e.g., congruence of figures), students used measuring tools in GEX to see if their diagrams satisfy those statements.

Students confirmed claims about incidences and inclusion-exclusions relationships from their observations of diagrams on the screen or didn't even explain that. On the other hand, students were not confident with their diagrams of which measurements were supposed to satisfy the given statement as much as they were about incidence and inclusion-exclusion relationships.

This is attributed to the fact that measurements of distance and angles in Klein model are not represented in the same way as in Euclidean geometry whereas the existence of intersections or inclusion-exclusion of figures are represented by the same pictorial feature of diagrams in both models. In other words, students didn't need a different way of interpreting diagrams with those properties in Klein model except points on the boundary of the disk. Students often checked the statements that claim about the existence of intersections or a figure included in another figure. For instance, students examined the congruence of triangles $\triangle APQ$ and $\triangle BPQ$ by using measurement tool in GEX. Avery and Kim doubted about the side length of the corresponding sides in those triangles, so they measured length of two pairs of sides. Deshi and Yao also measured and compared all three pairs of corresponding sides in those triangles.

Theme 2.2. Students used the drag feature used to identify counter examples. Students examined their diagrams to identify a counter example by students created independent diagrams representing arbitrary figures and dragged those diagrams to examine if the construction gives them a counter example. Most groups identified counter-examples in their constructions by dragging rays \overrightarrow{PR} or $\overrightarrow{PR'}$ that were moved away from the line L . They identified the independent diagram representing an arbitrary point that define those rays and explored them to find when there is no intersection with L . In addition, there was a group (Bing, Eric) who also changed the point P to search for more diverse cases of counter-examples. Note that P is an arbitrary point not on the line L so the construction was supposed to work no matter where students created it. After they identified a flaw in this construction, Bing and Eric continued investigation on $\angle RPR'$ by changing all the points R , P , and R' . They tried to see if there is a case of the angle where they never find a ray \overrightarrow{PR} meeting the line L (Figure 3.17).

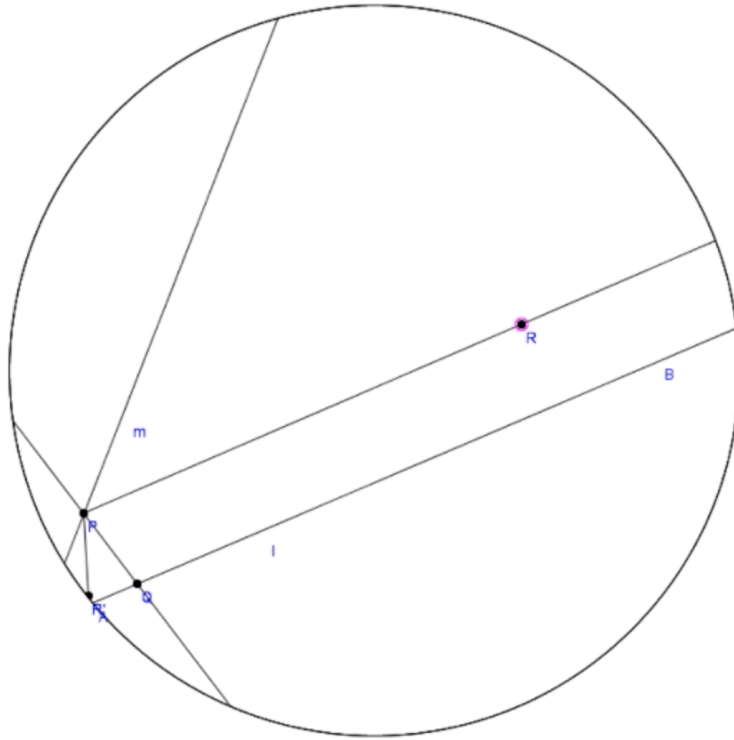


Figure 3.17. Bing and Eric's investigation on $\angle RPR'$

They moved the point P close to the left side of Klein disk toward its boundary. Once they placed P they dragged R or R' to see if there is any case where they cannot make the rays \overrightarrow{PR} or $\overrightarrow{PR'}$ meet the line L . They didn't explicitly make a statement in this recording, but they concluded that for any point P inside the disk, they can always find \overrightarrow{PR} for some point R such that the ray definitely meets the line L . It is notable that they accessed all the dimensions of variations (Leung, 2008) that are independent diagrams necessary to define $\angle RPR'$ so that, in theory, they could vary the diagram into any possible case. This shows that the students generated a conjecture from their interactions with GEX diagrams and tested it using the drag feature for exploring all possible cases of the diagram. This student work exemplifies an example of using the drag feature to facilitate students' mathematical reasoning in mathematization where

they generated a conjecture and tested it by accessing variations of particular diagrams representing arbitrary figures.

Theme 2.3. Students compared diagrams to figures in Euclidean geometry. Students compared diagrams of lines in Klein model to those in Euclidean geometry where lines would not be limited by a boundary. Due to the boundary of the disk that restrict lines from extending further, it is possible to have multiple parallel lines in Klein model that make Legendre's attempted proof fails. In their interpretation of diagrams for lines in Klein model, Deming and Tina characterized lines in Klein model that are stopped by the boundary of the disk and cannot keep going forever and ever until the \overrightarrow{PR} meets the line L (Figure 3.18).

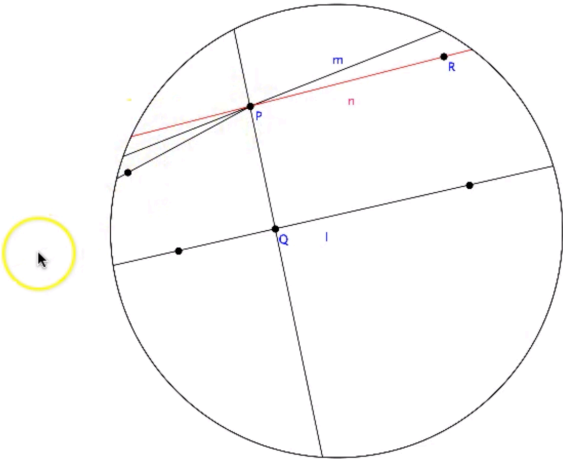
Snapshot (01:32)	Group	Deming, Tina
	DGE move	Guided dragging
	DGE feature(s)	Drag a diagram
	Transcript (01:14-01:38)	
	<p>Tina: (01:14) Step four, however, is what we're finding our problems and where we think the assumption is. He says that since line L passes through point Q, L must intersect one of the sides of the angle, but the problem is that that assumes that these sides of the angle just extend arbitrary arbitrarily long and far.</p> <p>Tina: (01:32) Whereas in the Klein model, they kind of get stopped at the edge of the disk. They have boundaries. They don't just keep going forever and ever.</p>	

Figure 3.18. Deming and Tina's identification of hidden assumption in Legendre's attempted proof of EPP in Task 2

In the above transcript, Deming and Tina articulated what they think of the hidden assumption in this argument where lines can be arbitrarily extended. They argued, however, that lines cannot go forever in Klein model because of the boundary of the disk and that this resulted

counter examples to the attempted proof. This indicates that what they think of a reason why this proof doesn't work in this model is the boundary of the disk that prohibit lines from getting extended until meet each other. In addition, their conjecture of the assumption came from how diagrams of lines would look like in usual Euclidean plane and how this proof would be successful if it was in a space where lines can be arbitrarily extended. In their written responses, the students were more explicit in comparing to Euclidean geometry as follows.

“The assumption is that you can extend a line segment infinitely in any direction. This holds in Euclidean geometry because it is one of Euclid's 5 postulates and leads to the feature that lines and planes extend outward. This does not hold in hyperbolic geometry because of the feature of the boundary of the disk, which “stops” the lines. Even where distance is defined with a logarithmic function, the line would not extend beyond the boundary, so an intersection is not guaranteed.” (Deming and Tina's written response in Task 2)

Note that Deming and Tina were aware of the particular way of measuring distance in Klein model where the measurement uses a logarithmic function that maps a chord of the unit disk to the infinity. This implies that what they meant by lines going forever outside the disk was not about the length of lines but the diagrams representing lines with infinite length in this model.

Task 3: Exploration of Limiting Parallel Lines in Klein Model

Task Design. The previous two tasks in the sequence provide students with experiences of close examinations of a hyperbolic geometry model that is necessary to conclude the independence of EPP and the consistency of hyperbolic geometry. This experience involves

constructing, verifying, and exploring the sophisticated structure of angle measurements in Klein model that is consciously designed to satisfy all the postulates of neutral geometry but EPP.

In Task 3, students use a particular construction feature in hyperbolic geometry mode of GEX that provides a pair of special parallel lines called *limiting parallel lines*. For a given line l and a point P not on l in Klein model, for instance, this feature creates two lines m and n that are pass P and do not meet l . This task is aimed at providing students with opportunities to make an initial sense of limiting parallel lines in Klein model of GEX that are new geometric objects to them. In this task, students are instructed to create limiting parallel lines based upon their examination of this given attempted proof from Task 2 and to respond the following three questions.

- Are the limiting parallel lines examples or counter-examples of $\angle RPR'$ in the Legendre's attempted proof?
- Using the limiting parallel lines, how can you separate examples or counter-examples of $\angle RPR'$ in the Legendre's proof?
- How can you describe the limiting parallel lines in the Klein model? Do you think the description is applicable in any models of hyperbolic geometry?

In the first question, students revisit their work in the previous task and determine if the limiting parallel lines meet the line l or not. They interpret what they create in Klein model using the creation feature for limiting parallel lines that are two lines that seem to meet l only on the Klein disk. Although students may see from the term, *limiting parallel lines*, that those lines are parallel lines, they may also see that those lines are different from other parallel lines from the term and/or from the diagrams thereof on the screen. It is followed by the last two questions

about an essential property of the limiting parallel lines that separate parallel and non-parallel lines (Figure 3.19).

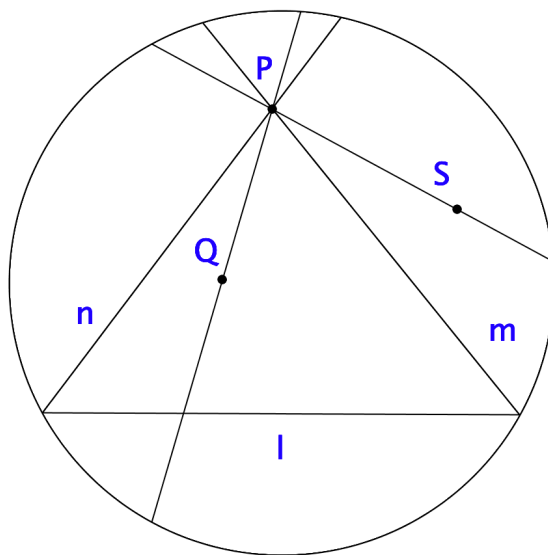


Figure 3.19. Limiting parallel lines m and n with respect to the line l and a point P not on l that separate parallel lines (e.g. \overrightarrow{PS}) and non-parallel lines (e.g., \overrightarrow{PQ}) to l

Description of the class. The class started with reflecting on student work on the previous task of analyzing Legendre's failed proof of EPP. I presented a summary of student responses collected from their submitted work. The summary included demonstration of counter-examples (e.g., dragging arbitrary lines not to meet the line L) that showed flaws in the fourth and sixth steps in his construction. Also, I recapped some of the students' comments made during the whole class discussion at the end of the last class. Those comments included how students connected their work to hyperbolic parallel postulate that was introduced at the beginning of the task sequence and how they emphasized the boundary of Klein disk in their argument.

In transition to the task of the class, I highlighted the implication of our work on analyzing proofs that have flaws and may fail in some sense. I reminded students of what they had or would have learned from their exploration of a pattern in the examples or non-examples

of Legendre's failed proof. The leading question of the day was presented: How can we extend our analysis and findings from Klein model to other models of hyperbolic geometry such as lines meeting at a boundary point of the disk. The question was deliberate in addressing the task objective on developing mathematical knowledge of hyperbolic geometry by mathematizing their observation in Klein model of GEX. Then, I introduced limiting parallel lines and how to create them in Klein model of GEX (Figure 3.20). This introduction was followed by the three focal questions for student's exploration of the limiting parallel lines in Klein model. Students were instructed to submit screencast presentation of their answers to those questions and a written answer to the third question in document.

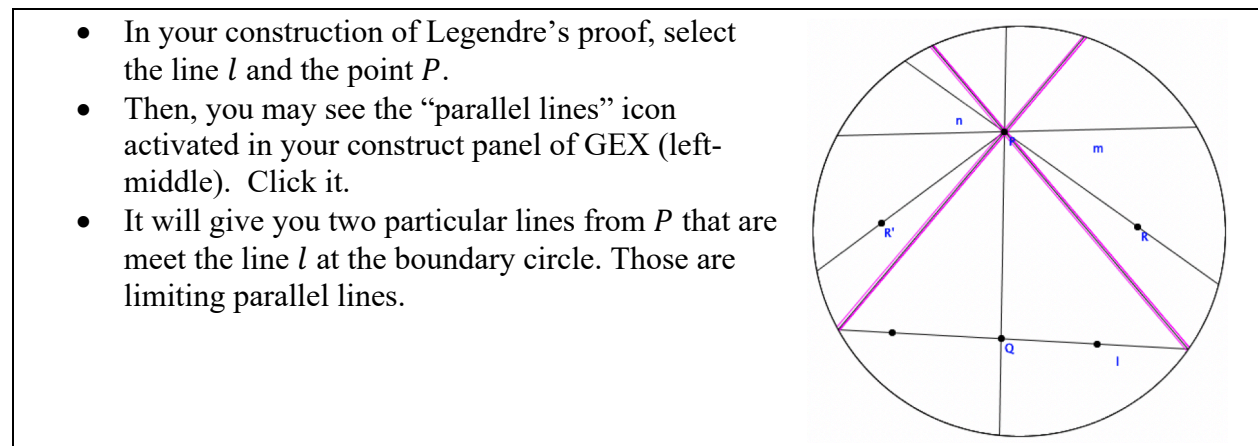


Figure 3.20. Introduction of limiting parallel lines in Klein model of GEX

Students' uses of DGEs. The participating groups used different DGE features in their enactment of Task 3 to answer the three questions given in the task that facilitate them to develop a concept of limiting parallel lines. Table 3.10 shows 9 DGE features and 9 DGE moves that were identified from the screencast presentations created by the seven participating groups (see the Appendix E for all the DGE features and DGE moves identified from the entire student data in this study).

Table 3.10. DGE features and DGE moves identified in the enactment of Task 3

DGE features	Number of occurrences	DGE moves	Number of occurrences
Move the cursor over diagrams	81	Point at a diagram	124
Select a diagram	52	Guided dragging	17
Drag a diagram	34	Place a diagram	7
Drag the cursor	10	Change the model	4
De-select diagrams	6	Create a diagram	4
Create a diagram	5	Undo the last step	3
Open a file	5	Wandering dragging	2
Undo	3	Dragging by accident	1
Switch the mode	2	Point at a feature icon	1
Total	198		163

In the screencast presentations created in the enacted Task 3, there were 198 DGE features and 163 DGE moves identified from 17 minutes 28 seconds of the recordings in total. There were 10 DGE features identified in the presentation that didn't have enough evidence to infer a purpose, so they were not associated with DGE moves. The frequent DGE features include "move the cursor over diagrams", "select a diagram", "drag a diagram", "drag a diagram". The frequent DGE moves include "point at a diagram", "guided dragging", "wandering dragging".

The numbers of occurrences indicate that the majority of students' uses of GEX were to point at diagrams by moving the cursor over diagrams, selecting or dragging diagrams while they collaborate to explore limiting parallel lines and arbitrary lines at the point P on the screen. In addition, there were a feature first used in the task sequence, "drag the cursor" when a group pointed at a particular part of limiting parallel lines in their presentation. In GEX, users can drag the cursor to make a rectangle that can enclose multiple diagrams to select them at once as the

users need. However, Yin and Yong dragged the cursor to make a rectangle that enclose one of the opposite rays of the two limiting parallel lines centered around the point P (Figure 3.21).

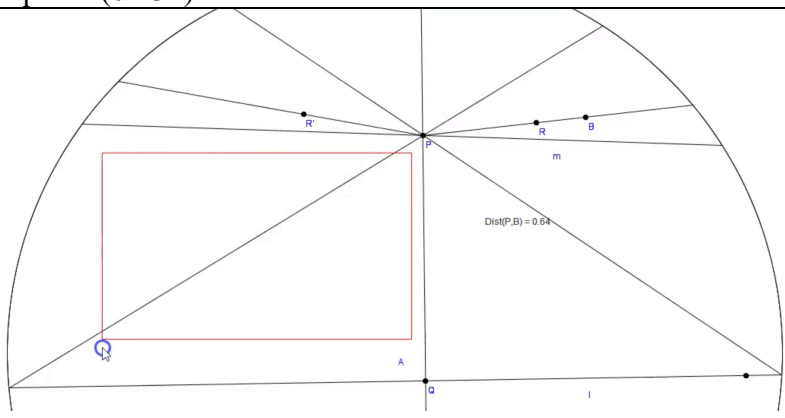
Snapshot (01:32)	Group	Yin, Yong
	DGE move	Point at a diagram
	DGE feature(s)	Drag the cursor
	Transcript (01:07-01:41)	
	<p>Yin: (01:07) So the limiting parallel lines are like the limits for the parallel lines like any points about, about that like actually between, between P and top part and P and the bottom part</p> <p>Yin: (01:32) which is like this one here. So, this right here and this right here.</p>	

Figure 3.21. Yin and Yong's response to the third question in Task 3

When they explained how the two limiting parallel lines separate parallel lines and non-parallel lines to the line L , they made rectangles by dragging the cursor to indicate the four rays of the two limiting parallel lines from the point P . This pointing move showed how the students identified the spaces where parallel line can be found (e.g., line m) in their construction of which boundaries are those four rays. At first, they selected limiting parallel lines that highlighted all the line including both opposite rays so they couldn't indicate one ray of the line unless they create a point on each side of the ray, for instance, say U and V , and refer to the points specifying rays \overrightarrow{PU} and \overrightarrow{PV} . So, they decided to drag the cursor to temporarily present rectangles that limit a space that only contains one side of the lines. This shows a wide range of different features used

by students, regardless of the original function of those features, in order for them to indicate particular diagrams as accurate and efficient as possible. In addition, it indicates that DGEs support student interacting with DGE diagrams not only an accuracy in diagrams that students can create (e.g., Hollebrands et al., 2010) but also an accuracy in their references to the diagrams in DGEs so that student can express their mathematical idea during their communications.

Another new feature used in Task 3 is to switch between different models of hyperbolic geometry when students responded the third question about applicability of their construction and description of limiting parallel lines in other models of hyperbolic geometry. Two groups (Deshi, Yao; Ying, Yong) either created another file of their construction of limiting parallel lines in Poincaré model or changed the mode of the model in GEX to show how their construction of limiting parallel lines worked in the way much the same as how they had presented in Klein model.

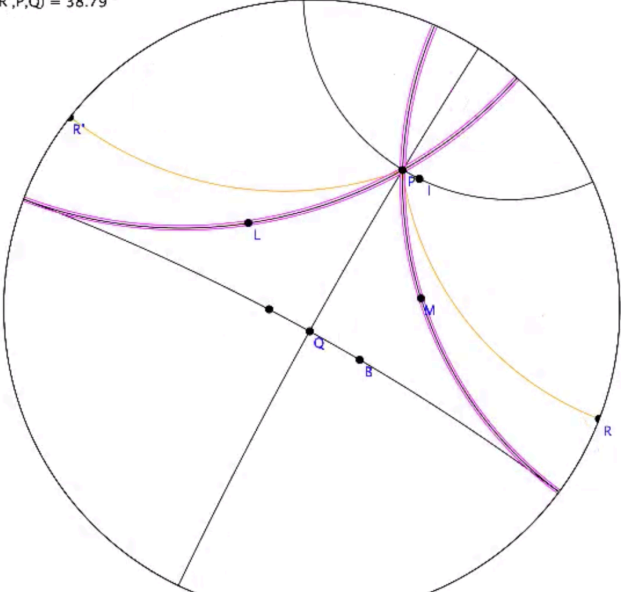
<div> <div>Snapshot (02:03)</div>  </div>	Group	Deshi, Yao
	DGE move	Change the model
	DGE feature(s)	Switch the mode
	<div>Transcript (01:58-02:12)</div> <div> <p>Deshi: (01:58) And this limiting [parallel] line is always working in our models because if I change to</p> <p>Deshi: (2:03) the Poincare disk it should work because this [the point R'] will also not intersect.</p> </div>	

Figure 3.22. Deshi and Yao's switch between Klein and Poincare models

At the end of their screencast recording, Deshi and Yao switched Klein model to Poincaré model in GEX and showed how their limiting parallel lines can still separate parallel lines in this model and that the rays \overrightarrow{PR} and $\overrightarrow{PR'}$ do not meet the line L (Figure 3.22). This shows that some students used multiple models of hyperbolic geometry provided in GEX to look for patterns or common characteristics between those models.

In their use of drag feature, most of the dragging moves were guided dragging (90%) when students demonstrated arbitrary lines (e.g., \overrightarrow{PR}) approaching to the limiting parallel lines. Students made those moves as they characterized limiting parallel lines in their terms by using the metaphor of limits of parallel or non-parallel line. Given that the students seemed to have enough evidence or intuitive sense to make their conjectures, wandering dragging appeared rarely (10%) when they show how they can freely move diagram for arbitrary lines.

Table 3.11. Summary of DGE features and DGE moves identified in each group in Task 3

Pairs or Triples	Duration (mm:ss)	Number of DGE features	Number of DGE moves
Avery, Kim	1:47	13	10
Bing, Eric, Jeremy	2:09	32	23
Ben, Tina	2:20	5	5
Deming, Jing	2:29	28	23
Deshi, Yao	2:15	36	34
Emily, Naomi, Odessa	2:01	12	11
Yin, Yong	4:27	50	48
Total	17:28		163

Table 3.11 shows the numbers of DGE feature and DGE moves identified from each group with different durations of screencast presentation. The numbers of moves relative to the duration of the recordings indicate the extent to which students interacted with GEX

constructions often in presenting their work and different types of presentations. For instance, Ben and Tina produced a screencast presentation with the duration of 2'20" while making five DGE moves, the least number among others in Task 3 regardless of the length of recordings. In their recording, their presentation was succinct and summative in the sense that they were well prepared to give a complete speech with rarely referencing diagrams on the screen. In that regard, most of their presentation was delivered by verbal explanation except when they characterized limiting parallel lines and demonstrated an arbitrary line approaching to limiting parallel line. On the other hand, Yin and Yong produced the longest recording (4'27") with the most moves (48 moves) among the group where they referred to diagrams on the screen more often than Ben and Tina. They were more explicit in indicating examples (e.g., parallel lines between limiting parallel lines) and relied on diagrams more than verbal language when specifically indicating figures in their explanation.

Students' engagement in mathematical reasoning on limiting parallel lines. In their responses to the three questions of Task 3, student groups reasoned about limiting parallel lines with respect to its properties in Klein model and possible way to describe the figure in an abstract way. They justified why limiting parallel lines are one of the counter-examples of Legendre's failed proof in Task 2, explained how those lines separate parallels and non-parallels with respect to a given line and a point not on the line, provided a description of limiting parallel lines that can be applicable to other possible models of hyperbolic geometry. In their presentations, they showed differences in ways of engaging in those reasoning about limiting parallel lines that entailed different uses of GEX (Table 3.12).

Table 3.12. Case themes of students' engagement in mathematical reasoning about limiting parallel lines in Task 3

Case themes
<p>Theme 3.1. Students used different warrants to explain why limiting parallel lines are parallel.</p> <ul style="list-style-type: none"> • Students argued that the limiting parallel lines do not meet the given line L with different warrants. • Some groups referred to the technology that does not identify intersections between the limiting parallel lines. • Other groups referred to the interpretation of points in Klein model (e.g., points inside Klein disk) with assumption that the limiting parallel lines meet L only at the boundary of the disk. <p>Theme 3.2. Students referred to different aspects of diagrams when describing how limiting parallel lines separate parallel and non-parallel lines.</p> <ul style="list-style-type: none"> • Some students described relationships between lines. • Some students identified particular region between diagrams inside the disk. • Some students compared measurements of angles. <p>Theme 3.3. Students demonstrated different approaches to characterize limiting parallel lines using diagrams and mathematical concepts.</p> <ul style="list-style-type: none"> • Some students described a procedure to construct the diagrams of the limiting parallel lines in Klein model. • Some students used the separation property of the limiting parallel lines to define them. • Some students brought in the concept of limiting mathematical objects. <p>Theme 3.4. Students made different uses of examples of hyperbolic models to argue for an applicability of their descriptions of the limiting parallel lines.</p> <ul style="list-style-type: none"> • Some students identified a feature common in hyperbolic models they observed in class, the existence of boundary circle. • Some students showed diagrams of the same construction that still works in two models of hyperbolic geometry they experienced in class.

Theme 3.1. Students used different warrants to explain why limiting parallel lines are parallel. In the first question of the task, students created limiting parallel lines of a given L that passes through a point P not on the line L using the parallel construction tool in GEX as instructed in class. They explored limiting parallel lines in Klein model of GEX and explained why they think those lines are parallel to the given line L . In their screencast presentation, three groups (Avery, Kim; Deming, Jing; Yin, Yong) selected a given line L and one of the two limiting parallel lines, then showed that the software didn't allow them to click the icon for the

intersection tool in GEX. For instance, Avery and Kim showed that there is no option activated to create an intersection in GEX when they selected the line L and the limiting parallel lines (Figure 3.23).

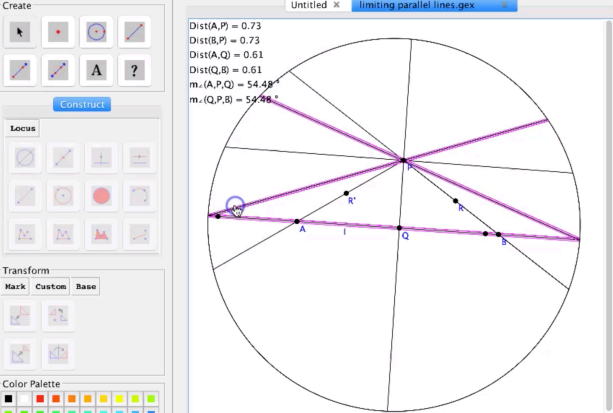
<p>Snapshot (00:25)</p> 	<p>Group</p> <p>DGE move</p> <p>DGE feature(s)</p> <p>Transcript (00:13-00:32)</p>	<p>Avery, Kim</p> <p>Create a diagram</p> <p>Select a diagram</p> <p>Kim: (00:13) So here we have a counter example of our angle RPR' intersecting line L. If we select line L and</p> <p>Kim: (00:25) our limiting parallel lines, there's no option to make a point of intersection in our GEX construction.</p>
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Figure 3.23. Avery and Kim's response to the first questions in Task 3

This means that GEX does not identify an intersection of those lines no matter what the diagrams look like to the students. The students took this response from the software (not allowing them to do further) as a warrant of non-existence of intersections. From there, they referred to the definition of parallel lines as non-intersecting lines and concluded that the limiting parallel line is parallel to the line L because they are not intersecting each other. In this example of student reasoning, the DGE feature of identifying intersections of multiple diagrams was the warrant for their claims that limiting parallel lines are parallel even though those lines seem to meet at a point on the boundary circle.

On the other hand, there were two groups (Ben, Tina; Deshi, Yao) who did not attempt to create an intersection between limiting parallel lines and L in their presentation. They assumed that those lines meet only at a point on the boundary circle of the model. They argued that those

lines do not intersect each other in this model since the only intersection of them lies on the boundary circle, which is not included in the space of the model. For instance, Ben and Tina pointed at where the line L and one of the limiting parallel lines seem to meet on the boundary circle using the cursor on the screen and clearly explained that those lines technically didn't intersect because boundary isn't included in Klein disk (Figure 3.24).

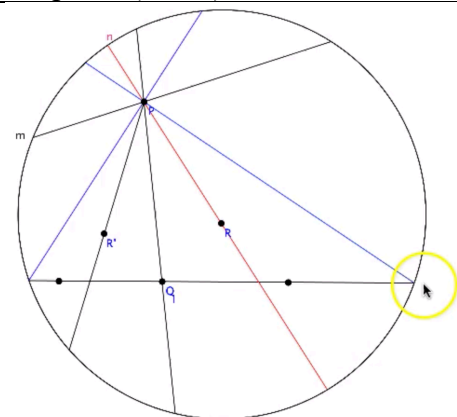
Snapshot (00:25)	Group	Ben. Tina
	DGE move	Point at a diagram
	DGE feature(s)	Move the cursor over diagrams
	Transcript (00:10-00:34)	
	<p>Tina: (00:10) So we think that the limiting parallel lines are possibly counter examples of, uh, Legendre's proof because the only potential intersection point is</p> <p>Tina: (00:25) over here out at the boundary line. And technically the boundary of the Klein disk isn't actually included in the Klein disk. So, they don't really intersect.</p>	

Figure 3.24. Ben and Tina's response to the first question in Task 3

In those groups' reasoning, the warrant for the claim is primarily based on their observation of diagrams in GEX and interpretation of those diagrams based on their knowledge the way of Klein model in representing points. In this regard, the technology provided a warrant—the non-existence of intersections—in the former example of student reasoning that may not need a further interpretation.

Theme 3.2. Students referred to different aspects of diagrams when describing how limiting parallel lines separate parallel and non-parallel lines. Students explained how limiting parallel lines separate parallels and non-parallels of L by describing different aspects of diagrams in their constructions. Students identified particular regions using relationship between diagrams

(Avery, Kim; Yin, Yong; Deshi, Yao), identified particular regions between diagrams using mathematical definitions of interior and/or exterior of angles (Bing, Eric, Jeremy; Ben, Tina; Emily, Naomi, Odessa), and compare measurements of angles (Deming, Jing). The followings are examples of those three types of student explanation of the separation of parallels and non-parallels. First, Deshi and Yao identified where they can place an arbitrary ray $\overrightarrow{PR'}$ on the screen to find its intersection with the line L or not in relation to diagrams of the limiting parallel lines (Figure 3.25).

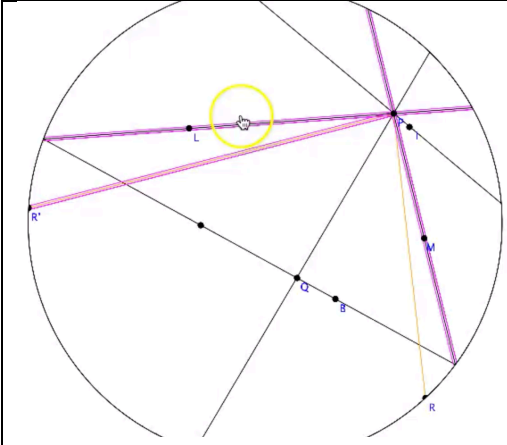
Snapshot (01:15)	Group	Deshi, Yao
	DGE move	Point at a diagram
	DGE feature(s)	Move the cursor over diagrams
	Transcript (00:34-00:58)	
	<p>Deshi: (00:34) In question 2, we try to take example or the counter example of this line so we can see this line is higher our... between the line the limiting line or this line and this line are examples of parallel lines. And between this line and...</p> <p>Deshi: (01:15) between these two limiting lines are the counter examples of the parallel lines intersect with the line L.</p>	

Figure 3.25. Deshi and Yao's response to the second questions of Task 3

In their presentations, they showed two examples of diagrams for $\overrightarrow{PR'}$ that represent particular regions inside Klein disk where the ray either intersects the line L or not. In specifying both regions for parallels and non-parallels, they used the same verbal explanations that those regions are between the limiting parallel lines. However, they specified those different regions by pointing at different parts of the limiting parallel lines. This example shows that GEX feature was used to support students making accurate descriptions to present their ideas when their

language or diagrams were not specific enough to express the ideas. Similarly, Avery and Kim also identified those regions for parallel lines and non-parallel by describing where the arbitrary rays \overrightarrow{PR} and $\overrightarrow{PR'}$ are underneath or above the limiting parallel lines. In this case, their verbal description relied on the diagrams on the screen in the moment that may not work if the entire construction flipped upside down. These groups were able to lighten the burden of using accurate and specific terms to specify figures by using relationships between diagrams on the screen.

On the other hand, Emily, Naomi, and Odessa identified those regions for parallel lines or non-parallel lines by using mathematical definitions of interior and exterior of angles (Figure 3.26).

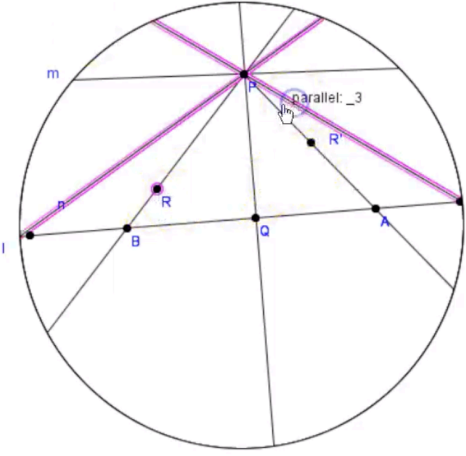
Snapshot (00:43)	Group	Emily, Naomi, Odessa
	DGE move	Point at a diagram
	DGE feature(s)	Select a diagram
	Transcript (00:17-00:58)	
	<p>Odessa: (00:17) Separating the examples and counter examples of Legendre's proof is time that this angle RPR', it meets this line L. Examples of this one. It doesn't meet this line L. So, in math terms, it's the RPR'</p> <p>Odessa: (00:43) in the interior of this angle, parallel lines, which is the intersection of the two half planes that contain this line L. And our counter examples are when they fall in the exterior of the angles that you don't fall in that intersection of the half planes.</p>	

Figure 3.26. Emily, Naomi, Odessa's response to the second question in Task 3

They also started with describing relationship between an arbitrary ray \overrightarrow{PR} and the limiting parallel line, but also recapped their demonstration using math terms, interior and exterior of angles, that were introduced earlier in class when they discussed plane separation postulate in Week 6. In brief, the interior of an angle $\angle XYZ$ is defined by the intersection of two

half planes of \overrightarrow{XY} and \overrightarrow{YZ} that contain each point Z and Y . In their construction, the diagrams of the limiting parallel lines may represent four different angles at the intersection of the lines P because there were no particular points on the lines except P that represent rays forming an angle. So, this group didn't directly use the definition as describe above but modified the definition in their explanation to specify one of the four angles that contain the line L . It is notable that they attempted to provide a description of the separation using mathematical terms they had used in class.

Theme 3.3. Students demonstrated different approaches to characterize limiting parallel lines using diagrams and mathematical concepts. Responding the third question in Task 3, students were asked to describe limiting parallel lines in a way that would be applicable to other possible models of hyperbolic geometry. They illustrated how to construct the diagrams of the limiting parallel lines in Klein model (Deming, Jing), used the property of the limiting parallel lines they explained in the second question (Ying, Yong; Bing, Eric, Jeremy; Emily, Naomi, Odessa), and brought in the idea of limit of mathematical objects (Avery, Kim; Ben, Tina; Bing, Eric, Jeremy; Deshi, Yao). To argue for an applicability of their descriptions of the limiting parallel lines, students identified a feature common in hyperbolic models they observed in class, the existence of boundary circle, and showed diagrams of the same construction that still works in another model of hyperbolic geometry.

For instance, Deming and Jing described how to construct the limiting parallel lines and said that the limiting parallels can be obtained by identifying the intersections between the line L and the boundary circle and connecting those intersections to the point P (Figure 3.27). Deming and Jing's description is a pictorial illustration of the diagrams on the limiting parallel lines in Klein model that explained how they would draw those diagrams in Euclidean plane. In their

uses of GEX, they pointed at diagrams including the limiting parallel lines, the line L , and the two intersecting points on the boundary circle by moving the cursor over diagrams. In their recording, there was no argument for an applicability of their description in other models.

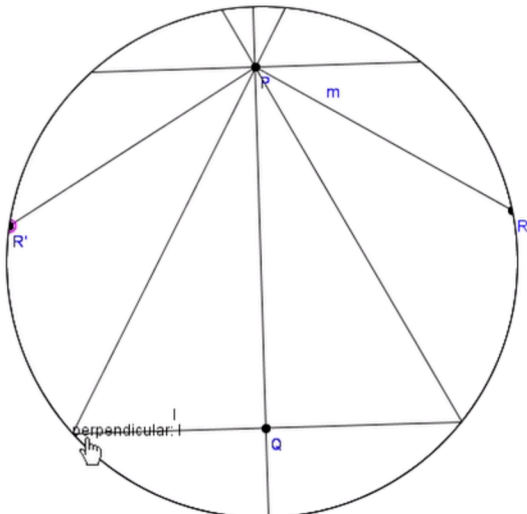
Snapshot (02:10)	Group	Deming, Jing
	DGE move	Point at a diagram
	DGE feature(s)	Move the cursor over diagrams
	Transcript (01:59-02:26)	
	<p>Jing: (01:59) For questions 3, we think that if we find the intersection between the boundary of the circle and the line L, Jing: (02:10) there are two points, then we connect these two points to point P, so then extend it. So, these two lines are limiting parallels.</p>	

Figure 3.27. Deming and Jing's description of the limiting parallel lines in Task 3

Whereas Deming and Jing attended to the pictorial aspect of diagrams, there were other groups who attended to diagrams for explaining the property of the limiting parallel lines separating parallels and non-parallels. Emily, Naomi, and Odessa attended to the diagrams of parallels and non-parallels of the line L separated by the limiting parallel lines (Figure 3.28). According to their description, the limiting parallel lines at a point P to a given line L are particular lines parallel to L that separate parallel lines and non-parallel lines to L passing thru P . This property allowed them to argue for an applicability to other models of hyperbolic geometry because there will be multiple parallel lines in any hyperbolic models.

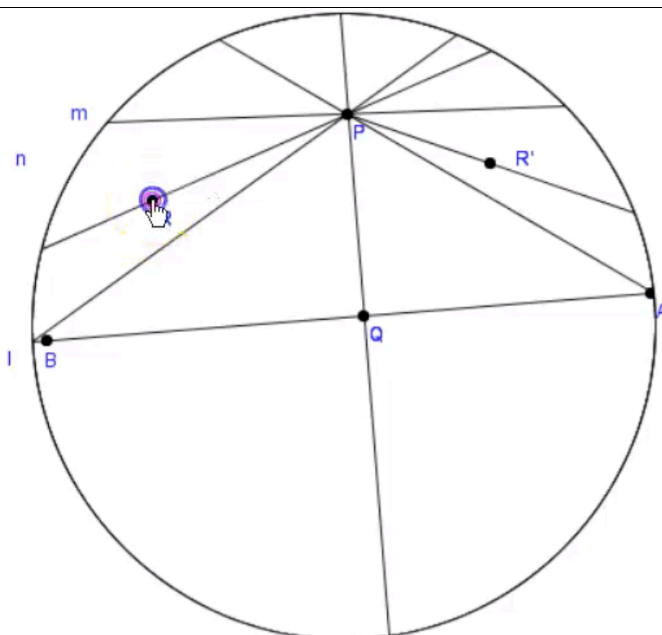
Snapshot (01:14)	Group	Emily, Naomi, Odessa
	DGE move	Guided dragging
	DGE feature(s)	Drag a diagram
	Transcript (00:59-02:00)	
	<p>Naomi: (00:59) The limiting parallel lines from question one, question two, you can say that it is the two parallel lines to the line L from point P that, um, that does not intersect the line L</p> <p>Naomi: (01:14) As you can see here, when the line is in the interior of the limiting parallel lines that, then it intersects the line L that it's in the exterior of the limiting parallel lines then it does not meet that point L. And we do think that this description is applicable in any models of hyperbolic geometry because in the hyperbolic geometry there is always unlimited amount of parallel lines thru a point not on the line, but there's only one or two lines that's the closest to that point.</p>	

Figure 3.28. Emily, Naomi, Odessa's description of the limiting parallel lines in Task 3

Bing, Eric, and Jeremy also described the limiting parallel lines using the separation property by demonstrating an arbitrary ray \overrightarrow{PR} moving across the limiting parallel lines (Figure 3.29). In this presentation, Bing, Eric, and Jeremy described the limiting parallel lines as boundaries of multiple parallel lines to the line L .

circle. For instance, Avery and Kim explained that the limiting parallel lines of a line L are lines that are arbitrarily close to L but do not intersect it

(Figure 3.30).

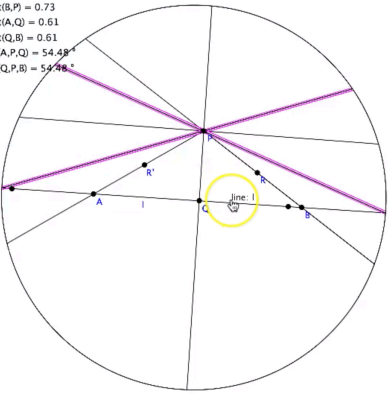
<p>Snapshot (01:16)</p> 	<p>Group</p> <p>Avery, Kim</p> <p>DGE move</p> <p>Point at a diagram</p> <p>DGE feature(s)</p> <p>Move the cursor over diagrams</p> <p>Transcript (01:06-01:21)</p>	<p>Avery: (01:06) We would call them the parallel lines that get arbitrarily close to line, L or whatever, whatever your line is that you're making a parallel line to. In this case it was line L.</p> <p>Avery: (01:16) So there're, there're those parallel lines, again, arbitrary close to line L without ever actually intersecting it. Um, and we do believe that this description is applicable in any models of hyperbolic geometry because all the models of hyperbolic geometry have that boundary circle. Um, and that's kind of the key of how they're constructed is.</p>
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Figure 3.30. Avery and Kim's description of the limiting parallel lines in Task 3

Although it was not clear in their verbal statement that the limiting parallel lines and L are arbitrarily close to each other, they seemed to indicate that those two lines are getting arbitrarily closer to each other as approaching to the boundary circle where they actually meet. It reminded of a graph of a function on the two-dimensional coordinate plane where the function goes to 0 as the independent variable x goes to the infinity. However, they emphasized that the limiting parallel lines do not ever actually intersect L although it gets arbitrarily close to L around the boundary of the circle. In addition, their argument for an applicability of their description indicated that they emphasized the role of the boundary circle in this model and possibly other

models of hyperbolic geometry. The boundary circle is the key feature of hyperbolic models for them. In the other example of this theme, Ben and Tina also described the limiting parallel lines in terms of limit, specifically the limit of the sequence of angle $\angle RPR'$ as the ray \overrightarrow{PR} approaches to the line L (Figure 3.31).

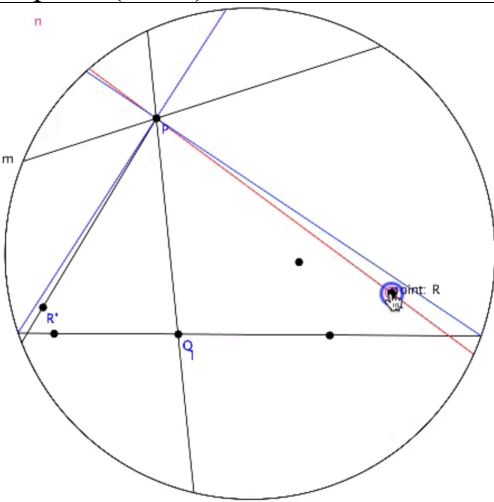
Snapshot (01:16)	Group	Ben, Tina
	DGE move	Guided dragging
	DGE feature(s)	Drag a diagram
	Transcript (01:12-01:21)	
<p>Tina: (01:12) So to describe the limiting parallel lines, we described them in terms of limit. As we see this angle, $\angle RPR'$ get larger and larger. We think it in terms of how it moves along our original line L.</p> <p>Tina: (01:26) We think of it in terms of approaching the edges of the line L limiting parallels or the point at which $\angle RPR'$ will no longer intersect this original line kind of right here. So, the limiting parallels or the point at which $\angle RPR'$ no longer intersects line L.</p>		

Figure 3.31. Ben and Tina's description of the limiting parallel lines in Task 3

Ben and Tina demonstrated $\angle RPR'$ getting larger as they moved \overrightarrow{PR} towards to the right limiting parallel line. According to their descriptions, limiting parallel lines are the limit of the \overrightarrow{PR} and $\overrightarrow{PR'}$ as $\angle RPR'$ gets larger. As the angle gets larger, the intersecting point between \overrightarrow{PR} and L approaches the boundary circle (or what they called the edge of L) where the line no longer intersects L . In this description, the boundary of the disk provided the representation of the limit point in this limiting process. With that in mind, it is not surprising that the boundary circle is the key feature of hyperbolic geometry for them that made their description applicable for other

hyperbolic models. They also mentioned that the angle measurement is the other key element in their description and exists in other models like Poincaré model.

In sum, students engaged in mathematical reasoning about limiting parallel lines in Task 3 where they justified why they are parallel to the given line, explore properties thereof, and described the figures in their own terms that can be applicable to other models of hyperbolic geometry. When they explained why limiting parallel lines are parallel to the given line, some groups relied on the software that does not identify intersection between lines while others reminded of how points are interpreted in Klein model and concluded that the limiting parallel lines are not intersecting the given line because they only meet outside of the set of all points. In the following questions, students explained how limiting parallel lines separate parallels and non-parallels by describing different aspects of diagrams in their constructions. Students identified particular regions using relationship between diagrams, identified particular regions between diagrams using mathematical definitions of interior and/or exterior of angles and compare measurements of angles. Responding the last question of the task, students described limiting parallel by illustrating how to construct the diagrams of the limiting parallel lines in Klein model, using the property of the limiting parallel lines they explained in the second question, and bringing in the idea of limit of mathematical objects. To argue for an applicability of their descriptions of the limiting parallel lines, students identified a feature common in hyperbolic models they observed in class, the existence of boundary circle and angle measurements, and showed diagrams of the same construction that still works in another model of hyperbolic geometry.

As the students engaged in the mathematical reasoning, DGEs provides students with warrants of their claims (e.g., identifying intersections), reference to specific mathematical

objects they mentioned, and tools to demonstrate mathematical situation (e.g., limiting process) in their presentations. In their ways of using DGE features for those purposes, there were differences in their interactions with diagrams in DGE ranging from attending to the pictorial aspect of the drawings to interpreting the meanings of what they see on the screen to connecting diagrams to their mathematical knowledge (e.g., limiting process).

Students' Engagement in Axiomatic Reasoning

In this section, I describe students' engagement in axiomatic reasoning that were identified across the three enacted tasks. As addressed in the theoretical framing of this study, axiomatic reasoning means making purposeful inferences about meta-mathematical objects including axioms, systems of axioms, and models of systems. The previous analyses of students' use of GEX and their engagement in mathematical reasoning informed of themes to describe how students reasoned about those meta-mathematical objects. In this perspective of axiomatic reasoning, students' enactment of the task sequence entailed distinguishing different systems of geometries, and characterizing limiting parallel lines applicable to all possible models of hyperbolic geometry (Table 3.13).

Table 3.13. Case themes of students' engagement in axiomatic reasoning in the task sequence

Case themes
<p>Theme 4.1. Students distinguished their perceptions of diagrams and mathematical knowledge in Klein model from their prior experience in Euclidean geometry.</p> <ul style="list-style-type: none"> • Some students referred to the different way of measuring figures in Klein model when accommodating their observations on diagrams in GEX. • Some students examined diagrams in Klein model to reassure the knowledge they had from prior experiences in Euclidean geometry. <p>Theme 4.2. Students characterized hyperbolic geometry using concrete models.</p> <ul style="list-style-type: none"> • Students characterized limiting parallel lines in hyperbolic geometry with using features common in two particular models they observed in class, Klein model and Poincaré model. • Students perceived features of hyperbolic geometry including parallelism and boundary of disks.

Theme 4.1. Students distinguished their perceptions of diagrams and mathematical knowledge in Klein model from their prior experience in Euclidean geometry. When students investigated Klein model, they tried to distinguish perceptions of diagrams and mathematical knowledge in Klein model from their prior experience in Euclidean geometry. First, students referred to the particular ways of measuring distance and angles in Klein model when they encountered unexpected or unwanted results from Klein model. For instance, Avery and Kim encountered diagrams of a perpendicular line in Klein model that didn't look like what they would expect to see if it were in Euclidean plane (See Figure 3.32).

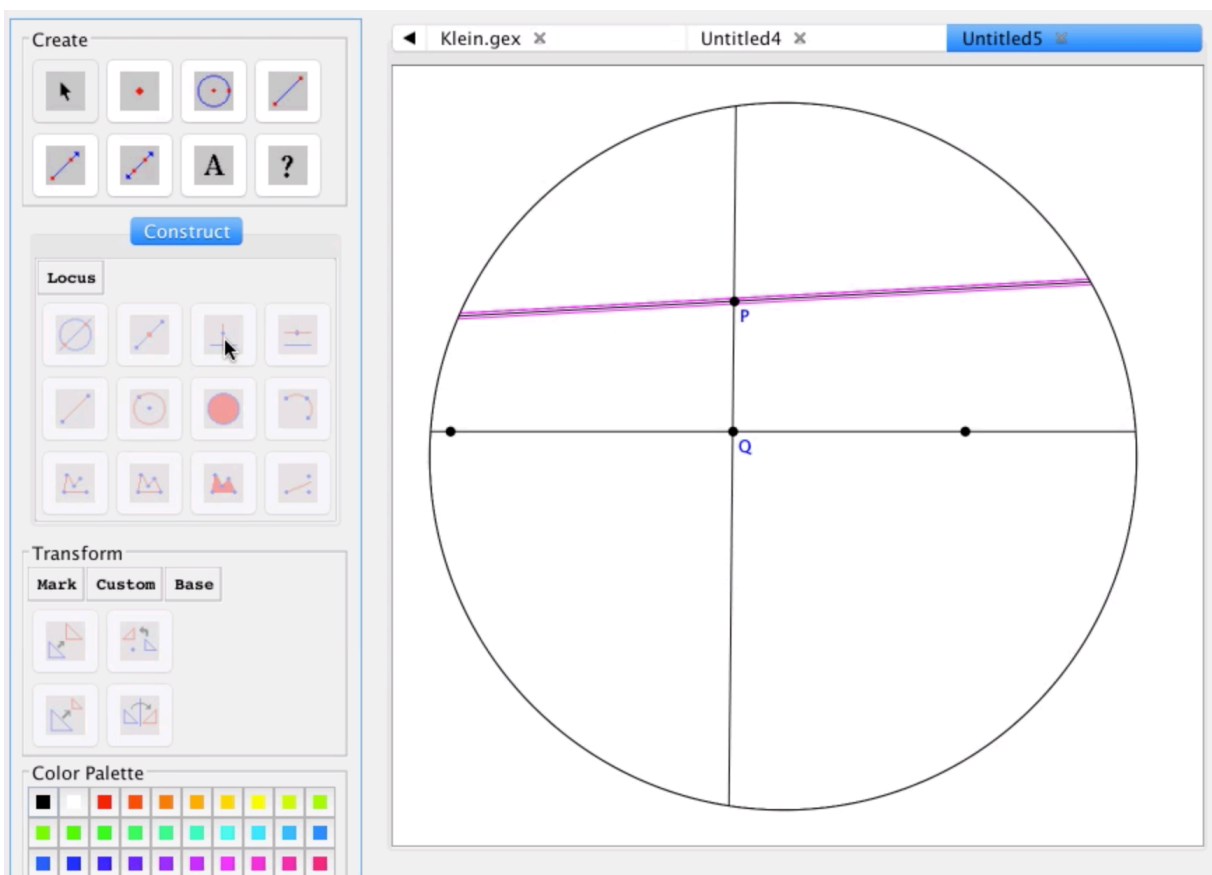


Figure 3.32. The perpendicular line (highlighted) to \overline{PQ} at P in Avery and Kim's construction of Legendre's attempted proof of EPP

In the screenshot above in Figure 3.32, Avery and Kim used the perpendicular construction tool in GEX (see the cursor on the icon) and obtained the highlighted line that was returned by the software. Their reactions to this perpendicular line in Klein model recorded in the video indicated that they were surprised about what they saw on the screen. But then soon after Avery reminded of Klein angle measurement that is not the usual angle measurement they expected to see.

Avery: Yes, yes, then perpendicular.

Kim: Baam [Kim clicked the icon for construction the perpendicular.]

Avery: Wow. Woohoo. It's got to be... it, it must be a perpendicular. It's got to be.

Both: (Giggling)

Avery: Remember this is like the orthogonal circles and stuff like that.

Kim: Yeah.

Avery: Alright. Okay.

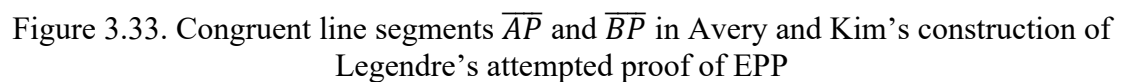
Kim: Yeah. It's probably?

Avery: It... It has to be.

Kim: Okay

They seemed convinced of this diagram of perpendicular line created by GEX tool when Avery reminded that Klein model has its own way of measuring angles using orthogonal circles different from Euclidean measurement. However, it wasn't easy for them to fully accept and feel confident that this is the perpendicular line they expected. This shows their intuitive sense of geometric diagrams (e.g., how two perpendicular lines would look like) that led this reaction to what GEX returned them when they executed the perpendicular construction tool of GEX in Klein model. They reminded of what they know about the angle measurement in Klein model

Later in the same task, Avery and Kim were doubt about two congruent line segments created by the reflection tool in GEX. When they completed the construction as described in the material, they didn't see any problems in their construction that included ray \overrightarrow{PR} intersecting l so tried to identify a hidden assumption that should have made their construction fail. The first statement they revisited to confirm in their construction is the congruence of the line segments \overline{AP} and \overline{BP} (See Figure 3.33 below).



What they saw on the screen didn't look congruent. So, they used the measuring tool of GEX to see if they are congruent. At this point, their conjecture of a hidden assumption is the congruence of line segments created by reflection although the construction didn't fail.

Kim: Should I measure these side lengths? Cause this $[\overline{AP}]$ doesn't look equivalent to that $[\overline{BP}]$.

Avery: Because what doesn't...

Kim: Like \overline{AP} doesn't look like \overline{BP} .

Avery: Yeah. Why don't we measure some of these guys?

Kim: Let's just test it.

Avery: Alright

Kim: Measure. Uh... man... What's wrong? Stop that [Kim selected a wrong diagram on the screen]. Distance from A to P . Let's try B to P . [When she saw the same length] Okay, bamboozled. So, what's the hidden assumption?

The diagrams in Klein model bamboozled the students. They were surprised a bit when GEX showed the same length measured for both line segments in their test on the congruence. Avery measured other corresponding pair of line segments in the triangles, \overline{AQ} and \overline{BQ} , following the conversation above, but those were turned out to be congruent as well. Their perceptions on length conflicted to those diagrams in Klein model, and that led them to use GEX tools to test their conjectures—the line segments reflected each other are not congruent.

To summarize both examples from this group, Avery and Kim encountered diagrams of perpendicular lines and congruent line segments in Klein model that conflicted the ways in which they were used to interpret those in Euclidean plane. To interpret those 'bamboozling diagrams', they recalled their previous work in construction Klein angle measurement function

or tested the diagrams using GEX tools. This indicates that students' experiences in GEX tasks and available tools provided resources (e.g., construction of Klein angle measurement function, measuring tools) for them to accommodate these conflicts when they were engaging in axiomatic reasoning. However, the students were not fully convinced with their test on the congruent line segments and concluded in their recording that the hidden assumption in this construction was the reflection which may not always produce congruent line segments in Klein model.

Another example of distinguishing between different geometries was students' conscious doubt about their knowledge from Euclidean geometry when they worked in Klein model. For instance, Deshi and Yao didn't assume Side-Angle-Side (SAS) triangle congruence in Klein model when they created two triangles $\triangle APQ$ and $\triangle BPQ$ as described in Legendre's attempted proof of Task 2. After they demonstrated that those triangles satisfy SAS condition ($\overline{AP} = \overline{BP}$, $\angle APQ = \angle BPQ$, \overline{PQ} shared), they didn't directly conclude the triangle congruence. Rather they said that it might not hold in Klein model and need to test the congruence of the other corresponding sides \overline{AQ} and \overline{BQ} . This indicates that the students distinguished different geometries where a known mathematical knowledge may or may not be applicable. The students nullified their knowledge about triangle congruence from Euclidean geometry and wanted to reassure that in Klein model. They used the measuring tool of GEX to check the other pair of corresponding sides, \overline{AQ} and \overline{BQ} , and confirmed that all three pairs of corresponding sides are congruent, which in turn, conclude that the triangles are congruent.

In sum, the analysis of students' screencast presentation in Klein model of hyperbolic geometry showed that the students reasoned about differences in hyperbolic and Euclidean geometries. Students encountered diagrams with measurements in Klein model that conflicted to their perceptions of those in Euclidean geometry. To accommodate those conflicts, they used

measuring tools in GEX and recalled the particular interpretation of angle measurement functions in Klein model. In addition, students doubted what they know about figures in Euclidean geometry and tried to confirm that knowledge in Klein model. As they engaged in axiomatic reasoning,

Theme 4.2. Students characterized hyperbolic geometry using concrete models. In the enacted task sequence, characteristics of hyperbolic geometry emerged as the students interacted with Klein model in GEX and explored diagrams thereof. In particular, they reasoned about diagrams in the construction of Legendre's attempted proof and limiting parallel lines of Klein model in Task 3. As reported in the previous section on students' enactment of Task 3, they described limiting parallel lines in different ways such as illustrating how to construct limiting parallel lines in Klein model, elaborating the property of the limiting parallel lines, and bringing in the concept of limiting mathematical objects. To argue for an applicability in each of their descriptions of limiting parallel lines to all possible models of hyperbolic geometry, they drew on prior experience with axioms and models of hyperbolic geometry in which they characterized using parallelism and boundary of disks.

Parallelism in hyperbolic geometry. Students were introduced two contrasting parallel postulates, Euclidean and hyperbolic parallel postulates at the beginning of the task sequence. The independence of those postulates was one of the major objectives of the task sequence presented to students in class. Some students referred to those different parallel postulates to describe and explain their observations of diagrams in hyperbolic model comparing to Euclidean plane in Task 2 and Task 3. For instance, there were two groups (Bing, Eric; Deming, Tina) who specifically mentioned Euclid's fifth postulate to explain why Legendre's construction didn't work in Klein model. In particular, they brought this to compare why the construction would

work in Euclidean but not in Klein model by interpreting the Euclid's fifth postulate to explain counter-examples of the attempted proof. In their reasoning, the construction fails in Klein model because the Euclid's fifth postulate didn't work. For instance, Bing and Eric wrote for Task 2 where they argued that Euclid's fifth postulate is the hidden assumption that Legendre had assumed in this construction, especially for the fourth step.

“[...] The hidden assumption is that the ray PR' does intersect line l in Klein's model even though they may not. In Euclidean geometry, this assumption will hold for the lines do stretch to infinity, thus the lines must intersect before infinity. In step 4, it is assuming that since Q is in the interior of angle RPR' (step 3), and l passes through Q , l must intersect one of the sides of angle RPR' by Euclid's V postulate (sum of the angles is less than 180, on one side of the intersecting line). ” (Bing and Eric's written response in Task 2)

In this response, Bing and Eric claimed that the hidden assumption is Euclid's V postulate that holds in Euclidean plane but not in Klein model with a boundary. In other student work in Task 3, students relied on the multiple parallel lines when they argued for an applicability of their descriptions of limiting parallel lines that separate parallels and non-parallel lines. When they claimed that their description is applicable in any hyperbolic models, the students referred to the hyperbolic parallel postulate that gives infinitely many parallel lines that could be separated from non-parallel lines in some ways.

Boundary in hyperbolic models. One of the key features of hyperbolic geometry that students perceived from the two particular models in class, Poincaré model and Klein model, was the existence of boundaries. Students observed those models before and during the task enactment and characterized hyperbolic geometry using features of diagrams common in both

models. This abstraction from the limited number of examples of hyperbolic models led them to characterize hyperbolic geometry with boundary circle of the disks. In fact, one of the most significant and explicit features common in those two models is the existence of the boundary circles. This individual characteristic common in both the models was considered a key feature of hyperbolic geometry so students assumed that there must be such boundaries in any models of hyperbolic geometry. It is shown in their responses to Task 3 that students described limiting parallel lines using the Klein disk and argued that their description would work for any model of hyperbolic geometry where they could find a boundary. As reported in the previous section of the students' enactment of Task 3, most groups used the boundary circle of Klein model to create a description applicable to any models of hyperbolic geometry. Four of them (Avery, Kim; Ben, Tina; Emily, Naomi, Odessa; Yin, Yong) specifically mentioned the existence of boundaries in any models of hyperbolic geometry in their written responses (See Appendix G). For instance, Ben and Tina described the limiting parallel lines using a sequence of lines intersecting the given line l and argued for its applicability in their screencast presentation as follows.

Tina: We do think that this description is applicable to other models of hyperbolic geometry because the main features that we used to describe this are the boundary circles, which we also have other models of hyperbolic geometry.

Though they described limiting parallel lines as limits of sequences of parallel lines earlier in their recording, they needed the boundary of the model to argue for the existence of limits of those sequence. Given their observations on the boundaries in Poincare and Klein models, they assumed that such boundary circles exist in other hyperbolic models where their description of limiting parallel lines would work as well.

In sum, students engaged in axiomatic reasoning about key feature of hyperbolic geometry when they argued for the applicability of their descriptions of limiting parallel lines. Two different types of characteristics of hyperbolic geometry emerged. The first characteristic is the hyperbolic parallelism that allowed them to assume multiple parallel lines in any models of hyperbolic geometry and particular lines that separate parallel and non-parallel lines as they observed in the earlier task. The other characteristic of hyperbolic geometry is the structural feature common in Poincare and Klein models, the boundary circles, as they identify limiting parallel lines as lines that meet the given line on the boundary circles. In both types, students' descriptions of limiting parallel lines stem from their observation and interaction with diagrams in GEX. However, it is notable that the first type entails students' attentions to the parallel postulates that distinguish hyperbolic geometry to Euclidean geometry whereas the other type resulted from abduction from their experience with two concrete models.

Summary of Findings

The analysis of the enacted task sequence of this study showed how the participating students used various features in GEX while engaging in reasoning on mathematical and meta-mathematical objects. I summarize the findings from the analyses of the enacted task sequence to answer the three research questions posed in this study.

Answers to RQ1: Students' uses of GEX. In all three of the tasks enacted in this study, the analysis of DGE features and DGE moves revealed students' purposeful uses of DGE features responding the tasks. Students used DGE features for different purposes as identified in various DGE moves they made in their screencast presentations. To point at a particular diagram, for instance, students moved the cursor to the diagram, clicked the diagram, or dragged the diagram. The most frequent uses of GEX in all three tasks was to point at particular diagrams on

the screen while presenting their work to the audience or discussing with peers. Students moved the cursor over the diagrams, selected the diagrams, dragged the diagrams, and dragged the cursor to enclose parts of diagrams when they engaged with reasoning about the figures represented by the diagrams. In some cases, such pointing moves in GEX were subsidiary to their verbal statements in the sense that their verbal descriptions were specific enough to identify particular diagrams on the screen without the pointing moves. The audience wouldn't need to watch the video to understand what the presenters meant. On the other hand, there were cases when students relied on pointing moves more than their verbal specification to specify geometric objects and phenomena such as certain parts of a diagram or a space that cannot be separated without adding new symbols or specific movements of diagrams by dragging (e.g., when a diagram moves to here). In the latter, pointing moves reduced students' cognitive loads to use precise definitions and symbols to identify particular diagrams as well as allowed them to express their mathematical ideas referring to movements, changes, and measurements of geometric objects that would not be captured in static pictures or written/verbal statements.

Enacting each task involving different natures of mathematical activities, students used DGE features to construct diagrams and investigate properties of diagrams. In particular, students used the drag feature to demonstrate variant or invariant properties of diagrams that represent arbitrary geometric figures. Students switched their dragging strategies between wandering and guided dragging as they generated, specified, and tested conjectures using empirical evidence obtained from concrete examples in DGEs responding the students' dragging. For instance, students dragged diagrams for arbitrary angles in Klein model to demonstrate that the diagrams keep preserving certain properties to meet the angle postulates in Task 1 or identified counter-examples of the given proof in Task 2. They used wandering dragging to

survey overall changes in diagrams and switched to guided dragging as they identified interesting geometric phenomena in the varying diagrams and tried to look for particular instances to demonstrate their conjectures responding given questions in the task.

Answers to RQ2: Students' engagement in mathematical reasoning and roles of GEX. In the enactment of the task sequence, students verified their construction of Klein angle measurement function with respect to the five angle postulates in the system of hyperbolic geometry, constructed and analyzed Legendre's attempted proof of EPP in Klein model, and develop the concept of limiting parallel lines in hyperbolic geometry. The analysis of this study revealed how students engaged in mathematical reasoning on the focal mathematical objects that were centered in each task (See Table 3.5, 3.9, 3.12).

In each case of mathematics reasoning, DGE moves associated with the identified case themes showed that students used DGEs to communicate their mathematical ideas and to examine mathematical statements describing properties of geometric objects within axiomatic systems and models of hyperbolic geometry. In Task 1, DGEs allowed students to create Klein model in Euclidean plane so that they interacted with diagrams by referring to diagrams and measurements as they verified their models with respect to the angle postulates of the hyperbolic geometry system. For instance, the existence and uniqueness of corresponding diagrams and measurements provided warrants to their arguments for well-definedness, injectivity, and surjectivity of Klein angle measurement function. The dragging feature of DGEs supported students investigating the range of the measurement function, especially when they generated, specified, and tested their conjectures on the angles with measurements close to 0° or 180° or local geometric phenomena in their construction such as what happened when the line passes through the center of the Klein disk. In Task 2, students transformed the written geometric

construction in Legendre's attempted proof and analyzed this working construction to identify counter-examples to EPP in Klein model and the hidden assumption in his proof. DGEs allowed students to critically assess each step of the proof by changing draggable points that represent arbitrary figures. This connects to students' characterization of limiting parallel lines in Klein model while DGEs provided them with draggable arbitrary lines that represent parallel and non-parallel lines separated by the limiting parallel lines.

Answers to RQ3: Students' engagement in axiomatic reasoning and roles of GEX. In the enactment of the task sequence, students engaged in axiomatic reasoning on axioms, models, and systems of axioms regarding Euclidean and non-Euclidean geometry (See Table 3.13). As they analyze Legendre's attempted proof that only works for Euclidean geometry, students distinguished their perceptions of diagrams and mathematical knowledge in Klein model from their prior experience in Euclidean geometry. They referred to the different way of measuring figures in Klein model when accommodating their observations on diagrams in GEX. DGE allowed students to examine properties of diagrams in Klein model to determine if the knowledge they had from prior experiences in Euclidean geometry also hold in hyperbolic geometry. When they characterized limiting parallel lines in hyperbolic geometry, students referred to perceived features common in two Klein model and Poincaré model. Those perceived features of hyperbolic geometry, parallelism and boundaries of models, were developed through their interactions with DGEs during the task enactment.

Discussion

Task Design for Effective Use of DGEs

The case themes of student engagement in mathematical reasoning and axiomatic reasoning while using DGEs in this study indicated that students may have different

interpretations of what they need to do for achieving the tasks. The differences in length, types, use of DGEs were affected by their interpretation of the nature of mathematical activities asked to respond the tasks. Understanding students' ways of interpreting and enacting tasks is essential part of reflecting and revising task designs. It is notable that the focus is not on forcing students to follow one particular pathway of enacting such tasks rather on maximizing the affordance of the tools for their own learning. In this regard, Ainley & Margolinas (2015) argued for an importance of understanding students' perceptions of mathematical tasks comparing to intentions of teachers and task designers and that such gaps between student perception and intentions in task design may result in missing opportunities to learn. They pointed out that it is important for teachers implementing tasks in class to recognize didactic contract and milieu that affect students' task enactment. To minimize such gaps, the authors argued for an importance of recognition of students' implicit rules developed previously and/or during the task enactment in class so that teachers understand unexpected responses from their students.

The findings of this study indicated that there exist the participating students' implicit rules in using DGEs in their argument, producing an acceptable presentation in the form of video recordings. For instance, some students (e.g., Yin and Yong) wanted to and were willing to share their process of construction and unfinished diagrams in the submitted presentation, whereas other students (e.g., Ben and Tina) created recordings as if they were verbally reading aloud their written arguments with necessary amount of using DGEs on their screen. This shows students executed their agencies when interpret what it means to produce screencast presentations to make argumentation ranging from experimental exploration to more formal mathematical proofs. Although the findings of this study didn't explicitly imply how such differences affect student learning, it is notable that the analysis revealed students' active engagement in making their own

meaning of those mathematical tasks in technology-rich environments and that the awareness of such difference would help teachers and task designers to better understand what students created responding the given tasks as addressed in Ainley and Margolinas (2015).

Another example of students' implicit rules is the extent to which they were thoroughly demonstrate concrete examples in their audiovisual argumentation. To verify the angle postulates in Klein model, the participants in this study used DGEs to possibly convince themselves, convince their peers, convince the audience of their presentations, or create mathematical products to achieve the given tasks providing with evidence of their knowledge and skills asked in this coursework. The literatures addressed this issue of different student usage of DGEs that are affected by types of tasks in which students are making meaning of given mathematical statements or claims. For instance, Hollebrands and colleagues (2010) identified different uses of DGEs involved in students' argumentation. Depending on types of arguments students produced, they used DGEs in different ways. In particular, the participants didn't use technology when provided explicit warrants for their justification and proof except reference to particular diagrams on the screen from which they reason. On the other hand, they used DGEs more actively when they were unsure as to the validity of a claim or to qualify (i.e., qualifier in Toulmin's framework) the claim with uncertainty by generating new data from measurement and dragging. Their studies indicate that students use DGEs purposefully and possibly as least as they think needed for their arguments. It connects to different purposes of proofs and roles of empirical evidences in DGEs (de Villiers, 1998). Olive (2000) distinguished students' convincing evidence from DGEs to a formal proof and explain that students would regard those evidence as a proof if they only see a verifying purpose of proof except other purposes suggested by de Villiers including explanation. In other words, students who already convinced of a statement by

authority of teachers or textbook as external convince (Harel & Sowder, 1998) may not see a need for more empirical evidence or formal proof for the statement. In that regard, the task of verification of Klein angle measurement function in this study wasn't effective in soliciting students to thoroughly engage in generalization of diagrams. Students with external conviction (Harel & Sowder, 1998) of the validity of the model didn't seem to need for additional evidence from more examples by dragging diagrams. One way to motivate such students would be to ask them further explanation why the function satisfies angle postulates or when their constructions would likely to fail as some of the groups (e.g., Bing & Eric) voluntarily investigated and reported in their recordings.

Using DGE for Supporting Mathematical Communication

The most frequent DGE moves identified from student work throughout the task enactment was the pointing moves that are students' purposeful behavior on the screen to indicate particular diagrams, measurements, or software tool icons. The analysis of audiovisual data of student work in GEX revealed that students made the pointing moves not only for presenting their work to the audience (e.g., instructor) but also to communicate with their peers while constructing figures in collaborative environment with sharing a device. In particular, the students were able to use less definitive terms (e.g., this, that, here) in their communication to indicate particular diagrams by pointing, selecting, dragging, and coloring them on the screen. Though one may argue that this can be done with a paper-and-pencil diagrams in printed materials, it is notable that this analysis implies that the students were able to interact with diagrams in DGE in ways much the same as how they would have done with printed diagrams and that this digital environment successfully functioned to transfer the traditional and important role of the pencil-and-paper diagrams. Moreover, additional features only available in DGE

allowed students to label, color, and rearrange diagrams more easily than ever. More importantly, students were able to communicate what and how they would like to change existing diagrams on the screen by dragging them without losing the geometric feature of the figures represented by those diagrams. The frequent uses of DGE for reference during collaborative work reassures an important role of DGE in supporting user's mathematical communications by amplifying and reorganizing ways of demonstrating their ideas using features that aren't easy or possible in traditional environment with printed diagrams.

Using Screencast Environment in the Task Enactment

Real-time response to dynamic changes in diagrams. Student reasoning captured in screencast presentation was affected by a particular technology environment used in this study where students manipulate diagrams on the screen and explained specific aspects of those varying diagrams in real-time. During the task enactment, students recorded their responses to given questions of tasks using screencast software. In the recordings, students were encouraged to use diagrams they had on the screen and to refer specific aspects of the diagrams as much as they need to produce their responses to the questions. Though the DGE features and DGE moves varied across groups and types of questions as reported in the analysis above, some groups used DGE features to make changes in the given constructions by creating, measuring, or dragging diagrams and referred to those changes in their verbal presentations. While referring to those dynamic changes made in their presentation, some groups encountered unexpected or undesired results that were not consistent with claims they were making in the presentations. In such cases, there were different types of responses to accommodate those inconsistent results. Some students paused their manipulations in GEX and verbal presentations for a while but ignored the inconsistent results and moved on without any further explanation. Also, there were some

students paused and attempted to accommodate the inconsistency by changing figures or interpreting figures in a different way. In some cases, students argued for technical limitations of software that may cause the results.

Different interpretations of creating screencast presentations. In addition, students seemed to have different interpretations of the task in terms of expectations and nature of the screencast presentation they were supposed to produce to complete the task. The analysis showed differences between each group's presentation of the ways in which they referred to visual representations while verbally explaining their ideas. In addition, students seemed to have different interpretation of what they were supposed to create to complete the tasks. For instance, Ben, Deming, and Tina produced relative short, succinct, and summative presentation to report their final products that consist of completed constructions and speech with well-formed sentences. There were no DGE feature or DGE moves made by accident or any cases they got unwanted or unexpected results from the software. Due to the limitation of the data collected in this study, all their work except the part recorded in their presentations were accessible in the analysis. So, the analysis doesn't inform any of those work outside the presentation, but at least, it implies what they wanted to include in the screencast presentation to represent their response to the given tasks. This implies that their interpretation of screencast presentation might prohibit them from interacting with diagrams while presenting their work because it might cause situations they didn't want to show in the recordings. On the other hand, other groups with longer recording often showed trial and errors in their manipulation of diagrams on the screen so their presentations included more active interaction with diagrams as well as their real-time response to what the software responded their manipulation. In sum, creating a screencast presentation to submit their work for the task was interpreted differently in groups and produced

different output in terms of their length, reference to diagrams, and interaction with dynamic diagrams. A question would be to see if how we can compare this difference in student interpretations of screencast presentation to their perception of written mathematical work such as well-constructed proof versus scratch work and how those interpretations informed us of student exercising their agencies during the task enactment where they can choose to go beyond simply answering the given task in a limited form.

Conclusion

In this chapter, I presented an empirical study on students' uses of DGEs and their engagement in mathematical reasoning and axiomatic reasoning while they enacted a task sequence in an axiomatic geometry course. The analysis of this study showed that students used DGEs for specific purposes with respect to the mathematical activities they engaged in responding the given tasks. DGEs provided students with experimental tools to model geometric figures and phenomena in hyperbolic geometry. Students reasoned about mathematical and meta-mathematical objects involved in the enacted tasks in which they used DGEs to express their mathematical ideas and to examine mathematical statements describing properties of geometric objects within axiomatic systems and models of hyperbolic geometry.

The analysis of students' uses of DGEs provided empirical evidence for pedagogical roles of technology on supporting students in communicating their mathematical ideas and engaging in advanced mathematical reasoning. These findings enrich and reassure the discussion on effective uses of DGEs for student learning in the extant literature. First, the pointing moves identified in this study indicated that DGEs were used for supporting students' verbal statements in the screencast presentation and expressing ideas in group discussions. Though some pointing moves were subsidiary to students' verbal statements so the audience wouldn't need to see the

diagrams. Other pointing moves play significant roles in completing verbal statements by specifying geometric objects and phenomena. The latter cases of pointing moves were identified when students wanted to point at certain parts of a diagram or a space that cannot be specified with existing labels or specific movements of diagrams by dragging (e.g., when a diagram moves to here). This implies that pointing moves support students' mathematical communications by providing alternative ways of referring to precise definitions and symbols and by allowing students to express their mathematical ideas referring to movements, changes, and measurements of geometric objects that would not be captured in static pictures or written/verbal statements. In the sense that using DGEs in group discussion and creating screencast presentations using DGEs allow students to communicate their ideas they may not be able to express without those tools, DGEs provide opportunities of new practices of mathematical communication. This finding of the study extends the discussion on pedagogical affordance of DGEs encompassing new practices of mathematics in the extant literature (e.g., Olive et al., 2010) and online platform supporting remote communication between GeoGebra users (Alqahtani & Powell, 2017).

Second, the dragging moves identified in this study support existing claims for pedagogical roles of DGEs on supporting students' advanced mathematical reasoning with students' uses of different drag modes (e.g., Arzarello et al., 2002; Leung, 2008). Some students in this study showed how they switched their dragging strategies between wandering and guided dragging as they generated, specified, and tested conjectures using empirical evidence obtained from concrete examples in DGEs responding to the students' dragging (e.g., Bing and Eric in Task 2). On the other hand, students who didn't take advantage of the drag feature of DGEs in this study were not successful in engaging in the task (e.g., Theme 1.1). These findings indicate that students would benefit from task design and enactment supporting their instrumentation

(Hollebrands & Okumuş, 2018) that allow students to appreciate DGEs and changes their uses of the tools for their own learning.

CHAPTER 4: What We Can See in Diagrams:

Epistemological Shifts in Geometric Models and Mathematical Proofs

Abstract

In this chapter, I address students' ontological and epistemological shifts that have been discussed in the existing literature of student learning of geometry. First, students make a shift in the ontological view of geometric models from Euclidean to non-Euclidean geometry, in which the geometric models are considered conscious artifacts of mathematical design. Second, students make a shift in the epistemological view of mathematical proofs from absolutism to fallibilism, in which proofs can be characterized with a variety of functions and forms. Drawing on the prior literature, I argue that making successful shifts can benefit students in axiomatic geometry and that such shifts can be facilitated by engaging in mathematical activities with supports of dynamic geometry environments. Examples of those activities that involve constructing and analyzing geometric proofs are illustrated.

Introduction

In the past decade, there have been notable shifts in research questions investigating technology in mathematics education (Sinclair, 2017). Shifting from closed questions such as examining an effect or a necessity of technology in student learning, researchers have been posing explorative and multifaceted questions such as designing suitable tasks for exploiting a given technology and establishing theories to better understand pedagogical roles of technology. Responding this call for attention in complexities of technology integrated mathematics classroom, I explored differentiated aspects of student learning in axiomatic geometry supported by technology and appropriate mathematical activities exploiting pedagogical roles of technology.

In this paper, axiomatic geometry means a study of geometry with a focus on the axiomatic approach to understand various geometries with different axiomatic systems including Euclidean and non-Euclidean geometries. In axiomatic geometry, students are usually expected to engage in formal reasoning independent from informal experience with visual representations. Researchers argued that this expectation in instructions may cause students' difficulties (e.g., Hollebrands, Conner, & Smith, 2010). This paper aims at improving this traditional approach to instructions by proposing theories of student learning and mathematical activities in a new environment supported by technology.

The purpose of this theoretical paper is to address students' ontological and epistemological shifts that have been discussed in the existing literature of mathematics education regarding student learning of axiomatic geometry. First, students make a shift in the ontological view of geometric models from Euclidean to non-Euclidean geometry, in which the geometric models are considered conscious artifacts of mathematical design. Second, students make a shift in the epistemological view of mathematical proofs from absolutism to fallibilism (Lakatos, 1976), in which proofs can be characterized with a variety of functions and forms. Drawing on the prior literature, I argue that making successful shifts can benefit students in axiomatic geometry and that such shifts can be facilitated by engaging in mathematical activities with supports of Dynamic Geometry Environments (DGEs).

A Shift in Ontology of Geometric Models for Axiomatic Geometry

In axiomatic geometry, students need to make a shift in ways of perceiving and interacting with geometric models. In this section, I draw on the existing literature addressing ontology of geometric models in Euclidean and non-Euclidean geometry. In particular, the literature indicates that making such shifts can be conceptualized as developing perceptual

representations of non-Euclidean models. I highlight a pedagogical benefit for this student development from engaging in DGE-activities such as exploring behaviors of dynamic figures and creating non-trivial models. At the end of this section, I illustrate a DGE-activity of creating a model of hyperbolic geometry.

Euclidean and Non-Euclidean Ontology

Understanding axiomatic systems and models involves knowing how these models are constructed with sophisticated mathematical designs, and why these models satisfy the given postulates of the systems. In the axiomatic approach to geometry, the essence of a geometry is determined by a set of selected postulates describing the properties of figures with set-theoretic definitions. But the figures can be also realized in a concrete model, which is consciously designed to satisfy the postulates of the geometry. In many cases, especially for non-Euclidean geometry, the models are not just natural outcomes resulting from physical observations of existing figures. Rather, they are conscious artifacts produced by creative and elaborative mathematical design. For example, Poincaré designed a model of hyperbolic geometry that allows infinitely many parallel lines to a given line and a point not on the line. In this model, a hyperbolic line is defined by an orthogonal circle intersecting the unit disk at right angles (Figure 4.1). The orthogonal circle (blue circle in Figure 4.1) created by Euclidean construction (red dotted lines in Figure 4.1) shows that the structure of the model is purposefully designed for the existence and uniqueness of a line passing through two given distinct points. For instance, the uniqueness of the orthogonal circle for a given two distinct points A and B inside the Poincaré disk (black circle in Figure 4.1) ensures the postulate in the system that states the uniqueness of lines between two distinct points.

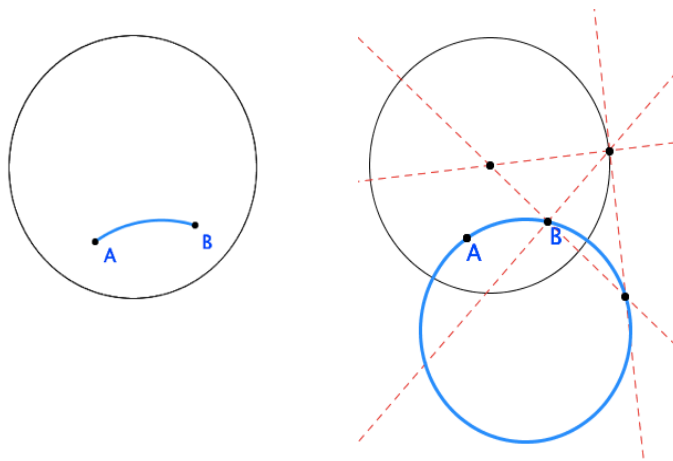


Figure 4.1. A hyperbolic line defined by two points A and B in a Poincaré model (left) and the underlying design of the line using the orthogonal circle (right)

This nature of non-Euclidean models indicates that students need to change their ways of perceiving the models, which have been developed through their prior experience in Euclidean geometry. This existing perception of geometric model from Euclidean geometry is described by Hegedus and Moreno-Armella (2011), who introduced a term, Euclidean ontology in their analysis of historical development of non-Euclidean geometry. With this ontology, when students draw a segment on paper, the drawing is the segment. This means that the paper-and-pencil diagram of a segment is an iconic model of the pure abstraction of all segments. A diagram is merely a mirror of an a priori mathematical object.

On the other hand, students cannot simply draw a hyperbolic line in Poincaré model on paper. The figural feature of the curved lines in the model (See Figure 4.1) cannot represent all hyperbolic lines because there is a model with straight lines such as Klein model. Without knowing the underlying mathematical structure of this conscious artifact, students may not be able to understand the relationship between the axiomatic systems and models. Therefore, students with Euclidean ontology in non-Euclidean geometry need to make a successful shift from to non-Euclidean ontology, in which geometric models are not simply representing figures,

rather being constructed, verified, and explored while students investigate the axiomatic nature of the geometry.

Conceptual Embodiment of Non-Euclidean Geometric Models

A process of making this shift in ontology of geometric model can be conceptualized in terms of conceptual embodiment of geometric models (Tall, 2008). The conceptual embodiment means a process of developing perceptual representations for abstract concepts of geometry and figures in non-Euclidean geometry given by set-theoretic definitions. According to his description, students “conceptually embody a geometric figure, such as a triangle consisting of three straight line-segments” (p. 7). This embodied figure does not mean a static mental image, rather a dynamic mathematical structure that realizes abstract concepts of geometry.

Tall (2018) argued that the conceptual embodiment for non-Euclidean geometry can be supported by providing a physical embodiment so that students can build perception of objects. In this regard, exploring behaviors of dynamic diagrams of figures in DGEs can facilitate students to develop perceptual representations of abstract concepts of the figures in non-Euclidean geometry. In this exploration, students can drag the figures and observe the dynamic responses to their manipulations. It enables students to build up their perceptual representations that they can access as engaging in formal reasoning. For example, Guven and Karatas (2009) showed that interactive diagrams of a DGE can support students' development of a conceptual embodiment of triangles in spherical geometry. In their study, the participants explored triangles on a sphere of a DGE and abstracted properties of spherical triangles that they did not see in Euclidean triangles. After the DGE exploration, the participants could access their perceptual representations of spherical triangles equipped with the properties in the following proving

activity where they were able to draw paper-and-pencil diagrams of spherical triangles in explaining the properties used in their proofs.

In addition, a mathematical activity of developing a new geometric model can be an opportunity for students to learn the axiomatic systems of the geometry as well as develop ownership in mathematics classroom. For example, Otten and Zin (2012) proposed a mathematical task for designing non-Euclidean models to provide authentic mathematical experience. However, the task did not involve an interactive model in their study which may provide a limited opportunity for building up perception of objects being explored in the model.

DGE-Construction of Poincaré Model

Constructing an interactive model of hyperbolic geometry in DGEs can be a mathematical activity for experiencing both exploration of dynamic figures and design of mathematical models highlighted in the prior studies (Güven & Karatas, 2009; Otten & Zin, 2012). In this activity, students can develop perceptual representation of hyperbolic figures that are equipped with the mathematics structures. This representation developed through DGE-activities can allow students to connect the observed properties of figures and mathematical structures. For example, students may recognize a hyperbolic line in Poincaré model becomes straight as the two distinct points getting close to a diameter of the unit disk (Figure 4.2). Knowing the underlying design of the lines in this model enables students to explain this observation by imagining that the radius of the orthogonal circle diverges as the two points lie on the diameter.

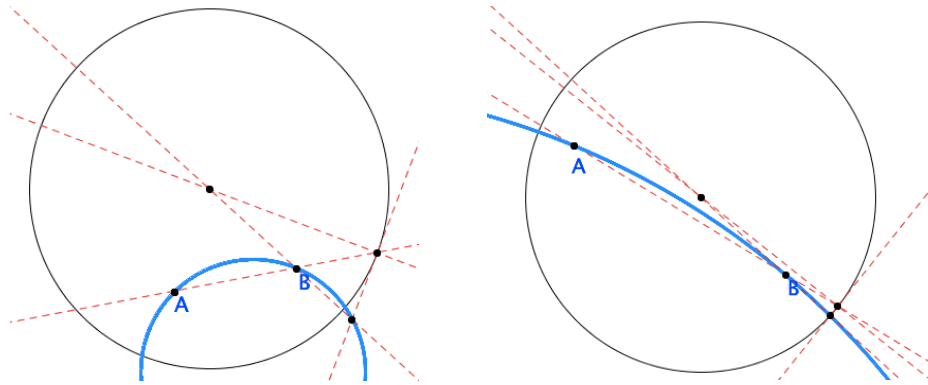


Figure 4.2. Hyperbolic lines in Poincaré model

Moreover, students can discover new concepts, which they have never seen in Euclidean geometry. For example, students can explore infinitely many parallel lines in hyperbolic geometry and discover particular set of parallel lines distinguished from others (Figure 4.3). Students can be guided to identify the limiting parallel lines M and N of the line L such that any line through the point A between M and N should meet the line L . In Euclidean geometry where we have the unique parallel line, different types of parallel lines have never been a focus of mathematical investigation for students.

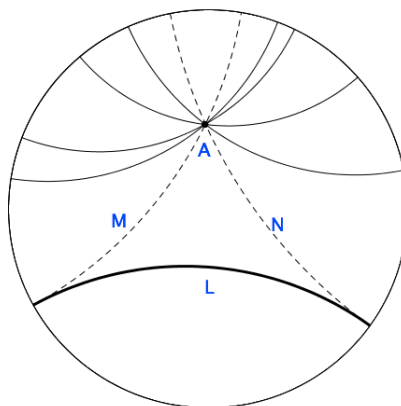


Figure 4.3. Limiting parallel lines M and N of L at A in Poincaré model

A Shift to Fallibilist View of Mathematical Proof

In the axiomatic approach to geometry, college students encounter mathematical proofs less absolute and deterministic than what they have seen in their prior experience. First, there exist mathematical statements that have different truth values in different geometries. For instance, the angle sum of a triangle is equal to the straight angle in Euclidean geometry, but not in either hyperbolic or elliptic geometry. In addition, there are mathematical statements describing the properties of figures which cannot be proved nor disproved in a given system, called axioms or postulates. Students are expected to understand why Euclidean parallel postulate is not provable in neutral geometry, and why the angle sum of a triangle is not determined in non-Euclidean geometry. In axiomatic geometry, proving is not for generating an absolute certainty of a given mathematical statement as usually pursued in the absolutist view of mathematics. Instead, students can benefit from analyzing why proofs work in one geometry but not in another. In this regard, the purpose of proving and its form in axiomatic geometry shift from the absolutist to the fallibilist view, which provide a variety of functions and forms of proofs.

Functions and Forms of Proofs in the Fallibilism

The literature indicates that the fallibilist view of proofs is appropriate for the various mathematical activities for developing axiomatic reasoning and investigating multiple axiomatic systems of non-Euclidean geometry. Researchers have compared the two contrasting views of mathematics, absolutism and fallibilism, in terms of how proofs are perceived differently in each view (de Villiers, 1998; Ernest, 1991, Öner, 2008). In the fallibilist view, also frequently referred to as quasi-empiricist (Lakatos, 1976), mathematics is considered a product of social processes in which its concepts and proofs can be revised (Ernest, 1991). This view contrasts to the absolutist

view, in which absolute certainty exists in mathematics. A proof in the absolutist view is a requirement for verification or conviction of mathematical knowledge. On the other hand, in the fallibilist view, personal conviction motivates one to seek a proof to convince oneself and/or intended audiences of the proof (Öner, 2008). As a quasi-empiricist, de Villiers (1998) argues that proving involves a variety of functions, including explanation, discovery, communication, and systematization, which constitute a broader set of mathematical activities. This set of various functions of proofs in the fallibilist view can also foster a variety of mathematical ways of knowing for students of axiomatic geometry.

In addition, these various functions of proofs in the fallibilist view can be promoted by an alternative form of proofs in exploring different results of the same proof in different geometries. In this exploration, DGEs enable students to transform a written proof into a form of geometric construction. This transformed proof in DGEs provides students with opportunities to analyze the logical deduction of the written proof by interacting with the dynamic construction. This proof analysis activity supported by DGE constructions facilitates students to discover hidden assumptions, logical flaws in proofs, and axiomatic feature of the geometry when they investigate a proof that may or may not work.

Robust Constructions

The new form of visual and interactive proofs in DGEs is called robust construction (Leung, 2008; Stylianides & Stylianides, 2005). Whereas a diagram in the written proof is a particular instantiation of generic objects from the abstract procedure of the construction, the constructed figures in DGEs serve as generic examples that can vary to represent any other particular example. Once a figure is constructed in DGEs, we can drag the components of the figure's diagram (e.g., three vertices of a triangle) to represent a particular kind of figure with

certain properties (e.g., equilateral, obtuse triangle, etc.). This generalizability of the robust constructions in DGEs enables us to transform written proofs to geometric constructions without loss of generality. In the following section, I will describe a DGE-activity of generating and analyzing robust construction of a proof for the exterior angle theorem, which is one of statements varying across different geometries.

Proof Analysis of Exterior Angle Theorem in Euclidean and Spherical Geometry

In high school geometry, the exterior angle theorem means that an exterior angle of a triangle equals the sum of its two opposite interior angles. The statement of this theorem indicates the relationship between an exterior angle and the opposite interior angles in a triangle. However, this relationship varies across different geometries. For example, in neutral geometry, students cannot assume that the angle sum of a triangle equals the straight angle, which in turn, we cannot prove the theorem without the Euclidean parallel postulate. Instead, they may be able to prove that the exterior angle is greater than either of the two opposite interior angles in neutral geometry. However, this weak version of exterior angle theorem does not hold in spherical geometry in general, especially for the particular cases of triangles. Regarding the extent to which we can conclude about properties of exterior angle in different geometries, analyzing a proof of the weak exterior angle theorem allow students to investigate the axiomatic natures of the geometries related to the proof.

In this proof analysis activity, students investigate why the (weak) exterior angle theorem holds in Euclidean, but not in spherical geometry. In this task, students create robust constructions of the same proof in the two different geometric models and explore the components of this construction that are equivalent to each step of the written proof (Figure 4.4). Students are expected to find a counter example in the robust construction in spherical geometry

by dragging the three vertices of the triangle ABC . They can see that the theorem fails when they stretch out the triangle enough to get two dotted lines intersecting each other, which never happens in Euclidean construction. This activity can enable students to figure out what feature of the spherical geometry – two distinct lines intersect at two distinct points – causes this inconsistent result between two geometries. Furthermore, this activity can lead student to experience developing new mathematical knowledge. Students generate their own definition of a family of triangles that do not satisfy the exterior angle theorem in spherical geometry then refine the original statement by adding a specified condition describing such triangles.

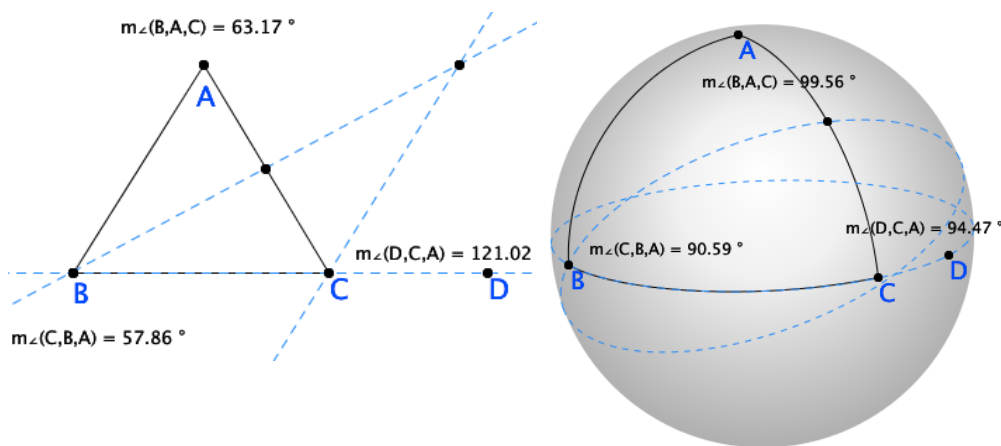


Figure 4.4. Robust constructions of the exterior angle theorem proof in Euclidean geometry (left) and spherical geometry (right)

Discussions

In this paper, I addressed students' ontological shift in geometric models and epistemological shift in mathematical proofs that can be differentiated learning goals for axiomatic geometry integrated with technology. The DGE-activities illustrated in this paper showed a potential of pedagogical role of technology in pursuing such goals in classrooms. One pedagogical implication from this theoretical exploration is an alternative way of mathematical

communication for supporting these activities. I propose a concept of proving as performing that can be considered one differentiated learning activity for technology-integrated mathematical practice. Then, I discuss pedagogical merits related to complexities of teaching practices in mathematics classroom. The discussion includes how technology can empower teachers and students in inquiry-oriented and discourse-based geometry classroom.

Proving as Performing: New Ways of Mathematical Communication

Adding to the existing discussion on new mathematical knowledge and practices in technology-rich environment (Olive et al., 2010), the new functions and forms of proof in geometry allow us to take advantage of technology to introduce new ways of communicating students' mathematics in geometry classrooms. The act of proving in this view is not just to produce a tangible material (e.g., written proofs or images of geometric constructions), but can be also to perform a proof transformed into the robust construction (Leung, 2008; Stylianides & Stylianides, 2005). For example, using screencast software, students can record their DGE-activities with verbal annotation in the form of video clips, showing how they created this generalizable geometric construction that is equivalent to the formal written proof.

One feature of this new way of mathematical communication by performing robust construction is that students can convey their formal reasoning involved in their thoughtful experiments of proof analysis. Students are expected to engage deductive reasoning to generate and understand robust construction of proofs. To see whether or not a given construction is robust enough to be generalized, they also need to identify the generic elements of the construction and test by dragging those elements (See Leung (2008) for a detailed description of the drag test). This implies an alternative way to support student development of formal reasoning in advance mathematics courses. Also, it indicates that researchers need to investigate

this new mathematical practice integrated with technology; how students interact with the technology while engaging in creating and analyzing proof constructions, how their ontology of geometric models and epistemology of proofs affect the engagement.

Pedagogical Implications to Teaching Practice

Screencast presentations of DGE practices are beneficial for implementing student-centered and discourse-oriented geometry classrooms. First of all, it provides instructors with effective and efficient ways for formative assessment of students' mathematical work in DGEs. Student-generated presentations afford a useful way for instructors to understand students' mathematics. Video clips allow instructors to listen while students are verbally explaining their reasoning with using the dynamic construction. Neither the details of students' interactions with DGEs, nor the mathematical arguments captured in video clips, are easily accessible for instructors in actual classrooms or in students' written work.

In addition, screencast presentation of DGE practices can empower teacher and students by establishing a mathematics register (Pimm, 1987) and enrich the communicative mathematical community in the classroom. The traditional meaning of *showing* or *justifying* students' work in advanced mathematics courses usually means to produce a formal proof, which is the unique and acceptable form of mathematical communication. Given the generalizability of robust construction, the idea of performing mathematical proofs allows students to communicate their formal reasoning with others by creating and sharing their presentations. Peer review on screencast presentations can encourage students to listen to peers' mathematical thoughts and formalize student-generated mathematics. The use of student-generated presentations can also promote students' agency in classroom by sharing their own mathematics, understanding their mathematical ideas, and interpreting these ideas into formal statements. Teachers can provide

screencast presentations of DGE practices as a supplementary material for their students. These materials serve as resources that support student-centered instruction to satisfy a need for teacher-centered lectures such as teacher's demonstrations.

Lastly, screencast presentations can support equitable teaching practices in discourse-based geometry classrooms for all students. Teaching practices allowing students to share their voices with each other and listen to them can support equitable power dynamics between students (e.g., Esmonde & Langer-Osuna, 2013). Also, the task of producing screencast presentations allows students to have a safe place to practice explaining and showing their work in DGEs, especially for those who are not able to actively engage in classroom discussions for some reason (e.g., gender, negative mathematical identities, introverted personality, English as Second Language students). Students can rehearse how to show their work to others and prepare for the class discussion when they produce screencast presentations. These experiences help students to feel more comfortable and confident in communicating with others in groups and sharing their mathematical thoughts.

CHAPTER 5: Conclusion

In this dissertation study, I reported the design, enactment, and reflection of a sequence of instructional tasks using DGEs for axiomatic geometry. As reported in the previous chapters, this reflective practice of teaching and research provided three strands of intellectual products with different perspectives to improve theories and practices. The practitioner paper in Chapter 2 illustrated the design of the task sequence in light of the mathematics on which the tasks are grounded and addressed practical issues observed during the enactment of the tasks in the actual classroom. It is followed in Chapter 3 by the empirical analysis of students' uses of DGEs and their engagement in mathematical reasoning and axiomatic reasoning showed that students used DGEs for specific purposes with respect to the mathematical activities they engaged in responding the given tasks. DGEs provided students with experimental tools to model geometric figures and phenomena in hyperbolic geometry. Students reasoned about mathematical and meta-mathematical objects involved in the enacted tasks in which they used DGEs to express their mathematical ideas and to examine mathematical statements describing properties of geometric objects within axiomatic systems and models of hyperbolic geometry. The case themes identified in this analysis provided pedagogical implications to revise the task sequence as well as opportunities to expand the ongoing discussion of alternative views of mathematics that has been rapidly changed in technology-rich environments as discussed in Chapter 4. It is important to note that those ideas addressed in the three chapters emerged and developed in reciprocal relationship between multiple agencies of myself as a researcher-practitioner.

In addition, writing was an opportunity to reflect on my work situated in the field where many researchers have made efforts on improving practices of teaching and learning of geometry in technology-rich environment. The process of writing forced me to elaborate and synthesize all

the ideas emerged around this journey in light of the extant literature. Those ideas include, but not limited to, affordances of technology in mathematics classroom, advanced mathematical reasoning, task design and enactment, and epistemology of mathematics. By adding a particular context, non-Euclidean geometry and axiomatic geometry, that have been less attended by the researchers, I hope that the existing discussion on relationships among those ideas is enriched and that connections between K-12 and post-secondary mathematics, between theories and practices, and between mathematics education are improved. In the following sections of this chapter, I discuss lessons I learned from this study that inform practical implication for revising the task sequence and the course design, implications for future research, and limitations of the analysis.

Reflection on the Task Sequence and the Course Design

In this section, I discuss critical reflection on the analysis of this study in a broader context of the entire course or the task sequence based on what I learned from the last three years of teaching experience.

Supporting students' instrumentation of DGEs. As many researchers have pointed out in the extant literature, it is not easy to introduce technology in classroom and expect students to use those tools effectively for their learning. Learning to use technology for supporting their own learning is not an easy task for students and not a natural outcome that teachers, task designers, and technology developers can take for granted. For instance, some researchers have pointed out that students need to appropriate DGEs and transform them into instruments to act upon mathematical objects such as geometric figures (Alqahtani & Powell, 2017; Hollebrands & Okumuş, 2018). It requires develop students' knowledge of how to use the tool and assign meaning to the tool, which happens through a process called instrumentation where students

change their uses of tools while assimilating or adapting a utilization scheme. So, designing a task sequence or a course highly integrated with technology involves strategic support for students to develop their own ways to use technology in mathematics classroom as an effective instrument. In this section, I discuss how the task sequence and the course design of this study have been refined over the last three years with consideration of such support for students' instrumental genesis. In particular, I focus on using DGEs to create robust constructions and use them to engage in formal reasoning.

One of the student difficulties with using DGEs common in the early stage of the course was to distinguish drawings and diagrams of geometric figures. When engaging in tasks for constructing geometric diagrams, students often created drawings that look exactly like the desired diagrams but do not necessarily represent figures defined by particular constructions. In the third task of Homework 1 (See Appendix B), for instance, some students created an arbitrary circle and created three points on the circle rather than creating three arbitrary points and using those independent points to construct a circle. Since the points are constructed to lie on the arbitrary circle, students cannot freely move those points out of the circle and cannot change the circle by dragging those points. The intended construction in the task is the circle that constructed to pass three arbitrary noncollinear points so that students can drag those points around while the circle dynamically change to follow the dragged points. As mentioned in Chapter 4, creating and using robust construction play key roles in engaging with formal reasoning (e.g., generalization of geometric properties) that are required for the tasks later in the course. In order to address this issue of drawings versus diagrams in class, I revised the course design over the last three years by adding different instructional interventions, (a) providing valid and invalid examples, (b) providing individual feedback on student work, and (c) explicitly

introducing the terms of robust construction and dimensions of variation (See more in Appendix B). Despite the difference in student population, student work on creating robust constructions have been advanced as the course design has been refined with those interventions. There were less students who submitted drawings instead of robust constructions in their homework. Later in the course, most students were able to create robust construction in their responses to DGE tasks as they were asked to represent figures defined by particular constructions.

However, students were not able to carry out drag tests on their robust constructions or did not always appreciate their draggable diagrams as engage in reasoning. This is because students did not have opportunities to reflect on their uses of robust constructions and to receive feedback from the software or the instructor. One example where students are relatively active in interacting with robust construction than other tasks is the verification of Exterior Angle Theorem (EAT) in the take-home midterm exam (See Chapter 4 or Appendix B). In this task, students were asked to find a counter example of EAT on spherical geometry. Students were motivated to change constructions of the proof for EAT as much as possible until they find a counter example and actively engaged in dragging dimensions of variables (three vertices of the given triangle) in their screencast presentation. In this case, the counter example they would find is a feedback for their drag test so that students seek for all the independent variable of the construction and drag them as much as possible. Such feedback embedded in the tasks would be beneficial for students to appreciate the drag feature of DGEs and the robust construction as well as to be able to use DGEs for further exploration as they move forward in the course.

Dual modes of interpreting geometric diagrams in Klein model. When students reasoned about incidence (e.g., intersections between figures) in Klein model, students were allowed to use Euclidean mode of reasoning: When they see two diagrams meet each other, there

exist an intersection between two figures. Despite a few exceptions such as points on the boundary circle of the disk, it is the exact same way of reasoning in which students rely on what they see in the diagrams to make conclusions about incidence in geometry. On the other hand, students were encouraged not to rely on their intuitive perception of measurements (e.g., distance, length, angle) when they interpret diagrams in Klein model. The student data of this study support the claim that students may see diagrams in Klein model much the same way as they have seen in Euclidean models and that this may cause a conflict that students need to resolve to make a valid interpretation of the hyperbolic model (e.g., Avery and Kim in Task 2). Thus, students were sometimes allowed to use the mode that they have been using for years in high school geometry whereas they were discouraged when reasoning about particular aspects of diagrams such as measurements. This implies that learning to make a valid interpretation of non-Euclidean geometric models is not just a shift from Euclidean to non-Euclidean ways of interpretation, rather it is to be able to switch between those two modes and know when they need to use which mode to make a valid interpretation when reasoning about diagrams they see.

Possible Revisions of the Task Sequence and the Course Design

The student work collected from actual classroom informed of limitations of the current task design and further suggestions for revise the task design. In particular, there were limitations of the questions and hyperbolic models used in this task that were revealed in the unexpected student responses and difference between mathematical experience intended in the task design to what the students actually enacted in the classroom. In the followings, I explain two practical implications to revise the task design by adding and modifying questions and models in the task design.

Authentic problem-solving experience for eliciting students' motivation. First, I suggest providing contexts and leading questions in the task sequence that provide students with authentic problem-solving experience. This can be done by addressing anticipated student questions that are reasonably posed in their early experiences with noble geometric models (e.g., why does Klein model have to use an angle measurement function other than Euclidean measurement?). Those questions will provide students with authentic context of the tasks and to deeply engage with geometric models as much as they are motivated to examine the axioms.

Given that how past mathematicians had struggled in and made efforts to discover, accept, and understand non-Euclidean geometry and its models (Öner, 2008), it may not be a feasible and effective goal in task design to provide students with authentic experience of discovering how to develop a hyperbolic model on their own in actual classrooms. As addressed in the literature on principles and practical guidelines for task design (e.g., Ainley & Margolinas, 2015; Komatsu & Jones, 2019), I believe the onus is on educators to identify essential components of such experience and to design tasks providing students with an environment where they can authenticate mathematical tasks situated in their contexts and take ownerships to solve the genuine problems by engaging in those mathematical practices that past mathematicians had engaged. In that regard, the task sequence provided students with accessible tools to explore non-Euclidean models but may miss some opportunities for them to ask why one had to make this model in this particular way.

In the current task design for construction and verification of Klein model in Task 1, some students didn't seem to be suspicious of the model or were not motivated to examine a possibility of failure in this model. They were given a particular way of creating a model of hyperbolic geometry that was named after a well-known mathematician. When they were asked

to create this model following a given instruction in class and to verify if this model truly satisfy all the postulates, students might not doubt about their final conclusion from this task—the model should be valid. The context of the task design and the way of introducing the task in class provided external conviction (Harel & Sowder, 1998) of the tasks that discourage students to critically investigate any example that disprove the validity of the model. For instance, the student examples illustrated in Task 1 shows that some students demonstrated how angles are measured in Klein model constructions they created following the given instructions but didn't generate detailed questions they identified and wanted to confirm while presenting their work. In addition, there was no direct feedback from the software to allows students to attend the 1-1 correspondence of the Klein angle measurement function.

Appropriate feedback from the software and/or teachers in classroom are important for design of instructional intervention as addressed in the literature (Hoyles & Lagrange, 2010). One way to encourage students to authenticate tasks is ask for teacher's role on addressing specific questions eliciting students' perceptions of hyperbolic models from the beginning and promoting students to follow with close examinations on the models. For example, students would be given a question such as why one cannot use the Euclidean measurements for distance and angles in a limited space inside a unit disk. The examples showed that students' early perceptions of Klein model relied on Euclidean perspective where a Klein disk is considered a unit circle that bounds straight line segments with finite length inside the circle. It would be closer to an early stage of student development of Klein model to ask what they see from the given diagram of Klein model and what would happen if figures are measured in Euclidean space. Such questions can lead students to further examinations where they would encounter some of the axioms in the system that would eventually result in counterexamples a model with

regular Euclidean measurements (Komatsu & Jones, 2019). This experience can provide students with more authentic context for them to probe other possibilities to measure figures.

Variations of hyperbolic models with respect to student response. Second, I also suggest providing a varied set of geometric models adaptive to the anticipated and actual student responses where they interact with given models to abstract the nature of the axiomatic system thereof. Given the models introduced in a consciously designed sequence, students investigate those models and generate naïve theories to describe features of hyperbolic geometry while new models introduced in the task sequence support or challenge their theories to be revised. It is a goal for class activities to provide students with experience of doing mathematics by generating, proving, and refuting conjectures to build up mathematical knowledge rather than forcing students to submit an end product in a limited class time.

Shifting from investigations on hyperbolic models to questions on axioms of hyperbolic geometry requires to get away from particular features of the models that are not necessarily applicable to other models. In the earlier version of the task sequence, one of the issues raised when students first studied Poincaré model of hyperbolic geometry was that they attributed what they see different from Euclidean plane (e.g., the fourth step in Legendre’s construction in Task 2) to the curved diagrams representing straight lines in Poincaré model. For instance, students would explain that EPP does not hold in Poincaré model or in hyperbolic geometry because lines are curved in there.

In the revised task design introduced in this paper, Klein model followed a few weeks after the students were introduced Poincaré model when they first studied non-Euclidean model that satisfies neutral geometry axioms. It helped students to see that both straight and curved lines are possible in models of hyperbolic geometry and that the representations would not matter

in studying the nature of hyperbolic geometry. However, the boundary circle of both models prohibited students from abstracting their observations on those models to describe properties of figures that are applicable in other models. In the student example in Task 2 above, the boundary of a space was considered a key feature of hyperbolic geometry beyond the two particular models they observed in class.

The student work from the revised task design using both Poincaré model and Klein model implied that the variation of models (or examples of any abstract concepts) for teaching may address what students would see from the given models to develop abstract concept of hyperbolic geometry. For instance, a possible response to the student work describe in Task 2 would be to provide another hyperbolic model without boundary or a Euclidean model with boundary. In that regard, a task sequence could be used to guide students to develop mathematical knowledge by providing a sequence of models or examples that are consciously chosen with respect to anticipated student development of the knowledge. In this experience, students are encouraged to engage in “conscious guessing” (Lakatos, 1976) that involves generating, proving, and refuting conjectures about relationships among those examples to elaborate their theories. Although this process does not guarantee an end product in practice, as highlighted by Lakatos (1976), engaging in this experience of doing mathematics provides a way for students to see a process of coming to know from “naive conjecture and counterexamples [that] do not appear in the fully fledged deductive structure” (p. 42).

Limitations of the Student-Generated Data in this Study

Most screencast recording data collected in this study, of which analysis were reported in Chapter 3, were the participating students’ final products submitted for their course evaluation. Unlike video recordings of student work collected in empirical studies in mathematics education,

those recordings in this study were produced by students who decided what they want to include in their final responses to the tasks. The student-created recordings induced inherent limitations of this study. First, the screencast presentation did not include what students did when they initiated the task before they started recording their work. This indicates that the data of this study did not inform how students initiated the task, interpreted what they were expected to do for successfully completing the task, and what trial-and-error they might go through before they were satisfied with what they had for creating their final answers to the task. Second, the participating students were able to choose what to report in their presentations by controlling the screencast software. When recording their presentations, for instance, some students paused and resumed the recording when they wanted to exclude or edit particular geometric diagrams on the screen or their prior verbal statements. Though students were asked to record from the beginning of their work in Task 2 as soon as they initiated the task, some students submitted recordings of their final responses to the task. Those limitations of the student data in this study indicates that the scope of the analysis is limited to what the participating students wanted to include in their final presentation rather than all of which actually happened in class.

On the other hand, this limitation of the student data clarifies the scope of the analysis. If it were a task-based interview or a teaching experiment, student data would be more comprehensive to include all the student work from the beginning. However, this study was conducted in a natural classroom setting where students engaged in the task sequence for their coursework. Since the screencast presentations were their genuine coursework submitted for their evaluation, it is reasonable to assume that the participants engaged in the tasks with greater fidelity in this natural classroom setting than in a research setting. Thus, it is notable when interpreting the student-created data in the natural classroom setting in this study that those

student work presented in the recordings are what they thought they were expected to show in their responses to the tasks. This implies there is another layer of students' interpretation of the tasks and their work between what they actually did in class and what researchers can make inference from interpreting the data. I hope readers can find places in this manuscript where this student-interpretation layer taken into account in the report of the data.

I am excited to continue this journey of studying the complexity around mathematical task design using technology with more specific focused on particular contexts such as roles of teachers and task designers within the context of teacher education, methodology in task design studies for practitioners, and new perspectives for exploring uncharted world with ever-changing technology around our students and teachers in future.

APPENDICES

APPENDIX A: Weekly Schedule of the Course

Table 6.1. Weekly schedule of the course

Week	Topics	Relevant Materials
1	Geometry and the Axiomatic Method	Textbook: Chap1
2	Euclidean Geometry / Euclidean Construction	Textbook: Chap2
3	Primitive concepts and postulates	Lecture note: Part 1
4	Distance and ruler postulate / Between-ness	Lecture note: Part 2, 3
5	Segments, rays, angles, and triangles	Lecture note: Part 4
6	Convexity and separation	Lecture note: Part 5
7	Angle measurement	Lecture note: Part 6
8	Congruence of triangle	Lecture note: Part 7
Spring Break (Mar 5 – 9)		
9	Parallel lines	Lecture note: Part 8
10	Parallel postulates	Lecture note: Part 9
11	Hyperbolic Geometry (1)	Textbook: Chap7
12	Hyperbolic Geometry (2)	Textbook: Chap7
13	Hyperbolic Geometry (3)	Textbook: Chap7
14	Elliptic Geometry	Textbook: Chap8
15	Transformational Geometry	Textbook: Chap5
Final Exam (Thursday, May 3, 12:45pm - 2:45pm)		

APPENDIX B: DGE Tasks for In-Class Activities and Homework of the Course

Week	Date	Description
1	01/12/18	Euclidean construction of equilateral triangles in GEX

Technology Investigation

Problem 1. Euclid's construction of equilateral triangle

According to the Euclid's work, construct an equilateral triangle and answer the following questions.

Q1. Measure the length and angle of the triangle. Is it an equilateral triangle?

Q2. Select the point A (or B) and drag it around. See if your construction still works, i.e., the triangle is still equilateral. Can you freely move around the point?

Q3. Select the point C, the intersection between the circles. Can you freely move around the point? If not, explain why.

Figure 6.1. DGE task for in-class activity in Week 1

Week	Date	Description
2	01/17/18	Euclid's first postulate in spherical model: Verifying a postulate in spherical model and exploring counter examples (lines intersecting at two antipodal points)

Tech Investigation

Spherical geometry (Exercise 1.6.7.)

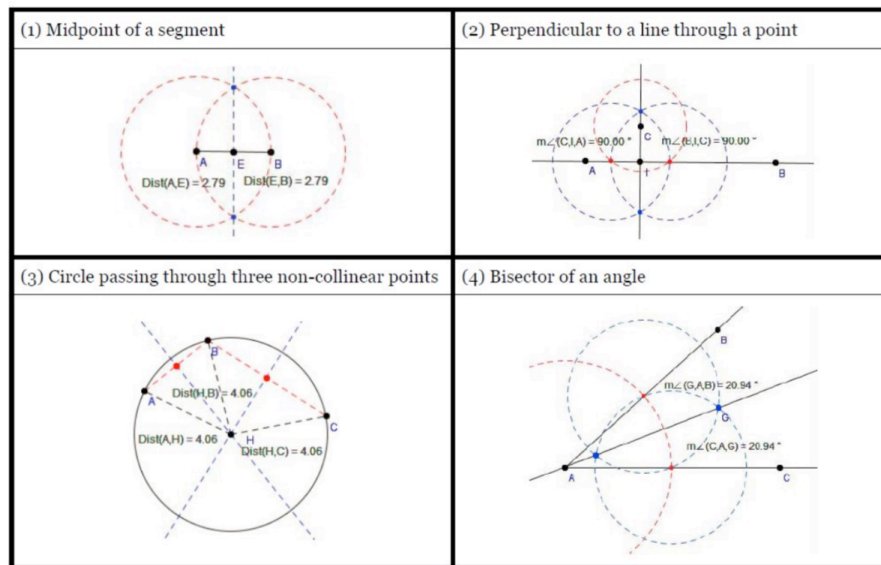
Suppose we re-interpreted the term point in Euclid's postulates to mean a point on a sphere and a line to be a part of a great circle on the sphere. A great circle is a circle on the sphere cut by a plane passing through the center of the sphere. Distance in this new geometry will be measured along lines (segments) on the sphere.

Q. Show that Euclid's first postulate holds for most pairs of points, but not all. Show that the second, third, and fourth postulates do hold in spherical geometry. Show that the fifth postulate (or equivalently Playfair's Postulate) does not hold.

Figure 6.2. DGE task for in-class activity in Week 2

Week	Due date	Description
2	01/19/18	Homework 1: Euclidean construction of equilateral triangles in GEX

HW1: Tech Assignment 1



HW1: Tech Assignment 2

A robust construction of an altitude of a triangle

- For a given triangle, construct an altitude of the triangle in GEX and verify that your construction works for an arbitrary triangle. Submit (1) *gex* file of your construction, and (2) video file of your verification.
- Once you create the GEX construction of the altitude, create a screencast to verify if your construction is robust. In your screencast, you need to identify the dimensions of variation and the invariant property of the construction.

See the instruction of the assignment posted in D2L.

Figure 6.3. DGE tasks for Homework 1 (Slide)

Tech Assignment 1: Euclidean Construction

Objective: To construct geometric figures using a dynamic geometry software, *Geometry Explorer* (GEX).

Geometry Explorer (GEX) provides the following four built-in constructions of figures on Euclidean geometry, which originally can be constructed by using only Euclidean tools, i.e., straight edge and compass.

(1) Midpoint of a segment		(2) Perpendicular to a line through a point	
(3) Circle (Arc) passing through three non-collinear points		(4) Bisector of an angle	

Problem: Construct the above four figures only by using lines, circles, and intersections in GEX.

You can use built-in constructions that you completed previously. For example, you can construct the second figure using the first one. Also, you can use *Edit* menu to label or color your figures and *Measure* menu to verify a distance or an angle in your construction as shown. Your constructions will not necessarily be identical to the examples, but your attempt to make use of the software in construction will be evaluated. If you have a problem with using the software, please bring your questions to the class.

<p>(1) Midpoint of a segment</p>	<p>(2) Perpendicular to a line through a point</p>
<p>(3) Circle passing through three non-collinear points</p>	<p>(4) Bisector of an angle</p>

All constructions will be saved as the form of *gex* file and uploaded to the D2L. Please label each construction using this format: **LastName_FigureNumber.gex** (e.g., Bae_Figure4.gex).

Figure 6.4. DGE tasks for Homework 1 (TA1 document)

Tech Assignment 2: Robust Construction

Objective: To create a screencast of your geometric construction on GEX

Like other dynamic geometry software, *Geometry Explorer* (GEX) provides an interactive environment, in which you can manipulate geometric construction with preserving dependencies between the objects included in the constructions. It allows you to change initial conditions of the construction by dragging components of it. In this course, you will be required to present your constructions on GEX by creating video clips of your exploration using a screen casting software, *Screencast-O-Matic* to capitalize on those experiences to achieve the course objectives.

Dimensions of variation

In a geometric construction of a figure, dimensions of variation are defined as the arbitrary figures given in the assumption of the initial condition of the construction. In the visual representation of a construction of GEX, the dimensions of variation are the components that you are supposed to create first as you start the construction. For example, the dimensions of variation in a circle passing through three non-collinear points are the three points, which are meant to be arbitrary.

Invariant property

Given a construction of a figure with dimensions of variation, a property of the figure is called invariant if the property holds no matter how the dimensions vary. If you have a valid construction in GEX, you need to drag around the dimensions to see if the figure still preserves the desired property of the construction. For example, as you drag around three points that define the circle in the above construction, you want to see if the circle exists and if its radius equals to the length of line segments from the center to each of the three points.

Problem: For a given triangle, construct an altitude of the triangle in GEX and verify that your construction works for an arbitrary triangle by dragging vertices. In your screencast, you need to identify the **dimensions of variation** and the **invariant property** of the construction and to show that your construction is robust enough to preserve the property as you change the dimensions.

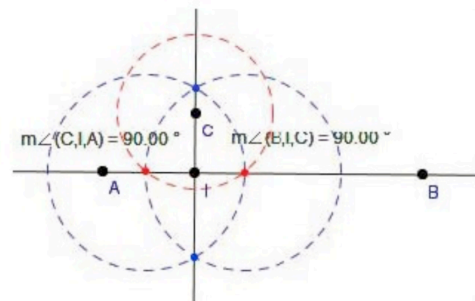
Your construction will be saved as the form of *gex* file and uploaded to the D2L. Please label the construction using this format: **LastName_TA2.gex** (e.g., Bae_TA2.gex). As a part of the assignment, you will also record your construction using *Screencast-O-Matic*. After recording your construction, you can either download the mp4 file of your recording or upload the recording to the server with a link. So, you can submit the mp4 file to D2L (**LastName_TA2.mp4**) or attach the link to your recording.

Figure 6.5. DGE tasks for Homework 1 (TA2 document)

Week	Date	Description
3	01/22/18	In-class activity to address student work submitted for HW1 that did not attend the key ideas of robust constructions and did not qualify criteria

Groupwork: HW1 Revision

Submit a robust construction of Figure 2 (perpendicular to a line thru a point not on the line), which is described in the statement below.



“For an arbitrary line AB and an arbitrary point C not on the line, there exists a unique line perpendicular to the line and passing thru the point.”

- Submit your group’s construction (.gex file) per group in D2L.
=> 1/22 Class Participation folder

HW1 Revision

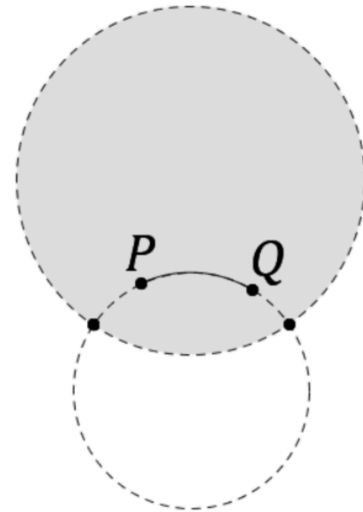
- In your group, share what you constructed for the figures in Homework 1. Discuss how to revise your constructions, if any.
- Submit your revised construction(s) or screencast presentation for the make-up.
(=> Homework 1 Revision folder)

Figure 6.6. DGE task for in-class activity in Week 3

Week	Date	Description
5	02/09/18	Introduction to Poincare model of hyperbolic geometry and discuss axioms and consequences from Euclidean and neutral geometry systems.

Poincare Model

- **Point** is any point interior to a given unit disk except the boundary circle.
- **Plane** is the set of all points within the disk is called the Poincare disk (shaded region).
- **Line** is an arc of an orthogonal circle within the Poincare' disk that meets the boundary the disk at right angles.



Poincare Model

- **Distance** between P and Q is

$$d(P, Q) = \left| \ln \frac{(PR)(QS)}{(PS)(QR)} \right|$$

where R, S are points where the hyperbolic line through P, Q meets the boundary circle, and (AB) represents Euclidean distance between A and B .

- **Angle** is determined and measured by tangent lines of hyperbolic lines as we did in Euclidean plane.

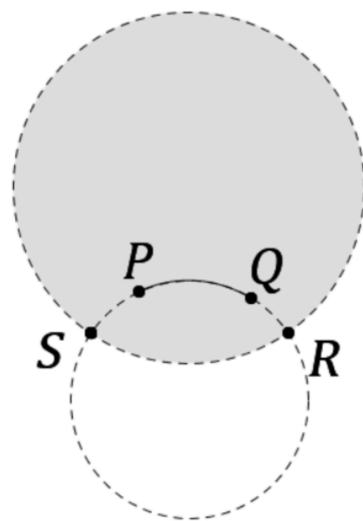


Figure 6.7. DGE task for in-class activity in Week 5 (Intro to Poincaré model)

Poincare Model

Question.

Does the Poincare model satisfy the postulates and theorems in our system?
Take a look at the lecture note and see which postulates you need close examination.

Pick some of the postulates in your group and discuss how you can verify this model satisfies the postulates by using the Poincare model in GEX.

Examples

- Does it always have a line passing through any given two distinct points? (P2)
- [Challenging question] Does the distance satisfy the P7? Do we have a coordinate system of the model satisfying P8?

Poincare Model

Group Project

Determine which of the following Euclidean properties holds in Poincare model of Geometry Explorer.

1. Rectangles can be constructed. (Try any construction you remember from high school geometry.)
2. The sum of the angles of a triangle is 180 degrees.
3. Euclid's construction of an equilateral triangle.
4. Given a line and a point not on the line there is a perpendicular to the line through that point.

Provide robust constructions to justify your answer.

Figure 6.8. DGE task for in-class activity in Week 3 (Questions)

Week	Due date	Description
6	02/16/18	Homework 3: Explore Poincare model by comparing Euclidean and hyperbolic geometry with respect to particular constructions discussed in the earlier group project.
<p style="text-align: center;">Technology Assignment 3: Spherical Model</p> <p>Objective: Try to interpret the sphere model of GEX with respect to the axiomatic system that we have been developing through the course.</p> <p>One of the distinguished features of <i>Geometry Explorer</i> (GEX) is that it allows us to explore lots of concrete models of different geometries including Euclidean and non-Euclidean geometries. There is a sphere model of spherical geometry in GEX (File > New > check the “Spherical” mode).</p> <p>In the perspective of axiomatic systems, the sphere model of GEX is an interpretation of the axiomatic system of spherical geometry though we don’t know yet what the system is. Let us consider the sphere model as a model of the system we are currently developing for Euclidean geometry. Suppose we interpret the undefined terms of S, L, and P of the system as follows.</p> <ul style="list-style-type: none"> • S is the set of all points on the sphere of the model. • L is the set of all great circles on the sphere where a great circle is defined as a circle on the sphere centered at the center of the sphere. • P is the set having one element, the entire sphere. (Note that the sphere does not mean to include the inside of the ball) <p>Following the instruction below, revise the system in Part 1 (Postulate 1-6) for spherical model of GEX.</p> <p>Problem 1. Identify which of Postulate 1-6 do not hold in the sphere model of GEX with respect to the interpretations of the undefined terms above.</p> <p><u>Create a screencast presentation of your answer to the problem.</u> In your presentation, you can show counter-examples of the postulates to support your arguments. Note that you are not supposed to create a robust construction. Also, you don’t submit the GEX file used in the presentation. <u>Submit your screencast presentation to Dropbox folder in D2L</u> with the label of LastName_TA3.mp4</p> <p>Problem 2. How can we modify Postulate 2 for the sphere model of GEX? What can we say about the existence/uniqueness of line(s) between two distinct points on the sphere? (Hint: It may need a definition of “antipodal points”)</p> <p>Evaluate the given (1) definition of antipodal points and (2) modified Postulate using the guiding questions below to make your decision in the evaluation.</p> <ul style="list-style-type: none"> • Does the modified postulate hold in the sphere model of GEX? • Does the modified postulate gives enough information about existence and uniqueness of lines containing two points? • Do the modified postulate and the definition use only the terms we are not given in the system at this point? 		

Figure 6.9. DGE task for in-class activity in Week 6

Week	Date	Description
7	02/23/18	Investigate power of a circle and orthogonal circles with their constructions in Euclidean geometry

Circle Inversion and Orthogonality

In this group project, we will investigate the following three concepts with respect to a given circle: (a) the power of a point, (b) the inverse of a point, and (c) orthogonal circles. This investigation will lead us to construct an orthogonal circle passing thru two distinct points that lie inside the given circle. Constructing this unique orthogonal circle is an essential part in design of hyperbolic geometric models including Poincaré model and Klein model.

Following the instruction in Section 2.7 of your textbook, you will create three geometric constructions in GEX illustrating the three concepts and answer Exercise 2.7.1 – 2.7.4. For your Class Participation today, you need to complete this project, submitting the following items in D2L Dropbox folder.

1. Project Report (1-page long Word document) per group

You need to compose a Project Report including your responses to Exercise 2.7.1 – 2.7.4 from the project.

2. GEX constructions (three .gex files) per group

You need to turn in three separate GEX constructions of (a) the power of a point P with respect to a circle c , (b) the inverse of a point P with respect to a circle c , and (c) the orthogonal circle passing thru two distinct points P and Q inside a circle c . The following figures from the section are examples of the constructions you may submit.

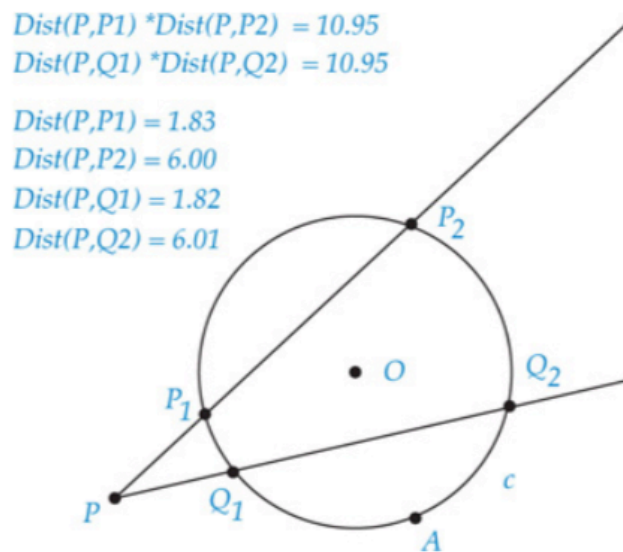


Figure 1. The power of a point P with respect to a circle c

Figure 6.10. DGE task for in-class activity in Week 7

Week	Date	Description
8	03/02/18	Analyze a proof for Exterior Angle Theorem (EAT) in Euclidean and spherical geometry
<p style="text-align: center;">Midterm Take-Home Exam</p> <p>In your Take-home exam, you will investigate a proof of the Exterior Angle Theorem (EAT) following the sequence of problems below that require GEX constructions and Screencast presentation. Consider the following statement of EAT from Section 2.1.</p> <div style="border: 1px solid black; padding: 10px; margin: 10px 0;"> <p>Exterior Angle Theorem (EAT) For a given triangle, if the one of the sides is extended, then the exterior angle produced is greater than either of the two interior and opposite angles.</p> </div> <p>In high school geometry, the “exterior angle theorem” usually means that the exterior angle equals the sum of the two opposite interior angles. Notice that this version is stronger than the one we have above in EAT with respect to the extent to which the theorem can tell us about the properties of exterior angle. In fact, the conclusion in high school version provides more information and implies the conclusion in EAT. In this regard, EAT can be considered a weak version of the high school version.</p> <p>On the other hand, EAT is also considered a primitive version of the theorem in the sense that we can see how it evolves while developing the foundation of Euclidean geometry in our class by introducing additional postulates. We can deduce the stronger version once we introduce all the postulates of Euclidean geometry, but EAT may be the best what we can draw from a given set of postulates. In the following sequence of the problems, we will investigate how a proof of EAT may or may not work in different geometries.</p> <p>Consider the following construction that illustrates a proof of EAT.</p> <div style="border: 1px solid black; padding: 10px; margin: 10px 0;"> <p>For a given triangle $\triangle ABC$, extend the side \overline{AC} and pick an arbitrary point D on the line \overleftrightarrow{AC} such that $A - C - D$. There is the midpoint of the side \overline{BC}. (Let’s call the midpoint E.) On the line \overleftrightarrow{AE}, there is a point F such that $\overline{AE} \cong \overline{EF}$ and $A - E - F$. Then, we have two congruent triangles $\triangle AEB$ and $\triangle FEC$. Then the angle $\angle EBA$ and $\angle ECF$ are congruent. Since $\angle BCD$ is greater than $\angle BCF$, $\angle BCD$ is greater than $\angle CBA$.</p> </div> <p>This construction shows how to prove that the exterior angle $\angle BCD$ is greater than $\angle BAC$. With similar construction with the other side of the triangle, we can also prove that $\angle BCD$ is greater than $\angle CBA$.</p>		

Figure 6.11. DGE task for Midterm Take-Home exam (Intro)

Problem 1 (10pt)

Note that this construction works in Euclidean geometry (even without Euclid's fifth postulate) but does not work in spherical geometry. In this problem, provide two GEX constructions illustrated in the previous page, (a) one in Euclidean geometry and (b) the other in spherical geometry.

Problem 2 (10pt)

While you drag around the three vertices of $\triangle ABC$ in your construction in spherical geometry, you will see in some cases that $\angle BCD$ is greater than $\angle CBA$. In this problem, create a screencast presentation to show such cases and explain "when" the construction works or not (i.e., $\angle BCD$ is greater than $\angle CBA$ or not). You need to be specific in describing when the construction fails in terms of what would be a condition that the construction works or fails.

In sum, you need to submit two robust constructions in Euclidean and spherical geometry (Problem 1(a) and 1(b)), and a screencast presentation explaining when the construction in spherical geometry works or fails (Problem 2).

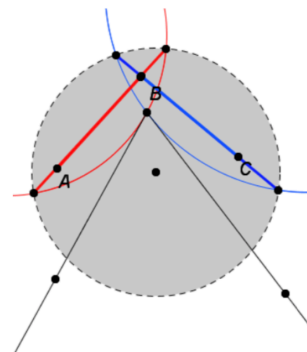
Figure 6.12. DGE task for Midterm Take-Home exam (Problems)

Week	Dates	Description
11	03/26/18 03/28/18	Construction and verification of Klein model of hyperbolic geometry (Task 1 in the task sequence of the study)

Angle measurement in Klein model

3/26 Class Participation

- Open the given GEX template (Klein.gex) including the unit disk and two lines AB and BC .
- Create orthogonal circles for the two lines AB and BC and the angle consisting of the two tangent lines of the chords, which defines the Klein angle $\angle ABC$.



3/26 Class Participation

- Using this GEX construction of Klein model, justify that this model satisfies Postulate 10 and 13 in our system.
- Create and submit screencast video of your group discussion including your justification.

Postulate 10 There exists a function $m: \mathcal{A} \rightarrow R$ where \mathcal{A} is the set of all angles and R is the set of real numbers.

Postulate 11 For every angle $\angle A$ in \mathcal{A} , $m\angle A$ is between 0 and 180.

Postulate 12 (Angle-Construction Postulate)

Let \overrightarrow{AB} be a ray on the edge of a half-plane H . For every number r between 0 and 180, there is exactly one ray \overrightarrow{AP} , with P in H , such that $m\angle PAB = r$.

Postulate 13 (Angle-Addition Postulate)

If D is in the interior of $\angle BAC$, then $m\angle BAC = m\angle BAD + m\angle DAC$

Figure 6.13. DGE task for in-class activity in Week 11 (Task 1 of the sequence on Day 1)

3/28 Class Participation (1)

- Using this GEX construction of Klein model, justify that this model satisfies Postulate 14 and any of the Postulates 10-13 you did not include in your previous work.
- Create screencast video of your group discussion including your justification.
- Based on your discussion, answer the following reflective questions: What do you think this GEX construction activity help you to understand the Klein model and our axiomatic systems?

Submit (a) screencast recording of your discussion, (b) GEX construction, (c) written answer to the reflective questions.

Figure 6.14. DGE task for in-class activity in Week 11 (Task 1 of the sequence on Day 2)

Week	Date	Description
11	03/30/18	Analysis of Legendre's attempted proof of Euclidean Parallel Postulate (EPP) (Task 2 in the task sequence of the study)
<p>3/30 CP (1): Legendre's Proof of EPP</p> <ol style="list-style-type: none"> 1. Open a Klein model of GEX: New - Hyperbolic - Model – Klein Disk <u>(start the screencast recording!)</u> 2. Create a geometric construction described in Legendre's proof (See the handouts), and identify a flaw in the written proof that fails in Klein model. <u>(End of the recording)</u> 3. Write a mathematical statement to describe the hidden assumption in his argument, and explain why this assumption holds in Euclidean but not in hyperbolic geometry. <p>Submit the followings to D2L Dropbox, 3/30 Class Participation (1)</p> <ol style="list-style-type: none"> (a) GEX construction of the proof in Klein model (.gex file), (b) screencast recording of your process (.mp4), and (c) written responses for the step 3 above (.doc). 		

Figure 6.15. DGE task for in-class activity in Week 11 (Task 2 of the sequence)

Week	Due date	Description
11	03/30/18	Homework 4: GEX construction problems on Lecture Note Part 6-7

Homework 4

In this homework, submit your written responses (scanned copy of hand-writing or electronic document) to the following problems. In addition, you need to submit GEX constructions for the first two technology problems.

Problem 1 – GEX Construction required

For an arbitrary triangle given in your GEX canvas of Poincare model, construct another triangle that meet the SSA condition, that means two corresponding sides are congruent and one of the angles not in between the sides are congruent.

Using the above GEX construction, determine if there is SSA (Side-Side-Angle) congruence theorem in our system or not. Justify your response by providing either proof of the theorem or a counter-example of the theorem.

Problem 2 – GEX Construction required

A new high school is to be built to serve four distant local communities A,B,C, and D represented below. In order to minimize transportation cost, it is decided that the school should be located so that the sum of the distances from the communities is as small as possible. Where should the school be located?



First, you need to provide a conjecture of such location for the school by using a GEX construction. For a given arbitrary four points in your GEX canvas of Euclidean plane, provide a formula (Measure feature) of the sum of the distances from the four points (A,B,C,D) and the point of your suggestion.

Then, justify your conjecture by showing that the sum of the distances is smaller than any other case. (Hint: Use the triangle inequality.)

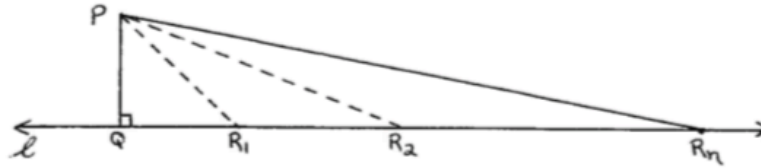
Figure 6.16. DGE task for Homework 4 (Problems 1 and 2)

Problem 3

Provide a formal statement of RHS (Right angle – Hypotenuse – Side) congruence of right triangles and prove the theorem.

Problem 4

Let $\overline{PQ} \perp l$, as in the diagram below. Then given any positive real number a , there is a point R on l such that $m\angle QRP < a$.



In order to prove the above statement, you need to justify each of the following steps. (Hint: You may need results in Part 8, the Archimedean Postulate and the corollary of Theorem 36 in Exercise 8.9 (the second version of the Exterior Angle Theorem).)

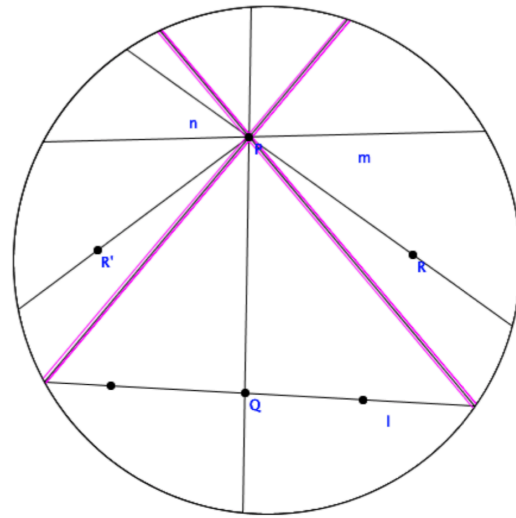
- (a) There exists a point R_1 on the line l such that $\overline{PQ} \cong \overline{QR_1}$.
- (b) $m\angle QR_1P \leq 45$.
- (c) There exists a point R_2 on the line l such that $Q - R_1 - R_2$ and $\overline{PR_1} \cong \overline{R_1R_2}$.
- (d) $m\angle QR_2P \leq \frac{1}{2}45$.
- (e) Continuing this way, there exists R_n on the line l such that $Q - R_{n-1} - R_n$ and $m\angle QR_nP < a$.

Figure 6.17. DGE task for Homework 4 (Problems 3 and 4)

Week	Date	Description
12	04/02/18	Exploration of limiting parallel lines in hyperbolic geometry (Task 3 in the task sequence of the study)

4/2 Class Participation: Limiting Parallels

- In your construction of Legendre's proof, select the line l and the point P .
- Then, you may see the "parallel lines" icon activated in your construct panel of GEX (left-middle). Click it.
- It will give you two particular lines from P that meet the line l at the boundary circle. Those are limiting parallel lines.



4/2 Class Participation: Limiting Parallels

Using your construction, answer the following questions.

- Q1. Are the limiting parallel lines examples or counter-examples of the angle RPR' in the Legendre's proof?
- Q2. Using the limiting parallel lines, how can you separate examples or counter-examples of angle RPR' in the Legendre's proof?
- Q3. How can you describe the limiting parallel lines in the Klein model? Do you think the description is applicable in any models of hyperbolic geometry?

Submit the followings to D2L Dropbox, **4/2 Class Participation (1)**

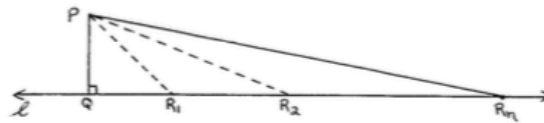
- (a) screencast presentation of your answers to all three questions (.mp4), and (b) written answers to the third question (.doc).

Figure 6.18. DGE task for in-class activity in Week 12 (Task 3 of the sequence)

Week	Date	Description
12	04/04/18	Construction of limiting parallel lines in hyperbolic geometry (Task 4 in the task sequence of the study)
12	04/06/18	
13	04/09/18	

Limiting Parallel Lines in Hyperbolic Geometry

Recall the iterative construction of triangles in Problem 4 of Homework 4 illustrated in the figure below.



For a given point P and a line l with the perpendicular line \overleftrightarrow{PQ} , this construction produced the sequence of triangles, $\Delta QPR_1, \Delta R_1PR_2, \Delta R_2PR_3, \dots, \Delta R_{n-1}PR_n, \dots$, where all the triangles are isosceles triangles. In this activity, we are aiming to produce an existential proof for the limiting parallel lines in hyperbolic geometry, which we observed in one particular model of the geometry, Klein model, in the previous GEX activity.

Observation

1. What did we observe from the triangle ΔQPR_n when we repeated the iteration? What was the claim that we made using this construction in Problem 4 of Homework 4?
2. Now, what can we expect to observe from the line $\overleftrightarrow{PR_n}$ if we repeat this construction? Would the line still intersect the line l for some large natural number n ?
3. How can you explain the relationship between the lines $\overleftrightarrow{PR_n}$ and the limiting parallel lines for the line l at P ?

Think – Include your answer to 4/4 Class Participation (Individual)

4. Do you think you can execute this iterative construction in either Klein model or Poincare model? Explain why.

Figure 6.19. DGE task for in-class activity in Week 12-13 (Task 4 of the sequence, Page 1)

Notice that we don't have a mathematical way to make the sequence of geometric figures (e.g., $\overleftrightarrow{PR_n}$) goes infinity as n goes infinity. Instead we have the rigorous definition for limit of a sequence when the sequence consists of real numbers in real analysis. That means it would be useful if we could identify a sequence of real numbers from the sequence of figures that can be used to *construct* the limiting parallel line.

Make Conjectures

Take a look at the angles $\angle QPR_n$ and there measurements $m\angle QPR_n$.

5. What do you think the sequence, $\{m\angle QPR_n\}_{n=1}^{\infty}$, convergent or divergent? If it converges, to where?

6. Do you think your answer would be changed in different geometries, Euclidean and hyperbolic geometry?

7. If the sequence, $\{m\angle QPR_n\}_{n=1}^{\infty}$, had a limit r for some real number r , do you think we can construct the limiting parallel line we observed in Klein model?

Think – Include your answer to 4/4 Class Participation (Individual)

8. Based on the previous GEX activity on Legendre's proof and the limiting parallel line, what do you think we have to show in order for a line \overleftrightarrow{PR} to be the limiting parallel line for l at P ? (Try to produce a formal statement to answer the question. You will eventually set up a solid statement that you have to prove).

Figure 6.20. DGE task for in-class activity in Week 12-13 (Task 4 of the sequence, Page 2)

Now, we want to show that there exists the limit of the sequence of real numbers, $\{m\angle QPR_n\}_{n=1}^{\infty}$, by using the following theorem.

Monotone convergence theorem

If a monotone increasing (or decreasing) sequence of real numbers is bounded, then the sequence has a finite limit.

Show that this theorem is applicable to the sequence, $\{m\angle QPR_n\}_{n=1}^{\infty}$, meeting the following two conditions included in the statement of the theorem.

Prove

9. The sequence, $\{m\angle QPR_n\}_{n=1}^{\infty}$, is a monotone increasing sequence, which means $m\angle QPR_{n-1} < m\angle QPR_n$ for all natural number n .

10. The sequence, $\{m\angle QPR_n\}_{n=1}^{\infty}$, is bounded, which means that there exists a real number that is greater than $m\angle QPR_n$ for any natural number n .

Note that the angle $\angle QPR$ formed by \overrightarrow{PQ} and \overrightarrow{PR} will be called *angle of parallelism* for l at P , where the ray \overrightarrow{PR} is constructed to produce $m\angle QPR = r$. (See also Exercise 7.2.3 in Section 7.2 of the textbook)

3

Figure 6.21. DGE task for in-class activity in Week 12-13 (Task 4 of the sequence, Page 3)

Now, we want to argue that this ray \overrightarrow{PR} is (a part of) one limiting parallel line with respect to the point P and the line l . That means this line separates the parallel (non-intersecting) and non-parallel (intersecting) lines thru P . Note that we can construct a ray $\overrightarrow{PR'}$ for the other limiting parallel line by constructing the sequence of triangles and corresponding angles on the other half plane of \overrightarrow{PQ} .

Prove

11. For any ray \overrightarrow{PS} where S is in the interior of the angle $\angle RPR'$, \overrightarrow{PS} intersects the line l .

12. For any ray \overrightarrow{PT} where T is not in the interior of the angle $\angle RPR'$, \overrightarrow{PT} does not intersect the line l .

Figure 6.22. DGE task for in-class activity in Week 12-13 (Task 4 of the sequence, Page 4)

Week	Date	Description
13	04/11/18	GEX constructions for theorems in the Lecture Note Part 9
<p>4/11 Class Participation (Group submission)</p> <p>Each group provides</p> <ol style="list-style-type: none"> (1) <u>written proofs</u> of the followings theorems in Euclidean geometry, and (2) <u>GEX constructions</u> for counter examples of the theorems in Poincare model. <p>Group 1 & 3: Thm 37-38 Group 2 & 5: Thm 39-41 Group 4 & 6: Thm 42-43</p>		
<p>Part 9. Parallel Postulates</p> <p>Thm 37 Given two lines and a transversal, if the lines are parallel, then each pair of alternate interior angles are congruent.</p> <p>Thm 38 Given two lines and a transversal, if the lines are parallel, then each pair of corresponding angles are congruent.</p> <p>Thm 39 In any triangle, the sum of the measures of the three angles of the triangle is equal to 180.</p> <p>Thm 40 The acute angles of a right triangle are complementary.</p> <p>Thm 41 (Euclidean Exterior Angle Theorem) For any triangle, the measure of an exterior angle is the sum of the measure of its two remote interior angles.</p> <p>Thm 42 (Transitivity of parallel lines) In a plane, any two lines parallel to a third line are parallel to each other.</p> <p>Thm 43 If a transversal is perpendicular to one of two parallel lines, it is perpendicular to the other.</p>		
<p>4/11 Class Participation (Group submission)</p> <p>Exercise 9.3 Parallelograms in hyperbolic geometry</p> <p>Definition 41. A parallelogram is a quadrilateral in which both pairs of opposite sides are parallel.</p> <p><u>Note:</u> Problem 2 in Homework 5 is about the statements (a) – (e) which may or may not hold in hyperbolic geometry.</p> <ul style="list-style-type: none"> • In your group, provides <u>a proof of the statement (f) in Euclidean geometry</u>. Specify when you are using known properties of parallelograms or properties of parallel lines. • Create <u>a GEX construction of your proof in Klein model</u> to determine if the statement holds in hyperbolic geometry. 		

Figure 6.23. DGE task for in-class activity in Week 13

Week	Due date	Description
13	04/13/18	Homework 5: GEX construction problems for the Lecture Note Part 8-9

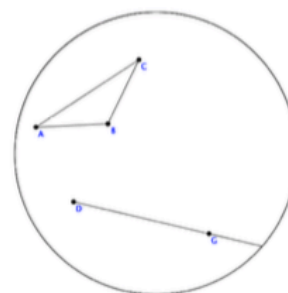
Homework 5

For each problem, read the instructions carefully and submit your responses in the attached list (e.g., GEX constructions, screencast presentations, or written responses).

Problem 1 – SAS (Side-Angle-Side) Postulate in Klein model

Given an arbitrary triangle $\triangle ABC$ and a ray \overrightarrow{DG} of Klein model in the given GEX file (HW5_Prob1.gex available in D2L), create a screencast presentation of constructing a triangle DEF that meets the followings SAS condition, where E is a point on the ray \overrightarrow{DG} .

$$\begin{aligned}\overline{AB} &\cong \overline{DE} \\ \angle BAC &\cong \angle EDF \\ \overline{AC} &\cong \overline{DF}\end{aligned}$$



In your presentation, (1) show how to construct the triangle $\triangle DEF$ that meets the SAS condition stated above and (2) show that $\triangle ABC \cong \triangle DEF$ by showing the other corresponding sides and angles are also congruent each other (i.e., $\overline{BC} \cong \overline{EF}$, $\angle CBA \cong \angle FED$, $\angle ACB \cong \angle DFE$). You will be able to move around any vertices of the triangle A, B, C and the points D and G of the ray \overrightarrow{DG} . At the end of your presentation, (3) show an example of the two congruent triangles that looks strange to you in the sense of congruent triangles in Euclidean plane and explain why.

- Submit your screencast presentation explaining (1)-(3) (.mp4) and the GEX construction (.gex).

Note

In this problem, you will learn how to rotate a figure for a given center of the rotation and a degree. To copy the angle $\angle BAC$ to the ray \overrightarrow{DG} , you can rotate the ray around the point D by $m\angle BAC$.

First, select the point D and go to Transform panel (lower left in your window) – Mark – Center. Then, select the angle $\angle BAC$ by clicking the points B, A , and C in this order (the counter-clockwise orientation) and go to Transform panel – Mark – Angle. Now you are all set up to get things rotated. Select the ray \overrightarrow{DG} and click the rotation icon activated in the Transform panel.

Figure 6.24. DGE task for Homework 5 (Page 1)

Problem 2 – Parallelograms in Hyperbolic Geometry

In Exercise 9.3 of your proof journal, you are asked to determine which of the known properties of parallelograms in Euclidean geometry are also true in hyperbolic geometry (See the properties (a) – (e) in Exercise 9.3 and ignore the last property-(f)). Note that we are given a definition of parallelograms as follows.

Definition 41. A parallelogram is a quadrilateral in which both pairs of opposite sides are parallel.

For each property you think valid only in Euclidean geometry, you need to provide a GEX construction of quadrilaterals in Klein model that violate the property. In your presentation, (1) show completed constructions of quadrilaterals in Klein model that are counter-examples of each property, and (2) specifically explain how the quadrilaterals violate the properties using relevant measurements.

Then, (3) submit written proofs of the properties and specify where you have to use the Euclidean Parallel Postulate. In addition, (4) submit written proof of the properties that you cannot find any counter-examples in Klein model. You cannot use any parallel postulate (or equivalent) here.

- Submit your screencast presentation explaining (1) and (2) (.mp4), the GEX construction (.gex), and a document of your written response to (3) and (4) (.doc).

Problem 3 – Construction of Poincare Model in Euclidean Plane

In this problem, you need to construct an angle and the angle measurement in a Poincare disk given in the GEX file (HW5_Prob3.gex) available on D2L. For given three non-collinear points A , B , and C in the file, you have to construct (1) two Poincare lines (orthogonal circles) from A to B and C , respectively. You may use the constructions 4.14-15 in Section 4.1 of the textbook to create the orthogonal circle passing thru arbitrary two points not on the boundary of the disk. Then, construct (2) two tangent lines at the intersection between the two orthogonal circles that define (3) $m\angle BAC$ in Poincare model.

- Submit the GEX construction (.gex) including (1), (2), and (3).

Example

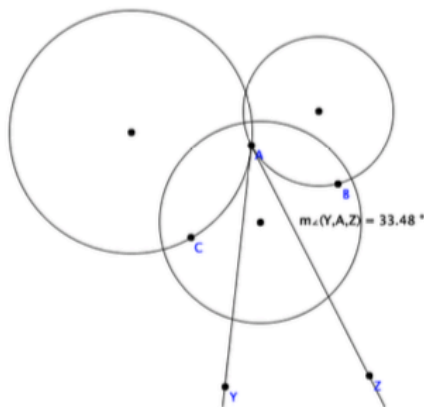


Figure 6.25. DGE task for Homework 5 (Page 2)

Week	Date	Description
14	04/20/18	Construct equilateral triangle in elliptic model and explore the range of angle sum of a triangle

Models of Elliptic Geometry

Section 8.3.3 Single Elliptic Geometry (Model)

4/20 Class Participation - Group

- Construct an equilateral triangle in the Sphere X-Y projection of GEX. (Hint: Use the Euclid's construction with two circles).
- Then submit your group's written responses to the following questions with the GEX construction being used.

Q1. Find the upper bound of the internal angles of the triangle.

Q2. What conjecture would you make as to the side length (in relation to the polar length) when the angles are right angles?

Figure 6.26. DGE task for in-class activity in Week 14

Week	Date	Description
15	04/25/18	Construct three reflections to transform a triangle to another

4/25 Class Participation (Individual)

Construct the three reflections that take $\triangle ABC$ to $\triangle PQR$ in the given GEX construction. (See the attached file, Exer5_2_4.gex.)

Figure 6.27. DGE task for in-class activity in Week 15

Week	Due date	Description
15	04/27/18	GEX construction problems for non-Euclidean geometry

Homework 6

Problem 1 – GEX required

In this problem, you will see that the definition of perpendicular lines in Klein model given in the textbook (Definition 7.6) is consistent with the angle measurement using orthogonal circles.

GEX construction: In your construction of Klein model in Euclidean plane (you may use the construction we created in class), create two perpendicular lines l and m in the model such that each line passes through the pole of the other line. Then, measure the angle between the two lines in Klein model by creating orthogonal circles and tangent lines from the chords. (The angle should be the right angle!)

Screencast presentation: Use the previous construction, create a Screencast presentation to justify the following statement.

“For a given chord and it’s pole in Klein model, an arbitrary chord passing through the pole (if extended outside the Klein disk) should meet the given chord at right angle.”

Problem 2 – GEX required

In this problem, you will explore common perpendicular lines of given two parallel lines in Klein model for two cases of parallels: (a) limiting parallels and (b) ultraparallels. You may use the concept of poles and how the author of the textbook define perpendicular lines with the concept.

Screencast presentation: Start with the given construction of Klein model in Euclidean plane with two arbitrary lines l and m (See the attached .gex file). Then, answer the following questions using the Klein model.

1. Justify that there is no common perpendicular line to two limiting parallel lines.
2. Justify that there is unique common perpendicular line to two ultraparallel lines. Use the concept of poles to show how to create the unique perpendicular.

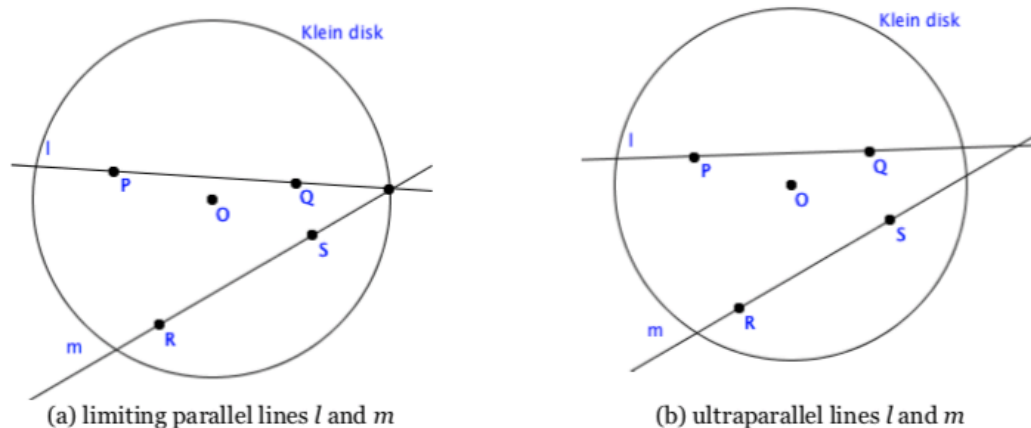


Figure 6.28. DGE task for Homework 6

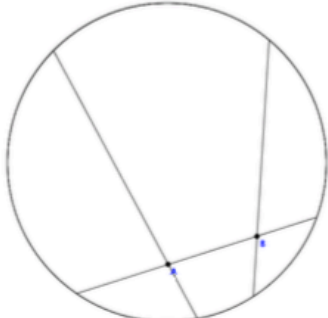
Week	Date	Description
Final		Problems in the final exam that involve DGE tasks
<p>Problem 1 (10pts)</p> <p>Younggon created a GEX screencast presentation to show that for any two parallel lines in Klein model there exists a common perpendicular line to the two parallel lines. Watch the presentation video (Final_Prob1.mp4) and answer the following questions.</p> <p>(1) <u>Explain why the construction is not appropriate to support his claim. In your answer, specify the generic elements (i.e., dimensions of variation) that are independent variables of this construction.</u></p> <p>(2) Rather it seems his construction implies another statement about perpendicular and parallel lines. <u>What would be a mathematical statement that he could have justified with using this construction?</u></p>		
		
		1

Figure 6.29. DGE task for Final exam (Problem 1)

Problem 3 (10pts) – GEX submission required

In Klein model, there is a special case of hyperbolic lines (i.e., chords inside the disk) of which we cannot construct orthogonal circles to measure angles. In this case, we are using the chord itself instead of tangent lines at orthogonal circles.

Using the attached .gex file of Klein disk on Euclidean plane, create an example of such lines that have no orthogonal circles and more than two perpendicular lines to this special line with the angle measurements. Then, answer the following questions. (Submit your .gex file to D2L Dropbox)

(1) Describe the special case of lines.

(2) Describe the perpendicular lines to the special line.

(3) Does this special line have a pole? Justify your answer.

Figure 6.30. DGE task for Final exam (Problem 3)

APPENDIX C: Analytic Framework of Different Dragging Strategies (Bae, 2017)

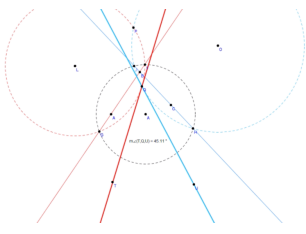
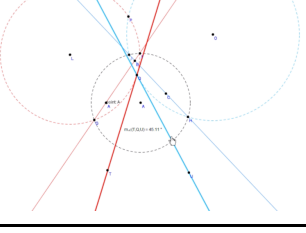
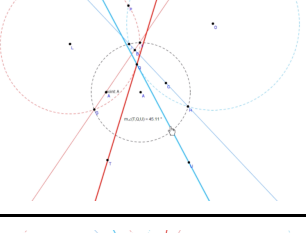
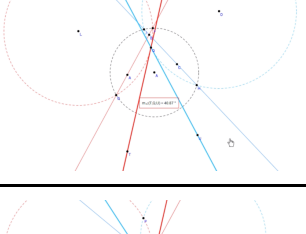
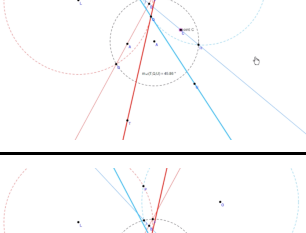
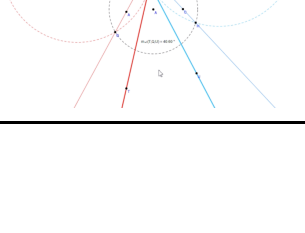
The followings are multiple dragging strategies with different purposes and the ways of dragging figures identified in Bae (2017). This framework was based on the analytic framework of variation in drag mode of DGE (Leung, 2008) which had been influenced by the original work on multi-modalities of drag feature of DGE done by Arzarello and colleagues (Arzarello et al., 1998, 2002). The dragging strategies include wandering dragging (WDR), guided dragging (GDR), putting-back dragging (BDR), inappropriate dragging (IDR), and non-dragging (NDR). Note that I adopted WDR and GDR for my dissertation study and created a new code, *dragging by accident*, mostly to integrate the two codes IDR and NDR.

- Wandering dragging means to drag around all the varying components of the constructions without strategic exploration, but survey overall changes of the construction or measurement.
- Guided dragging means when geometrical shape or numerical measurement guide students to determine their strategic dragging activities.
- Putting-back dragging means to drag a figure to put it back to its original place as given in the task before the exploration.
- Inappropriate dragging means to drag a component that result in an unintended variation of the construction.
- Non-dragging means an attempt to drag a non-draggable component in the construction.

Figure 6.31. Analytic framework of different dragging strategies

APPENDIX D: Example of the Analysis of Student Uses of GEX and Transcripts

Table 6.2. Example of the analysis of student uses of GEX and transcripts

Time	Screen shot	Transcript	DGE screen Description	DGE features	DGE moves: Purpose
00:00		E2: Okay, this is the Klein construction. We're going to show how postulate 10 through 13 hold.	The construction contains the two given lines AB and BC with the corresponding orthogonal circles and tangent lines. The corresponding diagram for each given line is colored same (red and blue)		
00:07		E2: (cont'd) So first, postulate 10, as you can see, we have A, B, and C right here.	Moved the cursor over the points A, B, C to indicate the diagram of the angle ABC.	Move the cursor on diagrams	Point a diagram: To indicate the angle ABC.
00:13		E2: So, if we move them around	Selected the point A and dragged it around nearby: The corresponding orthogonal circle, tangent line, and the measurement changed.	Drag a diagram	Wandering dragging: To show how construction changes and so as the measurement
00:16		E2: (cont'd) we get measured angle out	Dragged the measurement mTQU slightly down	Drag a measurement	Point a measurement: To indicate the measurement mTQU
00:18		E2: (cont'd) for any angle ??, we constructed, and	Selected the point B and dragged it around nearby: The corresponding orthogonal circle, tangent line, and the measurement changed.	Drag a diagram	Wandering dragging: To show how construction changes and so as the measurement
00:21		E2: (cont'd) we constructed these angles by making	Moved the cursor between two lines TQ and UQ	Move the cursor on diagrams	To indicate a diagram of angle TQU

APPENDIX E: DGE Feature and DGE Move Code Applications

Table 6.3. DGE feature and DGE move code applications

DGE-F-Id	Task	Day	Pair	Time stamp	Duration	Feature	Move
DGE-F-001	1	1	A1_K1	0:15	0:04	Move the cursor over diagrams	Point at a measurement
DGE-F-002	1	1	A1_K1	00:19	0:05	Move the cursor over diagrams	Point at a diagram
DGE-F-003	1	1	A1_K1	00:24	0:06	Move the cursor over diagrams	Point at a measurement
DGE-F-004	1	1	A1_K1	00:30	0:08	Move the cursor over diagrams	Point at a diagram
DGE-F-005	1	1	A1_K1	00:38	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-006	1	1	A1_K1	00:41	0:03	Move the cursor over diagrams	Point at a computation
DGE-F-007	1	1	A1_K1	00:44	0:07	Move the cursor over diagrams	Point at a diagram
DGE-F-008	1	1	A1_K1	00:51	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-009	1	1	A1_K1	00:53	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-010	1	1	B1_E2_J1	0:07	0:06	Move the cursor over diagrams	Point at a diagram
DGE-F-011	1	1	B1_E2_J1	0:13	0:03	Drag a diagram	Wandering dragging
DGE-F-012	1	1	B1_E2_J1	0:16	0:02	Drag a measurement	Point at a measurement
DGE-F-013	1	1	B1_E2_J1	0:18	0:03	Drag a diagram	Wandering dragging
DGE-F-014	1	1	B1_E2_J1	0:21	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-015	1	1	B1_E2_J1	0:24	0:02	Drag a diagram	Wandering dragging
DGE-F-016	1	1	B1_E2_J1	0:26	0:01	Drag a diagram	Not identified
DGE-F-017	1	1	B1_E2_J1	0:27	0:02	Drag a diagram	Wandering dragging
DGE-F-018	1	1	B1_E2_J1	0:29	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-019	1	1	B1_E2_J1	0:33	0:01	Select a diagram	Point at a diagram
DGE-F-020	1	1	B1_E2_J1	0:34	0:03	Drag a diagram	Wandering dragging
DGE-F-021	1	1	B1_E2_J1	0:37	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-022	1	1	B1_E2_J1	0:41	0:01	Drag a diagram	Not identified
DGE-F-023	1	1	B1_E2_J1	0:45	0:05	Drag a diagram	Guided dragging
DGE-F-024	1	1	B1_E2_J1	0:50	0:03	Drag a diagram	Guided dragging
DGE-F-025	1	1	B1_E2_J1	0:53	0:02	Drag a diagram	Guided dragging
DGE-F-026	1	1	B1_E2_J1	0:55	0:02	Drag a diagram	Guided dragging
DGE-F-027	1	1	B1_E2_J1	0:57	0:01	Drag a diagram	Place a diagram
DGE-F-028	1	1	B1_E2_J1	0:58	0:13	Drag a diagram	Guided dragging
DGE-F-029	1	1	B1_E2_J1	1:11	0:06	Move the cursor over diagrams	Not identified
DGE-F-030	1	1	B1_E2_J1	1:17	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-031	1	1	B1_E2_J1	1:18	0:01	Drag a diagram	Point at a diagram
DGE-F-032	1	1	B1_E2_J1	1:19	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-033	1	1	B1_E2_J1	1:22	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-034	1	1	B1_E2_J1	1:25	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-035	1	1	B1_E2_J1	1:28	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-036	1	1	B1_E2_J1	1:30	0:05	Drag a diagram	Place a diagram
DGE-F-037	1	1	B1_E2_J1	1:35	0:02	Drag a diagram	Place a diagram
DGE-F-038	1	1	B1_E2_J1	1:37	0:01	Drag a diagram	Place a diagram
DGE-F-039	1	1	B1_E2_J1	1:38	0:02	Drag a diagram	Place a diagram
DGE-F-040	1	1	B1_E2_J1	1:40	0:03	Drag a diagram	Guided dragging
DGE-F-041	1	1	B1_E2_J1	1:43	0:01	Drag a diagram	Guided dragging
DGE-F-042	1	1	B1_E2_J1	1:44	0:07	Drag a diagram	Guided Dragging
DGE-F-043	1	1	B1_E2_J1	1:51	0:03	Drag a diagram	Guided dragging
DGE-F-044	1	1	B1_E2_J1	1:54	0:01	Drag a diagram	Guided dragging
DGE-F-045	1	1	B1_E2_J1	1:55	0:05	Drag a diagram	Guided dragging
DGE-F-046	1	1	B1_E2_J1	2:00	0:02	Drag a diagram	Guided dragging
DGE-F-047	1	1	B1_E2_J1	2:02	0:03	Drag a diagram	Guided dragging
DGE-F-048	1	1	B1_E2_J1	2:05	0:04	Drag a label	Dragging by accident
DGE-F-049	1	1	B1_E2_J1	2:09	0:02	Drag a diagram	Guided dragging
DGE-F-050	1	1	B1_E2_J1	2:11	0:04	Drag a diagram	Guided dragging
DGE-F-051	1	1	B1_E2_J1	2:35	0:06	Select a diagram	Point at a diagram
DGE-F-052	1	1	B1_E2_J1	2:41	0:05	Move the cursor over diagrams	Point at a diagram
DGE-F-053	1	1	B1_E2_J1	2:46	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-054	1	1	B1_E2_J1	2:51	0:09	Move the cursor over diagrams	Not identified

Table 6.3 (cont'd). DGE feature and DGE move code applications

DGE-F-055	1	1	B1_E2_J1	3:00	0:03	Select a diagram	Point at a diagram
DGE-F-056	1	1	B1_E2_J1	3:03	0:03	Select a diagram	Point at a diagram
DGE-F-057	1	1	B1_E2_J1	3:06	0:02	Select a diagram	Point at a diagram
DGE-F-058	1	1	B1_E2_J1	3:08	0:02	Select a diagram	Point at a diagram
DGE-F-059	1	1	B1_E2_J1	3:10	0:09	Move the cursor over diagrams	Not identified
DGE-F-060	1	1	B1_E2_J1	3:19	0:04	Select a diagram	Point at a diagram
DGE-F-061	1	1	B1_E2_J1	3:23	0:03	Move the cursor over diagrams	Point at a measurement
DGE-F-062	1	1	B1_E2_J1	3:28	0:02	Select a diagram	Point at a diagram
DGE-F-063	1	1	B1_E2_J1	3:30	0:07	Move the cursor over diagrams	Point at a diagram
DGE-F-064	1	1	B1_E2_J1	3:37	0:02	Select a diagram	Point at a diagram
DGE-F-065	1	1	B1_E2_J1	3:39	0:02	Select a diagram	Point at a measurement
DGE-F-066	1	1	B1_E2_J1	3:41	0:06	Move the cursor over diagrams	Point at a diagram
DGE-F-067	1	1	B1_E2_J1	3:47	0:02	Select a diagram	Point at a diagram
DGE-F-068	1	1	B1_E2_J1	3:49	0:01	Select a diagram	Point at a measurement
DGE-F-069	1	1	B1_E2_J1	3:50	0:02	Select a diagram	Point at a computation
DGE-F-070	1	1	B1_E2_J1	3:52	0:01	Select a diagram	Point at a computation
DGE-F-071	1	1	B1_E2_J1	3:53	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-072	1	1	B2_D1_T1	0:06	0:14	Drag a diagram	Wandering dragging
DGE-F-073	1	1	B2_D1_T1	0:20	0:05	Move the cursor over diagrams	Point at a diagram
DGE-F-074	1	1	B2_D1_T1	0:25	0:05	Move the cursor over diagrams	Point at a measurement
DGE-F-075	1	1	B2_D1_T1	0:30	0:08	Drag a diagram	Wandering dragging
DGE-F-076	1	1	E1_L2	0:16	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-077	1	1	E1_L2	0:20	0:03	Move the cursor over diagrams	Point at a measurement
DGE-F-078	1	1	E1_L2	0:23	0:10	Drag a diagram	Wandering dragging
DGE-F-079	1	1	E1_L2	0:41	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-080	1	1	E1_L2	0:50	0:07	Move the cursor over diagrams	Point at a measurement
DGE-F-081	1	1	E1_L2	0:57	0:05	Move the cursor over diagrams	Point at a measurement
DGE-F-082	1	1	E1_L2	1:05	0:09	Drag a diagram	Wandering dragging
DGE-F-083	1	1	N1_O1	0:03	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-084	1	1	N1_O1	0:05	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-085	1	1	N1_O1	0:09	0:03	Move the cursor over diagrams	Point at a measurement
DGE-F-086	1	1	N1_O1	0:12	0:05	Move the cursor over diagrams	Not identified
DGE-F-087	1	1	N1_O1	0:23	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-088	1	1	N1_O1	0:26	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-089	1	1	N1_O1	0:30	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-090	1	1	N1_O1	0:32	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-091	1	1	N1_O1	0:34	0:02	Move the cursor over diagrams	Point at a measurement
DGE-F-092	1	1	N1_O1	0:38	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-093	1	1	N1_O1	0:41	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-094	1	1	N1_O1	0:43	0:08	Move the cursor over diagrams	Point at a measurement
DGE-F-095	1	1	N1_O1	0:51	0:06	Move the cursor over diagrams	Point at a diagram
DGE-F-096	1	1	N1_O1	0:57	0:05	Move the cursor over diagrams	Point at a measurement
DGE-F-097	1	1	N1_O1	1:02	0:02	Drag a diagram	Wandering dragging
DGE-F-098	1	1	N1_O1	1:04	0:04	Drag a diagram	Wandering dragging
DGE-F-099	1	1	Y2_Y3	0:00	0:07	Move the cursor over diagrams	Point at a diagram
DGE-F-100	1	1	Y2_Y3	0:07	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-101	1	1	Y2_Y3	0:09	0:01	Drag a diagram	Wandering dragging
DGE-F-102	1	1	Y2_Y3	0:12	0:04	Drag a diagram	Wandering dragging
DGE-F-103	1	1	Y2_Y3	0:16	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-104	1	1	Y2_Y3	0:17	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-105	1	1	Y2_Y3	0:19	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-106	1	1	Y2_Y3	0:24	0:03	Move the cursor over diagrams	Not identified
DGE-F-107	1	1	Y2_Y3	0:27	0:03	Drag a diagram	Wandering dragging
DGE-F-108	1	1	Y2_Y3	0:30	0:01	Drag a diagram	Wandering dragging
DGE-F-109	1	1	Y2_Y3	0:31	0:01	Move the cursor over diagrams	Wandering dragging
DGE-F-110	1	1	Y2_Y3	0:32	0:03	Drag a diagram	Wandering dragging
DGE-F-111	1	1	Y2_Y3	0:35	0:11	Drag a diagram	Wandering dragging
DGE-F-112	1	1	Y2_Y3	0:46	0:01	Move the cursor over diagrams	Not identified
DGE-F-113	1	1	Y2_Y3	0:56	0:05	Drag a diagram	Wandering dragging
DGE-F-114	1	1	Y2_Y3	1:01	0:11	Drag a diagram	Wandering dragging

Table 6.3 (cont'd). DGE feature and DGE move code applications

DGE-F-115	1	1	Y2_Y3	1:20	0:02	Drag a diagram	Wandering dragging
DGE-F-116	1	1	Y2_Y3	1:22	0:02	Drag a diagram	Wandering dragging
DGE-F-117	1	1	Y2_Y3	1:24	0:07	Create a diagram	Create a diagram
DGE-F-118	1	1	Y2_Y3	1:31	0:04	Create a diagram	Create a diagram
DGE-F-119	1	1	Y2_Y3	1:35	0:02	Move the cursor over diagrams	Measure a diagram
DGE-F-120	1	1	Y2_Y3	1:37	0:05	Select a diagram	Measure a diagram
DGE-F-121	1	1	Y2_Y3	1:42	0:01	De-select diagrams	Measure a diagram
DGE-F-122	1	1	Y2_Y3	1:45	0:01	Select a diagram	Measure a diagram
DGE-F-123	1	1	Y2_Y3	1:46	0:01	De-select diagrams	Measure a diagram
DGE-F-124	1	1	Y2_Y3	1:47	0:07	Select a diagram	Measure a diagram
DGE-F-125	1	1	Y2_Y3	1:54	0:01	Measure a diagram	Measure a diagram
DGE-F-126	1	1	Y2_Y3	1:55	0:02	Drag a measurement	Fit diagrams to the screen
DGE-F-127	1	1	Y2_Y3	1:57	0:03	Drag a measurement	Fit diagrams to the screen
DGE-F-128	1	1	Y2_Y3	2:00	0:03	Scroll the screen	Fit diagrams to the screen
DGE-F-129	1	1	Y2_Y3	2:03	0:01	De-select diagrams	Measure a diagram
DGE-F-130	1	1	Y2_Y3	2:04	0:06	Select a diagram	Measure a diagram
DGE-F-131	1	1	Y2_Y3	2:10	0:03	Measure a diagram	Measure a diagram
DGE-F-132	1	1	Y2_Y3	2:13	0:01	Drag a measurement	Fit diagrams to the screen
DGE-F-133	1	1	Y2_Y3	2:14	0:02	Drag a measurement	Fit diagrams to the screen
DGE-F-134	1	1	Y2_Y3	2:19	0:03	Drag a diagram	Not identified
DGE-F-135	1	1	Y2_Y3	2:22	0:03	Move the cursor over diagrams	Not identified
DGE-F-136	1	1	Y2_Y3	2:25	0:03	Move the cursor over diagrams	Not identified
DGE-F-137	1	1	Y2_Y3	2:28	0:01	Select a diagram	Not identified
DGE-F-138	1	1	Y2_Y3	2:29	0:02	Drag a diagram	Wandering dragging
DGE-F-139	1	1	Y2_Y3	2:31	0:01	Move the cursor over diagrams	Wandering dragging
DGE-F-140	1	1	Y2_Y3	2:32	0:08	Drag a diagram	Wandering dragging
DGE-F-141	1	2	A1_K1	00:10	0:05	Move the cursor over diagrams	Point at a diagram
DGE-F-142	1	2	A1_K1	0:15	0:03	Drag a diagram	Guided dragging
DGE-F-143	1	2	A1_K1	00:18	0:04	Drag a diagram	Guided dragging
DGE-F-144	1	2	A1_K1	00:30	0:05	Drag a diagram	Wandering dragging
DGE-F-145	1	2	A1_K1	00:35	0:01	Drag a diagram	Guided dragging
DGE-F-146	1	2	A1_K1	00:36	0:05	Drag a diagram	Guided dragging
DGE-F-147	1	2	A1_K1	00:41	0:05	Drag a diagram	Guided dragging
DGE-F-148	1	2	A1_K1	00:46	0:02	Drag a diagram	Guided dragging
DGE-F-149	1	2	A1_K1	00:58	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-150	1	2	A1_K1	00:59	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-151	1	2	A1_K1	01:04	0:02	Move the cursor over diagrams	Point at a measurement
DGE-F-152	1	2	A1_K1	01:06	0:02	Drag a diagram	Wandering dragging
DGE-F-153	1	2	A1_K1	01:08	0:03	Drag a diagram	Wandering dragging
DGE-F-154	1	2	A1_K1	01:11	0:04	Drag a diagram	Guided dragging
DGE-F-155	1	2	B1_E2_J1	0:05	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-156	1	2	B1_E2_J1	0:08	0:04	Move the cursor over diagrams	Not identified
DGE-F-157	1	2	B1_E2_J1	0:12	0:05	Move the cursor over diagrams	Point at a diagram
DGE-F-158	1	2	B1_E2_J1	0:17	0:01	Move the cursor over diagrams	Point at a measurement
DGE-F-159	1	2	B1_E2_J1	0:18	0:02	Drag a measurement	Point at a computation
DGE-F-160	1	2	B1_E2_J1	0:20	0:02	Move the cursor over diagrams	Not identified
DGE-F-161	1	2	B1_E2_J1	0:22	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-162	1	2	B1_E2_J1	0:26	0:05	Move the cursor over diagrams	Not identified
DGE-F-163	1	2	B1_E2_J1	0:31	0:02	Drag a diagram	Dragging by accident
DGE-F-164	1	2	B1_E2_J1	0:33	0:06	Drag a diagram	Wandering dragging
DGE-F-165	1	2	B1_E2_J1	0:39	0:03	Drag a diagram	Guided dragging
DGE-F-166	1	2	B1_E2_J1	0:42	0:08	Drag a diagram	Wandering dragging
DGE-F-167	1	2	B1_E2_J1	0:50	0:06	Move the cursor over diagrams	Not identified
DGE-F-168	1	2	B1_E2_J1	0:56	0:04	Drag a diagram	Not identified
DGE-F-169	1	2	B2_T1	0:09	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-170	1	2	B2_T1	0:13	0:01	Move the cursor over diagrams	Not identified
DGE-F-171	1	2	B2_T1	0:23	0:01	Move the cursor over diagrams	Point at a measurement
DGE-F-172	1	2	B2_T1	0:24	0:06	Move the cursor over diagrams	Not identified
DGE-F-173	1	2	B2_T1	0:30	0:09	Drag a diagram	Wandering dragging
DGE-F-174	1	2	D1_J2	0:11	0:02	Move the cursor over diagrams	Point at a diagram

Table 6.3 (cont'd). DGE feature and DGE move code applications

DGE-F-175	1	2	D1_J2	0:22	0:05	Move the cursor over diagrams	Point at a diagram
DGE-F-176	1	2	D1_J2	0:27	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-177	1	2	D1_J2	0:29	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-178	1	2	D1_J2	0:37	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-179	1	2	D1_J2	0:41	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-180	1	2	D1_J2	0:44	0:01	Move the cursor over diagrams	Point at a measurement
DGE-F-181	1	2	D1_J2	0:47	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-182	1	2	D1_J2	0:54	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-183	1	2	D1_J2	0:57	0:03	Move the cursor over diagrams	Point at a measurement
DGE-F-184	1	2	D1_J2	1:03	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-185	1	2	D1_J2	1:05	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-186	1	2	D1_J2	1:07	0:03	Move the cursor over diagrams	Point at a measurement
DGE-F-187	1	2	D1_J2	1:13	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-188	1	2	D2_Y1	0:00	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-189	1	2	D2_Y1	0:03	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-190	1	2	D2_Y1	0:05	0:01	Select a diagram	Point at a diagram
DGE-F-191	1	2	D2_Y1	0:06	0:01	Select a diagram	Point at a diagram
DGE-F-192	1	2	D2_Y1	0:07	0:01	De-select diagrams	Point at a diagram
DGE-F-193	1	2	D2_Y1	0:08	0:07	Move the cursor over diagrams	Point at a diagram
DGE-F-194	1	2	D2_Y1	0:15	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-195	1	2	D2_Y1	0:16	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-196	1	2	D2_Y1	0:17	0:09	Move the cursor over diagrams	Not identified
DGE-F-197	1	2	D2_Y1	0:26	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-198	1	2	D2_Y1	0:27	0:06	Move the cursor over diagrams	Point at a measurement
DGE-F-199	1	2	D2_Y1	0:33	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-200	1	2	D2_Y1	0:35	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-201	1	2	E1_L2	0:10	0:19	Drag a diagram	Wandering dragging
DGE-F-202	1	2	E1_L2	0:40	0:21	Drag a diagram	Wandering dragging
DGE-F-203	1	2	E1_L2	1:17	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-204	1	2	E1_L2	1:21	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-205	1	2	E1_L2	1:25	0:03	Move the cursor over diagrams	Not identified
DGE-F-206	1	2	E1_L2	1:28	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-207	1	2	E1_L2	1:30	0:07	Move the cursor over diagrams	Point at a measurement
DGE-F-208	1	2	E1_L2	1:37	0:21	Drag a diagram	Wandering dragging
DGE-F-209	1	2	N1_O1	0:00	0:07	Select a diagram	Point at a diagram
DGE-F-210	1	2	N1_O1	0:07	0:06	Drag a diagram	Wandering dragging
DGE-F-211	1	2	N1_O1	0:13	0:04	Drag a diagram	Guided dragging
DGE-F-212	1	2	N1_O1	0:28	0:04	Move the cursor over diagrams	Not identified
DGE-F-213	1	2	N1_O1	0:32	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-214	1	2	N1_O1	0:34	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-215	1	2	N1_O1	0:37	0:03	Drag a diagram	Dragging by accident
DGE-F-216	1	2	N1_O1	0:47	0:13	Drag a diagram	Wandering dragging
DGE-F-217	1	2	N1_O1	1:09	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-218	1	2	N1_O1	1:12	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-219	1	2	N1_O1	1:16	0:12	Move the cursor over diagrams	Point at a diagram
DGE-F-220	1	2	N1_O1	1:28	0:05	Move the cursor over diagrams	Point at a measurement
DGE-F-221	1	2	N1_O1	1:33	0:06	Drag a diagram	Wandering dragging
DGE-F-222	1	2	N1_O1	1:39	0:02	Move the cursor over diagrams	Point at a measurement
DGE-F-223	1	2	N1_O1	1:41	0:12	Drag a diagram	Wandering dragging
DGE-F-224	1	2	Y2_Y3	0:03	0:05	Move the cursor over diagrams	Not identified
DGE-F-225	1	2	Y2_Y3	0:08	0:06	Move the cursor over diagrams	Not identified
DGE-F-226	1	2	Y2_Y3	0:15	0:10	Drag a diagram	Not identified
DGE-F-227	1	2	Y2_Y3	0:25	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-228	1	2	Y2_Y3	0:27	0:07	Rescale the screen	Fit diagrams to the screen
DGE-F-229	1	2	Y2_Y3	0:34	0:00	Select a diagram	Point at a diagram
DGE-F-230	1	2	Y2_Y3	0:34	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-231	1	2	Y2_Y3	0:36	0:07	Label a diagram	Label a diagram
DGE-F-232	1	2	Y2_Y3	0:43	0:04	Select a measurement	Point at a measurement
DGE-F-233	1	2	Y2_Y3	0:47	0:05	Drag a diagram	Guided dragging
DGE-F-234	1	2	Y2_Y3	0:55	0:06	Drag a diagram	Guided dragging

Table 6.3 (cont'd). DGE feature and DGE move code applications

DGE-F-235	1	2	Y2_Y3	1:01	0:05	Move the cursor over diagrams	Point at a diagram
DGE-F-236	1	2	Y2_Y3	1:06	0:03	Drag a diagram	Guided dragging
DGE-F-237	1	2	Y2_Y3	1:09	0:04	Select a measurement	Point at a measurement
DGE-F-238	1	2	Y2_Y3	1:13	0:05	Drag a diagram	Guided dragging
DGE-F-239	1	2	Y2_Y3	1:18	0:03	Drag a diagram	Guided dragging
DGE-F-240	1	2	Y2_Y3	1:21	0:02	Select a measurement	Point at a measurement
DGE-F-241	1	2	Y2_Y3	1:23	0:09	Drag a diagram	Guided dragging
DGE-F-242	1	2	Y2_Y3	1:32	0:02	Drag a diagram	Place a diagram
DGE-F-243	1	2	Y2_Y3	1:34	0:03	Select a diagram	Point at a diagram
DGE-F-244	1	2	Y2_Y3	1:38	0:05	Drag a diagram	Wandering dragging
DGE-F-245	1	2	Y2_Y3	1:43	0:01	Select a measurement	Point at a measurement
DGE-F-246	1	2	Y2_Y3	1:44	0:12	Drag a diagram	Wandering dragging
DGE-F-247	1	2	Y2_Y3	1:56	0:09	Move the cursor over diagrams	Not identified
DGE-F-248	1	2	Y2_Y3	2:05	0:04	Move the cursor over diagrams	Not identified
DGE-F-249	1	2	Y2_Y3	2:09	0:07	Select a diagram	Point at a diagram
DGE-F-250	1	2	Y2_Y3	2:16	0:10	Rescale the screen	Fit diagrams to the screen
DGE-F-251	1	2	Y2_Y3	2:26	0:02	Select a diagram	Point at a diagram
DGE-F-252	1	2	Y2_Y3	2:28	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-253	1	2	Y2_Y3	2:29	0:03	Select a diagram	Point at a diagram
DGE-F-254	1	2	Y2_Y3	2:33	0:06	Move the cursor over diagrams	Point at a diagram
DGE-F-255	1	2	Y2_Y3	2:39	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-256	1	2	Y2_Y3	2:42	0:04	Select a diagram	Point at a diagram
DGE-F-257	1	2	Y2_Y3	2:53	0:04	Hide a diagram	Label a diagram
DGE-F-258	1	2	Y2_Y3	2:57	0:04	Select a diagram	Label a diagram
DGE-F-259	1	2	Y2_Y3	3:01	0:13	Un-hide a diagram	Label a diagram
DGE-F-260	1	2	Y2_Y3	3:14	0:04	Label a diagram	Label a diagram
DGE-F-261	1	2	Y2_Y3	3:18	0:04	Select a diagram	Point at a diagram
DGE-F-262	1	2	Y2_Y3	3:22	0:06	Create a diagram	Create a diagram
DGE-F-263	1	2	Y2_Y3	3:28	0:03	Create a diagram	Create a diagram
DGE-F-264	1	2	Y2_Y3	3:31	0:02	Undo	Undo last steps
DGE-F-265	1	2	Y2_Y3	3:33	0:01	Create a diagram	Create a diagram
DGE-F-266	1	2	Y2_Y3	3:37	0:06	Hide a diagram	Label a diagram
DGE-F-267	1	2	Y2_Y3	3:43	0:06	Un-hide a diagram	Label a diagram
DGE-F-268	1	2	Y2_Y3	3:49	0:03	Label a diagram	Label a diagram
DGE-F-269	1	2	Y2_Y3	3:54	0:12	Measure a diagram	Measure a diagram
DGE-F-270	1	2	Y2_Y3	4:06	0:02	Drag a measurement	Place a measurement
DGE-F-271	1	2	Y2_Y3	4:08	0:06	Scroll the screen	Fit diagrams to the screen
DGE-F-272	1	2	Y2_Y3	4:14	0:03	Select a diagram	Not identified
DGE-F-273	1	2	Y2_Y3	4:17	0:02	Move the cursor over diagrams	Point at a measurement
DGE-F-274	1	2	Y2_Y3	4:19	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-275	1	2	Y2_Y3	4:23	0:05	Move the cursor over diagrams	Point at a diagram
DGE-F-276	1	2	Y2_Y3	4:28	0:03	Select a diagram	Not identified
DGE-F-277	1	2	Y2_Y3	4:31	0:02	Select a diagram	Point at a diagram
DGE-F-278	1	2	Y2_Y3	4:33	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-279	1	2	Y2_Y3	4:37	0:02	Select a diagram	Point at a diagram
DGE-F-280	1	2	Y2_Y3	4:39	0:03	Select a diagram	Point at a diagram
DGE-F-281	1	2	Y2_Y3	4:42	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-282	1	2	Y2_Y3	4:46	0:08	Move the cursor over diagrams	Point at a diagram
DGE-F-283	1	2	Y2_Y3	4:54	0:01	Select a measurement	Point at a measurement
DGE-F-284	1	2	Y2_Y3	4:55	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-285	1	2	Y2_Y3	4:58	0:02	Select a measurement	Point at a measurement
DGE-F-286	1	2	Y2_Y3	5:00	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-287	1	2	Y2_Y3	5:02	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-288	1	2	Y2_Y3	5:05	0:05	Drag a diagram	Wandering dragging
DGE-F-289	1	2	Y2_Y3	5:10	0:01	Select a measurement	Point at a measurement
DGE-F-290	1	2	Y2_Y3	5:11	0:02	Select a measurement	Point at a measurement
DGE-F-291	1	2	Y2_Y3	5:13	0:03	Select a measurement	Point at a measurement
DGE-F-292	1	2	Y2_Y3	5:16	0:04	Select a measurement	Point at a measurement
DGE-F-293	1	2	Y2_Y3	5:20	0:01	Select a measurement	Point at a measurement
DGE-F-294	1	2	Y2_Y3	5:21	0:01	Select a measurement	Point at a measurement

Table 6.3 (cont'd). DGE feature and DGE move code applications

DGE-F-295	1	2	Y2_Y3	5:22	0:08	Drag a diagram	Wandering dragging
DGE-F-296	2	3	A1_K1	0:10	0:07	Move the cursor over diagrams	Not identified
DGE-F-297	2	3	A1_K1	0:19	0:03	Move the cursor over diagrams	Not identified
DGE-F-298	2	3	A1_K1	0:28	0:02	Move the cursor over diagrams	Not identified
DGE-F-299	2	3	A1_K1	0:30	0:02	Move the cursor over diagrams	Not identified
DGE-F-300	2	3	A1_K1	0:32	0:05	Create a diagram	Create a diagram
DGE-F-301	2	3	A1_K1	0:43	0:03	Create a diagram	Create a diagram
DGE-F-302	2	3	A1_K1	0:46	0:09	Label a diagram	Label a diagram
DGE-F-303	2	3	A1_K1	1:00	0:02	Select a diagram	Create a diagram
DGE-F-304	2	3	A1_K1	1:02	0:02	Create a diagram	Create a diagram
DGE-F-305	2	3	A1_K1	1:08	0:02	Create a diagram	Create a diagram
DGE-F-306	2	3	A1_K1	1:10	0:07	Label a diagram	Label a diagram
DGE-F-307	2	3	A1_K1	1:28	0:05	Select a diagram	Create a diagram
DGE-F-308	2	3	A1_K1	1:33	0:01	Create a diagram	Create a diagram
DGE-F-309	2	3	A1_K1	2:14	0:03	Move the cursor over diagrams	Not identified
DGE-F-310	2	3	A1_K1	2:17	0:04	Create a diagram	Create a diagram
DGE-F-311	2	3	A1_K1	2:21	0:07	Drag a diagram	Wandering dragging
DGE-F-312	2	3	A1_K1	2:28	0:09	Label a diagram	Label a diagram
DGE-F-313	2	3	A1_K1	2:44	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-314	2	3	A1_K1	2:46	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-315	2	3	A1_K1	2:50	0:05	Move the cursor over diagrams	Point at a diagram
DGE-F-316	2	3	A1_K1	2:55	0:06	Drag a diagram	Guided dragging
DGE-F-317	2	3	A1_K1	3:01	0:08	Label a diagram	Label a diagram
DGE-F-318	2	3	A1_K1	3:09	0:10	Create a diagram	Create a diagram
DGE-F-319	2	3	A1_K1	3:19	0:03	Drag a diagram	Wandering dragging
DGE-F-320	2	3	A1_K1	3:40	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-321	2	3	A1_K1	3:44	0:04	Move the cursor over diagrams	Not identified
DGE-F-322	2	3	A1_K1	3:48	0:04	Move the cursor over diagrams	Not identified
DGE-F-323	2	3	A1_K1	3:58	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-324	2	3	A1_K1	4:00	0:06	Select a diagram	Create a diagram
DGE-F-325	2	3	A1_K1	4:06	0:02	Select a diagram	Create a diagram
DGE-F-326	2	3	A1_K1	4:12	0:02	Select a diagram	Create a diagram
DGE-F-327	2	3	A1_K1	4:14	0:03	Mirror a diagram	Create a diagram
DGE-F-328	2	3	A1_K1	4:17	0:13	Label a diagram	Label a diagram
DGE-F-329	2	3	A1_K1	4:38	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-330	2	3	A1_K1	4:39	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-331	2	3	A1_K1	4:44	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-332	2	3	A1_K1	4:56	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-333	2	3	A1_K1	4:59	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-334	2	3	A1_K1	5:04	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-335	2	3	A1_K1	5:06	0:04	Create a diagram	Create a diagram
DGE-F-336	2	3	A1_K1	5:27	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-337	2	3	A1_K1	5:28	0:04	Create a diagram	Create a diagram
DGE-F-338	2	3	A1_K1	5:32	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-339	2	3	A1_K1	5:47	0:02	Create a diagram	Create a diagram
DGE-F-340	2	3	A1_K1	5:49	0:06	Label a diagram	Label a diagram
DGE-F-341	2	3	A1_K1	6:07	0:09	Select a diagram	Reflect a diagram
DGE-F-342	2	3	A1_K1	6:16	0:02	Mirror a diagram	Reflect a diagram
DGE-F-343	2	3	A1_K1	6:18	0:07	Label a diagram	Label a diagram
DGE-F-344	2	3	A1_K1	6:33	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-345	2	3	A1_K1	6:36	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-346	2	3	A1_K1	6:41	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-347	2	3	A1_K1	6:47	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-348	2	3	A1_K1	6:48	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-349	2	3	A1_K1	6:49	0:26	Move the cursor over diagrams	Point at a diagram
DGE-F-350	2	3	A1_K1	7:15	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-351	2	3	A1_K1	7:16	0:06	Move the cursor over diagrams	Point at a diagram
DGE-F-352	2	3	A1_K1	7:22	0:13	Select a diagram	Measure a diagram
DGE-F-353	2	3	A1_K1	7:35	0:05	Select a diagram	Measure a diagram
DGE-F-354	2	3	A1_K1	7:40	0:05	Select a diagram	Measure a diagram

Table 6.3 (cont'd). DGE feature and DGE move code applications

DGE-F-355	2	3	A1_K1	7:45	0:05	Select a diagram	Measure a diagram
DGE-F-356	2	3	A1_K1	7:50	0:02	Move the cursor over diagrams	Not identified
DGE-F-357	2	3	A1_K1	7:53	0:07	Select a diagram	Measure a diagram
DGE-F-358	2	3	A1_K1	8:00	0:10	Select a diagram	Measure a diagram
DGE-F-359	2	3	A1_K1	8:43	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-360	2	3	A1_K1	8:46	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-361	2	3	A1_K1	8:49	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-362	2	3	A1_K1	8:53	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-363	2	3	A1_K1	8:55	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-364	2	3	A1_K1	8:57	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-365	2	3	A1_K1	8:58	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-366	2	3	A1_K1	9:00	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-367	2	3	A1_K1	11:15	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-368	2	3	A1_K1(2)	0:07	0:06	Drag a diagram	Guided dragging
DGE-F-369	2	3	A1_K1(2)	0:13	0:08	Drag a diagram	Guided dragging
DGE-F-370	2	3	A1_K1(2)	0:21	0:06	Drag a diagram	Guided dragging
DGE-F-371	2	3	B1_E2	0:24	0:09	Create a diagram	Create a diagram
DGE-F-372	2	3	B1_E2	0:33	0:06	Create a diagram	Create a diagram
DGE-F-373	2	3	B1_E2	0:39	0:09	Label a diagram	Label a diagram
DGE-F-374	2	3	B1_E2	0:48	0:10	Hide a label	Hide a label
DGE-F-375	2	3	B1_E2	0:58	0:07	Label a diagram	Label a diagram
DGE-F-376	2	3	B1_E2	1:12	0:01	Select a diagram	Create a diagram
DGE-F-377	2	3	B1_E2	1:13	0:02	Select a diagram	Create a diagram
DGE-F-378	2	3	B1_E2	1:15	0:01	Create a diagram	Create a diagram
DGE-F-379	2	3	B1_E2	1:23	0:02	Create a diagram	Create a diagram
DGE-F-380	2	3	B1_E2	1:25	0:06	Label a diagram	Label a diagram
DGE-F-381	2	3	B1_E2	1:40	0:03	Select a diagram	Create a diagram
DGE-F-382	2	3	B1_E2	1:43	0:01	Create a diagram	Create a diagram
DGE-F-383	2	3	B1_E2	1:51	0:05	Label a diagram	Label a diagram
DGE-F-384	2	3	B1_E2	2:15	0:02	Create a diagram	Create a diagram
DGE-F-385	2	3	B1_E2	2:17	0:04	Label a diagram	Label a diagram
DGE-F-386	2	3	B1_E2	2:25	0:04	Select a diagram	Create a diagram
DGE-F-387	2	3	B1_E2	2:29	0:03	Create a diagram	Create a diagram
DGE-F-388	2	3	B1_E2	2:42	0:02	Drag a diagram	Wandering dragging
DGE-F-389	2	3	B1_E2	2:56	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-390	2	3	B1_E2	2:57	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-391	2	3	B1_E2	3:01	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-392	2	3	B1_E2	3:05	0:18	Select a diagram	Reflect a diagram
DGE-F-393	2	3	B1_E2	3:23	0:07	Select a diagram	Reflect a diagram
DGE-F-394	2	3	B1_E2	3:30	0:02	Create a diagram	Reflect a diagram
DGE-F-395	2	3	B1_E2	3:32	0:05	Label a diagram	Label a diagram
DGE-F-396	2	3	B1_E2	3:45	0:03	Create a diagram	Create a diagram
DGE-F-397	2	3	B1_E2	3:51	0:03	Drag a diagram	Wandering dragging
DGE-F-398	2	3	B1_E2	3:58	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-399	2	3	B1_E2	4:08	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-400	2	3	B1_E2	4:09	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-401	2	3	B1_E2	4:14	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-402	2	3	B1_E2	4:16	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-403	2	3	B1_E2	4:47	0:02	Drag a diagram	Wandering dragging
DGE-F-404	2	3	B1_E2	4:49	0:06	Drag a diagram	Wandering dragging
DGE-F-405	2	3	B1_E2	4:57	0:11	Drag a diagram	Guided dragging
DGE-F-406	2	3	B1_E2	5:21	0:01	Select a diagram	Point at a diagram
DGE-F-407	2	3	B1_E2	5:22	0:01	Select a diagram	Point at a diagram
DGE-F-408	2	3	B1_E2	6:05	0:07	Move the cursor over diagrams	Point at a diagram
DGE-F-409	2	3	B1_E2	6:12	0:02	Drag a diagram	Guided dragging
DGE-F-410	2	3	B1_E2	6:14	0:03	Drag a diagram	Guided dragging
DGE-F-411	2	3	B1_E2	6:17	0:01	Create a diagram	Create a diagram
DGE-F-412	2	3	B1_E2	6:18	0:08	Label a diagram	Label a diagram
DGE-F-413	2	3	B1_E2	6:30	0:03	Hide a diagram	Hide a diagram
DGE-F-414	2	3	B1_E2	6:33	0:02	Hide a diagram	Hide a diagram

Table 6.3 (cont'd). DGE feature and DGE move code applications

DGE-F-415	2	3	B1_E2	6:44	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-416	2	3	B1_E2	7:10	0:02	Select a diagram	Reflect a diagram
DGE-F-417	2	3	B1_E2	7:12	0:03	Mirror a diagram	Reflect a diagram
DGE-F-418	2	3	B1_E2	7:15	0:04	Label a diagram	Label a diagram
DGE-F-419	2	3	B1_E2	7:32	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-420	2	3	B1_E2	7:34	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-421	2	3	B1_E2	7:57	0:07	Move the cursor over diagrams	Not identified
DGE-F-422	2	3	B1_E2	8:04	0:02	Select a diagram	Point at a diagram
DGE-F-423	2	3	B1_E2	8:06	0:01	Select a diagram	Point at a diagram
DGE-F-424	2	3	B1_E2	8:07	0:02	Select a diagram	Point at a diagram
DGE-F-425	2	3	B1_E2	8:09	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-426	2	3	B1_E2	8:11	0:06	De-select diagrams	Not identified
DGE-F-427	2	3	B1_E2	8:17	0:04	Drag a diagram	Wandering dragging
DGE-F-428	2	3	B1_E2	8:21	0:04	Drag a diagram	Wandering dragging
DGE-F-429	2	3	B1_E2	8:25	0:05	Drag a diagram	Wandering dragging
DGE-F-430	2	3	B1_E2	8:31	0:02	Drag a diagram	Wandering dragging
DGE-F-431	2	3	B1_E2(2)	0:09	0:03	Drag a diagram	Wandering dragging
DGE-F-432	2	3	B1_E2(2)	0:13	0:07	Drag a diagram	Wandering dragging
DGE-F-433	2	3	B1_E2(2)	0:20	0:04	Drag a diagram	Guided dragging
DGE-F-434	2	3	B1_E2(2)	0:24	0:03	Drag a diagram	Wandering dragging
DGE-F-435	2	3	B1_E2(2)	0:27	0:07	Drag a diagram	Wandering dragging
DGE-F-436	2	3	B1_E2(2)	0:34	0:04	Drag a diagram	Guided dragging
DGE-F-437	2	3	B1_E2(2)	0:38	0:04	Drag a diagram	Wandering dragging
DGE-F-438	2	3	B1_E2(2)	0:47	0:01	Drag a diagram	Dragging by accident
DGE-F-439	2	3	B1_E2(2)	0:48	0:03	Drag a diagram	Guided dragging
DGE-F-440	2	3	B1_E2(2)	0:51	0:06	Drag a diagram	Guided dragging
DGE-F-441	2	3	B1_E2(2)	0:57	0:03	Drag a diagram	Guided dragging
DGE-F-442	2	3	B1_E2(2)	1:02	0:04	Drag a diagram	Wandering dragging
DGE-F-443	2	3	B1_E2(2)	1:07	0:05	Move the cursor over diagrams	Point at a diagram
DGE-F-444	2	3	B1_E2(2)	1:15	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-445	2	3	B1_E2(2)	1:21	0:03	Drag a diagram	Guided dragging
DGE-F-446	2	3	B1_E2(2)	1:31	0:02	Drag a diagram	Guided dragging
DGE-F-447	2	3	B1_E2(2)	1:33	0:04	Drag a diagram	Guided dragging
DGE-F-448	2	3	B1_E2(2)	1:37	0:03	Drag a diagram	Guided dragging
DGE-F-449	2	3	B1_E2(2)	1:40	0:03	Drag a diagram	Guided dragging
DGE-F-450	2	3	B1_E2(2)	1:43	0:06	Drag a diagram	Guided dragging
DGE-F-451	2	3	B1_E2(2)	1:49	0:02	Drag a diagram	Wandering dragging
DGE-F-452	2	3	B1_E2(2)	1:52	0:01	Drag a diagram	Wandering dragging
DGE-F-453	2	3	B1_E2(2)	1:53	0:06	Drag a diagram	Guided dragging
DGE-F-454	2	3	D1_T1	0:07	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-455	2	3	D1_T1	0:08	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-456	2	3	D1_T1	0:09	0:05	Move the cursor over diagrams	Point at a diagram
DGE-F-457	2	3	D1_T1	0:14	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-458	2	3	D1_T1	0:19	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-459	2	3	D1_T1	0:21	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-460	2	3	D1_T1	0:26	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-461	2	3	D1_T1	0:44	0:06	Select a diagram	Reflect a diagram
DGE-F-462	2	3	D1_T1	0:50	0:01	Mirror a diagram	Reflect a diagram
DGE-F-463	2	3	D1_T1	0:56	0:03	Create a diagram	Create a diagram
DGE-F-464	2	3	D1_T1	1:08	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-465	2	3	D1_T1	1:27	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-466	2	3	D1_T1	1:29	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-467	2	3	D1_T1	1:41	0:08	Drag a diagram	Guided dragging
DGE-F-468	2	3	D2_Y1	0:10	0:03	Move the cursor over diagrams	Point at a feature icon
DGE-F-469	2	3	D2_Y1	0:17	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-470	2	3	D2_Y1	0:23	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-471	2	3	D2_Y1	0:24	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-472	2	3	D2_Y1	0:27	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-473	2	3	D2_Y1	0:32	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-474	2	3	D2_Y1	0:33	0:05	Move the cursor over diagrams	Point at a diagram

Table 6.3 (cont'd). DGE feature and DGE move code applications

DGE-F-475	2	3	D2_Y1	0:38	0:01	Select a diagram	Point at a diagram
DGE-F-476	2	3	D2_Y1	0:39	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-477	2	3	D2_Y1	0:47	0:06	Select a diagram	Not identified
DGE-F-478	2	3	D2_Y1	0:53	0:01	Select a diagram	Point at a diagram
DGE-F-479	2	3	D2_Y1	0:54	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-480	2	3	D2_Y1	1:00	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-481	2	3	D2_Y1	1:01	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-482	2	3	D2_Y1	1:02	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-483	2	3	D2_Y1	1:05	0:05	Move the cursor over diagrams	Point at a diagram
DGE-F-484	2	3	D2_Y1	1:17	0:01	Drag a diagram	Point at a diagram
DGE-F-485	2	3	D2_Y1	1:28	0:04	Move the cursor over diagrams	Point at a feature icon
DGE-F-486	2	3	D2_Y1	1:36	0:10	Drag a diagram	Wandering dragging
DGE-F-487	2	3	D2_Y1	1:59	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-488	2	3	D2_Y1	2:00	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-489	2	3	D2_Y1	2:20	0:12	Drag a diagram	Wandering dragging
DGE-F-490	2	3	D2_Y1	2:32	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-491	2	3	D2_Y1	2:37	0:11	Drag a diagram	Guided dragging
DGE-F-492	2	3	D2_Y1	2:48	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-493	2	3	D2_Y1	3:12	0:15	Drag a diagram	Guided dragging
DGE-F-494	2	3	D2_Y1	3:29	0:03	Drag a diagram	Guided dragging
DGE-F-495	2	3	D2_Y1	3:46	0:03	Drag a diagram	Guided dragging
DGE-F-496	2	3	D2_Y1	3:51	0:03	Create a diagram	Create a diagram
DGE-F-497	2	3	D2_Y1	3:57	0:07	Label a diagram	Label a diagram
DGE-F-498	2	3	D2_Y1	4:09	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-499	2	3	D2_Y1	4:12	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-500	2	3	D2_Y1	4:18	0:11	Select a diagram	Reflect a diagram
DGE-F-501	2	3	D2_Y1	4:29	0:02	Select a diagram	Reflect a diagram
DGE-F-502	2	3	D2_Y1	4:31	0:03	Mirror a diagram	Reflect a diagram
DGE-F-503	2	3	D2_Y1	4:34	0:11	Create a diagram	Create a diagram
DGE-F-504	2	3	D2_Y1	4:52	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-505	2	3	D2_Y1	4:53	0:05	Move the cursor over diagrams	Point at a diagram
DGE-F-506	2	3	D2_Y1	4:58	0:01	Select a diagram	Create a diagram
DGE-F-507	2	3	D2_Y1	4:59	0:01	Select a diagram	Create a diagram
DGE-F-508	2	3	D2_Y1	5:00	0:01	Create a diagram	Create a diagram
DGE-F-509	2	3	D2_Y1	5:04	0:06	Label a diagram	Label a diagram
DGE-F-510	2	3	D2_Y1	5:14	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-511	2	3	D2_Y1	5:17	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-512	2	3	D2_Y1	5:19	0:06	Select a diagram	Measure a diagram
DGE-F-513	2	3	D2_Y1	5:25	0:09	Select a diagram	Measure a diagram
DGE-F-514	2	3	D2_Y1	5:42	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-515	2	3	D2_Y1	5:45	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-516	2	3	D2_Y1	5:50	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-517	2	3	D2_Y1	5:51	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-518	2	3	D2_Y1	5:52	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-519	2	3	D2_Y1	5:55	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-520	2	3	D2_Y1	5:57	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-521	2	3	D2_Y1	5:58	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-522	2	3	D2_Y1	5:59	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-523	2	3	D2_Y1	6:00	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-524	2	3	D2_Y1	6:08	0:02	Select a diagram	Not identified
DGE-F-525	2	3	D2_Y1	6:10	0:01	Select a diagram	Measure a diagram
DGE-F-526	2	3	D2_Y1	6:11	0:01	Select a diagram	Measure a diagram
DGE-F-527	2	3	D2_Y1	6:22	0:01	Measure a diagram	Measure a diagram
DGE-F-528	2	3	D2_Y1	6:25	0:02	Select a diagram	Measure a diagram
DGE-F-529	2	3	D2_Y1	6:27	0:01	Select a diagram	Measure a diagram
DGE-F-530	2	3	D2_Y1	6:30	0:01	Measure a diagram	Measure a diagram
DGE-F-531	2	3	D2_Y1	6:33	0:07	Move the cursor over diagrams	Point at a diagram
DGE-F-532	2	3	D2_Y1	6:46	0:05	Move the cursor over diagrams	Point at a diagram
DGE-F-533	2	3	D2_Y1	7:02	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-534	2	3	D2_Y1	7:04	0:02	Move the cursor over diagrams	Point at a diagram

Table 6.3 (cont'd). DGE feature and DGE move code applications

DGE-F-535	2	3	D2_Y1	7:06	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-536	2	3	D2_Y1	7:23	0:01	Select a diagram	Measure a diagram
DGE-F-537	2	3	D2_Y1	7:24	0:02	Select a diagram	Measure a diagram
DGE-F-538	2	3	D2_Y1	7:26	0:02	Select a diagram	Measure a diagram
DGE-F-539	2	3	D2_Y1	7:30	0:01	Measure a diagram	Measure a diagram
DGE-F-540	2	3	D2_Y1	7:31	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-541	2	3	D2_Y1	7:33	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-542	2	3	D2_Y1	7:35	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-543	2	3	D2_Y1	7:37	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-544	2	3	D2_Y1	7:41	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-545	2	3	D2_Y1	7:44	0:01	Select a diagram	Point at a diagram
DGE-F-546	2	3	D2_Y1	7:45	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-547	2	3	D2_Y1	7:46	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-548	2	3	D2_Y1	7:47	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-549	2	3	D2_Y1	7:50	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-550	2	3	D2_Y1	7:52	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-551	2	3	D2_Y1	7:58	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-552	2	3	N1_O1	0:00	0:08	Create a diagram	Create a diagram
DGE-F-553	2	3	N1_O1	0:08	0:10	Label a diagram	Label a diagram
DGE-F-554	2	3	N1_O1	0:18	0:04	Create a diagram	Create a diagram
DGE-F-555	2	3	N1_O1	0:22	0:06	Label a diagram	Label a diagram
DGE-F-556	2	3	N1_O1	0:32	0:04	Select a diagram	Create a diagram
DGE-F-557	2	3	N1_O1	0:36	0:02	Create a diagram	Create a diagram
DGE-F-558	2	3	N1_O1	0:38	0:03	Create a diagram	Create a diagram
DGE-F-559	2	3	N1_O1	0:44	0:08	Label a diagram	Label a diagram
DGE-F-560	2	3	N1_O1	0:54	0:01	Select a diagram	Not identified
DGE-F-561	2	3	N1_O1	0:57	0:02	Drag a diagram	Place a diagram
DGE-F-562	2	3	N1_O1	1:03	0:03	Select a diagram	Create a diagram
DGE-F-563	2	3	N1_O1	1:06	0:02	Select a diagram	Create a diagram
DGE-F-564	2	3	N1_O1	1:08	0:01	Create a diagram	Create a diagram
DGE-F-565	2	3	N1_O1	1:09	0:05	Label a diagram	Label a diagram
DGE-F-566	2	3	N1_O1	1:23	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-567	2	3	N1_O1	1:24	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-568	2	3	N1_O1	1:37	0:07	Create a diagram	Create a diagram
DGE-F-569	2	3	N1_O1	1:44	0:10	Label a diagram	Label a diagram
DGE-F-570	2	3	N1_O1	1:58	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-571	2	3	N1_O1	2:00	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-572	2	3	N1_O1	2:04	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-573	2	3	N1_O1	2:10	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-574	2	3	N1_O1	2:12	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-575	2	3	N1_O1	2:24	0:02	Create a diagram	Create a diagram
DGE-F-576	2	3	N1_O1	2:26	0:07	Label a diagram	Label a diagram
DGE-F-577	2	3	N1_O1	2:37	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-578	2	3	N1_O1	2:39	0:05	Select a diagram	Reflect a diagram
DGE-F-579	2	3	N1_O1	2:44	0:01	Select a diagram	Reflect a diagram
DGE-F-580	2	3	N1_O1	2:45	0:05	Mirror a diagram	Reflect a diagram
DGE-F-581	2	3	N1_O1	2:50	0:06	Label a diagram	Label a diagram
DGE-F-582	2	3	N1_O1	3:00	0:05	Create a diagram	Create a diagram
DGE-F-583	2	3	N1_O1	3:12	0:04	Select a diagram	Measure a diagram
DGE-F-584	2	3	N1_O1	3:16	0:03	Measure a diagram	Measure a diagram
DGE-F-585	2	3	N1_O1	3:19	0:08	Select a diagram	Measure a diagram
DGE-F-586	2	3	N1_O1	3:27	0:01	Measure a diagram	Measure a diagram
DGE-F-587	2	3	N1_O1	3:33	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-588	2	3	N1_O1	3:35	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-589	2	3	N1_O1	3:40	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-590	2	3	N1_O1	3:42	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-591	2	3	N1_O1	3:49	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-592	2	3	N1_O1	3:50	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-593	2	3	N1_O1	3:53	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-594	2	3	N1_O1	3:56	0:01	Move the cursor over diagrams	Point at a diagram

Table 6.3 (cont'd). DGE feature and DGE move code applications

DGE-F-595	2	3	N1_O1	3:58	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-596	2	3	N1_O1	4:02	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-597	2	3	N1_O1	4:05	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-598	2	3	N1_O1	4:07	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-599	2	3	N1_O1	4:12	0:03	Create a diagram	Create a diagram
DGE-F-600	2	3	N1_O1	4:15	0:09	Label a diagram	Label a diagram
DGE-F-601	2	3	N1_O1	4:30	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-602	2	3	N1_O1	4:32	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-603	2	3	N1_O1	4:35	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-604	2	3	N1_O1	4:38	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-605	2	3	N1_O1	4:41	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-606	2	3	N1_O1	4:44	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-607	2	3	N1_O1	4:46	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-608	2	3	N1_O1	4:47	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-609	2	3	N1_O1	4:51	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-610	2	3	N1_O1	4:57	0:02	Select a diagram	Reflect a diagram
DGE-F-611	2	3	N1_O1	4:59	0:01	Mirror a diagram	Reflect a diagram
DGE-F-612	2	3	N1_O1	5:00	0:09	Label a diagram	Label a diagram
DGE-F-613	2	3	N1_O1	5:13	0:06	Select a diagram	Measure a diagram
DGE-F-614	2	3	N1_O1	5:19	0:01	Measure a diagram	Measure a diagram
DGE-F-615	2	3	N1_O1	5:24	0:03	Select a diagram	Measure a diagram
DGE-F-616	2	3	N1_O1	5:27	0:02	Measure a diagram	Measure a diagram
DGE-F-617	2	3	N1_O1	5:29	0:06	Move the cursor over diagrams	Point at a measurement
DGE-F-618	2	3	N1_O1	5:35	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-619	2	3	N1_O1	5:38	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-620	2	3	N1_O1	5:40	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-621	2	3	N1_O1	5:50	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-622	2	3	N1_O1	5:51	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-623	2	3	N1_O1	5:54	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-624	2	3	N1_O1	5:58	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-625	2	3	N1_O1	6:00	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-626	2	3	N1_O1	6:04	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-627	2	3	N1_O1	6:06	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-628	2	3	N1_O1	6:10	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-629	2	3	N1_O1	6:14	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-630	2	3	N1_O1	6:15	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-631	2	3	N1_O1	6:19	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-632	2	3	N1_O1	6:20	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-633	2	3	N1_O1	6:25	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-634	2	3	N1_O1	6:30	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-635	2	3	N1_O1	6:33	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-636	2	3	N1_O1	6:49	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-637	2	3	N1_O1	6:50	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-638	2	3	N1_O1	6:51	0:05	Move the cursor over diagrams	Point at a diagram
DGE-F-639	2	3	N1_O1	6:56	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-640	2	3	N1_O1	6:59	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-641	2	3	N1_O1	7:01	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-642	2	3	N1_O1	7:09	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-643	2	3	N1_O1	7:10	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-644	2	3	N1_O1	7:11	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-645	2	3	N1_O1	7:16	0:05	Move the cursor over diagrams	Point at a diagram
DGE-F-646	2	3	N1_O1	7:21	0:09	Move the cursor over diagrams	Point at a diagram
DGE-F-647	2	3	N1_O1	7:30	0:06	Move the cursor over diagrams	Point at a diagram
DGE-F-648	2	3	N1_O1	7:36	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-649	2	3	N1_O1	7:39	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-650	2	3	N1_O1	7:42	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-651	2	3	N1_O1	7:43	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-652	2	3	N1_O1	7:47	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-653	2	3	N1_O1	7:48	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-654	2	3	N1_O1	7:51	0:07	Drag a diagram	Guided dragging

Table 6.3 (cont'd). DGE feature and DGE move code applications

DGE-F-655	2	3	N1_O1	7:58	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-656	2	3	N1_O1	8:00	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-657	2	3	N1_O1	8:04	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-658	2	3	N1_O1	8:06	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-659	2	3	N1_O1	8:11	0:04	Drag a diagram	Guided dragging
DGE-F-660	3	4	A1_K1	0:13	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-661	3	4	A1_K1	0:16	0:07	Move the cursor over diagrams	Point at a diagram
DGE-F-662	3	4	A1_K1	0:23	0:02	Select a diagram	Create a diagram
DGE-F-663	3	4	A1_K1	0:25	0:02	Select a diagram	Create a diagram
DGE-F-664	3	4	A1_K1	0:27	0:02	Create a diagram	Create a diagram
DGE-F-665	3	4	A1_K1	0:30	0:01	Create a diagram	Create a diagram
DGE-F-666	3	4	A1_K1	0:44	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-667	3	4	A1_K1	0:49	0:01	Select a diagram	Point at a diagram
DGE-F-668	3	4	A1_K1	0:50	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-669	3	4	A1_K1	0:52	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-670	3	4	A1_K1	1:18	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-671	3	4	A1_K1	1:30	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-672	3	4	A1_K1	1:42	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-673	3	4	B1_E2_J1	0:08	0:04	Drag a diagram	Place a diagram
DGE-F-674	3	4	B1_E2_J1	0:27	0:01	Drag a diagram	Place a diagram
DGE-F-675	3	4	B1_E2_J1	0:30	0:06	Move the cursor over diagrams	Point at a diagram
DGE-F-676	3	4	B1_E2_J1	0:36	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-677	3	4	B1_E2_J1	0:38	0:02	Select a diagram	Point at a diagram
DGE-F-678	3	4	B1_E2_J1	0:40	0:01	Select a diagram	Point at a diagram
DGE-F-679	3	4	B1_E2_J1	0:43	0:01	Select a diagram	Point at a diagram
DGE-F-680	3	4	B1_E2_J1	0:45	0:02	Drag a diagram	Point at a diagram
DGE-F-681	3	4	B1_E2_J1	0:48	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-682	3	4	B1_E2_J1	0:49	0:01	Drag a diagram	Point at a diagram
DGE-F-683	3	4	B1_E2_J1	0:53	0:01	Select a diagram	Point at a diagram
DGE-F-684	3	4	B1_E2_J1	0:55	0:01	Select a diagram	Point at a diagram
DGE-F-685	3	4	B1_E2_J1	0:56	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-686	3	4	B1_E2_J1	0:58	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-687	3	4	B1_E2_J1	1:00	0:01	Select a diagram	Point at a diagram
DGE-F-688	3	4	B1_E2_J1	1:04	0:02	Drag a diagram	Place a diagram
DGE-F-689	3	4	B1_E2_J1	1:06	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-690	3	4	B1_E2_J1	1:09	0:01	Select a diagram	Point at a diagram
DGE-F-691	3	4	B1_E2_J1	1:11	0:01	Select a diagram	Point at a diagram
DGE-F-692	3	4	B1_E2_J1	1:13	0:01	Select a diagram	Point at a diagram
DGE-F-693	3	4	B1_E2_J1	1:19	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-694	3	4	B1_E2_J1	1:21	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-695	3	4	B1_E2_J1	1:22	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-696	3	4	B1_E2_J1	1:26	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-697	3	4	B1_E2_J1	1:27	0:04	Drag a diagram	Guided dragging
DGE-F-698	3	4	B1_E2_J1	1:31	0:02	Drag a diagram	Guided dragging
DGE-F-699	3	4	B1_E2_J1	1:35	0:06	Drag a diagram	Guided dragging
DGE-F-700	3	4	B1_E2_J1	1:43	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-701	3	4	B1_E2_J1	1:45	0:04	Drag a diagram	Guided dragging
DGE-F-702	3	4	B1_E2_J1	2:01	0:01	Drag a diagram	Wandering dragging
DGE-F-703	3	4	B1_E2_J1	2:05	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-704	3	4	B1_E2_J1	2:06	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-705	3	4	B2_T1	0:09	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-706	3	4	B2_T1	0:24	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-707	3	4	B2_T1	0:44	0:01	Select a diagram	Point at a diagram
DGE-F-708	3	4	B2_T1	1:03	0:10	Drag a diagram	Guided dragging
DGE-F-709	3	4	B2_T1	1:19	0:23	Drag a diagram	Guided dragging
DGE-F-710	3	4	D1_J2	0:25	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-711	3	4	D1_J2	0:35	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-712	3	4	D1_J2	0:41	0:05	Create a diagram	Create a diagram
DGE-F-713	3	4	D1_J2	0:46	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-714	3	4	D1_J2	1:05	0:08	Drag a diagram	Guided dragging

Table 6.3 (cont'd). DGE feature and DGE move code applications

DGE-F-715	3	4	D1_J2	1:13	0:04	Drag a diagram	Guided dragging
DGE-F-716	3	4	D1_J2	1:19	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-717	3	4	D1_J2	1:20	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-718	3	4	D1_J2	1:21	0:05	Drag a diagram	Wandering dragging
DGE-F-719	3	4	D1_J2	1:26	0:05	Move the cursor over diagrams	Point at a diagram
DGE-F-720	3	4	D1_J2	1:31	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-721	3	4	D1_J2	1:33	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-722	3	4	D1_J2	1:39	0:03	Drag a diagram	Guided dragging
DGE-F-723	3	4	D1_J2	1:43	0:08	Drag a diagram	Guided dragging
DGE-F-724	3	4	D1_J2	1:52	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-725	3	4	D1_J2	1:55	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-726	3	4	D1_J2	1:56	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-727	3	4	D1_J2	2:02	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-728	3	4	D1_J2	2:05	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-729	3	4	D1_J2	2:08	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-730	3	4	D1_J2	2:10	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-731	3	4	D1_J2	2:11	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-732	3	4	D1_J2	2:14	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-733	3	4	D1_J2	2:15	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-734	3	4	D1_J2	2:16	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-735	3	4	D1_J2	2:18	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-736	3	4	D1_J2	2:20	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-737	3	4	D1_J2	2:22	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-738	3	4	D2_Y1	0:05	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-739	3	4	D2_Y1	0:07	0:01	Select a diagram	Point at a diagram
DGE-F-740	3	4	D2_Y1	0:08	0:01	Select a diagram	Point at a diagram
DGE-F-741	3	4	D2_Y1	0:11	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-742	3	4	D2_Y1	0:17	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-743	3	4	D2_Y1	0:21	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-744	3	4	D2_Y1	0:22	0:04	Move the cursor over diagrams	Point at a diagram
DGE-F-745	3	4	D2_Y1	0:27	0:02	Select a diagram	Point at a diagram
DGE-F-746	3	4	D2_Y1	0:29	0:01	Select a diagram	Point at a diagram
DGE-F-747	3	4	D2_Y1	0:43	0:04	Select a diagram	Point at a diagram
DGE-F-748	3	4	D2_Y1	0:47	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-749	3	4	D2_Y1	0:52	0:01	Select a diagram	Point at a diagram
DGE-F-750	3	4	D2_Y1	0:53	0:01	Select a diagram	Point at a diagram
DGE-F-751	3	4	D2_Y1	0:56	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-752	3	4	D2_Y1	0:57	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-753	3	4	D2_Y1	0:58	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-754	3	4	D2_Y1	0:59	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-755	3	4	D2_Y1	1:00	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-756	3	4	D2_Y1	1:02	0:04	Drag a diagram	Point at a diagram
DGE-F-757	3	4	D2_Y1	1:06	0:03	Drag a diagram	Place a diagram
DGE-F-758	3	4	D2_Y1	1:11	0:01	Select a diagram	Point at a diagram
DGE-F-759	3	4	D2_Y1	1:13	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-760	3	4	D2_Y1	1:15	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-761	3	4	D2_Y1	1:19	0:01	Drag a diagram	Point at a diagram
DGE-F-762	3	4	D2_Y1	1:20	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-763	3	4	D2_Y1	1:25	0:06	Drag a diagram	Place a diagram
DGE-F-764	3	4	D2_Y1	1:33	0:02	Select a diagram	Point at a diagram
DGE-F-765	3	4	D2_Y1	1:35	0:02	De-select diagrams	Point at a diagram
DGE-F-766	3	4	D2_Y1	1:37	0:01	Select a diagram	Point at a diagram
DGE-F-767	3	4	D2_Y1	1:44	0:01	Select a diagram	Point at a diagram
DGE-F-768	3	4	D2_Y1	1:45	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-769	3	4	D2_Y1	1:58	0:01	Select a diagram	Point at a diagram
DGE-F-770	3	4	D2_Y1	2:04	0:01	Switch the mode	Change the model
DGE-F-771	3	4	D2_Y1	2:05	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-772	3	4	D2_Y1	2:08	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-773	3	4	D2_Y1	2:14	0:01	Switch the mode	Change the model
DGE-F-774	3	4	E1_N1_O1	0:00	0:07	Move the cursor over diagrams	Point at a diagram

Table 6.3 (cont'd). DGE feature and DGE move code applications

DGE-F-775	3	4	E1_N1_O1	0:08	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-776	3	4	E1_N1_O1	0:10	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-777	3	4	E1_N1_O1	0:19	0:07	Move the cursor over diagrams	Not identified
DGE-F-778	3	4	E1_N1_O1	0:26	0:13	Drag a diagram	Guided dragging
DGE-F-779	3	4	E1_N1_O1	0:40	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-780	3	4	E1_N1_O1	0:41	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-781	3	4	E1_N1_O1	0:43	0:01	Select a diagram	Point at a diagram
DGE-F-782	3	4	E1_N1_O1	0:44	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-783	3	4	E1_N1_O1	0:46	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-784	3	4	E1_N1_O1	0:48	0:02	Move the cursor over diagrams	Point at a diagram
DGE-F-785	3	4	E1_N1_O1	0:50	0:05	Drag a diagram	Place a diagram
DGE-F-786	3	4	E1_N1_O1	1:02	0:02	Move the cursor over diagrams	Not identified
DGE-F-787	3	4	E1_N1_O1	1:13	0:21	Drag a diagram	Guided dragging
DGE-F-788	3	4	Y2_Y3	0:10	0:01	Select a diagram	Point at a diagram
DGE-F-789	3	4	Y2_Y3	0:11	0:01	Select a diagram	Point at a diagram
DGE-F-790	3	4	Y2_Y3	0:13	0:01	Select a diagram	Point at a diagram
DGE-F-791	3	4	Y2_Y3	0:14	0:03	Move the cursor over diagrams	Not identified
DGE-F-792	3	4	Y2_Y3	0:17	0:03	Move the cursor over diagrams	Point at a feature icon
DGE-F-793	3	4	Y2_Y3	0:24	0:04	De-select diagrams	Not identified
DGE-F-794	3	4	Y2_Y3	0:28	0:01	Select a diagram	Point at a diagram
DGE-F-795	3	4	Y2_Y3	0:34	0:01	Select a diagram	Point at a diagram
DGE-F-796	3	4	Y2_Y3	0:35	0:01	De-select diagrams	Not identified
DGE-F-797	3	4	Y2_Y3	0:40	0:01	Select a diagram	Point at a diagram
DGE-F-798	3	4	Y2_Y3	0:46	0:01	Select a diagram	Point at a diagram
DGE-F-799	3	4	Y2_Y3	0:49	0:05	Move the cursor over diagrams	Point at a diagram
DGE-F-800	3	4	Y2_Y3	1:11	0:06	Drag a diagram	Guided dragging
DGE-F-801	3	4	Y2_Y3	1:18	0:01	Select a diagram	Point at a diagram
DGE-F-802	3	4	Y2_Y3	1:22	0:01	Select a diagram	Point at a diagram
DGE-F-803	3	4	Y2_Y3	1:23	0:02	De-select diagrams	Not identified
DGE-F-804	3	4	Y2_Y3	1:25	0:02	Select a diagram	Point at a diagram
DGE-F-805	3	4	Y2_Y3	1:27	0:03	Drag the cursor	Point at a diagram
DGE-F-806	3	4	Y2_Y3	1:30	0:01	Select a diagram	Point at a diagram
DGE-F-807	3	4	Y2_Y3	1:31	0:02	Drag the cursor	Point at a diagram
DGE-F-808	3	4	Y2_Y3	1:34	0:04	Drag the cursor	Point at a diagram
DGE-F-809	3	4	Y2_Y3	1:39	0:02	Drag the cursor	Point at a diagram
DGE-F-810	3	4	Y2_Y3	1:43	0:08	Drag a diagram	Guided dragging
DGE-F-811	3	4	Y2_Y3	1:51	0:01	Drag a diagram	Place a diagram
DGE-F-812	3	4	Y2_Y3	1:53	0:01	Drag a diagram	Guided dragging
DGE-F-813	3	4	Y2_Y3	1:55	0:03	Move the cursor over diagrams	Point at a diagram
DGE-F-814	3	4	Y2_Y3	1:58	0:02	Drag a diagram	Not identified
DGE-F-815	3	4	Y2_Y3	2:02	0:05	Create a diagram	Create a diagram
DGE-F-816	3	4	Y2_Y3	2:08	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-817	3	4	Y2_Y3	2:09	0:02	Create a diagram	Create a diagram
DGE-F-818	3	4	Y2_Y3	2:11	0:05	Undo	Undo the last step
DGE-F-819	3	4	Y2_Y3	2:16	0:02	Undo	Undo the last step
DGE-F-820	3	4	Y2_Y3	2:18	0:02	Undo	Undo the last step
DGE-F-821	3	4	Y2_Y3	2:20	0:05	Drag a diagram	Dragging by accident
DGE-F-822	3	4	Y2_Y3	2:25	0:08	Drag a diagram	Guided dragging
DGE-F-823	3	4	Y2_Y3	2:39	0:05	Open a file	Change the model
DGE-F-824	3	4	Y2_Y3	2:44	0:01	Select a diagram	Point at a diagram
DGE-F-825	3	4	Y2_Y3	2:45	0:01	Select a diagram	Point at a diagram
DGE-F-826	3	4	Y2_Y3	2:46	0:01	Select a diagram	Point at a diagram
DGE-F-827	3	4	Y2_Y3	2:52	0:02	Select a diagram	Point at a diagram
DGE-F-828	3	4	Y2_Y3	2:54	0:01	Select a diagram	Point at a diagram
DGE-F-829	3	4	Y2_Y3	2:55	0:02	Select a diagram	Point at a diagram
DGE-F-830	3	4	Y2_Y3	2:57	0:08	Drag a diagram	Guided dragging
DGE-F-831	3	4	Y2_Y3	3:07	0:01	Select a diagram	Point at a diagram
DGE-F-832	3	4	Y2_Y3	3:08	0:01	Drag the cursor	Point at a diagram
DGE-F-833	3	4	Y2_Y3	3:09	0:01	Drag the cursor	Point at a diagram
DGE-F-834	3	4	Y2_Y3	3:10	0:01	Drag the cursor	Point at a diagram

Table 6.3 (cont'd). DGE feature and DGE move code applications

DGE-F-835	3	4	Y2_Y3	3:11	0:02	Drag the cursor	Point at a diagram
DGE-F-836	3	4	Y2_Y3	3:20	0:01	Select a diagram	Point at a diagram
DGE-F-837	3	4	Y2_Y3	3:21	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-838	3	4	Y2_Y3	3:22	0:01	Select a diagram	Point at a diagram
DGE-F-839	3	4	Y2_Y3	3:23	0:01	Move the cursor over diagrams	Point at a diagram
DGE-F-840	3	4	Y2_Y3	3:24	0:01	De-select diagrams	Not identified
DGE-F-841	3	4	Y2_Y3	3:25	0:02	De-select diagrams	Not identified
DGE-F-842	3	4	Y2_Y3	3:27	0:01	Select a diagram	Point at a diagram
DGE-F-843	3	4	Y2_Y3	3:32	0:06	Move the cursor over diagrams	Point at a diagram
DGE-F-844	3	4	Y2_Y3	3:38	0:04	Drag a diagram	Not identified
DGE-F-845	3	4	Y2_Y3	3:43	0:01	Open a file	Change the model
DGE-F-846	3	4	Y2_Y3	3:49	0:01	Open a file	Change the model
DGE-F-847	3	4	Y2_Y3	3:50	0:01	Open a file	Change the model
DGE-F-848	3	4	Y2_Y3	3:51	0:01	Open a file	Change the model
DGE-F-849	3	4	Y2_Y3	3:52	0:01	Select a diagram	Point at a diagram
DGE-F-850	3	4	Y2_Y3	3:56	0:02	Drag a diagram	Point at a diagram
DGE-F-851	3	4	Y2_Y3	3:58	0:02	Drag the cursor	Point at a diagram
DGE-F-852	3	4	Y2_Y3	4:00	0:01	Drag the cursor	Point at a diagram
DGE-F-853	3	4	Y2_Y3	4:01	0:02	Select a diagram	Point at a diagram
DGE-F-854	3	4	Y2_Y3	4:03	0:15	Select a diagram	Point at a diagram
DGE-F-855	3	4	Y2_Y3	4:18	0:01	Select a diagram	Point at a diagram
DGE-F-856	3	4	Y2_Y3	4:19	0:01	Select a diagram	Point at a diagram
DGE-F-857	3	4	Y2_Y3	4:20	0:02	Move the cursor over diagrams	Point at a diagram

APPENDIX F: Legendre's Attempted Proof of the Euclidean Parallel Postulate in Task 2

Below is Legendre's attempted proof of the other half of Euclidean Parallel Postulate. See if you can detect a flaw in the following proof and state the hidden assumption in the argument.

Given P not on line l . Drop perpendicular \overleftrightarrow{PQ} from P to l at Q . Let m be the line through P perpendicular to \overleftrightarrow{PQ} . Then m is parallel to l . Let n be any line through P distinct from m and \overleftrightarrow{PQ} . We must show that n meets l .

- 1) Let \overrightarrow{PR} be a ray of n between \overleftrightarrow{PQ} and a ray of m with endpoint P .
- 2) There is a point R' on the opposite side of \overleftrightarrow{PQ} from R such that $\angle QPR' \cong \angle QPR$.
- 3) Then Q lies in the interior of $\angle RPR'$.
- 4) Since line l passes through the point Q , l must intersect one of the sides of this angle.
- 5) If l meets \overrightarrow{PR} , then certainly l meets n .
- 6) Suppose l meets side $\overrightarrow{PR'}$ at a point A . Let B be the unique point on \overrightarrow{PR} such that $\overline{PA} \cong \overline{PB}$. Then $\triangle PQA \cong \triangle PQB$.
- 7) Hence $\angle PQB$ is a right angle, so that B lies on l (and n).

Figure 6.32. Legendre's attempted proof of the Euclidean Parallel Postulate in Task 2

APPENDIX G: Students' Written Responses in Task 2

Table 6.4. Students' written responses in Task 2

Avery, Kim
We found a hidden assumption in the original Legendre's construction through the omission of specific cases in which line n does not meet line l . This happens when the construction of line $PB=PA$ but ray PR doesn't intersect line l . Thus, the flaw lies in step 4 where we assume line l must intersect one side of the constructed angle $\angle RPR'$.
Bing, Eric
In Legendre's proof attempt, he says in step 6, "Suppose l meets side $(ray)PR'$ at point A ." But, in Klein's model, the lines are not infinitely long if we use Euclid's method of measurements rather than Klein's (because they are infinitely long in Klein's measurements), so as we saw in the construction, the ray PR' may or may not intersect line l due to the boundary of the Klein disc. The hidden assumption is that the ray PR' does intersect line l in Klein's model even though they may not. In Euclidean geometry, this assumption will hold for the lines do stretch to infinity, thus the lines must intersect before infinity. In step 4, it is assuming that since Q is in the interior of angle RPR' (step 3), and l passes through Q , l must intersect one of the sides of angle RPR' by Euclid's V postulate (sum of the angles is less than 180, on one side of the intersecting line). This goes back to the boundary issue where the Klein disc is limited to the circle and Euclidean geometry does not have such boundaries.
Deming, Tina
The assumption is that you can extend a line segment infinitely in any direction. This holds in Euclidean geometry because it is one of Euclid's 5 postulates and leads to the feature that lines and planes extend outward. This does not hold in hyperbolic geometry because of the feature of the boundary of the disk, which "stops" the lines. Even where distance is defined with a logarithmic function, the line would not extend beyond the boundary, so an intersection is not guaranteed.
Deshi, Yao
All of those are hold in both Euclidean and hyperbolic geometry excluding 4) because there exists another line that distinct with line m but still parallel to the line L because they will not intersect each other. So, when ray PR and PR' reach to those point, their end point which lie on the disk edge will above the endpoint of line l so they will not always intersect.
Naomi, Odessa
The flaw in the Legendre's Proof of EPP is that he based his proof on the assumption that the ray PR meets the line l . From our GEX construction, we can see that point R can be moved around so that Q still lies in the interior of $\angle RPR'$, but the line l does not intersect one of the sides of the angle (Step 4 fails). In addition, l does not always meet PR and thus, it does not always meet n (Step 5 fails). This assumption holds in Euclidean but not hyperbolic geometry because of the difference in the ways lines are defined in each of the models. In Euclidean geometry, lines are linear and two lines n and l that are not parallel will intersect indefinitely, no matter how n is moved around (as long as it is not parallel to l). So, this case where line l doesn't meet line n is never true in Euclidean geometry. In hyperbolic geometry, lines are on a sphere, so they act more like an arc. So, by the shape of them, the ends of the two lines could arc differently, and don't necessarily have to intersect.
Yin, Yong
In step 4, it says there are only 2 cases where line l must intersect either ray (PR') or ray (PR) ; however, we found that l might not intersect with both. In this case, there exists no triangle PQA or triangle PQB . Since they don't intersect, it means that they are all parallel to line l . And in Poincare model, there exists more than one parallel lines through a given point to the given line which is not the case in Euclidean geometry where there is only one. Hence, if the proof is in Euclidean geometry, it would be true.

APPENDIX H: Students' Written Responses in Task 3

Table 6.5. Students' written responses in Task 3

Avery, Kim
A limiting parallel line is a line comes arbitrarily close to the line you are making it parallel to without touching the line it is parallel to. We do believe that limiting parallel lines are applicable in any model of hyperbolic geometry because all models of hyperbolic geometry have a boundary circle. Limiting parallel lines intersect line l on the boundary circle. However, because the boundary circle is not part of our system, the lines are parallel to line l .
Bing, Eric, Jeremy
Q3: The limiting parallel lines in the Klein model basically acts as the boundaries of the area for point R where ray PR does not intersect line l . We can apply this similar description to other types of hyperbolic geometries for in hyperbolic geometry, there are no unique parallel lines, there are multiple parallel lines. (Q2: In other words, let us have a point M on line m on the same side of the perpendicular line to R and have a point N on the limiting parallel line on the same side of line m to R , then we can say if R is in the interior of angle NPM , then the proof fails.) Q3 (Resubmission): Let r be the limit for the angle measure PQR where R is a point on line l . If r exists, then that angle defines the limiting parallel line.
Ben, Tina
The limiting parallel lines are the limit for the lines RP and $R'P$. As they move toward the edges of line l from the interior of the limiting parallels, the limiting parallels are the boundary at which RP and $R'P$ no longer intersect the original line l . If the boundary circle were to exist within the Klein disk, line l and the limiting parallels would intersect on the boundary circle. We think this description is applicable in other models of hyperbolic geometry. For example, in the Poincare model we still have a boundary circle, we still have the possibility of multiple parallel lines, however the method of angle measurement is different. Therefore, we could use the adapted method of angle measurement, with the same description of limiting parallels.
Deshi, Yao
According to the graph, we can see the line that parallel to the given line is the closest parallel line to it, because the potential intersection point is when we go one step further, it will intersect with the given line. So that's sort of the limiting parallel line in the Klein model, and I think the description is applicable in any models of hyperbolic geometry, since for those two lines, they formed a normal triangle, so it will never intersect with each other as long as they are triangle, no matter what model we use, the edge of triangle would never intersect.
Emily, Naomi, Odessa
The limiting parallel lines in the Klein model shows the two closest line to l through the point P , that does not intersect the line l . This being said, the limiting parallel lines creates a boundary for Legendre's argument to be true. Every ray in the interior of the angle will meet the line l , and every ray in the exterior will not meet the line l . This exists in other hyperbolic geometry because there is always a boundary, so there are only 2 limiting parallel lines that are the closest to the given line without intersecting it (as the definition of a parallel line states).
Yin, Yong
The limiting parallel lines in Klein model are intersecting the original line l on the disk (not inside), which means they intersect l at the same point they would intersect the border of the model. So, they are not technically intersecting the original line l in the disk. In a hyperbolic model, same thing would happen when clicking a point and a line a for parallel line. The limiting parallel lines also intersect the line a on the disk, i.e. intersect at a point where the border, line a and the limiting parallel lines are concurrent. So, the description is applicable in this model as well.

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